The Local Chromatic Number

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The local chromatic number $\psi(G)$ of a graph G is a graph colouring parameter that is defined as

$$\psi(G) = \min_{c} \max_{v \in V(G)} |c(\bar{\Gamma}(v))|$$

where the minimum is taken over all proper colourings c of G and $\bar{\Gamma}(v)$ denotes the closed neighbourhood of a vertex v. So unlike the chromatic number of a graph G which is the minimum total number of colours required in a proper colouring of G, the local chromatic number is minimum number of colours that must appear in the closed neighbourhood of some vertex G in a proper colouring.

In this thesis we will examine basic properties of the local chromatic number, and techniques used to determine or bound it. We will examine a theory that was sparked by Lovász's proof [22] of the Kneser conjecture, using topological tools to give lower bounds on the chromatic number, and see how it is applicable to give lower bounds on the local chromatic number as well.

The local chromatic number lies between the fractional chromatic number and the chromatic number, and thus it is particularly interesting to study when the gap between these two parameters is large. We will examine the local chromatic number for specific classes of graphs, and give a slight generalization of a result by Simonyi and Tardos from [33] that gives an upper bound on the local chromatic number for a class of graphs called Schrijver graphs.

Finally we will discuss open conjectures about the chromatic number and investigate versions adapted to the local chromatic number.

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Chapter 1

Introduction

The local chromatic number is a graph parameter about colourings of graphs. Unlike the chromatic number which tells us how many colours must be used in total to colour a graph, the local chromatic number tell us how many colours must be used in a closed neighbourhood of a vertex. More formally, let us define an *m-colouring c* of a graph G to be a colouring using at most m colours, meaning that c can be seen as a map $c\colon V(G) \to [m]$ that assigns each vertex of G a colour. Here and in the following $[m]$ denotes the set $\{1, \dots, m\}$ of integers from 1 to m. A colouring is considered proper if no two adjacent vertices $u \sim v$ receive the same colour. Here $\sim$ denotes adjacency, and in the following we will assume colourings to be proper unless otherwise stated. Similarly, we call a vertex colouring *k-local* if in the closed neighbourhood $\bar{\Gamma}(v) = \{w \in V(G)\colon w \sim v\} \cup \{v\}$ of any vertex v at most k different colours are used.

Analogous to the chromatic number $\chi(G)$ being the smallest m for which an m-colouring of G exists, the *local chromatic number $\psi(G)$* is the smallest k such that a k-local colouring of G exists:

Definition 1.1. *The local chromatic number of a graph G is defined to be*

$$\psi(G) = \min_{c} \max_{v \in V(G)} |c(\bar{\Gamma}(v))|$$

where the minimum is taken over all proper colourings c of G.

Notice that it could also be expressed in terms of the neighbourhood $\Gamma(v)$ instead of the closed neighbourhood $\bar{\Gamma}(v)$ if the value of 1 is added at the end. The local chromatic number was first introduced and studied by Erdős, Füredi, Hajnal, Komjáth, Rödl, and Seress in [10].

One easy observation to make is that the local chromatic number can never be larger than the chromatic number, as any m-colouring is also an m-local colouring. Thus we get $\psi(G) \leq \chi(G)$. Less trivially though, we will see in Section 2.5 that the local chromatic number is bounded below by the fractional chromatic number $\chi_f(G)$, i.e. $\chi_f(G) \leq \psi(G)$, as shown by Körner, Pilotto and Simonyi in [20].

This makes it an interesting parameter to study, as there are various open conjectures about the chromatic number whose fractional versions are known to be true. Proving versions pertaining to the local chromatic number of these conjectures would give them further support. Yet as we will see in Chapter 2 there are graphs where the gap between the chromatic number and the local chromatic number is arbitrarily big. However also the gap between the fractional chromatic number and the local chromatic number can be arbitrarily big. So while local versions of these conjectures might not always give us much information about the chromatic number, they could still provide an improvement over fractional versions. Furthermore, with the local chromatic number lying between the fractional chromatic number χ_f and the chromatic number χ, it is particularly interesting to study it for graphs where the gap between χ_f and χ is large.

Chapter 2 is dedicated to studying the local chromatic numbers for such classes of graphs, along with basic properties of the local chromatic number and tools to gain information about it. Many of these have been summarized by Simonyi and Tardos in [33], and thus this chapter gives an exposition of a number of results from that paper. We will also see that just like for the chromatic number, determining the local chromatic number is NP-hard, and as a corollary of its relation to the fractional chromatic number that there are graphs with high girth and high local chromatic number. Finally we will see that graphs whose gap between the local chromatic number and the chromatic number is large need to use a high number of colours in total in their local colourings.

Chapter 3 gives an upper bound on the local chromatic number of Schrijver graphs. This new result is a slight generalization of a result given in [33] while still taking its main ideas from the original result.

In Chapter 4 we will first introduce topological tools and then use them to give a general lower bound on the local chromatic number in terms of a topological parameter which we will call the topological chromatic number. The groundbreaking idea to use topological tools to give lower bounds on the chromatic number was originally conceived by Lovász in 1978 to prove the Kneser conjecture [22] which at that point had been open for over 40 years. Over time the topological tools have been refined to give shorter variants of

Lovász' proof, and to develop a theory that can be applied more broadly. Matoušek's book "Using the Borsuk-Ulam Theorem" [25] gives a very thorough overview of this theory. This chapter will conclude with a combinatorial proof of the lower bound on the local chromatic number for Kneser graph. It is based on Matoušek's combinatorial proof of the Kneser conjecture [24], which in turn is inspired by the topological proofs of the conjecture.

Finally, Chapter 5 will investigate conjectures whose fractional versions or some other variants are known to be true, to see whether we can make statements about versions pertaining to the local chromatic number.

Chapter 2

Basic background on the local chromatic number

In this chapter we will survey various properties of the local chromatic number, beginning with a proof to show the existence of graphs with local chromatic number 3 and arbitrarily high chromatic number. In Section 2.1 we will proceed to show that determining the local chromatic number is NP-hard. Section 2.2 will provide us with a tool to give upper bounds on the local chromatic number. In Section 2.3 we will investigate the local chromatic number for various classes of graphs. Many of these results use topological parameters to give a lower bound on the local chromatic number. These techniques will be described in detail later in Chapter 4. In Section 2.4 we will see that k-local colourability can be expressed in terms of homomorphisms into some universal graph. We will furthermore see that 3-local colourings in graphs with $\psi(G) = 3$ but high chromatic number $\chi(G)$ must use a high number of colours in total. Finally, 2.5 will explain the relation between the local chromatic number and the fractional chromatic number.

We've already observed that the local chromatic number bounds the chromatic number from below. A natural question to ask is how far apart these two parameters can be. In [10] this question is answered by giving a class of graphs with local chromatic number 3 but with arbitrarily high chromatic number. A more detailed version of this proof is provided below. However in the same paper they also show that the total number of colours used in such 3-local colourings must be very high, see Section 2.4 for more details.

In order to prove our theorem, we will apply a hypergraph version of Ramsey's theorem. Our version is a special case of the theorem first proven by Ramsey in [29], and is stated below. A hypergraph $\mathcal{H}$ on a set of vertices V is simply a set of subsets of V. These subsets are called edges. If all the subsets have the same size r, then the hypergraph is called r-uniform. The complete r-uniform hypergraph on a vertex set V has all subsets of V of size r as edges.

Theorem 2.1 (Ramsey's theorem)**.** *Let r, c and k be positive integers. Then if n is large enough, any c-colouring of the edges of the r-uniform complete hypergraph on the vertex set $[n]$ will contain a monochromatic induced subhypergraph on k vertices. In other words, there is a set of k vertices such that all edges that are subsets of these vertices will have the same colour.* □

Now we can proceed to show the existence of graphs that are 3-locally colourable, but have arbitrarily high chromatic number. Henceforth, for a set S, let $\binom{S}{k}$ denote the set of k-subsets of S.

Theorem 2.2. *There are graphs with local chromatic number 3 and arbitrarily high chromatic number.*

Proof. For an integer n, define the graph G_n with $V(G_n) = \binom{[n]}{3}$ as the set of 3-subsets of $[n]$. Henceforth, assume $x_0 < x_1 < x_2$ when these variables appear as a vertex $\{x_0, x_1, x_2\}$. We define the edges as follows: $\{x_0, x_1, x_2\} \sim \{y_0, y_1, y_2\}$ if and only if $x_1 = y_0$ and $x_2 = y_1$ (or $y_1 = x_0$ and $y_2 = x_1$). This graph is called a *shift graph*. If we set $c(\{x_0, x_1, x_2\}) = x_1$, then this is a proper colouring. Furthermore it is a 3-local colouring, as the neighbours of $\{x_0, x_1, x_2\}$ receive the colours x_0 and x_2, see Figure 2.1.

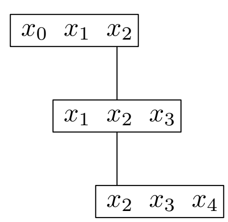


Figure 2.1: Neighbours of the vertex $\{x_1, x_2, x_3\}$ are of the form $\{x_0, x_1, x_2\}$ and $\{x_2, x_3, x_4\}$ ($x_0 < x_1$ and $x_4 > x_3$) and receive colours x_1 and x_3 respectively. As $\{x_1, x_2, x_3\}$ receives colour x_2 this shows that the colouring is proper and 3-local.

We shall now show that for any k there is an n such that $\chi(G_n) > k$. Let c be a proper k-colouring of G_n. Note that the vertex set of G_n is the edge set of the complete 3-uniform hypergraph $\mathcal{H}_n$ on the vertex set $[n]$, and we can view c as a colouring of the edges of $\mathcal{H}_n$. Now by Ramsey's theorem, if n is sufficiently large, we will find a set of 4 (or more if we wish) vertices $x_0 < x_1 < x_2 < x_3$ such that all the hypergraph edges induced by these vertices receive the same colour. In particular, $c(\{x_0, x_1, x_2\}) = c(\{x_1, x_2, x_3\})$. But these two vertices are adjacent in G_n, showing that the colouring c is not proper. $\square$

2.1 NP-completeness

Just as for most other chromatic parameters of graphs, the decision problems pertaining to the local chromatic number are hard:

Deciding whether there is a k-local m-colouring of a graph if $m \geq k \geq 3$ was shown to be NP-complete by Kun and Nešetřil in [21]. If $m \leq 2$ or $k \leq 2$, the problem can be decided in polynomial time: If a graph has an odd cycle, then in any colouring of that cycle one vertex will see two other colours in its neighbourhood, therefore no 2-local coloring can exist. This implies that the local chromatic number of a graph is at most 2 if and only if the graph is bipartite, but verifying whether a graph is bipartite can be done in linear time using breadth-first search for example.

In this section we will show that deciding whether a graph has a k-local colouring without any restriction on the total number of colours used is still NP-complete. The fact that the problem of k-colourability of graph is NP-complete for $k \geq 3$ is a well known folklore result. Using a similar proof we will now see that determining whether a graph has a k-local colouring for $k = 3$ is NP-complete. The case for $k \geq 3$ follows as an easy reduction to the case $k = 3$.

Theorem 2.3. *Given a graph G, it is NP-hard to decide whether G admits a 3-local colouring.*

Proof. Obviously the problem is in NP, as we can verify the validity of a colouring in polynomial time.

NP-hardness is shown by a reduction to 3-SAT (or SAT, which uses the same construction):

Assume we have a formula F in conjunctive normal form with variables $x_1, \ldots, x_k$. We construct a graph G associated with F that has a 3-local colouring if and only if there is an assignment of boolean values to the variables $x_1, \ldots, x_k$ that turns F into a true statement.

We start with a triangle of vertices t, f and a. For any proper colouring c, we may assume without loss of generality that the colours used on the triangle are called aux, $true$ and $false$ and $c(a) = aux$, $c(t) = true$ and $c(f) = false$. For each variable x_i we introduce

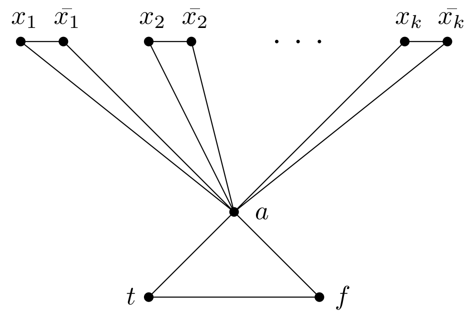


Figure 2.2: The basic setup of the vertices that correspond to variables.

two vertices x_i and $\bar{x}_i$, which are connected to a and to each other by edges, see Figure 2.2.

We associate x_i with the original variable x_i and $\bar{x_i}$ with the negation of the variable x_i. Note that in any 3-local colouring we can only use the colours *true* and $false$ in the neighbourhood of a, so for each pair x_i and $\bar{x_i}$ one of the vertices receives the colour *true* and the other one the colour $false$.

Figure 2.3 shows an or-gadget which we introduce to be used later:

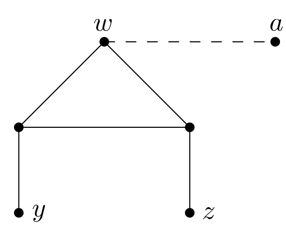


Figure 2.3: An or-gadget. The dashed edge means that the vertex w is connected to the vertex a of the main gadget. Assuming that y and z both already have a neighbour of colour *aux* somewhere, and are themselves and have a neighbour coloured from $\{true, false\}$, then w can only be coloured with *true* if one of y or z is coloured with *true*.

Assume y and z are vertices coloured from $\{true, false\}$ both of which already have a neighbour coloured *aux* and a neighbour coloured from $\{true, false\}$ outside of the or-gadget. We want to colour the rest of the or-gadget. Now if $c(y) = c(z) = false$ then the bottom two vertices of the triangle must receive the colours *aux* and *true* (and no new colours, as we already use two different colours in the neighbourhood of both y and z). Considering that the neighbours of the vertex receiving *aux* may only use two colours in total, now w must receive the colour $false$ as it already is adjacent to a vertex of colour *true*. However if one of y and z is coloured *true*, then it is possible to colour the gadget such that w receives colour *true* (or *false*, but no other colour because w is connected to a which receives colour *aux*). Notice that in either case w again has a neighbour coloured in *aux* and one coloured from $\{true, false\}$. So at the top of the triangle we get a vertex we can associate with $x \vee y$, as it can only be coloured with a boolean value, which can only be *true* if at least one of x and y was coloured in *true*.

Therefore we can combine these or-gadgets into clause-gadgets such that for each clause we get a vertex that can be coloured in *true* if and only if at least one of its parts is *true*. In the logic formula we have to force all clauses to become true in order for the entire formula to become true. So if we identify the last vertex of each clause-gadget with the vertex t which we coloured with *true* without loss of generality, then a valid 3-local colouring will give us an assignment of colours *true* or *false* to the vertices x_i and $\bar{x_i}$. This colouring corresponds to an assignment of boolean values to the variables that make the formula F become true. Similarly, an assignment to the boolean variables of F that makes F become true gives us a valid 3-local colouring of G. Notice that the size of G is linear in the size of the original formula, so we indeed have a polynomial reduction. See Figure 2.4 for an example.

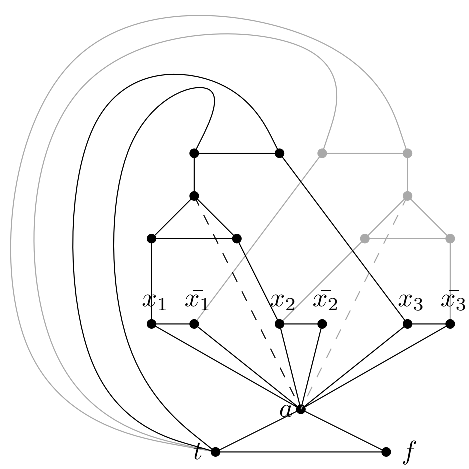


Figure 2.4: The graph corresponding to $(x_1 \vee x_2 \vee x_3) \wedge (\bar{x_1} \vee x_2 \vee \bar{x_3})$. The vertices and edges in grey correspond to the second clause of the formula.

$\square$

Notice that it easily follows that deciding whether there exists a k-local colouring for $k \geq 3$ is NP-complete as well:

Corollary 2.4. *Given a graph G, it is NP-hard to decide whether G admits a k-local colouring for $k \geq 3$.*

Proof. We reduce the problem to 3-local colouring. Assume we have a graph G for which we want to decide whether it is 3-locally colourable. We obtain G' by adding a copy of K_{k-3} to it and connecting all the vertices of K_{k-3} to all the vertices of G. By this we increase the local chromatic number by exactly $k-3$. Now determining whether G' has a k-local colouring will also determine whether G has a 3-local colouring. So we reduced 3-local colourability to k-local colourability, showing that the latter is at least as hard as the former, so by Theorem 2.3 the result follows. □

2.2 Wide colorings

In this section we will see a type of colourings that can be adapted into local colourings, therefore giving us a tool to give an upper bound on the local chromatic number.

A vertex colouring of a graph G is called *wide* if the end vertices of all walks of length 5 in G receive different colours. In particular wide colourings are proper, and graphs admitting wide colourings cannot contain 3-cycles or 5-cycles. Wide colourings are a special case of a type of colourings that have been examined in [16], but the term itself was coined by Simonyi and Tardos in [33] who used wide colourings to give upper bounds for the local chromatic number of specific classes of graphs. We will see some of these applications in Section 2.3.2 and 2.3.3. In general, if a graph has a wide t-colouring, then this colouring can be adapted to a $(\lfloor t/2 \rfloor + 2)$-local colouring, implying that in that case $\psi(G) \leq \lfloor t/2 \rfloor + 2$. The following proof of this fact was given in [33].

Lemma 2.5. *If a graph G has a wide colouring using t colours, then the local chromatic number $\psi(G) \leq \lfloor \frac{t}{2} \rfloor + 2$.*

Proof. Let c_0 be a wide t-colouring of G. We call a vertex v *troublesome* if too many colours are used in its neighbourhood, more specifically, if $c_0(\bar{\Gamma}(v)) \geq \lfloor \frac{t}{2} \rfloor + 2$. Let β be a new colour not yet used in the colouring, then we define a new colouring c as follows:

$$c(v) = \begin{cases} \beta & \text{if } v \text{ has a troublesome neighbour} \\ c_0(v) & \text{otherwise.} \end{cases}$$

Note that we did not add any new vertices to colour classes of colours that were already in the original colouring c_0, so these colour classes remain independent sets. To ensure that

c is a proper colouring, we only need to show that the colour class of β is an independent set. Assume not, then there are two adjacent vertices z and z' of colour β, each of which has a troublesome neighbour y and y' respectively (y and y' need not be distinct). Now in the original wide colouring that used t colours, each of y and y' had at least $\lfloor \frac{t}{2} \rfloor + 1$ different colours in its neighbourhood, so in particular there must be a colour α that is used in both neighbourhoods. Let the vertex of colour α adjacent to y be x, and the one adjacent to y' be x'. Then $x'y'z'zyx$ is a walk of length 5 between the vertices x' and x both of which were coloured with colour α, contradicting that c_0 was a wide colouring.

It remains to show that in each closed neighbourhood of a vertex, c uses at most $\lfloor \frac{t}{2} \rfloor + 2$ colours. In the closed neighbourhood of a troublesome vertex v, the new colouring c only uses β (on the neighbours) and $c_0(v)$ (on v itself), so $c(\bar{\Gamma}(v)) \leq 2$. Non-troublesome vertices v had at most $\lfloor \frac{t}{2} \rfloor + 1$ colours used in their neighbourhood in c_0. In c we increase this number by at most 1 by possibly introducing the new colour β in the neighbourhood of v and therefore $c(\bar{\Gamma}(v)) \leq \lfloor \frac{t}{2} \rfloor + 2$. Thus $\psi(G) \leq \lfloor \frac{t}{2} \rfloor + 2$.

□

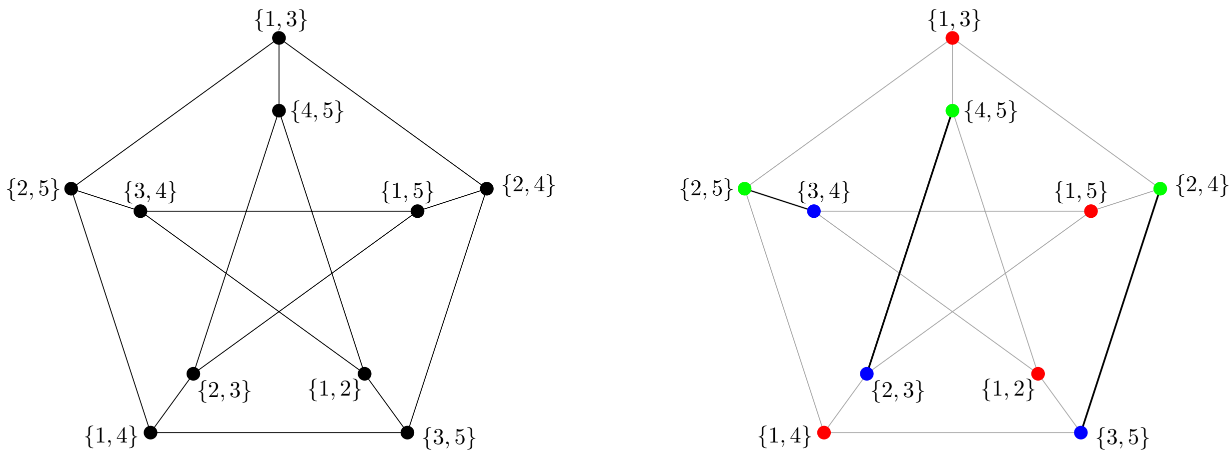
2.3 Results for specific classes of graphs

2.3.1 Kneser graphs

The *Kneser graph* $KG(n, k)$ has as vertex set all k-element subsets of $[n]$ which we denote with $\binom{[n]}{k}$, and two k-subsets are adjacent if and only if their intersection is empty.

Determining the chromatic number $\chi(KG(n, k))$ was a long-standing open problem, first conjectured by Kneser [19] to be $n - 2k + 2$ motivated by an explicit colouring giving this as an upper bound. The colouring is defined as follows: For a vertex v containing one of the elements $\{1, \ldots, n-2k\}$, set $c(v) = \min v$, i.e. the smallest element contained in the subset of $[n]$ that is v. The remaining vertices are k-subsets of $\{n - 2k + 1, \ldots, n\}$, which is a set of $2k$ elements. Thus for any vertex of this group there is only exactly one other vertex in this group which is adjacent to it, and therefore this group of vertices induces a perfect matching. But then we can 2-colour these vertices with the colours $n - 2k + 1$ and $n - 2k + 2$, thus giving us a $(n - 2k + 2)$-colouring. See Figure 2.5 for an example.

The lower bound and thus the conjecture was first proven in 1978 by Lovász using a novel approach involving topological methods, and in fact so far all known proofs make use of some variation of the Borsuk-Ulam theorem [4] (three variations of which are stated and used in later sections, see Theorem 2.12, Theorem 4.1 and Theorem 4.15). A mostly



(a) The graph $KG(5,2)$, which is also known as the Petersen graph.

(b) A 3-colouring of $KG(5,2)$ according to the description above. The vertices not containing the element 1 induce a perfect matching (black) and thus can be coloured with two colours green and blue.

Figure 2.5: Example of a Kneser graph and its 3-colouring.

combinatorial proof that however still takes ideas from the topological proofs, and based on the combinatorial proof by Matoušek [24], is given in Section 4.6. It also gives a lower bound for the local chromatic number for Kneser graphs of $\lceil \frac{t}{2} \rceil + 1$, where $t = \chi(KG(n,k)) = n - 2k + 2$. Alternatively, that same lower bound can also be obtained using topological properties of a simplicial complex called the box complex B_0 and then applying Theorem 4.17. Chapter 4 will elaborate more on this simplicial complex and theory behind it. Another lower bound that does not directly use topological methods is known and gives a better lower bound for certain parameters, as observed in [33]. Given a vertex $v \in KG(n,k)$, its neighbourhood consists of vertices that are k-subsets of $[n] \setminus v$. This is a set of size $n-k$, and thus the neighbourhood of v induces a copy of $KG(n-k,k)$. The chromatic number of that subgraph is $(n-k)-2k+2$, so at least $n-3k+3$ different colours are used in the closed neighbourhood of v in any colouring of $KG(n,k)$, thus we get a lower bound on the local chromatic number of $n-3k+3$. As of now however no non-trivial upper bounds are known in general.

2.3.2 Schrijver graphs

The *Schrijver graph* $SG(n,k)$ is the subgraph of $KG(n,k)$ induced by those k-element subsets that do not contain two consecutive integers modulo n, i.e. $V(SG(n,k)) = \{A \subseteq [n]\colon |A| = k, \forall i\colon \{i, i+1\} \not\subseteq A, \{n, 1\} \not\subseteq A\}$.

It was first introduced by Schrijver in [32] and shown to be vertex-critical, meaning that deleting any vertex will decrease its chromatic number. Just like for the Kneser graph, the same topological lower bound on the chromatic number can be obtained for $SG(n,k)$, thus showing that its chromatic number is $t = n-2k+2$ as well. For the same topological reason, we also have the lower bound for the local chromatic number of $\lceil \frac{t}{2} \rceil + 1$. In Theorem 3 of [33] a wide colouring on t colours is given assuming that $n \geq 4t^2 - 7t$ (which implicitly means that k has to be sufficiently close to $\frac{n}{2}$). By Lemma 2.5 this can be adapted into a $\lfloor \frac{t}{2} \rfloor + 2$-local colouring, providing an upper bound on the local chromatic number. This gives us that in these cases the topological lower bound on the local chromatic number is in fact tight if t is odd. There is still a gap of 1 between the lower and upper bound if t is even. In fact, Simonyi and Tardos remark that the bound on n can be improved to $n \geq 2t^2 - 4t + 3$, which will be elaborated on in Chapter 3. But for now we will prove the weaker result requiring $n \geq 4t^2 - 7t$. This proof for the upper bound is given in the next lemma and theorem, following the proof from [33].

Lemma 2.6. *If there is a walk of length $2s$ from vertex u to vertex v in $SG(n,k)$, then $|u \setminus v| \leq s(t-2)$ where $t = n - 2k + 2$.*

Proof. We will proceed by induction on s. For $s = 1$ we have a walk xyz. Now y is disjoint from x and z, therefore $z \subseteq [n] \setminus y$, thus

$$t - 2 = n - k - k = |([n] \setminus y) \setminus x| \geq |z \setminus x|.$$

Induction step: Let $x_0 x_1 \ldots x_{2s}$ be a walk of length $2s$. By induction hypothesis we have that $x_0 \setminus x_{2s-2} \leq (s-1)(t-2)$ and from the induction base we get $x_{2s-2} \setminus x_{2s} \leq t-2$. Altogether we get:

$$|x_0 \setminus x_{2s}| \leq |x_0 \setminus x_{2s-2}| + |x_{2s-2} \setminus x_{2s}| \leq (s-1)(t-2) + (t-2) = s(t-2).$$

□

The upper bound given in the following theorem equals the lower bound that follows from Theorem 4.17 if n is odd, thus determining the local chromatic number of $SG(n,k)$ if $n \geq t(4t-7)$ is odd.

Theorem 2.7. *If $t = n - 2k + 2 > 2$ and $n \geq t(4t - 7)$, then*

$$\psi(SG(n,k)) \leq \left\lfloor \frac{t}{2} \right\rfloor + 2.$$

Proof. We will provide a wide t-colouring c, then the result follows from Lemma 2.5.

Let $[n]$ be partitioned into intervals A_i of size $2p_i - 1$ for some p_i for $1 \leq i \leq t$, i.e. each A_i contains $2p_i - 1$ consecutive integers, such that $p_i \geq 2t - 3$ for all i. Notice that this is equivalent to ensuring that $2p_i - 1 \geq 4t - 7$, which is possible as we required $n \geq t(4t - 7)$.

Now notice

$$\begin{aligned} \sum_{i=1}^{t}(2p_i - 1) &= n \\ \sum_{i=1}^{t} 2p_i &= n + t \\ \sum_{i=1}^{t} p_i &= \frac{n+t}{2} \\ \sum_{i=1}^{t}(p_i - 1) &= \frac{n-t}{2} = \frac{n-(n-2k+2)}{2} = \frac{2k-2}{2} = k - 1. \end{aligned}$$

Thus for any k-element subset $x \subseteq [n]$ there must be an i such that x contains more than $p_i - 1$ elements from A_i, i.e. $|x \cap A_i| > p_i - 1$. Then let $c(x) = i$ for such an i where $|A_i \cap x| \geq p_i$. This is a proper colouring of $SG(n,k)$ (in fact, it is a proper colouring of $KG(n,k)$), as if there are two vertices x and y of colour i, then both contain at least p_i elements of A_i, meaning they cannot be disjoint due to $|A_i| = 2p_i - 1$.

We need to show that c as a colouring of $SG(n,k)$ is wide. For this purpose define C_i to be the unique subset of A_i of size p_i that does not contain two consecutive integers, i.e. C_i is the set containing the first, third, fifth, ... element from A_i. A vertex x in $SG(n,k)$ does not contain two consecutive integers. Then if $c(x) = i$ and therefore by definition $|x \cap A_i| \geq p_i$, we must in fact have $x \cap A_i = C_i$. We have $|C_i| = p_i$. Now consider a walk $x_0x_1x_2x_3x_4x_5$ of length 5 in $SG(n,k)$. Let $i = c(x_0)$. By Lemma 2.6 we then have $|x_0 \setminus x_4| \leq 2t - 4$. This means there are at most $2t - 4$ elements in x_0 that are not in x_4, and as $C_i \subseteq x_0$ in particular it means that there are at most $2t - 4$ elements in C_i that are not in x_4, i.e. $|x_4 \cap C_i| \geq |C_i| - (2t - 4) = p_i - (2t - 4) \geq 2t - 3 - (2t - 4) = 1$. This holds as we ensured at the beginning that $|C_i| = p_i \geq 2t-3$. So x_4 and C_i have a common element q, but that means that this element q cannot be in x_5 and therefore x_5 cannot have colour i. So the endpoints of any walk of length 5 cannot have the same colour, showing that the colouring c is in fact wide. □

A way to weaken the lower bound on n is outlined in Remark 4 from [33], whose result is restated in this thesis as Remark 3.1. In Chapter 3 we will prove a slight generalization of that result. A corollary that follows from this result gives us a generalization of Theorem 2.7 that covers a somewhat broader set of graphs. The proof is given in Section 3. This corollary already followed from Remark 3.1.

Corollary 3.3. *If $t = n - 2k + 2$ and $n \geq 2t^2 - 4t + 3$, then*

$$\psi(SG(n,k)) \leq \frac{t+3}{2} = \left\lceil \frac{t}{2} \right\rceil + 1.$$

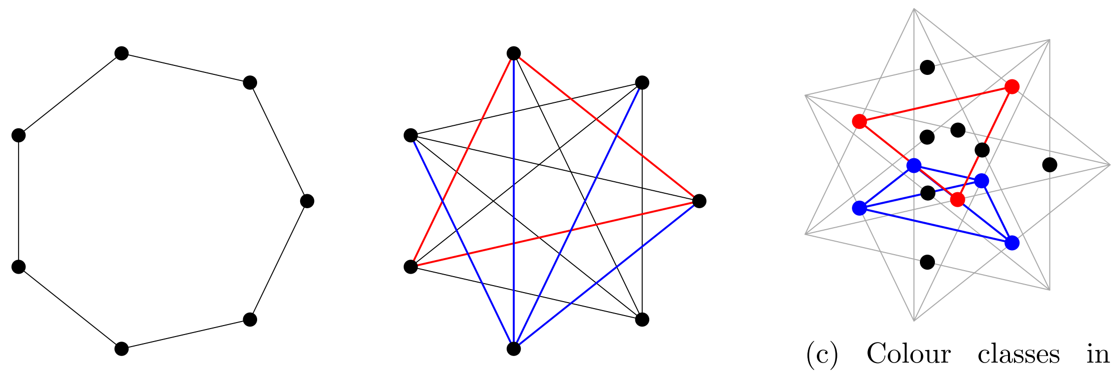
□

If t is odd, then with Theorem 4.19 we in fact get equality. The restriction on $n \geq 2t^2 - 4t + 3$ intuitively means that k has to be close enough to $\frac{n}{2}$. Notice that such a restriction is needed to some extent, as we will now show that for $k = 2$, $\psi(SG(n,2)) = \chi(SG(n,2))$. This result was first given in [33].

Theorem 2.8. *If $n \geq 4$ then $\psi(SG(n,2)) = \chi(SG(n,2)) = n-2$.*

Proof. For $n = 4$ the graph consists of a single edge and thus the statement is obviously true. For $n \geq 5$ assume the statement is false, and let c be a colouring that is $\psi(SG(n,2))$-local, and among such colourings uses the minimal number of colours in total. Observe that such a colouring must use at least $n-1$ colours in total. If it used exactly $n-2$ colours, but is at most $(n-3)$-local, then we can eliminate a colour a by recolouring each vertex of colour a to one of the remaining colours that is not used in its neighbourhood, thus obtaining a proper $(n-3)$-colouring, a contradiction.

Consider D_n, the complement of the cycle C_n. The cycle C_n can be envisioned as an n-gon, and in the same vein D_n can be envisioned as the diagonals of said n-gon. But those diagonals are exactly pairs of non-consecutive numbers between 1 and n modulo n, so they correspond to vertices in $SG(n,2)$. Two edges in D_n are incident (and their counterpart vertices in the line graph $L(D_n)$ are adjacent) to each other if and only if they are non-disjoint, while in $SG(n,2)$ vertices are adjacent when they are disjoint. Thus $SG(n,2)$ is isomorphic to the complement of the line graph of D_n, see Figure 2.6. An independent set in $SG(n,2)$ is a set of vertices that pairwise share an element. In D_n they correspond to edges that pairwise share an endpoint, however such sets of edges are either triangles or stars (i.e. induce $K_{1,m}$ for some $m \geq 0$).



(a) C_7 (b) Colour classes in D_7 (c) Colour classes in $L(D_7)$

Figure 2.6: A triangle colour class (red) and a star colour class (blue) as seen in D_7 and its line graph $L(D_7)$, which is the complement of $SG(n,2)$. A colour class in $SG(n,2)$ is an independent set, and thus corresponds to a clique in $L(D_7)$ which is a triangle or a star in D_7.

In a proper colouring of $SG(n,2)$ we say a colour class is *seen* by a vertex v if v is adjacent to some vertex from that colour class. In D_n this corresponds to an edge e seeing a colour class if and only if there is some edge in that colour class that is not incident to e. For a triangle colour class, every other edge in D_n (apart from the triangle itself) sees the colour class as for every edge e there is at least one edge in the triangle not incident to e. For a star colour class with at least 3 edges an edge e can be incident to all of them if and only if one of its endpoints is the center of the star. If we have a star colour class consisting of 2 edges uv and vw, then in addition to the edges having v as one endpoint, there is also the edge uw that shares an endpoint with both edges from the colour class. So in either of these cases a colour class is seen by all but at most $n-2$ edges in D_n. The only remaining case is if a colour class consists of a single edge x, but we will show that this case cannot occur. Recall that we have at least $n-1$ colour classes in total, of which x may see at most $n-4$. But this means we can choose a different colour for x to get a new colouring which still attains $\psi(SG(n,2)$-locality because for neighbours of x the colour of x disappears from their neighbourhood and is possibly replaced with a new colour. But this colouring uses 1 fewer colour in total, which contadicts the choice of our initial colouring, as we chose it to be minimal with respect to the total number of colours used.

We will now double count pairs (x, C) of vertices x and colour classes C seen by x. So with each colour class in $SG(n,2)$ being seen by all but at most $n-2$ vertices as we just showed, we have that for each colour class there are at least $(\binom{n}{2}-n)-(n-2)$ vertices seeing it. (Notice that $\binom{n}{2}-n$ is the number of vertices of $SG(n,2)$.) Recall that we observed that the colouring must use at least $n-1$ colours in total. Then the number of pairs (x,C) is at least $((\binom{n}{2}-n)-(n-2))(n-1)$. But also, due to the local chromatic number, each vertex is allowed to see at most $n-4$ colour classes, and thus the number of pairs (x,C) is at most $(\binom{n}{2}-n)(n-4)$.

However expanding these terms yields:

$$
\begin{aligned}
\left(\left(\binom{n}{2}-n\right)-(n-2)\right)(n-1) &\leq \left(\binom{n}{2}-n\right)(n-4) \\
\left(\frac{n(n-1)}{2}-2n+2\right)(n-1) &\leq \left(\frac{n(n-1)}{2}-n\right)(n-4) \\
\frac{n^2(n-1)}{2}-\frac{n(n-1)}{2}-2n^2+2n+2n-2 &\leq \frac{n^2(n-1)}{2}-\frac{4n(n-1)}{2}-n^2+4n \\
\frac{3}{2}(n^2-n)-2 &\leq n^2 \\
\frac{1}{2}n^2-\frac{3}{2}n-2 &\leq 0.
\end{aligned}
$$

This is a contradiction if $n \geq 5$. □

2.3.3 Generalized Mycielski graphs

We first define the *generalized Mycielski construction*: Let $\hat{P}_r$ be a path of length r with a loop attached to one end, and let its vertices be $\{0,1,\ldots,r\}$ where 0 is the end with the loop. Then for a graph G, the *Mycielskian* $M_r(G)$ is obtained by taking the direct product of G with $\hat{P}_r$, and then identifying all vertices that have the end vertex r of $\hat{P}_r$ as first coordinate, see Figure 2.7 for an example. The *direct product* $G_1 \times G_2$ of two graphs G_1 and G_2 is a new graph with vertex set $V(G_1)\times V(G_2)$ where two vertices (v_1,v_2) and (u_1,u_2) are adjacent if and only if $v_1 \sim u_1$ in G_1 and $v_2 \sim u_2$ in G_2. For $r=2$ this is the original contruction used by Mycielski [28] to increase the chromatic number of a graph by 1 while keeping the clique number fixed, and in particular showing that there are triangle-free graphs of arbitrarily high chromatic number.

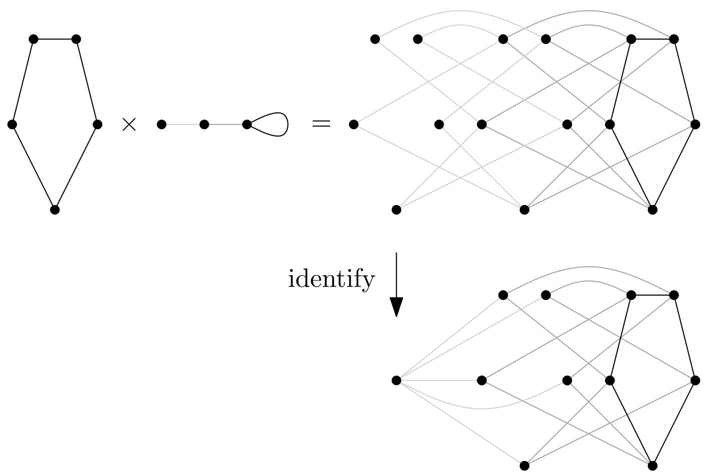


Figure 2.7: Example of a Mycielski construction for $r = 2$ and $G = C_5$, obtaining a 4-chromatic triangle-free graph.

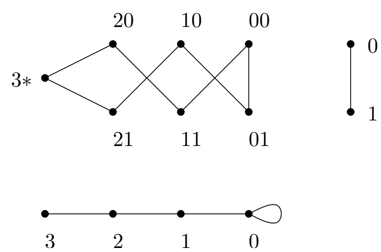
For a vector $r = (r_1, r_2, \dots, r_d)$ of length d and a graph G we use the notation $M_r^{(d)}(G)$ for $M_{r_d}(M_{r_{d-1}}(\dots(M_{r_1}(G))\dots))$, which is an iterated application of the generalized Mycielski construction.

We can also define the vertices and edges of this graph explicitly, see Figure 2.8 for an example: Each vertex corresponds to a sequence $a_d \dots a_2 a_1 u$ where $u \in V(G) \cup \{*\}$ and $a_i \in \{0, 1, \dots, r_i\} \cup \{*\}$ correspond to a vertex of G and $\hat{P}_{r_i}$ respectively, or the special character $*$. If $a_i = r_i$ for some i, then all the following characters a_j for $j < i$ and u in the sequence have to be $*$, and this is also the only instance when the character $*$ can appear. Intuitively, for each i, in the graph $M^{(i)}_{(r_1,\dots,r_i)}$ all vertices correspond to a pair consisting of a vertex from $\hat{P}_{r_i}$ and a vertex from $M^{(i-1)}_{(r_1,\dots,r_{i-1})}$. But as all vertices with $a_i = r_i$ are identified with each other, we cannot distinguish between different vertices from $M^{(i-1)}_{(r_1,\dots,r_{i-1})}$ as the second element of the pair when $a_i = r_i$, so we write a sequence of $*$s instead of a sequence representing a vertex from $M^{(i-1)}_{(r_1,\dots,r_{i-1})}$.

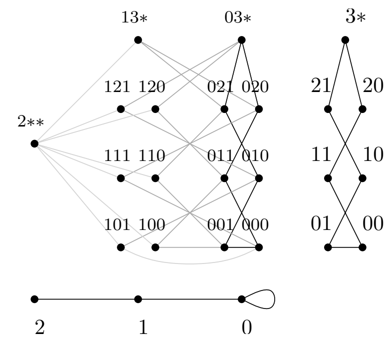
Two vertices $a_d \dots a_2 a_1 u$ and $a'_d \dots a'_2 a'_1 u'$ are connected by an edge if and only if the following two conditions hold:

1. $u = *$ or $u' = *$ or $\{u, u'\} \in E(G)$, and
2. For all i: $a_i = *$ or $a'_i = *$ or $\{a_i, a'_i\} \in E(\hat{P}_{r_i})$.

Just like for the chromatic number, it was proven in [33] that this construction increases



(a) $M_3(K_2)$ as a product of K_2 and $\hat{P}_3$ with vertices identified into $3*$.



(b) $M_2(M_3(K_2))$ as a product of $M_3(K_2)$ and $\hat{P}_2$ with vertices identified into $2**$.

Figure 2.8: The graph $M^{(2)}_{(3,2)}(K_2) = M_2(M_3(K_2))$ as the result of two Mycielski operations on K_2.

the local chromatic number by at most 1 for $r \geq 2$, while for $r = 2$ this construction increases the local chromatic number by exactly 1:

Theorem 2.9.

$$\begin{aligned} \psi(M_1(G)) &= \chi(G) + 1, \\ \psi(M_2(G)) &= \psi(G) + 1, \\ \psi(M_r(G)) &\leq \psi(G) + 1 \qquad \textit{for } r \geq 3. \end{aligned}$$

Proof. The first statement is trivial as $M_1(G)$ is just the graph G with a new vertex that is connected to all other vertices in G.

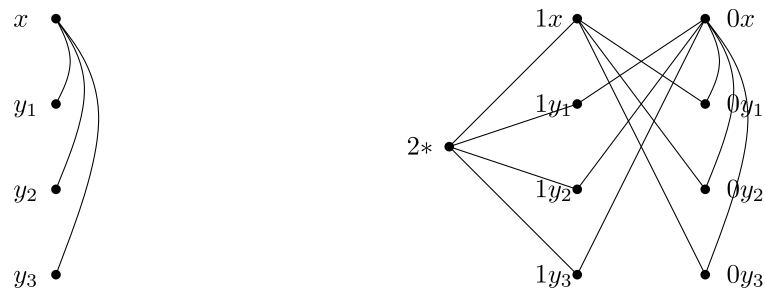
For the third statement consider a $\psi(G)$-local colouring c_0 of G. We define a new colouring c. Let α and β be two colours not used in c_0. For a vertex $u \in V(G)$, we set $c(0u) = c_0(u)$. For $a_1 \geq 1$ we set $c(a_1u) = \alpha$ if a_1 is odd and $c(a_1u) = \beta$ if a_1 is even. Similarly we set $c(r*) = \alpha$ if r is odd and $c(r*) = \beta$ if r is even. This is a $(\psi(G)+1)$-local colouring.

It remains to show that $\psi(M_2(G)) \geq \psi(G) + 1$. So consider a k-local colouring c of $M_2(G)$. We need to show that we can obtain a colouring c_0 of G from this that is $(k-1)$-local. Define c_0 as follows.

$$c_0(u) = \begin{cases} c(0u) & \text{if } c(0u) \neq c(2*) \\ c(1u) & \text{otherwise} \end{cases}$$

Note that c_0 is proper, as for two vertices $x \sim y$ in G we have $0x \sim 0y$ and $1x \sim 0y$. Furthermore we have that c_0 doesn't use the colour $c(2*)$, as $1u \sim 2*$ for all u, see Figure 2.9.

Now by definition of the local chromatic number there must be some vertex x in G that has at least $\psi(G) - 1$ different colours in its neighbourhood. First assume for some



(a) $\bar{\Gamma}(x)$ in G (b) Corresponding situation in $M_2(G)$.

Figure 2.9: The vertex x and its neighbours in G and how the situation translates into $M_2(G)$.

neighbour y of x we have $c_0(y) \neq c(0y)$. Then by the definition of the colouring c_0 for each neighbour $y \sim x$ in G either $1y$ or $0y$ has the same colour as y in $M_2(G)$. But both $1y$ and $0y$ are incident to $0x$ in $M_2(G)$, so $0x$ in its neighbourhood in $M_2(G)$ sees at least the colours that x sees in its neighbourhood in G, but $0x$ also sees $c(0y) = c(2*)$, which is a new colour. So $0x$ sees at least $\psi(G)$ colours in its neighbourhood.

In the other case we have $c_0(y) = c(0y)$ for all $y \sim x$. Then as $0y \sim 1x$ for all $y \sim x$ we have:

$$c(\Gamma_{M_2(G)}(1x)) = c(2*) \cup \bigcup_{0y \sim 1x} c(0y) = c(2*) \cup \bigcup_{y \sim x} c_0(y) = c(2*) \cup c_0(\Gamma_G(x)).$$

But this means that we use at least one more colour in the neighbourhood of $1x$ in $M_2(G)$ than in the neighbourhood of x in G.

□

Furthermore if we know that for a graph a wide t-colouring exists, it can in fact be shown that the Mycielskian $M_r(G)$ has a wide $(t+1)$-colouring if $r \geq 7$ [33]:

Lemma 2.10. *If G has a wide t-colouring and $r \geq 7$, then $M_r(G)$ has a wide $(t+1)$-colouring.*

Proof. Let c be a wide t-colouring in G. We introduce a new colour β. Then the following colouring of $M_r(G)$ is wide:

$$c_0(au) = \begin{cases} \beta & \text{if } a \in \{r, r-2, r-4\} \\ c(u) & \text{otherwise} \end{cases}$$

Assume we have a walk $a_0u_0, \ldots, a_5u_5$ of length 5. Now if one endpoint receives the colour β then the other endpoint cannot receive β because $a_0 \geq r-4 \geq 3$ and $a_5 \geq r-4 \geq 3$ have the same parity, but $|a_i - a_{i+1}| = 1$ unless $a_i = a_{i+1} = 0$. So assume none of the endpoints are coloured with β. Now unless $a_i = r$ for some i, we have that $u_0, \ldots, u_5$ is a walk in G, so in that case as c was wide, we cannot have that both endpoints receive the same colour. But if $a_i = r \geq 7$ for some i, then one of the endpoints must have colour β as $|a_i - a_{i+1}| = 1$ for all i in that case.

□

Using a more intricate recolouring procedure instead of wide colourings proves a similar bound in a more general setting. This proof was given in [33].

Theorem 2.11. *For a vector $r = (r_1, r_2, \ldots, r_d)$ with $r_i \geq 4$ for all i the following bound holds:*

$$\psi(M_r^{(d)}(G)) \leq \psi(G) + \left\lfloor \frac{d}{2} \right\rfloor + 2.$$

Proof. Let c_G be a $\psi(G)$-local colouring of G. Based on this colouring we will provide a colouring c_0 of $M_r^{(d)}(G)$, and then modify it into a colouring c that shows the desired bound. For convenience, let the colours used in c_G be $0, -1, -2, \ldots$. We define c_0 as follows:

$$c_0(a_d \ldots a_1 u) = \begin{cases} c_G(u) & \text{if } a_i \leq 2 \text{ for all } i \\ j & \text{if } a_i \leq 2 \text{ for all } i > j \text{ and } a_j \geq 3 \text{ is odd} \\ 0 & \text{if } a_i \leq 2 \text{ for all } i > j \text{ and } a_j \geq 4 \text{ is even for some } j. \end{cases}$$

Notice that this includes all vertices, in particular those that have $*$ as some entry, as the first $*$-entry a_i in the list is always preceded by an entry a_{i+1} of value $r_{i+1} \geq 4$. The same applies if u is the first $*$-entry of a vertex.

The colouring c_0 is a proper colouring. To see this, consider two adjacent vertices $x = a_d \ldots a_2 a_1 u$ and $x' = a'_d \ldots a'_2 a'_1 u'$. First observe that by definition of the adjacency relation we must have $u \sim u'$ in G, unless either of the two vertices is $*$. Similarly we must have $|a_i - a_i'| = 1$ unless $a_i = a_i' = 0$ or either of a_i and a_i' is $*$. Now assume that $c_0(x) = c_0(x')$. Recall that the definition of our colouring c_0 involved three cases. We will now proceed with a case analysis depending on which of the vertices x and x' were coloured via which case from the definition of c_0, and see that each case results in a contradiction, showing that c_0 must be a proper colouring. We have that the third case assigns the colour 0, the second case assigns positive colours and the first case assigns negative colours or the colour 0. So the cases to consider are as follows:

Case 1: Both x and x' were coloured via the first case from the definition of c_0. Then we must have $a_i \leq 2 < r_i$ and $a_i' \leq 2 < r_i$ for all i, and in particular $u \neq *$ and $u' \neq *$. Due to $x \sim x'$ this implies that $u \sim u'$ and thus $c_G(u) = c_0(x) = c_0(x') = c_G(u')$. But this is a contradiction to c_G being proper.

Case 2: Both vertices receive colour 0, one vertex via the first and the other via the third case. Without loss of generality assume that x is the vertex that received colour 0 via the first case. Then there is some j such that $a_j' \geq 4$. But $a_i \leq 2$ for all i and in particular $a_j \leq 2$. Therefore x and x' cannot be adjacent, a contradiction.

Case 3: Both x and x' receive colour j via the second case. Then a_j and a_j' are both odd and thus differ by an even number. This implies that x and x' cannot be adjacent, a contradiction.

Case 4: Both x and x' receive colour 0 via the third case. Let j be the smallest value such that $a_j \geq 4$, and j' the smallest value such that $a_{j'}' \geq 4$. Without loss of generality $j \leq j'$. If $j = j'$, then the difference between a_j and a_j' will be even, but neither a_j nor $a_{j'}'$ are 0, so x and x' cannot be adjacent. Otherwise $j < j'$, but then $a_j \geq 4$ and $a_j' \leq 2$, implying that x and x' cannot be adjacent as well.

Now that we have c_0, we adapt it to a new colouring c that achieves the desired locality. For this purpose we introduce a new colour β. In the following we consider β a special entity that is neither positive nor negative. Now set

$$c(a_d \dots a_1 u) = \begin{cases} \beta & \text{if } |\{i \colon a_i \text{ is odd}\}| \geq \lfloor \frac{d}{2} \rfloor + 1 \\ c_0(a_d \dots a_1 u) & \text{otherwise.} \end{cases}$$

Note that the colour class of β induces an independent set: if we have two vertices $x = a_d \dots a_2 a_1 u$ and $x' = a_d' \dots a_2' a_1' u'$ of colour β, then both of them have at least $\lfloor \frac{d}{2} \rfloor + 1$ odd entries a_i and a_i' respectively, so for some index i both a_i and a_i' must be odd, and thus x and x' cannot be adjacent. We already know that the other colour classes induce independent sets because c_0 was proper, so c is proper as well.

We need to show that the colouring c is $(\psi(G)+\lfloor\frac{d}{2}\rfloor+2)$-local. First we will see that a vertex can see at most $\psi(G)-1$ different (strictly) negative colours in its neighbourhood (in fact, this is true for c_0 as well). Fix a vertex $x=a_d\dots a_2a_1u$, and let $x'=a'_d\dots a'_2a'_1u'$ be a neighbour of x. Assume that x' receives a negative colour (via the first case in the definition of c_0), and thus $a'_i\le 2$ for all i and in particular $u'\ne *$. If $u=*$, we must have that some $a_i\ge 4$ and thus the two vertices cannot be adjacent because $a'_i\le 2$. So we must have $u\sim u'$ in G and thus the number of negative colours used in the neighbourhood of u can be at most $\psi(G)-1$, as c_G is $\psi(G)$-local. Apart from negative colours, we can also have the colour 0 and β in the neighbourhood of a vertex, making for a maximum of $\psi(G)+1$ different non-positive colours in the neighbourhood of a vertex. So we have to show that at most $\lfloor\frac{d}{2}\rfloor$ different (strictly) positive colours (i.e. colours from $\{1,\dots,d\}$) are used in the neighbourhood, together giving a total of at most $\psi(G)+1+\lfloor\frac{d}{2}\rfloor+1$ different colours in the closed neighbourhood of a vertex.

Assume we have a vertex $x=a_d\dots a_2a_1u$ with more than $\lfloor\frac{d}{2}\rfloor$ different positive colours in its neighbourhood. Note that if colour j appears in the neighbourhood of x, then we have that a neighbour x' of colour j must have $a'_j\ge 3$ odd. Then either we must have $a_j\ge 2$ is even, or $a_j=*$. The latter cannot occur however, because it'd imply some $a_i=r_i\ge 4$ for some $i>j$, and therefore $a'_i\ge 3$, which means that x' either receives colour 0 or i, both of which are different from j. So for each colour $j>0$ appearing in the neighbourhood of x we have $a_j\ge 2$ is even. Therefore more than $\lfloor\frac{d}{2}\rfloor$ entries of x are even and ≥ 2. Let $J=\{i\colon a_i\ge 2\text{ is even}\}$ be the set of indices of entries a_i that are even and ≥ 2, and let I be the greatest $\lfloor\frac{d}{2}\rfloor$ elements from J. Now assume there is a neighbour $x'=a'_d\dots a'_2a'_1u'$ of colour $j<\min I$. Then its entries a'_i for $i\in I$ cannot be $*$ as $j<i$, therefore $|a_i-a'_i|=1$ for those i, and thus each of these a'_i for $i\in I$ must be odd as the corresponding entry a_i in x is even and ≥ 2 by definition of I. Furthermore a'_j is odd, making for a total of at least $\lfloor\frac{d}{2}\rfloor+1$ odd entries in x'. But this means that x' has to be coloured in colour β. So we showed that indeed at most $\lfloor\frac{d}{2}\rfloor$ positive colours are used in the neighbourhood of a vertex, concluding the proof.

□

For lower bounds, again topological techniques have been used.

For the chromatic number, in Theorem 4.22 we will show that for e.g. Kneser graphs, Borsuk graphs and Schrijver graphs, the latter including complete graphs, odd cycles and K_2, the generalized Mycielski construction increases the chromatic number by exactly 1. We will see the definition of Borsuk graphs in the next section.

For the local chromatic number, we will see that if a certain topological lower bound is tight (e.g. for K_2, Schrijver graphs of odd chromatic number to which Corollary 3.3

applies, or certain Borsuk graphs), we can determine the local chromatic number of the graph obtained by applying the general Mycielski construction almost exactly via Theorem 4.23 for the lower bound and Theorem 2.11 for the upper bound. Sometimes we can even determine it exactly, as in Theorem 4.24.

With these tools to give lower and upper bounds for generalized Mycielski graphs, it is possible to determine the local chromatic numbers of many of them exactly or almost exactly. Corollary 14 of [33] gives an overview of these cases where the starting graph is K_2. It appears as Corollary 4.25 in Chapter 4. However one of the interesting open questions not covered there is: What happens to the local chromatic number when the parameter r in the iterations of the generalized Mycielski construction $M_r(G)$ is 3? For that case no better bounds than the trivial bound from repeated application of Theorem 2.9 are known.

2.3.4 Other classes of graphs

In this section we will summarize results about the local chromatic number for other classes of graphs.

The *Borsuk graph* $B(n, \alpha)$ for $0 < \alpha < 2$ is an infinite graph whose vertex set is the set of points on the sphere S^{n-1}, and two points are connected by an edge if and only if their distance is at least α.

We will now use the Lyusternik-Schnirelman [23] version of the Borsuk-Ulam theorem to show that the chromatic number of $B(n, \alpha)$ is at least $n + 1$.

Theorem 2.12 (Borsuk-Ulam)**.** *If the sphere S^{n-1} is covered by n open sets, then one of these sets must contain two antipodal points, i.e. a pair of points x and $-x$.* □

Proposition 2.13. $\chi(B(n, \alpha)) \geq n + 1$.

Proof. Assume the claim is false, meaning that the graph is n-colourable and let $c\colon S^{n-1} \to [n]$ be an n-colouring. Now let $\varepsilon = \frac{1}{3}(2 - \alpha)$. Then the open ε-neighbourhoods around each of the n colour classes define an open cover of S^{n-1}. So by the Borsuk-Ulam Theorem there is a colour c such that there are two antipodal points x and $-x$ contained in the ε-neighbourhood of that colour class. For each of these two points there is a vertex y and y' of that colour $c = c(y) = c(y')$ of distance at most ε from x and $-x$ respectively. Now x and $-x$ as antipodal points have distance 2 from each other, so by the triangle inequality y and y' have distance at least $2 - 2\varepsilon > \alpha$ from each other. But this means they are connected by an edge, so the colouring is not proper. □

In [33] Simonyi and Tardos give a lower bound on the local chromatic number of $\lceil \frac{n+3}{2} \rceil \leq \psi(B(n, \alpha))$, using topological tools. Furthermore they show if n is even and α is sufficiently close to 2, more specifically $2 - \frac{1}{25n+50} \leq \alpha < 2$, then this lower bound is tight. This is done by providing a wide $(n+1)$-colouring. So in this case the lower bound on the chromatic number that we observed in Proposition 2.13 is tight as well.

The local chromatic number of certain quadrangulations of non-orientable surfaces (called *odd* quadrangulations) have been investigated in [27]. For surfaces of genus at most 4 the local chromatic number is at least 4, while for surfaces of genus greater than 4 the local chromatic number is at most 3. This behaviour differs from the chromatic number which is 4 independent of the genus of the surface.

Gyarfas, Jensen and Stiebitz introduce a universal graph G_k for strong k-colourings in [16]. A strong k-colouring is a proper colouring on k colours such that the neighbourhood of each colour class is an independent set. Then G_k has the property that a graph G has a homomorphism $G \to G_k$ if and only if it is strongly k-colourable. The graph G_k is defined as follows:

The vertex set $V(G_k) = \{(x, A) \colon x \in [k], \emptyset \neq A \subseteq [k], x \in A\}$, and two vertices (x, A) and (y, B) are linked by an edge if $x \in B$, $y \in A$ and $A \cap B = \emptyset$.

In order to show that its chromatic number is k, they exhibit a homomorphism from the generalized Mycielski graph $M^{(k-2)}_{(4,\dots,4)}(K_2)$ into G_k. From Section 2.3.3 we know the local chromatic number of this Mycielski graph is at least $\lceil \frac{k}{2} \rceil + 1$. Therefore the same lower bound of $\lceil \frac{k}{2} \rceil + 1$ on the local chromatic number applies to G_k. However no non-trivial upper bound is known yet.

In [16] another more general set of classes of graphs is introduced. They define S^ℓ_k as the class of all those graphs that admit a k-colouring such that for each colour class X_i, the set of vertices of distance j from X_i for $j \leq \ell$ forms an independent set. Observe that for $\ell = 2$ this is exactly the class of graphs that admit wide k-colourings. If two vertices x and y at distance 3 or 5 receive the same colour, then the two center vertices of the path connecting x and y have x and y at distance 1 or 2 respectively, but are adjacent. Similarly, if the graph has a (chordless) 3- or 5-cycle containing a vertex x, then the pair of vertices along the cycle of distance 1 or 2 from x respectively will be adjacent.

It is mentioned that these classes S^ℓ_k of graphs have a homomorphism universal graph G^ℓ_k that is defined as follows: The vertices are pairs (i, A) where $i \in [k]$ and $A \subseteq [k]^\ell$ contains strings $x_1 \dots x_\ell$ with $x_1 \neq i$ and $x_t \neq x_{t-1}$ for all $t \geq 2$. Two vertices (i, A) and (j, B)

are linked by an edge if for all $x_1 \dots x_\ell \in A$ and $y_1 \dots y_\ell \in B$ we have that $jy_1 \dots y_{\ell-1} \in A$, $ix_1 \dots x_{\ell-1} \in B$ and $x_t \neq y_t$ for all t. So G_k^2 is the universal graph for wide colourings, thus any graph admitting a wide k-colouring has a homomorphism into G_k^2. We've seen in Theorem 2.7 that certain k-chromatic Schrijver graphs admit wide k-colourings. But these Schrijver graphs also have a lower bound of $\lceil \frac{k}{2} \rceil + 1$ for the local chromatic number, so the same lower bound applies to G_k^2. On the other hand, with G_k^2 itself admitting a wide k-colouring, Lemma 2.5 gives us an upper bound of $\lfloor \frac{k}{2} \rfloor + 2$, determining the local chromatic number for odd k and leaving a gap of 1 for even k.

We have one more class of universal graphs whose local chromatic number we know. They are universal graphs for k-local m-colourings and as such play a special role with respect to the local chromatic number. We will examine them in the following section.

2.4 Universal graphs

It is well known that that the property of m-colourability and fractional m-colourability can be expressed in terms of homomorphisms into universal graphs. A graph G is m-colourable if and only if there is a homomorphism $G \to K_m$, and is fractionally m-colourable if there are p and q with $m = \frac{p}{q}$ such that there is a homomorphism $G \to KG(p,q)$. A similar characterization can be done for k-local colourability.

Recall that a k-local m-colouring is an m-colouring such that no closed neighbourhood of a vertex receives more than k colours. In this section we will define a universal graph useful for characterizing graphs that admit k-local m-colourings. The following results and proofs were given by Erdős et al. in [10] and are rephrased in this section. Then we show that if a graph admits an 3-local m-colouring while its chromatic number is n, then m must be very large compared to n. More specifically, if a graph has chromatic number n and admits an 3-local m-colouring, then m must be at least $2^{\binom{n}{\lfloor n/2 \rfloor/4}(1+o(1))}$. On the other hand they also show that such graphs exist for $m = 2^{\binom{n}{\lfloor n/2 \rfloor/2}(1+o(1))}$. More generally, results for k-local m-colourings are given in [10] using the same techniques introduced below.

Definition 2.14. *We define the* universal graph $U(m,k)$ *as follows, first introduced by Erdős et al. in [10]:*

$$
\begin{aligned}
V(U(m,k)) &= \{(x,A)\colon x \in [m], A \subseteq [m], |A| \le k-1, x \notin A\}, \\
E(U(m,k)) &= \{\{(x,A),(y,B)\}\colon x \in B, y \in A\}.
\end{aligned}
$$

Theorem 2.15. *A graph G admits a k-local m-colouring if and only if there is a homomorphism from G into $U(m,k)$.*

Proof. "⇒": Let $c\colon V(G) \to [m]$ be a k-local m-colouring. Then the homomorphism is $v \mapsto (c(v), c(\Gamma(v)))$. This works as if $u \sim v$, then $u \in \Gamma(v)$ and thus $c(u) \in c(\Gamma(v))$, so edges are mapped to edges.

"⇐": Note that $U(m,k)$ admits a k-local m-colouring c by simply colouring each vertex (a, A) with colour a, as the neighbours of (a, A) will receive colours from A. Now assume we have a homomorphism f from G into $U(m,k)$. Then if we colour a vertex v with the colour $c(f(v))$, we get a k-local m-colouring of G: Two adjacent vertices will be mapped to two adjacent vertices that receive different colours in $U(m,k)$, so the colouring is proper. The closed neighbourhood of a vertex v gets mapped to a subset of the closed neighbourhood of $f(v)$ in $U(m,k)$, so the colouring is also k-local. □

This also gives us information about the chromatic number of $U(m,k)$ for certain m and k. We know that the Schrijver graph $SG(n,k)$ has chromatic number $t = n - 2k + 2$, and if $n \geq 2t^2 - 4t + 3$ then Corollary 3.3 gives us a $(\lceil \frac{t}{2} \rceil + 1)$-local $(t+1)$-colouring. So we have a homomorphism $SG(n,k) \to U(t+1, \lceil \frac{t}{2} \rceil + 1)$. But this means that the chromatic number of $U(t+1, \lceil \frac{t}{2} \rceil + 1)$ is at least t.

Definition 2.16. *Assume we have sets $A_{a,b} \subseteq [n]$ for $1 \leq a < b \leq m$. Then the system of these sets $\{A_{a,b}\colon 1 \leq a < b \leq m\}$ is called* (m,n,k)-independent *if and only if the following holds: For every $A \in \binom{[m]}{k}$ and every $a \in A$*

$$\bigcap_{\substack{b<a\\ b\in A}} A_{b,a} \setminus \bigcup_{\substack{b>a\\ b\in A}} A_{a,b} \neq \emptyset.$$

If $\{b \in A\colon b < a\} = \emptyset$, we define the (empty) intersection on the left to be $[n]$.

Lemma 2.17. *There is a graph G with $\chi(G) > n$ and that admits a k-local m-colouring, if and only if there are no (m,n,k)-independent systems.*

Proof. "⇒": Assume there is an (m,n,k)-independent system $\{A_{a,b}\colon 1 \leq a < b \leq m\}$. We will give an n-colouring c of $U(m,k)$, showing that its chromatic number is at most n. Because for every graph G that admits a k-local m-colouring there is a homomorphism $\psi\colon G \to U(m,k)$, then we can take an n-colouring c of $U(m,k)$ and then assign each vertex $v \in G$ the colour $c(\psi(v))$ to get a proper n-colouring of G.

So to obtain this n-colouring c, for a vertex (a, A) let

$$c(a, A) = \min\left(\bigcap_{\substack{b<a\\b\in A}} A_{b,a} \setminus \bigcup_{\substack{b>a\\b\in A}} A_{a,b}\right).$$

Now consider an edge from (a, A) to (b, B) and assume without loss of generality $a < b$. By definition this means that $b \in A$ and $a \in B$. First observe that $c(a, A) \notin A_{a,b}$, as $b > a$ and $b \in A$ and thus $A_{a,b}$ is part of the sets that get subtracted on the right. Next notice that by definition

$$c(b, B) = \min\left(\bigcap_{\substack{i<b\\i\in B}} A_{i,b} \setminus \bigcup_{\substack{i>b\\i\in B}} A_{b,i}\right).$$

As $a < b$ and $a \in B$ we have that $A_{a,b}$ is part of the intersection on the left, so in particular $c(b, B) \in A_{a,b}$. This shows that (a, A) and (b, B) receive different colours, and therefore the colouring c is proper.

"$\Leftarrow$": Now assume that every graph that admits a k-local m-colouring is n-colourable. So in particular we have a proper n-colouring c of $U(m, k)$. Now set $A_{a,b} = \{c(b, B) \colon B \ni a\}$ for $1 \leq a < b \leq m$. We claim that these sets give an (m, n, k)-independent system. If not, then by definition there is a set $A \in \binom{[m]}{k}$ and an $a \in A$ such that

$$\bigcap_{\substack{b<a\\b\in A}} A_{b,a} \setminus \bigcup_{\substack{b>a\\b\in A}} A_{a,b} = \emptyset.$$

Now first observe that

$$c(a, A \setminus \{a\}) \in \bigcap_{\substack{b<a\\b\in A}} \{c(a, B) \colon B \ni b\} = \bigcap_{\substack{b<a\\b\in A}} A_{b,a}$$

as for all $b \in A$ with $b < a$ we trivially have $b \in A \setminus \{a\}$ and therefore each of the sets $\{(a, B) \colon B \ni b\}$ contains $(a, A \setminus \{a\})$ in it, and this remains true when taking the images under c.

But this means that also

$$c(a, A \setminus \{a\}) \in \bigcup_{\substack{b>a\\b\in A}} A_{a,b} = \bigcup_{\substack{b>a\\b\in A}} \{c(b, B) \colon B \ni a\}$$

as this is the set we are subtracting to get the empty set. So there is some $B \ni a$ and $b \in A$ with $b > a$ such that $c(a, A \setminus \{a\}) = c(b, B)$. But because of $B \ni a$ and $b \in A \setminus \{a\}$ we have that $(a, A \setminus \{a\}) \sim (b, B)$ as vertices in $U(m, k)$, contradicting that the colouring was proper. $\square$

We will need one more definition before giving an upper bound on the number of colours required in a 3-local colouring of a graph with high chromatic number. In the following, for a set S let 2^S denote the set of subsets of S.

Definition 2.18. *A* Sperner family *(or antichain) $S \subseteq 2^{[n]}$ on $[n]$ is a collection of subsets of $[n]$ such that if $A \neq B$ where $A, B \in S$, then $A \not\subseteq B$. An intersecting Sperner family on $[n]$ is a Sperner family on $[n]$ in which the sets are pairwise non-disjoint, i.e. $A \cap B \neq \emptyset$ for any A and B from S. Define $S(n)$ to be the total number of intersecting Sperner families on $[n]$.*

Theorem 2.19. *There is a graph G with $\chi(G) > n$ and that admits an 3-local $(S(n)+1)$-colouring.*

Proof. We need to show there are no $(S(n)+1, n, 3)$-independent systems, then by Lemma 2.17 the result follows. So assume the contrary, i.e. there is such a system $S = \{A_{a,b} \colon 1 \leq a < b \leq S(n) + 1\}$. Now let S_j be the system of those sets $A_{i,j}$, $i < j$, that are minimal under inclusion for fixed j, i.e. the sets $A_{i,j}$ for which $A_{i',j} \not\subseteq A_{i,j}$ for all $i' < j$. Each S_j is a Sperner family as if we take two sets $A_{i,j} \neq A_{i',j}$, then we can't have $A_{i,j} \subseteq A_{i',j}$ due to minimality under inclusion. They are also intersecting, as if we set $A = \{i, i', j\} \in \binom{S(n)+1}{3}$ and $a = j$ we get by the definition of S being $(S(n) + 1, n, 3)$-independent that

$$\emptyset \neq \bigcap_{\substack{b<a \\ b\in A}} A_{b,a} = \bigcap_{b\in\{i,i'\}} A_{b,j} = A_{i,j} \cap A_{i',j}.$$

So each of the S_j is an intersecting Sperner family on $[n]$, meaning we have a total of $S(n) + 1$ intersecting Sperner families on $[n]$. But as the total number of intersecting Sperner families on $[n]$ is $S(n)$, they cannot all be different, so there must be some $i < j$ such that the corresponding Sperner families S_i and S_j are the same. Now consider $A_{i,j}$. Either there is some $A_{i',j} \subseteq A_{i,j}$ in S_j, or it is minimal under inclusion, in that case set $i' = i$. Now as $S_i = S_j$, the sets in these families are the same and thus there is some $k < i$ such that $A_{k,i} = A_{i',j}$. But then if we set $A = \{k, i, j\}$ and $a = i$, we get

$$\bigcap_{\substack{b<a \\ b\in A}} A_{b,a} \setminus \bigcup_{\substack{b>a \\ b\in A}} A_{a,b} = A_{k,i} \setminus A_{i,j} \subseteq A_{k,i} \setminus A_{i',j} = \emptyset$$

meaning that S was not $(S(n) + 1, n, 3)$-independent, a contradiction. $\square$

Erdős and Hindman showed in [11] that $S(n) = 2^{0.5\binom{n}{\lfloor 0.5n \rfloor}(1+o(1))}$, and therefore there exists a graph G with $\chi(G) > n$ and that admits an 3-local $(2^{0.5\binom{n}{\lfloor 0.5n \rfloor}(1+o(1))})$-colouring. The next theorem will show that this bound gives the correct order of magnitude in the exponent. Notice for that purpose that $\binom{n-2}{\lfloor 0.5(n-2) \rfloor} = 0.25\binom{n}{\lfloor 0.5n \rfloor}(1+o(1))$, so the lower bound given by Theorem 2.20 is $2^{0.25\binom{n}{\lfloor 0.5n \rfloor}(1+o(1))}$.

Theorem 2.20. *There are no graphs G with $\chi(G) > n$ that admit a 3-local 2^k-colouring for $k = \binom{n-2}{\lfloor 0.5(n-2) \rfloor}$.*

Proof. We will construct a $(2^k, n, 3)$-independent system, and then the result follows from Lemma 2.17. There are 2^k subsets of $\binom{[n-2]}{\lfloor 0.5(n-2) \rfloor}$. Enumerate them as X_i for $1 \leq i \leq 2^k$ such that $|X_i| \leq |X_j|$ for $i < j$. Let Y_i be the set obtained by adding the element n to each set in X_i. So we have $|Y_i| = |X_i|$ and thus $|Y_i| \leq |Y_j|$ for $i < j$ as well. As the Y_i are all distinct we can therefore choose $A_{i,j} \in Y_j \setminus Y_i$ for $i < j$. We need to show that this system of $A_{i,j}$, $1 \leq i < j \leq 2^k$ is $(2^k, n, 3)$-independent. So let $A = \{i, j, l\} \subseteq \binom{2^{[k]}}{3}$ with $i < j < l$ and $a \in A$. Let

$$S = \bigcap_{\substack{b<a \\ b \in A}} A_{b,a} \setminus \bigcup_{\substack{b>a \\ b \in A}} A_{a,b}.$$

We have to consider three cases to show that S is non-empty:

Case 1: $a = i$: $S = [n] \setminus (A_{i,j} \cup A_{i,l}) \ni n-1$ and is therefore non-empty, as $n-1$ is not contained in any Y_i and so not in any $A_{i,j}$ either.

Case 2: $a = j$: $S = A_{i,j} \setminus A_{j,l} = (A_{i,j} \setminus \{n\}) \setminus (A_{j,l} \setminus \{n\})$, where $A_{i,j} \setminus \{n\} \in X_j \setminus X_i$ and $A_{j,l} \setminus \{n\} \in X_l \setminus X_j$. This means that $A_{i,j} \setminus \{n\} \in X_j$ while $A_{j,l} \setminus \{n\} \notin X_j$. Therefore $A_{i,j} \neq A_{i,l}$ and thus $A_{i,j} \setminus A_{i,l} \neq \emptyset$, because both $A_{i,j}$ and $A_{i,l}$ are subsets of $[n-2]$ of the size $0.5\lfloor n-2 \rfloor$, i.e. the same size.

Case 3: $a = l$: $S = A_{i,l} \cap A_{j,l} \ni n$ therefore non-empty, as n is part of every Y_i and thus of every $A_{i,j}$.

□

Using (m, n, k)-independent systems as a tool, Erdős et al. show in [10] a few more results of the kind presented above. First they generalize the above result to r-local colourings for any $r \geq 3$.

Theorem 2.21. *There is a graph G with $\chi(G) > n$ and that admits an r-local 2^k-colouring for $k = 2n + 2^{\frac{n}{2^{r-3}}}$.*

There are no graphs G with $\chi(G) > n$ that admit an r-local 2^{2^k}-colouring for $k = \frac{n}{(r-1)2^{r-1}}$. □

Apart from a few more results similar to the above, it is shown in [10] that graphs with $\psi(G) = 3$ and high chromatic number can still have large girth.

Theorem 2.22. *For every g and n, there exists a graph G of girth at least g and with $\chi(G) > n$ and $\psi(G) = 3$.* □

The proof of the above theorem is probabilistic, and in fact the graph shown to exist is a subgraph of the shift graph on the vertex set $\binom{m}{3}$ for some m.

The paper [10] also examines the above questions for infinite graphs, however this topic is beyond the scope of this thesis.

2.5 Relation to fractional chromatic number

In this section we will define the fractional chromatic number and show that it gives a lower bound on the local chromatic number. The proof outlined here was given by Körner, Pilotto and Simonyi in [20]. Along the way we will use various results described in the book "Algebraic Graph Theory" [14] by Godsil and Royle.

Definition 2.23. *Let $\mathcal{I}(G)$ be the set of non-empty independent sets of G and $\mathcal{I}(G, v)$ be the set of non-empty independent sets of G containing v. A* fractional colouring *is a non-negative function $w\colon \mathcal{I}(G) \to \mathbb{R}_{\geq 0}$ assigning a weight to each independent set in G, that satisfies*

$$\sum_{A \in \mathcal{I}(G,v)} w(A) \geq 1$$

for all vertices v.

Then the fractional chromatic number *is defined as*

$$\chi_f(G) = \min_{w} \sum_{A \in \mathcal{I}(G)} w(A)$$

where the minimum is taken over all fractional colourings.

By Theorem 7.3.2 from [14] this is well defined (i.e. there is a fractional colouring achieving the infimum, implying that it is a minimum) and equal to

$$\min\left\{\frac{n}{k}\colon G \to KG(n,k)\right\},$$

where the relation $G \to H$ is defined as "there exists a homomorphism from G into H".

Lemma 2.24. *If there is a homomorphism $f\colon G \to H$, then $\chi_f(G) \le \chi_f(H)$.*

Proof. For a set A of vertices of H, write $f^{-1}(A) = \bigcup_{v\in A} f^{-1}(v)$, and $f^{-1}(\mathcal{I}(H,v)) = \bigcup_{A\in\mathcal{I}(H,v)}\{f^{-1}(A)\}$ for the set of preimages of the independent sets in $\mathcal{I}(H,v)$. Note that preimages of independent sets in H will be independent sets in G. So if we have a weight function w over all independent sets of H, then we obtain a weight function w' for G as follows: for $B \in \mathcal{I}(H)$ we assign $w'(f^{-1}(B)) = w(B)$. For the other independent sets $A \in \mathcal{I}(G)$ that are not preimages of independent sets in $B \in I(G)$ we assign $w'(A) = 0$. This is a valid fractional colouring, as all the independent sets in $f^{-1}(\mathcal{I}(H, f(v))$ contain v, and thus we get:

$$\sum_{A\in\mathcal{I}(G,v)} w'(A) \ge \sum_{A\in f^{-1}(\mathcal{I}(H,f(v))} w'(A) \ge \sum_{B\in\mathcal{I}(H,f(v))} w(B) \ge 1.$$

The total sum of $w'(A)$ over all independent sets $A \in \mathcal{I}(G)$ is the same as the total sum of $w(B)$ over all independent sets $B \in \mathcal{I}(H)$, as each independent set in H has at most one independent set in G as preimage. □

Furthermore note that $\chi_f(G) \ge \omega(G)$ where $\omega(G)$ denotes the size of the largest clique in G: Consider the largest clique $W = \{w_1, \dots, w_{\omega(G)}\}$ in a graph. Each independent set contains at most one of these vertices, so $\mathcal{I}(G, w_1), \dots, \mathcal{I}(G, w_{\omega(G)})$ together with the set of independent sets that do not contain any of the w_i is a partition of the independent sets, i.e. $\mathcal{I}(G, w_i)$ and $\mathcal{I}(G, w_j)$ are disjoint for $i \ne j$. Now the sum of $w(S)$ over all $S \in \mathcal{I}(G, w_i)$ has to be at least 1 for all $1 \le i \le \omega(G)$, therefore as the $\mathcal{I}(G, w_i)$ were disjoint, the total sum over $w(S)$ over all independent sets S has to be at least $\omega(G)$.

For a graph G, let $\alpha(G)$ denote the size of the largest independent set of G. We can give another lower bound on the fractional chromatic number. Let w be a fractional colouring of G that achieves the fractional chromatic number, i.e.

$$\chi_f(G) = \sum_{A\in\mathcal{I}(G)} w(A).$$

By summing up over all vertices and independent sets, we get

$$\alpha(G)\chi_f(G) = \sum_{A\in\mathcal{I}(G)} \alpha(G)w(A) \geq \sum_{v\in V(G)} \left(\sum_{A\in\mathcal{I}(G,v)} \alpha(G)w(A) \right) \geq \sum_{v\in V(G)} 1 = |V(G)|,$$

using the fact that each independent set contributes to the sum once for each vertex contained in it, so at most $\alpha(G)$ times. From this we get the bound $\chi_f(G) \geq \frac{|V(G)|}{\alpha(G)}$. For vertex transitive graphs this bound is in fact tight. This result is stated in the following lemma without proof.

Lemma 2.25 (Lemma 7.4.4 and Corollary 7.5.2 from [14])**.** *If a graph G is vertex transitive, then the fractional chromatic number equals $\frac{|V(G)|}{\alpha(G)}$.* □

Recall the definition of the universal graph $U(m,k)$ from Section 2.4:

$$\begin{aligned} V(U(m,k)) &= \{(x,A)\colon x\in[m], A\subseteq[m], |A|\leq k-1, x\notin A\}, \\ E(U(m,k)) &= \{\{(x,A),(y,B)\}\colon x\in B, y\in A\}. \end{aligned}$$

In the following we will need a subgraph $\bar{\mathrm{U}}(m,k)$ induced by the vertices

$$V(\bar{\mathrm{U}}(m,k)) = \{(x,A)\colon x\in[m], A\subseteq[m], |A| = k-1, x\notin A\}.$$

The only difference here is that we fix the size of A to be exactly $k-1$.

Theorem 2.15 implied that if there is a k-local m-colouring c of G, then there is a homomorphism from G into $U(m,k)$. This homomorphism was $v \mapsto (c(v), c(\Gamma(v)))$. We can adapt this homomorphism into a homomorphism into $\bar{\mathrm{U}}(m,k)$. We need to make sure that the set A_v in the image of $v \mapsto (c(v), A_v)$ has exactly $k-1$ elements. This is achieved by picking $A_v := c(\Gamma(v)) \cup B_v$ for some $B_v \subseteq [m]$ that is disjoint from $c(\Gamma(v))$ and $\{c(v)\}$ and that has size $|B_v| = k-1-|c(\Gamma(v))|$. This is a homomorphism because if we have an edge uv, then $u\in\Gamma(v)$ and $v\in\Gamma(u)$ and thus the images $(c(v), c(\Gamma(v))\cup B_v)$ and $(c(u), c(\Gamma(u))\cup B_u)$ are adjacent.

Lemma 2.26.

$$\chi_f(\bar{\mathrm{U}}(m,k)) = k.$$

Proof. "$\geq$": Note that for any given set $A \subseteq [m]$ of size k, the set of vertices $C = \{(x, A') \colon x \in A, A' = A \setminus \{x\}\}$ is a clique of size k. Therefore $\chi_f(\bar{\mathrm{U}}(m,k)) \geq \omega(\bar{\mathrm{U}}(m,k)) = k$.

"$\leq$": Note that $\bar{\mathrm{U}}(m,k)$ is vertex transitive, as any permutation of $[m]$ induces an automorphism on $\bar{\mathrm{U}}(m,k)$. So by Lemma 2.25 its fractional chromatic number equals $\frac{|V(\bar{\mathrm{U}}(m,k))|}{\alpha(\bar{\mathrm{U}}(m,k))}$. Now $S = \{(x, A) \colon \forall a \in A : x < a\}$ is an independent set in $\bar{\mathrm{U}}(m,k)$, as for two vertices (x, A) and (y, B) in S we may assume $x \leq y$, but then $x \notin B$ because for all $b \in B$ we have $b > y \geq x$. So $\chi_f(\bar{\mathrm{U}}(m,k)) = \frac{|V(\bar{\mathrm{U}}(m,k))|}{\alpha(\bar{\mathrm{U}}(m,k))} \leq \frac{|V(\bar{\mathrm{U}}(m,k))|}{|S|} = k$, as for every set $A \subseteq [m]$ of size k there are k vertices (x, A') with $|\{x\} \cup A'| = k$, and exactly one of them satisfies the property that x is smaller than all elements in A'. □

The previous lemmas and observations together give us the following theorem:

Theorem 2.27.

$$\chi_f(G) \leq \psi(G).$$

Proof. Let $\psi(G) = k$. Then there is some m such that there is a homomorphism from $G \to \bar{\mathrm{U}}(m,k)$. This gives us that $\chi_f(\bar{\mathrm{U}}(m,k)) \geq \chi_f(G)$. But $\chi_f(\bar{\mathrm{U}}(m,k))$ equals k by the previous lemma, so we get $\psi(G) = k = \chi_f(\bar{\mathrm{U}}(m,k)) \geq \chi_f(G)$. □

With these results we can now prove an easy corollary. A classic result by Erdős [9] shows the existence of graphs with arbitrarily high girth and high chromatic number. The probabilistic proof of this result actually shows that these graphs have the ratio $\frac{|V(G)|}{\alpha(G)}$ arbitrarily large. We've noted that this ratio is a lower bound for the fractional chromatic number. So this result also gives that there are graphs that have high girth and high fractional chromatic number. As just seen the fractional chromatic number is a lower bound for the local chromatic number, so these graphs also have a high local chromatic number.

Corollary 2.28. *There are graphs with arbitrarily high girth and local chromatic number.* □

Chapter 3

An upper bound for Schrijver graphs

In this chapter we will obtain a result that gives an upper bound on the local chromatic number of Schrijver graphs. Some ideas are similar to the proof of Theorem 2.7, however there are various refinements. Instead of finding a wide colouring that is transformed into a local colouring via Lemma 2.5 here a somewhat more intricate process is used on the initially found colouring.

The theorem presented here uses the same colouring procedure as presented in Remark 4 from [33] which is stated as follows:

Remark 3.1 (Remark 4 from [33])**.** *Let* $t = \chi(SG(n,k)) = n - 2k + 2$*. If* n *and* m *satisfy* $t \geq 2m + 3$ *and* $n \geq 8m^2 + 16m + 9$*, or if they satisfy* $t \geq 4m + 3$ *and* $n \geq 20m + 9$*, then* $\psi(SG(n,k)) \leq t - m$*.*

However a new analysis yields a weakening of the assumptions, so our result will in fact be a slight generalization. In the case $t \geq 4m + 3$ our requirement on n relative to m is slightly more demanding, though a note in our proof shows how the exact same bound could be obtained. However, if t is not too close to $2m + 3$ while still being smaller than $4m+3$ we give a weakening on the requirement on n relative to m. For example, our result gives that $\psi(SG(57,25)) \leq \chi(SG(57,25) - 2 = 7$ (here $t = 9, m = 2$), while this is not a consequence of Remark 3.1. Our generalization is as follows.

Theorem 3.2. *If* $t = \chi(SG(n,k)) = n - 2k + 2 > 2$ *and* $m > 0$ *are chosen in such a way that*

1. $t \geq 2m + 3$*, and*

2. $(t-2)(2\gamma-1)+\min(2\left\lceil\frac{t}{2}\right\rceil-1, 2\gamma-1)+1 \leq n$ *for some integer* $\gamma > 2+\frac{2m}{t-2m-2}$,

then

$$\psi(SG(n,k)) \leq t-m.$$

Proof. We will provide a t-colouring c_0, which is then adapted into a $(t-m)$-local colouring.

As before in the proof of Theorem 2.7, we partition $[n]$ into t disjoint consecutive intervals A_i for $1 \leq i \leq t$, i.e. each A_i contains consecutive integers, such that each A_i has odd size. We will define the sizes of A_i later. Define C_i as the unique largest subset of A_i that does not contain consecutive integers, i.e. C_i contains the first, third, fifth, ... element from A_i for $1 \leq i \leq t$. We now make specific choices for the sizes of $|A_i|$, or equivalently make choices for $|C_i|$ because $|A_i| = 2|C_i|-1$. Set $|C_i| = \gamma$ to be constant for $1 \leq i \leq t-2$, set $|C_{t-1}| = \min(\lceil\frac{t}{2}\rceil, \gamma)$, and $|C_t| \geq 1$, where γ is the constant from condition 2 in our assumption. These choices are possible as by condition 2 these sizes of A_i sum up to at most n, and by adjusting the size of C_t we can make them sum up to exactly n.

We define the t-colouring c_0 of $SG(n,k)$ very similarly to Theorem 2.7. This time for a vertex v (which is a k-subset of n) we define $c_0(v)$ to be the *smallest* (instead of any) index i for which $v \supseteq C_i$. By the same argument as in Theorem 2.7, $c_0(v)$ is well defined for all vertices v and c_0 is a proper t-colouring.

Let $b = t-m$. We will now obtain a b-local colouring using the assumed restrictions on m. The recolouring process to obtain a new colouring c works as follows. We introduce a new colour β. Consider a vertex y. Now y is recoloured with colour β if it has a neighbour x in whose neighbourhood there are at least $b-2$ different colours used that are smaller than $c_0(y)$. If there is no such x, set $c(y) = c_0(y)$. Note that we can easily see that now $|c(\bar{\Gamma}(v))| \leq b$ for any vertex v, as we only have the colour of v itself, possibly β, and at most $b-2$ colours from the original colours of c_0 in the neighbourhood of v, because any neighbour w of v that received a colour $c_0(w)$ greater than the lowest $b-2$ colours used in $\Gamma(v)$ got recoloured to colour $c(w) = \beta$. To see that c is proper we only need to ensure that the vertices of colour β form an independent set.

So assume that the new colour class β does not form an independent set. So there are recoloured vertices y and y' that are adjacent, and without loss of generality $c_0(y) < c_0(y')$. Let x and x' be respective neighbours that caused them to be recoloured. For a vertex v, define $I'(v) := \{j : v \cap C_j = \emptyset\}$, it is essentially the set of potential colours of neighbours of v, because in order to have a chance of receiving colour j a vertex must contain all numbers from C_j. For a vertex v that has at least $b-2$ different colours in its neighbourhood, let $I(v)$ be the smallest $b-2 = t-m-2$ indices from $I'(v)$. Notice that $I(x)$ and $I(x')$ are well-defined. In particular we have $\max I(x) < c_0(y)$ and $\max I(x') < c_0(y')$, because in x's neighbourhood there are at least $b-2$ colours that are smaller than $c_0(y)$, analogously for x' and y'. See Figure 3.1 for an example of these definitions.

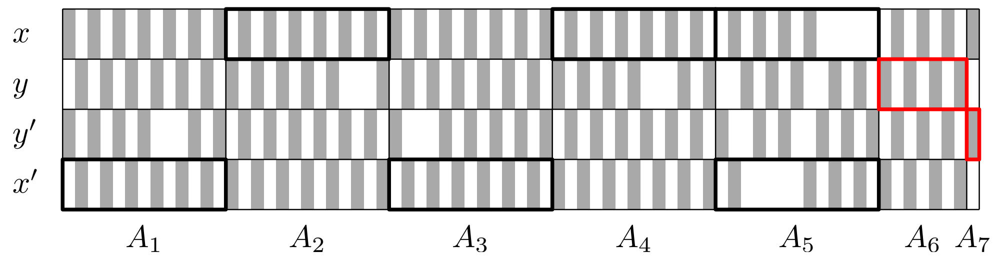


Figure 3.1: Example of a possible setup for $t = 7$ and $m = 2$ under the assumption that $x \sim y \sim y' \sim x'$. We have $|C_i| = \gamma = 7 > 2 + \frac{4}{1}$ for $1 \leq i \leq t-2$, $|C_{t-1}| = \lceil \frac{t}{2} \rceil$ and $|C_t| = 1$. This gives us $n = 73$ and $k = \frac{1}{2}(n-t-2) = 34$. Each row corresponds to a vertex, and each column represents a number from 1 to n, with a grey bar in column i in the row of a vertex v meaning that $i \in v$. Black boxes mark the blocks A_i for $i \in I(x)$ and $i \in I(x')$ respectively. The red boxes mark A_i for $i = c_0(y)$ and $i = c_0(y')$ respectively. Notice that y only contains $33 \neq k$ numbers, and in our proof we will show that there can not be a valid setup for these parameters in general.

For each i, $1 \leq i \leq t$, let $s_i = |x \cap A_i| + |y \cap A_i| + |y' \cap A_i| + |x' \cap A_i|$.

Then $s_i \leq 4|C_i| - 2$ because for each vertex v and each $1 \leq i \leq t$ we have $|v \cap A_i| \leq |C_i|$, and in fact if a vertex has $|v \cap A_i| = |C_i|$ for some i, then for a neighbour w of v only the numbers from $A_i \setminus C_i$ are left available to be used in w, so $|w \cap A_i| \leq |C_i| - 1$. With $x \sim y$ and $y' \sim x'$ the bound on s_i follows. Furthermore for a vertex v we have $\sum_{i=1}^{t} |v \cap A_i| = |v| = k$ and thus we get the equation $4k = \sum_{i=1}^{t} s_i$.

We will now consider various cases for the size of s_i. These cases are also illustrated in Figure 3.2.

Case 1: $i \in J_1 := (I(x) \cup \{c_0(y)\}) \cap I(x')$.

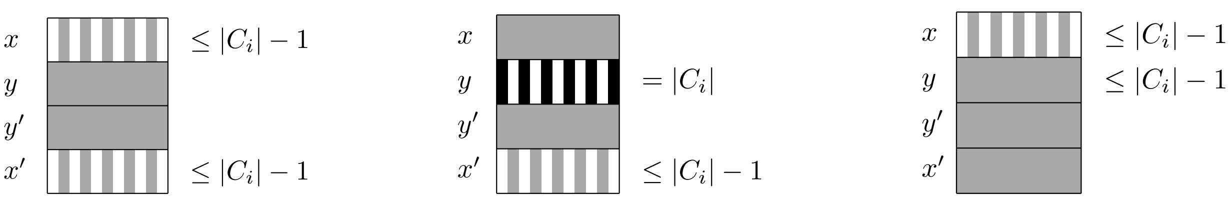
Case 1.1: $i \in I(x) \cap I(x')$. In this case we have that the numbers from C_i are not used by x nor x'. Each of these numbers can only be used by either y or y', but not both. So in total each number from C_i is used at most once. Each number from $A_i \setminus C_i$ can be used once between x and y, and once between y' and x', so at most twice in total. So we get $s_i \leq |C_i| + 2(|C_i| - 1) = 3|C_i| - 2$.

Case 1.2: $i = c_0(y)$ and $i \in I(x')$. In this case we have $y \cap A_i = C_i$, while the numbers from C_i are not used by x'. But as $x \sim y \sim y'$, the numbers from C_i can be used by neither

x nor y', so each of them is used exactly once (by y). Each of the numbers from $A_i \setminus C_i$ can be used at most once between y' and x', and possibly one more time by x, but not by y, so in total at most twice. So we get $s_i \leq |C_i| + 2(|C_i| - 1) = 3|C_i| - 2$.

Case 2: $i \in I(x) \triangle I(x')$ and $i \neq c_0(y)$, where $\triangle$ denotes the symmetric difference. Assume $i \in I(x)$, the argument for $i \in I(x')$ is symmetric. In this case we have that x doesn't use any numbers from C_i. Because $i \leq \max I(x) < c_0(y)$, we have that y cannot use all of the numbers from C_i. So we get $|y \cap A_i| \leq |C_i| - 1$. The vertices y' and x' can share each number at most once between them. So we get $s_i \leq 2(|C_i| - 1) + |A_i| = 4|C_i| - 3$. Let the set of these indices i be J_2.

Case 3: $i \notin I(x) \cup I(x')$. In this remaining case we can only give the trivial bound $s_i \leq 4|C_i| - 2$. Let the set of these indices i be J_3. In particular $t \in J_3$, because $\max I(x) < c_0(y) < c_0(y') \leq t$ and $\max I(x') < c_0(y') \leq t$.



(a) Case 1.1: $s_i \leq 3|C_i| - 2$

(b) Case 1.2: $s_i \leq 3|C_i| - 2$

(c) Case 2: $s_i \leq 4|C_i| - 3$

Figure 3.2: The vertices x, y, y', x restricted to block A_i in the various cases. A black bar means the number is used, a white bar means the number is not used by that vertex. A grey bar means that a case doesn't directly determine whether the number is used by that vertex or not.

Now $c_0(y) \notin I(x)$ because $\max I(x) < c_0(y)$. Therefore every index $1 \leq i \leq t$ belongs to exactly one of these cases, because it can be either in both (Case 1.1), one (Case 1.2 and Case 2) or none (Case 3) of the sets $I(x)$ and $I(x')$. So we get that $[t] = J_1 \cup J_2 \cup J_3$ where the union is disjoint. We then get:

$$\begin{aligned}
4k &= \sum_{i=1}^{t} s_i \\
&\leq \sum_{i\in J_1}(3|C_i|-2) + \sum_{i\in J_2}(4|C_i|-3) + \sum_{i\in J_3}(4|C_i|-2) \\
&= \sum_{i=1}^{t} 4|C_i| - \sum_{i\in J_1}|C_i| - 2|J_1| - 3|J_2| - 2|J_3| \\
&= 2(n+t) - \sum_{i\in J_1}|C_i| - 2|J_1| - 3|J_2| - 2|J_3|.
\end{aligned}$$

Here we are using that $\sum_{i=1}^{t} 4|C_i| = \sum_{i=1}^{t} 4 \cdot \frac{1}{2}(|A_i|+1) = 2(n+t)$.

We want to estimate the sizes of these three index sets J_1, J_2 and J_3. First we have $|I(x) \cup \{c_0(y)\}| = b-1$ and $|I(x')| = b-2$. We've already observed that $t \in J_3$, and thus none of these two sets contain t. Thus there are only $t-1$ slots to fit the indices from $|I(x) \cup c_0(y)|$ and $|I(x')|$ in. So we get that the size of the intersection of these two sets is $|J_1| \geq (b-1)+(b-2)-(t-1) = 2t-2m-3-(t-1) = t-2m-2$, which is greater than 0 because we have $t \geq 2m+3$ by condition 1 from our assumption. So let $|J_1| = t-2m-2+q$ for some integer $q \geq 0$. Next we estimate $|(I(x) \cup \{c_0(y)\}) \triangle I(x')|$. We get $|(I(x) \cup \{c_0(y)\}) \triangle I(x')| = |I(x) \cup \{c_0(y)\}| + |I(x')| - 2|(I(x) \cup \{c_0(y)\}) \cap I(x')| = (b-1)+(b-2)-2(t-2m-2+q) = 2t-2m-3-2t+4m+4-2q = 2m+1-2q$. Now $J_2 = (I(x) \triangle I(x')) \setminus \{c_0(y)\} = ((I(x) \cup \{c_0(y)\}) \triangle I(x')) \setminus \{c_0(y)\}$. Let $p = 1$ if $c_0(y) \in (I(x) \cup \{c_0(y)\}) \triangle I(x')$ and $p = 0$ if $c_0(y) \notin (I(x) \cup \{c_0(y)\}) \triangle I(x')$. With this definition we get $|J_2| = 2m+1-2q-p$. Finally $|J_3| = t-|J_1|-|J_2| = t-(t-2m-2+q)-(2m+1-2q-p) = 1+q+p$.

Continuing from above and using $p-1 \leq 0$ due to $p \in \{0,1\}$, we get

$$\begin{aligned}
4k &\leq 2(n+t) - \sum_{i\in J_1}|C_i| - 2|J_1| - 3|J_2| - 2|J_3| \\
&= 2(n+t) - 2(t-2m-2+q) - 3(2m+1-2q-p) - 2(1+q+p) - \sum_{i\in J_1}|C_i| \\
&= 2n-2m+(p-1)+2q - \sum_{i\in J_1}|C_i| \\
&\leq 2n-2m+2q - \sum_{i\in J_1}|C_i|.
\end{aligned}$$

Notice that by definition of t we have $4k = 2(n - t + 2) = 2n - 2t + 4$. Then the inequality from above becomes this, which we transform:

$$\begin{aligned} 2n - 2m + 2q - \sum_{i \in J_1} |C_i| &\geq 2n - 2t + 4 \\ \sum_{i \in J_1} |C_i| - 2q &\leq 2t - 2m - 4. \end{aligned}$$

So to reach a contradiction, we merely have to ensure that $\sum_{i \in J_1} |C_i| - 2q > 2t - 2m - 4$. We have that $\gamma > 2$ and $\lceil \frac{t}{2} \rceil > 2$ because $m \geq 1$ and $t \geq 2m + 3 \geq 5$, and because $t \notin J_1$ we get that $|C_i| > 2$ for all $i \in J_1$. Therefore $\sum_{i \in J_1} |C_i| - 2q \geq 3(t - 2m - 2) + q$. So the term on the left hand side is minimized for $q = 0$ and then we get $|J_1| = t - 2m - 2$.

Now if we have that the size of all the $|C_i|$ for $i \in J_1$ is constant (i.e. $|C_i| = \gamma$), then this inequality turns into

$$\begin{aligned} (t - 2m - 2)\gamma &\leq 2t - 2m - 4 \\ \gamma &\leq \frac{2t - 2m - 4}{(t - 2m - 2)} = 2 + \frac{2m}{t - 2m - 2}. \end{aligned}$$

But this directly contradicts our choice of $\gamma > 2 + \frac{2m}{t-2m-2}$.

(Note: In some cases, especially if $|J_1| = t - 2m - 2$ is big enough and there's a large gap between the lower bound on γ and the next integer greater than the bound, then one or a few of the block sizes can be decreased slightly. As long as $|C_i| \geq 2$ for all $1 \leq i \leq t - 1$ (to ensure that $q = 0$ minimizes the sum) and $\sum_{i \in J_1} |C_i| > 2t - 2m - 4$ holds for all valid choices of the index set J_1, we still get our contradiction. Thus in these cases we can lower the value of the sum over all $|A_i|$, and therefore a slight weakening of the lower bound assumed on n in condition 2 of the theorem is possible. However for simplicity a further analysis is omitted.)

So we just assumed that $|C_i| = \gamma$ for all $i \in J_1$. Because $t \notin J_1$, the only instance where this assumption is wrong is if the minimum in the expression for $|C_{t-1}| = \min(\lceil \frac{t}{2} \rceil, \gamma)$ takes the value $\lceil \frac{t}{2} \rceil$, and $t - 1 \in J_1$ meaning that Case 1 occurs in the interval A_{t-1}. This implies that $c_0(y) = t - 1$ because $\max I(x) < c_0(y) < c_0(y') \leq t$ and thus Case 1.1 cannot occur in the interval A_{t-1}, but Case 1.2 must occur instead.

We will examine the value of $|x'| + |y'|$. As $|v| = k$ for all vertices v, this value should be $2k$. Because Case 1.2 occurs in the interval A_{t-1}, we have $y \cap C_{t-1} = C_{t-1}$ and with $y \sim y'$ therefore $y' \cap C_{t-1} = \emptyset$. By Case 1.2 we also have that $t - 1 \in I(x')$, and thus $x' \cap C_{t-1} = \emptyset$.

So x' and y' can only use numbers from $A_{t-1} \setminus C_{t-1}$, but each of these $|C_{t-1}| - 1$ numbers can be used by at most one of them, therefore $|x' \cap A_{t-1}| + |y' \cap A_{t-1}| \leq |C_{t-1}| - 1$. For $1 \leq i \leq t-2$ we have in particular that $i < c_0(y')$, so we must have that $y' \cap A_i \neq C_i$, implying that $|y' \cap A_i| \leq |C_i| - 1$ for $1 \leq i \leq t-2$. Now for any vertex v and any index i, we cannot have $A_i \cap v = C_i$ and $A_{i+1} \cap v = C_{i+1}$ at the same time because then v would contain two consecutive integers (the biggest from A_i the smallest from A_{i+1}), which is not possible for a vertex of a Schrijver graph. Applying this to x', we get that $x' \cap A_i = C_i$ for only at most half of the indices $1 \leq i \leq t-2$; let this set of indices be K and we just saw that $|K| \leq \lceil \frac{t-2}{2} \rceil$. For the other half of the indices $|x' \cap A_i| \leq |C_i| - 1$ applies. We have $|[t-2] \setminus K| = (t-2) - |K| \geq \lfloor \frac{t-2}{2} \rfloor$. Together this gives us a total upper bound of

$$\begin{aligned}
2k &= |x'| + |y'| \\
&= \sum_{i=1}^{t} (|x' \cap A_i| + |y' \cap A_i|) \\
&\leq \left(\sum_{i \in K} (|C_i| + |C_i| - 1) \right) + \left(\sum_{i \in [t-2] \setminus K} (|C_i| - 1 + |C_i| - 1) \right) + (|C_{t-1}| - 1) + |A_t| \\
&\leq \left(\sum_{i=1}^{t-2} (2|C_i| - 1) \right) - \left(\sum_{i \in [t-2] \setminus K} 1 \right) + (|C_{t-1}| - 1) + |A_t| \\
&\leq \sum_{i=1}^{t-2} |A_i| - \left\lfloor \frac{t-2}{2} \right\rfloor \cdot 1 + (|C_{t-1}| - 1) + |A_t| \\
&= n - |A_{t-1}| - |A_t| - \left\lfloor \frac{t-2}{2} \right\rfloor + (|C_{t-1}| - 1) + |A_t| \\
&= n - \left\lfloor \frac{t-2}{2} \right\rfloor - |C_{t-1}|.
\end{aligned}$$

With the choice of $|C_{t-1}| = \lceil \frac{t}{2} \rceil > \lceil \frac{t-2}{2} \rceil$ this gives us

$$n - t + 2 = 2k \leq n - \left\lfloor \frac{t-2}{2} \right\rfloor - |C_{t-1}| < n - \left\lfloor \frac{t-2}{2} \right\rfloor - \left\lceil \frac{t-2}{2} \right\rceil = n - (t-2),$$

a contradiction.

□

By choosing a certain value for m, we can now obtain the following strengthening of Theorem 2.7.

Corollary 3.3 (restated)**.** *If* $t = n - 2k + 2$ *and* $n \geq 2t^2 - 4t + 3$*, then*

$$\psi(SG(n,k)) \leq \frac{t+3}{2} = \left\lceil \frac{t}{2} \right\rceil + 1.$$

Proof. Set $m = \frac{t-3}{2}$, then to apply Theorem 3.2 we need to check

1. $t \geq 2m + 3$, and
2. $(t-2)(2\gamma - 1) + \min(2\left\lceil \frac{t}{2} \right\rceil - 1, 2\gamma - 1) + 1 \leq n$ for some integer $\gamma > 2 + \frac{2m}{t-2m-2}$.

The first statement is obviously true as we have $t = 2m + 3$. For the second statement we have $2 + 2m/(t - 2m - 2) = 2m + 2$, so we can choose $\gamma = 2m + 3 = t$. Now notice that because t is odd we have $2\lceil t/2 \rceil - 1 = t \leq 2\gamma - 1$. Then we get as the value of the sum in the second condition

$$(t-2)(2t-1) + t + 1 = 2t^2 - 4t + 3 \leq n$$

by our assumption about n.

By Theorem 3.2 it follows that $\psi(SG(n,k)) \leq t - m = t - (t-3)/2 = (t+3)/2$, which is the desired bound.

□

If t from the above theorem is odd, then we have in fact equality: The lower bound $\psi(SG(n,k)) \geq \left\lceil \frac{t}{2} \right\rceil + 1$ is given in Theorem 4.19.

Chapter 4

A general lower bound

In this chapter we will develop topological techniques to define a topological chromatic number. The theory will culminate in the Zig-Zag theorem, showing us how the topological chromatic number is useful to give lower bounds on the local chromatic number and the chromatic number. We will see various applications of this theorem and of the theory developed in this chapter, and at the end see a combinatorial version of its proof which is inspired by topological ideas. Most of the definitions and basic material from this section are also described in Matoušek's book "Using the Borsuk-Ulam Theorem" [25].

In Section 4.1 we will begin with a brief introduction to simplicial complexes, then in Section 4.2 we will establish topological notions, define a simplicial complex called the box complex, and use it to define the topological chromatic number which we will then motivate by relating it to the chromatic number. We will find out in Section 4.3 that there are more simplicial complexes that are of interest. After introducing a generalization of the Borsuk-Ulam theorem, Section 4.4 gives the main result of this chapter, the Zig-Zag theorem which gives a general lower bound on the local chromatic number and the chromatic number of a graph. In Section 4.5 we will apply this theorem and the related theory developed in the previous sections to specific classes of graphs to give estimates for their local chromatic number. Finally, in Section 4.6 we will give a combinatorial proof of the main theorem of Section 4.4 for Kneser graphs.

4.1 Background on simplicial complexes

In this section we will give a short introduction to simplicial complexes so we can use them later together with topological tools. For a more gentle and thorough introduction refer to [25], which this section is based on.

An *abstract simplicial complex* K, henceforth just referred to as *simplicial complex*, is a hereditary set system. More precisely, we have a set $V(K)$ of *vertices*, and $K \subseteq 2^{V(K)}$ is a set of subsets of vertices called *simplices* that have the property that if $\sigma \in K$, then for all subsets $\sigma' \subseteq \sigma$ we must have that $\sigma' \in K$. We define the *dimension* $\dim(K) = \max\{|\sigma| - 1 \colon \sigma \in K\}$. For two simplicial complexes K and L, a *simplicial map* is a map $f\colon V(K) \to V(L)$ such that if $\sigma \in K$, we must have that $f(\sigma) \in L$, where $f(\sigma)$ denotes the set of images of the elements in σ.

We call a set of points $V = \{v_0, \ldots, v_k\} \subseteq \mathbb{R}^d$ for some d and k *affinely independent* if they have the property that the vectors $v_1 - v_0, \ldots, v_d - v_0$ are linearly independent. Then a *geometric simplex* σ is the convex hull of any set $k+1$ affinely independent points in $\mathbb{R}^d$ for some d and k. So $\sigma = \{\sum_{i=0}^k \alpha_i v_i \colon \alpha_i \in [0,1] \text{ for } i = 0, \ldots, k \text{ and } \sum_{i=0}^k \alpha_i = 1\}$. We call k the *dimension* of the simplex, and the points in V are called *vertices*. We also call a simplex of dimension k a *k-simplex*. The convex hull of an arbitrary subset of vertices of σ is called a *face* of σ. A *geometric simplicial complex* Δ is a set of geometric simplices that has following two properties: First, for every simplex $\sigma \in \Delta$, each of its faces is again a simplex in Δ. Second, the intersection of two simplices σ_1 and σ_2 from Δ is a face of both σ_1 and σ_2. The dimension of Δ is defined to be $\max\{\dim \sigma \colon \sigma \in \Delta\}$. For a simplicial complex Δ, we call the union of all its simplices the *body* of Δ and denote it with $||\Delta||$. The body $||\Delta||$ is a topological space with the topology coming from the metric in $\mathbb{R}^d$. For a topological space X we call a geometric simplicial complex Δ a *triangulation* of X if $||\Delta||$ is homeomorphic to X.

Assume we have an abstract simplicial complex K on a vertex set $V(K)$. Associate each vertex $x \in V(K)$ with a point $||x||$ in $\mathbb{R}^d$ for some d such that for each simplex $\sigma \in K$ the points associated with its vertices are affinely independent. Then in particular the convex combination of the points $\{||x|| \colon x \in \sigma\}$ will be a geometric simplex which we will denote with $||\sigma||$. If the set of the geometric simplices $\{||\sigma|| \colon \sigma \in K\}$ form a geometric simplicial complex, we will denote that complex as $||K||$ and call it a *geometric realization* of K. By the geometric realization theorem, we have that every d-dimensional abstract simplicial complex has a geometric realization in $\mathbb{R}^{2d+1}$, and another well known theorem states that all geometric realizations of a simplicial complex are homeomorphic, see e.g. [25] for these two theorems. So $||K||$ is well defined up to homeomorphism.

Given a simplicial map $f\colon V(K) \to V(L)$, we can extend it to a map $||f||\colon ||K|| \to ||L||$ via a process called *affine extension* as follows. Let x be a point in a simplex $||\sigma|| \in ||K||$ whose vertices are v_i for $0 \leq i \leq k$. By definition of a geometric simplex we can write $x = \sum_{i=0}^{k} \alpha_i v_i$ where $\alpha_i \in [0,1]$ for $i = 0, \dots, k$ and $\sum_{i=0}^{k} \alpha_i = 1$. Then define $||f(x)|| = \sum_{i=0}^{k} \alpha_i f(v_i)$. In the following, $||\cdot||$ will sometimes be implicit if it is clear from the context that we're viewing an abstract simplicial complex as its geometric realization (and thus as a topological space).

4.2 Topological notions

A $\mathbb{Z}_2$*-space* (T, ν) is a topological space T equipped with an *involution map* $\nu\colon T \to T$ that is continuous and has the property that $\nu^2 = \mathrm{id}$. For example, the n-dimensional unit sphere $(S^n, x \mapsto -x)$ equipped with the map that maps each point x to its antipodal point $-x$ is a $\mathbb{Z}_2$-space. When the involution is obvious from the context we will often suppress it in the notation, e.g. in the case $(S^n, x \mapsto -x)$ we will just write S^n. A $\mathbb{Z}_2$-space is called *free* if the involution has no fixed points. For example, the sphere S^n is a free $\mathbb{Z}_2$-space, while the closed n-dimensional unit ball $(B^n, x \mapsto -x)$ is not free because 0 is a fixed point. If a $\mathbb{Z}_2$-space T with involution ν admits a homeomorphism ϕ into a topological space S, then we can equip S with the involution $\mu = \phi \circ \nu \circ \phi^{-1}$ to turn S into a $\mathbb{Z}_2$-space because μ is continuous as a concatenation of continuous maps and $\mu^2 = \phi \circ \nu \circ \phi^{-1} \circ \phi \circ \nu \circ \phi^{-1} = \phi \circ \nu^2 \circ \phi^{-1} = \mathrm{id}$.

A continuous map $f\colon (S, \mu) \to (T, \nu)$ is called a $\mathbb{Z}_2$*-map* if the following diagram commutes, i.e. if $f \circ \mu = \nu \circ f$:

$$\begin{tikzcd}
(S, \mu) \arrow[r, "f"] \arrow[d, "\mu"] & (T, \nu) \arrow[d, "\nu"] \\
(S, \mu) \arrow[r, "f"] & (T, \nu)
\end{tikzcd}$$

We say that such a map *respects* the involutions of the topological spaces (S, μ) and (T, ν). If such a $\mathbb{Z}_2$-map exists between (S, μ) and (T, ν), then we write $(S, \mu) \to (T, \nu)$. Two Z_2-spaces (S, μ) and (T, ν) are called $\mathbb{Z}_2$*-equivalent* if we have both that $(S, \mu) \to (T, \nu)$ and $(T, \nu) \to (S, \mu)$. In that case we write $(S, \mu) \leftrightarrow (T, \nu)$.

We can transfer the notions of involutions and $\mathbb{Z}_2$-maps to simplicial complexes. A simplicial map $\nu\colon K \to K$ is called an *involution* if $\nu^2 = \mathrm{id}$. If K and L have involutions $\nu\colon K \to K$ and $\mu\colon L \to L$, then a simplicial map $f\colon K \to L$ is called a $\mathbb{Z}_2$-*map* if $f \circ \mu = \nu \circ f$. In that case we write $K \to L$, and we write $K \leftrightarrow L$ if in addition we have $L \to K$. If we have a simplicial map $\nu\colon K \to K$ which is an involution, then a straightforward computation verifies that the affine extension $||\nu||$ is an involution map of $||K||$, turning $(||K||, ||\nu||)$ in a $\mathbb{Z}_2$-space.

We furthermore define the index and the coindex of a $\mathbb{Z}_2$-space as

$$\begin{aligned} \mathrm{ind}(T) &= \min\{d \geq 0\colon T \to S^d\} \\ \mathrm{coind}(T) &= \max\{d \geq 0\colon S^d \to T\}. \end{aligned}$$

For our next lemma we need to make use of a version of the Borsuk-Ulam theorem again. This time we use the version by Borsuk from [4].

Theorem 4.1 (Borsuk-Ulam)**.** *For every continuous map $f\colon S^n \to \mathbb{R}^n$ there is an x with $f(x) = f(-x)$.* □

Lemma 4.2. $\mathrm{coind}(T) \leq \mathrm{ind}(T)$.

Proof. We have by definition of index and coindex that $S^{\mathrm{coind}(T)} \to T \to S^{\mathrm{ind}(T)}$. Note that any $\mathbb{Z}_2$-map $f\colon S^n \to S^m$ has the property $-f(x) = f(-x)$ by the definition of a $\mathbb{Z}_2$-map. We call a map with this property an *antipodal map*. If $m < n$, then S^m can be viewed as a subset of $\mathbb{R}^n$, so f is in fact a continuous map from S^n to $\mathbb{R}^n$. So by Theorem 4.1 f has a point x such that $f(x) = f(-x) = -f(x)$ and thus $f(x) = 0$. But $0 \notin S^m$, so such a map f cannot exist, and so $S^n \nrightarrow S^m$ when $m < n$. But as $S^{\mathrm{coind}(T)} \to S^{\mathrm{ind}(T)}$, this means that $\mathrm{coind}(T) \leq \mathrm{ind}(T)$. □

In the next paragraph we will define a simplicial complex on a graph G that is required to define the topological chromatic number and that has proven useful in determining information about colourability in certain instances. But first we have to define, for a graph G and disjoint vertex subsets S and T, the bipartite subgraph induced by S and T, denoted $G[S,T]$. So more precisely, the vertex set of $G[S,T]$ is $S \cup T$, and we add all the edges between S and T that were in G, i.e. the edge set is $\{uv \in E(G)\colon u \in S, v \in T\}$. We call S and T the *shores* of the bipartite graph. We allow S and T to be empty for the graph still to be considered a complete bipartite graph. For arbitrary subsets S and T of $V(G)$, we also define $S \uplus T := S \times \{1\} \cup T \times \{2\}$. With these two definitions, we're now ready to define the box complex of a graph.

Definition 4.3. *The* box complex $B_0(G)$ *of a graph* G *is a simplicial complex with vertex set* $V(G) \uplus V(G)$*, and the simplices are sets* $S \uplus T$ *for any* S *and* T *such that* $S \cap T = \emptyset$ *and* $G[S,T]$ *is a complete bipartite graph.*

Define the *shore swapping map* as $\nu_B \colon B_0(G) \to B_0(G)$, $S \uplus T \mapsto T \uplus S$. This map ν_B is a simplicial map and an involution, so as we have seen it follows that $(||B_0(G)||, ||\nu_B||)$ is a $\mathbb{Z}_2$-space with the affine extension of ν_B as involution.

The n-dimensional *crosspolytope* is the convex hull of $\{\pm e_1, \dots, \pm e_n\}$, the set of positive and negative standard basis vectors of $\mathbb{R}^n$. The $(n-1)$-dimensional *crosspolytope complex* $\diamond^{n-1}$ is a simplicial complex on vertices $V(\diamond^{n-1}) = \{\pm 1, \pm 2, \dots, \pm n\}$ where every subset that does not contain both i and $-i$ for some $1 \leq i \leq n$ is a simplex. Its geometric realization $|| \diamond^{n-1} ||$ is the boundary of the n-dimensional crosspolytope if we associate the vertices $\pm i$ with the points $\pm e_i$. The Figure 4.1 is an example for $n = 3$.

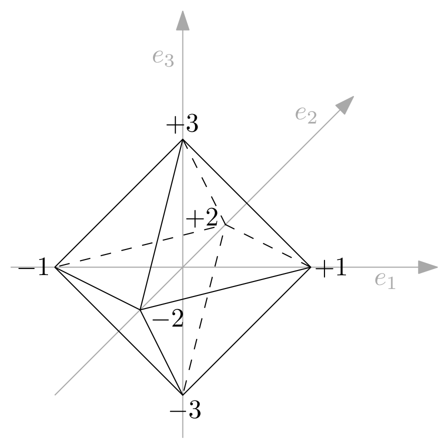


Figure 4.1: The geometric realization of the crosspolytope $\diamond^2$.

The simplicial complex $\diamond^{n-1}$ is equipped with an involution $\nu_\diamond \colon V(\diamond^{n-1}) \to V(\diamond^{n-1})$, $x \mapsto -x$. Its geometric realization $|| \diamond^{n-1} ||$ is homeomorphic to S^{n-1}, with the map that normalizes each point in $|| \diamond^{n-1} ||$ to have magnitude 1 as homeomorphism. This homeomorphism respects the existing involutions, and so in fact we have $(||\diamond^{n-1}||, ||\nu_\diamond||) \leftrightarrow (S^{n-1}, x \mapsto -x)$.

Lemma 4.4.

$$(B_0(K_n), \nu_B) \leftrightarrow (\diamond^{n-1}, \nu_\diamond).$$

Proof. For a set T, write $-T = \{-t \colon t \in T\}$. Let the vertex set of K_n be $\{1, \dots, n\}$. Notice that $B_0(K_n)$ contains $S \uplus T$ if and only if S and T are disjoint, as any disjoint pair gives a complete bipartite graph $G[S,T]$ in K_n. Then define $f\colon V(B_0(K_n)) \to V(\diamond^{n-1})$ via $(s,1) \mapsto +s$ and $(t,2) \mapsto -t$. This map is simplicial because a simplex $S \uplus T$ in $B_0(K_n)$ is mapped to $S \cup -T$, which is a simplex in $\diamond^{n-1}$ because S and T have to be disjoint for $S \uplus T$ to be a simplex in $B_0(K_n)$. An analogous argument works for the inverse. We have $\nu_\diamond(f(s,1)) = \nu_\diamond(+s) = -s = f(s,2) = f(\nu_B(s,1))$ and $\nu_\diamond(f(t,2)) = \nu_\diamond(-t) = +t = f(t,1) = f(\nu_B(t,2))$ and thus f is also a $\mathbb{Z}_2$-map. $\square$

Corollary 4.5.

$$||B_0(K_n)|| \leftrightarrow ||\diamond^{n-1}|| \leftrightarrow S^{n-1}.$$

Proof. We've already seen that $||\diamond^{n-1}|| \leftrightarrow S^{n-1}$ earlier. From Lemma 4.4 we have that $B_0(K_n) \to \diamond^{n-1}$ and $\diamond^{n-1} \to B_0(K_n)$, and thus via affine extension $||B_0(K_n)|| \to ||\diamond^{n-1}||$ and $||\diamond^{n-1}|| \to ||B_0(K_n)||$, showing the other $\mathbb{Z}_2$-equivalence. $\square$

We will now define a topological parameter and show that it gives a lower bound on the chromatic number of a graph. We should mention that this definition is non-standard, and in fact as we will see in Section 4.3 could be considered somewhat arbitrary as there are various other topological parameters giving lower bounds on the chromatic number.

Definition 4.6. *The* topological chromatic number $\mathrm{t}(G)$ *of a graph G is defined as*

$$\mathrm{t}(G) := \mathrm{coind}(||B_0(G)||) + 1.$$

We furthermore say a graph G is *topologically t-chromatic* (a notion introduced in [33]) if $\mathrm{t}(G) = \mathrm{coind}(||B_0(G)||) + 1 \geq t$. Note that this term is somewhat unusual as it indicates a lower bound on the parameter rather than an upper bound, which is more common when referring to colouring parameters. We will now motivate this definition by showing how it relates to the chromatic number $\chi(G)$ of a graph.

Theorem 4.7. $\mathrm{t}(G) \leq \chi(G)$.

Proof. Observe that a graph homomorphism $\phi\colon G \to H$ induces a simplicial map f from $B_0(G)$ to $B_0(H)$ that maps a simplex $S \uplus T$ to $f(S) \uplus f(T)$, because complete bipartite subgraphs in G get mapped to complete bipartite subgraphs in H. Note that $f(S) \cap f(T) = \emptyset$, as otherwise two vertices $s \in S$ and $t \in T$ would get mapped to the same vertex v, but s and t are adjacent so they have to be mapped to adjacent vertices. The map f furthermore respects shore swapping, i.e. $f \circ \nu_G = \nu_H \circ f$ for the involution maps ν_G and ν_H on $B_0(G)$ and $B_0(H)$ respectively. This map can be affinely extended to a $\mathbb{Z}_2$-map from $||B_0(G)||$ to $||B_0(H)||$, showing $||B_0(G)|| \to ||B_0(H)||$.

Now a graph G is $\chi(G)$-colourable if and only if there is a homomorphism $G \to K_{\chi(G)}$. By the note above, this implies that $||B_0(G)|| \to ||B_0(K_{\chi(G)})|| \leftrightarrow S^{\chi(G)-1}$ by Lemma 4.4. Therefore by the definition of the index, we have $\mathrm{ind}(||B_0(G)||) \leq \chi(G) - 1$ and by Lemma 4.2 we get in particular that $\mathrm{coind}(||B_0(G)||) \leq \chi(G) - 1$. Thus $\mathrm{t}(G) = \mathrm{coind}(||B_0(G)||) + 1 \leq \chi(G)$. □

The topological chromatic number gives us a useful tool to bound the chromatic number from below, as a lower bound on $\mathrm{t}(G)$ gives a lower bound on $\chi(G)$. Various classes have been shown to be topologically t-chromatic. For instance, it is known that for the Kneser graph $t(KG(n,k)) = n - 2k + 2$, which in particular implies Kneser's conjecture. For other graphs like Schrijver graphs, generalized Mycielski graphs and Borsuk graphs with certain parameters exact values or lower bounds for $\mathrm{t}(G)$ are known as well [33]. The first time topological methods have been used to give a lower bound on the chomatic number was in Lovász' proof of the Kneser conjecture [22]. Since then, other simplicial complexes have been used give alternative proofs of this conjecture and to study chromatic properties of graphs, of which we will give a short outline in the next section.

4.3 Other simplicial complexes

Before we proceed to relate the topological chromatic number and the local chromatic number to each other, we will give a short overview of some other topological parameters that have been used in the past and are mentioned in a few other sections of this thesis. We will see that calling $\mathrm{coind}(B_0(G)) + 1$ the topological chromatic number is a somewhat arbitrary choice. It gives a lower bound on the chromatic number, but in the same vein a number of other topological parameters could qualify for the same name. We made this choice as our focus in this thesis will be on the box complex B_0, while the other parameters will only be touched tangentially to help with outlining proofs of other results.

A survey of the various topological lower bounds and how they relate to each other is due to Matoušek and Ziegler [26]. Here we will only mention those complexes that are necessary for some result outlined in this thesis.

Before we start with more simplicial complexes, we have to define the *suspension* of a topological space. For a topological space S, the suspension $\mathrm{susp}(S)$ is the space obtained by taking $S \times [-1, 1]$ and identifying all the points $S \times \{1\}$ and all the points $S \times \{-1\}$ respectively. More precisely $S \times [-1, 1]$ is the topological space on the point set $\{(x, t) : x \in S, t \in [-1, 1]\}$ where the open sets are (possibly infinite) unions of sets of the form $A \times B$ where A is an open set in S and B is an open set in $[-1, 1]$. Next we define the map

$f\colon S \times [-1,1] \to \{-1\} \cup S \times (-1,1) \cup \{1\}$ that is the identity on $S \times (-1,1)$ and maps $(x, \pm 1)$ to ± 1 for $x \in S$. Then $\mathrm{susp}(S)$ is the image of f and its open sets are those sets whose preimages are open in $S \times [-1,1]$. If S has an involution ν, then $\mathrm{susp}(S)$ has the involution that maps $+1$ to -1 and vice versa, and otherwise maps $(x,t) \mapsto (\nu(x), -t)$.

Lemma 4.8. $\mathrm{susp}(S^t) \leftrightarrow S^{t+1}$.

Proof. We define the homeomorphism $f\colon \mathrm{susp}(S^t) \to S^{t+1}$ as follows. Set $f(1) = e_{t+1} = (0,\ldots,0,1)$, analogously $f(-1) = -e_{t+1} = (0,\ldots,0,-1)$, and finally map (x,t) to (x',t) where x' is a scalar multiple of x such that the magnitude of (x',t) is 1 while the sign of each coefficient remains the same. It is straightforward to check that this map respects the involutions. $\square$

Lemma 4.9. *For any $\mathbb{Z}_2$-space S with involution ν we have* $\mathrm{coind}(\mathrm{susp}(S)) \geq \mathrm{coind}(S)+1$ *and* $\mathrm{ind}(\mathrm{susp}(S)) \leq \mathrm{ind}(S) + 1$.

Proof. Let $\mathrm{coind}(S) = t$. Then by definition we have a $\mathbb{Z}_2$-map $f\colon S^t \to S$. Then the map $f'\colon \mathrm{susp}(S^t) \to \mathrm{susp}(S)$ defined by $(x,t) \mapsto (f(x),t)$ is a $\mathbb{Z}_2$-map as well. By Lemma 4.8 we have that $\mathrm{susp}(S^t) \leftrightarrow S^{t+1}$ and therefore in fact $S^{t+1} \to \mathrm{susp}(S^t) \to \mathrm{susp}(S)$. This implies $\mathrm{coind}(\mathrm{susp}(S)) \geq t+1$.

The proof for the index $\mathrm{ind}(\mathrm{susp}(S))$ works analogously. $\square$

The *restricted box complex* $B(G)$ of a graph G is a subcomplex of $B_0(G)$ that uses the following restriction: While in $B_0(G)$ any sets of the form $\emptyset \uplus T$ and $S \uplus \emptyset$ are simplices, in $B(G)$ these sets are only simplices if the vertices from T and S respectively have a common neighbour.

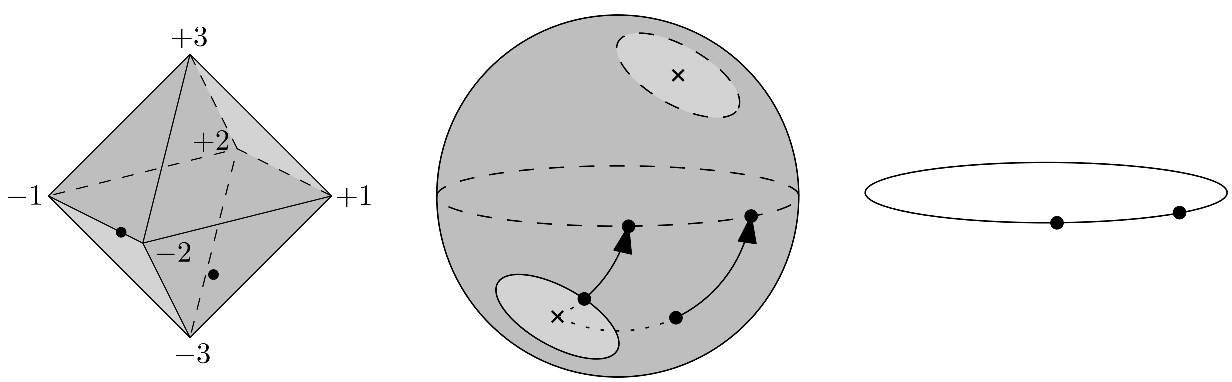
Similar to $B_0(K_n)$ being $\mathbb{Z}_2$-equivalent to the sphere S^{n-1} as we've already seen, we can make a statement about $B(K_n)$. The following is a proof sketch for Lemma 5.9.2 from Matoušek's book "Using the Borsuk-Ulam Theorem" [25] that was left as an exercise to the reader.

Lemma 4.10. $B(K_n) \leftrightarrow S^{n-2}$ *and thus* $\mathrm{coind}(B(K_n)) = \mathrm{ind}(B(K_n)) = n-2$.

Proof. Recall that $B_0(K_n) \leftrightarrow \diamond^{n-1}$, and thus we can view its vertices as $\{\pm 1, \cdots \pm n\}$, and its simplices as sets of vertices that don't contain two numbers of same absolute value. For the restricted box complex $B(K_n)$, translating its vertex set analogously, we have the additional condition that sets exclusively consisting of positive numbers, when these numbers are viewed as vertices in K_n, need to have a common neighbour. This always occurs unless our set is $\{+1, +2, \ldots, +n\}$. The same applies to sets exclusively consisting of negative numbers, so $\{+1, +2, \ldots, +n\}$ and $\{-1, -2, \ldots, -n\}$ are the only two sets that are simplices in $B_0(K_n)$ but not in $B(K_n)$.

First note that the restriction of $B(K_n)$ to those simplices that do not contain $+n$ or $-n$ is the same as $B_0(K_{n-1})$. By Corollary 4.5 we have $B_0(K_{n-1}) \leftrightarrow S^{n-2}$, therefore $S^{n-2} \leftrightarrow B_0(K_{n-1}) \to B(K_n)$.

Now we need to find a $\mathbb{Z}_2$-map from $B(K_n)$ into S^{n-2}. We will give a rough sketch how this map is defined. Figure 4.2 provides an example for the procedure outlined in the following. First notice that the body of $B(K_n)$ is homeomorphic to the $(n-1)$-dimensional sphere with two antipodal "holes", coming from the missing two simplices which are antipodal. A projection from the center of the body of $B(K_n)$ onto a surrounding $(n-1)$-dimensional sphere gives us one possible such homeomorphism that respects the Z_2-action. Then taking two antipodal points in the two antipodal holes, we can project the points along the great circles of S^{n-1} onto the intersection of S^{n-1} with the hyperplane through the origin whose last coordinate is 0, i.e. we are projecting onto S^{n-2}. This map is a contraction that again respects the $\mathbb{Z}_2$-action, and therefore $B(K_n) \to S^{n-2}$. $\square$



(a) The box complex $B(K_3)$, which differs from $B_0(K_3)$ by the missing simplices $\{+1, +2, +3\}$ and $\{-1, -2, -3\}$.

(b) The sphere S^2 with two "holes" corresponding to the two missing simplices in $B(K_3)$.

(c) The sphere S^1 as the image of a projection from the 2-dimensional sphere with two holes.

Figure 4.2: Mapping the restricted box complex $B(K_3)$ onto the sphere S^1 while respecting the Z_2-actions. Two points in $B(K_3)$ with their images in S^2 minus two holes and in S^1 are marked to illustrate the mappings.

Now in the same way as for the box complex $B_0(G)$, we have that if G has chromatic number t, then there is a homomorphism from G into K_t which translates into a $\mathbb{Z}_2$-map $B(G) \to B(K_n)$. With the lemma just proved we get $B(G) \to S^{t-2}$ and thus showing that $\operatorname{ind}(B(G)) \leq t-2$. This gives us the bound $\operatorname{ind}(B(G)) + 2 \leq \chi(G)$.

We need one more result due to Csorba in [6] which we will state without proof.

Theorem 4.11. *$B_0(G) \leftrightarrow \operatorname{susp}(B(G))$.* □

With this result and the above observations we can now get a chain of inequalities relating the indices and coindices of these two box complexes.

Lemma 4.12.

$$\chi(G) \geq \operatorname{ind}(B(G)) + 2 \geq \operatorname{ind}(B_0(G)) + 1 \geq \operatorname{coind}(B_0(G)) + 1 \geq \operatorname{coind}(B(G)) + 2.$$

Proof. We just explained the bound $\chi(G) \geq \operatorname{ind}(B(G)) + 2$. Theorem 4.11 and Lemma 4.9 imply that $\operatorname{coind}(B_0(G)) = \operatorname{coind}(\operatorname{susp}(B(G))) \geq \operatorname{coind}(B(G)) + 1$ and $\operatorname{ind}(B_0(G)) = \operatorname{ind}(\operatorname{susp}(B(G))) \leq \operatorname{ind}(B(G)) + 1$. Lemma 4.2 gives us that $\operatorname{ind}(B_0(G)) \geq \operatorname{coind}(B_0(G))$. Together this gives us the chain of inequalities. □

In [33] another complex referred to as the hom space $H(G)$ is used. This was first introduced by Babson and Kozlov as $\operatorname{Hom}(K_2, G)$ in [1]. By Remark 14 in Csorba [6] $\operatorname{Hom}(K_2, G)$ and $B(G)$ are $\mathbb{Z}_2$-homotopy equivalent, a notion that we will not define here, but that implies $\mathbb{Z}_2$-equivalence. So for our purposes with respect to $\mathbb{Z}_2$-equivalence, we can use $B(G)$ and $H(G)$ interchangeably. In particular we can use $B(G)$ in place of $H(G)$ with respect to $\mathbb{Z}_2$-equivalence when the latter is used in [33], and in fact we will limit our use to $B(G)$ in this thesis.

4.4 Relation between local and topological chromatic number

In this section we will establish a connection between the topological chromatic number and the local chromatic number. We will prove that if a graph G is topologically t-chromatic, then

$$\psi(G) \geq \left\lceil \frac{\mathrm{t}(G)}{2} \right\rceil + 1.$$

For this purpose we will introduce Ky Fan's Theorem, a generalization of the Borsuk-Ulam Theorem, and use it to prove the Zig-Zag Theorem which guarantees a large multicoloured complete bipartite subgraph in topologically t-chromatic graphs, from which we obtain the lower bound from above for the local chromatic number.

4.4.1 Ky Fan's Theorem

In this section we will present a generalization of the Borsuk-Ulam Theorem, and give a proof for it. This generalization is due to Ky Fan [12]. The proof is similar in spirit to the proofs of the Borsuk-Ulam Theorem that involve Tucker's lemma: We first prove a combinatorial lemma that generalizes Tucker's lemma, and then obtain Ky Fan's theorem by applying the lemma to a sufficiently fine triangulation of the sphere.

For $0 \leq k \leq n-1$ we define the *hemispheres* H_+^k and H_-^k as subsets of S^{n-1} as follows:

$$\begin{aligned} H_+^k &= \{x \in S^{n-1} \colon x_{k+1} \geq 0, x_{k+2} = \cdots = x_n = 0\} \\ H_-^k &= \{x \in S^{n-1} \colon x_{k+1} \leq 0, x_{k+2} = \cdots = x_n = 0\} \end{aligned}$$

Recall that a triangulation of B^n is a simplicial complex K whose body $||K||$ is homeomorphic to B^n. Let $\phi\colon ||K|| \to B^n$ be such a homeomorphism. We say a triangulation of S^{n-1} *respects hemispheres* if each $\phi^{-1}(H_+^k)$ and $\phi^{-1}(H_-^k)$ for $0 \leq k \leq n-1$ is a subcomplex of K (which triangulates that hemisphere). A triangulation K of B^n is *antipodal on the boundary* if for every simplex $\sigma \in K$ that is contained in the boundary, i.e. $\phi(\sigma) \in S^{n-1}$, we have that $\phi^{-1}(-\phi(\sigma))$ is a simplex of K and in the boundary as well. In order to state the next lemma, we need the following definition. For a simplicial complex K whose vertices are coloured with $\{\pm 1, \ldots, \pm k\}$, let $\alpha(j_1, j_2, \ldots, j_{n+1})$ denote the number of n-simplices whose vertices are coloured with colours $j_1, j_2, \ldots, j_{n+1}$. Here the j_i need not be distict, and for a simplex to be coloured with these colours it does not matter which vertex receives which colour as long as there is a 1-to-1 correspondence between the vertices and the colours.

Lemma 4.13 (Combinatorial Lemma). *Let K be a triangulation of B^n that respects hemispheres and is antipodal on the boundary. Let c be a colouring of its vertices with the colours $\{\pm 1, \ldots, \pm k\}$ (we call i and $-i$ opposite colours for some colour i), subject to the following two conditions:*

(i) Any two antipodal vertices v and $-v$ on the boundary receive opposite colours, i.e. $c(v) = -c(-v)$,

(ii) There is no 1-simplex $\{u, v\}$ that is coloured with opposite colours, i.e. no $c(u) = -c(v)$.

Then the number of multicoloured n-simplices whose colours have alternating signs when ordered by absolute value is odd. More formally

$$\sum_{1 \leq j_1 < \cdots < j_{n+1} \leq k} \big(\alpha(j_1, -j_2, \ldots, (-1)^n j_{n+1}) + \alpha(-j_1, j_2, \ldots, (-1)^{n+1} j_{n+1})\big) \equiv 1,$$

where $\equiv$ denotes equivalence modulo 2. In particular, $k \geq n+1$.

Proof. Let γ and δ be defined analogously to α, but for $(n-1)$-simplices, restricted to those in the northern hemisphere H_+^{n-1} and those not entirely contained in the boundary S^{n-1} respectively. More formally:

$\gamma(j_1, j_2, \ldots, j_n)$ is the number of $(n-1)$-simplices entirely contained in H_+^{n-1} whose vertices are coloured with colours $j_1, j_2, \ldots, j_n$.

$\delta(j_1, j_2, \ldots, j_n)$ is the number of $(n-1)$-simplices not entirely contained in S^{n-1} whose vertices are coloured with colours $j_1, j_2, \ldots, j_n$.

We proceed by induction, beginning with the induction step as the argument for the induction base uses parts of the induction step.

Induction Step: For fixed $1 \leq j_1 < \cdots < j_{n+1} \leq k$, we count the number of incidences of an n-simplex with an $(n-1)$-simplex (i.e. pairs where the $(n-1)$-simplex is a face of the n-simplex) where the vertices of the $(n-1)$-simplex are coloured with colours $j_1, -j_2, \ldots, (-1)^{n-1} j_n$.

Every $(n-1)$-simplex not entirely in the boundary is a face of exactly two n-simplices, while each $(n-1)$-simplex in H_+^{n-1} and each simplex in H_-^{n-1} is face of exactly one n-simplex. Note that to count the number of such simplices in H_-^{n-1} we can simply count the number of $(n-1)$-simplices in H_+^{n-1} that are coloured with the corresponding opposite colours, as our colouring is antipodal. As our triangulation respects hemispheres each simplex in S^{n-1} is contained in exactly one of the two hemispheres. So for the number of incidences we get:

$$\gamma(j_1, -j_2, \ldots, (-1)^{n-1} j_n) + \gamma(-j_1, j_2, \ldots, (-1)^n j_n) + 2\delta(j_1, -j_2, \ldots, (-1)^{n-1} j_n).$$

On the other hand, when counting n-simplices that use the colours $j_1, -j_2, \dots, (-1)^{n-1}j_n$ and one more last colour, we see that those n-simplices where one of the colours appears twice are part of two incidences, while n-simplices whose last colour is different from the $\pm j_i$ (note that by condition (ii) it cannot use two opposite colours, so we can exclude both j_i and $-j_i$) are part of one incidence, thus yielding a total:

$$2\sum_{i=1}^{n} \alpha(j_1, -j_2, \dots, (-1)^{n-1}, (-1)^{i-1}j_i) + \sum_{h \neq \pm j_i, \dots, \pm j_n} \alpha(j_1, -j_2, \dots, (-1)^{n-1}, h).$$

By setting these two equal, taking them modulo 2, and then summing them up over all n-tuples $1 \leq j_1 < \cdots < j_n \leq k$, we get

$$\begin{aligned} &\sum_{1 \leq j_1 < \cdots < j_n \leq k} \gamma(j_1, -j_2, \dots, (-1)^{n-1}j_n) + \gamma(-j_1, j_2, \dots, (-1)^{n}j_n) \\ \equiv &\sum_{1 \leq j_1 < \cdots < j_n \leq k} \; \sum_{h \neq \pm j_i, \dots, \pm j_n} \alpha(j_1, -j_2, \dots, (-1)^{n-1}j_n, h). \end{aligned} \tag{4.1}$$

Now the northern hemisphere H_+^{n-1} is homeomorphic to the ball B^{n-1}, where the homeomorphism projects each point in $H_+^{n-1} \subseteq \mathbb{R}^n$ down by setting its last coordinate to 0. Our triangulation when projected onto B^{n-1} is still antipodal and respects hemispheres, so we can apply the induction hypothesis. Note that on the left hand side we're in fact counting the number of multicoloured $(n-1)$-simplices in B^{n-1} whose colours have alternating signs when ordered by absolute values, so the induction hypothesis gives us that this number is odd. So we get

$$1 \equiv \sum_{1 \leq j_1 < \cdots < j_n \leq k} \; \sum_{h \neq \pm j_i, \dots, \pm j_n} \alpha(j_1, -j_2, \dots, (-1)^{n-1}j_n, h).$$

For given indices in our double sum, let j_- be the greatest and j_+ be the smallest index of the j_i subject to $j_- < |h| < j_+$, i.e. the two indices in the order $|h|$ fits between size-wise; notice that at least one of the two exists. If both exist, then j_- and j_+ have different signs in the α-expression. If h has the same sign as either of the two as they appear inside the α expression, then we will double count that particular α-expression, e.g. if j_- has the same sign as h, then we would double count $\alpha(j_1, -j_2, \dots, j_-, h, \dots, (-1)^{n-1})$ again as $\alpha(j_1, -j_2, \dots, h, j_-, \dots, (-1)^{n-1})$ when h takes the value of j_- and j_- the value of h and the remaining indices stay the same. There are only two instances where this doesn't occur, namely when $|h| < j_1$ and h is negative, and when $|h| > j_n$ and the sign of h is $(-1)^n$. So the equation from above simplifies to

$$\begin{aligned}
1 \equiv &\sum_{1\leq j_1<\cdots<j_n\leq k}\ \sum_{1\leq h<j_1} \alpha(-h, j_1, -j_2, \ldots, (-1)^{n-1} j_n) \\
+ &\sum_{1\leq j_1<\cdots<j_n\leq k}\ \sum_{j_n<h\leq k} \alpha(j_1, -j_2, \ldots, (-1)^{n-1} j_n, (-1)^n h) \\
= &\sum_{1\leq j_1<\cdots<j_{n+1}\leq k} \big(\alpha(j_1, -j_2, \ldots, (-1)^n j_{n+1}) + \alpha(-j_1, j_2, \ldots, (-1)^{n+1} j_{n+1})\big)
\end{aligned}$$

which is exactly the statement we wanted to prove.

Induction Base: Notice that equation (4.1) holds for $n = 1$, as at that point we have not made use of the induction hypothesis yet. So we have:

$$\sum_{1\leq j_1\leq k} \gamma(j_1) + \gamma(-j_1) \equiv \sum_{1\leq j_1\leq k}\ \sum_{h\neq\pm j_1} \alpha(j_1, h) \equiv \sum_{1\leq j_1\leq k}\ \sum_{1\leq h\neq j_1} \alpha(j_1, h) + \alpha(j_1, -h).$$

Notice that

$$\sum_{1\leq j_1\leq k}\ \sum_{1\leq h\neq j_1} \alpha(j_1, h) \equiv 0$$

as we're double-counting $\alpha(j_1, h)$ and $\alpha(h, j_1)$. Furthermore notice that

$$\sum_{1\leq j_1\leq k} \gamma(j_1) + \gamma(-j_1) = \sum_{1\leq |j_1|\leq k} \gamma(j_1) \equiv 1$$

as γ counts the number of 0-simplices in H_+^0, but H_+^0 is just a point containing a single vertex. So eventually we get

$$1 \equiv \sum_{1\leq j_1\leq k}\ \sum_{1\leq h\neq j_1} \alpha(j_1, -h) \equiv \sum_{1\leq j_1<j_2\leq k} \alpha(j_1, -j_2) + \alpha(-j_1, j_2)$$

which is the desired statement for the base case. □

In order to prove Ky Fan's theorem, we need one more theorem from basic topology which we will state without proof. It is mentioned e.g. in [17].

Lemma 4.14 (Lebesgue number lemma)**.** *Let C be a set of open sets covering a compact metric space M. Then there is a $\delta > 0$ such that every open ball of radius δ is entirely contained in one of the covering sets.* □

Theorem 4.15 (Ky Fan's Theorem)**.** *Let $\mathcal{A} = \{A_1, \dots, A_k\}$ be a system of open subsets of S^n such that $\bigcup_{A \in \mathcal{A}} A \cup -A = S^n$ and for all A we have $A_i \cap -A_i = \emptyset$. Then there are indices $i_1 < \cdots < i_{n+1}$ and a point $x \in S^n$ such that $(-1)^i x \in A_{i_j}$ for $j = 1, \dots, n+1$. In particular, $k \geq n+1$.*

Proof. Write $A_{-i} := -A_i$. As the A_i, $i \in \{\pm 1, \pm 2, \dots, \pm n\}$ are a cover of S^n, we can apply Lemma 4.14 to obtain a value $\delta > 0$ such that every open δ-ball is entirely contained in one of the A_i. Now let K be a triangulation of S^n that respects hemispheres and that is antipodal such that every simplex has diameter less than δ. To each vertex v of K, we assign a colour $c(v) \in \{\pm 1, \pm 2, \dots, \pm n\}$ satisfying the following two conditions:

(i) For any two antipodal vertices v and $-v$, we have $-c(v) = c(-v)$, i.e. they receive the same colour but with opposite sign,

(ii) The δ-ball around v is entirely contained in $A_{c(v)}$.

We now show that such a colouring exists: As the δ-ball around each vertex v is entirely contained in one of the A_i by the Lebesgue number lemma, we can satisfy (ii) and i is a valid choice for $c(v)$. Then the δ-ball around $-v$ is entirely contained in $-A_i = A_{-i}$, and so $-i$ is a valid choice for $c(-v)$, and so we can satisfy (i) as well. We need to verify that the two conditions for Lemma 4.13 are satisfied in order to apply it. The first condition follows from (i). To show the second condition holds, consider a vertex v of colour i. A 1-simplex $\{v, u\}$ for some vertex u has diameter less than δ, so it is entirely contained in the δ-ball around v, which was entirely contained in A_i by condition (ii). But then $u \in A_i$ and therefore $u \notin A_{-i}$, as A_i and A_{-i} are disjoint. This means u cannot be coloured with colour $-i$, and thus the second condition of Lemma 4.13 is satisfied as well.

So by Lemma 4.13, there is a simplex $\{v_1, \dots, v_{n+1}\}$ and colours $0 < i_1 < \cdots < i_{n+1}$ such that $c(v_j) = (-1)^j i_j$ for each j or $c(v_j) = (-1)^{j+1} i_j$ for each j. Assume the former, the latter works analogously. Then $v_j \in (-1)^j A_{i_j}$, and in particular, the δ-balls around v_j are contained in $(-1)^j A_{i_j}$. As the simplex had diameter less than δ, it is entirely contained in each of these δ-balls, showing that the intersection

$$\bigcap_{j=1}^{n+1} (-1)^j A_{i_j}$$

is non-empty. So any point x in this intersection has the property that $x \in (-1)^j A_{i_j}$ and equivalently $(-1)^j x \in A_{i_j}$ for $j = 1, \dots, n+1$, which is what we wanted to show. $\square$

4.4.2 Zig-Zag Theorem

We will now introduce the Zig-Zag Theorem, which will help us relate the local chromatic number to the topological chromatic number. It was first proven for Kneser graphs by Ky Fan [13], though not stated in terms of graphs.

Theorem 4.16 (Zig-Zag Theorem)**.** *Let G be a topologically t-chromatic graph, meaning that* $\operatorname{coind}(||B_0(G)||) + 1 \geq t$*, and let c be an arbitrary proper colouring of G. Then G contains a totally multicoloured $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$ subgraph.*

Proof. We have a $\mathbb{Z}_2$-map $f\colon S^{t-1} \to ||B_0(G)||$.

For $x \in S^{t-1}$, let $S_x \uplus T_x$ be the minimal simplex containing $f(x)$. Note that $S_{-x} = T_x$, as f is a $\mathbb{Z}_2$-map:

$$\begin{tikzcd}
x \arrow[r, mapsto, "f"] \arrow[d, mapsto] & S_x \uplus T_x \arrow[d, mapsto] \\
-x \arrow[r, mapsto, "f"] & S_{-x} \uplus T_{-x} = T_x \uplus S_x
\end{tikzcd}$$

For each colour i we define $A_i \subseteq S^{t-1}$ such that $x \in A_i$ if and only if S_x contains a vertex of colour i.

In order to apply Ky Fan's Theorem, we need to check three properties:

(i) A_i is open for all i,

(ii) $\bigcup_i A_i \cup -A_i = S^{t-1}$,

(iii) $A_i \cap -A_i = \emptyset$ for all i.

(i) Consider a point $x \in A_i$. There is some ε such that the ε-ball B_ε around $f(x)$ only intersects simplices that contain $S_x \uplus T_x$ as vertex subset. Due to the continuity of f, we can find a δ such that the image of the δ-ball B_δ around x is a subset of B_ε. Then for

$y \in B_\delta$, we have by definition of B_ε that $f(y)$ is in a simplex whose vertex set contains $S_x \uplus T_x$, and thus $y \in A_i$, as one of the vertices of S_x is coloured with i due to $x \in A_i$.

(ii) For each x, either S_x or $T_x = S_{-x}$ is non-empty. So there is a vertex v in S_x or S_{-x}. So either $x \in A_{c(v)}$, or $-x \in A_{c(v)}$ and thus $x \in -A_{c(v)}$.

(iii) Assume there is a point $x \in A_i \cap -A_i$, so in particular $-x \in A_i$. Then there is a vertex $v_S \in S_x$ of colour i, and a vertex $v_T \in S_{-x} = T_x$ of colour i. But by the definition of the box complex, $G[S_x, T_x]$ is a complete bipartite graph, and so there is an edge from v_S to v_T in G. But as both vertices received colour i, this means that the colouring was not proper, a contradiction.

Now that we ensured that all conditions for Ky Fan's Theorem are fulfilled, by applying the theorem we get indices $i_1 < \cdots < i_{n+1}$ and a point $x \in S^{t-1}$ such that $(-1)^j x \in A_{i_j}$. So there are vertices $z_j \in S_{(-1)^j x}$ that receive colour i_j. Now $S_{(-1)^j}x = S_x$ if j is even, and $S_{(-1)^j}x = S_{-x} = T_x$ if j is odd. The simplex $S_x \uplus T_x$ corresponds to a complete bipartite graph in G with shores S_x and T_x. Now z_j is coloured with i_j and is in S_x if j is even, and in T_x if j is odd. So the z_j induce a complete bipartite subgraph $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$ of G which is totally multicoloured, and the colours in increasing order alternate between the two shores. $\square$

Theorem 4.17. *If G is topologically t-chromatic, then*

$$\psi(G) \geq \left\lceil \frac{\mathrm{t}(G)}{2} \right\rceil + 1.$$

Proof. We just showed that we can find a large multicoloured complete bipartite graph in any topologically t-chromatic graph, and in particular if we pick a vertex in the smaller shore of it, then it has $\lceil \frac{\mathrm{t}(G)}{2} \rceil$ neighbours all with different colours, and thus we get the following bound on the local chromatic number: $\psi(G) \geq \lceil \frac{\mathrm{t}(G)}{2} \rceil + 1$. $\square$

This result prompts one question: are there graphs whose local chromatic number is arbitrarily higher than their topological chromatic number? Recall that in Corollary 2.28 we showed that there are graphs with high girth and local chromatic number. But these graphs do not contain 4-cycles, so they are $K_{2,2}$-free, so by the Zig-Zag theorem their topological chromatic number can be at most 3.

This also shows that the topological chromatic number can be arbitrarily smaller than the fractional chromatic number. On the other hand, the fractional chromatic number can be arbitrarily smaller than the topological chromatic number too: It is known that $\chi_f(KG(n,k)) = \frac{n}{k}$ [14], while $t(KG(n,k)) = n - 2k + 2$. So for instance setting $k = n^{(1-\varepsilon)}$, we get that the fractional chromatic number is $\Theta(n^\varepsilon)$, while the topological chromatic number is $\Theta(n)$.

4.5 Applications to specific classes of graphs

In this section we will use the topological tools that have been introduced to give some of the lower bounds on the local chromatic number that have been claimed in Section 2.3.

We begin with a result about a topological property relating to Schrijver graphs. This result was given by Schrijver in [32] and by Bárány in [2], cf. Proposition 8 in [33]. A proof of this theorem however is beyond the scope of this thesis.

Theorem 4.18. *For $t = n - 2k + 2$ we have* $\operatorname{coind}(B(SG(n, k))) + 2 \geq t$. □

Using this result, we can give a lower bound on the local chromatic number of Schrijver graphs.

Theorem 4.19. *For $t = n - 2k + 2$ we have $\psi(SG(n, k)) \geq \left\lceil \frac{t}{2} \right\rceil + 1$.*

Proof. By Theorem 4.18 and Lemma 4.12 we have the inequalities $\operatorname{coind}(B_0(SG(n, k))) + 1 \geq \operatorname{coind}(B(SG(n, k))) + 2 \geq t$. Then by Theorem 4.17 the result follows. □

Now we will move on to applications to generalized Mycielski graphs. Csorba proved in [5] a theorem on how the generalized Mycielski construction affects the restricted box complex of a graph, which again we will state without proof.

Theorem 4.20 ([5])**.** *For a graph G and $r \geq 1$, $B(M_r(G)) \leftrightarrow \operatorname{susp}(B(G))$.* □

With this theorem we can prove that an iteration of the generalized Mycielski construction increases the coindex of the restricted box complex of a graph by at least 1.

Corollary 4.21. *For a graph G and $r \geq 1$, $\operatorname{coind}(B(M_r(G))) \geq \operatorname{coind}(B(G)) + 1$.*

Proof. From Theorem 4.20 we know that $B(M_r(G))$ and $\operatorname{susp}(B(G))$ are $\mathbb{Z}_2$-equivalent and therefore $\operatorname{coind}(B(M_r(G))) = \operatorname{coind}(\operatorname{susp}(B(G)))$. Then Lemma 4.9 gives us the inequality $\operatorname{coind}(\operatorname{susp}(B(G))) \geq \operatorname{coind}(B(G)) + 1$. □

Recall that by Lemma 4.12 we have that $\chi(G) \geq \operatorname{coind}(B_0(G)) + 1 \geq \operatorname{coind}(B(G)) + 2$. If for a graph G this topological lower bound is tight, i.e. $\chi(G) = \operatorname{coind}(B(G)) + 2$, we get from Corollary 4.21 that the generalized Mycielski construction increases the chromatic number of G (and the topological lower bound) by at least 1. But the generalized Mycielski construction also increases the chromatic number by at most 1 (the proof is analogous to the proof in Theorem 2.9 for the local chromatic number), therefore we get the following theorem by repeated application of these observations.

Theorem 4.22. *For a graph G, $d \geq 1$ and a positive vector r of length d, if $\chi(G) = \operatorname{coind}(B(G)) + 2$, then $\chi(M_r^{(d)}(G)) = \chi(G) + d$.* □

As mentioned in [33], this topological bound is tight for e.g. Kneser graphs, Borsuk graphs and Schrijver graphs, the latter including complete graphs, odd cycles and K_2, so Theorem 4.22 is applicable to these classes of graphs.

We can obtain a similar but weaker theorem for the local chromatic number. From Theorem 4.17 we have $\psi(G) \geq \lceil \frac{\operatorname{coind}(B_0(G))+1}{2} \rceil + 1$, and thus $\psi(G) \geq \lceil \frac{\operatorname{coind}(B(G))+2}{2} \rceil + 1 = \lceil \frac{\operatorname{coind}(B(G))}{2} \rceil + 2$. If this lower bound is tight, we get the following lower bound for $\psi(M_r^{(d)}(G))$.

Theorem 4.23. *For a graph G, $d \geq 1$ and a positive vector r of length d, if $\psi(G) = \lceil \frac{\operatorname{coind}(B(G))}{2} \rceil + 2$, then*

$$\psi(M_r^{(d)}(G)) \geq \psi(G) + \left\lfloor \frac{d}{2} \right\rfloor.$$

If $\operatorname{coind}(B(G))$ is even and d is odd, we get

$$\psi(M_r^{(d)}(G)) \geq \psi(G) + \left\lfloor \frac{d}{2} \right\rfloor + 1.$$

Proof. By applying Theorem 4.17, Lemma 4.12 and Corollary 4.21 we get

$$\begin{aligned}
\psi(M_r^{(d)}(G)) &\geq \left\lceil \frac{\operatorname{coind}(B_0(M_r^{(d)}(G))) + 1}{2} \right\rceil + 1 \\
&\geq \left\lceil \frac{\operatorname{coind}(B(M_r^{(d)}(G)))}{2} \right\rceil + 2 \\
&\geq \left\lceil \frac{\operatorname{coind}(B(G)) + d}{2} \right\rceil + 2
\end{aligned}$$

If $\operatorname{coind}(B(G))$ is even and d is odd, we have $\left\lceil \frac{\operatorname{coind}(B(G))+d}{2} \right\rceil + 2 = \left\lceil \frac{\operatorname{coind}(B(G))}{2} \right\rceil + 2 + \left\lfloor \frac{d}{2} \right\rfloor + 1 = \psi(G) + \left\lfloor \frac{d}{2} \right\rfloor + 1$. Otherwise $\left\lceil \frac{\operatorname{coind}(B(G))+d}{2} \right\rceil + 2 = \left\lceil \frac{\operatorname{coind}(B(G))}{2} \right\rceil + 2 + \left\lfloor \frac{d}{2} \right\rfloor = \psi(G) + \left\lfloor \frac{d}{2} \right\rfloor$. □

This bound is good if the entries of r satisfy $r_i \geq 4$ for $1 \leq i \leq d$, as then we get a close upper bound of $\psi(M_r^{(d)}(G)) \leq \psi(G) + \lceil \frac{d}{2} \rceil + 2$ from Theorem 2.11.

In some cases we can in fact determine the local chromatic number exactly.

Theorem 4.24. *Let G be a graph that admits a wide t-colouring where $t = \mathrm{coind}(B(G))+2$ is even with $\psi(G) = \frac{t}{2}+1$. Let $d \geq 1$ odd and $r = (r_1, \ldots, r_d)$ be a positive vector with $r_i \geq 7$ for $1 \leq i \leq d$. Then*

$$\psi(M_r^{(d)}(G)) = \psi(G) + \left\lfloor \frac{d}{2} \right\rfloor + 1.$$

Proof. To get the upper bound, by Lemma 2.10, $M_r^{(d)}(G)$ admits a wide $(t+d)$-colouring. Then by Lemma 2.5 it follows that $\psi(M_r^{(d)}(G)) \leq \lfloor \frac{t+d}{2} \rfloor + 2 = \frac{t}{2} + \lfloor \frac{d}{2} \rfloor + 2 = \psi(G) + \lfloor \frac{d}{2} \rfloor + 1$.

From Theorem 4.23 we get the lower bound of $\psi(M_r^{(d)}(G)) \geq \psi(G) + \lfloor \frac{d}{2} \rfloor + 1$, which equals the upper bound.

□

Notice that Theorem 4.24 is applicable for $G = K_2$: the restricted box complex $B(K_2)$ has four vertices which we can identify with $+1, -1, +2$ and -2. Then the 1-simplices are $\{+1, -2\}$ and $\{-1, +2\}$, and therefore $S^1 \twoheadrightarrow B(K_2)$, implying that $\mathrm{coind}(B(K_2)) = 0$. So $t = 2$ is even, and K_2 obviously admits a wide 2-colouring. Therefore if d is odd, we can determine the exact local chromatic number of $\psi(M_r^{(d)}(K_2))$ for $r = (r_1, \ldots, r_d)$ with $r_i \geq 7$ for all i. The next corollary will summarize the results we know when the starting graph is K_2. It was previously given in [33].

Corollary 4.25.

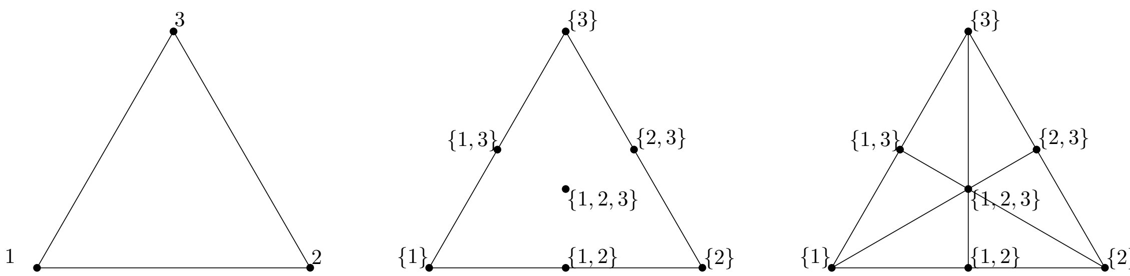
$$\psi(M_r^{(d)}(K_2)) = \begin{cases} \lceil \frac{d}{2} \rceil + 2 & \text{if } d \text{ is odd and } \forall i\colon r_i \geq 7 \\ \lceil \frac{d}{2} \rceil + 2 \text{ or } \lceil \frac{d}{2} \rceil + 3 & \text{if } \forall i\colon r_i \geq 4 \\ d+2 & \text{if } r_d = 1 \text{ or } \forall i\colon r_i = 2. \end{cases}$$

Proof. The first case is Theorem 4.24 applied to K_2. In the second case we obtain the lower bound from Theorem 4.23 and the upper bound from Theorem 2.11. In the case $r_d = 1$, we first get from Theorem 4.22 that $\chi(M_{(r_1,\ldots,r_{d-1})}^{(d-1)}(K_2)) = \chi(K_2) + (d-1) = d+1$. From Theorem 2.9 it follows that $\psi(M_r^{(d)}(K_2)) = \chi(M_{(r_1,\ldots,r_{d-1})}^{(d-1)}(K_2)) + 1 = d+2$. The case when $r_i = 2$ for all i follows from repeated application of Theorem 2.9. □

4.6 A combinatorial proof of the Zig-Zag Theorem for Kneser graphs

The aim of this section is to give a combinatorial proof of the Zig-Zag Theorem for Kneser graphs. That is, we will prove that for $t = n - 2k + 2$, any proper colouring of the Kneser graph $KG(n,k)$ contains a totally multicoloured copy of $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$. Notice that this implies that $\psi(KG(n,k)) \geq \lceil \frac{t}{2} \rceil + 1$, and as this bipartite subgraph has a total of $n-2k+2$ vertices it also implies that $\chi(KG(n,k)) \geq t$ and thus the Kneser conjecture. The proof takes its main ideas from the combinatorial proof of the Kneser conjecture by Matoušek in [24] and goes as follows.

Recall that the crosspolytope $\diamond^{n-1}$ has vertices $V(\diamond^{n-1}) = \{\pm 1, \ldots, \pm n\}$, and the simplices are those subsets of $V(\diamond^{n-1})$ that do not contain both $+i$ and $-i$ for any $1 \leq i \leq n$. Now let L^{n-1} be the first barycentric subdivision of $\diamond^{n-1}$: For the barycenter of each non-empty simplex in $\diamond^{n-1}$ we have a vertex in L^{n-1}, so non-empty simplices in $\diamond^{n-1}$ correspond to vertices in L^{n-1}, which are subsets of $V(\diamond^{n-1})$. The simplices in L^{n-1} are chains of vertices with respect to inclusion on the subsets that the vertices of L^{n-1} correspond to.



(a) The face $\{1,2,3\}$ of $\diamond^2$.

(b) The barycenters of its simplices.

(c) Its barycentric subdivision.

Figure 4.3: The barycentric subdivision of one of the faces of $\diamond^2$, which geometrically is the surface of an octahedron. The triangular faces in (a) and (c) except for the unbounded face are simplices as well.

Define a triangulation K^n of $\diamond^{n-1}$ as follows. We add another vertex corresponding to $\emptyset$ in the center of the crosspolytope and take the cone over it, i.e. in addition to the simplices in L^{n-1}, each simplex of L^{n-1} with $\emptyset$ added will be a simplex in K^n. So the simplices in K^n are still exactly chains of the sets the vertices correspond to.

Now assume we have a proper colouring c of the vertices of the Kneser graph $KG(n,k)$ using m colours. Let the colours used be $2k, 2k+1, \ldots, 2k+m-1$.

We will define a labelling λ on the vertices of K^n with labels from $\{\pm 1, \pm 2, \ldots, \pm(2k+m-1)\}$. Let "$\prec$" be a linear ordering on the subsets of $[n]$ that respects size, i.e. if $|A| < |B|$ then we must have $A \prec B$. Take a vertex v and recall that it corresponds to a subset of $\{\pm 1, \ldots, \pm n\}$. We define $\operatorname{pos}(v) = \{i \colon i \in v, i > 0\}$ as the values in v with positive sign and $\neg(v) = \{|i| \colon i \in v, i < 0\}$ as the values in v with negative sign. Notice that $\operatorname{pos}(v)$ and $\operatorname{neg}(v)$ are disjoint.

Assign a label to v as follows (here $|v|$ denotes the size of the set v):

Case 0: if $v = \emptyset$, set $\lambda(v) = +1$.

Case 1: If $|v| \leq 2k-2$:

$$\lambda(v) = \begin{cases} +(|v|+1) & \text{if } \operatorname{pos}(v) \succ \operatorname{neg}(v) \\ -(|v|+1) & \text{if } \operatorname{pos}(v) \prec \operatorname{neg}(v) \end{cases}$$

Case 2: If $|v| \geq 2k-1$:

Recall that c is the colouring of the vertices of $KG(n,k)$. For a subset A of $[n]$ on more than k elements, let $c(A)$ be the colour of the subset of A containing the k smallest elements of A. Then define

$$\lambda(v) = \begin{cases} +c(\operatorname{pos}(v)) & \text{if } \operatorname{pos}(v) \succ \operatorname{neg}(v) \\ -c(\operatorname{neg}(v)) & \text{if } \operatorname{pos}(v) \prec \operatorname{neg}(v) \end{cases}$$

Notice that this is well-defined: if $|v| \geq 2k-1$, then the bigger one of $\operatorname{pos}(v)$ and $\operatorname{neg}(v)$ has at least k elements.

We want to apply Ky Fan's Combinatorial Lemma, so we need to ensure that our labeling λ has two properties. First it must be antipodal on the boundary, and second the vertices of any 1-simplex must not receive opposite colours. Observe that this labeling is in fact antipodal on the boundary, as for a simplex σ we have $\operatorname{pos}(\sigma) = -\operatorname{neg}(-\sigma)$ and $\operatorname{neg}(\sigma) = -\operatorname{pos}(-\sigma)$, and therefore by the definition in Case 1 and Case 2 $-\sigma$ receives the opposite of the colour that σ receives.

Next we want to show that there is no 1-simplex whose vertices receive opposite labels. Assume there is and that the simplex is $\{A, B\}$ where A and B are subsets of $\{\pm 1, \ldots, \pm n\}$. We may assume $A \subset B$ as the simplices in K^n are chains. So in particular $\operatorname{pos}(A) \subseteq \operatorname{pos}(B)$

and $\operatorname{neg}(A) \subseteq \operatorname{neg}(B)$. Note that Case 1 assigns labels $\{\pm 2, \ldots, \pm 2k-1\}$ while Case 2 assigns $\{\pm 2k, \ldots, \pm m+2k-1\}$. So if A and B receive opposite colours, then the colours must have been assigned via the same case. As $|A| < |B|$, it cannot be Case 1 as the absolute values of the labels of A and B would differ. So the assignment must have occurred via Case 2. Due to the labels of A and B having opposite signs these labels must be $+c(\operatorname{pos}(A))$ and $-c(\operatorname{neg}(B))$ (or $-c(\operatorname{pos}(A))$ and $+c(\operatorname{neg}(B))$, for which the following arguments can be applied analogously). We know $\operatorname{pos}(B) \cap \operatorname{neg}(B) = \emptyset$. But also $\operatorname{pos}(B) \supseteq \operatorname{pos}(A)$, and therefore $\operatorname{pos}(A) \cap \operatorname{neg}(B) = \emptyset$. But then the two vertices in $KG(n,k)$ corresponding to the vertices in K^n receiving colours $c(\operatorname{pos}(A))$ and $c(\operatorname{neg}(B))$ are adjacent, a contradition to the colouring being proper.

Now with antipodality on the boundary and no 1-simplex having vertices of opposite colours we can apply Ky Fan's Combinatorial Lemma. This gives us a multicoloured n-simplex σ in K^n whose colours, when ordered by magnitude, have alternating signs. Now σ is a chain of vertices $v_0, \ldots, v_n$ with $|v_i| = i$. The first $2k-1$ vertices in this chain are assigned labels with absolute value less than $2k$, and half of them receive a positive and half of them receive a negative label. So for the remaining $n+1-(2k-1) = n-2k+2$ vertices also half of them have positive sign and half of them have negative sign. These remaining vertices were assigned labels via Case 2, and the labels correspond to colours $c(\operatorname{pos}(v_i))$ in half of the cases and $c(\operatorname{neg}(v_i))$ in the other half of the cases in the colouring of $KG(n,k)$ that are all different. Now notice that $\operatorname{pos}(v_i) \subseteq \operatorname{pos}(v_n)$ and $\operatorname{neg}(v_i) \subseteq \operatorname{neg}(v_n)$ for all i and therefore $\operatorname{pos}(v_i) \cap \operatorname{neg}(v_j) = \emptyset$ for all pairs i and j. Therefore those vertices v_i with label $-c(\operatorname{neg}(v_i))$ are all adjacent to any vertex v_j with label $+c(\operatorname{pos}(v_j)$, and therefore the vertices induce a complete bipartite subgraph in the Kneser graph with balanced shores and $n-2k+2$ vertices in total.

To summarize, given a proper colouring of the Kneser graph $KG(n,k)$, we showed that it contains a totally multicoloured copy of $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$ for $t = n-2k+2$. The proof we have just seen does not explicitly use a variation of the Borsuk-Ulam Theorem or any of the topological tools we've introduced. Nevertheless, it is inspired by topological proofs, using Ky Fan's Combinatorial Lemma as a combinatorial version of the Borsuk-Ulam Theorem, and applying it to the crosspolyope $\diamond^{n-1}$, which could be considered a combinatorial version of the sphere S^{n-1}.

Chapter 5

Conjectures and Conclusions

The chromatic number, in spite of being a very natural parameter of graphs and thus having been studied for a long time, is still not too well understood in various areas and leaves a lot of open questions to be solved. Many of them have been outstanding for decades with little progress towards their solution. However some of the conjectures that are still wide open have fractional or topological versions that are known to be true, giving support to the conjectures. With the local chromatic number lying between the fractional chromatic number and the chromatic number, and also topological parameters giving lower bounds to the local chromatic number, especially for those conjectures an interesting question to ask is: Are there versions pertaining to the local chromatic number that can be proven to be true? In this chapter we will present a few of these conjectures, and outline what topological or fractional versions of them are known to be true.

5.1 The Hedetniemi conjecture

There are various notions of graph products, taking two given graphs and turning them into a new graph according to certain rules. A natural question to ask is, if we know the chromatic numbers of the two given graphs, what can we say about the chromatic numbers of the resulting graph? The Hedetniemi conjecture asks this for the direct product of graphs. Recall from Section 2.3.3 that the *direct product* $F \times G$ of two graphs F and G is defined as follows: The vertex set of $F \times G$ is $V(F) \times V(G)$. Two vertices (u_1, v_1) and (u_2, v_2) are linked by an edge if $u_1 \sim u_2$ and $v_1 \sim v_2$. Figure 5.1 shows an example.

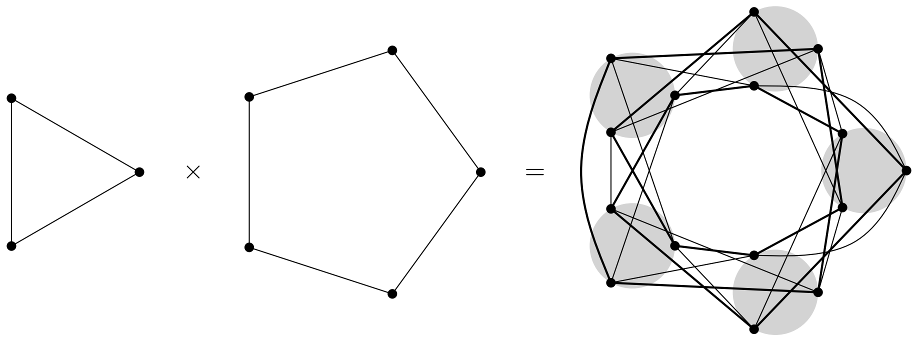


Figure 5.1: The direct product of two odd cycles contains an odd cycle. Here a 15-cycle as described in the text is marked with thick lines. In this case a shorter cycle of length 5 exists as well.

Then the Hedetniemi conjecture [18] states that

$$\chi(F \times G) = \min\{\chi(F), \chi(G)\}.$$

It was originally conjectured by Hedetniemi in 1966 [18], but remains unsolved. An extensive survey with partial results and variations, as well as evidence for and against the conjecture is due to Tardif [36].

The conjecture looks simple enough, and in fact one of the inequalities is easy to show: We may assume that $\chi(F) \leq \chi(G)$. Then the inequality $\chi(F \times G) \leq \min\{\chi(F), \chi(G)\} = \chi(F)$ is easily obtained by colouring each copy of F in $F \times G$ with the colouring of F, i.e. assing each vertex $(u, v) \in F \times G$ the colour that u receives in the colouring of F. Then the neighbours of (u, v) use the same set of colours as the neighbours of u in F. So this colouring is proper.

But furthermore this observation also tells us that the number of colours in a closed neighbourhood of a vertex (u, v) doesn't increase with this colouring, and thus $\psi(F \times G) \leq \min\{\psi(F), \psi(G)\}$. So one inequality also holds easily for the local chromatic number.

However, the other inequality is still wide open, and only partial results are known. If the chromatic numbers of the given graphs F and G are both at most 4, then the conjecture holds. Here the case $\chi(F) = \chi(G) = 4$ has been proven by El-Zahar and Sauer in [8]. The case $\min\{\chi(F), \chi(G)\} \leq 3$ is outlined below.

We have that $F \times G$ is non-empty if and only if both F and G are non-empty. So we get equality if $\chi(F) = \chi(G) = 2$ or $\psi(F) = \psi(G) = 2$ respectively, as then $F \times G$ will be non-empty and thus its (local) chromatic number will be at least 2. Moreover observe that

$F \times G$ has an odd cycle if and only if both F and G have an odd cycle, therefore we also have equality if $\chi(F) = \chi(G) = 3$ or $\psi(F) = \psi(G) = 3$ respectively, as then $F \times G$ will contain an odd cycle and thus its (local) chromatic number will be at least 3. Figure 5.1 provides an example, and in general, if we have C_n and C_m with vertex sets $\{0, 1, \ldots, n\}$ and $\{0, 1, \ldots, m\}$ respectively, then the sequence of vertices $(0, 0), (1, 1), \ldots, (nm, nm)$ where the first coordinate is always taken modulo n and the second modulo m forms a nm-cycle.

However beyond graphs of chromatic number at most 4, fractional and topological versions of the conjecture imply that the conjecture is true for certain other classes of graphs. In the following we will outline these versions and their implications.

5.1.1 Topological version

Next we will show that the Hedetniemi conjecture holds for a certain topological parameter that is also related to the chromatic number in a similar way as the topological chromatic number. We first need the following lemma, which is Lemma 4.4 from [33]:

Lemma 5.1. *A finite graph satisfies* $\operatorname{coind}(B(G)) \geq n - 1$ *if and only if there is an* $\alpha < 2$ *such that there is a graph homomorphism from the Borsuk graph* $B(m, \alpha)$ *into* G. □

Using this lemma, we can prove the Hedetniemi conjecture for this topological parameter. This proof was first given by Simonyi and Zsbán in [35].

Theorem 5.2.

$$\operatorname{coind}(B(F \times G)) = \min\{\operatorname{coind}(B(F)), \operatorname{coind}(B(G))\}.$$

Proof. "$\leq$": Let $k = \operatorname{coind}(B(F \times G))$. Then by Lemma 5.1 there is an α such that $B(k+1, \alpha) \to F \times G$. But we also have that $F \times G \to F$ and $F \times G \to G$ simply using the projection that maps a vertex $(v, w) \in V(F \times G)$ to $v \in V(F)$ and $w \in V(G)$ respectively. So we have $B(k+1, \alpha) \to F$ and $B(k+1, \alpha) \to G$ respectively and thus $\operatorname{coind}(B(F)) \geq k$ and $\operatorname{coind}(B(F)) \geq k$ and so in particular $\min\{\operatorname{coind}(B(F)), \operatorname{coind}(B(G))\} \geq k$.

"$\geq$": Let $k = \min\{\operatorname{coind}(B(F)), \operatorname{coind}(B(G))\}$. Then by Lemma 5.1 there exist an α_F and α_G such that there are homomorphisms $f\colon B(k+1, \alpha_F) \to F$ and $g\colon B(k+1, \alpha_G) \to G$. Let $\alpha = \max(\alpha_F, \alpha_G)$ and observe that by definition $B(k+1, \alpha_F)$ and $B(k+1, \alpha_G)$ are subgraphs of $B(k+1, \alpha)$ as $\alpha_F \leq \alpha$ and $\alpha_G \leq \alpha$, so the restrictions of f and g to $B(k+1, \alpha)$ are also homomorphisms. Now notice that (f, g) is a homomorphism from $B(k+1, \alpha)$ to

$F \times G$, since if two adjacent vertices u_1 and u_2 in $B(k+1, \alpha)$ are mapped to (v_1, w_1) and (v_2, w_2), then $v_1 = f(u_1) \sim f(u_2) \sim v_2$ and $w_1 = g(u_1) \sim g(u_2) \sim w_2$ and thus we must have $(v_1, w_1) \sim (v_2, w_2)$. So $B(k+1, \alpha) \to F \times G$ and so by the lemma we have that $\operatorname{coind}(B(F \times G)) \geq k$. □

Using the topological version of the Hedetniemi conjecture, we can now prove the (local) chromatic version of the Hedetniemi conjecture if the topological lower bound on the (local) chromatic number of the two initial graphs is tight.

For this purpose we need the inequality $\chi(G) \geq \operatorname{coind}(B(G)) + 2$ from Lemma 4.12. Then for graphs F and G where the topological bound is tight, meaning $\operatorname{coind}(B(F)) + 2 = \chi(F)$ and $\operatorname{coind}(B(F)) + 2 = \chi(F)$, we get

$$\begin{aligned}\chi(F \times G) &\geq \operatorname{coind}(B(F \times G)) + 2 \\ &= \min\{\operatorname{coind}(B(F)) + 2, \operatorname{coind}(B(G)) + 2\} \\ &= \min\{\chi(F), \chi(G)\}.\end{aligned}$$

Notice that we obtain the other inequality from the initial discussion of the Hedetniemi conjecture. So Hedetniemi's conjecture holds if e.g. F and G are both Kneser graphs.

A similar statement can be made for the local chromatic number. Again we need Lemma 4.12, this time for $\mathrm{t}(G) \geq \operatorname{coind}(B(G)) + 2$. Then by Theorem 4.17 we get

$$\psi(G) \geq \left\lceil \frac{\mathrm{t}(G)}{2} \right\rceil + 1 \geq \left\lceil \frac{\operatorname{coind}(B(G)) + 2}{2} \right\rceil + 1 = \left\lceil \frac{\operatorname{coind}(B(G))}{2} \right\rceil + 2.$$

Then if this bound on the right hand side for ψ is tight for the graphs F and G (e.g. if they are Schrijver graphs that Corollary 3.3 is applicable to), the local chromatic version of the Hedetniemi conjecture is true. We get

$$
\begin{aligned}
\psi(F\times G) &\geq \left\lceil \frac{\mathrm{t}(G)}{2} \right\rceil + 1 \\
&\geq \left\lceil \frac{\mathrm{coind}(B(F\times G))}{2} + 2 \right\rceil \\
&= \left\lceil \frac{\min\{\mathrm{coind}(B(F)), \mathrm{coind}(B(G))\}}{2} \right\rceil + 2 \\
&= \min\left\{ \left\lceil \frac{\mathrm{coind}(B(F))}{2} \right\rceil + 2, \left\lceil \frac{\mathrm{coind}(B(G))}{2} \right\rceil + 2 \right\} \\
&= \min\{\psi(F), \psi(G)\}.
\end{aligned}
$$

Again, we obtain the other inequality from the easy observation from above that the local chromatic number cannot increase when taking the direct product of the two graphs.

5.1.2 Fractional version

The Hedetniemi conjecture has recently been proven for the fractional chromatic number by Xuding Zhu [39], i.e.

$$\chi_f(F\times G) = \min\{\chi_f(F), \chi_f(G)\}.$$

Using this fact, again a version of the conjecture can be proven to be true similar to the one from the previous section that required the topological bound on the chromatic number to be tight. If the fractional lower bound on the chromatic number for the two graphs F and G is tight, meaning $\chi_f(F) = \chi(F)$ and $\chi_f(G) = \chi(G)$, we can get the same chain of inequalities to prove that the Hedetniemi conjecture is true in this case.

$$
\begin{aligned}
\chi(F\times G) &\geq \chi_f(F\times G) \\
&= \min\{\chi_f(F), \chi_f(G)\} \\
&= \min\{\chi(F), \chi(G)\}.
\end{aligned}
$$

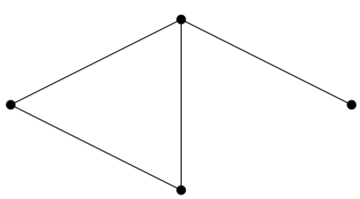
5.2 Behzad-Vizing conjecture

A different notion of colourings is that of a *total colouring*. Here we colour both vertices and edges, and a colouring is considered proper if no two adjacent vertices receive the same colour, but also no pair of a vertex with an incident edge or a pair of two incident edges receive the same colour either. For vertex colouring, there is the well known easy bound of $\chi(G) \leq \Delta(G) + 1$. For total colourings a similar bound has been conjectured, first independently by Behzad [3] and Vizing [38], namely that the total number of colours needed for a proper total colouring is at most $\Delta(G) + 2$.

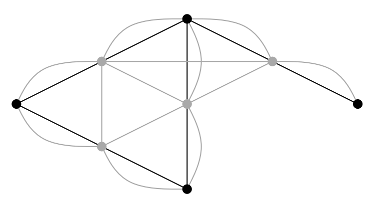
The problem of total-colouring a graph can be reduced to vertex-colouring an auxiliary graph $T(G)$ called the *total graph* of G, which is defined as follows: For each vertex and each edge in G we introduce a vertex in $T(G)$, i.e. the vertex set $V(T(G)) = V(G) \cup E(G)$. Two vertices in $T(G)$ are adjacent if they correspond to two adjacent vertices in G, or two incident edges, or a pair of an edge and a vertex that are incident to each other. More formally,

$$E(T(G)) = \{uv\colon \begin{array}{ll} u \in V(G), v \in V(G) & u \sim v \text{ or} \\ u \in V(G), v \in E(G) & u \text{ incident to } v \text{ or} \\ u \in E(G), v \in E(G) & u \text{ incident to } v\}. \end{array}$$

See Figure 5.2 for an example.



(a) An initial graph G.



(b) The graph $T(G)$, which contains G as a subgraph (in black). Grey vertices are those vertices corresponding to edges in G, and grey edges connect them to each other and other vertices.

Figure 5.2: Example of a construction of the auxiliary graph $T(G)$ from the graph G.

One easy observation is that $\Delta(T(G)) = 2\Delta(G)$. A vertex in G with Δ neighbours will still have these Δ neighbours in $T(G)$, but will have Δ additional neighbours corresponding to the incident edges in G. A vertex in $T(G)$ that corresponds to an edge e in G is incident to its 2 endpoints, each of which has degree at most $\Delta(G)$ and thus at most $\Delta(G) - 1$ other edges that are incident to e. So the degree of such a vertex in $T(G)$ is at most $2 + 2(\Delta(G) - 1) = 2\Delta(G)$. So the trivial bound on the chromatic number of $T(G)$ therefore is $\chi(T(G)) \leq 2\Delta(G) + 1$.

The Behzad-Vizing conjecture however makes statement that is a lot stronger, namely it is equivalent to saying $\chi(T(G)) \leq \Delta(G) + 2$.

Using this interpretation, we can adapt the conjecture to other parameters related to the chromatic number by replacing $\chi(T(G))$ with e.g. some topological parameter or the local chromatic number. The proof of the topological version was given in [35]. Unfortunately, unlike the previous result on Hedetniemi's conjecture this result does not give an easy consequence for the local chromatic number. The reason is that the proof uses a property of a topological parameter that does not apply to the local chromatic number, stated in the following theorem due to Csorba, Lange, Schurr and Wassmer [7].

Theorem 5.3 ($K_{l,m}$-theorem)**.** *If* $\mathrm{ind}(B(G)) + 2 \geq t$*, then* G *must contain a copy of the complete bipartite graph* $K_{l,m}$ *for all pairs* (l, m) *with* $l + m = t$*.* □

With this theorem we're ready to present the topological version of the Behzad-Vizing conjecture and its proof, due to [35].

Theorem 5.4.

$$\mathrm{ind}(B(T(G))) + 2 \leq \Delta(G) + 2.$$

Proof. Let $\Delta = \Delta(G)$. We will show that if $\Delta \geq 4$, then $T(G)$ cannot contain a complete bipartite $K_{2,\Delta+1}$-subgraph, implying by Theorem 5.3 that $\mathrm{ind}(B(T(G))) + 2 < \Delta(G) + 3$. If $\Delta \leq 3$, it was proven by Rosenfeld [31] and Vijayaditya [37] that in that case the original Behzad-Vizing conjecture holds, and as the chromatic number bounds the parameter $\mathrm{ind}(B(T(G))) + 2$ from above, it means that also the topological version holds.

So for the purpose of contradiction, assume that $T(G)$ contains a $K_{2,\Delta+1}$ and $\Delta \geq 4$. Let A and B denote the colour classes of this subgraph with $|A| = 2$ and $|B| = \Delta + 1$. Recall that the vertices of $T(G)$ were either from the vertex set V of G or from its edge set E. We will now do a case analysis and show that in either case we get a contradiction.

Case 1: A consists of two vertices u and v of G.

Case 1.1: B contains only vertices of G, i.e. $B \subseteq V$. Then v is neighbour to all vertices of B in the original graph G. This means that v has degree $\Delta + 1$ in G, a contradiction.

Case 1.2: B contains an edge $e \in E$. As u and v are adjacent to e in $T(G)$, they must be incident to e in G and thus $e = uv$ in G. Now as A and B induce a complete bipartite subgraph, there cannot be another edge in B, as it would have to be incident to both u and v as well and thus would be the same edge. So B contains Δ vertices. Now u must be adjacent to all of them, but u is also adjacent to v, meaning it has degree $\Delta + 1$, a contradiction.

Case 2: A consists of an edge e and a vertex v of G.

Case 2.1: v is an endpoint of e. Then B can contain at most one vertex u, as e has to be incident to all vertices in B, but is already incident to v and edges are incident to exactly two vertices. So B contains at least Δ edges. But v has to be incident to all these edges, but is already incident to e, so its degree is at least $\Delta + 1$, a contradiction.

Case 2.2: v is not an endpoint of e. Then B can contain at most two vertices. But also B can contain only at most two edges, as the edges of B have to be incident to v but also have to share an endpoint with $e = uw$ and thus these edges can only be vu and vw. But this means that $\Delta + 1 = |B| \leq 4$, a contradiction to $\Delta \geq 4$.

Case 3: A consists of two edges e and f of G.

Case 3.1: The two edges $e = vu$ and $f = vw$ share a common endpoint v. Then the only vertex in B can be v as it has to be incident to both e and f. So there must be at least Δ edges in B, all of which have to be incident to e and f. One of these edges can be uw, but the other at least $\Delta - 1$ edges must have v as an endpoint. But then v is incident to at least $\Delta - 1$ edges in B and two edges in A for a total of $\Delta + 1$. So its degree is at least $\Delta + 1$, a contradiction.

Case 3.2: The two edges share no common endpoint. Then B can contain only edges, as there are no vertices that are both incident to e and f. But there are only 4 possible edges sharing an endpoint with both e and f, thus $\Delta + 1 = |B| \leq 4$, contradicting $\Delta \geq 4$.

□

5.3 Concluding remarks

We have seen various interesting things about the local chromatic number. It lies between the fractional chromatic number and the chromatic number, and just as these two parameters is hard to determine. The techniques to bound the local chromatic number we have seen are mostly of topological nature for lower bounds, while we have wide colourings and homomorphisms into universal graphs for upper bounds. But all of these techniques have comparable analogues for the chromatic number, so as of now determining the local chromatic number seems to be no easier than determining the chromatic number. Knowing that the local chromatic number lies between the fractional chromatic number and the chromatic number gave us incentive to study the local chromatic number for classes of graphs with a large gap between fractional chromatic number and chromatic number. As we've seen, there are various results for classes like Schrijver graphs and Mycielski graphs, but for each of these classes there are still open problems about the local chromatic number for certain choices of the parameters.

A key to understanding the local chromatic number might be investigating what factors make it differ from the chromatic number. We saw in Section 2.4 that for graphs where these two parameters differ, the colourings that attain locality have to use a large amount of colours in total. The graphs given with this property in the corresponding proof, and the other classes like Schrijver and Mycielski graphs where we know of a gap between the local chromatic number and the chromatic number all seem to be rather large. Looking at these graphs raises (extremal) questions like: How large do these graphs have to be? Do they have to have a high maximum degree? Using computer aid to determine the chromatic number, the smallest graph that we know to have local chromatic number 3 and chromatic number 4 is the graph $\bar{U}(5,3)$, which is a regular subgraph of the universal graph $U(5,3)$ as defined in Section 2.5. It has 30 vertices, is 6-regular and vertex-critical, meaning that removing any vertex decreases its chromatic number. The smallest m for which $U(m,4)$ is not 4-colourable is $m = 6$. The subgraph $\bar{U}(6,4)$ has 60 vertices, is 18-regular and has chromatic number 5, but unlike $\bar{U}(5,3)$ it is not vertex critical, so it is not the smallest graph with local chromatic number 4 and chromatic number 5. Either way, the comparably large size of these graphs makes it hard to get an intuition for the factors that make the local chromatic number differ from the chromatic number. But as it seems that graphs where the local chromatic number and the chromatic number differ tend to be large, a reasonable conjecture with implications to other problems could be:

Conjecture 5.5. *If G is a graph and k an integer with $\chi(G) > k$ but $\psi(G) \leq k$, then $\Delta(G) \geq 2k$.*

If Conjecture 5.5 were to be true, then the local chromatic version of the Behzad-Vizing conjecture would be equivalent to the chromatic version. Assume not, then there is a graph G for which the local chromatic version holds while the chromatic version fails, which means that with $k = \Delta(G) + 2$ we get $\chi(T(G)) > k = \Delta(G) + 2$ while $\psi(T(G)) \leq k$. Our conjecture would then imply that $\Delta(T(G)) \geq 2k = 2\Delta(G) + 4$, but in Section 5.2 we have seen that $\Delta(T(G)) = 2\Delta(G)$, a contradiction.

Another conjecture for which this would have implications is the conjecture on ω, Δ and χ first conjectured by Reed in [30]. It states that $\chi(G) \leq \left\lceil \frac{\Delta(G)+1}{2} + \frac{\omega(G)}{2} \right\rceil$, where $\omega(G)$ is the size of the largest clique in G. Again, assume there is a graph G for which the local chromatic version holds while the chromatic version does not hold. Set $k = \chi(G) - 1$ to get $\chi(G) > k$. Then we have $k + 1 = \chi(G) > \left\lceil \frac{\Delta(G)+1}{2} + \frac{\omega(G)}{2} \right\rceil$ and therefore $\psi(G) \leq$

$\left\lceil \frac{\Delta(G)+1}{2} + \frac{\omega(G)}{2} \right\rceil \leq k$, because $k+1 > \left\lceil \frac{\Delta(G)+1}{2} + \frac{\omega(G)}{2} \right\rceil$ implies $k \geq \left\lceil \frac{\Delta(G)+1}{2} + \frac{\omega(G)}{2} \right\rceil$ due to integrality of both sides. Our conjecture would imply $\Delta(G) \geq 2k$ and thus $k+1 = \chi(G) > \left\lceil \frac{2k+1}{2} + \frac{\omega(G)}{2} \right\rceil \geq \left\lceil \frac{2k+3}{2} \right\rceil \geq k+2$ where we use that $\omega(G) \geq 2$ for any non-empty graph. This is a contradiction.

So it seems that if our intuition about graphs with a difference between the local chromatic number and the chromatic number requiring a large maximum degree is correct, then chromatic versions of some conjectures can be reduced to local chromatic versions, and thus a proof of the local chromatic version would suffice to prove the conjecture.

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