Bounded Control of the Kuramoto-Sivashinsky equation

by
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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Feedback control is used in almost every aspect of modern life and is essential in almost all engineering systems. Since no mathematical model is perfect and disturbances occur frequently, feedback is required. The design of a feedback control has been widely investigated in finite-dimensional space. However, many systems of interest, such as fluid flow and large structural vibrations are described by nonlinear partial differential equations and their state evolves on an infinite-dimensional Hilbert space. Developing controller design methods for nonlinear infinite-dimensional systems is not trivial.

The objectives of this thesis are divided into multiple tasks. First, the well-posedness of some classes of nonlinear partial differential equations defined on a Hilbert space are investigated. The following nonlinear affine system defined on the Hilbert space H is considered

$$\begin{aligned} &\dot{z}(t) = F(z(t)) + Bu(t), \quad t \geq 0 \\ &z(0) = z_0, \end{aligned}$$

where $z(t) \in H$ is the state vector and z_0 is the initial condition. The vector $u(t) \in \mathcal{U}$, where $\mathcal{U}$ is a Hilbert space, is a state-feedback control. The nonlinear operator $F : \mathcal{D} \subset H \to H$ is densely defined in H and the linear operator $B : \mathcal{U} \to H$ is a linear bounded operator. Conditions for the closed-loop system to have a unique solution in the Hilbert space H are given.

Next, finding a single bounded state-feedback control for nonlinear partial differential equations is discussed. In particular, Lyapunov-indirect method is considered to control nonlinear infinite-dimensional systems and conditions on when this method achieves the goal of local asymptotic stabilization of the nonlinear infinite-dimensional system are given.

The Kuramoto-Sivashinsky (KS) equation defined in the Hilbert space $L^2(-\pi, \pi)$ with periodic boundary conditions is considered.

$$\begin{aligned} &\frac{\partial z}{\partial t} = -\nu \frac{\partial^4 z}{\partial x^4} - \frac{\partial^2 z}{\partial x^2} - z\frac{\partial z}{\partial x}, \quad t \geq 0 \\ &z(0) = z_0(x), \end{aligned}$$

where the instability parameter $\nu > 0$. The KS equation is a nonlinear partial differential equation that is first-order in time and fourth-order in space. It models reaction-diffusion systems and is related to various pattern formation phenomena where turbulence or chaos appear. For instance, it models long wave motions of a liquid film over a vertical plane. When the instability parameter $\nu < 1$, this equation becomes unstable. This is shown by analyzing the stability of the linearized system and showing that the nonlinear C_0-semigroup corresponding to the nonlinear KS equation is Fréchet differentiable.

There are a number of papers establishing the stabilization of this equation via boundary control. In this thesis, we consider distributed control with a single bounded feedback control for the KS equation with periodic boundary conditions. First, it is shown that stabilizing the linearized KS equation implies local asymptotical stability of the nonlinear KS equation. This is done by establishing Fréchet differentiability of the associated nonlinear C_0-semigroup and showing that it is equal to the linear C_0-semigroup generated by the linearization of the equation. Next, a single state-feedback control that locally asymptotically stabilizes the KS equation is constructed. The same approach to stabilize the KS equation from one equilibrium point to another is used.

Finally, the solution of the uncontrolled/state-feedback controlled KS equation is approximated numerically. This is done using the Galerkin projection method to approximate infinite-dimensional systems. The numerical simulations indicate that the proposed Lyapunov-indirect method works in stabilizing the KS equation to a desired state. Moreover, the same approach can be used to stabilize the KS equation from one constant equilibrium state to another.

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This work is dedicated to my very precious boy Omar Al-ankar. His scribbles all over the draft copy of this thesis was inspiring. I would also like to dedicate this work to the memory of my late mother Khawla Musleh. It was her sincere wish for this dream to become a reality.

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Chapter 1

Introduction

The Kuramoto-Sivashinsky (KS) equation is a nonlinear partial differential equation (PDE) that is often related to turbulence phenomena in chemistry and combustion. For instance, it models the Bénard problem in an elongated box. The convection cell pattern developed from heating the plane horizon from below are modeled by the nonlinear KS equation *(See Figure 1.1)* [85].

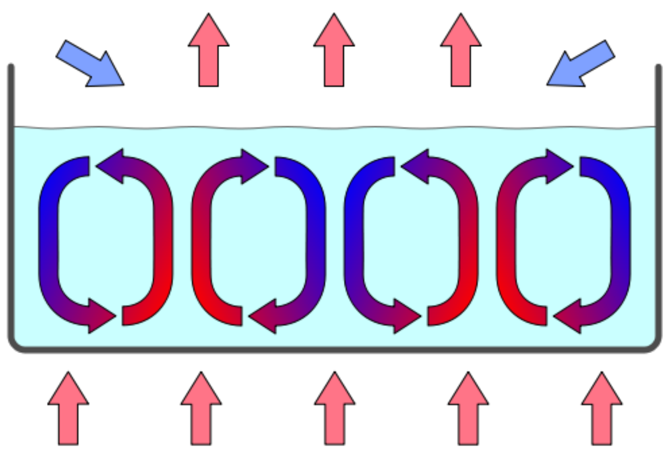

Figure 1.1: Bénard cells in a gravity field.

(http://en.wikipedia.org/wiki/Rayleigh-Bénard-convection)

In this thesis, the state-feedback control problem of the KS equation defined in $L^2(-\pi, \pi)$ is considered

$$\frac{\partial z}{\partial t} + \nu \frac{\partial^4 z}{\partial x^4} + \frac{\partial^2 z}{\partial x^2} + z\frac{\partial z}{\partial x} = BKz, \quad t \geq 0,$$

with periodic boundary conditions

$$\frac{\partial^n z}{\partial x^n}(-\pi,t) = \frac{\partial^n z}{\partial x^n}(\pi,t), \quad n = 0,1,2,3,$$

and initial condition

$$z(x,0) = z_0(x),$$

where $z \in L^2(-\pi,\pi)$ is the state of the system, $\nu > 0$ is the instability parameter, $-\nu\frac{\partial^4 z}{\partial x^4}$ is the dissipative term, $\frac{\partial^2 z}{\partial x^2}$ is the anti-dissipative term and $z\frac{\partial z}{\partial x}$ is the nonlinear term that can be interpreted as the energy transfer mechanism that transfers energy from low to high wave numbers [74]. The operators $B : \mathbb{C} \to L^2(-\pi,\pi)$ and $K : L^2(-\pi,\pi) \to \mathbb{C}$ are linear bounded operators. The operator K will be designed to force the solution to converge to a desired state. The nonlinear KS equation was first derived in 1978 by Kuramoto in one space dimension for the theoretical study of a turbulent state in a distributed chemical reaction system [50]. Then Sivashinsky [76, 77, 56] extended the equation to two dimension or more in his study of the propagation of a flame front to describe the combined influence of diffusion and thermal conduction of the gas.

The stability analysis of the dynamics of the KS equation analytically as well as numerically has attracted many researchers for years [6, 16, 17, 18, 19, 27, 35, 36, 37, 41, 46, 57, 65]. Analytical as well as numerical studies on the dynamics of the KS equation showed the existence of steady-state and periodic solutions and chaotic behaviour for very small values of the instability parameter [57]. This was done using the Lyapunov-indirect method. However, a counter example will be given in this thesis to show that the Lyapunov-indirect method does not hold for all infinite-dimensional systems in general. Furthermore, the Fréchet differentiability of the C_0-semigroup generated by the nonlinear operator of the infinite-dimensional system is essential for the Lyapunov-indirect method to hold. This work is carried out on the nonlinear KS equation.

There are a number of papers establishing the stabilization of the KS equation via boundary control. In this thesis, we consider a bounded state-feedback control to the KS equation with periodic boundary conditions. First, we show that stabilizing the linearized KS equation implies local exponential stability of the KS equation *(See Figure 1.3)*. This is done by establishing Fréchet differentiability of the associated semigroup and showing that it is equal to the semigroup generated by the linearization of the equation. Next, we construct a single state-feedback control that locally exponentially stabilizes the KS equation. Note that although the procedure used to design a state-feedback control to the KS equation is not new, the justification presented to show that this method actually works for the infinite-dimensional KS equation is new and original *(See Figure 1.2)*. This result can be generalized to a larger class of PDEs that generates Fréchet differentiable C_0-semigroups.

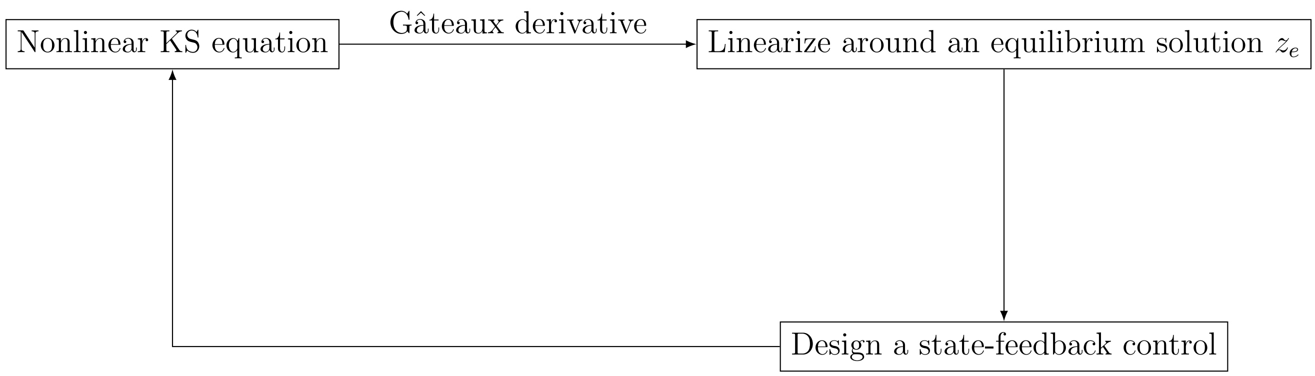

Figure 1.2: A bounded state-feedback control to the Kuramoto-Sivashinsky equation.

Finally, it is numerically verified that the method introduced earlier works in stabilizing the KS equation to a desired fixed point. Moreover, it can also be stabilized from one equilibrium solution to another. In conclusion, although the obtained stability result is local, it is very useful and can be used to drive the dynamics of the KS equation from one state to another.

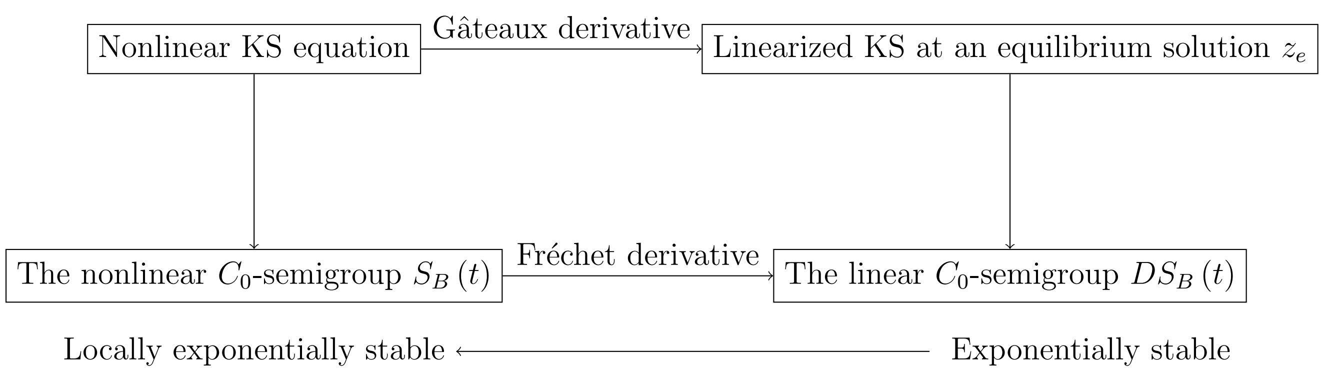

Figure 1.3: An approach to stabilize the nonlinear Kuramoto-Sivashinsky equation.

The thesis is organized as follows. In Chapter 2, the well-posedness problem of nonlinear partial differential equations (PDEs) defined on a Hilbert space is tackled and some original results are obtained for nonlinear affine infinite-dimensional systems. Chapter 3 reviews some stability analysis methods to analyze the stability of an equilibrium solution to nonlinear PDEs defined on a Hilbert space. In Chapter 4, we focus on stabilization techniques for nonlinear PDEs defined on a Hilbert space. Then, in Chapter 5, the above stability and stabilization techniques are used to analyze the stability of the nonlinear KS equation defined on the Hilbert space $L^2(-\pi, \pi)$ with periodic boundary conditions. A bounded state-feedback control is designed to locally exponentially stabilize the KS equation. Finally, in Chapter 5, some numerical simulations are presented. This is done using the classical Galerkin projection method.

Chapter 2

Well-posedness of nonlinear infinite-dimensional dynamical systems

In this chapter, the well-posedness of nonlinear systems defined on a Hilbert space H is investigated. The uncontrolled nonlinear system is considered in section 2.1 and some conditions on the operator are assumed in order to ensure the well-posedness of the system. Next, the perturbed system is considered. That is, the nonlinear system with a distributed control added to the system. Some original results are obtained to show the existence of a unique mild solution to the controlled system.

2.1 The well-posedness of partial differential equations defined on a Hilbert space

Consider the nonlinear abstract Cauchy problem defined on a complex Hilbert space H:

$$\begin{aligned} &\dot{z}(t) = F(z(t)), \quad t \geq 0 \\ &z(0) = z_0 \in H, \end{aligned} \tag{2.1}$$

where $z(t) \in H$ is the state vector, z_0 is an initial condition and $F : \mathcal{D} \subset H \to H$ is a nonlinear operator that is densely defined on H.

Definition 2.1.1. (*Nonlinear C_0-semigroup*)
A nonlinear C_0-semigroup defined on a Hilbert space H is a family of nonlinear operators $S(t) : H \to H$ that satisfies the following conditions:
- *$S(0)z = z$, for all $z \in H$.*

- $S(t+s)z = S(t)S(s)z$, *for all* $z \in H$, $s,t \geq 0$
- $t \mapsto S(t)z$ *is continuous in* t, *for all* $z \in H$

Definition 2.1.2. (*Nonlinear Contraction* C_0*-semigroup*)
A nonlinear contraction C_0*-semigroup* $S(t)$ *defined on a Hilbert space* H *is a nonlinear* C_0*-semigroup that satisfies*

$$\|S(t)z - S(t)w\| \leq \|z - w\| \text{ for all } t \geq 0,\ z, w \in H.$$

Definition 2.1.3. *[66, Definition 1.4.1]* (*Dissipative Operator*)
The operator $F : \mathcal{D} \subset H \to H$ *is a dissipative operator if*

$$Re\,\langle F(z), z\rangle \leq 0, \quad \text{for all } z \in \mathcal{D}.$$

Definition 2.1.4. *[61, Definition 2.5]* (*m-dissipative operator*)
An operator $F : \mathcal{D} \to H$ *that is densely defined on a Hilbert space* H *is m-dissipative if it is dissipative and the range* $\mathcal{R}(I - \lambda F) = H$ *for some* $\lambda > 0$.

Definition 2.1.5. *[61, Definition 3.2]*
The nonlinear operator F *is the generator of a nonlinear* C_0*-semigroup* $S(t)$ *if*

$$F(z) = \lim_{t \to 0^+} \frac{S(t)z - z}{t},$$

for all z *such that this limit exists.*

Theorem 2.1.6. *[58, Theorem 2.110]*
Let $S(t)$ *be a nonlinear contraction* C_0*-semigroup defined on a nonempty closed convex subset of a Hilbert space* H*. Then there exists a unique m-dissipative operator* F *that generates* $S(t)$*. Conversely, let* F *be an m-dissipative operator on* H*. Then there exists a unique* C_0*-semigroup* $S(t)$ *defined on* H *such that* F *is the generator of* $S(t)$*.*

Assume that the operator F in (2.1) is an m-dissipative operator, then it generates a unique contraction C_0-semigroup $S(t)$ defined on H. Hence, the abstract Cauchy problem (2.1) has a unique solution in H.

2.2 Well-posedness of nonlinear affine systems defined on a Hilbert space

In this section, we will consider the well-posedness of affine controlled systems defined on a Hilbert space H

$$\begin{aligned} \dot{z}(t) &= F(z(t)) + Bu(t), \quad t \geq 0 \\ z(0) &= z_0 \in \mathcal{D}, \end{aligned} \tag{2.2}$$

where $z(t) \in H$ is the state vector, z_0 is an initial condition. The vector $u(t) \in \mathcal{U}$ where $\mathcal{U}$ is a Hilbert space is a control input, the operator $F : \mathcal{D} \subset H \to H$ is a nonlinear operator that is densely defined on H and generates a nonlinear C_0-semigroup $S(t)$, $B : \mathcal{U} \to H$ a linear bounded operator. Consider a feedback control law defined by $u(t) = \phi(z(t))$ with $\phi : H \to \mathcal{U}$.

Definition 2.2.1. (*Locally Lipschitz Continuous Function*)
The function $\phi : H \to \mathcal{U}$ is locally Lipschitz continuous if for all $z_0 \in H$, there exists a neighbourhood $\mathcal{N} \subset H$ of z_0 and $M > 0$ such that

$$\|\phi(z) - \phi(w)\| \leq M\|z - w\|, \quad \text{for all } z, w \in \mathcal{N}.$$

Definition 2.2.2. (*Globally Lipschitz Continuous Function*)
The function $\phi : H \to \mathcal{U}$ is globally Lipschitz continuous if there exists $M > 0$ such that

$$\|\phi(z) - \phi(w)\| \leq M\|z - w\|. \quad \text{for all } z, w \in H.$$

Definition 2.2.3. *[29, Definition 3.1.1]* (*Strong Solution*)
Consider (2.2) on the Hilbert space H. The function $z(t)$ is a strong solution to (2.2) on $[0, \tau]$, if $z \in C^1([0, \tau]; H)$, $z(t) \in \mathcal{D}(F)$ for all $t \in [0, \tau]$ and $z(t)$ satisfies the initial value problem (2.2) for all $t \in [0, \tau]$.

Consider the following class of infinite-dimensional systems where the operator F in (2.2) is given by

$$F(z(t)) = A_1 z(t) + A_2(z(t)), \tag{2.3}$$

where $A_1 : \mathcal{D}_{A_1} \subset H \to H$ is a linear operator that generates a C_0-semigroup $T(t)$ and $A_2 : \mathcal{D}_{A_2} \subset H \to H$ is a nonlinear function. Finding a strong solution that satisfies (2.2) is often impossible, a weaker version of a solution is needed. Below is the definition of a mild solution.

Definition 2.2.4. *[88, Definition 10.6 & Proposition 10.11] (Mild Solution)*
Consider (2.2) on the Hilbert space H. The function $z \in L^p([0,\tau];H)$ for some $p \geq 1$ is a mild solution to (2.2), if it satisfies the integral

$$z(t) = z_0 + \int_0^t (F(z(s)) + Bu(s))\, ds,$$

where z_0 is the initial condition.

In [44, 66, 83], the existence of a unique solution to the above system, under some conditions, that will be mentioned in the next theorem, was proved.

Theorem 2.2.5. *[83, Theorem 6.1.1]*
Consider the uncontrolled system (2.2) with $u(t) = 0$, where the operator F is given by (2.3). Assume that the operator A_1 is an infinitesimal generator of a contraction C_0-semigroup $T(t)$ on H and $A_2 : H \to H$ is a nonlinear function that satisfies the following condition: For any $C > 0$, there exists a constant $K_C > 0$ such that, for every $z, w \in H$,

$$\begin{aligned} &\|A_2(z)\| \leq K_C, \\ &\|A_2(z) - A_2(w)\| \leq K_C\|z - w\|, \quad \text{for } \|z\| \leq C,\ \|w\| \leq C. \end{aligned} \tag{2.4}$$

If $\|z_0\| < C$, then there exists a unique mild solution $z(t) \in C([0, t_{max}], H)$, $t_{max} < \infty$ of the form

$$z(t) = T(t)z_0 + \int_0^t T(t-s) A_2(z(s))\, ds.$$

Corollary 2.2.6. *Consider the controlled system (2.2) where the operator F is given by (2.3) and satisfies the conditions in Theorem (2.2.5). Assume that the state-feedback control $\phi : H \to \mathcal{U}$ globally Lipschitz continuous and there exists $M > 0$ such that*

$$\|\phi(z)\| \leq M, \quad \text{for every } z(t) \in C([0, t_{max}], H),\ t_{max} < \infty.$$

Then there exists a unique mild solution $z(t) \in C([0, t_{max}], H)$, $t_{max} < \infty$ of the form

$$z(t) = T(t)z_0 + \int_0^t T(t-s)(A_2(z(s)) + B\phi(z(s)))\, ds.$$

Proof. The proof is straightforward using Theorem 2.2.5. $\square$

The existence of a unique strong solution for nonlinear systems with feedback depends on the properties of the feedback control law. In the following theorem, it is shown that if the state-feedback control $\phi(z)$ is Lipschitz continuous, then the above mild solution is the same as the strong solution.

Theorem 2.2.7. *[66, Theorem 6.1.6]*
Consider the uncontrolled system (2.2) with $u(t) = 0$, where the operator F is given by (2.3). Let A_1 be an infinitesimal generator of a C_0-semigroup $T(t)$ on a Hilbert space H. If A_2 is globally Lipschitz continuous, $z_0 \in \mathcal{D}_{A_1}$ and z is a mild solution to the system, then z is the strong solution for the system.

Corollary 2.2.8. *Consider the controlled system (2.2) where the operator F is given by (2.3). Let A_1 be an infinitesimal generator of a C_0-semigroup $T(t)$ on a Hilbert space H. If A_2 and the state-feedback control $\phi(\cdot)$ are globally Lipschitz continuous, $z_0 \in \mathcal{D}_{A_1}$ and z is a mild solution to the system, then z is the strong solution for the system.*

The following results will be used later in this thesis.

Proposition 2.2.9. *[94, Theorem 1.4.1] (Gronwall-Bellman Inequality)*
Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions with g non-negative. If $z : [a, b] \to \mathbb{R}$ is a continuous function satisfying

$$z(t) \leq f(t) + \int_a^t g(\tau) z(\tau) d\tau, \quad t \in [a, b]$$

then

$$z(t) \leq f(t) + \int_a^t f(s) g(s) e^{\int_s^t g(\tau) d\tau} ds.$$

In particular, if $f(\cdot) \equiv P$ is a constant, then

$$z(t) \leq P e^{\int_a^t g(\tau) d\tau}.$$

If, in addition $g(\cdot) \equiv G$ is a non-negative constant, then

$$z(t) \leq P e^{G(t-a)}.$$

Proposition 2.2.10. (*Young's Inequality*) *[70, Lemma 5.40]*
If $a, b \geq 0$, then

$$ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2,$$

where $\epsilon > 0$ is any positive constant in $\mathbb{R}$.

Proposition 2.2.11. (*Poincaré Inequality*) *[70, Lemma 1.8]*
If $z \in H^2_{periodic}[-\pi, \pi]$, then

$$\left\| \frac{\partial z}{\partial x} \right\| \leq \left\| \frac{\partial^2 z}{\partial x^2} \right\|.$$

Moreover, if $\int_{-\pi}^{\pi} z dx = 0$, then

$$\|z\| \leq \left\| \frac{\partial z}{\partial x} \right\|.$$

Proposition 2.2.12. (*Multiplicative Algebra*) *[85, Page 51]*
The space $H^m[-\pi, \pi]$ for $m \geq 1$ is a multiplicative algebra. That is, there exists a constant $c > 0$ such that if $z, y \in H^m[-\pi, \pi]$, then $z \cdot y \in H^m[-\pi, \pi]$ and

$$\|z \cdot y\|_{H^m} \leq c \ \|z\|_{H^m} \ \|y\|_{H^m},$$

where $\| \cdot \|_{H^m} = \| \cdot \| + \cdots + \left\| \frac{\partial^m}{\partial x^m} \cdot \right\|$.

Lemma 2.2.13. *Suppose that the nonlinear operator F generates a nonlinear C_0-semigroup $S(t)$ on a Hilbert space H. Let $\gamma \in \mathbb{R}$, then the nonlinear operator $(F - \gamma I)$ generates the nonlinear C_0-semigroup $e^{-\gamma t} S(t)$ on the Hilbert space H.*

Proof. The proof is shown using [9, Example 3.6] on rescaling semigroups. Denote the nonlinear C_0-semigroup generated by the operator $(F - \gamma I)$ by $S_\gamma(t)$. Using the exponential formula for generators of C_0-semigroup [61, Theorem 4.2],

$$\begin{aligned} S_\gamma(t) &= \lim_{n \to \infty} \left(I - \frac{t}{n} (F - \gamma I) \right)^{-n}, \\ &= \lim_{n \to \infty} \left(\frac{n}{t} \left(\frac{n}{t} I - (F - \gamma I) \right)^{-1} \right)^{n}, \end{aligned} \tag{2.5}$$

where the limit is uniform in t on bounded intervals. Defining $k = n + \gamma t$, the fraction $\frac{n}{t}$ can be written as

$$\frac{n}{t} = \frac{k - \gamma t}{t} = \left(1 - \frac{\gamma t}{k} \right) \frac{k}{t}. \tag{2.6}$$

Now, the terms of the sequence in (2.5) can be written as

$$\begin{aligned}\left(\frac{n}{t}\left(\frac{n}{t}I-(F-\gamma I)\right)^{-1}\right)^{n} &= \left(\frac{k-\gamma t}{t}\left(\frac{n+\gamma t}{t}I-F\right)^{-1}\right)^{k-\gamma t}, \\ &= \left(\frac{k-\gamma t}{t}\left(\frac{k}{t}I-F\right)^{-1}\right)^{k-\gamma t}, \\ &= \left(\frac{k-\gamma t}{t}\left(\frac{k}{t}I-F\right)^{-1}\right)^{k}\cdot\left(\frac{k-\gamma t}{t}\left(\frac{k}{t}I-F\right)^{-1}\right)^{-\gamma t}.\end{aligned}$$

Use (2.6) to obtain the following representation

$$\begin{aligned}\left(\frac{n}{t}\left(\frac{n}{t}I-(F-\gamma I)\right)^{-1}\right)^{n} &= \\ &\left(\left(1-\frac{\gamma t}{k}\right)\frac{k}{t}\left(\frac{k}{t}I-F\right)^{-1}\right)^{k}\left(\left(\frac{k}{t}-\gamma\right)\left(\frac{k}{t}I-F\right)^{-1}\right)^{-\gamma t}. \quad (2.7)\end{aligned}$$

Using [9, Lemma 2.11(ii)] and Crandall-Liggett [58, Theorem 2.115],

$$\begin{aligned}\lim_{k\to\infty}\left(\frac{k}{t}-\gamma\right)\left(\frac{k}{t}I-F\right)^{-1} &= \lim_{k\to\infty}\frac{k}{t}\left(\frac{k}{t}I-F\right)^{-1}-\lim_{k\to\infty}\gamma\left(\frac{k}{t}I-F\right)^{-1}, \\ &= I. \quad (2.8)\end{aligned}$$

Also, $\lim_{k\to\infty}\left(1-\frac{\gamma t}{k}\right)^{k}=e^{-\gamma t}$. This implies that the limit of (2.7) as $k\to\infty$ is

$$\lim_{k\to\infty}\left(\left(1-\frac{\gamma t}{k}\right)\frac{k}{t}\left(\frac{k}{t}I-F\right)^{-1}\right)^{k}\left(\left(\frac{k}{t}-\gamma\right)\left(\frac{k}{t}I-F\right)^{-1}\right)^{-\gamma t}=e^{-\gamma t}S(t). \ \square$$

Chapter 3

Stability analysis of nonlinear dynamical systems defined on a Hilbert space

In this chapter, the stability analysis of an equilibrium solution to a nonlinear dynamical system defined on a Hilbert space is presented. The first approach is by using Lyapunov direct method. That is, finding a Lyapunov function for the nonlinear system. Next, the Lyapunov indirect method is described and some examples that shows the failure of this approach in the stability analysis for infinite-dimensional systems are presented. Finally, some conditions on nonlinear infinite-dimensional system are suggested so that the Lyapunov indirect method can be used in analyzing the stability of an equilibrium solution to the nonlinear infinite-dimensional system.

3.1 Lyapunov direct method and LaSalle's invariance principle to analyze the stability of dynamical systems defined on a Hilbert space

Consider the following nonlinear system in a Hilbert space H

$$\begin{aligned} &\dot{z}(t) = F(z(t)), \quad t \geq 0 \\ &z(0) = z_0, \end{aligned} \tag{3.1}$$

where z_0 is the initial condition, the nonlinear operator $F : \mathcal{D}(F) \subset H \to H$ is densely defined on H with $F(z_e) = 0$. That is, z_e is an equilibrium solution to the nonlinear system. Assume that the above system is well-posed. In other words, the above system has a unique mild solution that can be written in terms of a nonlinear C_0-semigroup in H

$$z(t) = S(t)\, z_0,$$

where $S(t)$ is a nonlinear C_0-semigroup generated by the operator F.

Definition 3.1.1. (*Types of stability*)
Consider the nonlinear system (3.1).

1. *The equilibrium solution z_e to (3.1) is stable if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|z_0 - z_e\| < \delta$, then $\|z(t) - z_e\| < \varepsilon$, $t \geq 0$.*

2. *The equilibrium solution z_e to (3.1) is locally asymptotically stable if it is stable and there exists $\delta > 0$ such that if $\|z_0 - z_e\| < \delta$, then $\lim_{t\to\infty} z(t) = z_e$.*

3. *The equilibrium solution z_e to (3.1) is locally exponentially stable if there exists $\delta, \alpha, \beta > 0$ such that if $\|z_0 - z_e\| < \delta$, then $\|z(t) - z_e\| \leq \alpha \|z_0 - z_e\| e^{-\beta t}$, $t \geq 0$.*

4. *The equilibrium solution z_e to (3.1) is globally asymptotically stable if it is stable and for all $z_0 \in H$, we have $\lim_{t\to\infty} z(t) = z_e$.*

5. *The equilibrium solution z_e to (3.1) is globally exponentially stable if there exists $\alpha, \beta > 0$ such that for all $z_0 \in H$, we have $\|z(t) - z_e\| \leq \alpha \|z_0 - z_e\| e^{-\beta t}$, $t \geq 0$.*

6. *The equilibrium solution z_e to (3.1) is unstable if it is not stable.*

Definition 3.1.2. (*The orbit through z*)
Define for every $z \in H$, the orbit through z,

$$\gamma(z) = \cup_{t\geq 0} S(t) z, \tag{3.2}$$

where $S(t)$ is a nonlinear C_0-semigroup in H.

Definition 3.1.3. (*Pre-compact orbit*)
The orbit therough z, $\gamma(z)$ is pre-compact, if $\overline{\gamma(z)}$ is compact.

Definition 3.1.4. (*The ω-limit set*)
Define for every $z_0 \in H$, the ω-limit set,

$$\begin{aligned} \omega(z) &= \left\{ w \in H : w = \lim_{n\to\infty} S(t_n) z_0,\ t_n \to \infty \text{ as } n \to \infty \right\}, \\ &= \bigcap_{t\geq 0} \overline{\gamma(S(t) z_0)}, \end{aligned}$$

where $\gamma(z)$ is the orbit through z defined in (3.2).

Proposition 3.1.5. *[58, p.158]*
The ω-limit set is positively invariant. That is, $S(t)\,\omega(z) \subset \omega(z)$.

Proof. Let $y \in S(t)\,\omega(z)$, then

$$\begin{aligned} y &= S(t) \lim_{n\to\infty} S(t_n)\,z, \text{ for some } t_n \to \infty, \text{ as } n \to \infty. \\ &= \lim_{n\to\infty} S(t)\,S(t_n)\,z, \\ &= \lim_{n\to\infty} S(t+t_n)\,z \in \omega(z). \qquad \square \end{aligned}$$

Definition 3.1.6. *For $z \in H$ and $\Omega \subset H$, the distance from z to Ω is*

$$d(z,\Omega) = \inf_{w\in\Omega} \|z-w\|.$$

Theorem 3.1.7. *[58, Proposition 3.59 and Theorem 3.61]*
If for $z \in H$, $\gamma(z)$ is pre-compact, then the ω-limit set $\omega(z)$ is compact, not empty, connected and $\lim_{t\to\infty} d(S(t)\,z, \omega(z)) = 0$. In fact, $\omega(z)$ is the smallest closed set that $S(t)$ approaches. That is, if $S(t)\,z \to \Omega \subset H$ as $t \to \infty$, then $\omega(x) \subset \overline{\Omega}$.

The above theorem is used to characterize the stability of a dynamical system once the pre-compactness of the orbit is proven and the ω-limit set can is determined. Lyapunov direct method has been generalized to analyzing the stability of an equilibrium point to infinite-dimensional dynamical systems [7, 89, 92]. Below is the statement of the theorem.

Definition 3.1.8. (*Continuous Lyapunov Function*)
The functional $V : H \to \mathbb{R}^+$ is a continuous Lyapunov function if it is continuously differentiable with $V(z_e) = 0$ and for every $z \in H$,

$$\dot{V}(z) = \lim_{t\to 0} \frac{V(S(t)\,z) - V(z)}{t} \leq 0.$$

Definition 3.1.9. (*Radially unbounded*)
The functional $V : H \to \mathbb{R}^+$ is radially unbounded if

$$V(z) \to \infty \text{ as } \|z\| \to \infty.$$

Theorem 3.1.10. *[89, Theorem 3.6 & 3.7](Lyapunov Theorem)*
Consider the nonlinear dynamical system (3.1). If there exists a continuous Lyapunov functional $V : H \to \mathbb{R}$ such that

$$\dot{V}(z) \leq 0, \ \ z \neq z_e,$$

then, the equilibrium point z_e is locally stable. Moreover, if

$$\dot{V}(z) < 0, \ \ z \neq z_e,$$

then, the equilibrium point z_e is locally asymptotically stable. In addition, if the functional V is radially unbounded, then a global stability is achieved.

Example 3.1.11. (*Heat equation with Dirichlet Boundary conditions*)
Consider the 1-D heat equation with Dirichlet boundary conditions defined in the Hilbert space $L^2(0,1)$,

$$\dot{z}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t), \quad t \geq 0, \ \ x \in [0,1] \tag{3.3}$$

with Dirichlet boundary conditions

$$\begin{aligned} z(0,t) &= 0, \\ z(1,t) &= 0, \end{aligned}$$

and initial condition

$$z(x,0) = z_0(x) \in L^2(0,1).$$

The domain of the operator $\mathcal{D}\left(\frac{\partial^2}{\partial x^2}\right) = \left\{ z \in L^2(0,1) | \frac{\partial z}{\partial x^2}, \frac{\partial^2 z}{\partial x^2} \in L^2(0,1) \right\}$. It can be easily be shown that $z = 0$ is the only equilibrium solution to the above system. Now consider the following Lyapunov function candidate $V : L^2(0,1) \to \mathbb{R}^+$

$$V(z) = \frac{1}{2}\|z\|^2.$$

The Lyapunov derivative is

$$\begin{aligned} \dot{V}(z) &= Re\langle z, \dot{z} \rangle, \\ &= Re\langle z, \frac{\partial^2 z}{\partial x^2} \rangle. \end{aligned}$$

That is,

$$\begin{aligned}\dot{V}(z) &= Re\left\{ z\overline{\frac{\partial z}{\partial x}}\bigg|_0^1 - \int_0^1 \frac{\partial z}{\partial x}\overline{\frac{\partial z}{\partial x}}dx \right\}, \\ &= -\|\frac{\partial z}{\partial x}\|^2.\end{aligned}$$

Now, using Poincaré inequality (Proposition 2.2.11), we have for some constant $C > 0$

$$\dot{V}(z) \leq -C\|z\|^2 < 0, \ z \neq 0.$$

Since the Lyapunov function V is radially unbounded, Lyapunov Theorem (3.1.10) implies that the zero equilibrium solution to the 1-D heat equation is globally asymptotically stable. □

The strict negative definiteness condition on the Lyapunov derivative $\dot{V}(x)$ of Theorem 3.1.10 can be relaxed while ensuring the asymptotic stability of system (2.1) by using LaSalle's invariance principle.

Theorem 3.1.12. *[58, Theorem 3.64] (LaSalle's Invariance Principle)*
Let $V : H \to \mathbb{R}$ be a continuous Lyapunov function and $\omega(z)$ the largest invariant subset of $\mathcal{R} = \{z \in H : \dot{V}(z) = 0\}$. If the orbit $\gamma(z)$ is pre-compact, then

$$\lim_{t\to\infty} d(S(t)z, \omega(z)) = 0.$$

That is $z(t) \to \omega(z)$ as $t \to \infty$.

The following corollary is a straightforward consequence of LaSalle's Invariance Principle.

Corollary 3.1.13. *If the orbit $\gamma(z)$ is pre-compact and $\omega(z)$ contains only the equilibrium point $\{z_e\}$, then the equilibrium solution z_e to the nonlinear system is asymptotically stable.*

Hence, LaSalle's invariance principle can be used to characterize the asymptotic behaviour of system (3.1) if one is able to show the pre-compactness of the orbit $\gamma(x)$ and that $\omega(z) = \{z_e\}$, where z_e is the equilibrium solution. The orbit $\gamma(z)$ is pre-compact in many cases. The following result is useful.

Theorem 3.1.14. *[58, Theorem 3.65]*
Let $F : \mathcal{D} \to H$ be a nonlinear m-dissipative operator that is densely defined on a Hilbert space H. If the resolvent $(I - \lambda F)^{-1}$ is compact for some $\lambda > 0$, then the orbit $\gamma(z)$ is pre-compact for any $z \in \overline{\mathcal{D}}$.

Sketch of the proof.

1. Let $J_\lambda := (I - \lambda F)^{-1}$ for $\lambda > 0$ and show that for all $z \in R(I - \mu F)$ with $\mu > 0$,
$$J_\mu z = J_\lambda \left(\frac{\lambda}{\mu} z + \frac{\mu - \lambda}{\mu} J_\mu z \right).$$

2. Show that the operator J_μ is compact and for all $\lambda > \mu > 0$, $z \in \mathcal{D}(J_\lambda) \cap \mathcal{D}(J_\mu)$,
$$\|z - J_\mu z\| \leq \frac{\lambda}{\mu} \|z - J_\mu z\|.$$

3. Show that for every $z \in \mathcal{D}(F)$, $0 \leq t \leq n\lambda$ for $n = 1, 2, \cdots$,
$$\|S(t) z - J_{\frac{1}{n}} S(t) z\| \leq \frac{1}{n} \|F(z)\|,$$
where $S(t)$ is the C_0-semigroup generated by F.

4. Show that the orbit $\gamma(z)$ is bounded.

5. Show that the orbit $\gamma(z)$ is pre-compact provided that $\gamma(z)$ is bounded. That is, use Tychonov's theorem to show for any given sequence $\{\tau_m\} \subset \mathbb{R}^+$, there exists a subsequence $\{t_m\}$ such that for each n,
$$\|J_{\frac{1}{n}} S(t_{m+1}) z - J_{\frac{1}{n}} S(t_m) z\| < \frac{1}{m}, \text{ for all } m \geq 1. \quad \square$$

Many infinite-dimensional dynamical systems possess an infinite number of stable equilibrium solutions. The equilibrium solution to the dynamical system is determined by the initial condition.

Definition 3.1.15. *[92, Definition 2.6] (Stable Equilibrium Set)*
Consider the dynamical system (3.1). Let E be the set of all equilibrium solutions to the system. The set E is said to be stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $dist_H(z_0, E) < \delta$, then
$$dist_H(z(t), E) < \varepsilon. \quad t \geq 0$$

Definition 3.1.16. *[92, Definition 2.6] (Globally Asymptotically Stable Equilibrium Set)*
Let E be the set of all equilibrium solutions (3.1). The set E is said to be globally asymp-

totically stable if it is stable and for every $z_0 \in H$,

$$\lim_{t \to \infty} dist_H\left(z\left(t\right), E\right) = 0.$$

Example 3.1.17. (*Heat equation with Neumann Boundary conditions*)
Consider the 1-D heat equation with Neumann boundary conditions defined in the Hilbert space $L^2(0,1)$,

$$\dot{z}\left(x,t\right) = \frac{\partial^2 z}{\partial x^2}\left(x,t\right), \quad t \geq 0,\ x \in [0,1] \tag{3.4}$$

with Neumann boundary conditions

$$\begin{aligned} \frac{\partial z}{\partial x}\left(0,t\right) &= 0, \\ \frac{\partial z}{\partial x}\left(1,t\right) &= 0, \end{aligned}$$

and initial condition

$$z\left(x,0\right) = z_0\left(x\right) \in L^2(0,1).$$

The above system has equilibrium solutions of the form $z_e = c$, *where* $c \in \mathbb{R}$. *Define the set of equilibrium solutions*

$$E = \left\{c | c \in \mathbb{R}\right\}. \tag{3.5}$$

Note that the set E *is a closed invariant set. Now, consider the same Lyapunov function candidate* $V : L^2(0,1) \to \mathbb{R}^+$ *used in Example (3.1.11) and find the Lyapunov derivative to obtain*

$$\dot{V}\left(z\right) = -\|\frac{\partial z}{\partial x}\|^2 \leq 0.$$

Poincaré inequality (Proposition 2.2.11) cannot be used in this example due to different boundary conditions. Now, we shall consider using the LaSalle's invariance principle (3.1.12). That is, $\dot{V}\left(z\right) = 0$ *if and only if* $\frac{\partial z}{\partial x} = 0$. *This implies that*

$$z = c, \text{ where } c \in \mathbb{R}.$$

The Lyapunov derivative $\dot{V}\left(z\right) = 0$ *if and only if* z *is an element of* E *in (3.5).*

Using [29, Example 2.1.13], the resolvent of the operator $\frac{d^2}{dx^2}$ *is compact, therefore by Theorem (3.1.14), the orbit of the heat equation is pre-compact. Since the Lyapunov function*

V is radially unbounded, by Lyapunov Theorem (3.1.10) and LaSalle's invariance principle (3.1.12), the solution of the above heat equation will converge to the equilibrium set E. Hence, the equilibrium set E is globally asymptotically stable. □

The Lyapunov instability theorem for finite-dimensional systems generalizes to nonlinear infinite-dimensional systems.

Theorem 3.1.18. *[89, Theorem 3.8] (Lyapunov Instability Theorem)*
Consider the nonlinear system (3.1). Let z_e be the equilibrium solution to the system and let $V : \mathcal{D} \to \mathbb{R}^+$ be a continuously differentiable function such that $V(z_e) = 0$ and for every $z \neq z_e$, z in a neighbourhood of the equilibrium solution z_e, the Lyapunov derivative

$$\dot{V}(z) > 0.$$

Furthermore, assume that for every $\delta > 0$, there exists $z_0 \in \mathcal{D}$ such that $\|z_0 - z_e\| < \delta$ and $V(z_0) > 0$. Then the equilibrium solution z_e to (3.1) is unstable.

3.2 Lyapunov indirect method to analyze the stability of dynamical systems defined on a Hilbert space

First, we shall consider finite-dimensional dynamical systems defined in $\mathbb{R}^n$, for $n < \infty$

$$\begin{aligned} &\dot{z}(t) = f(z(t)), \quad t \geq 0 \\ &z(0) = z_0, \end{aligned} \tag{3.6}$$

where z_0 is the initial condition, the function $f : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$. Assume $f(z_e) = 0$, then z_e is an equilibrium solution to the nonlinear system. Below is the theorem statement for the Lyapunov indirect method for finite-dimensional systems.

Theorem 3.2.1. *[39, Theorem 3.19]*
Consider the nonlinear system defined in (3.6). Let

$$A = \left.\frac{\partial f}{\partial z}\right|_{z=z_e}.$$

Then the following statements hold:

1. *If $Re\lambda < 0$ for all $\lambda \in Spec(A)$, then the equilibrium solution z_e to (3.6) is exponentially stable.*

2. *If there exists* $\lambda \in Spec(A)$ *such that* $Re\lambda > 0$*, then the equilibrium solution* z_e *to (3.6) is unstable.*

Below is an example to illustrate the method.

Example 3.2.2. *[39, Example 3.15]*
Consider the dynamical system describing the motion of a simple pendulum with viscous damping

$$\begin{aligned} \dot{z_1}(t) &= z_2(t), \quad t \geq 0 \\ \dot{z_2}(t) &= -\tfrac{g}{l}\sin(z_1(t)) - z_2(t), \end{aligned} \tag{3.7}$$

and initial condition

$$\begin{aligned} z_1(0) &= z_{10}, \\ z_2(0) &= z_{20}, \end{aligned}$$

where g *is the acceleration due to gravity and* l *is the length of the pendulum. The above system (3.7) can be written in the form defined in (3.6) where* $z(t) \in \mathbb{R}^2$ *is defined by*

$$z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}.$$

and

$$\begin{aligned} f(z(t)) &= \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix}, \\ &= \begin{pmatrix} z_2(t) \\ \\ -\tfrac{g}{l}\sin(z_1) - z_2(t) \end{pmatrix}. \end{aligned}$$

The above system has infinitely many equilibrium solutions

$$z_e = \begin{pmatrix} n\pi \\ 0 \end{pmatrix}, \quad n = 0, \pm 1, \pm 2, \cdots.$$

Next, we shall analyze the stability of the equilibrium solution $(0,0)$*. First, linearize the* *system around the equilibrium solution. That is, compute the matrix A*

$$\begin{aligned} A &= \left.\frac{\partial f}{\partial z}\right|_{(0,0)}, \\ &= \left.\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} \end{pmatrix}\right|_{(0,0)}, \\ &= \left.\begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos(z_1) & -1 \end{pmatrix}\right|_{(0,0)}, \\ &= \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -1 \end{pmatrix}. \end{aligned}$$

The determinant of the matrix A is positive $|A| = \frac{g}{l}$ and the trace of the matrix A is negative $tr(A) = -1$. This implies that $Re\lambda < 0$ for all $\lambda \in Spec(A)$. Hence, using Lyapunov Indirect method (Theorem 3.2.1), the zero equilibrium solution $(0,0)$ to the nonlinear system (3.7) is exponentially stable.

Similarly, we can analyze the stability of the equilibrium solution $(\pi, 0)$ and use Theorem (3.2.1) to show that it is an unstable equilibrium solution. □

It will be natural to use the Lyapunov indirect method an analyze the stability of nonlinear infinite-dimensional dynamical systems. However, there are two issues that need tackling. First, how to linearize the nonlinear systems defined on a Hilbert space H. In other words, how to differentiate the nonlinear operator defined on the Hilbert space H. Second, what guarantees that the stability of the linearized infinite-dimensional system is the same as the nonlinear system. That is, if the linearized system is stable/unstable, then does the same conclusion apply to the original nonlinear system?

Consider the nonlinear infinite-dimensional system (3.1), where z_e is an equilibrium solution to the system. Now, we shall look at linearizing (3.1) at the equilibrium z_e. Below are two ways of differentiating nonlinear operators.

Definition 3.2.3. (*Fréchet Differentiable*)
Consider an operator $F : H \to H$ defined on a normed linear space H. F is Fréchet differentiable at z_0 if there exists a bounded linear operator $DF(z_0) : H \to H$ such that for all $h \in H$

$$\lim_{h \to 0} \frac{\|F(z_0 + h) - F(z_0) - DF(z_0)h\|}{\|h\|} = 0, \tag{3.8}$$

That is, there exists $\varepsilon > 0$ such that $\|h\| < \varepsilon$

$$F(z_0 + h) - F(z_0) = DF(z_0)\,h + \omega(z_0, h),$$

where

$$\frac{\|\omega(z_0, h)\|}{\|h\|} \to 0 \text{ as } \|h\| \to 0.$$

Moreover, F is Fréchet differentiable if it is Fréchet differentiable at every $z_0 \in \mathcal{D}(F)$.

Definition 3.2.4. (*Gâteaux Differentiable*)
Let $F : \mathcal{D}(F) \subset H \to H$ be an operator defined on a Hilbert space H. The operator F is Gâteaux differentiable at $z_0 \in \mathcal{D}(F)$ if there exists a linear operator $dF(z_0) : H \to H$ such that

$$\lim_{\varepsilon \to 0} \frac{F(z_0 + \varepsilon h) - F(z_0)}{\varepsilon} = dF(z_0)\,h,$$

where $h, (z_0 + \varepsilon h) \in \mathcal{D}(F)$.

Note that the Fréchet derivative is a very strict way of differentiation for nonlinear operators defined on a Hilbert space H. This is because such nonlinear operators are usually unbounded operators. Hence, we shall consider Gâteaux derivative to linearize nonlinear infinite-dimensional systems.

In general, the Lyapunov Indirect method can not be used to analyze the stability of a nonlinear infinite-dimensional systems. Below is an counter-example derived by Hans Zwart.

Example 3.2.5. *Consider the nonlinear system defined on the Hilbert space $l^2(\mathbb{C})$.*

$$\begin{aligned} \dot{z}(t) &= F(z(t)), \ t \geq 0 \\ z(0) &= z_0. \end{aligned} \tag{3.9}$$

where $z = (z_1, z_2, \cdots, z_n, \cdots)$, for $, n = 1, \cdots, \infty$ and

$$F(z(t)) \quad = \quad \begin{pmatrix} -z_1(t) + z_1^2(t) \\ \vdots \\ -\frac{1}{n} z_n(t) + z_n^2(t) \\ \vdots \end{pmatrix}. \ t \geq 0$$

That is,

$$F(z(t)) = -\begin{pmatrix} 1 & 0 & \dots & \\ 0 & \ddots & & \\ & & \frac{1}{n} & \\ & & & \ddots \end{pmatrix} z + \begin{pmatrix} z_1^2(t) \\ \vdots \\ z_n^2(t) \\ \vdots \end{pmatrix}. \tag{3.10}$$

Define the linear operator $A : l^2(\mathbb{C}) \to l^2(\mathbb{C})$

$$A = -\begin{pmatrix} 1 & 0 & \dots & \\ 0 & \ddots & & \\ & & \frac{1}{n} & \\ & & & \ddots \end{pmatrix}. \tag{3.11}$$

Hence, (3.10) can be written as

$$F(z) = Az + \begin{pmatrix} z_1^2(t) \\ \vdots \\ z_n^2(t) \\ \vdots \end{pmatrix}. \tag{3.12}$$

The exact solution of the above system (3.9) is $z \in l^2(\mathbb{C})$ *where*

$$z_n(t) = \frac{1}{n + c_n e^{\frac{1}{n}t}}, \tag{3.13}$$

and c_n *is determined by the initial condition* z_0*.*

The above system has infinitely many equilibrium solutions $z_e \in l^2(\mathbb{C})$ *since* $\dot{z} = 0$ *if and only if* $-\frac{1}{n}z_n + z_n^2 = 0$ *for* $n = 1, \cdots, \infty$*. This implies that* $z_n = 0, \frac{1}{n}$*.*

That is, an equilibrium solution to the above system $z_e \in l^2(\mathbb{C})$ *is the infinite vector with value* 0 *or* $\frac{1}{n}$ *in every* n^{th} *index. The set of equilibria is*

$$E = \left\{ z \in l^2 | z_n \in \left\{ 0, \frac{1}{n} \right\}, n = 1, \cdots, \infty \right\}.$$

In this example we shall look at the stability of the zero equilibrium solution $z_e = 0$*.*

First, let's use the Lyapunov Indirect method. To do so, we linearize the system (3.9) *around the equilibrium solution $z_e = 0$ to obtain*

$$\begin{aligned} \dot{z}(t) &= Az(t), \; t \geq 0 \\ z(0) &= z_0, \end{aligned} \tag{3.14}$$

where the operator A is defined in (3.11). The operator A generates an asymptotically stable C_0-semigroup. This is true, since

$$\begin{aligned} \lim_{t\to\infty} \|z(t) - z_e\| &= \lim_{t\to\infty} \|z(t)\|, \\ &= \lim_{t\to\infty} \left(\sum_{n=1}^{\infty} z_n^2(0) \, e^{-\frac{2}{n}t} \right)^{\frac{1}{2}}, \\ &= 0. \end{aligned}$$

Moreover, the C_0-semigroup generated by the operator A is not exponentially stable

$$e^{At} = \begin{pmatrix} e^{-t} & & & 0 \\ & \ddots & & \\ & & e^{-\frac{1}{n}t} & \\ 0 & & & \ddots \end{pmatrix}.$$

Furthermore,

$$\begin{aligned} \|e^{At}\| &= \sup_{\|z_0\|=1} \|e^{At} z_0\|, \\ &= \sup_{\|z_0\|=1} \left(\sum_{n=1}^{\infty} z_n^2(0) \, e^{-\frac{2}{n}t} \right)^{\frac{1}{2}}, \\ &= \left(\lim_{k\to\infty} e^{-\frac{2}{k}t} \right)^{\frac{1}{2}}, \\ &= 1. \end{aligned}$$

In the nonlinear system (3.9), choose components of the initial condition z_0 to be zero except in the n^{th} position to be $\frac{1}{n}$. That is,

$$z_0 = \left(0, \cdots, 0, \frac{1}{n}, 0, \cdots \right).$$

The solution is

$$z(t) = \begin{pmatrix} 0 \\ \vdots \\ \frac{1}{n} \\ \vdots \end{pmatrix}.$$

For any $\delta > 0$, choose n such that $\frac{1}{n} < \delta$. We have $\|z_0 - z_e\| = \frac{1}{n} < \delta$ and

$$\begin{aligned} \lim_{t\to\infty} \|z(t) - z_e\| &= \frac{1}{n}, \\ &\neq 0. \end{aligned}$$

Hence, the zero equilibrium solution z_e to the nonlinear system (3.9) is not asymptotically stable. $\square$

The above example illustrates that the Lyapunov Indirect method can not always be used for infinite-dimensional systems. That is, the zero equilibrium solution to the nonlinear system (3.9) is not asymptotically stable but it is asymptotically stable to the linearized system (3.14). However, the zero equilibrium solution is not exponentially stable and this was shown in the above example.

It will be interesting and useful to investigate when Lyapunov indirect method can be used to give information about the stability of a nonlinear infinite-dimensional system. This will be discussed below. The connection between the stability analysis of a nonlinear system and its linearization will be shown next.

Proposition 3.2.6. (*The Mean Value Theorem*) *[33, Theorem 8.5.4]*
Let X, Y be two Banach spaces, $G : \mathcal{D}(G) \subset X \to Y$ be a continuous mapping into Y. If G is Fréchet differentiable at every $z \in S \subset \mathcal{D}(G)$ where S is convex, then

$$\|G(z) - G(y)\| \leq \sup_{\eta\in S} \|DG(\eta)\| \cdot \|z - y\|, \quad z, y \in S,$$

where $DG(\eta)$ is the Fréchet derivative of G at η.

Smoller [80] considered the following system defined on a Banach space X

$$\begin{aligned} \dot{z} &= Az + f(z), \\ z(0) &= z_0, \end{aligned}$$

where $z(t) \in X$ is the state and z_0 is the initial condition. The operator $A : \mathcal{D}(A) \subset X \to X$ is a linear operator that generates a C_0-semigroup on X. The nonlinear operator $f : \mathcal{D}(f) \subset X \to X$ is a Fréchet differentiable locally Lipschitz continuous operator. In [80, Theorem 11.18], he showed that the nonlinear C_0-semigroup corresponding to the nonlinear system is continuously Fréchet differentiable.

Smoller [80, Theorem 11.22] showed that if the linearized system at an equilibrium solution generates an exponentially stable C_0-semigroup, then the nonlinear system generates a locally exponentially stable C_0-semigroup in a neighbourhood of that equilibrium. He used the Mean Value Theorem (Theorem 3.2.6) to show the result as well as the fact that the C_0-semigroup is continuously Fréchet differentiable. This is a quite strong condition. However, the idea of the proof is very useful and can be used to extend this result to a larger class of systems.

Theorem 3.2.7. *Consider the nonlinear system (3.1) defined on a Hilbert space H. Assume that the nonlinear operator $F : \mathcal{D}(F) \subset H \to H$ generates a nonlinear C_0-semigroup $S(t)$. Let z_e be an equilibrium solution to the above system (3.1) and suppose that $S(t)$ is Fréchet differentiable at z_e. If z_e is an exponentially stable equilibrium of the linearized system, then z_e is a locally exponentially stable equilibrium of the nonlinear system (3.1).*

Proof. The idea of the proof is similar to [30, Proposition 2.1] and [80, Theorem 11.22]. Let $T_{z_e}(t)$ be the Fréchet derivative of the nonlinear C_0-semigroup $S(t)$ at the equilibrium solution z_e. Since z_e is an exponentially stable equilibrium solution of the linearized system, then there exists $M \geq 1$ and $\gamma > 0$ such that for all $z_0 \in H$

$$\|T_{z_e}(t) z_0 - z_e\| \leq M e^{-\gamma t} \|z_0 - z_e\|, \ t \geq 0. \tag{3.15}$$

Using the definition of Fréchet derivative (Definition 3.2.3)

$$S(t) z_0 - S(t) z_e = T_{z_e}(t)(z_0 - z_e) + \omega(z_e, z_0 - z_e),$$

where

$$\lim_{\|z_0 - z_e\| \to 0} \frac{\|\omega(z_e, z_0 - z_e)\|}{\|z_0 - z_e\|} = 0.$$

Furthermore, since the C_0-semigroups $S(t)$ and $T_{z_e}(t)$ are continuous in t, then the function ω is continuous in t. Moreover, since $S(t)$ is Fréchet differentiable at z_e, then for any $\varepsilon_t > 0$, there exists $\delta > 0$ such that if $\|z_0 - z_e\| < \delta$, then for every $t > 0$,

$$\frac{\|\omega(z_e, z_0 - z_e)\|}{\|z_0 - z_e\|} < \varepsilon_t.$$

Since the function ω is continuous in t, there exists $\varepsilon > 0$ such that for $\tau \in [0, \bar{t}]$, $\bar{t} < \infty$, $\frac{\|\omega(z_e, z_0 - z_e)\|}{\|z_0 - z_e\|} < \varepsilon$ and

$$\begin{aligned}
\|S(\tau) z_0 - z_e\| &\leq \|T_{z_e}(\tau)(z_0 - z_e)\| + \|\omega(z_e, z_0 - z_e)\|, \\
&\leq M e^{-\gamma\tau}\|z_0 - z_e\| + \varepsilon\|z_0 - z_e\|, \\
&= \left(M e^{-\gamma\tau} + \varepsilon\right)\|z_0 - z_e\|, \\
&= C\|z_0 - z_e\|,
\end{aligned}$$

where $C = M + \varepsilon$.

Next show that z_e is locally exponentially stable to the nonlinear system (3.1). Choose $\bar{t} = \frac{\ln(4M)}{\gamma} > 0$, then using (3.15),

$$\begin{aligned}
\|T_{z_e}(\bar{t}) z_0 - z_e\| &\leq M e^{-\gamma\bar{t}}\|z_0 - z_e\|, \\
&\leq \frac{1}{4}\|z_0 - z_e\|. \qquad (3.16)
\end{aligned}$$

Furthermore, using the definition of Fréchet derivative (Definition 3.2.3),

$$\begin{aligned}
\lim_{\|z - z_e\| \to 0} \left\| \frac{S(\bar{t}) z_0 - S(\bar{t}) z_e - T_{z_e}(\bar{t}) z_0 + T_{z_e}(\bar{t}) z_e}{z_0 - z_e} \right\| &= \lim_{\|z - z_e\| \to 0} \left\| \frac{S(\bar{t}) z_0 - T_{z_e}(\bar{t}) z_0}{z_0 - z_e} \right\|, \\
&= 0.
\end{aligned}$$

Thus, there exists $\delta > 0$ such that if $\|z_0 - z_e\| < \delta$, then

$$\|S(\bar{t}) z_0 - T_{z_e}(\bar{t}) z_0\| \leq \frac{1}{4}\|z_0 - z_e\|. \qquad (3.17)$$

Using (3.16) and (3.17),

$$\begin{aligned}
\|S(\bar{t}) z_0 - z_e\| &= \|S(\bar{t}) z_0 - T_{z_e}(\bar{t}) z_0 + T_{z_e}(\bar{t}) z_0 - z_e\|, \\
&\leq \|S(\bar{t}) z_0 - T_{z_e}(\bar{t}) z_0\| + \|T_{z_e}(\bar{t}) z_0 - z_e\|, \\
&\leq \frac{1}{2}\|z_0 - z_e\|, \\
&= e^{-\ln 2}\|z_0 - z_e\|. \qquad (3.18)
\end{aligned}$$

Now, let $k > 0$ be an integer, then using the semigroup property and (3.18),

$$\begin{aligned}
\|S(k\bar{t})z_0 - z_e\| &= \|S^k(\bar{t})z_0 - z_e\|, \\
&= \|S(\bar{t})S^{k-1}(\bar{t})z_0 - z_e\|, \\
&\leq e^{-\ln 2}\|S^{k-1}(\bar{t})z_0 - z_e\|, \\
&\leq e^{-(\ln 2)k}\|z_0 - z_e\|.
\end{aligned} \tag{3.19}$$

Now, if $t > 0$ is given, let $k = \left[\frac{t}{\bar{t}}\right]$ and $\tau = t - k\bar{t}$. Then $\tau \in [0, \bar{t}]$ and using the semigroup property, (3.16) and (3.19),

$$\begin{aligned}
\|S(t)z_0 - z_e\| &= \|S(k\bar{t} + \tau)z_0 - z_e\|, \\
&= \|S(\tau)S(k\bar{t})z_0 - z_e\|, \\
&\leq C\|S(k\bar{t})z_0 - z_e\|, \\
&\leq Ce^{-(\ln 2)k}\|z_0 - z_e\|, \\
&\leq Ce^{-\alpha t}\|z_0 - z_e\|,
\end{aligned}$$

for $\alpha \leq \frac{\ln 2}{\bar{t}}$. This implies that the equilibrium solution z_e to the nonlinear system is locally exponentially stable. □

The above result is very important and will be used to analyze the stability of nonlinear infinite-dimensional systems. In the next theorem, we will show that if the linearized system at an equilibrium z_e is unstable, then the nonlinear system is unstable near that equilibrium point.

Theorem 3.2.8. *Let z_e be an equilibrium solution to the system (3.1). Assume that $S(t)$ is Fréchet differentiable at z_e and the derivative is given by $T_{z_e}(t)$. If the linearized system is unstable, then the nonlinear system (3.1) is unstable.*

Proof. The proof is done by contrapositive. Let z_e be a locally stable equilibrium solution to the nonlinear system (3.1). Using the definition of Fréchet differentiable (Definition (3.2.3)), there is $r > 0$ so that for all z_0

$$S(t)z_0 - S(t)z_e = T_{z_e}(t)(z_0 - z_e) + \omega(z_e, z_0 - z_e),$$

where $\omega(z_e, z_0 - z_e)$ satisfies

$$\lim_{\|z_0 - z_e\| \to 0} \frac{\|\omega(z_e, z_0 - z_e)\|}{\|z_0 - z_e\|} = 0. \tag{3.20}$$

Since $T_{z_e}(t)$ is a linear operator and z_e is an equilibrium solution,

$$S(t)z_0 - z_e = T_{z_e}(t)z_0 - z_e + \omega(z_e, z_0 - z_e). \tag{3.21}$$

The definition of locally stable equilibrium of the nonlinear system (Definition 3.1.1) implies that for any $\varepsilon > 0$, there exists $\delta > 0$ such that if

$$\|z_0 - z_e\| < \delta, \tag{3.22}$$

then

$$\|S(t)\, z_0 - z_e\| \leq \frac{\varepsilon}{2}, \text{ for all } t \geq 0, \tag{3.23}$$

Also, since

$$\lim_{\|z_0 - z_e\| \to 0} \frac{\|\omega(z_e, z_0 - z_e)\|}{\|z_0 - z_e\|} = 0, \tag{3.24}$$

there is $\hat{\delta}$, with $0 < \hat{\delta} < \delta$, such that if $\|z_0 - z_e\| \leq \hat{\delta}$, then

$$\frac{\|\omega(z_e, z_0 - z_e)\|}{\|z_0 - z_e\|} \leq \frac{\varepsilon}{2}.$$

Then, from (3.21)

$$\begin{aligned} \|T_{z_e}(t)\, z_0 - z_e\| &\leq \|\omega(z_e, z_0 - z_e)\| + \|S(t)\, z_0 - z_e\|, \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \\ &= \varepsilon. \end{aligned}$$

Thus, z_e is a stable equilibrium point of the linearization. This completes the proof. □

In summary, the Fréchet differentiability of the nonlinear C_0-semigroup corresponding to the nonlinear system plays an important role in analyzing the stability using Lyapunov indirect method. If the equilibrium solution of the linearized system around the equilibrium solution is exponentially stable, then the equilibrium solution to the nonlinear system is locally exponentially stable. Furthermore, if the equilibrium solution to the linearized system is unstable, then the nonlinear system is also unstable.

In addition, if the equilibrium solution to the linearized system is only asymptotically stable, then Lyapunov-indirect method can not be used and no conclusion about the stability of the equilibrium solution to the nonlinear system can be obtained. A counter example by Hans Zwart was presented (Example 3.2.5).

Chapter 4

Stabilization of partial differential equations

Feedback control is utilized in almost every aspect of life. Regulating the flow rate, temperature, concentration of substances in a reaction, amplifiers, modern commercial and military aircraft are some examples of control systems where feedback control may be used. Feedback control is essential in almost all engineering systems. Since no mathematical model is perfect and disturbances occur frequently, feedback is required to correct the errors. Also, feedback is used to drive the model to a desired state and hence change an unstable model to a stable one.

All feedback systems have common elements. *The plant* is the system being controlled. *The controller* is the system that affects the control. Measured outputs from the plant are fedback to the controller.

The design of a feedback control has been widely investigated in finite-dimensional space [23, 39, 43, 48, 62, 81], etc. However, many systems of interest, such as fluid flow and large structural vibrations are described by partial differential equations and their state evolves on an infinite-dimensional Hilbert space. Developing controller design methods for nonlinear infinite-dimensional systems is not trivial, although some results have been achieved *e.g.* [5, 15, 22, 23, 44, 58, 66]. Some previous work will be reviewed.

4.1 Background literature

In the last few years, stabilizing techniques and methods have been intensively investigated by many researchers [23, 39, 48, 58, 62, 66, 81] and others. It is used to drive the error between the measured and desired response to zero and drive the model to a desired state. Also it can be used to reduce the effects of disturbances to the system. There are many methods to stabilize a nonlinear finite-dimensional system such as feedback linearization [39, 43, 81] or back-stepping control [8, 48]. These two methods apply some coordinate transformation to produce an equivalent linear system that can be easily stabilized. The main advantage of the back-stepping technique is that it is a systematic way to construct a Lyapunov function for the closed loop system [48]. However, extending these methods to infinite-dimensional systems is not trivial, as a matter of fact, the coordinate transformation in feedback linearization and back-stepping involves repeated differentiation of the nonlinear terms which can cause problems in the nonlinear operators for coordinates transformation and control laws [8].

The stabilizing of linear infinite-dimensional systems has been widely explored *e.g.* [10, 12, 29, 53, 58, 63, 66]. In this section, a literature review about stabilizing nonlinear PDEs is presented *e.g.* [5, 21, 24, 26, 58]. The conventional approach to control infinite-dimensional systems is by approximating the model by a finite set of ODEs, then applying some linear/nonlinear control methods to the system to obtain the stability desired [22]. However, this method shows poor performance for systems where a large order system is needed to obtain a good finite-dimensional approximation. This will lead to a complicated control design. Hence it may be more convenient to control the original infinite-dimensional system.

Quasi-linear PDE systems arise in different models in chemical engineering. For instance, it models fluidized-bed reactor [21], fixed-bed reactor [82], plug-flow reactor [69] and pressure swing absorption processes. The eigenmodes of the partial differential operators in quasi-linear parabolic PDE systems can be divided into finite-dimensional slow modes and an infinite-dimensional stable fast complement [21]. The standard approach to stabilize such systems is by applying Galerkin's method to produce a system of ODEs that approximates the dynamics of the original PDE.

In [20], Christofides proposed a Lyapunov-based robust controller design method for a quasi-linear parabolic PDE system under the assumption that the measurements of the state variables are available. Later, in [21] Christofides and Baker extended the proposed method for which only a finite number of measurements of the output variable is available for the feedback. Their approach was to use Galerkin method to derive a system of ODEs that approximates the original system, then the new system is used to synthesize a robust feedback controller using Lyapunov's direct method. They used the assumption that the number of measurements is equal to the number of slow modes to obtain estimates for the states of the approximate ODE model from the measurements.

As for quasi-linear hyperbolic PDEs, the eigenmodes have almost the same energy and thus an infinite number of modes is needed to accurately describe the dynamics of the original system [22]. An alternative approach to stabilize such systems is by Lyapunov's direct method [90, 91]. In [3], Alonso and Ydstie constructed a Lyapunov functional candidate for quasi-linear PDE systems and showed that the closed loop systems is asymptotically stable. Moreover, in [22] Christofides and Daoutid considered first-order quasi-linear hyperbolic PDE systems with uncertain variables and un-modeled dynamics and were able to derive a necessary and sufficient conditions for the well-posedness of the system as well as to construct an explicit controller synthesis formula. The controller is constructed by using the Lyapunov's direct method and requires that there exists a known bounded function that captures the magnitude of the uncertain terms.

Slemrod [78] considered the feedback stabilizing problem of affine linear systems defined on a Hilbert space H,

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t), \\ z(0) &= z_0 \in H, \end{aligned} \tag{4.1}$$

where $z(t)$ is the state, $u(t)$ is a feedback control with $\|u(t)\| \leq r,\ r > 0$, the operator $A : \mathcal{D} \subset H \to H$ is an infinitesimal generator of a contraction C_0-semigroup $T(t)$, and $B : \mathcal{U} \to H$ is a linear bounded operator. Slemrod was able to show that the above system (4.1) can be stabilized. That is, there exists a feedback control law u such that $z(t) \to 0$ as $t \to \infty$. He derived the following feedback control law based on the "energy" stability method suggested by finite-dimensional systems where B^* denotes the adjoint operator

$$u(t) = \begin{cases} -r\frac{B^*z(t)}{\|B^*z(t)\|} & , \text{if } \|B^*z(t)\| \geq r \\ -B^*z(t) & , \text{if } \|B^*z(t)\| \leq r \end{cases}$$

In [15] Bounit and Hammouri have developed Slemrod's work and derived a nonlinear feedback control law that globally stabilizes the system (4.1) via arbitrary smooth state feedback. However, the compactness of the resolvent $(\lambda I - A)^{-1}$, for some $\lambda > 0$ is assumed. The proposed control law is given by

$$u(t) = -r\frac{B^*z(t)}{1 + \|B^*z(t)\|}.$$

Recently, feedback control design for nonlinear infinite-dimensional systems have been studied by several researchers *e.g.*[2, 5, 10, 22, 21, 23, 44, 45, 58, 66]. Many results on the asymptotic behaviour of the system are known [2, 58, 66, 83] where the dissipativity property of the system plays an important role in ensuring the well-posedness of the system.

In [26], the stability of one-dimensional nonlinear PDEs were investigated in which the system is linearized and a boundary control law is constructed to achieve the stability of the PDE. They showed that the stability is guaranteed if the Jacobian matrix of the boundary conditions satisfies a sufficient dissipativity condition. For example, in the Saint-Venant equations - which represent the dynamics of open water channels - the stability may be proved using a Lyapunov approach under the assumption that the bottom and friction slopes are sufficiently small. Later, in [10] they improved the result by giving a sufficient conditions for the exponential stability of the strong solution of the linearized Saint-Venant equations without any condition on the bottom and friction slopes. The stability analysis relies on finding a Lyapunov function for the system with strict negative definite Lyapunov derivative [25].

In [25], Coron et al. considered the boundary control problem of nonlinear hyperbolic PDEs. They designed a scheme where a boundary control is chosen such that the Lyapunov derivative of a Lyapunov function to the system is negative definite. Moreover, they showed that the derived boundary control guarantees the local convergence of the solution towards the desired state.

In addition, Krstic and Smyshyaev [49] considered the feedback boundary stabilization problem of first-order hyperbolic PDEs. They designed controllers using a backstepping techniques. That is, to use an integral transformation along with a boundary feedback in order to convert an unstable PDE into a delay line system that converges to zero in a finite time. In [49], their approach of the backstepping was to use invertible Voltera integral transformation together with the boundary feedback.

In [11], the stabilization of a non-relativistic charged particle in a one-dimensional infinite-dimensional square potential well is considered. An explicit feedback control law was constructed through a Lyapunov analysis to insure the stability of the system.

Furthermore, in [44] Ito and Kunisch considered the optimal control problem of semi-linear systems that are defined on a Hilbert space. Their approach was to construct a local control Lyapunov function such that a feedback control law of the form $u(t) = -\beta B^* z$, with $\beta > 0$, B a bounded linear operator and z the state, stabilizes the system. They were able to achieve their goal under the assumption that $A - \beta B B^*$, with A an infinitesimal generator of a C_0-semigroup, generates an exponentially stable C_0-semigroup. More on this result is presented later.

Kang and Ito [45] considered a control problem of semi-linear system that arises in fluid dynamics on a Hilbert space H:

$$\begin{aligned} &\dot{z}(t) = -\varepsilon A z(t) - F(z(t)) + Bu(t), \\ &z(0) = z_0 \in H, \end{aligned} \tag{4.2}$$

where $\varepsilon > 0$, A is a non-negative self-adjoint linear operator defined on H with $\mathcal{D}\left(A^{1/2}\right) \subset H$, B is a bounded linear operator, and F is a nonlinear operator that satisfies

$$\langle F(z) - F(z_e), z - z_e \rangle = 0,$$

for all $z \in \mathcal{D}\left(A^{1/2}\right)$ and $F(z)$ is Fréchet differentiable at z_e with derivative $F'(z_e) \in$

$\mathcal{L}(V, V^*)$, where z_e is the desired equilibrium state. Furthermore, they derived a feedback control law for system (4.2) based on nonlinear dynamic programming techniques that drives the solution to a desired equilibrium state z_e and enhances the energy dissipation effects on the given dynamics. The feedback control is given by

$$u(t) = \begin{cases} \frac{\varepsilon\langle Aw(t), w(t)\rangle - \sqrt{g_1(w(t)) + g_2(w(t))\|B^*w(t)\|^2}}{\|B^*w(t)\|^2} B^*w(t) & \text{, if } B^*w(t) \neq 0 \\ 0 & \text{, if } B^*w(t) = 0 \end{cases}$$

where

$$\begin{aligned} &w(t) = z(t) - z_e, \\ &g_1(w(t)) = (\varepsilon\langle Aw(t), w(t)\rangle)^2, \\ &g_2(w(t)) = \langle Aw(t), w(t)\rangle + \langle Qw(t), w(t)\rangle, \end{aligned}$$

with $Q \in \mathcal{L}(H)$ is a non-negative self-adjoint operator.

It is worth mentioning that finding a control Lyapunov function for an infinite-dimensional controlled system is very difficult and is often impossible since that there is no rule to follow to obtain such function. Hence, another approach to stabilize controlled infinite-dimensional systems is needed. This will be investigated in the next section.

4.2 A bounded state-feedback for affine infinite-dimensional dynamical systems: A linearization approach

In this section, we consider the controlled affine system (2.2) defined on a Hilbert space H and we shall find a bounded state-feedback control such that the closed loop system is locally exponentially stable. This is done by finding a state-feedback control to the linearized system at the equilibrium z_e using Gâteaux derivative (Definition 3.2.4). Then, the same state-feedback control is used to locally exponentially stabilize the nonlinear affine system (2.2). Conditions and justification of this approach are presented.

Consider the linearized controlled system (2.2) at the equilibrium solution z_e

$$\begin{aligned} &\dot{z} = dF(z_e)\, z + Bu, \\ &z(0) = z_0, \end{aligned} \tag{4.3}$$

where the operator $dF(z_e) : \mathcal{D} \to H$ is the Gâteaux derivative of the operator F at the equilibrium solution z_e and the actuator $B : \mathbb{C} \to H$ is defined by

$$Bu = bu, \tag{4.4}$$

with b an element in the Hilbert space H. Assume that the operator $dF(z_e)$ is a Riesz-spectral operator. Let $\{\phi_n\}$ be the eigenfunctions of $dF(z_e)$ and define

$$b_n =< b, \phi_n >, \text{ for } n = 1, 2, \cdots, \infty. \tag{4.5}$$

Let $N < \infty$ be the number of unstable eigenvalues of the linear operator $dF(z_e)$. Assume that

$$b_n \neq 0 \text{ for } n = 1, 2, \cdots, N. \tag{4.6}$$

Since the operator $dF(z_e)$ is a Riesz-spectral operator, then by [29, Theorem 5.2.10] the system $(dF(z_e), b(x))$ is exponentially stabilizable. That is, using [29, Definition 5.2.1] there exists a bounded linear feedback control operator $K : H \to \mathbb{C}$ with $Kz(t) =< K, z(t) >$, where $k \in L^2(-\pi, \pi)$ such that $dF(z_e) + b(x)K$ generates an exponentially stable C_0-semigroup, call it $T_{BK}(t)$. That is, the controlled linearized system (4.3) can be written as

$$\dot{z}(t) = dF(z_e)z(t) + b(x) < k, z(t) > . \tag{4.7}$$

Assume that the nonlinear C_0-semigroup $S(t)$ generated by the the nonlinear operator F in (2.2) is Frechét differentiable (see Definition 3.2.3). Moreover, the Fréchet derivative of $S(t)$ at z_e is the linear C_0-semigroup generated by the Gâteaux derivative of the nonlinear operator F at z_0. If z_e is an exponentially stable equilibrium solution to the linearized system (4.3). That is, the operator $dF(z_e) + b(x)$ generates an exponentially stable C_0-semigroup T_{BK}, then by Theorem 3.2.7, the nonlinear operator $F + b(x)$ generates a locally exponentially stable C_0-semigroup resulting a locally exponentially stable system (2.2).

The next theorem shows the existence of a finite-dimensional controller that locally stabilizes the nonlinear system (2.2).

Theorem 4.2.1. *Consider the nonlinear affine system (2.2) defined on a Hilbert space H where the nonlinear operator F generates a Fréchet differentiable C_0-semigroup $S(t)$ and the Fréchet derivative of $S(t)$ at an equilibrium point z_e is the linear C_0-semigroup generated by the linearized system at z_e.*

Let the operator $dF(z_e)$ in (4.3) be a Riesz-spectral operator in H and the actuator $b(x)$ satisfies condition (4.6). Then, a finite-dimensional controller stabilizes the linearized system (4.3) and locally stabilizes the nonlinear system (2.2).

Proof. Since assumption (4.6) holds and the linear operator $dF(z_e)$ is a Reisz-spectral operator, then by [29, Theorem 5.2.10], the linearized system $(dF(z_e), b(x))$ is stabilizable.

Using [29, Theorem 5.2.6], the linearized system $(dF(z_e), b(x))$ satisfies the spectrum decomposition assumption and a finite-dimensional controller can be designed to stabilize the linearized system. For completeness of the proof, this will be shown below.

For every $z \in H$, define the orthogonal projection operator of H onto the space spanned by the finite number of eigenvectors corresponding to the unstable eigenvalues of the linearized system (4.3), P by

$$Pz = \sum_{n=1}^{N} z_n \phi_n, \tag{4.8}$$

where $z_n =< z, \psi_n >$, ϕ_n is an orthonormal basis in the Hilbert space H, ψ_n is its adjoint operator for $n = 1, 2, \cdots, \infty$ and N is the number of unstable eigenvalues of the linearized system (4.3). Then the above projection induces the following decomposition of the Hilbert space H

$$H = H^+ \oplus Z = H^-, \text{ where } H^+ = PH \text{ and } H^- = (I - P) H. \tag{4.9}$$

Also, we have the following decompositions

$$dF(z_e) = \begin{pmatrix} dF(z_e)^+ & 0 \\ 0 & dF(z_e)^- \end{pmatrix}, \tag{4.10}$$

$$B = \begin{pmatrix} B^+ \\ B^- \end{pmatrix}, \tag{4.11}$$

where for every $z \in \mathcal{D}$ and $u \in \mathbb{C}$

$$\begin{aligned} &dF(z_e)^+ z = \textstyle\sum_{n=1}^{N} \lambda_n z_n \phi_n. \\ &B^+ u = u \textstyle\sum_{n=1}^{N} b_n \phi_n. \\ &dF(z_e)^- z = \textstyle\sum_{n=N+1}^{\infty} \lambda_n z_n \phi_n. \\ &B^- u = u \textstyle\sum_{n=N+1}^{\infty} b_n \phi_n. \end{aligned} \tag{4.12}$$

In fact, we have decomposed linearized system (4.3) as a vector sum of two subsystems. The first subsystem is the finite-dimensional system $\left(dF(z_e)^+, B^+\right)$ and the second subsystem is the infinite-dimensional system $\left(dF(z_e)^-, B^-\right)$.

Using [29, Theorem 5.2.6], one can find a stabilizing feedback operator $K = K_0 P$, where K_0 is a finite dimensional operator for $\left(\left(A - z_e \frac{\partial}{\partial z}\right)^+, B^+\right)$. That is, a finite-dimensional feedback can be used to stabilize the infinite-dimensional linearized system (4.3).

Furthermore, using Theorem (3.2.7), it is enough to use the finite-dimensional controller obtained earlier for a linearized system to locally exponentially stabilize the infinite-dimensional nonlinear system (2.2). □

There are many ways to design a bounded feedback control to the infinite-dimensional linearized system (4.3). It can be designed using LQR or H_∞ controllers where the goal is to minimize the cost functional corresponding to the system. This is done by solving the algebraic Riccati equation [29, 63]. Furthermore, a bounded feedback control can be designed so that the energy of the system decays as time grows resulting an asymptotically stable system [14, 78]. In [78], Slemrod considered the linear system similar to (4.3). His approach was based on the energy stability method.

Chapter 5

The Kuramoto-Sivashinsky equation

Consider the open-loop controlled Kuramoto-Sivashinsky (KS) equation with a single state-feedback control

$$\frac{\partial z}{\partial t} + \nu \frac{\partial^4 z}{\partial x^4} + \frac{\partial^2 z}{\partial x^2} + z\frac{\partial z}{\partial x} = b\left(x\right) u\left(t\right), \tag{5.1}$$

and periodic boundary conditions,

$$\frac{\partial^n z}{\partial x^n}\left(-\pi, t\right) = \frac{\partial^n z}{\partial x^n}\left(\pi, t\right), \quad n = 0, 1, 2, 3$$

and initial condition

$$z\left(x, 0\right) = z_0\left(x\right),$$

where $\nu > 0$ is the instability parameter, the vector $z \in L^2(-\pi, \pi)$ is the state of the system, the actuator $b\left(x\right) \in L^2(-\pi, \pi)$ has a finite rank and $u \in \mathbb{C}$ is a state-feedback control to the KS equation. We shall consider state-feedback control of the form $u\left(t\right) = Kz\left(t\right)$, where $K : L^2(-\pi, \pi) \to \mathbb{C}$ is defined by

$$Kz = \langle k, z\rangle, \tag{5.2}$$

with $k \in L^2(-\pi, \pi)$.

Definition 5.0.2. *For* $n \geq 1$

$$H^n_{periodic}(-\pi, \pi) := \left\{ z \in H^n(-\pi, \pi) | \frac{\partial^i z}{\partial x^i}\left(-\pi\right) = \frac{\partial^i z}{\partial x^i}\left(\pi\right) \text{ for } i = 0, \cdots, n-1 \right\}.$$

Define the linear operator $A : \mathcal{D}\left(A\right) \subset H^4(-\pi, \pi) \to L^2(-\pi, \pi)$ by

$$Az = -\nu \frac{\partial^4 z}{\partial x^4} - \frac{\partial^2 z}{\partial x^2}, \tag{5.3}$$

with

$$\mathcal{D}(A) = \left\{ z \in L^2(-\pi,\pi) | \frac{\partial^i z}{\partial x^i} \in L^2(-\pi,\pi) \text{ for } i = 1, \cdots, 4 \text{ and } \frac{\partial^i z}{\partial x^i}(-\pi) = \frac{\partial^i z}{\partial x^i}(\pi) \text{ for } i = 0, \cdots, 3 \right\}, \tag{5.4}$$

the nonlinear operator $J : \mathcal{D}(F) \subset H^1(-\pi,\pi) \to L^2(-\pi,\pi)$ by

$$J(z) = -z\frac{\partial z}{\partial x}, \tag{5.5}$$

with

$$\mathcal{D}(J) = \left\{ z \in L^2(-\pi,\pi) | \frac{\partial z}{\partial x} \in L^2(-\pi,\pi) \text{ and } z(-\pi) = z(\pi) \right\},$$

and the bounded linear operator $B : \mathbb{C} \to L^2(-\pi,\pi)$ by

$$Bu = b(x)\,u, \tag{5.6}$$

where $b(x) \in L^2(-\pi,\pi)$ has a finite rank.

Then, the feedback controlled KS equation (5.1) can be written in state-space form

$$\begin{aligned} \dot{z} &= Az + J(z) + BKz, \\ z(0) &= z_0. \end{aligned} \tag{5.7}$$

5.1 Background literature

The Kuramoto-Sivashinsky (KS) equation is a nonlinear partial differential equation (PDE) that is first-order in time and fourth-order in space. It was first introduced by Kuramoto [50] in one space dimension for the theoretical study of a turbulent state in a distributed chemical reaction system. Then Sivashinsky [56, 76, 77] extended the equation to two dimension or more in his study of the propagation of a flame front to describe the combined influence of diffusion and thermal conduction of the gas. The analysis of the KS equation in two space dimension or more is not complete yet as the semigroup is not defined everywhere in this case [85].

The KS equation is a mathematical model of reaction-diffusion systems and is related to various pattern formation phenomena where turbulence or chaos appear. For instance, it models long wave motions of the liquid film over a vertical plane, dendritic fronts in dilute binary alloys, unstable flame front, Belouzov-Zabotinskii reaction pattern and interfacial instabilities between two viscous fluids [4, 34, 38, 47, 54, 56, 72]. Also, it describes the feature of a nonlinear saturation mechanism of dissipative trapped ion modes [56]. Furthermore, the KS equation model is often used in the study of convective hydrodynamics, plasma confinement in toroidal devices, interfacial instabilities between two viscous fluids [34] and the bifurcation solutions of the Navier-Stokes equation [74].

Nicolaenko et al. [65] proved the existence and uniqueness of solutions to the KS equation under the assumption that solutions are odd and periodic. Then, Robinson [70] and Sell & You [74] showed the existence and uniqueness of the solution to the KS equation. Moreover, they showed the existence of global finite-dimensional attractor for some values of the instability parameter in the model.

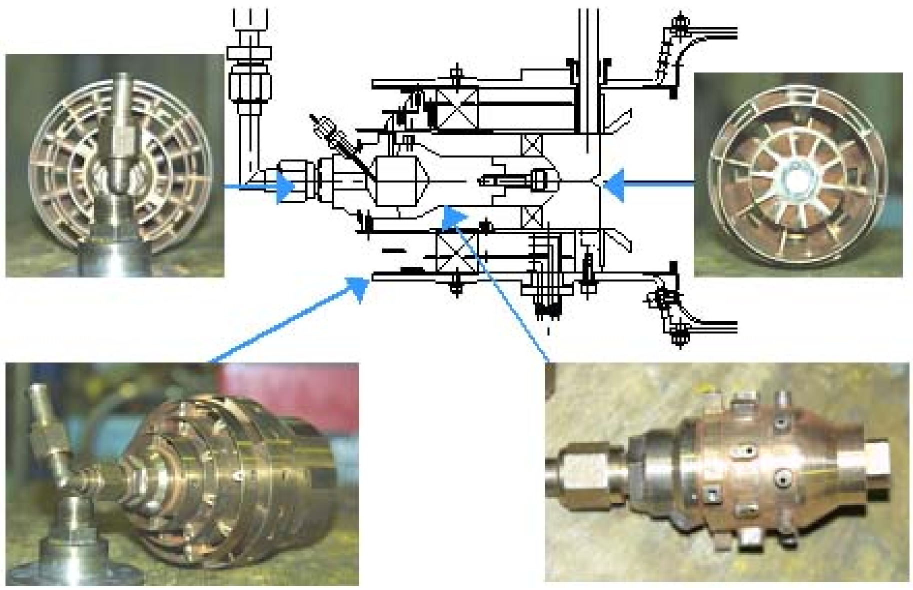

Figure 5.1: A prototype of a combustor.

(http://www.osakagas.co.jp/rd/sheet/126e.html)

An example of an experimental setup for the dynamics of the KS equation where control is needed is a combustor consisting of two concentric cylinders with a narrow gap filled with combustible gas (*see Figures 5.1 & 5.2*). In this case, without any control introduced to the system, the flame front will develop wrinkles governed by the KS dynamics.

$$\frac{\partial z}{\partial t} + \frac{\partial^4 z}{\partial x^4} + \mu \frac{\partial^2 z}{\partial x^2} + z\frac{\partial z}{\partial x} = 0, \quad x \in (0,1)\,, \quad t > 0,$$

where $\mu > 0$ is an anti-diffusion parameter. In order to control the flame front, one suggestion is to apply a distributed control to the system which is done through actuating the fuel supply all around the base of the combustor (*see Figure 5.3*). Another way is to apply a boundary control which will require fuel modulation only on a small section of the base of the combustor [56].

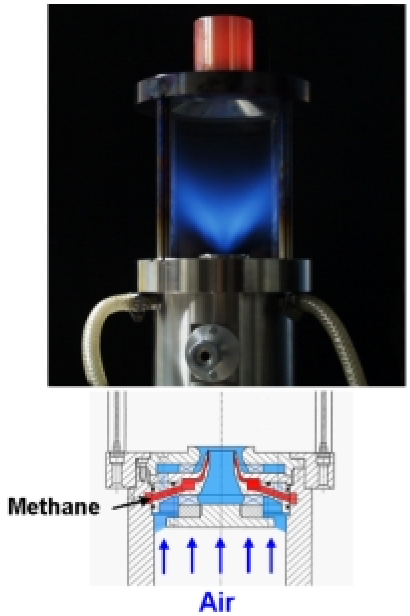

Figure 5.2: Gas turbine model combustor for swirled methane/air flames.

(http://www.dlr.de/vt/en/desktopdefault.aspx/tabid-3080/4657_read-15212/)

Another example of a physical experiment that is modelled by the KS equation where control is needed is a thin liquid film falling down a vertical plane. The height of the falling liquid in the vertical plane is regulated by the upwind gas flow or simply the air *(See Figure 5.4)*. This phenomenon occurs in both natural and industrial ways such as the flow of a thin raindrop down a windowpane under the action of gravity or the paint flow down a wet painted wall. This phenomenon causes an undesirable unsightly drip marks in the final dry coating process of the wall *(see Figure 5.5)*. Therefore, a control in needed in such application to regulate the film thickness at a desired constant value and as fast as possible to speed up the process. The dynamics is given by the following mathematical model

$$\begin{aligned}
&\tfrac{\partial z}{\partial t} + \nu \tfrac{\partial^4 z}{\partial x^4} + \tfrac{\partial^2 z}{\partial x^2} + z \tfrac{\partial z}{\partial x} = \textstyle\sum_{i=1}^{m} b_i u_i\left(t\right), \\
&\tfrac{\partial^n z}{\partial x^n}\left(-\pi, t\right) = \tfrac{\partial^n z}{\partial x^n}\left(\pi, t\right), \quad n = 0, 1, 2, 3, \qquad (5.8)\\
&z\left(x, 0\right) = z_0\left(x\right) \in L^2[-\pi, \pi],
\end{aligned}$$

where $z \in L^2[-\pi, \pi]$ is the state which represents the interface elevation that is assumed to be smaller than the film thickness, $\nu > 0$ is called the instability parameter which describes incipient instabilities of the system [57]. The parameter ν depends on a variety of parameters in the physical model such as the density of the liquid, the viscosity, the surface tension, the gas flow and the Reynolds number [20, 87]. For small values of the instability parameter ν, the solution of the system becomes oscillatory or unstable. The purpose of the control problem here is to regulate the film thickness at a desired constant

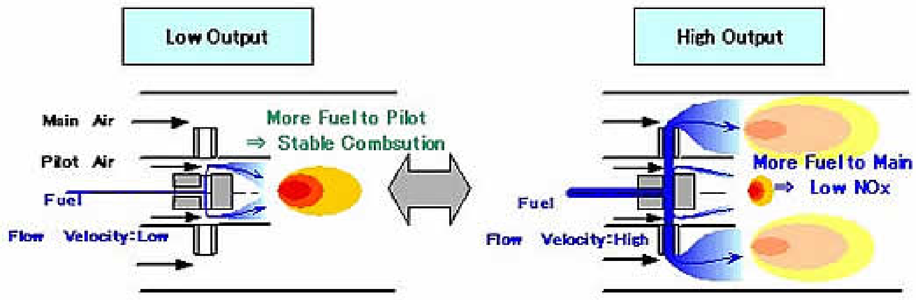

Figure 5.3: A diagram showing the distributed control concept to stabilize the flame front of a combustor.

(http://www.osakagas.co.jp/rd/sheet/126e.html)

value along the surface. This can be done through blowing or suction on the wall surface. That is by applying distributed control to the system given by the right-hand side of the above equation where m is the number of inputs, $u_i(t)$ is the i^{th} input and b_i is the i^{th} actuator which determines how the control action is computed by the i^{th} control input. That is, the actuator b_i is either blowing or suction with magnitude u_i [20, 34, 51, 54].

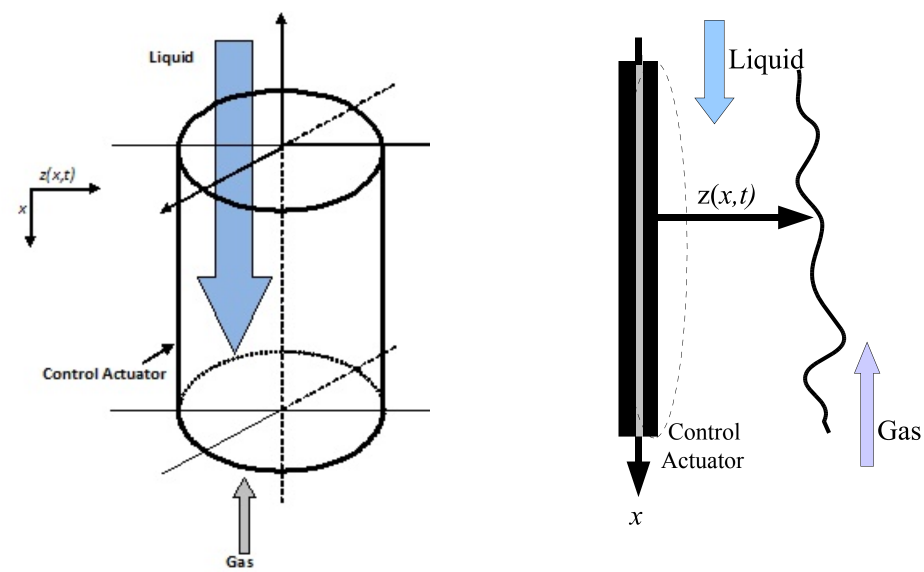

Figure 5.4: A thin liquid film falling down a vertical tube.

Many researchers studied the stability of the dynamics of the KS equation analytically as well as numerically [6, 16, 17, 18, 19, 27, 35, 36, 37, 41, 46, 57, 65]. The KS equation is a nonlinear PDE that can be regarded as an extension of the heat equation [56]. Furthermore, analytical as well as numerical studies on the dynamics of the KS equation showed the existence of steady-state and periodic solutions and chaotic behaviour for very small values of the instability parameter [57]. In [95], Zhang, Song and Axia considered the KS

Figure 5.5: The phenomenon of the flow of a thin paint over a wet painted wall.

(http://www.generalpaint.com/problem_solution_sagging)

equation with periodic boundary conditions and odd solutions and were able to show that the zero equilibrium solution to the KS equation is globally exponentially stable for certain values of the instability parameter. A more general result will be obtained in a later section.

Liu and Krstic [56] considered the following KS equation

$$\frac{\partial z}{\partial t} + \frac{\partial^4 z}{\partial x^4} + \mu \frac{\partial^2 z}{\partial x^2} + z\frac{\partial z}{\partial x} = 0, \quad x \in (0,1), \quad t > 0,$$

where $\mu > 0$ is a parameter with Dirichlet/Neumann boundary conditions. They showed that the closed loop system is well-posed by using the Banach contraction principle. In addition, they showed that the zero solution $z(x,t) \equiv 0$ of the KS equation under Dirichlet boundary conditions is unstable if $\mu > 4\pi^2$ and is asymptotically stable if $\mu < 4\pi^2$ (the case when $\mu = 4\pi^2$ was not investigated). Moreover, they showed that the zero solution to the KS equation under Neumann boundary conditions is unstable regardless to the value of μ. Finally, for $\mu < 4\pi^2$, they considered the boundary control problem of the KS equation with the following boundary conditions

$$\tfrac{\partial^2 z}{\partial x^2}(0) = u_1(t), \quad \tfrac{\partial^3 z}{\partial x^3}(0) = u_2(t),$$

$$\tfrac{\partial^2 z}{\partial x^2}(1) = u_3(t), \quad \tfrac{\partial^3 z}{\partial x^3}(1) = u_4(t),$$

where $u_1(t), \cdots, u_4(t)$ are boundary controllers. They designed a nonlinear boundary feedback control

$$u_1(t) = k\tfrac{\partial z}{\partial x}(0,t), \qquad u_2(t) = -kz(0,t) - z(0,t)^3,$$

$$u_3(t) = -k\tfrac{\partial z}{\partial x}(1,t), \quad u_4(t) = kz(1,t) + z(1,t)^3,$$

for a sufficiently large constant k and showed that the closed loop system is globally exponentially stable in $L^2(0,1)$ and globally asymptotically stable in $H^2(0,1)$. Moreover, they designed a simpler linear feedback control that guarantees the local stability of the system. The feedback is of the form

$$u_1(t) = k\tfrac{\partial z}{\partial x}(0,t), \qquad u_2(t) = -kz(0,t),$$

$$u_3(t) = -k\tfrac{\partial z}{\partial x}(1,t), \quad u_4(t) = kz(1,t).$$

However, In [56], Liu and Krstic claims that the boundary stabilization problem for the KS equation when $\mu > 4\pi^2$ is still an open problem and a different approach is needed to stabilize the system.

Koboyashi [47] considered the adaptive stabilization problem of the KS equation.

$$\frac{\partial z}{\partial t} + \nu\frac{\partial^4 z}{\partial x^4} + \mu\frac{\partial^2 z}{\partial x^2} + \gamma z\frac{\partial z}{\partial x} = 0, \quad x \in (0,1), \quad t > 0, \tag{5.9}$$

with boundary conditions

$$\tfrac{\partial z}{\partial x}(0) = 0, \quad \tfrac{\partial^3 z}{\partial x^3}(0) = u_1(t) + \theta^T v(t),$$

$$\tfrac{\partial z}{\partial x}(1) = 0, \quad \tfrac{\partial^3 z}{\partial x^3}(1) = -u_2(t),$$

where $\nu, \mu > 0$ are positive parameters, γ is a constant, $u_1(t)$ and $u_2(t)$ are input controllers, the vector $v(t)$ is a bounded disturbance vector function and θ is the l-dimensional unknown constant vector.

Under the existence of a bounded deterministic disturbances, Koboyashi was able to construct a nonlinear adaptive stabilizer by using the high-gain nonlinear output feedback and the mechanism of the unknown parameters in the system. Furthermore, his controller guarantees global asymptotic stability and the convergence of the system state to zero.

Dubljevic [34] introduced a boundary model predictive controller for the KS equation. His approach was to transform the original boundary control problem into an abstract boundary control problem which provides a model basis for synthesis of a finite-dimensional model modal predictive controller (MMPC). That is, a low dimensional model representation of the KS equation is used in the synthesis of a finite-dimensional MMPC in the full state feedback control realization. Moreover, the proposed boundary control guarantees asymptotic stability of the unstable KS equation.

Sakthivel and Ito [72] considered the robust boundary control problem of the KS equation (5.9) with boundary conditions

$$\frac{\partial z}{\partial x}(0,t) = 0, \qquad \frac{\partial z}{\partial x}(1,t) = 0.$$

$$\frac{\partial^3 z}{\partial x^3}(0,t) = u_1(t), \quad \frac{\partial^3 z}{\partial x^3}(1,t) = u_2(t),$$

where ν, μ, γ are positive constant, $\nu \in [\varepsilon_0 - \frac{1}{\gamma}, \varepsilon_0 + \frac{1}{\gamma}]$ for some fixed constants $\varepsilon_0, \gamma > 0$ and $u_1(t), u_2(t)$ are input controllers. They derived a robust boundary control using Lyapunov based stabilization that achieves the global asymptotic stability of the system in both L^2 and L^∞ spaces despite the uncertainty in the instability parameter ν. The derived control law can be implemented as a Neumann-like boundary control of the form

$$\begin{aligned} u_1(t) &= -\frac{\gamma}{\varepsilon_0\gamma - 1}\left[c_0 + \frac{c^2}{18c_0} z(0,t)^2\right] z(0,t). \\ u_2(t) &= \frac{\gamma}{\varepsilon_0\gamma - 1}\left[c_1 + \frac{c^2}{18c_1} z(1,t)^2\right] z(1,t), \end{aligned}$$

for $c_0, c_1 > 0$ positive constants. Moreover, since the above control is invertible, it can be implemented as Dirichlet-like boundary control.

Many researchers considered distributed stabilization of the KS equation with periodic boundary conditions *e.g.*[1, 4, 20, 54, 57, 79]. Periodic boundary conditions are often used with the KS equation model because it is suitable from the mathematical tractability point of view. That is, having periodic boundary conditions with the KS equation model removes irregularities and wave interactions since they correspond to the monochromatic wave description. This implies that waves retain their wave length as they grow in amplitude [34].

An approach that is commonly used to design a distributed control to the KS equation is by using Galerkin methods to produce a finite-dimensional system of ordinary differential equations (ODEs) [85]. Then, using the solution of the Hamilton-Jacobi-Bellman equation a feedback control is constructed to stabilize the system. However, the success of this method is not theoretically justified. That is, it is not proven that stabilizing the approximated system will guarantee stabilization of the original infinite-dimensional dynamical system.

In [54], Lee and Tran considered stabilization of the KS equation in one space dimension, with periodic boundary conditions. They designed both linear and nonlinear distributed feedback controls that stabilize the system. The dynamics of the system is given by (5.8). First, they applied some reduced-order methods such as the approximate inertial manifold and the proper orthogonal decomposition to the original system to obtain a system of ODEs that approximates the dynamics of the original system. Then, they constructed linear and nonlinear feedback control laws for the reduced system using the linear and nonlinear quadratic regulator methods which are first and second order approximation of the Hamilton-Jacobi-Bellman equation respectively. Finally, they numerically compared the two controllers and showed the better performance of the closed loop system with the nonlinear feedback control over the linear one.

Another approach that is used to stabilize infinite-dimensional nonlinear dynamical systems is by first linearizing the system then reducing the order of the system. For instance, in [4], Armaou and Christofides assumed that the linearized system around the zero solution is exactly controllable and were able to design an output feedback controller to the linearized reduced-order system that globally exponentially stabilizes the system if $0.25 < \nu < 1$. Hence, it will locally exponentially stabilize the nonlinear system [40, Theorem 5.1.1].

Lou and Christofides [57] computed the optimal locations of point control actuators and measurement sensors to the nonlinear output feedback control of the KS equation. They constructed a finite-dimensional approximation of the KS equation using the Galerkin method and designed a feedback control to the approximated system under the assumption that the number of measurement sensors is equal to the number of slow modes. Then, they obtained estimates for the states of the approximated finite-dimensional system for the KS equation from the measurements. These estimates are combined with the state feedback controllers to obtain an output feedback control. Furthermore, the optimal location of the measurement sensors is derived by minimizing a cost function of the estimation error in the closed loop infinite-dimensional system.

In [49], Krstic and Smyshlyaev used back-stepping techniques to stabilize the KS equation

$$\frac{\partial z}{\partial t} + \delta \frac{\partial^4 z}{\partial x^4} + \lambda \frac{\partial^2 z}{\partial x^2} + z \frac{\partial z}{\partial x} = 0,$$

where $\delta > 0$ and $\lambda \in \mathbb{R}$ are constant parameters. Their approach to stabilize the above nonlinear KS equation and drive the solution to the zero equilibrium solution $z = 0$ is by linearizing the KS equation around the equilibrium solution, then adding a new effect/term to the equation to make it trackable using back-stepping ideas.

$$\begin{aligned} \varepsilon \frac{\partial z}{\partial t} &= \frac{\partial^2 z}{\partial x^2} - \frac{\partial v}{\partial x}, \\ 0 &= \varepsilon \frac{\partial^2 v}{\partial x^2} + a \left(\gamma \frac{\partial u}{\partial x} - v \right), \end{aligned}$$

with boundary conditions

$$z(0) = 0, \qquad z(1) = 0,$$

$$\frac{\partial z}{\partial x}(0) = v(0), \quad \frac{\partial v}{\partial x}(0) = 0,$$

where $a = \frac{1}{\delta}$, $\varepsilon = \frac{\nu}{\delta}$, $\gamma = 1 + \varepsilon\lambda$ and u is the input controller. They designed an input-feedback control that converts the KS-like equation (The KS equation after adding the new term) into the exponentially stable heat equation

$$\nu\frac{\partial z}{\partial t} + \delta\frac{\partial^2 z}{\partial x^2} = 0,$$

$$\frac{\partial z}{\partial x}(0) = z(1) = 0,$$

where $\nu > 0$.

In conclusion, the boundary control problem of the KS equation has been widely explored [34, 47, 55, 56, 72]. However, little work has been done on the distributed control of the KS equation [4, 20, 49, 54]. The conventional approach that is used to design a distributed feedback control to the KS equation is not theoretically proven. That is, it is not shown that stabilizing the linearized KS equation will stabilize the nonlinear infinite-dimensional KS equation. In this thesis, it will be proved that such technique works in stabilizing the KS equation. Moreover, a single bounded state-feedback control will be designed that locally exponentially stabilizes the KS equation at a desired state.

5.2 The linearized Kuramoto-Sivashinsky equation

In this section, the feedback controlled KS equation (5.7) will be linearized at $z_0 \in \mathcal{D}(A)$, where the operator A is defined in (5.3). This is done by using the Gâteaux derivative (Definition 3.2.4).

We find the Gâteaux derivative of the nonlinear operator $J(z)$ defined in (5.5) at $z_0 \in \mathcal{D}(J)$. Using Definition 3.2.4, the Gâteaux derivative $dJ(z_0) : H^1[-\pi, \pi] \subset L^2[-\pi, \pi] \to L^2[-\pi, \pi]$

$$\begin{aligned} dJ(z_0)z &= \lim_{\varepsilon \to 0} \frac{J(z_0 + \varepsilon z) - J(z_0)}{\varepsilon}, \\ &= \lim_{\varepsilon \to 0} \frac{\varepsilon\left(\frac{\partial}{\partial x}(z_0 z) + \varepsilon z \frac{\partial z}{\partial x}\right)}{\varepsilon}, \\ &= \frac{\partial}{\partial x}(z_0 z). \end{aligned} \tag{5.10}$$

Hence, the linearized controlled system of the KS equation around z_0 is

$$\begin{aligned}\dot{z} &= dA(z_0)\,z - dJ(z_0)\,z + Bu(t)\,, \\ &= Az - \frac{\partial}{\partial x}(z_0 z) + Bu(t)\,,\end{aligned} \tag{5.11}$$

where A is defined in (5.3).

Let z_e be a constant function that does not depend on x, then z_e is an equilibrium solution to the KS equation. The Gâteaux derivative of the nonlinear operator J at z_e is

$$dJ(\tilde{z}) = z_e \frac{\partial z}{\partial x},$$

and the linearized open-loop controlled system of the KS equation around a constant function z_e is

$$\dot{z} = Az - z_e \frac{\partial z}{\partial x} + Bu(t)\,. \tag{5.12}$$

Theorem 5.2.1. *The operator* $\left(A - z_e \frac{\partial}{\partial x}\right)$*, where* z_e *is a constant function that does not depend on* x*, is a Riesz-spectral operator that has eigenvalues* $\lambda_n = -\nu n^4 + n^2 - inz_e$*,* $n \in \mathbb{Z}$ *and the corresponding eigenvectors* $\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$*.*

Proof. Let z_e be a constant function and consider the eigenvalue problem

$$A\phi_n - z_e \phi_n^{'} = \lambda_n \phi_n, \quad n \in \mathbb{Z}$$

subject to periodic boundary conditions

$$\frac{\partial^i \phi_n}{\partial x^i}(-\pi, t) = \frac{\partial^i \phi_n}{\partial x^i}(\pi, t)\,, \quad i = 0, 1, 2, 3, \tag{5.13}$$

Solving the above eigenvalue problem. The eigenvalues are $\lambda_n = -\nu n^4 + n^2 - inz_e$ and the corresponding eigenfunctions are $\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ for $n \in \mathbb{Z}$.

It is clear that the closure of $\{\lambda_n\}$ is totally disconnected and $\{\phi_n\}_{n=-\infty}^{\infty}$ form a Riesz basis in $L^2[-\pi, \pi]$. Therefore, by [29, Definition 2.3.4] the operator $\left(A - z_e \frac{\partial}{\partial x}\right)$ is a Riesz-spectral operator with eigenvalues $\{\lambda_n\}$ and corresponding eigenvectors $\{\phi_n\}$ for $n \in \mathbb{Z}$. □

Theorem 5.2.2. *Consider the linearized open-loop controlled KS equation at a constant function* z_e *defined in (5.12). The operator* $B : \mathbb{C} \to L^2(-\pi, \pi)$ *is a linear bounded operator and the control* $u(\cdot) \in L_p([0, T], \mathbb{C})$ *for some* $p \geq 1$*. The system is well-posed. That is,* *there exists a unique mild solution*

$$z(t) = T(t)\, z_0 + \int_0^t T(t-s)\, Bu(s)\, ds,$$

where $z_0 \in L^2(-\pi,\pi)$ *is the initial condition and* $T(t)$ *is the* C_0*-semigroup generated by the operator* $\left(A - z_e \frac{\partial}{\partial x}\right)$*.*

Proof. Using Theorem (5.2.1) that the operator $\left(A - z_e \frac{\partial}{\partial x}\right)$ is a Riesz spectral operator. That is, it generates a C_0-semigroup $T(t)$ on $L^2(-\pi,\pi)$.

Using [29, Definition 3.1.4]) the open-loop controlled system (5.12) has a unique mild solution

$$z(t) = T(t)\, z_0 + \int_0^t T(t-s)\, Bu(s)\, ds. \quad \square$$

5.3 The well-posedness of the controlled Kuramoto-Sivashinsky equation and some properties of the solution

In this section, we shall investigate the well-posedness of the nonlinear open-loop controlled KS equation (5.1), then investigate the Fréchet differentiability of the nonlinear C_0-semigroup corresponding to the nonlinear open-loop controlled system (5.1).

Definition 5.3.1. (*Compact* C_0*-semigroup*) *[74, Page 22]*
A C_0*-semigroup* $S(t)$ *in a Hilbert space* H *is said to be compact if for every bounded set* $B \subset H$*, there exits* $r = r(B)$ *with* $0 \le r < \infty$ *such that for every* $t > r$*, the set* $S(t)B$ *lies in a compact subset of* H*. That is, the set* $\overline{S(t)B}$ *is compact.*

Definition 5.3.2. *[74, Page 27]*
Let $S(t)$ *be a* C_0*-semigroup in* $M \subset H$ *where* H *is a Hilbert space and let* $M_1, M_2 \subset M$*. We say that* M_1 *attracts* M_2 *if for every* $\varepsilon > 0$*, there exists* $T \ge 0$ *such that*

$$dist_H(S(t)u, M_1) \le \varepsilon \text{ for every } t \ge T \text{ and } u \in M_2.$$

Definition 5.3.3. (*Global attractor*) *[74, Page 29]*
A set $\Psi \subset H$ *is said to be a global attractor for the* C_0*-semigroup* $S(t)$ *in a Hilbert space* H *if* Ψ *is a compact, invariant set in* H *and* Ψ *attracts* H*.*

The uncontrolled nonlinear KS equation (5.1) with $u(t) = 0$ is well-posed. That is, the uncontrolled KS equation has a unique strong solution [70, 74, 85]. In [74, Theorem 54.3], it is shown that the uncontrolled KS equation has a unique solution

$$z(t) = S(t) z_0 \in L^2\left([0,T]; H^2_{periodic}(-\pi,\pi)\right) \cap L^\infty\left([0,T]; L^2(-\pi,\pi)\right), \quad T < \infty,$$

where $z_0 \in H^2_{per}(-\pi,\pi)$ is the initial condition and $S(t)$ is a nonlinear C_0-semigroup. Moreover, in [74, Theorem 54.3] they showed that the C_0-semigroup is compact and that the KS equation has a global attractor in $H^2_{periodic}[-\pi,\pi]$ when the instability parameter $\nu = 1$. The global attractor when the instability parameter $\nu > 0$ will be investigated later in this chapter.

Lemma 5.3.4. *Consider the uncontrolled KS equation (5.1) with $u(t) = 0$. Let $S(t)$ be the C_0-semigroup generated by the nonlinear uncontrolled KS equation. Then,*

$$\|S(t) z_0\| \leq e^{\alpha t}\|z_0\|, \quad z_0 \in L^2[-\pi,\pi],$$

where $\alpha = \frac{1}{\sqrt{\nu}}$.

Proof. The uncontrolled KS equation is well-posed [74, Theorem 54.3] and the solution can be written as

$$z(t) = S(t) z_0,$$

where $S(t)$ is a nonlinear C_0-semigroup in $L^2[-\pi,\pi]$ and z_0 is the initial condition.

Multiply the uncontrolled KS equation by $\overline{z}(t)$ and integrate with respect to x over $[-\pi,\pi]$

$$\int_{-\pi}^{\pi} \overline{z}\frac{\partial z}{\partial t}dx = \int_{-\pi}^{\pi}\left(-\nu\frac{\partial^4 z}{\partial x^4} - \frac{\partial^2 z}{\partial x^2} - z\frac{\partial z}{\partial x}\right)\overline{z}dx. \tag{5.14}$$

We have

$$\begin{aligned}\int_{-\pi}^{\pi} \overline{z}\frac{\partial z}{\partial t}dx &= \int_{-\pi}^{\pi}\frac{1}{2}\frac{d}{dt}|z|^2dx, \\ &= \frac{1}{2}\frac{d}{dt}\|z\|^2,\end{aligned}$$

$$\begin{aligned}\int_{-\pi}^{\pi} -\nu\frac{\partial^4 z}{\partial x^4}\overline{z}dx &= -\nu\frac{\partial^3 z}{\partial x^3}\frac{\partial \overline{z}}{\partial x}\bigg|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\nu\frac{\partial^3 z}{\partial x^3}\frac{\partial \overline{z}}{\partial x}dx,\\ &= \nu\left|\frac{\partial^2 z}{\partial x^2}\right|^2\bigg|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\nu\left|\frac{\partial^2 z}{\partial x^2}\right|^2 dx,\\ &= -\nu\|\frac{\partial^2 z}{\partial x^2}\|^2,\end{aligned}$$

$$\begin{aligned}\int_{-\pi}^{\pi} -\frac{\partial^2 z}{\partial x^2}\overline{z}dx &= -\left|\frac{\partial z}{\partial x}\right|^2\bigg|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\left|\frac{\partial z}{\partial x}\right|^2 dx,\\ &= \|\frac{\partial z}{\partial x}\|^2,\end{aligned}$$

and

$$\int_{-\pi}^{\pi} -|z|^2\frac{\partial z}{\partial x} = -\frac{1}{3}|z|^3\bigg|_{-\pi}^{\pi} = 0.$$

Substitute the above in equation (5.14) to obtain

$$\frac{1}{2}\frac{d}{dt}\|z\|^2 = -\nu\|\frac{\partial^2 z}{\partial x^2}\|^2 + \|\frac{\partial z}{\partial x}\|^2.$$

Re-arrange the terms

$$\frac{1}{2}\frac{d}{dt}\|z\|^2 + \nu\|\frac{\partial^2 z}{\partial x^2}\|^2 = \|\frac{\partial z}{\partial x}\|^2. \tag{5.15}$$

Now, using Cauchy-Schwarz inequality and Young's inequality, we have

$$\begin{aligned}\|\frac{\partial z}{\partial x}\|^2 &= \langle\frac{\partial z}{\partial x}, \frac{\partial z}{\partial x}\rangle,\\ &= -\langle z, \frac{\partial^2 z}{\partial x^2}\rangle, \text{ (Integration by parts)}\\ &\leq |\langle z, \frac{\partial^2 z}{\partial x^2}\rangle|,\\ &\leq \|z\|\cdot\|\frac{\partial^2 z}{\partial x^2}\|,\\ &\leq \frac{1}{2\nu}\|z\|^2 + \frac{\nu}{2}\|\frac{\partial^2 z}{\partial x^2}\|^2. \end{aligned}\tag{5.16}$$

Use the above result in equation (5.15) and re-arrange the terms to obtain

$$\frac{d}{dt}\|z\|^2 + \nu\|\frac{\partial^2}{\partial x^2}\|^2 \leq \frac{1}{\nu}\|z\|^2, \tag{5.17}$$

and so,

$$\frac{d}{dt}\|z\|^2 \leq \frac{1}{\nu}\|z\|^2. \tag{5.18}$$

Using Gronwall's lemma 2.2.9, we have

$$\|z\|^2 \leq e^{\frac{1}{\nu}t}\|z_0\|^2, \quad t \geq 0,$$

and so since $z(t) = S(t)\, z_0$, the result follows. □

Now, we shall re-write the KS equation (5.7) in terms of different operators. This will be used in the proof of the theorem and lemmas below. First, re-write the operator A in (5.3)

$$Az = -\left(\hat{A} + R\right) z, \tag{5.19}$$

where $\hat{A} : \mathcal{D}(A) \to L^2[-\pi, \pi]$ is the linear operator

$$\hat{A}z = \nu\frac{\partial^4 z}{\partial x^4} \tag{5.20}$$

and $\mathcal{D}(A)$ is defined in (5.4). The operator $R : \mathcal{D}(R) \subset H^2[-\pi, \pi] \to L^2[-\pi, \pi]$ is

$$Rz = \frac{\partial^2 z}{\partial x^2}, \tag{5.21}$$

with

$$\begin{aligned}\mathcal{D}(R) &= H^2_{periodic},\\ &= \left\{z \in L^2[-\pi,\pi] | \frac{\partial^i z}{\partial x^i} \in L^2[-\pi,\pi] \text{ and } \frac{\partial^i z}{\partial x^i}(\pi) = \frac{\partial^i z}{\partial x^i}(\pi),\ \text{for } i = 0, 1\right\}.\end{aligned}$$

The controlled KS equation (5.7) can be written

$$\begin{aligned}&\dot{z} = -\hat{A}z - Rz - J(z) + BKz(t),\\ &z(0) = z_0,\end{aligned} \tag{5.22}$$

where the operator J is defined in (5.5) and the linear operator $BK : L^2(-\pi, \pi) \to$

$L^2(-\pi,\pi)$ is bounded. That is, there exists $M>0$ such that

$$\|BKz\,(t)\,\| \leq M\|z\,(t)\,\|, \text{ for all } z\,(t) \in L^2(-\pi,\pi).$$

It can be shown that the controlled KS equation has a unique strong solution.

Lemma 5.3.5. *For every $z \in H^2_{per}(-\pi,\pi)$,*

$$\|Rz\| \leq c_1\|z\|_{H^2},$$

for some constant $c_1>0$.

Proof. Let $z \in H^2_{per}(-\pi,\pi)$, then using the definition of the operator R, the triangle inequality and the fact that the operator BK is bounded,

$$\begin{aligned}
\|Rz\| &= \|\frac{\partial^2 z}{\partial x^2} - BKz\|, \\
&\leq \|\frac{\partial^2 z}{\partial x^2}\| + \|BKz\|, \\
&\leq \|\frac{\partial^2 z}{\partial x^2}\| + M\|z\|, \\
&\leq c_1\left(\|z\| + \|\frac{\partial^2 z}{\partial x^2}\|\right),
\end{aligned}$$

for an appropriate choice of $c_1>0$. This implies that

$$\begin{aligned}
\|Rz\| &\leq c_1\left(\|z\| + \|\frac{\partial z}{\partial x}\| + \|\frac{\partial^2 z}{\partial x^2}\|\right), \\
&= c_1\|z\|_{H^2}. \quad \square
\end{aligned}$$

Lemma 5.3.6. *For every $z, y \in H^2_{per}(-\pi,\pi)$,*

$$|\langle Rz, y\rangle| \leq c_1\|y\|\cdot\|z\|_{H^2},$$

where $c_1>0$.

Proof. Let $z, y \in H^2_{per}(-\pi,\pi)$, then using Cauchy-Schwarz inequality and Lemma 5.3.5,

$$\begin{aligned}
|\langle Rz, y\rangle| &\leq \|Rz\|\cdot\|y\|, \\
&\leq c_1\|y\|\cdot\|z\|_{H^2}. \quad \square
\end{aligned}$$

Lemma 5.3.7. *For every $z \in H^2_{per}(-\pi,\pi)$, the L^2-inner product $\langle J\,(z)\,, z\rangle = 0$.*

Proof. This result is straightforward using the definition of the inner product and the periodic boundary conditions,

$$\begin{aligned}\langle J(z), z\rangle &= \int_{-\pi}^{\pi} z\frac{\partial z}{\partial x}\overline{z}dx,\\ &= \frac{1}{3}|z|^3\Big|_{-\pi}^{\pi},\\ &= 0. \quad \square\end{aligned}$$

Lemma 5.3.8. *For every $z, y, w \in H^2_{per}(-\pi,\pi)$,*

1. $|\langle z\frac{\partial y}{\partial x}, w\rangle| \leq c_2\|z\|^{\frac{1}{2}} \cdot \|z\|^{\frac{1}{2}}_{H^2} \cdot \|y\|_{H^2} \cdot \|w\|$, *where* $c_2 > 0$.
2. $|\langle z\frac{\partial y}{\partial x}, w\rangle| \leq c_3\|y\|_{H^2} \cdot \|z\| \cdot \|w\|$, *where* $c_3 > 0$.

Proof. Let $z, y, w \in H^2_{per}(-\pi,\pi)$, then using the definition of the L^2-inner product, Cauchy-Schwarz inequality and Agmon's inequality [85, Page 50],

$$\begin{aligned}|\langle z\frac{\partial y}{\partial x}, w\rangle| &= \left|\int_{-\pi}^{\pi} z\frac{\partial y}{\partial x}\overline{w}ds\right|,\\ &\leq \|z\|_\infty \cdot \|\frac{\partial y}{\partial x}\| \cdot \|w\|,\\ &\leq c_2\|z\|^{\frac{1}{2}} \cdot \|z\|^{\frac{1}{2}}_{H^1} \cdot \|\frac{\partial y}{\partial x}\| \cdot \|w\|,\\ &\leq c_2\|z\|^{\frac{1}{2}} \cdot \|z\|^{\frac{1}{2}}_{H^2} \cdot \|y\|_{H^2} \cdot \|w\|.\end{aligned}$$

Furthermore, using the same inequalities as above,

$$\begin{aligned}\left|\int_{-\pi}^{\pi} z\frac{\partial y}{\partial x}\overline{w}ds\right| &\leq \|\frac{\partial y}{\partial x}\|_\infty \cdot \|z\| \cdot \|w\|,\\ &\leq c_3\|\frac{\partial y}{\partial x}\|^{\frac{1}{2}} \cdot \|\frac{\partial y}{\partial x}\|^{\frac{1}{2}}_{H^1} \cdot \|z\| \cdot \|w\|,\\ &\leq c_3\|y\|_{H^2} \cdot \|z\| \cdot \|w\|. \quad \square\end{aligned}$$

Theorem 5.3.9. *The feedback controlled KS equation with periodic boundary conditions has a unique strong solution*

$$z(t) \in C\left([0,T]; L^2(-\pi,\pi)\right) \cap L^2\left([0,T]; H^2_{per}(-\pi,\pi)\right), \quad 0 < T < \infty.$$

Proof. The proof is a special case of the result [59, Theorem 1.1]. The proof is presented below for completeness. The idea is to show the existence of a unique solution to the KS equation with periodic boundary conditions by using the Galerkin method.

First, solve the eigenvalue problem of the linear operator A with periodic boundary conditions to obtain the following orthonormal eigenfunctions that form a basis for $L^2(-\pi, \pi)$

$$\left\{ \phi_n(\cdot) = \frac{1}{\sqrt{2\pi}} e^{in\cdot} \right\}_{n\in\mathbb{Z}}.$$

Let $N \in \mathbb{N}$, $N < \infty$,

$$z_N(x,t) = \sum_{n=-N}^{n=N} a_n(t)\, \phi_n(x).$$

Substitute $z_N(x,t)$ into the KS equation,

$$\begin{aligned} &\tfrac{d}{dt} z_N(t) + A z_N(t) + R z_N(t) + P_N J(z_N(t)) = 0, \\ &z_N(0) = P_N z_0, \end{aligned} \tag{5.23}$$

where P_N is the projection in $L^2(-\pi,\pi)$ on the space spanned by $\{\phi_n\}_{-N}^{N}$. The above system has a unique solution. This can be shown by finding the inner product of the above system with $\{\phi_n\}_{-N}^{N}$ to obtain the finite-dimensional ODE system

$$\begin{aligned} &\tfrac{d}{dt} a(t) = \tilde{f}(a(t)), \\ &a(0) = a_0, \end{aligned}$$

where $a \in \mathbb{C}^{2N+1}$ and

$$\begin{aligned} &\tilde{f}(a_n) = (-\nu n^4 + n^2)\, a_n + \langle BK z_N + P_N J(z_N), \phi_n \rangle, \\ &a(0) = \langle z_0, \phi_n \rangle, \end{aligned}$$

for $n = -N, \cdots, N$.

The function $\tilde{f}$ is Lipschitz continuous. Let $z \in \mathbb{C}^{2N+1}$, choose a neighbourhood $N = \left\{ y \in \mathbb{C}^{2N+1} | \|y - z\| \le \frac{r}{2} \right\}$ for some $r > 0$, then for every $z_1, z_2 \in N$,

$$\|\tilde{f}(z_1) - \tilde{f}(z_2)\| \le \left(N^2 + M + r\right) \|z_1 - z_2\|.$$

Hence, by [39, Theorem 2.25], the finite-dimensional system (5.23) has a unique solution for $t \in [0,T]$ with $T < \infty$.

Now, take the L^2-inner product of (5.23) with z_N and use Lemma 5.3.7, Cauchy-Schwarz inequality and Young's inequality to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|z_N(t)\|^2 + \nu \|\frac{\partial^2 z_N}{\partial x^2}(t)\|^2 &= -\langle R z_N(t), z_N(t)\rangle - \langle J(z_N(t)), z_N(t)\rangle, \\ &= -\langle R z_N(t), z_N(t)\rangle + 0. \end{aligned} \tag{5.24}$$

That is,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|z_N\left(t\right)\|^2+\nu\|\frac{\partial^2 z_N}{\partial x^2}\left(t\right)\|^2 &\le |\langle Rz_N\left(t\right),z_N\left(t\right)\rangle|,\\ &\le \|Rz_N\left(t\right)\|\|z_N\left(t\right)\|,\\ &\le \|z_N\left(t\right)\|\cdot\|\frac{\partial^2 z_N}{\partial x^2}\left(t\right)\|+M\|z_N\left(t\right)\|^2,\\ &\le \frac{\nu}{2}\|\frac{\partial^2 z_N}{\partial x^2}\left(t\right)\|^2+\left(\frac{1}{2\nu}+M\right)\|z_N\left(t\right)\|^2.\end{aligned}$$

This implies that

$$\frac{d}{dt}\|z_N\left(t\right)\|^2+\nu\|\frac{\partial^2 z_N}{\partial x^2}\left(t\right)\|^2\le\left(\frac{1}{\nu}+2M\right)\|z_N\left(t\right)\|^2, \tag{5.25}$$

and so

$$\frac{d}{dt}\|z_N\left(t\right)\|^2\le\left(\frac{1}{\nu}+2M\right)\|z_N\left(t\right)\|^2,$$

which implies

$$\|z_N\left(t\right)\|\le\|z_N\left(0\right)\|e^{\left(\frac{1}{2\nu}+M\right)t}. \tag{5.26}$$

Hence,

$$\sup_{t\in[0,T]}\|z_N\left(t\right)\|\le\|z_N\left(0\right)\|e^{\left(\frac{1}{2\nu}+M\right)T},$$

and

$$z_N\left(t\right)\ \text{ is uniformly bounded in } L^\infty\left([0,T];L^2(-\pi,\pi)\right).$$

Furthermore, combining (5.46) and (5.26) and integrating with respect to T, we obtain

$$\|z_N\left(T\right)\|^2+\nu\int_0^T\|\frac{\partial^2 z_N}{\partial x^2}\left(t\right)\|^2dt\le\|z_N\left(0\right)\|^2e^{\left(\frac{1}{\nu}+2M\right)T},$$

and so,

$$\int_0^T\|\frac{\partial^2 z_N}{\partial x^2}\left(t\right)\|^2dt\le\frac{1}{\nu}\|z_N\left(0\right)\|^2e^{\left(\frac{1}{\nu}+2M\right)T}.$$

Hence,

$$\begin{aligned}
\int_0^T \|z_N(t)\|_{H^2}^2 dt &= \int_0^T \left(\|z_N(t)\|^2 + \|\frac{\partial z_N}{\partial x}(t)\|^2 + \|\frac{\partial^2 z_N}{\partial x^2}(t)\|^2 \right) dt \\
&\le \int_0^T \left(\|z_N(t)\|^2 + 2\|\frac{\partial^2 z_N}{\partial x^2}(t)\|^2 \right) dt \\
&\le \left(\frac{1}{\frac{1}{\nu} + 2M} + \frac{2}{\nu} \right) \|z_N(0)\|^2 e^{\left(\frac{1}{\nu}+2M\right)T}.
\end{aligned}$$

and so

$$z_N(t) \text{ is uniformly bounded in } L^2\left([0,T]; H^2_{per}(-\pi,\pi)\right).$$

Thus,

$$z_N(t) \text{ is uniformly bounded in } L^\infty\left([0,T]; L^2(-\pi,\pi)\right) \cap L^2\left([0,T]; H^2_{per}(-\pi,\pi)\right). \quad (5.27)$$

Let $H^{-2}_{per}(-\pi,\pi)$ indicate the dual space of $H^2_{per}(-\pi,\pi)$. The operator J can be extended to $J : H^2_{per} \to H^{-2}_{per}$ and using Lemma 5.3.8

$$\begin{aligned}
\|J(z)\|_{H^{-2}} &= \sup_{y\in H^2(-\pi,\pi), \|y\|_{H^2}\neq 0} \frac{|\langle J(z), y\rangle|}{\|y\|_{H^2}}, \\
&\le \sup_{y\in H^2(-\pi,\pi), \|y\|_{H^2}\neq 0} \frac{c_2\|z\|^{\frac{1}{2}} \cdot \|z\|_{H^2}^{\frac{3}{2}} \cdot \|y\|}{\|y\|_{H^2}}, \\
&\le \sup_{y\in H^2(-\pi,\pi), \|y\|_{H^2}\neq 0} \frac{c_2\|z\|^{\frac{1}{2}} \cdot \|z\|_{H^2}^{\frac{3}{2}} \cdot \|y\|_{H^2}}{\|y\|_{H^2}}, \\
&= c_2\|z\|^{\frac{1}{2}} \cdot \|z\|_{H^2}^{\frac{3}{2}}.
\end{aligned}$$

In addition,

$$\begin{aligned}
\|Az\|_{H^{-2}} &= \sup_{y\in H^2(-\pi,\pi), \|y\|_{H^2}\neq 0} \frac{|\langle Az, y\rangle|}{\|y\|_{H^2}}, \\
&= \sup_{y\in H^2(-\pi,\pi), \|y\|_{H^2}\neq 0} \frac{\nu\|\frac{\partial^2 z}{\partial^2 x}\| \cdot \|\frac{\partial^2 y}{\partial^2 x}\|}{\|y\|_{H^2}}, \\
&\le \sup_{y\in H^2(-\pi,\pi), \|y\|_{H^2}\neq 0} \frac{\nu\|z\|_{H^2} \cdot \|y\|_{H^2}}{\|y\|_{H^2}}, \\
&= \nu\|z\|_{H^2}.
\end{aligned}$$

And using Lemma 5.3.6

$$\begin{aligned}\|Rz\|_{H^{-2}} &= \sup_{y\in H^2(-\pi,\pi),\|y\|_{H^2}\neq 0} \frac{|\langle Rz, y\rangle|}{\|y\|_{H^2}},\\ &\leq \sup_{y\in H^2(-\pi,\pi),\|y\|_{H^2}\neq 0} \frac{c_1\|y\|\cdot\|z\|_{H^2}}{\|y\|_{H^2}},\\ &\leq \sup_{y\in H^2(-\pi,\pi),\|y\|_{H^2}\neq 0} \frac{c_1\|y\|_{H^2}\cdot\|z\|_{H^2}}{\|y\|_{H^2}},\\ &= c_1\|z\|_{H^2}.\end{aligned}$$

Therefore,

$$\int_0^T \left(\|Az_N(t)\|^2_{H^{-2}} + \|Rz_N(t)\|^2_{H^{-2}} + \|P_N J(z_N(t))\|^2_{H^{-2}}\right) dt < \infty.$$

Hence, Az_N, Rz_N, $J(z_N)$ and $P_N J(z_N)$ are bounded in $L^2\left([0,T], H^{-2}_{per}(-\pi,\pi)\right)$.
Since

$$\begin{aligned}\|\frac{dz_N}{dt}(t)\|_{H^{-2}} &\leq \|Az_N(t)\|_{H^{-2}} + \|Rz_N(t)\|_{H^{-2}} + \|P_N J(z_N(t))\|_{H^{-2}},\\ &\leq c_4\|z_N(t)\|_{H^2},\end{aligned}$$

then

$$\frac{dz_N}{dt}(t) \text{ is uniformly bounded in } L^2\left([0,T]; H^{-2}_{per}(-\pi,\pi)\right). \tag{5.28}$$

Now using the weak compactness theorem [70, Corollary 4.19], there exists a subsequence z_M such that

$$\begin{aligned} &z_M \to z \in L^2\left([0,T], H^2_{per}(-\pi,\pi)\right) \text{ weakly},\\ &z_M \to z \in L^\infty\left([0,T], L^2(-\pi,\pi)\right) \text{ weak-star},\\ &\tfrac{dz_M}{dt} \to \tfrac{dz}{dt} \in L^2\left([0,T]; H^{-2}_{per}(-\pi,\pi)\right) \text{ weakly}.\end{aligned} \tag{5.29}$$

Note that using [70, Corollary 4.19], the sequence $\frac{dz_M}{dt} \to \phi \in L^2\left([0,T]; H^{-2}_{per}(-\pi,\pi)\right)$ weakly. However, since the derivative operator is a closed operator, $\phi = \frac{dz}{dt}$.

Furthermore, from (5.28), (5.28) and using Aubin's compactness theorem [84, Theorem III.2.1],

$$z_M \to z \in L^2\left([0,T], L^2(-\pi,\pi)\right) \text{ strongly}. \tag{5.30}$$

That is,

$$\lim_{M\to\infty}\int_0^T \|z_M(t)-z(t)\|^2=0.$$

In order to show that $z(t)$ is a strong solution to the KS equation, let $y\in H^2_{per}(-\pi,\pi)$ be fixed and let $M\geq N$. We shall take the limit as $M\to\infty$ of the following

$$\langle\frac{d}{dt}z_M(t)+Az_M(t)+Rz_M(t)+P_MJ(z_M(t)),y(t)\rangle=0. \tag{5.31}$$

First,

$$\begin{aligned}
&\langle\tfrac{d}{dt}z_M(t),y(t)\rangle\to\langle\tfrac{d}{dt}z(t),y(t)\rangle \text{ as } M\to\infty,\\
&\langle Rz_M(t),y(t)\rangle\to\langle Rz(t),y(t)\rangle \text{ as } M\to\infty, \qquad (5.32)\\
&\langle Az_M(t),y(t)\rangle=\nu\langle\tfrac{\partial^2 z_M}{\partial x^2}(t),\tfrac{\partial^2 y}{\partial x^2}(t)\rangle\to\nu\langle\tfrac{\partial^2 z}{\partial x^2}(t),\tfrac{\partial^2 y}{\partial x^2}(t)\rangle=\langle Az(t),y(t)\rangle \text{ as } M\to\infty.
\end{aligned}$$

As for the nonlinear term, using integration by parts

$$\begin{aligned}
\langle J(z_M(t)),y(t)\rangle &= \int_{-\pi}^{\pi} z_M(t)\ \frac{\partial z_M}{\partial x}(t)\ \overline{y}(t)\,dx,\\
&= \frac{1}{2}|z_M|^2\ \overline{y}(t)\Big|_{-\pi}^{\pi}-\frac{1}{2}\int_{-\pi}^{\pi} z_M(t)\ \frac{\partial\overline{y}}{\partial x}\ \overline{z}_M(t)\,dx,\\
&= -\frac{1}{2}\langle z_M(t)\,\frac{\partial\overline{y}}{\partial x}(t),z_M(t)\rangle.
\end{aligned}$$

For any $y\in H^2_{per}(-\pi,\pi)$, the bilinear operator $\langle\cdot\frac{\partial\overline{y}}{\partial x}(t),\cdot\rangle : H^2_{per}\times L^2(-\pi,\pi)\to\mathbb{C}$ is continuous. Therefore,

$$\langle z_M(t)\,\frac{\partial\overline{y}}{\partial x}(t),z_M(t)\rangle\to\langle z(t)\,\frac{\partial\overline{y}}{\partial x}(t),z(t)\rangle \text{ as } M\to\infty.$$

Hence,

$$\langle J(z_M(t)),y(t)\rangle\to\langle J(z(t)),y(t)\rangle \text{ as } M\to\infty. \tag{5.33}$$

Therefore,

$$\begin{aligned}
&\langle\tfrac{d}{dt}z(t)+Az(t)+Rz(t)+J(z(t)),y(t)\rangle=0 \text{ for any } y(t)\in H^2_{per}(\pi,\pi),\\
&z(0)=z_0.
\end{aligned}$$

In addition, since

$$z(t) \in L^2\left([0,T]; H^2_{per}(-\pi,\pi)\right) \cap L^\infty\left([0,T]; L^2(-\pi,\pi)\right), \frac{dz}{dt}(t) \in L^2\left([0,T]; H^{-2}_{per}(-\pi,\pi)\right),$$

and using [85, Lemma II.3.1], the above system is equivalent to

$$\begin{aligned} &\tfrac{d}{dt} z(t) + Az(t) + Rz(t) + J(z(t)) = 0, \\ &z(0) = z_0. \end{aligned}$$

Moreover, using [85, Lemma II.3.2], the solution $z \in L^2\left([0,T]; H^2_{per}(-\pi,\pi)\right)$ is almost everywhere equal to a continuous function from $[0,T]$ into $L^2(-\pi,\pi)$. That is,

$$z(t) \ \in C\left([0,T]; L^2(-\pi,\pi)\right) \cap L^2\left([0,T]; H^2_{per}(-\pi,\pi)\right).$$

This implies that $z(t)$ is a strong solution of the KS equation.

Finally, we show that the solution is unique. It is worth mentioning that this result is not shown in [59, Theorem 1.1] although an outline can be found in [70, Exercise 17.7]. Suppose that $z(t), y(t)$ are two strong solutions of the KS equation where

$$z(t), y(t) \in L^2\left([0,T], H^2_{per}(-\pi,\pi)\right) \cap C\left([0,T]; L^2(-\pi,\pi)\right).$$

Let $w(t) = z(t) - y(t)$, then w satisfies

$$\begin{aligned} &\tfrac{\partial w}{\partial t}(t) + Aw(t) + Rw(t) + J(z(t)) - J(y(t)) = 0, \\ &w(0) = 0. \end{aligned} \tag{5.34}$$

Since

$$\begin{aligned} J(y(t)) - J(z(t)) &= y(t)\frac{\partial y}{\partial x}(t) - z(t)\frac{\partial z}{\partial x}(t), \\ &= y(t)\frac{\partial y}{\partial x}(t) - z(t)\frac{\partial y}{\partial x}(t) + z(t)\frac{\partial y}{\partial x}(t) - z(t)\frac{\partial z}{\partial x}(t), \\ &= (y(t) - z(t))\frac{\partial y}{\partial x}(t) + z(t)\frac{\partial}{\partial x}(y(t) - z(t)), \\ &= -w(t)\frac{\partial y}{\partial x}(t) - z(t)\frac{\partial w}{\partial x}(t), \end{aligned}$$

using Cauchy-Schwarz inequality, Agmon's inequality [85, page 45], Lemma 5.3.8 and

Poincaré inequality,

$$\begin{aligned}\langle J(y(t)) - J(z(t)), w\rangle &\leq |\langle w(t)\frac{\partial y}{\partial x}(t), w(t)\rangle| + |\langle z(t)\frac{\partial w}{\partial x}(t), w(t)\rangle|,\\ &= \left|\int_{-\pi}^{\pi} w(t)\frac{\partial y}{\partial x}(t)\overline{w}(t)\,dx\right| + \left|\int_{-\pi}^{\pi} z(t)\frac{\partial w}{\partial x}(t)\overline{w}(t)\,dx\right|.\end{aligned}$$

That is,

$$\begin{aligned}\langle J(y(t)) - J(z(t)), w\rangle &\leq \|w(t)\|\,\|\frac{\partial y}{\partial x}(t)\|_\infty\|w(t)\| + \|z(t)\|_\infty\|\frac{\partial w}{\partial x}(t)\|\,\|w(t)\|,\\ &\leq c_3\|\frac{\partial y}{\partial x}(t)\|^{\frac{1}{2}}\|\frac{\partial y}{\partial x}(t)\|^{\frac{1}{2}}_{H^1}\|w(t)\|^2 + c_2\|z(t)\|^{\frac{1}{2}}\|z(t)\|^{\frac{1}{2}}_{H^1}\cdot\\ &\quad \|w(t)\|\|\frac{\partial^2 w}{\partial x^2}(t)\|.\end{aligned}$$

Take the L^2-inner product of equation (5.34) with w and use Cauchy-Schwarz inequality, Lemma 5.3.8 and Young's inequality,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 + \nu\|\frac{\partial^2 w}{\partial x^2}(t)\|^2 &= -\langle\frac{\partial^2 w}{\partial x^2}(t), w(t)\rangle + \langle BKw(t), w(t)\rangle\\ &\quad +\langle J(y(t)) - J(z(t)), w\rangle,\\ &\leq \|\frac{\partial^2 w}{\partial x^2}(t)\|\cdot\|w(t)\| + \|BK\|\|w(t)\|^2 + c_3\|\frac{\partial y}{\partial x}(t)\|^{\frac{1}{2}}\cdot\\ &\quad \|\frac{\partial y}{\partial x}(t)\|^{\frac{1}{2}}_{H^1}\|w(t)\|^2 + c_2\|z(t)\|^{\frac{1}{2}}\|z(t)\|^{\frac{1}{2}}_{H^1}\|w(t)\|\cdot\\ &\quad \|\frac{\partial^2 w}{\partial x^2}(t)\|,\\ &\leq \nu\|\frac{\partial^2 w}{\partial x^2}(t)\|^2 + \left(\frac{1}{2\nu} + M + c_3\|\frac{\partial y}{\partial x}(t)\|^{\frac{1}{2}}\|\frac{\partial y}{\partial x}(t)\|^{\frac{1}{2}}_{H^1}\right.\\ &\quad \left.+\frac{c_2^2}{2\nu}\|z(t)\|\|z(t)\|_{H^1}\right)\|w(t)\|^2,\end{aligned}$$

where $\nu > 0$ is the instability parameter. This implies that

$$\frac{d}{dt}\|w(t)\|^2 \leq g(t)\|w(t)\|^2, \tag{5.35}$$

where

$$g\left(t\right)=\frac{1}{2\nu}+M+c_3\|\frac{\partial y}{\partial x}\left(t\right)\|^{\frac{1}{2}}\|\frac{\partial y}{\partial x}\left(t\right)\|^{\frac{1}{2}}_{H^1}+\frac{c_2^2}{2\nu}\|z\left(t\right)\|\|z\left(t\right)\|_{H^1}.$$

Since $z\left(t\right),y\left(t\right)\in L^2\left([0,T];H^2_{per}(-\pi,\pi)\right)\cap L^\infty\left([0,T];L^2(-\pi,\pi)\right)$, then

$$\sup_{t\in[0,T]}|g\left(t\right)|=Q<\infty,$$

for some $Q>0$. Finally, integrate (5.35) with respect to t and use Cauchy-Schwarz inequality, the above result and the fact that $w\left(0\right)=0$ to obtain

$$\begin{aligned}\|w\left(t\right)\|^2 &\leq \int_0^t g\left(t\right)\|w\left(t\right)\|^2dt,\\ &\leq Q\int_0^t \|w\left(s\right)\|^2dt.\end{aligned}$$

Using Gronwall's inequality (Proposition 2.2.9), $\|w\|^2\leq 0$, which shows the uniqueness of the solution to the KS equation. □

Note that if $z\left(t\right)$ is a solution to the feedback controlled KS equation (5.7), then for any $T>0$, $z\left(t\right)\in L^2\left([0,T];H^2_{per}\left(-\pi,\pi\right)\right)\cap L^\infty\left([0,T];L^2\left(-\pi,\pi\right)\right)$ and hence,

$$\|z\|_\infty=\text{ess}\sup_{t\in[0,T]}\|z\|<\infty.$$

We will now show that the C_0-semigroup $S_B\left(t\right)$ of controlled nonlinear KS equation (5.1) is Fréchet differentiable at any $y_0\in L^2[-\pi,\pi]$ and the Fréchet derivative is equal to the semigroup corresponding to the linearized system at y_0 given in (5.11). Note that if $b\left(x\right)=0$, then the uncontrolled KS equation is obtained.

Lemma 5.3.10. *For every $z,w\in H^1_{periodic}[-\pi,\pi]$,*

$$|\langle z\frac{\partial w}{\partial x},w\rangle|\leq\|z\|_\infty\;\|w\|\;\|\frac{\partial w}{\partial x}\|. \tag{5.36}$$

Proof. Let $z,w\in H^1_{periodic}[-\pi,\pi]$, then using Cauchy-Schwarz inequality [29, page 576], we obtain

$$\begin{aligned}
|\langle z\frac{\partial w}{\partial x}, w\rangle| &= \left|\int_{-\pi}^{\pi}\frac{\partial w}{\partial x}z\overline{w}dx\right|,\\
&\leq \|z\|_{\infty}\left|\int_{-\pi}^{\pi}\overline{w}\frac{\partial w}{\partial x}dx\right|,\\
&\leq \|z\|_{\infty}\ \|w\|\ \|\frac{\partial w}{\partial x}\|. \qquad \square
\end{aligned}$$

Lemma 5.3.11. *For every $y, w \in H^1_{periodic}[-\pi,\pi]$,*

$$|\langle w\frac{\partial y}{\partial x}, w\rangle| \leq 2\ \|y\|_{\infty}\ \|w\|\ \|\frac{\partial w}{\partial x}\|. \tag{5.37}$$

Proof. Let $y, w \in H^1_{periodic}[-\pi,\pi]$, then using integration by parts and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
|\langle w\frac{\partial y}{\partial x}, w\rangle| &= \left|\int_{-\pi}^{\pi} w\frac{\partial y}{\partial x}\overline{w}dx\right|,\\
&= \left|\int_{-\pi}^{\pi} y\left(w\frac{\partial \overline{w}}{\partial x}+\overline{w}\frac{\partial w}{\partial x}\right)dx\right|,\\
&\leq 2\ \|y\|_{\infty}\left|\langle w, \frac{\partial w}{\partial x}\rangle\right|,\\
&\leq 2\ \|y\|_{\infty}\ \|w\|\ \|\frac{\partial w}{\partial x}\|. \qquad \square
\end{aligned}$$

Lemma 5.3.12. *Define the nonlinear operator $G : H^2_{periodic}[-\pi,\pi] \to L^2[-\pi,\pi]$ by*

$$G(z) = Rz + J(z) - BKz, \tag{5.38}$$

where the operators R, J, B, K are defined in (5.21), (5.5), (5.6), (5.2), respectively. For each $z, y \in H^2_{periodic}[-\pi,\pi]$,

$$G(z) - G(y) = \frac{\partial}{\partial x}(y(z-y)) + (R - BK)(z-y) + J(z-y). \tag{5.39}$$

Furthermore,

$$\|J(z(t) - y(t))\| \leq \frac{c}{2}\ \|z-y\|_{H^1}, \tag{5.40}$$

where $c > 0$ is as in Proposition (2.2.12).

Proof. Let $z, y \in H^2_{periodic}[-\pi,\pi]$. Define $w = z - y$. Since the operator BK is a linear bounded operator, the there exists $M > 0$ such that $\|BK\| \le M$. Now, using the definitions of G,J in (5.38) and (5.5), respectively. We obtain

$$\begin{aligned} G(z) - G(y) &= w\frac{\partial y}{\partial x} + Rw - BKw + z\frac{\partial w}{\partial x}, \\ &= y\frac{\partial w}{\partial x} + w\frac{\partial y}{\partial x} + Rw - BKw + z\frac{\partial w}{\partial x} - y\frac{\partial w}{\partial x}, \\ &= y\frac{\partial w}{\partial x} + w\frac{\partial y}{\partial x} + Rw - BKw + w\frac{\partial w}{\partial x}, \end{aligned}$$

Next, use Poincaré inequality and Proposition (2.2.12) to obtain

$$\begin{aligned} \|J(z-y)\| &= \|(z-y)\left(\frac{\partial z}{\partial x} - \frac{\partial y}{\partial x}\right)\|, \\ &= \frac{1}{2}\|\frac{\partial}{\partial x}(z-y)^2\|, \\ &\le \frac{c}{2}\|z-y\|^2_{H^1}, \text{ (Proposition 2.2.12)} \end{aligned}$$

as was to be shown. □

The next result is the main theorem. It shows that the nonlinear C_0-semigroup corresponding to the open-loop controlled nonlinear KS equation (5.1) is Fréchet differentiable at every $z_0 \in L^2[-\pi,\pi]$ and the derivative is the linear C_0-semigroup corresponding to the linearized KS equation around $z := S_B(t)\,z_0$. The proof of the theorem is similar to Temam's approach to differentiability of semigroups [85, Section VI.8]. In this reference, it is shown that the nonlinear C_0-semigroup generated by the uncontrolled KS equation (5.1) with $b(x) = 0$ is Fréchet differentiable. However, he assumed that $\langle z\frac{\partial y}{\partial x}, z\rangle = 0$. In the next theorem, this assumption is not needed.

Theorem 5.3.13. *Consider the controlled KS equation (5.7). The nonlinear semigroup $S_B(t)$ is Fréchet differentiable at every $z_0 \in L^2[-\pi,\pi]$.*

Sketch of the Proof. The proof of this theorem is lengthy and complicated. Thus, below is a sketch of the proof to make it easier to follow.

1. (a) Bound on the integral $\int_0^t \|\frac{\partial^2 w}{\partial x^2}\|^2 ds$, where $w = z - y$, z, y are solutions to the KS equation (5.22) corresponding to different initial conditions z_0, y_0, respectively with $\|z_0 - y_0\| \le \varepsilon$ for some $\varepsilon > 0$.

   (b) Choose $r = (\|y_0\| + \varepsilon)\max\{1, e^{\left(\frac{1}{2\nu}+M\right)T}\} > 0$ and show that $\|z\|_\infty \le r$ and $\|y\|_\infty \le r$.

2. Linearize the KS equation around y, where y is a solution to the KS equation with initial condition y_0 for some $y_0 \in L^2[-\pi, \pi]$.

3. Show that the nonlinear C_0-semigroup $S_B(t)$ is Fréchet differentiable at y_0.

Proof of Theorem 5.3.13.

1. Consider the nonlinear controlled KS equation given by (5.7) with different initial conditions $z_0, y_0 \in L^2[-\pi, \pi]$

$$\begin{aligned} \dot{z}(t) &= -\hat{A}z(t) - G(z(t)), \quad z(0) = z_0, \\ \dot{y}(t) &= -\hat{A}y(t) - G(y(t)), \quad y(0) = y_0, \end{aligned} \tag{5.41}$$

where the operators $\hat{A}$ and G are given in (5.20) and (5.38), respectively.

Since the operator BK is bounded, then there exists $M > 0$ such that $\|BKz\| \leq M\|z\|$ for every $z \in L^2(-\pi, \pi)$. Using the same approach in Lemma 5.3.4. It can be shown that the L^2-norm of the solution $\|z(t)\| \leq e^{\left(\frac{1}{2\nu}+M\right)T}\|z_0\|$, where $\nu > 0$ and $t \in [0, T]$. Suppose $\|z_0 - y_0\| \leq \varepsilon$ for some $\varepsilon > 0$. Choose $r = (\|y_0\| + \varepsilon)\, e^{\left(\frac{1}{2\nu}+M\right)T}$. Then

$$\begin{aligned} \|y_0\| &\leq r - \varepsilon, \\ \sup_{t\in[0,T]} \|y(t)\| &\leq r, \\ \sup_{t\in[0,T]} \|z(t)\| &\leq \sup_{t\in[0,T]} e^{\left(\frac{1}{2\nu}+M\right)t}\left(\|z_0 - y_0\| + \|y_0\|\right), \\ &\leq r. \end{aligned}$$

Note that r does not depend on z_0. Subtracting the above two equations and letting $w(t) = z(t) - y(t)$,

$$\begin{aligned} &\dot{w}(t) + \hat{A}w(t) = -\left(G(z(t)) - G(y(t))\right), \\ &w(0) = z_0 - y_0 =: w_0. \end{aligned} \tag{5.42}$$

Moreover, it was shown in Lemma (5.3.12) that

$$G(z(t)) - G(y(t)) \;=\; Rw(t) + z(t)\frac{\partial w}{\partial x}(t) + w(t)\frac{\partial y}{\partial x}(t) - BKw(t). \tag{5.43}$$

Take the inner product of (5.42) with w to obtain

$$\langle \dot{w}(t), w(t)\rangle + \langle \hat{A}w(t), w(t)\rangle = -\langle (G(z(t)) - G(y(t))), w(t)\rangle.$$

That is,

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2+\nu\|\frac{\partial^2 w}{\partial x^2}(t)\|^2 \begin{array}{l} = -\langle (G(z(t))-G(y(t))),w(t)\rangle, \\ \leq |\langle (G(z(t))-G(y(t))),w(t)\rangle|. \end{array} \tag{5.44}$$

Using (5.43), Triangle inequality, Cauchy-Schwarz inequality, Lemma (5.3.10), Lemma (5.3.11), the Poincaré inequality and choosing $M \geq \|BK\|$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w(t)\|^2+\nu\|\frac{\partial^2 w}{\partial x^2}(t)\|^2 &\leq |\langle Rw(t),w(t)\rangle|+|\langle z(t)\frac{\partial w}{\partial x}(t),w(t)\rangle| \\
&\quad +|\langle w(t)\frac{\partial y}{\partial x}(t),w(t)\rangle|+|\langle BKw(t),w(t)\rangle|, \\
&\leq \|\frac{\partial^2 w}{\partial x^2}(t)\|\,\|w(t)\|+\|z(t)\|_\infty\|w(t)\|\,\|\frac{\partial w}{\partial x}(t)\| \\
&\quad +2\|y(t)\|_\infty\|w(t)\|\,\|\frac{\partial w}{\partial x}(t)\|+M\|w(t)\|^2, \\
&\leq (1+3r)\ \|w(t)\|\,\|\frac{\partial^2 w}{\partial x^2}(t)\|+M\|w(t)\|^2, \\
&= K_r\,\|w(t)\|\,\|\frac{\partial^2 w}{\partial x^2}(t)\|+M\|w(t)\|^2, \qquad (5.45)
\end{aligned}$$

where $K_r = 1+3r$.

Using Young's inequality from Proposition (2.2.10),

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2+\nu\|\frac{\partial^2 w}{\partial x^2}(t)\|^2 \leq \frac{\nu}{2}\|\frac{\partial^2 w}{\partial x^2}(t)\|^2+\left(\frac{K_r^2}{2\nu}+M\right)\|w(t)\|^2. \tag{5.46}$$

Multiplying the above inequality by 2 and re-arranging the terms, we obtain

$$\frac{d}{dt}\|w(t)\|^2+\nu\|\frac{\partial^2 w}{\partial x^2}(t)\|^2 \leq 2C_r\|w(t)\|^2, \tag{5.47}$$

where $C_r = \frac{K_r^2}{\nu}+M$ and so

$$\frac{d}{dt}\|w(t)\|^2 \leq 2C_r\|w(t)\|^2. \tag{5.48}$$

This implies that

$$\|w(t)\|^2 \leq \|w_0\|^2 e^{2C_r t}, \quad t \geq 0, \tag{5.49}$$

and so

$$\int_0^t \|w\,(s)\,\|^4 ds \le \frac{1}{4C_r}\|w_0\|^4 e^{4C_r t}, \quad t \ge 0. \tag{5.50}$$

Combine inequalities (5.48) and (5.49) to obtain

$$\frac{d}{dt}\|w\,(t)\,\|^2 \le 2C_r\|w_0\|^2 e^{2C_r t}.$$

Square the above inequality and integrate with respect to t to obtain

$$\begin{aligned}\int_0^t \left(\frac{d}{ds}\|w\,(s)\,\|^2\right)^2 ds \;&\le\; 4C_r^2\|w_0\|^4\int_0^t e^{4C_r s}ds,\\ &\le\; C_r\|w_0\|^4 e^{4C_r t}.\end{aligned} \tag{5.51}$$

Combine inequalities (5.47) and (5.49),

$$\frac{d}{dt}\|w\,(t)\,\|^2 + \nu\|\frac{\partial^2 w}{\partial x^2}\,(t)\,\|^2 \le 2C_r\|w_0\|^2 e^{2C_r t}. \tag{5.52}$$

Integrate (5.52) with respect to t to obtain

$$\|w\,(t)\,\|^2 - \|w_0\|^2 + \nu\int_0^t \|\frac{\partial^2 w}{\partial x^2}\,(s)\,\|^2 ds \le \|w_0\|^2 e^{2C_r t} - \|w_0\|^2,$$

which implies

$$\int_0^t \|\frac{\partial^2 w}{\partial x^2}\,(s)\,\|^2 ds \;\le\; \frac{1}{\nu}\|w_0\|^2 e^{2C_r t}. \tag{5.53}$$

Now, square inequality (5.52) and expand the perfect square on the left hand side to obtain

$$\left(\frac{d}{dt}\|w\,(t)\,\|^2\right)^2 + 2\nu\|\frac{\partial^2 w}{\partial x^2}\,(t)\,\|^2\cdot\frac{d}{dt}\|w\,(t)\,\|^2 + \nu^2\|\frac{\partial^2 w}{\partial x^2}\,(t)\,\|^4 \le 4C_r^2\|w_0\|^4 e^{4C_r t},$$

Re-arrange the terms and use Young's inequality ($|2a\cdot b| \le 2a^2 + \frac{1}{2}b^2$) to obtain

$$\nu^2\|\frac{\partial^2 w}{\partial x^2}\,(t)\,\|^4 \le 4C_r^2\|w_0\|^4 e^{4C_r t} - \left(\frac{d}{dt}\|w\,(t)\,\|^2\right)^2 - 2\nu\|\frac{\partial^2 w}{\partial x^2}\,(t)\,\|^2\cdot\frac{d}{dt}\|w\,(t)\,\|^2.$$

That is,

$$
\begin{aligned}
\nu^2 \| \frac{\partial^2 w}{\partial x^2}(t)\|^4 &\leq 4C_r^2 \|w_0\|^4 e^{4C_r t} + \left(\frac{d}{dt}\|w(t)\|^2\right)^2 + 2\nu \|\frac{\partial^2 w}{\partial x^2}(t)\|^2 \cdot \left|\frac{d}{dt}\|w(t)\|^2\right|, \\
&\leq 4C_r^2 \|w_0\|^4 e^{4C_r t} + \left(\frac{d}{dt}\|w(t)\|^2\right)^2 + 2\left(\frac{d}{dt}\|w(t)\|^2\right)^2 \\
&\quad + \frac{\nu^2}{2}\|\frac{\partial^2 w}{\partial x^2}(t)\|^4.
\end{aligned}
$$

Re-arrange the terms to obtain

$$
\frac{\nu^2}{2}\|\frac{\partial^2 w}{\partial x^2}(t)\|^4 \leq 4C_r^2 \|w_0\|^4 e^{4C_r t} + 3\left(\frac{d}{dt}\|w(t)\|^2\right)^2.
$$

Finally, integrate with respect to t and use inequality (5.51) to obtain

$$
\begin{aligned}
\frac{\nu^2}{2}\int_0^t \|\frac{\partial^2 w}{\partial x^2}(s)\|^4 ds &\leq C_r \|w_0\|^4 e^{4C_r t} + 3C_r \|w_0\|^4 e^{4C_r t}, \\
&= 4C_r \|w_0\|^4 e^{4C_r t}.
\end{aligned}
$$

Hence,

$$
\int_0^t \|\frac{\partial^2 w}{\partial x^2}(s)\|^4 ds \leq \frac{8C_r}{\nu^2}\|w_0\|^4 e^{4C_r t}. \tag{5.54}
$$

2. Next, use the Gâteaux derivative (5.10) to linearize the KS equation (5.42) around $y = S_B(t)\, y_0$

$$
\begin{aligned}
\dot{\overline{w}}(t) &= -\hat{A}\overline{w}(t) - R\overline{w}(t) - \tfrac{\partial}{\partial x}\left(y(t)\,\overline{w}(t)\right) + BK\overline{w}(t), \\
\overline{w}(0) &= w_0 := z_0 - y_0.
\end{aligned} \tag{5.55}
$$

Using [85, Theorem II.3.4] and [29, Lemma 3.1.5], the controlled linearized KS equation (5.55) has a unique strong solution

$$
\overline{w}(t) \in L^2\left(0, T; H^2_{periodic}[-\pi,\pi]\right) \cap L^\infty\left(0, T; L^2[-\pi,\pi]\right), \quad \text{for } t \leq T < \infty.
$$

That is, the solution can be written as

$$
\overline{w}(t) = T_B(t)\, w_0, \tag{5.56}
$$

where $T_B(t)$ is a C_0-semigroup on $L^2[-\pi,\pi]$.

3. Now, we will show that the nonlinear C_0-semigroup $S_B(t)$ is Fréchet differentiable at y_0 and $T_B(t)$ is its Fréchet derivative. Set $\phi = w - \overline{w}$ and use equations (5.42) and (5.55) and Lemma 5.3.12 to obtain

$$\begin{aligned}
\dot{\phi}(t) &= \dot{w}(t) - \dot{\overline{w}}(t), \\
&= -\hat{A}(w(t) - \overline{w}(t)) - (G(z(t)) - G(y(t))) + R\overline{w}(t) + \frac{\partial}{\partial x}(y(t)\overline{w}(t)) \\
&\quad -BK\overline{w}(t), \\
&= -\hat{A}\phi(t) - R\phi(t) - \frac{\partial}{\partial x}(y(t)\phi(t)) + BK\phi(t) - F(w(t)), \qquad \text{(Using (5.39))} \\
\phi(0) &= 0.
\end{aligned}$$

That is,

$$\begin{aligned}
&\dot{\phi}(t) + \hat{A}\phi(t) = -R\phi(t) - \tfrac{\partial}{\partial x}(y(t)\phi(t)) + BK\phi(t) - F(w(t)), \\
&\phi(0) = 0.
\end{aligned} \tag{5.57}$$

Take the L^2-inner product of the above system (5.57) with ϕ to obtain

$$\begin{aligned}
\langle\dot{\phi}(t), \phi(t)\rangle + \langle\hat{A}\phi(t), \phi(t)\rangle &= -\langle R\phi(t) + \frac{\partial}{\partial x}(y(t)\phi(t)) - BK\phi(t), \phi(t)\rangle \\
&\quad -\langle F(w(t)), \phi(t)\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\phi(t)\|^2 + \nu\|\frac{\partial^2\phi}{\partial x^2}(t)\|^2 &= -\langle R\phi(t) + \frac{\partial}{\partial x}(y(t)\phi(t)) - BK\phi(t), \phi(t)\rangle \\
&\quad -\langle F(w(t)), \phi(t)\rangle, \\
&\le |\langle R\phi(t) + \frac{\partial}{\partial x}(y(t)\phi(t)) - BK\phi(t), \phi(t)\rangle| \\
&\quad +|\langle F(w(t)), \phi(t)\rangle|.
\end{aligned} \tag{5.58}$$

Moreover, using Cauchy-Schwarz inequality, Lemma (5.3.10), Lemma (5.3.11) and Poincaré inequality (2.2.11), we have

$$\begin{aligned}
|\langle R\phi(t) + \frac{\partial}{\partial x}(y(t)\phi(t)) - BK\phi(t), \phi(t)\rangle| &\le |\langle R\phi(t), \phi(t)\rangle| + |\langle BK\phi(t), \phi(t)\rangle| \\
&\quad +|\langle y(t)\frac{\partial\phi}{\partial x}(t), \phi(t)\rangle| + |\langle\phi(t)\frac{\partial y}{\partial x}(t), \phi(t)\rangle|.
\end{aligned}$$

That is,

$$
\begin{aligned}
|\langle R\phi(t) + \frac{\partial}{\partial x}(y(t)\,\phi(t)) - BK\phi(t), \phi(t)\rangle| \leq M\|\phi(t)\|^2 + 3\|y(t)\|_\infty\|\phi(t)\| \cdot \\
\|\frac{\partial\phi}{\partial x}(t)\| + \|\phi(t)\|\;\|\frac{\partial^2\phi}{\partial x^2}(t)\|, \\
&\leq M\|\phi(t)\|^2 + (1 + 3\|y(t)\|_\infty)\|\phi(t)\|\,\|\frac{\partial^2\phi}{\partial x^2}(t)\|, \\
&\leq M\|\phi\|^2 + (1+3r)\|\phi(t)\|\,\|\frac{\partial^2\phi}{\partial x^2}(t)\|, \\
&= M\|\phi(t)\|^2 + K_r\,\|\phi(t)\|\;\|\frac{\partial^2\phi}{\partial x^2}(t)\|,
\end{aligned}
$$

where $K_r = 1 + 3r$.

Using the above result, Cauchy-Schwarz inequality, Young's inequality and Lemma (5.3.12), inequality (5.58) becomes

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\phi(t)\|^2 + \nu\|\frac{\partial^2\phi}{\partial x^2}(t)\|^2 &\leq K_r\,\|\phi(t)\|\;\|\frac{\partial^2\phi}{\partial x^2}(t)\| + M\|\phi(t)\|^2 + \|J(w(t))\| \cdot \\
&\quad \|\phi(t)\|, \\
&\leq \frac{\nu}{2}\|\frac{\partial^2\phi}{\partial x^2}(t)\|^2 + \left(\frac{K_r^2}{2\nu} + M\right)\|\phi(t)\|^2 + \frac{\nu}{2}\|J(w(t))\|^2 \\
&\quad + \frac{1}{2\nu}\|\phi(t)\|^2,
\end{aligned}
$$

and so

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\phi(t)\|^2 + \nu\|\frac{\partial^2\phi}{\partial x^2}(t)\|^2 &\leq \frac{\nu}{2}\|\frac{\partial^2\phi}{\partial x^2}(t)\|^2 + \left(\frac{K_r^2+1}{2\nu} + M\right)\|\phi(t)\|^2 \\
&\quad + \frac{\nu c^2}{8}\left(\|w(t)\|^2 + \|\frac{\partial^2 w}{\partial x^2}(t)\|^2\right)^2, \\
&\leq \frac{\nu}{2}\|\frac{\partial^2\phi}{\partial x^2}(t)\|^2 + \left(\frac{K_r^2+1}{2\nu} + M\right)\|\phi(t)\|^2 \\
&\quad + \frac{\nu c^2}{4}\left(\|w(t)\|^4 + \|\frac{\partial^2 w}{\partial x^2}(t)\|^4\right). \qquad (5.59)
\end{aligned}
$$

This implies,

$$\frac{d}{dt}\|\phi(t)\|^2+\nu\|\frac{\partial^2\phi}{\partial x^2}(t)\|^2 \leq \left(\frac{K_r^2+1}{\nu}+2M\right)\|\phi(t)\|^2+\frac{\nu c^2}{2}\Big(\|w(t)\|^4 + \|\frac{\partial^2 w}{\partial x^2}(t)\|^4\Big),$$

and so

$$\frac{d}{dt}\|\phi(t)\|^2 \leq \left(\frac{K_r^2+1}{\nu}+2M\right)\|\phi(t)\|^2+\frac{\nu c^2}{2}\left(\|w(t)\|^4+\|\frac{\partial^2 w}{\partial x^2}(t)\|^4\right) \quad (5.60)$$

Integrating with respect to t and using $\phi(0)=0$, inequalities (5.50) and (5.54), we obtain

$$\|\phi(t)\|^2 \leq \left(\frac{K_r^2+1}{\nu}+2M\right)\int_0^t \|\phi(s)\|^2 ds + \tilde{M}\|w_0\|^4 e^{4C_r t}, \quad (5.61)$$

where $\tilde{M}=\frac{\nu c^2}{2}\left(\frac{8C_r}{\nu^2}+\frac{1}{4C_r}\right)$.

Using Gronwall's lemma (Proposition 2.2.9) and $\phi(0)=0$, we obtain

$$\|\phi(t)\|^2 \leq \bar{C}^2\|w_0\|^4, \quad (5.62)$$

where $\bar{C}^2=\tilde{M}e^{4C_rT}+\frac{\tilde{M}\nu}{4C_r\nu-K_r^2+1-2M\nu}e^{\frac{4C_r\nu-K_r^2-1-2M\nu}{\nu}T}$, which implies that

$$\|\phi(t)\| \leq \bar{C}\|w_0\|^2, \quad t\in[0,T].$$

Using the definitions of ϕ, w

$$\|\phi(t)\| = \|w(t)-\overline{w}(t)\| = \|z(t)-y(t)-\overline{w}(t)\| \leq \bar{C}\|w_0\|^2 = \bar{C}\|z_0-y_0\|^2, \quad z_0\neq y_0.$$

That is,

$$\frac{\|z(t)-y(t)-\overline{w}(t)\|}{\|z_0-y_0\|} \leq \bar{C}\|z_0-y_0\|, \quad (5.63)$$

or,

$$\frac{\|S_B(t)z_0-S_B(t)y_0-T_B(t)w_0\|}{\|w_0\|} \leq \bar{C}\|w_0\|. \quad (5.64)$$

where $z_0 = y_0 + w_0$. Inequality (5.64) holds for every $z_0 \in L^2[-\pi, \pi]$ with $\|z_0 - y_0\| \le \varepsilon$ with $\varepsilon > 0$. Take the limit as $\|w_0\| \to 0$ to obtain

$$\lim_{\|w_0\| \to 0} \frac{\|S_B(t)(y_0 + w_0) - S_B(t)\, y_0 - T_B(t)\, w_0\|}{\|w_0\|} = \lim_{\|w_0\| \to 0} \bar{C}\|w_0\| = 0. \tag{5.65}$$

Thus, the nonlinear C_0-semigroup generated by the controlled KS equation, $S_B(t)$, is Fréchet differentiable. Moreover, the Fréchet derivative is the C_0-semigroup generated by the linearized KS equation, $T_B(t)$. □

5.4 The uncontrolled Kuramoto-Sivashinsky equation

In this section, the uncontrolled KS equation is considered. That is, equation (5.1) with $u(t) = 0$. First, a conservation law property is shown to hold for the uncontrolled KS equation. Then, equilibrium solutions of the uncontrolled KS equation are investigated and stability analyzed for different values of the instability parameter ν.

One of the properties of the uncontrolled KS equation (5.1) with $u(t) = 0$ is that it possesses a conservation law. That is, the integral $\int_{-\pi}^{\pi} z(t)\, dx$ is invariant under time evolution. This is shown in the next theorem.

Theorem 5.4.1. *Consider the uncontrolled KS equation (5.1) with $u(t) = 0$ with periodic boundary conditions. The uncontrolled system possesses a conservation law property. That is,*

$$\int_{-\pi}^{\pi} z(t)\, dx = C,$$

where $C \in \mathbb{R}$.

Proof. We start by integrating the uncontrolled KS equation (5.1) with $u(t) = 0$ with respect to the spatial variable x from $-\pi$ to π.

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{\partial z}{\partial t} dx &= -\nu \int_{-\pi}^{\pi} \frac{\partial^4 z}{\partial x^4} dx - \int_{-\pi}^{\pi} \frac{\partial^2 z}{\partial x^2} dx - \int_{-\pi}^{\pi} z \frac{\partial z}{\partial x} dx, \\
\frac{d}{dt} \int_{-\pi}^{\pi} z(t)\, dx &= -\nu \frac{\partial^3 z}{\partial x^3}\bigg|_{-\pi}^{\pi} - \frac{\partial z}{\partial x}\bigg|_{-\pi}^{\pi} - \frac{1}{2} z^2 \bigg|_{-\pi}^{\pi}, \\
\frac{d}{dt} \int_{-\pi}^{\pi} z(t)\, dx &= 0.
\end{aligned}$$

That is,

$$\int_{-\pi}^{\pi} z\,(t)\,dx = C, \tag{5.66}$$

where C is an integration constant. This completes the proof. $\square$

Note that the constant C in the above theorem represents the volume per unit circumference.

The KS equation has an infinite number of equilibrium points. In particular, any constant function is an equilibrium solution to the KS equation. Moreover, let z_e be an equilibrium solution to the uncontrolled KS equation, then from the conservation law property (5.66), we have

$$\int_{-\pi}^{\pi} z\,(t)\,dx = \int_{-\pi}^{\pi} z_e dx = 2\pi z_e. \tag{5.67}$$

Hence, the equilibrium solution of the uncontrolled KS equation is determined by the initial condition z_0. That is

$$z_e = \frac{1}{2\pi}\int_{-\pi}^{\pi} z_0 dx. \tag{5.68}$$

Furthermore, there are an infinite number of other functions that are also equilibrium solutions [52, 60]. There is no explicit form known for these equilibria although they can be approximated using Fourier truncated series. In [52], Lan and Cvitanović used the Newton descent method to numerically determine that such equilibria are unstable. In this thesis, we will denote an equilibrium solution by z_e and will be interested in the set of constant equilibrium solutions.

Define the closed invariant set Z_e to be the set of such equilibrium solutions to the uncontrolled KS equation

$$Z_e = \{z_e :\ z_e \text{ is a constant function}\}. \tag{5.69}$$

Stability of the equilibrium solutions to the uncontrolled KS equation ((5.1) with $u\,(t) = 0$) depends on the value of the instability parameter ν. In this section, it will be shown that if $\nu > 1$, then the set of all equilibrium solutions Z_e defined in (5.70) is globally exponentially stable. This is done using a continuous Lyapunov function and LaSalle's Invariance Principle.

Theorem 5.4.2. *Consider the uncontrolled KS equation (5.1) with $u\,(t) = 0$. If the instability parameter $\nu > 1$, then the set of equilibrium solutions Z_e defined in (5.70) is globally asymptotically stable.*

Proof. Consider the following positive definite function $V : \mathcal{D}(A) \subset L^2[-\pi, \pi] \to \mathbb{R}$

$$V(z) = \frac{1}{2}\|z\|^2, \tag{5.70}$$

It can be easily shown that the above function (5.70) is a continuous Lyapunov function (3.1.8) to the uncontrolled KS equation. The Lyapunov derivative is given by

$$\begin{aligned}
\dot{V}(z) &= Re\langle z, \dot{z}\rangle, \\
&= Re\left\{-\nu\langle z, \frac{\partial^4 z}{\partial x^4}\rangle - \langle z, \frac{\partial^2 z}{\partial x^2}\rangle - \langle z, z\frac{\partial z}{\partial x}\rangle\right\}, \\
&= -\nu\langle \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x^2}\rangle + \langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial x}\rangle - 0, \text{ (Periodic boundary conditions)} \\
&= -\nu\|\frac{\partial^2 z}{\partial x^2}\|^2 + \|\frac{\partial z}{\partial x}\|^2, \\
&\leq -(\nu - 1)\|\frac{\partial z}{\partial x}\|^2. \text{ (Poincaré inequality)} \\
&\leq 0,
\end{aligned}$$

since the instability parameter $\nu > 1$. Next we shall investigate when $\dot{V}(z)$ vanishes. That is,

$$-(\nu - 1)\|\frac{\partial z}{\partial x}\|^2 = 0.$$

So we have

$$\frac{\partial z}{\partial x} = 0.$$

Which implies that

$$z = C.$$

Next, we shall use LaSalle's Invariance Principle (Theorem 3.1.12) to show that the solution of the uncontrolled KS equation converges to the set of equilibrium solutions Z_e as $t \to \infty$. Since the C_0-semigroup generated by the uncontrolled KS equation is compact [74, Theorem 54.3], then the orbit $\gamma(z)$ is precompact for every $z \in \mathcal{D}(A)$. Therefore, by LaSalle's Invariance Principle (Theorem 3.1.12), the solution of the KS equation converges to the invariant set that contains all equilibrium solution Z_e defined in (5.70) resulting a globally asymptotically stable equilibrium set. □

If the instability parameter $\nu = 1$, then the Lyapunov derivative $\dot{V}(z) = 0$ for every $z \in \mathcal{D}(A)$. Hence, by Lyapunov Theorem (Theorem 3.1.10), the zero solution $z = 0$ is stable. In other words, the KS equation has a global attractor which is not surprising as Sell & You showed this result in [74, Theorem 54.3]. However, from the above analysis, we know what this global attractor is. Stability or instability of the equilibria for the KS equation when the instability parameter $\nu < 1$ needs to be determined.

We shall look at the linearized KS equation at the constant equilibrium z_e. The goal is to linearize the KS equation around an equilibrium z_e and analyze the stability of the system for different values of the instability parameter ν. This is done by means of the Gâteaux derivative. This has been done earlier. The linearized uncontrolled KS equation around the equilibrium solution z_e is

$$\dot{z} = Az - z_e \frac{\partial z}{\partial x}, \tag{5.71}$$

where A is defined in (5.3).

In Theorem 5.2.1, it was shown that the operator $\left(A - z_e \frac{\partial}{\partial x}\right)$ is a Riesz-spectral operator that has the distinct eigenvalues $\lambda_n = -\nu n^4 + n^2 - i z_e n$, $n \in \mathbb{Z}$, and the corresponding eigenvectors $\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$.

Note that the real part of an eigenvalue of the operator $\left(A - z_e \frac{\partial}{\partial x}\right)$ crosses the imaginary axis when

$$\nu = \frac{1}{n^2}, \quad n = \pm 2, ..., \pm\infty$$

Furthermore, since the operator $\left(A - z_e \frac{\partial}{\partial x}\right)$ is a Riesz-spectral operator, then by [29, Theorem 2.3.5 c] the spectrum determined growth assumption (SDGA) holds with growth bound ω_0 is given by

$$\omega_0 = \sup_{n \in \mathbb{Z}} Re\{\lambda_n\}.$$

Therefore, from the above analysis of the linearized KS equation around a constant equilibrium solution z_e. If $\nu > 1$, then all eigenvalues have strictly negative real part, resulting a stable linearized system and if $\nu < 1$, then the linearized system is unstable. The number of unstable eigenvalues depends on the value of the instability parameter ν which is a finite number. For a given $0 < \nu < 1$, choose N to be the smallest integer such that

$$N > \sqrt{\frac{1}{\nu}}. \tag{5.72}$$

The constant N is the number of unstable modes for the uncontrolled linearized KS equation at the equilibrium solution z_e.

Theorem 5.4.3. *Consider the feedback controlled KS equation (5.1). If the instability parameter $\nu > 1$, then the equilibrium solution to the KS equation is locally exponentially stable. If the instability parameter $\nu < 1$, then the KS equation is unstable.*

Proof. The stability analysis of the linearized uncontrolled KS equation discussed in earlier in this section shows that if $\nu > 1$, then all eigenvalues have strictly negative real part, resulting an exponentially stable linearized system around an equilibrium solution z_e. Using Theorem 3.2.7, the equilibrium solution $z_e \in \mathbb{R}$ is locally exponentially stable. This is not surprising as it was shown in Theorem 5.4.2 that the set of all equilibrium solution z_e is globally exponentially stable when $\nu > 1$.

Moreover, if $\nu < 1$, then the linearized system is unstable. Using Theorem 3.2.8, the nonlinear KS equation is unstable near the equilibrium z_e when $\nu < 1$. □

Finally, if the linearized KS equation around an equilibrium solution z_e is stabilizable, that is, there exists an input-feedback control such that the closed loop system generated an exponentially stable C_0-semigroup, then the same input feedback control can be used to locally stabilize the nonlinear controlled KS equation. This will be discussed next.

5.5 A bounded state-feedback control to the Kuramoto-Sivashinsky equation

In this section, we shall find a bounded state-feedback control to the KS equation (5.1) when the instability parameter $\nu \leq 1$. The feedback control is of the form $u(t) = Kz(t)$, where the bounded linear operator $K : L^2[-\pi, \pi] \to \mathbb{C}$. This is done by finding a stabilizing control to the linearized KS equation at an equilibrium z_e. Although the theory below holds for all equilibrium solutions of the KS equation, for simplicity only constant equilibrium solutions will be considered.

Consider the linearized controlled KS equation at z_e. Let $\{\phi_n, \psi_n\}$, where $\phi_0 = \frac{1}{\sqrt{2\pi}}$, $\phi_n(\cdot) = \frac{1}{\sqrt{\pi}}\cos(n\cdot)$ and $\psi_n(\cdot) = \frac{1}{\sqrt{\pi}}\sin(n\cdot)$ for $n = 1, \cdots, \infty$ be a basis in $L^2[-\pi, \pi]$ and define

$$\begin{aligned} b1_n &= < b, \phi_n >, \text{ for } n = 0, 1, \cdots, \infty. \\ b2_n &= < b, \psi_n >, \text{ for } n = 1, 2, \cdots, \infty. \end{aligned} \tag{5.73}$$

Define the state-feedback control $u(t) = Kz(t)$, where $Kz(t) = < k, z(t) >$, for an appropriate choice of $k \in L^2[-\pi, \pi]$.The controlled linearized KS equation can be written as

$$\dot{z} = Az - z_e \frac{\partial z}{\partial x} + b(x) < k, z > . \tag{5.74}$$

Now using Theorem (3.2.7), the nonlinear operator $A+BK+J$ generates a locally asymptotically stable C_0-semigroup for the nonlinear open-loop controlled KS equation (5.1) if there exists $K \in \mathcal{L}\left(L^2(-\pi,\pi),\mathbb{C}\right)$ such that $A - z_e\frac{\partial}{\partial x} + BK$ generates an exponentially stable C_0-semigroup.

The next theorem summarizes this approach to designing an state-feedback control for the nonlinear KS equation (5.7).

Theorem 5.5.1. *Consider the linearized KS equation around the constant state $\hat{z}$ not necessarily an equilibrium solution (5.11). Let $N_{unstable}$ is the number of unstable eigenvalues of the operator $\left(A - \hat{z}\frac{\partial}{\partial x}\right)$. Then there exists a finite-dimensional controller that locally stabilizes the unstable nonlinear KS equation (5.7) if*

$$\begin{aligned} &b1_n \neq 0 \text{ for } n = 0, 1, \cdots, N_{unstable}. \\ &b2_n \neq 0 \text{ for } n = 1, 2, \cdots, N_{unstable}. \end{aligned} \tag{5.75}$$

Proof. The linear operator $A - \hat{z}\frac{\partial}{\partial x}$ is a Reisz-spectral operator (Theorem (5.2.1)). Using [29, Theorem 5.2.10], $\left(\left(A - \hat{z}\frac{\partial}{\partial x}\right), B\right)$ is stabilizable if assumption (5.75) holds. Moreover, using [29, Theorem 5.2.6] a finite-dimensional stabilizing controller can be chosen to stabilize the infinite-dimensional system.

The C_0-semigroup generated by the nonlinear open-loop controlled KS equation is Fréchet differentiable (Theorem 5.3.13). This implies by using Theorem 3.2.7 that if the linearized KS equation at an equilibrium solution z_e generates an exponentially stable C_0-semigroup, then the controlled nonlinear KS equation with the same control generates a locally asymptotically stable C_0-semigroup. In other word, the equilibrium solution $\bar{z}$ to the controlled nonlinear KS equation is locally exponentially stable. □

The same approach can be used to show that there exists a controller that stabilizes the KS equation from one constant state to another. That is, if one desires the solution of the KS equation to converge to another constant equilibrium say $\hat{z}_e$. A state-feedback control can be found by stabilizing the linearized KS equation at $\hat{z}_e$. This is justified because the nonlinear C_0-semigroup corresponding to the nonlinear KS equation is Fréchet differentiable at any constant $\hat{z}_e \in L^2[-\pi,\pi]$ (see Theorem 5.3.13).

There are many ways to design a state-feedback controller that stabilizes linear infinite-dimensional systems [10, 12, 29, 53, 58, 63]. One approach is to design a bounded feedback controller that exponentially stabilizes the system

$$\begin{aligned} &\tfrac{dz}{dt}(t) = \tilde{A}z(t) + Bu(t), \\ &y(t) = Cz(t), \end{aligned}$$

where $\tilde{A}$ is a linear infinitesimal generator of a C_0-semigroup defined on a Hilbert space

H, the operators B and C are linear bounded operators defined on a Hilbert space H, is by designing a linear quadratic controller [29, 63, 93]. The idea is to minimize the cost functional corresponding to the system

$$\min J(z_0, u) = \int_0^\infty \left(\langle y(s), y(s) \rangle + \langle u(s), Ry(s) \rangle \right) ds,$$

where the linear operator R is a self-adjoint, coercive operator defined on the Hilbert space $\mathcal{U}$. Such optimal bounded input-feedback is of the form

$$u(t, z_0) = -R^{-1} B^* \Pi z(t),$$

where the linear operator $\Pi : H \to H$ is the solution of the algebraic Riccati equation (ARE)

$$\langle \tilde{A} z_1, \Pi z_2 \rangle + \langle \Pi z_1, \tilde{A} z_2 \rangle + \langle C z_1, C z_2 \rangle + \langle B^* \Pi z_1, R^{-1} B^* \Pi z_2 \rangle = 0, \text{ for } z_1, z_2 \in \mathcal{D}\left(\tilde{A}\right).$$

Another approach is by designing an H_∞ controller where the effect of the disturbance on the cost is considered instead of the initial condition. For more details see [63, Section 5]

Chapter 6

Galerkin Projection method to approximate infinite-dimensional systems

It is important to find a numerical approximation to the solution of nonlinear infinite-dimensional systems. This is because it is difficult and sometimes impossible to explicitly solve infinite-dimensional dynamical systems. In this thesis, the approximated solution to the nonlinear KS equation as well as the state-feedback controlled KS equation will be used to illustrate the theoretical results on the stability and the effect of the state-feedback control on the KS equation.

There are many choices for the numerical solution of partial differential equations (PDEs). For instance, the finite difference (FD) method or the finite element (FE) method. The FD method is the oldest and is based upon applications of a local Taylor series expansion to approximate differential equations [67]. In [42], Ide and Okada proposed an implicit FD scheme that preserves the energy properties such as energy dissipation, energy conservation, etc. of the PDE system. The method they proposed is called the discrete variational derivative method and is designed to compute and approximate solution for nonlinear PDEs with variable coefficients. Furthermore, they proved the stability, existence, uniqueness and convergence rate of $O\left(\Delta x^2 + \Delta t^2\right)$ of the solution.

In FD method a square grid of lines is used to construct the discretization of the PDE which will become a problem handling complex geometries in multiple dimensions. This issue motivates the use of an integral form of the PDE and hence development of the FE method [67].

Another method is also known as the Karhunen-Loève decomposition or the proper orthogonal decomposition. It is widely used in identifying and analyzing coherent structure in turbulent fluids, or for determining low-order models for complex dynamical systems [54, 71]. The idea of this method is to generate a basis of a finite-dimension for a given set of snapshots of the original system that spans the data optimally in L^2-sense. That is, a reliable solver is used to compute a priori ensemble solutions to the model which are called snapshots, then these snapshots are used to produce an optimal representation of the model. The main challenge is to find a good set of snapshots for the original system that captures the energy of the system.

In this thesis, we will use the Galerkin method to approximate infinite-dimensional systems (3.1). In particular, the uncontrolled and the state-feedback controlled KS equation. The Galerkin method is used to approximate the infinite-dimensional KS equation in terms of a finite-dimensional one [13, 31, 32]. The idea is to derive a system of ordinary differential equations (ODE's) that mimics the dynamics of the nonlinear uncontrolled/state-feedback controlled KS equation. More details on this are in the next section.

6.1 The Galerkin Projection method

The Galerkin Projection method provides a discrete algorithm to approximate the solution of infinite-dimensional PDEs. The idea is to produce a finite-dimensional ODE system that approximates the dynamics of the original PDE.

Consider the open-loop controlled infinite-dimensional system defined on a Hilbert space H

$$\begin{aligned} &\tfrac{dz}{dt} = Az(t) + Bu(t), \\ &z(0) = z_0, \end{aligned} \tag{6.1}$$

where the linear operator $A : \mathcal{D}(A) \subset H \to H$ is an infinitesimal generator of a C_0-semigroup $T(t)$ on H, the actuator $B : \mathcal{U} \to H$ is a linear bounded operator, $u \in \mathcal{U}$ is a controller which will be assumed to be finite ($u \in \mathbb{C}$), and $z_0 \in H$ is the initial condition. The system is well-posed [29, Definition 3.1.4]. That is, there exists a unique mild solution

$$z(t) = T(t) z_0 + \int_0^\infty T(t-s) Bu(s)\, ds.$$

Let $H_N \subset H$ be a finite subspace of the Hilbert space H equipped with the same norm of H. Define an orthogonal projection $P : H \to H_N$. This will be used to approximate the original system (6.1). The natural assumption for this approximation scheme is

$$\lim_{N\to\infty} \|Pz - z\| = 0, \quad \text{for every } z \in H.$$

Define $B_N = PB$ and $A_N : H_N \to H_N$ using some method. Then the infinite-dimensional system (6.1) is approximated by

$$\frac{dz}{dt} = A_N z(t) + B_N u(t), \qquad z(0) = Pz_0, \tag{6.2}$$

where the operator A_N generates a C_0-semigroup (matrix exponential), $T_N(t)$ on H_N.

The solution of the uncontrolled nonlinear KS equation ((5.7) with $u(t) = 0$) can be expanded using Fourier series expansion [64, Theorem 5.17.8]. Let $\phi_0 = \frac{1}{\sqrt{2\pi}}$, $\phi_n(\cdot) = \frac{1}{\sqrt{\pi}}\cos(n\cdot)$ and $\psi_n(\cdot) = \frac{1}{\sqrt{\pi}}\sin(n\cdot)$ for $n = 1, \cdots, \infty$, then $\{\phi_0, \phi_n, \psi_n\}_{n=1}^{\infty}$ form an orthonormal basis in the Hilbert space $L^2(-\pi, \pi)$. The solution of the nonlinear KS equation can be written as

$$z(x,t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(x) + \sum_{n=1}^{\infty} c_n(t)\psi_n(x). \tag{6.3}$$

That is,

$$z(x,t) \approx \sum_{n=0}^{N} a_n(t)\phi_n(x) + \sum_{n=1}^{N} c_n(t)\psi_n(x),$$

for sufficiently large N. Substituting the above approximated solution in the uncontrolled KS equation implies

$$\sum_{n=0}^{N} \dot{a}_n(t)\phi_n(x) + \sum_{n=1}^{N} \dot{c}_n(t)\psi_n(x) = \sum_{n=1}^{N}\left(-\nu n^4 + n^2\right)\left(a_n(t)\phi_n(x) + c_n(t)\psi_n(x)\right) + \alpha(x,t),$$

where

$$\alpha(x,t) = \left(\sum_{n=0}^{N} a_n(t)\phi_n(x) + \sum_{n=1}^{N} c_n(t)\psi_n(x)\right)\cdot\left(\sum_{n=1}^{N}\left(a_n(t)\phi_n'(x) + c_n\psi_n'(x)\right)\right). \tag{6.4}$$

That is,

$$\begin{aligned}
\dot{a}_n(t) &= (-\nu n^4 + n^2)a_n(t) + <\alpha(t), \phi_n>,\\
\dot{c}_n(t) &= (-\nu n^4 + n^2)c_n(t) + <\alpha(t), \psi_n>,\\
a_n(0) &= <z_0, \phi_n>,\\
c_n(0) &= <z_0, \psi_n>.
\end{aligned} \tag{6.5}$$

Using Runge-Kutta method to solve the above ODE system numerically from MATLAB, one can obtain an approximated solution to the uncontrolled nonlinear KS equation ((5.7) with $u(t) = 0$).

Let z_e be a constant equilibrium solution to the KS equation and let $N_{unstable}$ be the number of unstable modes of the linearized KS equation around z_e which is given in (5.72). Define the finite subspace of $Z^+ \subset L^2(-\pi,\pi)$ spanned by the eigenfunctions corresponding to the unstable eigenvalues of the linearized KS equation and define the orthogonal projection operator of $L^2(-\pi,\pi)$ onto Z^+, $P : L^2(-\pi,\pi) \to Z^+$,

$$Pz = \sum_{n=0}^{N_{unstable}} a_n \phi_n + \sum_{n=1}^{N_{unstable}} c_n \psi_n, \tag{6.6}$$

where $a_n =< z, \phi_n >$, $c_n =< z, \psi_n >$. The above projection induces the following approximated linearized system around a constant equilibrium solution z_e defined on Z^+

$$P\frac{dz}{dt} = P\left(Az - z_e\frac{dz}{dx}\right) + PBu(t), \tag{6.7}$$

where A, B are in (5.3), (5.6), respectively.

6.2 Numerical simulation

In this section, we will consider the uncontrolled nonlinear KS equation with instability parameter $\nu = \frac{1}{2}$

$$\frac{\partial z}{\partial t} = -\frac{1}{2}\frac{\partial^4 z}{\partial x^4} - \frac{\partial^2 z}{\partial x^2} - z\frac{\partial z}{\partial x}, \ x \in [-\pi,\pi], \ t \geq 0, \tag{6.8}$$

with periodic boundary conditions,

$$\frac{\partial^n z}{\partial x^n}(-\pi,t) = \frac{\partial^n z}{\partial x^n}(\pi,t), \quad n = 0,1,2,3$$

and initial condition

$$z_0(x) = \frac{1}{2}\cos\left(\frac{x}{10}\right)\cdot\left(1+\sin\left(\frac{x}{10}\right)\right). \tag{6.9}$$

Using the Galerkin method discussed earlier to approximate the solution to the uncontrolled KS equation, we use 9 eigenfunctions to approximate the solution

$$z(x,t) = \frac{1}{\sqrt{2\pi}}a_0(t) + \sum_{n=1}^{4}\frac{1}{\sqrt{\pi}}\left(a_n(t)\cos(nx) + c_n(t)\sin(nx)\right),$$

where $a_n(t), c_n(t)$ is the solution of the ODE system in Appendix A. Note that the KS equation model is real-valued and hence all computations are carried out in the real sense. Figure 6.1 is a 3-D landscape of the approximated solution. Clearly, the system is unstable.

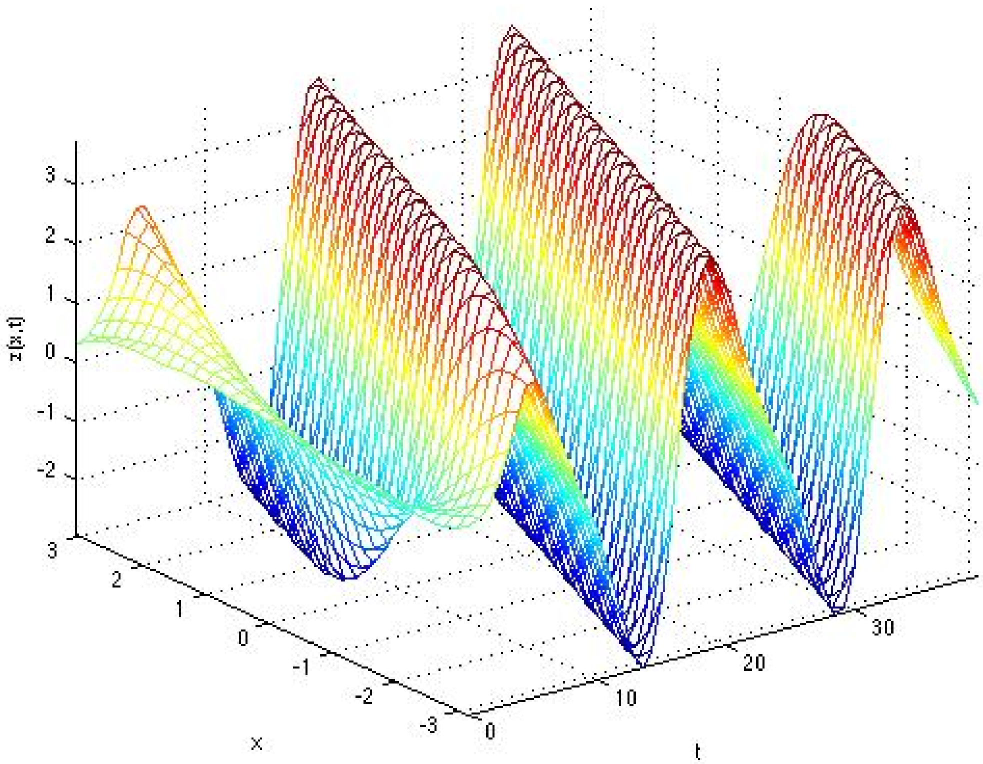

Figure 6.1: A 3-D landscape of the approximated uncontrolled solution of the nonlinear KS equation when $\nu = \frac{1}{2}$ with the initial condition given in (6.9).

In order to find the number of unstable eigenfunctions for the nonlinear KS equation with $\nu = \frac{1}{2}$, the result (5.72) is used. That is, the number of unstable eigenfunctions in the above KS equation is $2N_{unstable} + 1$, where $N_{unstable}$ is smallest integer such that

$$N_{unstable} \geq \sqrt{2}.$$

That is, $N_{unstable} = 2$ and hence, the number of unstable eigenvalues in the above KS equation (6.8) is equal to 5.

Next, we will consider controlling the KS equation to an equilibrium solution z_e. That is, we consider the following state-feedback controlled KS equation

$$\frac{\partial z}{\partial t} = -\frac{1}{2}\frac{\partial^4 z}{\partial x^4} - \frac{\partial^2 z}{\partial x^2} - z\frac{\partial z}{\partial x} + b(x)\,u(t)\,,\ x \in [-\pi, \pi],\ t \geq 0, \tag{6.10}$$

with actuator

$$b(x) = \frac{1}{0.3} \cdot I_{[\varepsilon-0.1,\varepsilon+0.2]}, \tag{6.11}$$

where $\varepsilon \in \mathbb{R}$ and

$$I_{[\varepsilon-0.1,\varepsilon+0.2]} = \begin{cases} 1, & \text{if } x \in [\varepsilon - 0.1, \varepsilon + 0.2] \\ 0, & \text{otherwise} \end{cases}$$

and $u(t) = -Kz(t)$, the operator $K : L^2[-\pi, \pi] \to \mathbb{R}$ is a linear bounded operator.

Note that condition (5.75) is satisfied. That is, the L^2-inner product $< b(x), \phi_n >\neq 0$ for $n = 0, 1, \cdots, 4$ and $< b, \psi_n >\neq 0$ for $n = 1, \cdots, 4$. Hence the linearized system is stabilizable. First, we shall consider the equilibrium solution $z_e = 0$ and $\varepsilon = 0$. The goal is to design a state-feedback control such that the solution of the KS equation converges to the equilibrium solution $z_e = 0$.

A state-feedback control can be designed to stabilize the KS equation using a number of approaches. A linear-quadratic approach was chosen. That is, a state-feedback control law $u(t) = -Ka(t)$ is designed to minimize the quadratic cost function

$$J(u) = \int_0^\infty \left(a^T a + u^T u\right) dt$$

subject to (6.7). This is done using LQR in MATLAB and a feedback control that stabilizes the linearized KS equation around $z_e = 0$

$$u(t) = -K \cdot \begin{pmatrix} a_0(t) \\ a_1(t) \\ a_2(t) \\ c_1(t) \\ c_2(t) \end{pmatrix}, \tag{6.12}$$

where $a_0(t), a_1(t), a_2(t), c_1(t), c_2(t)$ are the solutions of the ODE system given in Appendix B and

$$K = [-1.0000 \quad 4.7373 \quad 0.0486 \quad 35.6283 \quad 0.0054].$$

Figure 6.2 is a 3-D landscape of the controlled nonlinear KS equation to the equilibrium $z_e = 0$. The graph indicates that the proposed scheme of stabilizing the nonlinear KS equation worked in forcing the solution to converge to the desired equilibrium solution

$z_e = 0$.

Note that the number of unstable eigenfunctions is 5 and the state-feedback control is designed to stabilize these eigenfunctions only. It is worth mentioning that such finite-dimensional feedback controller is enough and using [63, Theorem 3.5], the approximated solution of the controlled KS equations converges to the exact solution of the KS equation. That is, the finite-dimensional controller K converges as the number of eigenfunctions chosen to approximate the solution increases. Note that the number of eigenfunctions used in the simulations is equal to 9 which is larger than the number of eigenfunctions used to design the controller, yet the feedback controller achieved the stabilizing goal and there was no spillover

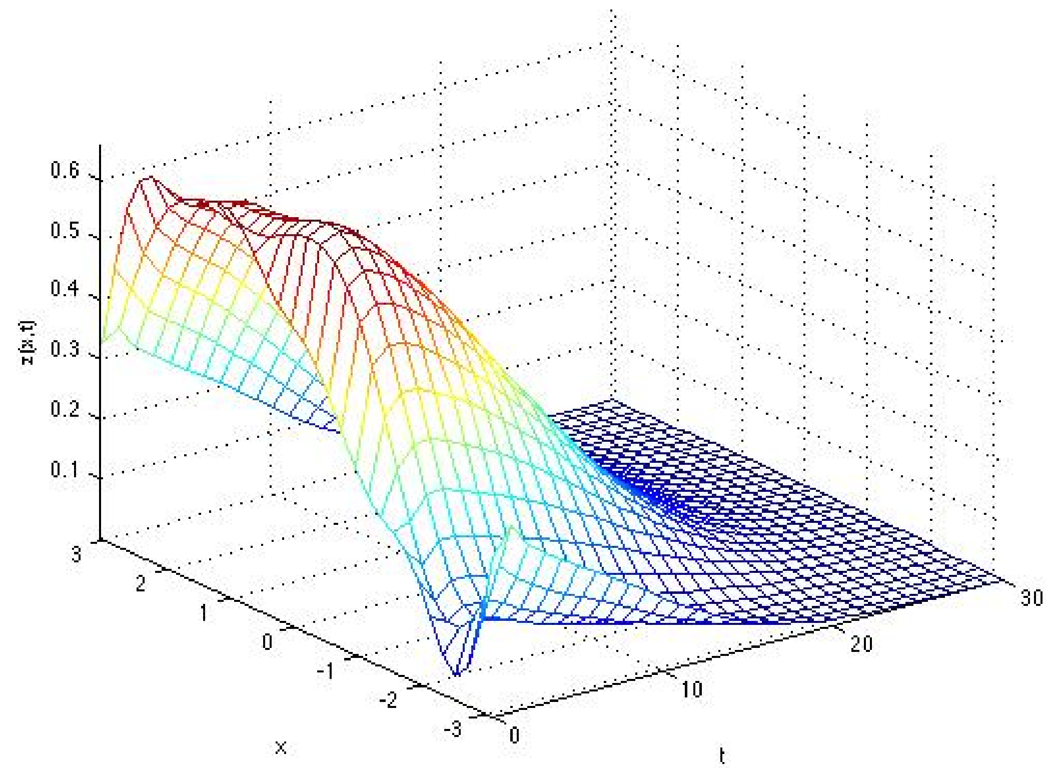

Figure 6.2: A 3-D landscape of the approximated controlled solution of the KS equation when $\nu = \frac{1}{2}$ and $z_e = 0$ with the initial condition given in (6.9) and actuator given by (6.11).

Next, we consider another equilibrium solution $z_e = 1$ and $\varepsilon = 1$. As above, a state-feedback control is obtained to exponentially stabilize the linearized KS equation around the equilibrium solution $z_e = 1$. This is again done by designing a state-feedback control law $u(t) = -Ka(t)$ that minimizes the quadratic cost function

$$J(u) = \int_0^\infty \left(a^T a + u^T u\right) dt$$

subject to (6.7). The feedback control (6.12) stabilizes the linearized KS equation around $z_e = 1$, where

$$K = [1.0000 \quad 5.2073 \quad -0.0121 \quad 1.5929 \quad 0.0397]. \tag{6.13}$$

Figure 6.3 is a 3-D landscape of the approximated controlled KS equation to the equilibrium solution $z_e = 1$ using 9 modes. Note that the state-feedback control (6.13) uses only 5 eigenfunctions which corresponds to the unstable eigenfunctions of the KS equation. The figure shows that the approximated solution of the KS equation converging towards the desired equilibrium solution $z_e = 1$ indicating that proposed scheme of stabilizing the nonlinear KS equation by stabilizing its linearized system at $z_e = 1$ worked in stabilizing the solution of the KS to the equilibrium solution $z_e = 1$.

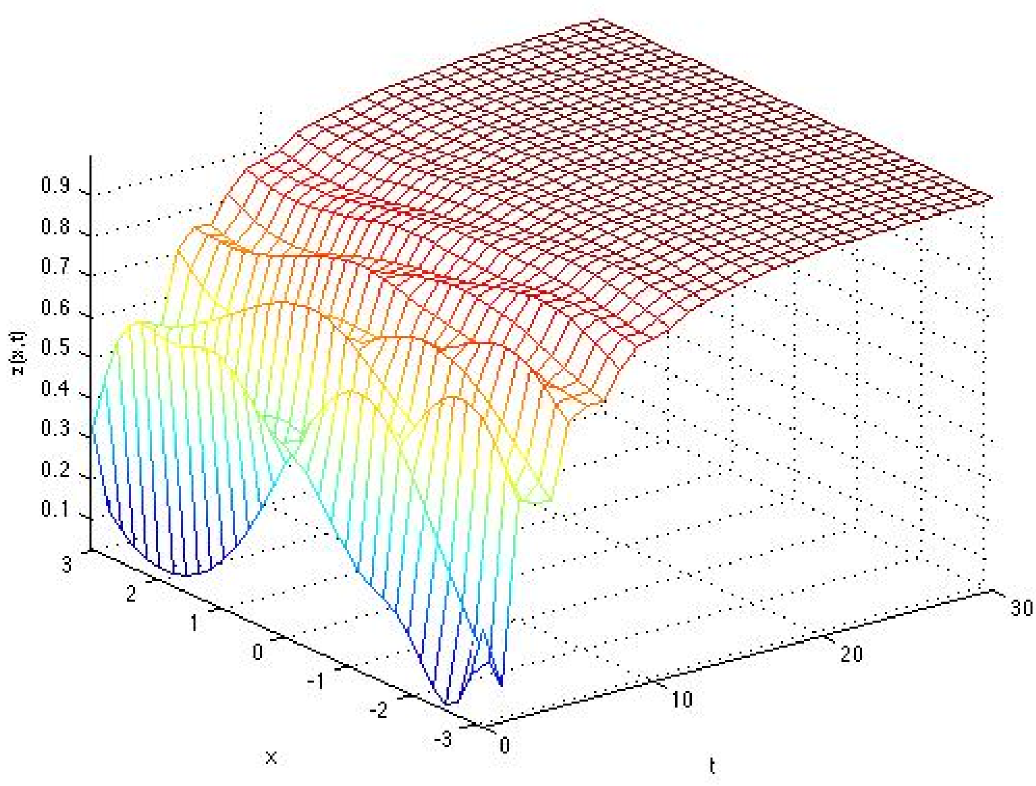

Figure 6.3: A 3-D landscape of the approximated controlled solution of the KS equation when $\nu = \frac{1}{2}$ and $z_e = 1$ with the initial condition given in (6.9) and actuator given by (6.11).

Moreover, we can use the same approach introduced earlier to stabilize the KS equation from one equilibrium state to another. For instance, $z_e = 1$ to $z_e = 2$. To do so, consider the following controlled KS equation

$$\frac{\partial z}{\partial t} = -\frac{1}{2}\frac{\partial^4 z}{\partial x^4} - \frac{\partial^2 z}{\partial x^2} - z\frac{\partial z}{\partial x} + b\left(x\right)u\left(t\right), \; x \in [-\pi, \pi], \; t \geq 0, \tag{6.14}$$

where $b\left(x\right)$ defined in (6.11) and

$$u\left(t\right) = \begin{cases} u_1\left(t\right), & t \in [0, 30], \\ u_2\left(t\right), & t > 30, \end{cases} \tag{6.15}$$

where for $t \in [0, 30]$, the state-feedback control introduced earlier in (6.13) is used to stabilize the KS equation to the first equilibrium solution $z_e = 1$. Then, for $t > 25$, the state-feedback control

$$u_2\left(t\right) = -K_2 \cdot \begin{pmatrix} a_0\left(t\right) \\ a_1\left(t\right) \\ a_2\left(t\right) \\ c_1\left(t\right) \\ c_2\left(t\right) \end{pmatrix},$$

where $a_0\left(t\right), a_1\left(t\right), a_2\left(t\right), c_1\left(t\right), c_2\left(t\right)$ are the solutions of the ODE system given in Appendix C and

$$K_2 = [1.0000 \quad 3.6731 \quad -0.0115 \quad 2.5365 \quad 0.0469].$$

Figure 6.4 is a 3-D landscape of the controlled nonlinear KS equation to the equilibrium solution $z_e = 1$, then from $z_e = 1$ to the equilibrium solution $z_e = 2$.

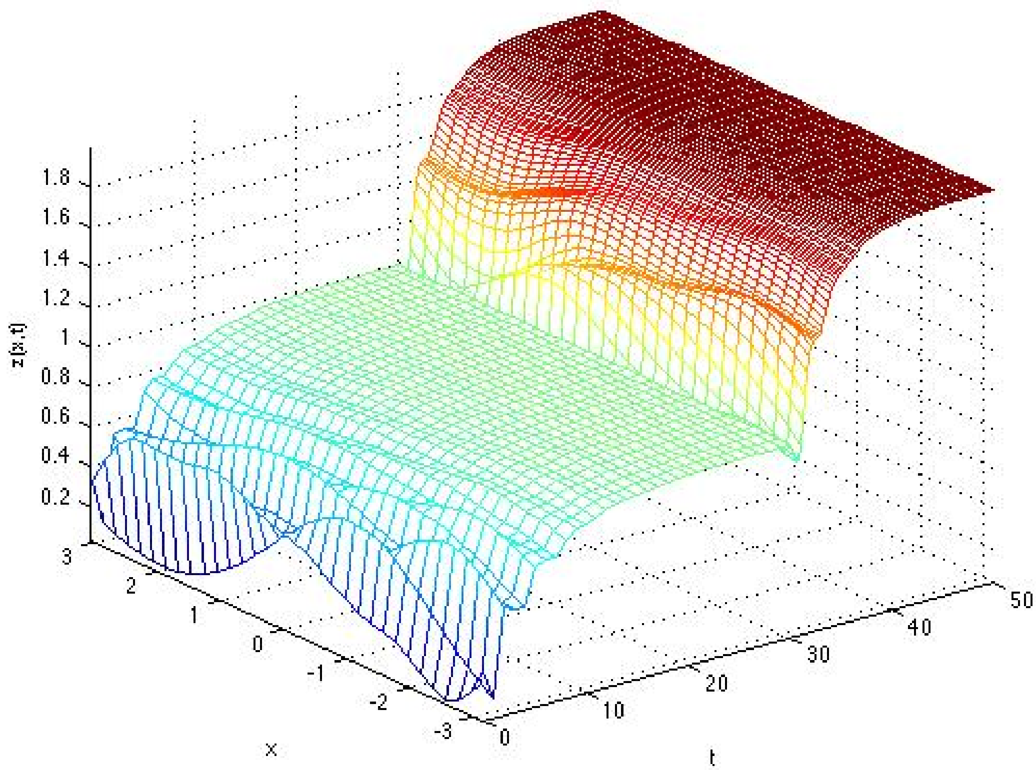

Figure 6.4: A 3-D landscape of the approximated controlled solution of the nonlinear KS equation when $\nu = \frac{1}{2}$ from the equilibrium $z_e = 1$ to another equilibrium $z_e = 2$ with initial condition (6.9) and actuator (6.11).

Chapter 7

Conclusions and Future work

In this thesis, the well-posedness of some classes of nonlinear partial differential equations defined on a Hilbert space are investigated. Next, Lyapunov indirect method is proposed to analyze the stability of these classes. However, the Fréchet differentiability of the generator is required for the method to hold.

The Kuramoto-Sivashinsky (KS) equation defined on the Hilbert space $L^2(-\pi, \pi)$ with periodic boundary conditions is considered.

$$\frac{\partial z}{\partial t} = -\nu \frac{\partial^4 z}{\partial x^4} - \frac{\partial^2 z}{\partial x^2} - z\frac{\partial z}{\partial x}, \quad t \geq 0$$

$$z\left(0\right) = z_0\left(x\right),$$

where the instability parameter $\nu > 0$. The KS equation has infinitely many equilibria. The set of all constant equilibrium solutions is analyzed. In particular, the set of all constant equilibria is globally exponentially stable when the instability parameter $\nu > 1$ and the zero equilibrium solution is Lyapunov stable when $\nu = 1$. This is shown using a Lyapunov function. Furthermore, the set of constant equilibria to the KS equation is unstable when $\nu < 1$. This result is obtained using Lyapunov indirect method and showing that the nonlinear operator corresponding to the KS equation generates a Fréchet differentiable C_0-semigroup.

A single bounded state-feedback control is designed for the KS equation with periodic boundary conditions when the instability parameter $\nu \leq 1$ to drive the solution of the KS equation to converge to a desired constant equilibrium solution. This is done by stabilizing the linearized KS equation around the constant equilibrium solution, then the same control is used to stabilize the nonlinear KS equation. This approach is proved to work by showing that the nonlinear C_0-semigroup generated by the state-feedback controlled KS equation is Fréchet differentiable and the derivative is the linear C_0-semigroup generated by the linearization of the equation. Simulations are presented to illustrate the effectiveness of the developed control scheme where the solution of the KS equation is driven to several constant equilibrium solutions and from one constant equilibrium solution to another.

Future work includes stabilizing the KS equation to any desired state not necessarily a constant function. Another research avenue deals with the development of an output-feedback control for the KS equation as the full state of infinite-dimensional systems is not accessible. Developing an adaptive control scheme for the KS equation when the instability parameter ν is unknown is another problem to consider.

Appendix A

The following is ODE system derived by MATLAB using the Galerkin projection method on the approximated solution of the uncontrolled KS equation (5.1) where $u(t) = 0$ and the instability parameter $\nu = \frac{1}{2}$ with periodic boundary conditions and initial condition defined in (6.9)

```
function ODEsystem = Uncontrolled(t,a)
ODEsystem=[0;
a(2)/2 - (7186705221432913*a(1)*a(6))/18014398509481984 - (5081767996463981
*a(2)*a(7))/18014398509481984 + (5081767996463981*a(3)*a(6))/18014398509481
984 - (1270441999115995*a(3)*a(8))/4503599627370496 + (1270441999115995*a(4
)*a(7))/4503599627370496 - (2540883998231991*a(4)*a(9))/9007199254740992 +
(2540883998231991*a(5)*a(8))/9007199254740992;
(10163535992927961*a(4)*a(6))/18014398509481984 - (7186705221432913*a(1)*a(
7))/9007199254740992 - (5081767996463981*a(2)*a(6))/9007199254740992 - (101
63535992927961*a(2)*a(8))/18014398509481984 - 4*a(3) - (5081767996463981*a(
3)*a(9))/9007199254740992 + (5081767996463981*a(5)*a(7))/9007199254740992;
(15245303989391943*a(5)*a(6))/18014398509481984 - (5390028916074685*a(1)*a(
8))/4503599627370496 - (15245303989391943*a(2)*a(7))/18014398509481984 - (1
5245303989391943*a(3)*a(6))/18014398509481984 - (15245303989391943*a(2)*a(9
))/18014398509481984 - (63*a(4))/2;
- 112*a(5) - (7186705221432913*a(1)*a(9))/4503599627370496 - (2032707198585
5923*a(2)*a(8))/18014398509481984 - (5081767996463981*a(3)*a(7))/4503599627
370496 - (20327071985855923*a(4)*a(6))/18014398509481984;
a(6)/2 + (7186705221432913*a(1)*a(2))/18014398509481984 + (5081767996463981
*a(2)*a(3))/18014398509481984 + (1270441999115995*a(3)*a(4))/45035996273704
96 + (2540883998231991*a(4)*a(5))/9007199254740992 + (5081767996463981*a(6)
*a(7))/18014398509481984 + (1270441999115995*a(7)*a(8))/4503599627370496 +
(2540883998231991*a(8)*a(9))/9007199254740992;
(5081767996463981*a(2)^2)/18014398509481984 + (10163535992927961*a(4)*a(2))
/18014398509481984 - 4*a(7) + (7186705221432913*a(1)*a(3))/9007199254740992
+ (5081767996463981*a(3)*a(5))/9007199254740992 + (10163535992927961*a(6)*a
(8))/18014398509481984 + (5081767996463981*a(7)*a(9))/9007199254740992 - (5
081767996463981*a(6)^2)/18014398509481984;
(5390028916074685*a(1)*a(4))/4503599627370496 - (63*a(8))/2 + (152453039893
91943*a(2)*a(3))/18014398509481984 + (15245303989391943*a(2)*a(5))/18014398
```

```
509481984 - (15245303989391943*a(6)*a(7))/18014398509481984 + (152453039893
91943*a(6)*a(9))/18014398509481984;
(5081767996463981*a(3)^2)/9007199254740992 - 112*a(9) + (7186705221432913*a
(1)*a(5))/4503599627370496 + (20327071985855923*a(2)*a(4))/1801439850948198
4 - (20327071985855923*a(6)*a(8))/18014398509481984 - (5081767996463981*a(7
)^2)/9007199254740992];
end
```

Appendix B

The following is ODE system derived by MATLAB using the Galerkin projection method on the approximated solution of the state-feedback controlled KS equation (5.1) to the equilibrium solution $z_e = 0$ and $z_e = 1$ with the instability parameter $\nu = \frac{1}{2}$, periodic boundary conditions and initial condition defined in (6.9)

```
function ODEsystem = Control2(t,a)
ze=0;

if ze==0
    %K=[ -1.0000    4.7373    0.0486   35.6283    0.0054]
    ODEsystem=[
    (7186705221432891*a(1))/18014398509481984 - (6291097799158453*a(2))/450
    3599627370496 - (2712474457236645*a(3))/144115188075855872;
    (2528202848011265*a(1))/4503599627370496 - (6600752716515019*a(2))/4503
    599627370496 - (3816873204979573*a(3))/144115188075855872 - (7186705221
432913*a(1)*a(6))/18014398509481984 - (5081767996463981*a(2)*a(7))/18014398
509481984 + (5081767996463981*a(3)*a(6))/18014398509481984 - (1270441999115
995*a(3)*a(8))/4503599627370496 + (1270441999115995*a(4)*a(7))/450359962737
0496 - (2540883998231991*a(4)*a(9))/9007199254740992 + (2540883998231991*a(
5)*a(8))/9007199254740992;
    (4980875193822341*a(1))/9007199254740992 - (8720316752741677*a(2))/4503
    599627370496 - (3626378816448035*a(3))/9007199254740992 - (71867052214
    32913*a(1)*a(7))/9007199254740992 - (5081767996463981*a(2)*a(6))/900719
    9254740992 - (10163535992927961*a(2)*a(8))/18014398509481984 + (1016353
    5992927961*a(4)*a(6))/18014398509481984 - (5081767996463981*a(3)*a(9))/
    9007199254740992 + (5081767996463981*a(5)*a(7))/9007199254740992;
    (15245303989391943*a(5)*a(6))/18014398509481984 - (5390028916074685*a(1
    )*a(8))/4503599627370496 - (15245303989391943*a(2)*a(7))/18014398509481
    984 - (15245303989391943*a(3)*a(6))/18014398509481984 - (15245303989391
    943*a(2)*a(9))/18014398509481984 - (63*a(4))/2;
    - 112*a(5) - (7186705221432913*a(1)*a(9))/4503599627370496 - (203270719
    85855923*a(2)*a(8))/18014398509481984 - (5081767996463981*a(3)*a(7))/45
    03599627370496 - (20327071985855923*a(4)*a(6))/18014398509481984;
    (7186705221432913*a(1)*a(2))/18014398509481984 - (5589455281664639*a(7)
    )/36893488147419103232 - (70491744655467*a(6))/140737488355328 + (50817
```

```
    67996463981*a(2)*a(3))/18014398509481984 + (1270441999115995*a(3)*a(4))
    /4503599627370496 + (2540883998231991*a(4)*a(5))/9007199254740992 + (50
    81767996463981*a(6)*a(7))/18014398509481984 + (1270441999115995*a(7)*a(
    8))/4503599627370496 + (2540883998231991*a(8)*a(9))/9007199254740992;
    (5081767996463981*a(2)^2)/18014398509481984 + (10163535992927961*a(4)*a
    (2))/18014398509481984 - (4451350885692725*a(6))/2251799813685248 - (73
    79249607963660109*a(7))/18446744073709551616 + (7186705221432913*a(1)*
    a(3))/9007199254740992 + (5081767996463981*a(3)*a(5))/9007199254740992
    + (10163535992927961*a(6)*a(8))/18014398509481984 + (5081767996463981*a
    (7)*a(9))/9007199254740992 - (5081767996463981*a(6)^2)/1801439850948198
    4;
    (5390028916074685*a(1)*a(4))/4503599627370496 - (63*a(8))/2 + (15245303
    989391943*a(2)*a(3))/18014398509481984 + (15245303989391943*a(2)*a(5))/
    18014398509481984 - (15245303989391943*a(6)*a(7))/18014398509481984 + (
    15245303989391943*a(6)*a(9))/18014398509481984;
    (5081767996463981*a(3)^2)/9007199254740992 - 112*a(9) + (71867052214329
    13*a(1)*a(5))/4503599627370496 + (20327071985855923*a(2)*a(4))/18014398
    509481984 - (20327071985855923*a(6)*a(8))/18014398509481984 - (50817679
    96463981*a(7)^2)/9007199254740992    ];
end

if ze==1
    % K=[1.0000    5.2073   -0.0121    1.5929    0.0397]
    ODEsystem=[
    (2785489162780709*a(3))/576460752303423488 - (584737779088081*a(2))/281
    474976710656 - (3593352610716427*a(1))/9007199254740992 - (572387991275
8521*a(5))/9007199254740992 - (2280268326253537*a(6))/144115188075855872;
    (7810911856391975*a(3))/2305843009213693952 - (2153477598973917*a(2))/2
    251799813685248 - (2519069261359359*a(1))/9007199254740992 - (125395169
    793359*a(5))/281474976710656 + (35629159516705917*a(6))/ - (71867052214
    32913*a(1)*a(6))/18014398509481984 - (5081767996463981*a(2)*a(7))/18014
    398509481984 + (5081767996463981*a(3)*a(6))/18014398509481984 - (127044
    1999115995*a(3)*a(8))/4503599627370496 + (1270441999115995*a(4)*a(7))/4
    503599627370496 - (2540883998231991*a(4)*a(9))/9007199254740992 + (2540
    883998231991*a(5)*a(8))/9007199254740992;
    (5054401834381525*a(1))/18014398509481984 + (3289963466920349*a(2))/225
    1799813685248 - (4615604081322752261*a(3))/1152921504606846976 + (80511
    96825089241*a(5))/18014398509481984 + (3207420384163213*a(6))/288230376
    151711744 + 2*a(7) - (7186705221432913*a(1)*a(7))/9007199254740992 - (5
    081767996463981*a(2)*a(6))/9007199254740992 - (10163535992927961*a(2)*a
    (8))/18014398509481984 + (10163535992927961*a(4)*a(6))/1801439850948198
    4 - (5081767996463981*a(3)*a(9))/9007199254740992 + (5081767996463981*a
    (5)*a(7))/9007199254740992;
    3*a(8) - (63*a(4))/2 - (5390028916074685*a(1)*a(8))/4503599627370496 -
    (15245303989391943*a(2)*a(7))/18014398509481984 - (15245303989391943*a(
```

```
    3)*a(6))/18014398509481984 - (15245303989391943*a(2)*a(9))/180143985094
    81984 + (15245303989391943*a(5)*a(6))/18014398509481984;
    4*a(9) - 112*a(5) - (7186705221432913*a(1)*a(9))/4503599627370496 - (20
    327071985855923*a(2)*a(8))/18014398509481984 - (5081767996463981*a(3)*a
    (7))/4503599627370496 -(20327071985855923*a(4)*a(6))/18014398509481984;
    (3404220556044299*a(3))/576460752303423488 - (7968788826789647*a(2))/22
    51799813685248 - (8783064020471533*a(1))/18014398509481984 - (699530621
    7558461*a(5))/9007199254740992 + (13854163466681671 9*a(6))/288230376151
    711744 + (7186705221432913*a(1)*a(2))/18014398509481984 + (508176799646
    3981*a(2)*a(3))/18014398509481984 + (1270441999115995*a(3)*a(4))/450359
    9627370496 + (2540883998231991*a(4)*a(5))/9007199254740992 + (508176799
    6463981*a(6)*a(7))/18014398509481984 + (1270441999115995*a(7)*a(8))/450
    3599627370496 + (2540883998231991*a(8)*a(9))/9007199254740992;
    (7186705221432913*a(1)*a(3))/9007199254740992 - (5625332660195887*a(2))
    /2251799813685248 - (2299143722917172781*a(3))/1152921504606846976 - (3
    441577764405647*a(5))/4503599627370496 - (2742098327751459*a(6))/144115
    188075855872 - 4*a(7) - (8642251502967453*a(1))/18014398509481984 + (10
    163535992927961*a(2)*a(4))/18014398509481984 + (5081767996463981*a(3)*a
    (5))/9007199254740992 + (10163535992927961*a(6)*a(8))/18014398509481984
    + (5081767996463981*a(7)*a(9))/9007199254740992 + (5081767996463981*a(2
    )^2)/18014398509481984 - (5081767996463981*a(6)^2)/18014398509481984;
    (5390028916074685*a(1)*a(4))/4503599627370496 - (63*a(8))/2 - 3*a(4) +
    (15245303989391943*a(2)*a(3))/18014398509481984 + (15245303989391943*a(
    2)*a(5))/18014398509481984 - (15245303989391943*a(6)*a(7))/180143985094
    81984 + (15245303989391943*a(6)*a(9))/18014398509481984;
    (5081767996463981*a(3)^2)/9007199254740992 - 4*a(5) - 112*a(9) + (71867
    05221432913*a(1)*a(5))/4503599627370496 + (20327071985855923*a(2)*a(4))
    /18014398509481984 - (20327071985855923*a(6)*a(8))/18014398509481984 -
    (5081767996463981*a(7)^2)/9007199254740992];
end
end
```

Appendix C

The following is ODE system derived by MATLAB using the Galerkin projection method on the approximated solution of the state-feedback controlled KS equation (5.1) from the equilibrium solution $z_e = 1$ to the equilibrium solution $z_e = 2$ with the instability parameter $\nu = \frac{1}{2}$, periodic boundary conditions and initial condition defined in (6.9)

```
function ODEsystem = Control_Again2(t,a)
ze=1;
ze_2=2;

%K=[ 1.0000    3.6731   -0.0115    2.5365    0.0469]
ODEsystem=[
(2655637397646189*a(3))/576460752303423488 - (6599366044267057*a(2))/450359
9627370496 - (898338152679111*a(1))/2251799813685248 - (4557270461222655*a(
6))/4503599627370496 - (5392944996471325*a(7))/288230376151711744;
(930848606086713*a(3))/288230376151711744 - (2374592847305897*a(2))/4503599
627370496 - (2519069261359371*a(1))/9007199254740992 + (11624777908191687*a
(6))/9007199254740992 - (7561296338298463*a(7))/576460752303423488 - (71867
05221432913*a(1)*a(6))/18014398509481984 - (5081767996463981*a(2)*a(7))/180
14398509481984 + (5081767996463981*a(3)*a(6))/18014398509481984 - (12704419
99115995*a(3)*a(8))/4503599627370496 + (1270441999115995*a(4)*a(7))/4503599
627370496 - (2540883998231991*a(4)*a(9))/9007199254740992 + (25408839982319
91*a(5)*a(8))/9007199254740992;
(2527200917190775*a(1))/9007199254740992 + (4641326840625337*a(2))/45035996
27370496 - (9230842864234245771*a(3))/2305843009213693952 + (64102465508899
93*a(6))/9007199254740992 + (2313428713717502369*a(7))/576460752303423488 -
(7186705221432913*a(1)*a(7))/9007199254740992 - (5081767996463981*a(2)*a(6)
)/9007199254740992 - (10163535992927961*a(2)*a(8))/18014398509481984 + (101
63535992927961*a(4)*a(6))/18014398509481984 - (5081767996463981*a(3)*a(9))/
9007199254740992 + (5081767996463981*a(5)*a(7))/9007199254740992;
6*a(8) - (63*a(4))/2 - (5390028916074685*a(1)*a(8))/4503599627370496 - (152
45303989391943*a(2)*a(7))/18014398509481984 - (15245303989391943*a(3)*a(6))
/18014398509481984 - (15245303989391943*a(2)*a(9))/18014398509481984 + (152
45303989391943*a(5)*a(6))/18014398509481984;
8*a(9) - 112*a(5) - (7186705221432913*a(1)*a(9))/4503599627370496 - (203270
71985855923*a(2)*a(8))/18014398509481984 - (5081767996463981*a(3)*a(7))/450
```

```
3599627370496 - (20327071985855923*a(4)*a(6))/18014398509481984;
(6491050505069841*a(3))/1152921504606846976 - (8536230205320317*a(2))/22517
99813685248 - (8783064020471575*a(1))/18014398509481984 - (1658880947960687
*a(6))/2251799813685248 - (3295430917468849*a(7))/144115188075855872 + (718
6705221432913*a(1)*a(2))/18014398509481984 + (5081767996463981*a(2)*a(3))/1
8014398509481984 + (1270441999115995*a(3)*a(4))/4503599627370496 + (2540883
998231991*a(4)*a(5))/9007199254740992 + (5081767996463981*a(6)*a(7))/180143
98509481984 + (1270441999115995*a(7)*a(8))/4503599627370496 + (254088399823
1991*a(8)*a(9))/9007199254740992;
(7186705221432913*a(1)*a(3))/9007199254740992 - (7935956652933133*a(2))/450
3599627370496 - (4605299034243676375*a(3))/1152921504606846976 - (548026895
2102399*a(6))/4503599627370496 - (289851674979378241*a(7))/7205759403792793
6 - (4321125751483747*a(1))/9007199254740992 + (10163535992927961*a(2)*a(4)
)/18014398509481984 + (5081767996463981*a(3)*a(5))/9007199254740992 + (1016
3535992927961*a(6)*a(8))/18014398509481984 + (5081767996463981*a(7)*a(9))/9
007199254740992 + (5081767996463981*a(2)^2)/18014398509481984 - (5081767996
463981*a(6)^2)/18014398509481984;
(5390028916074685*a(1)*a(4))/4503599627370496 - (63*a(8))/2 - 6*a(4) + (152
45303989391943*a(2)*a(3))/18014398509481984 + (15245303989391943*a(2)*a(5))
/18014398509481984 - (15245303989391943*a(6)*a(7))/18014398509481984 + (152
45303989391943*a(6)*a(9))/18014398509481984;
(5081767996463981*a(3)^2)/9007199254740992 - 8*a(5) - 112*a(9) + (718670522
1432913*a(1)*a(5))/4503599627370496 + (20327071985855923*a(2)*a(4))/1801439
8509481984 - (20327071985855923*a(6)*a(8))/18014398509481984 - (50817679964
63981*a(7)^2)/9007199254740992];
end
```

Appendix D

Below is the MATLAB code used to approximate the solution of the KS equation, design an state-feedback control and approximate the solution of the input-feedback controlled KS equation.

```
clear
% This code approximates the nonlinear controlled/uncontolled KS equation:
% \dot{z} = -\nu z_(4) - z_(2) -z z_x + Bu(t),
% z(0) = z_0.
% Periodic boundary conditions
% The solution z = \sum(0..N) a_n(t)\phi(x) + c_n(t)\psi(x).

% There will be 3 options:
% (1) To solve the uncontrolled KS equation.
% (2) To control to one equilibrium solution.
% (3) To control from one equilibrium to another.

% This is determined by "control_again"
% if control_again =
% -1: means no control at all.
% 0: means only one equilibrium
% 1: means controlling from equilibrium ze1 to ze2 to ze1

control_again =0;
ze = 0; % Equilibrium 1
ze_2 = 2; % Equilibrium 2.

% The instability parameter
nu = 0.5;

% The number of unstable modes
unstable = ceil(1/sqrt(nu));

% The number of modes used will be N+1, where N
N = unstable + 2;
nodes = N+1;
```

```
% Orthonormal Basis
% \phi = 1/\sqrt{pi} cos(nx) & \psi = 1/\sqrt{pi} sin(nx) for n=1..N and
% \phi_0 = 1/s\sqrt(pi)
syms x;
phi = [1/sqrt(2*pi)];
psi = [];
for n=1:N
    phi = [phi; (1/sqrt(pi))*cos(x*n)];
    psi = [psi; (1/sqrt(pi))*sin(x*n)];
end

%initial condition
%z0 = 2*cos(x/10) * (1+sin(x/10));
z0 = (0.5)*cos(x/3) * (1+sin(x/3));

% Initial condition for the ODE system resulted by taking L^2-innr product
% with phi_n and psi_n
if control_again ~= -1
    z0 = z0 - ze;
end
a0 = [double(int(z0*phi(1),-pi,pi))];
c0 = [];
for n=1:N
    integ1 = int(z0*phi(n+1),-pi,pi);
    integ2 = int(z0*psi(n),-pi,pi);
    a0=[a0 double(integ1)]; %row vector
    c0=[c0 double(integ2)]; %row vector
end


% Solving the ODE system resulted from performing the Galerkin Projection
tmax = 30; % Final time
dt=1; % Time step

if control_again==-1
    % Solving the ODE system: for the uncontrolled KS equation
    % a=[a0(t)...aN(t)] &c=[c1(t)...cN(t)] for different time step
    % t is a cloumn vector for different time steps

    % The ODE system written interms of a1..a4 and c1..c3
    syms a1; syms a2; syms a3; syms a4;
    syms a5; syms a6; syms a7; syms a8;
    syms a9;
    %system1 = GalerkinProjection22([a1 a2 a3 a4 a5 a6 a7 a8 a9])
```

```
    % solving the ODE system with the initial condition a0 & c0
    tmax=40; % Final time
    dt=0.75; % Time step
    [t,ac] = ode45('Uncontrolled2',[0:dt: tmax],[a0 c0]);
    %[t,a] = ode45(@GalerkinProjection1,[0:dt:tmax],b);
end %end control_again = -1

if control_again ~= -1
    % Solving the ODE system: for the controlled KS equation
    % a=[a0(t)...aN(t)] & c=[c1(t)...cN(t)] for different time step
    % t is a cloumn vector for different time steps

    % The ODE system written interms of a1..a4 and c1..c3
    syms a1; syms a2; syms a3; syms a4;
    syms a5; syms a6; syms a7; syms a8;
    syms a9;
    %system1 = GalerkinProjection22([a1 a2 a3 a4 a5 a6 a7 a8 a9])

    [t,ac] = ode45('Control2',[0:dt: tmax],[a0 c0]);
    %[t,a] = ode45(@GalerkinProjection1,[0:dt:tmax],b);
end % end control_again ~= -1


% writing the solution z using the variable x
% z is a column vactor for different time steps
z=[];
for n=1: length(t)
    sum=0;
    % sum(0..N) a_n*phi
    for i=1:N+1
        sum=sum+ac(n,i)*phi(i);
    end

    % sum (1..N) c_n*psi
    for i=1:N
        sum=sum+ac(n,i+N+1)*psi(i);
    end

    if control_again==-1
        ze=0;
    end

    z=[z; sum + ze ];  %adding the equilibrium
end
```

```
% Evaluating the solution for different x \in [-pi,pi]
dx=0.2;
xx=[-pi:dx: pi]; % row vector

if control_again == -1
    % the case of no control
    %------------------------------
    % To create an animation file:
    % Prepare new file
    %------------------------------
    fig=figure;
    VidObj = VideoWriter('KSE-no-control.avi');
    VidObj.FrameRate=5;
    open(VidObj);
    zz=subs(z,'x',xx);
    zz= zz';

    % Create an animaton for different time step and writing each frame to
    % the file
    % The first figure: For the animation
    figure
    f = floor(length(t)/2);
    for o=1:f
        TT=t(1:2*o);
        Zz = zz(:,2*o);
        %subplot(1,2,1) % 2D plot
        plot(xx,Zz','LineWidth',1.2);
        axis([-4 4 -5 5]);
        xlabel('x'), ylabel('z(x,t)');

        %subplot(1,2,2) % 3D plot
        %mesh(TT,xx,zz(:,1:2*o));shading interp, lighting phong, %axis tight
        %xlabel('t'); ylabel('x'); zlabel('z(x,t)');
        set(gca,'nextplot','replacechildren');

        %writing the frame into the file
        currFrame=getframe(gcf);
        writeVideo(VidObj,currFrame);
    end

    % the second window & last frame in animation:3D mesh of the whole graph
    mesh(t,xx,zz), shading interp, lighting phong, axis tight
    xlabel('t'); ylabel('x'); zlabel('z(x,t)');
    set(gca,'nextplot','replacechildren');
```

```
    %writing the frame into the file
    currFrame=getframe(gcf);
    writeVideo(VidObj,currFrame);
    close(VidObj);

    % Second figure
    figure
    mesh(t,xx,zz), shading interp, lighting phong, axis tight
    xlabel('t'); ylabel('x'); zlabel('z(x,t)');

end %end of control_again = -1

if control_again == 0
    % This means we are controlling to one equilbrium ze
    fig=figure;
    %-------------------------------
    % To create an animation file:
    % Prepare new file
    %-------------------------------
    VidObj = VideoWriter('KSE-control1.avi');
    VidObj.FrameRate=5;
    open(VidObj);

    % first figure: the 3D mesh
    %figure
    %zz= [subs(z(1)*ones(1,length(xx)),'x',xx)];
    %zz=[zz; subs(z(2:length(z)),'x',xx)];
    zz=subs(z,'x',xx);
    zz=zz';

    % Second figure
    %figure
    % Create an animaton for different time step and writing each frame to
    % the file
    f = floor(length(t)/2);
    for o=1:f
        TT=t(1:2*o);
        Zz = zz(:,2*o);
        %subplot(1,2,1) % 2D plot
        plot(xx,ze*ones(1,length(xx)),'g+--',xx,Zz','LineWidth',1.2);
        axis([-4 4 -2 7]);
        xlabel('x'), ylabel('z(x,t)');

        %subplot(1,2,2) % 3D plot
```

```
        %mesh(TT,xx,zz(:,1:2*o));shading interp, lighting phong,%axis tight
        %xlabel('t'); ylabel('x'); zlabel('z(x,t)');
        set(gca,'nextplot','replacechildren');


        %writing the frame into the file
        currFrame=getframe(gcf);
        writeVideo(VidObj,currFrame);
    end
    mesh(t,xx,zz), shading interp, lighting phong, axis tight
    xlabel('t'); ylabel('x'); zlabel('z(x,t)');


    set(gca,'nextplot','replacechildren');
    %writing the frame into the file
    currFrame=getframe(gcf);
    writeVideo(VidObj,currFrame);

    close(VidObj);

    % second figure
    figure
    mesh(t,xx,zz), shading interp, lighting phong, axis tight
    xlabel('t'); ylabel('x'); zlabel('z(x,t)');
end

if control_again==1
    % This means that we need to conrol the above controlled system from ze
    % to ze2 then to ze again

    % The approach will be similar to before. That is, linearize the system
    % around ze2, and do the chane of variables to control to the desired
    % ze2.
    % However, the initial condition will be different. It is going to be
    % z0 introduced earlier. When change of variales is done. That is,
    % w=z-ze_2, the initial condition will be w_0 = ze-ze_2. At the end to
    % get the solution to th original problem: z= w + ze_2.


    % For the ODE system:
    %--------------------
    % intial condition for the ODE system
    a0_2 = [double(int((ze-ze_2)*phi(1),-pi,pi))];
    c0_2 = [];
    for n=1:N
```

```
    integ1 = int((ze-ze_2)*phi(n+1),-pi,pi);
    integ2 = int((ze-ze_2)*psi(n),-pi,pi);
    a0_2 = [a0_2 double(integ1)]; % row vector
    c0_2 = [c0_2 double(integ2)]; % row vector
end


% Solving the ODE system
% Final time = the old final time + extra time1 + extra time2
tmax_2 = tmax + 20;
tmax_3 = tmax_2 + 20;
dt_2 = 0.2; % Time step

% Solving the ODE system: for the controlled KS equation
% a_2=[a0(t)...aN(t)] & c_2=[c1..cN] for different time step
% t_2 is a cloumn vector for different time steps starting by tmax
syms a1; syms a2; syms a3; syms a4;
syms a5; syms a6; syms a7; syms a8;
syms a9;
%system = GalerkinProjection22([a1 a2 a3 a4 a5 a6 a7 a8 a9])

[t_2,ac_2] = ode45('Control_Again2',[tmax:dt_2: tmax_2],[a0_2 c0_2]);
%[t_2,a_2] = ode45(@GalerkinProjection1,[tmax:dt_2: tmax_2],a0_2);

% To drive the system back to the first equilibrium, we shall use the
% same ODE system generated for part 1 of the control with inintial
% condition = -a0_2, - b0_2
%a00_2 = [double(int((ze)*phi(1),-pi,pi))];
%c00_2 = [];
%for n=1:N
%    integ1 = int((ze)*phi(n+1),-pi,pi);
%    integ2 = int((ze)*psi(n),-pi,pi);
%    a00_2 = [a00_2 double(integ1)]; % row vector
%    c00_2 = [c00_2 double(integ2)]; % row vector
%end

%[z1,z2]=size(ac_2);
%ac_00 = ac_2(z1,:) - [a00_2 c00_2];

%[t_22,ac_22] = ode45('Control3',[tmax_2:0.1: tmax_3],ac_00);


% writing the solution z using the variable x
% z is a column vactor for different time steps
% Here I will be adding more solutions to the z obtained earlier. That
```

```
% is, no need to start with z = [].
% NOte that the first loop will start from n=2 not 1 because it is
% already the the vector z.
for n=2: length(t_2)
    sum = 0;
    % sum(0..N) an*phi
    for i=1:(N+1)
        sum = sum + ac_2(n,i)*phi(i);
    end

    % sum(1..N) cn*psi
    for i=1:N
        sum=sum+ac_2(n,i+N+1)*psi(i);
    end

    z = [z ; sum + ze_2]; % adding the second equilibrium
end

%Returning to the first equilibrium ze
%for n=2: length(t_22)
%    sum = 0;
%    % sum(0..N) an*phi
%    for i=1:(N+1)
%        sum = sum + ac_22(n,i)*phi(i);
%    end

% sum(1..N) cn*psi
%    for i=1:N
%        sum=sum+ac_22(n,i+N+1)*psi(i);
%    end

%    z = [z ; sum+ ze]; % adding the second equilibrium
%end

% Evaluating the solution for different x \in [-pi,pi]
dx = 0.2;
xx=[-pi:dx: pi]; % row vector

T=[t];
for n=2:length(t_2)
    T=[T; t_2(n)];
end

%for n=2:length(t_22)
%    T=[T; t_22(n)];
```

```
%end

fig=figure;

%-------------------------------
% To create an animation file:
% Prepare new file
%-------------------------------
VidObj = VideoWriter('KSE-control2.avi');
VidObj.FrameRate=5;
open(VidObj);

% first figure: the 3D mesh
figure
zz=subs(z,'x',xx);
zz=zz';

% Create an animaton for different time step and writing each frame to
% the file
f = floor(length(T)/2);
for o=1:f
    TT=T(1:2*o);
    Zz = zz(:,2*o);
    %subplot(1,2,1) % 2D plot
    plot(xx,ze*ones(1,length(xx)),'g+--',xx,ze_2*ones(1,length(xx)),'r+
    --',xx,Zz','LineWidth',1.2);
    axis([-4 4 -2 5]);
    xlabel('x'), ylabel('z(x,t)');

    %subplot(1,2,2) % 3D plot
    %mesh(TT,xx,zz(:,1:2*o));shading interp, lighting phong,%axis tight
    %xlabel('t'); ylabel('x'); zlabel('z(x,t)');
    set(gca,'nextplot','replacechildren');

    %writing the frame into the file
    currFrame=getframe(gcf);
    writeVideo(VidObj,currFrame);
end
mesh(T,xx,zz), shading interp, lighting phong, axis tight
xlabel('t'); ylabel('x'); zlabel('z(x,t)');


set(gca,'nextplot','replacechildren');

%writing the frame into the file
```

```
    currFrame=getframe(gcf);
    writeVideo(VidObj,currFrame);
    close(VidObj);

    % second figure
    figure
    mesh(T,xx,zz), shading interp, lighting phong, axis tight
    xlabel('t'); ylabel('x'); zlabel('z(x,t)');
end
%===========================================================================
function ODE=GalerkinProjection22(a)
% In this function, there will be 3 options to obtain ODE system:
% (1) uncontrolled KS equation.
% (2) control to one equilibrium solution.
% (3) control from one equilibrium to another.

% This is determined by "control_again"
% if control_again =
% -1: means no control at all.
% 0: means only one equilibrium
% 1: means controlling from equilibrium 1 to the second ze2 then ti ze1

control_again = 1;
ze = 1; % Equilbrium 1
ze_2 = 2; % Equilibrium 2

% The instability parameter
nu=0.5;

% The number of unstable eigenvalues
unstable= ceil(1/sqrt(nu));

% The number of Modes used WILL BE (2N+1), where N is given by
N=unstable+2;

% Orthonormal Basis
% \phi = 1/\sqrt{pi} cos(nx) & \psi = 1/\sqrt{pi} sin(nx) for n=1..N and
% \phi_0 = 1/s\sqrt(pi)
syms x;
phi = [1/sqrt(2*pi)];
psi = [];
for n=1:N
    phi = [phi; (1/sqrt(pi))*cos(x*n)];
    psi = [psi; (1/sqrt(pi))*sin(x*n)];
end
```

```
% The linearized KS equation around an equilibrium ze \in R is given by
% \dot(a) = (A+dF(ze)) a(t) + Bu(t), where
% Az=- nu z_xxxx - z_xx,
% dF(ze)z= -ze z_x.

% The goal is to find K (row vector) such that (A+dF(ze)-BK) is Hurwitz and
% the controller of the form u(t) = -Ka(t). This is done using LQR where
% the cost function is j(u) = int_{0}{infty} a*Qa + u*Ru + 2a*NNu dt.
% The following choices are made: Q=I(N,N), R=1, NN=0

% When ze=0: it is enough to control a part(N+1 nodes) and use symmetry to
% control all unstable nodes (N+1).

% The matrix A1 {(2unstable+1)x(2unstable+1): Used when ze \neq 0, we have
% \dot{a_n} = (-\nu n^4 + n^2) a_n +n z_e b_n for n=0..unstable
% \dot{b_n} = (-\nu n^4 + n^2) b_n -n z_e a_n for n=1..unstable
A1=zeros(2*unstable+1,2*unstable+1);
A1(1,1)=0;
for n=1:unstable
    A1((n+1),(n+1)) = -1*nu *n^4 + n^2;
    A1((n+unstable+1),(n+unstable+1)) =  -1*nu *n^4 + n^2;
end

% terms from linearization at ze
for n=2:unstable+1
    A1(n+unstable, n) = -(n-1)*ze;
end

for n=1:unstable
    A1(n+1, n+unstable+1) = n*ze;
end


% The matrix A1 {(N+1)x(N+1): Used when ze=0
A2=zeros(unstable+1,unstable+1);
A2(1,1)=0;
for n=1:unstable
    A2(n+1,n+1) = -1*nu *n^4 + n^2 ;
end


%The matrix A =
%A1 when ze neq 0
```

```
%A2 when ze=0
if control_again~=-1
    if ze==0
        A=A2;
        Q=zeros(1+unstable,1+unstable);
        for n=1:(unstable+1)
            Q(n,n)=1;
        end
    else
        A=A1;
        Q=zeros(1+2*unstable,1+2*unstable);
        for n=1:(2*unstable+1)
            Q(n,n)=1;
        end
    end
end

% The actuator B: Computing bn which is a column vector <b(x),phi(n)>
epsilon1= 0.1;
epsilon2=0.2;
b=1/((epsilon1+epsilon2));
B=[];
% the appropriate size of bn:
if ze == 0
    B = [double(int(b*phi(1),ze - epsilon1,ze +epsilon2))];
    B2= [];
    for n=1:unstable
        integ1 = int(b*phi(n+1),ze - epsilon1,ze +epsilon2);
        integ2 = int(b*psi(n),ze - epsilon1,ze +epsilon2);
        B = [B ; double(integ1)];
        B2 = [B2 ; double(integ2)];
    end
else
    %z0 \neq 0
    for n=1:(unstable+1)
        integ=int(b*phi(n),ze-epsilon1,ze+epsilon2);
        B=[B; double(integ)];
    end
    for n=1:unstable
        integ=int(b*psi(n),ze-epsilon1,ze+epsilon2);
        B=[B; double(integ)];
    end
end

% R and NN from the cost function to perform the LQR
```

```
R=1;
NN=0;

% the case to control the KS equation to the first equilibrium point
if control_again ~= -1
    if ze==0
        [K1,S,e] = lqr(A,B,Q,R,NN);
        A11=A(2:unstable+1,2:unstable+1);
        Q11=Q(2:unstable+1,2:unstable+1);
        [K2,S,e] = lqr(A11,B2, Q11,R,NN);
        eig(A - B*K1)
        eig(A(2:unstable+1,2:unstable+1) - B2*K2)
        B = [B;B2];
        rank([B A1*B A1^2*B A1^3*B A1^4*B])
        K = [K1 K2]
        BK = B*K;
        eig(A1-BK)

    else
        % Computing the contrller u = -K.a(t), where K is a raw vector
        [K,S,e] = lqr(A,B,Q,R,NN);
        K
        BK=B*K;
        eig(A-BK)
    end
end

% The case to control the KS equation from one equilibrium point to
% another. I am assuming that the second equilibrium solution is never 0
% (to simplify the code).
if control_again ==1
    % The matrix A_2 (2N+1)x(2N+1)
    A_2=zeros(2*unstable+1,2*unstable+1);
    A_2(1,1)=0;
    for n=1:unstable
        A_2((n+1),(n+1)) = -1*nu *n^4 + n^2;
        A_2((n+unstable+1),(n+unstable+1)) = -1*nu *n^4 + n^2;
    end

    for n =1:unstable
        A_2((n+1),(n+unstable+1)) = n*ze_2;
        A_2((n+unstable+1),(n+1)) = -1*n*ze_2;
    end

    % Use the same actuator introduced earlier for controlling to the first
```

```
    % equilibrium solution
    epsilon1= 0.1;
    epsilon2=0.2;
    b=1/((epsilon1+epsilon2));
    B_2=[];
    for n=1:(unstable+1)
        integ=int(b*phi(n),ze-epsilon1,ze+epsilon2);
        B_2=[B_2;double(integ)];
    end
    for n=1:unstable
        integ=int(b*psi(n),ze-epsilon1,ze+epsilon2);
        B_2=[B_2; double(integ)];
    end

    % Q matrix (2N+1)x(2N+1)
    Q=zeros(1+2*unstable,1+2*unstable);
    for n=1:(2*unstable+1)
        Q(n,n)=1;
    end

    % Computing the second controller u_2 = -K_2.a(t), where K_2 is a row
    % vector
    [K_2,S_2,e_2] = lqr(A_2,B_2,Q,R,NN);
    K_2
    BK2=B_2*K_2;
end

% Now constructing the ODE system. Start with the linear part
if control_again==-1
    ODE=[0*a(1)];
    for n=1:N
        ODE=[ODE;(-1*nu*(n)^4+(n)^2)*a(n+1)];
    end
    for n=1:N
        ODE=[ODE;(-1*nu*(n)^4+(n)^2)*a(n+N+1)];
    end
    ODE
end

if control_again==0
    if ze == 0
        ODE1 = 0*a(1);
        %Adding control n=1..N+1
        for m=1:unstable+1
            ODE1=ODE1 - BK(1,m)*a(m);
```

```
    end
    ODE=[ODE1];
    % phi_n: n=2..N+1
    for n=1:N
        ODE1=(-1*nu*(n)^4+(n)^2)*a(n+1);
        % Adding the control
        if n<=unstable
            for m=1:unstable+1
                ODE1=ODE1 - BK(n+1,m)*a(m);
            end
        end
        ODE=[ODE; ODE1];
    end
    % psi_n: n=1..N
    for n=1:N
        ODE1=(-1*nu*(n)^4+(n)^2)* a(n+1+N);
        % Adding the control
        if n<=unstable
            for m=1:unstable
                ODE1 = ODE1 -BK(n+1+unstable,m+unstable+1)*a(m+N+1);
            end
        end
        ODE=[ODE;ODE1];
    end
else
    % controlling to one equilibrium point
    ODE1=(0)*a(1);
    % Adding the control
    for m=1:unstable+1
        ODE1=ODE1 - BK(1,m)*a(m);
    end
    for m=1:unstable
        ODE1 = ODE1 - BK(1,m+unstable+1)*a(m+N);
    end

    ODE=[ODE1];

    for n=1:N
        ODE1=(-1*nu*(n)^4+(n)^2)*a(n+1) + n*ze*a(n+N+1);
        % Adding the control
        if n<=unstable
            for m=1:unstable+1
                ODE1=ODE1 - BK(n+1,m)*a(m);
            end
            for m=1:unstable
```

```
                ODE1 = ODE1 - BK(n+1,m+unstable+1)*a(m+N);
            end
        end

        ODE=[ODE; ODE1];
    end

    for n=1:N
        ODE1=(-1*nu*(n)^4+(n)^2)* a(n+1+N) - n*ze*a(n+1);
        % Adding the control
        if n<=unstable
            for m=1:unstable+1
                ODE1=ODE1 - BK(n+1+unstable,m)*a(m);
            end
            for m=1:unstable
                ODE1 = ODE1 - BK(n+1+unstable,m+unstable+1)*a(m+N);
            end
        end

        ODE=[ODE;ODE1];
    end
  end
end

if control_again==1
    %controlling from one ze to ze_2
    ODE1=(0)*a(1);
    % Adding the control
    for m=1:unstable+1
            ODE1=ODE1 - BK2(1,m)*a(m);
        end
        for m=1:unstable
            ODE1 = ODE1 - BK2(1,m+unstable+1)*a(m+N+1);
        end

    ODE = [ODE1];

    for n=1:N
        ODE1=(-1*nu*(n)^4+(n)^2)*a(n+1) + n*ze_2*a(n+N+1);
        % Adding the control
        if n<=unstable
                for m=1:unstable+1
                    ODE1=ODE1 - BK2(n+1,m)*a(m);
                end
                for m=1:unstable
```

```
                ODE1 = ODE1 - BK2(n+1,m+unstable+1)*a(m+N+1);
            end
        end
        ODE=[ODE;ODE1];
    end

    for n=1:N
        ODE1=(-1*nu*(n)^4+(n)^2)*a(n+N+1) - n*ze_2*a(n+1);
        % Adding the control
        if n<=unstable
                for m=1:unstable+1
                    ODE1=ODE1 - BK2(n+1+unstable,m)*a(m);
                end
                for m=1:unstable
                    ODE1 = ODE1 - BK2(n+1+unstable,m+unstable+1)*a(m+N+1);
                end
        end

        ODE=[ODE;ODE1];
    end
end

% Adding the nonlinear term
% Buiding the nonlinear part with the inner product included
%<zz_x, phi_n>, n=1..N+1 and <zz_x,psi_n>, n=1..N
%M=[];

% start inner product with phi_n: n=0..N
for n=1:N+1
    % k:0..N , m:1..N
    for k=1:N+1
        for m=1:N
            integral=double(int(phi(k)*(-m)*psi(m)*phi(n),-pi,pi));
            if integral ~= 0
                ODE(n) = ODE(n) - integral*a(k)*a(m+1);
                %M=[M;k m+1 n integral];
            end
        end
    end

    % k:0..N  & m:1..N
    for k=1:N+1
        for m=1:N
            integral=double(int(phi(k)*(m)*phi(m+1)*phi(n),-pi,pi));
            if integral ~= 0
```

```
                ODE(n) = ODE(n) - integral*a(k)*a(m+N+1);
                %M=[M;k m+N+1 n integral];
            end
        end
    end

    %k:0..N  & m:1..N
    for k=1:N
        for m=1:N
            integral=double(int(psi(k)*(-1*(m))*psi(m)*phi(n),-pi,pi));
            if integral ~= 0
                ODE(n) = ODE(n) - integral*a(k+N+1)*a(m+1);
                %M=[M;k+N+1 m+1 n integral];
            end
        end
    end

    %k=1..N ,m:1..N
    for k=1:N
        for m=1:N
            integral=double(int(psi(k)*(m)*phi(m+1)*phi(n),-pi,pi));
            if integral ~= 0
                ODE(n) = ODE(n) - integral*a(k+N+1)*a(m+N+1);
                %M=[M;k+N+1 m+N+1 n integral];
            end
        end
    end
end


% start inner product with psi_n: n=1..N
for n=1:N
    % k:0..N , m:1..N
    for k=1:N+1
        for m=1:N
            integral=double(int(phi(k)*(-m)*psi(m)*psi(n),-pi,pi));
            if integral ~= 0
                ODE(n+N+1) = ODE(n+N+1) - integral*a(m+1)*a(k);
                %M=[M;k m+1 n+N+1 integral];
            end
        end
    end

    % k:0..N  & m:1..N
    for k=1:N+1
```

```
        for m=1:N
            integral=double(int(phi(k)*(m)*phi(m+1)*psi(n),-pi,pi));
            if integral ~= 0
                ODE(n+N+1) = ODE(n+N+1) - integral*a(m+N+1)*a(k);
                %M=[M;k m+N+1 n+N+1 integral];
            end
        end
    end

    %k:0..N  & m:1..N
    for k=1:N
        for m=1:N
            integral=double(int(psi(k)*(-1*(m))*psi(m)*psi(n),-pi,pi));
            if integral ~= 0
                ODE(n+N+1) = ODE(n+N+1) - integral*a(m+1)*a(k+N+1);
                %M=[M;k+N+1 m+1 n+N+1 integral];
            end
        end
    end

    %k=1..N ,m:1..N
    for k=1:N
        for m=1:N
            integral=double(int(psi(k)*(m)*phi(m+1)*psi(n),-pi,pi));
            if integral ~= 0
                ODE(n+N+1) = ODE(n+N+1) - integral*a(m+N+1)*a(k+N+1);
                %M=[M;k+N+1 m+N+1 n+N+1 integral];
            end
        end
    end
end

%[q1,q2]=size(M);
% Now building the ODE system with the nonlinear terms included
%for n=1:q1
%    n1 = M(n,3); % phi_n, psi_n
%    m = M(n,2); % a(i), b(i),
%    k = M(n,1) ;% a(j), b(j),
%    result = M(n,4); % the inner product

%    % the nonlinear term
%    %ODE(n1) = ODE(n1) - result*a(m)*a(k);
%end

end  %function
```

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