On Spectral Properties of the Grounded Laplacian Matrix

by

Mohammad Pirani

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Linear consensus and opinion dynamics in networks that contain stubborn agents are studied in this thesis. Previous works have shown that the convergence rate of such dynamics is given by the smallest eigenvalue of the grounded Laplacian induced by the stubborn agents. Building on those works, we study the smallest eigenvalue of grounded Laplacian matrices, and provide bounds on this eigenvalue in terms of the number of edges between the grounded nodes and the rest of the network, bottlenecks in the network, and the smallest component of the eigenvector for the smallest eigenvalue. We show that these bounds are tight when the smallest eigenvector component is close to the largest component, and provide graph-theoretic conditions that cause the smallest component to converge to the largest component. An outcome of our analysis is a tight bound for Erdos-Renyi random graphs and d-regular random graphs. Moreover, we define a new notion of centrality for each node in the network based upon the smallest eigenvalue obtained by removing that node from the network. We show that this centrality can deviate from other well known centralities. Finally we interpret this centrality via the notion of absorption time in a random walk on the graph.

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Chapter 1

Introduction

Collective behavior in networks of agents has been studied in a variety of communities including sociology, physics, biology, economics, computer science and engineering [15, 49]. There has been a great deal of research over the past several decades dedicated to the study of the structure and dynamics of networks. These investigations span multiple disciplines, and adopt diverse tools and perspectives, including combinatorial, probabilistic, game-theoretic, and algebraic approaches [5, 23, 27, 29]. A topic that has received particular interest is that of opinion dynamics and consensus in networks, where the agents repeatedly update their opinions or states via interactions with their neighbors [4, 13, 30]. For certain classes of interaction dynamics, various conditions have been provided on the network topology that guarantee convergence to a common state [33, 40, 47, 48].

It has been recognized that the *spectra* of graphs (i.e., the eigenstructure associated with certain matrix representations of the network) provide insights into both the topological properties of the underlying network and dynamical processes occurring on the network [7, 12]. The eigenvalues and eigenvectors of the Laplacian matrix of the graph, for example, contain information about the connectivity and community structure of the network [18, 34, 35, 38], and dictate the convergence properties of certain diffusion dynamics [40].

Aside from providing conditions under which convergence occurs, the question of what value the agents converge to is also of importance. In particular, the ability of certain individual agents in the network to excessively influence the final value can be viewed as both a benefit and a drawback, depending on whether those agents are viewed as leaders or adversaries. The effect of individual agents' initial values on the final consensus value has been studied in [24] [50]. When a subset of agents is fully stubborn (i.e., they refuse to update their value), it has been shown that under a certain class of linear update rules, the values of all other agents asymptotically converge to a convex combination of the stubborn agent's values [21]. Given the ability of individuals to influence linear opinion dynamics by keeping their values constant, a natural metric to consider is the speed at which the population converges to its final state for a given set of stubborn agents or leaders. The convergence rate is dictated by spectral properties of certain matrices; in continuous-time dynamics, this is the *grounded Laplacian* matrix [2]. There are various recent works that investigate the *leader selection* problem, where the goal is to select a set of leaders (or stubborn agents) to maximize the convergence rate [9, 10, 21, 44]. Similarly, one can consider the problem of leader selection in networks in the presence of noise [42], where the main goal is to minimize the steady state error covariance of the followers [16].

1.1 Laplacian Matrix

The graph Laplacian matrix is widely used in the analysis of synchronization dynamics in networked systems. The physical interpretation of the Laplacian matrix in those dynamics is that the state of each node (agent) evolves as a function of the differences between its current state and the states of its neighbors. There are variants of the Laplcian matrix such as the *normalized Laplacian matrix* [8]; in this thesis we concentrate on a special variant of the Laplacian matrix which we discuss next.

1.1.1 Grounded Laplacian Matrix

A variant of the Laplacian that has attracted attention in recent years is the *grounded Laplacian* or *Dirichlet Laplacian* matrix, obtained by removing certain rows and columns from the Laplacian. The spectral properties of the grounded Laplacian matrix were first studied in continuous-time diffusion dynamics, where the states of some of the nodes in the network are fixed at certain values, called *Dirichlet* boundary conditions. The grounded Laplacian forms the basis for the classical Matrix Tree Theorem (characterizing the number of spanning trees in the graph). The eigenvalues of the grounded Laplacian characterize the variance in the equilibrium values for noisy instances of such dynamics, and determine the rate of convergence to steady state [9, 42]. Recent years have seen the development of optimization algorithms to select "leader nodes" in the network in order to minimize the steady-state variance, to maximize the rate of convergence, or to quantify the effect of stubborn individuals in discrete-time versions of such dynamics [2, 9, 10, 16, 19, 21, 43, 44]. The notion of the grounded Laplacian matrix was introduced In [36] with some bounds on the elements of the inverse of this matrix. The elements of the inverse of the grounded

Laplacian have the following physical interpretation [45]. If we apply a one Ohm resistor to each edge of a given graph, the equivalent resistance between two nodes in the graph is called the *resistance distance* between those two nodes. The i-th diagonal element of the inverse of the grounded Laplacian matrix is the resistance distance between v_i and the grounded vertex [1]. The *effective resistance* of a vertex, or a set of vertices, in a network is the trace of the inverse of the grounded Laplacian matrix where those vertices are grounded, and is widely used in the literature on distributed control and estimation. For example, It is used in least-square estimation problems in trying to reconstruct global information from relative noisy measurements, and in the problem of formation control of multi-agent systems [2]. It has also been widely used in electrical network analysis and in the leader selection problem in stochastically forced consensus networks [16], [2], [22]. Another important notion which is derived from the inverse of the grounded Laplacian matrix is the *Kirchhoff index* which measures the sum of the resistance distances between each pair of nodes in the graph. As compared to the *Wiener index* (which is the sum of the topological distances (shortest paths) between each pair of nodes in the graph) the Kirchhoff index considers all of the possible paths between two nodes in the graph, which is useful in modeling random spreading effects in real world networks.

1.2 Related Works

Despite the large amount of work on the study of collective dynamics and consensus models, investigations of leader selection in networks for obtaining desired behavior is relatively new. There are various objectives that are desired to be obtained via the leader selection task. Among them, minimizing the convergence time [21] [9] [44] and minimizing the mean square deviation from consensus in stochatically forced networks [16] [19] are more common. Since both of the two leader selection problems deal with the spectrum of the grounded Laplacian matrix, we will do a brief literature review of the recent work on those topics.

Hao, Barooah and Veerman [26] discussed bounds for the smallest eigenvalue of the grounded Laplacian matrix for D-dimensional lattices. Barooah and Hespanha [2] introduced the notion of matrix valued effective resistance and discussed its applications in distributed control and estimation and provided some bounds for eigenvalues of grounded Laplacian matrices. Paterson and Bamieh [42], Fardad et al., [16] and Clark et al., [10] discussed selecting a set of leaders to minimize the steady state error covariance of the follower agents. Moreover they investigated the effect of increasing the number of leaders on this problem. In addition [16] analyzed the case when the leader is offended by noise

(noise corrupted leader selection) and the case of noise free leaders. Fitch and Leonard [19] showed that for the cases of a single leader or two leaders, the leaders minimizing the steady state error covariance are information central vertices. Ghaderi and Srikant [21] studied the effect of stubborn agents (with full or partial stubbornness) and the influence of the network structure on the rate of convergence in these networks. Clark et al., [9] discussed the leader selection problem in minimizing the convergence time. They provided an interpretation of the convergence time in terms of mixing time in a random walk on the graph.

1.3 Contributions of This Thesis

The major contribution of this thesis is characterizing the smallest eigenvalue of the grounded Laplacian matrices. This eigenvalue determines the rate of convergence in networks in the presence of stubborn agents. We provide graph-theoretic bounds on the smallest eigenvalue based on the number of edges leaving the grounded nodes, bottlenecks in the graph, and properties of the eigenvector associated with the eigenvalue. Our bounds become tighter as this eigenvector becomes more uniform; we provide bounds on the gap between the smallest and largest components of the eigenvector, leading to tight bounds for certain classes of graphs. In particular, our results allow us to characterize the smallest eigenvalue of the grounded Laplacian matrix for Erdos-Renyi random graphs, and provide scaling laws for the smallest eigenvalue in random regular graphs.

In addition to the above mentioned results, a contribution of this thesis is to introduce a new centrality metric, denoted by *grounding centrality*, which maximizes the convergence rate in networks in the presence of stubborn agents (leaders), and compare it with other well known centrality metrics. In particular we show that in certain graphs this centrality can arbitrarily deviate from other centralities. We provide a sufficient condition which certifies that grounding central vertex asymptotically converges to the vertex with maximum degree. Moreover we discuss an interpretation of grounding centrality based on absorption time in a random walk on a graph.

Chapter 2

Model

2.1 Notation

We use $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ to denote an undirected graph where $\mathcal{V}$ is the set of nodes (or vertices) and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. We will denote the number of vertices by n. The *neighbors* of node $v_i \in \mathcal{V}$ in graph $\mathcal{G}$ are given by the set $\mathcal{N}_i = \{v_j \in \mathcal{V} \mid (v_i, v_j) \in \mathcal{E}\}$. The *degree* of node v_i is $d_i = |\mathcal{N}_i|$, and the minimum and maximum degrees of the nodes in the graph will be denoted by d_{min} and d_{max}, respectively. If $d_{max} = d_{min} = d$, the graph is said to be d-regular. For a given set of nodes $S \subset \mathcal{V}$, the edge-boundary (or just boundary) of the set is given by $\partial S = \{(v_i, v_j) \in \mathcal{E} \mid v_i \in S, v_j \in \mathcal{V} \setminus S\}$. The isoperimetric constant of $\mathcal{G}$ is given by [7]

$$i(\mathcal{G}) \triangleq \min_{A \subset \mathcal{V}, |A| \leq \frac{n}{2}} \frac{|\partial A|}{|A|}.$$

Choosing A to be the vertex with the smallest degree yields the bound $i(\mathcal{G}) \leq d_{min}$.

We say a vertex $v_i \in \mathcal{V} \setminus \mathcal{S}$ is an α-vertex if $\mathcal{N}_i \cap \mathcal{S} \neq \emptyset$, and say v_i is a β-vertex otherwise.

2.2 Laplacian and Grounded Laplacian Matrices

The *adjacency matrix* for the graph is a matrix $A \in \{0, 1\}^{n \times n}$, where entry (i, j) is 1 if $(v_i, v_j) \in \mathcal{E}$, and zero otherwise. The *Laplacian matrix* for the graph is given by $L = D - A$, where D is the degree matrix with $D = diag(d_1, d_2, \ldots, d_n)$. For an undirected graph $\mathcal{G}$, the

Laplacian L is a symmetric matrix with real eigenvalues that can be ordered sequentially as $0 = \lambda_1(L) \leq \lambda_2(L) \leq \cdots \leq \lambda_n(L) \leq 2d_{max}$. The second smallest eigenvalue $\lambda_2(L)$ is termed the *algebraic connectivity* of the graph and satisfies the bound [7]

$$\lambda_2(L) \geq \frac{i(\mathcal{G})^2}{2d_{max}}. \tag{2.1}$$

We will designate a nonempty subset of vertices $\mathcal{S} \subset \mathcal{V}$ to be *grounded nodes*, and assume without loss of generality that they are placed last in the ordering of the nodes. We use α_i to denote the number of grounded nodes that v_i is connected to (i.e., $\alpha_i = |\mathcal{N}_i \cap \mathcal{S}|$). Removing the rows and columns of L corresponding to the grounded nodes S produces a *grounded Laplacian matrix* denoted by $L_g(S)$. When the set S is fixed and clear from the context, we will simply use the notation L_g to denote the grounded Laplacian. If $S = \{v_s\}$ we use the notation $L_g(s)$ to indicate the grounded Laplacian for the single grouded node v_s. For any given set S, we denote the smallest eigenvalue of the grounded Laplacian matrix by $\lambda(L_g(S))$ or simply λ.

When the graph $\mathcal{G}$ is connected, the grounded Laplacian matrix is a positive definite matrix and its inverse is a nonnegative matrix (i.e., a matrix whose elements are nonnegative) [36]. From the Perron-Frobenius (P-F) theorem, the eigenvector associated with the smallest eigenvalue of the grounded Laplacian can be chosen to be nonnegative (element-wise). Furthermore, when the grounding nodes do not form a vertex cut, the eigenvector associated with the smallest eigenvalue can be chosen to have all elements positive.

2.3 Applications to Consensus with Stubborn Agents

Consider a multi-agent system described by the connected and undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ representing the structure of the system, and a set of equations describing the interactions between each pair of agents. In the study of consensus and opinion dynamics [40], each agent $v_i \in \mathcal{V}$ starts with an initial scalar state (or opinion) $y_i(t)$, which evolves over time as a function of the states of its neighbors. A commonly studied version of these dynamics involves a continuous-time linear update rule of the form

$$\dot{y}_i(t) = \sum_{v_j \in \mathcal{N}_i} (y_j(t) - y_i(t)).$$

Aggregating the state of all of the nodes into the vector $Y(t) = \begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \end{bmatrix}^T$, the above equation produces the system-wide dynamical equation

$$\dot{Y} = -LY, \tag{2.2}$$

where L is the graph Laplacian. When the graph is connected, the trajectory of the above dynamical system satisfies $Y(t) \rightarrow \frac{1}{n}\mathbf{1}\mathbf{1}^T Y(0)$ (i.e., all agents reach consensus on the average of the initial values), and the asymptotic convergence rate is given by $\lambda_2(L)$ [40].

Now suppose that there is a subset $S \subset \mathcal{V}$ of agents whose opinions are kept constant throughout time, i.e., $\forall v_s \in S$, $\exists y_s \in \mathbb{R}$ such that $y_s(t) = y_s$ $\forall t \in \mathbb{R}_{\geq 0}$. Such agents are known as *stubborn agents* or *leaders* (depending on the context) [9, 21]. In this case the dynamics (2.2) can be written in the matrix form

$$\begin{bmatrix} \dot{Y}_F(t) \\ \dot{Y}_S(t) \end{bmatrix} = -\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} Y_F(t) \\ Y_S(t) \end{bmatrix}, \tag{2.3}$$

where Y_F and Y_S are the states of the followers and stubborn agents, respectively. Since the stubborn agents keep their values constant, the matrices L_{21} and L_{22} are zero. Thus, the matrix L_{11} is the grounded Laplacian for the system, i.e., $L_{11} = L_g(S)$. It can be shown that the state of each follower asymptotically converges to a convex combination of the values of the stubborn agents and that the rate of convergence is asymptotically given by λ, the smallest eigenvalue of the grounded Laplacian [9].

Similarly, one can consider discrete-time consensus dynamics (also known as DeGroot dynamics) with a set S of stubborn nodes, given by the update equation

$$Y_F(t+1) = A_g Y_F(t), \tag{2.4}$$

where $Y_F(t)$ is the state vector for the non-stubborn nodes at time-step t, and A_g is an $(n - |S|) \times (n - |S|)$ nonnegative matrix of the form

$$A_g(i,j) = \begin{cases} 1 - \frac{d_i}{k} & \text{if } i = j \\ \frac{1}{k} & \text{if } (v_i, v_j) \in \mathcal{E} \\ 0 & \text{otherwise,} \end{cases}$$

with constant $k \in (d_{max}, \infty)$ [31]. It is easy to see that $A_g = I - \frac{1}{k}L_g$, and once again, each non-stubborn node will converge asymptotically to a convex combination of the stubborn nodes' states. The largest eigenvalue of A_g is given by $\lambda_{max}(A_g) = 1 - \frac{1}{k}\lambda(L_g)$, and determines the asymptotic rate of convergence. Since the eigenvector corresponding to $\lambda_{max}(A_g)$ is the same as the eigenvector for $\lambda(L_g)$, our bounds on the smallest eigenvalue of the grounded Laplacian will readily translate to bounds on the largest eigenvalue of A_g.

As discussed in Section 1.1 , there have been various recent investigations of graph properties that impact the convergence rate for a given set of stubborn agents, leading to the development of algorithms to find approximately optimal sets of stubborn/leader agents to maximize the convergence rate [9, 21, 44]. The bounds provided in this thesis contribute to the understanding of consensus dynamics with fixed opinions by providing bounds on the convergence rate induced by any given set of stubborn or leader agents.

2.4 Grounding Centrality

There are various metrics for evaluating the importance of individual nodes in a network. Common examples include eccentricity (the largest distance from the given node to any other node), closeness centrality (the sum of the distances from the given node to all other nodes in the graph), degree centrality (the degree of the given node) and betweenness centrality (the number of shortest paths between all nodes that pass through the given node) [6] [39]. In addition to the above centrality metrics (which are purely based on position in the network), one can also derive centrality metrics that pertain to certain classes of dynamics occurring on the network. For example, [32] assigned a centrality score to each node based on its component in a left-eigenvector of the system matrix. Similarly, [17] studied discrete time consensus dynamics and proposed centrality metrics to capture the influence of forceful agents. The discussion on convergence rate induced by each node in the last section also lends itself to a natural dynamical centrality metric, defined as follows.

Definition 1 *Consider a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. The grounding centrality of each vertex $v_s \in \mathcal{V}$, denoted by $G(s)$, is $G(s) = \lambda(L_g(s))$. The set of grounding central vertices in the graph $\mathcal{G}$ is given by $GC(\mathcal{G}) = \operatorname{argmax}_{v_s \in \mathcal{V}} \lambda(L_g(s))$.*

According to the above definition, a grounding central vertex $v_s \in GC(\mathcal{G})$ is a vertex that maximizes the asymptotic convergence rate if chosen as a stubborn agent (or leader), over all possible choices of single stubborn agents.

It was shown in [21] and [44] that the convergence time in a network in the presence of stubborn agents (or leaders) is upper bounded by an increasing function of the distance from the stubborn agents to the rest of the network. In the case of a single stubborn agent, the notion of distance from that agent to the rest of the network is similar to that of closeness centrality and eccentricity. While this is a natural approximation for the grounding centrality (and indeed plays a role in the upper bounds provided in those papers), there are graphs where the grounding centrality can deviate from other well known centralities, as shown below.

Example 1 *A broom tree, $B_{n,\Delta}$, is a star S_Δ with Δ leaf vertices and a path of length $n - \Delta - 1$ attached to the center of the star, as illustrated in Fig. 2.1 [46].*

Define the closeness central vertex as a vertex whose summation of distances to the rest of the vertices is minimum, the degree central vertex as a vertex with maximum degree in *the graph and the center of the graph as a vertex with smallest eccentricity [39]. Consider the broom tree $B_{2\Delta+1,\Delta}$. By numbering the vertices as shown in Fig. 2.1, for $\Delta = 500$, we find (numerically) that the grounding central vertex is vertex 614, and the center of the graph is 750. The closeness and degree and betweenness central vertices are located at the middle of the star (vertex 501). The deviation of the grounding central vertex from the other centralities and the center of this graph increases as Δ increases.*

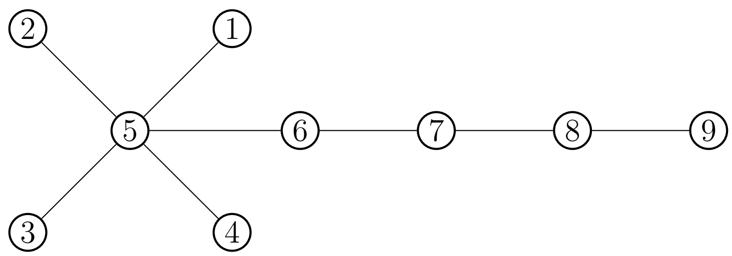

Figure 2.1: Broom tree with $\Delta = 4$, $n = 9$.

As discussed above, the problem of characterizing the grounding centrality of vertices using graph-theoretic properties is an ongoing area of research [2,9,21,44]. In the following chapter, we develop some bounds for $\lambda(L_g)$ by studying certain spectral properties of the grounded Laplacian.

Chapter 3

Bounds on the Smallest Eigenvalue of the Grounded Laplacian Matrix

The following theorem provides our core bounds on the smallest eigenvalue of the grounded Laplacian; in subsequent sections, we will characterize graphs where these bounds become tight.

Theorem 1 *Consider a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with a set of grounded nodes $S \subset \mathcal{V}$. Let λ denote the smallest eigenvalue of the grounded Laplacian L_g and let $\mathbf{x}$ be the corresponding nonnegative eigenvector, normalized so that the largest component is $x_{max} = 1$. Then*

$$\frac{|\partial S|}{n - |S|} x_{min} \leq \lambda \leq \min_{X \subseteq \mathcal{V} \setminus S} \frac{|\partial X|}{|X|} \leq \frac{|\partial S|}{n - |S|}, \tag{3.1}$$

where x_{min} is the smallest eigenvector component in $\mathbf{x}$.

Proof. From the Rayleigh quotient inequality [28], we have

$$\lambda \leq z^T L_g z, \tag{3.2}$$

for all $z \in \mathbb{R}^{n-|S|}$ with $z^T z = 1$. Let $X \subseteq \mathcal{V} \setminus S$ be the subset of vertices for which $\frac{|\partial X|}{|X|}$ is minimum, and assume without loss of generality that the vertices are arranged so that those in set X come first in the ordering. The upper bound $\min_{X \subseteq \mathcal{V} \setminus S} \frac{|\partial X|}{|X|}$ is then obtained by choosing $z = \frac{1}{\sqrt{|X|}} [\mathbf{1}_{1 \times |X|} \quad \mathbf{0}_{1 \times |\mathcal{V} \setminus S \cup X|}]^T$, and noting that the sum of all elements in the top $|X| \times |X|$ block of L_g is equal to the sum of the number of neighbors each vertex in X

has outside X (i.e., $|\partial X|$). The upper bound $\frac{|\partial S|}{n-|S|}$ readily follows by choosing the subset $X = \mathcal{V} \setminus S$.

For the lower bound, we left-multiply the eigenvector equation $L_g\mathbf{x} = \lambda\mathbf{x}$ by the vector consisting of all 1's to obtain

$$\sum_{i=1}^{n-|S|} \alpha_i x_i = \lambda \sum_{i=1}^{n-|S|} x_i, \tag{3.3}$$

where α_i is the number of grounded nodes in node v_i's neighborhood. Using the fact that the eigenvector is nonnegative, this gives

$$x_{min} \sum_{i=1}^{n-|S|} \alpha_i \leq \sum_{i=1}^{n-|S|} \alpha_i x_i = \lambda \sum_{i=1}^{n-|S|} x_i \leq \lambda(n-|S|)x_{max} = \lambda(n-|S|).$$

Since $\sum_{i=1}^{n-|S|} \alpha_i = |\partial S|$, the lower bound is obtained.

Remark 1 *For the case that* $|S| = 1$ *we have*

$$\frac{d_s x_{min}}{n-1} \leq \lambda \leq \frac{d_s}{n-1}, \tag{3.4}$$

where d_s *is the degree of the grounded node. Note that the smallest eigenvalue of the grounded Laplacian for* k *grounded nodes is always upper bounded by* k *(since* $|\partial S| \leq |S|(n-|S|)$*), with equality if and only if all grounded nodes connect to all other nodes (it is easy to see that the smallest eigenvector component* $x_{min} = 1$ *in this case).*

Example 2 *Consider the graph shown in Figure 3.1 consisting of two complete graphs on* $\frac{n}{2}$ *nodes, joined by a single edge. Suppose the black node in the figure is chosen as the grounded node. In this case, we have* $|\partial S| = \frac{n}{2} - 1$*, and the extreme upper bound in (3.1) indicates that* $\lambda \leq \frac{|\partial S|}{n-1} \approx \frac{1}{2}$ *for large* n*. Now, if we take* X *to be the set of all nodes in the left clique, we have* $|\partial X| = 1$ *and* $|X| = \frac{n}{2}$*, leading to* $\lambda \leq \frac{2}{n}$ *by the intermediate upper bound in (3.1).*

In the next section, we will characterize graphs under which x_{min} (the smallest eigenvector component) converges to 1, in which case the lower and upper bounds in (3.1) coincide and yield a tight characterization of λ. As seen in the above example, the presence of bottlenecks among the non-grounded nodes will cause x_{min} to go to zero; in certain graphs with good expansion properties, however, we will see that this will not occur.

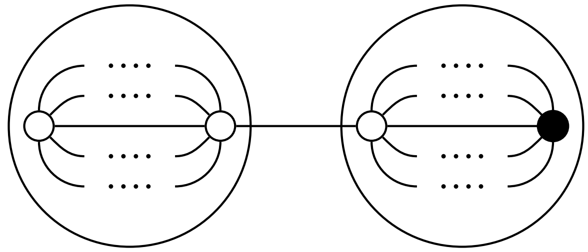

Figure 3.1: Two complete graphs, each with $\frac{n}{2}$ nodes, connected via a single edge. The grounded node is colored black.

3.1 The Behavior of the Smallest Eigenvector Component

In this section, we analyze the effect of the network structure on the behavior of the smallest eigenvector component x_{min}. We will provide conditions under which this component goes to 1 and stays bounded away from 0, respectively. This will then allow us to characterize the tightness of the bounds on the smallest eigenvalue in (3.1).

For a given subset $S \subset \mathcal{V}$ of grounded nodes, let $L_g(S)$ be the grounded Laplacian matrix with smallest eigenvalue λ and corresponding nonnegative eigenvector $\mathbf{x}$. We denote the i-th element of $\mathbf{x}$ by x_i. We write $L_g = \bar{L}+E$ where $\bar{L}$ is the $(n-|S|)\times(n-|S|)$ Laplacian matrix of the graph when we remove the grounded nodes and all of their incident edges. We assume that the graph corresponding to $\bar{L}$ is connected and denote the eigenvalues of $\bar{L}$ by $0 = \lambda_1(\bar{L}) < \lambda_2(\bar{L}) \leq ... \leq \lambda_{n-|S|}(\bar{L})$, with corresponding orthogonal eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-|S|}$. We take $\mathbf{v}_1 = \mathbf{1}$, and normalize all of the other eigenvectors so that $\|\mathbf{v}_i\| = 1$. Matrix E is a $(n-|S|) \times (n-|S|)$ diagonal matrix with the i-th diagonal element equal to α_i (the number of grounded neighbors of node v_i). There are various results in the literature that characterize the change in eigenvectors under modifications of matrix elements, including the commonly used Davis-Kahan theorems (which provide bounds on the angle between the original and perturbed eigenvectors) [14]. However, such bounds on the angle are not particularly useful in characterizing the behavior of the smallest component of the perturbed eigenvector.[1] We thus provide the following perturbation result bounding the smallest eigenvector component of $\mathbf{x}$ in terms of the

[1]For example, consider two $n \times 1$ vectors, the first of which consists of all entries equal to 1, and the second which has $n-1$ entries equal to 1 and the last component equal to 0. The angle between these two vectors goes to 0 as n increases, but the smallest component of the second vector is always 0.

number of grounded nodes, the number of edges they have to the other nodes, and the connectivity of the graph induced by the non-grounded nodes. The proof of the lemma starts in a similar manner to the proof of standard perturbation results [14], but the latter half of the proof leverages the explicit nature of the perturbations to obtain a bound on the smallest eigenvector component (i.e., this result can be viewed as providing a bound on the ∞-norm of the difference between the original and perturbed eigenvectors, as opposed to a bound on the angle between the vectors).

Lemma 1 *Let $\bar{L}$ be an $(n-|S|)\times(n-|S|)$ Laplacian matrix and $E = diag\left(\alpha_1, \alpha_2, \ldots, \alpha_{n-|S|}\right)$, where $0 \le \alpha_i \le |S|$ for all $1 \le i \le n-|S|$. Let $\mathbf{x}$ be the nonnegative eigenvector corresponding to the smallest eigenvalue of $L_g = \bar{L} + E$ normalized so that $||\mathbf{x}||_\infty = 1$. Then the smallest eigenvector component of $\mathbf{x}$ satisfies*

$$x_{min} \ge 1 - \frac{2\sqrt{|S||\partial S|}}{\lambda_2(\bar{L})}, \tag{3.5}$$

where $|\partial S| \triangleq \sum_{i=1}^{n-|S|} \alpha_i$.

Proof. The eigenvector equation for L_g is given by

$$L_g\mathbf{x} = (\bar{L} + E)\mathbf{x} = \lambda\mathbf{x}. \tag{3.6}$$

Project the eigenvector $\mathbf{x}$ onto the subspace spanned by $\mathbf{v}_1$ to obtain $\mathbf{x} = \gamma\mathbf{1} + \mathbf{d}$, where $\mathbf{d}$ is orthogonal to $\mathbf{v}_1$ and $\gamma = \frac{\mathbf{1}^T\mathbf{x}}{n-|S|}$. Thus we can write

$$\mathbf{d} = \sum_{i=2}^{n-|S|} \delta_i \mathbf{v}_i, \tag{3.7}$$

for some real numbers $\delta_2, \delta_3, \ldots, \delta_{n-|S|}$. Substituting into (3.6) and rearranging gives

$$\bar{L}\mathbf{d} = \underbrace{(\lambda I - E)\mathbf{x}}_{\triangleq \mathbf{z}}, \tag{3.8}$$

Multiplying both sides of (3.8) by $\mathbf{1}^T$ yields $0 = \mathbf{1}^T\mathbf{z}$, and thus $\mathbf{z}$ is also orthogonal to $\mathbf{v}_1$. Writing $\mathbf{z} = \sum_{i=2}^{n-|S|} \varphi_i \mathbf{v}_i$ for some constants $\varphi_2, \varphi_3, \ldots, \varphi_{n-|S|}$ and substituting this and (3.7) into (3.8), we have

$$\bar{L}\mathbf{d} = \sum_{i=2}^{n-|S|} \delta_i \bar{L}\mathbf{v}_i = \sum_{i=2}^{n-|S|} \delta_i \lambda_i(\bar{L})\mathbf{v}_i = \sum_{i=2}^{n-|S|} \varphi_i \mathbf{v}_i, \tag{3.9}$$

which gives $\delta_i = \frac{\varphi_i}{\lambda_i(\bar{L})}$ by the linear independence of the eigenvectors $\mathbf{v}_2, \dots, \mathbf{v}_{n-|S|}$. Thus we can write $\mathbf{d} = \sum_{i=2}^{n-|S|} \frac{\varphi_i}{\lambda_i(\bar{L})}\mathbf{v}_i$ with 2-norm given by

$$\|\mathbf{d}\|^2 = \sum_{i=2}^{n-|S|} \left(\frac{\varphi_i}{\lambda_i(\bar{L})}\right)^2 \leq \frac{1}{\lambda_2(\bar{L})^2} \sum_{i=2}^{n-|S|} \varphi_i^2 = \frac{||\mathbf{z}||^2}{\lambda_2(\bar{L})^2}. \tag{3.10}$$

From the definition of $\mathbf{z}$ in (3.8), we have

$$\begin{aligned} \|\mathbf{z}\|^2 &= \sum_{i=1}^{n-|S|} (\lambda - \alpha_i)^2 x_i^2 \leq \sum_{i=1}^{n-|S|} (\lambda - \alpha_i)^2 \\ &= (n - |S|)\lambda^2 - 2\lambda|\partial S| + \sum_{i=1}^{n-|S|} \alpha_i^2. \end{aligned}$$

Applying (3.1), $|\partial S| \leq |S|(n - |S|)$, and the fact that $\alpha_i \leq |S|$ for all $1 \leq i \leq n - |S|$, we obtain

$$\|\mathbf{z}\|^2 \leq (n - |S|)\frac{|\partial S|^2}{(n - |S|)^2} - 2\lambda|\partial S| + |S||\partial S| \leq 2|S||\partial S|.$$

Combining this with (3.10) yields

$$\|\mathbf{d}\|^2 \leq \frac{2|S||\partial S|}{\lambda_2(\bar{L})^2}. \tag{3.11}$$

Next, from $\mathbf{d} = \mathbf{x} - \gamma\mathbf{1}$ we have

$$\|\mathbf{d}\|^2 \geq (x_{max} - \gamma)^2 + (\gamma - x_{min})^2 = (1 - \gamma)^2 + (\gamma - x_{min})^2. \tag{3.12}$$

The right hand side of (3.12) achieves its minimum when $\gamma = \frac{1+x_{min}}{2}$. Substituting this value and rearranging gives

$$x_{min} \geq 1 - \sqrt{2}\|d\| \geq 1 - \frac{2\sqrt{|S||\partial S|}}{\lambda_2(\bar{L})},$$

as required.

The above result, in conjunction with Theorem 1, allows us to characterize graphs where the bounds in (3.1) become asymptotically tight.

Theorem 2 *Consider a sequence of connected graphs $\mathcal{G}_n$, $n \in \mathbb{Z}_+$, where n indicates the number of nodes. Consider an associated sequence of grounded nodes S_n, $n \in \mathbb{Z}_+$. Let $\bar{L}_n$ denote the Laplacian matrix induced by the non-grounded nodes in each graph $\mathcal{G}_n$, and let λ_n denote the smallest eigenvalue of the grounded Laplacian for the graph $\mathcal{G}_n$ with grounded set S_n. Then:*

1. *If* $\limsup_{n\to\infty} \frac{2\sqrt{|S_n||\partial S_n|}}{\lambda_2(\bar{L}_n)} < 1$, *then* $\lambda_n = \Theta\left(\frac{|\partial S_n|}{n-|S_n|}\right)$.

2. *If* $\lim_{n\to\infty} \frac{\sqrt{|S_n||\partial S_n|}}{\lambda_2(\bar{L}_n)} = 0$, *then* $(1-o(1))\frac{|\partial S_n|}{n-|S_n|} \leq \lambda_n \leq \frac{|\partial S_n|}{n-|S_n|}$.

In the next sections, we will apply this result to study the smallest eigenvalue of the grounded Laplacian of Erdos-Renyi and d-regular random graphs.

3.2 Some Properties of the Eigenvector Corresponding to $\lambda(L_g)$

In this section we provide some properties of the eigenvector elements corresponding to $\lambda(L_g)$ when there is a grounded node. We will refer to the neighbours of the grounded node as α-vertices and all other non-grounded nodes as β-vertices. In the following proposition we show that the eigenvector component corresponding to a β-vertex is bigger than the average value of the eigenvector components of its neighbors. Moreover we show that this is reversed for an α-vertex.

Proposition 1 *Let $S = \{v_s\}$ be a single grounded node, and $\mathbf{x}$ be the nonnegative eigenvector corresponding to the smallest eigenvalue of $L_g(s)$. For each vertex $v_i \in \mathcal{V} \setminus \{v_s\}$, define $a_{v_i} = \frac{\sum_{v_j \in \mathcal{N}_i \setminus \{v_s\}} x_j}{|\mathcal{N}_i \setminus \{v_s\}|}$ if $|\mathcal{N}_i \setminus \{v_s\}| > 0$, and $a_{v_i} = 0$ otherwise. Then for each β-vertex v_i we have $x_i > a_{v_i}$ and for each α-vertex v_i we have $x_i \leq a_{v_i}$.*

Proof. Rearranging the eigenvector equation for vertex v_i, we have $x_i = \frac{\sum_{v_j \in \mathcal{N}_i \setminus \{v_s\}} x_j}{d_i - \lambda}$. From Remark 1 we know that $0 < \lambda \leq 1$. Thus we have

$$\frac{\sum_{v_j \in \mathcal{N}_i \setminus \{v_s\}} x_j}{d_i} < x_i = \frac{\sum_{v_j \in \mathcal{N}_i \setminus \{v_s\}} x_j}{d_i - \lambda} \leq \frac{\sum_{v_j \in \mathcal{N}_i \setminus \{v_s\}} x_j}{d_i - 1}. \tag{3.13}$$

Since we have $a_{v_i} = \frac{\sum_{v_j \in \mathcal{N}_i \setminus \{v_s\}} x_j}{d_i}$, $\forall v_i \in \mathcal{V} \setminus \{\mathcal{N}_s \cup \{v_s\}\}$ and $a_{v_i} = \frac{\sum_{v_j \in \mathcal{N}_i \setminus \{v_s\}} x_j}{d_i - 1}$, $\forall v_i \in \mathcal{N}_s$, according to (3.13) we have $x_i > a_{v_i}$, $\forall v_i \in \mathcal{V} \setminus \{\mathcal{N}_s \cup \{v_s\}\}$ and $x_i \leq a_{v_i}$, $\forall v_i \in \mathcal{N}_s$.

According to the above proposition, the eigenvector element of an α-vertex is less than the average value of its neighbors' eigenvector entries. Thus it does not have the maximum eigenvector component among its neighbors. Similarly the eigenvector component of a β-vertex is greater than the average value of its neighbors' components and it does not have the minimum eigenvector component among its neighbors.

Corollary 1 *For any β-vertex v, there is a decreasing sequence of eigenvector components starting from v that ends at an α-vertex.*

Proof. Since each β-vertex has a neighbor with smaller eigenvector component, starting from any β-vertex there is a path consisting of vertices that have decreasing eigenvector components. If this sequence does not finish at an α-vertex it finishes at a β-vertex. There exists another vertex in the neighborhood of that vertex with smaller eigenvector component. Thus the decreasing sequence must finish at one of the α-vertices.

This leads to the following corollary; a vertex is said to be in the i-th layer if its shortest path to the grounded node has length i.

Corollary 2 *The minimum eigenvector component in layers i and j, where $i > j$, occurs in layer j.*

Proof. Let v and $\bar{v}$ be the vertices with minimum eigenvector components among the vertices in layers i and j respectively. According to Corollary 1 there is a path starting from v and ending at an α-vertex, making a decreasing sequence of eigenvector components. Since $i > j$ this path contains a vertex v' in layer j. Thus according to Corollary 1 we have $x_v \geq x_{v'} \geq x_{\bar{v}}$ which proves the claim. As a result the global minimum eigenvector component belongs to one of the α-vertices.

3.3 Applications to Random Graphs

Here we show that the bounds provided in the previous section for λ can be applied to determine the consensus rate in Erdos-Renyi random graphs and random regular graphs when the set of grounded nodes are chosen randomly.

3.3.1 Erdos-Renyi Random Graphs

Definition 2 *An Erdos-Renyi (ER) random graph, denoted $\mathcal{G}(n,p)$, is a graph on n nodes where each possible edge between two distinct vertices is present independently with probability p (which could be a function of n). Equivalently, an ER random graph can be viewed as a probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$, where the sample space Ω_n consists of all possible graphs on n nodes, the σ-algebra $\mathcal{F}_n$ is the power set of Ω_n, and the probability measure $\mathbb{P}_n$ assigns a probability of $p^{|\mathcal{E}|}(1-p)^{\binom{n}{2}-|\mathcal{E}|}$ to each graph with $|\mathcal{E}|$ edges.*

Definition 3 *For an ER random graph, we say that a property holds asymptotically almost surely if the probability of the set of graphs with that property (over the probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$) goes to 1 as $n \to \infty$. For a given graph function $f : \Omega_n \to \mathbb{R}_{\geq 0}$ and another function $g : \mathbb{N} \to \mathbb{R}_{\geq 0}$, we say $f(\mathcal{G}(n,p)) \leq (1 + o(1))g(n)$ asymptotically almost surely if there exists some function $h(n) \in o(1)$ such that $f(\mathcal{G}(n,p)) \leq (1+h(n))g(n)$ with probability tending to 1 as $n \to \infty$.*

We start by showing the following bounds on the degrees and isoperimetric constants of such graphs; while there exist bounds on these quantities for specific forms of p (e.g., [3, 11, 41]), we have been unable to find a clear statement of bounds for the wide range of probability functions considered by the following lemma and thus we provide a proof here.

Lemma 2 *Consider the Erdos-Renyi random graph $\mathcal{G}(n,p)$, where the edge probability p satisfies $\limsup_{n\to\infty} \frac{\ln n}{np} < 1$. Fix any $\epsilon \in (0, \frac{1}{2}]$. There exists a positive constant α (that depends on p) such that the minimum degree d_{min}, maximum degree d_{max} and isoperimetric constant $i(\mathcal{G})$ satisfy*

$$\alpha np \leq i(\mathcal{G}) \leq d_{min} \leq d_{max} \leq np \left(1 + \sqrt{3}\left(\frac{\ln n}{np}\right)^{\frac{1}{2}-\epsilon}\right).$$

asymptotically almost surely.

Proof. The degree bounds are readily obtained from classical concentration inequalities. Specifically, let d denote the degree of a given vertex. Note that d is a Binomial random variable with parameters $n-1$ and p, with expected value $\mathbb{E}[d] = (n-1)p$. Now, for any $0 < \beta \leq \sqrt{3}$ we have[2] [37]

$$\mathbf{Pr}(d \geq (1+\beta)\mathbb{E}[d]) \leq e^{\frac{-\mathbb{E}[d]\beta^2}{3}}.$$

[2] The statement of this concentration inequality in [37] has $0 < \beta \leq 1$, but the improved upper bound of $\sqrt{3}$ can be obtained from the same proof *mutatis mutandis*.

Choose $\beta = \sqrt{3}\left(\frac{\ln n}{np}\right)^{\frac{1}{2}-\epsilon}$, which is at most $\sqrt{3}$ for probability functions satisfying the conditions in the lemma and for sufficiently large n. Substituting into the above expression, we have

$$\begin{aligned}\mathbf{Pr}(d \geq (1+\beta)\mathbb{E}[d]) &\leq e^{-(n-1)p\left(\frac{\ln n}{np}\right)^{1-2\epsilon}} \\ &= O\left(e^{-\ln n\left(\frac{\ln n}{np}\right)^{-2\epsilon}}\right).\end{aligned}$$

To show that the maximum degree is smaller than the given bound asymptotically almost surely, we show that all vertices have degree less than the given bound with probability tending to 1. By the union bound, the probability that at least one vertex has degree larger than $(1+\beta)\mathbb{E}[d]$ is upper bounded by

$$n\mathbf{Pr}(d \geq (1+\beta)\mathbb{E}[d]) = O\left(e^{\ln n-\ln n\left(\frac{\ln n}{np}\right)^{-2\epsilon}}\right).$$

Since $\limsup_{n\to\infty} \frac{\ln n}{np} < 1$, the above expression goes to zero as $n \to \infty$, proving the upper bound on the maximum degree.

We will now show the lower bound for $i(\mathcal{G})$. Specifically, we will show that for p satisfying the properties in the lemma, almost every graph has the property that all sets of vertices of size s, $1 \leq s \leq \lfloor \frac{n}{2} \rfloor$, have at least αsnp edges leaving that set, for some constant α that we will specify later. For any specific set $\mathcal{S}$ of vertices of size s, the probability that $\mathcal{S}$ has $\lfloor \alpha snp \rfloor$ or fewer edges leaving the set is $\sum_{j=0}^{\lfloor \alpha snp \rfloor} \binom{s(n-s)}{j} p^j (1-p)^{s(n-s)-j}$. Let E_s denote the event that at least one set of vertices of size s has $\lfloor \alpha snp \rfloor$ or fewer edges leaving the set. Then

$$\mathbf{Pr}\left[E_s\right] \leq \binom{n}{s} \sum_{j=0}^{\lfloor \alpha snp \rfloor} \binom{s(n-s)}{j} p^j (1-p)^{s(n-s)-j}. \tag{3.14}$$

Note that for $1 \leq j \leq \lfloor \alpha snp \rfloor$,

$$\begin{aligned}\frac{\binom{s(n-s)}{j} p^j (1-p)^{s(n-s)-j}}{\binom{s(n-s)}{j-1} p^{j-1} (1-p)^{s(n-s)-j+1}} &= \frac{s(n-s)-j+1}{j}\frac{p}{1-p} \\ &\geq \frac{s(n-s)-\alpha snp}{\alpha snp}\frac{p}{1-p} \\ &\geq \frac{1-2\alpha p}{2\alpha}\frac{1}{1-p} \\ &\geq \frac{1}{2\alpha},\end{aligned}$$

for $s \leq \lfloor \frac{n}{2} \rfloor$ and $2\alpha < 1$ (which will be satisfied by our eventual choice of α). Thus, there exists some constant $r > 0$ such that

$$\sum_{j=0}^{\lfloor \alpha snp \rfloor} \binom{s(n-s)}{j} p^j (1-p)^{s(n-s)-j} \leq r \binom{s(n-s)}{\lfloor \alpha snp \rfloor} p^{\lfloor \alpha snp \rfloor} (1-p)^{s(n-s)-\lfloor \alpha snp \rfloor}.$$

Substituting into (3.14) and using the fact that $\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$, we have

$$\begin{aligned} \mathbf{Pr}\left[E_s\right] &\leq r \left(\frac{ne}{s}\right)^s \left(\frac{s(n-s)ep}{\alpha snp}\right)^{\alpha snp} e^{-p(s(n-s)-\alpha snp)} \\ &\leq r e^{s \ln \frac{ne}{s}} \left(\frac{e}{\alpha}\right)^{\alpha snp} e^{-p(s(n-s)-\alpha snp)} \\ &= r e^{sh(s)}, \end{aligned} \tag{3.15}$$

where

$$h(s) = 1 + np \underbrace{\left(\frac{\ln n}{np} + \alpha - \alpha \ln \alpha + \alpha p - 1\right)}_{\Gamma(\alpha)} + ps - \ln s. \tag{3.16}$$

Noting that $h(s)$ is decreasing in s until $s = \frac{1}{p}$ and increasing afterwards, we have

$$\begin{aligned} h(s) &\leq \max\left\{h(1), h\left(\frac{n}{2}\right)\right\} \\ &= \max\left\{1 + p + np\Gamma(\alpha), 1 + \ln 2 + np\left(\Gamma(\alpha) - \frac{\ln n}{np} + \frac{1}{2}\right)\right\}. \end{aligned}$$

From (3.16), $\Gamma(\alpha)$ is increasing in α, for $\alpha < 1$ with $\Gamma(0) = \frac{\ln n}{np} - 1$ being negative and bounded away from 0 for sufficiently large n (by the assumption on p from the statement of the lemma). Thus, there exists some sufficiently small positive constant α such that $h(s) \leq -\bar{\alpha} np$ for some constant $\bar{\alpha} > 0$ and for sufficiently large n. Thus (3.15) becomes $\mathbf{Pr}\left[E_s\right] \leq r e^{-s\bar{\alpha} np}$ for sufficiently large n.

By the union bound, the probability that $i(\mathcal{G}) < \alpha np$ is upper bounded by the sum of the probabilities of the events E_s for $1 \leq s \leq \lfloor \frac{n}{2} \rfloor$. Using the above expression, we have

$$\sum_{s=1}^{\lfloor \frac{n}{2} \rfloor} \mathbf{Pr}[E_s] \leq r \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor} e^{-s\bar{\alpha} np} \leq r \sum_{s=1}^{\infty} e^{-s\bar{\alpha} np} = r \frac{e^{-\bar{\alpha} np}}{1 - e^{-\bar{\alpha} np}}$$

which goes to 0 as $n \to \infty$. Thus, we have $i(\mathcal{G}) \geq \alpha np$ asymptotically almost surely.

Remark 2 *Note that the probability functions captured by the above lemma include the special cases where p is a constant and where $p(n) = \frac{c \ln n}{n}$ for constant $c > 1$. The above results generalize the bounds on the degrees and the isoperimetric constant in [3, 11, 41] where probability functions of the form $\frac{c \ln n}{n}$ were studied, although the bounding constants provided in those works will be generally tighter than the ones provided above due to the special case analysis. Further note that when $\ln n = o(np)$ the upper bound on the maximum degree becomes $np(1 + o(1))$.*

The above lemma, together with the lower bound (2.1), immediately leads to the following corollary.

Corollary 3 *Consider the Erdos-Renyi random graph $\mathcal{G}(n, p)$, where the edge probability p satisfies $\limsup_{n\to\infty} \frac{\ln n}{np} < 1$. Then there exists a positive constant γ (that depends on p) such that the algebraic connectivity $\lambda_2(\mathcal{G})$ satisfies $\lambda_2(L) \geq \gamma np$ asymptotically almost surely.*

With the above results in hand, we are now in place to prove the following fact about the smallest eigenvalue of the grounded Laplacian matrix for Erdos-Renyi random graphs.

Theorem 3 *Consider the Erdos-Renyi random graph $\mathcal{G}(n, p)$, where the edge probability p satisfies $\limsup_{n\to\infty} \frac{\ln n}{np} < 1$. Let S be a set of grounded nodes chosen uniformly at random with $|S| = o(\sqrt{np})$. Then the smallest eigenvalue λ of the grounded Laplacian satisfies*

$$(1 - o(1))|S|p \leq \lambda \leq (1 + o(1))|S|p$$

asymptotically almost surely.

Proof. For probability functions satisfying the conditions in the theorem, Lemma 2 indicates for any set S of grounded nodes, $|\partial S| \leq |S|d_{max} \leq \beta|S|np$ asymptotically almost surely for some positive constant β. Let $\bar{L}$ be the Laplacian matrix for the graph induced by the non-grounded nodes (i.e., the graph obtained by removing all grounded nodes and their incident edges). From [18], we have $\lambda_2(\bar{L}) \geq \lambda_2(L) - |S|$. Combining this with Corollary 3, we obtain

$$\frac{\sqrt{|S||\partial S|}}{\lambda_2(\bar{L})} \leq \frac{|S|\sqrt{\beta np}}{\gamma np - |S|} = o(1)$$

asymptotically almost surely when $|S| = o(\sqrt{np})$. From Theorem 1 and Lemma 1, we have $(1 - o(1))\frac{|\partial S|}{n-|S|} \leq \lambda \leq \frac{|\partial S|}{n-|S|}$ asymptotically almost surely.

Next, consider the random variable $|\partial S|$; there are $|S|(n-|S|)$ possible edges between S and $\mathcal{V} \setminus S$, each appearing independently with probability p, and thus $|\partial S|$ is a Binomial random variable with $|S|(n-|S|)$ trials. For all $0 < \alpha < 1$ we have the concentration inequalities [37]

$$\begin{aligned}
\mathbf{Pr}(|\partial S| \geq (1+\alpha)\mathbb{E}[|\partial S|]) &\leq e^{\frac{-\mathbb{E}[|\partial S|]\alpha^2}{3}} \\
\mathbf{Pr}(|\partial S| \leq (1-\alpha)\mathbb{E}[|\partial S|]) &\leq e^{\frac{-\mathbb{E}[|\partial S|]\alpha^2}{2}}.
\end{aligned} \tag{3.17}$$

We know that $\mathbb{E}[|\partial S|] = |S|(n-|S|)p$. Consider $\alpha = \frac{1}{\sqrt[4]{\ln n}}$ which causes the upper bound in the first expression to become $exp(-\frac{|S|(n-|S|)p}{3\sqrt{\ln n}})$. Since $|S|(n-|S|)$ is lower bounded by $n-1$ and noting that $np > \ln n$ for sufficiently large n, the bounds in (3.17) asymptotically go to zero. Thus we have

$$(1-o(1))|S|(n-|S|)p \leq |\partial S| \leq (1+o(1))|S|(n-|S|)p,$$

asymptotically almost surely. Substituting into the bounds for λ, we obtain the desired result.

3.3.2 Random d-Regular Graphs

We now consider random d-regular graphs, defined as follows, and characterize the smallest eigenvalue of the grounded Laplacian for such graphs.

Definition 4 *For any $n \in \mathbb{N}$, let $d = d(n) \in \mathbb{N}$ be such that $3 \leq d < n$ and dn is an even number. Define $\Omega_{n,d}$ to be the set of all d-regular graphs on n nodes, possibly with self-loops and multiple edges between nodes. Define the probability space $(\Omega_{n,d}, \mathcal{F}_{n,d}, \mathbb{P}_{n,d})$, where the sigma-algebra $\mathcal{F}_{n,d}$ is the power set of $\Omega_{n,d}$, and $\mathbb{P}_{n,d}$ is a uniform probability distribution on $\mathcal{F}_{n,d}$. An element of $\Omega_{n,d}$ drawn according to $\mathbb{P}_{n,d}$ is called a random d-regular graph, and denoted by $\mathcal{G}_{n,d}$ [5].*

Let $\lambda'_1(A) \leq \lambda'_2(A) \leq ... \leq \lambda'_n(A)$ be the eigenvalues of the adjacency matrix of any given graph $\mathcal{G}$; note that $\lambda'_n(A) = d$ for d-regular graphs. Define

$$\lambda'(\mathcal{G}) = \max\{|\lambda'_1(A)|, |\lambda'_{n-1}(A)|\}. \tag{3.18}$$

For uniform probability distributions $\mathbb{P}_{n,d}$, it was shown in [20] that for any $\epsilon > 0$,

$$\lambda'(\mathcal{G}_{n,d}) \leq 2\sqrt{d-1} + \epsilon, \tag{3.19}$$

asymptotically almost surely. As the Laplacian for the graph is given by $L = D - A = dI - A$, for any $\epsilon > 0$, the algebraic connectivity of a random d-regular graph satisfies

$$\lambda_2(L) \geq d - 2\sqrt{d-1} - \epsilon, \tag{3.20}$$

asymptotically almost surely. On the other hand we know that $\lambda_2(\bar{L}) \geq \lambda_2(L) - |S|$ [18]. Thus for a random d-regular graph on n nodes with a single grounded node, we have

$$\frac{2\sqrt{|S||\partial S|}}{\lambda_2(\bar{L})} = \frac{2\sqrt{d}}{\lambda_2(\bar{L})} \leq \frac{2\sqrt{d}}{\lambda_2(L) - 1} \leq \frac{2\sqrt{d}}{d - 2\sqrt{d-1} - \epsilon - 1} < 1, \tag{3.21}$$

for sufficiently large d and sufficiently small ϵ. From Theorem 1 and Lemma 1, we obtain the following result.

Theorem 4 *Let $\mathcal{G}$ be a random d-regular graph on n vertices with a single grounded node. Then for sufficiently large d, the smallest eigenvalue of the grounded Laplacian satisfies*

$$\lambda = \Theta\left(\frac{d}{n}\right)$$

asymptotically almost surely.

Chapter 4

Interpretation via Absorption Time in Random Walks on Graphs

The convergence properties of linear consensus dynamics with stubborn agents are closely related to certain properties of random walks on graphs, including mixing times, commute times, and absorption probabilities [9,10,21,50]. In this section we discuss the relationship between grounding centrality and the expected absorption time of an absorbing random walk on the underlying graph. To this end, we first review some properties of the inverse of the grounded Laplacian matrix.

4.1 Some Properties of the Inverse of Grounded Laplacian

As discussed earlier, when the graph $\mathcal{G}$ is connected, for any grounded node $v_s \in \mathcal{V}$, the inverse of the grounded Laplacian matrix L_g^{-1} exists and is a nonnegative matrix. In this case, an alternative definition for the grounding centrality of v_s from the one in Definition 1 is that it is the maximum eigenvalue of L_g^{-1}, with $GC(\mathcal{G}) = \text{argmin}_{v_s \in \mathcal{V}} \lambda_{max}(L_g^{-1}(s))$ where $L_g^{-1}(s)$ is the inverse of the grounded Laplacian formed by removing rows and columns corresponding to the vertex v_s from the Laplacian matrix. We know that the eigenvector corresponding to the largest eigenvalue of L_g^{-1} is the same as the eigenvector for the smallest eigenvalue of L_g. Thus this eigenvector can be chosen to be nonnegative, and strictly positive if v_s is not a cut vertex.

One of the consequences of the P-F theorem applied to L_g^{-1} is that the largest eigenvalue satisfies

$$\lambda_{\max}(L_g^{-1}) \leq \max_i \{[L_g^{-1}]_i \mathbf{1}\}, \tag{4.1}$$

where $[L_g^{-1}]_i$ is the i-th row of L_g^{-1}. Let

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_{n-1} \end{bmatrix}^T$$

be the eigenvector corresponding to $\lambda_{\max}(L_g^{-1})$. The element x_i in this eigenvector is associated with the i-th vertex. As this eigenvector has nonnegative elements, we can normalize it such that $\max_i x_i = 1$. Defining $\mathbf{x}_{min} = x_{min}\mathbf{1}$, since all of the elements of L_g^{-1} and $\mathbf{x}$ are nonnegative, and we have $\mathbf{x} \geq \mathbf{x}_{min}$ elementwise, we have

$$[L_g^{-1}]_i \mathbf{x}_{min} \leq [L_g^{-1}]_i \mathbf{x} = \lambda_{max}(L_g^{-1}) x_i.$$

Since $0 \leq x_i \leq 1$ we have $[L_g^{-1}]_i \mathbf{x}_{min} \leq \lambda_{max}(L_g^{-1})$. Combined with (4.1), this gives

$$\max_i \{[L_g^{-1}]_i \mathbf{x}_{min}\} \leq \lambda_{\max}(L_g^{-1}) \leq \max_i \{[L_g^{-1}]_i \mathbf{1}\}. \tag{4.2}$$

By minimizing over all choices of grounded nodes $v_s \in \mathcal{V}$ from (4.2) we have

$$\min_s \max_i \{[L_g^{-1}(s)]_i \mathbf{x}_{min}\} \leq \min_s \lambda_{\max}(L_g^{-1}(s)) \leq \min_s \max_i \{[L_g^{-1}(s)]_i \mathbf{1}\}. \tag{4.3}$$

As $x_{min} \to 1$ the upper bound and the lower bound of (4.3) approach $\min_s \lambda_{\max}(L_g^{-1})$.

Equations (4.2) and (4.3) provide bounds on the grounding centrality of each vertex in the graph and the grounding centrality of vertices in $GC(\mathcal{G})$, respectively. We now relate the bounds in (4.3) to an absorbing random walk on the underlying graph.

4.2 Relationship of Grounding Centrality to Absorbing Random Walk on Graphs

We start with the following preliminary definitions about absorbing Markov chains.

Definition 5 *A Markov chain is a sequence of random variables $Y_1, Y_2, Y_3, \ldots$ with the property that given the present state, the future and past states are independent. Mathematically*

$$Pr(Y_{n+1} = y | Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n) = Pr(Y_{n+1} = y | Y_n = y_n). \tag{4.4}$$

A state y_i of a Markov chain is called absorbing if it is impossible to leave it, i.e., $Pr(Y_{n+1} = y_i | Y_n = y_i) = 1$. A Markov chain is absorbing if it has at least one absorbing state and if from every state it is possible to go to an absorbing state. A state that is not absorbing is called a transient state [25].

If there are r absorbing states and t transient states, the transition matrix will have the canonical form

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}, \quad P^n = \begin{bmatrix} Q^n & \bar{R} \\ 0 & I \end{bmatrix}, \tag{4.5}$$

where $Q_{t\times t}$, $R_{t\times r}$ and $\bar{R}_{t\times r}$ are some nonzero matrices, $0_{r\times t}$ is a zero matrix and $I_{r\times r}$ is an identity matrix. The first t states are transient and the last r states are absorbing. The probability of going to state x_j from state x_i is given by entry p_{ij} of matrix P. Furthermore entry (i, j) of the matrix P^n is the probability of being in state x_j after n steps when the chain is started in state x_i.

The *fundamental matrix* for P is given by [25]

$$N = \sum_{j=0}^{\infty} Q^j = (I - Q)^{-1}. \tag{4.6}$$

The entry n_{ij} of N gives the expected number of time steps that the process is in the transient state x_j when it starts from the transient state x_i. Furthermore the i-th entry of $N\mathbf{1}$ is the expected number of steps before the chain is absorbed, given that the chain starts in the state x_i. In the context of a random walk on a given graph $\mathcal{G}$ containing one absorbing vertex v_s, the probability of going from transient vertex v_i to the transient vertex v_j is $P_{ij} = A_{ij}/d_i$ where A is the adjacency matrix and d_i is the degree of v_i. Thus the matrix Q in (4.5) becomes $Q = D_g^{-1}(s)A_g(s)$ where $A_g(s)$ and $D_g(s)$ are the grounded degree and grounded adjacency matrix, respectively (obtained by removing the rows and columns corresponding to the absorbing state v_s from those matrices).

To relate the absorbing walk to the grounded Laplacian, note that $L_g^{-1}(s) = (D_g(s) - A_g(s))^{-1} = (I - D_g^{-1}(s)A_g(s))^{-1}D_g^{-1}(s)$. Comparing to (4.6), we have

$$N_s = L_g^{-1}(s)D_g(s). \tag{4.7}$$

where the index s denotes that vertex v_s is an absorbing state. This leads to the following result.

Proposition 2 *Given graph $\mathcal{G}$ and a grounded node $v_s \in \mathcal{V}$, let d_{max} and d_{min} denote the maximum and minimum degrees of vertices in $\mathcal{V} \setminus \{v_s\}$, respectively. Then*

$$\frac{1}{d_{max}} \max_i \{[N_s]_i \mathbf{x}_{min}\} \leq \lambda_{max}(L_g^{-1}(s)) \leq \frac{1}{d_{min}} \max_i \{[N_s]_i \mathbf{1}\}, \tag{4.8}$$

where $[N_s]_i \mathbf{1}$ is the expected absorption time of a random walk starting at $v_i \in \mathcal{V} \setminus \{v_s\}$ with absorbing vertex v_s.

Proof. Substituting (4.7) into (4.2) gives

$$\begin{aligned} \frac{1}{d_{max}} \max_i \{[N_s]_i \mathbf{x}_{min}\} \leq \max_i \{[N_s D_g^{-1}(s)]_i \mathbf{x}_{min}\} \leq \lambda_{\max}(L_g^{-1}(s)) \\ \leq \max_i \{[N_s D_g^{-1}(s)]_i \mathbf{1}\} \leq \frac{1}{d_{min}} \max_i \{[N_s]_i \mathbf{1}\}, \end{aligned} \tag{4.9}$$

which proves the claim.

Remark 3 *Taking the minimum over all possible choices of absorbing vertex in (4.8) gives*

$$\frac{1}{d_{max}} \min_s \max_i \{[N_s]_i \mathbf{x}_{min}\} \leq \min_s \lambda_{max}(L_g^{-1}(s)) \leq \frac{1}{d_{min}} \min_s \max_i \{[N_s]_i \mathbf{1}\}. \tag{4.10}$$

In d-regular random graphs with sufficiently large d, the discussion in Section 3.3 shows that the the grounding central vertex can be approximated by vertex that if is chosen as the absorbing vertex, the maximum expected absorption time in the random walk on $\mathcal{G}$ is minimized.

Chapter 5

Conclusion and Future Work

We analyzed spectral properties of the grounded Laplacian matrix in the context of linear consensus dynamics with stubborn agents. We defined a natural centrality metric based upon the smallest eigenvalue of the grounded Laplacian, and provided bounds on this centrality using graph-theoretic properties.

An avenue for future research is to analyze the behavior of the other eigenvalues of the grounded Laplacian matrix. In particular as discussed in Section 1.1 analyzing the trace of the inverse of the grounded Laplacian matrix is of interest in the context of effective resistance of nodes in the network. A leader selection problem that should be considered for future work is to give a sufficient condition for a vertex or a set of vertices to minimize both convergence time and effective resistance simultaneously.

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