Extensions of Signed Graphs

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Given a signed graph (G, Σ) with an embedding on a surface S, we are interested in "extending" (G, Σ) by adding edges and splitting vertices, such that the resulting graph has no embedding on S. We show (assuming 3-connectivity for (G, Σ)) that there are a small number of minimal extensions of (G, Σ) with no such embedding, and describe them explicitly. We also give conditions, for several surfaces S, for an embedding of a signed graph on S to extend uniquely. These results find application in characterizing the signed graphs with no odd-K_5 minor.

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Chapter 1

Introduction

A *signed graph* is a pair (G, Σ), where G is a graph, and $\Sigma \subseteq E(G)$. A subset $F \subseteq E(G)$ is said to be *even* (or *odd*) if $|F \cap \Sigma|$ is even (or odd, respectively). We will refer to the "evenness" or "oddness" of a set $F \subseteq E$ as the *parity* of F.

For an unsigned graph H with $e \in E(G)$, $H \setminus e$ denotes the graph obtained from H by deleting e, while H/e denotes the graph obtained from H by contracting e. Let (G, Σ) be a signed graph and $e \in E(G)$. We define the signed graph $(G, \Sigma) \setminus e$ as $(G \setminus e, \Sigma \setminus e)$. We define $(G, \Sigma)/e$ as $(G \setminus e, \emptyset)$ if e is an odd loop of (G, Σ), and as $(G \setminus e, \Sigma)$ if e is an even loop of (G, Σ). If e is not a loop of (G, Σ), we define $(G, \Sigma)/e$ as $(G/e, \Sigma')$, where Σ' is a signature of (G, Σ) that does not contain e. We will say that $(G, \Sigma) \setminus e$ is obtained from (G, Σ) by *deleting* e, and that $(G, \Sigma)/e$ is obtained from (G, Σ) by *contracting* e. If (H, Γ) is a signed graph such that (G, Σ) is obtained from (H, Γ) by contracting or deleting a sequence of edges, then (G, Σ) is a *minor* of (H, Γ), and (H, Γ) is a *major* of (G, Σ).

An *odd-K_n* is the signed graph $(K_n, E(K_n))$, and we will call a signed graph *odd-K_n free* if it has no odd-K_n minor. The goal of this thesis is to take steps toward a structural characterization of signed graphs with no odd-K_5 minor. Many structural results have been proven for classes of graphs defined by excluded minors. For example, graphs with no K_4 minor are series-parallel graphs. Also, Wagner [13] characterized the structure of planar graphs with no K_5 minor. It is natural to try to extend these results to signed graphs.

A signed graph is called *bipartite* if it has no odd cycles. A *blocking vertex* of signed graph (G, Σ) is a vertex $v \in V(G)$ such that deleting v from (G, Σ) renders (G, Σ) bipartite. Similarly, a *blocking pair* of (G, Σ) is a pair of vertices $u, v \in V(G)$ such that deleting u, v

from (G, Σ) renders (G, Σ) bipartite. Gerards [2] gave a structural characterization of odd-K_4 free signed graphs. He proved that any odd-K_4 free signed graph can be constructed by pasting together signed graphs with a blocking vertex, planar signed graphs with two odd faces, and two special graphs in a particular way.

Currently, no such characterization theorem exists for the class of all odd-K_5 free signed graphs (although Conforti and Gerards recently gave a structure theorem for a subclass of odd-K_5 free signed graphs [1]). These signed graphs are significant in the study of multi-commodity flow problems [4, 5].

A general strategy to prove a characterization theorem for odd-K_5 free signed graphs was outlined in [7]. We can define decomposition operations for a signed graph (G, Σ) with the property that (G, Σ) is odd-K_5 free if and only if each of its parts is odd-K_5 free. A signed graph is *irreducible* if it is 3-connected and loopless. We define some classes of odd-K_5 signed graphs to be *basic*; we can think of irreducible basic signed graphs as the building blocks with which we hope to generate reducible odd-K_5 free signed graphs. The following are basic classes of odd-K_5 free signed graphs (G, Σ) [7]:

(B1) (G, Σ) has a blocking pair;

(B2) G is planar.

We can also define some basic classes of odd-K_5 free signed graphs topologically. To do so, we will need to define a few surfaces. The *projective plane* is the surface obtained from a disc by identifying opposite points on the boundary of the disc. The *pinched projective plane* is obtained from the projective plane by identifying two distinct points on the projective plane, to form a *pinch point*. Note that the pinched projective plane is not technically a surface; we will refer to it as a *pinched surface*. The *Klein bottle* is the surface obtained from a rectangle by identifying one pair of opposite sides with the same orientation, and identifying the other pair of opposite sides with a twist.

For a signed graph (G, Σ) with embedding Π in a (possibly pinched) surface S, we will use $\Pi(G')$ to denote the embedded subgraph G' of G. A cycle of (G, Σ) is called Π-*facial* if it is a facial cycle of $\Pi(G)$. We will say that (G, Σ) is Π-*embedded* in S. An *even-face embedding* Π of a signed graph (G, Σ) on a surface (or pinched surface) S is an embedding of (G, Σ) on S where for every Π-facial cycle C of (G, Σ) is even. A signed graph (G, Σ) is *apex with two odd faces* if for some $v \in V(G)$, $(G \setminus v, \Sigma \setminus \delta(v))$ has a planar embedding with exactly two odd faces. The following are topological classes of odd-K_5 free signed graphs:

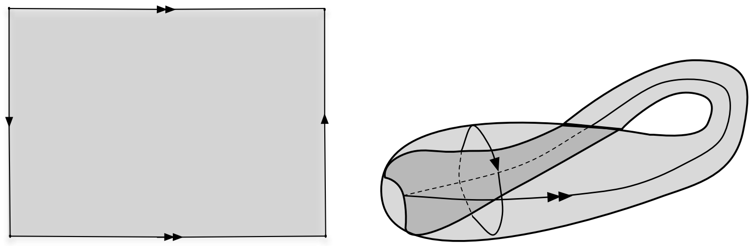


Figure 1.1: The Klein bottle

(B3) (G, Σ) has an even-face embedding on the pinched projective plane;

(B4) (G, Σ) has an even-face embedding on the Klein bottle ;

(B5) (G, Σ) is apex with exactly two odd faces.

We wish to prove that every irreducible odd-K_5 free signed graph either is in a basic class, or belongs to a highly structured class of signed graphs that we can fully describe. We will explain our strategy, based on the method outlined in [7]. A set of signed graphs $\mathcal{U}$ is *unavoidable* if every odd-K_5 free signed graph that is irreducible but not basic has a minor in $\mathcal{U}$. A general proof strategy is to find an unavoidable set $\mathcal{U}$ and then for each $(G, \Sigma) \in \mathcal{U}$ prove that the irreducible, non-basic, odd-K_5 free signed graphs with a minor (G, Σ) can be fully described. The success of such a strategy hinges on our ability to find such a set $\mathcal{U}$ where none of the signed graphs are in a basic class.

We will describe how to find such an unavoidable set $\mathcal{U}$. A 3-connected signed graph (G, Σ) is *minimally blocking pair free* if (G, Σ) has no blocking pair, but every 3-connected minor of (G, Σ) has a blocking pair. We have the following conjecture from [7]:

Conjecture. *There exist finitely many minimally blocking pair free signed graphs.*

Then if the conjecture holds, the set of minimally blocking pair free signed graphs is an unavoidable set $\mathcal{U}_1$ of irreducible signed graphs with no blocking pair. We then construct an unavoidable set $\mathcal{U}_2$ where every signed graph in $\mathcal{U}_2$ is irreducible, blocking pair free and non-planar as follows. For each signed graph (G, Σ) in $\mathcal{U}_1$, we find the set of all loopless, 3-connected, non-planar, odd-K_5 free signed graphs that contain (G, Σ) as a minor. To do this, we need an "escape" theorem for planar signed graphs – a result characterizing the minimal non-planar signed graphs containing a specific planar signed graph (G, Σ) as a minor. We prove such a theorem in Chapter 2.

Suppose we next try to construct an unavoidable set $\mathcal{U}_3$, where every signed graph in $\mathcal{U}_3$ is irreducible, blocking pair free, non-planar, and has no even-face embedding on the Klein bottle. We use a similar strategy as for finding $\mathcal{U}_2$. However, because a 3-connected signed graph may have more than one even-face embedding on the Klein bottle, this case is slightly more complicated. We proceed as follows: For each signed graph $(G, \Sigma) \in \mathcal{U}_2$, we try to find the set of all signed graphs minimally containing (G, Σ) that have no even-face embedding on the Klein bottle. We do this by generating all even-face embeddings of (G, Σ) on the Klein bottle, then (for each embedding) finding all non-equivalent ways to minimally "break" the even-face embedding by adding edges or splitting vertices. For example, in Figure 1.2, we break the embedding of (G, Σ) by adding two crossing edges e and f in a single face. To execute this procedure, we need another escape theorem, one telling us how to minimally break a given even-face embedding of a graph on the Klein bottle. In Chapter 2, we will see that our escape theorem for the planar case also gives us the result we need for embeddings on other surfaces, so long as the embedded graph (G, Σ) has representativity at least 3.

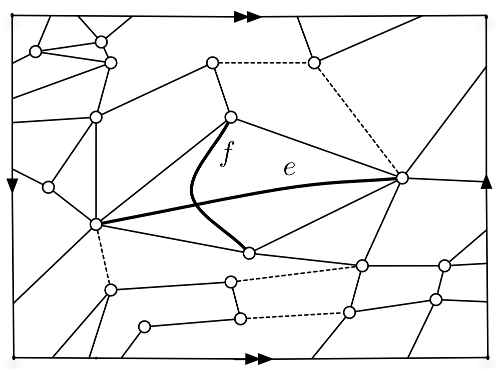


Figure 1.2: Breaking an even-face embedding on the Klein bottle. Odd edges are dotted, even edges are solid, added edges are bold.

There remains, however, a problem that we need to deal with. When we break one even-face embedding of (G, Σ) on the Klein bottle, it is possible that the resulting graph (H, Γ) still has an even-face embedding on the Klein bottle. For example, in Figure 1.2, the edges of $E(H) \setminus E(G)$ are placed such that the embedding is broken. However, in Figure 1.3, we see in that drawing the added edge e in a different face of (G, Σ) gives an even-face embedding of (H, Γ) on the Klein bottle. So we cannot add (H, Γ) to the unavoidable set $\mathcal{U}_3$. Instead, we must add all the minimal odd-K_5 free signed graphs containing (H, Γ) that do not have an even-face embedding on the Klein bottle. Unfortunately, this means

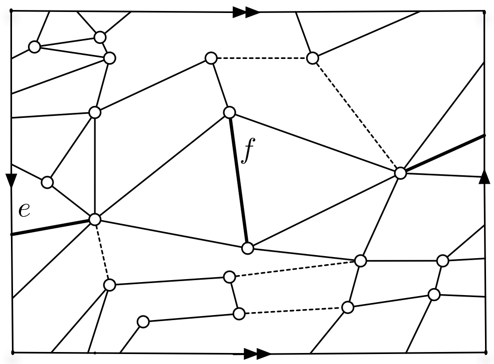


Figure 1.3: Breaking one even-face embedding may not create a graph with no even-face embedding on the Klein bottle

we need to repeat the process we just performed for (G, Σ) on (H, Γ) (find all even-face embeddings on the Klein bottle, and break each one).

In this case, although we have broken one embedding of (G, Σ), we have not really gained much ground toward finding $\mathcal{U}_3$. We would therefore like to avoid this scenario while constructing the unavoidable set $\mathcal{U}_3$. Notice that the key problem in this example is that edge e can be added to the embedding of (G, Σ) in two different ways, such that adding just e in either way does not break the embedding. In fact, our problem arises exactly when an edge in $E(H) \setminus E(G)$ can be placed in the surface in more than one way without breaking the embedding. It follows that to avoid our problem we need a "stabilizer" result for the Klein bottle – sufficient conditions to guarantee that every time we add an edge or split a vertex in an embedding of (G, Σ), the new edge can be added to the embedding of (G, Σ) in at most one way without breaking the embedding. In Section 3, we prove such a result for each of our topological classes. Since we need only consider irreducible signed graphs, we will assume in all of our escape and stabilizer results that the signed graphs we work with are simple and 3-connected.

We need to take a look at the hypotheses of our stabilizer theorems. Consider, for example, the stabilizer result for graphs with an even-face embedding on the Klein bottle:

Theorem 3.2.7 *Let (G, Σ) be a simple non-bipartite 3-connected signed graph with no blocking vertex or blocking pair. Suppose (G, Σ) has an even-face embedding on the Klein bottle. If (G, Σ) is non-planar, and does not have an even-face embedding on the projective plane or on the pinched projective plane, then (G, Σ) extends uniquely.*

Notice that to apply this stabilizer theorem, we need to guarantee that the signed graphs we try to extend do not have an even-face embedding on the projective plane or on the pinched projective plane, in addition to having no blocking pair and being non-planar. This means that, using our strategy, we cannot find a set of unavoidable signed graphs that have no even-face embedding on the Klein bottle until we already have an unavoidable set of signed graphs where every member is non-planar, has no blocking pair, and does not have an even-face embedding on the projective plane or pinched projective plane. In this case, the stabilizer theorem suggests an order in which we need to consider the basic classes.

We note here that it may seem redundant to consider graphs with an even-face embeddings on the projective plane and graphs with an even-face embedding on the pinched projective plane separately, as the first set of graphs is clearly a subset of the second. However, the reason for this is apparent from the stabilizer theorem for signed graphs with an even-face embedding on the pinched projective plane. First, we need a definition. Let x_1, x_2, y_1, y_2 be distinct points on a sphere. The *double-pinched sphere* is the pinched surface obtained from the sphere by identifying x_1 with y_1, and x_2 with y_2, to form two distinct pinch points. We now state the stabilizer theorem for signed graphs with an even-face embedding on the pinched projective plane.

Theorem 3.2.4. *Let (G, Σ) be a simple non-bipartite 3-connected signed graph with no blocking vertex of blocking pair. Suppose (G, Σ) has an even-face embedding Π on the pinched projective plane, where the pinch point is not contained in $\Pi(G, \Sigma)$. Suppose G is non-planar, and (G, Σ) has no even-face embedding on the projective plane or on the double-pinched sphere. Then (G, Σ) extends uniquely.*

In this case, the theorem tells us that considering signed graphs with an even-face embedding on the projective plane before signed graphs with an even-face embedding on the pinched projective plane will actually help us, a fact that is not at all obvious without this result. Similarly, the stabilizer result for apex signed graphs with two odd faces alerts us to the fact that we should additionally consider signed graphs with an even-face embedding on the double-pinched sphere.

There is one type of degenerate signed graphs which is not dealt with in this thesis. Let (G, Σ) be a signed graph with embedding Π on a pinched surface. Let $u, v \in V(G)$, where $\Pi(v)$ coincides with a pinch point. We will say that a pair of Π-faces F_1, F_2 of (G, Σ) is *bad* if both F_1, F_2 contain both u, v. Note note that if (G, Σ) has a bad pair of Π-faces, then Π cannot extend uniquely – it is easy to see that an edge uv may be added to G such that uv lies in either F_1 or F_2.

In our stabilizer results for signed graphs with an even-face embedding on a pinched surface, we assume in this thesis that the embedding we are given does not have a bad pair

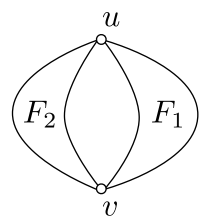


Figure 1.4: A bad pair of faces

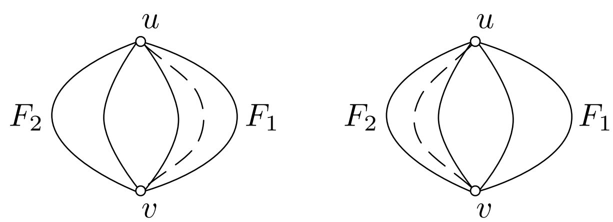


Figure 1.5: Adding an edge in two different ways

of faces. It is worth noting, however, that to fully implement our strategy we must find a stabilizer result for embeddings with a bad pair of faces. We plan to study this case in future work.

Assuming, then, that we are not given an embedding of a signed graph with a bad pair of faces, Figure 1.6 illustrates which classes of signed graphs we need to consider in our strategy outlined above, and an idea of the order in which they should be considered. An arrow from class A to class B indicates that class A must be considered before class B.

Notice that the digraph in the above diagram is acyclic. It is therefore possible to order our classes such that, when we try to find a set of unavoidable signed graphs that is outside the first i classes, the conditions for unique extension in the next class to be considered are met (so long as we do not encounter a bad pair of faces). One such order is the following:

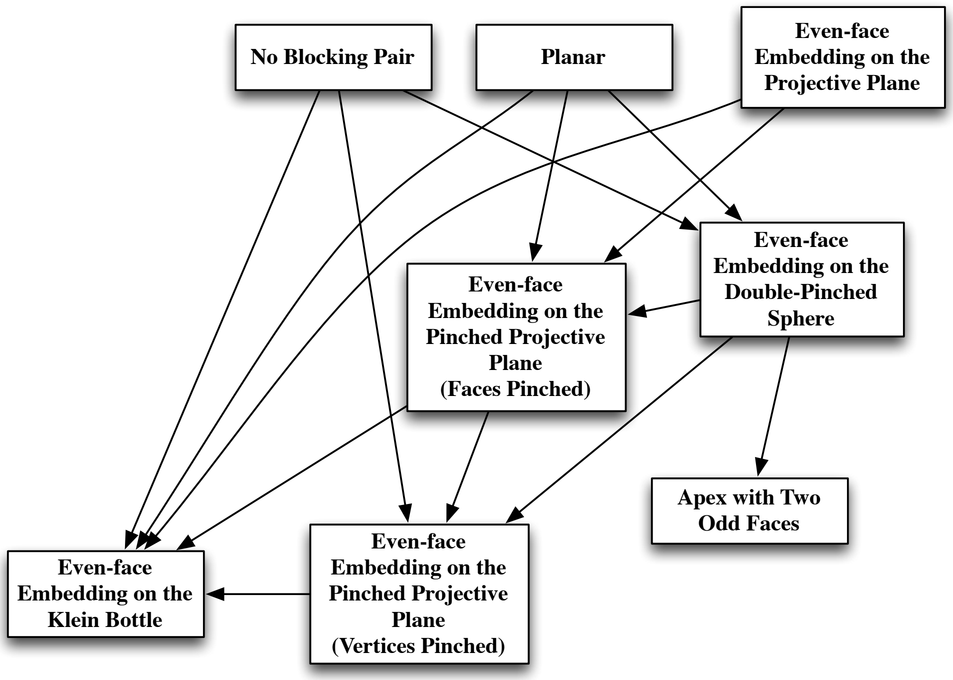


Figure 1.6: An ordering of our basic classes

1. Planar
2. No blocking pair
3. Projective plane
4. Double-pinched sphere
5. Pinched projective plane – faces pinched
6. Pinched projective plane – vertices pinched
7. Klein bottle
8. Apex with two odd faces.

We will close this section with a few conventions. In this thesis all graphs are finite, and may have loops or parallel edges unless stated otherwise. Paths and circuits have no "repeated" vertices or edges, and the *length* of a path is the number of edges it contains. We will use the term *cycle* of G to refer to a subgraph of G in which every vertex has even degree. This usage is consistent with terminology in matroid theory. For a path P, we will consider the notation P to refer to the set of edges in the path. When we refer specifically to the set of vertices of the path, we will use the notation $V(P)$. We use this same convention for circuits and for cycles.

Chapter 2

Escape

Let S be a surface. A *non-contractible closed curve* s in S is a closed curve such that removing a sufficiently small neighbourhood of s from S does not separate S into two parts. The *representativity* of a Π-embedded signed graph (G, Σ) in a surface S is the minimum number k such that a non-contractible curve in S intersects $\Pi(G, \Sigma)$ in exactly k points.

Let (G, Σ) be a signed graph Π-embedded on S with representativity at least 3. We are interested in determining the minimal signed graphs (H, Γ) that "contain" (G, Σ), but have no embedding on S that extends from Π, or a "similar" embedding of (G, Σ). Specifically, we ask: what are the minimal structures that can be added to (G, Σ) to obtain a graph with no such embedding?

In Section 2.1, we state the answer to this problem in the case where G is simple, and in the case where we allow G to have some parallel edges. In this first statement, we describe the extensions of (G, Σ) in terms of "bridges". These descriptions will be restated in Section 2.2 in terms of paths, triads, and facial circuits. In Section 2.1, for a signed graph (G, Σ) with an even-face embedding on S, we also characterize the "minimal extensions" of (G, Σ) that have no even-face embedding on S. We present this result as an application of the main theorem. In Section 2.3, we state and prove a result on bridges that is crucial to the proof of Theorem 2.1.1. The results of Section 2.1 will be proved in Sections 2.4 and 2.5, and the validity of the statements in Section 2.2 will be proved in Section 2.6.

2.1 Main Results

We will say that a signed graph is a *subdivision* of another if the first can be obtained from the second by replacing each edge by a non-zero length path with the same ends and parity, where the paths are disjoint, except possibly for shared ends. Throughout the paper, we will consider a signed graph (H, Γ), which contains as a subgraph a subdivision of a signed graph (G, Σ). We will formalize this idea of "containment" by extending the definition in [11] to signed graphs:

Let (G, Σ) and (H, Γ) be signed graphs. A mapping η with domain $V(G) \cup E(G)$ is called a *homeomorphic embedding* of (G, Σ) into (H, Γ) if for every two vertices v, v' and every two edges e, e' of (G, Σ),

(i) $\eta(v)$ is a vertex of (H, Γ), and if v, v' are distinct then $\eta(v)$, $\eta(v')$ are distinct,

(ii) if e has ends v, v' then $\eta(e)$ is a path of (H, Γ) with ends $\eta(v)$, $\eta(v')$, and otherwise disjoint from $\eta(V(G))$, where the parity of e is the same as that of $\eta(e)$, and

(iii) if e, e' are distinct, then $\eta(e)$ and $\eta(e')$ are edge-disjoint, and if they have a vertex in common, then this vertex is an end of both.

We will use $\eta : (G, \Sigma) \hookrightarrow (H, \Gamma)$ to denote "η is a homeomorphic embedding of (G, Σ) into (H, Γ)". If K is a subgraph of G, we use $\eta(K)$ to denote the subgraph of H consisting of vertices $\eta(v)$, where $v \in V(G)$, and all vertices and edges that belong to $\eta(e)$ for some $e \in E(G)$.

Now, it is likely that for a pair of signed graphs (G, Σ) and (H, Γ) there is more than one way to choose a homeomorphic embedding of (G, Σ) into (H, Γ). It is also reasonable to assume that not every homeomorphic embedding will have desirable properties. It follows that we need a well-defined way to transform a given homeomorphic embedding $\eta : (G, \Sigma) \hookrightarrow (H, \Gamma)$ into a different homeomorphic embedding $\eta' : (G, \Sigma) \hookrightarrow (H, \Gamma)$. We will call this process *rerouting*, and will shortly define two different circumstances in which we may reroute.

We say that a subset $\Sigma' \subseteq E(G)$ is a *signature* of (G, Σ) if (G, Σ) and (G, Σ') have the same set of even cycles. The following is well-known:

Remark. *Σ' is a signature of (G, Σ) if $\Sigma \,\Delta\, \Sigma'$ is a cut of G.*

Throughout the rest of the paper, whenever we consider a subpath of $\eta(e)$ for a particular $e \in E(G)$, we will assume that Γ has been replaced with an equivalent signature Γ'

such that $\eta(e) \cap \Gamma' = \emptyset$. (We will say in this case that (H, Γ) has been *resigned*.) We will take a moment to justify this convention. Let $B \subseteq E(H)$, and let $H[B] = (V(H), B)$. It is well known that if $(H[B], \Gamma \cap B)$ has no odd cycle, then there exists a signature Γ' of (H, Γ) such that $\Gamma' \cap B = \emptyset$.

Remark. *Let P be a path in signed graph (H, Γ). Then (H, Γ) can be resigned such that every edge of P is even.*

For a homeomorphic embedding $\eta : (G, \Sigma) \hookrightarrow (H, \Gamma)$, we will call a path Q in (H, Γ) having at least one edge an *η-path* if its ends and only its ends belong to $\eta(G)$.

Now, we describe our process for rerouting: Let (G, Σ) and (H, Γ) be signed graphs, and let $\eta : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be a homeomorphic embedding. Let $e \in E(G)$, and assume every edge of $\eta(e)$ is even.

Let Q be an η-path with endpoints on $\eta(e)$. Let P be the subpath of $\eta(e)$ with ends the ends of Q, and suppose Q is even. Let $\eta'(e)$ be the path obtained from $\eta(e)$ by replacing P with Q, and let $\eta'(x) = \eta(x)$ for all $x \in V(G) \cup E(G) - \{e\}$. Then $\eta' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ is a homeomorphic embedding, and we say that η' is obtained from η by *rerouting P along Q*.

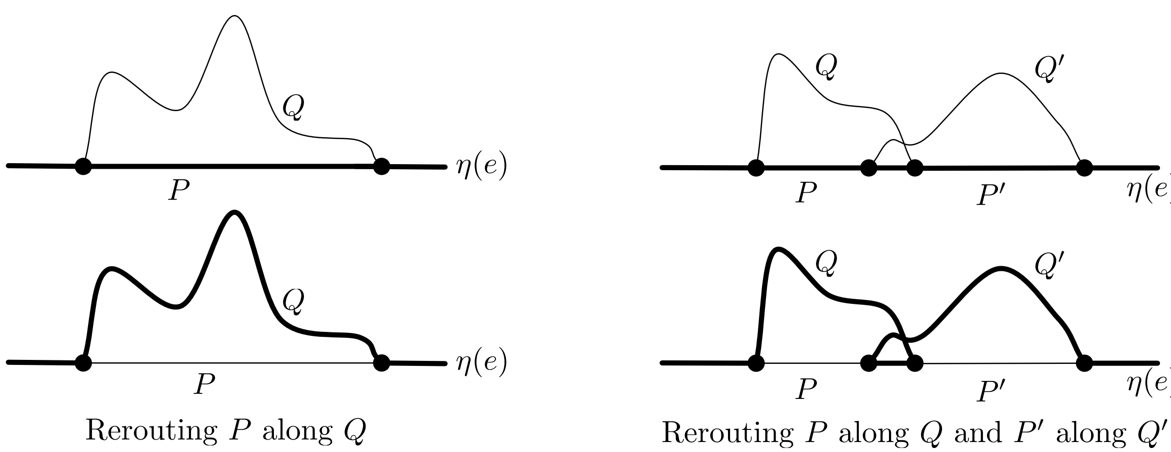


Figure 2.1: Examples of rerouting.

Let Q and Q' be η-paths with both ends on $\eta(e)$, such that Q and Q' are internally disjoint and share at most one endpoint. Let P be the subpath of $\eta(e)$ whose endpoints are the endpoints of Q, and let P' be the subpath of $\eta(e)$ whose endpoints are the endpoints of Q'. Suppose Q, Q' are both odd, $P \not\subseteq P'$ and $P' \not\subseteq P$. Let $\eta'(e)$ be the path given by $(\eta(e) - P\Delta P') \cup Q \cup Q'$, and let $\eta'(x) = \eta(x)$ for all $x \in V(G) \cup E(G) - \{e\}$. Then $\eta' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ is a homeomorphic embedding, and we say that η' was obtained from η by *rerouting* P *along* Q *and* P' *along* Q'.

We will say that two homeomorphic embeddings η and η' are *parallel* if one can be obtained from the other by a series of reroutings.

If η is a homeomorphic embedding of (G, Σ) into (H, Γ), an η-*bridge* is a connected subgraph B of (H, Γ) with $E(B) \cap E(\eta(G)) = \emptyset$ such that either

(i) $|E(B)| = 1$, $E(B) = \{e\}$ say, and both ends of e are in $V(\eta(G))$, or

(ii) for some component C of $H \setminus V(\eta(G))$, $E(B)$ consists of all edges of H with at least one end in $V(C)$.

It follows that every edge of H not in $\eta(G)$ belongs to a unique η-bridge. We say that a vertex v of H is an *attachment* of an η-bridge B if
$v \in V(\eta(G)) \cap V(B)$. We say that an η-bridge B is *unstable* if there exists an edge $e \in E(G)$ such that $V(B) \cap V(\eta(G)) \subseteq V(\eta(e))$, and otherwise we say that it is *stable*.

We will need a way to describe relationships between η-bridges, particularly between the locations of their attachments in the image of a particular edge of (G, Σ). Later, we will see that these relationships affect whether or not a collection of unstable η-bridges can be added to an embedding of $\eta(G, \Sigma)$ in a planar way.

Let B be an unstable η-bridge with all attachments in $\eta(e)$, for some $e \in E(G)$. Let P be the minimal subpath of $\eta(e)$ containing the attachments of B. If some η-bridge $A \neq B$ has an attachment in the interior of P, we say that A *has an attachment under* B, and that B is *over an attachment of* A.

Let B, B' be unstable η-bridges, and let P, P' be the minimal subpaths of $\eta(e)$ containing all the attachments of B, B', respectively. If $V(P) \cap V(P') \neq \emptyset$, we say B and B' *intersect*. If B is over an attachment of B' and B' is over an attachment of B, we say B and B' *cross*. Note that if two unstable η-bridges cross, they cannot be embedded in the same induced face of $\eta(G, \Sigma)$ in any embedding of (H, Γ).

Next, we will explain what we mean when we say that two embeddings are "similar". We will say that two embeddings Π_1, Π_2 of (G, Σ) in S are *closely related* if one can be obtained from the other by swapping the positions of e_1 and e_2 in S, for some number of pairs e_1, e_2 of parallel edges of G. (Note that under this definition, an embedding is closely related to itself.)

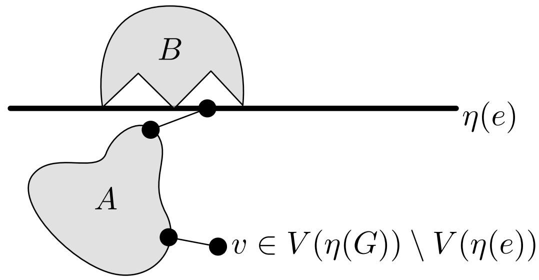


Figure 2.2: "B is over an attachment of A"

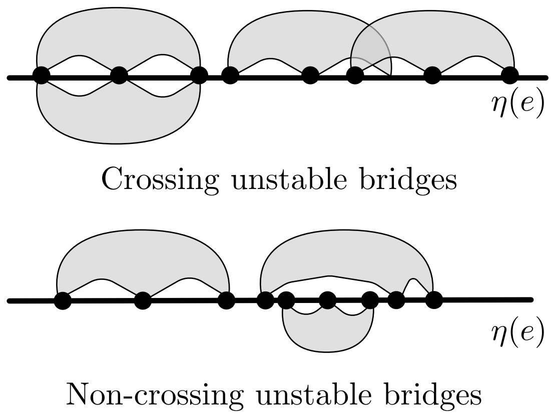


Figure 2.3: Examples of crossing and non-crossing unstable bridges.

We will call circuit C in G a Π-*potential facial circuit* if it is a facial circuit in an embedding of G that is closely related to Π. If $\eta : (G, \Sigma) \hookrightarrow (H, \Gamma)$ is a homeomorphic embedding, we will say that a circuit $\eta(C)$ is a Π-potential facial circuit in $\eta(G)$ if and only if C is a Π-potential facial circuit in G. We will describe potential facial circuits in greater detail in Section 2.4.

We are almost ready to state the main results, but first need three more definitions.

Let $x_1, x_2, x_3 \in V(\eta(G))$, let $x \in V(H) \setminus V(\eta(G))$, and let P_1, P_2, P_3 be three paths in (H, Γ) such that P_i has ends x and x_i. Suppose further that any two of the P_i intersect only in x, and that each is disjoint from $V(\eta(G)) - \{x_1, x_2, x_3\}$. In those circumstances we say that the triple P_1, P_2, P_3 is an η-*triad*. The vertices x_1, x_2, x_3 are its *feet*.

Let Π be the given embedding of G, and let C be a Π-potential facial circuit in G. Let P_1 and P_2 be two disjoint η-paths with ends x_1, y_1 and x_2, y_2, respectively, such that x_1, x_2, y_1, y_2 belong to $V(C)$ and occur on C in the order listed. In those circumstances we say that the pair P_1, P_2 is an *η-cross*. We also say that it is an *η-cross in C*. We say that x_1, x_2, y_1, y_2 are the *feet* of the cross. We say that the cross is *special* if for $i = 1, 2$ there is no $e \in e(G)$ such that P_i has both ends in $V(\eta(e))$.

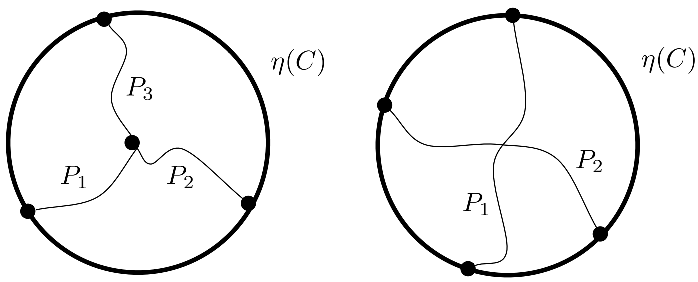


Figure 2.4: An η-triad in C (left) and an η-cross in C (right)

In Section 2.4, we prove the following result:

Theorem 2.1.1. *Let (G, Σ) be a 3-connected signed graph with $|V(G)| \geq 5$, such that G is simple, and (G, Σ) is Π-embedded on a surface S with representativity at least 3. Let (H, Γ) be a signed graph, let $\eta : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be a homeomorphic embedding. If (H, Γ) has no embedding on S that extends from an embedding closely related to Π, then there exists a homeomorphic embedding $\eta' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ parallel to η such that one of the following conditions holds:*

(a1) there exists an η'-path such that no Π-potential facial circuit of $\eta'(G, \Sigma)$ includes both of its ends, or

(a2) there exists a special η'-cross, or

(a3) there exists a separation (X, Y) of (H, Γ) of order at most three such that $|\eta(V(G)) \cap X - Y| \leq 1$ and $H|X$ does not have a drawing in a disk with $X \cap Y$ drawn on the boundary of the disk, or

(a4) there exists an η'-triad such that for every pair of its feet, some Π-potential facial circuit of $\eta'(G, \Sigma)$ contains both of them, but no Π-potential facial circuit of $\eta'(G, \Sigma)$ contains all feet of the triad, or

(a5) for some $e \in E(G)$ there exist three pairwise crossing unstable η'-bridges with attachments on $\eta'(e)$, or

(a6) for some $e \in E(G)$ there exist crossing unstable η'-bridges B_1, B_2 with attachments on $\eta'(e)$, and an η'-path $\bar{P}$ with endpoints w, z such that w is under both B_1 and B_2, and $z \in V(\eta'(C)) \setminus V(\eta'(e))$, where C is a Π-potential facial circuit of G that contains e, or

(a7) for some $e \in E(G)$ there exist crossing unstable η'-bridges B_1, B_2 with attachments on $\eta'(e)$, and η'-paths $\bar{P}_1, \bar{P}_2$ with endpoints w_1, z_1 and w_2, z_2, respectively, where w_i is under B_i but not under B_{3-i} and $z_1, z_2 \in V(\eta'(C))$, where C is a Π-potential facial circuit of G that contains e, or

(a8) for some $e \in E(G)$ there exists an unstable η'-bridge B with all attachments on $\eta'(e)$, and η'-paths $\bar{P}_1, \bar{P}_2$ with endpoints w_1, z_1 and w_2, z_2, respectively, where w_i is under B and $z_i \in V(\eta'(C_i)) \setminus V(\eta'(e))$ for $i = 1, 2$, where C_1, C_2 are Π-potential facial circuits of G containing e, and C_1, C_2 share at most two vertices.

(a9) for some $e \in E(G)$, where (H, Γ) has been resigned such that every edge of $\eta'(e)$ is even, there exists an odd η'-path Q with endpoints x, y, and there exist η'-paths $\bar{P}_1, \bar{P}_2$ with endpoints w_1, z_1 and w_2, z_2, respectively, such that x, y are distinct vertices of $\eta'(e)$, w_1 and w_2 are internal vertices of $\eta'(e)[x, y]$, and $z_i \in V(\eta'(C_i)) \setminus V(\eta'(e))$ for $i = 1, 2$, where C_1, C_2 are distinct potential facial circuits of G that contain e which share at most two vertices, or

(a10) for some $e \in E(G)$, where (H, Γ) has been resigned such that every edge of $\eta'(e)$ is even, there exist odd η'-paths Q_1 and Q_2 with endpoints x_1, y_1 and x_2, y_2, respectively, and there exist η'-paths $\bar{P}_1, \bar{P}_2$ with endpoints w_1, z_1 and w_2, z_2, respectively, such that $e \in E(G)$, $x_1, w_1, x_2, y_1, w_2, y_2$ occur on $\eta'(e)$ in that order, where w_1, x_2 may coincide and y_1, w_2 may coincide, and z_1, z_2 are in $V(\eta'(C)) \setminus V(\eta'(e))$ for some Π-potential facial circuit C of G that contains e.

In Section 2.2 we will make outcomes (a5)-(a8) explicit in terms of η'-paths and triads.

A signed graph (G, Σ) is *simple* if G is loopless, and any pair of parallel edges of G differ in parity. Now, we will state the result for the case where (G, Σ) is simple, but G need not be simple. This theorem, will be proved in Section 2.5.

Theorem 2.1.2. *Let (G, Σ) be a simple 3-connected signed graph with $|V(G)| \geq 5$. Suppose (G, Σ) is Π-embedded on a surface S with representativity at least 3. Let (H, Γ) be a signed*

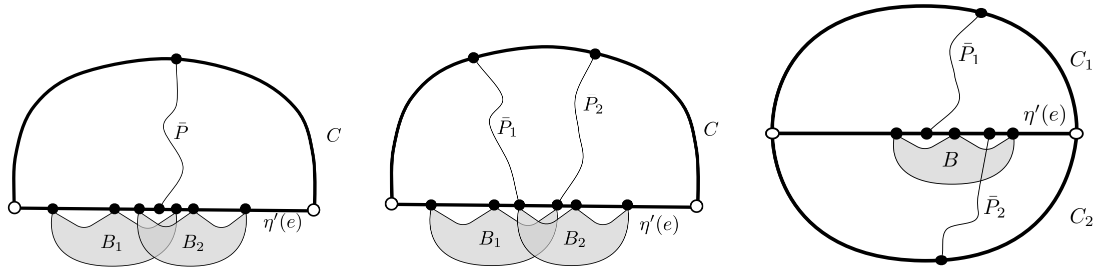


Figure 2.5: Examples of (a6) (left), (a7) (centre), (a8) (right).

graph, and let $\eta : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be a homeomorphic embedding. If there is no embedding of (H, Γ) on S that extends from an embedding closely related to Π, then there exists a homeomorphic embedding $\eta' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ parallel to η such that one of the following conditions holds:

(b1) one of outcomes (a1)-(a8), or (a10) from Theorem 2.1.1, or

(b2) outcome (a9) from Theorem 2.1.1 holds, and there are no two edges $e_1, e_2 \in E(G)$ such that x, y are the endpoints of both $\eta'(e_1), \eta'(e_2)$, or

(b3) there exists a pair of parallel edges e_1, e_2 of (G, Σ) with endpoints x, y, an odd η'-path Q with endpoints $\eta'(x), \eta'(y)$, and η'-paths $\bar{P}_1, \bar{P}_2, \bar{P}_3$ such that $\bar{P}_i$ has endpoints w_i, z_i for $i = 1, 2, 3$, where w_i is an internal vertex of $\eta'(e_i)$ for $i = 1, 2$, and $z_i \in V(\eta'(C_i)) \setminus V(\eta'(e_i))$ for $i = 1, 2$, where C_i is a Π-potential facial circuit of G containing e_i but not e_{3-i} and C_1, C_2 intersect in exactly two vertices, w_3 is an internal vertex of $V(\eta'(e_1))$, and z_3 is an internal vertex of $V(\eta'(e_2))$, or

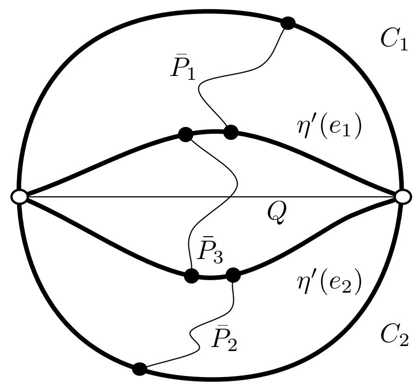
(b4) there exists a pair of parallel edges e_1, e_2 of (G, Σ), and η'-paths $\bar{P}_1$, $\bar{P}_2$ with endpoints w_1, z_1 and w_2, z_2 such that w_i is an internal vertex of $\eta'(e_i)$ for $i = 1, 2$, and $z_1, z_2 \in V(\eta'(C_1 \setminus \{e_1\}))$ where C_1 is a Π-potential facial circuit of G containing e_1, or

(b5) there exists a pair of parallel edges e_1, e_2 of (G, Σ), and η'-paths $\bar{P}_1$, $\bar{P}_2$ with endpoints w_1, z_1 and w_2, z_2 such that w_i is an internal vertex of $\eta'(e_i)$ for $i = 1, 2$, $z_1 \in V(\eta'(e_1'))$, and $z_2 \in V(\eta'(e_2'))$, where e_1', e_2' are parallel edges of G and $e_1' \in C_1$ for some Π-potential facial circuit C_1 of G containing e_1, or

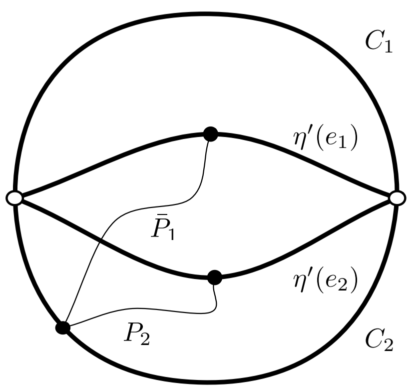
(b6) there exists a pair of parallel edges e_1, e_2 of (G, Σ), η'-paths $\bar{P}_1, \bar{P}_2$ with endpoints w_1, z_1 and w_2, z_2, respectively, such that w_1, w_2 are internal vertices of $\eta'(e_1)$, $z_i \in$

$V(\eta'(C_i \setminus \{e_i\}))$ for $i = 1, 2$, where C_i is a Π-potential facial circuit of G containing e_i but not e_{3-i}, and C_1, C_2 intersect in exactly two vertices.

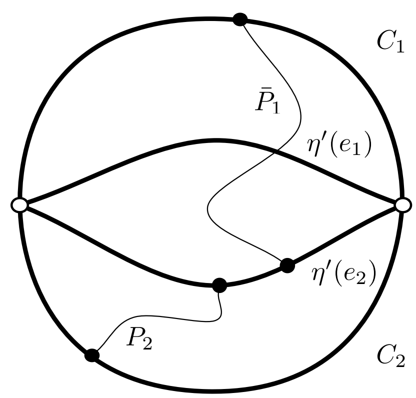
Note that when G is simple, Theorem 2.1.2 is equivalent to Theorem 2.1.1.



Ex. of $(b3)$



Ex. of $(b4)$



Ex. of $(b6)$

Figure 2.6: Examples of outcomes from Theorem 2.1.2.

We finish this section by stating our result for even-face embeddings. We will prove this result in Section 2.6 as a corollary of Theorem 2.1.2.

Corollary 2.1.3. *Let (G, Σ) be a simple 3-connected signed graph with $|V(G)| \geq 5$. Additionally, suppose (G, Σ) is Π-embedded on surface S with representativity* *at least 3, such that every* Π*-face of* (G, Σ) *is even. Let* (H, Γ) *be a signed graph, let* $\eta : (G, \Sigma) \hookrightarrow (H, \Gamma)$ *be a homeomorphic embedding, and suppose* (H, Γ) *has no even-face embedding on* S *that extends from an embedding closely related to* Π*. Suppose further that* (H, Γ) *is almost simple with respect to* η*. Then there exists a homeomorphic embedding* $\eta' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ *parallel to* η *such that one of the following conditions holds:*

(e1) one of (a1), (a3) from Theorem 2.1.1, or

(e2) there exists a special η'*-cross* P_1, P_2 *in a* Π*-facial circuit* C *of* (G, Σ) *such that, if* (H, Γ) *is resigned such that every edge of* $\eta'(C)$ *is even, neither* P_1 *nor* P_2 *is odd, or*

(e3) there exists an η'*-triad such that for every pair of its feet, some* Π*-facial circuit of* $\eta'(G)$ *contains both of them, but no* Π*-facial circuit of* $\eta'(G)$ *contains all feet of the triad, and if* (H, Γ) *is resigned such that every edge of the triad is even, no* η'*-path in the triad divides a* Π*-facial circuit into two odd paths, or*

(e4) there exists an η'*-path* $\bar{P}$ *with both endpoints* w, z *in* $V(\eta'(C))$*, for some* Π*-facial circuit* C *of* G*, where if* (H, Γ) *is resigned such that every edge of* $\eta'(C)$ *is even, then* $\bar{P}$ *is odd.*

2.2 Explicit Description of obstructions

The reader may recall that in the preceding results, several of the outcomes are stated in terms of η'-bridges. In this section, we will restate these outcomes in terms of η'-paths and η'-triads of specified parities. The lemmas given in this section will be proved in Section 2.6.

For a homeomorphic embedding $\xi : (G, \Sigma) \hookrightarrow (H, \Gamma)$, we will say that a ξ-triad T is *odd* if for some pair x, y of its feet, the path in T from x to y is odd.

Lemma 2.2.1. *In the statement of Theorem 2.1.1, outcome (a8) can be replaced with the following:*

There exists $e \in E(G)$ *such that, when* (H, Γ) *is resigned such that every edge of* $\eta'(e)$ *is even, one of the following holds:*

(c1) there exist odd η'*-paths* Q_1 *and* Q_2 *with endpoints* x_1, y_1 *and* x_2, y_2*, respectively, and there exists an* η'*-path* $\bar{P}$ *with endpoints* w, z *such that* x_1, x_2, w, y_1, y_2 *occur on* $\eta'(e)$ *in that order, and* $y \in V(\eta'(C)) \setminus V(\eta'(e))$*, where* C *is a* Π*-potential facial circuit of* G *that contains* e*, or*

(c2) there exists an odd η'-path Q_1 with endpoints x, y_1, an odd η'-triad T_2 with feet x, v, y_2, and an η'-path $\bar{P}$ with endpoints w, z, such that x, w, v, y_1, y_2 occur on $\eta'(e)$ in that order, where w, v may coincide, and $z \in V(\eta'(C)) \backslash V(\eta'(e))$, where C is a Π-potential facial cycle of G that contains e, or

(c3) there exists an odd η'-path Q_1 with endpoints x_1, y_1, an odd η'-triad T_2 with feet x_2, v, y_2, and an η'-path $\bar{P}$ with endpoints v, z, such that x_2, x_1, v, y_1, y_2 occur on $\eta'(e)$ in that order, and $z \in V(\eta'(C)) \setminus V(\eta'(e))$, where C is a Π-potential facial circuit of G that contains e, or

(c4) there exist two odd η'-triads T_1 and T_2 with feet x, v_1, y and x, v_2, y, and there exists an η'-path $\bar{P}$ with endpoints w, z, such that $x, \{v_1, v_2, w\}, y$ occur on $\eta'(e)$ in that order, where v_1, v_2, w may coincide, and $z \in V(\eta'(C)) \setminus V(\eta'(e))$, where C is a Π-potential facial circuit of G that contains e.

Lemma 2.2.2. *In the statement of Theorem 2.1.1, (a5) can be replaced by the following:*

There exists $e \in E(G)$ such that, when (H, Γ) is resigned such that every edge of $\eta'(e)$ is even, one of the following holds:

*(d1) there exist odd η'-paths Q_1 and Q_2 with endpoints x_1, y_1 and x_2, y_2, respectively, and an odd η'-triad T_3 with feet x_2, v, y_3 such that $x_1, x_2,$
$\{v, y_1\}, y_2, y_3$ occur on $\eta'(e)$ in this order, where v, y_1 may coincide, and there exists an η'-path $\bar{P}$ with endpoints x_2, z where $z \in V(\eta'(C))) \setminus V(\eta'(e))$ where C is a Π-potential facial circuit C of G that contains e, or*

*(d2) there exists an odd η'-path Q_1 with endpoints x_1, y_1, and odd η'-triads T_2 and T_3 with feet x_1, v_1, y_2 and x_2, v_2, y_3, respectively, such that $x_2, x_1,$
$\{v_1, v_2\}, y_1, \{y_2, y_3\}$ occur on $\eta'(e)$ in that order, where y_2, y_3 may coincide, y_3, y_1 may coincide, y_2, y_1 may not coincide, and there exists an η'-path $\bar{P}$ with endpoints x_1, z such that $z \in V(\eta'(C)) \setminus V(\eta'(e))$, where C is a Π-potential facial circuit of G that contains e, or*

*(d3) there exists an odd η'-path Q_1 with endpoints x_1, y_1, and odd η'-triads T_2 and T_3 with feet x_2, v_2, y_2 and x_2, v_3, y_2, respectively, where $x_1, x_2,$
$\{y_1, v_2, v_3\},$
y_2 occur on $\eta'(e)$ in that order, y_1, v_2, v_3 may coincide, and there exists an η'-path $\bar{P}$ with endpoints x_2, z where $z \in V(\eta'(C)) \setminus V(\eta'(e))$, where C is a Π-potential facial circuit of G that contains e, or*

(d4) for $i = 1, 2, 3$ there exist odd η'-triads T_1 with feet x_1, v_1, y_1, T_2 with feet x_2, v_2, y_2, and T_3 with feet x_2, v_3, y_2, such that $x_1, x_2, \{v_1, v_2, v_3\}, y_2, y_1$ occur on $\eta'(e)$ in that order, where v_1, v_2, v_3 may coincide, y_1, y_2 may coincide, and there exists an η'-path $\bar{P}$ with endpoints x_2, z, where $z \in V(\eta'(C)) \setminus V(\eta'(e))$, where C is a Π-potential facial circuit of G that contains e.

Lemma 2.2.3. *We can remove (a7) from the statement of Theorem 2.1.1.*

Lemma 2.2.4. *We can remove (a8) from the statement of Theorem 2.1.1.*

We now note that Theorem 2.1.2 (and consequently Theorem 2.1.1 as well) is in fact an if and only if statement; i.e., if one of the outcomes listed occurs, then (H, Γ) has no embedding on S that extends from an embedding closely related to Π. The proof of this result appears in Section 2.6.

Theorem 2.2.5. *Let (G, Σ) be a simple signed graph Π-embedded in a surface S with representativity at least 3, and with $|V(G)| \geq 5$. Let (H, Γ) be a signed graph, and let $\eta : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be a homeomorphic embedding. If there exists a homeomorphic embedding $\eta' : G \hookrightarrow H$ such that one of (a1)-(a4) or (a10) of Theorem 2.1.1, (b2), (b3), (b4)-(b6) of Theorem 2.1.2, (c1)-(c4) of Lemma 2.2.1, or (d1)-(d4) of Lemma 2.2.2 holds for η', then H has no embedding on S that extends from an embedding closely related to Π.*

We pause here to remark that when $\Gamma = \emptyset$, Theorem 2.1.1 implies the main theorem of [11], up to a slight relaxation of outcome (a2) and the requirement that $|V(G)| \geq 5$. This assertion is easy to verify:

Since outcomes (a9), (a10) of Theorem 2.1.1 and the outcomes of Lemmas 2.2.1 and 2.2.2 all involve paths of two different parities in (H, Γ), it is easy to see that none of these can occur when every edge of (H, Γ) is even. Then, by Lemmas 2.2.1, 2.2.2, 2.2.3, 2.2.4, one of outcomes (a1)-(a4) holds for a homeomorphic embedding η' parallel to η, as desired.

2.3 A result on Unstable Bridges

As we go on, our argument will depend on being able to "eliminate" certain types of unstable bridges from our homeomorphic embedding, by means of rerouting. To complete the proofs of Sections 2.4 and 2.5, we will need a theorem describing which unstable bridges can be eliminated, and describing the behaviour of the unstable bridges that cannot be eliminated. This section is devoted to the statement and proof of such a result.

It will become apparent that there are different types of unstable η-bridges; namely, those we can certainly do away with, and those we cannot count on being able to get rid of. We will briefly give a notation, and then put names to these different types of unstable bridges.

Let P be a path, and let x, y be vertices in P. Then $P[x, y]$ denotes the subpath of P with endpoints x, y.

Now, let B be an unstable η-bridge with attachments on $\eta(e)$, where $\eta(e) \cap \Gamma = \emptyset$. If there exists an odd η-path in B, then B is *bad with respect to* η. If every η-path in B is even, B is *good with respect to* η. Let A be an η-bridge with an attachment z under B. If there exists no even η-path Q in B with endpoints x, y, where $z \in V(\eta(e)[x, y]) \setminus \{x, y\}$, then B is *bad with respect to* A. Otherwise, B is *good with respect to* A.

If B is an unstable η-bridge over an attachment of a stable η-bridge, we will say that B is *type-1*. If B crosses a type-1 η-bridge, but is not itself type-1, we will stay that B is *type-2*.

We need one more definition, related to homeomorphic embeddings. Let (G, Σ) be a signed graph, (H, Γ) be a loopless signed graph, and $\eta : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be a homeomorphic embedding. Suppose that for any pair u, v of vertices of H, there are at most two edges e, f of H with endpoints u, v, and that in this case e and f differ in parity, and $e, f \in \eta(E(G))$. Then we will say that (H, Γ) is *almost simple* with respect to η. Shortly, we will give an example to demonstrate the necessity of this definition.

Theorem 2.3.1. *Let (G, Σ) be a signed graph, let (H, Γ) be a 3-connected signed graph, and let $\eta : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be a homeomorphic embedding. Suppose (H, Γ) is almost simple with respect to η. Then there exists a homeomorphic embedding $\eta' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ parallel to η such that every unstable η'-bridge is bad with respect to η'. Furthermore, for each $e \in E(G)$, $\eta'(e)$ contains the attachments of at most one maximal set $\mathcal{B}$ of pairwise intersecting η'-bridges. For each $B \in \mathcal{B}$, let P_B be the minimal subpath of $\eta'(e)$ containing all attachments of B. Let x, y be the endpoints of the path $P = \bigcup_{B \in \mathcal{B}} P_B$, and let $Z = \bigcap_{B \in \mathcal{B}} V(P_B)$. Then we can say further that*

(B1) some $z \in Z \setminus \{x, y\}$ is an attachment of a stable η'-bridge,

(B2) no $z \in (V(P) \setminus Z) \setminus \{x, y\}$ is an attachment of a stable η'-bridge, and

(B3) every $B \in \mathcal{B}$ is bad with respect to every stable η'-bridge with an attachment in $Z \setminus \{x, y\}$.

We remark here that "(H,Γ) is almost simple" is a necessary hypothesis. Let $(G,\Sigma) = (K_6, \emptyset)$, where the vertices of G are $v_1, v_2, ..., v_6$. Let (H,Γ) and $\eta : (G,\Sigma) \hookrightarrow (H,\Gamma)$ be as shown below, where the thick edges are in $E(\eta(G))$ and large vertices are in $\eta(V(G))$, the thin edges and small vertices represent edges and vertices of (H,Γ) that are not in $E(\eta(G))$ or $\eta(V(G))$, respectively, odd edges are dotted, and even edges are solid.

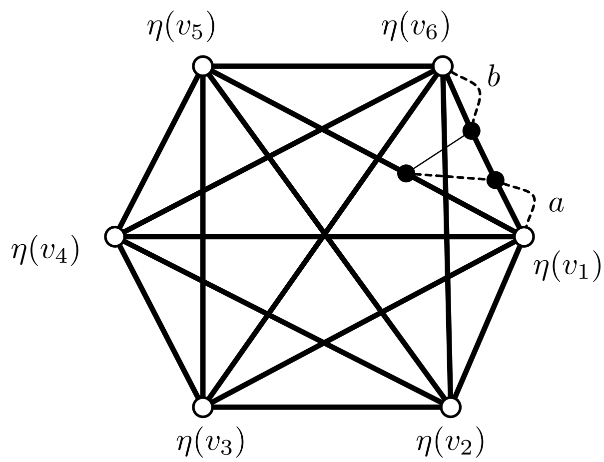


Figure 2.7: (H,Γ) is not almost simple with respect to (G,Σ).

Now, each vertex of (G,Σ) has degree 5. Since the vertices of $\eta(V(G))$ are exactly the vertices of (H,Γ) with degree at least 5, we must have $\eta(V(G)) = \xi(V(G))$ for any homeomorphic embedding $\xi : (G,\Sigma) \hookrightarrow (H,\Gamma)$. We may therefore assume, up to some level of equivalence, that $\eta(v_i) = \xi(v_i)$ for all $v_i \in V(G)$. Then we must also have $\eta(E(G) \setminus \{v_1v_3, v_1v_2\}) = \xi(E(G) \setminus \{v_1v_3, v_1v_2\})$. Note that in order to choose two internally disjoint paths from $\eta(v_3)$ to $\eta(v_1)$ and from $\eta(v_1)$ to $\eta(v_2)$ in (H,Γ) without using an edge in $\eta(E(G) \setminus \{v_1v_3, v_1v_2\}) = \xi(E(G) \setminus \{v_1v_3, v_1v_2\})$, we must take $\eta(v_1v_3)$ as one of the paths. So we must have $\xi(v_1v_3) = \eta(v_1v_3)$. Also, we must have either $\xi(v_1v_2) = \eta(v_1v_2)$, or we obtain $\xi(v_1v_2)$ from $\eta(v_1v_2)$ by rerouting along a or b (or both). But in each of these cases, there are two non-intersecting unstable ξ-bridges with attachments in $\xi(v_1v_2)$, giving a counterexample to the weakened version of the theorem.

We will now prove Theorem 2.3.1. The method of proof is as follows: using our definitions of good and bad η-bridges, we will first prove several lemmas stating that for a "most preferred" homomorphic embedding η', (H,Γ) does not contain certain types of η'-bridges. To finish, we will show that η' satisfies Theorem 2.3.1.

We begin by formalizing our notion of preference among homeomorphic embeddings. Let $n = |V(H)|$. For a homeomorphic embedding $\xi : (G, \Sigma) \hookrightarrow (H, \Gamma)$ and an integer $i = 1, 2, ..., n$, let a_{2n+i} be the number of stable ξ-bridges B with $|V(B)| = i$, let $a_n + i$ be the number of type-1 ξ-bridges B with $|V(B)| = i$, and let a_i be the number of type-2 ξ-bridges B with $|V(B)| = i$. We say that $(a_{3n}, a_{3n-1}, ...a_1)$ is the *trace* of ξ. Now if $\xi' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ is another homeomorphic embedding with trace $(a'_{3n}, a'_{3n-1}, ..., a'_1)$ we say that ξ is *preferred* to ξ' if there exists an integer $i \in \{1, 2, ..., 3n\}$ such that $a_i > a'_i$ and $a_j = a'_j$ for all $j \in \{i + 1, i + 2, ..., 3n\}$.

As we remarked above, the theorem does not hold without the hypothesis that (H, Γ) is almost simple with respect to η. Consequently, we will need to know what this hypothesis means for (H, Γ) with respect to a "most preferred" embedding η' parallel to η. The following lemma provides an answer:

Lemma 2.3.2. *Let (G, Σ) be a signed graph, let (H, Γ) be a 3-connected signed graph, and let $\xi : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be a homeomorphic embedding. Suppose (H, Γ) is almost simple with respect to ξ. Let ξ' be a homeomorphic embedding parallel to ξ such that no homeomorphic embedding parallel to ξ is preferred to ξ'. Then (H, Γ) is almost simple with respect to ξ'.*

Proof. Suppose (H, Γ) is not almost simple with respect to ξ'. Then there exist a pair of parallel edges f_1, f_2 of (H, Γ) such that $f_1 \notin \xi'(E(G))$. Since (H, Γ) is almost simple with respect to ξ, f_1 and f_2 differ in parity, and $f_1 = \xi(e_1)$ and $f_2 = \xi(e_2)$ for some edges $e_1, e_2 \in E(G)$. Then $\xi'(e_1)$ is not an edge of (H, Γ). Since ξ' is parallel to ξ, $\xi'(e_1)$ and $\xi(e_1)$ have the same endpoints. Denote these endpoints by u and v. Then $\xi'(e_1)$ is a path with endpoints u, v, and has at least one internal vertex.

Since (H, Γ) is 3-connected, there exists some internal vertex w of $\xi'(e_1)$ such that w is the endpoint of an ξ'-path P in (H, Γ) whose other endpoint is in $V(\xi'(G, \Sigma)) \setminus \xi'(e_1)$. Now, consider the homeomorphic embedding $\xi'' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ obtained from ξ' by rerouting $\xi'(e_1)$ along f_1. Let $(a'_{3n}, a'_{3n-1}, ..., a'_1)$ be the trace of ξ', and let $(a''_{3n}, a''_{3n-1}, ..., a''_1)$ be the trace of ξ''.

Let B be a stable ξ'-bridge with an attachment in $\xi'(e_1)$, such that $k = |V(B)|$ is maximum among all such bridges. (We know such a stable ξ'-bridge exists, since P is obviously contained in one.) If B' is a stable ξ'-bridge with more vertices than B, then B' has no attachments in $\xi'(e_1)$ and so is a stable ξ''-bridge. So $a''_i \geq a'_i$ for all $i = 2n + k + 1, 2n + k + 2, ..., 3n$.

The vertices of B are a proper subset of the vertices of some stable ξ''-bridge A that contains P. Let $l = |V(A)|$. Then $l > k$, and $a''_{2n+l} > a'_{2n+l}$. Thus ξ'' is preferred to ξ', a contradiction.

□

The proof of the theorem will also require some results relating rerouting to the trace of an embedding. Specifically, we will need to know when rerouting in a given embedding produces a new embedding that is preferred to the original. Lemmas 2.3.3, 2.3.4, 2.3.5, and 2.3.6 provide these results.

Lemma 2.3.3. *Let B be an unstable ξ-bridge with attachments in $\xi(e)$ for some $e \in E(G)$, such that B is over an attachment of stable ξ-bridge B_S. If B is good with respect to B_S, then there exists a homeomorphic embedding $\xi' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ parallel to ξ such that ξ' is preferred to ξ.*

Proof. We may assume $\xi(e) \cap \Gamma = \emptyset$. Then there exists by definition an even ξ-path Q in B with endpoints x, y such that the subpath P of $\xi(e)$ with endpoints x and y contains an attachment of B_S.

Let $\xi' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be the homeomorphic embedding obtained from ξ by rerouting P along Q. Let $(a_{3n}, a_{3n-1}, ..., a_1)$ be the trace of ξ, and let $(a'_{3n}, a'_{3n-1}, ..., a'_1)$ be the trace of ξ'. Let T be a stable ξ-bridge with an attachment in the interior of P, such that $|V(T)|$ is maximum among such stable bridges. Let $k = |V(T)|$. Then any stable ξ-bridge with more vertices than T has no attachment in the interior of P, and so is a stable ξ'-bridge. It follows that $a'_j \geq a_j$ for all $j = 2n+k+1, 2n+k+2, ..., 3n$. Now, $V(T)$ is a proper subset of the vertices of some stable ξ'-bridge T', where T' is not an ξ-bridge. Let $l = |V(T')|$. Then $l > k$ and $a'_{2n+l} > a_{2n+l}$. Thus ξ' is preferred to ξ. □

Lemma 2.3.4. *Let $e \in E(G)$ such that $\xi(e) \cap \Gamma = \emptyset$. Let B, B', be unstable ξ-bridges with attachments on $\xi(e)$ such that B, B' are bad with respect to ξ. Let Q be an odd ξ-path in B with endpoints x, y, and let Q' be an odd ξ-path in B' with endpoints x', y'. Let $P = \xi(e)[x, y]$, and let $P' = \xi(e)[x', y']$. Suppose there exists a stable ξ-bridge B_S with an attachment z in the interior of $P \Delta P'$. Then there exists a homeomorphic embedding ξ' parallel to ξ such that ξ' is preferred to ξ.*

Proof. Let $\xi' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be the homeomorphic embedding obtained from ξ by rerouting P along Q and P' along Q'. Let B'_S be a stable ξ-bridge with an attachment in the interior of $P \Delta P'$, such that $|V(B'_S)|$ is maximum among such stable ξ-bridges. Let $(a_{3n}, a_{3n-1}, ...a_1)$ be the trace of ξ, and let $(a'_{3n}, a'_{3n-1}, ..., a'_1)$ be the trace of ξ'. Let $k = |V(B'_S)|$. Every stable ξ-bridge with more vertices than B'_S has no attachment in the interior of $P \Delta P'$, and so is a stable ξ'-bridge. It follows that $a'_j \geq a_j$ for $j = 2n + k +$

$1, 2n+k+2, ..., 3n$. Now, the vertices of B'_S are a proper subset of a stable ξ'-bridge T, where T is not a ξ-bridge. Let $l = |V(T)|$. Then $l > k$, and $a'_{2n+l} > a_{2n+l}$. Thus ξ' is preferred to ξ. □

Lemma 2.3.5. *Let $e \in E(G)$. Let B, B', be unstable ξ-bridges with attachments on $\xi(e)$ such that B and B' cross, and B is type-1. Suppose further that B' is type-2, and B' is good with respect to ξ. Then there exists a homeomorphic embedding ξ' parallel to ξ such that ξ' is preferred to ξ.*

Proof. We may assume that $\xi(e) \cap \Gamma = \emptyset$. Then there exists an even ξ-path Q in B' with endpoints x, y such that the subpath P of $\xi(e)$ with endpoints x and y contains an attachment of B.

Let $\xi' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be the homeomorphic embedding obtained from ξ by rerouting P along Q. Let $(a_{3n}, a_{3n-1}, ..., a_1)$ be the trace of ξ, and let $(a'_{3n}, a'_{3n-1}, ..., a'_1)$ be the trace of ξ'. Let T be a type-1 ξ-bridge with an attachment in the interior of P, such that $|V(T)|$ is maximum among such type-1 bridges. Let $k = |V(T)|$. Since B' is type-2, no stable ξ-bridge has an attachment in the interior of P, and so every stable ξ-bridge is a stable ξ'-bridge. Also, any type-1 ξ-bridge with more vertices than T has no attachment in the interior of P, and so is a type-1 ξ'-bridge. It follows that $a'_j \geq a_j$ for all $j = n+k+1, n+k+2, ..., 3n$. Now, $V(T)$ is a proper subset of the vertices of some ξ'-bridge T', where T' is not an ξ-bridge. Since no stable ξ-bridge had an attachment in the interior of P, T' is a type-1 ξ'-bridge. Let $l = |V(T')|$. Then $l > k$ and $a'_{n+l} > a_{n+l}$. Thus ξ' is preferred to ξ.

□

Lemma 2.3.6. *Let $e \in E(G)$. Let B, B', be unstable ξ-bridges with attachments on $\xi(e)$ such that B and B' cross, and B is type-2. Suppose further that B' is neither type-1 nor type-2, and B' is good with respect to ξ. Then there exists a homeomorphic embedding ξ' parallel to ξ such that ξ' is preferred to ξ.*

Proof. We may assume that $\xi(e) \cap \Gamma = \emptyset$. Then there exists an even ξ-path Q in B' with endpoints x, y such that the subpath P of $\xi(e)$ with endpoints x and y contains an attachment of B.

Let $\xi' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be the homeomorphic embedding obtained from ξ by rerouting P along Q. Let $(a_{3n}, a_{3n-1}, ..., a_1)$ be the trace of ξ, and let $(a'_{3n}, a'_{3n-1}, ..., a'_1)$ be the trace of ξ'. Since B is a type-2 xi-bridge with an attachment in the interior of P, we may choose T to be a type-2 ξ-bridge with an attachment in the interior of P, such that $|V(T)|$ is maximum among such type-2 bridges. Let $k = |V(T)|$. Since B' is not type-1, no stable ξ-bridge has an attachment in the interior of P, and so every stable ξ-bridge is a stable ξ'-bridge. Also, since B' is not type-2, no type-1 ξ-bridge has an attachment in the interior of P, and so every type-1 ξ-bridge is a type-1 ξ'-bridge. Finally, any type-2 ξ-bridge with more vertices than T has no attachment in the interior of P, and so is a type-2 ξ'-bridge. It follows that $a'_j \geq a_j$ for all $j = k+1, k+2, ..., 3n$. Now, $V(T)$ is a proper subset of the vertices of some ξ'-bridge T', where T' is not an ξ-bridge. Since no stable ξ-bridge or type-1 ξ-bridge had an attachment in the interior of P, T' is a type-2 ξ'-bridge. Let $l = |V(T')|$. Then $l > k$ and $a'_l > a_l$. Thus ξ' is preferred to ξ.

□

The final tool we will need for the proof is a description of the "placement" of unstable bridges on the image of a particular edge of (G, Σ), relative to the attachments of the stable bridges:

Lemma 2.3.7. *Let $\xi : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be a homeomorphic embedding, where (H, Γ) is 3-connected and is almost simple with respect to ξ. Let B be an unstable ξ-bridge with attachments on $\xi(e)$, such that B is not type-1. Then there exists a sequence $B = B_1, B_2, ..., B_k$ of unstable ξ-bridges with attachments in $\xi(e)$ such that*

(i) B_i crosses B_{i-1} and B_{i+1}, $i \in 2, 3, ..., k-1$,

(ii) B_k is type-1, and

(iii) for $i, j \in [k]$, $i \neq j$, B_i and B_j cross only if i and j are consecutive .

Proof. We define a graph F such that $V(F)$ is the set of unstable ξ-bridges with attachments on $\xi(e)$, and two vertices of F are adjacent if and only if the corresponding bridges cross.

Suppose the result is false. Then for some component C of F, no bridge in C is over an attachment of a stable bridge. Let P be the minimal subpath of $\xi(e)$ containing all attachments of the unstable bridges in C, and denote its endpoints x, y. We will show that $\{x, y\}$ is a 2-separation of (H, Γ).

Since (H, Γ) is 3-connected and almost simple with respect to ξ, and since C contains at least one ξ-bridge, P must contain an internal vertex w. By choice of C, every internal vertex of P is under an unstable ξ-bridge in C. Since no bridge in C is over an attachment of a stable ξ-bridge, every path in (H, Γ) from w to $\xi(G) \setminus P$ uses a vertex of $\xi(e) \setminus P$.

In particular, every path in (H, Γ) from w to a vertex of $\xi(e) \setminus P$ uses one of x or y – otherwise, this path would be part of an unstable ξ-bridge B, such that B crossed a bridge in C. But then C would not be a component of F, a contradiction. It follows that every path in (H, Γ) from w to $(G, \Sigma) \setminus P$ uses x or y, and so (H, Γ) has a 2-separation. This completes the proof. □

Proof of Theorem 2.3.1

Let $\eta' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be a homeomorphic embedding parallel to η such that no homeomorphic embedding parallel to η is preferred to η'. We claim that η' is as desired.

Suppose for a contradiction that there exists a good unstable η'-bridge B_1 with attachments in $\eta(e)$ for some $e \in E(G)$. By Lemma 2.3.2, (H, Γ) is almost simple with respect to η' and so Lemma 2.3.7 applies. Let $B_1, B_2, ..., B_k$ be a sequence as in Lemma 2.3.7, and let B_S be a stable η'-bridge with an attachment under B_k. By Lemma 2.3.3, B_k is bad with respect to B_S. Since B_i does not cross B_k for $i \in [k-2]$, it follows from Lemma 2.3.4 that $B_1, B_2, ..., B_{k-2}$ are good with respect to η'. By Lemma 2.3.5, we may also assume that B_{k-1} is bad with respect to η'. If $k \geq 3$, then Lemma 2.3.6 applied to B_{k-1} and B_{k-2} tells us that there exists a homeomorphic embedding $\eta'' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ such that η'' is preferred to η' – a contradiction. It follows that every unstable η'-bridge is bad, proving the first part of the theorem.

We now prove the "furthermore". First, suppose for a contradiction that for some $e \in E(G)$, $\eta'(e)$ contains the attachments of two distinct maximal sets $\mathcal{B}_1$ and $\mathcal{B}_2$ of pairwise intersecting η'-bridges. As we just proved, every η'-bridge in $\mathcal{B}_1$ or $\mathcal{B}_2$ is bad. By Lemma 2.3.7, there exists a bridge B in one of $\mathcal{B}_1$ or $\mathcal{B}_2$ such that B is over an attachment z of a stable η'-bridge. Without loss of generality, we assume that $B \in \mathcal{B}_1$. If some unstable η'-bridge $B' \in \mathcal{B}_2$ does not cross B, then we can reroute along B and B' as in Lemma 2.3.4 to get a homeomorphic embedding $\eta'' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ preferred to η' – a contradiction. So B must cross every η'-bridge in $(\mathcal{B}_1 \cup \mathcal{B}_2) \setminus \{B\}$. Let B_1, B_2 be distinct bridges such that $B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2$. Then (by assumption that $\mathcal{B}_1$, $\mathcal{B}_2$ are maximal) B_1, B_2 do not intersect, and we may assume that for some $i = 1, 2$, z is not under B_i and is not an attachment of B_i. Then we can reroute along B and B_i as in lemma 2.3.4 to get a homeomorphic embedding $\eta'' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ that is preferred to η' – a contradiction. So $\eta'(e)$ contains the attachments of at most one maximal set $\mathcal{B}$ of pairwise intersecting unstable η'-bridges.

Let $\mathcal{B}$ be such a set of pairwise intersecting η'-bridges with attachments on $\eta'(e)$. Suppose $z \in (V(P) \setminus Z) \setminus \{x, y\}$ is an attachment of a stable η'-bridge. Then there exist unstable η'-bridges $B, B' \in \mathcal{B}$ such that z is under B, but is not under $\bar{B}$. Rerouting along B and B' as in Lemma 2.3.4 gives a homeomorphic embedding $\eta'' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ that is preferred to η' –a contradiction. Thus (B2) holds. By Lemma 2.3.7, there exists a bridge $B \in \mathcal{B}$ such that B is over an attachment z of a stable η'-bridge. Then $z \in V(P) \setminus \{x, y\}$. This, together with (B2), gives (B1). By Lemma 2.3.3, (B3) holds as well. This completes the proof.

□

2.4 Proof of Theorem 2.1.1

In this section we prove Theorem 2.1.1, which specifies the "minimal non-planar extensions" of a signed graph (G, Σ) Π-embedded in a surface S with representativity at least 3, where G is simple. The reader should note, however, that most of the lemmas stated in this section do not assume that G is simple, and will be applied again in the next section. We begin with a result from [11]:

Remark. *Let G be a simple, 3-connected planar graph, with planar embedding Π. Let C_1, C_2 be two distinct Π-facial circuits of G. Then C_1, C_2 intersect in a complete graph on at most two vertices (possibly the null graph).*

We will need an analogue of that applies when (G, Σ) is Π-embedded in an arbitrary surface S with representativity at least 3, and when G may have parallel edges. This is supplied by the following:

Lemma 2.4.1. *Let (G, Σ) be a simple 3-connected signed graph. Let S be a surface, and let Π be an embedding of (G, Σ) in S with representativity at least 3. Then, if G is simple, two distinct Π-facial circuits of G intersect in a complete graph on at most two vertices (possibly the null graph). If G is not simple, two Π-facial circuits of G may additionally intersect exactly in two adjacent vertices.*

Proof. Suppose for a contradiction that there exist distinct Π-facial circuits C_1, C_2 of G that intersect in two non-adjacent vertices. Call these vertices x, y. Let s_1 be a curve in S with endpoints x, y, and whose interior is interior to C_1. Let s_2 be a curve in S with endpoints x, y, and whose interior is interior to C_2. Let $s = s_1 \cup s_2$.

If s is homologous to zero, then s separates (G, Σ) into two parts (by the Jordan Curve Theorem: the part inside s, and the part outside s). Since x, y are not adjacent, there is at least one vertex in each of these parts. So (G, Σ) has a vertex 2-separation, contradicting the connectedness of (G, Σ).

If s is not homologous to zero, then (G, Σ) has representativity at most 2 (by definition of representativity). This completes the proof. $\square$

The following generalizes Remark (1) of the proof of Theorem (3.4), in [11]:

Lemma 2.4.2. *Let (G, Σ) be a simple 3-connected signed graph with $|V(G)| \geq 5$. Suppose (G, Σ) is Π-embedded on surface S with representativity at least 3. Let (H, Γ) be a signed graph, and let $\xi : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be a homeomorphic embedding such that (H, Γ) is almost simple with respect to ξ. Suppose neither of (a1), (a4) hold for ξ. Let B be a stable ξ-bridge. Then there exists a Π-potential facial circuit C of G such that all attachments of B are in $V(\xi(C))$. Furthermore, if C, C' are distinct Π-potential facial circuits containing all attachments of B, then $C \,\Delta\, C'$ is the union of some number of pairs of parallel edges of G. Furthermore, for each edge $e \in C \,\Delta\, C'$, $V(\xi(e))$ contains no attachment of B.*

Proof. We will use the same methods as in [11]. Let ξ and B be as stated, and let A be the set of all attachments of B. Since (a1) does not hold for ξ, we deduce that for every pair of elements $a_1, a_2 \in A$ there exists a Π-potential facial circuit C in G such that $a_1, a_2 \in V(\xi(C))$. Since (a4) does not hold for ξ, we deduce that the same holds for every triple of elements of A.

Now, let $k \geq 3$ be an integer such that for every k-element subset A' of A there exists a Π-potential facial circuit C in G such that $A' \subseteq V(\xi(C))$. We shall prove that the same holds for every $(k+1)$-subset of A. To this end, suppose for a contradiction that $a_1, a_2, ..., a_{k+1}$ are distinct elements of A such that $a_1, a_2, ..., a_{k+1} \in V(\xi(C))$ for no Π-potential facial circuit C of G. For $i = 1, 2, ..., k+1$ let C_i be a Π-facial circuit of G such that $V(\xi(C_i))$ includes all of $a_1, a_2, ..., a_{k+1}$ except a_i. Then these circuits are pairwise distinct. Since a_1 and a_2 belong to both $V(\xi(C_3))$ and $V(\xi(C_4))$, there exists by Lemma an edge $e_{12} \in E(G)$ such that C_3, C_4 intersect either in e_{12} or in the endpoints of e_{12}. Similarly, there is an edge $e_{ij} \in E(G)$ such that $a_i, a_j \in V(\xi(e_{ij}))$ for all distinct integers $i, j = 1, 2, ..., k+1$. Now for all $i = 1, 2, ..., k+1$, the vertex a_i is an end of $\xi(e_{ij})$, for otherwise the edges $e_{ij}(j \in \{1, 2, ..., k+1\} - \{i\})$ would all be equal, implying that $a_1, a_2, ..., a_{k+1}$ all belong to $V(\xi(C_t))$ for all $g = 1, 2, ..., k+1$, a contradiction. Thus there exist vertices $u_1, u_2, ..., u_{k+1} \in V(G)$ such that $\xi(u_i) = a_i$. It follows that $\{u_1, u_2, ..., u_{k+1}\}$ is the vertex-set of a complete subgraph of G.

Note that any two of the Π-potential facial circuits C_i, C_j share $k-1$ vertices. Notice that deleting one edge from each pair of parallel edges of G gives a graph $\mathrm{si}(G)$ containing a Π-facial circuit C'_i with $V(C'_i) = V(C_i)$, and a Π-facial circuit C'_j with $V(C'_j) = V(C_j)$. By Lemma 2.4.1, it follows that $k \leq 3$. So we must in fact have $k = 3$. Then G' is isomorphic to K_4. Since G is 3-connected, we see that $|V(G)| = 4$ and G contains K_4 as a subgraph, for otherwise it is not true that for every triple of elements of A there is a peripheral circuit C of G such that $V(\xi(C))$ includes the triple. But $|V(G)| \geq 5$. This gives the contradiction.

It follows inductively that there exists a Π-facial circuit C of G such that $A \subseteq V(\xi(C))$. From Lemma 2.4.1 and the definition of a stable ξ-bridge, it follows that C is unique up to possibly exchanging parallel edges that do not contain an attachment of B.

□

The above remark suggests that if we can somehow "remove" the unstable η'-bridges in H, the proof will be much easier. Our next result allows us to do exactly that:

Lemma 2.4.3. *Let (G, Σ) be a simple 3-connected signed graph with $|V(G)| \geq 5$. Let (G, Σ) be Π-embedded on S with representativity at least 3. Let (H, Γ) be a signed graph, and let $\xi : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be a homeomorphic embedding. Suppose (H, Γ) has no embedding on S that extends from an embedding closely related to Π, and that none of (a1) - (a8) from Theorem 2.1.1 hold for ξ. Let $\bar{H}$ be the graph obtained from H by deleting the unstable ξ-bridges. Then $\bar{H}$ has an embedding on S that extends from an embedding closely related to Π if and only if H does.*

Proof. Clearly, if $\bar{H}$ is has no embedding on S that extends from an embedding closely related to Π, then neither does H. Now suppose H has no such embedding, and suppose by way of contradiction that $\bar{H}$ does. For each $e \in E(G)$, let $\mathcal{B}_e$ be the set of unstable ξ-bridges with attachments on $\xi(e)$. We use the following claim:

(1). *There exists an edge $f \in E(G)$ such that $\bar{H}$ together with the unstable bridges in B_f has no embedding on S extending from an embedding closely related Π.*

To prove (1), we first note that since H has no such embedding there exists a minimal set $\{f_1, f_2, ..., f_k\}$ of edges of G such that $\bar{H}$ together with the unstable bridges in $\cup_{i=1}^{k} \mathcal{B}_{f_i}$ has no such embedding. By way of contradiction, suppose $k \geq 2$. Let $\bar{H}_1$ be the graph given by $\bar{H}$ together with the unstable bridges in $\cup_{i=1}^{k-1} \mathcal{B}_{f_i}$, and let $\bar{H}_2$ be the graph given by $\bar{H}$ together with the bridges in $\mathcal{B}_{f_k}$. By minimality, $\bar{H}_i$ has an embedding Π_i extending from Π, for $i = 1, 2$. By Lemma 2.4.2, for each stable ξ-bridge B_S of $\bar{H}$ there is a unique Π-potential facial circuit C of G such that B_S can be drawn in $\xi(C)$, up to possibly exchanging parallel edges of C, where for an exchanged edge e $V(\xi(e))$ contains no attachment of B_S. We may thus assume that the restrictions of Π_1, Π_2 to $\bar{H}$ are closely related. Now, it is clear that no bridge in $\cup_{i=1}^{k-1} \mathcal{B}_{f_i}$ crosses a bridge in $\mathcal{B}_{f_k}$. It follows that these bridges can be added to $\Pi_2(\bar{H}_2)$ without crossings. But this gives an embedding of $\bar{H}$ together with the unstable bridges in $\cup_{i=1}^{k}\mathcal{B}_{f_i}$ in B_S that extends from an embedding closely related to Π, contradicting our choice of $\{f_1, f_2, ...f_k\}$. This completes the proof of (1).

Let $f \in E(G)$ be an edge with this property. Let Π' be an embedding of $\bar{H}$ that extends from an embedding closely related to Π. From Theorem 2.3.1, we know that the bridges in $\mathcal{B}_f$ are pairwise intersecting. For each $B \in \mathcal{B}_f$, let P_B be the minimal subpath of $\xi(f)$ containing all attachments of B. Let $Z = \cap_{B\in\mathcal{B}} V(P_B)$. By the claim, some stable ξ-bridge B_S has an attachment $z \in Z$.

Since (a5) does not hold for ξ, any set of pairwise crossing bridges in $\mathcal{B}_f$ contains at most two bridges. Suppose Z is a vertex, say x. Then there is some stable ξ-bridge B_S with x as an attachment. Let C_1 and C_2 be the Π-potential facial circuits of G containing f, such that $\xi(C_1)$ contains all the attachments of B_S.

Since (a1), (a6) do not hold for ξ, no two unstable bridges over x cross. It follows that all the unstable bridges in $\mathcal{B}_f$ that are over x can be added to $\Pi'(\bar{H})$ in the Π-face of $\xi(G)$ bounded by $\xi(C_2)$, without producing any crossing edges. Call this set of bridges $\mathcal{A}$. Note that the bridges in $\mathcal{B} \setminus \mathcal{A}$ are pairwise non-crossing, for otherwise there would be three pairwise crossing bridges in $\mathcal{B}$. Furthermore, none of these bridges is over x. It follows that we can add the bridges of $\mathcal{B}\setminus\mathcal{A}$ to $\Pi'(\bar{H})$ in the face of $\xi(G)$ bounded by $\xi(C_1)$ without crossings. But then $\bar{H}$ together with $\mathcal{B}_f$ has an embedding in S that extends from Π', a contradiction. So Z is not a vertex.

Then $\cap_{B\in\mathcal{B}} P_B$ is a path, say with endpoints x and y. Suppose $\mathcal{B}_f$ contains two crossing bridges B_1 and B_2. We may assume that x is an endpoint of P_{B_1} (and is under B_2), and y is an endpoint of P_{B_2} (and is under B_1). Let C_1 and C_2 be the Pi-facial circuits of G that contain f. Since we assume (a1), (a6) do not hold for ξ, there is no stable bridge with an attachment under both B_1 and B_2. Since by Theorem 2.3.1 some stable ξ-bridge has an attachment in Z, we may assume without loss of generality that there exists a stable ξ-bridge B_{S1} with x as an attachment. We may assume by Remark 2.4.2 that all attachments of B_{S1} are in $\xi(C_1)$. Since x is under B_2, B_2 has an attachment interior to $\xi(f)$, and (a1), (a8) do not hold for ξ, we may assume that there is no stable bridge B_S with x as an attachment such that all attachments of B_S are in $\xi(C_2)$.

Suppose no stable ξ-bridge has an endpoint in $Z \setminus \{x\}$. Since we assume (a6) does not hold for ξ, no two unstable bridges over x cross. It follows that all the unstable bridges in $\mathcal{B}_f$ that are over x can be added to $\Pi'(\bar{H})$ in the Π-face of $\xi(G)$ bounded by $\xi(C_2)$ without producing any crossing edges. Call this set of bridges $\mathcal{A}$. Note that the bridges in $\mathcal{B}_f \setminus \mathcal{A}$ are pairwise non-crossing, for otherwise we would have three pairwise crossing bridges in $\mathcal{B}_f$. Furthermore, none of these bridges is over x. So we can add the bridges of $\mathcal{B}_f \setminus \mathcal{A}$

to $\Pi'(\bar{H})$ in the Π-face of $\xi(G)$ bounded by $\xi(C_1)$ without crossings. But then $\bar{H}$ together with the bridges in $\mathcal{B}_f$ has an embedding in S that extends from Π', a contradiction.

It follows that there exists a stable ξ-bridge B_{S2} with an attachment in $Z \setminus x$. Since we established earlier that no stable bridge has an attachment in $Z \setminus \{x, y\}$, B_{S2} must have y as an attachment. Since (a7) does not occur, we may assume that every stable bridge with y as an attachment has all its attachments in $V(\xi(C_2))$. Let $\mathcal{A}$ be the set of ξ-bridges in $\mathcal{B}_f$ that are over x. Since (a1) and (a7) do not hold for ξ, none of the bridges in $\mathcal{A}$ cross. It follows that the bridges in $\mathcal{A}$ can be added $\Pi'(\bar{H})$ in the Π face of $\xi(G)$ bounded by C_2, without crossings. Similarly, the bridges in $\mathcal{B}_f \setminus \mathcal{A}$ can be added without crossings in the Π-face of $\xi(G)$ bounded by C_1. But then $\bar{H}$ together with $\mathcal{B}_f$ has an embedding in S that extends from Π', a contradiction. So $\mathcal{B}_f$ does not contain a pair of crossing bridges.

Suppose no two unstable bridges in $\mathcal{B}_f$ cross. By Theorem 2.3.1, there exists a stable ξ-bridge B_S with an attachment $x \in Z$. By Remark 2.4.2, we may assume that all the attachments of B_S are contained in $\xi(C_1)$. It is easy to see that every point in Z is under some bridge in B_f. Then, since (a1), (a8) do not hold for ξ, it follows that every stable ξ-bridge with an attachment in Z has all attachments in $\xi(C_1)$. Since no two bridges in $\mathcal{B}_f$ cross, we can add all of these bridges to $\Pi'(\bar{H})$ in the face of $\xi(G)$ bounded by C_2, without crossings. But then $\bar{H}$ together with the bridges in $\mathcal{B}_f$ has an embedding in S that extends from Π' - a contradiction.

□

Let $e \in E(G)$, let z, w be the ends of $\eta(e)$, and let P_1, P_2 be two disjoint η-paths in H with ends x_1, y_1 and x_2, y_2, respectively, such that $z, x_1, x_2, y_1, w \in V(\eta(e))$ occur on $\eta(e)$ in the order listed, and $y_2 \notin V(\eta(e))$. Let P_3 be a path disjoint from $V(\eta(G)) - \{y_2\}$ with one end $x_3 \in V(P_1)$ and the other $y_3 \in V(P_2)$. We say that the triple P_1, P_2, P_3 is an *η-tripod*, and that the paths $\eta(e)[z, x_1]$, $\eta(e)[y_1, w]$ and $P_2[y_2, y_3]$ are its *legs*.

Lemma 2.4.4. *Let (G, Σ) be a simple 3-connected signed graph with $|V(G)| \geq 5$. Suppose that (G, Σ) has an embedding on a surface S with representativity at least 3. Let (H, Γ) be a signed graph, and let $\xi : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be a homeomorphic embedding. Suppose (H, Γ) has no embedding on S that extends from an embedding closely related to Π. Suppose none of (a1) - (a8) from Theorem 2.1.1 holds for ξ, and that for some $e \in E(G)$ there exists a ξ-tripod. Then one of (a9), (a10), and (b3) holds for a homeomorphic embedding parallel to ξ. Furthermore, if G is simple, then one of (a9), (a10) holds for a homeomorphic embedding parallel to ξ.*

Proof. We choose a homeomorphic embedding ξ' parallel to ξ and a ξ' tripod P_1, P_2, P_3 such that the sum of the lengths of the tripod's legs is minimum. Let $e, x_1, y_1, x_2, y_2, x_3, y_3$

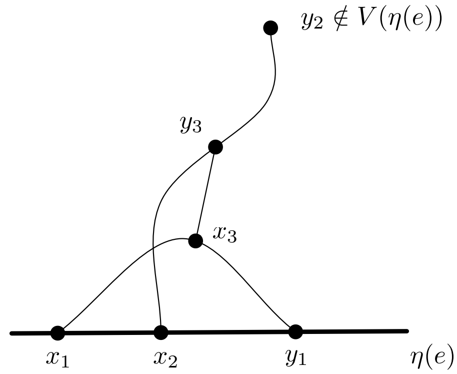


Figure 2.8: An η-tripod.

be as in the definition of a tripod. By possibly resigning, we may assume that every edge of $\xi'(e)$ is even.

Let X' be the vertex-set of $\xi(e)[x_1, y_1] \cup P_2[x_2, y_3] \cup P_1 \cup P_3$, and let $Y' = V(\xi(G)) - (X' - \{x_1, y_1, y_3\})$. If there is no path between X' and Y' in $H \setminus \{x_1, y_1, y_3\}$, then there exists a separation (X, Y) of order three with $X' \subseteq X$ and $Y' \subseteq Y$ (and hence $X \cap Y = \{x_1, y_1, y_3\}$). Then ξ and (X, Y) satisfy (a3), a contradiction. Thus there exists a path P in $H \setminus \{x_1, y_1, y_3\}$ with ends $x \in X'$ and $y \in Y'$.

Suppose P_1 is even. Let ξ' be obtained from ξ by rerouting $\xi(e)[x_1, y_1]$ along P_1; then $\xi(e)[x_1, y_1]$, $P_3 \cup P_2[y_3, y_2]$, $P_2[x_2, y_3]$ is a ξ'-tripod with the same legs. Thus there is symmetry between $\xi(e)[x_1, y_1] \cup P_2[x_2, y_3]$ and $P_1 \cup P_3$, and we may assume that $x \in V(P_1) \cup V(P_3) - \{x_1, y_1, y_3\}$. By the minimality of the legs, $y \notin V(\xi(e)) \cap V(P_2)$.

Since the vertices x_2, y_2, y are attachments of a stable ξ'-bridge, by Remark 2.4.2 there exists a potential facial circuit C in G such that $x_2, y_2, y \in V(\xi(C))$. Since x_2 is an internal vertex of $\xi'(e)$, we see that C must be a potential facial circuit of G that contains e; otherwise, P_2 is a ξ'-path satisfying (a1). Since $y \neq y_2$ (because $y \notin V(P_2)$), $P_1 \cup P_2 \cup P_3 \cup P$ includes a special ξ'-cross in C, a contradiction.

Now suppose P_1 is odd. If $x \in V(P_1) \cup V(P_3)$, we proceed as above. Now, suppose $x \in V(\xi(e)[x_1, y_1]) \cup V(P_2[x_2, y_3])$.

Then $y \notin P_2$ (by minimality of legs). Again, $y_2 \in V(\xi'(C))$ for some potential facial circuit C of G containing e, for otherwise P_2 is a ξ-path satisfying (a1). Suppose $y \in V(\xi'(C)) \setminus V(\xi'(e))$. Since $y \notin P_2$, $y \neq y_2$. Then $P_1 \cup P_2 \cup P_2 \cup P$ contains a special ξ-cross in C_1. So we may assume $y \notin V(\xi(C_1)) \setminus V(\xi(e))$.

Suppose $x \in V(P_2[x_2, y_3])$. Suppose $y \notin V(\xi(C)) \setminus V(\xi(e))$, i.e. $y \in V(\xi(e))$. Without loss of generality, we may assume $y \in V(\xi(e)[y_1, q])$, where q is an endpoint of $\xi(e)$ such that $x_1 \notin V(\xi(e)[y_1, q])$. $P_2[x_2, x] \cup P$ is even, we can reroute $\xi(e)[x_2, y]$ along $P_2[x_2, x] \cup P$ to get a homeomorphic embedding $\xi' : (G, \Sigma) \hookrightarrow (H, \Gamma)$. But then $P_1 \cup \xi(e)[y_1, y], P_2[x, y_2], P_3$ is a ξ'-tripod with sum of legs smaller than the original tripod, a contradiction. So $P_2[x_2, x] \cup P$ must be odd.

Then we can reroute $\xi(e)[x_1, x_2]$ and $\xi(e)[x_2, y]$ along $P_1, P_2[x_2, x] \cup P$, respectively. This gives a homeomorphic embedding $\xi' : (G, \Sigma) \hookrightarrow (H, \Gamma)$. Suppose we resign (H, Γ) such that every edge of $\xi'(e)$ is even. Then $\xi(e)[x_1, x_2]$, $\xi(e)[y_1, y]$ are odd ξ'-paths with endpoints x_1, u_1, x_2, y occurring on $\xi'(e)$ in that order, and $P_3 \cup P_2[y_2, x]$ is a subset of a ξ'-bridge with an attachment under each path. It follows that (a10) holds for ξ-prime, a contradiction.

Now suppose $x \in V(\xi(e)[x_1, y_1])$. Without loss of generality, we may assume $x \in V(\xi(e)[x_2, y_1]$. Suppose $y \in V(\xi(e))$. If P is even, we can reroute $\xi(e)[x, y]$ along P to give a homeomorphic embedding $\xi' : (G, \Sigma) \hookrightarrow (H, \Gamma)$. Suppose $y \in V(\xi(e)[y_1, q])$, where q is an endpoint of $\xi(e)$ such that $x \notin V(\xi(e)[y_1, q])$. Then $\xi(e)[y, y_1] \cup P_1$, P_2, P_3 is a ξ'-tripod with sum of legs smaller than the original tripod, a contradiction. Now suppose $y \in V(\xi(e)[y_1, r])$, where r is an endpoint of $\xi(e)$ such that $x \notin V(\xi(e)[y_1, r])$. Then $\xi(e)[y, x_1] \cup P_1$, $P_2 \cup \xi(e)[x, x_2]$, P_3 is a ξ'-tripod with sum of legs smaller than the original tripod, a contradiction. So P is odd.

Now, P is part of some stable ξ-bridge, B_S. We may assume that B_S does not contain P_1, P_2, P_3, for then we are in one of the cases treated above. Then there must exist a path P' with endpoint x' in $V(P) \setminus \{x, y\}$ and other endpoint y' in $V(\xi(G)) \setminus (V(X) \cup V(\xi(e)) \cup V(P_2[y_3, y_2]))$. If $y' \in V(C)$, then there is a special ξ-cross in C. So we may assume $y \notin V(C)$.

Then we must have $y' \in V(\xi(C') \setminus V(\xi(e))$, where $C' \neq C$ is a potential facial circuit of G containing e, for otherwise $P[x, x'] \cup P'$ is a ξ-path satisfying (a1).

Then P_1 is an odd ξ-path with endpoints on $\xi(e)$, and $P_2, P[x, x'] \cup P'$ are ξ-paths with one endpoint in $\xi(e)[x_1, y_1] \setminus \{x_1, y_1\}$, and the other endpoint in $V(C) \setminus V(\xi(e))$, $V(C') \setminus V(\xi(e))$, respectively. Suppose x_1, y_1 are the endpoints of $\xi(e)$, e, f are a pair of multiple edges in G, and $C' = \{e, f\}$. Then $\{x_1, y_1, y_3\}$ remains a 3-separation unless there exists another path from X to Y. By our previous work, we may assume this path does not have an endpoint in X'. So the path must have one endpoint x'' in $V(\xi'(f))$. Then the other endpoint y'' of the path must be in $V(\xi'(C'')) \setminus V(\xi'(f))$, for some potential facial circuit C'' of G. We may assume C'' and C are not related, for otherwise one of (b4), (b5) of Theorem 2.1.2 holds for ξ'. Then this gives (b3) of Theorem 2.1.2. Otherwise,

$P_1, P_2, P[x, x'] \cup P'$ satisfy (a9). This completes the case analysis.

□

The following is proved in [11], although the corrseponding statement in [11] is slightly weaker. As it is consequently not obvious that our statement follows from [11], we repeat the proof here.

Lemma 2.4.5. *Let (G, Σ) be a simple 3-connected signed graph Π-embedded on a surface S with representativity at least 3. Let (H, Γ) be a non-planar signed graph. Let $\xi : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be a homeomorphic embedding. Let C be a Π-potential facial circuit of G, and let H_C be the union of $\xi(C)$ and the stable ξ-bridges with all attachments in $\xi(C)$. If H_C has no planar embedding in which $\xi(C)$ bounds a face, then (H, Γ) has a ξ-tripod, a special ξ-cross in C, or a separation (X, Y) satisfying (a3).*

We will need the following result from [12]:

Remark 2.4.6. *Let G be a graph, and let C be a circuit in G. Then one of the following conditions holds:*

(i) the graph G has a planar embedding in which C bounds a face,

(ii) there exists a separation (A, B) of G of order at most three such that $V(C) \subseteq A$ and $G|B$ does not have a drawing in a disc with the vertices in $A \cap B$ drawn on the boundary of the disc,

(iii) there exist two disjoint paths in G with ends $s_1, t_1 \in V(C)$ and $s_2, t_2 \in V(C)$, respectively, and otherwise disjoint from C such that the vertices s_1, s_2, t_1, t_2 occur on C in the order listed.

Proof of Lemma 2.4.5 By Remark 2.4.6, either there exists a separation (A, B) of H_C of order at most 3 such that $V(\eta(C)) \subseteq A$ and $G|B$ does not have a drawing in a disc with the vertices of $A \cap B$ drawn on the boundary of the disc, or there exists an η-cross P_1, P_2 in C.

In the first case, $(B, A \cup V(H) \setminus B)$ is a separation of H satisfying (a3). Now suppose the second case occurs. Let u_1, v_1 be the ends of P_i. Suppose there exists an edge $e \in E(G)$ such that $u_1, v_1, u_2, v_2 \in V(\eta(e))$. Since the η-bridge containing P_1 is stable, there exists a path P between P_1 and a vertex $v \in V(\eta(G)) \setminus V(\eta(e))$, disjoint from $V(\eta(G)) \setminus \{v\}$. It follows that $P_1 \cup P_2 \cup P$ includes an η-cross whose feet are not contained in $\eta(e)$ for any $e \in E(G)$. Let P_1', P_2' denote this cross.

For $i = 1, 2$, let x_i, y_i denote the ends of P_i'. Suppose P_1, P_2 is not special. Then we may assume $x_1, y_1 \in V(\eta(e))$ for some $e \in E(G)$. Then one of x_2, y_2 belongs to $V(\eta(e))$ and the other does not. We may therefore assume that $x_2 \in V(\eta(e))$; then x_1, x_2, y_1 occur on $\eta(e)$ in the order listed.

For $i = 1, 2$ let B_i be the η-bridge containing P_i. If $B_1 = B_2$, then there exists a path P_3' as in the definition of a tripod. Now suppose we may assume $B_1 \neq B_2$. Since B_1 is stable there exists a path P_3 in B_1 with one end in $V(P_1) - \{x_1, y_1\}$ and the other end $z \in V(\eta(G)) - V(\eta(e))$. If $z \neq y_2$, then $P_1' \cup P_2' \cup P_3'$ includes a special cross, and if $z = y_2$, then P_1', P_2', P_3' is an η-tripod in (H, Γ).

□

We are now ready to proceed with the proof of the main theorem.

Proof of Theorem 2.1.1

By induction on $|V(H)| + |E(H)|$. We may assume that (H, Γ) is almost simple with respect to η. If it is not, let (H', Γ') be the underlying almost simple graph of (H, Γ). The result then follows by applying the inductive hypothesis to (H', Γ'). We may also assume that (H, Γ) is 3-connected. If not, there exists a separation (A, B) of (H, Γ) of order at most 2, such that $A - B$ and $B - A$ are both non-empty. We pick such a separation of smallest possible order. Since (G, Σ) is 3-connected, we may assume (without loss of generality) that $\eta(V(G)) \subseteq A$. If the order of the separation is 1, let (J, Δ) be the restriction of (H, Σ) to A; otherwise, let (J, Δ) be obtained from the restriction of (H, Γ) to A by adding an even edge and an odd edge joining the two elements of $A \cap B$. Then η can be modified in the obvious way to give a homeomorphic embedding $\eta' : (G, \Sigma) \hookrightarrow (J, \Delta)$. If (J, Δ) is planar, then the restriction of (H, Γ) to B does not have an embedding in the disc with the vertices of $B \cap A$ on the boundary of the disc (since (H, Γ) is non-planar). Then the separation (A, B) satisfies (a3). So we may assume (J, Δ) is non-planar. Then the result follows by applying the inductive hypothesis to (J, Δ).

Thus we may assume that (H, Γ) is almost simple with respect to η, and is 3-connected. Suppose by way of contradiction that there does not exist a homeomorphic embedding parallel to η such that one of (a1) -(a10) holds. Then by Lemma 2.4.3, we may assume that all η-bridges are stable – otherwise, the result follows by applying the inductive hypothesis to the graph $(\bar{H}, \Gamma \cap E(\bar{H}))$ of Lemma 2.4.3.

For every peripheral circuit C of G let H_C be the union of $\eta(C)$ and all stable η-bridges B whose attachments are included in $V(\eta(C))$. Since (H, Γ) has no embedding on S that extends from Π, there exists some Π-facial circuit C of H such that H_C does not have a planar drawing with C bounding the infinite region. By Lemma 2.4.5, may assume that for some $e \in E(G)$ there exists an η-tripod. But then by Lemma 2.4.4, one of (a9), (a10)holds for some homeomorphic embedding $\eta' : (G, \Sigma) \hookrightarrow (H, \Gamma)$ parallel to η, a contradiction. This completes the proof of Theorem 2.1.1.

□

2.5 Proof of Theorem 2.1.2

In this section we prove the main result for a Π-embedded signed graph (G, Σ) where G need not be simple. We will approach the proof as follows: First, we will reduce the given graphs (G, Σ) and (H, Γ) using a process we will define as "zipping", such that G is rendered simple, and some important aspects of the relationship between (G, Σ) and (H, Γ) are preserved. If after this reduction (H, Γ) still has no embedding in S extending from an embedding of G closely related to Π, then we will apply Theorem 2.1.1 to get the result. Otherwise, we will complete the proof using methods similar to those in the proof of Theorem 2.1.1.

We will begin by defining our reduction operation, and describing its properties.

Let e_1, e_2 be a pair of parallel edges of G. By possibly adding vertices of degree 2 to one of $\eta(e_1), \eta(e_2)$, we may assume that $\eta(e_1)$ and $\eta(e_2)$ have the same length. Let $x_1, x_2, ..., x_k$ denote the vertices of $\eta(e_1)$, and let $y_1, y_2, ..., y_k$ denote the vertices of $\eta(e_2)$, occurring on $\eta(e_1), \eta(e_2)$ in that order, where $x_1 = y_1$ and $x_k = y_k$. Let H' be the graph obtained from H as follows: Delete every η-bridges whose attachments are contained in $V(\eta(e_1)) \cap V(\eta(e_2))$. Replace $\eta(e_1)$ and $\eta(e_2)$ by an even path $\eta(e)$, and let $v_1, v_2, ..., v_k$ be the vertices of $\eta(e)$, occurring on $\eta(e)$ in that order, where $v_1 = x_1$ and $v_k = x_k$. For each remaining η-bridge of H, replace an attachment x_i or y_i of B with v_i, $i = 1, 2, ..., k$. We will say that H' was obtained from H by *zipping* e_1 and e_2, and will use $z(H, \Gamma)$ to denote the graph obtained from (H, Γ) by zipping all pairs of parallel edges of (G, Σ).

Let $\mathrm{si}(G, \Sigma)$ denote the graph obtained from (G, Σ) by deleting the odd edge in each pair of parallel edges. We will use $\mathrm{si}(G)$ to refer to the underlying unsigned graph of $\mathrm{si}(G, \Sigma)$, and will denote the induced embedding of $\mathrm{si}(G)$ from Π by Π.

We would like to develop a correspondence between the Π-potential facial circuits of (G, Σ), and the Π-faces of $\mathrm{si}(G, \Sigma)$. Since G possesses Π-faces of degree 2 but $\mathrm{si}(G, \Sigma)$ does not, we cannot find such a correspondence for every Π-potential facial cycle of (G, Σ). We will thus limit the rest of our investigation to those Π-potential facial circuits of (G, Σ) that contain more that two edges.

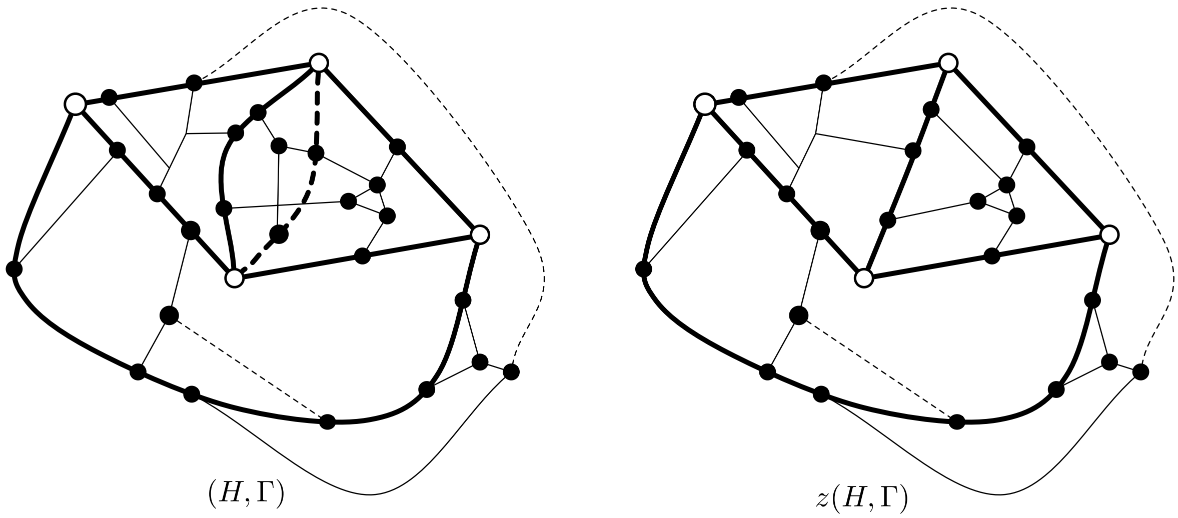


Figure 2.9: Zipping (H, Γ). Thick edges and large vertices belong to $\eta(G)$, dotted edges are odd.

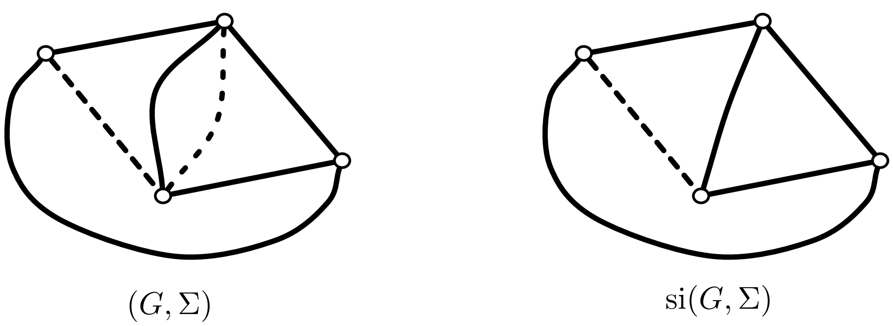


Figure 2.10: Simplifying (G, Σ). Dotted edges are odd.

Let C be such a Π-potential facial circuit of (G, Σ). Suppose $e_1, ..., e_i$ are the edges of C that are odd and belong to a parallel pair in G. Then $\mathrm{si}(G, \Sigma)$ contains a unique cycle C' where $C' \cap C = C \setminus \{e_1, ..., e_i\}$. Note that for every Π-facial cycle C' of $\mathrm{si}(G, \Sigma)$ there exists a (not necessarily unique) cycle C of (G, Σ) that corresponds to C' in this way.

We will formalize this idea defining a function f from the Π-potential facial circuits of (G, Σ) with more than two edges to the Π-facial circuits of $\mathrm{si}(G, \Sigma)$. Suppose e_1, e_2 are parallel edges in G, where e_1 is the even edge, and consequently is in $E(\mathrm{si}(G))$. Define $f(e_i) := e_1$, for $i = 1, 2$. For a set of edges $F \subseteq E(G)$, let $f(F) = \{f(e) : e \in F\}$, and for a facial cycle C' of $\mathrm{si}(G)$, let $f^{-1}(C') = \{C : C$ is a facial circuit of G, and $f(C) = C'\}$. We will also define $f(\eta(F)) = \eta(f(G))$ and $f^{-1}(\eta(F)) = \eta(f_{-1}(F))$, for any $F \subseteq E(G)$.

We will say that two Π-potential facial circuits C_1, C_2 of (G, Σ) are *related* if $f(C_1) = f(C_2)$. (Note that a Π-potential facial circuit is related to itself.) It is easy to see that if two Π-potential facial circuits are related, then they have all vertices in common. The converse also holds:

Lemma 2.5.1. *Let G be a 3-connected graph Π-embedded on surface S with representativity at least 3. If two Π-potential facial circuits C_1, C_2 intersect in two non-adjacent vertices, then C_1 and C_2 are related.*

Proof. Suppose by way of contradiction that C_1, C_2 intersect in two non-adjacent vertices x, y, but are not related. Then C_2 contains some vertex that is not in C_1, and C_1 contains some vertex that is not in C_2. Let $f(C_1) = C_1'$ and $f(C_2) = C_2'$. Then C_i, C_i' contain the same vertices for $i = 1, 2$, and so C_1', C_2' are Π-facial circuits of $\mathrm{si}(G, \Sigma)$ that intersect in two non-adjacent vertices. Then by Lemma 2.4.1 we have $C_1' = C_2'$. □

Given a set $\mathcal{S}$ of related Π-potential facial circuits of G, we also wish to know how many circuits of $\mathcal{S}$ can be Λ-facial circuits of G, for an embedding Λ closely related to Π. The following gives an answer:

Lemma 2.5.2. *Let (G, Σ) be a simple, 3-connected signed graph. Suppose C_1, C_2 are distinct related Π-potential facial circuits in G. Let S be a surface. Then there is no embedding Λ of G on S with representativity at least 3 such that C_1, C_2 are both Λ-facial circuits of G.*

Proof. Suppose for a contradiction that C_1, C_2 are both Λ facial circuits of G, for some embedding Λ closely related to Π. By Lemma 2.4.1, we may assume C_1, C_2 intersect in either a single vertex, or in two adjacent vertices. By the definition of related circuits, $V(C_1) = V(C_2)$. So each of C_1, C_2 contains exactly two vertices. Since (G, Σ) is 3-connected, we must have that $C_1 = C_2$, a contradiction. □

Suppose that (G, Σ) is a simple, 3-connected signed graph Π-embedded on surface S. Let (H, Γ) be a signed graph, and let $\xi : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be a homeomorphic embedding. We remark here that in this case, modifying ξ in the obvious way gives a homeomorphic embedding from $\mathrm{si}(G,\Sigma)$ into (H,Γ). We will denote this homeomorphic embedding by ξ as well.

In the following lemmas, we will use zipping as a tool to deduce structural characteristics of (G,Σ) and (H,Γ) with respect to ξ. First, however, we need to know whether zipping a pair of edges of G in (H,Γ) can create new unstable bridges. The answer is negative:

Lemma 2.5.3. *Let (G,Σ) is a simple, 3-connected signed graph Π-embedded on surface S. Let (H,Γ) be a signed graph, and let $\xi : (G,\Sigma) \hookrightarrow (H,\Gamma)$ be a homeomorphic embedding. Suppose (H,Γ) has no unstable ξ-bridges. Then zipping a pair of parallel edges of G in (H,Γ) gives a graph with no unstable ξ-bridges.*

Proof. Let e_1, e_2 be a pair of parallel edges of G. Suppose we zip these edges in (H,Γ). It is easy to see that any ξ-bridge of (H,Γ) that has an attachment outside of $V(\xi(e_1)) \cup V(\xi(e_2))$ remains stable. By the definition of zipping, any ξ-bridge with all attachments in $V(\xi(e_1) \cup V(\xi(e_2)$ is deleted, and so is not an ξ-bridge in the resulting graph. This proves the Lemma. □

For signed graphs (G,Σ) and (H,Γ) with homeomorphic embedding $\xi : (G,\Sigma) \hookrightarrow (H,\Gamma)$ and a circuit C of (G,Σ), we will use H_C to denote the union of $\xi(C)$ and the stable ξ-bridges whose attachments are contained in $V(\xi(V))$.

Lemma 2.5.4. *Let (G,Σ) be a simple 3-connected signed graph Π-embedded on surface S. Let (H,Γ) be a signed graph, and let $\xi : (G,\Sigma) \hookrightarrow (H,\Gamma)$ be a homeomorphic embedding. Suppose (H,Γ) has no embedding on S that extends from an embedding closely related to Π, that every ξ-bridge of (H,Γ) is stable, and that (b4), (b6) do not hold for (H,Γ) and ξ. Let e_1, e_2 be a pair of parallel edges of G, and let (H',Γ') be the graph obtained from (H,Γ) by zipping e_1, e_2. Then either $H_{\{}e_1, e_2\}$ does not have a planar embedding in which $\xi(e_1 \cup e_2)$ bounds a face, or (H',Γ') has no embedding on S that extends from an embedding closely related to Π.*

Proof. Suppose (H',Γ') has an embedding on S that extends from an embedding closely related to Π. Suppose $e_1 \in \Sigma$, and let G' be the graph obtained from G by deleting e_1. Note that the Π-potential facial circuits of G that do not contain one of e_1, e_2 are exactly the Π-potential facial circuits of G' that do not contain e_2. Let C'_1, C'_2 denote the Π-facial circuits of G' that contain e_2. Now consider the set $\mathcal{C}$ of Π-potential facial circuits of G that contain one of e_1, e_2 such that for $C \in \mathcal{C}$ we have $C \setminus \{e_1, e_2\} \in \{C'_1 \setminus \{e_2\}, C'_2 \setminus \{e_2\}$. One such circuit is $\{e_1, e_2\}$.

Then $|\mathcal{C}| = 4$, and the elements of $\mathcal{C}$ are C_1, C_1', C_2, C_2', where $C_1 \setminus \{e_1\} = C_1' \setminus \{e_2\}$, and $C_2 \setminus \{e_1\} = C_2' \setminus \{e_2\}$. Note that in any embedding of G, $\{e_1, e_2\}$ is a facial circuit. Since each of e_1, e_2 is in exactly two facial circuits in any embedding of G, exactly one of C_1, C_2 and exactly one of C_1', C_2' is a Π-facial circuit of G . Also, by Lemma 2.5.2, for $i = 1, 2$ exactly one of C_i, C_i' is a Π-facial circuit of G. It follows that either C_1 and C_2' or C_2 and C_1' are Π-facial circuits in G.

Since (b6) does not occur, then there are no two ξ-bridges B_1, B_2 of (H'', Γ'') such that all the attachments of B_1 are contained in $V(\xi(C_1'))$ but not $V(\xi(C_2'))$ and all attachments of B_2 are contained in $V(\xi(C_2'))$ but not $V(\xi(C_1'))$. Also, we cannot have all attachments of B_1 in $V(\xi(C_1))$ but not $V(\xi(C_2))$ and all attachments of $B_2 \in V(\xi(C_2))$ but not $V(\xi(C_1))$. Since (b4) does not occur, then there are no two ξ-bridges B_1, B_2 of (H'', Γ'') such that all the attachments of B_1 are contained in $V(\xi(C_i))$ but not $V(\xi(C_{2+i}))$ and all the attachments of B_2 are contained in $V(\xi(C_{2+i}))$ but not $V(\xi(C_i))$ for some $i = 1, 2$.

Let $\mathcal{B}_1$ be the set of all ξ-bridges B of (H', Γ') whose attachments are contained in C_1', and let A_1 denote the set of all attachments of these bridges in (H, Γ). Similarly, let $\mathcal{B}_2$ be the set of all ξ-bridges B of (H', Γ')whose attachments are contained in C_2', and let A_2 denote the set of all attachments of these bridges in (H, Γ). It follows from the above that either $A_1 \subseteq V(\xi(C_1))$ and $A_2 \subseteq V(\xi(C_2'))$, or $A_1 \subseteq V(\xi(C_1'))$ and $A_2 \subseteq V(\xi(C_2))$. We may assume that the first case occurs. Since (H', Γ') has an embedding on S that extends from an embedding closely related to Π', we are able to draw the bridges of $\mathcal{B}_1$ in $\xi(C_1)$ without crossings, and to draw the bridges of $\mathcal{B}_2'$ in $\xi(C_2')$ without crossings. If we can do the same for the ξ-bridges of H with all attachments in $V(\xi(\{e_1, e_2\}))$, then (H, Γ) has an embedding in S that extends from Π – a contradiction. Thus $H_{\{e_1,e_2\}}$ has no planar embedding in which $\xi(\{e_1, e_2\})$ bounds a face. □

Lemma 2.5.5. *Let (G, Σ) be a simple 3-connected signed graph Π-embedded on surface S. Let (H, Γ) be a signed graph, and let $\xi : (G, \Sigma) \hookrightarrow (H, \Gamma)$ be a homeomorphic embedding. Suppose (H, Γ) has no embedding on S that extends from an embedding closely related to Π, that every ξ-bridge of (H, Γ) is stable, and that (b4), (b5), (b6) do not hold for (H, Γ) and ξ. Then there exists a Π-potential facial circuit C of G such that H_C has no planar embedding in which $\xi(C)$ bounds a face.*

Proof. Suppose H_C has a planar embedding in which $\xi(C)$ bounds a face, for each Π-potential facial circuit of G comprised of two parallel edges. Since (b5) does not hold for (H, Γ) and ξ, we cannot create a ξ-bridge satisfying (b4) by zipping pairs of parallel edges of G. It is easy to see that zipping pairs of parallel edges of G can never create a ξ-bridge satisfying (b6). Also, by Lemma 2.5.3 we see that zipping pairs of parallel edges of G

cannot create an unstable ξ-bridge. Then by repeatedly zipping a pair of parallel edges and applying Lemma 2.5.4, we see that $z(H, \Gamma)$ has no embedding on S that extends from an embedding closely related to Π. By Lemma 2.4.2, there exists a Π-facial circuit C of G such that H_C has no planar embedding where $\xi(C)$ bounds a face. Then C is the required Π-potential facial circuit of G. $\square$

Proof of Theorem 2.1.2

By induction on $|V(H)| + |E(H)|$. We may assume that (H, Γ) is almost simple with respect to η. If it is not, let (H', Γ') be the underlying almost simple graph of (H, Γ). The result then follows by applying the inductive hypothesis to (H', Γ'). We may also assume that (H, Γ) is 3-connected. If not, there exists a separation (A, B) of (H, Γ) of order at most 2, such that $A - B$ and $B - A$ are both non-empty. We pick such a separation of smallest possible order. Since (G, Σ) is 3-connected, we may assume (without loss of generality) that $\eta(V(G)) \subseteq A$. If the order of the separation is 1, let (J, Δ) be the restriction of (H, Σ) to A; otherwise, let (J, Δ) be obtained from the restriction of (H, Γ) to A by adding an even edge and an odd edge joining the two elements of $A \cap B$. Then η can be modified in the obvious way to give a homeomorphic embedding $\eta' : (G, \Sigma) \hookrightarrow (J, \Delta)$. If (J, Δ) is planar, then the restriction of (H, Γ) to B does not have an embedding in the disc with the vertices of $B \cap A$ on the boundary of the disc (since (H, Γ) is non-planar). Then the separation (A, B) satisfies (a3). So we may assume (J, Δ) is non-planar. Then the result follows by applying the inductive hypothesis to (J, Δ).

Thus we may assume that (H, Γ) is almost simple with respect to η, and is 3-connected. Suppose by way of contradiction that there does not exist a homeomorphic embedding parallel to η such that one of (b1) -(b6) holds. Then by Lemma 2.4.3, we may assume that all η-bridges are stable – otherwise, the result follows by applying the inductive hypothesis to the graph $(\bar{H}, \Gamma \cap E(\bar{H}))$ of Lemma 2.4.3.

Then by Lemma 2.5.5, there exists a Π-potential facial circuit C of G such that H_C does not have a planar embedding with $\eta(C)$ bounding a face. It follows by Lemma 2.4.5 that (H, Γ) contains an η-tripod. But then by Lemma 2.4.4, one of (a9), (a10), and (b3) holds for a homeomorphic embedding parallel to η, a contradiction. $\square$

2.6 Some outstanding proofs

In this section we give the proofs that have been bypassed to this point. We will begin with the proof of Corollary 2.1.3, stated in Section 2.1. The following Lemma will be useful:

Lemma 2.6.1. *Any even-face embedding of (G, Σ) that is closely related to Π can be obtained by resigning on a cut X of (G, Σ), where X contains only parallel pairs of edges of G, up to relabelling within pairs of parallel edges.*

Proof. Suppose we obtain embedding Λ from Π by exchange the positions of e_1, e_2, for some number of pairs of parallel edges $e_1, e_2 \in E(G)$. Let $\mathcal{P}$ denote the set of pairs of parallel edges affected by the exchange. Since (G, Σ) is simple, e_1, e_2 differ in parity for each pair of parallel edges $\{e_1, e_2\} \in \mathcal{P}$. It follows that Λ is an even-face embedding of (G, Σ) only if for each Π-facial circuit C of G, $|C \cap \{e : \{e, f\} \in \mathcal{P}\}|$ is even. Consequently, there exists a set of closed curves s on S such that each closed curve intersects G in $\cup_i = 1^m\{e, f\}$, where $\{e, f\}_i \in \mathcal{P}$ for $i = 1, ..., m$. Furthermore, we can choose these curves such that no two curves intersect the same edge. Since the edges intersected by each of these curves corresponds to a cut, it follows that $\Lambda(G, \Sigma)$ can be obtained from $\Pi(G, \Sigma)$ by resigning on the union of these disjoint cuts, and relabeling. □

Proof of Corollary 2.1.3

By Lemma 2.6.1, it suffices to show that (H, Γ) has no embedding on S that extends from the even-face embedding Π of (G, Σ).

Suppose first that (H, Γ) has an embedding Π' on S that extends from Π, but that no such even-face embedding exists. We proceed by induction on $|E(H)|$. We may assume that deleting any edge in $E(H) \setminus E(\eta(G))$ from (H, Γ) gives a graph with an even-face embedding Λ' on S extending from an embedding Λ closely related to Π, for otherwise the result follows by induction on the smaller graph. Since adding edges to Π-embedded signed graph $\eta(G, \Sigma)$ cannot decrease the number of odd faces, it follows that Λ is an even-face embedding of $\eta(G, \Sigma)$, and hence of (G, Σ).

We will assume that (H, Γ) is Π'-embedded on S, and that Π' is such that the number of odd faces is minimum over all embeddings Π' of (H, Γ) that extend from an embedding of (G, Σ) closely related to Π. Now, let $e \in E(H) \setminus E(\eta(G))$. Suppose e is in Π'-facial cycles C_1 and C_2 of (H, Γ). Since deleting e from (H, Γ) gives a graph with an even-face embedding Λ' extending from an even-face embedding Λ closely related to Π, both of C_1, C_2 are odd. Since we suppose that the number of odd Π'-faces of (H, Γ) is minimum, we can assume C_1, C_2 are the only odd facial cycles in (H, Γ).

Now, suppose some edge $f \in E(H) \setminus E(\eta(G))$ lies does not lie in both C_1 and C_2. If f lies in two even faces, then deleting f creates a larger even face from these face. If f lies in an even face and an odd face, then deleting f creates a larger odd face from these two faces. In either case, deleting f does not create a graph with an even face embedding on S, contradicting our choice of (H,Γ). Thus we may assume that all edges of $E(H)\setminus E(\eta(G))$ lie in both C_1 and C_2. Then the edges of $E(H)\setminus E(\eta(G))$ form a path, P. So $C=(C_1\cup C_2)\setminus P$ is a Λ-facial cycle of $\eta(G,\Sigma)$, and is therefore even. Since C_1, C_2 are both odd, it follows that P must be odd, when (H,Γ) has been resigned such that every edge of C is even. Thus (e4) holds for (H,Γ) and η.

Now suppose (H,Γ) has no embedding (even-face or otherwise) on S that extends from Π. Then Theorem 2.1.2 applies. If one of (a1), (a3)holds, we are done. Note that (G,Σ) cannot contain a pair of parallel edges, since two parallel edges of a simple 3-connected signed graph must be the boundary of an odd face in any embedding of the graph. (In this case, (G,Σ) has no even-face embedding). It follows that (b3)-(b6) do not hold.

We will now consider the case where one of the other outcomes holds. If (a2) holds, then resigning C such that every edge is even shows that both paths in the cross must also be even, or (e4) holds. This gives (e2). If (a4) holds, we may similarly assume that no path in the triad is odd when the facial cycle containing its endpoints is resigned to be even. (Otherwise, (e4) holds.) This gives (e3). Since by Lemmas 2.2.1 - 2.2.4, outcomes (a5)-(a8) of Theorem 2.1.1 can be described as a structure containing an η'-path as described in (e4), and since (a9), (a10) contain such a path by definition, if any of these outcomes occur then (e4) also occurs.

This completes the proof.

□

Notice that if (e4) holds, (H,Γ) has no even-face embedding in S that extends from an embedding closely related to Π, as adding the specified η'-path Q to $\Pi(G,\Sigma)$ creates an odd face. It follows from Theorem 2.2.5 that the converse of Corollary 2.1.3 also holds.

Recall that in Section 2.1, we stated several outcomes of Theorem 2.1.1 in terms of bridges. Later, in Section 2.2, we gave lemmas describing these outcomes explicitly in terms of odd η-paths and odd η-triads. In this section, we give the proofs of these Lemmas. We will use slightly weaker versions of these lemmas to prove the results in Section 2.2, and will then give a complete proof of the weaker versions.

Proof of Lemmas 2.2.1 - 2.2.4

Let (G,Σ) be a simple signed graph Π-embedded on surface S with representativity at least 3. Let (H,Γ) be a signed graph, let $\xi:(G,\Sigma)\hookrightarrow(H,\Gamma)$ be a homeomorphic embedding, and suppose (H,Γ) is almost simple with respect to ξ. Suppose further that ξ satisfies Theorem 2.3.1, and that (H,Γ) has no embedding on S that extends from the given embedding of (G,Σ). Suppose further that (a1) does not hold for ξ. Then the following hold:

Lemma 2.6.1. *Suppose (a6) of Theorem 2.1.1 holds for ξ. Then one of (c1)-(c4) of Lemma 2.2.1 also holds.*

Lemma 2.6.2. *Suppose (a5) of Theorem 2.1.1 holds for ξ. Then one of (d1)-(d4) of Lemma 2.2.2 also holds, or (a6) of Theorem 2.1.1 holds for ξ.*

Lemma 2.6.3. *If (a8) of Theorem 2.1.1 holds for ξ, then so does (a9).*

Lemma 2.6.4. *If (a7) of Theorem 2.1.1 holds for ξ, then so does (a10).*

Applying these Lemmas, we see that by assuming none of (a1), (c1)-(c4), (d1)-(d4), (a9), (a10) hold, it follows that none of (a5)-(a8) hold when ξ satisfies Theorem 2.3.1. Recall that in the proofs which required the assumption that none of (a5)-(a8) hold, namely those of Lemma 2.4.4, Lemma 2.5.3, Theorem 2.1.1, and Theorem 2.1.2, we assume that the homeomorphic embedding satisfies Theorem 2.3.1 and that (a1) does not hold. It follows that our results remain true when we replace (a5)-(a8) in the statement of Theorem 2.1.1 as described in Lemmas 2.2.1 - 2.2.4. □

We will now prove Lemmas 2.6.1 - 2.6.4.

Proof of Lemma 2.6.1

Proof. Suppose (a6) holds, i.e. for some $e \in E(G)$ there exist crossing bridges B_1, B_2 with attachments on $\xi(e)$, and a ξ-path $\bar{P}$ with endpoints w, z such that w is under both B_1, B_2 and $z \in V(\xi(C)) \setminus V(\xi(e))$ for some potential facial circuit C of G that contains e.

Let P_1 be the minimal subpath of $\xi(e)$ containing all attachments of B_1, and let P_2 be the minimal subpath of $\xi(e)$ containing all attachments of B_2. For $i = 1, 2$. let x_i, y_i denote the endpoints of P_i. Let Q_1 be a ξ-path in B_1 with endpoints x_1, y_1. Note that by Theorem 2.3.1, Q_i is odd.

If x_1, x_2, y_1, y_2 are distinct vertices of $\xi(e)$, then we may assume x_1, x_2, y_1, y_2 occur on $\xi(e)$ in that order, or in the order x_2, x_1, y_1, y_2. In the first case, w is an internal vertex of $\xi(e)[x_2, y_1]$ (by choice of $\bar{P}$). Let Q_2 be the ξ-path in B_2 with endpoints x_2, y_2. Then by Theorem 2.3.1 Q_2 is odd, and Q_1, Q_2 satisfy (c1). In the second case, by the definition of crossing bridges, B_2 has an attachment v under B_1, i.e. v is an internal vertex of $\xi(e)[x_1, y_1]$. So B_2 contains a triad, T_2, with feet x_2, v, y_2. By Theorem 2.3.1, the path in T_2 from x_2 to y_2 is odd, and so T_2 is odd. By choice of $\bar{P}$, w is an internal vertex of $\xi(e)[x_1, y_1]$. If $w = v$, then (c3) holds. Otherwise, we may assume w is an internal vertex of $\xi(e)[x_1, v]$. Let Q_2 be the ξ-path in B_2 with endpoints x_2, v. By Theorem 2.3.1, Q_2 is odd. Then Q_1 and Q_2 satisfy (c1).

Now, suppose x_1, x_2 coincide, but y_1, y_2 are distinct. We may assume $x = x_1, y_1, y_2$ occur on $\xi(e)$ in that order. By definition of crossing bridges, B_2 must have an attachment v under B_1; i.e. v is an internal vertex of $\xi(e)[x, y_1]$). By choice of $\bar{P}$, w is an internal vertex of $\xi(e)[x, y_1]$. Then x, v, w, y_1, y_2 occur on $\xi(e)$ in that order, or in the order x, w, v, y_1, y_2, where w, v may coincide. In the first case, let Q_2 be a ξ-path in B_2 with endpoints v, y_2. By Theorem 2.3.1, Q_2 is odd. if w, v are distinct, then Q_1 and Q_2 satisfy (c1). Now suppose $w \notin V(\xi(e)[v, y_1]) \setminus v)$. Let T_2 be the ξ-tripod in B_2 with feet x, v, y_2. By Lemma 2.3.1, T_2 is odd, and so (c2) holds.

Otherwise, Suppose $x_1 = x_2 = x$ and $y_1 = y_2 = y$ for $x, y \in V(\xi(e))$. By the definition of crossing bridges, for $i = 1, 2$ B_i must have an attachment v_i that is an internal vertex of $\xi(e)[x, y]$. By choice of $\bar{P}$, w is an internal vertex of $\xi(e)[x, y]$. We may assume that $x, \{v_1, v_2, w\} y$ occur on $\xi(e)$ in that order, where v_1, v_2, w may coincide. For $i = 1, 2$, Let T_i be the ξ-triad in Bi with feet x, v_1, y; by Theorem 2.3.1, T_i is odd. Then T_1, T_2 satisfy (c4). This completes the proof.

□

Proof of Lemma 2.6.2

Proof. Suppose (a5) holds, i.e. for some $e \in E(G)$ there exist three pairwise crossing unstable bridges B_1, B_2, B_3 with attachments in $\xi(e)$.

For $i = 1, 2, 3$, let P_i be the minimal subpath of $\xi(e)$ containing all the attachments of B_i, and denote the endpoints of P_i by x_i, y_i. By Theorem 2.3.1, P_i is odd for $i = 1, 2, 3$. Note that by definition of a stable ξ-bridge, any stable ξ-bridge with an attachment w in $\xi(e)$ contains a ξ-path $\bar{P}$ with endpoints w, z where $z \notin \xi(e)$. Furthermore, since we assume (a1) does not hold for ξ, there exists a potential facial circuit C of G containing e such that $z \in V(\xi(C)) \setminus V(\xi(e))$. So to show that the path $\bar{P}$ described in the outcomes of Lemma 2.2.2 exists, it suffices to show that there exists a stable ξ-bridge with an attachment w in the specified location. We now proceed to the case analysis.

By Theorem 2.3.1, some stable ξ-bridge has an attachment $w \in V(P_1) \cap V(P_2) \cap V(P_3)$. It is easy to see that if z is an internal vertex of $P_1 \cap P_2 \cap P_3$, then w is under B_1, B_2 and B_3, and so (a6) holds. Suppose w is an endpoint of $P_1 \cap P_2 \cap P_3$. Suppose w is an endpoint of exactly one of P_1, P_2, P_3, say P_1. Then w is under both B_2 and B_3, and (a6) holds.

Now we need only consider the cases where w is an endpoint of $P_1 \cap P_2 \cap P_3$, and w is an endpoint of at least two of P_1, P_2, P_3. We will assume that $w = x_1 = x_2$. By Theorem 2.3.1, we may assume further that w is not an endpoint of $P_1 \cup P_2 \cup P_3$.

Note first that we cannot have $z = x_1 = x_2 = x_3$ or $z = x_1 = x_2 = y_3$, for then either B_1, B_2, B_3 are not pairwise crossing, or z is an endpoint of $P_1 \cup P_2 \cup P_3$. So we may assume $z \neq x_3, y_3$.

If $y_1 = y_2 = y_3 = y$, then x_3, z, y must occur on $\xi(e)$ in that order. (Otherwise, z is an endpoint of $P_1 \cup P_2 \cup P_3$.) Then by the definition of crossing bridges, for $i = 1, 2, 3$ B_i has an attachment v_i which is an internal vertex of $\xi(e)[w, y]$, and so B_i contains a ξ-triad T_i with feet x_i, v_i, y_i. By Theorem 2.3.1, each T_i is odd, and so (d4) holds.

Now suppose $y_1 = y_2 = y$, $y_3 \neq y$. We may assume that x_3, w, y, y_3 occur on $\xi(e)$ in this order, or in the order x_3, w, y_3, y. In the first case, by the definition of crossing bridges, for $i = 1, 2, 3$ B_i has an attachment v_i which is an internal vertex of $\xi(e)[w, y]$, and so B_i contains a ξ-triad T_i with feet x_i, v_i, y_i. By Theorem 2.3.1, each T_i is odd, and so (d4) holds. In the second case, by the definition of crossing bridges, B_1, B_2 must have attachments v_1, v_2, respectively, which are internal vertices of $\xi(e)[z, y]$, and so B_i contains a ξ-triad T_i with feet x_i, v_i, y_i for $i = 1, 2$. By Theorem 2.3.1, T_1, T_2 are odd. Then (d3) holds.

Now suppose $y_1 = y_3 = y$, $y_2 \neq y$. We may assume that x_3, w, y, y_2 occur on $\xi(e)$ in this order, or in the order x_3, w, y_2, y. In the first case, by definition of crossing bridges, for $i = 2, 3$ B_i has an attachment v_i which is an internal vertex of $\xi(e)[w, y]$, and so contains a ξ-triad T_i with feet x_i, v_i, y_i. By Theorem 2.3.1, T_2, T_3 are odd, and so (d2) holds. A similar argument gives (d2) in the second case as well.

Finally, suppose y_1, y_2, y_3 are distinct. We may assume x_3, w, y_1, y_2, y_3 occur on $\xi(e)$ in this order, in the order x_3, z, y_1, y_3, y_2, or the order x_3, w, y_3, y_1, y_2. In the first case, by the definition of crossing bridges, B_2, B_3 must have attachments v_2, v_3, respectively, in $\xi(e)[w, y_1]$. So for $i = 1, 2$, B_i contains a ξ-triad T_i with feet $x_i, v_i.y_i$. By Theorem 2.3.1, T_1, T_2 are odd. This gives (d2). By a similar argument, (d2) also holds in the second case. In the third case, by the definition of crossing bridges, B_2 must have an attachment v_2 which is an internal vertex of $\xi(e)[w, y_1]$, and so B_2 contains a ξ-triad T_2. By Theorem 2.3.1, T_2 is odd. Then (d1) holds. This completes the proof. $\square$

Proof of Lemma 2.6.3 Suppose (a8) holds for ξ, i.e. for some $e \in E(G)$, there exists an unstable ξ-bridge B with all attachments on $\xi(e)$, and ξ-paths $\bar{P}_1, \bar{P}_2$ with endpoints w_1, z_1 and w_2, z_2, respectively, where w_i is under B and $z_i \in V(\xi(C_i)) \setminus V(\xi(e))$ for $i = 1, 2$, where C_1, C_2 are potential facial circuits of G containing e, and C_1, C_2 share at most two vertices.

Let P be the minimal subpath of $\xi(e)$ containing all attachments of B, and let x, y be its endpoints. Let Q be a ξ-path in B with endpoints x, y. Since ξ-path $\bar{P}_1$ does not have both endpoints in $\xi(e)$, $\bar{P}_1$ is contained in a stable ξ-bridge with an attachment under B. Then by Theorem 2.3.1, Q is odd. Furthermore, by the definition of $\bar{P}_1, \bar{P}_2$, w_1, w_2 must be internal vertices of $\xi(e)[x, y]$. This completes the proof. □

Proof of Lemma 2.6.4 Suppose (a7) holds for ξ, i.e. for some $e \in E(G)$ there exist crossing unstable bridges B_1, B_2 with attachments on $\xi(e)$, and ξ-paths $\bar{P}_1, \bar{P}_2$ with endpoints w_1, z_1 and w_2, z_2, respectively, where w_i is under B_i and $z_i \in V(\xi(C)) \setminus V(\xi(e))$ for $i = 1, 2$, where C is a potential facial circuit of G that contains e.

For $i = 1, 2$, let P_i be the minimal subpath of $\xi(e)$ containing all attachments of B_i, and let x_i, y_i denote the endpoints of P_i. We may assume x_1, x_2, y_1, y_2 occur on $\xi(e)$ in that order. By choice of $\bar{P}_1, \bar{P}_2$, $w_1 \in V(\xi(e)) \setminus \{x_1\}$, and $w_2 \in V(\xi(e)) \setminus \{y_2\}$. Since $\bar{P}_1, \bar{P}_2$ each have one endpoint in $V(\xi(e))$ and one endpoint not in $V(\xi(e))$, each is contained in a stable ξ-bridge. For $i = 1, 2$, let Q_i be the ξ-path in B_i with endpoints x_i, y_i. Then it follows from Theorem 2.3.1 that Q_1, Q_2 are both odd. This completes the proof. □

Now we will prove Theorem 2.2.5, stated in Section 2.1. Our strategy will be to consider an embedding Λ closely related to Π, and consider adding each structure listed in Theorem 2.1.2 to $\Lambda(G, \Sigma)$. (Note that we will use Lemmas 2.2.1 - 2.2.4 to replace outcomes (a5)-(a8) with more explicit descriptions of the structures added.) We will argue in each case that the resulting graph is non-planar.

Proof of Theorem 2.2.5

We will proceed by case analysis:

Case 1: (a1) occurs.
Let P be an η'-path as in (a1). Let Λ be an embedding of (G, Σ) closely related to Π. Suppose we can add P to $\Lambda(\eta'(G, Sigma))$ without crossings. Then P must lie in some Λ-face of $\eta'(G, \Sigma)$, and both endpoints of P must lie in the same Λ-facial circuit C. But C is a Π-facial circuit of G containing both endpoints of P, contradicting our choice of P.

Case 2: (a2) occurs.
Suppose P_1, P_2 is a special cross in Π-facial cycle C of $\eta'(G, \Sigma)$. Let Λ be an embedding of (G, Σ) closely related to Π. Suppose we can add P_1, P_2 to $\Lambda(\eta'(G, \Sigma))$ without crossings. Let C_1 be the Λ-facial cycle of G containing the endpoints of P_1, and let C_2 be the Λ-facial cycle of G containing P_2. By Lemmas 2.4.1 and the definition of a special η'-cross, each of C_1, C_2 is unique. If C_1, C_2 coincide, it is clear that we cannot add P_1, P_2 to $\Lambda(\eta'(G, \Sigma))$ without crossings.

So C_1, C_2 must be related facial circuits of G. But then by Lemma 2.5.2, C_1, C_2 cannot both be Λ-facial circuits of G – contradiction.

Case 3: (a3) occurs.
Let (X,Y) be a separation of H as in (a3), and suppose we have an embedding Π' of H in S that extends from an embedding closely related to Π. Then (by definition of a separation) there exists a simple closed curve s in the plane that intersects H exactly in the vertices of $X \cap Y$. Then C separates H into two parts: D_X, comprised of C and the part of S containing $H[X]$; and D_Y, comprised of C and the part of S containing $H[Y]$. Since Π has representativity at least 3, Π' also has representativity at least 3, and s is a contractible curve. (If s is not contractible, then Π' has representativity 0.) It follows that at least one of D_X and D_Y are homeomorphic to a disc. If S is the plane, then both D_X, D_Y are homeomorphic to a disc. Now suppose S is not the plane. Then, since $|\eta(V(G)) \cap X - Y| \leq 1$ and $|V(G)| \geq 5$, we see that D_Y cannot be homeomorphic to a disc - if so G is planar. So in either case, D_X is homeomorphic to a disc. Then the embedding of H induces an embedding of both $H[X]$ in the disc, with the vertices of $X \cap Y$ on the boundary of the disc. This contradicts the choice of (X,Y).

Case 4: (a4) occurs.
Let T be a triad as in (a4), and let Λ be an embedding of (G,Σ) closely related to Π. Suppose we can add T to $\Lambda(G,\Sigma)$ without crossings. Then the feet of T are contained in a a Λ-facial circuit C of G. But then C is a Π-potential facial circuit of G containing the feet of T – contradiction.

Case 5: (a10) occurs.
Let Λ be an embedding of (G,Σ) closely related to Π. Suppose we can add $Q_1, Q_2, \bar{P}_1, \bar{P}_2$ to $\Lambda(G,\Sigma)$ without crossings. Let C_i be the Λ-facial circuit of G such that $\eta'(C_i)$ contains both w_i, z_i for $i = 1,2$. Then C_1, C_2, are Π-related, and by Lemma 2.5.2 must coincide. We may therefore assume that $z_1, z_2 \in V(\eta'(C' \setminus e))$, for a Λ-facial circuit of G containing e. Let C'' be the other Λ-facial circuit of G containing e. Since $\bar{P}_1$ separates x_1, y_1 in $\eta'(C')$, it is easy to see that Q_1 must lie in $\eta'(C'')$. But then $s = C' \setminus \eta'(e)[x_1, y_1] \cup Q_1$ is a simple closed contractible curve in S. Furthermore, x_2 is on one side of s, and y_2 is on the other. It follows that Q_2 cannot be added without crossings.

Case 5: (b2) occurs.
Let Λ be an embedding of (G,Σ) closely related to Π. Let C', C'' be the Λ-facial circuits of G containing e. (Since G has no edge parallel to e, C', C'' are unique.) Suppose $Q, \bar{P}_1, \bar{P}_2$ can be added to $\Lambda(G,\Sigma)$ without crossings. Then $z_1, z_2 \in V(\eta(()C' \cup C'') \setminus e))$. We may assume that $z_1 \in \eta'(C' \setminus e)$, $z_2 \in \eta'(C'' \setminus e)$, for otherwise C', C'' are Π-related facial circuits – which is impossible, by Lemma 2.5.2. Let v_1, v_2 denote the endpoints of e. By Lemma 2.4.1, C', C'' are the only Λ-facial circuits of G that contain both v_1, v_2. It follows that regardless of whether x, y are endpoints of e, Q is contained either in C' or C''. But $\bar{P}_1$ separates x, y in C', and $\bar{P}_2$ separates x, y in C''. It follows that Q cannot be added without

crossings.

Case 6: (b3) occurs.
Let Λ be an embedding of (G, Σ) closely related to Π. Let C_1 be a Λ-facial circuit of G containing e_1, and let C_2 be a Λ-facial circuit containing e_2, such that $C_1, C_2 \neq \{e_1, e_2\}$. By Lemma 2.5.2, we may assume $C_1 \cap C_2 = e$. We may assume further $z_i \in V\eta(C_i \setminus e_i)$ for $i = 1, 2$. Note that by Lemma 2.4.1, $C_1, C_2, \{e_1, e_2\}$ are the only Λ-facial circuits of G that contain both x, y. It follows that Q must lie in one of these circuits. But for each $i = 1, 2$, $\bar{P}_i$ separates x, y in C_i. Furthermore, $\bar{P}_3$ separates x, y in $\{e_1, e_2\}$. It follows that $Q1$ cannot be added without crossings.

Case 7: (b4) occurs.
Let Λ be an embedding of (G, Σ) closely related to Π. Let C_1 be a Λ-facial circuit of G containing e_1, and let C_2 be a Λ-facial circuit containing e_2, such that $C_1, C_2 \neq \{e_1, e_2\}$. We may assume that $z_1, z_2 \in V(C_1 \setminus e_1)$. By Lemma 2.4.1, C_1, C_2 intersect in exactly the end-vertices of e_1, e_2. Note that $s = \Lambda(C_2 \setminus e_2 \cup e_1)$ is a separating cycle in S. Note that the interior of $\Lambda(e_2)$ is on one side of s, and $\Lambda(C_1 \setminus e_1)$ is on the other. Then the endpoints $\Lambda(w_1), \Lambda(z_1)$ of $\bar{P}_1$ are separated by $\Lambda(\eta'(C_2 \setminus e_2 \cup e_1))$ in S. So $\bar{P}_1$ cannot be added to $\Lambda(G, \Sigma)$ without crossings.

Case 8: (b5) occurs.
Notice that in this case, there exists a path (not an η'-path) $\bar{P}_2'$ with endpoint $w_2 \in V(\eta; (e_2))$ and endpoint z_2' on C_1 (we may take z_2' to be an endpoint of $\eta(e_2')$, and $\bar{P}_2' = \bar{P}_2 \cup \eta(e_2')[z_2, z_2']$). Then following the proof for (b4), substituting $\bar{P}_2'$ for $\bar{P}_2$ gives the result.

Case 9: (b6) occurs.
Let Λ be related to Π. Let C_1 be a Λ-facial circuit of G containing e_1, and let C_2 be a Λ-facial circuit containing e_2, such that $C_1, C_2 \neq \{e_1, e_2\}$. We may assume that $z_1 \in V(C_1 \setminus e_1)$, and that $z_2 \in V(C_2 \setminus e_2)$. Note that $s = \Lambda(C_1 \setminus e_1 \cup e_2)$ is a separating cycle in S. Note that the interior of $\Lambda(e_1)$ is on one side of s, and $\Lambda(C_2 \setminus e_2)$ is on the other. Then the endpoints $\Lambda(w_2), \Lambda(z_2)$ of $\bar{P}_2$ are separated by $\Lambda(\eta'(C_1 \setminus e_1 \cup e_2))$ in S. So $\bar{P}_2$ cannot be added to $\Lambda(G, \Sigma)$ without crossings.

Case 10: (c1) occurs.
Let Λ be an embedding of (G, Σ) closely related to Π. Suppose we can add $Q_1, Q_2, \bar{P}$ to $\Lambda(G, \Sigma)$ without crossings. Let C_1, C_2 be the Λ-facial circuits of G that contain e. We may assume $x \in V(\eta'(C_1))$. Then $\bar{P}$ separates x_1, y_1 in C_1. It follows that Q_1 must be drawn in C_2. But $\bar{P}$ also separates x_2, y_2 in C_1, and Q_1 separates x_2, y_2 in C_2. It follows that Q_2 cannot be added without crossings.

Case 11: (c2) occurs.
Let Λ be an embedding of (G, Σ) closely related to Π. Suppose we can add $Q_1, T, \bar{P}$ to

$\Lambda(G, \Sigma)$ without crossings. Let C_1, C_2 be the Λ-facial circuits of G containing e. We may assume that $z \in V(\eta'(C \setminus e))$. Then $\bar{P}$ (the path) separates x, y_1 in C_1, and so Q_1 is drawn in C_2. Similarly, the path from x to y_2 in T must be drawn in C_2. Let a be the degree 3 vertex of T. Then $s = Q_1 \cup \Lambda(\eta'(e)[x, y_1])$ is a contractible curve in S, such that v is on one side of s and a is on the other. So the path in T from a to v cannot be drawn without crossing s.

Case 12: (c3) occurs.
The proof is nearly identical to that of the previous case.

Case 13: (c4) occurs.
Let Λ be an embedding of (G, Σ) closely related to Π. Suppose we can add $T_1, T_2, \bar{P}$ to $\Lambda(G, \Sigma)$ without crossings. Let C_1, C_2 be the Λ-facial circuits of G containing e. We may assume that $z \in V(\eta'(C \setminus e))$. Then $\bar{P}$ separates x, y in C_1. Since by Lemma 2.4.1 C_1, C_2 are the only Λ-facial circuits of G containing both x, y, the paths L_1, L_2 in T_1, T_2 (respectively) with endpoints x, y are drawn in C_2. Let a be the degree 3 vertex of T_1. We may assume (by possibly relabeling T_1, T_2) that $s = L_1 \cup \Lambda(\eta'(e)[x, y])$ is a contractible curve in S, such that v is on one side of s and a is on the other. So the path in T from a to v cannot be drawn without crossing s.

Case 14: (d1) occurs.
Let Λ be an embedding of (G, Σ) closely related to Π. Suppose we can add $Q_1, Q_2, T_3, \bar{P}$ to $\Lambda(G, \Sigma)$ without crossings. Let C_1, C_2 be the Λ-facial circuits of G containing e. We may assume that $z \in V(\eta'(C \setminus e))$. Then $\bar{P}$ separates x_1, y_1 in C_1. It follows that Q_1 must be drawn in C_2. Then Q_1 separates x_2, y_2 and x_2, y_3 in C_2. It follows that both Q_1 and the path in T_3 from x_2 to y_3 must be drawn in C_1. Let a be the degree 3 vertex of T_3. Then $s = Q_2 \cup \Lambda(\eta'(e)[x_2, y_2])$ is a contractible curve in S, such that v is on one side of s and a is on the other. So the path in T_3 from a to v cannot be drawn without crossing s.

Case 15: (d2) occurs.
Let Λ be an embedding of (G, Σ) closely related to Π. Suppose we can add $Q_1, T_2, T_3, \bar{P}$ to $\Lambda(G, \Sigma)$ without crossings. Let C_1, C_2 be the Λ-facial circuits of G containing e. We may assume that $z \in V(\eta'(C \setminus e))$. Then $\bar{P}$ separates x_2, y_3 in C_1. It follows that the path in T_3 with endpoints x_2, y_3 must be drawn in C_2. Then the path in T_3 from v_2 to y_3 separates x_1, y_1 and x_1, y_2 in C_2. It follows that both Q_1 and the path in T_2 from x_1 to y_2 must be drawn in C_1. Let a be the degree 3 vertex of T_2. Then $s = Q_1 \cup \Lambda(\eta'(e)[x_1, y_1])$ is a contractible curve in S, such that v_1 is on one side of s and a is on the other. So the path in T_2 from a to v_1 cannot be drawn without crossing s.

Case 16: (d3) occurs.
Let Λ be an embedding of (G, Σ) closely related to Π. Suppose we can add $Q_1, T_2, T_3, \bar{P}$ to

$\Lambda(G, \Sigma)$ without crossings. Let C_1, C_2 be the Λ-facial circuits of G containing e. We may assume that $z \in V(\eta'(C \setminus e))$. Then $\bar{P}$ separates x_1, y_1 in C_1. It follows that Q_1 must be drawn in C_2. Then Q_1 separates x_2, y_2 in C_2. It follows that the paths L_2, L_3 in T_2, T_3, respectively, with endpoints x_2, y_2 must be drawn in C_1. Let a be the degree 3 vertex of T_2. Note that $s = L_3 \cup \Lambda(\eta'(e)[x_2, y_2])$ is a contractible curve in S. By possibly relabeling T_2, T_3, we may assume that a is on one side of s and v_2 is on the other. So the path in T_2 from a to v_2 cannot be drawn without crossing s.

Case 17: (d4) occurs.
Let Λ be an embedding of (G, Σ) closely related to Π. Suppose we can add $T_1, T_2, T_3, \bar{P}$ to $\Lambda(G, \Sigma)$ without crossings. Let C_1, C_2 be the Λ-facial circuits of G containing e. We may assume that $z \in V(\eta'(C \setminus e))$. Then S separates x_1, v_1 in C_1. It follows that the path L_1 in T_1 from x_1 to v_1 must be drawn in C_2. Then L_1 separates x_2, y_2 in C_2. It follows that the paths L_2, L_3 in T_2, T_3, respectively, with endpoints x_2, y_2 must be drawn in C_1. Let a be the degree 3 vertex of T_2. Note that $s = L_3 \cup \Lambda(\eta'(e)[x_2, y_2])$ is a contractible curve in S. By possibly relabeling T_2, T_3, we may assume that a is on one side of s and v_2 is on the other. So the path in T_2 from a to v_2 cannot be drawn without crossing s.

$\square$

Chapter 3

Stabilizer

In this chapter, we consider signed graphs (G, Σ) in the following topological classes:

- (G, Σ) has an even-face embedding on the projective plane;
- (G, Σ) has an even-face embedding on the torus;
- (G, Σ) has an even-face embedding on the Klein bottle;
- (G, Σ) has an even-face embedding on the pinched projective plane;
- (G, Σ) has an even-face embedding on the double-pinched sphere;
- (G, Σ) is apex with two odd faces;

where the *pinched projective plane* is the projective plane with a pair of distinct points identified (forming a *pinch point*), and the *double-pinched sphere* is the sphere with two pairs of distinct points identified (to form two separate pinch points).

For a signed graph (G, Σ) in a topological class $\mathcal{C}$, we will need to refer to an embedding Π of (G, Σ) that meets the conditions for membership in $\mathcal{C}$. In this case, we will say that Π is an *embedding of* (G, Σ) *in* $\mathcal{C}$.

Let (G, Σ) be a signed graph in topological class $\mathcal{C}$, and let (H, Γ) be a signed graph in $\mathcal{C}$ that contains (G, Σ) as a minor, where (G, Σ) and (H, Γ) are both "sufficiently connected". We will use *edge-addition* to refer to the inverse operation of edge deletion, and *vertex-splitting* to refer to the inverse operation of edge contraction. Then a natural question to ask, given an embedding Π of (G,Σ) in $\mathcal{C}$, is whether Π can be extended by adding edges or splitting vertices to yield two different embeddings of (H,Γ) in $\mathcal{C}$. If for embedding Π the answer to this question is "no" for every major (H,Γ) of (G,Σ), we will say that Π *extends uniquely* in $\mathcal{C}$. If every embedding of (G,Σ) in $\mathcal{C}$ extends uniquely, we will say that (G,Σ) *extends uniquely* in $\mathcal{C}$. Our goal in this chapter is, for each topological class listed above, to give sufficient conditions for a signed graph (G,Σ) in that class to extend uniquely. Moreover, we desire to give these conditions in terms of (G,Σ).

In Section 3.1, we will state our main results. In Section 3.2, we will give some necessary definitions and define our problem on unique extension more precisely. In Section 3.3 we will prove the result for signed graphs with an even-face embedding on the projective plane. In Sections 3.4-3.9 we will prove the result for signed graphs with an even-face embedding on the torus, the Klein bottle, the pinched projective plane, or the double-pinched sphere. We will prove the result for apex signed graphs with two odd faces in Section 3.10.

3.1 Overview of results

We will now state the main theorems of the thesis.

Theorem 3.1.1. *Let (G,Σ) be a simple, 3-connected signed graph with an even-face embedding in the projective plane, such that (G,Σ) is non-bipartite. Then (G,Σ) extends uniquely.*

Theorem 3.1.2. *Let (G,Σ) be a simple, 3-connected signed graph with an even-face embedding on the torus, such that G is not planar and (G,Σ) and has no blocking pair or blocking vertex. Then (G,Σ) extends uniquely.*

Let (G,Σ) be a signed graph with embedding Π on a pinched surface. Let $u,v \in V(G)$, where $\Pi(v)$ coincides with a pinch point. As stated in the Introduction, we will say that a pair of Π-faces F_1,F_2 of (G,Σ) is *bad* if both F_1,F_2 contain both u,v.

Theorem 3.1.3. *Let (G,Σ) be a simple non-bipartite 3-connected signed graph with no blocking vertex of blocking pair. Suppose (G,Σ) has an even-face embedding on the double-pinched sphere, and that G does not contain a bad pair of faces for any such embedding. Then (G,Σ) extends uniquely if G is non-planar.*

Theorem 3.1.4. *Let (G,Σ) be a simple non-bipartite 3-connected signed graph with no blocking vertex of blocking pair. Suppose (G,Σ) has an even-face embedding Π on the* *pinched projective plane, where the pinch point is not contained in* $\Pi(G,\Sigma)$*. Suppose* G *is non-planar, and* (G,Σ) *has no even-face embedding on the projective plane or on the double-pinched sphere. Then* (G,Σ) *extends uniquely.*

Theorem 3.1.5. *Let* (G,Σ) *be a simple non-bipartite 3-connected signed graph with no blocking vertex or blocking pair. Let* Π *be an even-face embedding of* (G,Σ) *on the pinched projective plane, where the pinch point contained in* $\Pi(G,\Sigma)$*. Suppose that* (G,Σ) *has no even-face embedding* Λ *on the pinched projective plane where the pinch point is not in* $\Lambda(G,\Sigma)$*, and that* (G,Σ) *has no embedding on the double-pinched sphere. Suppose also that* (G,Σ) *does not contain a bad pair of* Λ*-faces for any even-face embedding* Λ *of* (G,Σ) *on the pinched projective plane. Then* (G,Σ) *extends uniquely.*

Theorem 3.1.6. *Let* (G,Σ) *be a simple non-bipartite 3-connected signed graph with no blocking vertex or blocking pair. Suppose* (G,Σ) *has an even-face embedding on the Klein bottle. If* (G,Σ) *is non-planar, and does not have an even-face embedding on the projective plane or on the pinched projective plane, then* (G,Σ) *extends uniquely.*

Let G be a graph and let $X \subseteq E(G)$. We write $\mathcal{B}_G(X)$ for $V_G(X)\cap V_G(\bar{X})$. Suppose that $\mathcal{B}_G(X) = \{u_1,u_2\}$ for some $u_1,u_2 \in V(G)$. Let G' be the graph obtained by identifying vertices u_1,u_2 of $G[X]$ with vertices u_2,u_1 of $G[\bar{X}]$, respectively. Then G' is obtained from G by a *Whitney flip* on X. Suppose G is a planar graph, and Π_1, Π_2 are embeddings of G. Let G_1^* be the Π_1-dual of G, and let G_2^* be the Π_2-dual of G. if G_2^* can be obtained from G_1^* by a Whitney flip, we will say that Π_1, Π_2 are related by a *dual Whitney flip.* For an apex graph G with apex embeddings $\Pi = (\lambda, a)$, $\Pi' = (\lambda', a)$, we will say that Π' is obtained from Π by a dual Whitney flip if λ and λ' are related by a dual Whitney flip.

Theorem 3.1.7. *Let* (G,Σ) *be a loopless signed graph with no blocking vertex. Let* $\Pi = (\lambda, v)$ *be an apex embedding of* (G,Σ) *with exactly two odd faces, and let* (H,Γ) *be an extension of* (G,Σ)*. Suppose* (G,Σ) *has no even-face embedding on the double-pinched sphere. Then any two extensions of* Π *are related by dual Whitney flips. Furthermore, if* (H,Γ) *is an extension of* (G,Σ)*, then every apex embedding of* (H,Γ) *with two odd faces is an extension of some apex embedding of* (G,Σ) *with two odd faces.*

3.2 Making the problem precise

Our goal for this section is to make our problem on unique extension precise. To that end, we will begin with some assumptions. Throughout this chapter, we will assume that the signed graphs we consider are non-bipartite, and have neither a blocking vertex nor a blocking pair. (Usually we state these assumptions explicitly). The reasoning for these assumptions follows from the application given in Section **??**.

We denote by ecycle(G,Σ) the set of all even cycles of (G,Σ). It can be verified that ecycle(G,Σ) is the set of cycles of a binary matroid (with ground set $E(G)$), which we call the even cycle matroid of (G,Σ). We identify ecycle(G,Σ) with that matroid. We will use ecycle$^*(G,\Sigma)$ to denote the dual of ecycle(G,Σ). The *even-cycle space* of a signed graph (G,Σ) is the binary vector space whose elements are the characteristic vectors (mod 2) of the even cycles of (G,Σ). We will say that a set K of cycles of (G,Σ) *generates* the set ecycle(G,Σ) of even cycles of (G,Σ) if the characteristic vectors (mod 2) of the cycles in K generate the even-cycle space of (G,Σ).

We will need the following result relating properties of (G,Σ) to the connectedness of ecycle(G,Σ):

Remark 3.2.1. *Let (G,Σ) be a simple signed graph with no blocking vertex or blocking pair. If G is 3-connected (up to parallel edges), then ecycle(G,Σ) is 3-connected.*

Proof. Let r (resp. r') denote the rank function for $cycle(G)$ (resp. $ecycle(G,\Sigma)$). The connectivity function for $cycle(G)$ (resp. $ecycle(G,\Sigma)$) is defined as $\lambda(X) := r(X_1) + r(X_2) - r(E(G)) + 1$ (resp. $\lambda'(X_1) := r'(X_1) + r'(X_2) - r'(E(G)) + 1$) for all partitions X_1, X_2 of $E(G)$. We now assume that X_1, X_2 is a partition of $E(G)$ where $|X_1|, |X_2| \geq 2$. We need to show that $\lambda'(X_1) \geq 3$. For $i = 1, 2$ let c_i denote the number of components of $G[X_i]$. We use the following notation, for $X \subseteq E(G)$, $V(X)$ is the set of vertices that are an endpoint of an edge of X and $G[X]$ is the graph with edges X and vertices $V(X)$.

Claim 1: $\lambda(X_1) = |V(X_1) \cap V(X_2)| - c_1 - c_2 + 2$.

Proof. For $i = 1, 2$, $r(X_i)$ is equal to the size of the largest forest in $G[X_i]$, i.e. $|V(X_i)| - c_i$. Similarly, $r(E(G)) = |V(G)| - 1$. Thus $\lambda(X_1) = |V(X_1)| - c_1 + |V(X_2)| - c_2 - (|V(G)| - 1) + 1$ which yields the result. $\Diamond$

For $i = 1, 2$ set $p_i = 1$ if $(G[X_i], \Sigma \cap X_i)$ is non-bipartite and set $p_i = 0$ otherwise.

Claim 2: $\lambda'(X_1) = |V(X_1) \cap V(X_2)| - c_1 - c_2 + 1 + p_1 + p_2$.

Proof. Observe that for $i = 1, 2$, $r'(X_i) = r(X_i) + 1$ when $(G[X_i], \Sigma \cap X_i)$ is non-bipartite, and $r'(X_i) = r(X_i)$ otherwise (as in the former case a maximal forest of $(G[X_i], \Sigma \cap X_i)$ together with a single edge where the unique cycle is odd is an independent set in $ecycle(G,\Sigma)$). Since (G,Σ) is non-bipartite (it has no blocking pair), $r'(E(G)) = r(E(G)) + 1$. Hence, $\lambda'(X_1) = \lambda(X_1) + p_1 + p_2 - 1$. Now the result follows from Claim 1. $\Diamond$

Note that every vertex in $V(X_1) \cap V(X_2)$ is in exactly one component of $G[X_1]$ and exactly one component of $G[X_2]$. Thus we can construct a bipartite (multi-)graph H where the vertices of H correspond to components of $G[X_1], G[X_2]$ and we have k parallel edges joining a pair of components from $V(X_1)$ and $V(X_2)$ whenever these components have k vertices in common. Then $|V(H)| = c_1 + c_2$ and $|E(H)| = |V(X_1) \cap V(X_2)|$. Suppose for a contradiction now that $\lambda'(X_1) \leq 2$. Then

Claim 3: $|E(H)| \leq |V(H)| + 1 - p_1 - p_2$.

Proof. Since $2 \geq \lambda'(X_1)$, we have by Claim 2,

$$2 \geq |V(X_1) \cap V(X_2)| - c_1 - c_2 + 1 + p_1 + p_2 = |E(H)| - |V(H)| + 1 + p_1 + p_2$$

and the result follows. $\Diamond$

Claim 4: (a) H is connected and bridgeless. (b) If H has a 2-edge separation then one of the sides is a single vertex and the corresponding component of $G[X_1], G[X_2]$ is a single edge.

Proof. If H has a bridge then G has a cut-vertex, a contradiction as H is 3-connected. If H has a 2-edge separation then G has a 2-vertex separation u, v. Then one of the sides has to consist of the single edge uv. $\Diamond$

Because of Claim 4(a) $|E(H)| \geq |V(H)|$. Consider first the case where equality holds. Then H is a collection of disjoint even cycles. Because of Claim 4(a) H is a single cycle C. Moreover, because of Claim 4(b) $|C| = 2$ and $|X_i| = 1$ for some $i \in \{1, 2\}$ a contradiction.

Thus we may assume that $|E(H)| > |V(H)|$ and hence by Claim 3 that $|E(H)| = |V(H)| + 1$ and that $p_1 = p_2 = 0$. Hence, $(G[X_i], \Sigma \cap X_i)$ is bipartite for $i = 1, 2$. Suppose first that H has a cut vertex v. Let H_1, H_2 be obtained from H by splitting on v. Then for $i = 1, 2$, $|V(H_i)| = |E(H_i)|$ and H_i is bridgeless. Thus H_i is a cycle C_i. Moreover, by Claim 4(b) C_i consists of two edges e_i, f_i. Thus H is the graph with vertices v, u_1, u_2 and edges $e_1 = vu_1, f_1 = vu_1, e_2 = vu_2, f_2 = vu_2$. We may assume that $G[X_1]$ consists of a single component corresponding to v and that $G[X_2]$ consists of two components corresponding to u_1 and u_2. By Claim 4(b) the components corresponding to u_1, u_2 must be single edges say g_1, g_2. Since $(G[X_1], \Sigma \cap X_1)$ is bipartite, there exists a signature such that all edges of (G, Σ) except possibly g_1, g_2 are even. Hence, (G, Σ) has a blocking pair, a contradiction.

Thus we may assume that H is 2-connected and there is an ear decomposition of H which consists of a circuit C and a path P that is internally disjoint from C and where the endpoints of P are in C. Thus H consists of 3 internally disjoint paths P_1, P_2, P_3 with endpoints say s, t. By Claim 4(b), $|P_i| \leq 2$. Moreover, as H is bipartite, either $|P_1| = |P_2| = |P_3| = 1$ or $|P_1| = |P_2| = |P_3| = 2$. Consider the former case. Then $G[X_1]$ and $G[X_2]$ are connected and $V(X_1) \cap V(X_2)$ is a set of three vertices u_1, u_2, u_3. It can then be readily checked that we can resign so that all the odd edges are incident to a single vertex among u_1, u_2, u_3. But then clearly (G, Σ) has a blocking vertex, a contradiction. Finally, consider the case where $|P_1| = |P_2| = |P_3| = 2$. Then we may assume $G[X_1]$ consists of two components $G[X_1'], G[X_1'']$ and, by Claim 4(b), that $G[X_2]$ consists of three independent edges e_1, e_2, e_3 each with one endpoint in $G[X_1']$ and the other in $G[X_1'']$. Again it can then be readily checked that we can resign so that all the odd edges are contained in exactly one of e_1, e_2, e_3 and (G, Σ) has a blocking vertex, a contradiction. $\square$

Let S be a surface. A closed curve s in S is said to be *onesided* if left and right interchange along S. Otherwise, s is said to be *twosided*. For a Π-embedded graph G in a surface S, the Π-onesided cycles of G are orientation-reversing curves in S, and the Π-twosided cycles of G are orientation-preserving curves in S.

Let (H, Γ) be a Π'-embedded graph in (possibly pinched) surface S, and $(G, \Sigma) = (H, \Gamma) \setminus I / J$ a minor of (H, Γ). Suppose that deleting the edges of $\Pi'(I)$ from $\Pi'(H, \Gamma)$ and contracting the edges of $\Pi'(J)$ in S gives a drawing of (G, Σ) in S without crossing edges. Then this drawing corresponds to an embedding Π of (G, Σ). We will say that Π is the *induced embedding* of (G, Σ) from Π'. If additionally $E(H) \setminus E(G)$ contains no loops, we will say that the embedding Π' of (H, Γ) is an *extension* of the embedding Π of (G, Σ).

For apex graphs, our terminology for embeddings differs somewhat from the terminology given in Chapter 1 for embeddings of graphs on surfaces. We define an *apex embedding* of (G, Σ) as a pair $\Pi = (\lambda, v)$ where $v \in V$, and λ is a planar embedding of $(G - v, \Sigma \setminus \delta_G(v))$. We will call λ the *planar part* of the embedding, and will say that v is an *apex vertex* of (G, Σ). We will additionally call the λ-faces of $(G - v, \Sigma \setminus \delta_G(v))$ the Π-*faces* of G. Let (G, Σ) be a signed graph with apex embedding $\Pi = (\lambda, a)$, and let (H, Γ) be a major of (G, Σ) obtained by a sequence of vertex-splittings and edge-additions. Suppose (H, Γ) has apex embedding $\Pi' = (\lambda', a')$. Let A be the set of vertices of H that are obtained by applying a sequence of vertex-splittings to a, and let $\gamma(A)$ denote the set of edges of H with at least one endpoint in A. We will say that Π' is an *extension* of Π if $a' \in A$, and $\lambda' | (H - A, \Gamma \setminus \gamma(A))$ is an extension of λ.

When for an embedding Π of a simple, 3-connected signed graph (G, Σ) in class $\mathcal{C}$ no two extensions of Π in $\mathcal{C}$ are embeddings of the same simple, 3-connected signed graph (H, Γ), we will say that Π *extends uniquely* in $\mathcal{C}$. If every embedding of (G, Σ) in $\mathcal{C}$ extends uniquely, we will say that (G, Σ) *extends uniquely* in $\mathcal{C}$. We will generally omit the phrase "in $\mathcal{C}$".

We can now restate our problem more precisely: Given a simple, 3-connected signed graph (G, Σ) in a class $\mathcal{C}$ listed above, what are sufficient conditions on (G, Σ) such that (G, Σ) extends uniquely in $\mathcal{C}$? We will state the answer for each class $\mathcal{C}$ in the next section.

3.3 Even-face embedding on the projective plane

In this section we prove Theorem 3.1.1, using results from matroid theory. The *cut matroid* $\text{cut}(G)$ of (unsigned) graph G is the matroid with ground set $E(G)$ whose circuits are the one-vertex cuts of G. If a matroid M is the cut matroid of some graph H, we will say that M is *co-graphic*. The following result about graphic matroids from [14] is well-known:

Remark. *Let M be a 3-connected matroid. If M is co-graphic, then there is a unique graph G such that $M = \text{cut}(G)$.*

In topology, two closed curves in a surface are said to be *homologous mod 2* if they bound a region in the surface (we will generally abbreviate "homologous mod 2" to "homologous"). A contractible closed curve in a surface is said to be *homologous to 0*. Note that any two homologous non-contractible closed curves in a surface S differ by a symmetric difference of facial cycles. It follows that in an even-face embedding of a signed graph (G, Σ), any two non-contractible cycles with the same homology type have the same parity.

The following is from [10]: Given an embedding Π of a connected graph G, we define the Π-*dual graph* G^* and its embedding Π^*, called the *dual embedding* of Π, as follows. The vertices of G^* correspond to the Π-facial cycles of G. The edges of G^* are in bijective correspondence $e \mapsto e^*$ with the edges of G, and the edge e^* joins the vertices corresponding to the Π-facial cycles containing e. If C is a Π-facial cycle and w its vertex of G^*, then $\Pi(v)$ lies inside $\Pi(C)$ on the surface.

Proof of Theorem 3.1.1 Suppose (G, Σ) does not extend uniquely. Then there exists an even-face embedding Π of (G, Σ) in the projective plane with distinct extensions Π_1 and Π_2 such that Π_1, Π_2 are embeddings of the same simple, 3-connected signed graph (H, Γ). Now we consider a set of cycles of (H, Γ) that generates $\text{ecycle}(H, \Gamma)$. We may assume that (H, Γ) contains some odd cycle (otherwise, (H, Γ) is bipartite, and hence (G, Σ) is bipartite, contradicting our choice of (G, Σ)). Since (G, Σ) is not planar, (G, Σ) (and hence (H, Γ)) contains a Π-onesided cycle. Note that this cycle is both Π_1-onesided and Π_2-onesided in H. Since the projective plane has only one homology type of non-contractible curve (the one-sided curves), it follows that some Π_i-onesided cycle of (H, Γ) is odd, for $i = 1, 2$. Hence every Π_i-onesided cycle of (H, Γ) is odd, for $i = 1, 2$. Consequently, ecycle(H, Γ) is generated by both the Π_1-facial cycles of H and the Π_2-facial cycles of H, and so we can describe ecycle(H, Γ) as the matroid whose cycles are exactly the Π_i-facial circuits of H, for either $i = 1$ or $i = 2$.

Let H_1^* be the Π_1-dual of H, and let H_2^* be the Π_2-dual of H. Consider the cut matroids cut(H_1^*) and cut(H_2^*). Since the one-vertex cuts of H_i^* are exactly the Π_i-facial cycles of (H, Γ) for $i = 1, 2$, it is easy to see that cut$(H_1^*) =$ ecycle$(H, \Gamma) =$ cut(H_2^*).

Note that both cut(H_1^*), cut(H_2^*) are cographic, and 3-connected. But since Π_1, Π_2 are distinct embeddings of (H, Γ), H_1^* and H_2^* are distinct, contradicting the Remark. So Π extends uniquely, and so does (G, Σ). $\square$

3.4 Even-face embeddings on more complicated surfaces

Let $\mathcal{S}$ be the set whose members are the torus, Klein bottle, pinched projective plane, and double-pinched sphere. In this section we will use $\mathcal{C}$ to denote the class of signed graphs with an even-face embedding on some particular $S \in \mathcal{S}$. Note that the results in Sections 3.4-3.9 apply for all $S \in \mathcal{S}$, so are not concerned at the moment with distinguishing between these classes. Note also that $\mathcal{S}$ is the largest set of surfaces to which the results of Sections 3.8.1-3.9 apply; thus we have opted to provide a stabilizer theorem for signed graphs with an even-face embedding on the torus, despite the fact that such signed graphs are not odd-K_5 free.

We will begin in Section 3.5 with a natural construction for an object dual to a signed graph embedded on $S \in \mathcal{S}$, which we will call a "mate". In Section 3.6, we will use even-cycle matroids to describe when a signed graph $(G, \Sigma) \in \mathcal{C}$ extends uniquely. This result will be in terms of the mate of (G, Σ). We will pause briefly to give some necessary background in topology in Section 3.7, and in Section 3.8 we will develop another general result, but this time in terms of (G, Σ). Finally, in Section 3.9 we will prove Theorems 3.1.2 - 3.1.5.

3.5 Mates of signed graphs

Let (G, Σ) be Π-embedded in surface S, and let Σ^* be (the edge set of) an even Π-non-contractible cycle of (G, Σ). Let G^* denote the Π-dual of (G, Σ). Then the signed graph (G^*, Σ^*) is the Π-*mate* of (G, Σ).

In proving the theorem, we will need to understand the relationship between the even-cycle matroid of a signed graph (G, Σ) embedded in S, and the even-cycle matroid of its mate. We give the following Lemma:

Lemma 3.5.1. *Let (G, Σ) be a simple, 3-connected signed graph with an embedding Π on a surface $S \in \mathcal{S}$, such that $\Sigma \neq \emptyset$, and Π contains at least one cycle of each homology type in S. Let M be the even cycle matroid of (G, Σ), and let M^* be the even-cycle matroid of the Π-mate (G^*, Σ^*) of (G, Σ). Then M^* is the dual of M.*

A *cocircuit* of a matroid M is a minimal subset X of the ground set E of M such that $r(E \setminus X) < r(E)$, where r denotes the rank function of M.

Proof of Lemma 3.5.1 From [8], the cocircuits of ecycle(G^*, Σ^*) are the minimal cuts of G^*, together with the signatures of (G^*, Σ^*). Since any signature of (G^*, Σ^*) differs from Σ^* by a cut of G^*, we then see that the cocircuits of ecycle(G^*, Σ^*) are generated by the one-vertex cuts of G^* together with any one signature of (G^*, Σ^*).

Now, notice that the facial cycles of (G, Σ) correspond exactly to the one-vertex cuts of G^*, and any signature of (G^*, Σ^*) corresponds to an even non-contractible cycle of (G, Σ). Since ecycle(G, Σ) is generated by its facial cycles together with any one even non-contractible cycle, we see that ecycle$(G, \Sigma) = $ ecycle$^*(G^*, \Sigma^*)$. $\square$

3.6 A Result from Even-Cycle Matroids

In this section we prove a general result on unique extension in terms of the mate of a signed graph, which will be used as a basis for our work in Sections 3.8 and 3.9.

Let (G, Σ) be a signed graph Π-embedded in surface S. We say that a pair of edges e, f incident to a vertex v of (G, Σ) are Π-*consecutive* if e and f are met consecutively when traversing some boundary component of a sufficiently small neighbourhood of v in S. A blocking pair u, v of (G, Σ) is called Π- *consecutive* in if in some embedding Π of (G, Σ), for some signature Σ' of (G, Σ) where $\Sigma' \subseteq \Delta_G(u) \cup \Delta_G(v)$, all edges of $\Sigma' \cap \delta_G(v)$ are Π-consecutive and all edges of $\Sigma' \cap \delta_G(u)$ are Π-consecutive. Similarly, a blocking vertex v

of (G, Σ) is called Π-*consecutive* if the edges of $\Sigma' \cap \delta_G(v)$ are Π-consecutive. We can now state the main theorem for this section:

Theorem 3.6.1. *Let (G, Σ) be a simple, 3-connected signed graph with even-face embedding Π on surface S, where (G, Σ) is not bipartite, has no blocking vertex or blocking pair, and G contains four pairwise Π-non-homologous cycles. Suppose Π extends to distinct even-face embeddings Π_1 and Π_2 of simple, 3-connected signed graph (H, Γ) on S. Then the Π-mate (G^*, Σ^*) of (G, Σ) has either a Π-consecutive blocking vertex, or a Π-consecutive blocking pair u, v where neither u nor v is a blocking vertex.*

The content of this section is largely derived from [6], and the above theorem is an easy consequence of its results. We will thus spend much of this section giving the necessary definitions and results from [6], and will finish with a proof of Theorem 3.6.1.

We will start with some background on matroids. Let N be a matroid, and let X be a subset of the elements of N. Then $N \setminus X$ denotes the matroid obtained from N by *deleting* e, and N/X denotes the matroid obtained from N by *contracting* e. If $M = N \setminus I/J$ for some subsets I, J of the elements of N, then we say that M is a *minor* of N. Even-cycle matroids have the following property with respect to minor operations (see [6], for instance):

Remark 3.6.2. $\mathrm{ecycle}(G, \Sigma) \setminus I/J = \mathrm{ecycle}((G, \Sigma) \setminus I/J)$.

For a matroid N, we will use the notation N^* to refer to the dual of N. It is well known that for subsets I, J of the elements of a matroid N, we have

$$(N \setminus I/J)^* = N^* \setminus J/I.$$

Let G, G' be graphs. If G' can be obtained from G by a sequence of Whitney flips, we say that G and G' are *equivalent*. Let (G, Σ) be a signed graph. We say that (G, Σ) is a *representation* of the matroid $\mathrm{ecycle}(G, \Sigma)$. Note that $\mathrm{ecycle}(G, \Sigma)$ may have distinct representations (G, Σ) and (G', Σ'), where G and G' are not equivalent. In this case, we call (G, Σ) and (G', Σ') *siblings*.

If M is a minor of a matroid N then N is a *major* of M. Consider an even cycle matroid N with a representation (H, Γ). Let I and J be disjoint subsets of $E(N)$ and let $M := N \setminus I/J$. Let $(G, \Sigma) := (H, \Gamma) \setminus I/J$ It follows from Remark 3.6.2 that (H, Γ) is a representation of N that contains (G, Σ) as a minor. We will say that (H, Γ) is an *extension* to N of the representation (G, Σ) of M, or that (G, Σ) *extends* to N.

Consider a matroid N and let $M := N \setminus I/J$ be a minor of N. If $J = \emptyset$ and $|I| = 1$ then N is a *column major* of M. If $I = \emptyset$ and $|J| = 1$, then N is a *row major* of M. A set $\mathbb{F}$ of representations of an even cycle matroid N is *closed under equivalence* if, for every $(G, \Sigma) \in \mathbb{F}$ and (G', Σ') equivalent to (G, Γ), we have that $(G', \Gamma') \in \mathbb{F}$. We will say that $\mathbb{F}$ is an *equivalence class* of M if additionally for every two distinct signed graphs (G, Σ) and (G', Σ') in $\mathbb{F}$, G and G' are equivalent.

Let $\mathbb{F}$ be an equivalence class of a 3-connected even cycle matroid M and let N be a 3-connected major of M. We say that $\mathbb{F}$ is *row stable* (resp. *column stable*) if for all 3-connected row (resp. column) majors N of M, the set of extensions of $\mathbb{F}$ to N is an equivalence class.

In [6] the hypothesis "not graphic" is used in place of 3-connectivity for N, and the hypotheses "not graphic" and "has no loop and no co-loop" are used in place of 3-connectivity for M. We note here that the results we cite from [6] still hold with this modified definition of stability, and that we modify hypotheses in this way whenever we state results from [6].

Remark 3.6.3. *Every equivalence class of an even-cycle matroid is column stable.*

Consider a pair of equivalent graphs G_1 and G_2. Suppose that, for $i = 1, 2$, we have $\alpha_i \subseteq \delta_{G_i}(v_i) \cup \text{loop}(G_i)$ for some $v_i \in V(G_i)$. Then for $i = 1, 2$, let H_i be obtained from G_i by splitting v_i into v_i^- and v_i^+ according to α_i and let $T_i = \{v_i^-, v_i^+\}$.

If H_1 is not equivalent to H_2, then there is a unique pair of signatures Σ_1 and Σ_2 (up to signature exchanges) [6] such that $\text{ecycle}(H_1, \Sigma_1) = \text{ecycle}(H_2, \Sigma_2)$. We say, in that case, that (H_1, Σ_1) and (H_2, Σ_2) are *split siblings*. Observe that, in the previous definition, if Ω is a loop of G_1, G_2 contained in $\alpha_1 \cap \alpha_2$, then for $i = 1, 2$, Ω has endpoints v_i^-, v_i^+ in H_i. We will refer to split siblings with such an edge as Ω-*split siblings*.

We say that a tuple $\mathbb{T} = (G_1, v_1, \alpha_1, G_2, v_2, \alpha_2)$, where G_1, G_2 are 2-connected (up to loops), is a *split-template* if the following conditions hold:

(a) G_1 and G_2 are equivalent graphs;

(b) for $i = 1, 2$, $v_i \in V(G_i)$;

(c) for $i = 1, 2$, $\alpha_i \subseteq \delta_{G_i}(v_i) \cup \text{loop}(G_i)$.

We say that the split siblings (H_1, Σ_1) and (H_2, Σ_2) defined in the previous paragraph *arise* from the split-template $\mathbb{T}$.

Remark 3.6.4. *Let $\mathbb{T} = (G_1, v_1, \alpha_1, G_2, v_2, \alpha_2)$ be a split-template and let (H_1, Σ_1) and (H_2, Σ_2) be split siblings that arise from $\mathbb{T}$. Then, up to signature exchange, we have $\Sigma_1 = \Sigma_2 = \alpha_1 \,\Delta\, \alpha_2$.*

Remark 3.6.5. *Let M be a 3-connected even-cycle matroid and let $\mathbb{F}$ be an equivalence class of M. Let N be a 3-connected row major of M. Let Ω denote the unique element in $E(N) - E(M)$. Suppose that the set $\mathbb{F}'$ of extensions of $\mathbb{F}$ to N is non-empty. Then $\mathbb{F}'$ is either an equivalence class or the union of two equivalence classes $\mathbb{F}_1$ and $\mathbb{F}_2$ and any $(H_1, \Sigma_1) \in \mathbb{F}_1$ and $(H_2, \Sigma_2) \in \mathbb{F}_2$ are Ω-split siblings.*

Proof of Theorem 3.6.1
Let (G, Σ) be a simple, 3-connected non-bipartite signed graph with no blocking vertex or blocking pair, and with even-face embedding Π on surface (or pseudo-surface) S, such that G contains four non-Π-homologous cycles. Let (G^*, Σ^*) be the Π-mate of (G, Σ). Suppose there exist distinct extensions Π_1, Π_2 of Π such that Π_1, Π_2 are both embeddings of simple, 3-connected signed graph (H, Γ). For $i = 1, 2$, let (H_i^*, Γ_i^*) be the mate of (H, Γ) with respect to embedding Π_i.

Then $\text{ecycle}(H_i^*, \Gamma_i^*) = \text{ecycle}^*(H, \Gamma)$ for $i = 1, 2$. Let this matroid be denoted by N, and let $M = \text{ecycle}(G^*, \Sigma^*)$. By Lemma 3.2.1, both N and M are 3-connected. It is easy to see that N is a major of M, and so there exists some element Ω of N such that either $N \setminus \Omega = M$ or $N/\Omega = M$. Since N has two inequivalent representations (H_1^*, Γ_1^*) and (H_1^*, Γ_2^*), Remark 3.6.3 tells us that the latter case must occur. So N is a row major of M. Then Remark 3.6.5 tells us that either H_1^*, H_2^* are equivalent, or $(H_1^*, \Gamma_1^*), (H_2^*, \Gamma_2^*)$ are Ω-split siblings.

Suppose H_1^*, H_2^* are equivalent. Since H is 3-connected, H_i^* has 2-separations only at the endpoints of induced paths of length 2, for $i = 1, 2$. It is easy to see performing a Whitney flip on one of these 2-separations gives a graph isomorphic to H_i^*. So H_1^*, H_2^* cannot be distinct and related by this operation. Also, 3-connectivity of H implies that H_i^* has 1-separations only at pendant vertices for $i = 1, 2$. Suppose H_1^* can be obtained from H_2^* by detaching a pendant edge e of H_1^*, and re-attaching the same edge to a different vertex. Note that in this case, we must have $\Omega = e$. But then Ω is a loop of $E(H, \Gamma) \setminus E(G, \Sigma)$, contradicting the definition of an extension of an embedding.

Then $(H_1^*, \Gamma_1^*), (H_2^*, \Gamma_2^*)$ must be Ω-split siblings. Since

$$(H_1^*, \Gamma_1^*)/\Omega = (H_2^*, \Gamma_2^*)/\Omega = (G^*, \Sigma^*),$$

these siblings arise from some split template $\mathbb{T} = (G^*, v_1, \alpha_1, G^*, v_2, \alpha_2)$. Because of Remark 3.6.4, we may assume (after possibly a signature exchange) that $\Gamma_1^* = \Gamma_2^* = \alpha_1 \,\Delta\, \alpha_2$. Since

$G^* = H_i^*/\Omega$ for $i = 1, 2$, $\alpha_1 \Delta \alpha_2$ is also a signature of (G^*, Σ^*). Since $\alpha_i \subseteq \delta_{G^*}(H_i^*)$, it follows that either v_1, v_2 is a blocking pair of (G^*, Σ^*), or one of v_1, v_2 is a blocking vertex of (G^*, Σ^*). Furthermore, since the embedding of each H_i^*, $i = 1, 2$ extends from the embedding of G^* after splitting v_i relative to α_i for each $i = 1, 2$, we see that we have either a Π-consecutive blocking pair or a Π-consecutive blocking vertex. This completes the proof. $\square$

3.7 Some Topology

The *cycle space* of a graph G, denoted cycle(G), is the binary vector space whose elements are the characteristic vectors of cycles of G. We will say that a set $\mathcal{C}_1$ of cycles of G is *generated* by a set $\mathcal{C}_2$ of cycles of G if the characteristic vectors of the cycles of $\mathcal{C}_1$ can be written as linear combinations of the characteristic vectors of the cycles in $\mathcal{C}_2$. Suppose G is Π-embedded in a surface S. Let $F(G, \Pi)$ denote the number of Π-facial cycles of G. Let $p(G, \Pi)$ be the maximum size of a set P of Π-non-homologous, Π-non-contractible cycles in G whose characteristic vectors are linearly independent.

Now, it is easy to see that any set of $F(G, \Pi) - 1$ Π-facial cycles of G generates every Π-contractible cycle of G. Furthermore, any smaller set of Π-facial cycles of G fails to generate every non-contractible cycle of G. (In particular, no Π-facial cycle excluded by such a smaller set can be generated by the cycles in the set.) It is also clear that the maximum set P of Π-non-contractible cycles described above generates a Π-non-contractible cycle of every homology type, and that a smaller set does not have this property. Since any two Π-non-contractible cycles of G with the same homology type differ by a symmetric difference with a Π-contractible cycle, it follows that a set of $F(G, \Pi) - 1$ facial cycles of G, together with the cycles of P, generate cycle(G). So $\dim(\text{cycle}(G)) = F(G, \Pi) - 1 + p(G, \Pi)$.

The *cut space* of graph G, denoted cut(G) is the binary vector space generated by the characteristic vectors of the one-vertex cuts of G. Consequently, $\dim(\textit{cycle}(G))$ is one less than the number of vertices of G. The cycle space and cut space of G are orthogonal complements, and so dim(cycle(G)) + dim(cycle(G)) is a constant.

Lemma 3.7.1. *For each S in $\mathcal{S}$, a graph embedded in S has at most four pairwise non-homologous cycles.*

Proof. The torus and Klein bottle both have homology group $\mathbb{Z}_2 \times \mathbb{Z}_2$ mod 2. This group has 4 elements, i.e. these surfaces both have four homology types of curve, and any graph on these surfaces contains at most 4 pairwise non-homologous cycles.

Now consider a graph G embedded on a (possibly pinched) surface S. Suppose we pinch two points of S corresponding to two vertices of G. Then the resulting graph G' (with embedding Π') has one fewer vertices than G, but the number of faces and edges remain unchanged. Consequently, $\dim(\mathrm{cut}(G')) = \dim(\mathrm{cut}(G)) - 1$, and so $\dim(\mathrm{cycle}(G')) = \dim(\mathrm{cycle}(G)) + 1$. Since $F(G', \Pi') = F(G, \Pi)$, we must have $h(G', \Pi') = h(G, \Pi) + 1$.

Now suppose we obtain graph G' (with embedding Π' by pinching two points of S that lie inside different faces of G. Then the number of Π'-faces of G' is one less than the number of Π-faces of G, and the number of edges and vertices remain unchanged. So $\dim(\mathrm{cut}(G')) = \dim(\mathrm{cut}(G))$, and $\dim(\mathrm{cycle}(G')) = \dim(\mathrm{cycle}(G'))$. Since the $F(G', \Pi') = F(G, \Pi) - 1$, we must have $h(G', \Pi') = h(G, \Pi) + 1$.

Now, the sphere has homology group $\{0\}$ mod 2, i.e. a graph embedded on the sphere has at most one homology type of cycle. If G is embedded in the double-pinched sphere where one pinch point is in a face of G and the other is in a vertex of G, it follows from the above that the homology group of G mod 2 has up to two more generators than that of a graph embedded in the sphere. So the homology group of G mod 2 is a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$, and G contains at most 4 pairwise non-homologous cycles.

The projective plane has homology group $\mathbb{Z}_2$ mod 2. So a graph embedded in the projective plane has up to 2 homology types of cycles (contractible cycles and one-sided cycles). From the above, we see that if G is embedded in the pinched projective plane with the pinch point either in a face of G or a vertex of G, then the homology group of G mod 2 has one more generator than that of a graph embedded in the projective plane, i.e. the homology group of G mod 2 is a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$. So G contains at most 4 pairwise non-homologous cycles. $\square$

Let (G, Σ) be Π-embedded in a surface $S \in \mathcal{S}$. We will assume that $\Pi(G, \Sigma)$ has four pairwise Π-non-homolgous cycles. Let S_1 and S_2 be two non-homologous cycles in $\Pi(G, \Sigma)$, neither of which is homologous to 0. Let $S_3 = S_1 \Delta S_2$. Then S_1, S_2, S_3 are Π-non-homologous cycles, none of which is homologous to 0. Then we can say the following:

Remark 3.7.2. *Exactly one of S_1, S_2, S_3 is even.*

Proof. Since the symmetric difference of two odd cycles is even, it is clear that at least one of S_1, S_2, S_3 must be even. Suppose two of these cycles are even, say S_1 and S_2. Then S_3 is also even, since the symmetric difference of two even cycles is even. Since we have an even-face embedding of (G, Σ), and since any cycle homologous to S_i can be obtained by taking the symmetric difference of facial cycles and S_i, $i = 1, 2, 3$, every cycle in (G, Σ) must be even. Then $\Sigma = \emptyset$, contradicting our choice of (G, Σ). $\square$

Recall that if Π is an even-face embedding of (G, Σ), then any two Π-homologous cycles of (G, Σ) have the same parity. This leads us to the following:

Remark 3.7.3. *The mate (G^*, Σ^*) of (G, Σ) is unique up to resigning.*

Proof. It is clear that the choice of G^* is unique. By Remark 3.7.2, the choice of Σ^* is unique up to symmetric difference of facial cycles in G, i.e. up to resigning on cuts of G^*. □

3.8 A Result on Unique Extension

In this section, we will give a general result on uniqueness of extension for (G, Σ), in terms of properties of (G, Σ). We will obtain this result by using Theorem 3.6.1 of Section 3.6, along with some facts from topology. Our main result for this section is the following:

Theorem 3.8.1. *Let (G, Σ) be a simple, non-bipartite 3-connected signed graph, where (G, Σ) contains neither a blocking vertex nor a blocking pair. Suppose (G, Σ) has an even-face embedding Π on a (possibly pinched) surface S in $\mathcal{S}$, such that G contains four pairwise non-Π-homologous cycles. If Π does not extend uniquely in S, then one of the following occurs:*

- *(s1) There exists a Π-non-contractible curve s in S such that s intersects G in exactly two vertices x, y, and faces F_1, F_2 of G such that each contains a segment of s. Furthermore, every cycle of (G, Σ) homologous to s is even.*
- *(s2) There exists a Π-non-contractible curve s in S such that s intersects G in exactly one vertex w, and a face F of G such that s lies in F and w is met twice when traversing the boundary of F. Furthermore, every cycle of (G, Σ) homologous to s is even.*
- *(s3) There exist two non-homologous, non-contractible curves s_1, s_2 in S such that for $i = 1, 2$, s_i intersects G in exactly one vertex w_i and there exists a face F_i of G such that s_i lies in F_i and w_i is met twice when traversing the boundary of F_i. Furthermore, for $i = 1, 2$, every cycle of (G, Σ) homologous to s_i is odd.*

Before we prove Theorem 3.8.1 we need some preliminary results:

Lemma 3.8.2. *Let (G, Σ) be a signed graph with even-face embedding Π on surface $S \in \mathcal{S}$, such that G contain four pairwise Π-non-homologous cycles. Let (G^*, Σ^*) be the Π-mate of (G, Σ). Then Σ' is a signature of (G^*, Σ^*) if and only if $\Sigma^* \,\Delta\, \Sigma'$ is Π-contractible.*

Proof. Since any two equivalent signatures differ by a cut,

$$\begin{aligned}\Sigma^* \,\Delta\, \Sigma' &= \delta_{G^*}(\{u_1, ..., u_k\}), \text{ where } u_i \in V(G^*) \text{ for all } i \in [k],\\ &= \delta_{G^*}(u_1) \,\Delta \ldots \Delta\, \delta_{G^*}(u_k).\end{aligned}$$

By the construction of (G^*, Σ^*),This is a symmetric difference of Π-facial cycles of G and hence is Π-contractible. This proves the result. □

Lemma 3.8.3. *Let (G, Σ) be a signed graph with even-face embedding Π on a surface $S \in \mathcal{S}$. Let S_1, S_2, S_3 be Π-non-homologous cycles, none of which is Π-homologous to 0, such that $S_1 \Delta S_2 = S_3$, where S_3 is even. Let (G^*, Σ^*) be the Π-mate of (G, Σ), with $\Sigma^* = S_3$. Then a minimal signature of (G^*, Σ^*) is one of the following:*

1. *a circuit Σ' homologous to S_3, or*
2. *a cycle $\Sigma' = D_1 \mathbin{\dot\cup} D_2$,*

where D_1 is a circuit homologous to S_1, D_2 is a circuit homologous to S_2, and D_1, D_2 share at most one vertex.

Proof. Suppose Σ' is a minimal signature. We may write

$$\Sigma' = B_1 \mathbin{\dot\cup} B_2 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} B_k,$$

where each B_i is a circuit, and their disjoint union is homologous to S_3. We may assume that for all $i \in [k]$ B_i is not homologous to S_3 – otherwise, by minimality $B_i = \Sigma'$, and we have case (1).
Similarly, we may assume there do not exist distinct $i, j \in [k]$ such that B_i is homologous to S_1, B_j is homologous to S_2. (Otherwise, we have case (2).) However, it is easy to see that if neither of these cases occurs, the disjoint union of the B_i cannot be homologous to S_3; then by Lemma 3.8.2, the disjoint union cannot be a signature of (G^*, Σ^*).

Now we show that in case (2), D_1 and D_2 share at most one vertex. Suppose not. Let u and v be distinct vertices in both D_1 and D_2. Then there are four distinct uv-paths in $D_1 \mathbin{\dot\cup} D_2$, and so $D_1 \mathbin{\dot\cup} D_2$ contains $\binom{2}{4} = 6$ distinct cycles that each contain both u and v. Since there are only 4 homology types in the torus, some two of these cycles, say C_1 and C_2, must have the same homology type. Then $C_1 \Delta C_2 \neq \emptyset$, and $C_1 \Delta C_2$ is homologous to 0. It follows that $(D_1 \mathbin{\dot\cup} D_2) \Delta (C_1 \Delta C_2)$ is homologous to S_3, and is contained in $D_1 \mathbin{\dot\cup} D_2 = \Sigma'$. This contradicts the minimality of Σ'. □

Proof of Theorem
By contrapositive.

Let (G, Σ) be a simple, 3-connected, non-bipartite signed graph with no blocking vertex or blocking pair. Suppose (G, Σ) has an even-face embedding Π on $S \in \mathcal{S}$, where Π does not extend uniquely. By Theorem 3.6.1, the Π-mate (G^*, Σ^*) of (G, Σ) contains a consecutive blocking pair u, v (where it is possible that one of u, v is a blocking vertex). We may assume that Σ^* is a minimal signature of (G^*, Σ^*), and that $\Sigma^* \subseteq \delta_{G^*}(u) \cup \delta_{G^*}(v)$. Let F_u and F_v be the faces of G associated with vertices u and v of G^*, respectively. By duality, in G the edges of Σ^* are consecutive along the boundaries of F_u and F_v.

Case 1: $\Sigma^* \cap \delta_{G^*}(u)$ is a path P_u in G.
Since Σ^* is a circuit (by Lemma 3.8.3), $\Sigma^* \cap \delta_{G^*}(v)$ is a path P_v of G, and P_u, P_v have the same endpoints endpoints x, y, where $x \neq y$. Then x and y both lie on the boundary of both F_u and F_v. So there exists a path s_u with interior in F_u, and a path s_v with interior in F_v, such that s_u, s_v both have endpoints x, y. Let $s = s_u \cup s_v$. Then s is a non-contractible curve in S that intersects (G, Σ) in exactly two vertices, and lies in two distinct faces of (G, Σ). Furthermore, s is Π-homologous to Σ^*, and hence every cycle of (G, Σ) Π-homologous to s is even.

Case 2: $\Sigma^* \cap \delta_{G^*}(u)$ is a cycle C_u.
We may assume that $\delta_{G^*}(u) \cap \Sigma^* \neq \emptyset$. We cannot have $\delta_{G^*}(u) \subseteq \Sigma^*$; otherwise Σ^* is not minimal. So there exists a circuit C such that $C \cup C_u$ is the boundary of of F_u, and there exists a closed curve s_u which lies inside F_u and intersects (G, Σ) exactly at x, where x is met twice when traversing the boundary of F_u. Then s_u is homologous to $\Sigma^* \cap \delta_{G^*}(u)$. Furthermore, by Lemma 3.8.3 s_u is non-contractible.

If $\delta_{G^*}(v) \cap \Sigma$ is empty, then $\Sigma^* = \delta_{G^*}(u) \cap \Sigma^* = C_u$, so s_u is homologous to Σ^*, and every cycle of (G, Σ) homologous to s is even. This (s2) holds. If $\delta_{G^*}(v) \cap \Sigma$ is non-empty, then $\delta_{G^*}(v) \cap \Sigma^*$ is also a cycle C_v in G – otherwise, Σ^* contains vertices of odd degree and is not a cycle. Then a similar argument gives a non-contractible curve s_v Π-homologous to $\Sigma^* \cap \delta_{G^*}(v)$ which lies inside F_v and intersects (G, Σ) exactly at a vertex y, where y is met twice when traversing the boundary of F_v. Furthermore, Lemma 3.8.3 tells us that C_u, C_v are non-contractible and non-homologous. By Lemma 3.7.2, C_u, C_v are both odd, and so every cycle of (G, Σ) homologous to one of s_u, s_v is odd. This gives (s3) $\square$

3.9 Specific results by surface

In this section, we will apply Theorem 3.8.1 to each $S \in \mathcal{S}$ to give a more specific result unique extension of signed graphs with an even-face embedding on S. We will handle each surface in its own subsection. First, we will give some definitions and results that will prove helpful later.

In topology, two objects are said to be *homeomorphic* if one can be deformed into the other by a continuous, invertible mapping. Let G be a graph Π-embedded in (possibly pinched) surface S. We say that Π is a *cellular embedding* if every Π-face of G is homeomorphic to a disc. (Note that an embedding of a graph G in a pinched surface is cellular only if every pinch point is a vertex of G.)

Lemma 3.9.1. *Suppose G is Π-embedded on a (non-pinched) surface S, where Π is cellular. Then every closed curve in S is homologous (mod 2) to a cycle of G.*

Let graph G be Π-embedded on a (possibly pinched) surface S. Suppose s is a curve in surface S and intersects G only in vertices. For each vertex v of G such that v is on s (it is possible that there are no such vertices), we split v into two vertices v_1 and v_2, where v_1 is incident with the edges of $\delta(v)$ on the right-hand side of s and v_2 is incident with the edges of $\delta(v)$ on the left-hand side of s. (If s is onesided, there may be some ambiguity as to the right and left side of s. However, in any case we still obtain the same graph after splitting, up to relabelling some of the split vertices.) This produces a new graph $\hat{G}$ which is $\hat{\Pi}$-embedded on S such that s does not intersect $\hat{G}$. Now we remove from S a small neighbourhood of s, to obtain an embedding $\hat{\Pi}'$ of $\hat{G}$ on a bordered surface S'. We will say that $\hat{G}$ with embedding $\hat{\Pi}'$ on S' was obtained by *cutting along s*.

Now, suppose we sew a disc onto each boundary component of S'. If s was twosided, we now have an embedding $\hat{\Pi}'$ of $\hat{G}$ on a surface of genus one less than that of S (we have effectively removed a handle from S). If s was onesided, we now have an embedding $\hat{\Pi}'$ of $\hat{G}$ on a surface of non-orientable genus one less than that of S (we have effectively removed a cross-cap from S).

3.9.1 Extensions of signed graphs on the torus

In this section, we will prove Theorem 3.1.2. We will begin with some useful lemmas.

Lemma 3.9.2. *Let (G, Σ) be a signed graph with an even-face embedding Π on the torus. Suppose G does not contain three pairwise Π-non-homologous, Π-non-contractible cycles. Then G is planar.*

Proof. Note that if G contains two Π-non-homologous, Π-non-contractible cycles C_1 and C_2, then $C_1 \Delta C_2$ is a Π-non-contractible cycle that is not Π-homologous to C_1 or C_2. Thus G contains at most one Π-homology type of Π non-contractible cycle. It follows that there is some non-contractible closed curve C in the torus such that G has no cycle Π-homologous to C. Then by Lemma 3.9.1, G is not cellular, and so there is some Π-face F of G that is not homeomorphic to a disc. Since G is embedded on the torus, F must be homeomorphic to a cylinder. Let s be a non-contractible closed curve that lies inside F and does not intersect $\Pi(G)$. By cutting along s, we obtain an embedding of G on an open-ended cylinder. Sewing a disc onto each end, we obtain an embedding of G on the sphere. Hence G is planar.

$\square$

Lemma 3.9.3. *Let s be a non-contractible closed curve in the torus. Then every non-contractible cycle of G that is not homotopic to s intersects s.*

Proof. We may assume (possibly perturbing s) that s intersects G only in vertices of G. Cutting along s, we obtain an embedding of $\hat{G}$ in an open-ended cylinder. Now, $\hat{G}$ has only only one homology type of non-contractible curve – those that are homotopic to a boundary component of the cylinder. So any cycle of $\hat{G}$ that is not homologous to zero is homotopic to s. Thus every non-contractible cycle of G that is not homotopic to s intersects s. $\square$

Proof of Theorem 3.1.2

Let (G, Σ) be a simple, 3-connected signed graph, such that G is not planar and (G, Σ) has no blocking pair or blocking vertex. Let Π be an even-face embedding of (G, Σ) on the torus. By Lemma 3.9.2, G contains four pairwise Π-non-homologous cycles. Let S_1, S_2, S_3 be Π-non-homologous, Π-non-contractible cycles of G. By Lemma 3.7.2 we may assume that S_3 is even.

Suppose by way of contradiction that Π does not extend uniquely. Then by Theorem 3.8.1 one of (s1), (s2) or (s3) occurs. Suppose first that either (s1) or (s2) occurs. Then there exists a non-contractible curve s in the torus such that s intersects G in at most two vertices, and s is homotopic to S_3. By Lemma 3.9.3, every odd cycle of (G, Σ) contains a vertex of (G, Σ) that lies in s. Since there are at most two such vertices, (G, Σ) has a blocking pair or a blocking vertex, a contradiction.

Now suppose (s3) occurs. We may assume s_1 is homologous to S_1 and s_2 is homologous to S_2. By Lemma 3.9.3, every cycle of G homologous to S_i intersects s_{3-i}, for $i = 1, 2$. Since each s_i contains exactly one vertex w_i, this implies that every odd cycle of (G, Σ)

intersects one of the w_i. So (G, Σ) contains a blocking pair, a contradiction. This completes the proof. $\square$

3.9.2 Extensions of signed graphs on the double-pinched sphere

We consider a signed graph (G, Σ) with even-face embedding Π on the double-pinched sphere such that every Π-face of (G, Σ) is even. Note any pinch point that is not contained in $\Pi(G)$ lies in a single face of $\Pi(G)$, which is not homeomorphic to a disc. We will call this face the *pinched face* of $\Pi(G)$. The boundary of the pinched face is composed of two components (which may intersect), such that each component differs from the other by the symmetric difference of all other facial cycles of G. Note that "unpinching" this pinch point gives an embedding of G in the pinched sphere in which each boundary component of the pinched face is itself a facial cycle.

Suppose some pinch point x is contained in $\Pi(G, \Sigma)$, but is not a vertex of $\Pi(G, \Sigma)$. Then x is contained in the interior of $\Pi(e)$, for some $e \in E(G)$. Let s be the curve in the double-pinched sphere with endpoints x, y, where $y = \Pi(v)$ for an endpoint of v of e, such that s is contained in $\Pi(e)$. Then it is easy to see that contracting s in the double-pinched sphere gives an embedding of (G, Σ) on the double-pinched sphere where x and $\Pi(v)$ coincide. We may thus assume without loss of generality that any pinch point is either a vertex of $\Pi(G)$, or is not contained in $\Pi(G, \Sigma)$. We then have three different possibilities for the placement of the pinch points – either both are vertices of $\Pi(G)$, one is a vertex of $\Pi(G)$ and the other is not contained in $\Pi(G)$, or neither is contained in $\Pi(G)$. As we will see, the proof is easy if both pinch points are in $\Pi(G)$, or if neither is. It will remain, then, only to consider the case where exactly one pinch point is a vertex of $\Pi(G)$.

As with the torus, we must first have a guarantee that for any embedding Π of (G, Σ) on the double-pinched sphere, the mate of (G, Σ) with respect to Π is well-defined.

Lemma 3.9.4. *Let (G, Σ) be a signed graph with an even-face embedding Π on the double-pinched sphere, where one pinch point is a vertex of $\Pi(G)$, and the other pinch point is not contained in $\Pi(G)$. Suppose G does not have three pairwise Π-non-homologous, Π-non-contractible cycles. Then either G is planar, or (G, Σ) has a blocking vertex.*

Proof. Note that G contains only one homology type of Π-non-contractible cycle, and that all Π-non-contractible cycles of R are odd. Let C be a Π-non-contractible cycle in G. If C uses a pinch point x in the surface, where x is a vertex of G, then every odd cycle of G uses x. So x is a blocking vertex. Otherwise, any cycle of G that contains x is contractible.

Then un-pinching both pinch points gives an embedding of G on the sphere, and G is planar. □

We can now prove Theorem 3.1.3.

Proof of Theorem 3.1.3

Proof. Let (G, Σ) be a simple, 3-connected non-planar signed graph with no blocking vertex or blocking pair, and let Π be an even-face embedding of (G, Σ) on the double-pinched sphere, such thats (G, Σ) does not contain a bad pair of Π-faces. We have three different possibilities for the placement of the pinch points. First, suppose neither pinch point of the surface is a vertex of $\Pi(G)$. Then G has an embedding on the sphere and hence is planar, a contradiction. Now, suppose both pinch points are vertices of $\Pi(G)$, say u and v. Then every odd cycle in $\Pi(G, \Sigma)$ contains one of u, v, and u, v is a blocking pair, another contradiction. Now we will consider the case where exactly one pinch point is a vertex of $\Pi(G)$. By Lemma 3.9.4, we may assume that G contains four pairwise Π-non-homologous cycles.

By way of contradiction, suppose Π does not extend uniquely. Then by Theorem 3.8.1, one of (s1), (s2), (s3) occurs.

Suppose (s1) occurs. Then there exists a non-contractible curve s in the double-pinched sphere such that s intersects G in exactly two vertices x, y, and s is Π-homologous to an even cycle in (G, Σ). Furthermore, there exist faces F_1, F_2 of G such that each contains a segment of s. Suppose s is homologous to a boundary component of the pinched face of G. By the description of F_1, F_2, cutting the double-pinched sphere along s separates G into two pieces. If both pieces contain vertices of G, then G is not 3-connected. So one piece must contain only edges. In particular, there must be two edges e_1, e_2 with endpoints x and y, such that e_1, e_2 form a boundary component of the pinched face. Since ecycle(G, Σ) is 3-connected, we may assume e_1 is even, and e_2 is odd. Then these two edges form an odd cycle homologous to s. But by hypothesis, every cycle of (G, Σ) homologous to s is even – contradiction. It follows that s must contain the pinch point that is a vertex of (G, Σ). We may therefore assume that x is a pinch point. It follows that there exist two faces F_1, F_2 of G such that x, y are both in F_1 and in F_2. Thus F_1, F_2 is a bad pair of Π-faces, contradicting our choice of Π.

Now suppose (s2) occurs. Then there exists a non-contractible curve s in the double-pinched sphere such that s intersects G in exactly one vertex w, and s is homologous to an even cycle in (G, Σ). Furthermore, there exists a face F of G such that s lies in F and w is met twice when traversing the boundary of F. As in the previous case, s cannot be homologous to the boundary of a pinched face of G. So s must contain the pinch point; in particular, w is the pinched vertex. Consider the graph (G', Σ) obtained from (G, Σ) by un-pinching the pinch point. Notice that (G', Σ) has an embedding (not an even-face embedding) in the sphere in which s is an arc in the sphere disjoint from G' except at its endpoints. Furthermore, the endpoints of s are the vertices obtained by un-pinching the pinch point. Then contracting s to a point gives an embedding of (G, Σ) in the sphere; i.e. (G, Σ) is planar, a contradiction.

Finally, suppose (s3) occurs. Then there exist two non-homologous, non-contractible curves s_1, s_2 in the double-pinched sphere such that, for $i = 1, 2$, s_i intersects G in exactly one vertex w_i and s_i is Π-homologous to an odd cycle of (G, Σ). Furthermore, for each $i = 1, 2$ there exists a face F_i of G such that s_i lies in F_i and w_i is met twice when traversing the boundary of F_i. Since we have two non-homologous, non-contractible curves, and two of the homology types of non-contractible curves in G contain the pinch point, one of s_1, s_2 must contain the pinch point. A similar argument to the above shows that G is planar. So in each case, we arrive at a contradiction. Thus Π extends uniquely.

□

3.9.3 Extensions of graphs on the pinched projective plane

Let (G, Σ) be a 3-connected signed graph with an even-face embedding Π in the pinched projective plane. We will begin with the case where the pinch point is not contained in $\Pi(G)$.

Lemma 3.9.5. *Let (G, Σ) be a signed graph with an even-face embedding Π on the pinched projective plane, where the pinch point is not in $\Pi(G)$. Suppose Π does not have three pairwise Π-non-homologous, Π-non-contractible cycles. Then either G has an even-face embedding on the projective plane, or G is planar.*

Proof. Note that G contains at most one homology type of Π-non-contractible cycle, and every odd cycle of G is Π-non-contractible. Let C be a Π-non-contractible cycle. Suppose C is Π-onesided. Then unpinching the pinch point of the surface does not create any odd faces (as $\Pi(G)$ has no cycles homotopic to a boundary component of the pinched face), and so (G, Σ) has an even-face embedding on the projective plane. Otherwise, G contains no Π-onesided cycles, and un-pinching the pinch point gives an embedding of G on a disc. Hence G is planar. □

Proof of Theorem 3.1.4

Let Π be an even-face embedding of (G, Σ) on the pinched projective plane, such that the pinch point is not in $\Pi(G)$. Suppose (G, Σ) is non-planar and does not have an even-face embedding on the projective plane or on the double-pinched sphere. Then by Lemma 3.9.5, G contains four pairwise Π-non-homologous cycles. Suppose by way of contradiction that Π does not extend uniquely. By Theorem 3.8.1, one of (s1), (s2), (s3) holds for Π.

Suppose (s1) or (s2) occurs. Then there exists a Π-non-contractible curve s in the pinched projective plane such that s intersects G in exactly one or two vertices and s is Π-homologous to an even cycle in G. If s is Π-twosided, then cutting the pinched projective plane along s separates G into two pieces. If both pieces contain a vertex of G, then G is not 3-connected. So one piece must contain only edges of G. In particular, G contains a cycle C homologous to s where C is either two parallel edges differing in parity, or an odd loop. But every cycle homologous to s must be even – a contradiction. So s is Π-onesided in this case. If (s3) occurs, we may assume one of the s_1, s_2 is orientation-reversing; let this curve be denoted s. So in any of the three cases, we have a Π-onesided curve s in the pinched projective plane that intersects G in at most two vertices. We will begin by unpinching the pinch point in the surface to get an embedding $\bar{\Pi}$ of G in the projective plane with two odd faces.

Suppose s intersects G in exactly one vertex x. Then s lies in a single $\bar{\Pi}$-face F of (G, Σ), and x occurs twice on the boundary of F. Let (G', Σ) be the graph with embedding Π' obtained from (G, Σ) by unpinching the pinch point and cutting along s. Capping the boundary component of this surface, we see that Π' is an embedding of (G', Σ') on the sphere, such that some Π'-face of G' contains both vertices x_1, x_2 obtained from splitting x. Then we can continuously deform the sphere to identify x_1, x_2 as a single vertex, x. This gives us an embedding Π' of (G, Σ) on the sphere, contradicting the non-planarity of G.

Now suppose s intersects G in exactly two vertices x and y. Split y into two vertices y_1 and y_2, where y_1 is incident with the edges of $\delta(y)$ on the left-hand side of s and y_2 is incident with the edges of $\delta(y)$ on the right-hand side of s. Let the resulting embedded graph be denoted $\hat{G}$, with embedding $\hat{P}i$. Let F_1, F_2 denote the $\bar{\Pi}$-faces of G that contain s. Note that the $\hat{\Pi}$-faces of $\hat{G}$ are identical to the $\bar{\Pi}$-faces of G', except that in $\hat{G}$ faces F_1, F_2 have been replaced by a single face F whose boundary contains exactly the edges in the boundaries of F_1 and F_2. If F_1, F_2 have the same parity, then F is even. If F_1, F_2 differ in parity, then F is odd. It follows that $(\hat{G}, \Sigma)$ has two odd $\hat{\Pi}$-faces.

Note that s intersects $\hat{G}$ exactly in vertex x, and so we can cut along s and cap the boundary component of the resulting bordered surface to obtain a graph $(\hat{G}', \Sigma)$ with embedding $\hat{\Pi}'$, as in the first case. As before, we can modify this graph to obtain an embedding Π' of $(\hat{G}, \Sigma)$ on the sphere.

Note that the $\hat{\Pi}'$-faces of $(\hat{G}, \Sigma)$ are identical to the $\hat{\Pi}$-faces of $(\hat{G}, \Sigma)$, except that F is replaced by two faces with boundaries C_1 and C_2. Note that $C_1 \cap C_2 = \emptyset$, and that each of C_1, C_2 is $\hat{\Pi}$-onesided. Furthermore, at least one of C_1, C_2 is $\hat{\Pi}$-homologous to s (and hence is even). If both C_1, C_2 are even, then F was also even. If one of C_1, C_2 is odd, then F was odd. In either case, the number of odd $\hat{\Pi}'$-faces of (G, Σ) is the same as the number of odd $\hat{\Pi}$-faces of (G, Σ). So (G, Σ) has two odd $\hat{\Pi}'$-faces.

Then pinching together the two odd faces, as well as pinching together y_1 and y_2 (to identify them as a single vertex y) gives an even-face embedding of (G, Σ) on the double-pinched sphere. □

Now we will consider the case where the pinch point is in $\Pi(G)$. As in the previous section, we may assume that the pinch point is a vertex of $\Pi(G)$.

Lemma 3.9.6. *Let (G, Σ) be a signed graph with an even-face embedding Π on the pinched projective plane, where the pinch point is a vertex of $\Pi(G)$. Suppose G does not have three pairwise Π-non-homologous, Π-non-contractible cycles. Then G has an even-face embedding on the pinched projective plane where the pinch point is not in $\Pi(G)$, or (G, Σ) has a blocking vertex.*

Proof. Note that G contains only one homology type of Π-non-contractible cycle, and that all Π-non-contractible cycles of (G, Σ) are odd. Let C be a Π-non-contractible circuit of G. Suppose C contains the pinch point, x. Then every odd cycle of (G, Σ) contains the pinch point, and x is a blocking vertex of (G, Σ). Otherwise, no Π-non-contractible circuit of (G, Σ) uses the pinch point. Perturbing $\Pi(G, \Sigma)$ gives an even-face embedding of (G, Σ) in the pinched projective plane with the pinch point not in $\Pi(G)$. □

Proof of Theorem 3.1.5 Let (G, Σ) be a simple, 3-connected signed graph with an even-face embedding Π on the pinched projective plane, where the pinch point is contained in $\Pi(G)$, and G has no bad pair of Π-faces. Suppose (G, Σ) has no even-face embedding Λ on the pinched projective plane where the pinch point is not in $\Lambda(G, \Sigma)$ or on the double-pinched sphere, and that (G, Σ) has no blocking vertex. For a contradiction, suppose Π does not extend uniquely. By Lemma 3.9.6, G contains four pairwise non-homologous cycles. Then by Theorem 3.8.1 one of (s1), (s2), (s3) occurs.

Suppose (s1) occurs. Then there exists a non-contractible curve s in the pinched projective plane such that s intersects $\Pi(G)$ in exactly two vertices x, y, and s is homologous to an even cycle in G. Furthermore, there exist faces F_1, F_2 of G such that each contains a segment of s. If s contains the pinch point, then F_1, F_2 is a pair of bad faces. If s does not contain the pinch point, then every cycle of G that does not contain the pinch point is even. It follows that every odd cycle of (G, Σ) contains the pinch point, and so (G, Σ) has a blocking vertex, a contradiction.

Now suppose (s2) or (s3) occurs. Then there exists a Π-non-contractible curve s that intersects G exactly once in vertex x. First, suppose x is the pinch point. Consider the graph (G', Σ) obtained from (G, Σ) by un-pinching the pinch point (i.e. splitting x into vertices x_1, x_2 relative to the pinch point). Notice that (G', Σ) has an embedding (perhaps not an even-face embedding) in the projective plane in which s is an arc in the projective plane disjoint from G', except at its endpoints x_1, x_2. Then contracting s to a point gives an embedding of (G, Σ) in the projective plane, with at most two odd faces. So (G, Σ) has an embedding in the pinched projective plane, where the pinch point lies in two faces of (G, Σ) – a contradiction.

Now suppose s does not contain the pinch point. Let F be the Π-face of G containing s and let x be the vertex on s. Cutting along s and sewing a disc onto the boundary created gives a graph G' with an embedding Π' on the pinched sphere, where some Π' face of G' contains both vertices x_1, x_2 obtained from splitting x. Note that the Π'-faces of G' are identical to the Π-faces of G. Now, we deform the pinched sphere such that x_1, x_2 are re-identified into a single vertex x to give an embedding Π'' of (G, Σ) on the sphere. There are two Π''-faces of (G, Σ) that are not Π-faces of (G, Σ), namely those created by identifying x_1, x_2. Pinching these faces together gives an even-face embedding of (G, Σ) in the double-pinched sphere, a contradiction. Thus Π extends uniquely. □

3.9.4 Extensions of signed graphs on the Klein bottle

Let (G, Σ) be a signed graph with an even-face embedding on the Klein bottle.

Lemma 3.9.7. *Let (G, Σ) be a signed graph with an even-face embedding Π on the Klein bottle. Suppose G does not contain four pairwise Π-non-homologous cycles. Then G has an even-face embedding on the projective plane, or G is planar.*

Proof. Note that if G contains two Π-non-homologous, Π-non-contractible cycles C_1 and C_2, then $C_1 \,\Delta\, C_2$ is a Π-non-contractible cycle that is not Π-homologous to C_1 or C_2. Thus G contains at most one Π-homology type of Π non-contractible cycle. It follows that there is some non-contractible closed curve C in the Klein bottle such that G has no cycle Π-homologous to C. Then by Lemma 3.9.1, Π is not cellular, and so there is some Π-face F of G that is not homeomorphic to a disc. Since G is embedded on the Klein bottle, F must be homeomorphic to either a cylinder or a Möbius band. Let s be a non-contractible closed curve that lies inside F and does not intersect $\Pi(G)$. By cutting along s, we obtain an embedding of G on either an open-ended cylinder or a Möbius band. In the first case, sewing a disc onto each end of the cylinder gives us an embedding of G on the sphere. Hence G is planar. In the second case, sewing a disc onto the boundary of the Möbius band gives an embedding Π' of (G, Σ) on the projective plane. Since every Π'-face of (G, Σ) is a Π-face of (G, Σ), Π' is an even-face embedding. $\square$

Lemma 3.9.8. *Let G be a graph Π-embedded on the Klein bottle. Then every non-contractible-twosided curve in the Klein bottle intersects every Π-onesided cycle of G.*

Proof. Let s be an twosided curve in the Klein bottle. By possibly perturbing s, we may assume s intersects $\Pi(G)$ only in vertices. Cutting along s gives a graph G' embedded in an open-ended cylinder band with embedding Π'. Since the cylinder is orientable, it is clear that G' contains no Π'-onesided cycles. Thus every Π'-onesided cycle of G intersects s. $\square$

Proof of Theorem 3.1.6

Let (G, Σ) be a simple 3-connected signed graph with an even-face embedding Π on the Klein bottle. Suppose G is non-planar, and does not have an even-face embedding on the projective plane or the pinched projective plane. Suppose also that (G, Σ) has no blocking vertex or blocking pair.

For a contradiction, suppose Π does not extend uniquely. By Lemma 3.9.7, G contains four pairwise Π-non-homologous cycles, and Theorem 3.8.1 applies. Then one of (s1), (s2), (s3) of Theorem 3.8.1 occurs.

Suppose one of (s1), (s2) occurs. Then there exists a Π-non-contractible curve s in the Klein bottle such that s intersects G in exactly one or two vertices and s is Π-homologous to an even cycle in G. Suppose s is Π-twosided. Then by Lemma 3.9.8 every Π-onesided cycle of G, and hence every odd cycle of G, intersects s. Thus every odd cycle of G intersects a vertex on s, and G has either a blocking pair or a blocking vertex – a contradiction. So s is Π-onesided in this case. If (s3) occurs, we may assume one of the s_1, s_2 is orientation-reversing; let this curve be denoted s. So in any of the three cases, we have a Π-onesided curve s in the pinched projective plane that intersects G in at most two vertices.

Suppose s intersects G in exactly one vertex x. Then s lies in a single Π-face F of (G, Σ), and x occurs twice on the boundary of F. Let C_1, C_2 denote the two cycles homologous to s that make up the boundary of F. Let (G', Σ) be the graph with embedding Π obtained from (G, Σ) by cutting along s. Capping the boundary component of this surface, we see that Π is an embedding of (G', Σ') on the projective plane, such that some Π'-face of G'

contains both vertices x_1, x_2 obtained from splitting x. Then we can continuously deform the projective plane to identify x_1, x_2 as a single vertex, x. This process replaces F by two faces with boundaries C_1, C_2. Identifying a point in the interior of both of these faces (to pinch the two faces) gives an embedding Π' of (G, Σ) on the pinched projective plane with two pinched faces, where the Π'-faces of G are exactly the Π-faces of G. It follows that Π' is an even-face embedding of (G, Σ) on the pinched projective plane, a contradiction.

Now suppose s intersects G in exactly two vertices x, y (i.e. suppose (s1) occurs). Then s lies in two Π-faces F_1, F_2 of (G, Σ), and every cycle of (G, Σ) homologous to s is even. Let P_1, P_2 be the two x,y-paths in the boundary of F_1, and let P_3, P_4 be the two x,y-paths in the boundary of F_2. Notice that each of $P_i \cup P_j$ is then an even Π-cycle in (G, Σ) for $i = 1, 2$ and $j = 3, 4$.

Let (G', Σ) be the graph with embedding Π' obtained from (G, Σ) by cutting along s. Capping the boundary component of this surface, we see that Π is an embedding of (G', Σ') on the projective plane, such that some Π'-face of G' contains both vertices x_1, x_2 obtained from splitting x and both vertices y_1, y_2 obtained from splitting y. Then we can continuously deform the projective plane to identify x_1, x_2 as a single vertex, x. This process replaces F_1, F_2 by two faces F_1', F_2'. Without loss of generality, we may assume that the boundary of F_1' is given by $P_1 \cup P_3$, and the boundary of F_1' is given by $P_2 \cup P_4$. It follows that F_1', F_2' are even Π'-faces of G'. Since all other Π'-faces of G' are also Π-faces of G, it follows that Π' is an even-face embedding of (G', Σ). Then pinching y_1, y_2 together gives an even-face embedding of (G, Σ) on the pinched projective plane, a contradiction. It follows that Π extends uniquely. $\square$

3.10 Extensions of apex signed graphs with two odd faces

In this section, we will prove Theorem 3.1.7, i.e. we will give sufficient conditions for an apex graph with two odd faces to extend uniquely. Recall that a signed graph (G, Σ) is apex with two odd faces if for some $v \in V(G)$, there exists a planar embedding of $(G - v, \Sigma \setminus \delta_G(v))$ with exactly two odd faces. We will begin with a series of lemmas, describing the effects of a single vertex-splitting or edge-addition on an apex graph with two odd faces.

Lemma 3.10.1. *Let (G', Σ') be a signed graph, and let $\Pi' = (\lambda', a')$ be an apex embedding of (G', Σ') with exactly two odd faces. Suppose (G', Σ') does not have an even-face embedding* *on the double-pinched sphere. Then any signed graph (H, Γ) obtained from (G', Σ') by splitting the apex vertex a' admits at most one embedding that extends from Π'.*

Proof. Suppose we obtain (H, Γ) from (G', Σ') by splitting the apex vertex a' of (G', Σ') into vertices a_1 and a_2. Suppose (H, Γ) admits two distinct apex embeddings Π_1, Π_2 with exactly two odd faces that extend from Π'. Then we must have $\Pi_1 = (\lambda_1, a_1)$, and $\Pi_2 = (\lambda_2, a_2)$. Since both $(H - a_1, \Gamma \setminus \delta_H(a_1))$ and $(H - a_2, \Gamma \setminus \delta_H(a_2))$ are both planar, $N(a_1) - a_2$ is contained in the boundary of some face F_1 of G', and $N(a_2) - a_1$ is contained in the boundary of some face F_2 of G'. Then it is easy to see that we can embed G' on the pinched sphere, where a' is the pinch point. We know all but exactly two λ'-faces F_1', F_2' of $G' - a'$ are even. We may identify identify points x_1, x_2 on the surface of the pinched sphere, where x_i is interior to $F-_i$, to obtain an embedding Λ of (G', Σ') on the double-pinched sphere such that the boundaries of F_1' and F_2' form the boundary of the pinched face. Suppose this embedding of (G', Σ') in the double-pinched sphere has some odd face. Then a Λ-face F' of (G', Σ') containing a' is odd; otherwise, (G', Σ') has an odd Π'-face. Without loss of generality, we may assume that the boundary of F' is a cycle of (H, Γ) containing a_1. But then F' is a λ_2-face of $H - a_2$, and $H - a_2$ has 3 odd λ_2-faces. This contradicts our choice of λ_2. So we have an even-face embedding of (G', Γ') on the double-pinched sphere. Since (G, Σ) is a subgraph of (G', Σ'), (G, Σ) also has an even-face embedding on the double-pinched sphere – contradiction. □

Lemma 3.10.2. *Let (G', Σ') be a signed graph, and let $\Pi' = (\lambda', a')$ be an apex embedding of (G', Σ') with exactly two odd faces. Suppose (H, Γ) is obtained from (G', Γ') by undeleting an edge between apex vertex a' of G and vertex $v \neq a'$ in $V(G)$. Then (H, Γ) has a unique apex embedding that extends from Π'.*

Proof. Suppose (H, Γ) is obtained from (G', Γ') by undeleting an edge between apex vertex a' and vertex $v \neq a'$. Since a' is not split by this operation, and since $(H - v, \Gamma \setminus \delta_H(a'))$ is unchanged, (H, Γ) must also have apex embedding $(\lambda', a') = \Pi'$. So the unique apex embedding of (H, Γ) that extends from Π' is Π'. □

Lemma 3.10.3. *Let (G', Σ') be a loopless, non-bipartite apex graph with exactly two odd faces and no blocking vertex. Let (H, Γ) be obtained from (G', Σ') by splitting a vertex x of G', or by adding an edge e to G. Suppose (H, Γ) has apex embedding $\hat{\Pi} = (\hat{\lambda}, \hat{a})$. Then there exists an apex embedding of (G', Σ') with exactly two odd faces and apex vertex a, such that either $\hat{a} = a$, or $\hat{a}$ is obtained by splitting a. Furthermore, $\hat{\Pi}$ is an extension of Π.*

Proof. We first consider the case where (H, Γ') was obtained by splitting a vertex x of G'. Let x_1, x_2 denote the vertices of H obtained from splitting x. Note that $(H-\hat{a}, \Gamma \setminus \delta_H(\hat{a}))$ is $\hat{\lambda}$-embedded in the plane with exactly two odd faces. By possibly resigning, we may assume the edge x_1x_2 is even. Notice that contracting or deleting an even edge in an embedded signed graph has no effect on the number of odd faces the signed graph contains. Suppose first that x_1, x_2 are both in $(H - \hat{a}, \Gamma \setminus \delta_H(\hat{a}))$. Then deforming the plane to contract edge x_1x_2 gives an embedding of (G', Σ') with apex vertex $\hat{a}$ and exactly two odd faces.

Now suppose $x_1 = \hat{a}$. We can contract edge x_1x_2 such that the resulting vertex x is the apex vertex of the resulting graph. This gives an apex embedding Π' of (G', Σ') with x as the apex vertex. Suppose F' is an odd $\hat{\lambda}$-face of (H, Γ) containing x_2. If the second odd $\hat{\lambda}$-face of (H, Γ) does not contain x_2, then (G', Σ') has exactly two odd Π'-faces – an impossibility. So the second odd face of (H, Γ) also contains x_2. Then we see that (G', Σ') has no odd Π'-faces, and so the planar part of (G', Σ') is bipartite. It follows that either (G', Σ') is bipartite, or x is a blocking vertex of (G', Σ'). This contradicts our choice of (G', Σ'). It follows that no $\hat{\lambda}$-face of (H, Γ) containing x_2 is odd, and apex embedding Π' of (G', Σ') has exactly two odd faces.

Now we consider the case where (G', Σ') was obtained by adding an edge e to G. By possibly resigning, we may assume that e is even. Then deleting e from the apex embedding of (H, Γ) gives an apex embedding of (G', Σ') with apex vertex $\hat{a}$, and exactly two odd faces. This completes the proof.

□

We will also need the following result relating two planar embeddings of a graph, proved in [14]:

Lemma 3.10.4. *Let G be a planar graph. Then any two planar embeddings of G are related by a sequence of dual Whitney flips.*

Proof of Theorem 3.1.7

Suppose $(G, \Sigma) \in \mathcal{C}$ does not extend uniquely. Then there exists an apex embedding Π of (G, Σ) and a signed graph (H, Γ) such that (G, Σ) is a subgraph of (H, Γ), (H, Γ) admits apex embeddings Π_1, Π_2 that are both extensions of Π, and no minor of (H, Γ) admits two such embeddings. It is easy to see that there exists a 3-connected major (G', Σ') of (G, Σ) such that (G', Σ') admits only one apex embedding $\Pi' = (\lambda', a')$ extending from Π, and (H, Γ) is obtained from (G', Σ') by a single vertex-splittng or by a single edge-addition.

By Lemmas 3.10.1, 3.10.2, (H, Γ) must be obtained from (G', Σ') by splitting a vertex in $G' - a'$, or adding an edge between two vertices $u, v \in V(G') - \backslash a'$. It follows that

$\Pi_1 = (\lambda_1, a')$, and $\Pi_2 = (\lambda_2, a')$. Furthermore, we see that λ_1, λ_2 are distinct planar embeddings of the same signed graph $(H - a', \Gamma \setminus \delta_H(a'))$. By Lemma 3.10.4, it follows that $\lambda_1(H), \lambda_2(H)$ are related by a sequence of dual Whitney flips.

It remains to prove the "furthermore" of the theorem. We will complete the proof inductively. Suppose (G', Σ') is an extension of (G, Σ) (possibly equal to (G, Σ)). Let (H, Γ) be an extension of (G', Σ') obtained by a single vertex-splitting or a single edge-addition. By Lemma 3.10.3, every apex embedding of (H, Γ) with exactly two odd faces is an extension of some apex embedding of (G', Σ') with exactly two odd faces. It follows inductively that every apex embedding of (H, Γ) is an extension of some apex embedding of (G, Σ). This completes the proof of the theorem. □

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