Spectral Aspects of Cocliques in Graphs

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis considers spectral approaches to finding maximum cocliques in graphs. We focus on the relation between the eigenspaces of a graph and the size and location of its maximum cocliques.

Our main result concerns the computational problem of finding the size of a maximum coclique in a graph. This problem is known to be NP-Hard for general graphs. Recently, Codenotti et al. showed that computing the size of a maximum coclique is still NP-Hard if we restrict to the class of circulant graphs. We take an alternative approach to this result using quotient graphs and coding theory. We apply our method to show that computing the size of a maximum coclique is NP-Hard for the class of Cayley graphs for the groups $\mathbb{Z}_p^n$ where p is any fixed prime.

Cocliques are closely related to equitable partitions of a graph, and to parallel faces of the eigenpolytopes of a graph. We develop this connection and give a relation between the existence of quadratic polynomials that vanish on the vertices of an eigenpolytope of a graph, and the existence of elements in the null space of the Veronese matrix. This gives a us a tool for finding equitable partitions of a graph, and proving the non-existence of equitable partitions. For distance-regular graphs we exploit the algebraic structure of association schemes to derive an explicit formula for the rank of the Veronese matrix. We apply this machinery to show that there are strongly regular graphs whose τ-eigenpolytopes are not prismoids.

We also present several partial results on cocliques and graph spectra. We develop a linear programming approach to the problem of finding weightings of the adjacency matrix of a graph that meets the inertia bound with equality, and apply our technique to various families of Cayley graphs. Towards characterizing the maximum cocliques of the folded-cube graphs, we find a class of large facets of the least eigenpolytope of a folded cube, and show how they correspond to the structure of the graph. Finally, we consider equitable partitions with additional structural constraints, namely that both parts are convex subgraphs. We show that Latin square graphs cannot be partitioned into a coclique and a convex subgraph.

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Chapter 1

Introduction

Algebraic graph theory is the study of graphs using their symmetries and associated matrices. The symmetries of a graph have implications for its structure, and the adjacency matrix encodes everything about the graph in matrix form. Naturally these objects can be studied algebraically. There are many classical applications of the spectral analysis of matrices to graphs [4]. The spectrum of the adjacency matrix of a graph can be used to: bound the chromatic number of a graph; determine whether a graph is bipartite; determine whether a graph is connected; bound the diameter of a graph; bound the size of a maximum clique and the size of a maximum coclique; and more. The spectrum of a graph is closely linked to its structural properties.

In this thesis we will focus on the relation between the spectrum of a graph and the size (and location) of its maximum cocliques. There are two main bounds on the size of a maximum coclique in a graph that arise from its spectrum, the inertia bound and the ratio bound. Graphs that meet the ratio bound have the additional property that the partition of the graph into a maximum coclique and its complement is equitable. Equality in the ratio bound also implies that eigenvectors for the least eigenvalue of the graph can be constructed from the characteristic vectors of its maximum cocliques. This is an example of structural information encoded by an eigenspace of a graph.

The connection between the least eigenspace of a graph and its maximum cocliques was used by Godsil and Newman [17] to show that for some graphs it is possible to characterize the maximum cocliques using the spectrum. Recently Godsil and Meagher used this approach to prove an Erdős-Ko-Rado type theorem for elements of the symmetric group [15]. Their main result characterizes the maximum intersecting families of symmetric group elements. They achieve this by characterizing the maximum cocliques in a family of graphs. Their proof utilizes the least eigenspace of these graphs to characterize their maximum cocliques. This algebraic argument can be viewed geometrically.

From an eigenspace of a graph, we can construct a natural family of combinatorially equivalent polytopes. These eigenpolytopes are closely related to the theory of graph representations [13], [14]. Godsil and Meagher [16] give a proof of the Erdős-Ko-Rado Theorem, and proofs of Erdős-Ko-Rado type theorems for elements of the symmetric group and binary strings, based on the eigenpolytopes of the associated families of graphs. Again, the central method is the classification of maximum cocliques through the combinatorial properties of the eigenpolytopes. Eigenpolytopes also geometrically encode information about the equitable partitions of a graph. The ability to characterize extremal objects by analyzing the eigenpolytopes of a graph is a promising aspect of this theory.

The connection between the eigenspaces of a graph and its structure are particularly strong for distance-regular graphs. Distance-regular graphs are a class of graphs with rich algebraic structure. They give rise to association schemes and provide an ideal environment to apply algebraic arguments. Distance-regular graphs are well-studied objects in algebraic graph theory. They arise naturally from the study of highly regular objects. The classification of distance-regular graphs based on their parameters, and the study of individual families of distance-regular graphs is a vibrant area of study. In this thesis we will see distance-regular graphs in several contexts.

1.1 Main Results

Our main result concerns the computational problem of finding the size of a maximum coclique in a graph. This problem is known to be NP-Hard for general graphs. In [7], Codenotti et al. showed that computing the size of a maximum coclique is still NP-Hard if we are restricted to the class of circulant graphs. This is a surprising result, as we might expect that assuming additional structure would decrease the hardness of the problem. In Chapter 3 we will examine the method from [7], and extract the key tools used in the proof. The auxiliary graph construction employed by Codenotti et al. can readily be applied to other classes of graphs. However, the rest of their proof cannot. We depart from [7] by using quotient graphs and coding theory, and outline a method for proving complexity results analogous to the result in [7] that can be applied to other classes of graphs. In particular, we apply our method to show that computing the size of a maximum coclique is NP-Hard for the class of Cayley graphs for the groups $\mathbb{Z}_p^n$ where p is any fixed prime (Theorem 3.0.2). We also look at the spectrum of the auxiliary graphs we employ in the proof, and draw conclusions about their structure (Corollary 3.13.4).

Cocliques are closely tied to equitable partitions of graphs. For example, if a graph is ratio tight, then each maximum coclique gives an equitable partition of the graph into two parts. Equitable partitions are also closely related to the face lattices of the eigenpolytopes of the graph. Each equitable partition corresponds to a pair of parallel faces of some eigenpolytope of the graph. This implies that an equitable partition into two parts gives a partition of the vertices of an eigenpolytope into parallel faces. Chapter 5 focuses on this connection. We note that the existence of parallel faces of a polytope that partition its vertex set corresponds to the existence of quadratic polynomials that vanish on the vertices of the polytope. Our second main result is to re-formulate the existence of these quadratic polynomials to the existence of elements in the null space of the Veronese matrix derived from the eigenspace of the graph. This gives us a tool for finding equitable partitions of a graph, and for establishing that there are no equitable partitions of a specific form. We show that for distance-regular graphs, the parameters of the association scheme give an explicit formula for the rank of the Veronese matrix (Lemma 5.7.3). Strongly regular graphs are distance-regular graphs with minimum non-trivial diameter. We show that in the case of strongly regular graphs, the rank of the Veronese matrix can be determined exactly (Corollary 5.8.1). We also present some computations of the rank of the Veronese matrix. We use these calculations to show that there are strongly regular graphs whose τ-eigenpolytopes are not prismoids.

We also present several partial results on cocliques and graph spectra. In Chapter 2 we will look at the inertia bound for graphs. The inertia bound gives an upper bound on the size of a maximum coclique in a graph, based on the number of non-negative or non-positive eigenvalues of a weighted adjacency matrix. For a single graph, there are a wide array of valid weighted adjacency matrices. This makes finding the optimal value of the inertia bound difficult. It is an open question whether there exist graphs that do not have a weighted adjacency matrix that gives equality in the inertia bound. We focus on this problem, and consider what can be said for various families of Cayley graphs. By using weightings based on the connection set of a Cayley graph, we are able to find inertia tight weightings for a large number of cyclic interval graphs. To extend these computational results, we develop a linear program to compute the optimal weightings that use the connection set of the graph. We apply this linear programming approach to the circulant graphs, and the cubelike graphs. The data we obtained suggests that this method does not produce inertia tight weightings for all Cayley graphs.

While we do not know whether or not there are graphs that do not meet the inertia bound, we have examples of families of graphs that meet the bound without weighting the adjacency matrix. The folded-cube graphs are an example of a family of graphs that meet the inertia bound with their unweighted adjacency matrices. For graphs that meet the ratio bound, we are able to characterize the maximum cocliques by analyzing the least eigenspace, or least eigenpolytope. For graphs that meet the inertia bound, it is less clear how the cocliques are related to the eigenpolytopes. In Chapter 4 we will consider the least eigenpolytopes of the folded-cube graphs. Folded-cube graphs are distance-regular graphs, and have a natural family of "canonical" maximum cocliques derived from the distance partitions of the graph. Towards showing that the canonical cocliques are exactly the maximum cocliques of a folded-cube graph, we find a class of large facets of the least eigenpolytope (Lemma 4.10.3). We also show that our geometric analysis can be used to derive dual eigenvalues of the association scheme corresponding to the folded cube (Corollary 4.11.2).

Higman and Haemers [21] considered equitable partitions of strongly regular graphs into two parts. They derived spectral conditions on strongly regular graphs that have equitable partitions where one part is a coclique, and the other part is strongly regular. In Chapter 6 we see how this result can be applied to rule out equitable partitions of strongly regular graphs. We also consider equitable partitions with additional structural constraints, namely that both parts are convex subgraphs. We show that Latin square graphs cannot be partitioned into a coclique and a convex subgraph (Lemma 6.4.1). We will also see how the ideas explored in the thesis apply to examples of strongly regular graphs.

Chapter 2

Cayley Graphs and The Inertia Bound

The inertia bound is a bound on the size of a largest coclique in a graph. It was first introduced by Cvetković in 1971 [4], and uses the number of non-negative or non-positive eigenvalues of a matrix as an upper bound on the size of a coclique. We get a valid bound using the spectrum of the adjacency matrix of the graph. However, we also get a valid bound by using weightings of the adjacency matrix. As a consequence, it is difficult to determine the best possible value of the bound.

Given a bound, we have the following natural question. What can be said about those graphs that meet the bound with equality? What structural properties lead to equality in the bound, and what deductions can we make given a graph that meets the bound with equality? In the case of the inertia bound, the answer is not known. We do not have a characterization of the graphs that are inertia tight. Moreover, as a consequence of the difficulty of determining the best possible value of the inertia bound, it is unknown whether or not *there exist* graphs that are not inertia tight.

In this chapter we will look at the question of whether or not there exist graphs that are not inertia tight. As we will see, the search space of valid weighted adjacency matrices grows very quickly with the size of our graph. So in order to make any headway, we will restrict ourselves to graphs that have additional structure, namely Cayley graphs. The vertex-transitivity of Cayley graphs, together with the fact that we have nice formulae for their eigenvalues, makes them good candidates for investigating this problem.

The Andrásfai graphs are a family of circulant graphs that are inertia tight. They generalize naturally to the cyclic interval graphs, but it is not known whether these graphs have inertia tight weightings. We will present some computational evidence that they are inertia tight. Our method is to assign weights to the generators of the graph. This method has the advantage of generating weighted adjacency matrices that have easily expressible eigenvalues.

We will see that by weighting the generators of a Cayley graph, we can formulate the problem of finding an optimal weighting as a linear program. This allows for efficient computations. We apply our linear program to the circulant graphs on at most 32 vertices, and the cubelike graphs on 32 vertices. We are able to find inertia tight weightings for graphs that are not amenable to other ad hoc weighting methods.

2.1 Graphs

Graphs are well-studied combinatorial objects. For background on basic graph theoretic concepts, we refer to reader to Diestel [10]. For algebraic graph theory, we will follow Godsil and Royle [18]. We have attempted to use standard terminology throughout this thesis.

A *graph* $X = (V, E)$ consists of a set $V = V(X)$ of *vertices*, together with a multiset $E = E(X)$ of 2-multisets of elements of V called the *edges* of X. A *loop* is an edge containing one vertex, for example $\{x, x\}$, and a *multiple edge* is an edge that appears more than once in E. A graph is *simple* if it does not contain any loops or multiple edges. We will refer to an edge $\{x, y\}$ in the abbreviated form xy.

An *arc* is an element of $V \times V$. We also refer to arcs as *directed edges*, and we view the edge $\{x, y\}$ as being equivalent to the two opposite arcs (x, y) and (y, x). A graph is *directed* if E is a subset of $V \times V$, and *undirected* if for each arc of E, its reverse arc is also an element of E. For the purposes of this thesis, graphs are simple and undirected. We will have occasion to consider loops and directed edges when working with Cayley graphs. However, we mention them only to eliminate them from consideration. All of our results apply only for undirected graphs.

A *subgraph* of a graph $X = (V, E)$ is a graph $Y = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$. If $V' \subseteq V$, and $E' \subseteq E$ is the set of all edges in E that are subsets of V', then $Y = (V', E')$ is an *induced subgraph* of X. We denote the subgraph of X induced by the vertices V' as $X[V']$.

Let x be a vertex of a graph X. If $\{x, y\}$ is an edge of X, then we say that y is *adjacent* to x or that y is a *neighbour* of x. We refer to the set of all neighbours of x as the *neighbourhood* of x. The size of the neighbourhood of x is the *degree* of x, or $\deg(x)$. We say a graph is *regular* if every vertex has the same degree. The *valency* of a regular graph is the degree of its vertices. A graph with valency k is k-regular.

A *walk* is a sequence of vertices $x_1 \dots x_m$ so that each x_{i+1} is adjacent to x_i. A *path* is a walk with no repeated vertices. We say that a graph is *connected* if any two vertices in X are joined by a path. The *distance* between two vertices in a connected graph X is the length of a shortest path between them. The *diameter* of a graph, $\operatorname{diam}(X)$, is the greatest distance between two vertices.

2.2 Spectra and Interlacing

Let X be a graph on n vertices. The *adjacency matrix* of X is a $n \times n$ matrix $A(X)$ with entries 0 and 1. The rows and columns of $A(X)$ are both indexed by a fixed ordering of $V(X)$. The xy-entry of $A(X)$ is equal to the number of arcs from x to y in X. Since our graphs are simple, $A(X)$ is a 01-matrix. Also since our graphs are undirected, $A(X)$ is a symmetric matrix.

Let A be a $n \times n$ complex-valued matrix, $z \in \mathbb{C}^n$ be a vector, and $\theta \in \mathbb{C}$. If $Az = \theta z$, we say that θ is an *eigenvalue* for A with *eigenvector* z. We also refer to z as a θ-eigenvector for A. The multiset of eigenvalues of a matrix is its *spectrum*. When A is the adjacency matrix of a graph X, we refer to the eigenvalues, eigenvectors and spectrum of A as the eigenvalues, eigenvectors and spectrum of X respectively.

We have the following facts about the spectrum of a graph, taken from Godsil and Royle [18]. We omit the proofs.

2.2.1 Lemma (Lemma 8.4.2 in [18])**.** *The eigenvalues of a real symmetric matrix A are real numbers.* □

2.2.2 Theorem (Theorem 8.4.5 in [18])**.** *Let A be a real symmetric $n \times n$ matrix. Then $\mathbb{R}^n$ has an orthonormal basis consisting of eigenvectors of A.* □

These two results imply that if X is a graph, then the eigenvalues of X are all real. We take the *multiplicity* of an eigenvalue θ to be the dimension of the space spanned by the eigenvectors for θ. From the above theorem we have that the sum of the multiplicities of the eigenvalues of X is n. Therefore we can order the spectrum of a graph X from largest to smallest as

$$\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n.$$

We also write the spectrum of X as follows. Let Θ be the set of distinct eigenvalues of X, and let m_θ be the multiplicity of θ, then the spectrum of X is

$$\{(\theta)^{m_\theta} \, : \, \theta \in \Theta\}.$$

2.2.3 Example. If X is a k-regular graph, then every row of $A = A(X)$ contains exactly k entries with value 1. Therefore if $\mathbf{1}$ is the n-dimensional all-ones vector, $A\mathbf{1} = k\mathbf{1}$, and $\mathbf{1}$ is a k-eigenvector of X.

2.2.4 Example. If X is a graph with n vertices, and no edges, then $A(X)$ is the zero matrix. So for any real n-dimensional vector z, $Az = 0z$. Thus the eigenvalues of X are $\theta_i = 0$ for all $1 \leq i \leq n$.

We use interlacing to relate the spectrum of a graph to the spectrum of its induced subgraphs. Let A be an $n \times n$ real symmetric matrix with spectrum

$$\theta_1(A) \geq \ldots \geq \theta_n(A),$$

and let B be an $m \times m$ real symmetric matrix with spectrum

$$\theta_1(B) \geq \ldots \geq \theta_m(B).$$

The eigenvalues of B *interlace* the eigenvalues of A if

$$\theta_i(A) \geq \theta_i(B) \geq \theta_{n-m+i}(A)$$

for all $1 \leq i \leq m$.

If A is a square matrix, a *principal submatrix* of A is a matrix that can be obtained from A by removing some subset of the rows of A, together with the corresponding subset of columns of A (i.e., if we remove the ith row, then we also remove the ith column). The eigenvalues of a principal submatrix of A interlace the eigenvalues of A.

2.2.5 Theorem (Theorem 9.1.1 in [18])**.** *Let A be a real symmetric matrix and let B be a principal submatrix of A. Then the eigenvalues of B interlace the eigenvalues of A.* □

Note that if X is a graph and $A = A(X)$, then the principal submatrices of A correspond to induced subgraphs of X. This follows as if B is a principal submatrix of A, then B is obtained by deleting the rows and columns of A indexed by some subset of $V(X)$. Thus if Y is an induced subgraph of X, then the eigenvalues of Y interlace the eigenvalues of X.

2.3 The Inertia Bound

A *coclique* in a graph X is a subset S of the vertices of X so that there are no edges of X joining any two vertices in S. Cocliques are also referred to as *independent sets*. Note that if S is a coclique, then the induced subgraph $X[S]$ is a graph on $|S|$ vertices with no edges. From Example 2.2.4 we recall that the eigenvalues of $X[S]$ are all 0. Let

$$\theta_1 \geq \ldots \geq \theta_n$$

be the eigenvalues of X, and let $|S| = m$. Then by Theorem 2.2.5 we see that

$$\theta_i \geq 0 \geq \theta_{n-m+i}$$

for all $1 \leq i \leq m$. The *independence number* of a graph X is the size of its maximum cocliques. We denote the independence number of X by $\alpha(X)$. From the above observation we can derive a bound on the size of a coclique in a graph, and hence a bound on $\alpha(G)$.

Given a real symmetric matrix A, we denote the number of positive eigenvalues of A by $n^+(A)$, the number of negative eigenvalues of A by $n^-(A)$ and the number of zero eigenvalues of A by $n^0(A)$. So if A is a $n \times n$ matrix,

$$n = n^+(A) + n^0(A) + n^-(A).$$

The ordered triple

$$(n^+(A), n^0(A), n^-(A))$$

is called the *inertia* of the matrix A (the term "inertia" was coined by Sylvester in 1852 due to the fact that this property is invariant for congruence classes of matrices).

We can use the inertia of a weighted adjacency matrix of X to bound its independence number. The following theorem is given as Theorem 9.6.3 in Godsil and Royle [18], we also present the proof from [18].

2.3.1 Theorem (The Inertia Bound)**.** *Let X be a graph on n vertices with adjacency matrix A. Let B be a real symmetric $n \times n$ matrix such that $B[i,j] = 0$ whenever $A[i,j] = 0$. Then*

$$\alpha(X) \leq \min\{n - n^+(B), n - n^-(B)\}$$

Proof. Let S be a coclique in X. Since B has the property that $B[i,j] = 0$ whenever $A[i,j] = 0$, the principal submatrix of B indexed by the elements of S is the zero matrix. Thus the eigenvalues of $X[S]$ interlace the eigenvalues of B.

Let $\theta_1 \geq \ldots \geq \theta_n$ be the eigenvalues of B and let $|S| = m$. Then we have that

$$\theta_i \geq 0 \geq \theta_{n-m+i}$$

for all $1 \leq i \leq m$. In particular, $0 \leq \theta_i$ for $1 \leq i \leq m$ implies that B has at least m non-negative eigenvalues. So

$$m \leq n - n^-(B).$$

Finally we note that $-B$ is a real symmetric $n \times n$ matrix, and $-B[i,j] = 0$ whenever $A[i,j] = 0$. If z is a θ-eigenvector for B, then $-Bz = -\theta z$. So the eigenvalues of $-B$ are $-\theta_n \geq \ldots \geq -\theta_1$. Applying our interlacing argument again we conclude that

$$m \leq n - n^-(-B).$$

But $n^-(-B) = n^+(B)$, so we have that

$$m \leq n - n^+(B).$$

Therefore for any coclique S in X,

$$|S| \leq \min\{n - n^+(B), n - n^-(B)\},$$

and we have the result. □

The inertia bound gives an important connection between the spectrum of a graph and its structural properties. It is originally due to Cvetković, and is sometimes referred to as the Cvetković bound [4]. The bound can also be derived using the Witt index of a matrix (for example, Elzinga and Gregory use this formulation [11],[12]).

2.4 Inertia Tight Graphs

We call the matrix B in Theorem 2.3.1 a *weighted adjacency matrix* for the graph X. The entries of B correspond to weighting the edges of the graph with real numbers. If there is a weighted adjacency matrix B for X so that equality holds in the inertia bound, then we say that X is *inertia tight*.

2.4.1 Example. The Petersen Graph is the graph on the 2-subsets of $\{1,2,3,4,5\}$, where two sets are adjacent if and only if they are disjoint. The spectrum of the Petersen Graph is

$$\{(3)^1, (1)^5, (-2)^4\}.$$

By using the regular adjacency matrix of the Petersen Graph, Theorem 2.3 implies that the size of a largest coclique is at most 4. The sets

$$\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}$$

form a coclique of size 4, so the Petersen Graph is inertia tight.

A graph X is *bipartite* if there is a partition of the vertex set of X into two cocliques. Let X be a bipartite graph and (S,T) be a partition of $V(X)$ into cocliques. Note that every edge of X joins a vertex of S to a vertex of T. If we order the vertices of $V(X)$ so that every vertex of S comes before every vertex of T, then the adjacency matrix $A(X)$ of X takes the form

$$A(X) = \begin{pmatrix} 0 & P \\ P^T & 0 \end{pmatrix},$$

where P is a $|S| \times |T|$ matrix. Now if z is a θ-eigenvector for $A(X)$, we can write $z = (x,y)$ where x, y are $|S|, |T|$-dimensional respectively. So $A(X)z = \theta z$ implies that

$$\begin{pmatrix} 0 & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Py \\ P^T x \end{pmatrix} = \theta \begin{pmatrix} x \\ y \end{pmatrix},$$

and so

$$\begin{pmatrix} 0 & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -Py \\ P^T x \end{pmatrix} = -\theta \begin{pmatrix} x \\ -y \end{pmatrix}.$$

Thus if θ is an eigenvalue of X, then $-\theta$ is also an eigenvalue of X.

Now if we suppose that X is a bipartite graph and 0 is not an eigenvalue of X, then the inertia of X is $(n/2, 0, n/2)$ and the inertia bound implies that $\alpha(X) \leq n/2$. However, since X is bipartite, $\alpha(X) \geq n/2$. Therefore X is inertia tight.

Of course, it is possible that a bipartite graph X is inertia tight with respect to its unweighted adjacency matrix, and has 0 as an eigenvalue.

2.4.2 Example. Let P_3 be the path on 3 vertices. The spectrum of P_3 is

$$\{\sqrt{2}, 0, -\sqrt{2}\}.$$

The inertia bound gives $\alpha(P_3) \leq 2$, which is tight.

2.4.3 Example. Let C_4 be the cycle on 4 vertices. The spectrum of C_4 is

$$\{2, (0)^2, -2\}.$$

The inertia bound gives $\alpha(C_4) \leq 3$, but $\alpha(C_4) = 2$, so if we use the unweighted adjacency matrix, the resulting bound is not tight.

These two examples show that if 0 is an eigenvalue of bipartite graph X, then we cannot guarantee that the inertia bound is tight if we use the unweighted adjacency matrix $A(X)$. However, we can apply the bound for all weighted adjacency matrices B. So we cannot conclude that these graphs are not inertia tight. Let us look at C_4 again.

2.4.4 Example. The adjacency matrix for C_4 is

$$A(C_4) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

We represent a general weighted adjacency matrix for C_4 as

$$B(a, b, c, d) = \begin{pmatrix} 0 & a & 0 & d \\ a & 0 & b & 0 \\ 0 & b & 0 & c \\ d & 0 & c & 0 \end{pmatrix}.$$

Now if we set $a = c = 1$ and $b = d = 0$ we have that the spectrum of $B(1, 0, 1, 0)$ is

$$\{(1)^2, (-1)^2\}.$$

Thus C_4 is inertia tight. Note that the weighting $B(1, 0, 1, 0)$ is the adjacency matrix of a subgraph of C_4 corresponding to two disjoint edges.

This example hints at a strategy for proving that all bipartite graphs are inertia tight.

2.4.5 Proposition. *If X is a bipartite graph, then there is a weighted adjacency matrix B for X that meets the inertia bound.*

Proof. Let M be a maximum matching of X. Define B to be the weighted adjacency matrix for X that results from assigning every edge in M weight 1, and the remaining edges weight 0. The matrix B is the adjacency matrix for

$$|M|K_2 \cup (n - 2|M|)K_1$$

and has spectrum

$$\{(-1)^{|M|}, 0^{n-2|M|}, 1^{|M|}\}.$$

So the inertia of B is $(|M|, n - 2|M|, |M|)$ and by the inertia bound we have $\alpha(X) \leq n - |M|$.

Now by König's Theorem we have that for any bipartite graph, the size of a maximum matching plus the size of a maximum coclique is equal to the total number of vertices in X. Thus $\alpha(X) = n - |M|$ and B is an inertia tight weighting. $\square$

For non-bipartite graphs we use a similar strategy to find inertia tight weightings. A *clique* in a graph X is a set of vertices T so that every pair of vertices in T is joined by an edge. Cliques and cocliques are "dual" objects. For a graph X we denote by $\overline{X}$ the *complement* of X, defined as the graph on $V(X)$ where two vertices are adjacent if and only if they are not adjacent in X. So cocliques in X are cliques in $\overline{X}$. The size of a maximum clique in X is the *clique number* of X which we denote by $\omega(X)$.

If S is coclique in X and T is a clique in X, then $|S \cap T| \leq 1$. This is easily seen, as if $s, t \in S \cap T$ then s is simultaneously adjacent to and not adjacent to t. Now suppose we have a partition of $V(X)$ into m cliques, $T_1, \dots, T_m$. Since the sets T_i are disjoint, and $|S \cap T_i| \leq 1$ for each $1 \leq i \leq m$, we have that $|S| \leq m$. If S is a maximum coclique, this implies that $\alpha(X) \leq m$.

Let X and Y be graphs with disjoint vertex sets. Then the *union* $X \cup Y$ of X and Y is the graph

$$(V(X) \cup V(Y), E(X) \cup E(Y)).$$

If $A(X)$ and $A(Y)$ are the adjacency matrices of X and Y respectively, then the adjacency matrix of $X \cup Y$ is

$$\begin{pmatrix} A(X) & 0 \\ 0 & A(Y) \end{pmatrix}.$$

If x is a θ-eigenvector for X, then $(x, 0)$ is a θ-eigenvector for $X \cup Y$. Likewise if y is a θ-eigenvector for Y, then $(0, y)$ is a θ-eigenvector for $X \cup Y$. So the spectrum of $X \cup Y$ is the union of the spectra of X and Y.

Returning to our clique partition, suppose T is a clique on m vertices. Then the adjacency matrix of T is $J_m - I_m$ where J_m is the $m \times m$ matrix will all values 1, and I_m is the $m \times m$ identity matrix. The spectrum of J_m is $\{m, (0)^{m-1}\}$, and every vector is a 1-eigenvector of I_m, so it follows that the spectrum of T is $\{m-1, (-1)^{m-1}\}$.

Now consider our graph X partitioned into cliques $T_1, \dots, T_m$. We obtain the weighted adjacency matrix B by weighting all of the edges in each T_i with value 1 and all other edges of X with value 0. Let $m_i = |T_i|$ for each i. By the above notes we see that each $m_i - 1$ is an eigenvalue of B, and -1 is an eigenvalue of B with multiplicity

$$\sum_{i=1}^{m} (m_i - 1) = n - m.$$

Therefore the inertia bound gives $\alpha(X) \leq m$.

So we can try to prove inertia tightness by finding a partition $T_1, \dots, T_m$ of X into cliques with m as small as possible. An obvious strategy is to try to make the cliques T_i as large as possible. Since each $T_i \leq \omega(X)$, the best possible outcome is for each T_i to be a maximum clique, in which case

$$\alpha(X) \leq n/\omega(X).$$

2.4.6 Example. Denote the set $\{1, 2, \ldots, n\}$ by $[n]$. Let S_n be the symmetric group of order n. The elements of S_n are permutations of the set $[n]$. For $\alpha \in S_n$ we denote the image of i under α as $\alpha(i)$. We say that two permutations $\alpha, \beta \in S_n$ are *non-intersecting* if $\alpha(i) \neq \beta(i)$ for each $1 \leq i \leq n$. Let X_n be the graph with vertex set S_n where $\alpha, \beta \in S_n$ are adjacent if and only if they are non-intersecting.

Now, X_n has $n!$ vertices. If we fix $1 \leq i, j \leq n$ and let $S_{i,j}$ be the set of permutations α so that $\alpha(i) = j$, then $S_{i,j}$ is a coclique in X_n. The size of $S_{i,j}$ is $(n-1)!$ as every bijection between $[n] \setminus i$ and $[n] \setminus j$ gives an element in $S_{i,j}$ and vice versa. Thus $\alpha(X) \geq (n-1)!$. If α is any cycle of order n in S_n, then

$$T_\alpha = \{\alpha, \alpha^2, \ldots, \alpha^n\}$$

is a clique in X_n of size n. Thus $\omega(X) \geq n$.

The Clique-Coclique bound for Cayley graphs implies that $\alpha(X)\omega(X) \leq n!$. So we have that

$$n! \leq \alpha(X)\omega(X) \leq n!,$$

and we have equality everywhere.

In fact, the clique T_α is a subgroup of S_n. Thus the cosets of T_α partition S_n. Moreover, if T is a coset of T_α, then T is also a clique. To see this note that T is obtained by composing each element of T_α by some $\beta \in S_n$. Thus

$$\alpha^i(a) \neq \alpha^j(a)$$

for $i \neq j$ and $1 \leq a \leq n$ implies

$$\alpha^i(\beta(a)) \neq \alpha^j(\beta(a))$$

for $i \neq j$ and $1 \leq a \leq n$. Since β is a permutation of $[n]$ this implies that the elements of T are pairwise non-intersecting.

Therefore, there is a weighted adjacency matrix B for X_n that achieves tightness in the inertia bound.

We have seen examples of graphs that are inertia tight by virtue of their unweighted adjacency matrix, and examples of graphs that are inertia tight by virtue of a weighted adjacency matrix. Can we find a weighted adjacency matrix for every graph that meets the inertia bound? This is an important, and open, question. Can we always meet the inertia bound, and if not how bad can the bound be?

For the remainder of this chapter we will address these questions. Specifically we will look at the inertia bound applied to Cayley graphs.

2.5 Cayley Graphs

Cayley graphs are a class of vertex-transitive graphs constructed using groups. The Cayley graphs we will be working with in this thesis will all be constructed using finite Abelian groups. For that reason our definitions are specifically for finite Abelian groups. However, our definitions generalize easily to non-Abelian groups, as do the basic facts in this section and Section 2.6. We take our treatment of Cayley graphs largely from Godsil and Royle [18].

Let G be a finite Abelian group with group operation $+$, and let C be a subset of G. The *Cayley graph* $X = X(G, C)$ is defined on G by the following adjacency rule. Given $a, b \in G$, the arc (a, b) is a directed edge of X if and only if $b - a \in C$.

Note that if the identity element, 0, of G is in C, then (a, a) is an edge of X for all $a \in G$. So X is loopless if $0 \notin C$, and every vertex is the end of a loop otherwise. Also note that we have defined Cayley graphs as directed graphs. Define

$$-C = \{-c \,:\, c \in C\}.$$

If $C = -C$, then for $a, b \in G$, $b - a \in C$ implies

$$-(b - a) = a - b \in C$$

and both (a, b) and (b, a) are in $E(X)$. Thus X is an undirected graph.

For the purposes of this thesis, we will always take Cayley graphs to be loopless and undirected.

2.5.1 Example. $X = X(\mathbb{Z}_m, \{1, m-1\})$ is a Cayley graph for the integers modulo m. Each $i \in \mathbb{Z}_m$ is adjacent to $i - 1$ and $i + 1$ in $\mathbb{Z}_m$. So X is the cycle on m vertices. Cayley graphs for the groups $\mathbb{Z}_m$ are called *circulants*. We will see this class of Cayley graphs in this chapter, and in Chapter 3.

2.5.2 Example. For a vector space V, let e_i denote the ith standard basis vector of V. The graph

$$X = X(\mathbb{Z}_2^3, \{e_i \,:\, 1 \leq i \leq 3\})$$

is a Cayley graph for the group $\mathbb{Z}_2^3$. It is easily seen that X is isomorphic to the cube graph. Cayley graphs for the groups $\mathbb{Z}_2^m$ are called *cubelike graphs*. We will revisit cubelike graphs in Section 2.13.

Given graphs X, Y, the map $f : V(X) \to V(Y)$ is an *isomorphism* if f is a bijection between $V(X)$ and $V(Y)$, and $ab \in E(X)$ if and only if

$$f(a)f(b) \in E(Y).$$

An isomorphism is an *automorphism* if $X = Y$.

Note that the identity map on $V(X)$ is an automorphism of X. Also, if f is an automorphism of X, then its inverse f^{-1} is also an automorphism of X.

Define the composition of two functions $f : A \to B$ and $g : B \to C$ as $f \circ g : A \to C$ given by

$$(f \circ g)(x) = g(f(x)).$$

If f, g are automorphisms of X, then $f \circ g$ is an automorphism of X. This shows that the set of automorphisms of a graph X is a group with group operation $\circ$. We denote the *automorphism group* of X as $\mathrm{Aut}(X)$.

Note that if $X = X(G, C)$ is a Cayley graph, then we can define an automorphism of X for every element of G. For $g \in G$ define $f_g : G \to G$ as

$$f_g(h) = h + g.$$

The map $f_g \in \mathrm{Aut}(X)$ as f_g is a bijection with $f_g^{-1} = f_{-g}$. It also preserves the directed edges of X as $ab \in E(X)$ if and only if

$$b - a = (b + g) - (a + g) \in C$$

if and only if $f(a)f(b) \in E(X)$. Thus

$$\{f_g \,:\, g \in G\} \subseteq \mathrm{Aut}(X).$$

A graph X is *vertex transitive* if for any two vertices $a, b \in V(X)$, there is an automorphism $f \in \mathrm{Aut}(X)$ so that $f(a) = b$. Every Cayley graph is vertex transitive. This follows as if $a, b \in G$, then $b - a \in G$ and so $f_{b-a} \in \mathrm{Aut}(X)$. So f_{b-a} is an automorphism of X with

$$f_{b-a}(a) = a + (b - a) = b.$$

2.6 Spectra of Cayley Graphs

Let G be an Abelian group, and let $X = X(G, C)$ be a Cayley graph for G. We can use the linear characters of G to determine the spectrum of X. We refer the reader to Isaacs [22] for background on group representations and characters (all of the results we will need are found in Chapter 2 of [22]).

A *linear character* $\chi : G \to \mathbb{C}$ is a homomorphism from G to $\mathbb{C} \setminus \{0\}$. If G is an Abelian group, the linear characters of G are exactly the irreducible characters of G. The set H of linear characters of G forms a group with group operation defined by

$$(\chi \circ \phi)(g) = \chi(g)\phi(g).$$

Moreover, H is isomorphic to G, and in particular, there are $|G|$ linear characters for G.

Let χ be a linear character for G. Consider the vector $z_\chi \in \mathbb{C}^G$ where $z_\chi[g] = \chi(g)$ for all $g \in G$. Let $A = A(X)$ be the adjacency matrix for X. Now consider Az_χ. The g-entry of Az_χ is the inner product of the g-row of A with z_χ. So

$$(Az_\chi)[g] = \langle \overline{A_g}, z_\chi \rangle = \sum_{h \in G} \overline{A[g,h]} z_\chi[h] = \sum_{h \in G} \overline{A[g,h]} \chi(h).$$

But since A is the adjacency matrix of a graph, $A[g, h] = 1$ exactly when g and h are adjacent, and 0 otherwise. Since X is a Cayley graph with connection set C, we have that g is adjacent to h exactly when $h - g \in C$. To state this another way, the neighbours of g are exactly

$$\{g + c \,:\, c \in C\}.$$

So
$$(Az_\chi)[g] = \sum_{c\in C} \chi(g+c) = \sum_{c\in C} \chi(g)\chi(c) = \chi(g)\sum_{c\in C}\chi(c) = z_\chi[g]\sum_{c\in C}\chi(c).$$
Thus we have shown that z_χ is an eigenvector for X with eigenvalue
$$\sum_{c\in C}\chi(c).$$

Now suppose that χ and ϕ are distinct linear characters of G. Consider the vectors z_χ and z_ϕ. Since χ and ϕ are distinct irreducible characters, they are orthogonal with respect to the inner product
$$\langle \chi, \phi\rangle = (1/|G|)\sum_{g\in G}\chi(g)\overline{\phi(g)}.$$
However, this implies that z_χ and z_ϕ are orthogonal vectors in $\mathbb{C}^n$. Thus the set of linear characters of G gives an orthogonal set of n vectors in $\mathbb{C}^n$,
$$\{z_\chi \,:\, \chi \text{ is a linear character of } G\}$$
that form a basis of $\mathbb{C}^n$. As a consequence we have that
$$\left\{\sum_{c\in C}\chi(c) \,:\, \chi \text{ is a linear character of } G\right\}$$
is the spectrum of $X = X(G, C)$.

2.6.1 Example. Let $G = \mathbb{Z}_n$ and $X = X(G, C)$, so X is a circulant. Let $\omega \in \mathbb{C}$ be a primitive nth root of unity (i.e., $\omega \in \mathbb{C}$ is such that $\omega^n = 1$, and $\omega^k \neq 1$ for any $k < n$). For $1 \le i \le n$ define the character χ_i by
$$\chi_i(g) = (\omega^i)^g.$$
It is easily checked that the χ_i are distinct linear characters of G. So the eigenvalues of X are
$$\left\{\sum_{c\in C}(\omega^i)^c \,:\, 1 \le i \le n\right\}.$$

If we let
$$\omega = \cos(2\pi/n) + i\sin(2\pi/n),$$
then
$$\Re(\omega^j) = \cos(2\pi j/n)$$
and since $-C = C$, we have that
$$\sum_{c\in C}(\omega^i)^c = \sum_{c\in C}\cos(2\pi ic/n).$$
Thus the spectrum of X is
$$\left\{\sum_{c\in C}\cos(2\pi i/n) \,:\, 1 \le i \le n\right\}.$$

2.7 Andrásfai Graphs

The Andrásfai graphs are a class of Cayley graphs that meet the inertia bound with their unweighted adjacency matrices. We will also see that they are triangle-free circulants whose maximum cocliques are exactly the neighbourhoods of the vertices.

For a non-negative integer k we define the *kth Andrásfai graph* as $\mathrm{And}(k) = X(\mathbb{Z}_{3k-1}, C)$. Here the connection set C is the set of integers $0 \leq i \leq 3k-2$ so that i is congruent to 1 modulo 3. We immediately see that $\mathrm{And}(k)$ is a circulant on $3k-1$ vertices with degree k.

To see that $\mathrm{And}(k)$ is triangle-free, consider the neighbours of 0. Suppose $3i+1$ and $3j+1$ are adjacent neighbours of 0. Then

$$(3i+1) - (3j+1) = 3(i-j) \in C,$$

which is a contradiction. We can also easily show that $\mathrm{And}(k)$ has diameter 2. Note that we can partition the vertex set of $\mathrm{And}(k)$ into the residue classes modulo 3. We have C_0 the vertices with residue 0, C the vertices with residue 1, and C_2 the vertices with residue 2. If $3i+2 \in C_2$, then $3i+2 = (3i+1)+1$. Since $1 \in C$, $3i+2$ is adjacent to $3i+1 \in C$, which is adjacent to 0. Thus every vertex in C_2 is at distance 2 from 0. If $3i \in C_0$, then $3i$ is adjacent to $3i+1$ which is adjacent to 0, so again every vertex in $C_0 \setminus \{0\}$ is at distance 2 from 0. Thus $\mathrm{And}(k)$ has diameter 2.

We give the following argument from Godsil [19] to show that $\mathrm{And}(k)$ meets the inertia bound.

2.7.1 Lemma (Lemma 6.5.1 from [19]). *$\alpha(\mathrm{And}(k)) = k$, and $\mathrm{And}(k)$ is inertia tight.*

Proof. Since $\mathrm{And}(k)$ is triangle-free, and the valency of $\mathrm{And}(k)$ is k, the neighbourhood of any vertex of $\mathrm{And}(k)$ is a coclique of size k. Moreover, since $\mathrm{And}(k)$ has diameter 2, these cocliques are maximal. Thus $\alpha(\mathrm{And}(k)) \geq k$. To prove the reverse, we show that $\mathrm{And}(k)$ meets the inertia bound with its unweighted adjacency matrix.

Following Example 2.6.1, the spectrum of $\mathrm{And}(k)$ is

$$\left\{ \sum_{c \in C} \omega^{ic} \,:\, 0 \leq i \leq 3k-2 \right\},$$

where ω is a primitive $(3k-1)$th root of unity. Using the connection set for $\mathrm{And}(k)$ we have

$$\sum_{c \in C} \omega^{ic} = \sum_{j=0}^{k-1} \omega^{i(3j+1)} = \omega^i \sum_{j=0}^{k-1} (\omega^{3i})^j.$$

Now evaluating this sum we see

$$\sum_{j=0}^{k-1} (\omega^{3i})^j = \frac{\omega^{3ik}-1}{\omega^{3i}-1} = \frac{(\omega^i)^{3k}-1}{\omega^{3i}-1} = (\omega^i - 1)(\omega^{3i}-1)^{-1}.$$

Therefore

$$\sum_{c \in C} \omega^{ic} = \omega^i(\omega^i - 1)(\omega^{3i} - 1)^{-1} = \omega^i(\omega^{2i} + \omega^i + 1)^{-1} = (\omega^i + 1 + \omega^{-i})^{-1}.$$

In order to bound $\alpha(\mathrm{And}(k))$ using the inertia bound, we need to determine either the number of strictly negative or strictly positive eigenvalues. We will use the number of strictly negative eigenvalues. We want to determine the values j for which $(\omega^j + 1 + \omega^{-j})^{-1} < 0$.

Let

$$\omega = \cos(2\pi/(3k-1)) + i\sin(2\pi/(3k-1))$$

where i is the imaginary unit. Now

$$(\omega^j + 1 + \omega^{-j})^{-1} = (1 + 2\cos(2\pi j/(3k-1)))^{-1}.$$

In order for

$$\cos(2\pi j/(3k-1)) < -1/2$$

we must have

$$\frac{2\pi}{3} < \frac{2\pi j}{3k-1} < \frac{4\pi}{3}.$$

Thus

$$(3k-1)/3 < j < (6k-2)/3$$

and $k \le j \le 2k-1$. So there are exactly k values of j for which

$$\sum_{c \in C} \omega^{jc} < 0.$$

Thus Theorem 2.3.1 implies that $\alpha(\mathrm{And}(k)) \le k$, and we have the result. $\square$

We have seen that $\alpha(\mathrm{And}(k)) = k$, and that the neighbourhood of any vertex is a maximum coclique in $\mathrm{And}(k)$. In fact, the reverse is also true. If S is a coclique of size k in $\mathrm{And}(k)$, then there is some $i \in \mathbb{Z}_{3k-1}$ so that S is the set of neighbours of i. This is not difficult to show, but we omit the proof (see Lemma 6.10.2 in Godsil and Royle [18]).

2.8 Cyclic Interval Graphs

Andrásfai graphs are a subfamily of a larger family of circulant graphs. To see how we can generalize the definition of $\mathrm{And}(k)$, we first give an alternative definition of the Andrásfai graphs.

The following proposition is essentially Exercise 39 in Chapter 6 of Godsil and Royle [18].

2.8.1 Proposition. *$\mathrm{And}(k)$ is isomorphic to the graph $X = X(\mathbb{Z}_{3k-1}, C')$, where $C' = \{i : k \le i \le 2k-1\}$.*

Proof. Recall that $\mathrm{And}(k) = X(\mathbb{Z}_{3k-1}, C)$ where C is the subset of $\mathbb{Z}_{3k-1}$ consisting of the integers that are congruent to 1 modulo 3. We define the function $f : V(X) \to V(\mathrm{And}(k))$ by

$$f(i) = 3(i-k) + 1 \pmod{3k-1}.$$

First we show that f is a bijection. Let $g : V(\mathrm{And}(k)) \to V(X)$ be defined as $g(a) = ka$. Since $3k$ is congruent to 1 modulo $3k-1$,

$$g(f(i)) = g(3(i-k)+1) = k(3(i-k)+1) = (3k)(i-k) + k = i.$$

Thus g is the inverse of f, and f is a bijection from $\mathbb{Z}_{3k-1}$ to itself.

Finally, we show that f is a graph homomorphism between $\mathrm{And}(k)$ and X. Note that $f(C') = C$, as $i \in C'$ implies $k \leq i \leq 2k-1$, so $1 \leq f(i) \leq 3k-2$. Thus $f(i)$ is congruent to 1 modulo 3. Also note that f is a linear function. This completes the proof, as if i, j are neighbours in X, then there is some $a \in C'$ so that $j - i = a$. Now

$$f(j-i) = f(j) - f(i) = f(a) \in C,$$

so $f(i), f(j)$ are adjacent in $\mathrm{And}(k)$. □

This alternative definition of $\mathrm{And}(k)$ lends itself more readily to generalization.

The *cyclic interval graph* $C(n,r)$ is the Cayley graph $C(n,r) = X(\mathbb{Z}_n, C)$ where

$$C = \{i : r \leq i \leq n-r\}.$$

Proposition 2.8.1 shows that $\mathrm{And}(k) \cong C(3k-1,k)$. The name "cyclic interval" comes from the fact that these graphs have an alternative construction.

The graph $C(n,r)$ is isomorphic to the graph on the set of cyclic shifts of $[r]$ in $[n]$ where two sets are adjacent if and only if they are disjoint. To see these graphs are isomorphic, consider the function that maps $i \in [n]$ to the cyclic shift of $[r]$ that begins with i. This set is adjacent to the cyclic shifts that begin with $i + r \leq j \leq i + n - r$ (working modulo n). Thus this map is an isomorphism. We will use these two definitions of $C(n,r)$ interchangeably (but it will be clear which we are using).

We are interested in determining whether or not $C(n,r)$ meets the inertia bound. So we need to determine $\alpha(C(n,r))$. First we note that as a consequence of the Pigeonhole Principle, if $2r > n$, then any two cyclic shifts of $[r]$ will intersect non-trivially. Thus $C(n,r)$ is the empty graph on n vertices when $2r > n$. If $2r \leq n$, then $C(n,r)$ will have some edges. In order for S to be a coclique in $C(n,r)$, it must be the case that the elements of S are pairwise intersecting. Thus the sets S_i consisting of the cyclic shifts of $[r]$ that contain i form cocliques of size r in $C(n,r)$. The following lemma demonstrates that the sets S_i are exactly the maximum cocliques of $C(n,r)$.

2.8.2 Lemma (Lemma 7.7.1 in [18])**.** *If $2r \leq n$, then $\alpha(C(n,r)) = r$. Moreover, the maximum cocliques are exactly the sets S_i for $1 \leq i \leq n$.*

Proof. Let T be a maximum coclique in $C(n,r)$. Since $C(n,r)$ is a Cayley graph, it is vertex transitive. So without loss of generality we assume that $[r] \in T$. Define the sets

$$T_i = \{A \in T \,:\, i \in A\}.$$

Note that every $A \in T$ is either in T_1 or T_r. This must be the case as if $A = [r]$, then $a \in T_1$ and T_r. If $A \neq [r]$, then $A \cap [r] \neq \emptyset$ implies that there is some least $1 \leq i \leq r$ so that $i \in A$. If $i = 1$, then $A \in T_1$. If $i > 1$, then since A is a cyclic shift of $[r]$, we see that $r \in A$ and $A \in T_r$. Thus

$$|T| = |T_1| + |T_r| - 1.$$

Suppose that $1 \leq i \leq r$ is the smallest number so that every $A \in T_r$ contains $i \in A$. Since each $A \in T_1$ intersects each $B \in T_r$, each $B \in T_1$ contains i. Thus

$$|T_1| \leq r - i + 1,$$

and $|T_r| \leq i$. Therefore $|T| \leq r$, and $\alpha(C(n,r)) = r$. Moreover, in order to achieve equality we must have

$$|T_1| = r - i + 1,$$

so T_1 contains all of the cyclic shifts whose largest element is in the range $[i,r]$. Likewise we must have T_r the set of all cyclic shifts whose least element is in the range $[1,i]$. Thus $T = S_i$. □

We can determine the spectrum of $C(n,r)$ as follows. Recall that $C(n,r) = X(\mathbb{Z}_n, C)$ with $C = \{r, \ldots, n-r\}$. Let ω be a primitive nth root of unity. Then the character χ_i defined by

$$\chi_i(a) = \omega^{ia}$$

gives an eigenvector for $C(n,r)$ with eigenvalue

$$\sum_{c \in C} \chi_i(c) = \sum_{c=r}^{n-r} \omega^{ic} = \omega^{ir} \frac{\omega^{i(n-2r+1)}}{\omega^i - 1}.$$

We can apply the trick we used to prove Lemma 2.7.1 to derive a more useful formula for the eigenvalues of $C(n,r)$ for specific values of n and r.

2.9 Triangle-Free Cyclic Interval Graphs

The graphs $C(n,r)$ are triangle free for

$$(n+1)/3 \leq r \leq n/2.$$

We have seen that when $(n+1)/3 = r$, the cyclic interval graph $C(n,r)$ meets the inertia bound with its unweighted adjacency matrix. When $r = n/2$, the inertia bound is tight for $C(n,r)$ using its unweighted adjacency matrix. In this case, $C(n,r) \cong rK_2$, and has spectrum

$$\{(-1)^r, (1)^r\}.$$

So $\alpha(C(n,r)) \leq r$ by the inertia bound, and by choosing one vertex per copy of K_2 we have a coclique that meets this bound.

What happens for $(n+1)/3 < r < n/2$? In this case we computed the inertia bound value for all values of r and n so that $2 \leq r \leq 14$. Of those 104 graphs, 64 meet the inertia bound with their unweighted adjacency matrices and are inertia tight. That leaves 40 unaccounted for. We saw in Section 2.4 that one strategy for weighting the adjacency matrix of a graph in order to achieve equality in the inertia bound is to partition the graph into cliques.

A clique in $C(n,r)$ is a set of non-overlapping intervals of $[n]$. Since each interval has length r, clearly

$$\omega(C(n,r)) = \lfloor n/r \rfloor.$$

If $n = rj + a$ for $0 \leq a < r$, then we can partition the vertices of $C(n,r)$ into r cliques of size j, together with a cliques of size 1. The spectrum of K_j is $\{(-1)^{j-1}, j-1\}$ and the spectrum of K_1 is $\{0\}$. So the spectrum of $rK_j \cup aK_1$ is

$$\{(-1)^{r(j-1)}, (0)^a, (j-1)^r\},$$

and the inertia bound gives $\alpha(C(n,r)) \leq r + a$. Thus if $a = 0$, we can partition the vertices of $C(n,r)$ into r cliques with n/r vertices, and $C(n,r)$ is inertia tight. However, if $a \neq 0$, then

$$\alpha(C(n,r)) = r < r + a,$$

and this weighting does not show that $C(n,r)$ is inertia tight (if indeed it is).

For $(n+1)/3 < r < n/2$, we see that $n = 2r + a$ for some $0 < a < r - 1$. Thus the clique approach fails to show that our 40 stubborn graphs are inertia tight. In order to show that these graphs are inertia tight, we use another method.

2.10 Generator Weightings

Given a graph X, in order to find an optimal bound in Theorem 2.3.1, we are looking for a real symmetric matrix B with the property that $B[i,j] = 0$ whenever $A(X)[i,j] = 0$. We want to find such a matrix B so that the number of strictly negative eigenvalues of B is as small as possible. Valid matrices B correspond exactly to edge weightings of the graph X where there are no restrictions on the weights we can assign to each edge. As such the search space of valid matrices is very large. However, when X is a Cayley graph, there is a natural class of edge weightings of X.

Let $X = X(G,C)$ be a Cayley graph for a finite Abelian group G. Then C is inverse-closed (i.e., $-C = C$), and every edge $e \in E(X)$ has a corresponding inverse pair of generators in C. This follows as $e = \{x, y\} \in E(X)$ implies that for $x, y \in G$, both $x - y$ and $y - x$ are in C. So there is some $c \in C$ with $x = y + c$. Note that every vertex $x \in G$ is incident with an edge corresponding to c for each $c \in C$, the edge $\{x, x + c\}$. So there are two edges incident with x corresponding to the inverse pair $\{c, -c\}$ (unless $c = -c$, in which case x is incident with one edge corresponding to the inverse "pair" $\{c, c\}$). Thus if we partition C into inverse pairs (or singletons)

$$C = \{c_1, -c_1\} \cup \cdots \cup \{c_k, -c_k\},$$

we see that X decomposes into Cayley graphs $X_i = X(G, \{c_i, -c_i\})$ each of which is either a 2-regular graph or a 1-regular graph.

We can use this decomposition of X to construct a weighted adjacency matrix B. Let the vertices of X be ordered as $V(X) = \{x_1, \ldots, x_n\}$. If

$$C = \{c_1, -c_1\} \cup \cdots \cup \{c_k, -c_k\},$$

define the *generator weighting* $\{\alpha_1, \ldots, \alpha_k\}$ of X to be the matrix B with entries

$$B[i, j] = \begin{cases} \alpha_l, & \text{if } x_i - x_j \in \{c_l, -c_l\} \\ 0, & \text{if } x_i \text{ is not adjacent to } x_j. \end{cases}$$

We determine the eigenvalues of B as follows.

Let χ be a linear character of G, and define $z_\chi \in \mathbb{C}^n$ by $z_\chi(x_i) = \chi(x_i)$. We have shown that z_χ is an eigenvector for X with eigenvalue

$$\sum_{c \in C} \chi(c).$$

Consider Bz_χ. The x_i entry of Bz_χ is

$$Bz_\chi[i] = \sum_{j=1}^{n} B[i, j] z_\chi[j] = \sum_{c \in C} \alpha_c \chi(c),$$

where α_c is the weight corresponding to the generator $c \in C$. So calculating the eigenvalues of a weighted adjacency matrix B corresponding to a generator weighting is achieved in much the same way as calculating the eigenvalues of X. Our generator weightings correspond exactly to weightings of the character sums over the connection set.

Using this method we were able to assign generator weightings for each of the 40 cyclic interval graphs from the previous section that achieve equality in the inertia bound. This shows that all of the cyclic interval graphs with $(n + 1)/3 \leq r \leq n/2$ and $2 \leq r \leq 14$ are inertia tight.

To achieve this we were able to find suitable weightings by inspection. But ideally we would like to be able to find optimal generator weightings for a large set of graphs, or even a class of graphs, quickly. In the next section we show that we can employ a linear program to make this task easier.

2.11 A Linear Program

In this section we will see how to construct a linear program to find generator weightings of a Cayley graph that result in optimal inertia bound values. Our method does not require more linear programming or integer programming than can be found in Appendix A and Chapter 6 of Cook et al. [8]. We let $X = X(G, C)$ be a Cayley graph for a finite Abelian group G.

Let $C = C_1 \cup \cdots \cup C_k$ be a partition of the connection set C so that each C_i is inverse closed. Our method will work for any such partition, but in order to consider all possible generator weightings we will assume that each C_i has size 1 or 2 (i.e., each C_i consists of some $g \in G$ and its inverse). Let $\alpha \in \mathbb{R}^k$ be a vector of generator weights, and define

$$B_\alpha = \sum_{i=1}^{k} \alpha_i A(X(G, C_i)).$$

So B_α is the weighted adjacency matrix obtained from $A(X)$ by weighting the edges of X with generator weights α. Recall that the eigenvalues of B_α are

$$\left\{ \sum_{i=1}^{k} \alpha_i \left(\sum_{c \in C_i} \chi(c) \right) : \chi \text{ is a linear character of } G \right\}.$$

Let

$$(n^+(B_\alpha), n^0(B_\alpha), n^-(B_\alpha))$$

be the inertia of B_α. We want to find a weighting α that minimizes

$$\min\{n - n^-(B_\alpha), n - n^+(B_\alpha)\}.$$

For each $\alpha \in \mathbb{R}^k$ we can compute the eigenvalues of B_α explicitly. Let $\chi_1, \ldots, \chi_n$ be an ordering of the linear characters of G, and let

$$v_j = \sum_{i=1}^{k} \alpha_i \sum_{c \in C_i} \chi_j(c)$$

be the jth eigenvalue of X (according to the ordering of the characters). Now we can express our problem as a mathematical program (P).

$$\begin{array}{llrcll}
(P) & \min & s & & & \\
 & \text{s.t.} & v_j & = & \sum_{i=1}^{k} \alpha_i \sum_{c \in C_i} \chi_j(c) & \text{for } 1 \le j \le n \\
 & & a_j & = & \begin{cases} 1, \text{if } v_j > 0 \\ 0, \text{else} \end{cases} & \text{for } 1 \le j \le n \\
 & & b_j & = & \begin{cases} 1, \text{if } v_j < 0 \\ 0, \text{else} \end{cases} & \text{for } 1 \le j \le n \\
 & & s & \ge & \min\{n - \mathbf{1}^T a, n - \mathbf{1}^T b\} & \\
 & & \alpha & \in & \mathbb{R}^k &
\end{array}$$

Here $\mathbf{1}$ is the vector with all entries equal 1, and a and b are the vectors with $a[j] = a_j$ and $b[j] = b_j$ respectively. This program computes the best possible inertia bound value over all possible generator weightings of X. However, in order to use (P) to compute an optimal weighting for a given graph, we need to transform it into a linear program.

The program (P) is close to being a linear program. We only need to replace the definition of a_j and b_j with linear constraints, and replace the constraint

$$s \geq \min\{n - \mathbf{1}^T a, n - \mathbf{1}^T b\}$$

with a linear constraint.

The second task is easily accomplished. We introduce a decision variable $d \in \{0, 1\}$ and use the constraints

$$\begin{aligned} s &\geq n(1-d) - \mathbf{1}^T a \\ s &\geq nd - \mathbf{1}^T b. \end{aligned}$$

Since the values of a, b and $\mathbf{1}$ are non-negative,

$$0 \leq \mathbf{1}^T a, \quad \mathbf{1}^T b \leq n.$$

Thus when $d = 1$, the first inequality is weaker than the second, and when $d = 0$ the reverse is true. So if

$$\min\{n - \mathbf{1}^T a, n - \mathbf{1}^T b\} = n - \mathbf{1}^T a,$$

setting $d = 0$ results in $s = n - \mathbf{1}^T a$ as desired. The other case is handled likewise.

The first task is more troublesome. Consider the constraint

$$a_j = \begin{cases} 1, \text{if } v_j > 0 \\ 0, \text{else} \end{cases}$$

for some $1 \leq j \leq n$. If we let M be some large fixed real number, then the constraints

$$\begin{aligned} Ma_j - v_j &\geq 0 \\ Ma_j - v_j &\leq M \end{aligned}$$

enforce the conditions: $0 < v_j \leq M$ implies $a_j = 1$; and, $-M \leq v_j < 0$ implies $a_j = 0$. However, when $v_j = 0$, $a_j = 0$ and $a_j = 1$ both satisfy these two constraints. In order to fix this problem, we let $\epsilon > 0$ be some small fixed real number. The constraints

$$\begin{aligned} Ma_j - v_j &\geq 0 \\ Ma_j - v_j &\leq M - \epsilon \end{aligned}$$

enforce the conditions

$$a_j = \begin{cases} 1, \text{if } \epsilon \leq v_j \leq M \\ 0, \text{if } -(M - \epsilon) \leq v_j \leq 0 \end{cases} \quad .$$

Thus we have eliminated the possibility that $a_j = 0$ or 1 are both feasible values.

The price of these modifications is that we have reduced the space of feasible solutions to our program. By adding the last set of constraints, we have constrained each eigenvalue to lie in the interval $[-M, M]$ for some positive real number M. This restriction does not affect the optimal value of our program. If α is a generator weighting and B_α is the resulting weighted adjacency matrix, then for any positive real number a, the matrix aB_α is a weighted adjacency matrix. Moreover, θ is an eigenvalue for B_α if and only if $a\theta$ is an eigenvalue for aB_α, so B_α and aB_α have the same inertia, and thus give the same value in the inertia bound. Finally, we note that

$$aB_\alpha = B_{a\alpha},$$

and so aB_α is also a generator weighting of X. Therefore, if α is an optimal solution to (P), there is some $a \in \mathbb{R}$ so that $a\alpha$ is an optimal solution to P, and the eigenvalues of $B_{a\alpha}$ are all contained in $[-M, M]$.

Adding ϵ results in a more serious restriction. In fact, the last two constraints imply that the eigenvalues v_j all lie in the set

$$S = [-(M - \epsilon), 0] \cup [\epsilon, M].$$

It is possible that there are optimal solutions α to (P) so that there is no scalar multiple of B_α whose eigenvalues all lie in the set S. Thus by using these constraints we are not only reducing the number of feasible solutions, but we may also be changing the optimal value of our original program.

The resulting linear program is still useful for searching for optimal generator weightings of X. In particular, given a graph X, our program may find a generator weighting that meets the inertia bound. But, if there is a gap between $\alpha(X)$ and the value returned by the program, it does not give a certificate that better generator weightings do not exist.

We give our final program as a mixed linear integer program, $(MILP)$.

$$\begin{array}{lllll}
(MILP) & \max & -s & & \\
& \text{s.t.} & v_j &=& \sum_{i=1}^k \alpha_i \sum_{c \in C_i} \chi_j(c) \quad \text{for } 1 \le j \le n \\
& & Ma_j - v_j &\ge& 0 \quad \text{for } 1 \le j \le n \\
& & Ma_j - v_j &\le& M - \epsilon \quad \text{for } 1 \le j \le n \\
& & Mb_j + v_j &\ge& 0 \quad \text{for } 1 \le j \le n \\
& & Mb_j + v_j &\le& M - \epsilon \quad \text{for } 1 \le j \le n \\
& & s &\ge& n(1-d) - \mathbf{1}^T a \\
& & s &\ge& nd - \mathbf{1}^T b \\
& & a &\in& \{0,1\}^n \\
& & b &\in& \{0,1\}^n \\
& & d &\in& \{0,1\} \\
& & \alpha &\in& \mathbb{R}^k
\end{array}$$

Note that in $(MILP)$ we have the constraint $\alpha \in \mathbb{R}^k$, and some of our eigenvalues v_j may have irrational values. However, in practice, we will use a rational approximation to the eigenvalues of our matrix, and we will constrain α to be a rational vector, $\alpha \in \mathbb{Q}^k$. We also have a choice of M and ϵ when we compute solutions to $(MILP)$. For our computations we used $M = 10^6$ and $\epsilon = 10$. A ratio of $\epsilon = M/10^5$ seemed to work best with the software we used, and $\epsilon = 10$ (as opposed to $\epsilon = 1$ or 10^{-1}) gave weightings that were clearest to read.

2.12 Computational Findings

In his PhD thesis, Elzinga [11] used a computer search to show that all graphs on at most 10 vertices have a weighted adjacency matrix that meets the inertia bound. He extended these calculations to show that all vertex-transitive graphs on at most 12 vertices have a weighted adjacency matrix that meets the inertia bound. The method Elzinga employs uses α-critical graphs.

A graph G is *α-critical* if for all edges $e \in E(G)$, $\alpha(G \setminus e) > \alpha(G)$. In order to show that the inertia bound is met by all graphs, it would suffice to prove that all α-critical graphs are inertia tight. This follows, as if G is not α-critical, then there is some subset $S \subseteq E(G)$ for which $\alpha(G \setminus S) = \alpha(G)$ and $G \setminus S$ is α-critical. Now if $G \setminus S$ is inertia tight, then we can extend an inertia tight weighting of the edges of $G \setminus S$ to an inertia tight weighting of the edges of G simply by assigning weight 0 to the edges in S.

Elzinga [11] used a computer to find the α-critical graphs on at most 10 vertices, then found optimal weightings by inspection. In some cases, a non-trivial weighting was required to meet the inertia bound.

2.12.1 Example. The Paley graphs are a family of Cayley graphs defined as follows. For a prime power q with q congruent to 1 modulo 4, the *Paley graph of order* q is $P(q) = X(GF(q), C)$ where $GF(q)$ is the finite field with q elements, and C is the set of non-zero squares in $GF(q)$. For example if $q = 5$, then $GF(5)$ is the set of integers modulo 5, and the non-zero squares are 1 and 4. Thus $P(5)$ is the 5-cycle.

The graph $P(13)$ is the Cayley graph on $\mathbb{Z}_{13}$ with connection set

$$\{1, 3, 4, 9, 10, 12\}.$$

We can partition the vertices of $P(13)$ according to their distance from 0. Since $P(13)$ has diameter 2 this partition has three parts,

$$\{0\}, \quad \{1, 3, 4, 9, 10, 12\}, \quad \{2, 5, 6, 7, 8, 11\}.$$

The last set of vertices induces a graph isomorphic to the triangular prism graph (two disjoint triangles joined by a perfect matching), and thus has independence number 2. So any coclique in $P(13)$ containing 0 has size at most 3. Since $P(13)$ is vertex transitive, we have that $\alpha(P(13)) = 3$. Can we find a weighting that shows $P(13)$ is inertia tight?

Note that the neighbourhood of 0 in $P(13)$ is a 6-cycle. So the size of a largest clique in $P(13)$ is 3. We can partition $P(13)$ into four copies of K_3, together with one copy of K_1, as

$$\{0,9,10\},\quad \{1,2,5\},\quad \{3,4,7\},\quad \{8,11,12\},\quad \{6\}.$$

The resulting weighted adjacency matrix has spectrum

$$\{(-1)^8,(0)^1,(2)^4\},$$

and implies that $\alpha(P(13)) \leq 5$. We can try replacing $\{3,4,7\}$ and $\{6\}$ with $\{3,6\}$ and $\{4,7\}$. Now our partition consists of three copies of K_3 and two copies of K_2. The resulting spectrum is

$$\{(-1)^8,(1)^2,(2)^3\}$$

and gives the same value in the inertia bound. So $\alpha(P(13)) \leq 5$ is the best bound achievable by a partition of $P(13)$ into cliques.

Elzinga considered $P(13)$ and found by inspection a weighting of the generators that gives $\alpha(P(13)) \leq 4$. His weighting assigns value 1 to the edges corresponding to the generators $\{1,12\}$ and weight -1 to all of the other edges.

We ran $(MILP)$ using the eigenvalues of $P(13)$. The program found an optimal weighting with weight 0.31234 assigned to generators $\{1,12\}$, weight 0.16395 assigned to generators $\{3,10\}$ and weight 0.0237 assigned to $\{4,9\}$. The objective value of this solution is 4. So we have strong evidence that $\alpha(P(13)) \leq 4$ is the best bound obtainable by a weighting of the generators of $P(13)$.

The question of whether $P(13)$ is inertia tight is still open.

We were interested in approaching the question of whether there are graphs that are not inertia tight by using the mixed integer linear program $(MILP)$. The linear program corresponding to a given Cayley graph is relatively small. If $X(G,C)$ has n vertices, then $(MILP)$ has $3n+2$ variables and a number of constraints linear in n. So the resulting programs can be solved very efficiently. Our calculations were performed using the mathematics software system Sage [30] run on a laptop computer.

We generated the complete set of circulant graphs on n vertices for $23 \leq n \leq 32$. From these graphs we considered the non-bipartite connected graphs and computed their independence numbers, and inertia bound values. For graphs that did not meet the inertia bound with their unweighted adjacency matrix, we ran $(MILP)$ to find a weighting of the generators that gave an optimal tightening of the inertia bound over weightings of the generators.

In our calculations we omitted the graphs that are inertia tight using their unweighted adjacency matrix. We also omit the graphs X for which the clique number $\omega(X)$ and independence number $\alpha(X)$ satisfy $\omega(X)\alpha(X) = |V(X)|$, as those graphs can be partitioned into $\alpha(X)$ copies of $K_{\omega(X)}$, and are thus inertia tight.

Table 2.12 is a summary of the data. The columns give: the size of the graphs n; the total number of graphs on n vertices; the number that are inertia tight using the unweighted adjacency matrix; the number for which $\omega(X)\alpha(X) =$

$|V(X)|$; and finally the results of running $(MILP)$ on the graphs that are not inertia tight and for which $\omega(X)\alpha(X) < |V(X)|$. The entries in the final column are in the form $\{a : b\}$ where a is the difference between the optimal value of $(MILP)$ and $\alpha(X)$ (i.e., the slack in the inertia bound value) and b is the number of graphs that achieve that difference.

Table 2.1: Circulant LP Data Summary

n	Total	I.T.	$\omega\alpha = n$	$(MILP)$ Differences
23	186	6	0	$\{0:176\}, \{1:3\}, \{2:1\}$
24	1234	7	583	$\{0:439\}, \{1:158\}, \{2:37\}, \{3:10\}$
25	419	5	76	$\{0:324\}, \{1:13\}, \{2:1\}$
26	1358	7	25	$\{0:1103\}, \{1:206\}, \{2:17\}$
27	919	9	110	$\{0:753\}, \{1:42\}, \{2:5\}$
28	3027	8	643	$\{0:1762\}, \{1:541\}, \{2:70\}, \{3:3\}$
29	1180	7	0	$\{0:1100\}, \{1:65\}, \{2:8\}$
30	8621	12	2677	$\{0:4791\}, \{1:1024\}, \{2:89\}, \{3:10\}$
31	2190	5	0	$\{0:2005\}, \{1:165\}, \{2:15\}$
32	8244	3	1422	$\{0:4671\}, \{1:1753\}, \{2:369\}, \{3:16\}, \{4:10\}$

From the data we see that the worst possible inertia value slack is growing with n. This shows that neither the method of weighting the generators of X, nor the method of partitioning X into disjoint cliques performs well as n increases. So in order to show that all of the circulants for $23 \leq n \leq 32$ vertices are inertia tight, we will need to find another approach.

2.13 Cubelike Graphs

Cubelike graphs are Cayley graphs for the groups $\mathbb{Z}_2^n$. We also looked at the inertia bound values given by weighting the generators of a cubelike graph. In particular, we looked at the cubelike graphs on 32 vertices. Our thanks to Gordon Royle for providing a user-friendly list of the non-isomorphic cubelike graphs on 32 vertices.

There are 1372 non-isomorphic cubelike graphs on 32 vertices. We again restricted our attention to those that are non-bipartite and connected. Of these 1304 graphs, only 6 meet the inertia bound with their unweighted adjacency matrix. That leaves 1298 graphs, 1177 of which have the property that $\omega(X)\alpha(X) = 32$.

We ran $(MILP)$ on the remaining 121 cubelike graphs on 32 vertices. We found that 41 of these had generator weightings that gave equality in the inertia bound, 41 had optimal generator weightings that gave an inertia bound value of $\alpha(X) + 1$ and the remaining 39 had optimal generator weightings that gave an inertia bound value of $\alpha(X) + 2$.

From this data we see that our linear programming approach performs better for the cubelike graphs on 32 vertices than for the circulants on 32 vertices. However, we still have 80 cubelike graphs on 32 vertices for which inertia tightness is not determined.

2.14 Open Problems

The central open problem for the inertia bound is whether or not there exist graphs that have no inertia tight weighting. We have seen that this problem is very difficult. Even for $P(13)$, a graph on 13 vertices with considerable structure, ruling out inertia tight weightings is computationally infeasible. The computational tools we have developed in this chapter suggest some "easier" open problems.

We can constrain the question of whether or not inertia loose graphs exist to the question: given a subset of weighted adjacency matrices, is it possible to find a weighting that is inertia tight? For example, we saw that partitioning a graph into disjoint cliques gives inertia tight weightings for Cayley graphs with $\alpha(X)\omega(X) = |V(X)|$. However, we also were able to show that $P(13)$ does not have a partition into cliques that gives equality in the inertia bound. We can ask the same question for generator weightings. Is there a Cayley graph for which there is no inertia tight generator weighting? Our data suggests that these graphs do exist. However, the results of our linear program do not constitute a proof.

Chapter 3

Computational Complexity of Maximum Coclique

In this chapter we consider the computational complexity of the Maximum Coclique problem. The *Maximum Coclique problem* is the problem of finding a coclique of maximum size in a given graph. It is well known that for the class of all graphs, the Maximum Coclique problem is NP-Hard. That is, given the assumption that P$\neq$NP, there is no algorithm that solves the Maximum Coclique problem for all graphs and runs in polynomial time.

Throughout this chapter, we use very little complexity theory. We will only rely on a few facts about the classes of computational problems (such as P, NP, NP-Complete and NP-Hard). The class P consists of those problems that can be solved in *polynomial time* (i.e., for each $\Pi \in$P there is an algorithm that solves each instance of Π in time bounded by a fixed polynomial in the size of the input). In order to show that a problem Π_1 is NP-hard, we will *reduce* a problem Π_2 that is known to be NP-Hard to Π_1. By this we mean that we will show that given an oracle Ω that solves Π_1 in polynomial time, we can describe an algorithm that solves Π_2 in polynomial time. For a more rigourous treatment of this subject we refer to Papadimitriou [29].

The Maximum Coclique problem is equivalent to the Maxclique problem. This follows from the fact that cocliques in a graph X are cliques in $\overline{X}$, the complement of X. In this chapter we will formulate our results in terms of cliques, and the Maxclique problem. But our results apply equally for cocliques, and the Maximum Coclique problem.

Finding the clique number $\omega(X)$ of a given graph X is a hard problem in general. However, we can make this problem easy by restricting the class of graphs we are working with. For instance, if we look at the Maxclique problem on the class of forests (i.e., acyclic graphs), there is a very fast algorithm that returns a clique of maximum size. If a forest X has an edge, we return the edge, if not we return any vertex. Since X contains no cycles, it contains no triangles and so $\omega(X) = 1$ or 2. This is a very trivial example, however it leads us to a more general question. Given a class $\mathcal{X}$ of graphs, what is the computational complexity of the Maxclique problem restricted to $X \in \mathcal{X}$?

In [7], Codenotti et al. consider this problem where $\mathcal{X}$ is the class of circulant graphs (i.e., Cayley graphs for the groups $\mathbb{Z}_m$). The class of circulants is a very restricted class of graphs, but Codenotti et al. were able to prove the following result.

3.0.1 Theorem (Theorem 1 in [7])**.** *The Maxclique problem restricted to circulant graphs is NP-Hard.* □

Despite the additional structure of circulant graphs, the computational complexity of the Maxclique problem is the same as for the class of all graphs.

This result is counter-intuitive, as Cayley graphs are vertex transitive. So in order to find a clique of maximum size in a Cayley graph X, it suffices to pick any vertex $x \in V(X)$, and try to find a maximum clique in the neighbourhood of x. However, the result of Codenotti et al. shows that for circulants the extra regularity is not helpful.

In this chapter we develop a new method to prove the following similar theorem.

3.0.2 Theorem. *For a fixed prime p, the Maxclique problem restricted to Cayley graphs for the family of groups $\mathbb{Z}_p^n$ is NP-hard.* □

While Codenotti et al. are able to rely on a result in additive number theory, this fails for the class of Cayley graphs we consider. Our method employs quotient graphs and coding theory to prove Theorem 3.0.2. As a result, our method has greater potential for generalization, and application to different classes of Cayley graphs.

3.1 3-Sum Respecting Assignments

In order to show that the Maxclique problem is at least as hard for a class of Cayley graphs as it is for the class of all graphs, we exhibit a polynomial-time reduction. Towards our specific construction, we first develop a general construction of an auxiliary Cayley graph whose cliques are related to the cliques of an input graph. This is a generalization of the method used by Codenotti et al. in [7]. Our auxiliary graph will grow exponentially with the size of the input, we will fix this later.

Given a graph X, and an Abelian group G, let $\eta : V(X) \to G$ be an assignment of group elements to the vertices of X. For convenience, we will assume that $V(X) = [n]$, and we denote $\eta(i)$ by g_i. For a positive integer k, we say that η is a *k-sum respecting assignment* if the sums

$$\sum_{s \in S} g_s$$

are distinct for all multisets S of elements of $[n]$ of size k.

Note that if η is k-sum respecting, then the k-sums kg_i must all be distinct. In particular, at most one g_i can have order k. We also have that if η is k-sum respecting, then η is also i-sum respecting for all $i \leq k$. To see this, assume S and T are two multisets of elements of $[n]$ of size i with

$$\sum_{t\in T} g_t = \sum_{s\in S} g_i.$$

Extend S, T to S', T' respectively by adding $k-i$ copies of g_1 to each. Now,

$$\sum_{s\in S'} g_s = (k-i)g_1 + \sum_{s\in S} g_s = (k-i)g_1 + \sum_{t\in T} g_t = \sum_{t\in T'} g_t$$

which gives us a contradiction.

3.1.1 Example. Let G be an Abelian group, and X be a graph on n vertices. The direct product G^n is an Abelian group. Let $g \in G \setminus \{0\}$ and let $g_i \in G^n$ be the element with g in the ith component and 0 in all other components. Define $\eta(i) = g_i$. It is easy to see that the k-sums of distinct elements of $\{g_i : 1 \leq i \leq n\}$ are all distinct for any $1 \leq k \leq n$. Thus if g has order k, then η is a $(k-1)$-sum respecting assignment.

3.1.2 Example. Let $G = \mathbb{Z}_2$, and X be a graph on n vertices. The direct product G^k is an Abelian group for any $k \geq 1$. Since every element of G has order at most 2, any assignment $\eta : V(X) \to G^k$ will be at most 1-sum respecting.

We now show that 3-sum respecting assignments can be used to construct auxiliary Cayley graphs with the same clique number as our starting graph. The ideas and proofs are slight adaptations of those demonstrated by Codenotti et al. in [7].

First we construct a Cayley graph for G from our given graph X. Given a graph X and an assignment η of elements of an Abelian group G to the vertices of X, we define the set C as,

$$C = \{g_i - g_j \,:\, i \text{ is adjacent to } j \text{ in } X\}.$$

Define the Cayley graph Γ to be $\Gamma = X(G, C)$. Since $g_i - g_j \in C$ implies that $g_j - g_i \in C$, our graph Γ is indeed a graph (and not a directed graph). This definition is valid for all assignments η, however from here on we assume that η is 3-sum respecting.

Note that since η is 3-sum respecting, it is also 2-sum respecting. This implies that the differences $g_i - g_j$ are all distinct, as

$$g_i - g_j = g_k - g_l$$

implies

$$g_i + g_l = g_k + g_j$$

and we have a contradiction. Thus our connection set C has size $2|E(X)|$. Moreover, η is also 1-sum distinct, so all of the elements g_i are distinct, and $g_i - g_j \neq 0$ for any i and j. Thus $0 \notin C$ and Γ is loopless.

For the remainder of this section we assume that Γ is constructed from X using a 3-sum respecting assignment. The following pair of lemmas show $\omega(\Gamma) = \omega(X)$.

3.1.3 Lemma. *If S is a clique in X and $i \in S$, then*

$$T = \{g_j - g_i \,:\, j \in S\}$$

is a clique in Γ containing 0. Moreover, $|S| = |T|$.

Proof. Since S is a clique in X, $g_j - g_k \in C$ for all distinct $j, k \in S$. In particular, $g_j - g_i \in C$ for all $j \in S$. So $0 \in T$, and 0 is adjacent to all other elements of T in Γ.

Now suppose $j, k \in S \setminus \{i\}$. We have

$$(g_i - g_j) + (g_j - g_k) = g_i - g_k.$$

Since $g_j - g_k \in C$, we have that $g_i - g_j$ is adjacent to $g_i - g_k$ in Γ. Thus T is a clique in Γ containing 0.

To see that $|S| = |T|$ we only need the differences $g_i - g_j$ to be distinct for all $j \in S$. We have already noted that this is true when η is 3-sum respecting. □

3.1.4 Lemma. *If S is a clique in Γ, then there is a clique in X with size $|S|$.*

Proof. If S is a clique in Γ, then we can translate S to a clique T in Γ with $|S| = |T|$ and $0 \in T$. To do this we simply take $s \in S$ and set

$$T = \{s' - s \,:\, s' \in S\}.$$

So from here on we assume that $0 \in S$.

Consider $g_i - g_j$ adjacent to $g_k - g_l$ in the neighbourhood of 0. Since these vertices are adjacent, we must have some $g_s - g_t \in C$ so that

$$(g_i - g_j) + (g_s - g_t) = g_k - g_l.$$

Rearranging we see that

$$g_i + g_s + g_l = g_j + g_t + g_k.$$

Since our assignment is 3-sum respecting, we conclude that the sets of indices on either side of the equation are the same, $\{i, s, l\} = \{j, t, k\}$. Since none of the vertices we started with were 0, we see that $i \neq j$, $k \neq l$ and $s \neq t$. So there are two possibilities, either $(i, s, l) = (k, j, t)$, or $(i, s, l) = (t, k, j)$. In the first case we have that $g_i - g_j$ is adjacent to $g_i - g_l$ by the element $g_j - g_l$ of C. We see that ijl forms a triangle in X. Likewise, the second case gives triangle ijk in X. So every triangle in Γ containing 0 corresponds to a triangle in X.

Now suppose that ijk is a triangle in X. We have that

$$\pm(g_i - g_j), \quad \pm(g_j - g_k), \quad \pm(g_i - g_k)$$

are all in C, and thus are vertices in the neighbourhood of 0. Note that by the previous paragraph, the adjacencies between these vertices are the solutions to the "equation"

$$\pm(g_i - g_j) \pm (g_j - g_k) = \pm(g_i - g_k)$$

(more precisely, the assignments of signs so that the resulting equation is valid). Which assignments are valid?

Consider the equation

$$(g_i - g_j) + (g_j - g_k) = -(g_i - g_k).$$

Rearranging the terms we have that $2g_i = 2g_k$, and the fact that η is 2-sum respecting implies that $i = k$. This contradicts the fact that we started with a triangle in X. The other assignments with a mixture of positive and negative signs can easily be seen to lead to contradictions as well.

The only valid assignments of signs to the three terms are to make each term positive, or to make each term negative. From the first assignment we have

$$(g_i - g_j) + (g_j - g_k) = g_i - g_k$$

which gives edges

$$\{g_i - g_j, g_i - g_k\}, \quad \{g_j - g_k, g_i - g_k\}, \quad \{g_i - g_j, g_k - g_j\}.$$

From the second we have

$$-(g_i - g_j) - (g_j - g_k) = -(g_i - g_j)$$

which gives edges

$$\{g_j - g_i, g_k - g_i\}, \quad \{g_k - g_j, g_k - g_i\}, \quad \{g_j - g_i, g_j - g_k\}.$$

These six vertices and six edges form an induced 6-cycle in the neighbourhood of 0. Thus every triangle in X corresponds to an induced 6-cycle in Γ.

Finally, consider the elements of S. Suppose $s_e, s_f \in S \setminus \{0\}$ where e, f are edges of X. Since $0, s_e, s_f$ is a triangle, there is an edge g in X so that e, f, g are the edges of a triangle in X. Thus there is some vertex p in X so that p is shared by e and f. Now suppose we have $s_e, s_f, s_g \in S \setminus \{0\}$ so that the vertex p shared by e and f is different from the vertex q shared by f and g. Since e and g also share a vertex, we must have that e, f, g are the edges of a triangle in X. But as we have shown, this triangle corresponds to a 6-cycle in Γ containing s_e, s_f, s_g, a contradiction. Thus there is some vertex p in X so that p is an end of e for all $s_e \in S \setminus \{0\}$. So without loss of generality, every element of $S \setminus \{0\}$ has the form $g_p - g_q$, and

$$\{p\} \cup \{q \,:\, g_p - g_q \in S\}$$

is a clique in X with size $|S|$. $\square$

From these two lemmas we immediately have the following corollary.

3.1.5 Corollary. $\omega(X) = \omega(\Gamma)$. □

Note that Lemmas 3.1.3 and 3.1.4 are constructive. They both provide methods of finding a maximum clique in one graph, given a maximum clique in the other. Moreover, each of these implicit algorithms runs in time polynomial in the size of their inputs.

In [7] this construction is used to show that the Maxclique problem is NP-Hard for circulants. The proof relies on a result of Bose and Chowla (Theorem 1 in [1]) that guarantees a 3-sum respecting assignment of elements of $\mathbb{Z}_m$ to the vertices of X where m is polynomial in the number of vertices of X. After applying this result to guarantee a favourable assignment, the remainder of the proof uses the construction above to obtain an auxiliary graph Γ, a circulant, and reduce the problem of finding a maximum clique in X to finding a maximum clique in Γ.

3.2 The Graph Γ

In this section we assume that Γ is constructed from X using a 3-sum respecting assignment. There are more relations between Γ and X aside from their cliques. To start, we note that X is an induced subgraph of Γ.

3.2.1 Proposition. *The vertices $\{g_i : 1 \leq i \leq n\}$ induce a subgraph of Γ isomorphic to X.*

Proof. Let $S = \{g_i : 1 \leq i \leq n\}$, and let $f : S \to V(X)$ be defined as $f(g_i) = i$. Clearly f is a bijection between S and $V(X)$.

Suppose g_i is adjacent to g_j in Γ. Then there is some $g_s - g_t \in C$ so that

$$g_i + (g_s - g_t) = g_j.$$

Rearranging we have

$$g_i + g_s = g_j + g_t.$$

Since η is 2-sum respecting we must have that $\{i, s\} = \{j, t\}$. Given that $i \neq j$ the only possibility is that $s = j$, $t = i$, and $g_j - g_i \in C$. Thus $\{i, j\} \in E(X)$ and i is adjacent to j in X. Thus f is a bijective homomorphism. □

As a direct consequence of Proposition 3.2.1 we have that $\omega(\Gamma) \geq \omega(X)$. So we could have cited Proposition 3.2.1 instead of Lemma 3.1.3 in the proof of Corollary 3.1.5. However, the constructive nature of Lemma 3.1.3 is important, as we want to show that we can easily construct a maximum clique in X given a maximum clique in our auxiliary graph Γ.

Given a graph X, the *line graph* of X, denoted $L(X)$, is the graph on $E(X)$ where two edges are adjacent if and only if they share an endpoint. Let $T(X)$ be the graph on $E(X)$ where two edges are adjacent if and only if they both lie in a triangle in X. The graph $T(X)$ is a subgraph of the line graph $L(X)$.

Let X and Y be graphs, and $h : X \to Y$ be a homomorphism. If for all $y \in V(Y)$ the map induced by h from the neighbours of a vertex in $h^{-1}(y)$ to the neighbours of y is a bijection, then h is a *local isomorphism*. The map h is a *covering map* if h is a local isomorphism and a surjection. We say that X is a *cover* of Y. If $|h^{-1}(y)| = r$ for all $y \in V(Y)$, we say that X is an *r-fold cover* of Y.

3.2.2 Lemma. *The neighbourhood of* 0 *in* Γ *is a 2-fold cover of* $T(X)$.

Proof. Let $\Gamma[0]$ denote the neighbours of 0 in Γ. Define $h : \Gamma[0] \to V(T(X))$ by

$$h(g_i - g_j) = h(g_j - g_i) = \{i, j\}.$$

From the definition of C we see that h is clearly a surjection, and that

$$|h^{-1}(\{i, j\})| = 2$$

for all edges $\{i, j\}$ in X. It remains to show that h is a homomorphism, and a local isomorphism.

Recall from the proof of Lemma 3.1.4 that $g_i - g_j$ is adjacent to $g_k - g_l$ in $\Gamma[0]$ if and only if either $k = i$ or $j = l$, and ijl or ijk respectively is a triangle in X. Thus if $g_i - g_j$ is adjacent to $g_i - g_l$, then $\{i, j\}$ and $\{i, l\}$ are edges of X that lie in a triangle. So $\{i, j\}$ is adjacent to $\{i, l\}$ in $T(X)$. (The case $g_i - g_j$ adjacent to $g_k - g_j$ is similar.) Therefore h is a homomorphism.

Finally, if $\{i, j\}$ is an edge of X, then

$$h^{-1}(\{i, j\}) = \{g_i - g_j, g_j - g_i\}.$$

Consider the map induced by h between the neighbours of $g_i - g_j$ and the neighbours of $\{i, j\}$. If $\{i, j\}$ is adjacent to $\{i, l\}$ in $T(X)$, then $g_i - g_j$ is adjacent to $g_i - g_l$ and

$$h(g_i - g_l) = \{i, l\}$$

so h induces a surjection. If $g_k - g_l$ and $g_s - g_t$ are both neighbours of $g_i - g_j$ in Γ, then

$$h(g_k - g_l) = \{k, l\}$$

and

$$h(g_s - g_t) = \{s, t\}.$$

If $\{k, l\} = \{s, t\}$, then either $k = s$ or $k = t$. If $k = s$, then

$$g_k - g_l = g_s - g_t.$$

If $k = t$, then

$$g_k - g_l = -(g_s - g_t).$$

However, in order to be a neighbour of $g_i - g_j$ we must have that either $k = i$ or $l = j$, and either $s = i$ or $t = j$. If $k = i$, then $t = i \neq j$ so we must have $s = i = t$ which is a contradiction. The other cases give similar contradictions. Thus h induces an injection, and the induced map is a bijection. $\square$

3.3 Cayley Graphs for $\mathbb{Z}_m^n$

The groups $\mathbb{Z}_m^n$ where m and n are positive integers are Abelian groups, so we can apply the construction from Section 3.1 using these groups. We are working towards proving that the Maxclique problem is NP-Hard for the class of Cayley graphs on $\mathbb{Z}_m^n$ where m is a fixed prime. For now we let m be some fixed positive integer.

We saw in Example 3.1.1 that if we are given a graph X, we can naturally construct an assignment η of elements of G^n to $V(X)$ where n is the number of vertices of X and G is an Abelian group. Let $G = \mathbb{Z}_m$, and take the element $g = 1$. Now as in Example 3.1.1 we take $g_i \in \mathbb{Z}_m^n$ to be the element with 1 in the ith component and 0 in every other component. Since 1 has order m in $\mathbb{Z}_m$, the resulting assignment is $(m-1)$-sum respecting. So if we construct Γ as in Section 3.1.5, we will have a Cayley graph for $\mathbb{Z}_m^n$ with $\omega(\Gamma) = \omega(X)$ provided that $m \geq 4$.

This is a good start for proving a reduction of the Maxclique problem on general graphs to the Maxclique problem on the class of Cayley graphs for the groups $\mathbb{Z}_m^n$. However, our assignment results in a graph Γ on m^n vertices, which is exponential in the number of vertices of X. In order to address this problem, we will use a quotient graph. First we address the cases $m = 3$ and $m = 2$.

3.4 Cayley Graphs for $\mathbb{Z}_2^n$

Let $m = 2$, and consider a graph X with auxiliary graph Γ constructed as usual. Since every element g_i has order 2, we are not guaranteed that the 3-sums or 2-sums of the elements g_i are distinct. For example $2g_i = 2g_j$ for all $1 \leq i, j \leq n$. However, if we add the restriction that the summands are distinct, then we can conclude that the sums are distinct.

3.4.1 Proposition. *If $m = 2$, the 2-sums $g_i + g_j$ for distinct $1 \leq i, j \leq n$ are distinct. The 3-sums $g_i + g_j + g_k$ where $|\{i, j, k\}| = 1, 3$ are distinct.*

Proof. If $i \neq j$, then $g_i + g_j$ has exactly two non-zero components, i and j. If

$$g_i + g_j = g_k + g_l$$

then g_k+g_l has non-zero components i and j and we conclude that $\{i, j\} = \{k, l\}$.

If $|\{i, j, k\}| = 3$ then $g_i + g_j + g_k$ has exactly three non-zero components i, j, k. If $|\{i, j, k\}| = 1$ then

$$g_i + g_j + g_k = 3g_i = g_i.$$

Now clearly if $|\{i, j, k\}| = 1, 3$ and $|\{r, s, t\}| = 1, 3$, then

$$g_i + g_j + g_k = g_r + g_s + g_t$$

implies that $\{i, j, k\} = \{r, s, t\}$. □

Since each g_i has order 2 in $\mathbb{Z}_2^n$, $g_i = -g_i$ for all $1 \leq i \leq n$. Also

$$g_i - g_j = g_i + g_j = -g_i + g_j,$$

and our connection set C is

$$C = \{g_i + g_j : i \text{ and } j \text{ are adjacent in } X\}.$$

So $\Gamma = X(C, \mathbb{Z}_2^n)$ is $|E(X)|$-regular (as opposed to $2|E(X)|$-regular). We show that the properties in Proposition 3.4.1 are enough to guarantee that $\omega(\Gamma)$ and $\omega(X)$ are closely related.

We begin by noting that unlike the case $m \geq 4$, we will not be able to conclude that $\omega(\Gamma) = \omega(X)$ for all graphs X.

3.4.2 Proposition. *If Y is a cubelike graph, then $\omega(Y) \neq 3$.*

Proof. Let $Y = X(\mathbb{Z}_2^n, C)$, and suppose that Y contains a triangle. Without loss of generality, we consider the triangle $\{0, a, b\}$ where $a, b \in C$. Since a and b are adjacent in Y, we have that $a + b \in C$. Thus $a, b, a + b$ are all adjacent to 0, and since $a, b \in C$, a and b are both adjacent to $a + b$. Therefore $\{0, a, b, a + b\}$ is a clique of size 4 in Y. □

This proposition shows that if $\omega(X) = 3$, we will not be able to translate cliques of maximum size in Γ into maximum cliques of the same size in X. However, we can show that this is the only problematic case for $m = 2$. We start with a simple observation.

3.4.3 Proposition. *Let $m = 2$ and $k \geq 4$. Suppose $0, h_1, \ldots, h_k$ is a clique in Γ where $h_i = g_i^{(1)} + g_i^{(2)}$. Then there is some $g \in \{g_i : 1 \leq i \leq n\}$ so that for each $1 \leq i \leq k$, there is $j \in \{1, 2\}$ with $g = g_i^{(j)}$.*

Proof. Since h_i and h_j are adjacent for each $1 \leq i, j \leq k$, we have

$$g_i^{(1)} + g_i^{(2)} + g_s + g_t = g_j^{(1)} + g_j^{(2)}$$

for some $g_s + g_t \in C$. From the proof of Proposition 3.4.2 we see that we must have either

$$g_i^{(1)} \in \{g_j^{(1)}, g_j^{(2)}\} \text{ or } g_i^{(2)} \in \{g_j^{(1)}, g_j^{(2)}\}.$$

Thus

$$|\{g_i^{(1)}, g_i^{(2)}\} \cap \{g_j^{(1)}, g_j^{(2)}\}| = 1$$

for all h_i and h_j.

Suppose that no such element g exists. Consider $h_1 = g_1^{(1)} + g_1^{(2)}$. For each $h_i \neq h_1$ we have that either

$$g_1^{(1)} \in \{g_i^{(1)}, g_i^{(2)}\} \text{ or } g_1^{(2)} \in \{g_i^{(1)}, g_i^{(2)}\}.$$

Let S_1 be the subset of $\{h_2, \ldots, h_k\}$ so that each $h_i \in S_1$ is of the form $g_1^{(1)} + g_i$, and let S_2 be defined analogously. Since $k \geq 4$, one of S_1, S_2 is not a singleton. Without loss of generality let $|S_1| > 1$.

Since g does not exist, S_2 is non-empty. We have $h_i, h_j \in S_1$ and $h_a \in S_2$. So

$$h_i = g_1^{(1)} + g_i, \quad h_j = g_1^{(1)} + g_j, \quad \text{and} \quad h_a = g_1^{(2)} + g_a$$

where $g_a \neq g_1^{(1)}$ and $g_i, g_j \neq g_1^{(2)}$. Thus only one of h_i and h_j can be adjacent to h_a, a contradiction. □

3.4.4 Lemma. *Let $m = 2$. If $\omega(X) = 3$, then $\omega(\Gamma) = 4$. Otherwise, $\omega(\Gamma) = \omega(X)$.*

Proof. Suppose that S is a clique in X. Then fix $i \in S$ and let

$$T = \{g_i + g_j \,:\, j \in S\}.$$

Then T is a subset of the vertices of Γ containing 0. Since i is adjacent to every other vertex in S, the sums $g_i + g_j \in C$ for all $j \in S \setminus \{i\}$. Thus 0 is adjacent to every $g_i + g_j \in T$ with $i \neq j$. Also, if $g_i + g_j \in T$ and $g_i + g_k \in T$, then since $j, k \in S$ we have that $g_j + g_k \in C$, and so $g_i + g_j$ is adjacent to $g_i + g_k$. Therefore T is a clique in Γ with $|T| = |S|$, and $\omega(X) \leq \omega(\Gamma)$.

Now suppose S is a clique in Γ. Assume that $|S| \geq 5$. Without loss of generality $0 \in S$. By Proposition 3.4.3 there is a vertex i of X so that every element of $S \setminus \{0\}$ is of the form $g_i + g_j$. Thus the vertices

$$\{i\} \cup \{j \,:\, g_i + g_j \in S\}$$

form a clique in X of size $|S|$. Thus $\omega(X) = \omega(\Gamma)$.

This also shows that if $\omega(X) = 3$, then $\omega(\Gamma) = 4$. This follows as if $\omega(X) = 3$, then $\omega(\Gamma) \geq 3$, and by Proposition 3.4.2 $\omega(\Gamma) \geq 4$. However, if $\omega(\Gamma) > 4$, then by the remarks above we see that $\omega(X) > 4$ which is a contradiction. Therefore $\omega(\Gamma) = 4$.

In the case $\omega(X) = 2$, X has edges but is triangle-free. Thus the neighbourhood of 0 in Γ contains no edges and so $\omega(\Gamma) = 2$. The case $\omega(X) = 1$ is trivial. □

Recall from Lemma 3.2.2 that when $m \geq 4$, the neighbourhood of 0 in Γ is a 2-fold cover of $T(X)$. In the $m = 2$ case, $|C| = |E(X)|$, and $\Gamma[0]$ is isomorphic to the graph $T(X)$.

3.5 Cayley Graphs for $\mathbb{Z}_3^n$

Now we consider the case $m = 3$. Similar to the case $m = 2$, we no longer have that our assignment is 3-sum respecting. However, the assignment is 2-sum respecting, so the graph Γ is a $2|E(X)|$-regular Cayley graph for $\mathbb{Z}_3^n$, and has connection set

$$C = \{g_i - g_j \,:\, i \text{ is adjacent to } j \text{ in } X\}$$

as usual. In this case, the assignment is almost 3-sum respecting. The only problem is the fact that $3g_i = 0$ for all i.

3.5.1 Proposition. *For $m = 3$, if $g_i + g_j + g_k = g_r + g_s + g_t$, then either $\{i,j,k\} = \{r,s,t\}$ or $|\{i,j,k\}| = |\{r,s,t\}| = 1$.*

Proof. Suppose that

$$g_i + g_j + g_k = g_r + g_s + g_t,$$

and at least one of $|\{i,j,k\}|$ and $|\{r,s,t\}|$ is not 1. First note that if $|\{i,j,k\}| = 2$, then $g_i + g_j + g_k$ has exactly two non-zero components, and if $|\{i,j,k\}| = 3$, then $g_i + g_j + g_k$ has exactly three non-zero components. So

$$|\{i,j,k\}| = |\{r,s,t\}| \neq 1$$

(as if one of the sets has size 1, the corresponding sum evaluates to 0).

We have two cases. First suppose that

$$|\{i,j,k\}| = |\{r,s,t\}| = 2.$$

Now $g_i + g_j + g_k$ has exactly two non-zero components, one takes 1, and the other takes value 2. The only way that this is possible is if $\{i,j,k\} = \{r,s,t\}$ as sets, and as multi-sets.

Finally, suppose that

$$|\{i,j,k\}| = |\{r,s,t\}| = 3.$$

In this case $g_i + g_j + g_k$ has exactly three non-zero components, each of which have value 1, and it is easy to see that we must have $\{i,j,k\} = \{r,s,t\}$. $\square$

From this proposition we see that Lemma 3.1.3 applies when $m = 3$, and we have that $\omega(X) \leq \omega(\Gamma)$. However, as in the case $m = 2$, we cannot conclude that $\omega(X) = \omega(\Gamma)$ in all cases. We begin by proving an analogue to Lemma 3.1.4. We will follow the proof of Lemma 3.1.4 very closely, modifying it where necessary.

3.5.2 Lemma. *If S is a clique in Γ with $|S| \geq 4$, then there is a clique in X with size $|S|$.*

Proof. Suppose the S is a clique in Γ with $|S| \geq 4$. Without loss of generality we assume that S contains 0.

First we note that since every non-zero element of $\mathbb{Z}_3^n$ has order 3, we have that $3(g_i - g_j) = 0$ for all $1 \leq i,j \leq n$. So for all $g_i - g_j \in C$ we have $3(g_i - g_j) = 0$, or

$$(g_i - g_j) + (g_i - g_j) = -(g_i - g_j).$$

Since $g_i - g_j \in C$ we have that $g_i - g_j$ is adjacent to $-(g_i - g_j)$ for all $g_i - g_j \in C$. These edges are a perfect matching in $\Gamma[0]$.

Consider $g_i - g_j$ adjacent to $g_k - g_l$ in $\Gamma[0]$. Since these vertices are adjacent, we must have some $g_s - g_t \in C$ so that

$$(g_i - g_j) + (g_s - g_t) = g_k - g_l.$$

Rearranging we see that

$$g_i + g_s + g_l = g_j + g_t + g_k.$$

From Proposition 3.5.1 we have that either $\{i, s, l\} = \{j, t, k\}$ or $|\{i, s, l\}| = |\{j, t, k\}| = 1$. In the latter case, we have an edge of the form $\{g_i - g_j, g_j - g_i\}$ as discussed above. So we assume that $\{i, s, l\} = \{j, t, k\}$. Since $i \neq j$, $k \neq l$ and $s \neq t$, there are two possibilities: either $(i, s, l) = (k, j, t)$; or $(i, s, l) = (t, k, j)$.

In the first case we have that $g_i - g_j$ is adjacent to $g_i - g_l$ by the element $g_j - g_l$ of C. We see that ijl forms a triangle in X. Likewise, the second case gives triangle ijk in X. So every triangle containing 0 in Γ corresponds either to a triangle in X, or an edge in X.

Now suppose that ijk is a triangle in X. We have that

$$\pm(g_i - g_j), \quad \pm(g_j - g_k), \quad \pm(g_i - g_k)$$

are all in C, and thus are vertices in the neighbourhood of 0. Note that by the previous paragraph, the adjacencies between these vertices are the edges $\{g_i - g_j, g_j - g_i\}$, together with the solutions to the "equation"

$$\pm(g_i - g_j) \pm (g_j - g_k) = \pm(g_i - g_k)$$

(more precisely, the assignments of signs so that the resulting equation is valid). We determine the valid assignments.

Consider the equation

$$(g_i - g_j) + (g_j - g_k) = -(g_i - g_k).$$

Rearranging the terms we have that $2g_i = 2g_k$, and the fact that η is 2-sum respecting implies that $i = k$. This contradicts the fact that we started with a triangle in X. The other assignments with a mixture of positive and negative signs can easily be seen to lead to contradictions as well.

The only valid assignments of signs to the three terms are to make each term positive, or to make each term negative. From the first assignment we have

$$(g_i - g_j) + (g_j - g_k) = g_i - g_k$$

which gives edges

$$\{g_i - g_j, g_i - g_k\}, \quad \{g_j - g_k, g_i - g_k\}, \quad \{g_i - g_j, g_k - g_j\}.$$

From the second we have

$$-(g_i - g_j) - (g_j - g_k) = -(g_i - g_j)$$

which gives edges

$$\{g_j - g_i, g_k - g_i\}, \quad \{g_k - g_j, g_k - g_i\}, \quad \{g_j - g_i, g_j - g_k\}$$

(together with the edges $\{g_i - g_j, g_j - g_i\}$). These six vertices and nine edges form an induced copy of $K_{3,3}$ in the neighbourhood of 0. Thus every triangle in X corresponds to an induced $K_{3,3}$ in Γ.

In fact, we have also shown that in

$$\Gamma[0] \setminus \big\{\{g_i - g_j, g_j - g_i\} \,:\, i \text{ is adjacent to } j \text{ in } X\big\},$$

the vertices $g_i - g_j$ and $g_j - g_i$ are at distance 3. Therefore in $\Gamma[0]$, there is no triangle containing an edge of the form $\{g_i - g_j, g_j - g_i\}$. Thus if S is a clique containing 0, and $|S| \geq 4$, then S contains at most one of $g_i - g_j$ and $g_j - g_i$ for all $\{i,j\} \in E(X)$. Thus the remainder of the proof goes through as normal.

Suppose $s_e, s_f \in S \setminus \{0\}$ where e, f are edges of X. Since $0 s_e s_f$ is a triangle, there is an edge g in X so that e, f, g are the edges of a triangle in X. Thus there is some vertex p in X so that p is shared by e and f. Now suppose we have $s_e, s_f, s_g \in S \setminus \{0\}$ so that the vertex p shared by e and f is different from the vertex q shared by f and g. Since e and g also share a vertex, we must have that e, f, g are the edges of a triangle in X. But as we have shown, this triangle corresponds to a 6-cycle in Γ containing s_e, s_f, s_g, a contradiction. Thus there is some vertex p in X so that p is an end of e for all $s_e \in S \setminus \{0\}$. So without loss of generality, every element of $S \setminus \{0\}$ has the form $g_p - g_q$, and

$$\{p\} \cup \{q \,:\, g_p - g_q \in S\}$$

is a clique in X with size $|S|$. □

Lemma 3.5.2 allows us to show exactly how the cliques of X correspond to the cliques of Γ.

3.5.3 Lemma. *Let $m = 3$. If $\omega(X) = 2$, then $\omega(\Gamma) = 3$. Otherwise $\omega(\Gamma) = \omega(X)$.*

Proof. We have already seen that $\omega(X) \leq \omega(\Gamma)$. Suppose that $\omega(X) \geq 4$. Then from Lemma 3.5.2, we have that $\omega(X) \geq \omega(\Gamma)$ and hence $\omega(X) = \omega(\Gamma)$.

If $\omega(X) = 3$, then $\omega(\Gamma) \geq 3$. But again by Lemma 3.5.2 we cannot have $\omega(\Gamma) \geq 4$, so $\omega(\Gamma) = 3$.

If $\omega(X) = 2$, then $\omega(X)$ has some edge $\{i,j\} \in E(X)$. So in $\Gamma[0]$ we will have an edge between $g_i - g_j$ and $g_j - g_i$ giving a triangle in Γ containing 0. Thus $\omega(\Gamma) = 3$.

The case $\omega(X) = 1$ is trivial. □

When $m = 3$, our assignment is not 3-sum respecting, and as a result, the neighbourhoods of Γ do not form a 2-fold cover of $T(X)$. However, we can prove a modification of that result.

Let $\Gamma[0]$ denote the graph induced by the neighbours of 0 in Γ, and let

$$M = \big\{\{g_i - g_j, g_j - g_i\} \,:\, i \text{ is adjacent to } j \text{ in } X\big\}.$$

3.5.4 Lemma. *$\Gamma[0] \setminus M$ is a 2-fold cover of $T(X)$.*

Proof. Define $h : \Gamma[0] \to V(T(X))$ by

$$h(g_i - g_j) = h(g_j - g_i) = \{i, j\}.$$

From the definition of C we see that h is clearly a surjection, and that

$$|h^{-1}(\{i, j\})| = 2$$

for all edges $\{i, j\}$ in X. It remains to show that h is a homomorphism, and a local isomorphism.

We have seen that $g_i - g_j$ is adjacent to $g_k - g_l$ in $\Gamma[0]$ if and only if: either $k = i$ or $j = l$ and ijl or ijk respectively is a triangle in X; or $l = i$ and $k = j$. In the first case, if $g_i - g_j$ is adjacent to $g_i - g_l$, then $\{i, j\}$ and $\{i, l\}$ are edges of X that lie in a triangle. So $\{i, j\}$ is adjacent to $\{i, l\}$ in $T(X)$. (The case $g_i - g_j$ adjacent to $g_k - g_j$ is similar.) In the second case, the resulting edge is an element of M, and does not need to be considered. Therefore h is a homomorphism.

Now we show h is a local isomorphism. If $\{i, j\}$ is an edge of X, then

$$h^{-1}(\{i, j\}) = \{g_i - g_j, g_j - g_i\}.$$

Consider the map induced by h between the neighbours of $g_i - g_j$ and the neighbours of $\{i, j\}$. If $\{i, j\}$ is adjacent to $\{i, l\}$ in $T(X)$, then $g_i - g_j$ is adjacent to $g_i - g_l$ and $h(g_i - g_l) = \{i, l\}$ so h induces a surjection.

If $g_k - g_l$ and $g_s - g_t$ are both neighbours of $g_i - g_j$ in $\Gamma[0] \setminus M$, then $h(g_k - g_l) = \{k, l\}$ and $h(g_s - g_t) = \{s, t\}$. If $\{k, l\} = \{s, t\}$, then either $k = s$ or $k = t$. If $k = s$, then

$$g_k - g_l = g_s - g_t.$$

If $k = t$, then

$$g_k - g_l = -(g_s - g_t).$$

However, in order to be a neighbour of $g_i - g_j$ we must have that either $k = i$ or $l = j$, and either $s = i$ or $t = j$. If $k = i$, then $t = i \neq j$ so we must have $s = i = t$ which is a contradiction.

The other cases give similar contradictions. Thus h induces an injection, and the induced map is a bijection. □

3.6 Quotient Graphs

In order to reduce the Maxclique problem on general graphs to the Maxclique problem on Cayley graphs for the groups $\mathbb{Z}_m^n$, we need to be able to construct an auxiliary graph $\Gamma = X(\mathbb{Z}_m^n, C)$ from a graph X so that: we can find a maximum clique in X efficiently given a maximum clique in Γ; and, the size of Γ is bounded by a polynomial in the size of X. In Section 3.3 we saw that we could construct an auxiliary graph Γ with the first property. However, the graphs we constructed had size exponential in the size of the input graph. In order to construct a suitable auxiliary graph for X we will make use of quotient graphs.

Let X be a graph, and let Π be a partition of $V(X)$. The *quotient graph* X/Π is the graph on the cells of Π with adjacency defined as follows. For A, B cells of Π, A and B are adjacent if and only if there are vertices $a \in A$ and $b \in B$ so that a and b are adjacent in X.

If X is a Cayley graph for a group G, then we can construct partitions using the subgroups of G. If H is a subgroup of G, then the cosets of H partition the elements of G into cells of equal size. We denote the partition induced by H as Π_H. If X is a Cayley graph for the group G, then we denote the quotient graph of X with respect to the partition Π_H as $X_H = X/\Pi_H$.

Using a partition of G into cosets of H, rather than an arbitrary partition, gives Π_H additional structure. If $\Pi = \{\pi_1, \dots, \pi_k\}$ is a partition of the vertices of a graph X we call Π an *equitable partition* if for every $x \in \pi_i$, the number of neighbours of x in π_j depends only on i and j. For Π_H we have the following result.

3.6.1 Proposition. *If H is a subgroup of G, then Π_H is an equitable partition of any Cayley graph X for G.*

Proof. Let $\Pi_H = \{\pi_1, \dots, \pi_k\}$ with $\pi_1 = H$. Let C be the connection set of $X = X(G, C)$. In X, vertices x and y are adjacent if an only if $y - x \in C$. We have that H is a subgroup of G, so for $x, y \in H$, we have $y - x \in H$. So x and y are adjacent in X if and only if $y - x \in C \cap H$. Since H is closed under the group operation we have that $X[\pi_1]$ is a $|H \cap C|$-regular subgraph.

Consider the edges between π_1 and π_j in X for any $j \neq 1$. The cell π_j is a coset of H, so without loss of generality $\pi_j = H + \alpha$. For $x, y \in H$, note that x is adjacent to $y + \alpha \in \pi_j$ if and only if $y - x + \alpha \in C$. For every $z + \alpha \in \pi_j \cap C$ we have that $x + z + \alpha \in \pi_j$ is a neighbour of x, so x has $|\pi_j \cap C|$ neighbours in π_j. Thus the number of neighbours of $x \in \pi_1$ in π_j depends only on the index j.

Note that for any $\alpha \in G$, the map $f_\alpha : G \to G$ given by $f_\alpha(x) = x + \alpha$ is an automorphism of X. Since f_α is clearly a bijection, we only need to check that f_α preserves edges. We have that $x, y \in X$ are adjacent if and only if $y - x \in C$. But

$$y - x = f_\alpha(y) - f_\alpha(x),$$

so $x, y \in X$ are adjacent if and only if $f_\alpha(x), f_\alpha(y) \in X$ are adjacent, and f_α is an isomorphism.

Finally we consider the edges between π_i and π_j in X. If $\pi_i = H + \alpha$, then $f_{-\alpha}(\pi_i) = H$. Thus the number of neighbours of $x \in \pi_i$ in π_j is the number of neighbours of $x - \alpha \in H$ in $\pi_{j'} = \pi_j - \alpha$ and depends only on the index j'. This completes the proof. $\square$

We can also show that X_H is a Cayley graph for the quotient group G/H. Given a group G and a subgroup H of G, the *quotient group* G/H is the group on the cosets of H in G with group operation defined as follows. For each coset of H, H', we take a representative $\alpha \in G$ so $H' = H + \alpha$. Given cosets $H + \alpha$ and $H + \beta$ we define

$$(H + \alpha) + (H + \beta) = H + (\alpha + \beta).$$

It is straightforward to show that this operation is well-defined, and defines a group.

3.6.2 Proposition. *$X_H = X(G/H, C')$ where $C' = \{H + g \,:\, g \in C\}$.*

Proof. X_H is a graph on G/H (the cosets of H in G). It remains to show that the edges of X_H are described exactly by C'.

Let $g \in C$ be an element of the connection set of X. Then for any $x \in G$, x is adjacent to $x + g$ in X. Thus the coset of H containing x is adjacent to the coset of H containing $x + g$ in X_H. Let $x \in H + \alpha$ and $x + g \in H + \beta$. Now there is $h_x, h_y \in H$ so $x = h_x + \alpha$ and $x + g = h_y + \beta$. Combining these we have

$$h_x + \alpha + g = h_y + \beta$$

and therefore

$$H + \alpha + (H + g) = H + \beta.$$

Now suppose that $H + \alpha$ is adjacent to $H + \beta$ in X_H. Then there are some $h_a, h_b \in H$ so that $h_a + \alpha \in H + \alpha$ is adjacent to $h_b + \beta \in H + \beta$ in X. Thus there is some $g \in C$ so that

$$h_a + \alpha + g = h_b + \beta.$$

Again we have that

$$H + \alpha + (H + g) = H + \beta$$

where $g \in C$. Therefore $X_H = X(G/H, C')$. □

Note that in the preceding proof we may have that $|C'| \neq |C|$, and X_H may have loops even if X does not.

We are working with an auxiliary graph $\Gamma = X(\mathbb{Z}_m^n, C)$. To construct a "small" graph from Γ we can find a "large" subgroup H of $\mathbb{Z}_m^n$ and use Γ_H. Since in Theorem 3.0.2 we are working with the groups $\mathbb{Z}_m^n$ with m prime, we assume from now on that $m = p$ is prime.

For p prime, $\mathbb{Z}_p$ is a finite field, and $\mathbb{Z}_p^n$ is a vector space. In this case a subgroup H of the additive group $\mathbb{Z}_p^n$ is a sub-vector space (or subspace). As above H induces a partition Π_H of $\mathbb{Z}_p^n$, and we denote the quotient graph of Γ with respect to Π_H as Γ_H. We begin with a brief discussion of the subspaces of $\mathbb{Z}_p^n$.

3.7 Codes

In this section we give a brief account of linear codes. Coding theory is a vast subject, and we will only need a very small sample. All of the material in this section is standard. We will follow MacWilliams and Sloane [26] for the coding theory we need.

Suppose we are given a finite field $GF(q)$ where q is a prime power. Given an integer n, a *q-ary linear code* (or *code*) is a subspace D of the vector space $GF(q)^n$ for some integer n. The *block length* of D is the length of the vectors in D, or the dimension of the ambient space $GF(q)^n$, n. The *size* of D is the number of vectors in D, and is equal to q^k where k is the dimension of D as a subspace of $GF(q)^n$.

In order to preserve clique information in our quotient graphs, we will need codes with a specific distance property. For our purposes, the *distance* (or *Hamming distance*) between two vectors in $x, y \in GF(q)^n$ is the number of indices $1 \leq i \leq n$ for which $x_i \neq y_i$. We denote the distance between x and y by $d(x, y)$. Given a code D, the *minimum distance* (or *distance*) of D is the minimum distance between any two elements of D,

$$d = \min\{d(x, y) \,:\, x, y \in D\}.$$

When working with codes it is often more useful to work with the weight of the codewords, rather than the distance between them. The *weight* of a vector $x \in D$, $w(x)$, is the number of indices $1 \leq i \leq n$ so that $x_i \neq 0$. So $0 \in D$ is the unique codeword with weight zero, and all other codewords have non-zero weight. Now if $x, y \in D$, since D is a subspace, $x - y \in D$. Moreover, if $z \in GF(q)^n$

$$d(x, y) = d(x - z, y - z)$$

as $x_i \neq y_i$ if and only if $x_i - z_i \neq y_i - z_i$. So

$$d(x, y) = d(x - y, 0) = w(x - y).$$

Therefore the distance of D can also be expressed as the minimum weight of a non-zero codeword,

$$d = \min\{w(x) \,:\, x \in D \setminus \{0\}\}.$$

As we will see in the next section, by constraining the distance of a code D in $\mathbb{Z}_p^n$, we will be able to deduce properties of the cliques in the quotient graph Γ_D.

3.8 Codes in Γ

Our goal is to find a code D in $V(\Gamma) = \mathbb{Z}_p^n$ so that $\omega(X)$ can easily be computed from the quotient graph Γ_D. We can achieve this by constraining the distance d of D. For now we assume that $\Gamma = X(\mathbb{Z}_p^n, C)$ is constructed as usual, and p is any prime (including $2, 3$).

3.8.1 Proposition. *If D has distance $d \geq 3$, then D is a coclique in* Γ.

Proof. Consider any element of $|D \cap C|$. Since D has distance $d \geq 3$, all non-zero elements of D have weight at least 3. But $g_i - g_j$ has weight 2 for all $i \neq j$. Thus $|D \cap C| = 0$, and D is a coclique in Γ. □

It follows from Propositions 3.6.1 and 3.8.1 that Π_D gives a partition of Γ into cocliques. This immediately implies that any clique in Γ gives a corresponding clique in Γ_D of equal size. So we have $\omega(\Gamma_D) \geq \omega(\Gamma)$. We also have the following immediate corollary.

3.8.2 Corollary. *If $d \geq 3$, then Γ_D is a Cayley graph for the group $\mathbb{Z}_m^n/D$ with connection set*

$$C' = \{(D + g_i) - (D + g_j) \, : \, i \text{ is adjacent to } j \text{ in } X\},$$

and the map $f(g) = D + g$ gives a bijection between C and C'.

Proof. It follows immediately from Proposition 3.6.2 that C' is the connection set for Γ_D. The distance condition ensures that the cosets $D + (g_i - g_j)$ are distinct for all i and j. Thus f is a bijection between C and C' and $|C| = |C'|$. □

Further restricting d gives us more information about Γ_D.

3.8.3 Proposition. *If D has distance $d \geq 5$, then Γ_D contains an induced copy of X.*

Proof. Since $d \geq 5$, there is no i so that $g_i \in D$ (as g_i has weight 1). Moreover, if $D+g$ is a coset of D, then for indices $i \neq j$ we cannot have both $g_i, g_j \in D+g$. This follows as otherwise there are $\alpha, \beta \in D$ so that $\alpha + g = g_i$ and $\beta + g = g_j$. Thus

$$g_i - g_j = \alpha - \beta \in D.$$

However, $g_i - g_j$ has weight $2 < 5$, contradicting the distance of D. Thus $D + g_i$ is a vertex of G_D for each $1 \leq i \leq v$. We show that the vertices

$$\{D + g_i \, : \, 1 \leq i \leq v\}$$

give an induced copy of X in Γ_D.

Consider adjacent vertices i, j in X. We have that $g_i - g_j \in C$, and g_i and g_j are connected by an edge in Γ. We also have that $g_i \in D + g_i$ and $g_j \in D + g_j$, so $D + g_i$ and $D + g_j$ are connected by an edge in Γ_D. So X is a subgraph of Γ_D.

Now suppose that i, j are non-adjacent vertices of X. Suppose that $D + g_i$ and $D + g_j$ are connected by an edge in Γ_D. Then we have $\alpha, \beta \in D$ and $g_a - g_b \in C$ so that

$$\alpha + g_i + g_a - g_b = \beta + g_j,$$

or

$$g_i + g_a - g_b - g_j = \beta - \alpha \in D.$$

But the weight of the left-hand side of the equation is at most 4 and $d \geq 5$ so we have a contradiction.

Therefore X is an induced subgraph of Γ_D. □

Proposition 3.8.3 immediately implies that $\omega(X) \leq \omega(\Gamma_D)$.

Finally, if we increase the distance of D again, we can show that Γ_D will have the same maximum clique size as X, with the exceptions from Section 3.3. Our approach is to show that the elements of C' in each case satisfy the same properties as those of C with respect to 2-sums and 3-sums. As a result, the proofs in Sections 3.1 (Corollary 3.1.5 in particular) and 3.3 (Lemmas 3.4.4 and 3.5.3 in particular) will apply unchanged.

3.8.4 Lemma. *Let D be a code with distance $d \geq 7$. If $p \geq 4$, then $\omega(\Gamma_D) = \omega(X)$. If $p = 3$, then $\omega(\Gamma_D) = \omega(X)$ unless $\omega(X) = 2$, in which case $\omega(\Gamma_D) = 3$. If $p = 2$, then $\omega(\Gamma_D) = \omega(X)$ unless $\omega(X) = 3$, in which case $\omega(\Gamma_D) = 4$.*

Proof. We begin by assuming that $p \geq 4$. In this case we show that the 3-sums of elements of C' are all distinct. Let $D + (g_i + g_j + g_k)$ and $D + (g_a + g_b + g_c)$ be cosets of D for any $\{i, j, k\} \neq \{a, b, c\}$. Suppose that

$$D + (g_i + g_j + g_k) = D + (g_a + g_b + g_c).$$

Then we have $\alpha, \beta \in D$ so that

$$\alpha + g_i + g_j + g_k = \beta + g_a + g_b + g_c,$$

and as a result,

$$g_i + g_j + g_k - g_a - g_b - g_c = \alpha - \beta \in D.$$

This gives an immediate contradiction as the weight of the left-hand side of this equation is at most 6 and at least 2, while $d \geq 7$. Therefore we have that $\omega(\Gamma_D) = \omega(X)$.

Now assume that $p = 2$. In this case we need to show that the cosets $D + (g_i + g_j)$ are distinct when $i \neq j$, and that the cosets $D + (g_i + g_j + g_k)$ are distinct when $|\{i, j, k\}| = 1, 3$. First suppose that $|\{i, j\}| = |\{a, b\}| = 2$, $\{i, j\} \neq \{a, b\}$ and

$$D + (g_i + g_j) = D + (g_a + g_b).$$

Then we have $\alpha, \beta \in D$ so that

$$\alpha + g_i + g_j = \beta + g_a + g_b,$$

and as a result,

$$g_i + g_j - g_a - g_b = \alpha - \beta \in D.$$

This gives an immediate contradiction as the weight of the left-hand side of this equation is at most 4 and at least 2, while $d \geq 7$.

Secondly, suppose that $|\{i, j, k\}| = 1$ or 3, $|\{a, b, c\}| = 1$ or 3, and $\{i, j, k\} \neq \{a, b, c\}$. Again we assume that

$$D + (g_i + g_j + g_k) = D + (g_a + g_b + g_c).$$

Then we have $\alpha, \beta \in D$ so that

$$\alpha + g_i + g_j + g_k = \beta + g_a + g_b + g_c,$$

and as a result,

$$g_i + g_j + g_k - g_a - g_b - g_c = \alpha - \beta \in D.$$

In this case we have that the weight of $g_i + g_j + g_k$ and $g_a + g_b + g_c$ is at most 3, and so the weight of the left-hand side of the equation is at most 6, and at least 2. This contradicts $d \geq 7$. Therefore we have that when $p = 2$, $\omega(\Gamma_D) = \omega(X)$ unless $\omega(X) = 3$ in which case $\omega(\Gamma_D) = 4$.

Finally we consider the case $p = 3$. In this case we need to show that the cosets $D+(g_i+g_j+g_k)$ and $D+(g_a+g_b+g_c)$ are distinct unless either $\{i, j, k\} = \{a, b, c\}$ or $|\{i, j, k\}| = |\{a, b, c\}| = 1$. We assume that $\{i, j, k\} \neq \{a, b, c\}$ and that at least one of the sets contains more than one element. Suppose that we have

$$D = (g_i + g_j + g_k) = D + (g_a + g_b + g_c).$$

Then we have $\alpha, \beta \in D$ so that

$$\alpha + g_i + g_j + g_k = \beta + g_a + g_b + g_c,$$

and as a result,

$$g_i + g_j + g_k - g_a - g_b - g_c = \alpha - \beta \in D.$$

In this case we again have that the weight of the left-hand side of the equation is at most 6, and at least 2. Thus we have that when $p = 3$, $\omega(\Gamma_D) = \omega(X)$ unless $\omega(X) = 2$ in which case $\omega(\Gamma_D) = 3$. □

So we have proven that given a code with large enough distance, the resulting quotient graph will have the desired maximum clique size. To finish our reduction, it remains to show that we can find a code efficiently that has $d \geq 7$, and is large enough so that Γ_D has size polynomial in the size of X. In the next section we give the construction of such codes.

3.9 Goppa Codes

To construct codes with the properties we desire, we will use Goppa codes. Goppa codes are a family of linear codes. Their construction is similar to the construction of BCH codes. In this section we give a brief description of how to construct a Goppa code. Again we follow MacWilliams and Sloane [26]. See Chapter 12, Section 3 of [26] for a more complete treatment of Goppa codes.

A Goppa code is a linear code over the finite field $GF(q)$. In order to specify the code we need two ingredients: a polynomial $G(x)$ whose coefficients are elements of $GF(q^m)$; and a set $L \subseteq GF(p^m)$ of non-roots of $G(x)$. The polynomial $G(x)$ is called the *Goppa polynomial*.

Let $L = \{\alpha_1, \ldots, \alpha_n\}$ be a subset of the non-roots of $G(x)$. Given a vector $a \in GF(q)^n$, we define the rational function

$$R_a(x) = \sum_{i=1}^{n} a_i/(x - \alpha_i).$$

The *Goppa code* $D(G, L)$ is the set of all vectors $a \in GF(q)^n$ such that $R_a(x) = 0$ in the polynomial ring $GF(q^m)[x]/G(x)$. Note that $D(G, L)$ is a q-ary linear code with block length n.

A linear code D is a subspace of a vector space $GF(q)^n$; so D has a basis, and can be expressed as the row space of a matrix B. We call B the *generator matrix* of D. If D has rank k, and block length n, then B is a $k \times n$ matrix with elements from $GF(q)$. We can convert B into reduced row-echelon form, and so we may assume that B takes the form

$$B = [I_k|A]$$

where A is a $k \times (n - k)$ matrix over $GF(q)$. The *dual code* of D is the code defined by the generator matrix

$$H = [-A^T|I_{n-k}].$$

Since H is a $(n - k) \times n$ matrix over $GF(q)$, the code generated by H has rank $n - k$ and block length n. Note that

$$BH^T = HB^T = 0,$$

so $D = \ker(H^T)$. In coding applications, the matrix H is used to check whether or not a received word is a codeword. The matrix H is called the *parity check matrix* for the code D.

Let $L = \{\alpha_1, \ldots, \alpha_n\}$. The parity check matrix for the Goppa code $D(G, L)$ is constructed as follows. Let H' be the matrix whose entries are defined as

$$H'[i, j] = \alpha_j^i G(\alpha_j)^{-1}$$

for $1 \leq i \leq r$ and $1 \leq j \leq n$. The matrix H' is a $r \times n$ matrix with entries in $GF(q^m)$. Then H is the matrix whose entries are obtained by replacing $H'[i, j]$ with the column vector in $GF(q)^m$ corresponding to $H'[i, j] \in GF(q^m)$. Now H is a $rm \times n$ matrix with entries in $GF(q)$.

In this case H may not have full rank. However, we can still use H as the parity check matrix for $D(G, L)$. We can also construct a matrix B from H with $\mathrm{row}(B) = \mathrm{null}(H)$, so B is a generator matrix for $D(G, L)$. Since $\mathrm{rk}(H) + \mathrm{null}(H) = n$ and $\mathrm{rk}(H) \leq rm$, it follows that the rank of $D(G, L)$ is $k \geq n - rm$.

Finally, we need some information on the distance of Goppa codes. In general, if the Goppa polynomial $G(x)$ has rank r, then the code $D(G, L)$ will have distance $d \geq r + 1$ (see [26] Chapter 12, Theorem 1). Note that for both the rank and distance of $D(G, L)$ the specific values will depend on the polynomial chosen to construct the code. However, the bounds $k \geq n - rm$ and $d \geq r + 1$ will suffice for our application.

3.10 Proof of Main Theorem

We are now ready to complete our proof of Theorem 3.0.2. Recall that given a graph X on n vertices, and a prime p we can construct as in Section 3.3 a Cayley graph $\Gamma(\mathbb{Z}_p^n, C)$ with the property that $\omega(X)$ is easily computed from $\omega(\Gamma)$. In the case where $p > 4$, we have from Corollary 3.1.5 that $\omega(X) = \omega(\Gamma)$.

If $p = 2$, we have from Lemma 3.4.4 that $\omega(X) = \omega(\Gamma)$ unless $\omega(\Gamma) = 4$, in which case $\omega(X) = 3$ or 4. So we can find $\omega(X)$ simply by checking to see if X contains any cliques of size 4. The brute-force approach to this problem is to simply check all $\binom{n}{4}$ subsets of $V(X)$ to see if the subgraph induced by these vertices is complete. Since

$$\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)(n-4)}{4!} \leq n^4$$

for $n \geq 0$, this algorithm runs in time polynomial in n. Thus we can recover $\omega(X)$ from $\omega(\Gamma)$ in time polynomial in n.

If $p = 3$, then from Lemma 3.5.3 we have that $\omega(X) = \omega(\Gamma)$ unless $\omega(\Gamma) = 3$, in which case $\omega(X) = 2$ or 3. The same approach as we used for $p = 2$ shows that we can recover $\omega(X)$ from $\omega(\Gamma)$ in time polynomial in n.

If we are given an oracle that solves the Maxclique problem in polynomial time for Cayley graphs for the class of groups $\mathbb{Z}_p^n$, then we can solve for $\omega(\Gamma)$, and find $\omega(X)$ in polynomial time. The problem is that the oracle runs in polynomial time in the size of Γ, which is exponential in the size of X. So in order for this to work, we need to construct an auxiliary graph Γ' whose cliques are the same as the cliques of Γ, and has size polynomial in n. To do this we will construct Γ' by taking a quotient graph of Γ.

Recall from Lemma 3.8.4 that if we take D to be a p-ary linear code with distance $d \geq 7$ and block length n, then the quotient graph Γ_D whose vertices are the cells of the partition of $\mathbb{Z}_p^n$ given by the cosets of D has $\omega(\Gamma_D) = \omega(\Gamma)$. The size of Γ_D is p^{n-k} where k is the rank of D. Thus given a code D with distance $d \geq 7$, and rank k so that p^{n-k} is bounded by a polynomial in n, Γ_D will be an auxiliary graph that matches our requirements.

There is a Goppa code that satisfies our requirements. Take Goppa polynomial $G(x) = x^6$. For any $m \geq 1$, $G(x)$ is a polynomial with coefficients in $GF(p^m)$ and non-roots $GF(p^m) \setminus \{0\}$. So we can let L be any subset of non-zero elements of $GF(p^m)$. Let $D(G, L)$ be the Goppa code constructed from $G(x)$ and L. From Section 3.9 we have that $D(G, L)$ will have distance $d \geq 7$, block length $|L|$ and rank $k \geq |L| - 6m$ where $|L| \leq p^m - 1$.

From the above discussion, we want the block length of $D(G, L)$ to be n, the number of vertices of X. We also want the rank of $D(G, L)$ to satisfy $p^{n-k} \leq f(n)$ for all $n \geq N$, where $f(x)$ is a polynomial in x and N is some fixed integer. The block length of $D(G, L)$ is $|L|$, and L can be any subset of $GF(p^m) \setminus \{0\}$, so we can choose any such L with $|L| = n$ provided $n \leq p^m - 1$. We rearrange the constraint $p^{n-k} \leq f(n)$ as $k \geq n - \log_p f(n)$. In order to ensure this inequality is satisfied, we want to choose m so that

$$k \geq n - 6m \geq n - \log_p f(n),$$

or $m \leq \log_p f(n)/6$.

3.10.1 Lemma. *Let $f(x) = x^{12}$. There is an integer N so that for all $n \geq N$, there is some integer m with $n \leq p^m - 1$ and $m \leq \log_p f(n)/6$.*

Proof. The inequality $m \leq \log_p f(n)/6$ can be expressed as $m \leq \log_p n^2$, or $p^m \leq n^2$. Also the condition $n \leq p^m - 1$ is equivalent to $n < p^m$ as all the quantities are integers.

Take $N = p^2$. Now for any $n \geq N$, we have $\log_p n \geq 2$, and so the interval $(\log_p n, 2\log_p n]$ contains an integer. Let m be the largest such integer. $\square$

By Lemma 3.10.1 we can choose m to be the largest integer in $(\log_p n, 2\log_p n]$. Then we can take L to be a set of non-zero elements of $GF(p)^m$ of size n, and the Goppa code $D(G, L)$ will have rank $k \geq n - \log_p f(n)$.

We have almost completed the proof. This shows that there exists an auxiliary graph Γ_D for X so that the $\omega(X)$ can be computed in polynomial time from $\omega(\Gamma_D)$ and Γ_D has size polynomial in n. All that remains to show is that we can construct Γ_D in polynomial time. The construction we have outlined so far involves constructing Γ, and then forming Γ_D as a quotient of Γ. However, that construction involves constructing a graph with p^n vertices, and does not run in polynomial time.

In order to get around this problem, we note that a Cayley graph is specified by its connection set. By Corollary 3.8.2 we have that Γ_D is a Cayley graph $\Gamma_D = X(\mathbb{Z}_p^{n-k}, C')$. So in order to show that we can construct Γ_D efficiently, we show how to compute C' from C efficiently.

We have already seen an explicit description of a parity check matrix H for $D(G, L)$ in Section 3.9. Using H we can recover a generator matrix B for $D(G, L)$ so that $D(G, L) = \text{row}(B)$. From the rows of B we can find a basis $\{\beta_1, \ldots, \beta_k\}$ for $D(G, L)$ and extend this basis to a basis for $\mathbb{Z}_p^n$,

$$\{\beta_1, \ldots, \beta_k, \beta_{k+1}, \ldots, \beta_n\}.$$

Now for any $\alpha \in \mathbb{Z}_p^n$, α can be written uniquely as

$$\alpha = \sum_{i=1}^{n} a_i \beta_i$$

where the a_i are elements of $\mathbb{Z}_p$. Furthermore, in the quotient space $\mathbb{Z}_p^n / D(G, L)$, α lies in the coset

$$D + \left(\sum_{i=k+1}^{n} a_i \beta_i \right).$$

Thus the elements of the connection set

$$C = \{g_i - g_j \, : \, i \text{ is adjacent to } j \text{ in } X\}$$

can be expressed using our basis, and we set

$$C' = \left\{ \sum_{i=k+1}^{n} a_i\beta_i \,:\, \sum_{i=1}^{n} a_i\beta_i \in C \right\}.$$

This is the last piece of the proof of Theorem 3.0.2. We give a summary of the proof below.

3.10.2 Theorem (Theorem 3.0.2)**.** *For a fixed prime p, the Maxclique problem restricted to Cayley graphs for the family of groups $\mathbb{Z}_p^n$ is NP-hard.*

Proof. To show that the Maxclique problem is NP-Hard for the class of Cayley graphs for the groups $\mathbb{Z}_p^n$ where p is a fixed prime, we give a polynomial time reduction from the problem on the class of all graphs. We assume that we are given an oracle Ω that solves the Maxclique problem on any graph $X(\mathbb{Z}_p^n, C)$ in time polynomial in p^n (the size of the input graph).

Let p be a fixed prime. We are given a graph X on n vertices. Suppose $n < p^2$. In this case we simply solve for $\omega(X)$ exhaustively. Since there are only finitely many graphs with less than p^2 vertices, this has no effect on our result.

Assume $n \geq p^2$. By Lemma 3.10.1, we can choose m to be the largest integer in $(\log_p n, 2\log_p n]$ (i.e., set $m = \lfloor 2\log_p n \rfloor$).

Construct the field $GF(p^m)$ by finding an irreducible polynomial f of degree m over the field $GF(p)$, and representing $GF(p^m)$ as $GF(p)[x]/\langle f(x)\rangle$ where $\langle f(x)\rangle$ is the ideal generated by $f(x)$. This can be done in time polynomial in m [7], and hence in time polynomial in n.

We choose a subset $L \subseteq GF(p^m) \setminus \{0\}$ with $|L| = n$ as follows. Let $\alpha \in GF(p^m)$ be a primitive element. We can find α by calculating a^i for all $1 \leq i \leq p^m - 1$ and $a \in GF(p^m)$. This involves checking at most $p^m \leq n^2$ elements, each of which requires at most n^2 multiplications in $GF(p^m)$, so this can be accomplished in polynomial time. Set

$$L = \{\alpha^i \,:\, 1 \leq i \leq n\}.$$

Set $G(x) = x^6$, and consider the Goppa code $D(G, L)$. We construct a check matrix H for $D(G, L)$ as in Section 3.9. We set

$$H'[i,j] = \alpha_j^i G(\alpha_j)^{-1}$$

for $1 \leq i \leq r$ and $1 \leq j \leq n$. Since H' is a $r \times n$ matrix, with $r < n$, this involves at most n^2 calculations, each of which involves $O(n^2)$ computations in $GF(p^m)$. We obtain a $rm \times n$ matrix H from H' by replacing each entry of H' with a vector in $GF(p)^m$ corresponding to its entry in $GF(p^m)$. Again this requires at most n^2 replacement operations, each of which is constant time, as given a polynomial $\alpha \in GF(p^m)$ (recall that we are using the representation $GF(p^m) = GF(p)[x]/\langle f(x)\rangle$) we replace α with the vector of its coefficients in $GF(p)$.

From H we construct a basis for $\mathbb{Z}_p^n$. We have that H is a $rm \times n$ matrix whose rows span a space of dimension $n-k$, and whose null space has dimension

k. We take $\{\beta_1, \ldots, \beta_k\}$ to be a basis for null$(H) = D(G, L)$, and $\{\beta_{k+1}, \ldots, \beta_n\}$ to be a basis for row(H). Now $B = \{\beta_1, \ldots, \beta_n\}$ is a basis for $\mathbb{Z}_p^n$. We can find B by converting H into reduced row-echelon form in time polynomial in n (as H is at most $n \times n$).

For each $1 \le i \le n$ let g_i be the ith standard basis vector in $\mathbb{Z}_p^n$. Let

$$C = \{g_i - g_j \, : i \text{ is adjacent to } j \text{ in } X\}.$$

Each element of C can be uniquely expressed as a sum of elements of B. So we set

$$C' = \left\{ \sum_{i=k+1}^{n} a_i\beta_i \, : \, \sum_{i=1}^{n} a_i\beta_i \in C \right\}.$$

Describing the elements of C' is done in constant time, given the elements of C. For each $\alpha \in \mathbb{Z}_p^n$, to write α as a sum of elements of B, we solve the matrix equation $Bx = \alpha$. This can be done in polynomial time for each $\alpha \in C$, so in total we solve $O(n^2)$ equations to find C'.

From C' we construct the graph $\Gamma' = X(\mathbb{Z}_p^{n-k}, C')$. Recall that we chose m so that p^{n-k} is polynomial in n. So constructing the vertices of Γ' and the at most

$$2\binom{p^{n-k}}{2}$$

arcs between them can be done in polynomial time. Once we have Γ', we run our Maxclique oracle to compute $\omega(\Gamma')$. By assumption Ω runs in time polynomial in the size of the input graph, which is polynomial in n. So Ω returns $\omega(\Gamma')$ in time polynomial in n.

Finally, we compute $\omega(X)$ from $\omega(\Gamma')$. If $p \ge 4$, Corollary 3.1.5 gives us that $\omega(X) = \omega(\Gamma')$, so our computation takes constant time. If $p = 2$, then Lemma 3.4.4 gives us that either $\omega(X) = \omega(\Gamma')$, or that $\omega(\Gamma') = 4$ and $\omega(X) = 3$ or 4. We check the 4-subsets of $V(X)$ exhaustively for cliques to determine whether $\omega(X) = 3$ or 4. This takes $O(n^4)$, so the entire procedure runs in polynomial time. Likewise for $p = 3$, Lemma 3.5.3 gives a polynomial time method to compute $\omega(X)$ from $\omega(\Gamma')$. $\square$

As a final note, we point out that in our proof, if our oracle Ω returns a maximum clique in the auxiliary graph Γ', then the proofs of Corollary 3.1.5 and Lemmas 3.4.4 and 3.5.3 give a method for finding a maximum clique in X in polynomial time.

3.11 Generalizing to Direct Powers

An immediate question raised by Theorem 3.0.2 is whether we can generalize to other families of Cayley graphs. Most naturally, we would hope to be able to generalize to Cayley graphs for the groups $\mathbb{Z}_m^n$ where m is any fixed integer. In fact, we can do better. In this section we show that we can generalize Theorem 3.0.2 to the class of Cayley graphs for the groups G^n where G is any finite group. The proof follows easily from the case where $G = \mathbb{Z}_p$ for some prime p.

3.11.1 Theorem. *Let G be a finite group. The Maxclique problem restricted to Cayley graphs for the family of groups G^m is NP-hard.*

Proof. We are given a graph X on n vertices, and want to construct an auxiliary graph that is a Cayley graph for a group of the form G^m. There is some prime p that divides the order of G. This implies that there is a subgroup H of G so that $H \cong \mathbb{Z}_p$.

We apply the construction from the previous section to construct a graph $\Gamma_D = X(\mathbb{Z}_p^m, C)$. Recall that our construction ensures that p^m is polynomially bounded in n, Γ_D can be constructed in time bounded by a polynomial in n, and $\omega(X)$ can be calculated from $\omega(\Gamma_D)$ in time polynomial in n.

Since $C \subseteq \mathbb{Z}_p^m$ the isomorphism between $\mathbb{Z}_p^m$ and H^m maps C to $C' \subseteq H^m$. Consider the graph $\Gamma' = X(G^m, C')$. Since $C' \subseteq H$, the graph Γ' consists of $(|G|/|H|)^m$ connected components, each of which is isomorphic to Γ_D (i.e. we have one copy of Γ_D for each coset of H^m in G^m). Therefore either $\omega(X) = \omega(\Gamma')$, or $p = 2, 3$ and we have the usual caveats.

Moreover, we know that $p^m \leq f(n)$ where $f(x)$ is some polynomial in x. There is some integer α so that $|G| \leq p^\alpha$. Thus

$$(|G|/|H|)^m \leq (p^{\alpha-1})^m = (p^m)^{\alpha-1} \leq (f(n))^{\alpha-1}$$

and the size of Γ' is bounded by a polynomial in n. This completes the proof. $\square$

3.12 Spectra of Γ and Γ_D

In this section we determine the spectrum of Γ and Γ_D. Since $\mathbb{Z}_p^m$ is a finite Abelian group, and Γ and Γ_D are both Cayley graphs for some $\mathbb{Z}_p^m$, we will use the characters of $\mathbb{Z}_p^m$ to find the eigenvalues of Γ and Γ_D. We refer back to Section 2.6 for more details.

Suppose G is a finite Abelian group, and χ_i are the linear characters of G for $1 \leq i \leq n$. Let m be a positive integer, and consider the group G^m whose elements are the m-tuples of elements of G, and

$$(a_1, \ldots, a_m) \circ_m (b_1, \ldots, b_m) = (a_1 \circ b_1, \ldots, a_m \circ b_m)$$

where $\circ$ is the group operation of G. For $a \in [n]^m$, define the function $\phi_a : G^m \to \mathbb{C}$ by

$$\phi_a(b) = \prod_{i=1}^{m} \chi_{a_i}(b_i).$$

Since each χ_i is a linear character of G, each ϕ_a is a linear character of G^m. Moreover,

$$\{\phi_a \,:\, a \in [n]^m\}$$

is the set of all linear characters of G^m.

Recall that the linear characters of $\mathbb{Z}_p$ are the characters $\chi_i : \mathbb{Z}_p \to \mathbb{C}$ defined by $\chi_i(a) = \omega^{ia}$ where ω is a primitive pth root of unity. So the spectrum of $X(\mathbb{Z}_p^n, C)$ is

$$\left\{ \sum_{c \in C} \left(\prod_{i=1}^{n} \omega^{a_i c_i} \right) \, : \, a \in \mathbb{Z}_p^n \right\}.$$

Let X be a graph, and p be a fixed odd prime. Let $\Gamma = X(\mathbb{Z}_p^n, C)$ and Γ_D be defined as usual. The elements of C have the form $g_i - g_j$ where g_i and g_j are standard basis vectors of $\mathbb{Z}_p^n$. Also, if $ij \in E(X)$, then both $\pm(g_i - g_j)$ are in C. So if $a \in \mathbb{Z}_p^n$, we have that

$$\begin{aligned}
\phi_a(C) &= \sum_{g_i - g_j \in C} \prod_{l=1}^{n} \left(\omega^{a_l(g_i - g_j)} \right) \\
&= \sum_{g_i - g_j \in C} \omega^{a_i - a_j} \\
&= \sum_{ij \in E(X)} \left(\omega^{a_i - a_j} + \omega^{a_j - a_i} \right) \\
&= \sum_{ij \in E(X)} 2\Re(\omega^{a_i - a_j})
\end{aligned}$$

is an eigenvalue of Γ (the order of i and j does not matter as $\Re(\omega^{\alpha}) = \Re(\omega^{-\alpha})$). If we let

$$\omega = \cos(2\pi/p) + i \sin(2\pi/p),$$

then we see the spectrum of Γ is

$$\left\{ \sum_{ij \in E(X)} 2\cos(2\pi(a_i - a_j)/p) \, : \, a \in \mathbb{Z}_p^n \right\}.$$

So we get an eigenvalue for every assignment of elements of $\mathbb{Z}_p$ to the vertices of X, and the eigenvalue we obtain depends on the differences of our assignment values on adjacent vertices.

3.13 The Ratio Bound

We can use the spectrum of Γ to draw some conclusions about its structure. In particular we will look at the cocliques of Γ. We have already seen in the construction of Γ_D from Γ that Γ has large cocliques. In order to bound the size of a coclique in Γ, we could apply the inertia bound as we saw in Chapter 2. However, we will use a different bound on $\alpha(G)$.

3.13.1 Theorem (Theorem 3.5.2 in Brouwer and Haemers [4]). *Let Y be a k-regular graph on n vertices with least eigenvalue τ. Then*

$$\alpha(Y) \leq \frac{n}{1 - k/\tau}. \qquad \square$$

Theorem 3.13.1 is referred to as either the *Ratio Bound* or the *Hoffman Bound*. We omit the proof and direct the reader to [4] for a proof based on eigenvalue interlacing. We will refer to graphs that meet the ratio bound as *ratio tight*.

Without finding the exact value of $\alpha(\Gamma)$ we can still show that Γ is not ratio tight. If the right-hand side of the inequality in Theorem 3.13.1 is not integral, then the inequality cannot be tight. Note that $\cos(2\pi i/p)$ is minimum when $i = (p-1)/2$ or $(p+1)/2$ (provided p is odd). Thus if τ is the least eigenvalue of Γ, we have that

$$\tau \geq 2|E(X)|\cos(\pi(p-1)/p),$$

and equality is achieved if we can assign elements of $\mathbb{Z}_p$ to $V(X)$ so that the difference on any edge is either $(p-1)/2$ or $(p+1)/2$.

3.13.2 Lemma. *If p is odd, and X is bipartite, then Γ does not meet the ratio bound.*

Proof. Let X have partition (A, B). Let $a \in \mathbb{Z}_p^n$ be the assignment where each vertex of A is assigned $(p-1)/2$ and each vertex of B is assigned 0. Then for every edge $ij \in E(X)$,

$$a_i - a_j \in \{(p-1)/2, (p+1)/2\}.$$

Thus the least eigenvalue of Γ is

$$\tau = 2|E(X)|\cos(\pi(p-1)/p).$$

Since $\cos(\pi(p-1)/p)$ is irrational, τ is irrational and Γ cannot meet the ratio bound. $\square$

If $p = 3$ we can show that τ must be irrational independent of the structure of X.

3.13.3 Lemma. *If $p = 3$, and X has at least one edge, then Γ does not meet the ratio bound.*

Proof. If $p = 3$, then the eigenvalues of Γ have the form

$$a + 2b\cos(2\pi/3)$$

where $0 \leq a, b \leq |E(X)|$ are integers with $a + b = |E(X)|$. Since $\cos(2\pi/3)$ is irrational, any eigenvalue with $b > 0$ is irrational. Since Γ has negative eigenvalues, and all of the negative eigenvalues of Γ are irrational, it follows that τ is irrational and Γ is not ratio tight. $\square$

We remark that this analysis becomes more difficult for $p > 3$, despite the fact that we can still express all of the eigenvalues as integer sums of $\cos(2\pi i/p)$. This follows as there are values of i for which $\cos(2\pi i/p)$ is both irrational and positive, and both irrational and negative. So we need more information about the edges of X to determine whether τ is irrational or not.

We note that if Γ is a bipartite graph, then the least eigenvalue of Γ is $-|E(X)|$. Also, both partite sets of Γ are cocliques, so $\alpha(\Gamma) \geq |\Gamma|/2$. So we have

$$\frac{|\Gamma|}{2} \leq \alpha(\Gamma) \leq \frac{|\Gamma|}{1-k/(-k)} = \frac{|\Gamma|}{2}.$$

Thus Γ (and every regular bipartite graph) is ratio tight. So we have the following corollary.

3.13.4 Corollary. *If X is bipartite and p is odd, or if $p = 3$, then Γ is not bipartite.* □

Finally, we consider the special case $p = 2$. For $p = 2$, the elements of $\mathbb{Z}_2^n$ all have order 2. In particular, each element of the connection set has order 2. Therefore, in this case the eigenvalues of Γ are

$$\left\{ \sum_{ij \in E(X)} (-1)^{a_i + a_j} \,:\, a \in \mathbb{Z}_p^n \right\}.$$

Each eigenvalue arises from an assignment of the values 0 and 1 to the vertices of X. If P is a partition of $V(X)$ into two sets, let $B(P)$ be the number of edges of X with an end in each part of P. So given an assignment $a \in \mathbb{Z}_2^n$, we have a corresponding partition P, and

$$\sum_{ij \in E(X)} (-1)^{a_i + a_j} = -B(P) + (|E(X)| - B(P)) = |E(X)| - 2B(P).$$

If X is bipartite, then there is a partition of $V(X)$ so $B(P) = |E(X)|$. Now we have $\tau = -|E(X)|$, and since Γ is $|E(X)|$ regular, the ratio bound gives

$$\alpha(\Gamma) \leq \frac{2^n}{1 - |E(X)|/(-|E(X)|)} = 2^{n-1}.$$

In this case we see that Γ meets the ratio bound if and only if Γ is bipartite.

We constructed Γ_D as a quotient graph of Γ using an equitable partition of the vertices. This implies that the eigenvalues of Γ_D are eigenvalues of Γ. Moreover, if θ is an eigenvalue of Γ_D with multiplicity m, then the multiplicity of θ as an eigenvalue of Γ is at least m (Theorem 9.3.3 in Godsil and Royle [18]). However, the eigenvalues of Γ_D are even harder to get a handle on than those of Γ. In particular, the deductions we have made about the least eigenvalue of Γ do not apply to Γ_D.

3.14 Open Problems

The largest open problem raised in this chapter is the question of whether Theorem 3.0.2 can be extended to other classes of Cayley graphs. The method we have presented in this chapter for constructing reductions may be applicable to other classes of Cayley graphs.

In their paper [7], Codenotti et al. also consider the colouring problem. Given a graph X, a *colouring* is an assignment of colours to the vertices of X so that adjacent vertices are not assigned the same colour. The *chromatic number* $\chi(X)$ of X is the smallest number of colours required to construct a valid colouring of X. For general graphs X, computing $\chi(X)$ is known to be NP-Hard. Codenotti et al. show that computing $\chi(X)$ is still an NP-Hard problem if we restrict X to be a circulant graph (Theorem 4 in [7]). Their proof does not apply the construction used to prove their Maxclique result. It is not immediately clear how to prove this result for the class of Cayley graphs for the groups $\mathbb{Z}_m^n$ where m is a fixed integer.

Chapter 4

Eigenpolytopes of Folded Cubes

In this chapter we present some results on the folded-cube graphs. Folded cubes are a family of distance-regular cubelike graphs. As we will see, they provide another example of a family of graphs that meet the inertia bound with their unweighted adjacency matrix.

The proof that the folded cubes are inertia tight uses the fact that we can find a natural family of maximum cocliques for the folded cubes. These canonical cocliques come from the distance partition of the graph with respect to a vertex. We will show that the canonical cocliques can be obtained in two different ways. This leads to the question of whether or not the canonical cocliques are exactly the maximum cocliques of the folded cubes.

To address this question, we follow a geometrical method of Godsil and Meagher [15, 16]. Namely, there are distance-regular graphs for which the maximum cocliques are closely related to the facets of the τ-eigenpolytope (a polytope derived from the τ-eigenspace of a graph). We will derive two families of facets for the τ-eigenpolytope of the folded cube. These facets do not give us a characterization of the maximum cocliques, however they are related to the structure of the graph and are interesting in their own right.

4.1 Folded Cubes

For the basic properties of the folded-cube graphs, we follow Section 6.9 in Godsil [19] and Section 9.D in Brouwer et al. [3].

The *n-dimensional hypercube* is the cubelike graph $H_n = X(\mathbb{Z}_2^n, C)$ with connection set

$$C = \{e_i :, 1 \leq i \leq n\}$$

where e_i is the ith standard basis vector of $\mathbb{Z}_2^n$. This is one standard definition of the n-dimensional hypercube. Alternatively, we can define H_n as the graph on the subsets of $[n]$ where S and T are adjacent if and only if there is some element $i \in [n]$ so that the size of the symmetric difference of S and T is 1 (i.e., $|S \triangle T| = 1$). To see that these definitions define the same graph, we note that we can view the vectors in $\mathbb{Z}_2^n$ as the characteristic vectors of the subsets of $[n]$ and that addition in $\mathbb{Z}_2^n$ corresponds exactly to the symmetric difference operation.

The hypercube H_n is a n-regular graph on 2^n vertices. We can partition H_n into sets A and B where $a \in A$ if a has even weight, and $b \in B$ if b has odd weight. It is easily seen that the edges of H_n connect elements of A to elements of B, thus H_n is bipartite. If we consider the vertex $0 \in \mathbb{Z}_2^n$, we see that the distance between 0 and x is exactly the weight of x. Thus the diameter of H_n is n, and there is exactly one vertex at distance n from 0 (the vertex $\mathbf{1}$). Since is a Cayley graph, H_n is vertex transitive, and every vertex x has a unique vertex y so that x and y are at distance n in H_n.

The *folded cube of order* n is the graph G_n obtained from H_n by identifying the vertices of H_n that are at maximum distance from each other. By this we mean that G_n is the quotient graph $H_n/\mathcal{P}$ where $\mathcal{P}$ is the partition of the vertices of H_n into pairs of vertices that are at distance n from each other. For each $x \in \mathbb{Z}_2^n$, the vertex at maximum distance from x in H_n is the vertex $x + \mathbf{1}$. Viewed as subsets of $[n]$, the vertex S is at maximum distance in H_n from the vertex $\overline{S}$. Thus the vertices of G_n are the partitions of $[n]$ into two parts, $(S, \overline{S})$. Two partitions are adjacent in G_n if one part of each in adjacent in H_n. So if $(S, \overline{S})$ and $(T, \overline{T})$ are partitions, without loss of generality they are adjacent if $S = T \cup \{i\}$ for some $1 \leq i \leq n$. This implies that $\overline{S} = \overline{T} \setminus \{i\}$, and there are exactly two edges between $(S, \overline{S})$ and $(T, \overline{T})$ in H_n. So the partition $\mathcal{P}$ is an equitable partition of H_n. This also implies that G_n is a n-regular graph.

4.1.1 Example. When $n = 3$, the hypercube H_n is the familiar cube graph. The 8 vertices of the cube can be partitioned into 4 pairs of vertices at distance 3 as

$$\{000, 111\}, \quad \{001, 110\}, \quad \{010, 101\}, \quad \{100, 011\}.$$

The folded-cube graph G_3 in this case is K_4, as 000 is adjacent to the vertices of weight 1 in H_3, and any vertex of weight 2 is adjacent to the vertices of weight 1 that are not in the same pair.

4.1.2 Example. When $n = 4$, the hypercube H_n is the tesseract. The 16 vertices of the tesseract can be partitioned into 8 pairs of vertices at distance 4 as

$$\begin{aligned}
&\{0000, 1111\};\\
&\{1000, 0111\}, \quad \{0100, 1011\}, \quad \{0010, 1101\}, \quad \{0001, 1110\};\\
&\{1100, 0011\}, \quad \{1001, 0110\}, \quad \{1010, 0101\}.
\end{aligned}$$

We have arranged the vertices of G_4 suggestively. The vertex $\{0000, 1111\}$ is adjacent to each vertex in the second row. We can also check that every vertex in the second row is adjacent to every vertex in the third row. Since the vectors in each row have the same parity, there are no edges between vertices on the same row. Thus G_4 is $K_{4,4}$ the complete bipartite graph on partite sets of size 4.

4.2 Structural Properties of G_n

Every vertex of G_n is a partition $(S, \overline{S})$ of $[n]$. When n is odd, we will identify $(S, \overline{S})$ with the set that has size less than $n/2$. When n is even, we will identify $(S, \overline{S})$ with the set that has size less than $n/2$ where possible (i.e., we have a bijection between the sets of size $< n/2$ and the partitions that contain one part of size $< n/2$, and in the even case we also have the partitions into parts of size $n/2$). Consider the vertex $\emptyset$ in G_n. The neighbours of $\emptyset$ are the sets $\{i\}$ for $1 \leq i \leq n$. If S is a subset of $[n]$ with size at most $n/2$, then the distance between $\emptyset$ and the vertex corresponding to S (either S or $(S, \overline{S})$) is $|S|$. Since H_n is vertex transitive, we see that G_n is also vertex transitive. Thus the diameter of G_n is $\lfloor n/2 \rfloor$. We break our further discussion of the structure of G_n into two cases: n even; and n odd.

4.2.1 Proposition. *If n is even, then G_n is bipartite.*

Proof. Suppose that n is even, and let $n = 2r$. In this case the vertices of G_n are the subsets of $[n]$ with size less than r together with the partitions of $[n]$ into two sets of size r. For $0 \leq i \leq r$, let Γ_i be the set of vertices at distance i from $\emptyset$. We have that for $0 \leq i \leq r-1$, the set Γ_i consists of the subsets of $[n]$ of size i. If $(S, \overline{S})$ and $(T, \overline{T})$ are partitions with $|S| = |T| = i$, then $|\overline{T}| = n - i$ and

$$n = 2r = 2(r-1) + 2 \geq 2i + 2,$$

so $n - i \geq i + 2$ and S is not adjacent to T or $\overline{T}$ in H_n. Therefore for each $0 \leq i \leq r-1$, the set Γ_i is a coclique in G_n. Also, the set Γ_r is a coclique in G_n. This follows directly as if $(S, \overline{S}) \in \Gamma_r$, then $|S| = |\overline{S}| = r$ and since S is not adjacent to any vertex of size r in H_n, we have that Γ_r is a coclique. This shows that we can partition G_n into $A = \cup_{i \text{ odd}} \Gamma_i$ and $B = \cup_{i \text{ even}} \Gamma_i$ each of which is a coclique. Therefore G_n is bipartite when n is even. □

Now suppose that n is odd and let $n = 2r + 1$. In this case the vertices of G_n are the subsets of $[n]$ with size at most r. Again we consider the vertex $\emptyset$ in G_n. As for the case when n is even, we can partition the vertices into the sets Γ_i for $0 \leq i \leq r$ where Γ_i is the set of subsets of $[n]$ with size i. We have that Γ_i consists of the vertices at distance i from $\emptyset$, and that for $0 \leq i \leq r-1$, the set Γ_i is a coclique in G_n. The difference between the even and odd case is the set Γ_r. For n odd, Γ_r is the set of partitions of $[n]$ into $(S, \overline{S})$ where $|S| = r$. Since sets of size r cannot be adjacent in H_n, if we have an edge between $(S, \overline{S})$ and $(T, \overline{T})$ in Γ_r, then there is an edge between S and $\overline{T}$ in H_n. So there is some $1 \leq i \leq n$ with $S = \overline{T} \setminus \{i\}$. Thus $S \cap T = \emptyset$. So in G_n, the subgraph induced by the vertices in Γ_r is the graph on the r-subsets of $[n]$ where S and T are adjacent if and only if they are disjoint. This is the *Kneser graph* $K_{2r+1:r}$. So in the case $n = 2r + 1$, we have an odd cycle through $\emptyset$. For example, $[r]$ and $\{r+1, \ldots, 2r\}$ are adjacent vertices in Γ_r, take the path from $\emptyset$ to each of these vertices formed by adding each element in turn. This results in a cycle of length n. Moreover, since G_n is vertex transitive, n is the length of a shortest odd cycle in G_n, or the *odd girth* of G_n.

We can see immediately that since G_n is bipartite when n is even, the maximum cocliques will be the sets of size $n/2$ that arise from the unique 2-colouring of G_n. The odd case is much more interesting, and we spend the remainder of the chapter looking at the maximum cocliques in G_n for n odd. In both cases we can say much more about the structure of G_n. First we will show that the odd order folded cubes are inertia tight.

4.3 The Inertia Bound

The inertia bound, Theorem 2.3.1, relates the size of a maximum coclique in a graph to its spectrum. We show that the folded cubes for odd n are inertia tight using the unweighted adjacency matrix of G_n. The proof we present is an expanded version of proof given by Godsil in Section 6.10 of [19]. We assume for the remainder of this section that n is odd.

We saw in Section 3.12 that if $X = X(\mathbb{Z}_2^n, C)$ is a cubelike graph, then the eigenvalues of X are computed as follows. Since -1 is the only primitive 2nd root of unity,

$$\left\{ \sum_{c \in C} \left(\prod_{i=1}^{n} (-1)^{a_i c_i} \right) \,:\, a \in \mathbb{Z}_2^n \right\}$$

is the spectrum of X. Thus the spectrum of the n-dimensional hypercube H_n is

$$\left\{ \sum_{i=1}^{n} (-1)^{a_i} \,:\, a \in \mathbb{Z}_2^n \right\}.$$

However for $a \in \mathbb{Z}_2^n$, a_i is either 0 or 1, and we have that

$$\sum_{i=1}^{n} (-1)^{a_i} = (n - \operatorname{wt}(a))1 + \operatorname{wt}(a)(-1) = n - 2\operatorname{wt}(a).$$

Therefore the eigenvalues of H_n are $n - 2i$ with multiplicity $\binom{n}{i}$ for $0 \leq i \leq n$. We will use the same strategy to derive the eigenvalues of G_n. First we need to show that G_n is a Cayley graph for $\mathbb{Z}_2^{n-1}$.

Let C be the connection set

$$C = \{e_1, \ldots, e_{n-1}\} \cup \{\mathbf{1}\},$$

and define $X_n = X(\mathbb{Z}_2^{n-1}, C)$.

4.3.1 Proposition. $X_n \cong G_n$.

Proof. In Section 4.1 we constructed G_n by taking a quotient of H_n. Equivalently we can construct G_n from H_{n-1} by adding a perfect matching to H_{n-1} joining the unique pairs of vertices at distance $n-1$. To see that these definitions are equivalent, let G_n be constructed from H_n, and let G'_n be constructed from H_{n-1}. The vertices of G_n are pairs of elements of $\mathbb{Z}_2^n$ of the form $\{x, y\}$

where $x = \mathbf{1} - y$. The vertices of G'_n are elements of $\mathbb{Z}_2^{n-1}$. Let f be the function between $V(G'_n)$ and $V(G_n)$ defined as follows. For $x \in \mathbb{Z}_2^n$, we have that x_n is either 0 or 1. Moreover, for each $\{x, y\} \in V(G_n)$ exactly one of x and y has nth coordinate 0. We map $x \in \mathbb{Z}_2^{n-1}$ to the element of $V(G_n)$ that contains $(x, 0)$. Now f is clearly a bijection between $V(G_n)$ and $V(G'_n)$. To see that f is a homomorphism, note that if $x \in \mathbb{Z}_2^{n-1}$ and $1 \leq i \leq n-1$, then $f(x + e_i)$ contains $(x + e_i, 0)$ and so is adjacent to the pair containing $(x, 0)$. This accounts for the edges of G'_n corresponding to H_{n-1}. However, x is also adjacent to $x + \mathbf{1}$ in G'_n. Now

$$f(x + \mathbf{1}) = \{(x + \mathbf{1}, 0), (x, 1)\}$$

and thus x is adjacent to $f(x)$ via the generator e_n. Therefore f is an isomorphism between G_n and G'_n.

We have constructed G_n from H_{n-1} by adding an edge between the pairs of vertices at maximum distance. Equivalently we add an edge between every vertex x of H_{n-1} and the vertex $x + \mathbf{1}$. Since $H_{n-1} = X(\mathbb{Z}_2^{n-1}, C \setminus \{\mathbf{1}\})$ this shows that $G_n \cong X_n$. □

The spectrum of X_n is

$$\left\{ \sum_{c \in C} \left(\prod_{i=1}^{n} (-1)^{a_i c_i} \right) \,:\, a \in \mathbb{Z}_2^{n-1} \right\}.$$

Now for $a \in \mathbb{Z}_2^{n-1}$ the eigenvalue of X_n corresponding to a is

$$\theta_a = \sum_{c \in C} \left(\prod_{i=1}^{n-1} (-1)^{a_i c_i} \right) = \sum_{i=1}^{n-1} (-1)^{a_i} + \prod_{i=1}^{n-1} (-1)^{a_i}.$$

Therefore we have

$$\begin{aligned} \theta_a &= (n - \mathrm{wt}(a) - 1)1 + \mathrm{wt}(a)(-1) + (-1)^{\mathrm{wt}(a)} \\ &= n - 2\,\mathrm{wt}(a) + ((-1)^{\mathrm{wt}(a)} - 1). \end{aligned}$$

When $\mathrm{wt}(a)$ is even, $(-1)^{\mathrm{wt}(a)} - 1 = 0$, and when $\mathrm{wt}(a)$ is odd $(-1)^{\mathrm{wt}(a)} - 1 = -2$. Thus

$$\theta_a = \begin{cases} n - 2\,\mathrm{wt}(a), & \text{if } \mathrm{wt}(a) \text{ even} \\ n - 2(\mathrm{wt}(a) + 1), & \text{if } \mathrm{wt}(a) \text{ is odd.} \end{cases}$$

If we let $\mathrm{wt}(a) = 2i$, then we have that $\theta_a = n - 4i$ is an eigenvalue of X_n. There are $\binom{n-1}{2i}$ such vectors a that give us this eigenvalue. If we let $\mathrm{wt}(a) = 2i - 1$, then we have that $\theta_a = n - 4i$ is an eigenvalue if X_n. There are $\binom{n-1}{2i-1}$ such vectors a that give us this eigenvalue. Therefore for $0 \leq i \leq \lfloor n/2 \rfloor$ we have that $n - 4i$ is an eigenvalue of X_n with multiplicity

$$\binom{n-1}{2i} + \binom{n-1}{2i-1} = \binom{n}{2i}.$$

If we let $n = 2r + 1$. Then the eigenvalues of G_n are $2r + 1 - 4i$ for $0 \leq i \leq r$.

4.3.2 Lemma. *For n odd, G_n is inertia-tight.*

Proof. Note that since the eigenvalues of G_n are all odd, 0 is not an eigenvalue of G_n. So the inertia bound implies that

$$\alpha(G_n) \leq \min\{n^-(G_n), n^+(G_n)\}$$

where $n^-(G_n)$ is the number of negative eigenvalues of G_n and $n^+(G_n)$ is the number of positive eigenvalues of G_n. The first step in showing the G_n is inertia tight is to calculate $n^-(G_n)$ and $n^+(G_n)$.

We start with $n^+(G_n)$. The eigenvalue $2r+1-4i$ is positive when

$$\begin{aligned} 2r+1-4i &> 0 \\ r/2+1/4 &> i \\ \lfloor r/2 \rfloor &\geq i. \end{aligned}$$

So we have a positive eigenvalue for $0 \leq i \leq \lfloor r/2 \rfloor$. Consequently, the eigenvalue $2r+1-4i$ is negative when $\lfloor r/2 \rfloor + 1 \leq i \leq r$. To calculate $n^+(G_n)$ and $n^-(G_n)$ we simply sum the corresponding multiplicities. So we have

$$n^+(G_n) = \sum_{i=0}^{\lfloor r/2 \rfloor} \binom{2r+1}{2i},$$

and

$$n^-(G_n) = \sum_{i=\lfloor r/2 \rfloor+1}^{r} \binom{2r+1}{2i}.$$

To determine whether $\alpha(G_n)$ is equal to $\min\{n^-(G_n), n^+(G_n)\}$ we attempt to find a coclique in G_n with size $\min\{n^-(G_n), n^+(G_n)\}$. To do this, recall that we can partition the vertices of G_n into sets Γ_i for $0 \leq i \leq r$. The set Γ_i is the set of vertices at distance i from $\emptyset$. We saw in the previous section, that with the exception of Γ_r, the sets Γ_i are all cocliques in G_n. So we can construct a large coclique in G_n by taking either the sets Γ_i for even i, or for odd i, depending on the parity of r.

Suppose that $r = 2j+1$ is odd. In this case the set

$$\bigcup_{i=0}^{j} \Gamma_{2i}$$

is a coclique in G_n. Therefore

$$\alpha(G_n) \geq \sum_{i=0}^{j} |\Gamma_{2i}| = \sum_{i=0}^{j} \binom{2r+1}{2i}.$$

We also have that

$$\alpha(G_n) \leq n^+(G_n) = \sum_{i=0}^{\lfloor r/2 \rfloor} \binom{2r+1}{2i} = \sum_{i=0}^{j} \binom{2r+1}{2i}.$$

Therefore, when r is odd, G_n is inertia tight.

Suppose that $r = 2j$ is even. Now instead of the even Γ_i we take instead the odd Γ_i. The set

$$\bigcup_{i=0}^{j-1} \Gamma_{2i+1}$$

is a coclique in G_n. Therefore

$$\alpha(G_n) \geq \sum_{i=0}^{j-1} |\Gamma_{2i+1}| = \sum_{i=0}^{j-1} \binom{2r+1}{2i+1}.$$

Now we use the binomial coefficient identity $\binom{n}{i} = \binom{n}{n-i}$. We have that

$$\begin{aligned}
\sum_{i=0}^{j-1} \binom{2r+1}{2i+1} &= \sum_{i=0}^{j-1} \binom{4j+1}{4j-2i} \\
&= \sum_{i=j-1}^{0} \binom{4j+1}{4j-2i} \\
&= \sum_{i=j+1}^{2j} \binom{4j+1}{2i} \\
&= n^{-}(G_n).
\end{aligned}$$

Therefore we see that $\alpha(G_n) = n^{-}(G_n)$ and G_n is inertia tight. □

4.4 Canonical Cocliques

To show that G_n is inertia tight when n is odd, we found a family of maximum cocliques using the distance partition from $\emptyset$. Since G_n is vertex transitive, the distance partition from any vertex gives a maximum coclique in G_n. We call these the canonical cocliques in G_n. In this section we show that there is an alternative construction for these cocliques.

Let $n = 2r + 1$. Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_r$ be the partition of the vertices of G_n where Γ_i consists of the vertices at distance i from $\emptyset$. We have seen that when r is odd, $\cup_{i \text{ even}} \Gamma_i$ is a maximum coclique in G_n; and when r is even, $\cup_{i \text{ odd}} \Gamma_i$ is a maximum coclique in G_n. Note that since G_n is vertex transitive, we can partition G_n into distance sets $\Gamma_i(v)$ for $0 \leq i \leq r$ where $\Gamma_i(v)$ is the set of vertices of G_n at distance i from v. The sets $\Gamma_i(v)$ are all cocliques, and $|\Gamma_i(v)| = |\Gamma_i(\emptyset)|$. Thus for any vertex v, either $\cup_{i \text{ even}} \Gamma_i(v)$ or $\cup_{i \text{ odd}} \Gamma_i(v)$ is a maximum coclique in G_n (depending on the parity of r).

In fact, these cocliques are distinct for distinct vertices u and v. We will prove this in the case when r is odd (the even case is analogous). We need to show that

$$\bigcup_{i \text{ even}} \Gamma_i(v) \neq \bigcup_{i \text{ even}} \Gamma_i(u).$$

Consider the distance between u and v in G_n. If u and v are at an odd distance from each other, then $u \in \Gamma_{2i+1}(v)$ for some i. Thus $u \notin \cup_{i \text{ even}} \Gamma_i(v)$, and the two cocliques are distinct. Now suppose that u and v are at an even distance from each other. Then $u \in \Gamma_{2i}(v)$ for some i. Note that $\Gamma_r(v)$ contains edges of G_n. Also, there are vertices in Γ_r that are at distance $r - 2i$ from u. Let $x \in \Gamma_r$ be such a vertex, and consider $y \in \Gamma_r$ so that y is adjacent to x. Now if y is at distance $r - 2i$ from u, then there is a closed walk of length $2(r-2i)+1 = 2r+1-4i$ containing x, y and u. Thus this closed walk contains an odd cycle of length at most $2r + 1 - 4i$. This contradicts the fact that the odd girth of G_n is $2r + 1$. Therefore y is at distance $r - 2i + 1$ from u and r from v. Thus $y \in \cup_{i \text{ even}} \Gamma_i(u)$ and $y \notin \cup_{i \text{ even}} \Gamma_i(v)$, and we have the result.

So for each vertex v of G_n, the distance partition of G_n corresponding to v gives us a distinct maximum coclique. We refer to this set of 2^{n-1} cocliques as the *canonical cocliques* of G_n. Are these the only maximum cocliques in G_n?

Note that the distance partition corresponding to $\emptyset$ gives us another natural coclique. Assume that $r = 2j + 1$ is odd. The sets Γ_i for $i \neq r$ are all cocliques in G_n. So the set

$$\bigcup_{i=0}^{j-1} \Gamma_{2i+1}$$

is a coclique in G_n. We can augment this coclique by adding a coclique in Γ_r. The subgraph of G_n induced by the vertices of Γ_r is isomorphic to the Kneser graph $K_{n:r}$. The maximum cocliques of $K_{n:r}$ are exactly the sets

$$S_i = \{S \subseteq [n] \,:\, |S| = r, i \in S\}$$

for $1 \leq i \leq n$ by the Erdős-Ko-Rado Theorem. So for any $1 \leq i \leq n$, the set

$$T_i = S_i \bigcup \left(\bigcup_{i=0}^{j-1} \Gamma_{2i+1} \right)$$

is a coclique in G_n. Since the sets Γ_{2i+1} are disjoint (and disjoint from S_i), to find $|T_i|$ we simply sum their cardinalities,

$$|T_i| = \binom{n-1}{r-1} + \sum_{i=0}^{j-1} |\Gamma_{2i+1}| = \binom{n-1}{r-1} + \sum_{i=0}^{j-1} \binom{n}{2i+1}.$$

We show that T_i is a maximum coclique by showing that is has the same size as the canonical coclique T corresponding to $\emptyset$. The size of T is

$$\begin{aligned}
|T| &= \sum_{i=0}^{j} |\Gamma_{2i}| \\
&= \binom{n}{0} + \sum_{i=1}^{j} \binom{n}{2i} \\
&= \binom{n-1}{0} + \sum_{i=1}^{j} \left(\binom{n-1}{2i} + \binom{n-1}{2i-1} \right) \\
&= \sum_{i=0}^{r-1} \binom{n-1}{i}.
\end{aligned}$$

By manipulating our expression for $|T_i|$ we see that

$$\begin{aligned}
|T_i| &= \binom{n-1}{r-1} + \sum_{i=0}^{j-1} \binom{n}{2i+1} \\
&= \binom{n-1}{r-1} + \sum_{i=0}^{j-1} \left(\binom{n-1}{2i+1} + \binom{n-1}{2i} \right) \\
&= \binom{n-1}{r-1} + \sum_{i=0}^{r-2} \binom{n-1}{i} \\
&= \sum_{i=0}^{r-1} \binom{n-1}{i} \\
&= |T|.
\end{aligned}$$

Therefore for each $1 \leq i \leq n$, the set T_i is a maximum coclique of G_n. We can construct the analogous cocliques when r is even, and the same method shows that these are maximum cocliques for those graphs.

We have seen that by examining the distance partition of G_n, we can find two families of maximum cocliques. The canonical cocliques, and the cocliques of the form T_i. However, these families are in fact the same.

4.4.1 Lemma. *The set of cocliques of G_n of the form T_i is exactly the set of canonical cocliques.*

Proof. We prove this for $r = 2j$. For each $0 \leq i \leq n$, let

$$T_i = S_i \bigcup \left(\bigcup_{i=0}^{j-1} \Gamma_{2i}(\emptyset) \right)$$

be the coclique described above for the distance partition corresponding to $\emptyset$.

Let v be a neighbour of $\emptyset$, and let

$$T = \bigcup_{i=0}^{j-1} \Gamma_{2i+1}(v)$$

be the canonical coclique corresponding to v. Since v is adjacent to $\emptyset$, there is some $1 \leq a \leq n$ so that v corresponds to the partition containing $\{a\}$ (we will refer to v as a from here). We show that $T = T_a$.

Let x be any vertex in Γ_{2i} with $i < j$. Suppose that x is not at odd distance from v. Then there is a minimum length path P from v to x of even length. If $\emptyset$ is not on P, then $\emptyset P$ is a minimum length path between $\emptyset$ and x of odd length, contradicting $x \in \Gamma_{2i}$. Therefore $\emptyset$ lies on P. Suppose the edge $\emptyset a$ is not an edge of P. Then P has the form

$$P = aP_1 b\emptyset b' P_2 x$$

where a, b, b' are distinct neighbours of $\emptyset$. Now the path $a\emptyset b' P_2 x$ is a path from a to x that is shorter than P, a contradiction. Therefore P has the form $a\emptyset P' x$. Since P is a shortest path from a to x, the subpath P' is a shortest path from $\emptyset$ to x. However, P' is an odd length path which again contradicts $x \in \Gamma_{2i}$. Therefore, the vertices in Γ_{2i} for $i < j$ are all vertices at odd distance from a, and

$$\bigcup_{i=0}^{j-1} \Gamma_{2i}(\emptyset) \subset \bigcup_{i=0}^{j-1} \Gamma_{2i+1}(a).$$

Finally, note that every partition in Γ_r consists of a set of size r and a set of size $r+1$. We identify each partition with the smaller of its parts. So for every $S \in S_a$, the set S has size r, and $a \in S$. Thus we can write S as $S = \{a\} \cup S'$ where $S' = \{s_1, \ldots, s_{r-1}\}$. We construct a path from a to S by adding the elements s_i in turn. This results in a path of length $r-1$ from a to S. Note that the ith vertex of this path is at distance $i+1$ from $\emptyset$. Therefore it is a minimum length path between a and S, and $S \in \Gamma_{r-1}(a)$. Thus $S_i \subset \Gamma_{r-1}(a)$ and $T_a \subseteq T$. Since $|T| = |T_a|$ we conclude that $T = T_a$.

This shows that for all $1 \leq i \leq n$, the coclique T_i is the canonical coclique corresponding to $\{i\}$. Since G_n is vertex transitive, this shows that the coclique $T_i(v)$ is canonical for all vertices v and $1 \leq i \leq n$. The proof is identical for the case $r = 2j+1$. □

So far we have identified a canonical set of cocliques, and another natural family that gives an alternative definition of the canonical cocliques. In order to explore the possibility of a different class of maximum cocliques, we further develop the structure of G_n.

4.5 Distance-regular Graphs

Distance-regular graphs are a class of graphs with a strong algebraic structure. We present some of the basic theory of distance-regular graphs following Godsil [13].

Let X be a graph, and let x, y be vertices in X. Keeping with our previous notation, we will denote the distance between x and y in X as $d(x, y)$, and we will denote the set of vertices in X at distance i from x as $\Gamma_i(x)$. We will denote the diameter of X by d.

A connected graph X is *distance regular* if there are constants a_i, b_i and c_i for all $0 \leq i \leq d$ with the following property. For all vertices $x, y \in V(X)$, if $d(x, y) = i$, then

$$\begin{aligned} |\Gamma_{i-1}(x) \cap \Gamma_1(y)| &= c_i, \\ |\Gamma_i(x) \cap \Gamma_1(y)| &= a_i, \\ |\Gamma_{i+1}(x) \cap \Gamma_1(y)| &= b_i. \end{aligned}$$

4.5.1 Example. The hypercube graph H_n is distance regular for all n. To see this, let $0 \leq i \leq n$ and let $x, y \in V(H_n)$ be arbitrary vertices with $d(x, y) = i$. Now let S_x and S_y be the subsets of $[n]$ with characteristic vectors x and y respectively. Then $S = S_x \triangle S_y$ has $|S| = i$. The neighbours of y are the vectors $y + e_j$ as $1 \leq j \leq n$. We have two cases, either $j \in S$ or $j \notin S$. If $j \in S$, then $d(y + e_j, x) = i - 1$ and

$$y + e_j \in \Gamma_{i-1}(x) \cap \Gamma_1(y).$$

If $j \notin S$ then $d(y + e_j, x) = i + 1$ and

$$y + e_j \in \Gamma_{i+1}(x) \cap \Gamma_1(y).$$

Therefore

$$\begin{aligned} |\Gamma_{i-1}(x) \cap \Gamma_1(y)| &= i, \\ |\Gamma_i(x) \cap \Gamma_1(y)| &= 0, \\ |\Gamma_{i+1}(x) \cap \Gamma_1(y)| &= n - i. \end{aligned}$$

Since x and y were arbitrary, $c_i = i$, $a_i = 0$ and $b_i = n - i$, and H_n is distance regular.

For a distance-regular graph X, we refer to the numbers b_i, a_i, c_i for $0 \leq i \leq d$ as the *parameters* of X. Note that if $i = 0$, then we have that $c_i = 0$, $a_i = 0$ and $b_i = \deg(x)$. Thus X is a regular graph with valency $k = b_0$. Also, if $i = d$ then $b_i = 0$. If x and y are vertices at distance i, then the neighbours of y are partitioned into those as distance $i - 1, i$ and $i + 1$ from x. Therefore for each $0 \leq i \leq d$, we have that

$$b_i + a_i + c_i = k = b_0.$$

So we can derive the a_i values given the b_i and c_i values. In order to compactly express the parameters of X we define its *intersection array* to be the following array of parameters,

$$\begin{pmatrix} b_0 = k & b_1 & b_2 & \dots & b_d = 0 \\ c_0 = 0 & c_1 = 1 & c_2 & \dots & c_d \end{pmatrix}.$$

There is another set of parameters that we associate with X. Consider the distance partition of X with respect to a vertex x. Let $n_i = |\Gamma_i(x)|$ be the size of the ith distance set in the partition. For each $y \in \Gamma_i(x)$, the number of neighbours of y in $\Gamma_{i+1}(x)$ is b_i. Also, for each $y \in \Gamma_{i+1}(x)$ the number of neighbours of y in $\Gamma_i(x)$ is c_{i+1}. Therefore the number of edges between $\Gamma_i(x)$ and $\Gamma_{i+1}(x)$ is $n_i b_i = n_{i+1} c_{i+1}$. Thus $n_{i+1} = n_i b_i / c_{i+1}$ depends only on n_i and the parameters of X. Thus we can derive the values n_i from $n_0 = 1$ and the parameters of X. Therefore the values n_i are the same for all vertices x of X, and the distance partition consists of sets of the same size no matter which vertex we start with.

We can prove that the folded n-cube G_n is a distance-regular graph for all n by deriving its parameters.

4.5.2 Lemma. *The folded cube G_n is distance regular for all $n \geq 1$. The size of the ith distance set in any distance partition of G_n is $n_i = \binom{n}{i}$.*

If $n = 2r$ then the parameters of G_n are $c_i = i$ for $0 \leq i \leq r-1$ and $c_r = n$, and $b_i = n - i$ for $0 \leq i \leq r-1$ and $b_r = 0$.

If $n = 2r+1$ then the parameters of G_n are $c_i = i$ for $0 \leq i \leq r$, and $b_i = n - i$ for $0 \leq i \leq r-1$ and $b_r = 0$.

Proof. Recall that G_n is vertex transitive. So, if x and y are vertices in G_n with $d(x,y) = i$, then there is some automorphism σ of G_n so that $\sigma(x) = \emptyset$. Moreover,

$$i = d(x,y) = d(\sigma(x), \sigma(y)) = d(\emptyset, \sigma(y)).$$

Therefore it suffices to show the distance regularity condition for $x = \emptyset$.

Suppose that y is any vertex of G_n with $d(\emptyset, y) = i$. Then we have $y = \{y_1, \ldots, y_i\}$. If $0 \leq i < \lfloor n/2 \rfloor$, then y has no neighbours x with $|x| = i$. The number of neighbours x of y with $|x| = i-1$ is the number of ways we can remove an element from y, or i. Similarly the number of neighbours x of y with $|x| = i+1$ is $n - i$.

Let $n = 2r$ and y be a vertex of G_n with $d(\emptyset, y) = r$. Then y is a partition of $[n]$ into two parts of size r. The number of neighbours of y that are partitions of $[n]$ into two sets of size r is 0, as we have already seen. Now for each $i \in [n]$ one of the parts of y contains i. Without loss of generality $y = (y_1, y_2)$ and $i \in y_1$. Now the partition $(y_1 \setminus \{i\}, y_2 \cup \{i\})$ (denoted $y_1 \setminus \{i\}$) is at distance $i-1$ from $\emptyset$. Performing this operation for each i gives a distinct neighbour of y so $c_r = n$.

Let $n = 2r+1$ and y be a vertex of G_n with $d(\emptyset, y) = r$. Then y is a subset of $[n]$ of size r. As above we see that y has r neighbours of size $r-1$ in G_n. Two vertices at distance r from $\emptyset$ are adjacent if and only if they are disjoint. Thus there are $n - r$ neighbours of y at distance r from $\emptyset$.

It remains to note that $n_i = \binom{n}{i}$ as we have already seen. This completes the proof that G_n is distance regular with the given parameters. □

For a graph X (not necessarily distance regular) and $0 \leq i \leq d$, we define the ith distance matrix of X as $A_i(X)$ where $A_i(X)[x,y] = 1$ if $d(x,y) = i$ and 0 otherwise. The matrices $A_i(X)$ are symmetric matrices with entries 0 and 1, so each $A_i(X)$ defines a graph on $V(X)$. We let X_i be the graph with adjacency matrix $A(X_i) = A_i(X)$. Then X_i is the *ith distance graph of* X; the graph where two vertices are adjacent if and only if they are at distance i in X. Note that $A_0(X)$ is the identity matrix, and so X_0 is the graph on $V(X)$ where every vertex has a loop edge and there are no other edges. Every other X_i is a graph with no loops or multiple edges.

Consider the matrices $A_i(X)$ and $A_j(X)$. The entries of their product $A_i(X)A_j(X)$ have a combinatorial interpretation. If $x, y \in V(X)$, then the (x, y)-entry of $A_i(X)A_j(X)$ is the inner product of the x-row of $A_i(X)$ and the y-row of $A_j(X)$ and is equal to the number of vertices in X that are at distance i from x and at distance j from y. So

$$(A_i(X)A_j(X))[x, y] = |\Gamma_i(x) \cap \Gamma_j(y)|.$$

We can translate the definition of distance regularity into a condition on distance matrices. Let X be a distance-regular graph with parameters a_i, b_i and c_i for $0 \le i \le d$. Then

$$A_i(X)A_1(X) = c_i A_{i-1}(X) + a_i A_i(X) + b_i A_{i+1}(X).$$

This follows from the definition of distance regularity, as if $d(x, y) = i$ and $j < i - 1$ or $j > i + 1$, then $|\Gamma_j(x) \cap \Gamma_1(y)| = 0$.

Since $A_0(X)$ is the identity matrix, we have that $A_i(X)A_0(X) = A_i(X)$ for all $0 \le i \le d$. So if X is distance regular, then the product of $A_i(X)$ with either $A_0(X)$ or $A_1(X)$ is a linear combination of distance matrices. In fact, much more is true.

4.5.3 Lemma (Lemma 2.1 in Chapter 11 of [13])**.** *Let X be a connected graph with diameter d. Then X is distance regular if and only if $A_i(X)A_j(X)$ is a linear combination of distance matrices for all $0 \le i, j \le d$.*

Proof. If $A_i(X)A_j(X)$ is a linear combination of distance matrices for all $0 \le i, j \le d$, then $A_i(X)A_1(X)$ is a linear combination of distance matrices for all $0 \le i \le d$. As above we note that the coefficient of $A_j(X)$ for $j \notin \{i-1, i, i+1\}$ must be 0 in this linear combination, and we take c_i, a_i, b_i to be the coefficients of $A_{i-1}(X), A_i(X), A_{i+1}(X)$ respectively. Therefore X is distance regular.

Now suppose that X is distance regular. Then for all $1 \le i \le d+1$, we have

$$A_{i-1}(X)A_1(X) = c_{i-1}A_{i-2}(X) + a_{i-1}A_{i-1}(X) + b_{i-1}A_i(X).$$

By rearranging this equation we can express $A_i(X)$ as a linear combination of $A_{i-2}(X), A_{i-1}(X)$ and $A_{i-1}(X)A_1(X)$. Therefore by induction, we can express $A_i(X)$ as a polynomial p of degree i in $A_1(X)$ for all $0 \le i \le d$. Therefore for $0 \le i, j \le d$, we have

$$A_i(X)A_j(X) = p(A_1(X))A_j(X).$$

Each term of the form $A_1(X)^\alpha A_j(X)$ can be iteratively expanded to a linear combination of distance matrices. Therefore $A_i(X)A_j(X)$ is a linear combination of distance matrices. $\square$

So the combinatorial structure of distance-regular graphs gives us a strong algebraic structure between their distance matrices.

4.6 Association Schemes

Association schemes are extremely important in algebraic combinatorics. They provide an indispensable tool for working with distance-regular graphs. In this section we give a very brief introduction to association schemes. We focus on the basic definitions that we will need in later sections. We follow Brouwer et al. [3] and Godsil [13].

Let X be a set of size n, and let R_i be a relation on X for $0 \leq i \leq d$. Let $\{R_0, \ldots, R_d\}$ satisfy the following properties:

(a) $\{R_0, \ldots, R_d\}$ is a partition of $X \times X$;

(b) $R_0 = \{(x, x) : x \in X\}$;

(c) each R_i is a symmetric relation;

(d) for $(x, y) \in R_k$, the number of elements $z \in X$ with $(x, z) \in R_i$ and $(z, y) \in R_j$ depends only on the indices i, j, k.

Then the relations R_i form an *association scheme with d classes.*

Note that we can represent each relation R_i with a $n \times n$ matrix A_i with entries 0 and 1. We translate the conditions (a)-(d) in the definition of an association scheme to give conditions on $\{A_0, \ldots, A_d\}$. The matrices $\{A_0, \ldots, A_d\}$ satisfy the following properties:

(a) $\sum_{i=0}^{d} A_i = J$ (where J is the $n \times n$ matrix with all values 1);

(b) $A_0 = I$ (where I is the $n \times n$ identity matrix);

(c) A_i is a symmetric matrix;

(d) for $0 \leq i, j, k \leq d$ there exist constants $p_{i,j}^k$ so that $A_i A_j = \sum_{k=0}^{d} p_{i,j}^k A_k$.

The numbers $p_{i,j}^k$ are called the *intersection numbers* of the scheme.

Note that (d) implies that the product of any of the A_i matrices can be written as a linear combination of $\{A_0, \ldots, A_d\}$. Thus these matrices generate a matrix algebra $\mathcal{A}$. This algebra is referred to as the *Bose-Mesner algebra* of the association scheme. Furthermore, from (a) we have that $\{A_0, \ldots, A_d\}$ is a linearly independent set, and thus is a basis for the Bose-Mesner algebra of the scheme.

Define the operation $\circ$ on the set of $n \times n$ real matrices to be entrywise multiplication. So

$$(A \circ B)[i, j] = A[i, j]B[i, j]$$

for all $1 \leq i, j \leq n$. We will refer to this operation as *Schur multiplication.* Note that since the A_i are 01-matrices, $A_i \circ A_i = A_i$, and from (a) we have that

$A_i \circ A_j = 0$ when $i \neq j$. Thus Schur multiplication by A_i on $\mathcal{A}$ is an idempotent operation, and we refer to $\{A_0, \ldots, A_d\}$ as the basis of *Schur idempotents* for $\mathcal{A}$.

From our definition of an association scheme, property (d) implies not only that the numbers $p_{i,j}^k$ exist, but also that $p_{i,j}^k = p_{j,i}^k$. Therefore, we see that

$$A_i A_j = \sum_{k=0}^{d} p_{i,j}^k A_k = \sum_{k=0}^{d} p_{j,i}^k A_k = A_j A_i.$$

Thus the Bose-Mesner algebra is a $d+1$-dimensional commutative algebra of symmetric matrices with constant diagonal. Since the matrices $\{A_0, \ldots, A_d\}$ commute, they are simultaneously diagonalizable. As a result we can find a second basis for $\mathcal{A}$.

4.6.1 Theorem (Theorem 2.1 from Chapter 12 of [13])**.** *Let $\{A_0, \ldots, A_d\}$ be the Schur idempotents of the Bose-Mesner algebra $\mathcal{A}$ of a d-class association scheme. Then there is a set of idempotent matrices $\{E_0, \ldots, E_d\}$ and real numbers $p_i(j)$ for $0 \leq i, j \leq d$ satisfying:*

(a) $\sum_{i=0}^{d} E_i = I$ (where I is the $n \times n$ identity matrix);

(b) $A_i E_j = p_i(j) E_j$ for all $0 \leq i, j \leq d$;

(c) $E_0 = (1/n)J$ (where J is the $n \times n$ matrix with all values 1);

(d) $\{E_0, \ldots, E_d\}$ is an orthogonal basis for $\mathcal{A}$. □

We omit the proof of Theorem 4.6.1 and refer the reader to [13].

For all i and j, we have that the column space of E_j is the $p_i(j)$ eigenspace of A_i. As such we refer to the values $p_i(j)$ as the *eigenvalues* of the scheme. The dimension of the $p_i(j)$ eigenspace is denoted as m_j, the *multiplicity* of $p_i(j)$ (since the eigenspaces are common to all of the matrices in $\mathcal{A}$ the multiplicity does not depend on i).

There is a duality between the basis of Schur idempotents $\{A_0, \ldots, A_d\}$ and the basis of idempotents $\{E_0, \ldots, E_d\}$. The matrices E_j are elements of $\mathcal{A}$, and so we can express each E_i in terms of the Schur idempotents,

$$E_j = (1/n) \sum_{i=0}^{d} q_j(i) A_i$$

(here the factor of $1/n$ is included by convention for the sake of convenience). We refer to the numbers $q_j(i)$ as the *dual eigenvalues* of $\mathcal{A}$, as they satisfy a dual relation to point (b) in Theorem 4.6.1. Namely,

$$E_j \circ A_i = (1/n) \sum_{k=0}^{d} q_j(i) A_k \circ A_i = (1/n) q_j(i) A_i.$$

We also have a set of parameters that are dual to the intersection numbers of the scheme. Since each E_j can be expressed as a linear combination of the matrices A_i, and the A_i are Schur idempotent, the matrix $E_i \circ E_j \in \mathcal{A}$ for all i, j. So we can express $E_i \circ E_j$ in terms of the matrices E_i, and there are constants $q_{i,j}^k$ such that

$$E_i \circ E_j = \sum_{k=0}^{d} q_{i,j}^k E_k.$$

The constants $q_{i,j}^k$ are the *Krein parameters* of the scheme.

Define the matrix P to be the matrix of eigenvalues of $\mathcal{A}$. So $P[i,j] = p_i(j)$ for all $0 \leq i, j \leq d$. Likewise define the matrix Q to be the matrix of dual eigenvalues of $\mathcal{A}$. Note that

$$A_i = A_i \sum_{j=0}^{d} E_j = \sum_{j=0}^{d} p_i(j) E_j.$$

Combining this identity for each A_i with the equation

$$E_j = (1/n) \sum_{i=0}^{d} q_j(i) A_i,$$

we see that

$$E_i = (1/n) \sum_{k=0}^{d} \sum_{j=0}^{d} q_i(j) p_j(k) E_k$$

and therefore $(1/n)\sum_{j=0}^{d} q_i(j)p_j(k)$ is 1 if $i = k$ and 0 otherwise. Thus P and Q are related by $PQ = (1/n)I$. This relation shows that we can compute the dual eigenvalues from the eigenvalues of $\mathcal{A}$ and vice versa. With a little more work we can show that if we are given only P (or only Q) we can recover all of the parameters of the scheme (i.e., the intersection numbers, eigenvalues, multiplicities, Krein parameters, dual eigenvalues and dual multiplicities).

Finally we give the connection between distance-regular graphs and association schemes. From our original definition of an association scheme, the relations R_i each define a graph. Let X_i be the ith graph of the scheme, and note that $A(X_i) = A_i$. Therefore, if X is a distance-regular graph, and we let X_i be the ith distance graph of X for $0 \leq i \leq d$, Lemma 4.5.3 implies that the relations X_i are an association scheme.

In the proof of Lemma 4.5.3 we saw that each A_i could be expressed as a polynomial of degree i in A_1. Schemes with this property are called *metric* or *P-Polynomial* schemes. If X is a distance-regular graph, then the distance graphs of X form a metric association scheme. Conversely we have the following lemma.

4.6.2 Lemma (Lemma 3.1 from Chapter 12 of [13]). *If $\{R_0, \ldots, R_d\}$ is a metric association scheme, then the relations R_i are the distance relations of a distance-regular graph.* □

So metric association schemes are exactly the schemes that come from graphs that are distance regular. From Lemma 4.5.2 we have that the folded n-cube G_n is a distance-regular graph for all n. Therefore the i-distance graphs of G_n for $0 \leq i \leq \lfloor n/2 \rfloor$ form a metric association scheme.

4.7 Polytopes

In this section we will look at polytopes constructed from the eigenspaces of a graph, and the connections between the structure of the graph and the structure of its eigenpolytopes. We start with a quick introduction to polytopes. We will follow Grünbaum [20]. We will skip over the most basic definitions, and refer the reader to [20]

A subset C of $\mathbb{R}^n$ is *convex* if for all $x, y \in C$ and $0 \leq \lambda \leq 1$, the point $\lambda x + (1 - \lambda)y$ is in C. Given a set of point $S \subseteq \mathbb{R}^n$, the *convex hull* of S is the set

$$\mathrm{conv}(S) = \left\{ \sum_{i=1}^{k} \alpha_i s_i \,:\, s_i \in S, \sum_{i=1}^{k} \alpha_i = 1 \right\}.$$

A *hyperplane* in $\mathbb{R}^n$ is a set of the form

$$\{x \in \mathbb{R}^n \,:\, h^T x = a\}$$

where $h \in \mathbb{R}^n$ and $a \in \mathbb{R}$. An *open halfspace* is a subset of $\mathbb{R}^n$ of the form

$$\{x \in \mathbb{R}^n \,:\, h^T x < a\}.$$

Note that for every hyperplane H, we have a partition of $\mathbb{R}^n$ given by H together with the two open halfspaces corresponding to H. If C is a convex subset of $\mathbb{R}^n$, and H is a hyperplane, then H *cuts* C if both of the open halfspaces corresponding to H contain a point of C. If $H \cap C \neq \emptyset$ and H does not cut C, then H is a *supporting hyperplane* of C.

If C is a closed convex subset of $\mathbb{R}^n$, then C is the intersection of the closed halfspaces that contain C. Equally, C is the intersection of the closed halfspaces that contain C for which the corresponding hyperplane is a supporting hyperplane of C.

Let C be a convex subset of $\mathbb{R}^n$. An *extreme point* of C is a point $x \in C$ so that for all $y, z \in C$ and $0 < \lambda < 1$, if $x = \lambda y + (1 - \lambda)z$, then $x = y = z$. We refer to the set of extreme points of C as $ext(C)$. For a supporting hyperplane H, the subset $H \cap C$ is called a *face* of C. We also take $\emptyset$ and C to be faces of C; they are referred to as *improper faces*. If $\{x\}$ is a singleton face, then $x \in ext(C)$.

A subset K of $\mathbb{R}^n$ is a *polyhedral set* if K is the intersection of finitely many closed halfspaces in $\mathbb{R}^n$. Since closed halfspaces are convex closed sets, polyhedral sets are closed and convex. Let $\mathcal{F}(K)$ be the set of faces of K. The set $\mathcal{F}(K)$ is finite, and with the subset relation $\subseteq$, it forms a lattice. The singleton faces are called the *vertices* of K, and the maximal proper faces of K are called the *facets* of K.

A subset S of $\mathbb{R}^n$ is *affinely independent* if there is no non-trivial affine combination of the elements of S that gives 0. That is, there are no $s_i \in S$ and $\lambda_i \in \mathbb{R}$ (with some $\lambda_i \neq 0$) so that $\sum_{i=1}^k \lambda_i s_i = 0$ and $\sum_{i=1}^k \lambda_i = 0$. An *affine combination* of elements of S is a point of the form $\sum_{i=1}^k \lambda_i s_i$ where the $\lambda_i \in \mathbb{R}$ satisfy $\sum_{i=1}^k \lambda_i = 1$. The *affine hull* of S is $\operatorname{aff}(S)$ and in the set of all affine combinations of elements in S. The affine hull of a set is an example of an affine space. Every affine subspace of $\mathbb{R}^n$ is a translation of a linear subspace of $\mathbb{R}^n$. Thus the *dimension* of an affine subspace is the dimension of the linear space we obtain through translation. We define the *dimension of S* to be $\dim(S) = \dim(\operatorname{aff}(S))$.

A polyhedral set C that is compact (i.e., closed and bounded) is called a *polytope*. A polytope is equal to the convex hull of its vertices. The faces of dimension 1 of C are called the *edges* of C. Note that the vertices and edges of a polytope define a graph embedded in $\mathbb{R}^n$. We will refer to this as the *graph* of the polytope.

4.7.1 Example. Let $C \subset \mathbb{R}^3$ be defined as

$$\begin{aligned} C \quad = \quad & \operatorname{conv}\{(0,0,0),(0,0,1),(0,1,0),(1,0,0), \\ & \qquad (1,1,0),(1,0,1),(0,1,1),(1,1,1)\}. \end{aligned}$$

The polytope C is the 3-dimensional cube. Its graph is isomorphic to the hypercube H_3. The facets of C are the 6 faces obtained by taking the convex hull of the four vertices (a, b, c) where one of a, b, c has constant value.

Let P and Q be polytopes. We say that P and Q are *combinatorially equivalent* if there is a map $\phi : \mathcal{P} \to \mathcal{Q}$ that is an inclusion preserving bijection. So ϕ is an isomorphism between the face lattices of P and Q. Combinatorially equivalent polytopes have the same number of faces of dimension k for all integers k.

4.8 Eigenpolytopes

We can construct polytopes from the eigenspaces of graphs. Let X be a graph on n vertices, and let θ be an eigenvalue of X with multiplicity m. The θ-eigenvectors of X span a subspace of $\mathbb{R}^n$ of dimension m. Let $\{u_1, \dots, u_m\}$ be an orthonormal basis of the θ-eigenspace of X. Define the $n \times m$ matrix U_θ to be the matrix whose ith column is u_i. The rows of U_θ are indexed by $V(X)$, and we let the row corresponding to $x \in V(X)$ be denoted $x(\theta)$. The θ-eigenpolytope of X is

$$P_\theta = \operatorname{conv}\{x(\theta) : x \in V(X)\}.$$

Note that by our definition, the polytope P_θ depends on our choice of basis vectors u_i. However, if $\{u'_1, \dots, u'_m\}$ is an orthonormal basis for the θ-eigenspace of X, then there is a linear transformation of $\mathbb{R}^n$ that maps each u'_i to u_i. Therefore, the polytopes obtained from these bases are combinatorially equivalent. Since we are only concerned with the combinatorial structure of these polytopes, we ignore the specific choice of basis.

Eigenpolytopes are closely related to the study of *representations* or *Euclidean representations* of graphs. In [14], Godsil studied the eigenpolytopes of distance-regular graphs, focusing on the eigenpolytope corresponding to the second largest eigenvalue of X. We summarize some observations on eigenpolytopes from [14], and on representations of distance-regular graphs from Chapter 13 of [13].

Let X be a distance-regular graph. Let θ be an eigenvalue of X, and define U_θ as above. Then the matrix $E_\theta = U_\theta U_\theta^T$ is the matrix representing orthogonal projection onto the θ-eigenspace of X. Thus E_θ is a matrix idempotent for the scheme corresponding to X, and can be written as a linear combination of the distance matrices $\{A_0, \dots, A_d\}$ of X. Therefore the inner product $\langle x(\theta), y(\theta)\rangle$ of any two rows of U_θ depends only on the distance between x and y in X. In particular, this implies that $\langle x(\theta), x(\theta)\rangle$ is independent of $x \in V(X)$, and is equal to m/n where m is the multiplicity of θ. So the vertices of P_θ all lie on a sphere in $\mathbb{R}^m$ centred at 0.

We hope to use eigenpolytopes to derive information about the structure of X. We have that each vertex of X gives a point $x(\theta)$ in $\mathbb{R}^m$. The fact that all of these points lie on a sphere implies that each $x(\theta)$ is a vertex of P_θ. We would like the map between vertices of X and vertices of P_θ to be a bijection. We will see that in the cases we are interested in, this is the case.

A distance-regular graph X is *antipodal* if the dth distance graph X_d is a union of cliques. Equally X is antipodal if the vertices at distance d from u are pairwise at distance d. From [13] we have the following characterization of eigenpolytopes P_θ for which $V(P_\theta) = V(X)$.

4.8.1 Lemma (Lemma 3.1 from Chapter 13 of [13])**.** *Let X be a distance-regular graph with diameter d and valency $k > 2$. For θ an eigenvalue of X, the map $x \to x(\theta)$ between $V(X)$ and $V(P_\theta)$ is not a bijection if and only if:*

(a) $\theta = k$;

(b) $\theta = -k$; or,

(c) X is antipodal, and there is an even number of eigenvalues of X greater than θ. □

Our application of eigenpolytopes will be for eigenvalues $\theta \neq k$, and to graphs that are not bipartite. So the only potential problem will be for distance-regular graphs that are antipodal.

Let $h \in \mathbb{R}^n$. For each $a \in \mathbb{R}$, the sets

$$H_a = \{x \in \mathbb{R}^n \,:\, h^T x = a\}$$

are hyperplanes that partition $\mathbb{R}^n$. If P is an eigenpolytope in $\mathbb{R}^m$ and $h \in \mathbb{R}^m$, then there is some $a \in \mathbb{R}$ so that $P \cap H_a \neq \emptyset$. It follows that by finding the maximum and minimum values of a so that $P \cap H_a \neq \emptyset$, we find faces of P.

Since P is the convex hull of the rows of a matrix U, if $h \in \mathbb{R}^m$, then $Uh = z$ and

$$\begin{aligned} F_{min} &= \{x \in P : h^T x = z_{min}\} \\ F_{max} &= \{x \in P : h^T x = z_{max}\} \end{aligned}$$

are a pair of parallel faces of P.

Moreover, since $z = Uh$ is a linear combination of θ-eigenvectors of X, it follows that z is a θ-eigenvector. Since the columns of U are a basis for the θ-eigenspace, we have that every θ-eigenvector z can be obtained in this way. Therefore if z is a θ-eigenvector for X, then the vertices of X on which z takes its maximum value, and the vertices of X for which z takes its minimum value, are the vertex sets of parallel faces of the θ-eigenpolytope.

Finally, we note that equitable partitions give information about the eigenpolytopes of a graph X. In [13] it is noted that if $\mathcal{P}$ is an equitable partition of X, then there are eigenvectors of X that are constant on the cells of $\mathcal{P}$. If z is an eigenvector that is constant on the cells of $\mathcal{P}$, then the vertices on which z takes its maximum (minimum) value are the union of some subset of cells of $\mathcal{P}$. Therefore there is some eigenvalue θ for which z gives a pair of parallel faces of P_θ whose vertex sets are unions of cells in $\mathcal{P}$.

Since the folded n-cube G_n for odd n is not bipartite, and not antipodal, Lemma 4.8.1 implies that the map between vertices of G_n and vertices of the eigenpolytope P_θ is a bijection for all $\theta \neq n$. In particular, we will be looking at the eigenpolytope P_τ where τ is the least eigenvalue of G_n. In that case we have a bijection between $V(G_n)$ and $V(P_\tau)$.

4.9 The τ-Eigenspace

Recall that we are trying to find a characterization for the maximum cocliques in the folded n-cube G_n for odd n. We approach this problem by looking at the eigenpolytope P_τ where τ is the least eigenvalue of G_n. This approach has precedents.

In [15] and [16], Godsil and Meagher use the τ-eigenspace to characterize the maximum cocliques in two families of graphs. We have seen that the maximum cocliques in the Kneser graph $K_{n:k}$ for $2k \leq n$ are characterized by the Erdős-Ko-Rado Theorem. Using the facets of a polytope related to the τ-eigenspace of $K_{n:k}$, Godsil and Meagher were able to characterize the maximum cocliques of $K_{n:k}$ thereby giving a proof of the Erdős-Ko-Rado Theorem. (We will see this example in more detail in Section 6.3.)

In [15] this method is used again to prove an Erdős-Ko-Rado type theorem for intersecting families of the symmetric group. Let S_n be the symmetric group on n elements. Recall from Example 2.4.6 that two permutations $\alpha, \beta \in S_n$ are said to *intersect* if there is some $1 \leq i \leq n$ so that $\alpha(i) = \beta(i)$. Let X_n be the graph on S_n where $\alpha, \beta \in S_n$ are adjacent if and only if α and β are non-intersecting. Then the maximum cocliques in X_n are exactly the maximum intersecting families of S_n. The graph X_n is a Cayley graph for S_n, and X_n is a graph in the *conjugacy class scheme* for S_n. The polytope given by the resulting τ-eigenvectors of X_n is the well-studied perfect matchings polytope of the complete bipartite graph $K_{n,n}$. Again, the facets of this polytope provide a key piece of the proof that the canonical intersecting families of S_n are exactly the maximum intersecting families of S_n. (We will look at this example in more detail in Section 5.3.)

There are two differences between the folded-cube graphs G_n and the graphs in each of these examples. For the Kneser graph and the derangement graph, we have that the size of a maximum coclique meets the ratio bound and the clique-coclique bound respectively. Both of these bounds give relations between the characteristic vectors of a maximum coclique and τ-eigenvectors. In the case of the folded n-cube, G_n is neither ratio nor clique-coclique tight. We have seen that G_n is inertia tight, but the tightness in the inertia bound does not give a relation between the characteristic vectors of maximum cocliques and τ-eigenvectors. The second difference is that for both the Kneser graphs and the derangement graphs, the τ-eigenpolytopes are well-studied combinatorial polytopes. The structure of the face lattices of these polytopes is well-known. In our application, there is no existing theory of the face lattice of any of the eigenpolytopes of G_n.

For the remainder of this chapter we will look at the structure of P_τ for G_n with n odd. We end this section with an example.

4.9.1 Example. Take $n = 7$ and consider the folded 7-cube G_7. The least eigenvalue of G_7 is -5 with multiplicity 7. Let P_{-5} be the (-5)-eigenpolytope for G_7. We have that P_{-5} is a 7-dimensional polytope in $\mathbb{R}^7$. The number of faces of P_{-5} is given by the array:

$$\begin{pmatrix} -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 64 & 672 & 2240 & 2800 & 1624 & 532 & 78 & 1 \end{pmatrix}$$

where the ith column contains i and the number of faces of dimension i (by convention we take the dimension of $\emptyset$ to be -1).

In particular, P_{-5} has 64 vertices, and 78 facets. The facets of P_{-5} are of two types. There are 64 facets each containing 7 vertices. These facets are the neighbourhoods of each vertex in G_7. The remaining 14 facets each contain 32 vertices.

The 14 large facets can be grouped into 7 pairs of facets, each pair partitioning the vertex set of P_{-5}. Each pair of facets is a parallel pair, so P_{-5} is a *prismatoid* (we explore prismatoids in Section 5.4). The subgraph of G_7 induced by the vertex set of a facet with 32 vertices is a matching with 16 edges (so the induced subgraphs of a parallel pair give a perfect matching of G_7).

Let the graph of P_{-5} be denoted $G(P_{-5})$. The adjacency matrix of $G(P_{-5})$ is A_2 (the 2-distance matrix in the scheme corresponding to G_7). Also, the maximum cocliques of G_7 correspond exactly to the sets $u \cup \Gamma_1(u)$ for u a vertex of $G(P_{-5})$. In $G(P_{-5})$ the vertices in $\Gamma_1(u)$ induce a copy of the line graph of K_7.

Using Sage [30] we were able to take a close look at P_τ for the folded 7-cube. However, as n increases, the number of vertices of G_n increases exponentially, and the number of faces of P_τ increases dramatically. So we are not able to explore beyond $n = 7$. However, we can generalize some of the findings of the above example to all odd n.

4.10 Facets

Let $n = 2r + 1$, and consider the folded n-cube G_n. Let $\Gamma_0, \Gamma_1, \dots, \Gamma_r$ be the distance partition of G_n with respect to $\emptyset$. We have seen that each Γ_i is a coclique of size $\binom{n}{i}$ for $0 \leq i \leq r-1$, and Γ_r is isomorphic to $K_{n:r}$. For each $j \in [n]$ we define the *canonical j-matching of G_n* to be the following perfect matching. If $x, y \notin \Gamma_r$, then x is matched to y if and only if $x = y \cup \{j\}$ or $x = y \setminus \{j\}$. For $x \in \Gamma_r$, if $j \in x$, then x is matched to $x \setminus \{j\}$. This leaves the elements of Γ_r that do not contain j. For each such x, there is a unique r-set y such that $x \cap y = \emptyset$ these edges complete our perfect matching. (Note that this is a slightly complicated definition of the matching given by the edges corresponding to the element e_j of the connection set of G_n viewed as a cubelike graph.)

Note that we can partition the canonical j-matching into two induced matchings of equal size, $M = M_0 \cup M_1$. For r even, we take M_0 to be the matching edges joining Γ_i to Γ_{i+1} for i even, together with the induced matching in Γ_r, and we take M_1 to be the matching edges joining Γ_i to Γ_{i+1} for i odd. For r odd, we take M_0 to be the matching edges joining Γ_i to Γ_{i+1} for i even, and we take M_1 to be the matching edges joining Γ_i to Γ_{i+1} for i odd together with the induced matching in Γ_r. To see that $|M_0| = |M_1|$ note that the number of matching edges joining Γ_i to Γ_{i+1} is the same as the number of i subsets that do not contain j, $\binom{n-1}{i}$. So (when r is even),

$$|M_0| = \sum_{i=0}^{r/2-1} \binom{n-1}{2i} + 1/2\binom{n-1}{r}$$

and

$$|M_1| = \sum_{i=0}^{r/2-1} \binom{n-1}{2i+1}.$$

We see that $|M_0|$ is half of the number of even subsets of $[n-1]$ and $|M_1|$ is half of the number of odd subsets of $[n-1]$, thus $|M_0| = |M_1|$. The odd case follows similarly. The existence of M_0 and M_1 characterizes the canonical matchings.

4.10.1 Lemma. *If M is a perfect matching in G_n, and M can be partitioned into $M = M_0 \cup M_1$ with $|M_0| = |M_1|$ and M_0, M_1 both induced, then M is canonical.*

Proof. We assume that r is even, and note that the proof for r odd is analogous. Without loss of generality the matching edge covering $\emptyset$ lies in M_0. Since M_0 is induced, this implies that M_0 covers exactly one vertex in Γ_1, we refer to this vertex as $\{j\}$.

4.10.2 Proposition. *For $0 \leq i \leq r-2$, the matching edges of M matching elements of Γ_i to elements of Γ_{i+1} are exactly those that matching the elements u of Γ_i that do not contain j to the elements $u \cup \{j\}$ in Γ_{i+1}. Moreover these edges all lie in M_α where $\alpha \equiv i$ (mod 2).*

Proof. We proceed by induction. We have already established the base case. Suppose that the claim holds for all $0 \leq i' < i$. Without loss of generality, $i \equiv 0 \pmod 2$ (the identical argument applies in the odd case).

Let $A \subseteq \Gamma_i$ be the set of i-subsets that contain j, and $B = \Gamma_i \setminus A$. Since the claim holds for $i-1$, the vertices in A are covered by matching edges in M_1 matching A to a subset of Γ_{i-1}. Since M is a perfect matching, B is matched to a subset B' of Γ_{i+1}.

Suppose that some edge $uv \in M$ with $u \in B$ and $v \in B'$ lies in M_1. Since B is the set of i-subsets that do not contain j, the neighbours of B in Γ_{i-1} do not contain j, and are exactly the $(i-1)$-subsets matched to A by M_1. Thus the neighbours of u in Γ_{i-1} are all covered by edges in M_1, contradicting our assumption that M_1 is induced. Thus $uv \in M_0$.

Now suppose we have some $u \in B$ and $v \in B'$ so that $uv \in M_0$ and $v \neq u \cup \{j\}$. Since $j \notin u$ and $j \notin v$, there are no neighbours of v in A. But the neighbours of A in Γ_{i+1} is the set of $(i+1)$-subsets of $[n]$ that contain j, which has the same size as B'. Thus there must be some $v' \in \Gamma_{i+1} \setminus B'$ with neighbours in A. Since these vertices are covered by edges in M_1, this again contradicts M_1 induced. Thus each $u \in B$ is matched to $u \cup \{j\} \in B'$ by M_0. □

By Proposition 4.10.2 the matching edge $\emptyset\{j\}$ determines all of the matching edges between Γ_i and Γ_{i+1} for $0 \leq i \leq r-2$ and the partite set to which they belong. It remains to consider the matching edges between Γ_{r-1} and Γ_r, and the matching edges in Γ_r.

Since Γ_r is not a coclique in G_n, we cannot apply directly the previous argument. However, we can use the first part directly. Namely, we know that the subset $A \subseteq \Gamma_{r-1}$ consisting of the $(r-1)$-subsets containing j is covered by edges of M_0 between Γ_{r-2} and Γ_{r-1}. We also know that the edges matching $B = \Gamma_{r-1} \setminus A$ to $B' \subseteq \Gamma_r$ are all edges in M_1. It remains to show that $u \in B$ is matched to $u \cup \{j\} \in B'$.

Note that the size of B' is $\binom{n-1}{r-1}$. Since M_0 is an induced matching, we have that B' is a coclique in $K_{n:r}$ of maximum size. By the Erdős-Ko-Rado Theorem, we have that B' is a canonical coclique, and therefore there is some $1 \leq j' \leq n$ so that B' is the set of r-subsets that contain j'.

Suppose that $j' \neq j$. Now B is the set of $(r-1)$-subsets that do not contain j, and B' is the set of r-subsets that contain j'. Partition B into $B = B_0 \cup B_1$ where

$$\begin{aligned} B_0 &= \{u \in B : j' \notin u\}, \\ B_1 &= \{u \in B : j' \in u\}. \end{aligned}$$

Likewise, partition B' into $B' = B'_0 \cup B'_1$ where

$$\begin{aligned} B'_0 &= \{v \in B' : j \notin v\}, \\ B'_1 &= \{v \in B' : j \in v\}. \end{aligned}$$

Now consider B'_1. Since $j \in v$ for all $v \in B'_1$, the neighbours of v in B consist only of $v \setminus \{j\}$. Since $j' \in v$, we see that $v \setminus \{j\} \in B_1$. Thus we must match $v \in B'_1$ to $v \setminus \{j\} \in B_1$. Since

$$|B_1| = |B'_1| = \binom{n-2}{r-2},$$

B_1 is matched to B'_1 in this way. Likewise, if $u \in B_0$, then $j, j' \notin u$, and $u \cup \{j'\}$ is the unique neighbour of u in B'_0.

Thus the M_1 edges matching B to B' match B_0 to B'_0 and B_1 to B'_1. However, in G_n there are edges between B_0 and B'_1 and edges between B_1 and B'_0. So M_1 cannot be induced, a contradiction. Therefore $j' = j$, and M_1 matches $u \in B$ to $u \cup \{j\} \in B'$.

Finally, we have that the remaining matching edges in Γ_r must be M_0 edges. In $K_{n:r}$, the complement of any maximum coclique is a perfect matching. Thus these are exactly the M_0 edges, and M is canonical. □

For each $1 \leq i \leq n$, the i-matching gives an equitable partition of the vertices of G_n into two parts of size $|V|/2$. From Section 4.8 we recall that this partition yields parallel faces of an eigenpolytope of G_n. Next we show that they are facets of the τ-eigenpolytope.

4.10.3 Lemma. *For $1 \leq i \leq n$ the i-matching gives a parallel pair of facets of the τ-eigenpolytope of G_n.*

Proof. Let M be the i-matching of G_n, and M_0 and M_1 be the partition of M into equal-sized induced matchings. Let V_0 be the subset of V covered by M_0 and V_1 be the subset of V covered by M_1. The partition (V_0, V_1) is equitable, each $u \in V_0$ has 1 neighbour in V_0 and $n-1$ neighbours in V_1, and each $u \in V_1$ has 1 neighbour in V_1 and $n-1$ neighbours in V_0.

Let z be the vector in $\mathbb{R}^{V(G_n)}$ defined by

$$z_u = \begin{cases} 1, & \text{if } u \in V_0 \\ -1, & \text{if } u \in V_1 \end{cases}$$

for all $u \in V(G_n)$. Letting A be the adjacency matrix of G_n, we see that

$$Az = -(n-2)z.$$

So z is a τ-eigenvector for G_n.

Let U be a matrix whose columns are an orthonormal basis for the τ-eigenspace of G_n. Since z is a τ-eigenvector, there is a vector h so that $Uh = z$. The τ-eigenpolytope of G_n is

$$P_\tau = \text{conv}\{x(\tau) : x \in V(G_n)\}.$$

So

$$\begin{aligned} F_1 &= \operatorname{conv}\{v \in V(P_\tau) \,:\, h^T v = 1\}, \\ F_{-1} &= \operatorname{conv}\{v \in V(P_\tau) \,:\, h^T v = -1\} \end{aligned}$$

are a pair of parallel faces of P_τ that partition the vertices.

Let $z^{(i)}$ be the τ-eigenvector corresponding to the i-matching. We show that

$$B = \{z^{(i)} \,:\, 1 \leq i \leq n\}$$

is an orthogonal basis for the τ-eigenspace.

4.10.4 Proposition. *For $i \neq j$, $z^{(i)}$ is orthogonal to $z^{(j)}$.*

Proof. Recall the entries of $z^{(i)}$,

$$z_u^{(i)} = \begin{cases} 1, & \text{if either } |u| \text{ is even and } i \notin u, \text{ or } |u| \text{ is odd and } i \in u \\ -1, & \text{if either } |u| \text{ is even and } i \in u, \text{ or } |u| \text{ is odd and } i \notin u. \end{cases}$$

For $i \neq j$,

$$\langle z^{(i)}, z^{(j)} \rangle = \sum_{v \in V} z_v^{(i)} z_v^{(j)},$$

where

$$z_v^{(i)} z_v^{(j)} = \begin{cases} -1, & |v| \text{ even and } i \notin v,\ j \in v, \text{ or } |v| \text{ odd and } i \in v,\ j \notin v \\ 1, & |v| \text{ even and } i, j \notin v, \text{ or } |v| \text{ odd and } i, j \in v \\ 1, & |v| \text{ even and } i, j \in v, \text{ or } |v| \text{ odd and } i, j \notin v \\ -1, & |v| \text{ even and } i \in v,\ j \notin v, \text{ or } |v| \text{ odd and } i \notin v,\ j \in v. \end{cases}$$

The map on subsets of $[n]$ that maps v to $v \setminus \{i\}$ if $i \in v$ and v to $v \cup \{i\}$ if $i \notin v$ maps

$$\{v \,:\, i, j \in v \text{ or } i, j \notin v\} \to \{v \,:\, i \notin v, j \in v \text{ or } i \in v, j \notin v\}$$

bijectively. However, the vertices of G_n are the subsets of $[n]$ of size at most r. If we modify the map so that when $|v| = r$,

$$v \to \begin{cases} \overline{v \cup \{i\}}, \text{when } i \notin v \\ v \setminus \{i\}, \text{when } i \in v, \end{cases}$$

then the map is a bijection on the vertices of G_n. (Note that this bijection is exactly the i-matching M.)

This map pairs sets with different z-product values. Thus $\langle z^{(i)}, z^{(j)} \rangle = 0$. □

Since B is an orthogonal set of vectors, it is linearly independent. Since the multiplicity of τ as an eigenvalue of G_n is n, the vectors $z^{(i)}$ form an orthogonal basis for the τ-eigenspace.

It remains to show that the parallel faces corresponding to V_0 and V_1 in the i-matching are facets of P_τ. Since the $z^{(i)}$ are an orthogonal basis for the τ-eigenspace, we can take the vectors $z^{(i)}$ to be the columns of the matrix U used to define P_τ.

Fix $1 \leq i \leq n$ and let M be the i-matching of G_n. Let (V_0, V_1) be the partition of V induced by M. For each vector $z^{(j)} \in B$, we define $y^{(j)}$ to be the restriction of $z^{(j)}$ to the vertices in V_0. So $y^{(i)}$ has all entries equal to 1. The rows of the matrix U' formed by eliminating the V_1 rows from U are the vertices in one of the faces corresponding to the i-matching.

4.10.5 Proposition. *For $j \neq k$, $y^{(j)}$ is orthogonal to $y^{(k)}$.*

Proof. We proceed similarly to the computation of $\langle z^{(i)}, z^{(j)} \rangle$.

To begin, we assume that $i \notin \{j, k\}$.

For $j \neq k$,

$$\langle y^{(j)}, y^{(k)} \rangle = \sum_{v \in V_0} y_v^{(j)} y_v^{(k)},$$

where

$$y_v^{(j)} y_v^{(k)} = \begin{cases} -1, & |v| \text{ even and } j \notin v,\, k \in v, \text{ or } |v| \text{ odd and } j \in v,\, k \notin v \\ 1, & |v| \text{ even and } j, k \notin v, \text{ or } |v| \text{ odd and } j, k \in v \\ 1, & |v| \text{ even and } j, k \in v, \text{ or } |v| \text{ odd and } j, k \notin v \\ -1, & |v| \text{ even and } j \in v,\, k \notin v, \text{ or } |v| \text{ odd and } j \notin v,\, k \in v. \end{cases}$$

Recall, that V_0 is the set of subsets of $[n]$ of size at most r that either are even and do not contain i, or are odd and do contain i. So we can remove i from the odd subsets in V_0, and treat V_0 as two copies of the even subsets of $[n-1]$ of size at most r. Thus from the preceding case analysis, $y_v^{(j)} y_v^{(k)} = 1$ for exactly twice the number of even subsets of $[n-1]$ that have size at most r and either contain both j and k, or neither. Likewise, $y_v^{(j)} y_v^{(k)} = -1$ for exactly twice the number of even subsets of $[n-1]$ that have size at most r and either contain exactly one of j and k. These numbers are equal, so we conclude that $\langle y^{(j)}, y^{(k)} \rangle = 0$.

Now suppose that $j = i$. In this case, $y_v^{(i)} = 1$ for all $v \in V_0$. So

$$\langle y^{(i)}, y^{(k)} \rangle = \sum_{v \in V_0} y_v^{(k)}.$$

Now,

$$y_v^{(k)} = \begin{cases} 1, & |v| \text{ even, } k \notin v, \text{ or } |v| \text{ odd, } k \in v \\ -1, & |v| \text{ even, } k \in v, \text{ or } |v| \text{ odd, } k \notin v. \end{cases}$$

Again, from the argument above, $y_v^{(k)}$ takes values 1 and -1 equally often, so $\langle y^{(i)}, y^{(k)} \rangle = 0$. $\square$

Since $\{y^{(j)} : 1 \le j \le n\}$ is an orthogonal set of vectors, it is a linearly independent set of vectors. Therefore the matrix U' has rank n, and the rows of U' span a space of dimension n. So the corresponding face has affine dimension $n-1$. Since $P_\tau \subseteq \mathbb{R}^n$, this implies that these faces are facets of P_τ. □

We have shown that there are n pairs of parallel facets of P_τ that partition the vertices of P_τ into equal parts. These partitions correspond to the n canonical perfect matchings of G_n. This generalizes one of the families of facets of P_{-5} of G_7 from Example 4.9.1 that we found via computer. It also shows that the polytope P_τ is a prismatoid for every odd-order folded cube.

4.11 Dual Eigenvalues

In this section we see that the other family of facets of P_{-5} of G_7 that we saw in Example 4.9.1 also generalizes to a family of facets of P_τ of G_n for all odd n. To prove this generalization we will derive some of the dual eigenvalues of the scheme corresponding to G_n.

Let U be a matrix whose columns form an orthonormal basis for the τ-eigenspace, and set $E_\tau = UU^T$. As we have seen, we can construct an orthonormal basis for the τ-eigenspace from the i-matchings of G_n. Let A_i be the i-distance matrix of G_n.

4.11.1 Lemma. *If U' is a matrix whose columns are the vectors $z^{(i)}$, then $U'U'^T = \sum_{i=0}^{r}(-1)^i(n-2i)A_i$.*

Proof. Note that if U'_v is the v-row of U', then

$$U'_v[i] = \begin{cases} 1, & \text{if } |v| \text{ is even and } i \notin v, \text{ or if } |v| \text{ is odd and } i \in v \\ -1, & \text{if } |v| \text{ is even and } i \in v, \text{ or if } |v| \text{ is odd and } i \notin v. \end{cases}$$

Suppose that v is a subset of $[n]$. Then v is at distance $|v|$ from $\emptyset$. Since $U'_\emptyset[i] = 1$ for all i,

$$\langle U'_\emptyset, U'_v \rangle = \sum_{i=1}^{n} U'_v[i] = \begin{cases} n - 2|v|, & \text{if } |v| \text{ is even} \\ 2|v| - n, & \text{if } |v| \text{ is odd.} \end{cases}$$

So,

$$\langle U'_\emptyset, U'_v \rangle = (-1)^{d(\emptyset,v)}(n - 2d(\emptyset, v)).$$

Since G_n is vertex transitive, we have that

$$\langle U'_u, U'_v \rangle = (-1)^{d(u,v)}(n - 2d(u, v))$$

for all $u, v \in V$. This completes the proof. □

Note that the values $(-1)^i(n-2i)$ for $0 \leq i \leq r$ are the eigenvalues of G_n (which we have seen represented as $n-4i$ for $0 \leq i \leq r$).

Recall that for the association scheme corresponding to G_n, the *dual eigenvalues* are the values $q_j(k)$, where $E_j = 1/2^{n-1}\sum_{k=0}^{r} q_j(k)A_k$. By convention we order the matrix idempotents of the scheme $E_0, \ldots, E_r$ so that E_i corresponds to the ith eigenvalue of G_n (where the eigenvalues of G_n are ordered from largest to smallest). So $E_r = E_\tau$ where τ is the least eigenvalue of G_n.

4.11.2 Corollary. *The dual eigenvalues $q_r(k)$ are the eigenvalues of G_n.*

Proof. The matrix $E_\tau = E_r$ is obtained by taking $E_r = UU^T$ where the columns of U are the normalized columns of U'. Since each $z^{(i)}$ has squared length 2^{n-1},

$$U = (1/\sqrt{2^{n-1}})U'$$

and

$$E_r = (1/2^{n-1})U'U'^T = (1/2^{n-1})\sum_{k=0}^{r}(-1)^k(n-2k)A_k.$$

So $q_r(k) = (-1)^k(n-2k)$. □

We can use Corollary 4.11.2 to find another set of facets of P_τ.

Let θ be an eigenvalue of G_n and let $x \in V(G_n)$. Let U be a matrix whose columns are a orthonormal basis for the θ-eigenspace of G_n. Define the vector $z_x \in \mathbb{R}^{V(G_n)}$ to be the vector with entries $z_x[y] = \langle x(\theta), y(\theta)\rangle$ for all $y \in V(G_n)$. Note that $z_x = Ux(\theta)^T$, and as such z_x is a θ-eigenvector of G_n. Therefore we have faces

$$\begin{aligned} F_{max} &= \operatorname{conv}\{y(\theta) : z_x[y] \text{ is maximum}\}, \\ F_{min} &= \operatorname{conv}\{y(\theta) : z_x[y] \text{ is minimum}\} \end{aligned}$$

of the eigenpolytope P_θ. Using this eigenvector we can show that the neighbourhood of any vertex of G_n gives a facet of P_τ.

4.11.3 Corollary. *For any $x \in V(G_n)$, let $\Gamma_1(x)$ be the neighbours of x in G_n. Then $\{y(\tau) : y \in \Gamma_1(x)\}$ is the vertex set of a facet of P_τ.*

Proof. Let $x \in V(G_n)$ and define the τ-eigenvector z_x as above. Note that $E_r = UU^T$, and so since $z_x = Ux(\tau)^T$, we have that z_x is the x column of E_r. From Corollary 4.11.2 we have

$$E_r = (1/2^{n-1})\sum_{k=0}^{r}(-1)^k(n-2k)A_k.$$

Therefore, the y-entry of z_x depends only on the distance between x and y in G_n, and

$$z_x[y] = (-1)^{d(x,y)}(n-2d(x,y)).$$

This implies that

$$\begin{aligned} F_{max} &= \operatorname{conv}\{x(\tau)\}, \\ F_{min} &= \operatorname{conv}\{y(\tau) : d(x,y) = 1\}. \end{aligned}$$

Therefore we see that the vertices $y(\tau)$ for $\Gamma_1(x)$ are the vertices of a face of P_τ.

We show that this face is a facet of P_τ when $x = \emptyset$. Since G_n is vertex transitive, it follows that the neighbourhood of each vertex gives a distinct facet of P_τ. Let $z^{(i)}$ be the basis vectors for the τ-eigenspace of G_n constructed as in Lemma 4.10.3, and let U' be the matrix formed by taking the $z^{(i)}$ as columns. Recall that

$$z^{(i)}[x] = \begin{cases} 1, \text{ if } |x| \text{ is even and } i \notin x, \text{ or } |x| \text{ is odd and } i \in x \\ -1, \text{ if } |x| \text{ is even and } i \in x, \text{ or } |x| \text{ is odd and } i \notin x. \end{cases}$$

For each $x \in \Gamma_1(\emptyset)$, the size of x is 1, and exactly one of the neighbours contains i for any given $1 \leq i \leq n$. Thus the submatrix of U' indexed by the rows corresponding to the elements of $\Gamma_1(\emptyset)$ is similar to $-J + 2I$. Therefore it has rank n, and the space spanned by its rows has affine dimension $n-1$. Thus the face of P_τ with vertices $x(\tau)$ for $x \in \Gamma_1(\emptyset)$ is a facet of P_τ. $\square$

4.12 Open Problems

In Example 4.9.1 we saw that the facets of P_{-5} for G_7 are exactly the facets given by Lemma 4.10.3 and Corollary 4.11.3. We were able to show that these facets are also present in the τ-eigenpolytope of the folded n-cube for all odd n. However, we were not able to show that these are all of the facets of these polytopes. Nor were we able to show that there are not other faces of the τ-eigenpolytope that contain a large proportion of the vertices of P_τ (as is the case for the τ-eigenpolytopes for the Kneser and derangement graphs). There may be other equitable partitions of G_n into two parts that give parallel faces of the τ-eigenpolytope, or eigenpolytopes for other eigenvalues. All of these questions are as yet unanswered.

We started this chapter with the goal of characterizing the maximum cocliques of the folded cubes. That question is still open. We used a computer to confirm that the canonical cocliques of G_n are exactly the maximum cocliques of G_n for $n \leq 9$. We found that the distance partition of G_n gives us the canonical cocliques and the canonical matchings, and were able to relate the canonical matchings to facets of the τ-eigenpolytope. But it is still unclear how to apply information about the τ-eigenpolytope to obtain information about the maximum cocliques of G_n.

Chapter 5

Veronesian Rank

In Chapter 4 we saw a connection between cocliques in graphs and parallel faces of their eigenpolytopes. In this chapter we will look at polytopes that have the property that their vertex set can be partitioned into two parallel faces.

We will begin by taking a closer look at one of the polytopes from Section 4.9. In that section we mentioned that the Erdős-Ko-Rado Theorem for elements of the symmetric group can be proved by analysing the faces of a polytope. We give an overview of this proof, and see exactly how the faces of polytopes are used to draw conclusions about the cocliques of a class of graphs.

Polytope proofs of Erdős-Ko-Rado type theorems rely on the eigenpolytope of a graph having the property that its vertex set can be partitioned into two parallel faces. Polytopes with this property are called prismoids. For distance regular graphs, prismoid decompositions of eigenpolytopes correspond to equitable partitions of the graph into two parts.

There is also an algebraic aspect to prismoids. The property of being a prismoid corresponds to the existence of quadratic polynomials that vanish on the points of the polytope. The space of solutions to these quadratics is a subspace of the null space of a matrix associated with the polytope. This translates the geometric problem of determining whether a polytope is a prismoid into the algebraic problem of finding vectors in the null space of a matrix.

The Veronese matrix gives us a new tool for characterizing the equitable partitions of a graph. We show that the equitable partitions of a graph into two parts form a subset of the null space of the Veronese matrix. For distance regular graphs, we can use the parameters of the association scheme of the graph to find the rank of this space explicitly. We show that this formula applied to strongly regular graphs gives the null space of the Veronese matrix exactly, and rules out the existence of equitable partitions of specific strongly regular graphs into two parts.

5.1 Intersecting Permutations

In Section 4.9 we mentioned that polytopes could be used to prove a version of the Erdős-Ko-Rado Theorem for the symmetric group. We begin by taking a more detailed look at that argument. Recall that S_n is the symmetric group on n elements, and permutations $\alpha, \beta \in S_n$ are said to *intersect* if there is some $1 \leq i \leq n$ so that $\alpha(i) = \beta(i)$. We note that the sets

$$I_{p,q} = \{\alpha \in S_n \,:\, \alpha(p) = q\}$$

give natural intersecting families in S_n (note that the $I_{p,q}$ are the cosets of point stabilizers in S_n). The Erdős-Ko-Rado Theorem for intersecting families of the symmetric group characterizes the intersecting families of maximum size.

5.1.1 Theorem. *If I is an intersecting family of S_n, then $|I| \leq (n-1)!$. Moreover, if $|I| = (n-1)!$, then there are $1 \leq p, q \leq n$ so that $I = I_{p,q}$.* □

Theorem 5.1.1 was first proved by Cameron and Ku in [6]. Several proofs of this theorem have been given. Wang and Zhang in [34] give a very short elementary proof. Larose and Malvenuto prove a graph-theoretic formulation of Theorem 5.1.1 by characterizing the maximum cocliques in a family of graphs. In [15] Godsil and Meagher give a characterization of the cocliques in the derangement graph using an algebraic argument. Their argument can be re-formulated using the τ-eigenpolytope of the derangement graph. We give a brief overview of this argument.

Let X_n be the graph on S_n where $\alpha, \beta \in S_n$ are adjacent if and only if α and β are non-intersecting. Recall that the cocliques of X_n are exactly the intersecting families of S_n. So the statement of Theorem 5.1.1 is equivalent to: $\alpha(X_n) = (n-1)!$ and the maximum cocliques of X_n are exactly the sets $I_{p,q}$ for $1 \leq p, q \leq n$.

The graph X_n is known as the *derangement graph* of S_n (see also Example 2.4.6). A *derangement* is an element α of S_n so that $\alpha(i) \neq i$ for any $1 \leq i \leq n$. Let $D \subseteq S_n$ be the set of all derangements of S_n. Then D does not contain the identity permutation, and D is inverse closed. Note that for all $\alpha, \beta \in S_n$, the automorphism $f : S_n \to S_n$ defined as $f(\sigma) = \sigma(\alpha^{-1}\beta)$ maps α to β. Moreover, if $\alpha^{-1}\beta$ is a derangement, then α and β are non-intersecting. Now we see that X_n is a Cayley graph for S_n with $X_n = X(S_n, D)$. Therefore, if $d(n)$ is the number of derangements in S_n, then X_n is a $d(n)$-regular, vertex-transitive graph.

Since S_n is not an Abelian group, finding the eigenvalues of X_n is a little more difficult than for the other Cayley graphs we have seen. However, Ku and Wales [23] have shown that the least eigenvalue of X_n is

$$\tau = -d(n)/(n-1).$$

Therefore we can find $\alpha(X_n)$ using the ratio bound. Recall from Theorem 3.13.1 that for any k-regular graph Y on n vertices with least eigenvalue τ,

$$\alpha(Y) \leq \frac{n}{1 - k/\tau}.$$

In our case we have X_n is a $d(n)$-regular graph on $n!$ vertices with least eigenvalue $-d(n)/(n-1)$, and thus

$$\alpha(X_n) \leq \frac{n!}{1 - d(n)/(-d(n)/(n-1))} = \frac{n!}{n} = (n-1)!.$$

We have that for all $1 \leq p, q \leq n$, the sets $I_{p,q}$ are cocliques in X_n with size $(n-1)!$. Therefore $\alpha(X_n) = (n-1)!$ and X_n is ratio-tight.

5.2 Eigenvectors

Theorem 3.13.1 can be extended to give τ-eigenvectors for ratio-tight graphs. We give a proof of this extension from Meagher and Spiga [28].

5.2.1 Theorem. *Let Y be a k-regular graph on n vertices with least eigenvalue τ. Let S be a maximum coclique in Y, and let v_S be the characteristic vector of S. Then*

$$\alpha(Y) \leq \frac{n}{1 - k/\tau}$$

and if equality holds, then $v_S - |S|/|Y|\mathbf{1}$ is a τ-eigenvector for Y.

Proof. Let A be the adjacency matrix for Y. Consider the matrix

$$M = A - \tau I - (k - \tau)/nJ.$$

Note that $\mathbf{1}$ is an eigenvector for M with eigenvalue 0. Now suppose that z is an θ-eigenvector for A that is orthogonal to $\mathbf{1}$. Then

$$Mz = (\theta - \tau)z,$$

and z is a $(\theta - \tau)$-eigenvector for M. Since A is the adjacency matrix of a k-regular graph, k is the largest eigenvalue of A, and $\tau < 0$. Therefore the spectrum of M is

$$\{0\} \cup \{\theta - \tau \,:\, \theta \neq k \text{ is an eigenvalue of } A\},$$

and M is positive semi-definite.

Since v_S is the characteristic vector of a coclique of Y, we have that

$$v_S^T A v_S = 0.$$

Since M is positive semi-definite,

$$\begin{aligned} 0 \leq v_S^T M v_S &= v_S^T A v_S - \tau v_S^T I v_S - \frac{k - \tau}{n} v_S^T J v_S \\ &= -\tau|S| - \frac{k - \tau}{n}|S|^2. \end{aligned}$$

This implies that

$$|S| \leq n/(1 - k/\tau)$$

and the bound in the theorem follows.

Now if Y meets the bound, then we have a coclique S so that $v_S^T M v_S = 0$. Since M is positive semi-definite this implies that v_S is a 0-eigenvector for M and we have that

$$0 = Av_S - \tau v_S - \frac{k-\tau}{n}\mathbf{1},$$

and therefore $v_S - |S|/n\mathbf{1}$ is a τ-eigenvector for Y. □

5.3 A Polytope

If we let $v_{p,q}$ be the characteristic vector of $I_{p,q}$ then from Theorem 5.2.1 we have that the vectors $v_{p,q} - \mathbf{1}/n$ are τ-eigenvectors for X_n. In fact, these vectors are a basis of the τ-eigenspace (see Godsil and Meagher [15]).

Define the matrix M as the matrix with the vectors $v_{p,q}$ as its columns for $1 \leq p, q \leq n$. The rows of M are indexed by the elements of S_n. So for each $1 \leq p, q \leq n$, the entries of the α-row M_α of M are 1 if $\alpha(p) = q$ and 0 otherwise. Therefore M_α contains exactly n entries with value 1, and so $\mathbf{1}$ is in the column space of M. Therefore if z is a τ-eigenvector for X_n, then $z \in \text{col}(M)$.

Define P_n to be the polytope

$$P_n = \text{conv}\{M_\alpha \,:\, \alpha \in S_n\}.$$

While P_n is not an eigenpolytope according to our definition (i.e., the columns of M do not form an orthonormal basis of an eigenspace of X_n), it is closely related to the τ-eigenspace of X_n. Note that the polytope formed by taking the convex hull of the rows of $M - 1/nJ$ is a translation of P_n and thus is combinatorially equivalent to P_n. This polytope is the convex hull of a matrix whose columns form a basis for the τ-eigenspace of X_n (but again, not an orthonormal basis).

Suppose z is the characteristic vector of a maximum coclique in X_n. Then since $z - 1/n\mathbf{1}$ is a τ-eigenvector for X_n, there is some h so that $Mh = z$. Therefore the rows of M indexed by the maximum-valued entries of z contain the vertex set of a face F_1 of P_n. Likewise the rows of M indexed by the minimum-valued entries of z contain the vertex set of a face F_2 of P_n. Since z is 01-valued, we see that F_1 and F_2 partition the rows of M. We also note that these faces are parallel.

The complete bipartite graph $K_{n,n}$ is the bipartite graph with partite sets $A = B = [n]$ where each $a \in A$ is adjacent to each $b \in B$. Note that if $\alpha \in S_n$, then the edges $\{i, \alpha(i)\}$ for $1 \leq i \leq n$ are a perfect matching in $K_{n,n}$. In $\mathbb{R}^{E(K_{n,n})}$, the characteristic vector of this perfect matching is the row of M indexed by α, M_α. Therefore P_n is the polytope in $\mathbb{R}^{E(K_{n,n})}$ defined as the convex hull of the characteristic vectors of the perfect matchings of $K_{n,n}$. This polytope is called the *perfect matchings polytope* of $K_{n,n}$ and is denoted $PM(K_{n,n})$.

For general graphs Y, the perfect matchings polytope $PM(Y)$ is a well-studied object. In the case when Y is bipartite , Birkhoff's Theorem (Theorem

6.12 in Cook et al. [8]) gives a description of $PM(Y)$ as an intersection of half-spaces. If S is a subset of $E(Y)$ and $x \in \mathbb{R}^{E(Y)}$ we let

$$x(S) = \sum_{s \in S} x_s.$$

For a vertex y of Y, we denote the edges incident with y by $\delta(y)$.

5.3.1 Theorem (Birkhoff's Theorem)**.** *If Y is a bipartite graph, then*

$$PM(Y) = \{x \in \mathbb{R}^{E(Y)} : x_e \geq 0, \forall e \in E(Y); x(\delta(y)) = 1, \forall y \in V(Y)\}. \qquad \square$$

Theorem 5.3.1 implies that the faces of $PM(Y)$ are exactly the sets

$$F_e = \{x \in PM(Y) : x_e = 0\}$$

where $e \in E(Y)$. Therefore the facets of $PM(Y)$ all have this form.

Since $K_{n,n}$ is bipartite, Theorem 5.3.1 applies to $PM(K_{n,n}) = P_n$. Now we recall that F_1 and F_2 are faces that partition the vertices of P_n. Since every face lies in a facet, we have that F_1 and F_2 both lies in facets of P_n. In particular, there is some F_e so that $F_2 \subseteq F_e$. This implies that $x_e = 1$ for all $x \in F_1$. Since the edges of $K_{n,n}$ correspond to pairs (p, q) we conclude that z indexes all of the rows M_α for which $\alpha(p) = q$ for some $1 \leq p, q \leq n$. This proves that the maximum cocliques of X_n are exactly the sets $I_{p,q}$.

This proof gives an example of how we can apply polytopes to characterize cocliques of graphs. In particular, we have an example of a class of polytopes, $PM(K_{n,n})$ whose vertices can be partitioned into parallel faces (one of which is a facet). We take a closer look at polytopes with this property in the next section.

5.4 Prismoids and Prismatoids

Following Grünbaum [20], a *prismoid* is a d-dimensional polytope P with the following property. There are two $(d-1)$-dimensional parallel hyperplanes containing polytopes P_1 and P_2 so that P is the convex hull of the vertices of P_1 and P_2, or

$$P = \mathrm{conv}(P_1 \cup P_2).$$

As we saw in the previous section, the perfect matchings polytope of the graph $K_{n,n}$ is an example of a prismoid. Note that there is no restriction on the polytopes P_1 and P_2 other than that they lie in parallel hyperplanes.

5.4.1 Example. Let P be a polytope in $\mathbb{R}^n$. Define $P' \subseteq \mathbb{R}^{n+1}$ by

$$P' = \{(x, 0) : x \in P\}.$$

Then P' is a polytope in $\mathbb{R}^{n+1}$, and P' is combinatorially equivalent to P. Let $y \in P$ be an arbitrary point in P. Now

$$P'' = \mathrm{conv}(P' \cup \{(y, 1)\})$$

is a prismoid in $\mathbb{R}^{n+1}$. Prismoids of this form are called *pyramids*.

Pyramids give an extreme example of a prismoid with parallel faces of different dimension. In this example the parallel faces of P'' are P' and $\{(y, 0)\}$. So P'' is partitioned into a single vertex, and a facet.

5.4.2 Example. The n-dimensional hypercube gives another example of a prismoid. Let H_n be the polytope in $\mathbb{R}^n$ defined as the convex hull of the 01-vectors in $\mathbb{R}^n$. Then for any $1 \leq i \leq n$ we have that

$$\begin{aligned} F_0 &= \{x \in H_n : x_i = 0\} \\ F_1 &= \{x \in H_n : x_i = 1\} \end{aligned}$$

are parallel faces of H_n that partition its vertices. Moreover, each of F_0 and F_1 is isomorphic to the hypercube H_{n-1}. Therefore both F_0 and F_1 are facets of H_n.

If P is a prismoid whose vertices are partitioned into parallel faces (P_1, P_2) where P_1 and P_2 are facets of P, then we call P a *prismatoid*. There is not a large amount of literature on prismatoids. However, recently Santos [31] used prismatoids to construct an infinite family of counterexamples to the Hirsch Conjecture. Santos and his coauthors in [27] and [32] further explore prismatoids with a fixed combinatorial width.

The τ-eigenpolytope of the folded-cube graph G_n for n odd gives another example of a prismatoid. Recall from Chapter 4 that the folded cube G_n is a distance-regular graph. The τ-eigenpolytope of G_n is the polytope P_τ defined by the convex hull of the rows of a matrix U_τ whose columns form an orthonormal basis for the τ-eigenspace of G_n. In Lemma 4.10.3 we saw that for each $1 \leq i \leq n$, the canonical i-matching M_i of G_n gave a natural partition of the vertices of G_n into two sets of equal size. The corresponding partition of the rows of U_τ gives two parallel facets in P_τ. Therefore P_τ is a prismatoid.

We also found another set of facets of P_τ. From Corollary 4.11.3 we have that if x is a vertex of G_n, and $\Gamma(x)$ is the set of neighbours of x, then the rows of U_τ corresponding to the vertices $\Gamma(x)$ give the vertex set of a facet of P_τ. In the proof of Corollary 4.11.3 we saw that the supporting hyperplane that defines this facet is parallel to a supporting hyperplane of P_τ that contains only the vertex of P_τ corresponding to x. However, P_τ *is not* a pyramid, as the facet defined by $\Gamma(x)$ together with the vertex corresponding to x do not contain all of the vertices of P_τ.

To see this, recall that the matrix $E_\tau = U_\tau U_\tau^T$ is a matrix idempotent in the association scheme defined by the distance matrices of G_n. Therefore E_τ is a linear combination of distance matrices A_i, and thus the (x, y)-entry of E_τ depends only on the distance between x and y in G_n. Moreover in Corollary 4.11.2 we derived the dual eigenvalues corresponding to E_τ, which give the coefficients of this linear combination. Since the coefficients for the A_i are all distinct, the (x, y)-entry and (x, z)-entry of E_τ are distinct if $d(x, y) \neq d(x, z)$ in G_n. Finally, let the vertices of P_τ corresponding to $\Gamma(x)$ be $\{y_1, \ldots, y_n\}$, and let the face defined by their convex hull be F. Since these vertices are affinely independent, if $z \in F$, then there are $\alpha_i \geq 0$ so that

$$\sum_{i=1}^{n} \alpha_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i y_i = z.$$

Therefore if z is a vertex of P_τ that is not one of the y_i, and $z \in F$, then z can be expressed as an affine combination of the y_i for some coefficients α_i. Now, abusing notation by setting the dual eigenvalues of E_τ to be $q_\tau(i)$,

$$\begin{aligned}
\langle x, z \rangle &= \langle x, \sum_{i=1}^{n} \alpha_i y_i \rangle \\
&= \sum_{i=1}^{n} \alpha_i \langle x, y_i \rangle \\
&= \sum_{i=1}^{n} \alpha_i q_\tau(1) \\
&= q_\tau(1).
\end{aligned}$$

So we see that there are no vertices of P_τ in F other than the y_i.

In general, we are interested in the face lattice of a polytope, as we are trying to relate the combinatorial structure of the polytope to the combinatorial structure of the graphs they are constructed from. In terms of the face lattice, prismoids and prismatoids are not especially interesting classes of polytopes. If P is a polytope, and F and F' are faces of the polytope that partition the vertices, then there is a polytope P' that is combinatorially equivalent to P for which the faces corresponding to F and F' are parallel (see [20, p. 38]). So for general polytopes, we are interested only in the vertices that belong to individual faces. However, we are not working with general polytopes. We are looking at eigenpolytopes of graphs (more specifically, distance-regular graphs). For these polytopes, the existence of parallel faces that partition the vertices has an important combinatorial interpretation.

5.5 Equitable Partitions

We have defined a prismoid as a polytope whose vertices can be partitioned into parallel faces. Let $P \subseteq \mathbb{R}^n$ be a polytope, and

$$H = \{x \in \mathbb{R}^n \,:\, x^T h \leq \alpha\}$$

be a supporting hyperplane for P. Since P is a polytope, P is a bounded subset of $\mathbb{R}^n$. Therefore there is some $\beta \in \mathbb{R}$ so that $x^T h \geq \beta$ for all $x \in P$. Letting β be the least such number, we have that

$$H' = \{x \in \mathbb{R}^n \,:\, x^T h \geq \beta\}$$

is a supporting hyperplane of P. Note that H and H' are parallel hyperplanes, and $F = H \cap P$ and $F' = H' \cap P$ define faces of P. We refer to $h \in \mathbb{R}^n$ as the *direction* of H and H'. If P is a prismoid, and F and F' partition the vertex set of P, then we call h a *prismoid direction* of P. Likewise if F and F' are facets, we refer to h as a *prismatoid direction* of P.

5.5.1 Example. The hypercube $H_n \subseteq \mathbb{R}^n$ is a prismatoid. For each $1 \leq i \leq n$ we let e_i be the ith standard basis vector of $\mathbb{R}^n$, and define

$$\begin{aligned} P_i &= \{x \in \mathbb{R}^n \,:\, x^T e_i \leq 1\} \\ P_i' &= \{x \in \mathbb{R}^n \,:\, x^T e_i \geq 0\}. \end{aligned}$$

Now P_i and P_i' are supporting hyperplanes for H_n, and the faces $F_i = P_i \cap H_n$ and $F_i' = P_i' \cap H_n$ are parallel facets of H_n. Thus each e_i is a prismatoid direction of H_n. So it is possible for a polytope to be a prismatoid with respect to several direction vectors.

Let X be a connected k-regular graph. Let θ be an eigenvalue of X, and let U be a matrix whose columns form an orthonormal basis for the θ-eigenspace of X. Take P to be the θ-eigenpolytope of X (so P is the convex hull of the rows of U). If z is a θ-eigenvector of X, then there is some h so that $Uh = z$. As we have seen, the maximum and minimum entries of z index the vertices of parallel faces of P. If we suppose that z has two distinct entries α and β, then

$$\begin{aligned} F_a &= \{x \in P \,:\, x^T h = \alpha\} \\ F_b &= \{x \in P \,:\, x^T h = \beta\} \end{aligned}$$

are parallel faces of P that partition its vertex set. Thus h is a prismoid direction of P. This direction vector gives an equitable partition of X.

5.5.2 Lemma. *Let P be the θ-eigenpolytope of a k-regular connected graph X, and suppose*

$$\begin{aligned} F_a &= \{x \in P \,:\, x^T h = \alpha\} \\ F_b &= \{x \in P \,:\, x^T h = \beta\} \end{aligned}$$

are parallel faces of P that partition its vertex set. If $A, B \subseteq V(X)$ are the subsets of $V(X)$ corresponding to the vertices of F_a and F_b, then (A, B) is an equitable partition of X.

Proof. Let A be the adjacency matrix of X. Then for each $x \in V(X)$, the x-row of A is the characteristic vector of the set of neighbours of x. If z is a θ-eigenvector for X, then $Az = \theta z$. Thus for each $x \in V(X)$,

$$\theta z_x = [Az]_x = A_x^T z = \sum_{xy \in E(X)} z_y.$$

Now suppose that z is a θ-eigenvector for X with two entries, α and β. Let A and B be the subsets of $V(X)$ so $z_a = \alpha$ for all $a \in A$ and $z_b = \beta$ for all

$b \in B$. Now we apply the above identity to an arbitrary vertex $x \in A$. We have the following system of equations:

$$\begin{aligned} \theta\alpha &= |A \cap \Gamma(x)|\alpha + |B \cap \Gamma(x)|\beta \\ k &= |A \cap \Gamma(x)| + |B \cap \Gamma(x)|. \end{aligned}$$

Since θ, α, β and k are all known, we can solve for $|A \cap \Gamma(x)|$ and $|B \cap \Gamma(x)|$. We find that

$$|A \cap \Gamma(x)| = \frac{k\beta - \theta\alpha}{\beta - \alpha}, \quad \text{and} \quad |B \cap \Gamma(x)| = \frac{\alpha(\theta - k)}{\beta - \alpha}.$$

Since $x \in A$ was chosen arbitrarily, we see that the number of neighbours of $x \in A$ that lie in A and the number that lie in B are independent of x. A similar calculation shows that if we consider $x \in B$, then the number of neighbours of x that lie in A and the number that lie in B are independent of x. Therefore (A, B) is an equitable partition of X. □

Note that we only assumed that $\alpha \neq \beta$ and did not consider the case where one of α or β is 0. Suppose $\alpha = 0$, so z is the characteristic vector of a subset A of X (appropriately scaled). Then for $x \in A$ we have that $0 = |B \cap \Gamma(x)|$ which implies that X is not connected. Therefore this does not happen. Also, this implies that if we are considering connected graphs, then the eigenvectors of X are never characteristic vectors of our graph. This is the reason why we needed to consider shifts of eigenvectors in Section 5.3.

Lemma 5.5.2 shows that by finding prismoid directions of an eigenpolytope of X, we can find equitable partitions. In our applications we have been trying to classify combinatorial structures using the face lattice of a polytope. The face lattice gives us all of the prismoid directions of a polytope. We want to know if this in turn gives us all of the possible equitable partitions of our graph into two parts. Our next lemma shows that while equitable partitions into two parts and prismoid directions are strongly related, we do not have a bijection between them.

5.5.3 Lemma. *Let X be a k-regular connected graph, and let (A, B) be an equitable partition of the vertices of X. Then there is an eigenvector z for X that is constant on A and on B (but takes distinct values on each).*

Proof. We follow Godsil [14]. Let P be an equitable partition of X. If A is the adjacency matrix of X, then the space of vectors in $\mathbb{R}^{E(X)}$ that are constant on the cells of P is invariant under A. This follows as

$$[Az]_x = \sum_{xy \in E(X)} z_y,$$

and the number of neighbours of x in each of the cells of P depends only on the cell of P containing x. Since this subspace is A-invariant, it contains eigenvectors of A. Since X is k-regular, the vector $\mathbf{1}$ is a k-eigenvector for X that is constant on the cells of P. If P has more than one cell, than the space of vectors constant on the cells of P is not spanned by $\mathbf{1}$. Thus there are eigenvectors that are constant on the cells of P and take on more than one value. This proves the result. $\square$

As a result, we have that an equitable partition (A, B) gives a prismoid direction for *some* eigenpolytope of X. However, it may not be the one we are interested in. If X is a distance-regular graph, then we have the following corollary.

5.5.4 Corollary. *If X is distance regular with ith distance graph X_i, and (A, B) is an equitable partition of X, then (A, B) is an equitable partition of X_i.*

Proof. Since (A, B) is an equitable partition of X, by Lemma 5.5.3 we have that there is an eigenvector z for X that is constant on the vertices in A and B. Since X is distance regular, its distance graphs all have the same eigenspaces. Thus the entries of z partition the vertices of some eigenpolytope of X_i. So by Lemma 5.5.2 we have that (A, B) is an equitable partition of X_i. $\square$

Note that the proof of Lemma 5.5.3 implies that if P is an equitable partition of X, then there is some eigenpolytope Q of X containing two parallel faces whose vertex sets are cells of P. Moreover, if h is the direction of these faces, and $P_1, \dots, P_m$ are the cells of P, then there are values $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ so that

$$Q_i = \{x \in Q \,:\, x^T h = \alpha_i\}$$

gives a partition of the vertices of Q and the vertices contained in Q_i correspond exactly to the vertices in P_i.

Finally we mention that for the applications we have seen, we are looking for cocliques in graphs that are vertex transitive, and graphs that are distance regular. In the case of the folded-cube graphs for example, we have that G_n is distance regular. As a result, the distance partitions of G_n are equitable partitions, and the canonical cocliques we identified are derived from the distance partition of the graph. Lemmas 5.5.2 and 5.5.3 suggest that prismoid directions in eigenpolytopes give a promising tool for characterizing cocliques with an associated equitable partition.

5.6 The Veronese Matrix

We have seen that prismoid directions give parallel faces of a polytope. Moreover, if $P \subseteq \mathbb{R}^m$ is a polytope defined as the convex hull of the rows of a $n \times m$ matrix M, and $Mh = z$, then the parallel hyperplanes

$$H_\alpha = \{x \in \mathbb{R}^m \,:\, x^T h = \alpha\}$$

partition the vertices of P according to the entries of z. Let the distinct entries of z be $\alpha_1, \dots, \alpha_n$. Then for every row M_j of M, we have that

$$\prod_{i=1}^{n} (M_j^T h - \alpha_i) = 0.$$

Therefore the rows M_j give solutions to the equation

$$\prod_{i=1}^{n}(x^T h - \alpha_i) = 0.$$

The left-hand side of this equation is a polynomial of degree n in the variables $x_1, \ldots, x_m$. We let $S \subseteq \mathbb{R}^m$ be the set of solutions to this equation.

We would like to use this polynomial to derive information about the partitions of the vertices of P. For $s \in S$ all we know is that $s^T h = \alpha_i$ for some $1 \leq i \leq n$. This is not a particularly useful piece of information. So we change our point of view. Consider the set of polynomial equations

$$\left\{\prod_{i=1}^{n}(M_j^T x - \alpha_i) = 0 \,:\, 1 \leq j \leq n\right\}. \tag{5.6.1}$$

Now h gives a solution to each of these equations simultaneously. The intersection of the sets of solutions to the individual equations in (5.6.1) for all $1 \leq j \leq n$ is a subset S of $\mathbb{R}^m$. Now for every $s \in S$ we have that $Ms = z_s$ where the set of distinct entries of z_s is a subset of $\{\alpha_1, \ldots, \alpha_n\}$. Therefore the vectors in S are all direction vectors that give partitions of the vertices of P into at most n parts.

If we suppose that P is a prismoid, then there is some prismoid direction h. Let α and β be the distinct entries of $z = Mh$. Now every row M_j of M satisfies

$$(M_j^T h - \alpha)(M_j^T h - \beta) = 0, \tag{5.6.2}$$

and is therefore h is a solution to the quadratic polynomials

$$\left\{(M_j^T x - \alpha)(M_j^T x - \beta) = 0 \,:\, 1 \leq j \leq n\right\}.$$

As we have already seen, each simultaneous solution to these quadratic polynomials corresponds to a direction vector that gives a partition of the vertices of P into at most two parts. We want to use this set of solutions to classify the prismoid directions of P. However, the solutions to these quadratic polynomials only give us prismoid directions h for which the ratio of the entries of $z = Mh$ is α/β (or its inverse).

Note that if we expand the quadratic in Equation (5.6.2) we have

$$\begin{aligned}
0 &= (M_j^T h - \alpha)(M_j^T h - \beta) \\
&= (M_j^T h)^2 - (\alpha + \beta) M_j^T h + \alpha\beta \\
&= \sum_{1 \leq a,b \leq n} h_a h_b [M_j]_a [M_j]_b - \sum_{1 \leq a \leq n} (\alpha + \beta) h_a [M_j]_a + (\alpha\beta)1.
\end{aligned}$$

Let M^j be the jth column of M. From α, β and h we can derive the coefficients of a linear combination of $\mathbf{1}$, the columns M^a and the Schur products of columns $M^a \circ M^b$ that give the zero vector. Thus α, β and h give us elements of the null space of a matrix.

Given a $n \times m$ matrix M, define the *Veronese matrix* of M to be the $n \times \binom{m+1}{2}$ matrix $v(M)$ whose columns are the Schur products

$$\{M^i \circ M^j \;:\; 1 \leq i \leq j \leq m\}.$$

The Veronese matrix of M is named for the Veronese map found in algebraic geometry. The *Veronese map* of order d is the map Φ_d defined by

$$\Phi_d(x_0, \ldots, x_n) = (\ldots, y, \ldots),$$

where y runs over all possible monomials in the x_i of degree d [9]. So our matrix $v(M)$ comes from the Veronese map of order 2.

The solutions to our system of quadratic polynomials correspond to the elements in the null space of the matrix

$$M_{ver} = [\mathbf{1} \,|\, M \,|\, v(M)].$$

The dimensions of M_{ver} are $n \times \binom{m+2}{2}$. We define the *Veronesian rank* of M to be $\mathrm{rk}(M_{ver})$. So the null space of M_{ver} contains the space of prismoid directions of P. By the rank-nullity theorem, the null space of M_{ver} is $\binom{m+2}{2} - \mathrm{rk}(M_{ver})$. So by calculating the Veronesian rank of M we obtain information about the prismoid directions of P.

We conclude this section by giving an alternative matrix with the same rank as M_{ver}. If M is a $n \times m$ matrix, then we define the matrix M' to be the $n \times m^2$ matrix whose columns are the Schur products

$$\{M^i \circ M^j \;:\; 1 \leq i, j \leq m\}.$$

The matrix M' has the same set of columns as $v(M)$, but with some repetition (as M' contains columns $M^i \circ M^j$ and $M^j \circ M^i$). In practice we will find M' easier to work with than $v(M)$.

5.6.1 Proposition. *Given a matrix M, the Veronesian rank of M is*

$$\mathrm{rk}(M_{ver}) = \mathrm{rk}([\mathbf{1} \,|\, M]').$$

Proof. Consider the matrix $[\mathbf{1} \,|\, M]'$. The columns of this matrix are exactly $\mathbf{1}$, the columns of M (each appearing twice) and the columns of $[M]'$. Therefore the set of columns of $[\mathbf{1} \,|\, M]'$ is exactly the set of columns of M_{ver} (with some repetition). It follows that the column spaces of these matrices are the same, and thus

$$\mathrm{rk}([\mathbf{1} \,|\, M]') = \mathrm{rk}(M_{ver}). \qquad \square$$

So we can use the matrix $[\mathbf{1} \,|\, M]'$ to calculate the Veronesian rank of M.

5.7 Association Schemes

Recall from Section 4.6 that the Bose-Mesner algebra $\mathcal{A}$ of an association scheme is the matrix algebra generated by the basis $\{A_0, \ldots, A_d\}$. The matrices A_i

are adjacency matrices of graphs on n vertices and are simultaneously diagonalizable. The matrix idempotents of $\mathcal{A}$ are the orthogonal projectors onto the d eigenspaces of the matrices A_i. We denote the matrix idempotents as $E_0, \ldots, E_d$.

The matrix idempotents of $\mathcal{A}$ form a basis for $\mathcal{A}$. The Bose-Mesner algebra is closed under Schur multiplication, so the matrices $E_i \circ E_j$ are elements of $\mathcal{A}$, and can be uniquely expressed as a linear combination of $\{E_0, \ldots, E_d\}$. The Krein parameters of the scheme are the constants $q_{i,j}^k$ so that

$$E_i \circ E_j = \sum_{k=0}^{d} q_{i,j}^k E_k.$$

The Krein parameters satisfy an important set of inequalities known as the *Krein conditions.*

5.7.1 Theorem (Theorem 2.3.2 in Brouwer et al. [3])**.** *For $1 \leq i, j, k \leq d$, the Krein parameters $q_{i,j}^k$ are non-negative.* □

Suppose that U is a matrix whose columns are an orthonormal basis for the ith eigenspace of the matrices A_j. We let the dimension of the ith eigenspace be m_i. So U is a $n \times m_i$ matrix. Note that E_i can be derived from U as $UU^T = E_i$. The Veronesian rank of U is entirely determined by the Krein parameters and eigenvalues of the scheme.

5.7.2 Example. Let $i = 0$, and U be a matrix whose columns are an orthonormal basis for the 0th eigenspace. Recall that $E_0 = 1/nJ$. Therefore $U = (1/\sqrt{n})\mathbf{1}$, and from Proposition 5.6.1 we have that the Veronesian rank of U is

$$\operatorname{rk}([\mathbf{1} \,|\, U]') = \operatorname{rk}([\mathbf{1} \,|\, (1/\sqrt{n})\mathbf{1} \,|\, (1/\sqrt{n})\mathbf{1} \,|\, (1/n)\mathbf{1}]) = 1.$$

So the Veronesian rank of U is m_0 when $i = 0$.

If $i > 0$ we can express the Veronesian rank of U as a sum of the multiplicities of the eigenvalues of the scheme. Define σ_j as

$$\sigma_j = \begin{cases} 0, & \text{if } q_{i,i}^j = 0 \\ 1, & \text{else.} \end{cases} \tag{5.7.1}$$

5.7.3 Lemma. *Assume $i > 0$, and let U be a matrix whose columns form an orthonormal basis of the ith eigenspace of $\mathcal{A}$. Then*

$$\operatorname{rk}(U_{ver}) = m_0 + m_i + \sum_{k \neq 0, i} \sigma_k m_k.$$

Proof. From Proposition 5.6.1 we have that

$$\operatorname{rk}(U_{ver}) = \operatorname{rk}([\mathbf{1} \,|\, U]').$$

For any matrix M we have that

$$\mathrm{rk}(M) = \mathrm{rk}(M^T M) = \mathrm{rk}(MM^T),$$

therefore

$$\mathrm{rk}([\mathbf{1}\,|\,U]') = \mathrm{rk}([\mathbf{1}\,|\,U]'([\mathbf{1}\,|\,U]')^T).$$

We let

$$V = [\mathbf{1}\,|\,U]'([\mathbf{1}\,|\,U]')^T$$

and compute the (a, b) entry of V.

We denote the jth row of a matrix M by M_j. Recall that the columns of M' are the Schur products of all pairs of columns of M. So,

$$\begin{aligned} V_{a,b} = ([\mathbf{1}\,|\,U]'([\mathbf{1}\,|\,U]')^T)_{a,b} &= \langle [\mathbf{1}\,|\,U]'_a, [\mathbf{1}\,|\,U]'_b \rangle \\ &= \langle [\mathbf{1}\,|\,U]_a, [\mathbf{1}\,|\,U]_b \rangle^2 \\ &= 1 + 2\langle U_a, U_b\rangle + \langle U_a, U_b\rangle^2. \end{aligned}$$

Since $UU^T = E_i$, we have that $\langle U_a, U_b\rangle = [E_i]_{a,b}$. Therefore

$$\begin{aligned} V &= J + 2E_i + E_i \circ E_i \\ &= (n + q^0_{i,i})E_0 + (2 + q^i_{i,i})E_i + \sum_{k\neq 0,i} q^k_{i,i} E_k. \end{aligned}$$

To calculate the rank of V, we consider its eigenvalues. Suppose that v is a vector in the jth eigenspace of $\mathcal{A}$. Then since the matrices E_l are orthogonal projections onto the lth eigenspace, we have $E_l v = 0$ for $l \neq j$, and $E_j v = v$. Thus,

$$Vv = \begin{cases} (n + q^0_{i,i})v, & \text{if } j = 0 \\ (2 + q^i_{i,i})v, & \text{if } j = i \\ q^j_{i,i} v, & \text{else,} \end{cases}$$

whenever v is the in jth eigenspace of $\mathcal{A}$. Since V can be expressed as a linear combination of the matrix idempotents, it is an element of the Bose-Mesner algebra. Therefore the eigenspaces of $\mathcal{A}$ contain all of the eigenvectors of V. Therefore, the spectrum of V is

$$\{n + q^0_{i,i}, 2 + q^i_{i,i}\} \cup \{q^j_{i,i} \,:\, j \neq 0, i\},$$

and the multiplicity of the eigenvalue given by v is equal to the dimension of the eigenspace to which it belongs.

It follows that $\mathrm{rk}(U_{ver})$ is the sum of the ranks of the matrices E_k for which the corresponding eigenvalue is non-zero. From Theorem 5.7.1 we have that $q^k_{i,j} \geq 0$ for all $1 \leq i, j, k \leq d$. So $(n + q^0_{i,i}) > 0$ and $(2 + q^i_{i,i}) > 0$. Therefore

$$\mathrm{rk}(U_{ver}) = m_0 + m_i + \sum_{k\neq 0,i} \sigma_k m_k,$$

as required. $\square$

As we saw in Chapter 4, if X is a distance-regular graph, and $X_0, \dots, X_d$ are the ith distance graphs of X, then the adjacency matrices $A_i = A(X_i)$ give a metric association scheme. Therefore Lemma 5.7.3 can be used to calculate the Veronesian rank of a matrix U whose columns are an orthonormal basis of an eigenspace of X. Since the eigenpolytope P is defined to be the convex hull of the rows of U, this gives us the dimension of the null space of U_{ver} which is related to the prismoid directions of P. In the next section, we will see that the Veronesian rank of U can be calculated exactly for distance-regular graphs with small diameter.

5.8 Strongly Regular Graphs

We give a brief treatment of strongly regular graphs, and compute the Veronesian ranks associated with their eigenvalues. We will revisit strongly regular graph again in Chapter 6 where our treatment will be more thorough. Our source for the basic theory of strongly regular graphs is Brouwer and van Lint [5] and Chapter 10 of Godsil and Royle [18].

A *strongly regular graph* is a distance-regular graph X with diameter 2. We will let X be a strongly regular graph on n vertices with valency k. The distance matrices of X are A_0, A_1 and A_2 and they generate a 2-class metric association scheme. Let the matrix idempotents of this scheme be E_1, E_1 and E_2. Note that our definition implies that X is connected, and so X has three eigenvalues, $k \geq \theta \geq \tau$. We let E_1 be the orthonormal projector onto the θ-eigenspace of A_1, and E_2 be the projector onto the τ-eigenspace of A_1. We let the rank of E_1 be m_θ, and the rank of E_2 be m_τ.

We will see in Chapter 6 that we can derive the values of θ, τ and their multiplicities from the parameters of X. However, for the material in this section we only need to find some of the Krein parameters in terms of the eigenvalues and dual eigenvalues of the scheme. Recall that the eigenvalues of the scheme are the values $p_i(j)$ so that

$$A_i = \sum_{i=0}^{d} p_i(j) E_j = p_i(0)E_0 + p_i(1)E_1 + p_i(2)E_2.$$

Since the matrices E_i sum to the identity matrix, and A_0 is the identity matrix, we have that $p_0(j) = 1$ for all j. We have defined $p_1(0) = k$, $p_1(1) = \theta$ and $p_1(2) = \tau$. The values $p_2(0), p_2(1), p_2(2)$ are the eigenvalues of A_2.

Since X has diameter 2, A_2 is the adjacency matrix of the complement of X. Therefore

$$A_2 = J - I - A_1,$$

and we can compute the eigenvalues of A_2 using the eigenvectors of A_1. The vector $\mathbf{1}$ spans the k-eigenspace of A_1, and

$$A_2\mathbf{1} = J\mathbf{1} - I\mathbf{1} - A_1\mathbf{1} = (n-1-k)\mathbf{1}$$

shows that $n-k-1$ is an eigenvalue of A_2. If v is a σ-eigenvector of A_1 that is orthogonal to $\mathbf{1}$, then

$$A_2 v = Jv - Iv - A_1 v = (-1-\sigma)v,$$

and v is a $(-1-\sigma)$-eigenvector of A_2. Therefore we have

$$p_2(0) = n-k-1, \quad p_2(1) = -1-\theta, \quad p_2(2) = -1-\tau.$$

Recall that the matrix of eigenvalues is the matrix P with $P[i,j] = p_i(j)$. Thus

$$P = \begin{pmatrix} 1 & k & n-k-1 \\ 1 & \theta & -1-\theta \\ 1 & \tau & -1-\tau \end{pmatrix}.$$

The dual eigenvalues of the scheme are the values $q_i(j)$ so that

$$E_j = (1/n)\sum_{i=0}^{d} q_j(i) A_i.$$

Recall that the matrix of dual eigenvalues Q, given by $Q[i,j] = q_i(j)$ has the property that $PQ = 1/nI$. Therefore we can find Q by inverting P, so

$$Q = 1/nP^{-1} = \frac{1}{n(\theta-\tau)} \begin{pmatrix} \theta-\tau & -k-(n-1)\tau & k+(n-1)\theta \\ \theta-\tau & n-k+\tau & k-n-\theta \\ \theta-\tau & \tau-k & k-\theta \end{pmatrix}.$$

The Krein parameters, eigenvalues and dual eigenvalues of an association scheme are related by several identities. We will use the following identities from Lemma 2.3.1 in [3]:

(i) $q_{0,j}^k = \delta_{jk}$

(iii) $q_{i,j}^k = q_{j,i}^k$

(v) $\sum_{i=0}^d q_{i,j}^k = m_j$

(viii) $p_i(k)q_j(i) = \sum_{l=0}^d q_{j,l}^k p_i(l)$

for all $0 \le i,j,k \le d$, where $\delta_{ab} = 1$ if $a = b$ and 0 otherwise.

5.8.1 Corollary. *Let X be a strongly regular graph, and $U(\sigma)$ be a matrix whose columns form an orthonormal basis for the σ-eigenspace of G. Then*

$$\operatorname{rk}(U(\theta)_{ver}) = \operatorname{rk}(U(\tau)_{ver}) = n.$$

Proof. We will prove this corollary for $U = U(\theta)$, as the proof for $U(\tau)$ is very similar.

To compute $\mathrm{rk}(U_{ver})$ we use Lemma 5.7.3. We have that

$$\mathrm{rk}(U_{ver}) = m_0 + m_1 + \sigma m_2$$

where σ is 0 if $q^2_{1,1} = 0$, and 1 otherwise. We assume that $\sigma = 0$ and derive a contradiction.

Let $q^2_{1,1} = 0$. From identity (i) we have that $q^2_{0,1} = 0$. Now identity (v) gives us that

$$q^2_{1,0} + q^2_{1,1} + q^2_{1,2} = m_1$$

which implies that $q^2_{1,2} = m_1$. Also identity (iii) gives us that $q^2_{1,0} = 0$. Finally, we apply identity (viii) with $i = 0, j = 1$ and $k = 2$, so

$$p_0(2)q_1(0) = q^2_{1,0}p_0(0) + q^2_{1,1}p_0(1) + q^2_{1,2}p_0(2).$$

Using the matrices P and Q above, we see that

$$(n - k - 1)(1/n) = 0(1) + 0(k) + m_1(n - k - 1),$$

which simplifies to $1/n = m_1$. This is a contradiction as m_1 is a positive integer.

Therefore $q^2_{1,1} > 0$ and $\sigma = 1$. Thus

$$\mathrm{rk}(U_{ver}) = m_0 + m_1 + m_2 = n,$$

as required. □

5.9 The Absolute Bound

Lemma 5.7.3 gives us a tool for calculating the Veronesian rank of U when U is constructed from an eigenspace of a distance-regular graph. In this section we will use a classical bound on the sums of multiplicities of an association scheme to bound the Veronesian rank of U.

Let $\mathcal{A}$ be the Bose-Mesner algebra of an association scheme with matrix idempotents E_i and multiplicities m_i for $0 \leq i \leq d$. For $0 \leq i, j, k \leq d$, let $q^k_{i,j}$ be the Krein parameters of the scheme. The absolute bound gives a bound on the sums of multiplicities m_i for which the associated Krein parameters are non-zero. We present the proof given in Brouwer et al. [3] as Theorem 2.3.3.

Fix $0 \leq i, j \leq d$. For $0 \leq k \leq d$, define

$$\sigma_k = \begin{cases} 0 & \text{if } q^k_{i,j} = 0 \\ 1 & \text{else.} \end{cases}$$

5.9.1 Theorem (The Absolute Bound)**.** *For all* $0 \leq i, j \leq d$*,*

$$\sum_{k=0}^{d} \sigma_k m_k \leq \begin{cases} m_i m_j & \text{for } i \neq j \\ m_i(m_i + 1)/2 & \text{for } i = j. \end{cases}$$

Proof. Consider the matrix $E_i \circ E_j$. By definition we have

$$E_i \circ E_j = \sum_{k=0}^{d} q_{i,j}^k E_k.$$

Since each E_i is the orthogonal projector onto the ith eigenspace of the matrices in $\mathcal{A}$, and the E_i are pairwise orthogonal, we have that $\operatorname{rk}(E_i \circ E_i)$ is equal to the sum of the ranks of the matrices $q_{i,j}^k E_k$. Therefore

$$\operatorname{rk}(E_i \circ E_j) = \sum_{k=0}^{d} \sigma_k m_k.$$

Now we derive a bound on $\operatorname{rk}(E_i \circ E_j)$. We have two cases. Suppose first that $i \neq j$. For matrices A, B let $A \otimes B$ be the standard Kronecker product. Note that the entries of $E_i \otimes E_j$ are pairwise products of the entries of E_i and E_j. It follows that $E_i \circ E_j$ is a submatrix of $E_i \otimes E_j$, and therefore $\operatorname{rk}(E_i \circ E_j) \leq \operatorname{rk}(E_i \otimes E_j)$. Since $\operatorname{rk}(A \otimes B) = \operatorname{rk}(A) \operatorname{rk}(B)$ we have that $\operatorname{rk}(E_i \circ E_j) \leq m_i m_j$ as required.

Now suppose that $i = j$. If U is a $n \times m_i$ matrix whose columns are an orthonormal basis for the ith eigenspace of $\mathcal{A}$, then $E_i = UU^T$. The columns of E_i are linear combinations of the columns of U. The entries of $E_i \circ E_i$ are the squares of the entries of E_i, and the columns of E_i are the Schur squares of the columns of E_i. Therefore the columns of E_i are linear combinations of the Schur squares of the columns of U, together with the Schur products of distinct columns of U. Thus

$$\operatorname{rk}(E_i \circ E_i) \leq m_i + \binom{m_i}{2} = m_i(m_i + 1)/2. \qquad \square$$

Since we are interested in the null space of the Veronese matrix of U, it is natural to look for examples where U_{ver} has small rank. Specifically we would like to find distance-regular graphs X with U a matrix whose columns form a basis for the eigenspace of the least eigenvalue of X and for which $\operatorname{rk}(U_{ver})$ is as large as possible. Recall that if X has n vertices, and the dimension of the eigenspace is m, then U is a $n \times m$ matrix, and U_{ver} is a $n \times \binom{m+2}{2}$ matrix. So if we want the rank of U_{ver} to be as close to the number of columns of U_{ver} as possible, we need n to be sufficiently large by comparison to m. In general, since

$$n = \sum_{i=0}^{d} m_i,$$

this is not particularly restrictive.

5.9.2 Corollary. *Let U be a matrix whose columns are an orthonormal basis for the ith eigenspace of a distance-regular graph X. If $q_{i,i}^i \neq 0$, then*

$$\operatorname{rk}(U_{ver}) \leq \binom{m_1 + 2}{2} - m_i - 1$$

and equality is achieved exactly when $q_{i,i}^k \neq 0$ for all $0 \leq k \leq d$, and the absolute bound holds with equality. If $q_{i,i}^i = 0$, then

$$\mathrm{rk}(U_{ver}) \leq \binom{m_1+2}{2} - 1$$

and equality is achieved exactly when $q_{i,i}^k \neq 0$ for all $k \neq i$, and the absolute bound holds with equality.

Proof. Suppose that U is a matrix whose columns are an orthonormal basis for the ith eigenspace. We can use Theorem 5.9.1 to give an upper bound on $\mathrm{rk}(U_{ver})$. From Lemma 5.7.3 we have that

$$\mathrm{rk}(U_{ver}) = m_0 + m_i + \sum_{k \neq 0,i} \sigma_k m_k.$$

From Lemma 2.3.1 (ii) in [3] we have that the Krein parameters satisfy $q_{i,j}^0 = \delta_{ij} m_i$. Therefore $q_{i,i}^0 = m_i \neq 0$. So we can simplify our expression to

$$\mathrm{rk}(U_{ver}) = m_i + \sum_{k \neq i} \sigma_k m_k.$$

Now if we assume that $q_{i,i}^k \neq 0$ for all $0 \leq k \leq d$, then Theorem 5.9.1 implies that

$$\begin{aligned}
\mathrm{rk}(U_{ver}) &= m_i + \sum_{k \neq i} \sigma_k m_k \\
&= \sum_{k=0}^{d} \sigma_k m_k \\
&\leq \binom{m_i+1}{2} = \binom{m_i+2}{2} - m_i - 1.
\end{aligned}$$

Therefore, if we make the additional assumption that the absolute bound holds with equality, we have that the rank of the null space of U_{ver} is $m_i + 1$.

If we add the additional assumption that $q_{i,i}^i = 0$, then the same argument shows that

$$\mathrm{rk}(U_{ver}) \leq \binom{m_1+2}{2} - 1$$

and if the absolute bound holds with equality, then the rank of the null space of U_{ver} is 1. Since these assumptions maximize the Veronesian rank of U, we have that U_{ver} never has full column rank, and the best possible result is $\mathrm{rk}(U_{ver}) = \binom{m_1+2}{2} - 1$. □

The second bound in Corollary 5.9.2 has an alternative derivation. Recall from Section 4.8 that if U is a matrix whose columns are an orthonormal basis for the ith eigenspace of a distance-regular graph X, then the rows of U all lie on a sphere in $\mathbb{R}^{m_i}$ centred at the origin. Specifically, U_j is the jth row of U, then

$$\langle U_j, U_j \rangle = m_i/n$$

for all $1 \leq j \leq n$. Therefore, the entries of each U_j satisfy the quadratic

$$\sum_{i=1}^{m_i} x_i^2 = m_i/n.$$

This quadratic gives an element of the null space of U_{ver}, and thus $\mathrm{rk}(U_{ver}) \leq \binom{m_1+2}{2} - 1$. This quadratic is an example of an element of the null space of U_{ver} that does not give us a prismoid direction of the corresponding eigenpolytope.

5.10 Computational Results

We have seen that the null space of the Veronese matrix is related to the structure of the eigenpolytopes of a graph. The simplest examples are given by matrices whose Veronese matrix has a null space of low dimension. If we are given a matrix M, constructing M_{ver} and finding its rank is computationally easy, while extracting information about its null space is more difficult.

We used Sage [30] to generate computational data on the Veronese ranks of matrices from the eigenspaces of a number of graphs. Our data set consisted of 4973 graphs in total, most of which were either distance regular or vertex transitive. Restricting ourselves to eigenvalues with multiplicity at least 3, we found 588 graphs with Veronese null space of rank 1. Of these graphs, there are a few interesting examples.

5.10.1 Example. The Desargues graph (found on pg 418 of Brouwer et al. [3]) is a distance-regular graph of diameter 5 on 20 vertices. It is both bipartite and antipodal. Its intersection array is

$$\{3, 2, 2, 1, 1; 1, 1, 2, 2, 3\},$$

and its spectrum is

$$\{(3)^1, (2)^4, (1)^5, (-1)^5, (-2)^4, (-3)^1\}.$$

Since the graph is bipartite, its spectrum is symmetric around 0. Let θ be an eigenvalue, U the matrix we obtain from the θ-eigenspace, and V the matrix we obtain from the $(-\theta)$-eigenspace. For each of the eigenvalues of the Desargues graph $\mathrm{rk}(U_{ver}) \neq \mathrm{rk}(V_{ver})$. When $\theta = 2$, the matrix U_{ver} has 15 columns, and $\mathrm{rk}(U_{ver}) = 14$.

The other two interesting examples are strongly regular graphs: the Schläfli graph, and the McLaughlin graph. We will return to those in the next chapter.

The data we generated was helpful in suggesting the results in Section 5.8. However, we were not able to make many inferences from the computations.

Chapter 6

Strongly Regular Graphs

Strongly regular graphs are distance-regular graphs of the smallest (interesting) diameter. We have seen connections between cocliques in distance-regular graphs and their equitable partitions. In this chapter we will take a closer look at the equitable partitions of strongly regular graphs. We will also consider equitable partitions of strongly regular graphs into two parts that have additional structural properties.

In [21], Higman and Haemers characterized the strongly regular graphs with strongly regular partitions. These are equitable partitions into two parts each of which is either a clique, a coclique or a strongly regular graph. When one of the cells is a coclique, there are implications for the inertia bound and the ratio bound. We will give a brief account of the theory in [21], and see examples of graphs that have strongly regular partitions, and a family of graphs that has no equitable partitions into two parts. The criteria for a graph to have a strongly regular partition give restrictions on the spectrum of the graph.

Following the idea of adding additional structure to equitable partitions, we also look at convex subgraphs. We present a theorem of Lambeck classifying the convex subgraphs of a family of distance-regular graphs. We will see that this classification leads to partitions of distance-regular graphs into two parts, both of which are convex. These partitions are connected to partitions of the Kneser graphs $K_{n:k}$ into a maximum coclique and its complement.

Finally we summarize some computations on strongly regular graphs. These computations bring together the ideas from all of the preceding chapters of the thesis. We look at the eigenpolytopes of two non-isomorphic strongly regular graphs with the same parameter set, and see that the parameters of a distance-regular graph do not determine the combinatorial types of its eigenpolytopes.

6.1 Strongly Regular Graphs

Recall in Section 5.8 we defined strongly regular graphs as distance-regular graphs of diameter 2. In this section we will expand the basic theory of strongly regular graphs that we will use later in the chapter. We will follow Brouwer and van Lint [5] and Chapter 10 in Godsil and Royle [18].

Let X be a strongly regular graph on n vertices. We take the valency of X to be k. The association scheme corresponding to X has Schur idempotents A_0, A_1, A_2 where $A_0 = I$, $A_1 = A(X)$ and A_2 is the distance 2 relation on X. We also have matrix idempotents E_0, E_1, E_2 where E_i is the orthogonal projector onto the ith eigenspace of X. We take $E_0 = 1/nJ$ to project onto the k-eigenspace of dimension 1. By convention we will take E_1 to project onto the θ-eigenspace of dimension m_θ, and E_2 to project onto the τ-eigenspace of dimension m_τ where $k > \theta > \tau$.

Recall from Section 4.5 that if we let $\{\Gamma_0(x), \Gamma_1(x), \Gamma_2(x)\}$ be the distance partition of X with respect to some vertex x, then there are constants a_i, b_i, c_i with the following property. For all vertices $x, y \in V(X)$, if $d(x, y) = i$, then

$$\begin{aligned} |\Gamma_{i-1}(x) \cap \Gamma_1(y)| &= c_i, \\ |\Gamma_i(x) \cap \Gamma_1(y)| &= a_i, \\ |\Gamma_{i+1}(x) \cap \Gamma_1(y)| &= b_i. \end{aligned}$$

We also saw that since $k = a_i + b_i + c_i$ for each i, the parameters of a distance-regular graph can be compressed into its intersection array. The intersection array of X is

$$\begin{pmatrix} k & b_1 & 0 \\ 0 & 1 & c_2 \end{pmatrix}.$$

Note that the only "unknowns" in this intersection array are the parameters b_1 and c_2, and that those two values determine all of the a_i values. We will take our notation from [18] and denote these parameters as $a = a_1 = k - b_1 - 1$ and $c = c_2$. Along with n and k, these values complete the *parameters of a strongly regular graph*, which are given in the form $(n, k; a, c)$. We refer to X as an $(n, k; a, c)$ strongly regular graph.

The parameters a and c have the following property (as a specialization of the parameters of a distance-regular graph). Let x and y be any two vertices of X. Then the number of common neighbours of x and y is a if x and y are adjacent, and c if x and y are not adjacent. Strongly regular graphs are often defined as a k-regular graphs with parameters a and c that satisfy the given property. Since there is no provision for connectedness, a strongly regular graph is called *primitive* if both X and $\overline{X}$ are connected graphs, and *imprimitive* otherwise. Lemma 10.1.1 in [18] classifies the imprimitive strongly regular graphs as graphs X that are isomorphic to mK_{k+1} for some integer m (or graphs X for which $\overline{X}$ is isomorphic to mK_{k+1}). We have already assumed that our graph X is connected, for the remainder of this section we also assume that X is primitive.

Since the Schur idempotents sum to the matrix J with all entries 1, we have that

$$A_2 = J - I - A_1 = A(\overline{X}).$$

Therefore, $\overline{X}$ is a strongly regular graph that generates the same association scheme as X. If X is a $(n,k;a,c)$ strongly regular graph, then $\overline{X}$ is a $(n,\overline{k};\overline{a},\overline{c})$ strongly regular graph where:

$$\begin{aligned}\overline{k} &= n-k-1;\\ \overline{a} &= n-2-2k+c;\\ \overline{c} &= n-2k+a.\end{aligned}$$

As we have already seen, the spectrum of $\overline{X}$ is

$$\{(n-k-1)^1, (-1-\tau)^{m_\tau}, (-1-\theta)^{m_\theta}\}.$$

We can use the parameters of X to compute θ and τ in terms of a and c. Let $A_1 = A$ and consider the (i,j)-entry of A^l. By induction we see that this is equal to the number of walks in X from the ith vertex to the jth vertex of length l. If x and y are vertices of X, then the number of walks of length 2 from x to y can be computed as follows. If $x=y$, then every edge incident to x gives a walk of length 2. If x and y are adjacent, then they share a common neighbours, and thus there are a walks of length 2 from x to y. Similarly, if x and y are non-adjacent then there are c walks of length 2 from x to y. Therefore

$$A^2 = kI + aA + c(J-I-A),$$

or

$$A^2 - (a-c)A - (k-c)I = cJ.$$

Let z be a σ-eigenvector of A. If $\sigma = k$, then z is a multiple of $\mathbf{1}$; otherwise, z is orthogonal to $\mathbf{1}$. Assume that $\sigma \neq k$, so $Jz = 0$. Now $A^2z = \sigma^2 z$, and

$$0 = cJz = A^2z - (a-c)Az - (k-c)Iz = \sigma^2 z - (a-c)\sigma z - (k-c)z.$$

Since z is not the zero vector, we have that σ is a root of the quadratic

$$x^2 - (a-c)x - (k-c).$$

Therefore

$$\sigma = \frac{(a-c) \pm \sqrt{(a-c)^2 + 4(k-c)}}{2}.$$

We have assumed that X is primitive, and as a result, $c < k$. Thus the two values of σ are both non-zero and have opposite signs. So we have $k > \theta > 0 > \tau$, and θ, τ can be written as a function of the parameters k, a, c.

We can take this one step further and derive the multiplicities m_θ and m_τ from $(n,k;a,c)$. We have that the multiplicities of the eigenvalues of X sum to n. We also have that the trace of A_1 is the sum of its eigenvalues. Since A_1 has zero diagonal, its eigenvalues sum to zero. Thus we have the system of equations:

$$\begin{aligned}n &= 1 + m_\theta + m_\tau\\ 0 &= k(1) + \theta(m_\theta) + \tau(m_\tau).\end{aligned}$$

Solving this system we find that

$$m_\theta = -\frac{(n-1)\tau + k}{\theta - \tau}, \quad \text{and} \quad m_\tau = \frac{(n-1)\theta + k}{\theta - \tau}.$$

Therefore we can derive m_θ and m_τ from the parameters n, k, a, c.

Given an $(n, k; a, c)$ strongly regular graph X, we can derive all of the constants we are interested in without considering the structure of X. Unfortunately X is not determined by its parameters in general, so we cannot draw any immediate conclusions about the cocliques or other structure of X from its parameters. The preceding material does show that parameters of strongly regular graphs are easily derived, which is not the case for distance-regular graphs in general. Also, implicit in our presentation are a number of feasibility conditions on the parameter sets $(n, k; a, c)$. For example m_θ and m_τ must be integers. These conditions (and others) can be applied to give a list of feasible parameter sets of strongly regular graphs.

Andries Brouwer's website [2] contains a list of feasible parameter sets, together with the graphs that achieve those parameter sets where they are known. Not all strongly regular graphs are determined by their parameter sets. As we will see later, there are two non-isomorphic $(16, 6; 2, 2)$ strongly regular graphs. The strongly regular graphs on $n < 16$ vertices are all unique [2]. We conclude this section by revisiting the Petersen graph (Example 2.4.1).

6.1.1 Example. The Petersen graph P is the Kneser graph $K_{5:2}$. Its vertices are the 2-subsets of [5], and two subsets are adjacent if and only if they are disjoint. So P is a 3-regular graph on 10 vertices. If x and y are adjacent vertices of P, then $|x \cup y| = 4$, and thus x and y have 0 common neighbours. If x and y are not adjacent, then $|x \cap y| = 1$, and $|x \cup y| = 3$. Thus there is a unique vertex z so that $x \cap z = y \cap z = \emptyset$, and x and y have 1 common neighbour. This also shows that P is connected. Therefore P is a $(10, 3; 0, 1)$ strongly regular graph.

Using the formulae above, we see that the spectrum of P is $\{(3)^1, (1)^5, (-2)^4\}$. From the spectrum of P we see that $\alpha(P) \leq 4$ by the inertia bound, and $\alpha(P) \leq 4$ by the ratio bound. As we have already seen, the Erdős-Ko-Rado Theorem characterizes the maximum cocliques of P. They are the families of 2-subsets that contain a common element (e.g., $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}$). So we have that the Petersen graph is ratio tight, and inertia tight using its unweighted adjacency matrix.

Let P_τ be the (-2)-eigenpolytope of P, and let P_θ be the 1-eigenpolytope of P. For a polytope of dimension m, let f_i denote the number of faces of dimension i, and define the *f-vector* of the polytope to be the vector $(f_0, \ldots, f_m)$. The f-vectors of P_τ and P_θ are

$$\begin{aligned} f(P_\tau) &= (1, 10, 30, 30, 10, 1), \\ f(P_\theta) &= (1, 10, 45, 90, 75, 22, 1). \end{aligned}$$

For P_τ: the faces of dimension 1 are the vertices of P; the faces of dimension 2 are the non-edges of P; and, the faces of dimension 3 are all of the cocliques of size 3. The facets of P_τ come in parallel pairs. They are the maximum cocliques of P and their complements. Each coclique gives an equitable partition of P into two parts.

For P_θ, we consider the 22 facets. There are 12 facets of P_θ with 5 vertices. These are the 6 equitable partitions of P into disjoint 5-cycles. The remaining 10 facets are the second neighbourhoods of the vertices of P. So they are paired with the vertices of P and result from the distance partitions of P.

Note that the characteristic vectors of the cocliques of size 4 in the Petersen graph give a basis for the τ-eigenspace. Specifically, if S is a coclique of size 4, and v_S is the characteristic vector of S, then $z_S = v_S - (3/5)\mathbf{1}$ is a τ-eigenvector of P. Let U be the matrix formed by taking the vectors z_S as columns. Then the convex hull of the rows of U is a polytope that is combinatorially equivalent to the τ-eigenpolytope of P. Moreover, the null space of U_{ver} has dimension at least 5 as each of the canonical cocliques gives a prismoid direction of the resulting polytope. Computing the Veronesian rank of U_{ver}, we have from Corollary 5.8.1 that $\mathrm{rk}(U_{ver}) = 10$, and U_{ver} has 15 columns. Therefore the vectors corresponding to the prismoid directions of the τ-eigenpolytope of P span the null space of U_{ver}.

6.2 Strongly Regular Partitions

We have been interested in equitable partitions of graphs. In Section 5.5 we showed that if X is regular, and P is an equitable partition of $V(X)$ into two parts, then P corresponds to a prismoid direction of an eigenpolytope of X. Higman and Haemers [21] looked at equitable partitions of strongly regular graphs with additional structure. They were interested in partitions of a strongly regular graph X into parts P_1, P_2 so that the subgraphs $X[P_1]$ and $X[P_2]$ are also strongly regular. In this section we give a summary of some of the results from [21].

Let X_0 be a strongly regular graph. Let (P_1, P_2) be a partition of $V(X_0)$, and let the subgraphs induced by this partition be $X_1 = X_0[P_1]$ and $X_2 = X_0[P_2]$. If X_1 and X_2 are both regular graphs, the partition (P_1, P_2) is equitable. If X_1 and X_2 are both one of a strongly regular graph, a clique, or a coclique, then we call this partition *strongly regular*.

We let X_i be a graph on n_i vertices. If X_i is regular, the valency of X_i is k_i. If X_i is strongly regular then the parameters of X_i are $(n_i, k_i; a_i, c_i)$. If X_i is strongly regular, a clique, or a coclique, then the spectrum of X_i is $\{k_i^1, \theta_i^{m_{\theta_i}}, \tau_i^{m_{\tau_i}}\}$ (note that k_i, θ_i and τ_i are not necessarily distinct).

Note that we can order the vertices of X_0 so that

$$A(X_0) = A_0 = \begin{pmatrix} A_1 & C \\ C^T & A_2 \end{pmatrix},$$

where $A_1 = A(X_1)$, $A_2 = A(X_2)$ and C is the incidence matrix given by the edges of X_0 between P_1 and P_2. Higman and Haemers apply a basic interlacing argument to show the following theorem.

6.2.1 Theorem (Theorem 2.2 in [21]). *Suppose X_0 is strongly regular, and X_1 is regular. Then*
$$\tau_0 \le \frac{k_1 n_0 - k_0 n_1}{n_0 - n_1} \le \theta_0$$
Moreover the partition is equitable if and only if equality holds in one of the inequalities. □

Note that the ratio bound for strongly regular graphs is an immediate consequence of Theorem 6.2.1. If X_1 is a coclique, then it is a 0-regular graph. Thus we have
$$\tau_0 \le \frac{-k_0 n_1}{n_0 - n_1} \le \theta_0$$
and so
$$n_1 \le \frac{n_0}{1 - k_0/\tau_0}.$$
The implication of Theorem 6.2.1 for graphs that are ratio tight is that the partition is equitable with $k_2 = k_1 - k_1 + \tau_0$. From this fact it is easy to show that if v_1 is the characteristic vector of P_1, then $v_1 - n_1/n_0\mathbf{1}$ is a τ_0-eigenvector for X_0.

We can derive the inertia bound for strongly regular graphs as well.

6.2.2 Theorem (Theorem 2.3 in [21]). *If X_0 is a strongly regular graph and X_1 is a coclique, then $n_1 \le \min\{m_{\theta_0}, m_{\tau_0}\}$.* □

Theorem 6.2.2 is proved by considering the matrix
$$A = A_0 - (k_0 - \tau_0)/n_0 J - \tau_0 I.$$
The matrix A has rank m_{τ_0} and the A_1 block of A_0 becomes a non-singular submatrix of A. So we have $n_1 \le m_{\tau_0}$. An analogous construction for θ_0 completes the proof.

Haemers and Higman go on to show that if X_0 and X_1 are strongly regular, and the partition is regular, then the spectrum of X_2 is determined. As an easy corollary, this gives five simple conditions on the eigenvalues of X_0 and X_1 that determine whether the partition is strongly regular. Both of these results are lengthy to state, and we will not need to appeal them directly, so we omit them (they are Theorem 2.4 and Corollary 2.5 in [21] for the interested reader). These results do admit a very nice corollary in the case where X_1 is a coclique.

6.2.3 Theorem (Theorem 2.6 in [21]). *Let X_0 be a strongly regular graph, and X_1 be a coclique. Then*
$$n_1 = m_{\tau_0} = \frac{n_0}{1 - k_0/\tau_0}$$
if and only if X_2 is strongly regular. □

So in order for the partition to be strongly regular and X_1 to be a coclique, X_1 must be a coclique in X_0 that meets both the ratio bound and the inertia bound (with the unweighted adjacency matrix of X_0). Theorem 6.2.3 allows us to eliminate feasible parameter sets for strongly regular graphs that cannot have a strongly regular partition where one part is a coclique.

For example, we saw in Example 6.1.1 that the Petersen graph P meets both the ratio bound, and the inertia bound. Therefore the equitable partitions of P given by the maximum cocliques are strongly regular partitions. In this case the strongly regular graph X_2 is isomorphic to $3K_2$, and so is not primitive.

The Petersen graph is the Kneser graph $K_{5:2}$. As we have seen, the Kneser graph $K_{n:k}$ is a distance-regular graph. When $k = 2$, the diameter of $K_{n:2}$ is 2, and $K_{n:2}$ is a strongly regular graph. The parameters of $K_{n:2}$ are

$$\left(\binom{n}{2}, \binom{n-2}{2}; \binom{n-4}{2}, \binom{n-3}{2}\right),$$

and the spectrum of $K_{n:2}$ is

$$\left\{\binom{n-2}{2}^1, (1)^{n(n-3)/2}, (3-n)^{n-1}\right\}.$$

From the Erdős-Ko-Rado Theorem we know that the maximum cocliques in $K_{n:2}$ are the sets $\{X \in V(K_{n:2}) : i \in X\}$ for all $1 \leq i \leq n$. So $\alpha(K_{n:2}) = n-1$. We see immediately that $n - 1 = m_\tau$, so $K_{n:2}$ is inertia tight. Also,

$$\frac{\binom{n}{2}}{1 - \binom{n-2}{2}/(3-n)} = n - 1,$$

so $K_{n:2}$ is also ratio tight. Thus Theorem 6.2.3 implies that if S is a maximum coclique in $K_{n:2}$, then the partition $(S, \overline{S})$ is strongly regular, and $\overline{S}$ induces a strongly regular subgraph of $K_{n:2}$. Working out the parameters we find that $\overline{S}$ induces a copy of $K_{n-1:2}$ in $K_{n:2}$.

We saw in Section 6.1 that the complement of a strongly regular graph is a strongly regular graph. If (A, B) is an equitable partition of a graph X, then (A, B) is also an equitable partition of $\overline{X}$. Thus the strongly regular partition $(S, \overline{S})$ of $K_{n:2}$ is also a strongly regular partition of $\overline{K_{n:2}}$ into a clique of size $n - 1$, and a subgraph isomorphic to $\overline{K_{n-1:2}}$. The complement of the Kneser graph $K_{n:2}$ is the *Johnson graph* $J_{n:2}$. This strongly regular partition of the Johnson graph $J_{n:2}$ has an additional interesting property, it is an equitable partition of $J_{n:2}$ into convex subgraphs. We will look at convex partitions in the next two sections.

The graphs $K_{n:2}$ give an infinite family of strongly regular graphs with strongly regular partitions. We end this section with an example of an infinite family of graphs with no equitable partitions into two parts. In Example 2.12.1 we defined the Paley graphs as family of Cayley graphs for $GF(q)$ where q a prime power congruent to 1 modulo 4. Recall that two vertices of $P(q)$ are adjacent if and only if their difference is a non-zero square in $GF(q)$. The graph $P(q)$ is strongly regular with parameters

$$\left(q, \frac{q-1}{2}; \frac{q-5}{4}, \frac{q-1}{4}\right),$$

and spectrum

$$\left\{\left(\frac{q-1}{2}\right)^1, \left(\frac{-1+\sqrt{q}}{2}\right)^{m_\theta}, \left(\frac{-1-\sqrt{q}}{2}\right)^{m_\tau}\right\}$$

where $m_\theta = m_\tau = (q-1)/2$. Consider the case where $q = p^1$ for some prime p. We have that θ and τ are both irrational numbers, so $P(p)$ cannot be ratio tight. As a result, Theorem 6.2.3 implies that there is no equitable partition of $P(p)$ into a coclique and a strongly regular graph. In fact, we can rule out all equitable partitions of $P(p)$ with two cells.

Let (A, B) be a partition of the vertices of $P(p)$. If (A, B) is equitable, then there are constants a, b, c, d so that for each $x \in A$, there are a neighbours of x in A, and c neighbours of x in B, and for each $x \in B$ there are d neighbours of x in A and b neighbours of x in B. We count the number of edges $E(A, B)$ joining A to B in $P(p)$. We have that

$$|E(A, B)| = c|A| = d|B|.$$

We also have that $|A| + |B| = p$, and that

$$a + c = b + d = (p-1)/2.$$

Using the first two equations, we calculate the size of A as

$$|A| = p/(c+d).$$

Since $|A| \in \mathbb{Z}$, and p prime, we conclude that $c + d$ is either p or 1. Since $P(p)$ is connected, $c, d \geq 1$, and so $c + d = p$. Finally, we have that

$$a + c + b + d = p - 1,$$

which implies that $a + b = -1$, a contradiction. Therefore the Paley graph $P(p)$ has no equitable partitions with two parts. In fact, this is a corollary of a more general lemma.

6.2.4 Lemma. *Let X be a k-regular graph on p vertices for p prime. If $k < p/2$, then X has no equitable partitions with two cells.*

Proof. As in the preceding paragraph, we let (A, B) be an equitable partition with constants a, b, c, d. We can show that $c + d = p$, which contradicts the fact that

$$a + c + b + d = 2k < p. \qquad \square$$

6.3 Convex Subgraphs

Let X be a connected graph, and Y be a subgraph of X. Since Y is a subgraph, for any $x, y \in Y$, we have $d_Y(x,y) \geq d_X(x,y)$. We call Y a *geodetic* subgraph if for all $x, y \in Y$, $d_Y(x,y) = d_X(x,y)$. For $x, y \in X$, we define

$$C_X(x,y) = \{z \in X \,:\, d_X(x,z) + d_X(z,y) = d_X(x,y)\}.$$

Clearly, $C_X(x,y)$ is the union of all of the shortest x,y-paths in X. If $C_X(x,y) \subseteq Y$ for all $x, y \in Y$ with $d_X(x,y) = t$, then Y is *t-convex*. We call Y a *convex* subgraph of X if Y is t-convex for all $t \leq \operatorname{diam}(X)$.

In his thesis [24], Lambeck looked at characterizing the convex subgraphs of families of distance-regular graphs. One of the families he considered is the Johnson graphs. The *Johnson graph* $J_{n:k}$ is the graph on the k-subsets of $[n]$ where two k-subsets are adjacent if and only if they intersect in $k-1$ elements. The Johnson graph $J_{n:k}$ is the maximum distance graph of the Kneser graph $K_{n:k}$. If $\mathcal{A}$ is the Bose-Mesner algebra of the association scheme generated by the distance relations on $K_{n:k}$, then $A_d = A(J_{n:k})$. This scheme is called the *Johnson scheme*.

Lambeck proved that the convex subgraphs of $J_{n:k}$ are the graphs induced by the following "intervals." Let $A \subseteq B \subseteq [n]$, and set

$$I(A,B) = \{S \in V(J_{n:k}) \,:\, A \subseteq S \subseteq B\}.$$

6.3.1 Theorem (Proposition 5.7 from [24])**.** *The non-complete convex subgraphs of $J_{n:k}$ are exactly the subgraphs induced by the sets $I(A,B)$ where $A \subseteq B \subseteq [n]$.* □

Note that the subgraph of $J_{n:k}$ induced by $I(A,B)$ is isomorphic to a Johnson graph,

$$J_{n:k}[I(A,B)] \cong J_{|B|-|A|:k-|A|}.$$

Recall that in Section 4.8 we mentioned in passing that the Erdős-Ko-Rado Theorem can be proved by analysing the cocliques of the Kneser graph $K_{n:k}$. The proof follows the format of the proof of the Erdős-Ko-Rado Theorem for S_n given in Section 5.3. We define the canonical intersecting families of k-subsets of $[n]$ to be the sets S_i of all k-subsets containing $1 \leq i \leq n$. The Theorem states that the sets S_i are exactly the maximum cocliques of $K_{n:k}$. Since $K_{n:k}$ is a ratio tight graph, the characteristic vectors of the sets S_i give τ-eigenvectors as in Theorem 5.2.1.

If we let U be the matrix whose columns are the characteristic vectors of the families S_i, then we can show that the column space of U is the τ-eigenspace of $K_{n:k}$. Let P be the polytope defined as the convex hull of the rows of U. From Godsil and Meagher [16] we have that the vertex sets of the faces of P are exactly the sets $I(A,B)$. Each of the partitions $(S_i, \overline{S_i})$ of $V(K_{n:k})$ gives a partition of the vertices of P into parallel faces. These partitions are also equitable partitions of $J_{n:k}$ into two convex subgraphs. However, the equitable partition of $K_{n:k}$ is not into convex subgraphs, as one of the graphs is a coclique, and the other subgraph is not convex.

Recall the τ-eigenpolytope of the Petersen graph from Example 6.1.1. The Petersen graph is the Kneser graph $K_{5:2}$. The sets $I(A,B)$ give all of the faces of the τ-eigenpolytope of $K_{5:2}$. When $A = B = \emptyset$, we have $I(A,B) = \{\emptyset\}$, the face of dimension -1. When $A = B = \{i,j\}$, we have $I(A,B) = \{\{i,j\}\}$ giving the vertices. When $A = \{i\}$, and $B = \{i,j,k\}$,

$$I(A,B) = \{\{i,j\},\{i,k\}\}$$

giving all of the cocliques of size 2 as the edges of the polytope. When $A = \emptyset$ and $B = \{i,j,k\}$,

$$I(A,B) = \{\{i,j\},\{i,k\},\{j,k\}\},$$

and when $A = \{i\}$ and $B = \{i,j,k,l\}$,

$$I(A,B) = \{\{i,j\},\{i,k\},\{i,l\}\}.$$

These are all of the cocliques of size 3, and the vertex sets of the faces of dimension 3. When $A = \{i\}$ and $B = [5]$, then $I(A,B)$ is the ith canonical coclique, and gives a facet with 4 vertices. When $A = \emptyset$ and $B = \{i,j,k,l\}$, $I(A,B)$ is the set of all of the 2-subsets of B. This is a set of size 6 and gives the vertex set of a facet parallel to the facet with 4 vertices formed by the canonical coclique with $A = [5] \setminus B$. The last face is the entire polytope, given by $A = \emptyset$ and $B = [5]$.

Finally, we note that the polytope P is not an eigenpolytope of $K_{n:k}$ as we defined them in Chapter 4. However, P is combinatorially equivalent to the τ-eigenpolytope of $K_{n:k}$. To see this, note that P is the convex hull of a matrix U whose columns are a basis for the τ-eigenspace of $K_{n:k}$. Therefore there is a matrix M so that $U' = UM$ and the columns of U' form an orthonormal basis for the τ-eigenspace. Since M is an affine mapping from $\mathbb{R}^{m_\tau}$ to itself, M maps P to the polytope P' defined as the convex hull of the rows of U', and P' is combinatorially equivalent to P (see Grünbaum [20, p. 38]). Therefore the polytope P has the same face lattice as the τ-eigenpolytope of $K_{n:k}$, but the angles between the vertices are not the same, and we cannot apply the parameters of the Johnson scheme directly to derive geometrical information about P.

The Johnson graphs give a family of examples of graphs with partitions into two convex subgraphs. In this case the partitions of X are exactly the partitions in the d-distance graph X_d of X given by maximum cocliques. For strongly regular graphs X, the diameter of X is 2, and the maximum distance graph is $\overline{X}$, which is strongly regular. So we would hope to find examples of families of strongly regular graphs that can be partitioned into convex subgraphs.

6.4 Convex Partitions

For a graph X, we define a *convex partition* of X to be an equitable partition (A,B) where both A and B induce either a clique or a convex subgraph of X.

Let X be a $(n, k; a, c)$ strongly regular graph. If (A, B) is a convex partition, and B is not a clique, then B has diameter 2. From the definition of convexity, we see that if $x, y \in B$ are at distance 2 in X, then B contains $C(x, y)$. Since X is strongly regular, x, y have c common neighbours, and each of these neighbours must be in B. Thus if B is a $(n_B, k_B; a_B, c_B)$ strongly regular graph, then $c_B = c$.

However, B convex does not imply B strongly regular, as if $x, y \in B$ are adjacent vertices of X, then the convexity assumption does not constrain the vertices in the neighbourhoods of both x and y. The only exception is if X is a triangle-free strongly regular graph. In this case $a = 0$, and adjacent vertices x and y in X (and thus in B) have no common neighbours. Therefore if X is a triangle-free strongly regular graph, and X_1 is a strongly regular subgraph of X, then X_1 is convex if and only if $c_1 = c$.

In Section 6.1 we saw that the Johnson graphs $J_{n:2}$ are strongly regular graphs with strongly regular convex partitions. In the remainder of this section, we introduce another family of strongly regular graphs, and show that they have no convex partitions.

A *Latin square of order* n is an $n \times n$ array with entries in $[n]$ so that every row and column contains each $1 \leq i \leq n$. For example, if G is a group of order n, we can order the elements of G as $\{g_1, \ldots, g_n\}$. Now define the $n \times n$ array A by letting the (i, j)-entry of A be k, where $g_i \circ g_j = g_k$. Since G is a group, $g_k \neq g_i \circ g_l$ for any $l \neq j$. Therefore A is a Latin square of order n. So Latin squares exist for each positive integer.

We can define a graph on the cells of a Latin square by specifying an adjacency relation. If A is a Latin square, define the *Latin square graph* $X(A)$ to be the graph on the triples (a, b, c) where $1 \leq a, b \leq n$, and c is the (a, b)-entry of A. Two triples (a, b, c) and (a', b', c') are adjacent in $X(A)$ if and only if either $a = a'$, $b = b'$, or $c = c'$ (but not all three). Note that if $a = a'$, then equality in either of the remaining coordinates implies equality in all three. We have that $X(A)$ has n^2 vertices, and degree $3(n-1)$.

The Latin square A can be visualized as an orthogonal array. Let OA be a $k \times n^2$ array with entries in Ω for $|\Omega| = n$. The array OA is an *orthogonal array* if the columns of the $2 \times n^2$ subarray given by any two rows contain all of the possible ordered pairs of elements of Ω. So by writing the triples (a, b, c) from A as the columns of a $3 \times n^2$ array, we have an orthogonal array. Note that if OA is an orthogonal array, then we can take three rows of OA in any order, and the triples (a, b, c) corresponding to the columns of the resulting subarray define a Latin square. In particular we see that permuting the entries of the triples (a, b, c) of a Latin square result in another Latin square.

Let A be a Latin square and $X(A)$ the graph as defined above. Note that permuting the entries of A results in a Latin square A' with $X(A) = X(A')$. Now if (a, b, c) is adjacent to (a', b', c') in $X(A)$ we can say without loss of generality that $a = a'$. Consider the common neighbours of (a, b, c) and (a, b', c'). Both vertices are adjacent to all triples (a, x, y), the vertex (x, b', c) and the vertex (x, b, c'). Thus the intersection of the neighbourhoods of (a, b, c) and (a, b', c') has size n. Suppose (a, b, c) and (a', b', c') are non-adjacent vertices of $X(A)$, so

$a \neq a'$, $b \neq b'$ and $c \neq c'$. Now if (x, y, z) is a common neighbour of (a, b, c) and (a', b', c') then (x, y, z) shares a coordinate of each vertex, and these coordinates are distinct. There are 3 ways to choose two coordinates of (x, y, z) and 2 ways to choose which coordinate to match with (a, b, c), therefore there are 6 possibilities. Thus (a, b, c) and (a', b', c') share 6 neighbours. Since we made no assumptions about the vertices of $X(A)$, we have that these values are constant for all pairs of vertices in $X(A)$. Therefore $X(A)$ is a strongly regular graph.

Latin square graphs are a well-studied family of strongly regular graphs. The graph $X(A)$ has parameters

$$(n^2, 3(n-1); n, 6),$$

and spectrum

$$\{3(n-1)^1, (n-3)^{3(n-1)}, (-3)^{(n-1)(n-2)}\}.$$

As we have seen the multiplication table of any group G gives a Latin square graph. For $n = 4$, the cyclic group $\mathbb{Z}_4$, and the Klein 4-group give rise to non-isomorphic Latin square graphs with the same parameters (see Section 10.4 in Godsil and Royle [18]).

We can use the spectrum of $X(A)$ to bound the size of a maximum coclique in the usual ways. The ratio bound gives

$$\alpha(X(A)) \leq \frac{n^2}{1 - 3(n-1)/(-3)} = n,$$

and the inertia bound gives

$$\alpha(X(A)) \leq \min\{3(n-1), (n-1)(n-2)\}.$$

Since

$$n < \min\{3(n-1), (n-1)(n-2)\},$$

for $n \geq 4$, we see that the inertia bound cannot be tight for most Latin square graphs.

A coclique in $X(A)$ is a set of triples (a, b, c) so that no two share a common entry. This corresponds to a set of cells of A none of which are in the same row or column, and none of which share the same entry. These are called *partial transversals* of A (or *transversals* if they have size n). The graph $X(A)$ is ratio tight if and only if A has a transversal. So if A has a transversal, then $X(A)$ has a coclique S of size n and the partition $(S, \overline{S})$ is equitable. Since $X(A)$ is not inertia tight, Theorem 6.2.3 implies that $X(A)$ has no equitable partition into a coclique and a strongly regular graph.

Latin square graphs give an example of a family of graphs with no convex partitions.

6.4.1 Lemma. *If $X(A)$ is a Latin square graph of order $n \geq 3$, then $X(A)$ has no convex partitions.*

Proof. Let X be a Latin square graph, and suppose that (X_1, X_2) is an equitable partition of X. To show that this partition is not convex, we proceed by contradiction. We begin by assuming that X_1 is a clique.

The vertices of X are triples (a, b, c). Suppose that the vertices of X_1 do not all share a common coordinate. Then $|X_1| \leq 3$, and the partition (X_1, X_2) is not equitable. So, without loss of generality, we can assume that the vertices of X_1 are all of the form (a, x, y). Let $(a, b, c) \in X_1$, and consider the vertices (a', b, c') and (a'', b', c) where a, a', a'' distinct. Since $a', a'' \neq a$, these are vertices in X_2. We can take $b \neq b'$ and $c \neq c'$, so we have that (a', b, c') is adjacent to (a, b, c) and (a, b, c) is adjacent to (a'', b', c). But (a', b, c') is not adjacent to (a'', b', c). This contradicts the assumption that X_2 is convex.

Finally we assume that X_1 and X_2 are both convex subgraphs with diameter 2. Suppose that X_1 and X_2 induce a partition of the rows of the Latin square (i.e., there is a partition (S, T) of $[n]$ so that $(a, b, c) \in X_1$ if and only if $a \in S$ and $(a, b, c) \in X_2$ if and only if $a \in T$). Since neither of X_1 and X_2 is a clique, we have $|S|, |T| \geq 2$. Let $(a, b, c), (a', b', c') \in X_1$ be vertices at distance 2, so $a \neq a'$, $b \neq b'$ and $c \neq c'$. Consider the vertices in X_2 of the form (x, b, y). If $(x, b, c') \in X_2$, then X_1 is not convex. Thus none of the elements of X_2 have the form (x, b, c') (note that this implies $|S|, |T| \geq 3$). Following the same reasoning we have that there are no vertices in X_2 with the form (x, b', c). Now consider a vertex $(x, y, c) \in X_2$. So (a, b, c) is adjacent to (x, y, c). Since $|S| \geq 3$ we have some $(x', y, c') \in X_1$ with $x' \neq a$, and again we conclude that X_1 is not convex. Therefore the partition (X_1, X_2) also induces a partition of the third coordinates of the vertices. But this contradicts the existence of S, T.

Thus for some $1 \leq a \leq n$, the set of vertices of the form (a, x, y) contains elements in both X_1 and X_2. Consider $(a, b, c) \in X_1$ and $(a, b', c') \in X_2$. Now for $x \neq a$, the vertex (x, b', c) is either in X_1 or X_2. Suppose $(x, b', c) \in X_2$. Then if $(x', b', y) \in X_1$ for any other triple with 2nd component b', we have that (a, b, c) is adjacent to (x, b', c) which is adjacent to (x', b', y). But $x' \neq a$ and $y \neq c$, so X_1 is not convex. Thus either every element of the form (x', b', y) is in X_2, or $(x, b', c) \in X_1$. First suppose that every element of the form (x', b', y) is in X_2. Then X_2 contains all of the elements with 2nd component b'. This implies that X_1 is not convex. So we must have that $(x, b', c) \in X_1$. Therefore X_1 contains all of the vertices of the form (x, y, c) where $(a, y, z) \in X_2$. Thus the partition (X_1, X_2) induces a partition of the triples by their 3rd component. The first part our argument shows (by the fact that we can permute the coordinates of the triples without altering X) that this leads to a contradiction. This completes the proof. □

6.5 Computations on Strongly Regular Graphs

The problem of looking for equitable partitions of graphs computationally is difficult as the number of possible partitions of the vertices is large. We saw in Section 6.2 that Higman and Haemers [21] were able to find conditions of the spectra of a strongly regular graph that determine whether or not they have strongly regular partitions. We saw that the Kneser graphs $K_{n:2}$ are both inertia tight and ratio tight, and so they have equitable partitions into a maximum coclique and a strongly regular graph. In [21] the authors give a table, excluding the graphs $K_{n:2}$, of feasible parameter sets of strongly regular graphs up to 300 vertices that have strongly regular partitions. Interestingly there are only two feasible parameters sets that admit a partition into a strongly regular graph and a coclique. These are $(126, 60; 33, 24)$ and $(261, 84; 39, 21)$, and there are no strongly regular graphs with those parameters. So we see that there are no strongly regular graphs on at most 300 vertices that are both ratio and inertia tight.

We used Sage [30] to compute the ratio bound and inertia bound values for a large set of graphs, along with the size of their maximum cocliques. Our data set included some interesting strongly regular graphs on a large number of vertices (such as the Higman-Sims graph), but was composed mostly of the 4391 strongly regular graphs compiled by Ted Spence [33]. These graphs are strongly regular graphs on at most 64 vertices (but the list does not contain *all* strongly regular graphs on at most 64 vertices). Of these graphs only 578 did not meet the ratio bound, and so do not have equitable partitions of the form $(S, \overline{S})$ where S is a maximum coclique.

More interestingly, there were a very small number of inertia tight graphs. The Kneser graphs $K_{n:2}$ are inertia tight, as we have already seen. Apart from these graphs we have the Clebsch graph, the Schläfli graph, the Higman-Sims graph, the McLaughlin graph, and the graph induced by the vertices of the McLaughlin graph at distance two from a given vertex. We will return to the Schläfli graph and the McLaughlin graph in Section 6.7.

The Clebsch graph has parameters $(16, 5; 0, 2)$. We have already defined this graph as the folded 5-cube, and seen that it is inertia tight. The Clebsch graph is an example of a triangle-free strongly regular graph (i.e., a strongly regular graph with $a = 0$). The triangle-free strongly regular graphs that meet the inertia bound with their unweighted adjacency matrices are C_5, the Petersen graph ($K_{5:2}$), the Clebsch graph and the Higman-Sims graph. The Higman-Sims graph is the unique strongly regular graph with parameters $(100, 22; 0, 6)$. For C_5, the Clebsch graph and the Higman-Sims graph, the maximum cocliques are exactly the vertex neighbourhoods. The remaining triangle-free strongly regular graphs are: the Hoffman-Singleton graph, the Gewirtz graph and the M_{22} graph [5]. None of these meet the inertia bound with their unweighted adjacency matrix.

We were able to do some computations on the eigenpolytopes of these graphs. However our results are very limited by the computing power available. We were able to work with polytopes with at most 28 vertices, and of dimension at most 10. This limited us to eigenpolytopes of graphs on at most 28 vertices, and for eigenvalues with multiplicity at most 10. We give some interesting findings on these polytopes in the final two sections of this chapter.

6.6 Strongly Regular Graphs on 16 Vertices

There are 2 feasible parameter sets for strongly regular graphs on 16 vertices. These are $(16,5;0,2)$ and $(16,6;2,2)$. Both are realized by strongly regular graphs. The first parameter set is the, now familiar, Clebsch graph. The other parameter set is the parameter set for the complement of a Latin square graph on 16 vertices. There are two non-isomorphic graphs that realize the second parameter set, the Shrikhande graph (the complement of the Latin square graph obtained from the multiplication table of the group $\mathbb{Z}_4$), and the line graph of $K_{4,4}$ (the complement of the Latin square graph obtained from the multiplication table of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$).

Ignoring the valency, we look at the θ and τ eigenpolytopes for both of these graphs. For the Shrikhande graph, the f-vectors of the τ and θ eigenpolytopes are

$$\begin{aligned} f(P_\tau) &= (1,16,120,528,1440,2464,2608,1622,524,64,1), \\ f(P_\theta) &= (1,16,96,236,272,144,28,1). \end{aligned}$$

For $L(K_{4,4})$ the f-vectors of the τ and θ eigenpolytopes are

$$\begin{aligned} f(P_\tau) &= (1,16,120,528,1392,2176,1968,978,240,24,1), \\ f(P_\theta) &= (1,16,48,68,56,28,8,1). \end{aligned}$$

As we can see simply from the f-vectors, while the parameters of the schemes corresponding to these graphs are the same (and thus the angles between the vertices of the corresponding eigenpolytopes are the same), the combinatorial structure of the polytopes is different. Both of these graphs are ratio-tight, but fail to meet the inertia bound with their ordinary adjacency matrix.

6.7 The Schläfli Graph

The Schläfli graph gives an example of a graph that is inertia tight, but has no equitable partitions into two parts. It is the unique strongly regular graph X with parameters $(27,10;1,5)$ and spectrum $\{10^{(1)}, 1^{(20)}, -5^{(6)}\}$ (note that as with some named strongly regular graphs, some authors refer to X as the complement of the Schläfli graph). The maximum cocliques of X have size 6, so it is inertia tight, however it is not ratio tight. There are 72 distinct maximum cocliques in X.

As we mentioned in Section 6.5, we were only able to work with eigenpolytopes of dimension at most 10. So we were only able to look at the faces of the τ-eigenpolytope of X. The τ-eigenpolytope has f-vector

$$f(P_\tau) = (1,27,216,720,1080,648,99,1).$$

The 99 facets consist of the 72 maximum cocliques, together with the 27 vertex neighbourhoods. Recall that in Section 5.10 we mentioned that we computed the Veronesian rank of the matrices whose columns form orthonormal bases of the eigenspaces of X. If U is a matrix whose columns are an orthonormal basis for the τ-eigenspace of X, then the dimension of the null space of U_{ver} is 1. This implies that the τ-eigenpolytope of X is not a prismoid.

The maximum cocliques of X are the cells of equitable partitions, just not equitable partitions with two cells. The facets of the τ-eigenpolytope corresponding to the maximum cocliques of X form 36 pairs of parallel facets. Each of these pairs gives two parts of an equitable partition (A, B, C) where A and C are maximum cocliques and B is a set of 15 vertices. The graph $X[B]$ induced by the non-coclique part is isomorphic to $K_{6:2}$ which is strongly regular. So we are close to a strongly regular partition of X.

The Schläfli graph shares some properties with the McLaughlin graph. The McLaughlin graph is the unique $(275, 112; 30, 56)$ strongly regular graph and has spectrum $\{112^1, 2^{252}, (-28)^{22}\}$. As we mentioned in Section 6.5, the McLaughlin graph meets the inertia bound with its unweighted adjacency matrix. It is also not ratio tight. If we let U be a matrix whose columns form an orthonormal basis for the τ-eigenspace of the McLaughlin graph, then the null space of U_{ver} has dimension 1. Thus the τ-eigenpolytope of the McLaughlin graph is not a prismoid.

The existence of these graphs argues against the possibility of connecting inertia tightness of distance-regular graphs to prismoid directions in their eigenpolytopes.

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