The Cycling Property for the Clutter of Odd *st*-Walks

by

Ahmad Abdi

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2014

Abstract

A binary clutter is cycling if its packing and covering linear program have integral optimal solutions for all Eulerian edge capacities. We prove that the clutter of odd st-walks of a signed graph is cycling if and only if it does not contain as a minor the clutter of odd circuits of K_5 nor the clutter of lines of the Fano matroid. Corollaries of this result include, of many, the characterization for weakly bipartite signed graphs [5], packing two-commodity paths [7, 11], packing T-joins with small $|T|$, a new result on covering odd circuits of a signed graph, as well as a new result on covering odd circuits and odd T-joins of a signed graft.

Acknowledgements

I would like to thank the brilliant Professor Bertrand Guenin for his support, without his everlasting assistance and guidance, this work would not have been possible.

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Chapter 1

Introduction

A *clutter* $\mathcal{C}$ is a finite collection of sets, over some finite ground set $E(\mathcal{C})$, with the property that no set in $\mathcal{C}$ is contained in, or is equal to, another set of $\mathcal{C}$. This terminology was first coined by Edmonds and Fulkerson [2]. A *cover* B is a subset of $E(\mathcal{C})$ such that $B \cap C \neq \emptyset$, for all $C \in \mathcal{C}$. The *blocker* $b(\mathcal{C})$ is the clutter of the minimal covers. It is well known that $b(b(\mathcal{C})) = \mathcal{C}$ ([9, 2]). A clutter is *binary* if, for any $C_1, C_2, C_3 \in \mathcal{C}$, their symmetric difference $C_1 \triangle C_2 \triangle C_3$ contains, or is equal to, a set of $\mathcal{C}$. Equivalently, a clutter is binary if, for every $C \in \mathcal{C}$ and $B \in b(\mathcal{C})$, $|C \cap B|$ is odd ([9]). It is therefore immediate that a clutter is binary if and only if its blocker is.

Let $\mathcal{C}$ be a clutter and $e \in E(\mathcal{C})$. The *contraction* $\mathcal{C}/e$ and *deletion* $\mathcal{C} \setminus e$ are clutters on the ground set $E(\mathcal{C}) - \{e\}$ where $\mathcal{C}/e$ is the collection of minimal sets in $\{C - \{e\} : C \in \mathcal{C}\}$ and $\mathcal{C} \setminus e := \{C : e \notin C \in \mathcal{C}\}$. Observe that $b(\mathcal{C}/e) = b(\mathcal{C}) \setminus e$ and $b(\mathcal{C} \setminus e) = b(\mathcal{C})/e$. Contractions and deletions can be performed sequentially and the result does not depend on the order. A clutter obtained from $\mathcal{C}$ by a sequence of deletions E_d and a sequence of contractions E_c ($E_d \cap E_c = \emptyset$) is called a *minor* of $\mathcal{C}$ and is denoted $\mathcal{C} \setminus E_d / E_c$.

Given edge-capacities $w \in \mathbb{Z}_+^{E(\mathcal{C})}$ consider the linear program

$$(P) \quad \begin{cases} \min & \sum(w_e x_e : e \in E(\mathcal{C})) \\ \text{s.t.} & x(C) \geq 1, \quad C \in \mathcal{C} \\ & x_e \geq 0, \quad e \in E(\mathcal{C}), \end{cases}$$

and its dual

$$(D) \quad \begin{cases} \max & \sum(y_C : C \in \mathcal{C}) \\ \text{s.t.} & \sum(y_C : e \in C \in \mathcal{C}) \leq w_e, \quad e \in E(\mathcal{C}) \\ & y_C \geq 0, \quad C \in \mathcal{C}. \end{cases}$$

A clutter is said to be *ideal* if, for every edge-capacities $w \in \mathbb{Z}_+^{E(\mathcal{C})}$, (P) has an optimal solution that is integral. A beautiful result of Lehman [10] states that a clutter is ideal if and only if its blocker is. Edge-capacities $w \in \mathbb{Z}_+^{E(\mathcal{C})}$ are said to be *Eulerian* if, for every B and B' in $b(\mathcal{C})$, $w(B)$ and $w(B')$ have the same parity. Seymour [14] calls a binary clutter *cycling* if, for every Eulerian edge-capacities $w \in \mathbb{Z}_+^{E(\mathcal{C})}$, (P) and (D) both have optimal solutions that are integral. It can be readily checked that if a clutter is cycling (or ideal) then so are all its minors ([14, 15]). Therefore, one can characterize the class of cycling clutters by excluding minor-minimal clutters that are not in this class. In this paper, we will only focus on binary clutters.

$\mathcal{O}_5$ is the clutter of the odd circuits of K_5. Let $\mathcal{L}_7$ be the clutter of the lines of the Fano matroid, i.e. $E(\mathcal{L}_7) = \{1, 2, 3, 4, 5, 6, 7\}$ and

$$\mathcal{L}_7 := \{\{1, 2, 7\}, \{3, 4, 7\}, \{5, 6, 7\}, \{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}.$$

Let $\mathcal{P}_{10}$ be the collection of the postman sets of the Petersen graph, i.e. sets of edges which induce a subgraph whose odd degree vertices are the (odd degree) vertices of the Petersen graph. Observe that the four clutters $\mathcal{O}_5, b(\mathcal{O}_5), \mathcal{L}_7, \mathcal{P}_{10}$ are binary, and moreover, it can be readily checked that none of these clutters is cycling. Hence, if a binary clutter is cycling then it cannot have any of these clutters as a minor. The following excluded minor characterization is predicted.

Conjecture 1.1 (Cycling Conjecture)**.** *A binary clutter is cycling if, and only if, it has none of the following minors:* $\mathcal{O}_5, b(\mathcal{O}_5), \mathcal{L}_7, \mathcal{P}_{10}$.

The Cycling Conjecture, as stated, can be found in Schrijver [13]. However, this conjecture was first proposed by Seymour [14] and then edited by A.M.H. Gerards and B. Guenin. It is worth mentioning that this conjecture contains the *four color theorem* [16]. None of our results in this paper have any apparent bearings on this theorem.

Consider a finite graph G, where parallel edges and loops are allowed. A *cycle* of G is the edge set of a subgraph of G where every vertex has even degree. A *circuit* of G is a minimal cycle, and a *path* is a circuit minus an edge. We define an *st-path* as follows: if $s \neq t$ then it is a path where s and t are the degree one vertices of the path; otherwise, when $s = t$ then it is just the singleton vertex s. Let Σ be a subset of its edges. The pair (G, Σ) is called a *signed graph.* We say a subset S of the edges is *odd* (resp. *even*) in (G, Σ) if $|S \cap \Sigma|$ is odd (resp. even). Let s, t be vertices of G. We call a subset of the edges of (G, Σ) an *odd st-walk* if it is either an odd st-path, or it is the union of an even st-path P and an odd circuit C where P and C share at most one vertex. Observe that when $s = t$ then an odd st-walk is simply an odd circuit. It is easy to see that clutters of odd

st-walks are closed under taking minors. As is shown in [6] the clutter of odd st-walks is binary, and it does not have a minor isomorphic to $b(\mathcal{O}_5)$ or $\mathcal{P}_{10}$. In this paper, we verify the Cycling Conjecture for this class of binary clutters:

Theorem 1.2. *A clutter of odd st-walks is cycling if, and only if, it has no $\mathcal{O}_5$ and no $\mathcal{L}_7$ minor.*

1.1 Restating Theorem 1.2

One can view Theorem 1.2 as a packing and covering result. We first the following definition: we say that two edges of a signed graph are *parallel* if they have the same end-vertices as well as the same sign. Now let $(G = (V, E), \Sigma)$ be a signed graph without any parallel edges, and choose $s, t \in V$. Let $\mathcal{C}$ be the clutter of the odd st-walks, over the ground set E, and choose edge-capacities $w \in \mathbb{Z}_+^E$. An *odd st-walk cover of* (G, Σ) is simply a cover for $\mathcal{C}$. When there is no ambiguity, we refer to an odd st-walk cover as just a cover.

Proposition 1.3 ([6]). *If a subset of the edges is a minimal cover then it is either an st-bond (a minimal st-cut) or it is of the form $\Sigma \triangle C$, where C is a cut with s and t on the same shore.*

The minimal covers of the latter form above are called *signatures*. Notice that if Σ' is a signature, then (G, Σ) and (G, Σ') have the same clutter of odd st-walks.

Reset (G, Σ) as follows: replace each edge e of (G, Σ) with w_e parallel edges. The *packing number* $\nu(G, \Sigma)$ of (G, Σ) is the maximum number of pairwise (edge-)disjoint odd st-walks. A dual parameter to the packing number is the *covering number* $\tau(G, \Sigma)$, which records the minimum size of a cover of (G, Σ). Consider a packing of $\nu(G, \Sigma)$ pairwise disjoint odd st-walk and a cover of size $\tau(G, \Sigma)$. As the cover intersects every odd st-walk in the packing it follows that $\tau(G, \Sigma) \geq \nu(G, \Sigma)$. A natural question arises: when does equality hold? Theorem 1.2 gives sufficient conditions for a signed graph to satisfy $\tau(G, \Sigma) = \nu(G, \Sigma)$. To elaborate, observe that $\tau(G, \Sigma)$ is the value of (P) and $\nu(G, \Sigma)$ is the value of (D). For w to be Eulerian is to say that every two minimal covers of (G, Σ) have the same parity. Therefore, Proposition 1.3 implies the following.

Remark 1.4. *Edge-capacities w are Eulerian if, and only if,*

(1) $s = t$ and the degree of every vertex is even, or

(2) $s \neq t$, $\deg(s) - |\Sigma|$ and the degree of every vertex in $V - \{s, t\}$ are even.

We call such signed graphs *st-Eulerian*.

Just like how we defined minor operations for clutters, we now define minor operations for signed graphs. Let $e \in E$. Then the minor operations for $\mathcal{C}$ correspond to the following minor operations for (G, Σ): (1) *delete* e: replace (G, Σ) by $(G \setminus e, \Sigma - \{e\})$, (2) *contract* e: replace (G, Σ) by $(G/e, \Sigma')$, where Σ' is a signature of (G, Σ) that does not use the edge e. Observe that vertices s and t move to wherever the edge contractions take them, and if s and t are ever identified then we say $s = t$. A signed graph (H, Γ) *is a minor of* (G, Σ) if it is isomorphic to a signed graph obtained from (G, Σ) by a sequence of edge deletions, edge contractions, and possibly deletion of isolated vertices and switching s and t. Note that if (H, Γ) is a minor of (G, Σ), then the clutter of odd st-walks of (H, Γ) is a minor of the clutter of odd st-walks of (G, Σ).

The two special clutters $\mathcal{O}_5$ and $\mathcal{L}_7$ that appear in Theorem 1.2 have the following representations: $\mathcal{O}_5$ is the clutter of odd st-walks of $\widetilde{K_5} := (K_5, E(K_5))$ where $s = t$ is one of the five vertices, and $\mathcal{L}_7$ is the clutter of odd st-walks of the signed graph F_7 with $s \neq t$, as shown in Figure 1.1. Observe that $\tau(\widetilde{K_5}) = 4 > 2 = \nu(\widetilde{K_5})$ and $\tau(F_7) = 3 > 1 = \nu(F_7)$. We can now restate Theorem 1.2 as follows, and in fact, we will prove this restatement instead of the original one:

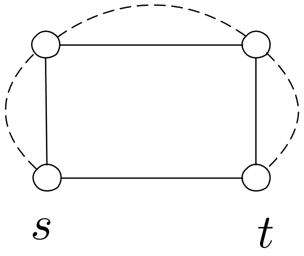

Figure 1.1: Signed graph F_7: a representation of $\mathcal{L}_7$. Dashed edges are odd.

Theorem 1.5. *Let (G, Σ) be a signed graph with $s, t \in V(G)$. If (G, Σ) is an st-Eulerian signed graph that does not contain $\widetilde{K_5}$ or F_7 as a minor then $\tau(G, \Sigma) = \nu(G, \Sigma)$.*

1.2 Generalizing Theorem 1.5

Let $(G = (V, E), \Sigma)$ be a signed graph with $s, t \in V$. Suppose (G, Σ) is an st-Eulerian signed graph that does not contain $\widetilde{K_5}$ or F_7 as a minor. If $s \neq t$ let τ_{st} be the size of a minimum st-bond, otherwise let $\tau_{st} := \tau(G, \Sigma)$. Observe that $\tau_{st} \geq \tau(G, \Sigma)$ as every st-bond is also a cover. Add $\tau_{st} - \tau(G, \Sigma)$ odd loops to (G, Σ) to obtain another st-Eulerian signed graph (G', Σ'). Since neither $\widetilde{K_5}$ nor F_7 contain an odd loop, it follows that (G', Σ') also does not contain $\widetilde{K_5}$ or F_7 as a minor. Observe that $\tau(G', \Sigma') = \tau(G, \Sigma) + (\tau_{st} - \tau(G, \Sigma)) = \tau_{st}$ and so by Theorem 1.2, one can find a packing of τ_{st} pairwise disjoint odd st-walks in (G', Σ'). In (G, Σ) this packing corresponds to a collection of τ_{st} pairwise disjoint elements, at least (and therefore exactly) $\tau(G, \Sigma)$ of which are odd st-walks and the remaining elements are even st-paths. Therefore, we get the following generalization of Theorem 1.5.

Theorem 1.6. *Let (G, Σ) be a signed graph with $s, t \in V(G)$. Suppose that (G, Σ) is an st-Eulerian signed graph that does not contain $\widetilde{K_5}$ or F_7 as a minor. Then there exists a collection of $\tau_{st}(G, \Sigma)$ pairwise disjoint elements, $\tau(G, \Sigma)$ of which are odd st-walks and the remaining elements are even st-paths.*

To obtain another generalization of Theorem 1.5 and a counterpart to Theorem 1.6, let τ_Σ be the size of a minimum signature. Observe that $\tau_\Sigma \geq \tau(G, \Sigma)$ and that $\tau(G, \Sigma) = \min\{\tau_{st}, \tau_\Sigma\}$. In contrast to above, this time we add $\tau_\Sigma - \tau(G, \Sigma)$ even edges between s and t to (G, Σ) to obtain another st-Eulerian signed graph (G', Σ'). Notice, however, that we can no longer guarantee that (G', Σ') contains no $\widetilde{K_5}$ or F_7 minor. Observe that this is true if, and only if, (G, Σ) does not contain $\widetilde{K_5}, \widetilde{K_5}^0, \widetilde{K_5}^1, \widetilde{K_5}^2, \widetilde{K_5}^3$ or F_7^- as a minor, where

(1) for $i \in \{0, 1, 2, 3\}$, $\widetilde{K_5}^i$ is the signed graph obtained from splitting a vertex, and its incident edges, of $\widetilde{K_5}$ into two vertices s, t, where s has degree i and t has degree $4 - i$, and

(2) F_7^- is the signed graph obtained from F_7 by deleting the edge between s and t.

Note that if we add an even edge to any of these signed graphs, then a $\widetilde{K_5}$ or an F_7 appears as a minor. It can be readily checked that if (G, Σ) does not contain any of these five signed graphs as a minor, then (G', Σ') contains no $\widetilde{K_5}$ or F_7 minor. Observe now that $\tau(G', \Sigma') = \tau(G, \Sigma) + (\tau_\Sigma - \tau(G, \Sigma)) = \tau_\Sigma$ and so by Theorem 1.2, one can find a packing of τ_Σ pairwise disjoint odd st-walks in (G', Σ'). In (G, Σ) this packing corresponds to a collection of τ_Σ pairwise disjoint elements, $\tau(G, \Sigma)$ of which are odd st-walks and the remaining elements are odd circuits. Thus, the following counterpart to Theorem 1.6 is obtained.

Theorem 1.7. *Let* (G, Σ) *be a signed graph with* $s, t \in V(G)$. *Suppose that* (G, Σ) *is an st-Eulerian signed graph that does not contain* $\widetilde{K_5}, \widetilde{K_5}^0, \widetilde{K_5}^1, \widetilde{K_5}^2, \widetilde{K_5}^3$ *or* F_7^- *as a minor. Then in* (G, Σ) *there exists a collection of* $\tau_\Sigma(G, \Sigma)$ *pairwise disjoint elements,* $\tau(G, \Sigma)$ *of which are odd st-walks and the remaining elements are odd circuits.*

Chapter 2

Applications of Theorem 1.2

In this section, we discuss some applications of Theorem 1.2. Observe that a cycling clutter is also ideal. As a corollary, we get the following theorem:

Corollary 2.1 (Guenin [6]). *A clutter of odd st-walks is ideal if, and only if, it has no $\mathcal{O}_5$ and no $\mathcal{L}_7$ minor.*

When $s = t$ an odd st-walk is just an odd circuit. A signed graph is said to be *weakly bipartite* if the clutter of its odd circuits is ideal. The clutter of odd circuits does not contain an $\mathcal{L}_7$ minor [6]. Hence, we get the following two results as corollaries of Theorem 1.2:

Corollary 2.2 (Guenin [5]). *A signed graph is weakly bipartite if, and only if, it has no $\widetilde{K_5}$ minor.*

Corollary 2.3 (Geelen and Guenin [3]). *A clutter of odd circuits is cycling if, and only if, it has no $\mathcal{O}_5$ minor.*

Observe that $2w$ is Eulerian for any $w \in \mathbb{Z}_+^{E(G)}$. As a result, the following result follows as a corollary of Theorem 1.2:

Theorem 2.4. *Suppose that $\mathcal{C}$ is a clutter of odd st-walks without an $\mathcal{O}_5$ or an $\mathcal{L}_7$ minor. Then, for any edge-capacities $w \in \mathbb{Z}_+^{E(G)}$, the linear program (P) has an optimal solution that is integral and its dual (D) has an optimal solution that is half-integral.*

To obtain more applications of Theorem 1.2, we will turn to its restatement Theorem 1.5, and naturally try to find nice classes of signed graphs without a $\widetilde{K_5}$ or an F_7 minor.

2.1 Signed graphs without $\widetilde{K_5}$ and F_7 minor

Let (G, Σ) be a signed graph with $s, t \in V$. Observe that if $s = t$ then (G, Σ) has no F_7 minor, and there are many classes of such signed graphs without a $\widetilde{K_5}$ minor. For instance, whenever G is planar or $|\Sigma| = 2$, (G, Σ) does not contain a $\widetilde{K_5}$ minor. Other classes of such signed graphs can be found in [4, 3]. In this section, we focus only on signed graphs (G, Σ) with distinct $s, t \in V$.

A *blocking vertex* is a vertex v whose deletion removes all the odd cycles, and a *blocking pair* is a pair of vertices $\{u, v\}$ whose deletion removes all the odd cycles.

Remark 2.5. *The following classes of signed graphs with $s \neq t$ do not contain $\widetilde{K_5}$ or F_7 as a minor:*

(1) *signed graphs with a blocking vertex,*

(2) *signed graphs where $\{s, t\}$ is a blocking pair,*

(3) *plane signed graphs with at most two odd faces,*

(4) *signed graphs that have an even face embedding on the projective plane, and s and t are connected with an odd edge,*

(5) *signed graphs where every odd st-walk is connected, and*

(6) *plane signed graphs with a blocking pair $\{u, v\}$ where s, u, t, v appear on a facial cycle in this cyclic order.*

Observe that class (5) contains (2) and (4). We will apply Theorem 1.5 to the first three classes, and in the first two cases, we obtain quite well-known results. However, the third class will yield a new and interesting result on packing odd circuit covers. Notice that one can even apply the generalization Theorem 1.6 to these classes.

Observe further that the signed graphs in (1) and (2) do not contain $\widetilde{K_5}^0, \widetilde{K_5}^1, \widetilde{K_5}^2, \widetilde{K_5}^3$ or F_7^- as a minor either, so one may even consider applying Theorem 1.7 to these classes. We leave it to the reader to find out what Theorems 1.6 and 1.7 applied to these classes imply.

2.1.1 Class (1): packing T-joins with $|T| = 4$

Let H be a graph with vertex set W, and choose an even vertex subset T. A *T-join* of H is an edge subset whose odd degree vertices are (all) the vertices in T. A *T-cut* of H is an edge subset of the form $\delta(U)$ where $U \subseteq W$ and $|U \cap T|$ is odd. Observe that the blocker of the clutter of minimal T-joins is the clutter of minimal T-cuts.

Corollary 2.6. *Let H be a graph and choose a vertex subset T of size 4. Suppose that every vertex of H not in T has even degree and that all the vertices in T have degrees of the same parity. Then the maximum number of pairwise disjoint T-joins is equal to the minimum size of a T-cut.*

Proof. Suppose that $T = \{s, t, s', t'\}$. Identify s' and t' to obtain G, and let $\Sigma = \delta_H(s')$. Then the signed graph (G, Σ) contains a blocking vertex $s't'$, and so it belongs to class (1). By Remark 1.4 (G, Σ) is st-Eulerian. Theorem 1.2 then implies that $\tau(G, \Sigma) = \nu(G, \Sigma)$. However, observe that an odd st-walk of (G, Σ) is a T-join of H, and a T-join in H contains an odd st-walk of (G, Σ). Hence, $\tau(G, \Sigma) = \nu(G, \Sigma)$ implies that the maximum number of pairwise disjoint T-joins is equal to the minimum size of a T-cut. $\square$

This result is actually true for any even vertex subset T of size at most 8 [1].

2.1.2 Class (2): packing two-commodity paths

Corollary 2.7 (Hu [7], Rothschild and Whinston [11])**.** *Let H be a graph and choose two pairs (s_1, t_1) and (s_2, t_2) of vertices, where $s_1 \neq t_1$, $s_2 \neq t_2$, all of s_1, t_1, s_2, t_2 have the same parity, and all the other vertices have even degree. Then the maximum number of pairwise disjoint paths, that are between s_i and t_i for some $i = 1, 2$, is equal to the minimum size of an edge subset whose deletion removes all s_1t_1- and s_2t_2-paths.*

Proof. Identify s_1 and s_2, as well as t_1 and t_2 to obtain G, and let $\Sigma = \delta_H(s_1) \triangle \delta_H(t_2)$. Let $s := s_1s_2 \in V(G)$ and $t := t_1t_2 \in V(G)$. Then the signed graph (G, Σ) has $\{s, t\}$ as a blocking pair, and so it belongs to class (2). Again by Remark 1.4 (G, Σ) is st-Eulerian. Therefore, by Theorem 1.2 we get that $\tau(G, \Sigma) = \nu(G, \Sigma)$. However, observe that an odd st-walk of (G, Σ) is an s_it_i-path of H, for some $i = 1, 2$, and such a path in H contains an odd st-walk of (G, Σ). Thus, $\tau(G, \Sigma) = \nu(G, \Sigma)$ proves the corollary. $\square$

2.1.3 Class (3): packing odd circuit covers

Theorem 2.8. *Let (H, Σ) be a plane signed graph with exactly two odd faces and choose distinct $g, h \in V(H)$. Let (G, Σ) be the signed graph obtained from identifying g and h in H, and suppose that every two odd circuits of (G, Σ) have the same parity. Then in (G, Σ) the maximum number of pairwise disjoint odd circuit covers is equal to the size of a minimum odd circuit.*

(Here an odd circuit cover is simply a cover for the clutter of odd circuits.) As the reader may be wondering, what is the rationale behind the rather strange construction of (G, Σ) above? Interestingly, the clutter of minimal odd circuit covers is binary, and so the Cycling Conjecture predicts an excluded minor characterization for when this clutter is cycling. As we did with the clutter of odd st-walks, one can restate the Cycling Conjecture for the clutter of odd circuit covers as follows:

> *(?) for signed graphs (G, Σ) without a $\widetilde{K_5}$ minor such that every two odd circuits have the same parity, the maximum number of pairwise disjoint odd circuit covers is equal to the minimum size of an odd circuit. (?)*

The construction in the statement of Theorem 2.8 yields a signed graph (G, Σ) that has no $\widetilde{K_5}$ minor, and Theorem 2.8 verifies the restatement above for these classes of signed graphs.

Proof. Let H^* be the plane dual of H, and let P be an odd gh-path in (H, Σ). Let s and t be the two odd faces of (H, Σ). Consider the plane signed graph (H^*, P); note that this signed graph has precisely two odd faces, namely g and h, and so it belongs to class (3). In particular, (H^*, P) contains no $\widetilde{K_5}$ and F_7 minor. Since every two odd circuits of (G, Σ) have the same parity, it follows from Remark 1.4 that (H^*, P) is st-Eulerian. So Theorem 1.2 applies and we have $\tau(H^*, P) = \nu(H^*, P)$.

We claim that an odd cycle of (G, Σ) is an odd st-walk cover of (H^*, P), and vice-versa. Let L be an odd cycle of (G, Σ). If L is an odd cycle of (H, Σ) then L separates the two odd faces s and t, and so it is an st-cut in (H^*, P). Otherwise, L is an odd gh-path and so $L \triangle P$ is an even cycle of (H, Σ). However, an even cycle in (H, Σ) is a cut in (H^*, P) having s and t on the same shore. Hence, L is of the form $P \triangle \delta(U)$ where $s, t \in U \subseteq V(H^*)$. Therefore, in either cases, L is an odd st-walk cover of (H^*, P). Similarly, one can show that an odd st-walk cover of (H^*, P) is an odd cycle of (G, Σ). Therefore, since $b(b(\mathcal{C})) = \mathcal{C}$ for any clutter $\mathcal{C}$, it follows that an odd circuit cover of (G, Σ) is an odd st-walk of (H^*, P), and vice-versa.

Hence, $\tau(H^*, P)$ is the minimum size of an odd circuit of (G, Σ), and $\nu(H^*, P)$ is the maximum number of pairwise disjoint odd circuit covers of (G, Σ). Since $\tau(H^*, P) = \nu(H^*, P)$ the result follows. ☐

In the next section, the restatement Theorem 1.5 delivers a packing and covering result for a very intriguing class of binary clutters.

2.2 Clutter of odd circuits and odd T-joins

Let $(G = (V, E), \Sigma)$ be a signed graph, and let $T \subseteq V$ be a subset of even size. We call the triple (G, Σ, T) a *signed graft*. Let $\mathcal{C}$ be the clutter over the ground set E that consists of odd circuits and minimal odd T-joins of (G, Σ, T). This minor-closed class of such clutters is fairly large. For instance, if $T = \emptyset$ then $\mathcal{C}$ is the clutter of odd circuits, and if Σ is a T-cut then $\mathcal{C}$ is the clutter of T-joins.

Remark 2.9. *$\mathcal{C}$ is a binary clutter.*

Proof. Take any three elements C_1, C_2, C_3 of $\mathcal{C}$. If an even number of C_1, C_2, C_3 are odd circuits, then $C_1 \triangle C_2 \triangle C_3$ is an odd T-join and so it contains an element of $\mathcal{C}$. Otherwise, an odd number of C_1, C_2, C_3 are odd circuits, and so $C_1 \triangle C_2 \triangle C_3$ is an odd cycle and so it contains an element of $\mathcal{C}$. Since this is true for all C_1, C_2, C_3 in $\mathcal{C}$, it follows from definition that $\mathcal{C}$ is binary. ☐

Remark 2.10. *Minimal covers of $\mathcal{C}$ are of the form $\Sigma \triangle \delta(U)$, where $U \subseteq V$ and $|U \cap T|$ is even.*

Proof. Let B be a minimal cover of $\mathcal{C}$. Then B intersects every odd circuit of (G, Σ), and so $B \triangle \Sigma = \delta(U)$ for some $U \subseteq V$. The preceding remark showed $\mathcal{C}$ is binary, and so B intersects every odd T-join in an odd number of edges, so $|U \cap T|$ must be even. ☐

The result below follows as a corollary of Theorem 1.5.

Theorem 2.11. *Let (G, Σ, T) be a plane signed graft with exactly two odd faces that has no minor isomorphic to $\widetilde{F_7}$. Let $\mathcal{C}$ be the clutter of odd circuits and minimal odd T-joins, and suppose that every two elements of $\mathcal{C}$ have the same size parity. Then the maximum of pairwise disjoint minimal covers of $\mathcal{C}$ is equal to the minimum size of an element of $\mathcal{C}$.*

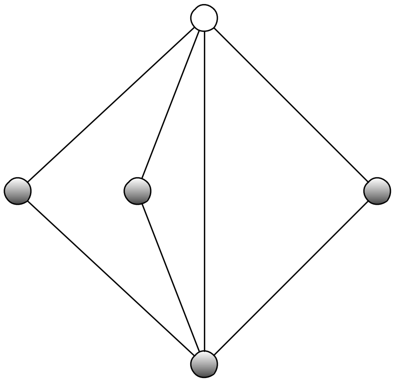

Figure 2.1: Signed graft $\widetilde{F}_7$, where all edges are odd and shaded vertices are in T. For this signed graft, $\mathcal{C} = b(\mathcal{C}) \cong \mathcal{L}_7$.

Proof. Let G^* be the plane dual of G, and let P be an odd T-join in (G, Σ, T). Let s and t be the two odd faces of (G, Σ, T). Since (G, Σ, T) has no minor isomorphic to $\widetilde{F}_7$, it follows that the signed graph (G^*, P) contains no F_7 minor, and since it is planar, it has no $\widetilde{K_5}$ minor either. Since every two elements of $\mathcal{C}$ have the same parity, it follows that (G^*, P) is st-Eulerian. So by Theorem 1.5, $\tau(G^*, P) = \nu(G^*, P)$.

We claim that $\mathcal{C}$ is the clutter of odd st-walk covers of (G^*, P), and vice-versa. Let $C \in \mathcal{C}$. If C is an odd circuit of (G, Σ, T), then C is an st-cut of G^*. Otherwise, C is an odd T-join and so $C \triangle P$ is an even cycle of (G, Σ). Thus, $C = P \triangle \delta(U)$ for some $U \subseteq V(G^*) - \{s, t\}$, i.e. C is a signature of (G^*, P).

Hence, $\tau(G^*, P)$ is the minimum size of an element of $\mathcal{C}$, and $\nu(G^*, P)$ is the maximum number of pairwise disjoint covers of $\mathcal{C}$. Since $\tau(G^*, P) = \nu(G^*, P)$ the result follows. $\square$

Observe that this theorem is, in fact, a generalization of Theorem 2.8.

Chapter 3

Overview of the Proof of Theorem 1.5

We start with an st-Eulerian signed graph (G,Σ) that does not *pack*, i.e. $\tau(G,\Sigma) > \nu(G,\Sigma)$, and we will look for either of the *obstructions* $\widetilde{K_5}$, F_7 as a minor.

Among all the st-Eulerian non-packing weighted minors of (G,Σ), we pick one (G',Σ') with smallest $\tau(G',\Sigma')$, smallest $|V(G')|$ and largest $|E(G')|$, in this order of priority. Such a non-packing weighted minor exists. Indeed, if an edge has sufficiently many parallel edges, then it may be contracted while keeping (G',Σ') non-packing and $\tau(G',\Sigma')$ unchanged. Reset $(G,\Sigma) := (G',\Sigma')$ and let $\tau := \tau(G,\Sigma)$, $\nu := \nu(G,\Sigma)$. By identifying a vertex of each (connected) component with s, if necessary, we may assume that G is connected. (Notice that none of the obstructions $\widetilde{K_5}$, F_7 have a cut-vertex.)

Remark 3.1. *There do not exist* $\tau - 1$ *pairwise disjoint odd* st*-walks in* (G,Σ)*.*

Proof. Suppose otherwise. Remove some $\tau-1$ pairwise disjoint odd st-walks in (G,Σ). Observe that what is left is an odd $\{s,t\}$-join because $|\Sigma|$, $\deg(s), \deg(t)$ and τ all have the same parity and all vertices other than s,t have even degree. Hence, since every odd $\{s,t\}$-join contains an odd st-walk, one can actually find τ pairwise disjoint odd st-walks in (G,Σ), contradicting the fact that (G,Σ) is non-packing. $\square$

Let B be a cover of (G,Σ) of size τ. Choose an edge Ω as follows. If $s=t$ then let $\Omega \in E - B$, and since label s is irrelevant to our problem in this case, we may as well assume $\Omega \in \delta(s)$. Otherwise, when $s \neq t$, let $\Omega \in \delta(s) \cup \delta(t) - B$. Indeed, if such an edge does not exist, then $\delta(s) \cup \delta(t)$ is contained in the minimum cover B, implying that $\delta(s) \cup \delta(t) = \delta(s) = \delta(t)$, but this cannot be the case as G is connected and non-packing. We may assume that Ω is incident to s. Let s' be the other end-vertex of Ω. Add two parallel edges Ω_1, Ω_2 to Ω to obtain (K, Γ); this st-Eulerian signed graph must pack since $\tau(K, \Gamma) = \tau$ as B is also a minimum cover for (K, Γ), $V(K) = V(G)$ but $|E(K)| > |E(G)|$. Hence, (K, Γ) contains a collection $\{L_1, L_2, \ldots, L_\tau\}$ of pairwise disjoint odd st-walks. Observe that all of Ω, Ω_1 and Ω_2 must be used by the odd st-walks in $\{L_1, L_2, \ldots, L_\tau\}$, say by L_1, L_2, L_3, since otherwise one finds at least $\tau - 1$ disjoint odd st-walks in (G, Σ), which is not the case by the preceding remark. As a result, the sequence $(L_1, L_2, L_3, \ldots, L_\tau)$ corresponds to an *Ω-packing of odd st-walks* in (G, Σ), described as follows:

(1) $L_1, \ldots, L_\tau$ are odd st-walks in (G, Σ),

(2) $\Omega \in L_1 \cap L_2 \cap L_3$ and $\Omega \notin L_4 \cup \cdots \cup L_\tau$, and

(3) $(L_j - \{\Omega\} : 1 \leq j \leq \tau)$ are pairwise disjoint subsets.

We may assume that $(L_1, \ldots, L_\tau)$ covers a minimal subset of edges, amongst all the Ω-packings of odd st-walks.

For an odd st-walk L, we say that a minimal cover B is a *mate* of L if $|B - L| = \tau - 3$.

Lemma 3.2. *Let L be an odd st-walk such that $(G, \Sigma) \setminus L$ contains at least $\tau - 3$ pairwise disjoint odd st-walks. Then L has a mate.*

Observe that if $L \subseteq L_1 \cup L_2 \cup L_3$ or $L \in \{L_4, \ldots, L_\tau\}$, then $(G, \Sigma) \setminus L$ contains at least $\tau - 3$ pairwise disjoint odd st-walks.

Proof. The signed graph $(G, \Sigma) \setminus L$ packs since it is st-Eulerian and $\tau((G, \Sigma) \setminus L) < \tau$. Let B' be one of its minimum covers. By our assumption, $\tau((G, \Sigma) \setminus L) \geq \tau - 3$. Since both (G, Σ) and $(G, \Sigma) \setminus L$ are st-Eulerian it follows that $\tau((G, \Sigma) \setminus L)$ and τ have different parities, and so $\tau((G, \Sigma) \setminus L)$ is either $\tau - 3$ or $\tau - 1$. However, observe that the latter is not possible due to Remark 3.1 and the fact that $(G, \Sigma) \setminus L$ packs. As a result $|B'| = \tau((G, \Sigma) \setminus L) = \tau - 3$. It is now clear that $B' \cup L$ contains a mate for L. □

Choose an integer $3 \leq m \leq \tau$ and rearrange $L_4, \ldots, L_\tau$ such that $L_{m+1}, \ldots, L_\tau$ are the connected odd st-walks. So each L_j, $4 \leq j \leq m$, is the vertex-disjoint union of an odd circuit C_j and an even st-path P_j, and each of $L_{m+1}, \ldots, L_\tau$ is either an odd st-path, or the union of an odd circuit C and an even st-path P such that C and P have a vertex in common. Let $H := L_1 \cup L_2 \cup L_3 \cup \bigcup_{j=4}^{m} P_j$ and orient the edges in H so that each P_j, $1 \leq j \leq m$, is a directed st-path, and every odd circuit C_j, $1 \leq j \leq 3$ (if any), is a directed circuit. We call an odd st-walk *directed* if it is either an odd directed st-path, or it is the union of an even directed st-path and a directed odd circuit. By our terminology, the three odd st-walks L_1, L_2 and L_3 in H are directed.

We call T a *transversal* of a collection of sets if T picks exactly one element from each of the sets.

Remark 3.3. *Let B be a mate of L_i, for some $1 \leq i \leq 3$. If B is a signature then $B \cap E(H) = B \cap L_i$.*

Proof. Since $|B - L_i| = \tau - 3$ it follows that $\Omega \in B$ and $B - L_i$ is a transversal of $L_4, \ldots, L_\tau$. However, as B is a signature, we get that $|B \cap C_j|$ is odd for all $4 \leq j \leq m$ implying that $B \cap P_j = \emptyset$ for all $4 \leq j \leq m$. Since $B \cap L_k = \{\Omega\}$ for $k \in [3] - \{i\}$, and $\Omega \in L_i$, it follows that $B \cap E(H) = B \cap L_i$. $\square$

Let (G', Σ') be a minor of (G, Σ) and let H' be a directed graph obtained by orienting edges in a subgraph of G', where (G', Σ') and H' are minimal subject to

(M1) $E(G) - E(G') \subseteq E(H \setminus \Omega)$, and $E(H') \subseteq L_1 \cup L_2 \cup L_3 \cup \bigcup_{j=4}^{m} P_j$,

(M2) there exist m edge subsets in H' that are pairwise disjoint except possibly at Ω, exactly three of which contain Ω which are directed odd st-walks, and the remaining $m - 3$ edge subsets are even directed st-paths,

(M3) for any directed odd st-walk L of H' for which $(G', \Sigma') \setminus L$ contains $\tau - 3$ pairwise disjoint odd st-walks, there exists an odd st-walk cover B of (G', Σ') such that $|B - L| = \tau - 3$, and

(M4) there is no odd st-walk cover for (G', Σ') of size $\tau - 2$.

Note that these conditions are satisfied by (G, Σ) and H, so (G', Σ') and H' are well-defined. As in (M2) let $L'_1, L'_2, L'_3, P'_4, P'_5, \ldots, P'_m$ be m edge subsets of H' that are pairwise disjoint except possibly at Ω, L'_1, L'_2 and L'_3 are directed odd st-walks that contain Ω, and $P'_4, \ldots, P'_m$ are even directed st-paths that do not contain Ω. We make the following three assumptions about the choice of $L'_1, L'_2, L'_3, P'_4, P'_5, \ldots, P'_m$, in this order of priority:

(A1) $L'_1 \cup L'_2 \cup L'_3 \cup \bigcup_{j=4}^{m} P'_j$ is a minimal edge subset among all possible choices for the m edge subsets as in (M2), (A2) the number of non-simple odd st-walks amongst L'_1, L'_2, L'_3 is maximum among all possible choices for the m edge subsets, and

(A3) $H' = L'_1 \cup L'_2 \cup L'_3 \cup \bigcup_{j=4}^{m} P'_j$.

For notational convenience, let $V' := V(G'), E' := E(G')$ and reset $L_i := L'_i$, $L_j := C_j \cup P'_j$ and $P_j := P'_j$ for all $1 \leq i \leq 3$ and $4 \leq j \leq m$. By identifying a vertex of each component with s, if necessary, we may assume that

(A4) H' is connected.

For each $1 \leq i \leq 3$, choose a minimal mate B_i for L_i as in (M3). Observe that for each $1 \leq i \leq 3$ since $|B_i - L_i| = \tau - 3$, $B_i - L_i$ must be a transversal of $\{L_4, \ldots, L_\tau\}$ and $\Omega \in B_i$. Keep in mind that each of B_1, B_2, B_3 is either an st-bond or a signature, and complicating matters more, each of L_1, L_2, L_3 is either simple or non-simple. Even more, if L_i, $1 \leq i \leq 3$ is non-simple then Ω could be in either C_i or P_i. The various combinations of the possibilities for L_1, L_2, L_3, B_1, B_2, B_3 and where the edge Ω is sitting only makes the problem of finding the obstructions more complex. However, as we will see in the following lemma, various combinations for L_1, L_2 and L_3 restricts the possibilities for B_1, B_2, B_3 and where Ω is sitting. Recall that s and s' are the end-vertices of Ω.

Lemma 3.4. *One of the following holds:*

Template (I): *L_1, L_2 and L_3 are simple,*

Template (II): *at least one of L_1, L_2, L_3 is non-simple, and whenever L_k is non-simple for some $1 \leq k \leq 3$, then $\Omega \in C_k$,*

Template (III): *at least two of L_1, L_2, L_3 are non-simple, and $\Omega \in P_1 \cap P_2 \cap P_3$.*

Proof. We will show that if (I) or (II) does not hold, then (III) must hold. In other words, we assume that at least one L_k of L_1, L_2, L_3 is non-simple and $\Omega \notin C_k$, and we will show that at least two of L_1, L_2, L_3 must be non-simple and $\Omega \in P_1 \cap P_2 \cap P_3$.

We may assume that $k = 1$. Then $\Omega \in P_1$. We first show that $\Omega \in P_2 \cap P_3$. Notice that, for $i = 2, 3$, $B_i \cap L_1 = \{\Omega\}$ and $\Omega \notin C_1$, implying that $B_i \cap C_1 = \emptyset$. Hence, B_2 and B_3 cannot be signatures, i.e. they are st-bonds. Hence, since $B_2 \cap L_3 = B_3 \cap L_2 = \{\Omega\}$ and B_2 intersects any circuit an even number of times, it follows that $\Omega \notin C_2 \cup C_3$ and so $\Omega \in P_2 \cap P_3$.

It remains to show that L_2 and L_3 cannot both be simple odd st-walks. Suppose otherwise. Choose minimal vertex subsets $U_i \subseteq V' - \{t\}$ such that $\delta(U_i) = B_i$, for $i = 2, 3$. Let $U := U_2 \cap U_3$ and $B := \delta(U)$. Note $B \subseteq B_2 \cup B_3$, and B is an st-cut so it is a cover of (G', Σ'), implying $|B| \geq \tau$. We will obtain a contradiction to (M4) by showing that $|B| = \tau - 2$.

Take $1 \leq i \leq \tau$. We will show that $|B \cap L_i| = 1$. If $i \notin \{2, 3\}$ then $|B \cap L_i| \leq |B_2 \cap L_i| + |B_3 \cap L_i| = 2$, and since $|B \cap L_i|$ is odd it follows that $|B \cap L_i| = 1$. Otherwise, $i \in \{2, 3\}$. Since $\Omega \in B \cap L_i$ and $s \in U$, we get that $s' \notin U_2 \cup U_3$. We claim that $B \cap L_i = \{\Omega\}$. If not, then there exists a vertex $u \in V(L_i) \cap U - \{s\}$. But then $L_i[s', u] \cap B_{5-i} \neq \emptyset$, which cannot be the case as $L_i \cap B_{5-i} = \{\Omega\}$. (Here $L_i[s', u]$ denotes the subpath in L_i between s' and u.) As a result, $|B| = |B \cap (\bigcup_{i=1}^{\tau} L_i)| = 1 + \sum_{i=4}^{\tau} |B \cap L_i| = \tau - 2$, a contradiction. □

Observe that the case where $s = t$ is under template (II). We find either of the obstructions $\widetilde{K_5}, F_7$ as a minor starting from one of the templates (I),(II) and (III). The template having the least structure is (I) and the one with the most structure is (III). Hence, as the reader may expect, finding the obstructions is most difficult when (I) occurs and it is least difficult for (III). The proof is split into eight parts, four of which are spent to find an obstruction in (I), three parts are taken by (II), and the remaining part considers (III).

3.1 Template (I): Parts (1)-(4)

Suppose we are given Template (I). There are three main factors that extensively split the proof into four parts. The first factor is whether $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ contains an odd cycle.

In Part (1) we assume that $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ does contain an odd cycle. In this case, we show that exactly one of B_1, B_2, B_3, say B_3, is an st-bond. Therefore, $B_3 = \delta(U_3)$ for some vertex subset $U_3 \subseteq V$. We then move on to showing that $P_1 \cup P_2$ contains two odd cycles, and then a disentangling argument allows us to assume that P_1, P_2 and P_3 do not pairwise intersect "wildly". Then connectivity inside $G'[U_3]$ gives us a path R connecting s to $VP_3 - \{s\}$, see Figure 3.1. (Hereinafter, $VQ := V(Q)$ for a path, circuit or an $\{s, t\}$-join Q.) At this point, once R is contracted the signed graph in Figure 3.2 is obtained, which clearly is F_7.

For the other three parts, we assume that $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ is bipartite. Now a second factor comes into effect, and that is whether or not the following holds:

(X1) *no odd st-dipath of $(H', \Sigma' \cap E(H'))$ has a mate which is an st-bond.*

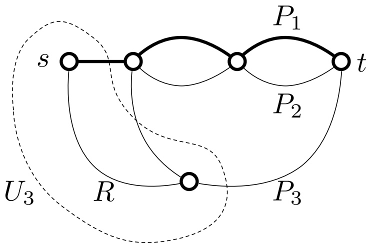

Figure 3.1: Towards an F_7 minor. Bold edges are odd.

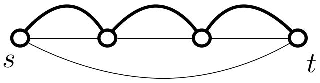

Figure 3.2: Signed graph F_7, where the bold edges are odd.

In Part (2) we assume that (X1) holds. If vertex t lies on every odd circuit of $(H', \Sigma' \cap E(H'))$, then a disentangling argument allows us to assume that P_1, P_2, P_3 and P_4 (whose existence is also proved) do not intersect wildly in H', see Figure 3.3; the path P_4 is contracted to identify s and t, and then carefully chosen paths in (G', Σ') are added to obtain the signed graph in Figure 3.4, which evidently is $\widetilde{K_5}$. Otherwise, there is an odd circuit C of $(H', \Sigma' \cap E(H'))$ that avoids t. In this case, we find a vertex v common to both P_1 and P_2 that is closest to t, followed by two vertex disjoint paths Q and P, one connecting s' to v and the other connecting s to t, see Figure 3.5. Then, after considering the mates B_1 and B_2, we are able to find a path R in (G', Σ'), vertex disjoint from $P \cup Q \cup C$, connecting $VP_1[v,t] - \{v,t\}$ to $VP_2[v,t] - \{v,t\}$. We then contract paths R and Q to obtain F_7, as in Figure 3.2.

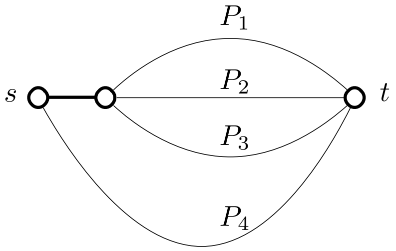

Figure 3.3: Towards a $\widetilde{K_5}$ minor. The bold edge is odd.

For the remaining two parts, we assume that (X1) does not hold. So there exists a

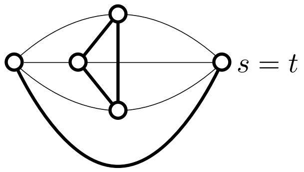

Figure 3.4: Signed graph $\widetilde{K_5}$, where the bold edges are odd.

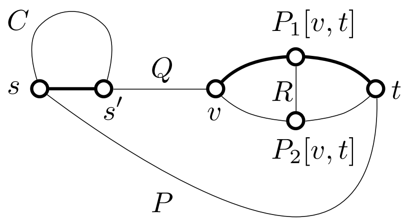

Figure 3.5: Towards an F_7 minor. The bold edges are odd.

simple odd st-walk L of $(H', \Sigma' \cap E(H'))$ owning an st-bond B as a mate, i.e. B is a cover of (G', Σ') such that $|B - L| = \tau - 3$. It turns out that we may assume $L = L_1$ and $B = B_1$. Choose $U_1 \subseteq V - \{t\}$ so that $B_1 = \delta(U_1)$, see Figure 3.6. Let $u \neq s$ and w be, respectively, the closest and furthest vertices on P_1 from s that lie inside U_1. Let $C_1 := P_1[s, u]$, $Q_1 := P_1[w, t]$, and $F' := (P_1 \cap G'[U_1]) \cup C_1 \cup Q_1 \cup \bigcup_{j=2}^{m} P_j$. The third factor presents itself: whether or not

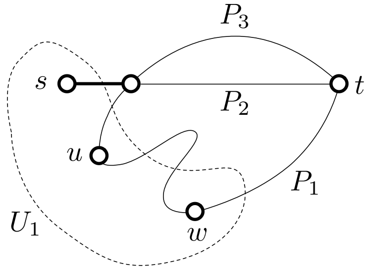

Figure 3.6: The bold edge is odd.

(X2) *for every even st-dipath P in $(F', \Sigma' \cap E(F'))$, $V(P) \cap V(C_1) \subseteq U_1$.*

In Part (3) we assume that (X2) holds, and in Part (4) (X2) does not hold. In either parts, both obstructions can be present as minors.

Here is a summary of the four parts:

Part (1): Template (I) holds, and $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ is not bipartite.

Part (2): Template (I) holds, $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ is bipartite, and (X1) holds.

Part (3): Template (I) holds, $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ is bipartite, (X1) does not hold, and (X2) holds.

Part (4): Template (I) holds, $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ is bipartite, (X1) does not hold, and (X2) does not hold.

3.2 Template (II): Parts (5)-(7)

Suppose now we are given Template (II). A natural question to ask is how many of L_1, L_2, L_3 are non-simple odd st-walks, and the three possibilities form the three parts taken by Template (II).

In Part (5) we assume that all of L_1, L_2, L_3 are non-simple. If $s = t$ then we appeal to a lemma by Geelen and Guenin [3] to find a $\widetilde{K_5}$ minor. Otherwise, an even st-path P is carefully chosen and contracted to identify s and t and then the same lemma comes to the rescue, see Figure 3.7. In Part (6) we assume that two of L_1, L_2, L_3 are non-simple. In this part, an F_7 minor is found in a similar manner as it was constructed in Part (2) (see Figures 3.5 and 3.2). In Part (7) we assume that only one of L_1, L_2, L_3 is non-simple. This part turns out to be more complex than the other two parts, and both obstructions can in fact appear as minors.

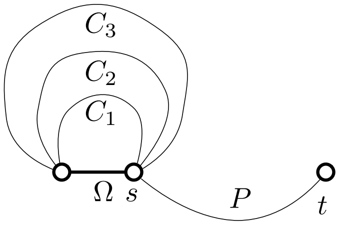

Figure 3.7: The bold edge is odd.

Here is a summary of the three parts:

Part (5): Template (II) holds, and all of L_1, L_2, L_3 are non-simple.

Part (6): Template (II) holds, and two of L_1, L_2, L_3 are non-simple.

Part (7): Template (II) holds, and one of L_1, L_2, L_3 is non-simple.

3.3 Template (III): Part (8)

In the last part, Part (8), we assume that Template (III) is given.

Part (8): Template (III) holds.

That is, at least two of L_1, L_2, L_3, say L_1 and L_2, are non-simple, and $\Omega \in P_1 \cap P_2 \cap P_3$. Here we find an F_7 minor. However, our arguments for this part will be presented in a more general setting, as we will refer to this part as a subroutine in the other parts of the proof. For this part, we assume

(M1') (G', Σ') contains τ odd $\{s,t\}$-joins $L_1, L_2, L_3, \ldots, L_\tau$ that are pairwise disjoint except possibly at Ω,

(M2') $\Omega \in L_j$ if and only if $1 \le j \le 3$,

(M3') L_1, L_2 and L_3 are odd st-walks, at least two of which are non-simple,

(M4') for every odd st-walk $L \subseteq L_1 \cup L_2 \cup L_3$, there exists an odd st-walk cover B of (G', Σ') such that $|B - L| = \tau - 3$,

(M5') there is no odd st-walk of (G', Σ') of size $\tau - 2$.

We may assume that L_1, L_2 and L_3 as above have a minimal union amongst all possible choices for L_1, L_2, L_3 in (G', Σ'). As before let B_i be a of L_i, for $1 \le i \le 3$. We first show that the covers B_1, B_2 and B_3 are all st-bonds. Then if U_i is the shore of B_i containing s ($1 \le i \le 3$) we prove that $U_1 \subsetneq U_2 \subsetneq U_3$. To explain the ideas of the proof more transparently, suppose that L_3 is non-simple as well. As we will see, this implies that $\Omega \in P_3$. Then we show that $P_i \cap B_i = \{\Omega\}$ and $|C_i \cap B_i| \ge 2$ for $1 \le i \le 3$. Moreover, we will see that $V(C_1) \subseteq U_2$, $V(C_2) \subseteq U_3 - U_1$ and $V(C_3) \in V - U_1 - U_2$. Then for the final argument we will show that there is enough connectivity in each of the pieces $U_1, U_2 - U_1, U_3 - U_2$ and $V - U_3$ so as to connect s to $V(C_1) \cap U_1$, $V(C_1) \cap (U_2 - U_1)$ to $V(C_2) \cap (U_2 - U_1)$, $V(C_2) \cap (U_3 - U_2)$ to $V(C_3) \cap (U_3 - U_2)$, and $V(C_3) \cap (V - U_3)$ to t in each of the respective pieces, see Figure 3.8. Then appropriate edges are contracted and an F_7 minor, as in Figure 3.2 is found.

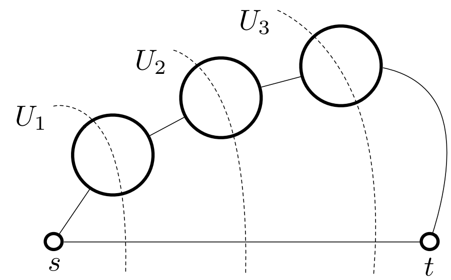

Figure 3.8: Towards an F_7 minor. The three bold circuits are odd.

Chapter 4

Some lemmas

For the sake of notational convenience, we denote $X - (Y_1 \cup Y_2 \cup \cdots \cup Y_n)$ by $X - Y_1 - Y_2 - \cdots - Y_n$, for sets $X, Y_1, Y_2, \ldots, Y_n$.

4.1 Basic lemmas

Lemma 4.1. *Let (F, Γ) be a signed graph with distinguished vertices s, t that is non-packing, and let B be a minimal cover. Then B cannot be both a signature and an st-bond.*

Proof. Suppose not, and let B be a minimal cover that is both a signature and an st-bond. Let P be an st-path. Then $|B \cap P|$ is odd as B is an st-cut, and so P is odd as B is also a signature. Hence, every st-path is odd and so every odd st-walk is an st-path, and vice-versa. Therefore, Menger's theorem implies that (F, Γ) packs, a contradiction. □

Lemma 4.2. *Let $(F = (V, E), \Gamma)$ be a signed graph with distinguished vertices s, t that is st-Eulerian, and let Ω be an edge incident to s. Suppose that $\tau(F, \Gamma) > \tau - 2$ for some integer $\tau \geq 3$. Let Q_1 and Q_2 be (edge-)disjoint paths such that $\Omega \notin Q_1 \cup Q_2$. Suppose there exist minimal covers S_1, S_2 such that*

(1) for $i \in \{1, 2\}$, $\Omega \in S_i$ and $|S_i - Q_i - \{\Omega\}| = \tau - 3$,

(2) $S_1 \cap Q_2 = S_2 \cap Q_1 = \emptyset$, and

(3) S_1 and S_2 are signatures.

Suppose further that there exists a collection $\mathcal{L}$ *of* $\tau - 3$ *pairwise disjoint edge subsets of* $E - Q_1 - Q_2 - \{\Omega\}$*, each of which is either an odd circuit or an odd st-path. Choose a vertex subset* U *of* $V - \{s, t\}$ *such that* $\delta(U) = S_1 \triangle S_2$*. Then there exists a path in* $F[U] - S_1 - S_2$ *between* $V(Q_1)$ *and* $V(Q_2)$*.*

Proof. Observe first that, for each $i = 1, 2$, $S_i - Q_i - \{\Omega\}$ is a transversal of $\mathcal{L}$, as every element of $\mathcal{L}$ is odd and $|\mathcal{L}| = |S_i - Q_i - \{\Omega\}|$.

We next show that there exists a path in $F[U]$ between $V(Q_1)$ and $V(Q_2)$. Suppose not. Then there exists $U_0 \subseteq U$ such that $V(Q_1) \cap U \subseteq U_0$, $V(Q_2) \cap U \cap U_0 = \emptyset$ and $\delta(U_0) - \delta(U) = \emptyset$. Let $S := S_1 \triangle \delta(U_0)$ which is another cover of (F, Γ). We claim that $|S| = \tau - 2$. Observe that $S \cap Q_1 = \emptyset$ as $S_1 \cap Q_1 = \delta(U_0) \cap Q_1$. Moreover, for every $L \in \mathcal{L}$, $|S \cap L| = 1$. Indeed, if $L \cap \delta(U_0) = \emptyset$ then $S \cap L = S_1 \cap L$ and so $|S \cap L| = 1$. Otherwise, if $L \cap \delta(U_0) \neq \emptyset$ then $|L \cap \delta(U_0)| = 2$ as $L \cap \delta(U_0) = L \cap \delta(U) = (L \cap S_1) \cup (L \cap S_2)$ since L is either a circuit or an *st*-path. Therefore, since $S \subseteq \{\Omega\} \cup Q_1 \cup (\bigcup(L : L \in \mathcal{L}))$, it follows that $|S| = 1 + |\mathcal{L}| = \tau - 2$, as claimed. However, $\tau - 2 = |S| \geq \tau(F, \Gamma) > \tau - 2$, a contradiction. So there exists a path P in $F[U]$ between $V(Q_1)$ and $V(Q_2)$.

To finish the proof, note that if $e \in S_1 \cup S_2$ is an edge of $F[U]$ then e must be in both S_1 and S_2 as $e \notin \delta(U) = S_1 \triangle S_2$. Thus if $P \cap (S_1 \cap S_2) = \emptyset$ then we are done. Otherwise, let $e \in P \cap S_1 \cap S_2$. Then e must belong to an element L of $\mathcal{L}$. Observe that L must be a circuit lying completely in $F[U]$ since otherwise, for some $i \in \{1, 2\}$, $|L \cap S_i| > 1$. But then one can bypass the edge e by rerouting P through $L \setminus e$. By repeatedly applying this operation, we will end up with a path in $F[U] - S_1 - S_2$ between $V(Q_1)$ and $V(Q_2)$, as desired. ☐

Lemma 4.3. *Let* $(F = (V, E), \Gamma)$ *be a signed graph with distinguished vertices* s, t *that is st-Eulerian, and let* Ω *be an edge incident to* s*. Suppose that* $\tau(F, \Gamma) > \tau - 2$ *for some integer* $\tau \geq 3$*. Let* Q_1 *and* Q_2 *be odd st-paths such that* $\Omega \in Q_1 \cap Q_2$*. Suppose there exist minimal covers* S_1, S_2 *such that*

(1) $\Omega \in S_1 \cap S_2$ *and* $|S_i - Q_i| = \tau - 3$ *for* $i = 1, 2$*,*

(2) $S_1 \cap Q_2 = S_2 \cap Q_1 = \{\Omega\}$*, and*

(3) S_1 *is an st-bond and* S_2 *is a signature.*

Suppose further that there exists a collection $\mathcal{L}$ *of* $\tau - 3$ *pairwise disjoint odd* $\{s, t\}$*-joins of* $E - Q_1 - Q_2$*. Choose a vertex subset* U *of* $V - \{t\}$ *such that* $\delta(U) = S_1$*. Then there exists a path in* $F[U] - S_2$ *between* s *and* $V(Q_1) - \{s\}$*.*

Proof. As above, observe that, for $i = 1, 2$, $S_i - Q_i$ has size $\tau - 3$ and so is a transversal of $\mathcal{L}$, as every element of $\mathcal{L}$ contains an odd st-walk. Suppose, for a contradiction, there is no path in $F[U] - S_2$ connecting s and $V(Q_1) - \{s\}$. Then there exists $U_0 \subseteq U$ such that $V(Q_1) \cap U_0 = \{s\}$ and $\delta(U_0) - \delta(U) \subseteq S_2$.

Let $S := \delta(U_0)$. Then S is a cover of (F, Γ) contained in $S_1 \cup S_2$. We claim that $|S| = \tau - 2$. Observe that $S \cap Q_1 = \{\Omega\}$ and, since $\Omega \in Q_2$ and $S_1 \cap Q_2 = \{\Omega\}$, we also have $S \cap Q_2 = \{\Omega\}$. Moreover, for every $L \in \mathcal{L}$, $|L \cap S| \leq |L \cap S_1| + |L \cap S_2| = 2$ and so $|L \cap S| = 1$ as $|L \cap S|$ is odd. Therefore, since $S \subseteq \{\Omega\} \cup (\bigcup(L : L \in \mathcal{L}))$, it follows that $|S| = 1 + |\mathcal{L}| = \tau - 2$, as claimed. However, $\tau - 2 = |S| \geq \tau(F, \Gamma) > \tau - 2$, a contradiction. $\square$

Lemma 4.4. *Let $(F = (V, E), \Gamma)$ be a signed graph with distinguished vertices s, t that is st-Eulerian, and let Ω be an edge incident to s. Suppose that $\tau(F, \Gamma) > \tau - 2$ for some integer $\tau \geq 3$. Let $(L_1, L_2, \ldots, L_\tau)$ be an Ω-packing of odd $\{s, t\}$-joins, where L_1 and L_2 are connected. Suppose that B_1, B_2 are minimal covers such that, for $i = 1, 2$, $|B_i - L_i| = \tau - 3$. Then B_1, B_2 cannot both be st-bonds.*

Proof. Observe, for $i = 1, 2$, $\Omega \in B_i$ and that $B_i - L_i$ is a transversal of $L_4, \ldots, L_\tau$. Suppose, for a contradiction, that both B_1 and B_2 are st-bonds. For $i = 1, 2$, choose minimal vertex subsets $U_i \subseteq V - \{t\}$ such that $\delta(U_i) = B_i$. Let $U := U_1 \cap U_2$ and $B := \delta(U)$. Note that B is an st-cut, in particular, it is a cover, and that $B \subseteq B_1 \cup B_2$. We will show that $|B| = \tau - 2$. Take $1 \leq i \leq \tau$. If $i \neq 1, 2$ then $|B \cap L_i| \leq |B_1 \cap L_i| + |B_2 \cap L_i| = 2$, and since $|B \cap L_i|$ is odd it follows that $|B \cap L_i| = 1$.

Take $j \in \{1, 2\}$. Suppose that s, s' are the end-vertices of Ω. Since $\Omega \in B \cap L_j$ and $s \in U$, we get that $s' \notin U_1 \cup U_2$. We claim that $B \cap L_j = \{\Omega\}$. If not then there exists a vertex $u \in VL_j \cap U - \{s\}$. Since L_j is connected, $L_j[s', u] \cap B_{3-j} \neq \emptyset$ (here $L_j[s', u]$ denotes the subpath of L_j between s' and u). But then $L_j \cap B_{3-j} \supsetneq \{\Omega\}$, which is not the case.

As a result, $|B| = \left|B \cap \left(\bigcup_{j=1}^{\tau} L_j\right) = \right| = 1 + \sum_{j=4}^{\tau} |B \cap L_j| = \tau - 2$. However, $\tau - 2 = |B| \geq \tau(F, \Gamma) > \tau - 2$, a contradiction. $\square$

Lemma 4.5. *Let (F, Γ) be a signed graph with distinguished vertices s, t that is st-Eulerian, and let Ω be an edge incident to s. Suppose that $\tau(F, \Gamma) > \tau - 2$ for some integer $\tau \geq 3$. Let $(L_1, L_2, \ldots, L_\tau)$ be an Ω-packing of odd $\{s, t\}$-joins, and suppose that, for each $1 \leq i \leq 3$, there exists a minimal cover B_i such that $|B_i - L_i| = \tau - 3$. If s has degree one in $F[L_1 \cup L_2 \cup L_3]$, then it cannot be the case that one of L_1, L_2, L_3 is a non-simple odd st-walk and the other two are simple odd st-walks.*

Proof. Suppose otherwise. We may assume that L_3 is a non-simple odd st-walk and the two odd st-walks L_1 and L_2 are simple. Choose $1 \leq i \leq 2$. Since $|B_i - L_i| = \tau - 3$, it follows that $B_i - L_i$ is a transversal of $L_4, \ldots, L_\tau$, and $B_i \cap L_1 = \{\Omega\}$. Let C be the odd circuit contained in L_3. As s has degree one in $F[L_1 \cup L_2 \cup L_3]$ it follows that $B_i \cap C = \emptyset$, implying that B_i is an st-bond. Therefore, B_1 and B_2 are st-bonds, contradicting Lemma 4.4. □

Lemma 4.6. *Let (F, Γ) be a signed graph with distinguished vertices s, t that is st-Eulerian, and let Ω be an edge incident to s. Suppose that $\tau(F, \Gamma) > \tau - 2$ for some integer $\tau \geq 3$. Let $(L_1, L_2, \ldots, L_\tau)$ be an Ω-packing of odd $\{s, t\}$-joins, where*

(1) L_1 is a simple odd st-walk and L_4 is contains an even st-path P_4,

(2) there exists a vertex $v \in V(L_1) \cap V(P_4) - \{s, t\}$, and

(3) there exist minimal covers B_1, B_4 of (F, Γ) such that $|B_1 - \{\Omega\} - L_1[v, t]| = \tau - 3$ and $|B_4 - \{\Omega\} - P_4[v, t]| = \tau - 3$.

Then B_1, B_4 cannot both be st-bonds.

Proof. Suppose not. Choose minimal vertex subsets $U_1, U_4 \subseteq V - \{t\}$ such that $B_i = \delta(U_i)$ for $i = 1, 4$. Let $U := U_1 \cap U_4$ and $B := \delta(U)$, which is an st-cut. We will show that $|B| = \tau - 2$.

Observe that $|B_1 - L_1| \geq \tau - 3$ as each of $L_4, L_5, \ldots, L_\tau$ contain an st-path, and so $B_1 \cap L_j \neq \emptyset$ for all $4 \leq j \leq \tau$. However, $|B_1 - L_1| \leq |B_1 - \{\Omega\} - L_1[v, t]| = \tau - 3$, and so $|B_1 - L_1| = \tau - 3$, $\Omega \in B_1$ and

$$(*) \quad B_1 \cap L_1[s, v] = \{\Omega\}, |B_1 \cap P_4[s, v]| = 1.$$

Note that since $|B_1 \cap L_4| = 1$, $(*)$ implies that $B_1 \cap P_4[v, t] = \emptyset$.

Let $L_1' := L_1[s, v] \cup P_4[v, t]$ and $L_4' := (L_4 - P_4) \cup (P_4[s, v] \cup L_4[v, t])$. Similarly, $|B_4 - L_1'| \geq \tau - 3$ as each of $L_4', L_5, \ldots, L_\tau$ contain an st-path, and so $B_4 \cap L_4' \neq \emptyset$ and $B_4 \cap L_j \neq \emptyset$ for all $5 \leq j \leq \tau$. However, $|B_4 - L_1'| \leq |B_4 - \{\Omega\} - P_4[v, t]| = \tau - 3$, and so $|B_4 - L_1'| = \tau - 3$ and

$$(**) \quad B_4 \cap L_1[s, v] = \{\Omega\}, |B_4 \cap P_4[s, v]| = 1.$$

Again, since $|B_4 \cap L_4'| = 1$, $(**)$ implies that $B_4 \cap L_1 = \{\Omega\}$.

Take $1 < i \leq \tau$ such that $i \neq 4$. Then $|B \cap L_i| \leq |B_1 \cap L_i| + |B_4 \cap L_i| = 2$, and since $|B \cap L_i|$ is odd it follows that $|B \cap L_i| = 1$. We claim that $B \cap L_1 = \{\Omega\}$. If not then there exists a vertex $u \in VL_1 \cap U - \{s\}$. Since L_1 is an odd st-path, $L_1[s', u] \cap \delta(U_4) \neq \emptyset$, which is not the case as $B_4 \cap L_1 = \{\Omega\}$. Lastly, we show that $|B \cap L_4| = 1$. Observe that $B \cap L_4 = B \cap P_4$ since $(B_1 \cup B_4) \cap (L_4 - P_4) = \emptyset$. Notice that $(*)$ and $(**)$ imply that $|B \cap P_4[s, v]| = 1$. Moreover, $B \cap P_4[v, t] = \emptyset$ since $B_1 \cap P_4[v, t] = \emptyset$. Hence, $|B \cap L_4| = |B \cap P_4| = 1$.

As a result, $|B| = \left| B \cap \left(\bigcup_{j=1}^{\tau} L_j \right) = \right| = 1 + \sum_{j=4}^{\tau} |B \cap L_j| = \tau - 2$. However, $\tau - 2 = |B| \geq \tau(F, \Gamma) > \tau - 2$, a contradiction. □

Lemma 4.7. *Let (F, Γ) be a signed graph that is the union of pairwise disjoint xy-paths $Q_1, \ldots, Q_n$, for some distinct vertices $x, y \in V(F)$. If $Q_i \cup Q_j$ contains no odd cycle, for all $i, j \in \{1, \ldots, n\}$, then (F, Γ) is bipartite.*

Proof. We proceed by induction on n. For $n = 1$ the lemma is trivial. Choose $n \geq 2$, and assume that the statement holds for all $2 \leq k < n$. Then $Q_1 \cup \cdots \cup Q_{n-1}$ is bipartite, and so there exists a signature $\Gamma' \subseteq Q_n$. Choose a vertex subset $U \subseteq V(F) - \{x\}$ such that $\Gamma' \cap Q_n = \delta(U) \cap Q_n$. Since $Q_n \cup Q_j$ is bipartite, for all $1 \leq j \leq n-1$, it follows that $y \notin U$, and $U \cap V(Q_j) = \emptyset$. As a result, $\Gamma' = \delta(U)$, implying that (F, Γ'), and therefore (F, Γ), is bipartite. □

4.2 The Intersection Lemma

For an acyclic graph F and two of its vertices x, y, we say that $x \succeq y$ if there is a yx-dipath. If $x \neq y$ then $x \succ y$. For an edge subset E' of $E(F)$, we say $x \succeq_{E'} y$ if there is a yx-dipath in E'. Similarly, if $x \neq y$ then $x \succ_{E'} y$.

Lemma 4.8 (Intersection Lemma)**.** *Let F be an acyclic directed graph and let s, t be distinct vertices of F. Suppose further that F is the union of st-dipaths $Q_1, \ldots, Q_m$. For every $j \in \{1, \ldots, m\}$, let $v_j \succ s$ be the closest vertex to s on Q_j that also lies on Q_i, for some $i \in \{1, \ldots, m\} - \{j\}$. Then there exists an index $i \in \{1, \ldots, m\}$ such that $v_i \preceq v_1$, and whenever $v_i \in V(Q_j)$, $v_i = v_j$.*

Proof. Suppose otherwise. Then for each $i \in \{1, 2, \ldots, m\}$ such that $v_i \preceq v_1$, there exists $f(i) \in \{1, 2, \ldots, m\} - \{i\}$ such that $v_i \in V(Q_{f(i)})$ but $v_{f(i)} \prec v_i$. But then $v_1 \succ v_{f(1)} \succ v_{f(f(1))} \succ v_{f(f(f(1)))} \succ \cdots$. However, this not possible since there are only finitely many vertices in F and F is acyclic, a contradiction. □

4.3 The $\widetilde{K_5}$ Lemma

Lemma 4.9 ([3]). *Let (F, Γ) be a signed graph and let $\Omega = \{x, y\}$ be an edge of F, for some distinct vertices x and y of F. Suppose that every odd circuit cover of (F, Γ) has length more than $\tau - 2$, for some integer $\tau \geq 3$. Suppose that $(C_1, C_2, C_3, \ldots, C_\tau)$ is a sequence of odd circuits of (F, Γ) such that*

(1) $\Omega \in C_1 \cap C_2 \cap C_3$ *but* $\Omega \notin \bigcup_{j=4}^{\tau} C_j$,

(2) $(C_j - \{\Omega\} : 1 \leq j \leq \tau)$ *are pairwise edge-disjoint,*

(3) the three xy-paths $P_j := C_j - \{\Omega\}$, $j = 1, 2, 3$ are pairwise internally vertex-disjoint, and

(4) for every $j \in \{1, 2, 3\}$, there exists a minimal odd circuit cover B_j such that $|B_j - C_j| = \tau - 3$.

Then (F, Γ) has a $\widetilde{K_5}$ minor.

4.4 The Reduction Lemma

The following is essentially due to Geelen and Guenin [3]. However, the proof we provide here is slightly different than theirs and makes use of Menger's theorem.

Lemma 4.10. *Let (F, Γ) be a signed graph with distinguished vertices s, t. (It is possible that $s = t$.) Let $\Omega = \{s, s'\}$ be an edge of F, for some vertex $s' \in V(F) - \{s, t\}$. Suppose that $\tau(F, \Gamma) > \tau - 2$ for some integer $\tau \geq 3$. Let $(L_1, L_2, L_3, \ldots, L_\tau)$ be a sequence of odd $\{s, t\}$-joins such that*

(R1) $\Omega \in L_1 \cap L_2 \cap L_3$ *and* $\Omega \notin L_4 \cup \ldots \cup L_\tau$,

(R2) $(L_j - \{\Omega\} : 1 \leq j \leq \tau)$ *are pairwise disjoint,*

(R3) if $s = t$, then L_1, L_2, L_3 are odd circuits, and otherwise when $s \neq t$, L_1, L_2, L_3 are odd st-paths,

(R4) if L'_1, L'_2 and L'_3 are odd st-walks in $L_1 \cup L_2 \cup L_3$ that use Ω and are pairwise disjoint except at Ω, then $L'_1 \cup L'_2 \cup L'_3 = L_1 \cup L_2 \cup L_3$,

(R5) for every odd st-walk $L \subseteq L_1 \cup L_2 \cup L_3$*, there exists an odd st-walk cover* B *of* (F, Γ) *for which* $|B - L| = \tau - 3$*, and*

(R6) whenever L *and* B *satisfy the following, then* B *is a signature:* $L \subseteq L_1 \cup L_2 \cup L_3$ *is an odd st-walk and* B *is an odd st-walk cover of* (F, Γ) *such that* $|B - L| = \tau - 3$*, and for every other odd st-walk cover* B' *of* (F, Γ) *such that* $|B' - L| = \tau - 3$*, we have* $B' \cap L \not\subseteq B \cap L$*.*

Then there exists a minor (F', Γ') *of* (F, Γ) *and odd st-walks* $L_1', L_2', L_3' \subseteq L_1 \cup L_2 \cup L_3$ *such that*

(R1') $E(F) - E(F') \subseteq L_1 \cup L_2 \cup L_3 - \{\Omega\}$*,*

(R2') if $s = t$ *then* L_1', L_2', L_3' *are odd circuits, otherwise when* $s \neq t$*,* L_1', L_2', L_3' *are odd st-paths.*

(R3') $L_1' - \{\Omega\}, L_2' - \{\Omega\}$ *and* $L_3' - \{\Omega\}$ *are pairwise internally vertex-disjoint* $s't$*-paths,*

(R4') for $j \in \{1, 2, 3\}$*, there exists a signature* B_j' *of* (F', Γ') *for which* $|B_j' - L_j'| = \tau - 3$*, and*

(R5') there is no odd st-walk cover of (F', Γ') *of size* $\tau - 2$*, i.e.* $\tau(F', \Gamma') > \tau - 2$*.*

Before we prove the lemma, let us make the following definition. Let L be an odd st-walk contained in $L_1 \cup L_2 \cup L_3$. An odd st-walk cover B of (F, Γ) is said to be an *internally minimal mate of* L if $|B - L| = \tau - 3$, and for any other odd st-walk cover B' of (F, Γ) such that $|B' - L| = \tau - 3$, we have $B' \cap L \not\subseteq B \cap L$. So condition (R6) of the lemma can be rephrased as follows: for every odd st-walk $L \subseteq L_1 \cup L_2 \cup L_3$ and every internally minimal mate B of L, B is a signature.

Proof. We will now proceed to prove the lemma. Let $H := L_1 \cup L_2 \cup L_3 - \{\Omega\}$, and for each $i \in \{1, 2, 3\}$, let B_i be an internally minimal mate for L_i, which exists by (R5) and is a signature by (R6).

Claim 1. $\{\Omega\}$ *is a signature for* $(H \cup \{\Omega\}, (EH \cup \{\Omega\}) \cap \Gamma')$*.*

Proof of Claim. Let $J := H \cup \{\Omega\}$. We will proceed by finding a vertex subset $U \subseteq V(F) - \{s, t\}$ such that $(B_3 \triangle \delta(U)) \cap EJ = \{\Omega\}$. Let $U \subseteq VL_3 - \{s, t\}$ be the unique subset for which $L_3 \cap \delta(U) = L_3 \cap B_3 - \{\Omega\}$. Observe that $B_1 \cap (L_2 \cup L_3) = \{\Omega\}$, and so $L_2 \cup L_3 - \{\Omega\}$ is bipartite, which in turn implies $U \cap VL_2 = \emptyset$. Similarly, $U \cap VL_1 = \emptyset$. Therefore, $\delta(U) \cap EJ = \delta(U) \cap L_3 = B_3 \cap EJ - \{\Omega\}$ and so

$$(B_3 \triangle \delta(U)) \cap EJ = (B_3 \cap EJ) \triangle (B_3 \cap EJ - \{\Omega\}) = \{\Omega\},$$

as claimed. ◇

For each L_j, $1 \leq j \leq 3$ let $P_j := L_j - \{\Omega\}$, which is an $s't$-path, whether $s = t$ or not. Then $H = P_1 \cup P_2 \cup P_3$, and orient the edges of H so that each P_i, $1 \leq i \leq 3$ is a directed $s't$-path. Observe that Claim 1 implies that $(H, EH \cap \Gamma)$ is bipartite.

Claim 2. *H is acyclic.*

Proof of Claim. Suppose otherwise, and let C be a directed circuit in H. As H is acyclic, we can find three pairwise disjoint $s't$-dipaths P'_1, P'_2, P'_3 in $H \setminus C$. However, by Claim 1, each $P'_j \cup \{\Omega\}$, $1 \leq j \leq 3$ is an odd st-walk, contradicting the minimality assumption (R4) of Lemma 4.10. ◇

Let (F', Γ') be a minor of (F, Γ), and let H' be a directed subgraph of (F', Γ'), where (F', Γ') and H' are minimal subject to

(i) $E(F) - E(F') \subseteq P_1 \cup P_2 \cup P_3$, and $E(H') \subseteq P_1 \cup P_2 \cup P_3$

(ii) H' is acyclic and there exist three disjoint $s't$-dipaths in H',

(iii) for every $s't$-dipath P' of H', $P \cup \{\Omega\}$ is an odd st-walk, and there exists an odd st-walk cover B' of (F', Γ') such that $|B' - P'| = \tau - 3$,

(iv) for every $s't$-dipath P' of H' and every internally minimal mate B' of P', B' is a signature, and

(v) there is no odd st-walk cover of (F', Γ') of size $\tau - 2$.

Observe that (F, Γ) and H satisfy all of the five properties, so (F', Γ') and H' are well-defined. Let P'_1, P'_2 and P'_3 be three disjoint $s't$-dipaths of H' as in *(ii)*, and for each $1 \leq j \leq 3$ let $L'_j := P'_j \cup \{\Omega\}$, which is an odd st-walk by *(iii)*. We claim that (F', Γ') and

L_1', L_2', L_3' satisfy (R1')-(R5'), and this will finish the proof of Lemma 4.10. It is clear that (R1'), (R2'), (R4') and (R5') hold. We are then left with (R3').

We need to prove that P_1', P_2' and P_3' are pairwise internally vertex-disjoint. Suppose, for a contradiction, that this is not the case. We will obtain a contradiction by showing that (F', Γ') and H' were *not* a minimal choice subject to $(i)-(v)$. For each $1 \leq i \leq 3$, let $v_i \neq t$ be the closest vertex to t on P_i' that also lies on another P_j'. We may assume that $v_1 \neq s'$, so $s' \succ v_1 \succ t$. Since H is acyclic by Claim 2, the Intersection Lemma implies that there exists $v_i \succeq v_1$ such that whenever $v_i \in V(P_j')$ then $v_i = v_j$. We may again assume that $i = 1$. Let I be the set of all indices j in $\{1, 2, 3\}$ such that $v_1 = v_j$. Note that $1 \in I$ and $|I| \geq 2$.

Claim 3. *For every $i \in I$, there exists an odd st-walk cover B' of (F', Γ') such that $|B' - P_i'[v_i, t] - \{\Omega\}| = \tau - 3$.*

Proof of Claim. Suppose otherwise. By symmetry, we may assume that there is no odd st-walk cover B' of (F', Γ') such that $|B' - P_1'[v_1, t] - \{\Omega\}| = \tau - 3$. Let $(F'', \Gamma'') := (F', \Gamma') \setminus P_1'[v_1, t] / \cup (P_j'[v_j, t] : j \in I, j \neq 1)$ and $H'' := H' \setminus P_1'[v_1, t] / \cup (P_j'[v_j, t] : j \in I, j \neq 1)$. It is clear that (F'', Γ'') and H'' still satisfy (i) and (ii). We claim that $(iii)-(v)$ also hold, thereby contradicting the minimality of (F', Γ') and H'. We may assume that $2 \in I - \{1\}$.

To prove (iii), let P'' be an $s't$-dipath of H''. Then $P'' \cup P_2'[v_2, t]$ contains an $s't$-dipath of H', and since $v_1 = v_2$, it follows that $P'' \cup P_1'[v_1, t]$ also contains an $s't$-dipath of H'. Hence, since (iii) holds for (F', Γ') and H", there exists an odd st-walk cover B' of (F', Γ') such that $|B' - (P'' \cup P_1'[v_1, t]) - \{\Omega\}| = \tau - 3$. Let $B'' := B - P_1'[v_1, t]$, which is an odd st-walk cover for (F'', Γ''). Then $|B'' - P''| = \tau - 3$ and this proves (iii) for (F'', Γ'') and H''.

To prove (iv), let B'' be an internally minimal mate for P'', for some $s't$-dipath P'' of (F'', Γ''). Let P' be the $s't$-dipath of (F', Γ') contained in $P'' \cup P_1'[v_1, t]$. Then $B'' \cup P_1'[v_1, t]$ contains an internally minimal mate B' of P' in (F', Γ'). Observe that it must be the case that $B'' \subseteq B'$, and since B' is a signature for (F', Γ') by (iv), it follows that B'' too is a signature for (F'', Γ'').

It remains to prove (v). If there were an odd st-walk cover B'' of (F'', Γ'') of size $\tau - 2$, then $B' := B'' \cup P_1'[v_1, t]$ would be an odd st-walk cover of (F', Γ'), but $|B' - P_1'[v_1, t] - \{\Omega\}| = |B'' - \{\Omega\}| = \tau - 3$, a contradiction since we assumed such B' does not exist. This finishes the proof of the claim. $\diamond$

Claim 4. *There is no cut-vertex in H' separating s' from $\{v_1, t\}$.*

Proof of Claim. Suppose otherwise. Let $v \in V(H') - \{s', t\}$ be a cut-vertex of H' separating s' from $\{v_1, t\}$. Then $v \in V(P'_i)$ for every $i \in \{1, 2, 3\}$. Let $R'_i := P'_i[v, t]$ for $i \in \{1, 2, 3\}$. One of the following must hold:

(1) for every vt-dipath R' in $R'_1 \cup R'_2 \cup R'_3$, there is an odd st-walk cover B' such that $|B' - R' - \{\Omega\}| = \tau - 3$, or

(2) there exists a vt-dipath R' in $R'_1 \cup R'_2 \cup R'_3$ for which there is no odd st-walk cover B' such that $|B' - R' - \{\Omega\}| = \tau - 3$.

If (1) holds then let $(F'', \Gamma'') := (F', \Gamma')/\cup(P'_j[s', v] : 1 \le j \le 3)$ and $H'' := H'/\cup(P'_j[s', v] : 1 \le j \le 3)$. It can be readily checked that $(i), (ii), (iv)$ and (v) still hold, and by assumption (1), (iii) also holds for (F'', Γ'') and H''. However, this cannot be the case by the minimality of (F', Γ') and H'. Hence, (2) holds. By the acyclicity of H', we may assume that $R' = R'_1$. Then let $(F'', \Gamma'') := (F', \Gamma') \setminus R'_1/(R'_2 \cup R'_3)$ and $H'' := H' \setminus R'_1/(R'_2 \cup R'_3)$. Again, it is clear that (i) and (ii) still hold. Likewise to the proof of the preceding claim, (iii) and (iv) hold, and by assumption (2), (v) also holds for (F'', Γ'') and H''. But this is a contradiction to the minimality of (F', Γ') and H'. This finishes the proof of the claim. $\diamond$

Hence, by Menger's theorem, there exists two directed paths P and P''_3 that have only vertex s' in common, P is from s' to v_1, and P''_3 is from s' to t. Let $P''_i := P'_i[v_i, t]$ for $i = 1, 2$, and let $(F'', \Gamma'') := (F', \Gamma')/P$ and $H'' := P''_1 \cup P''_2 \cup P''_3$. Notice that P''_1, P''_2 and P''_3 are pairwise internally vertex-disjoint $s't$-dipaths in (F'', Γ''). It is clear that $(i), (ii), (iv)$ and (v) still hold for (F'', Γ'') and H''. Moreover, by Claim 3, for each $j \in \{1, 2\}$, there exists an odd st-walk cover B'' of (F'', Γ'') such that $|B'' - P''_i - \{\Omega\}| = \tau - 3$. Moreover, P''_3 is also an $s't$-dipath for (F', Γ') and so by (iii) applied to (F', Γ') and H', there exists an odd st-walk cover B' of (F', Γ') such that $|B' - P''_3 - \{\Omega\}| = \tau - 3$. However, B' is also an odd st-walk cover for (F'', Γ''), and so (iii) holds for (F'', Γ'') and H''. This is however a contradiction to the minimality of (F', Γ') and H'.

Hence, P'_1, P'_2 and P'_3 are pairwise internally vertex-disjoint, proving the last needed piece (R3'). This finishes the proof of the Reduction Lemma. $\square$

4.5 The Mate Lemma

Lemma 4.11. *Let* $(F = (V, E), \Gamma)$ *be a connected signed graph with distinguished vertices* s, t *that is st-Eulerian. Let* m *and* τ *be integers so that* $3 \leq m \leq \tau$, $(L_1, L_2, L_3, \ldots, L_\tau)$ *be a sequence of odd* $\{s,t\}$*-joins, and* $(B_1, \ldots, B_m)$ *be a sequence of minimal covers such that*

(1) $\Omega \in L_1 \cap L_2 \cap L_3$ *and* $\Omega \notin L_4 \cup \ldots \cup L_\tau$,

(2) $(L_j - \{\Omega\} : 1 \leq j \leq \tau)$ *are pairwise disjoint,*

(3) L_1, L_2 *and* L_3 *are odd st-walks, and for each* $j \in \{4, \ldots, m\}$, L_j *contains a disconnected odd st-walk* $C_j \cup P_j$, *and each of* $L_{m+1}, \ldots, L_\tau$ *contains a connected odd st-walk, and*

(4) $|B_j - L_j| = \tau - 3$ *for all* $1 \leq j \leq m$.

Suppose further that $\tau(F, \Gamma) > \tau - 2$. *If* $|B_j - P_j - \{\Omega\}| = \tau - 3$ *for all* $1 \leq j \leq m$, *then* B_i *is an st-bond for some* $1 \leq i \leq m$.

Proof. Suppose that $|B_j - P_j - \{\Omega\}| = \tau - 3$ for all $1 \leq j \leq m$. If L_j is non-simple and $\Omega \in P_j$, for some $j \in \{1, 2, 3\}$ then for any $i \in \{1, 2, \ldots, m\} - \{j\}$, $B_i \cap L_j = \{\Omega\}$ and so $B_i \cap C_j = \emptyset$, implying in turn that B_i must be an *st*-bond, and so we are done.

Otherwise,

whenever $L_j \in \{L_1, L_2, L_3\}$ is non-simple, $\Omega \in C_j$ and so in particular, $s \in VC_j$.

Suppose, for a contradiction, that none of $B_1, B_2, \ldots, B_m$ is an *st*-bond, so they are all signatures. We will find an odd *st*-walk cover for (F, Γ) of size $\tau - 2$, which would yield a contradiction as $\tau(F, \Gamma) > \tau - 2$.

For all distinct $i, j \in \{1, 2, \ldots, m\}$, choose $U_{ij} \subseteq V - \{s, t\}$ such that $\delta(U_{ij}) = B_i \triangle B_j$. Take distinct $i, j, k \in \{1, \ldots, m\}$. Observe that

$$\delta(U_{ij} \triangle U_{jk} \triangle U_{ki}) = (B_i \triangle B_j) \triangle (B_j \triangle B_k) \triangle (B_k \triangle B_i) = \emptyset.$$

Since F is connected and $s, t \notin U_{ij} \cup U_{jk} \cup U_{ki}$, it then follows that $U_{ij} \triangle U_{jk} \triangle U_{ki} = \emptyset$ and so, in particular, $U_{ij} \cap U_{jk} \cap U_{ki} = \emptyset$.

For each $i \in \{1, 2, \ldots, m\}$ and $\emptyset \neq A \subseteq \{1, 2, \ldots, m\} - \{i\}$, let

$$S_i^A := \bigcap_{j \in A} U_{ij}.$$

Observe that

$$\delta(S_i^A) \subseteq \cup_{j\in\{i\}\cup A} B_j.$$

Indeed, if $e = \{u, v\} \in \delta(S_i^A)$ with $v \notin S_i^A$, then $v \notin U_{ij}$ for some $j \in A$ and so $e \in \delta(U_{ij})$, implying that $e \in B_i$ or $e \in B_j$. Furthermore, for all distinct $i, j, k \in \{1, \ldots, m\}$ and A such that $\{j, k\} \subseteq A \subseteq \{1, \ldots, m\} - \{i\}$, since $S_i^A \subseteq U_{ij} \cap U_{ik}$ and $U_{ij} \cap U_{ik} \cap U_{jk} = \emptyset$, it follows that

$$S_i^A \cap U_{jk} = \emptyset.$$

Take $i \in \{1, 2, \ldots, m\}$ and $\emptyset \neq A \subseteq \{1, 2, \ldots, m\} - \{i\}$.

Claim 1. *$P_i \cap \delta(S_i^A) = P_i \cap B_i - \{\Omega\}$, and $P_j \cap \delta(S_i^A) = \emptyset$ for all $j \in \{1, \ldots, m\} - \{i\}$ such that $A - \{j\} \neq \emptyset$.*

Proof of Claim. To see why $P_i \cap \delta(S_i^A) = P_i \cap B_i - \{\Omega\}$, notice that $P_i \cap \delta(U_{ik}) = P_i \cap B_i - \{\Omega\}$ and $s, t \notin U_{ik}$, for any $k \in A$. Moreover, since $P_j \cap \delta(U_{ik}) = \emptyset$ for any $k \in A - \{j\}$, it follows that $P_j \cap \delta(S_i^A) = \emptyset$. This proves the claim. ◇

Claim 2. *If $L \in \{L_j : m < j \leq \tau\}$ and $L \cap \delta(S_i^A) \neq \emptyset$, then $|L \cap \delta(S_i^A)| = 2$ and $|L \cap \delta(S_i^A) \cap B_i| = 1$.*

Proof of Claim. Take $L \in \{L_j : m < \tau \leq \tau\}$ such that $L \cap \delta(S_i^A) \neq \emptyset$. We may assume that $i = 1$. Notice that L is a *connected* odd $\{s, t\}$-join, and so we are able to write $L = (s = v_0, e_1, v_1, e_2, \ldots, e_p, v_p = t)$. Choose $1 \leq i < k \leq p$ such that $e_i, e_k \in \delta(S_1^A)$ with $v_i, v_{k-1} \in S_1^A$. As $|L \cap B_1| = 1$ we may assume that $L[s, v_i] \cap B_1 = \emptyset$. Since $v_i \in U_{1j}$ and $s \notin U_{1j}$ for all $j \in A$, we have that $L[s, v_i] \cap \delta(U_{1j}) \neq \emptyset$. However, $L[s, v_i] \cap B_1 = \emptyset$, so $L[s, v_i] \cap B_j \neq \emptyset$ for all $j \in A$.

We claim that $e_k \in B_1$. As $v_k \notin S_1^A$, there exists $j \in A$ such that $v_k \notin U_{1j}$ and so $e_k \in \delta(U_{1j})$. However, $|L \cap B_j| = 1$ and $L[s, v_i] \cap B_j \neq \emptyset$, implying that $L[v_{k-1}, t] \cap B_j = \emptyset$. Hence, $e_k \notin B_j$ and so $e_k \in B_1$. Since $|L \cap B_j| = 1$ for all $j \in \{1\} \cup A$, it follows that $L \cap \delta(S_1^A) = \{e_i, e_k\}$ and $L \cap \delta(S_1^A) \cap B_1 = \{e_k\}$, as claimed. ◇

For the next claim, let $C_j := \emptyset$ if L_j contains no odd circuit, for $1 \leq j \leq 3$.

Claim 3. *If* $C \in \{C_j : 1 \leq j \leq m\}$ *and* $C \cap \delta(S_i^A) \neq \emptyset$ *then* $|C \cap \delta(S_i^A)| = 2$. *Moreover, if* $C \cap \delta(S_i^A) \subseteq B_j \cup B_k$ *for distinct* $j, k \in A$, *then* $V(C) \subseteq U_{ij} \cup U_{ik}$.

Proof of Claim. Assume $C \cap \delta(S_i^A) \neq \emptyset$ for some $C \in \{C_j : 1 \leq j \leq m\}$. By symmetry, we may assume that $i = 1$. Write $C = (v_0, e_1, v_1, e_2, \ldots, e_p, v_p = v_0)$.

Suppose there exist $1 \leq i < k \leq p$ such that $e_i, e_k \in \delta(S_1^A) - B_1$ with $v_i, v_{k-1} \notin S_1^A$. Assume that $e_i \in B_j$ for some $j \in A$. Since $e_i \notin B_1$, we get that $e_i \in \delta(U_{1j})$. Because $v_{i-1} \in S_1^A \subseteq U_{1j}$, it follows that $v_i \notin U_{1j}$. Since $|C \cap B_j| = 1$ it follows that $e_k \notin \delta(U_{1j})$. However, $v_k \in S_1^A \subseteq U_{1j}$, so $v_{k-1} \in U_{1j}$. Therefore, since $v_i \notin U_{1j}$ but $v_{k-1} \in U_{1j}$, it follows that $C[v_i, v_{k-1}] \cap \delta(U_{1j}) \neq \emptyset$, for $C[v_i, v_{k-1}] = (v_i, e_{i+1}, \ldots, e_{k-1}, v_{k-1})$. However, $C \cap B_j = \{e_i\}$ and so $C[v_i, v_{k-1}] \cap B_1 \neq \emptyset$.

As a result, if there exist $1 \leq i < k \leq p$ such that $e_i, e_k \in \delta(S_1^A) - B_1$ with $v_i, v_{k-1} \notin S_1^A$, then $C[v_i, v_{k-1}] \cap B_1 \neq \emptyset$. Therefore, as $|C \cap B_1| = 1$, we get that $|C \cap \delta(S_i^A)| = 2$. This proves the first part of the claim.

For the second part, assume $C \cap \delta(S_1^A) = \{e, f\}$ where $e \in B_j$ and $f \in B_k$. If $e \in B_1$ then $C \cap \delta(U_{1j}) = \emptyset$, but $V(C) \cap S_1^A \neq \emptyset$ and $S_1^A \subseteq U_{1j}$, implying that $V(C) \subseteq U_{1j} \subseteq U_{1j} \cup U_{1k}$, and so we are done. Similarly, if $f \in B_1$ then $V(C) \subseteq U_{1k} \subseteq U_{1j} \cup U_{1k}$, and we are again done. Otherwise, $\{e, f\} \cap B_1 = \emptyset$. Write $C = (v_0, e_1, v_1, e_2, \ldots, e_p, v_p = v_0)$ for some $v_0 \in S_1^A$, and assume that $e = e_i$, $f = e_l$ for some $1 \leq i < l \leq p$ where $v_i, v_{l-1} \notin S_1^A$. As $e \in B_j - B_1$ it follows that $e \in \delta(U_{1j})$, and since $v_{i-1} \in S_1^A \subseteq U_{1j}$, we get $v_i \notin U_{1j}$. Also, as $|C \cap B_j| = 1$, we have $f \notin B_j$. This, together with the facts that $f \notin B_1$ and $v_l \in S_1^A \subseteq U_{1j}$, implies that $v_{l-1} \in U_{1j}$.

Observe that $v_i \notin U_{1j}$, $v_{l-1} \in U_{1j}$ and $|C \cap B_1| = 1$ imply that there exists a unique edge $e_r \in B_1 \cap C$ where $i < r < l$, and $v_r, v_{r+1}, \ldots, v_{l-1} \in U_{1j}$. Similarly, we have $v_i, v_{i+1}, \ldots, v_{r-1} \in U_{1k}$. Furthermore, note that $v_0, v_1, \ldots, v_{i-1}, v_l, v_{l+1}, \ldots, v_{p-1} \in S_1^A \subseteq U_{1j} \cap U_{1k}$. Therefore, $V(C) = \{v_0, v_1, \ldots, v_{p-1}\} \subseteq U_{1j} \cup U_{1k}$, as claimed. This finishes the proof of the claim. ◇

For every $k \geq 1$, let $[k]$ denote $\{1, 2, \ldots, k\}$. Consider the $m - 2$ sets in

$$\mathcal{S} := \left\{ S_j^{[j-1]} : 3 \leq j \leq m \right\}.$$

We call a circuit $C \in \{C_j : 1 \leq j \leq m\}$ *bad* for $S := S_i^{[i-1]} \in \mathcal{S}$ if $|C \cap \delta(S)| = 2$ but $C \cap \delta(S) \cap B_i = \emptyset$. Let C be a bad circuit for $S = S_i^{[i-1]} \in \mathcal{S}$ (if any), and assume that $C \cap \delta(S) \subseteq B_j \cup B_k$ for distinct $j, k \in \{1, 2, \ldots, i-1\}$. Then by Claim 3 we have that

$V(C) \subseteq U_{ij} \cup U_{ik}$ and so $s \notin V(C)$ and $V(C) \cap S_\ell^{[\ell-1]} = \emptyset$ for any $m \geq \ell > i$ (since $(U_{ij} \cup U_{ik}) \cap S_\ell^{[\ell-1]} = \emptyset$). Thus, in particular, C is *not* bad for $S_\ell^{[\ell-1]}$ for any $m \geq \ell > i$, and $C \notin \{C_1, C_2, C_3\}$, since $s \in V(C_j)$ if $C_j \neq \emptyset$ for some $j \in \{1,2,3\}$. Thus each $C \in \{C_j : 1 \leq j \leq m\}$ is bad for at most one set in $\mathcal{S}$ and every bad circuit is in $\{C_j : 4 \leq j \leq m\}$. Therefore, since $|\mathcal{S}| = m-2$ and there are at most $m-3$ bad circuits, there exists $S := S_i^{[i-1]} \in \mathcal{S}$ which has no bad circuit.

Let $B := B_i \triangle \delta(S)$. Then B is an odd st-walk cover. We claim that $|B| = \tau - 2$, which would yield a contradiction, thereby finishing the proof of Lemma 4.11. Observe that $B \subseteq \bigcup_{j=1}^{\tau} L_j$.

Take $m < j \leq \tau$. If $L_j \cap \delta(S) = \emptyset$, then $|L_j \cap B| = |L_j \cap B_i| = 1$. Otherwise by Claim 2, $|L_j \cap \delta(S)| = 2$ and $|L_j \cap \delta(S) \cap B_i| = 1$, implying that $|L_j \cap B| = |L_j \cap (B_i \triangle \delta(S))| = 1$.

Next take $1 \leq j \leq m$. We claim that $|L_j \cap B| = 1$. By Claim 1, $P_j \cap B = P_j \cap (B_i \triangle \delta(S))$ is either $\emptyset$ (if $C_j \neq \emptyset$) or $\{\Omega\}$ (if $C_j = \emptyset$). We now consider $C_j \cap B$. If $C_j \cap \delta(S) = \emptyset$ then $C_j \cap B = C_j \cap B_i$. Otherwise, $C_j \cap \delta(S) \neq \emptyset$. Then by Claim 3 and the fact that C_j is not bad for S, it follows that $|C_j \cap \delta(S)| = 2$ and $|C_j \cap \delta(S) \cap B_i| = 1$. As a result, $|C_j \cap B| = |C_j \cap (B_i \triangle \delta(S))| = 1$. Hence, $|L_j \cap B| = 1$, as claimed.

Notice that $L_j \cap B = \{\Omega\}$ for $j \in \{1,2,3\}$. Thus, $|B| = 1 + \sum_{j=4}^{\tau} |B \cap L_j| = \tau - 2$, a contradiction as $\tau - 2 = |B| \geq \tau(F, \Gamma) > \tau - 2$. This finishes the proof of the Mate Lemma. $\square$

4.6 The Shore Lemma

Lemma 4.12. *Let $(F = (V,E), \Gamma)$ be a connected signed graph with distinguished vertices s, t that is st-Eulerian. Let m and τ be integers so that $3 \leq m \leq \tau$, $(L_1, L_2, L_3, \ldots, L_\tau)$ be a sequence of odd $\{s,t\}$-joins, and $(B_1, \ldots, B_m)$ be a sequence of minimal covers such that*

(1) $\Omega \in L_1 \cap L_2 \cap L_3$ and $\Omega \notin L_4 \cup \ldots \cup L_\tau$,

(2) $(L_j - \{\Omega\} : 1 \leq j \leq \tau)$ are pairwise disjoint,

(3) L_1, L_2 and L_3 are simple odd st-walks, and for each $j \in \{4, \ldots, m\}$, L_j contains a disconnected odd st-walk $C_j \cup P_j$, and each of $L_{m+1}, \ldots, L_\tau$ contains a connected odd st-walk, and

(4) $|B_j - L_j| = \tau - 3$ and $|B_k - P_k - \{\Omega\}| = \tau - 3$, for all $1 \leq j \leq 3 < k \leq m$,

(5) B_1 *is an* st*-bond and* $B_1 = \delta(U_1)$ *for some* $U_1 \subseteq V - \{t\}$*, but* $B_2, B_3, \ldots, B_m$ *are signatures,*

(6) for every st*-bond* B *such that* $|B - L_1| = \tau - 3$*, we have* $B \cap L_1 \not\subseteq B_1 \cap L_1$*, and*

(7) $B_j \cap P_j \cap EF[U_1] = \emptyset$ *for all* $j \in \{4, 5, \ldots, m\}$*.*

Suppose further that $\tau(F, \Gamma) > \tau - 2$*. Then there exists a path between* s *and every connected component of* $L_1 \cap F[U_1]$ *in* $F[U_1] \setminus \bigcup_{j=2}^{m} B_j$*.*

Proof. As in the proof for the Mate Lemma, for all distinct $i, j \in \{2, 3, \ldots, m\}$, choose $U_{ij} \subseteq V - \{s, t\}$ such that $\delta(U_{ij}) = B_i \triangle B_j$. For each $i \in \{2, 3, \ldots, m\}$ and $\emptyset \neq A \subseteq \{2, 3, \ldots, m\} - \{i\}$, let $S_i^A := \bigcap_{j \in A} U_{ij}$. As before, $\delta(S_i^A) \subseteq \cup_{j \in \{i\} \cup A} B_j$. Also, $S_i^A \cap U_{jk} = \emptyset$ for all distinct $i, j, k \in \{2, 3, \ldots, m\}$ and A such that $\{j, k\} \subseteq A \subseteq \{2, 3, \ldots, m\} - \{i\}$.

Take $i \in \{2, 3, \ldots, m\}$ and $\emptyset \neq A \subseteq \{2, 3, \ldots, m\} - \{i\}$. Then the following three statements hold, and the proofs are exactly the same as the proofs for the Mate Lemma.

Claim 1. $P_i \cap \delta(S_i^A) = P_i \cap B_i - \{\Omega\}$*, and* $P_j \cap \delta(S_i^A) = \emptyset$ *for all* $j \in \{1, \ldots, m\} - \{i\}$ *such that* $A - \{j\} \neq \emptyset$*.*

Claim 2. *If* $L \in \{L_j : m < j \leq \tau\}$ *and* $L \cap \delta(S_i^A) \neq \emptyset$*, then* $|L \cap \delta(S_i^A)| = 2$ *and* $|L \cap \delta(S_i^A) \cap B_i| = 1$*.*

Claim 3. *If* $C \in \{C_j : 4 \leq j \leq m\}$ *and* $C \cap \delta(S_i^A) \neq \emptyset$ *then* $|C \cap \delta(S_i^A)| = 2$*. Moreover, if* $C \cap \delta(S_i^A) \subseteq B_j \cup B_k$ *for distinct* $j, k \in A$*, then* $V(C) \subseteq U_{ij} \cup U_{ik}$*.*

For every $k \geq 2$, let $[k]'$ denote $\{2, 3, \ldots, k\}$. Consider the $m - 3$ sets in

$$\mathcal{S} := \left\{ S_j^{[j-1]'} : 4 \leq j \leq m \right\}.$$

As before, we call a circuit $C \in \{C_j : 4 \leq j \leq m\}$ *bad* for $S := S_i^{[i-1]'} \in \mathcal{S}$ if $|C \cap \delta(S)| = 2$ but $C \cap \delta(S) \cap B_i = \emptyset$. Let C be a bad circuit for $S = S_i^{[i-1]'} \in \mathcal{S}$ (if any), and assume that $C \cap \delta(S) \subseteq B_j \cup B_k$ for distinct $j, k \in \{2, 3, \ldots, i-1\}$. Then by Claim 3, we have that $V(C) \subseteq U_{ij} \cup U_{ik}$ and so $V(C) \cap S_\ell^{[\ell-1]'} = \emptyset$ for any $m \geq \ell > i$ (since $(U_{ij} \cup U_{ik}) \cap S_\ell^{[\ell-1]'} = \emptyset$). Thus, in particular, C is *not* bad for $S_\ell^{[\ell-1]'}$ for any $m \geq \ell > i$.

Thus each $C \in \{C_j : 4 \leq j \leq m\}$ is bad for at most one set in $\mathcal{S}$.

Claim 4. *There is a bad circuit for every $S \in \mathcal{S}$.*

Proof of Claim. Suppose otherwise. Choose an $S := S_i^{[i-1]'} \in \mathcal{S}$ that has no bad circuit. Let $B := B_i \triangle \delta(S)$. Then B is an odd st-walk cover. We claim that $|B| = \tau - 2$, which would yield a contradiction. Observe that $B \subseteq \bigcup_{j=1}^{\tau} L_j$.

Take $m < j \leq \tau$. If $L_j \cap \delta(S) = \emptyset$ then $|L_j \cap B| = |L_j \cap B_i| = 1$. Otherwise by Claim 2, $|L_j \cap \delta(S)| = 2$ and $|L_j \cap \delta(S) \cap B_i| = 1$, implying that $|L_j \cap B| = |L_j \cap (B_i \triangle \delta(S))| = 1$.

Next take $1 \leq j \leq m$. We claim that $|L_j \cap B| = 1$. By Claim 1, $P_j \cap B = P_j \cap (B_i \triangle \delta(S))$ is either $\emptyset$ (if $4 \leq j \leq m$) or $\{\Omega\}$ (if $1 \leq j \leq 3$). We now consider $C_j \cap B$. If $C_j \cap \delta(S) = \emptyset$ then $C_j \cap B = C_j \cap B_i$. Otherwise, $C_j \cap \delta(S) \neq \emptyset$. Then by Claim 3 and the fact that C_j is not bad for S, it follows that $|C_j \cap \delta(S)| = 2$ and $|C_j \cap \delta(S) \cap B_i| = 1$. As a result, $|C_j \cap B| = |C_j \cap (B_i \triangle \delta(S))| = 1$. Hence, $|L_j \cap B| = 1$, as claimed.

Notice that $L_j \cap B = \{\Omega\}$ for $j \in \{1, 2, 3\}$. Thus, $|B| = 1 + \sum_{j=4}^{\tau} |B \cap L_j| = \tau - 2$, a contradiction as $\tau - 2 = |B| \geq \tau(F, \Gamma) > \tau - 2$. ◇

Therefore, since $|\mathcal{S}| = m - 3$, it follows that there is a one-to-one correspondence between the circuits and the elements of $\mathcal{S}$, where every circuit is bad for its corresponding element of $\mathcal{S}$. Hence, by Claim 3, we get that

$$\bigcup_{j=4}^{m} V(C_j) \subseteq \bigcup_{i,j \in [m]'} U_{ij}.$$

Claim 5. *Take an edge $e \in E$ with both ends in $V - \bigcup_{i,j \in [m]'} U_{ij}$. If $e \in B_l$ for some $l \in \{2, 3, \ldots, m\}$, then $e \in \bigcap_{k \in [m]'} B_k$.*

Proof of Claim. Since e has both ends in $V - \bigcup_{i,j \in [m]'} U_{ij}$, it follows that $e \notin \delta(U_{kl}) = B_k \triangle B_l$ for any $k, l \in \{2, 3, \ldots, m\}$, proving the claim. ◇

To prove the lemma, we need to show that there exists a path in $F[U_1] \setminus \bigcup_{j=2}^{m} B_j$ between s and every connected component of $L_1 \cap F[U_1]$. Suppose, for a contradiction, that this is not true. Then, in particular, there is no path in $F\left[U_1 - \bigcup_{i,j\in[m]'} U_{ij}\right] \setminus \bigcup_{j=2}^{m} B_j$ between s and some connected component of $L_1 \cap F[U_1]$. Hence, there exists a vertex subset $U \subseteq U_1 - \bigcup_{i,j\in[m]'} U_{ij}$ with $s \notin U$ and $L_1 \cap \delta(U) \neq \emptyset$ such that $\delta(U) - B_1 - \delta\left(\bigcup_{i,j\in[m]'} U_{ij}\right) \subseteq \bigcup_{l\in[m]'} B_l$. By Claim 5, it follows that $\delta(U) - B_1 - \delta\left(\bigcup_{i,j\in[m]'} U_{ij}\right) \subseteq \bigcap_{l\in[m]'} B_l$.

Let

$$B := B_1 \triangle \delta(U) = \delta(U_1 \triangle U).$$

We claim that $L_1 \cap B \subsetneq L_1 \cap B_1$ and $|B - L_1| = \tau - 3$, contradicting assumption (6) of Lemma 4.12. Observe that

$$L_1 \cap B = L_1 \cap (B_1 \triangle \delta(U)) = (L_1 \cap B_1) \triangle (L_1 \cap \delta(U)) \subsetneq L_1 \cap B_1$$

because $\emptyset \neq L_1 \cap \delta(U) \subseteq L_1 \cap B_1$.

It remains to show that $|B - L_1| = \tau - 3$. Since $B \subseteq \bigcup_{j=1}^{\tau} L_j$, we can proceed by showing that $B \cap L_2 = B \cap L_3 = \{\Omega\}$ and $|B \cap L_j| = 1$ for all $4 \leq j \leq \tau$. Note that $L_2 \cap \delta(U_1) = L_3 \cap \delta(U_1) = \{\Omega\}$, and so $L_2 \cap \delta(U) = L_3 \cap \delta(U) = \emptyset$, since L_2 and L_3 are simple. Hence, $L_2 \cap B = L_3 \cap B = \{\Omega\}$. Observe then that $C_k \cap \delta(U) = \emptyset$ for all $4 \leq k \leq m$, as $U \cap \bigcup_{i,j\in[m]'} U_{ij} = \emptyset$ but $V(C_k) \subseteq \bigcup_{i,j\in[m]'} U_{ij}$.

Take $j \in \{4, 5, \dots, \tau\}$. If $L_j \cap \delta(U) = \emptyset$ then $L_j \cap B = L_j \cap B_1$, and so $|L_j \cap B| = 1$. Otherwise, $L_j \cap \delta(U) \neq \emptyset$. Write $(s = v_0, e_0, v_1, \dots, e_n, v_n = t)$ for L_j if $j \notin \{4, \dots, m\}$, and for P_j otherwise. Note that, for $4 \leq j \leq m$, $L_j \cap \delta(U) = (s = v_0, e_0, v_1, \dots, e_k, v_k = t) \cap \delta(U)$ since $C_j \cap \delta(U) = \emptyset$. Note further that $P_j \cap (B_2 \cap B_3 \cap \dots \cap B_m) = \emptyset$, for all $4 \leq j \leq m$.

We claim that $|L_j \cap \delta(U)| = 2$ and $|L_j \cap \delta(U) \cap B_1| = 1$. Suppose that $e_i, e_k \in L_j \cap \delta(U)$ for some $0 \leq i < k \leq n$ with $v_i, v_{k+1} \notin U$. We will prove that $e_k \in B_1$, and note that once this is proved, it then easily follows that $|L_j \cap \delta(U)| = 2$ and $|L_j \cap \delta(U) \cap B_1| = 1$. Thus, it remains to show that $e_k \in B_1$.

Since $|L_j \cap \delta(U_1)| = 1$, it must be that $v_i \in U_1$. If $v_i \notin \bigcup_{a,b\in[m]'} U_{ab}$, then we must have that $e_i \in \bigcap_{\ell\in[m]'} B_\ell$. So $m < j \leq \tau$, and since $L_j \cap B_k = \{e_i\}$ for all $k \in \{2, 3, \dots, m\}$, it follows that $e_k \in B_1$. Otherwise, we have that $v_i \in \bigcup_{a,b\in[m]'} U_{ab}$. Then $e_i \in \bigcup_{a\in[m]'} B_a$ and so $e_i \in B_p$, for some $p \in \{2, 3, \dots, m\}$. Suppose, for a contradiction, that $e_k \notin B_1$. So $v_{k+1} \in U_1$. Notice that $v_{k+1} \in \bigcup_{a,b\in[m]'} U_{ab}$ since otherwise $e_k \in \bigcap_{\ell\in[m]'} B_\ell$, implying that $m < j \leq \tau$, but then $L_j \cap B_p \supseteq \{e_i, e_k\}$, which cannot be the case. As $v_{k+1} \in \bigcup_{a,b\in[m]'} U_{ab}$, it follows that $e_k \in B_q$ for some $q \in \{2, 3, \dots, m\}$. If e_k is also in B_p, then $|L_j \cap B_p| \geq 2$ implying that $4 \leq j = p \leq m$, which cannot be possible by assumption (7) (since $e_i, e_k \in EF[U_1]$). Hence, $e_k \notin B_p$ and similarly, $e_i \notin B_q$. In fact, since $e_i \in B_p \cap EF[U_1]$ and

$e_k \in B_q \cap EF[U_1]$, it follows from assumption (7) that $j \notin \{p, q\}$. Since $e_k \in B_q - B_p$ and $e_i \in B_p - B_q$, $\{e_i, e_k\} \subseteq L_j \cap \delta(U_{pq})$. Hence, since $v_{i+1} \in U$ and $U \subseteq U_1 - U_{pq}$, we get that $v_i \in U_{pq}$. But $s \notin U_{pq}$ and so there exists an edge e_r such that $0 \leq r < i$ and $e_r \in \delta(U_{pq})$. Hence, $L_j \cap \delta(U_{pq}) \supseteq \{e_r, e_i, e_k\}$ implying that $j \in \{p, q\}$, a contradiction. Therefore, it must be the case that $e_k \in B_1$.

Therefore, $|L_j \cap \delta(U)| = 2$ and $|L_j \cap \delta(U) \cap B_1| = 1$ and so

$$|L_j \cap B| = |L_j \cap (B_1 \triangle \delta(U))| = 1,$$

as claimed. As a result, $L_1 \cap B \subsetneq L_1 \cap B_1$ and $|B - L_1| = \tau - 3$, contradicting assumption (6). This finishes the proof of the Shore Lemma. □

4.7 The Linkage Lemma

Let H be a graph, and take distinct vertices s, s', t' and t. We are interested in characterizing when we can find a pair of vertex-disjoint paths, one between s and t and the other between s' and t'. The following lemma is due to Seymour [17], and it is stated similarly as in [8].

Lemma 4.13 ([17]). *There is no pair of vertex-disjoint paths P_{st} and $P_{s't'}$ in H, where P_{st} is an st-path and $P_{s't'}$ is an $s't'$-path, if and only if H can be obtained as follows:*

(L1) place a circuit C on the boundary S^1 of the unit disc, and the circuit contains the vertices s, s', t, v_1 in this cyclic order,

(L2) add vertices to the interior of the disc, and triangulate the resulting graph inside the disc to get K,

(L3) for every facial triangle T, consider an arbitrary graph K_T such that $V(K_T) \cap V(K) = V(T)$,

(L4) take the union $K \cup \bigcup_T K_T$, and delete some edges to obtain H.

We call the vertices of H in K *pinned*. We assume that every vertex on the boundary S^1 is pinned. We draw H on the unit disc so that $H|K$ (H restricted to K) agrees with the given plane drawing of K, and each $H|K_T$ is drawn (though not necessarily a plane drawing) in the interior of the facial triangle T.

Remark 4.14. *Let P be an xy-path of H, where x and y are distinct vertices of H on S^1. Suppose that $v \in V(P)$ is not pinned, that is, $v \in V(P) \cap (V(K_T) - V(T))$ for some T. Then P contains a subpath $P[u, w]$ where u, w are distinct (pinned) vertices of T and $P[u, w]$ is a path contained in K_T that contains v.*

The previous remark allows us to define, for every xy-path (x and y on S^1), two *pinned sides*: take an xy-path P as above. Then P restricted to the pinned vertices induces an xy-path $\widetilde{P}$ in K, and as K is drawn on the plane, it divides the pinned vertices into two sides, where the pinned vertices on P are considered to be on both sides.

Remark 4.15. *Let P be an xy-path of H, where x and y are distinct vertices of H on S^1. If u is a pinned vertex off of P, then all the vertices in $V(T)$, and therefore $H|K_T$, lie in the same side as u if $u \in V(T)$.*

The preceding remark shows that the notion of *sides* to an xy-path P can be generalized to H, and not just K. Note that if $V(T) \subseteq V(P)$ for some K_T, then $H|K_T$ is thought of as being on both sides.

Remark 4.16. *Let P be an xy-path of H, where x and y are distinct vertices of H on S^1. Take vertices u and v of H that lie on different sides of P, and let Q be a uv-path in H. Then Q and P must intersect on a pinned vertex.*

Chapter 5

The Proof

5.1 Part (1)

Recall that L_1, L_2, L_3 are simple and $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ contains an odd cycle. Notice $L_i = P_i$ for $i \in \{1, 2, 3\}$.

Claim 1. *Exactly one of B_1, B_2, B_3 is an st-bond.*

Proof of Claim. Observe that Lemma 4.4 implies that at most one of B_1, B_2, B_3 is an *st*-bond. So the only thing we need to show is that not all of B_1, B_2, B_3 are signatures. Suppose otherwise. Observe that, for all $k \in \{1, 2, 3\}$, $B_k \cap E(H') = B_k \cap P_k$, as $B_k \cap P_j = \emptyset$ for all $j \in \{4, \ldots, m\}$ and $B_k \cap P_i = \{\Omega\}$ for all $i \in \{1, 2, 3\} - \{k\}$. Take distinct $i, j \in \{1, \ldots, m\}$ and take $k \in \{1, 2, 3\} - \{i, j\}$. Then $B_k \cap (P_i \cup P_j - \{\Omega\}) = \emptyset$ and so $P_i \cup P_j - \{\Omega\}$ is bipartite. Since this is true for all such i and j, it follows from Lemma 4.7 that $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ is bipartite, contrary to our assumption. ◇

By symmetry, we may assume that B_3 is an *st*-bond and B_1, B_2 are signatures.

Claim 2. *$P_1 \cup P_2$ contains an odd cycle, and $P_1 \cup P_3$ and $P_2 \cup P_3$ are bipartite.*

Proof of Claim. Take distinct $i, j \in \{1, \ldots, m\}$ such that $\{i, j\} \neq \{1, 2\}$. Take $k \in \{1, 2\} - \{i, j\}$ and notice that $B_k \cap (P_i \cup P_j - \{\Omega\}) = \emptyset$ as B_k is a signature. So $P_i \cup P_j - \{\Omega\}$

is bipartite. In particular, $P_1 \cup P_3$ and $P_2 \cup P_3$ are bipartite. If $P_1 \cup P_2 - \{\Omega\}$ is also bipartite then $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ is bipartite by Lemma 4.7, which is not the case. Hence, $P_1 \cup P_2$ contains an odd cycle. ◇

Claim 3. *Take two distinct vertices $u, v \in V(P_i) \cap V(P_j)$, for some distinct $i, j \in \{1, 2, 3\}$. If $u \prec_{P_i} v$ and $v \prec_{P_j} u$, then $\{i, j\} = \{1, 2\}$.*

Proof of Claim. Suppose that $u \prec_{P_i} v$ and $v \prec_{P_j} u$, but $\{i, j\} \neq \{1, 2\}$. We assume that $i = 1, j = 3$ and the other cases such as $i = 2, j = 3$ or $i = 3, j = 1$ can be treated similarly. Let $L'_1 := P_1[s, u] \cup P_3[u, t]$ and $L'_3 := P_3[s, v] \cup P_1[v, t]$, which are connected $\{s, t\}$-joins. Then $L'_1 \cap B_2 = L'_3 \cap B_2 = \{\Omega\}$, implying that both L'_1 and L'_3 are odd. However, $L'_1 \cup L_2 \cup L'_3 \cup \bigcup_{j=4}^{m} P_j$ contradicts the minimality of $L_1 \cup L_2 \cup L_3 \cup \bigcup_{j=4}^{m} P_j$ by (A1). Hence $\{i, j\} = \{1, 2\}$, finishing the proof. ◇

As a corollary, $P_i \cup P_3$ is acyclic for $i = 1, 2$. Let $F' := P_1 \cup P_2 \cup P_3$. Let (G'', Σ'') be a minor of (G', Σ') and let F'' be a directed graph obtained by orienting edges in a subgraph of G', where (G'', Σ'') and F'' are minimal subject to

(F1) $E(G') - E(G'') \subseteq E(F' \setminus \Omega)$, and $E(F'') \subseteq P_1 \cup P_2 \cup P_3$,

(F2) there exist three odd st-dipaths P''_1, P''_2, P''_3 in F'' that are pairwise disjoint except at Ω,

(F3) $P''_i \cup P''_3$ is bipartite and acyclic for $i = 1, 2$, but $P''_1 \cup P''_2$ contains an odd cycle,

(F4) for any odd st-walk L of F'' there exists a cover B of (G'', Σ'') such that $|B - L| = \tau - 3$, and

(F5) there is no cover for (G'', Σ'') of size $\tau - 2$.

Note that these conditions are satisfied by (G', Σ') and F', so (G'', Σ'') and F'' are well-defined. We may assume that $F'' = P''_1 \cup P''_2 \cup P''_3$. Let B''_i be a minimal cover of (G'', Σ'') such that $|B''_i - P''_i| = \tau - 3$, whose existence is guaranteed by (F4). Since $P''_1 \cup P''_2$ contains an odd cycle, it follows that B''_3 is an st-bond, and so by Lemma 4.4, B''_1, B''_2 are signatures. For the sake of notational ease, reset $P_i := P''_i$ and $B_i := B''_i$ for all $1 \leq i \leq 3$.

Choose minimal vertex subsets $U_{12} \in V(G'') - \{s, t\}$ and $U_3 \subseteq V(G'') - \{t\}$ so that $B_1 \triangle B_2 = \delta(U_{12})$ and $B_3 = \delta(U_3)$. Since $P_1 \cup P_2$ contains an odd cycle, it follows that

$VP_1 \cap VP_2 \cap U_{12} \neq \emptyset$.

Claim 4. *P_3 is internally vertex-disjoint from P_1 and P_2.*

Proof of Claim. Suppose otherwise. Let $v_1 \neq s', s$ and $v_2 \neq t$ be, respectively, the closest vertices to s' and t on P_3 that lie on $P_1 \cup P_2$. Observe that $v_1 \preceq_{P_3} v_2$ and $v_1, v_2 \notin U_{12}$. We may assume that $v_2 \in VP_2$. Suppose that $v_1 \in VP_k$ for some $k \in \{1, 2\}$.

We claim that there is an odd cycle in $P_1 \cup P_2$ that avoids either $P_k[s', v_1]$ or $P_2[v_2, t]$. Suppose for a contradiction that this is not the case. Let $y \in VP_1 \cap VP_2 \cap U_{12}$. Since each of $P_1[s', y] \cup P_2[s', y]$ and $P_1[y, t] \cup P_2[y, t]$ contains an odd cycle, and since $P_2 \cup P_3$ is acyclic, it follows that $k \neq 2$, and so $k = 1$. For every odd cycle intersects $P_1[s', v_1]$ and $P_2[v_2, t]$, it follows that $y \in VP_1[s', v_1]$ and $y \in VP_2[v_2, t]$.

Let $C'_1 := P_1[y, v_1] \cup P_3[v_1, v_2] \cup P_2[v_2, y]$, $P'_1 := P_1[s, y] \cup P_2[y, t]$, $L'_2 := P_2[s, v_2] \cup P_3[v_2, t]$ and $L'_3 := P_3[s, v_1] \cup P_1[v_1, t]$. Let $L'_1 := C'_1 \cup P'_1$, which is a non-simple odd $\{s, t\}$-join. Note further that L'_2 and L'_3 are also odd $\{s, t\}$-joins. By Lemma 4.5 therefore, at least two of L'_1, L'_2, L'_3 are non-simple, and so by Part (8), (G'', Σ'') contains an F_7 minor, implying that (G, Σ) contains an F_7 minor, which is a contradiction.

Hence, there is an odd cycle in $P_1 \cup P_2$ that avoids either $P_k[s', v_1]$ or $P_2[v_2, t]$. Suppose w.l.o.g. that there is an odd cycle in $P_1 \cup P_2$ that avoids $P_2[v_2, t]$.

Observe that if there is a cover B of (G'', Σ'') such that $|B - P_2[v_2, t] - \{\Omega\}| = \tau - 3$, then B must be an st-cut, since there is an odd cycle of F'' avoiding $P_2[v_2, t]$ (and Ω), which in turn is in conflict with Lemma 4.4 as B_3 is an st-cut. Therefore, there is no cover B of (G'', Σ'') such that $|B - P_2[v_2, t] - \{\Omega\}| = \tau - 3$.

Let $(G''', \Sigma''') := (G'', \Sigma'') \setminus P_2[v_2, t]/P_3[v_2, t]$ and $F''' := F'' \setminus P_2[v_2, t]/P_3[v_2, t]$. It is easily seen that (F1),(F2) and (F4) hold for (G''', Σ''') and F'''. We just showed that (F5) holds as well. Moreover, since there is an odd cycle in $P_1 \cup P_2$ avoiding $P_2[v_2, t]$, it follows that (F3) holds as well for (G''', Σ''') and F''', contradicting the minimality of (G'', Σ'') and F''.

Thus, P_3 is internally vertex-disjoint from P_1 and P_2, as claimed. ◇

Claim 5. *If there is a directed circuit C in $P_1 \cup P_2$ then C is even.*

Proof of Claim. Suppose otherwise. Decompose $P_1 \cup P_2 \setminus C$ into the union of two $\{s, t\}$-joins P'_1 and L'_2. We may assume that P'_1 is even and L'_2 is odd. Let $L'_1 := C \cup P'_1$, which is a non-simple odd $\{s,t\}$-join. Hence, applying Lemma 4.5, followed by the argument of Part (8), we get that (G'',Σ''), and so (G,Σ), has an F_7 minor, a contradiction. $\diamond$

By Lemma 4.3, there exists a path R in $G''[U_3]\setminus B_1$ between s and VP_3. After contracting all the directed even circuits in $P_1\cup P_2$, it is easily seen that $P_1\cup P_2\cup P_3\cup R$ has an F_7 minor. But then (G,Σ) has an F_7 minor, a contradiction. As a result, Part (1) is not feasible.

5.2 Three lemmas for Parts (2)-(4)

In this section, we provide three lemmas that will be needed for Parts (2)-(4). Recall that for these three parts, L_1,L_2,L_3 are simple and $(H'\setminus\Omega,\Sigma'\cap E(H'\setminus\Omega))$ is bipartite.

Lemma 5.1. *An st-path P in $(H'\setminus\Omega,\Sigma'\cap E(H'\setminus\Omega))$ is odd if and only if $\Omega\in P$.*

Proof. Let P be an st-path of H'. Then $P\triangle P_1$ is an even cycle if and only if P is odd. However, as $(H'\setminus\Omega,\Sigma'\cap E(H'\setminus\Omega))$ is bipartite, it follows that $P\triangle P_1$ is even if and only if $\Omega\notin P\triangle P_1$. So P is odd if and only if $\Omega\in P$, as claimed. $\square$

Lemma 5.2. *H' is acyclic.*

Proof. Suppose otherwise, and let C be a directed circuit in H'. Clearly $\Omega\notin C$, and so one can find m st-dipaths $P'_1,P'_2,P'_3,\dots,P'_m$ in $H'\setminus C$ such that $\Omega\in P'_1\cap P'_2\cap P'_3$, $\Omega\notin P'_4\cup\cdots P'_m$ and $(P'_j-\{\Omega\}:1\le j\le m)$ are pairwise disjoint. By Lemma 5.1 it follows that P'_1,P'_2,P'_3 are odd and $P'_4,\dots,P'_m$ are even. But then $(P'_1,P'_2,P'_3,P'_4\cup C_4,\dots,P'_m\cup C_m,L_{m+1},\dots,L_\tau)$ is an Ω-packing, contradicting the minimality of $(L_1,L_2,L_3,\dots,L_\tau)$ by (A1). $\square$

Lemma 5.3. *Every odd st-dipath P in $(H',\Sigma'\cap E(H'))$ has a mate, i.e. there exists a cover B of (G',Σ') such that $|B-P|=\tau-3$.*

Proof. Since H' is acyclic, after rerouting $P_1,P_2,P_3,\dots,P_m$ in H', if necessary, we may assume that $P=P_1$, and so by (M3) P has a mate. $\square$

5.3 Part (2)

Recall that L_1, L_2, L_3 are simple odd st-walks, $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ is bipartite, and the following holds:

> (X1) *no odd st-dipath of $(H', \Sigma' \cap E(H'))$ has a mate which is an st-bond.*

We will use lemmas from §5.2. Observe that (X1), together with Lemma 5.3, implies that every odd st-path of $(H', \Sigma' \cap E(H'))$ has only, and at least one, signature mates.

Claim 1. *$m \geq 4$.*

Proof of Claim. (X1) implies in particular that B_1, B_2, B_3 are all signatures, and so by the Mate Lemma and Lemma 4.1 it follows that $m \geq 4$. ◇

Claim 2. *There is an odd circuit C in H' that avoids t.*

Proof of Claim. Suppose otherwise. Then $VP_4 \cap (VP_1 \cup VP_2 \cup VP_3) = \{s, t\}$, since otherwise $P_i[s, v] \cup P_4[s, v]$ is an odd cycle in H' that avoids t where $v \in VP_4 \cap VP_i - \{s, t\}$ for some $1 \leq i \leq 3$. Now contract the even st-dipath to identify s and t, then apply the Reduction Lemma, followed by the $\widetilde{K_5}$ Lemma, to obtain a $\widetilde{K_5}$ minor. This implies that (G', Σ'), and so (G, Σ), contains a $\widetilde{K_5}$ minor, a contradiction. ◇

For each $1 \leq j \leq m$, let $v_j \neq t$ be the closest vertex to t on P_j that also lies on P_i, for some $i \in \{1, \ldots, m\} - \{j\}$. By the Intersection Lemma there exists an index $i \in \{1, \ldots, m\}$ such that whenever $v_i \in VP_j$ for some $j \in \{1, \ldots, m\}$ then $v_i = v_j$. Let I be the set of all indices j such that $v_i = v_j$. Note that $i \in I$ and $|I| \geq 2$. Recall that the end-vertices of Ω are s and s'.

Claim 3. *There exists a directed path in H' from s' to v_i.*

Proof of Claim. Suppose not. Let $(G'', \Sigma'') := (G', \Sigma') / \cup (P_j[v_j, t] : j \in I)$ and $H'' := H' / \cup (P_j[v_j, t] : j \in I)$. It is clear that (M1), (M2) and (M4) still hold for (G'', Σ'') and H''. Moreover, by our assumption, (M3) also holds as there is no odd st-dipath in H' that uses an edge of $\cup(P_j[v_j, t] : j \in I)$. However, this contradicts the minimality of (G', Σ')

and H'. ◇

So by rerouting $P_1, P_2, \ldots, P_m$ in H', if necessary, we may assume that $i = 1$.

Claim 4. *For each $j \in I$, there exists a cover B of (G', Σ') such that $|B - P_j[v_j, t] - \{\Omega\}| = \tau - 3$.*

Proof of Claim. Suppose otherwise. Then there exists $j \in I$ such that $|B - P_j[v_j, t] - \{\Omega\}| > \tau - 3$, for all covers B of (G', Σ'). Let $(G'', \Sigma'') := (G', \Sigma') \backslash P_j[v_j, t] / \cup (P_k[v_k, t] : k \in I, k \neq j)$ and $H'' := H' \setminus P_j[v_j, t] / \cup (P_k[v_k, t] : k \in I, k \neq j)$. It is clear that (M1) and (M2) still hold for (G'', Σ'') and H''. Moreover, by our hypothesis, (M4) also holds. Moreover, (M3) also holds true, for if P is an odd st-dipath of H'' then $P \cup P_j[v_j, t]$ contains an odd st-dipath of H'. However, this contradicts the minimality of (G', Σ') and H'. ◇

Claim 5. *There do not exist vertex-disjoint paths P and Q in H', where P is between s and t and Q is between s' and v_1.*

Proof of Claim. Suppose, for a contradiction, there exist vertex-disjoint paths P and Q in H', where P is between s and t and Q is between s' and v_1. Take $j \in I \setminus \{1\}$. Note that $(P \cup Q) \cap (P_1[v_1, t] \cup P_j[v_j, t]) = \emptyset$. By Claim 3 we can choose minimal covers B_1 and B_j such that $|B_1 - P_1[v_1, t] - \{\Omega\}| = \tau - 3 = |B_j - P_j[v_j, t] - \{\Omega\}|$. By (X1) both of B_1 and B_j are signatures. So Remark 3.3 implies that $(B_1 - P_1[v_1, t] - \{\Omega\}) \cap EH' = \emptyset = (B_j - P_j[v_j, t] - \{\Omega\}) \cap EH'$. Now choose a minimal $U_{1j} \subseteq VG' \setminus \{s, t\}$ so that $\delta(U_{1j}) = B_1 \triangle B_j$. Then by Lemma 4.2, there exists a path R in $G'[U_{1j}]$ between $VP_1[v_1, t]$ and $VP_j[v_j, t]$ that is disjoint from B_1. Note that $VR \cap (VP \cup VQ \cup VC) = \emptyset$. It is now easily seen that $C \cup P \cup Q \cup P_1[v_1, t] \cup P_j[v_j, t] \cup R$ has an F_7 minor. This implies that (G', Σ'), and hence (G, Σ), has an F_7 minor, a contradiction. ◇

Claim 5, together with the Linkage Lemma, implies that H' can be obtained as follows:

(L1) place a circuit C on the boundary S^1 of the unit disc, and the circuit contains the vertices s, s', t, v_1 in this cyclic order,

(L2) add vertices to the interior of the disc, and triangulate the resulting graph inside the disc to get K, (L3) for every facial triangle T, consider an arbitrary graph K_T such that $V(K_T) \cap V(K) = V(T)$,

(L4) take the union $K \cup \bigcup_T K_T$, and delete some edges to get H'.

Consider the st-path P_1 in the drawing. Observe that s and t lie on different sides of the path $P_1[s', v_1]$. Consider the set Γ_{P_1} of pinned vertices that lie strictly inside the side of $P_1[s', v_1]$ that contains s. As H' is acyclic, we may assume that the set Γ_{P_1} is minimal over all possible odd st-dipaths P_1 in H'.

Note that every P_j, $j \in \{4, \dots, m\}$, is an st-path. So for every such j, there exists a pinned vertex u_j that lies on both P_j and $P_1[s', v_1]$; we may assume that u_j is the closest such vertex to t on P_j. Note that this implies that $u_i = v_i$ for all $i \in I$.

For each $j \in \{4, \dots, m\}$ let $R_j := P_j[u_j, t]$ and $Q_j := P_1[s, u_j] \cup R_j$. For $j \in [3] - I$ let $R_j := P_j[u_j = v_j, t]$ and $Q_j := P_1[s, u_j = v_j] \cup R_j$, and for $j \in \{1, 2, 3\} - I$ let $R_j := P_j[s', t]$ and $Q_j := P_j$. By the Mate Lemma and (X1), we get that there exists $k \in \{1, \dots, m\}$ such that $|B - R_k - \{\Omega\}| > \tau - 3$, for all covers B of (G', Σ'). Notice that $k \notin I$ due to Claim 4.

Claim 6. *Take an odd st-dipath P in H'. Suppose that $VP \cap VP_1[u_k, t] \neq \{t\}$ and let u be the closest vertex to s on P that lies on $P_1[u_k, t]$. Then there exists a vertex $v \in VP[s, u]$ that lies on R_k.*

Proof of Claim. Suppose not. Then in particular $u \notin VR_k = VP_k[u_k, t]$. As H' is acyclic it follows that $u \notin VP_k[s, u_k]$, and so $u \notin VP_k$. It is now easily seen that u lies strictly inside the side of P_k which contains v_1. As P is odd it follows that $s' \in VP$. Consider the subpath $P[s', u]$ of P. Since s' lies on a different side of P_k than that of u, Remark 4.16 implies that $P[s', u]$ and P_k share a pinned vertex w, say, and suppose that w is the closest such vertex to u. By our hypothesis, $w \notin VR_k = VP_k[u_k, t]$. Hence, $w \in VP[s, u_k] - \{u_k\}$. Let w' be the closest vertex to w in $P[s', w]$ that lies on $P_1[s', u_k]$. Then $P_1' := P_1[s, w'] \cup P[w', w] \cup P_k[w, u_k] \cup P_1[u_k, t]$ contradicts the minimality of P_1 as $\Gamma_{P_1'} \subseteq \Gamma_{P_1} - \{w\}$. Hence, there exists a vertex $v \in VP[s, u]$ that lies on R_k, proving the claim. ◇

Now let $(G'', \Sigma'') := (G', \Sigma') \setminus R_k / P_1[u_k, t]$ and let H'' be obtained from $H' \setminus R_k / P_1[u_k, t]$ after deleting all the outgoing arcs at t. We claim that (G'', Σ'') and H'' satisfy (M1)-(M4), therefore contradicting the minimality of (G', Σ') and H'. It is clear that (M1) holds and that H'' is acyclic. The choice of R_k implies that (M4) holds as well. To show

(M3) holds, let Q be an odd st-dipath of H''. Let P be an st-dipath of H' contained in $Q \cup P_1[u_k, t]$. If $P \cap P_1[u_k, t] = \emptyset$ then clearly (M3) holds. Otherwise, define u as in the preceding claim, and find v as found above. Choose B' to a cover of (G', Σ') such that $|B' - P[s, v] - R_k[v, t]| = \tau - 3$. Then $B' - R_k$ is a cover of (G'', Σ'') for which $|(B' - R_k) - Q| = \tau - 3$. This proves (M3) holds. Using the preceding claim again shows that (M2) also holds. However, this contradicts the minimality of (G', Σ') and H'. Therefore, Part (2) is not feasible.

5.4 Setup and a lemma for Parts (3) and (4)

In this section, we provide the setup needed to initiate Parts (3) and (4), as well as a lemma that will be frequently referenced. For these two parts, L_1, L_2, L_3 are simple, $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ is bipartite, and (X1) does not hold.

Since (X1) does not hold, there exists an odd st-dipath Q in $(H', \Sigma' \cap E(H'))$ and an st-bond S such that $|S - Q| = \tau - 3$. Recall that a cover S is said to be an internally minimal mate for Q if $|S - Q| = \tau - 3$, and there is no other cover S' such that $|S' - Q| = \tau - 3$ and $S' \cap Q \subsetneq S \cap Q$. A cover S is said to be an *internally minimal st-bond mate* for Q if S is an st-bond and $|S - Q| = \tau - 3$, and there is no other st-bond S' such that $|S' - Q| = \tau - 3$ and $S' \cap Q \subsetneq S \cap Q$. The closest non-$\Omega$ edge on Q to Ω that is in $S \cap Q$ is what we call the *head of* (Q, S).

Let e and f be two distinct edges of H'. We say e *precedes* f if there exists an st-dipath in H' containing both e and f but, on the path, e is closer to s than f. Observe that since H' is acyclic, there cannot exist two edges mutually preceding one another, and therefore, there cannot be an infinite sequence of edges in H', where each edge of the sequence is preceded by the next edge. This observation allows us to pick an odd st-dipath P in H' that satisfies the following:

(B1) there exists an internally minimal st-bond mate B for P, and

(B2) for any odd st-path P' in H' and any internally minimal st-bond mate B' for P' (if any), the head of (P', B') does *not* precede the head of (P, B).

We may assume by rerouting $P_1, P_2, P_3, \ldots, P_m$ in H', if necessary, that $P = P_1$. Reset $B_1 := B$ and choose a minimal vertex subset $U_1 \subseteq V(G') - \{t\}$ such that $B_1 = \delta(U_1)$.

Let $u \neq s$ and w be, respectively, the closest and furthest vertices on P_1 from s that lie inside U_1. Let u' (resp. w') be the neighboring vertex of u (resp. w) on P_1 that lies outside U_1. Let $C_1 := P_1[s,u]$ and $Q_1 := P_1[w,t]$ followed by $Q_j := P_j, \forall 2 \leq j \leq m$ and $F' := (P_1 \cap G'[U_1]) \cup C_1 \cup \bigcup_{j=1}^{m} Q_j$. Let $\widetilde{R} := P_1 \cap G'[U_1]$.

Lemma 5.4. *Let P be an odd st-dipath in $(F', \Sigma' \cap E(F'))$ such that $V(P) \cap U_1 = \{s\}$, and let B be an internally minimal mate for P. Then B is not an st-bond.*

Proof. Suppose otherwise. Let $R := P_1[u', w']$. Observe that $P \cap P_1[u,w] = \emptyset$ as $P \subseteq E(F')$ and $E(F') \cap P_1[u,w] = \emptyset$. Hence, since $V(P) \cap U_1 = \{s\}$, it follows that $P \cap R = \emptyset$.

Let $P'_1 := P$, and decompose $H' \setminus (P - \{\Omega\})$ into st-dipaths $P'_2, P'_3, \ldots, P'_m$, keeping R in one of the st-dipaths P'_j, where $\Omega \in P'_1 \cap P'_2 \cap P'_3$, $\Omega \notin P'_4 \cup \cdots \cup P'_m$ and $(P'_i - \{\Omega\} : 1 \leq i \leq m)$ are pairwise disjoint. Set $B'_1 := B$ and $B'_j := B_1$.

If $j \in \{2,3\}$ then st-cuts B'_1 and B'_j contradict Lemma 4.4. Otherwise, we may assume that $j = 4$ and there exists a vertex $v \in V(P'_4) \cap V(P'_1)$ between s' and u' (so v may be s' or u'). We may assume such a v exists since $R \subseteq P_1$. Notice that $P'_1[s,v] \cap B'_1 = \{\Omega\}$, since otherwise the head of (P'_1, B'_1) would precede the head $\{u', u\}$ of (P_1, B_1), which is not possible by (B2). Hence, $|B'_1 - \{\Omega\} - P'_1[v,t]| = |B'_1 - P'_1| = \tau - 3$.

We claim that $|B'_4 - \{\Omega\} - P'_4[v,t]| = \tau - 3$. Observe that

$$|B'_4 - \{\Omega\} - P'_4[v,t]| \geq |B'_4 - (P'_1[s,v] \cup P'_4[v,t])| \geq \tau - 3$$

as $P'_4[s,v] \cup P'_1[v,t], P'_5, P'_6, \ldots, P'_m, L_{m+1}, \ldots, L_\tau$ are disjoint edge-subsets of $E(G') - (P'_1[s,v] \cup P'_4[v,t])$ and each contain an st-dipath. However,

$$|B'_4 - \{\Omega\} - P'_4[v,t]| \leq |B_1 - \{\Omega\} - R| = \tau - 3,$$

and so $|B'_4 - \{\Omega\} - P'_4[v,t]| = \tau - 3$, as claimed. But this is a contradiction to Lemma 4.6. Hence, B is not an st-bond. □

Consider the following statement:

(X2) *for every even st-dipath P in $(F', \Sigma' \cap E(F'))$, $V(P) \cap V(C_1) \subseteq U_1$.*

In Part (3) we assume (X2) is true, and in Part (4) we assume (X2) does not hold.

5.5 Part (3)

The setup for this part is provided in §5.4. Recall that L_1, L_2, L_3 are simple, $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ is bipartite, (X1) does not hold, and (X2) holds.

Let (G'', Σ'') be a minor of (G', Σ') and let F'' be a directed graph obtained by orienting edges in a subgraph of G', where (G'', Σ'') and F'' are minimal subject to

(F1) $E(G') - E(G'') \subseteq E(F''[V - U_1])$, and $E(F'') \subseteq E(F')$,

(F2) for each even st-dipath P in F'', $V(P) \cap V(C_1) \subseteq U_1$,

(F3) F''_{uw} is acyclic, where F''_{uw} is obtained from F'' after identifying the two vertices u, w, and there exist m st-dipaths in F''_{uw} that are disjoint except possibly at Ω, exactly three of which contain Ω and exactly one of these contains the identified vertex uw,

(F4) for any odd st-dipath P of F'' such that $V(P) \cap U_1 = \{s\}$, there exists a signature B of (G'', Σ'') such that $|B - P| = \tau - 3$, and

(F5) there is no cover of (G'', Σ'') of size $\tau - 2$.

Note that these conditions are satisfied by (G', Σ') and F', so (G'', Σ'') and F'' are well-defined. By identifying a vertex of each component with s, if necessary, we may assume that G'' is connected. Now let $\left(Q''_j\right)_{j=1}^m$ be m st-dipaths in F''_{uw} that are pairwise disjoint except possibly at Ω, $\Omega \in Q''_j$ if and only if $j \in \{1, 2, 3\}$, and Q''_1 uses the identified vertex uw in F''_{uw}. Note that $F''_{uw} = \widetilde{R} \cup \bigcup_{j=1}^m Q''_j$. For the sake of notational ease, let $Q_j := Q''_j$ for all $2 \le j \le m$, and define $Q_1 := Q''_1[uw, t]$ and $C_1 := Q''_1[s, uw]$. Observe that $Q_2, \ldots, Q_m$ are st-dipaths in F'', and that C_1 and Q_1 have endvertices s, u and w, t, respectively. Observe that P is an odd st-dipath in F'' if and only if $\Omega \in P$, since $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ is bipartite.

For each Q_j other than Q_1, let $v_j \neq t$ be the closest vertex to t on Q_j that also lies on another Q_i, $i \in \{1, \ldots, m\} - \{j\}$. Let $v_1 \neq t$ be the closest vertex to t in $VQ_1 \cup \{s\}$ that also lies on another Q_i, $i \in \{2, \ldots, m\}$. Then by the Intersection Lemma there exists $i \in \{1, \ldots, m\}$ such that whenever $v_i \in VQ_j$ then $v_i = v_j$. Let I be the set of all indices $j \in \{1, \ldots, m\}$ such that $v_j = v_i$. Note that $i \in I$ and $|I| \geq 2$. We may assume $v_i \notin U_1$ (otherwise, remove all paths Q_j, $j \in I$ temporarily and reapply the Intersection Lemma). This implies that $VQ_j[v_j, t] \cap U_1 = \emptyset$ for all $j \in I$.

In Part (3.1) we assume $VC_1 \cap VQ_j[v_j, t] = \emptyset$, for all $j \in I$. In Part (3.2) we assume $VC_1 \cap VQ_j[v_j, t] \neq \emptyset$, for some $j \in I$.

5.5.1 Part (3.1): $VC_1 \cap VQ_j[v_j, t] = \emptyset$ for all $j \in I$

Claim 1. *$Q_i[v_i, t]$ is contained in an odd st-dipath of $(F'', \Sigma'' \cap E(F''))$ that intersects U_1 at only s.*

Proof of Claim. Suppose otherwise. Then let $(G''', \Sigma''') := (G'', \Sigma'')/\cup (Q_j[v_j, t] : j \in I)$ and $F''' := F''/\cup(Q_j[v_j, t] : j \in I)$. Clearly, (F1), (F3) and (F5) still hold for (G''', Σ''') and F'''. Since $VQ_j[v_j, t] \cap VC_1 = \emptyset$ for all $j \in I$, it follows that (F2) also holds. Furthermore, our assumption implies that (F4) also holds, a contradiction to the minimality of (G'', Σ'') and F''. ◇

This claim allows us to assume that $i = 2$.

Claim 2. *For each $j \in I$ there exists a minimal cover B_j such that $|B_j - Q_j[v_j, t] - \{\Omega\}| = \tau - 3$.*

Proof of Claim. Suppose otherwise. Then there is no cover B such that $|B - Q_i[v_i, t] - \{\Omega\}| = \tau - 3$, for some $i \in I$. Then let $(G''', \Sigma''') := (G'', \Sigma'') \setminus Q_i[v_i, t]/\cup (Q_j[v_j, t] : j \in I, j \neq i)$ and $F''' := F'' \setminus Q_i[v_i, t]/\cup (Q_j[v_j, t] : j \in I, j \neq i)$. Clearly, (F1), (F3) and (F4) still hold for (G'', Σ'') and H''. Since $VQ_j[v_j, t] \cap VC_1 = \emptyset$ for all $j \in I$, it follows that (F2) also holds. Our assumption implies that (F5) also holds, a contradiction to the minimality of (G'', Σ') and F''. ◇

We may assume that each B_j, $j \in I$ is an internally minimal mate of $Q_j[v_j, t]$. Observe that each B_j is a signature, by Lemma 5.4. Pick $k \in I - \{2\}$, and choose a minimal $U \subseteq V(G'') - \{s, t\}$ such that $\delta(U) = B_2 \triangle B_k$.

Claim 3. *Suppose there exists a path in $G''[U] \setminus B_k$ between VQ_2 and VQ_k for which there is a vertex-disjoint path in $G''[U_1] \setminus B_k$ between s and every component of $\widetilde{R}$. Then (G'', Σ'') contains an F_7 minor.*

Proof of Claim. To prove the claim, we first show the following.

Subclaim 3.1. *There exist two vertex-disjoint paths P and Q in F'', where Q is between s' and v_2, and P is between t and either of s, w.*

Proof of Subclaim 3.1. We will treat U_1 as a vertex, and in order to prove the subclaim, it suffices to find two vertex-disjoint paths P and Q in F'', where Q is between s' and v_2, and P is between t and U_1. Suppose for a contradiction that this is not possible. Then the Linkage Lemma implies that F'' can be obtained as follows:

- (L1) place a circuit C on the boundary S^1 of the unit disc, and the circuit contains the vertices U_1, s', t, v_2 in this cyclic order,
- (L2) add vertices to the interior of the disc, and triangulate the resulting graph inside the disc to get K,
- (L3) for every facial triangle T, consider an arbitrary graph K_T such that $V(K_T) \cap V(K) = V(T)$,
- (L4) take the union $K \cup \bigcup_T K_T$, and delete some edges to get F''.

Consider the st-path Q_2 in the drawing. Observe that s (which is now merged with U_1) and t lie on different sides of the path $Q_2[s', v_2]$. Consider the set Γ_{Q_2} of pinned vertices that lie strictly inside the side of $Q_2[s', v_2]$ that contains s. As F'' is acyclic, we may assume that the set Γ_{Q_2} is minimal over all possible odd st-dipaths Q_2 in F''.

Note that every Q_j, $j \in \{1, \dots, m\} - \{2, 3\}$, is an st-path. Hence, for every $j \in \{1, \dots, m\} - \{2, 3\}$, there exists a pinned vertex u_j that lies on both Q_j and $Q_2[s', v_2]$; we may assume that u_j is the closest such vertex to t on Q_j. Note that this implies that $u_i = v_i$ for all $i \in I$.

For each $j \in \{1, \dots, m\} - \{2, 3\}$ let $R_j := Q_j[u_j, t]$ and $S_j := Q_2[s, u_j] \cup R_j$. For $j \in \{2, 3\} \cap I$ let $R_j := Q_j[u_j = v_j, t]$ and $S_j := Q_2[s, u_j = v_j] \cup R_j$, and for $j \in \{2, 3\} \setminus I$ let $R_j := Q_j[s', t]$ and $S_j := Q_j$. By the Mate Lemma and Lemma 5.4, we get that there exists an R_k such that $|B - R_k - \{\Omega\}| > \tau - 3$, for all covers B of (G'', Σ''). Notice that $k \notin I$ and that $k \notin \{2, 3\}$.

Subclaim 3.1.1. *Take an odd st-dipath P in $(F'', \Sigma'' \cap E(F''))$ that has intersection $\{s\}$ with U_1. Suppose that $VP \cap VQ_2[u_k, t] \neq \{t\}$ and let u be the closest vertex to s on P that lies on $Q_2[u_k, t]$. Then there exists a vertex $v \in VP[s, u]$ that lies on R_k.*

Proof of Subclaim 3.1.1. Suppose not. Then in particular $u \notin VR_k = VQ_k[u_k, t]$. As F'' is acyclic it follows that $u \notin VQ_k[s, u_k]$, and so $u \notin VQ_k$. It is now easily seen that u

lies strictly inside the side of Q_k which contains v_2. As P is odd it follows that $s' \in VP$. Consider the subpath $P[s', u]$ of P. Since s' lies on a different side of Q_k than that of u, Remark 4.16 implies that $P[s', u]$ and Q_k share a pinned vertex w, say, and suppose that w is the closest such vertex to u. By our hypothesis, $w \notin VR_k = VQ_k[u_k, t]$. Hence, $w \in VP[s, u_k] - \{u_k\}$. Let w' be the closest vertex to w in $P[s', w]$ that lies on $Q_2[s', u_k]$. Then $Q_2' := Q_2[s, w'] \cup P[w', w] \cup Q_k[w, u_k] \cup Q_2[u_k, t]$ contradicts the minimality of Q_2 as $\Gamma_{Q_2'} \subseteq \Gamma_{Q_2} - \{w\}$. Hence, there exists a vertex $v \in VP[s, u]$ that lies on R_k. This finishes the proof of Subclaim 3.1.1.

Now let $(G''', \Sigma''') := (G'', \Sigma'') \backslash R_k / Q_2[u_k, t]$ and let F''' be obtained from $F'' \backslash R_k / Q_2[u_k, t]$ after deleting all the outgoing arcs at t. We claim that (G''', Σ''') and F''' satisfy (F1)-(F5), therefore contradicting the minimality of (G'', Σ'') and F''. It is clear that (F1) and (F3) hold. The choice of R_k implies that (F5) holds as well. To prove (F2) holds, it suffices to show that $VC_1 \cap VQ_2[u_k, t] = \emptyset$. If not, then by the preceding claim, we get that $VC_1 \cap VR_k \neq \emptyset$, a contradiction as R_k belongs to the even st-path Q_k and $VQ_k \cap VC_1 \subseteq U_1$ by (F2). Hence, $VC_1 \cap VQ_2[u_k, t] = \emptyset$ and so (F2) still holds for (G''', Σ''') and F'''. Lastly, to show (F4) holds, let Q be an odd st-dipath of F'''. Let P be an st-dipath of F'' contained in $Q \cup Q_2[u_k, t]$. If $P \cap Q_2[u_k, t] = \emptyset$ then clearly (F4) holds. Otherwise, define u as in the preceding claim, and find v as found above. Choose B' to a cover of (G'', Σ'') such that $|B' - P[s, v] - R_k[v, t]| = \tau - 3$. Then $B' - R_k$ is a cover of (G''', Σ''') for which $|(B' - R_k) - Q| = \tau - 3$. This proves (F4) holds. However, this contradicts the minimality of (G'', Σ'') and F''.

End of Proof of Subclaim 3.1

We are now ready to prove Claim 3. Recall that $k \in I - \{2\}$, that the both of B_2 and B_k are signatures, and $\delta(U) = B_2 \triangle B_k$. Let R be a shortest path in $G''[U] \setminus B_k$ between VQ_2 and VQ_j, as given in the statement of Claim 3. By our assumption, there exist paths R_1 and R_2 in $G''[U_1] \setminus B_k$ between s and u, and s and w, respectively, that are vertex-disjoint R. (Note that $\widetilde{R} \cap (B_2 \cup B_k) = \emptyset$.) It is now easily seen that $R_1 \cup R_2 \cup P \cup Q \cup C_1 \cup Q_2[v_2, t] \cup Q_j[v_j, t] \cup R$ has an F_7 minor. This concludes the proof of Claim 3. ◇

However, (G, Σ) does not have an F_7 minor, and so the assumption of Claim 3 cannot be true. Hence, in particular,

> $(*)$ for a connected component K of $\widetilde{R}$, there is no path in $G[U_1] \setminus \bigcup_{j \in I} B_j$ between s and K,

and also that

> ($**$) $U_1 \cap U \neq \emptyset$, and there is no path in $G''[U - U_1] \setminus B_k$ between VQ_2 and VQ_k.

Observe that the Shore Lemma, together with ($*$), implies that $m \geq 4$.

Claim 4. *There exist vertex disjoint paths P and Q in F'', where P is between s and t and Q is between s' and v_2.*

Proof of Claim. Suppose for a contradiction that this is not possible. Then the Linkage Lemma implies that F'' can be obtained as follows:

(L1) place a circuit C on the boundary S^1 of the unit disc, and the circuit contains the vertices s, s', t, v_2 in this cyclic order,

(L2) add vertices to the interior of the disc, and triangulate the resulting graph inside the disc to get K,

(L3) for every facial triangle T, consider an arbitrary graph K_T such that $V(K_T) \cap V(K) = V(T)$,

(L4) take the union $K \cup \bigcup_T K_T$, and delete some edges to get F''.

Consider the st-path Q_2 in the drawing. Observe that s and t lie on different sides of the path $Q_2[s', v_2]$. Consider the set Γ_{Q_2} of pinned vertices that lie strictly inside the side of $Q_2[s', v_2]$ that contains s. As F'' is acyclic, we may assume that the set Γ_{Q_2} is minimal over all possible odd st-dipaths Q_2 in F'' that have intersection $\{s\}$ with U_1.

Note that every Q_j, $j \in \{4, \ldots, m\}$, is an st-path. So for every $j \in \{4, \ldots, m\}$, there exists a pinned vertex u_j that lies on both Q_j and $Q_2[s', v_2]$; we may assume that u_j is the closest such vertex to t on Q_j. Note that this implies that $u_i = v_i$ for all $i \in I$. For each $j \in \{4, \ldots, m\}$ let $R_j := Q_j[u_j, t]$ and $S_j := Q_2[s, u_j] \cup R_j$. For $j \in \{2, 3\} \cap I$ let $R_j := Q_j[u_j = v_j, t]$ and $S_j := Q_2[s, u_j = v_j] \cup R_j$, and for $j \in \{2, 3\} - I$ let $R_j := Q_j[s', t]$ and $S_j := Q_j$.

By the Shore Lemma, along with ($*$) and Lemma 5.4, we get that there exists an R_k such that $|B - R_k - \{\Omega\}| > \tau - 3$, for all covers B of (G'', Σ''). Notice that $k \notin I$ and that

$k \notin \{2, 3\}$.

Subclaim 4.1. *Take an odd st-dipath* P *in* $(F'', \Sigma'' \cap E(F''))$ *that has intersection* $\{s\}$ *with* U_1. *Suppose that* $VP \cap VQ_2[u_k, t] \neq \{t\}$ *and let* u *be the closest vertex to* s *on* P *that lies on* $Q_2[u_k, t]$. *Then there exists a vertex* $v \in VP[s, u]$ *that lies on* R_k.

Proof of Subclaim 4.1. Suppose otherwise. Then in particular $u \notin VR_k = VQ_k[u_k, t]$. As F'' is acyclic it follows that $u \notin VQ_k[s, u_k]$, and so $u \notin VQ_k$. It is now easily seen that u lies strictly inside the side of Q_k which contains v_2. As P is odd it follows that $s' \in VP$. Consider the subpath $P[s', u]$ of P. Since s' lies on a different side of Q_k than that of u, Remark 4.16 implies that $P[s', u]$ and Q_k share a pinned vertex w, say, and suppose that w is the closest such vertex to u. By our hypothesis, $w \notin VR_k = VQ_k[u_k, t]$. Hence, $w \in VP[s, u_k] - \{u_k\}$. Let w' be the closest vertex to w in $P[s', w]$ that lies on $Q_2[s', u_k]$. Then $Q'_2 := Q_2[s, w'] \cup P[w', w] \cup Q_k[w, u_k] \cup Q_2[u_k, t]$ contradicts the minimality of Q_2 as $\Gamma_{Q'_2} \subseteq \Gamma_{Q_2} - \{w\}$. Hence, there exists a vertex $v \in VP[s, u]$ that lies on R_k. *End of Proof of Subclaim 4.1*

Now let $(G''', \Sigma''') := (G'', \Sigma'') \backslash R_k / Q_2[u_k, t]$ and let F''' be obtained from $F'' \backslash R_k / Q_2[u_k, t]$ after deleting all the outgoing arcs at t. We claim that (G''', Σ''') and F''' satisfy (F1)-(F5), therefore contradicting the minimality of (G'', Σ'') and F''. It is clear that (F1) and (F3) hold. The choice of R_k implies that (F5) holds as well. To prove (F2) holds, it suffices to show that $VC_1 \cap VQ_2[u_k, t] = \emptyset$. If not, then by the preceding claim, we get that $VC_1 \cap VR_k \neq \emptyset$, a contradiction as R_k belongs to the even st-path Q_k and $VQ_k \cap VC_1 \subseteq U_1$ by (F2). Hence, $VC_1 \cap VQ_2[u_k, t] = \emptyset$ and so (F2) still holds for (G''', Σ''') and F'''. Lastly, to show (F4) holds, let Q be an odd st-dipath of F'''. Let P be an st-dipath of F'' contained in $Q \cup Q_2[u_k, t]$. If $P \cap Q_2[u_k, t] = \emptyset$ then clearly (F4) holds. Otherwise, define u as in the preceding claim, and find v as found above. Choose B' to a cover of (G'', Σ'') such that $|B' - P[s, v] - R_k[v, t]| = \tau - 3$. Then $B' - R_k$ is a cover of (G''', Σ''') for which $|(B' - R_k) - Q| = \tau - 3$. This proves (F4) holds. However, this contradicts the minimality of (G'', Σ'') and F''. This finally finishes the proof of Claim 4. ◇

Claim 5. *There is no odd circuit* C *in* F'' *that avoids the vertex* t.

Proof of Claim. Suppose otherwise. Pick $j \in I - \{2\}$. Recall that the both of B_2 and B_j are signatures. Choose a minimal $U \subseteq V - \{s, t\}$ so that $\delta(U) = B_2 \triangle B_j$. By Lemma 4.2, there exists a shortest path R in $G''[U] \setminus B_i$ between VQ_2 and VQ_j.

Observe that $VP \cap VR = VQ \cap VR = VC \cap VR = \emptyset$. It is now easily seen that $C \cup P \cup Q \cup Q_2[v_2, t] \cup Q_j[v_j, t] \cup R$ has an F_7 minor. However, (G, Σ) has no such minor, a contradiction. $\diamond$

Since $m \geq 4$ it follows that every even st-path Q_j, $4 \leq j \leq m$, is internally vertex-disjoint from Q_2 and Q_3. So $I \subseteq \{1, 2, 3\}$, and Q_4 is internally vertex-disjoint from Q_2 and Q_3.

We will now analyze $(**)$. Since there is no path in $G''[U - U_1] \setminus B_k$ between VQ_2 and VQ_k, it follows that $U - U_1$ partitions into two sets Y_2 and Y_k such that

$$VQ_i \cap U \subseteq Y_i \text{ and } \delta(Y_i) - (\delta(U) \cup \delta(U_1)) = \emptyset \text{ for } i = 2, k.$$

Claim 6. *The following hold:*

(1) $(U_1 \cap U) \cap (VQ_2 \cup VQ_k) = \emptyset$,

(2) $\delta(Y_i) \cap Q_i \neq \emptyset$ *and* $\delta(Y_i) \cap Q_j = \emptyset$ *for all* i, j *such that* $\{i, j\} = \{2, k\}$,

(3) $\delta(Y_i) \subseteq B_2 \cup B_k \cup \delta(U_1)$ *for* $i = 2, k$, *and*

(4) $\delta(Y_1) \cap \delta(Y_2) = \emptyset$.

Proof of Claim. (1) follows from the fact that $s \notin U_1 \cap U$, $Q_i \cap \delta(U_1) = \{\Omega\}$ for $i = 2, k$. For (2) fix $i \in \{2, k\}$ and let j be the other index. As $VQ_i \cap U \subseteq Y_i$ it follows that $Q_i \cap \delta(Y_j) = \emptyset$. Note that $\delta(U) \cap Q_i \neq \emptyset$. However,

$$Q_i \cap \delta(U) \subseteq (Q_i \cap \delta(Y_i)) \cup (Q_i \cap \delta(Y_j)) \cup (Q_i \cap \delta(U_1 \cap U)) = Q_i \cap \delta(Y_i)$$

and so $Q_i \cap \delta(Y_i) \neq \emptyset$. (3) follows directly from definition. We show (4) by contradiction. Suppose that $e \in \delta(Y_1) \cap \delta(Y_2)$. Then e has endvertices x, y where $x \in Y_1$ and $y \in Y_2$. Therefore, $e \in \delta(U) \cup \delta(U_1)$. However, $x, y \notin U_1$, which implies that $e \in \delta(U)$, a contradiction as $x, y \in U$. $\diamond$

Claim 7. *For each* $i \in \{2, k\}$, *there exists* $\tau \geq p(i) \geq m + 1$ *such that* $|L_{p(i)} \cap \delta(Y_i)| = 2$ *and* $L_{p(i)} \cap \delta(Y_i) \cap B_i = \emptyset$, *and there is a shortest (possibly empty) path* R_i *in* $G''[Y_i]$ *between* VQ_i *and* $VL_{p(i)}$ *such that* $R_i \cap B_i = \emptyset$.

Proof of Claim. We may assume that $i = 2$. Let $U' \subseteq Y_2$ be the largest component of $G''[Y_2]$ containing a vertex of Q_2. Note that $\delta(U') = \delta(U') \cap \delta(Y_2) \subseteq B_2 \cup B_k \cup \delta(U_1)$, and as U' contains a vertex of Q_2, it follows that $\delta(U') \cap Q_2 \cap B_2 = \delta(U') \cap Q_2 \neq \emptyset$. Let $B := B_2 \triangle \delta(U')$. Then $B \cap Q_2 \subsetneq B_2 \cap Q_2$ and $B \cap Q_3 = B \cap (Q_1 \cup C_1) = \{\Omega\}$ as $\delta(Y_2) \cap Q_3 = \delta(Y_1) \cap (Q_1 \cup C_1) = \{\Omega\}$. Moreover, it is easily seen that $|B \cap L_j| = 1$ for all $4 \leq j \leq m$. Therefore, by the minimality of B_2, there exists $\tau \geq p(2) \geq m+1$ such that $\delta(U') \cap L_{p(i)} \neq \emptyset$ and $\delta(U') \cap L_{p(2)} \cap B_2 = \emptyset$. Hence, $|\delta(Y_2) \cap L_{p(2)}| \geq 2$ and so $|\delta(Y_2) \cap L_{p(2)}| = 2$.

Moreover, note that if $e \in B_2 \cap EG''[U']$ then $e \in B_2 \cap B_k$, and so $e \in C_j$ for some $m \geq j \geq 4$, where $C_j \subseteq EG''[U']$. Hence, there is a path Q in $G''[U']$ between the end-vertices of e such that $Q \cap B_2 = \emptyset$. This observation implies that there is a shortest (possibly empty) path R_2 in $G''[U']$ (in particular, $G''[Y_2]$) between VQ_2 and $VL_{p(2)}$ such that $R_2 \cap B_2 = \emptyset$. ◇

Claim 8. *$p(1)$ and $p(2)$ are distinct.*

Proof of Claim. Suppose otherwise. Then $|L_{p(2)} \cap \delta(Y_2)| = 2 = |L_{p(2)} \cap \delta(Y_k)|$, and so as $\delta(Y_2) \cap \delta(Y_k) = \emptyset$ and $\delta(Y_2) \cup \delta(Y_k) \subseteq B_2 \cup B_k \cup \delta(U_1)$, it follows that $|L_{p(2)} \cap B_i| > 1$ for some $i \in \{2, k\}$ or $|L_{p(2)} \cap \delta(U_1)| > 1$, a contradiction either way. ◇

Suppose that $R_2 : [u_2, v_2]$ for $u_2 \in VQ_2 \cap Y_2$ and $v_2 \in VL_{p(2)}$ where $L_{p(2)}[s, v_2]$ is internally vertex-disjoint from $Q_2[s, u_2]$. Also, assume that $R_k : [u_k, v_k]$ for $u_k \in VQ_k \cap Y_k$ and $v_k \in VL_{p(k)}$ where $L_{p(k)}[v_k, t]$ is internally vertex-disjoint from $Q_k[u_k, t]$. Note that $R_2 \cap B_2 = R_k \cap B_2 = \emptyset$. Observe that $VL_{p(2)}[s, v_2] \subseteq U \cup U_1 - Y_k$ and $VL_{p(k)}[v_k, t] \subseteq Y_k \cup (V - U_1 - U)$. We may assume that $L_{p(2)}[s, v_2]$ and $L_{p(k)}[v_k, t]$ are paths (otherwise, replace them with the longest paths contained in them). Next let $Q : [u, v]$ be the shortest path in $G''[U]$ between $VQ_2 \cup VL_{p(2)}[s, v_2] \cup R_2$ and $VQ_k \cup VL_{p(k)}[v_k, t] \cup R_k$ such that $Q \cap B_1 = \emptyset$. Now observe that $Q_2 \cup L_{p(2)}[s, v_2] \cup R_2 \cup Q_k \cup L_{p(k)}[v_k, t] \cup R_k \cup Q \cup Q_4$ contains an F_7 minor. But this implies that (G'', Σ''), and therefore (G, Σ), has an F_7 minor, which is a contradiction.

Therefore, Part (3.1) is not possible.

5.5.2 Part (3.2): $VC_1 \cap VQ_j[v_j, t] \neq \emptyset$ for some $j \in I$

We may assume that $VC_1 \cap VQ_i[v_i, t] \neq \emptyset$. Observe that this implies that $Q_i[v_i, t]$, and therefore $Q_j[v_j, t]$ for all $j \in I$, is not contained in any even st-path of F'', due to (X2).

Hence, $I \subseteq \{1,2,3\}$, and since H' is acyclic by Lemma 5.2, it follows that $I = \{2,3\}$.

The following claim easily follows from the acyclicity of H'.

Claim 1. *Q_j is internally vertex-disjoint from Q_2 and Q_3, for $j = 1$ and all $4 \leq j \leq m$.*

Claim 2. *If $m = 3$ then (G'', Σ'') has a $\widetilde{K_5}$ minor.*

Proof of Claim. Suppose that $m = 3$ and set $R_1 := C_1, R_2 := Q_2, R_3 := Q_3$ and $R_4 := Q_1$. For $j \in \{1,2,3\}$, let u_j be the closest vertex to s' in $V(R_j) - \{s, s'\}$ that also lies on another R_i, $i \in \{1,2,3\} - \{j\}$. By the Intersection Lemma there exists $i \in \{1,2,3\}$ such that whenever $u_i \in VR_j$ then $u_i = u_j$. Let $J \subseteq \{1,2,3\}$ be the set of all indices j such that $u_j = u_i$. Note that $i \in J$ and $|J| \geq 2$.

Subclaim 2.1. *For each $j \in J$, there exists a minimal cover B_j of (G'', Σ'') such that $|B_j - R_j[s, u_j]| = \tau - 3$.*

Proof of Subclaim 2.1. Suppose otherwise. Let $(G''', \Sigma''') := (G'', \Sigma'') \setminus R_j[s', u_j]/ \cup (R_k[s', u_k] : k \in J, k \neq j)$ and $F''' := F'' \setminus R_j[s', u_j]/ \cup (R_k[s', u_k] : k \in J, k \neq j)$, and now it is easily seen that (G''', Σ''') and F''' satisfy $(F1) - (F5)$. However, this contradicts the minimality of (G'', Σ'') and F''. *End of Proof of Subclaim 2.1*

We may assume that each B_j, $j \in J$ is an internally minimal mate of $R_j[s, u_j]$. Observe that Lemma 5.4 implies that each $B_j, j \in J$ is a signature.

Subclaim 2.2. *There are two internally vertex-disjoint directed paths P and Q in F'', where Q is from u_i to t and P is from s' to t.*

Proof of Subclaim 2.2. Suppose otherwise. Then by Menger's theorem, there exists a vertex $v \notin \{s, s', t\}$, whose removal from F'' leaves no $s't$-dipath behind. So, in particular, $v \in VR_j$ for $j \in \{2,3\}$. Observe, further, that the assumption that $VC_1 \cap VQ_j[v_j, t] \neq \emptyset$ for some $j \in I$, implies that $v \in VR_1$.

If there exists an $s'v$-dipath R' in F'' for which there is no cover B of (G'', Σ'') such that $|B - R' - \{\Omega\}| = \tau - 3$, then delete R' and contract all the other $s'v$-dipaths in F'' to get a more minimal instance (G''', Σ''') and F''', which is not possible.

Otherwise, for evey $s'v$-dipath R' in F'', there is a cover B of (G'', Σ'') such that $|B - R' - \{\Omega\}| = \tau - 3$. After applying the Reduction Lemma on $\bigcup_{j=1}^{3} R_j[s, v]$, if necessary, we may assume that $(R_j[s', v] : 1 \leq j \leq 3)$ are pairwise internally vertex-disjoint. For each $1 \leq j \leq 3$, let D_j be an internally minimal mate of $R_j[s, v]$. (So, in particular, each D_j is a cover of (G'', Σ'') such that $|D_j - R_j[s, v]| = \tau - 3$.) Observe that Lemma 5.4 implies that each D_j is a signature. By the Shore Lemma, there is a path R in $G''[U_1] \setminus \cup(D_j : 1 \leq j \leq 3)$ between s and VR_4 (note $m = 3$). Now applying the $\widetilde{K_5}$ Lemma to $(R \cup \bigcup_{j=1}^{4} R_j)/ \cup (R_j[v, t] : 1 \leq j \leq 3)/(R \cup R_4)$ gives us a $\widetilde{K_5}$ minor, which implies in turn that (G, Σ) has a $\widetilde{K_5}$ minor, a contradiction. *End of Proof of Subclaim 2.2*

So, in particular, $|J| = 2$. Let $R'_1 := P \cup \{\Omega\}$ and let R'_2, R'_3 be the two paths $R_j[s, u_j], j \in J$. For each $1 \leq j \leq 3$, Let D_j be an internally minimal mate for R'_j. (So, in particular, each D_j is a cover of (G'', Σ'') such that $|D_j - R'_j| = \tau - 3$.) Lemma 5.4 implies that each D_j is a signature, and so by the Shore Lemma, there exists a path R in $G''[U_1] \setminus \cup(D_j : 1 \leq j \leq 3)$ between s and VR_4. Now contract the two paths $R_j[u_j, t], j \in J$ and apply the $\widetilde{K_5}$ Lemma to $(R \cup R_4 \cup R'_1 \cup R'_2 \cup R'_3)/(R \cup R_4)$ to obtain a $\widetilde{K_5}$ minor. Hence, (G'', Σ'') contains a $\widetilde{K_5}$ minor, and this finishes the proof of Claim 2. ◇

However, (G, Σ) does not contain a $\widetilde{K_5}$ minor, and so $m \geq 4$. Set $R_1 := C_1, R_2 := Q_2$ and $R_3 := Q_3$. As above, for $j \in \{1, 2, 3\}$, let u_j be the closest vertex to s' in $V(R_j) - \{s, s'\}$ that also lies on another R_i. By the Intersection Lemma there exists $i \in \{1, 2, 3\}$ such that whenever $u_i \in VR_j$ then $u_i = u_j$. Let $J \subseteq \{1, 2, 3\}$ be the set of all indices j such that $u_j = u_i$. Note that $i \in J$ and $|J| \geq 2$.

Claim 3. *For each $j \in J$, there exists a minimal cover B_j of (G'', Σ'') such that $|B_j - R_j[s, u_j]| = \tau - 3$.*

Proof of Claim. Suppose otherwise. Let $(G''', \Sigma''') := (G'', \Sigma'') \setminus R_j[s', u_j]/ \cup (R_k[s', u_k] : k \in J, k \neq j)$ and $F''' := F'' \setminus R_j[s', u_j]/ \cup (R_k[s', u_k] : k \in J, k \neq j)$, and now it is easily seen that (G''', Σ''') and F''' satisfy $(F1) - (F5)$. However, this contradicts the minimality of (G'', Σ'') and F''. ◇

We may assume that each B_j, $j \in J$ is an internally minimal mate of $R_j[s, u_j]$. Observe that Lemma 5.4 implies that each $B_j, j \in J$ is a signature.

Claim 4. *There are two internally vertex-disjoint directed paths P and Q in F'', where Q is from u_i to t and P is from s' to t.*

Proof of Claim. Suppose otherwise. Then by Menger's theorem, there exists a vertex $v \notin \{s, s', t\}$, whose removal from F'' leaves no $s't$-dipath behind. So, in particular, $v \in VR_j$ for $j \in \{2, 3\}$. Observe, further, that the assumption that $VC_1 \cap VQ_j[v_j, t] \neq \emptyset$ for some $j \in I$, implies that $v \in VR_1$.

If there exists an $s'v$-dipath R' in F'' for which there is no cover B of (G'', Σ'') such that $|B - R' - \{\Omega\}| = \tau - 3$, then delete R' and contract all the other $s'v$-dipaths in F'' to get a more minimal instance (G''', Σ''') and F''', which is not possible.

Otherwise, for evey $s'v$-dipath R' in F'', there is a cover B of (G'', Σ'') such that $|B - R' - \{\Omega\}| = \tau - 3$. After applying the Reduction Lemma on $\bigcup_{j=1}^{3} R_j[s, v]$, if necessary, we may assume that $(R_j[s', v] : 1 \leq j \leq 3)$ are pairwise internally vertex-disjoint. For each $1 \leq j \leq 3$, let D_j be an internally minimal mate of $R_j[s, v]$. So, in particular, each D_j is a cover of (G'', Σ'') such that $|D_j - R_j[s, v]| = \tau - 3$. Observe that Lemma 5.4 implies that each B_j is a signature. Now applying the $\widetilde{K_5}$ Lemma to $(Q_4 \cup \bigcup_{j=1}^{3} R_j / \cup(R_j[v, t] : 1 \leq j \leq 3)/Q_4$ gives us a $\widetilde{K_5}$ minor, which implies in turn that (G, Σ) has a $\widetilde{K_5}$ minor, a contradiction.$\diamond$

So, in particular, $|J| = 2$. Let $R'_1 := P \cup \{\Omega\}$ and let R'_2, R'_3 be the two paths $R_j[s, u_j], j \in J$. Let D_j be an internally minimal mate of R'_j, for each $1 \leq j \leq 3$. So, in particular, D_j is a cover of (G'', Σ'') such that $|D_j - R'_j| = \tau - 3$. Lemma 5.4 implies that each D_j is a signature. Now contract the two paths $R_j[u_j, t], j \in J$ and apply the $\widetilde{K_5}$ Lemma to $(Q_4 \cup R'_1 \cup R'_2 \cup R'_3)/Q_4$ to obtain a $\widetilde{K_5}$ minor. Hence, (G'', Σ''), and therefore (G, Σ), contains a $\widetilde{K_5}$ minor, a contradiction.

As a result, Part (3.2) is not feasible, which is turn implies that Part (3) is not possible.

5.6 Part (4)

The setup for this part is provided in §5.4. Recall that L_1, L_2, L_3 are simple, $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ is bipartite, and neither of (X1), (X2) hold. The argument for this part is partially similar to that of Part (3). Since (X2) does hold there exists an even st-dipath P in $(F', \Sigma' \cap E(F'))$ such that $V(P) \cap V(C_1) - U_1 \neq \emptyset$. In particular, $m \geq 4$. After rerouting $C_1, Q_1, Q_2, \ldots, Q_m$, if necessary, we may assume that $P = Q_4$. Let $x \in V(Q_4) \cap V(C_1) - U_1$

and let $R := Q_4[s,x] \cup C_1[x,u]$. Redefine Q_1, Q_4 and F'' as follows: $Q_4 := Q_1$, $Q_1 := C_1[s,x] \cup Q_4[x,t]$, and $F'' := \bigcup_{j=1}^{m} Q_j$.

Let (G'', Σ'') be a minor of (G', Σ') and let F'' be a directed graph obtained by orienting edges in a subgraph of G', where (G'', Σ'') and F'' are minimal subject to

(F1) $E(G') - E(G'') \subseteq E(F''[V - U_1])$, and $E(F'') \subseteq E(F')$,

(F2) F'' is acyclic and there exist m directed paths in F'' that are disjoint except possibly at Ω, exactly three of which contais Ω; $m-1$ of these paths are st-dipaths and the remaining one is a wt-dipath,

(F3) for any odd st-dipath P of F'' such that $V(P) \cap U_1 = \{s\}$, there exists a signature B of (G'', Σ'') such that $|B - P| = \tau - 3$, and

(F4) there is no cover of (G'', Σ'') of size $\tau - 2$.

Note that these conditions are satisfied by (G', Σ') and F', so (G'', Σ'') and F'' are well-defined. By identifying a vertex of each component with s, if necessary, we may assume that G'' is connected. Now let $\left(Q''_j\right)_{j=1}^{m}$ be m directed paths in F'' that are pairwise disjoint except possibly at Ω, $\Omega \in Q''_j$ if and only if $j \in \{1,2,3\}$, and Q''_4 is a wt-dipath. For the sake of notational ease, reset $Q_j := Q''_j$ for all $1 \leq j \leq m$. Observe that P is an odd st-dipath in F'' if and only if $\Omega \in P$, since $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ is bipartite.

For each Q_j other than Q_4, let $v_j \neq t$ be the closest vertex to t on Q_j that also lies on another Q_i, $i \in \{1, \ldots, m\} - \{j\}$. Let $v_4 \neq t$ be the closest vertex to t in $VQ_4 \cup \{s\}$ that also lies on another Q_i, $i \in \{1, \ldots, m\} - \{4\}$. Then by the Intersection Lemma there exists $i \in \{1, \ldots, m\}$ such that whenever $v_i \in VQ_j$ then $v_i = v_j$. Let I be the set of all indices $j \in \{1, \ldots, m\}$ such that $v_j = v_i$. Note that $i \in I$ and $|I| \geq 2$. We may assume $v_i \notin U_1$ (otherwise, remove all paths Q_j, $j \in I$ temporarily and reapply the Intersection Lemma). This implies that $VQ_j[v_j, t] \cap U_1 = \emptyset$ for all $j \in I$.

Claim 1. *For each $j \in I$, $Q_j[v_j, t]$ is contained in an odd st-dipath of $(F'', \Sigma'' \cap E(F''))$ that intersects U_1 at only s.*

Proof of Claim. Suppose not. Then let $(G''', \Sigma''') := (G'', \Sigma'')/\cup(Q_j[v_j,t] : j \in I)$ and $F''' := F''/\cup(Q_j[v_j,t] : j \in I)$. Clearly, (F1), (F2) and (F4) still hold for (G''', Σ''') and F'''. Furthermore, our assumption implies that (F3) also holds, a contradiction to the minimality of (G'', Σ'') and F''. ◇

We may therefore assume that $i = 1$.

Claim 2. *For each $j \in I$, there exists a minimal cover B_j such that $|B_j - Q_j[v_j, t] - \{\Omega\}| = \tau - 3$.*

Proof of Claim. Suppose otherwise. Then there is no cover B such that $|B - Q_i[v_i, t] - \{\Omega\}| = \tau - 3$, for some $i \in I$. Then let $(G''', \Sigma''') = (G'', \Sigma'') \setminus Q_i[v_i, t] / \cup (Q_j[v_j, t] : j \in I, j \neq i)$ and $F''' = F'' \setminus Q_i[v_i, t] / \cup (Q_j[v_j, t] : j \in I, j \neq i)$. Clearly, (F1)-(F3) still hold for (G'', Σ'') and H''. Our assumption implies that (F4) also holds, a contradiction to the minimality of (G'', Σ') and F''. ◇

We may assume that each B_j, $j \in I$ is an internally minimal mate of $Q_j[v_j, t]$. Observe that each B_j is a signature, by Lemma 5.4.

Claim 3. *There exists an odd circuit C in $F'' \cup R$ such that $VC \cap VQ_j[v_j, t] - \{v_j\} = \emptyset$ for all $j \in I$.*

Proof of Claim. Suppose otherwise. Then

(1) $I \subseteq \{1, 2, 3, 4\}$,

(2) $VR \cap VQ_i - \{s\} \subseteq \bigcup_{j \in I} VQ_j[v_j, t] - \{v_j\}$, for all $i \in \{1, 2, 3\}$, and

(3) every odd st-dipath of F'' is internally vertex-disjoint from Q_j, for any $5 \leq j \leq m$.

Moreover, by the definition of R and the existence of x, it follows that $VR \cap VQ_k[v_k, t] - \{v_k\} \neq \emptyset$ for some $k \in I$. Choose $y \in VR \cap VQ_k[v_k, t] - \{v_k\}$ such that $R[s, y]$ is internally vertex-disjoint from each Q_i, $i \in \{1, 2, 3\}$. Notice that the acyclicity of H', by Lemma 5.2, implies that $I \subseteq \{1, 2, 3\}$ and that $VQ_4 \cap VQ_j = \{t\}$ for all $j \in I$. Hence, we may assume that $k = 1$.

For $j \in \{1, 2, 3\}$, let u_j be the closest vertex to s' in $VQ_j - \{s, s'\}$ that also lies on another Q_i. By the Intersection Lemma there exists $i \in \{1, 2, 3\}$ such that whenever $u_i \in VQ_j$, for some $j \in \{1, 2, 3\}$, then $u_i = u_j$. Let $J \subseteq \{1, 2, 3\}$ be the set of all indices j

such that $u_j = u_i$. Note that $i \in J$ and $|J| \geq 2$.

Subclaim 3.1. *For each $j \in J$, there exists a minimal cover B of (G'', Σ'') such that $|B - Q_j[s, u_j]| = \tau - 3$.*

Proof of Subclaim 3.1. Suppose otherwise. Let $(G''', \Sigma''') := (G'', \Sigma'') \setminus Q_j[s', u_j] / \cup (Q_k[s', u_k] : k \in J, k \neq j)$ and $F''' := F'' \setminus Q_j[s', u_j] / \cup (Q_k[s', u_k] : k \in J, k \neq j)$, and now it is easily seen that (G''', Σ''') and F''' satisfy (F1)-(F4). However, this contradicts the minimality of (G'', Σ'') and F''. *End of Proof of Subclaim 3.1*

Subclaim 3.2. *There are two internally vertex-disjoint directed paths P and Q in F'', where Q is from u_i to t and P is from s' to t.*

Proof of Subclaim 3.2. Suppose otherwise. Then by Menger's theorem, there exists a vertex $v \notin \{s, s', t\}$, whose removal from F'' leaves no $s't$-dipath behind. So, in particular, $v \in VQ_j$ for $j \in \{1, 2, 3\}$.

If there exists an $s'v$-dipath R' in F'' for which there is no cover B of (G'', Σ'') such that $|B - R' - \{\Omega\}| = \tau - 3$, then delete R' and contract all the other $s'v$-dipaths in F'' to get a more minimal instance (G''', Σ''') and F''', which is not possible.

Otherwise, for evey $s'v$-dipath R' in F'', there is a cover B of (G'', Σ'') such that $|B - R' - \{\Omega\}| = \tau - 3$. After applying the Reduction Lemma on $\bigcup_{j=1}^{3} Q_j[s, v]$, if necessary, we may assume that $(Q_j[s', v] : 1 \leq j \leq 3)$ are pairwise internally vertex-disjoint. For each $1 \leq j \leq 3$, let D_j be an internally minimal mate of $Q_j[s, v]$. So, in particular, D_j is a cover of (G'', Σ'') such that $|D_j - Q_j[s, v]| = \tau - 3$. Observe that Lemma 5.4 implies that each B_j is a signature. Now applying the $\widetilde{K_5}$ Lemma to $(R[s, y] \cup \bigcup_{j=1}^{3} Q_j / \cup (Q_j[v, t] : 1 \leq j \leq 3) / R[s, y]$ gives us a $\widetilde{K_5}$ minor, which implies in turn that (G, Σ) has a $\widetilde{K_5}$ minor, a contradiction.

End of Proof of Subclaim 3.2

So, in particular, $|J| = 2$. By rerouting P or Q, if necessary, we may assume that $Q_1[v_1, t] \cap P = Q_1[v_1, t] \cap Q = \emptyset$. Let $R_1 := P \cup \{\Omega\}$ and let R_2, R_3 be the two paths $Q_j[s, u_j], j \in J$. For each $1 \leq j \leq 3$, let D_j be an internally minimal mate of R_j. So, in particular, D_j is a cover of (G'', Σ'') such that $|D_j - R_j| = \tau - 3$. Lemma 5.4 implies that each D_j is a signature. Now contract the two paths $Q_j[u_j, t], j \in J$ and apply the $\widetilde{K_5}$ Lemma to $(R[s, y] \cup Q_1[y, t] \cup R_1 \cup R_2 \cup R_3)/(R[s, y] \cup Q_1[y, t])$ to obtain a $\widetilde{K_5}$ minor.

Hence, (G'', Σ''), and therefore (G, Σ), contains a $\widetilde{K_5}$ minor, a contradiction. This finishes the proof of Claim 3. $\diamond$

Claim 4. *Suppose there exists a path in $G''[U_1] \setminus \bigcup_{j \in I} B_j$ between s and every connected component of $\widetilde{R}$. Then (G'', Σ'') contains an F_7 minor.*

Proof of Claim. We will first prove the following.

Subclaim 4.1. *There exist two vertex-disjoint paths P and Q in F'', where Q is between s' and v_1, and P is between t and either of s, w.*

Proof of Subclaim 4.1. We will treat U_1 as a vertex, and in order to prove the lemma, it suffices to find two vertex-disjoint paths P and Q in F'', where Q is between s' and v_1, and P is between t and U_1. Suppose for a contradiction that this is not possible. Then the Linkage Lemma implies that F'' can be obtained as follows:

(L1) place a circuit C on the boundary S^1 of the unit disc, and the circuit contains the vertices U_1, s', t, v_1 in this cyclic order,

(L2) add vertices to the interior of the disc, and triangulate the resulting graph inside the disc to get K,

(L3) for every facial triangle T, consider an arbitrary graph K_T such that $V(K_T) \cap V(K) = V(T)$,

(L4) take the union $K \cup \bigcup_T K_T$, and delete some edges to get F''.

Now consider the st-path Q_1 in the drawing. Observe that s (which is now merged with U_1) and t lie on different sides of the path $Q_1[s', v_1]$. Consider the set Γ_{Q_1} of pinned vertices that lie strictly inside the side of $Q_1[s', v_1]$ that contains s. As F'' is acyclic, we may assume that the set Γ_{Q_1} is minimal over all possible odd st-dipaths Q_1 in F''.

Note that every Q_j, $j \in \{2, \dots, m\}$, is an st-path (note s and U_1 are merged). Thus, for every $j \in \{2, \dots, m\}$, there exists a pinned vertex u_j that lies on both Q_j and $Q_1[s', v_1]$; we may assume that u_j is the closest such vertex to t on Q_j. Note that this implies that $u_i = v_i$ for all $i \in I$.

For each $j \in [m] - \{1\}$ let $R_j := Q_j[u_j, t]$ and $S_j := Q_1[s, u_j] \cup R_j$. For $j \in \{1, 2, 3\} \cap I$ let $R_j := Q_j[u_j = v_j, t]$ and $S_j := Q_1[s, u_j = v_j] \cup R_j$, and for $j \in \{1, 2, 3\} - I$ let $R_j := Q_j[s', t]$

and $S_j := Q_j$. By the Mate Lemma and Lemma 5.4, we get that there exists an R_k such that $|B - R_k - \{\Omega\}| > \tau - 3$, for all covers B of (G'', Σ''). Notice that $k \notin I$ and that $k \notin \{1, 2, 3\}$.

Subclaim 4.1.1. *Take an odd st-dipath P in F'' that has intersection $\{s\}$ with U_1. Suppose that $VP \cap VQ_1[u_k, t] \neq \{t\}$ and let u be the closest vertex to s on P that lies on $Q_1[u_k, t]$. Then there exists a vertex $v \in VP[s, u]$ that lies on R_k.*

Proof of Subclaim 4.1.1. Suppose otherwise. Then in particular $u \notin VR_k = VQ_k[u_k, t]$. As F'' is acyclic it follows that $u \notin VQ_k[s, u_k]$, and so $u \notin VQ_k$. It is now easily seen that u lies strictly inside the side of Q_k which contains v_1. As P is odd it follows that $s' \in VP$. Consider the subpath $P[s', u]$ of P. Since s' lies on a different side of Q_k than that of u, Remark 4.16 implies that $P[s', u]$ and Q_k share a pinned vertex w, say, and suppose that w is the closest such vertex to u. By our hypothesis, $w \notin VR_k = VQ_k[u_k, t]$. Hence, $w \in VP[s, u_k] - \{u_k\}$. Let w' be the closest vertex to w in $P[s', w]$ that lies on $Q_1[s', u_k]$. Then $Q_1' := Q_1[s, w'] \cup P[w', w] \cup Q_k[w, u_k] \cup Q_1[u_k, t]$ contradicts the minimality of Q_1 as $\Gamma_{Q_1'} \subseteq \Gamma_{Q_1} - \{w\}$. Hence, there exists a vertex $v \in VP[s, u]$ that lies on R_k. This finishes the proof of Subclaim 4.1.1.

Now let $(G''', \Sigma''') := (G'', \Sigma'') \backslash R_k / Q_1[u_k, t]$ and let F''' be obtained from $F'' \backslash R_k / Q_1[u_k, t]$ after deleting all the outgoing arcs at t. We claim that (G''', Σ''') and F''' satisfy (F1)-(F4), therefore contradicting the minimality of (G'', Σ'') and F''. It is clear that (F1) and (F2) hold. The choice of R_k implies that (F4) holds as well. Lastly, to show (F3) holds, let Q be an odd st-dipath of F'''. Let P be an st-dipath of F'' contained in $Q \cup Q_1[u_k, t]$. If $P \cap Q_1[u_k, t] = \emptyset$ then clearly (F3) holds. Otherwise, define u as in the preceding claim, and find v as found above. Choose B' to a cover of (G'', Σ'') such that $|B' - P[s, v] - R_k[v, t]| = \tau - 3$. Then $B' - R_k$ is a cover of (G''', Σ''') for which $|(B' - R_k) - Q| = \tau - 3$. This proves (F3) holds. However, this contradicts the minimality of (G'', Σ'') and F''. *End of Proof of Subclaim 4.1*

We are now ready to prove Claim 4. Choose $i \in I - \{1\}$, and let $U \subseteq V - \{s, t\}$ be a minimal vertex subset such that $\delta(U) = B_1 \triangle B_i$. By Lemma 4.2, there exists a shortest path R' in $G''[U] \setminus B_1$ between VQ_1 and VQ_i. By the assumption of Claim 4, there exist a path R'' in $G''[U_1] \setminus (B_2 \cup B_i)$ between s and w. (Note that $\widetilde{R} \cap (B_2 \cup B_i) = \emptyset$.) It is now easily seen that $C \cup R'' \cup P \cup Q \cup Q_1[v_1, t] \cup Q_i[v_i, t] \cup R'$ has an F_7 minor. This concludes the proof of Claim 4. ◇

However, (G, Σ) does not have an F_7 minor, and so the assumption of Claim 4 cannot be true, i.e.

> $(*)$ for a connected component K of $\widetilde{R}$, there is no path in $G[U_1] \setminus \bigcup_{j \in I} B_j$ between s and K.

Observe that the Shore Lemma, together with $(*)$, implies that $m \geq 5$.

Claim 5. *There exist vertex disjoint paths P and Q in F'', where P is between s and t and Q is between s' and v_1.*

Proof of Claim. Suppose for a contradiction that this is not possible. Then the Linkage Lemma implies that F'' can be obtained as follows:

(L1) place a circuit C on the boundary S^1 of the unit disc, and the circuit contains the vertices s, s', t, v_1 in this cyclic order,

(L2) add vertices to the interior of the disc, and triangulate the resulting graph inside the disc to get K,

(L3) for every facial triangle T, consider an arbitrary graph K_T such that $V(K_T) \cap V(K) = V(T)$,

(L4) take the union $K \cup \bigcup_T K_T$, and delete some edges to get F''.

Now consider the st-path Q_1 in the drawing. Observe that s and t lie on different sides of the path $Q_1[s', v_1]$. Consider the set Γ_{Q_1} of pinned vertices that lie strictly inside the side of $Q_1[s', v_1]$ that contains s. As F'' is acyclic, we may assume that the set Γ_{Q_1} is minimal over all possible odd st-dipaths Q_1 in F'' that have intersection $\{s\}$ with U_1.

Note that every Q_j, $j \in \{5, \ldots, m\}$, is an st-path. So for every $j \in \{5, \ldots, m\}$, there exists a pinned vertex u_j that lies on both Q_j and $Q_1[s', v_1]$; we may assume that u_j is the closest such vertex to t on Q_j. Note that this implies that $u_i = v_i$ for all $i \in I$. For each $j \in \{5, \ldots, m\}$ let $R_j := Q_j[u_j, t]$ and $S_j := Q_1[s, u_j] \cup R_j$. For $j \in \{1, 2, 3\} \cap I$ let $R_j := Q_j[u_j = v_j, t]$ and $S_j := Q_1[s, u_j = v_j] \cup R_j$, and for $j \in \{1, 2, 3\} - I$ let $R_j := Q_j[s', t]$ and $S_j := Q_j$.

By the Shore Lemma, along with $(*)$ and Lemma 5.4, we get that there exists an R_k such that $|B - R_k - \{\Omega\}| > \tau - 3$, for all covers B of (G'', Σ''). Notice that $k \notin I$ and that $k \notin \{1, 2, 3\}$.

Subclaim 5.1. *Take an odd st-dipath P in F'' that has intersection $\{s\}$ with U_1. Suppose that $VP \cap VQ_1[u_k, t] \neq \{t\}$ and let u be the closest vertex to s on P that lies on $Q_1[u_k, t]$. Then there exists a vertex $v \in VP[s, u]$ that lies on R_k.*

Proof of Subclaim 5.1. Suppose not. Then in particular $u \notin VR_k = VQ_k[u_k, t]$. As F'' is acyclic it follows that $u \notin VQ_k[s, u_k]$, and so $u \notin VQ_k$. It is now easily seen that u lies strictly inside the side of Q_k which contains v_1. As P is odd it follows that $s' \in VP$. Consider the subpath $P[s', u]$ of P. Since s' lies on a different side of Q_k than that of u, Remark 4.16 implies that $P[s', u]$ and Q_k share a pinned vertex w, say, and suppose that w is the closest such vertex to u. By our hypothesis, $w \notin VR_k = VQ_k[u_k, t]$. Hence, $w \in VP[s, u_k] - \{u_k\}$. Let w' be the closest vertex to w in $P[s', w]$ that lies on $Q_1[s', u_k]$. Then $Q_1' := Q_1[s, w'] \cup P[w', w] \cup Q_k[w, u_k] \cup Q_1[u_k, t]$ contradicts the minimality of Q_1 as $\Gamma_{Q_1'} \subseteq \Gamma_{Q_1} - \{w\}$. Hence, there exists a vertex $v \in VP[s, u]$ that lies on R_k. *End of Proof of Subclaim 5.1*

Now let $(G''', \Sigma''') := (G'', \Sigma'') \backslash R_k / Q_1[u_k, t]$ and let F''' be obtained from $F'' \backslash R_k / Q_1[u_k, t]$ after deleting all the outgoing arcs at t. We claim that (G''', Σ''') and F''' satisfy (F1)-(F4), therefore contradicting the minimality of (G'', Σ'') and F''. It is clear that (F1) and (F2) hold. The choice of R_k implies that (F4) holds as well. Lastly, to show (F3) holds, let Q be an odd st-dipath of F'''. Let P be an st-dipath of F'' contained in $Q \cup Q_1[u_k, t]$. If $P \cap Q_2[u_k, t] = \emptyset$ then clearly (F3) holds. Otherwise, define u as in the preceding claim, and find v as found above. Choose B' to a cover of (G'', Σ'') such that $|B' - P[s, v] - R_k[v, t]| = \tau - 3$. Then $B' - R_k$ is a cover of (G''', Σ''') for which $|(B' - R_k) - Q| = \tau - 3$. This proves (F3) holds. However, this contradicts the minimality of (G'', Σ'') and F''. This finally finishes the proof of Claim 5. $\diamond$

Pick $i \in I - \{1\}$. Recall that the both of B_1 and B_i are signatures. Choose a minimal $U \subseteq V - \{s, t\}$ so that $\delta(U) = B_1 \triangle B_i$. By Lemma 4.2, there exists a shortest path R' in $G''[U] \setminus B_1$ between VQ_1 and VQ_i. Observe that $VP \cap VR' = VQ \cap VR' = VC \cap VR' = \emptyset$. It is now easily seen that $C \cup P \cup Q \cup Q_1[v_1, t] \cup Q_i[v_i, t] \cup R$ has an F_7 minor. But then (G, Σ) has an F_7 minor. However, this is not possible, implying that Part (4) is not possible.

5.7 A lemma for Parts (5)-(7)

In this section, we provide a lemma that is frequently referenced in Parts (5)-(7). Recall that at least one of L_1, L_2, L_3 is non-simple and whenever L_i, $1 \leq i \leq 3$, is non-simple then $\Omega \in C_i$.

Lemma 5.5. *Let $P_j \in \{P_1, P_2, \dots, P_m\}$ be an even st-path, and let B be a minimal odd st-walk cover of (G', Σ') such that $|B - P_j - \{\Omega\}| = \tau - 3$. Then B cannot be an st-bond.*

Proof. After rearranging $P_1, P_2, \dots, P_m$, if necessary, we may assume that $j \in \{1, 2, 3\}$. Notice that $B \cap L_j \neq \emptyset$ for all $4 \leq j \leq \tau$, and since $|B - P_j - \{\Omega\}| = \tau - 3$, it then follows that $B \cap C_j = \{\Omega\}$. Therefore, since C_j is a circuit and $B \cap C_j$ has odd size, it follows that B cannot be an st-bond. $\square$

5.8 Part (5)

Recall that all of L_1, L_2, L_3 are non-simple, and $\Omega \in C_1 \cap C_2 \cap C_3$. For this part, we will use the lemma stated in §5.7.

We will first show that $s = t$. Suppose otherwise. Lemma 5.5, together with the Mate Lemma, ensures that there exists $P_j \in \{P_1, P_2, \dots, P_m\}$ for which there is no odd st-walk cover B of (G', Σ') such that $|B - P_j - \{\Omega\}| = \tau - 3$. In particular, P_j must be an even st-path – due to (M3). After rearranging $P_1, P_2, P_3, \dots, P_m$, if necessary, we may assume that $j = 1$. Observe that the minimality of $L_1, L_2, L_3, P_4, \dots, P_m$ by (A1) shows that each C_i, $1 \leq i \leq 3$, and each P_j, $1 \leq j \leq m$, are vertex disjoint except at s. Let $(G'', \Sigma'') := (G', \Sigma') \setminus P_1 / \bigcup_{j=2}^{m} P_j$ and $H'' := H' \setminus P_1 / \bigcup_{j=2}^{m} P_j$. Observe that $s = t$ in (G'', Σ''), and that (G'', Σ'') and H'' satisfy all of (M1)-(M4), which is in contradiction with the minimality of (G', Σ') and H'.

Thus, $s = t$. Applying the Reduction Lemma, followed by the $\widetilde{K_5}$ Lemma, gives us a $\widetilde{K_5}$ minor for (G', Σ'). But then (G, Σ) has a $\widetilde{K_5}$ minor, which is not possible. So Part (5) is not feasible.

5.9 Part (6)

Recall that exactly two, say L_1 and L_2, of L_1, L_2, L_3 are non-simple, and $\Omega \in C_1 \cap C_2 \cap P_3$. For this part, we will use the lemma stated in §5.7.

Claim 1. *Each of B_1, B_2 and B_3 is a signature.*

Proof of Claim. Let $i \in \{1,2,3\}$, and take $j \in \{1,2\} - \{i\}$. Notice that $B_i \cap C_j = \{\Omega\}$, and so B_i must be a signature. $\Diamond$

Claim 2. *$\{\Omega\}$ is a signature for $(H', \Sigma' \cap E(H'))$.*

Proof of Claim. We will prove this by finding a vertex subset $U \subseteq V(G') - \{s,t\}$ such that $(B_3 \triangle \delta(U)) \cap EH' = \{\Omega\}$. Let $U \subseteq VP_3 - \{s,t\}$ be the unique subset for which $P_3 \cap \delta(U) = P_3 \cap B_3 - \{\Omega\}$. We will show that $U \cap VL_i = U \cap VP_j = \emptyset$ for all $i \in \{1,2\}$ and $4 \leq j \leq m$. Observe that $B_1 \cap (L_2 \cup P_3) = \{\Omega\}$, and so $L_2 \cup P_3 - \{\Omega\}$ is bipartite, which in turn implies $U \cap VL_2 = \emptyset$. Similarly, $U \cap VL_1 = \emptyset$. Furthermore, for all $4 \leq j \leq m$, $B_1 \cap (P_j \cup P_3) = \{\Omega\}$ implying that $P_j \cup P_3 - \{\Omega\}$ is bipartite, and so $U \cap VP_j = \emptyset$. Therefore, $\delta(U) \cap EH' = \delta(U) \cap P_3 = B_3 \cap EH' - \{\Omega\}$ and so

$$(B_3 \triangle \delta(U)) \cap EH' = (B_3 \cap EH') \triangle (B_3 \cap EH' - \{\Omega\}) = \{\Omega\},$$

as claimed. $\Diamond$

Claim 3. *$H' \setminus (C_1 \cup C_2 - \{\Omega\})$ is acyclic.*

Proof of Claim. Suppose otherwise, and let C be a directed circuit in $H' \setminus (C_1 \cup C_2 - \{\Omega\})$. Clearly $\Omega \notin C$, and so one can find m pairwise disjoint st-dipaths $P'_1, P'_2, P'_3, \ldots, P'_m$ in $H' \setminus (C_1 \cup C_2 - \{\Omega\}) \setminus C$ such that $\Omega \in P'_3$. Let $L'_i := C_i \cup P'_i$, for $i \in \{1,2\}$, and $L'_3 := P'_3$. By Claim 2, each of L'_1, L'_2, L'_3 is odd, and that each P'_j, $4 \leq j \leq m$, is even. However, this contradicts the minimality of $L_1, L_2, L_3, P_4, \ldots, P_m$ by (A1). $\Diamond$

Claim 4. *Every even st-dipath P of Σ'-signed subgraph $H' \setminus (C_1 \cup C_2 - \{\Omega\})$ is vertex-disjoint from C_1 and C_2 except at s.*

Proof of Claim. By rerouting $P_1, P_2, P_3, \ldots, P_m$ in $H' \setminus (C_1 \cup C_2 - \{\Omega\})$, if necessary, we may assume that $P = P_1$, and it is therefore clear that P and C_1 are vertex-disjoint except at s. Similarly, P and C_2 are vertex-disjoint except at s. $\Diamond$

Claim 5. *For every odd st-dipath P of Σ'-signed subgraph $H' \setminus (C_1 \cup C_2 - \{\Omega\})$, there* *exists a cover B of (G', Σ') such that $|B - P| = \tau - 3$.*

Proof of Claim. By rerouting $P_1, P_2, P_3, \ldots, P_m$ in $H' \setminus (C_1 \cup C_2 - \{\Omega\})$, if necessary, we may assume that $P = P_3$, and by (M3) such B exists. ◇

For each P_j let $v_j \neq t$ be the closest vertex to t on P_j that also lies on another P_i, $i \in \{1, \ldots, m\} - \{j\}$. Then by the Intersection Lemma there exists $v_i \succeq v_3$ such that whenever $v_i \in VP_j$ then $v_i = v_j$. Let I be the set of all indices $j \in \{1, \ldots, m\}$ such that $v_j = v_i$. Note that $i \in I$ and $|I| \geq 2$, and we may assume that $i \neq 3$.

Lemma 5.5, together with the Mate Lemma, implies that there exists $P_j \in \{P_1, \ldots, P_m\}$ for which there is no cover B such that $|B - P_j - \{\Omega\}| = \tau - 3$. By Claim 5 we get that $j \neq 3$. After rerouting $P_1, \ldots, P_m$ in $H' \setminus (C_1 \cup C_2 - \{\Omega\})$, if necessary, we may assume that $j = 1$.

Claim 6. *$v_3 \neq s$.*

Proof of Claim. Suppose otherwise. Then, for some $j \in I$, $P_j[v_j, t]$ is not contained in an odd st-path of Σ'-signed subgraph $H' \setminus (C_1 \cup C_2 - \{\Omega\})$. Let $(G'', \Sigma'') := (G', \Sigma') \setminus P_1/P_2$ and $H'' := H' \setminus P_1/P_2$. Then it is easily seen that (M1) and (M2) are still satisfied by (G'', Σ'') and H''. Our choice of P_1 implies that (M4) also holds, and since $v_3 = s$, it follows that (M3) holds as well, subsequently contradicting the minimality of (G', Σ') and H'. ◇

Observe that Claim 6, together with the fact that $v_i \succeq v_3$, implies that, for every $j \in I$, $P_j[v_j, t]$ is contained in an odd st-dipath of Σ'-signed subgraph $H' \setminus (C_1 \cup C_2 - \{\Omega\})$. It also implies that $v_i \neq s$, and so $P_j[v_j, t]$ is contained in an even st-dipath of Σ'-signed subgraph $H' \setminus (C_1 \cup C_2 - \{\Omega\})$, for all $j \in I$, and so it is vertex-disjoint from C_1 and C_2.

Claim 7. *For each $j \in I$, there exists a cover B of (G', Σ') such that $|B - P_j[v_j, t] - \{\Omega\}| = \tau - 3$.*

Proof of Claim. Suppose otherwise. Then, for some $j \in I$, there is no cover B of (G', Σ') such that $|B - P_j[v_j, t] - \{\Omega\}| = \tau - 3$. Let $(G'', \Sigma'') := (G', \Sigma') \setminus P_j[v_j, t] / \cup (P_k[v_k, t] : k \in I, k \neq j)$ and $H'' := H' \setminus P_j[v_j, t] / \cup (P_k[v_k, t] : k \in I, k \neq j)$. It is now easily seen that (1)-(4) still hold for (G'', Σ'') and H'', contradicting the minimality of (G', Σ') and H'. ◇

Notice that, as a corollary, $1 \notin I$.

Claim 8. *There exists an* $s'v_i$*-dipath* Q *in* $H' \setminus (C_1 \cup C_2 - \{\Omega\})$ *that is vertex-disjoint from* P_1*.*

Proof of Claim. Suppose otherwise. Choose $v \in VP_1$ to be the closest vertex to s for which there is a vv_i-dipath R in $H' \setminus (C_1 \cup C_2 - \{\Omega\})$ with $\Omega \notin R$ and $VR \cap VP_1 = \{v\}$. Note that $P_1[s,v] \cup R \cup P_i[v_i,t]$ is an even st-dipath in $H' \setminus (C_1 \cup C_2 - \{\Omega\})$ and so by Claim 4, $VR \cap VC_1 = VR \cap VC_2 \subseteq \{s\}$. Now let $(G'', \Sigma'') := (G', \Sigma') \setminus P_1[v,t]/(R \cup P_i[v_i,t])$ and $H'' := H' \setminus P_1[v,t]/(R \cup P_i[v_i,t])$. Clearly, (M1) and (M2) still hold for (G'', Σ'') and H''. Our choice of P_1 implies that (M4) holds as well. We will now show that (M3) holds as well.

Let P be an odd st-dipath of $(H'', \Sigma'' \cap E(H''))$. Then P is a dipath in H' from s to a vertex $w \in \{t\} \cup VR$ and $\Omega \in P$. If $w = t$ then (M3) clearly holds. Otherwise $w \in VR$ and so by our assumption, it follows that $VP[s', w] \cap VP_1 \neq \emptyset$. By our choice of v, it follows that $VP[s', w] \cap VP_1[v, t] \neq \emptyset$. Choose $w' \in VP[s', w] \cap VP_1[v, t]$, and let B be a cover of (G', Σ') such that $|B - (P[s, w'] \cup P_1[w', t])| = \tau - 3$. Let $B' := B - P_1[w', t]$, this is a cover for (G'', Σ'') that satisfies $\tau - 3 \leq |B' - P| \leq |B' - P[s, w']| = \tau - 3$, and so (M3) holds.

Next let L be a non-simple directed odd st-walk of H'', and let C and Q be, respectively, the odd directed circuit and the even st-dipath contained in it. If $Q = \emptyset$, following the exact same approach as above on C (rather than P) shows that (M3) holds. Otherwise, C is still an odd directed circuit in H' and Q is a dipath in H' from s to a vertex $w \in \{t\} \cup VR$ and $\Omega \notin Q$.

If $w = t$ then (M3) clearly holds. Otherwise $w \in VR$ and so by our choice of v, it follows that $VQ[s, w] \cap VP_1[v, t] \neq \emptyset$. Choose $w' \in VQ[s, w] \cap VP_1[v, t]$, and let B be a cover of (G', Σ') such that $|B - (C \cup Q[s, w'] \cup P_1[w', t])| = \tau - 3$. Let $B' := B - P_1[w', t]$, this is a cover for (G'', Σ'') that satisfies $\tau - 3 \leq |B' - L| \leq |B' - C \cup Q[s, w']| = \tau - 3$, and so (M3) holds.

Thus, (M3) also holds for (G'', Σ'') and H'', contradicting the minimality of (G', Σ') and H'. $\diamond$

Pick $j \in I - \{i\}$, and choose minimal covers B_i and B_j such that $|B_i - P_i[v_i, t] - \{\Omega\}| = |B_j - P_j[v_j, t] - \{\Omega\}| = \tau - 3$. Since $B_i \cap C_1 = B_j \cap C_1 = \{\Omega\}$ it follows that both B_i and B_j are signatures. Choose a minimal $U \subseteq V - \{s, t\}$ so that $\delta(U) = B_i \triangle B_j$. By Lemma 4.2, there is a shortest path R in $G'[U] \setminus B_i$ between VP_i and VP_j. Observe that

$VC_1 \cap U = VP_1 \cap U = VQ \cap U = \emptyset$. It is now easily seen that $C_1 \cup Q \cup P_i[v_i, t] \cup P_j[v_j, t] \cup R \cup P_1$ has an F_7 minor. But then (G, Σ) has an F_7 minor, which is not possible. Thus, Part (6) is not feasible.

5.10 Part (7)

Recall that exactly one, say L_1, of L_1, L_2, L_3 is non-simple, and $\Omega \in C_1 \cap P_2 \cap P_3$. For this part, we will use the lemma stated in §5.7.

Claim. $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ *does not contain an odd cycle.*

Proof of Claim. Suppose, for a contradiction, that $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ does contain an odd cycle. We will yield a contradiction by showing that (G', Σ'), and therefore (G, Σ), must have an F_7 minor. Recall that B_1, B_2, B_3 are minimal covers of (G, Σ) such that $|B_j - L_j| = \tau - 3$, for all $1 \leq j \leq 3$.

Subclaim 1. *B_2 and B_3 are signatures but, B_1 is an st-bond.*

Proof of Subclaim 1. Observe that, for $i = 2, 3$, $B_i \cap C_1 = \{\Omega\}$ and so B_i cannot be an st-bond. It remains to show that B_1 is an st-bond. Suppose otherwise. Observe that, for all $k \in \{1, 2, 3\}$, $B_k \cap E(H') = B_k \cap L_k$, as $B_k \cap P_j = \emptyset$ for all $j \in \{4, \ldots, m\}$ and $B_k \cap L_i = \{\Omega\}$ for all $i \in \{1, 2, 3\} - \{k\}$. Take $j \in \{2, 3, \ldots, m\}$ and $k \in \{2, 3\} - \{j\}$. Then $B_k \cap (L_1 \cup P_j - \{\Omega\}) = \emptyset$ and so $P_i \cup P_j - \{\Omega\}$ is bipartite. Since this is true for all $j \in \{2, 3, \ldots, m\}$ and since B_1 is a signature, it follows that $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ is bipartite, contrary to our assumption.

End of Proof of Subclaim 1

Subclaim 2. *$P_2 \cup P_3$ contains an odd cycle, and $L_1 \cup P_2 - \{\Omega\}$ and $L_1 \cup P_3 - \{\Omega\}$ are bipartite.*

Proof of Subclaim 2. Take distinct $i, j \in \{2, 3, \ldots, m\}$ such that $\{i, j\} \neq \{2, 3\}$. Take $k \in \{2, 3\} - \{i, j\}$. Then $B_k \cap (P_i \cup P_j - \{\Omega\}) = \emptyset$ and $B_k \cap (L_1 \cup P_j - \{\Omega\}) = \emptyset$, because B_k is a signature. So $L_1 \cup P_j - \{\Omega\}$ and $P_i \cup P_j - \{\Omega\}$ are bipartite, for all $i, j \in \{2, 3, \ldots, m\}$ such that $\{i, j\} \neq \{2, 3\}$. In particular, $L_1 \cup P_2 - \{\Omega\}$ and $L_1 \cup P_3 - \{\Omega\}$ are bipartite. If

$P_2 \cup P_3$ is also bipartite, then $(H' \setminus \Omega, \Sigma' \cap E(H' \setminus \Omega))$ is bipartite as before, which is not the case. Hence, $P_2 \cup P_3$ contains an odd cycle. *End of Proof of Subclaim 2*

Subclaim 3. *Take two distinct vertices* $u, v \in VL_i \cap VL_j - \{s\}$*, for some distinct* $i, j \in \{1, 2, 3\}$*. If* $u \prec_{L_i} v$ *but* $v \prec_{L_j} u$ *then* $\{i, j\} = \{2, 3\}$*.*

Proof of Subclaim 1.3. Suppose that $u \prec_{L_i} v$ but $v \prec_{L_j} u$, but $\{i, j\} \neq \{2, 3\}$. We assume that $i = 2, j = 1$ and the other cases such as $i = 1, j = 3$ or $i = 3, j = 1$ can be treated similarly. Let $L_2' := L_2[s, u] \cup L_1[u, t]$ and $L_1' := L_1[s, v] \cup L_2[v, t]$, which are connected $\{s, t\}$-joins. Then $L_2' \cap B_3 = L_1' \cap B_3 = \{\Omega\}$, implying that L_1' and L_2' are both odd. However, this contradicts the minimality of $L_1 \cup L_2 \cup L_3 \cup \bigcup_{j=4}^{m} P_j$ by (A1). Hence $\{i, j\} = \{2, 3\}$. *End of Proof of Subclaim 3*

Therefore, $L_1 \cup L_i - \{\Omega\}$ is ayclic, for $i = 2, 3$. Let $F' := L_1 \cup L_2 \cup L_3$. Let (G'', Σ'') be a minor of (G', Σ') and let F'' be a directed graph obtained by orienting edges in a subgraph of G', where (G'', Σ'') and F'' are minimal subject to

(F1) $E(G') - E(G'') \subseteq E(F' \setminus \Omega)$, and $E(F'') \subseteq L_1 \cup L_2 \cup L_3$,

(F2) there exist three directed odd st-walks L_1'', L_2'', L_3'' in F'' that are pairwise disjoint except at Ω, where L_1'' is non-simple and $\Omega \in C_1''$, and L_2'', L_3'' are simple,

(F3) $L_1'' \cup L_i'' - \{\Omega\}$ is bipartite and acyclic for $i = 2, 3$, but $L_2'' \cup L_3''$ contains an odd cycle,

(F4) for any odd st-walk L of F'' there exists a cover B of (G'', Σ'') such that $|B - L| = \tau - 3$, and

(F5) there is no cover for (G'', Σ'') of size $\tau - 2$.

Note that these conditions are satisfied by (G', Σ') and F', so (G'', Σ'') and F'' are well-defined. We may assume that $F'' = L_1'' \cup L_2'' \cup L_3''$. Let B_i'' be a minimal cover of (G'', Σ'') such that $|B_i'' - L_i''| = \tau - 3$, whose existence is guaranteed by (F4). Since $L_2'' \cup P_3''$ contains an odd cycle, it follows that B_1'' is an st-bond, and since $B_2'' \cap C_1'' = B_3'' \cap C_1'' = \{\Omega\}$, B_2'', B_3'' are signatures. For the sake of notational ease, reset $L_i := L_i''$ and $B_i := B_i''$ for all $1 \leq i \leq 3$.

Choose a minimal vertex subset $U_{23} \in V(G'') - \{s, t\}$ such that $B_2 \triangle B_3 = \delta(U_{23})$. Since $P_2 \cup P_3$ contains an odd cycle, it follows that $VP_2 \cap VP_3 \cap U_{23} \neq \emptyset$.

Subclaim 4. *L_1 is internally vertex-disjoint from P_2 and P_3.*

Proof of Subclaim 4. We first show that C_1 is internally vertex-disjoint from P_2 and P_3. Suppose otherwise. Let $v \neq s', s$ be the closest vertex to s' on C_1 that lies on $P_2 \cup P_3$. We may assume that $v \in VP_2$.

We claim that there is an odd cycle in $P_2 \cup P_3$ that avoids $P_2[s, v]$. Suppose for a contradiction that this is not the case. Let $y \in VP_2 \cap VP_3 \cap U_{23}$. Since every odd cycle intersects $P_2[s', v]$, it follows that $y \in VP_2[s', v]$. Let $C'_1 := P_2[s, v] \cup C_1[v, t]$ and $P'_2 := C_1[s, v] \cup P_2[v, t]$. Let $C := C'_1[s', y] \cup P_3[s', y]$, which is an odd cycle in $C'_1 \cup P_3 - \{\Omega\}$. Notice that C'_1 is an odd circuit and P'_2 is an odd st-dipath in F''. Hence, by (F4) there is a cover B of (G'', Σ'') such that $|B - P'_2| = \tau - 3$. Then $B \cap (C'_1 \cup P_3) = \{\Omega\}$, and so B must be a signature as $B \cap C'_1 = \{\Omega\}$. But $B \cap (C'_1 \cup P_3) = \{\Omega\}$, implying that $B \cap C = \emptyset$, a contradiction since C is odd. Hence, there is an odd cycle in $P_2 \cup P_3$ that avoids $P_2[s, v]$.

Observe that if there is a cover B of (G'', Σ'') such that $|B - P_2[s, v]| = \tau - 3$, then B must be an st-cut, since there is an odd cycle of F'' avoiding $P_2[s, v]$, a contradiction to Lemma 5.5. Therefore, there is no cover B of (G'', Σ'') such that $|B - P_2[s, v]| = \tau - 3$. Let $(G''', \Sigma''') := (G'', \Sigma'') \setminus P_2[s', v]/C_1[s', v]$ and $F''' := F'' \setminus P_2[s', v]/C_1[s', v]$. It is easily seen that (F1),(F2) and (F4) hold for (G''', Σ''') and F'''. We just showed that (F5) holds as well. Moreover, since there is an odd cycle in $P_2 \cup P_3$ avoiding $P_2[s, v]$ (and L_1), it follows that (F3) holds as well for (G''', Σ''') and F''', contradicting the minimality of (G'', Σ'') and F''. Thus, C_1 is internally vertex-disjoint from P_2 and P_3.

We next show that P_1 is internally vertex-disjoint from P_2 and P_3. Suppose otherwise. Let $u \neq t$ be the closest vertex to t on P_1 that lies on $P_2 \cup P_3$. We may assume that $u \in VP_2$.

We claim that there is an odd cycle in $P_2 \cup P_3$ that avoids $P_2[u, t]$. Suppose for a contradiction that this is not the case. Let $y \in VP_2 \cap VP_3 \cap U_{23}$. Since every odd cycle intersects $P_2[u, t]$, it follows that $y \in VP_2[u, t]$. Let $P'_1 := P_1[s, u] \cup P_2[u, t]$ and $P'_2 := P_2[s, u] \cup P_1[u, t]$. Let $C := P'_1[y, t] \cup P_3[y, t]$, which is an odd cycle in $P'_1 \cup P_3 - \{\Omega\}$. Notice that P'_2 is an odd st-dipath in F'', so by (F4) there is a cover B of (G'', Σ'') such that $|B - P'_2| = \tau - 3$. Then $B \cap (C_1 \cup P'_1 \cup P_3) = \{\Omega\}$, and so B must be a signature as $B \cap C'_1 = \{\Omega\}$. But $B \cap (P'_1 \cup P_3) = \{\Omega\}$, implying that $B \cap C = \emptyset$, a contradiction since C is odd. Hence, there is an odd cycle in $P_2 \cup P_3$ that avoids $P_2[u, t]$.

Observe that if there is a cover B of (G'', Σ'') such that $|B - P_2[u, t] - \{\Omega\}| = \tau - 3$, then B must be an st-cut, since there is an odd cycle of $P_2 \cup P_3$ avoiding $P_2[u, t]$, a contradiction to Lemma 5.5. Therefore, there is no cover B of (G'', Σ'') such that $|B - P_2[u, t] - \{\Omega\}| = \tau - 3$. Let $(G''', \Sigma''') := (G'', \Sigma'') \setminus P_2[u, t]/P_1[u, t]$ and $F''' := F'' \setminus P_2[u, t]/P_1[u, t]$. It is easily seen that (F1),(F2) and (F4) hold for (G''', Σ''') and F'''. We just showed that (F5) holds as well. Moreover, since there is an odd cycle in $P_2 \cup P_3$ avoiding $P_2[s, v]$, it follows that (F3) holds as well for (G''', Σ''') and F''', contradicting the minimality of (G'', Σ'') and F''. Thus, P_1 is internally vertex-disjoint from P_2 and P_3.

Thus, L_1 is internally vertex-disjoint from P_2 and P_3, as claimed.

End of Proof of Subclaim 4

Subclaim 5. *If there is a directed circuit C in $P_2 \cup P_3$ then C is even.*

Proof of Subclaim 5. Suppose otherwise. Decompose $P_2 \cup P_3 \setminus C$ into the union of two $\{s, t\}$-joins P'_2 and L'_3. We may assume that P'_2 is even and L'_3 is odd. Let $L'_2 := C \cup P'_2$, which is a non-simple odd $\{s, t\}$-join. But then $\Omega \in C_1$ but $\Omega \notin C$, a contradiction to Lemma 4.5.

End of Proof of Subclaim 5

After contracting all the directed even circuits in $P_2 \cup P_3$, it is easily seen that $L_1 \cup P_2 \cup P_3$ has an F_7 minor. But then (G, Σ) has an F_7 minor, a contradiction. This finishes the proof of Claim. $\diamond$

Observe that Claim implies that

(∗) $H' \setminus \Omega$ is acyclic.

Suppose otherwise, and let C be a directed circuit in $H' \setminus \Omega$. Then one can find m st-dipaths $P'_1, P'_2, P'_3, \dots, P'_m$ and a directed circuit C'_1 in $H' \setminus C$ that are pairwise disjoint except possibly at Ω, and where $\Omega \in C'_1 \cap P'_2 \cap P'_3$ and $\Omega \notin P'_1 \cup P'_4 \cup P'_5 \cup \cdots \cup P'_m$. However, Claim implies that $L'_1 := C'_1 \cup P'_1$, $L'_2 := P'_2, L'_3 := P'_3$ are directed odd st-walks, and $P'_4, \dots, P'_m$ are even st-dipaths, contradicting the minimality of $L_1, L_2, L_3, P_4, \dots, P_m$ given by (A1).

Notice that (∗) implies that

(a) every even st-dipath Q in $H' \setminus (C_1 - \{\Omega\})$ is vertex-disjoint from C_1 except at s, and

(b) for every odd st-dipath P in $H' \setminus (C_1 - \{\Omega\})$, there exists a cover B of (G', Σ') such that $|B - P| = \tau - 3$.

To see (a), by rerouting $P_1, P_2, P_3, \dots, P_m$ in $H' \setminus (C_1 - \{\Omega\})$, if necessary, we may assume that $Q = P_1$, and it is therefore clear that Q and C_1 are vertex-disjoint except at s. To see (b), by rerouting $P_1, P_2, P_3, \dots, P_m$ in $H' \setminus (C_1 - \{\Omega\})$, if necessary, we may assume that $P = P_2$, and by (M3) such B exists.

For each P_j, $1 \le j \le m$, let $v_j \neq t$ be the closest vertex to t on P_j that also lies on another P_i, $i \in \{1, \dots, m\} - \{j\}$. Then by the Intersection Lemma there exists $v_i \succeq v_3$ such that whenever $v_i \in VP_j$ then $v_i = v_j$. Let I be the set of all indices $j \in \{1, \dots, m\}$ such that $v_j = v_i$. Note that $i \in I$ and $|I| \ge 2$. We may assume that $i \neq 1$. There are two possibilities based on whether or not $VC_1 \cap VP_j[v_j, t] = \emptyset$ for all $j \in I$.

5.10.1 Part (7.1): $VC_1 \cap VP_j[v_j, t] = \emptyset$ for all $j \in I$

Observe that since $v_i \succeq v_3$ and $v_3 \neq s$, it follows that, for every $j \in I$, $P_j[v_j, t]$ is contained in an odd st-dipath of $H' \setminus (C_1 - \{\Omega\})$.

Claim 1. *For each $j \in I$, there exists a cover B of (G', Σ') such that $|B - P_j[v_j, t] - \{\Omega\}| = \tau - 3$.*

Proof of Claim. Suppose otherwise. Then, for some $j \in I$, there is no cover B of (G', Σ') such that $|B - P_j[v_j, t] - \{\Omega\}| = \tau - 3$. Let $(G'', \Sigma'') := (G', \Sigma') \setminus P_j[v_j, t] / \cup (P_k[v_k, t] : k \in I, k \neq j)$ and $H'' := H' \setminus P_j[v_j, t] / \cup (P_k[v_k, t] : k \in I, k \neq j)$. It is now easily seen that (M1) and (M2) still hold for (G'', Σ'') and H''. By our assumption, (M4) holds as well, and since $VC_1 \cap VP_j[v_j, t] = \emptyset$ for all $j \in I$, it follows that (M3) also holds, contradicting the minimality of (G', Σ') and H'. ◇

Lemma 5.5, together with the Mate Lemma, implies that there exists $P_j \in \{P_1, \dots, P_m\}$ for which there is no cover B such that $|B - P_j - \{\Omega\}| = \tau - 3$. By (b) and Claim 1, we get that $j \notin \{2, 3\} \cup I$. After rerouting $P_1, \dots, P_m$ in $H' \setminus (C_1 - \{\Omega\})$, if necessary, we may assume that $j = 1$.

Claim 2. *There exists an $s'v_i$-dipath Q in $H' \setminus (C_1 - \{\Omega\})$ that is vertex-disjoint from P_1.*

Proof of Claim. Suppose otherwise. Choose $v \in VP_1$ to be the closest vertex to s for which there is a vv_i-dipath R in $H' \setminus (C_1 - \{\Omega\})$ with $\Omega \notin R$ and $VR \cap VP_1 = \{v\}$. Note that $P_1[s, v] \cup R \cup P_i[v_i, t]$ is an even st-dipath in $H' \setminus (C_1 - \{\Omega\})$ and so by (a), $VR \cap VC_1 \subseteq \{s\}$.

Now let $(G'', \Sigma'') := (G', \Sigma') \setminus P_1[v,t]/(R \cup P_i[v_i,t])$ and $H'' := H' \setminus P_1[v,t]/(R \cup P_i[v_i,t])$. Clearly, (M1) and (M2) still hold for (G'', Σ'') and H''. Our choice of P_1 implies that (M4) holds as well. We will now show that (M3) holds as well.

Let P be an odd st-dipath of H''. Then P is a dipath in H' from s to a vertex $w \in \{t\} \cup VR$ and $\Omega \in P$. If $w = t$ then (M3) clearly holds. Otherwise $w \in VR$ and so by our assumption, it follows that $VP[s', w] \cap VP_1 \neq \emptyset$. By our choice of v, it follows that $VP[s', w] \cap VP_1[v,t] \neq \emptyset$. Choose $w' \in VP[s', w] \cap VP_1[v,t]$, and let B be a cover of (G', Σ') such that $|B - (P[s,w'] \cup P_1[w',t])| = \tau - 3$. Let $B' := B - P_1[w',t]$, this is a cover for (G'', Σ'') that satisfies $\tau - 3 \leq |B' - P| \leq |B' - P[s,w']| = \tau - 3$, and so (M3) holds.

Next let L be a non-simple directed odd st-walk of H'', and let C and Q be, respectively, the odd directed circuit and the even st-dipath contained in it. If $Q = \emptyset$, following the exact same approach as above on C (rather than P) shows that (M3) holds. Otherwise, C is still an odd directed circuit in H' and Q is a dipath in H' from s to a vertex $w \in \{t\} \cup VR$ and $\Omega \notin Q$.

If $w = t$ then (M3) clearly holds. Otherwise $w \in VR$ and so by our choice of v, it follows that $VQ[s,w] \cap VP_1[v,t] \neq \emptyset$. Choose $w' \in VQ[s,w] \cap VP_1[v,t]$, and let B be a cover of (G', Σ') such that $|B - (C \cup Q[s,w'] \cup P_1[w',t])| = \tau - 3$. Let $B' := B - P_1[w',t]$, this is a cover for (G'', Σ'') that satisfies $\tau - 3 \leq |B' - L| \leq |B' - C \cup Q[s,w']| = \tau - 3$, and so (M3) holds.

Thus, (M3) also holds for (G'', Σ'') and H'', contradicting the minimality of (G', Σ') and H'. ◇

Pick $j \in I - \{i\}$, and choose minimal covers B_i and B_j such that $|B_i - P_i[v_i,t] - \{\Omega\}| = |B_j - P_j[v_j,t] - \{\Omega\}| = \tau - 3$. Since $B_i \cap C_1 = B_j \cap C_1 = \{\Omega\}$ it follows that both B_i and B_j are signatures. Choose a minimal $U \subseteq V - \{s,t\}$ so that $\delta(U) = B_i \triangle B_j$. Then by Lemma 4.2, there exists a shortest path R in $G'[U] \setminus B_i$ between VP_i and VP_j. Observe that $VC_1 \cap U = VP_1 \cap U = VQ \cap U = \emptyset$. It is now easily seen that $C_1 \cup Q \cup P_i[v_i,t] \cup P_j[v_j,t] \cup R \cup P_1$ has an F_7 minor. But then (G, Σ) has an F_7 minor, which is not possible. Hence, Part (7.1) is not possible.

5.10.2 Part (7.2): $VC_1 \cap VP_j[v_j,t] \neq \emptyset$ for some $j \in I$

We may assume that $VC_1 \cap VP_i[v_i,t] \neq \emptyset$. Therefore, (a) implies that $P_i[v_i,t]$, and so every $P_j[v_j,t]$ ($j \in I$), is not contained in any even st-dipath of $H' \setminus (C_1 - \{\Omega\})$. Hence, $I = \{2,3\}$ and $VP_j \cap VP_k = \{s,t\}$ for all $j \in [m] - \{2,3\}$ and $k \in \{2,3\}$. Let F' be the signed subgraph of H' induced by $L_1 \cup L_2 \cup L_3$.

Let (G'', Σ'') be a minor of (G', Σ') and let F'' be a directed graph obtained by orienting edges in a subgraph of G'', where (G'', Σ'') and F'' are minimal subject to

(F1) $E(G') - E(G'') \subseteq E(F') - (P_1 \cup \delta(s))$, and $E(F'') \subseteq E(F')$,

(F2) $F'' \setminus \Omega$ is acyclic, and there exist three pairwise disjoint $s't$-dipaths in F'', exactly one of which uses s,

(F3) for any $s't$-dipath Q of F'' that avoids s, there exists a cover B of (G'', Σ'') such that $|B - Q - \{\Omega\}| = \tau - 3$, and

(F4) there is no cover for (G'', Σ'') of size $\tau - 2$.

Note that these conditions are satisfied by (G', Σ') and F', so (G'', Σ'') and F'' are well-defined. By identifying a vertex of each component with s, if necessary, we may assume that G'' is connected. Now let $(Q_j)_{j=1}^{3}$ be pairwise disjoint $s't$-dipaths in F'', as in (F2), such that $s \in VQ_j$ if and only if $j = 1$. Note that $P_1 \subsetneq Q_1$, $Q_j \cup \{\Omega\}$ is an odd st-path for $j = 2, 3$, and $Q_1 \cup \{\Omega\}$ is a non-simple odd st-walk (note it is possible that $t \in VQ_1$). We may assume that $F'' \setminus \Omega = \bigcup_{j=1}^{3} Q_j$. Notice $(F'' \setminus \Omega, \Sigma'' \cap E(F'' \setminus \Omega))$ is bipartite.

Claim 1. *Let P be an odd st-dipath of F'', and let B be a cover of (G'', Σ'') such that $|B - P| = \tau - 3$. Then B is a signature.*

Proof of Claim. Since $F'' \setminus \Omega$ is acyclic, we may assume that $P = Q_2$, and so since $B \cap (\{\Omega\} \cup Q_1[s', s]) = \{\Omega\}$, it follows that B is a signature. ◇

For each Q_j let $v_j \neq s'$ be the closest vertex to s' on Q_j that also lies on another Q_i, $i \in \{1, 2, 3\} - \{j\}$. Then by the Intersection Lemma, there exists $i \in \{1, 2, 3\}$ such that whenever $v_i \in VQ_j$ then $v_i = v_j$. Let I be the set of all indices j in $\{1, 2, 3\}$ such that $v_j = v_i$. Note that $i \in I$ and $|I| \geq 2$.

Claim 2. *There exist internally vertex-disjoint paths Q and R in F'' that do not use s, and where Q is an $s't$-dipath and R is a v_it-dipath.*

Proof of Claim. Suppose otherwise. Then there exists a vertex v in F'' for which there is no $s't$-dipath in $F'' - \{s, v\}$. In particular, $v \in VQ_2 \cap VQ_3$.

In the first case, assume that $v \in VQ_1$. If there exists an $s'v$-dipath R' in F'' for which there is no cover B of (G'', Σ'') such that $|B - R' - \{\Omega\}| = \tau - 3$, then delete R' and contract all the other $s'v$-dipaths in F'' to get (G''', Σ''') and F''' that satisfy all of (F1)-(F4), which is not possible by the minimality of (G'', Σ'') and F''. Otherwise, contract all the vt-dipaths in F'', and then apply the $\widetilde{K_5}$ Lemma on $(\{\Omega\} \cup P_1 \cup (Q_i[s', v] : i \in [3]))/P_1$ to obtain a $\widetilde{K_5}$ minor, which cannot be the case since (G, Σ) has no such minor.

Hence, $v \notin VQ_1$. Then $(VQ_2[v, t] \cup VQ_3[v, t]) \cap VQ_1[s', s] = \emptyset$. If there exists a vt-dipath R' in F'' for which there is no cover B of (G'', Σ'') such that $|B - R' - \{\Omega\}| = \tau - 3$, then delete R' and contract all the other vt-dipaths in F'' to get (G''', Σ''') and F''' that satisfy all of (F1)-(F4), which is not possible by the minimality of (G'', Σ'') and F''.

Otherwise, for $i = 2, 3$, let B_i be a cover of (G'', Σ'') such that $|B_i - Q_i[v, t] - \{\Omega\}| = \tau - 3$. By Claim 1, both B_2 and B_3 are signatures. Let $U \subseteq V(G'') - \{s, t\}$ be a minimal subset such that $\delta(U) = B_2 \triangle B_3$, and let R be a shortest path in $G''[U] \setminus B_3$ between VQ_2 and VQ_3. Note that $VQ_1 \cap U = \emptyset$. Now it is easily seen that $\{\Omega\} \cup Q_1 \cup Q_2 \cup Q_3[v, t] \cup R$ has an F_7 minor, a contradiction since (G, Σ) has no such minor. ◇

Therefore, in particular, $|I| = 2$.

Claim 3. *For each $j \in I$, there exists a cover B of (G'', Σ'') such that $|B - Q_j[s', v_j] - \{\Omega\}| = \tau - 3$.*

Proof of Claim. If not, delete $Q_j[s', v_j]$ and contract the other path $Q_k[s', v_k]$, $k \in I - \{j\}$, to get (G''', Σ''') and F''' that satisfy all of (F1)-(F4), a contradiction to the minimality of (G'', Σ'') and F''. ◇

Now apply the $\widetilde{K_5}$ Lemma to $(\{\Omega\} \cup P_1 \cup Q \cup R \cup (Q_i[v_i, t] : i \in I))/(R \cup P_1)$ to obtain a $\widetilde{K_5}$ minor, a contradiction since (G, Σ) has no such minor. Hence, Part (7.2), and therefore Part (7), is not possible.

5.11 Part (8)

Recall that at least two, say L_1 and L_2, of L_1, L_2, L_3 are non-simple, and $\Omega \in P_1 \cap P_2 \cap P_3$. We will finish off the proof by showing that (G', Σ'), and therefore (G, Σ), contains an F_7 minor. Observe that all of B_1, B_2 and B_3 are st-bonds. Indeed, take $B_i \in \{B_1, B_2, B_3\}$, and choose $j \in \{1, 2\} - \{i\}$. Then $B_i \cap L_j = \{\Omega\}$ and so, since $\Omega \notin C_j$, it follows that $B_i \cap C_j = \emptyset$. However, C_j is an odd circuit, and so B_i cannot be a signature.

We now abandon our earlier criterion for the choice of B_i being minimal, and we assume instead that, for every $i \in \{1, 2, 3\}$,

> $(*)$ $B_i = \delta(U_i)$ is an st-cut such that $|B_i - L_i| = \tau - 3$, and $U_i \subseteq V - \{t\}$ is minimal among all possible choices of B_i. In other words, B_i is shorewise minimal.

Claim 1. *For all $i \in \{1, 2, 3\}$, $G''[U_i]$ is connected.*

Proof of Claim. Suppose otherwise. Then there exists a vertex subset $U \subseteq U_i - \{s\}$ for which $\delta(U) \subseteq \delta(U_i)$. Let $B := B_i \triangle \delta(U) = \delta(U_i \triangle U) = \delta(U_i - U)$, which is another st-cut for which $|B - L_i| = \tau - 3$, contradicting the shorewise minimality of B_i. ◇

Claim 2. *Whenever L_i is non-simple, for some $i \in \{1, 2, 3\}$, then $P_i \cap B_i = \{\Omega\}$.*

Proof of Claim. Suppose that L_i is non-simple for some $i \in \{1, 2, 3\}$. We may assume that $i = 1$. Suppose, for a contradiction, that $\{\Omega\} \subsetneq P_1 \cap B_1$. By (M4') there exists a cover B such that $|B - C_1 - P_2| = \tau - 3$. Since $B \cap C_2 = \emptyset$, it follows that B is an st-cut. So $B = \delta(U)$ for some $U \subseteq V - \{t\}$. Note that $U_1 \not\subseteq U$ since $P_1 \cap \delta(U_1) \supsetneq \{\Omega\}$ but $P_1 \cap \delta(U) = \{\Omega\}$. Now let $U_1' := U_1 \cap U \subsetneq U_1$, and let $B_1' := \delta(U_1')$. We claim that $|B_1' - L_1| = \tau - 3$, and this will contradict the shorewise minimality of B_1.

Observe first that $B_1' \subseteq B_1 \cup B$. For any $L_r \in \{L_3, L_4, \ldots, L_\tau\}$, we know that $|B_1' \cap L_r| \leq |B_1 \cap L_r| + |B \cap L_r| = 2$ and since $|B_1' \cap L_r|$ is odd, it follows that $|B_1' \cap L_r| = 1$. Moreover, $B_1' \cap C_2 \subseteq (B_1 \cap C_2) \cup (B \cap C_2) = \emptyset$ and so $B_1' \cap C_2 = \emptyset$. Since $U_1' \subsetneq U_1$ and $\delta(U_1) \cap P_2 = \{\Omega\}$, it follows that $B_1' \cap P_2 = \{\Omega\}$. Combining the two equalities yields $B_1' \cap L_2 = \{\Omega\}$. Hence, since $B_1' \subseteq B \cup B_1 \cup \bigcup_{j=1}^{\tau} L_j$, it follows that $|B - L_1| = \tau - 3$, as claimed, but this contradicts the shorewise minimality of B_1. Hence, $P_i \cap B_i = \{\Omega\}$ whenever L_i is non-simple for some $i \in \{1, 2, 3\}$. ◇

Claim 3. *There is a rearrangement i_1, i_2, i_3 of $1, 2, 3$ such that $U_{i_1} \subseteq U_{i_2} \subseteq U_{i_3}$.*

Proof of Claim. Choose distinct $i, j \in \{1, 2, 3\}$ and let k be the other index in $\{1, 2, 3\}$. We will show that either $U_i \subseteq U_j$ or $U_j \subseteq U_i$, and since this is true for all such i, j, it will follow that there is a rearrangement i_1, i_2, i_3 of $1, 2, 3$ such that $U_{i_1} \subseteq U_{i_2} \subseteq U_{i_3}$.

Suppose, for a contradiction, that neither $U_i \subseteq U_j$ nor $U_j \subseteq U_i$ is true. Let $U := U_i \cap U_j$, which is strictly contained in U_i and U_j, and let $U' := U_i \cup U_j$. Similarly as above, $\delta(U) \subseteq B_i \cup B_j$, and $|\delta(U) \cap L_r| = 1$ for all $L_r \subset \{L_k, L_4, \ldots, L_\tau\}$.

Since $\Omega \in \delta(U)$ and $P_i \cap B_j = P_j \cap B_i = \{\Omega\}$, it follows that $P_i \cap \delta(U) = P_j \cap \delta(U) = \{\Omega\}$. However, since U is strictly contained in U_i and U_j, the shorewise minimality of B_i and B_j therefore implies that $\delta(U) \cap L_j \neq \{\Omega\}$ and $\delta(U) \cap L_i \neq \{\Omega\}$. Hence, L_i and L_j are non-simple, and $\delta(U) \cap C_j \neq \emptyset$ and $\delta(U) \cap C_i \neq \emptyset$. Thus, since $C_i \cap \delta(U_j) = C_j \cap \delta(U_i) = \emptyset$, it follows that $C_i \subseteq G'[U_j]$ and $C_j \subseteq G'[U_i]$, and so $C_i \cup C_j \subseteq G'[U']$.

Next consider $\delta(U')$. It is again the case that $\delta(U') \subseteq B_i \cup B_j$ and $|\delta(U') \cap L_r| = 1$ for all $L_r \in \{L_k, L_4, \ldots, L_\tau\}$. Since $C_i \cup C_j \subseteq G'[U']$, and since $P_i \cap B_i = P_j \cap B_j = \{\Omega\}$ by Claim 2, it follows that $\delta(U') \cap L_i = \delta(U') \cap L_j = \{\Omega\}$. However, $\delta(U') \subseteq B_i \cup B_j \subseteq \bigcup_{j=1}^{\tau}$ and so $|\delta(U')| = \tau - 2$, a contradiction to (M5'). Consequently, either $U_i \subseteq U_j$ or $U_j \subseteq U_i$.$\diamond$

Note that if $C_3 = \emptyset$, then $P_3 \cap B_3 \neq \{\Omega\}$ and so we cannot have $U_3 \subseteq U_1$ or $U_3 \subseteq U_2$. Moreover, notice that U_1, U_2 and U_3 are pairwise different, because B_1, B_2 and B_3 are pairwise different. Therefore, we may assume that $U_1 \subsetneq U_2 \subsetneq U_3$.

Claim 4. *Suppose that C and C' are two disjoint odd cycles such that $C \cup C' = C_1 \cup C_2$ and $C' \cap \delta(U_1) \neq \emptyset$. Then C and C' are odd circuits, $VC \subseteq V - U_1$ and $VC \nsubseteq U_2$.*

Proof of Claim. Observe first that C and C' are odd circuits due to the minimality of L_1, L_2, L_3. Let $L_1' = C' \cup P_1$ and $L_2' = C \cup P_2$. By (M4') there exist covers B_1' and B_2' such that $|B_1' - L_1'| = |B_2' - L_2'| = \tau - 3$. Since $B_1' \cap C = B_2' \cap C' = \emptyset$ it follows that B_1' and B_2' are st-bonds, and so there exist $U_1', U_2' \subseteq V - \{t\}$ such that $B_i' = \delta(U_i')$ for $i = 1, 2$. Suppose further that, for $i = 1, 2$, B_i' and U_i' are chosen under $(*)$, so that U_i' is minimal. So by Claims 2 and 3, $B_i' \cap P_i = \{\Omega\}$ for $i = 1, 2$, and either $U_1' \subseteq U_2' \subseteq U_3$ or $U_2' \subseteq U_1' \subseteq U_3$.

Let $W := U_1' \cap U_2' \in \{U_1', U_2'\}$ and consider $\delta(W \cap U_1)$. Observe that $\delta(W \cap U_1) \subseteq \delta(W) \cup B_1$, and since $VC_2 \subseteq V - U_1$, $\delta(W \cap U_1) \cap C_2 = \emptyset$. Hence, $|\delta(W \cap U_1) - L_1| = \tau - 3$, and so by the minimality of U_1, it follows that $W \cap U_1 = U_1$. Similarly, by reversing the roles of L_1, L_2, L_3 by L_1', L_2', L_3, one can show that $W \cap U_1 = W$ and so $W = U_1$. However, $\emptyset \neq C' \cap \delta(U_1) = C' \cap \delta(W)$ and so $U_1 = W = U_1'$ and $U_1' \subseteq U_2'$. But $VC \subseteq U - U_1' = U - U_1$, as claimed.

Next consider $\delta(U_2' \cup U_2)$, which is a subset of $B_2' \cup B_2$. We know that $VC' \subseteq U_2'$. If $VC \subseteq U_2$ then $VC_1 \cup VC_2 = VC \cup VC' \subseteq U_2' \cup U_2$, implying that $\delta(U \cup U_2) \cap (C_1 \cup C_2) = \emptyset$

and so $|\delta(U \cup U_2)| = \tau - 2$, a contradiction to (M5'). Hence $VC \nsubseteq U_2$, as claimed. $\diamond$

Observe that in the proof above, the minimality of B_2 and U_2 under $(*)$ was not used at all. This fact will come in handy later.

Claim 5. *Assume that $C_3 = \emptyset$. Suppose that C is an odd cycle and L is an odd $\{s,t\}$-join disjoint from C such that $C \cup L = C_2 \cup P_3$. Then C is an odd circuit, L is an odd st-walk and $L \cap \delta(U_2) = \{\Omega\}$.*

Proof of Claim. Due to the minimality of L_1, L_2, L_3, it trivially follows that C is an odd circuit and L is an odd st-walk. Let $L'_2 := C \cup P_2$ and $L'_3 := L$. By (M4') there exist covers B'_2 and B'_3 such that $|B'_2 - L'_2| = |B'_3 - L'_3| = \tau - 3$. Since $B'_i \cap C_1 = \emptyset$, it follows that B'_i is an st-bond, for $i = 2, 3$. So there exist $U'_2, U'_3 \subseteq V - \{t\}$ such that $B'_i = \delta(U'_i)$ for $i = 2, 3$. Suppose further that, for $i = 2, 3$, B'_i and U'_i are chosen under $(*)$, so that U'_i is minimal. By Claim 3 we get that $U'_2 \subseteq U'_3$. Now consider $\delta(U'_2 \cap U_2)$. Observe that $\delta(U'_2 \cap U_2) \subseteq B'_2 \cup B_2$, and since $VL_3 - \{s\} \subseteq V - U_2$, $\delta(U'_2 \cap U_2) \cap L_3 = \{\Omega\}$. Thus $|\delta(U' \cap U_2) - L_3| = \tau - 3$, and so by the minimality of U_2, it follows that $U'_2 \cap U_2 = U_2$. Similarly, by reversing the roles of L_1, L_2, L_3 by L_1, L'_2, L'_3, one can show that $U'_2 \cap U_2 = U'_2$ and so $U'_2 = U_2$. But $\{\Omega\} = L'_3 \cap \delta(U'_2) = L \cap \delta(U_2)$, as claimed. $\diamond$

Claim 6. *Assume that $C_3 = \emptyset$. Suppose that $L \subseteq P_2 \cup L_3$ is an odd $\{s,t\}$-join. Then L is an odd st-walk, $L \triangle P_2 \triangle L_3$ is an even st-path, and $L \cap \delta(U_3) = L_3 \cap \delta(U_3)$.*

Proof of Claim. Due to the minimality of L_1, L_2, L_3, it trivially follows that L is an odd st-walk and $L \triangle P_2 \triangle L_3$ is an even st-path. Let $L'_2 := C_2 \cup (L \triangle P_2 \triangle L_3)$ and $L'_3 := L$. By (M4') there exist covers B'_2 and B'_3 such that $|B'_2 - L'_2| = |B'_3 - L'_3| = \tau - 3$. Since $B'_i \cap C_1 = \emptyset$, it follows that B'_i is an st-bond, for $i = 2, 3$. So there exist $U'_2, U'_3 \subseteq V(G') - \{t\}$ such that $B'_i = \delta(U'_i)$ for $i = 2, 3$. Suppose further that, for $i = 2, 3$, B'_i and U'_i are chosen under $(*)$, so that U'_i is minimal. By Claim 3 we get that $U'_2 \subseteq U'_3$. Consider $\delta(U'_3 \cap U_3)$. Observe that $\delta(U'_3 \cap U_3) \subseteq B'_3 \cup B_3$, and since $VP_2 - \{s\} \subseteq V - U_3$ and $VC_2 \subseteq U'_3 \cap U_3$, $\delta(U'_3 \cap U_3) \cap L_2 = \{\Omega\}$. Thus, $|\delta(U'_3 \cap U_3) - L_3| = \tau - 3$ and so by the minimality of U_3, it follows that $U'_3 \cap U_3 = U_3$. Similarly, by reversing the roles of L_1, L_2, L_3 by L_1, L'_2, L'_3, one can show that $U'_3 \cap U_3 = U'_3$, and so $U'_3 = U_3$. The result now easily follows. $\diamond$

This is an immediate corollary of Claim 6.

Claim 7. *Assume that* $C_3 = \emptyset$. *Suppose that* $L \subseteq P_2 \cup L_3$ *is an odd* $\{s,t\}$*-join. Suppose that* $w \in VP_2 \cap VL_3$. *Then either* $L_3[s', w] \subseteq G'[V - U_3]$ *and* $L_3[s', w] \cup P_2[s', w]$ *is an even cycle, or* $L_3[w, t] \subseteq G'[V - U_3]$ *and* $L_3[w, t] \cup P_2[w, t]$ *is an even cycle.*

The following is the last ingredient needed to find an F_7 minor. Let L_0 be the singleton vertex $\{s\}$, $U_0 := \emptyset$ and $B_0 := \emptyset$.

Claim 8. *Take* $v \in VL_{j+1} \cap (U_{j+1} - U_j)$ *for some* $j \in \{0, 1, 2\}$. *Let* U *be the component of* $G'[U_{j+1} - U_j]$ *containing* v. *Then* $VL_j \cap U \neq \emptyset$.

Proof of Claim. Suppose otherwise. Observe that $\delta(U) \subseteq B_j \cup B_{j+1}$, $\delta(U) \cap L_{j+1} \neq \emptyset$ and $\delta(U) \cap L_j = \emptyset$. But then $|\delta(U_{j+1} - U) - L_{j+1}| = \tau - 3$ since $\delta(U_{j+1} - U) = \delta(U_{j+1} \triangle U) = B_{j+1} \triangle \delta(U)$, contradicting the choice of B_{j+1}, U_{j+1} under $(*)$. ◇

Claim 9. (G', Σ') *has an* F_7 *minor.*

Proof of Claim. Here $\mathbf{X}$ denotes the image of object X after contraction and/or deletion is applied in (G', Σ'). If X is not affected under the operation, then $X = \mathbf{X}$; repeatedly applying contraction and/or deletion resets $\mathbf{X}$.

By Claim 8, there exists a shortest path Q_0 in $G'[U_1]$ between s and VC_1. Suppose that x is the other end-vertex of Q_0. Contract Q_0, and then contract all the edges in $\mathbf{C_1}$ that are not incident with $\mathbf{s}$. At this stage, $\mathbf{C_1}$ is an odd circuit with two edges and two vertices $\mathbf{s}$ and, say, u.

We claim that $\mathbf{C_1}$ is the only odd circuit in $(\mathbf{C_1} \cup \mathbf{C_2}) \cap \mathbf{G'}[\mathbf{U_2}]$. Suppose otherwise. Let $\mathbf{C} \subseteq (\mathbf{C_1} \cup \mathbf{C_2}) \cap \mathbf{G'}[\mathbf{U_2}]$ be an odd circuit different from $\mathbf{C_1}$. Notice that $\mathbf{s} \notin V\mathbf{C}$. Let C be *an* inverse image of $\mathbf{C}$ under the contractions such that C avoids the vertex x, it is odd and $C \subseteq (C_1 \cup C_2) \cap G'[U_2]$. By Claim 4 then, since $VC \subseteq U_2$, we must have that $C \cap \delta(U_1) = C_1 \cap \delta(U_1)$ and $C \cap EG'[U_1] = C_1 \cap EG'[U_1]$. So $x \in VC$, which is not the case. Hence, $\mathbf{C_1}$ is the only odd circuit in $(\mathbf{C_1} \cup \mathbf{C_2}) \cap \mathbf{G'}[\mathbf{U_2}]$.

By Claim 8 there is a shortest path Q_1 in $\mathbf{G'}[\mathbf{U_2}]$ between $V\mathbf{C_1}$ and $V\mathbf{C_2}$ that does not use the vertex $\mathbf{s}$. (It is possible that $Q_1 = \emptyset$.) Contract Q_1 and note that $\mathbf{C_1}$ is still the only odd circuit in $(\mathbf{C_1} \cup \mathbf{C_2}) \cap \mathbf{G'}[\mathbf{U_2}]$. Let $\mathbf{C_2'}$ be an odd circuit contained in $\mathbf{C_2}$ that uses the vertex $\mathbf{u}$. Let U be the union of the components of $G'[U_3 - U_2]$ that contain a vertex of of $V\mathbf{C_2'}$.

We claim that $U \cap VL_3 \neq \emptyset$. Suppose not. Then there exists an inverse image C' of $\mathbf{C_2'}$ under the contractions such that $x \notin VC'$, C' is odd and $C' \subseteq (C_1 \cup C_2) \cap G'[U_2 \cup U]$. Let $U_2' := U_2 \cup U = U_2 \triangle U$, and note that for $B_2' := \delta(U_2') = \delta(U_2) \triangle \delta(U)$, we have $|B_2' - L_2| = \tau - 3$. By Claim 4 then, since $VC' \subseteq U_2'$, we must have that $C' \cap \delta(U_1) = C_1 \cap \delta(U_1)$ and $C' \cap EG'[U_1] = C_1 \cap EG'[U_1]$. But then $x \in VC'$, which is not the case. Thus, $U \cap VL_3 \neq \emptyset$, and so there exists a shortest path Q_2 between $V\mathbf{C_2'}$ and VL_3 in $G'[U_3 - U_2]$.

Now contract all the edges in $\mathbf{C_2'}$ that are not incident with u. At this stage, $\mathbf{C_2} := \mathbf{C_2'}$ is an odd circuit with exactly two edges and two vertices $\mathbf{u}$ and, say, v. Similarly as above, by using Claim 4 and Claim 5 this time though, we obtain that $\mathbf{C_2}$ is the only odd circuit in $(\mathbf{C_2} \cup \mathbf{L_3}) \cap \mathbf{G'}[\mathbf{U_3}]$. Notice that $\mathbf{s}, \mathbf{u} \notin V\mathbf{Q_2}$. Contract $\mathbf{Q_2}$ and note that $\mathbf{C_2}$ is still the only odd circuit in $(\mathbf{C_2} \cup \mathbf{L_3}) \cap \mathbf{G'}[\mathbf{U_3}]$.

First assume that L_3 is non-simple. Let $\mathbf{C_3'}$ be an odd circuit contained in $\mathbf{C_3}$ that uses the vertex $\mathbf{v}$. Let U' be the union of the components of $G'[V - U_3]$ that contain a vertex of $V\mathbf{C_3'}$. As before, by Claim 4 it follows that $t \in U'$. So there is a shortest path Q_3 in $G'[V - U_3]$ between $V\mathbf{C_3'}$ and VP_3. Contract all the edges in $\mathbf{C_3'}$ that are not incident with $\mathbf{v}$ and all the edges in $P_3 \cup Q_3 - \{\Omega\}$. Now $\mathbf{C_3} := \mathbf{C_3'}$ is an odd circuit with exactly two edges and two vertices $\mathbf{v}$ and $\mathbf{t}$. Note that Ω is an even edge between $\mathbf{s}$ and $\mathbf{t}$. As a result, $\mathbf{C_1} \cup \mathbf{C_2} \cup \mathbf{C_3} \cup \{\mathbf{\Omega}\}$ is isomorphic to F_7.

Next assume that L_3 is simple. In the first case, assume that there exists an odd st-path $\mathbf{P_3'}$ in $\mathbf{P_3}$ that uses the vertex $\mathbf{v}$. Contract all the edges in $P_2 - \{\Omega\}$ and all the edges in $\mathbf{P_3'}$ that are not incident with $\mathbf{v}$. By Claim 7, $\mathbf{P_3'}$ contains an odd st-walk that consists of the even edge $\mathbf{\Omega}$, which is between $\mathbf{s}$ and $\mathbf{t}$, and an odd circuit D_3 of length two between $\mathbf{v}$ and $\mathbf{t}$. It is clear that $\mathbf{C_1} \cup \mathbf{C_2} \cup D_3 \cup \{\mathbf{\Omega}\}$ is isomorphic to F_7.

In the remaining case, there exists an odd circuit C_3' in $\mathbf{P_3}$ that uses the vertex $\mathbf{v}$. Let U'' be the union of the components of $G'[V - U_3]$ that contain a vertex of VC_3'. Similarly as before, we know that $t \in U''$. According to Claim 7, $VP_2 \cap VC_3' = \emptyset$. Now let Q_3 be a shortest path in $G'[V - U_3]$ between VC_3' and VP_2. Now contract all the edges in C_3' that are not incident with $\mathbf{v}$ and all the edges in $(\mathbf{P_3} \cup Q_3) \cap G'[V - U_3]$. $\mathbf{C_3'}$ is now an odd circuit with exactly two edges and two vertices $\mathbf{v}$ and $\mathbf{t}$, and $\mathbf{\Omega}$ is an even edge between $\mathbf{s}$ and $\mathbf{t}$. So $\mathbf{C_1} \cup \mathbf{C_2} \cup \mathbf{C_3'} \cup \{\mathbf{\Omega}\}$ is isomorphic to F_7.

Therefore, (G', Σ'), and therefore (G, Σ), has a minor isomorphic to F_7, proving Claim 9.

◇

However, (G, Σ) does not have such a minor, proving that Part (8) is not possible either. This finally finishes the proof of Theorem 1.5.

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