Cardinality Constrained Robust Optimization Applied to a Class of Interval Observers

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Observers are used in the monitoring and control of dynamical systems to deduce the values of unmeasured states. Designing an observer requires having an accurate model of the plant — if the model parameters are characterized imprecisely, the observer may not provide reliable estimates. An interval observer, which comprises an upper and lower observer, bounds the plant's states from above and below, given the range of values of the imprecisely characterized parameters, i.e., it defines an interval in which the plant's states must lie at any given instant.

We propose a linear programming-based method of interval observer design for two cases: 1) only the initial conditions of the plant are uncertain; 2) the dynamical parameters are also uncertain. In the former, we optimize the transient performance of the interval observers, in the sense that the volume enclosed by the interval is minimized. In the latter, we optimize the steady state performance of the interval observers, in the sense that the norm of the width of the interval is minimized at steady state.

Interval observers are typically designed to characterize the widest interval that bounds the states. This thesis proposes an interval observer design method that utilizes additional, but still-incomplete information, that enables the designer to identify tighter bounds on the uncertain parameters under certain operating conditions. The number of bounds that can be refined defines a class of systems. The definition of this class is independent of the specific parameters whose bounds are refined.

Applying robust optimization techniques, under a cardinality constrained model of uncertainty, we design a single observer for an entire class of systems. These observers guarantee a minimum level of performance with respect to the aforementioned metrics, as we optimize the worst-case performance over a given class of systems. The robust formulation allows the designer to tune the level of uncertainty in the model. If many of the uncertain parameter bounds can be refined, the nominal performance of the observer can be improved, however, if few or none of the parameter bounds can be refined, the nominal performance of the observer can be designed to be more conservative.

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Nomenclature

$-\top$ Matrix inverse transpose

$\bar{e}_{\ell_1}$ Mapping from L and $\mathcal{A}$ to an upper bound on $\|\bar{e}\|_1$

$\binom{n}{k_1,\ldots,k_m}$ The multinomial coefficient

$\binom{n}{k}$ n choose k

$\|\bar{e}\|_1$ Upper bound of the ℓ_1-norm of the steady state supremum of the interval error

$\lceil k \rceil$ The ceiling of k

$\mathbf{col}(v_1, v_2)$ Stacked column vector of v_1 and v_2

δ_{ij} The Kronecker delta

$\mathbf{diag}(v)$ Diagonal matrix with elements of v along the diagonal

$\Delta\overline{A}$ Matrix of admissible elementwise perturbations to the matrix $\overline{A}^{\downarrow}$

$\Delta\overline{x}_0$ Vector of admissible elementwise perturbations to $\overline{x}_0^{\downarrow}$

$\Delta\underline{A}$ Matrix of admissible elementwise perturbations to the matrix $\underline{A}^{\uparrow}$

$\Delta\underline{x}_0$ Vector of admissible elementwise perturbations to $\underline{x}_0^{\uparrow}$

$\lfloor k \rfloor$ The floor of k

Γ The stacked vector of cost and constraint protection levels

Γ_0 The cost protection level

Γ_i The protection level for the ith constraint

$\mathbf{\Gamma}$	The vector of constraint protection levels
$\hat{x}^l$	Lower state estimate
$\hat{x}^u$	Upper state estimate
$\hat{x}^{u,l}$	The pair $(\hat{x}^u, \hat{x}^l)$
$\mathbb{Z}$	The integers
$\mathbf{J}$	The set of sets of indices of uncertain constraint coefficients
$\|e\|_1$	The ℓ_1-norm of the interval error
$\mathcal{A}$	The set of admissible state matrix pairs for a given set of constraint protection levels
$\mathcal{F}^L_{KM}$	The set of feasible L matrices for the known model problems
$\mathcal{F}^L_{RUM}$	The set of feasible L matrices for the robust uncertain model problem
$\mathcal{F}^L_{UM}$	The set of feasible L matrices for the uncertain model problem
$\mathcal{J}$	The set of all admissible sets of indices of uncertain constraint coefficients
$\mathcal{L}_{KM}$	The set of optimal L matrices for the known model interval observer
$\mathcal{L}_{RKM}$	The set of optimal L for the robust known model interval observer
$\mathcal{L}_{RUM}$	The set of optimal L matrices for the robust uncertain model interval observer
$\mathcal{L}_{UM}$	The set of optimal L matrices for the uncertain model interval observer
$\mathcal{P}(S)$	The power set of S
$\mathcal{P}_k(S)$	The set of members of $\mathcal{P}(S)$ with cardinality no greater than k
$\mathcal{X}$	The set of all initial conditions effected by perturbing no more than Γ_0 initial conditions bounds
μ	Arithmetic mean
$\overline{A}^{\downarrow}$	Elementwise tighter upper bound on A
$\overline{x}_0^{\downarrow}$	Tighter upper bound on $x(0)$

$\overline{\xi}$	Upper bound on ξ
$\overline{A}$	Elementwise upper bound on A
$\overline{x}_0$	Upper bound on $x(0)$
Ω_0	Cost perturbation
Ω_i	Constraint perturbation for the ith constraint
$\mathbf{1}_n$	The column vector of 1s
ϕ	Function that retains negative elements and maps positive elements to 0
Ψ	Mapping from indices of uncertain cost coefficients to the set of all admissible initial conditions
ψ	Function that retains positive elements and maps negative elements to 0
$\mathbb{R}$	The real numbers
$\mathbf{S}$	The set of sets of indices of perturbed constraint coefficients
σ	Standard deviation
$\succeq, \preceq$	Elementwise nonstrict inequality, with strict inequality holding for at least one element
$\top$	Matrix transpose
$\underline{\xi}$	Lower bound on ξ
$\underline{A}$	Elementwise lower bound on A
$\underline{x}_0$	Lower bound on $x(0)$
$\underline{A}^{\uparrow}$	Elementwise tighter lower bound on A
$\underline{x}_0^{\uparrow}$	Tighter lower bound on $x(0)$
ε	An arbitrarily small positive constant
$\varnothing$	The empty set
$\widehat{E}$	Matrix of admissible constraint coefficient perturbations

Ξ Mapping from constraint coefficient perturbation indices to upper and lower state matrix pairs

ξ Nonlinearity in plant model

$\mathbf{0}_{m \times n}$ The matrix of 0s

$\mathbf{0}_n$ The column vector of 0s

A The state matrix

$A^{u,l}$ The pair (A^u, A^l)

b Vector on right-hand side of constraints

C The output matrix

c Cost vector

D The vector of admissible cost coefficient perturbations associated with Q

d Vector of admissible cost coefficient perturbations

E Constraint coefficient matrix

e Interval error

e^l Lower estimation error

e^u Upper estimation error

e_{ℓ_1} Map from L and $\mathcal{X}$ to $\|e\|_1$

I The identity matrix of contextually appropriate dimensions

J The set of sets of indices of uncertain coefficients

J_0 The set of indices of uncertain cost coefficients

J_i The set of indices of uncertain constraint coefficients in constraint i

L Observer gain matrix

Q Stacked vector of decision variables q and dummy variables q'

q Vector of decision variables

S_0 The set of indices of perturbed cost coefficients

S_i The set of indices of maximally perturbed coefficients in the ith constraint

t_i The index of the partially perturbed coefficient in the ith constraint

U The bit length of the binary encoding of the vectors c, b, and matrix E

x Plant state vector

y The output vector

KM Known model

KMIO Known model interval observer

LP Linear program

RKMIO Robust known model interval observer

UM Uncertain model

UMIO Uncertain model interval observer

Chapter 1

Introduction

1.1 Motivation

Suppose that you are a chemist who wishes to perform a large number of similar experiments, e.g., for system identification [1], [2], or in vitro testing of a new drug. Suppose that there is some base value, or perhaps a range of values, that is common to each experiment, e.g., the biomass of a particular bacterium, or the concentrations of particular chemical species. In each experiment, you will change certain parameters, specifically, the initial conditions, e.g., chemical species concentrations. In other words, there is a baseline set of maxima and minima for the initial conditions of your experiments, but in specific instances of the experiment, you vary specific parameters beyond the baseline extrema, but still within another set of known bounds that you have defined.

Now suppose that you have deployed an autonomous robot in an unknown environment. The robot depends on its sensor readings for its operation. But suppose some of its sensors sustain damage [3], making their readings unreliable. Rather than being able to trust the original sensor measurements, there is now a range of true state values given a reading from a damaged sensor. The body of the robot could also sustain damage [4], thereby modifying its dynamical parameters. If it is infeasible to access the robot for repair, the control systems of the robot must take into account the change in dynamics. If the robot is still serviceable, there must be some range of values in which the new dynamical parameters must lie, even if the new values are not known precisely.

Lastly, consider the following example, which we revisit later in this thesis.

Example 1.1.1. Consider the dynamical system

$$\begin{aligned}\dot{x} &= Ax \\ y &= Cx,\end{aligned} \tag{1.1}$$

where $A \in \mathbb{R}^{2\times 2}$, $C \in \mathbb{R}^{1\times 2}$, $x \in \mathbb{R}^2$, and $y \in \mathbb{R}$.

We wish to generate estimates of the state $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$ for the system (1.1). Specifically, we seek a range of estimates that contains the true state value at any given time $t \geq 0$. Suppose that we have identified an interval that is guaranteed to contain the initial conditions $x(0)$,

$$\begin{bmatrix} -2 \\ -1 \end{bmatrix} \leq x(0) \leq \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

But under certain operating conditions, we can refine the upper and lower bounds on the initial conditions of particular states,

$$\begin{aligned} x_1(0) \geq 0, \quad x_1(0) \leq 1.5 \\ x_2(0) \geq 0, \quad x_2(0) \leq 0.5, \end{aligned}$$

where it is possible that only a subset of upper and lower bounds may be refined. Consequently, depending on the situation, there are multiple possible intervals that might be certain to contain $x(0)$.

Suppose also that the coefficients of the state matrix A are uncertain. As with the initial conditions, we know an interval in which each coefficient must lie,

$$\begin{bmatrix} -3.3 & -1.1 \\ 0.9 & -4.4 \end{bmatrix} \leq A \leq \begin{bmatrix} -2.7 & -0.9 \\ 1.1 & -3.6 \end{bmatrix},$$

and tighter upper and lower bounds on particular coefficients under certain circumstances,

$$\begin{aligned} A_{11} &\geq -3.1, & A_{11} &\leq -2.9 \\ A_{21} &\geq 0.99, & A_{21} &\leq 1.01 \\ A_{12} &\geq -1.05, & A_{12} &\leq -0.95 \\ A_{22} &\geq -4.1, & A_{22} &\leq -3.9. \end{aligned}$$

As with the initial conditions, not every upper or lower bound on a particular coefficient of A may be refined in every situation.

These parameter bounds can be used to dynamically characterize upper and lower bounds on the state x at any given time $t \geq 0$. By using the refined bounds on the uncertain parameters, when applicable, we can generate improved bounds on the states. However, note the combinatorial nature of the upper and lower parameter bounds; as we discuss later, a system of small dimensions admits a relatively large number of parameter bound combinations. We propose a method of observer design such that a single observer will exhibit optimal performance, with respect to particular objectives, for all systems in the class defined by the number of unrefined parameter bounds.

$\triangle$

In each of these scenarios, the dynamical parameters of the system are uncertain, but lie within a known interval. Whether by design, or exposure to deleterious environmental conditions, these intervals can change in predictable ways, i.e., they are perturbed by *a priori* known amounts. The problem we address is the design of *interval observers* whose upper and lower estimates are guaranteed to bound the true states, if at most k of the plant's initial conditions and dynamical parameters have been perturbed. We wish to design these observers such that the size of the interval between the upper and lower observers, as measured by specific norms, is optimal. The optimality of these interval observers does not depend on which specific k coefficients are perturbed.

1.2 Interval Observers

Given a plant to be monitored or controlled, it is not always possible to directly measure all of its states. An observer is a dynamical system that uses an internal model of the plant, and the measured outputs, to generate state estimates. A classic observer (e.g., Luenberger, high-gain) generates a single dynamical estimate for each plant state. The difference between the observer's estimates and the plant's states is called the *estimation error*. Ideally, the norm of the estimation error tends to zero as time tends to infinity. However, the limit of the estimation error is dependent upon the accuracy of the plant model. If the plant model is imprecise, or if it is characterized by uncertain parameters, the classic observer may generate unreliable estimates.

An interval observer is a pair of observers whose dynamics and initial conditions are defined such that their trajectories characterize upper and lower bounds on the state values at any given instant. The upper observer is designed using values of the uncertain parameters such that its estimates bound the true states from above at any given instant. The lower observer is designed using values of the uncertain parameters such that its estimates bound the true states from below at any given instant. The interval between the upper and lower observers is guaranteed to contain the true state values. Ideally, the distance between the upper and lower observers converges to zero asymptotically, but the form of the uncertain dynamics may preclude this.

Interval observers are useful in situations where the plant dynamics are highly uncertain, or sensor readings are unreliable. Despite having access to only the basic structure of the plant's dynamics, and imprecise parameter values, an interval observer characterizes bounds on the values of the plant's states. Classic observers may be augmented with an interval observer to bound its state estimates, or to monitor its convergence [5].

The bounded observation problem was first addressed using recursive methods in [6]; a time-varying ellipsoid was used to identify a set in the state space that is guaranteed to contain the plant's states. Bounding ellipsoids were also used in [7]. Set-based state estimate bounding for uncertain systems was extended to parallelotopes in [8] and zonotopes in [9]. Dynamical interval observers were first described in [10], wherein they were designed for a monotone system, specifically a wastewater management treatment model. Interval observer theory was extended to non-monotone systems in [11]. Necessary and sufficient conditions for the existence of a class of interval observers for a class of non-monotone systems were identified in [12], [13] in terms of linear constraints; the observers proposed therein are optimal with respect to various norms of the difference between estimates of the upper and lower observers. The work of [12], [13] was extended to positive linear systems with time-varying delays in [14]. The theory in [12] was also extended to systems with uncertain outputs in [15]. Necessary and sufficient conditions were identified in [16] for the existence of a class of interval observers for continuous and discrete-time positive linear systems, described in terms of linear matrix inequalities. It was shown in [5] that, using a time-varying change of coordinates, a class of interval observers can be constructed for any 2-dimensional, stable, detectable system.

Interval observers have seen much success in in the area of biological processes, particularly wastewater management systems [17], [18], [19], [11], [20]. They have also been applied to population dynamics [18], algae cultures [21], robotics [22], circuits [15], fault-detection [23], pharmacokinetics [13], and localization of autonomous underwater vehicles [24]. They have also been used for feedback control [25], [26].

1.3 Robust Optimization

Constrained optimization problems involve the minimization, or maximization, of an objective function subject to constraints, which define a set of feasible solutions. It is assumed in the construction and solution of constrained optimization problems that the parameters are known; there is no possibility of variation, unless somehow explicitly accounted for in the model. However, even relatively small parametric perturbations can result in far-from-optimal, or even highly infeasible solutions [27], as illustrated in [28].

The goal of robust optimization is to generate solutions that, under a particular model of uncertainty, are deterministically feasible. A problem is made robust against a particular model of uncertainty, at the expense of increased nominal cost, in exchange for guaranteed feasibility under what is called the budget of uncertainty [27]. As this thesis uses only LPs, we limit this survey to robust linear optimization models of uncertainty.

An ellipsoidal uncertainty model [29], [30] considers elementwise perturbations in the coefficients of the nominal problem. The budget of uncertainty under this model is the norm of the vector comprising the elementwise coefficient perturbations. The robust problem under the ellipsoidal model is not linear, but a second order cone program. A special case of the ellipsoidal model of uncertainty is polyhedral uncertainty [27]. Under this model of uncertainty, the perturbations of the coefficients are bounded by linear constraints, effecting a combinatorial set of admissible perturbed coefficients. The budget of uncertainty under the polyhedral model is the elementwise bounds on the perturbed coefficients. An attractive property of the polyhedral model of uncertainty is that an LP's robust formulation is linear.

The norm model of uncertainty [31] characterizes the perturbed coefficients in terms of norms of the vector comprising the elementwise perturbations of the constraint matrix. The budget of uncertainty under this model is the upper bound of this norm. Depending on the norm used, the robust problem is either a second order cone program, or a linear program.

Lastly, the model of uncertainty used in this thesis is cardinality constrained uncertainty [32], [33]. Under this model of uncertainty, the solution is guaranteed to be feasible if no more than k parameters are subjected to bounded perturbations. The budget of uncertainty is the number of coefficients protected against perturbation.

Robust optimization techniques have seen varied application, including antenna design [34], truss topology design [35], constrained stochastic linear-quadratic control [36], circuit design [37], [38], wireless channel power control [39], and portfolio optimization [40]. The reader is referred to [27] for a comprehensive review of the robust optimization literature.

1.4 Proposed Approach

In [12], necessary and sufficient conditions for the existence of a class of interval observers for a particular class of systems are identified and expressed as linear constraints. Two cases are examined: 1) no model uncertainty; 2) model uncertainty. These constraints are used to define linear programs (LPs) used to design optimal interval observers. In the case of no model uncertainty, the cost function of the LP is the ℓ_1-norm of the difference between the upper and lower estimates, and is a function of the difference between the upper and lower bounds on the initial conditions; in the case of model uncertainty, the cost represents an upper bound on the ℓ_1-norm of the steady state supremum of the difference between the upper and lower estimates, and is a function of the difference between the maximum and minimum values of the uncertain dynamical parameters.

We propose a method of interval observer design that leverages additional, but still-incomplete, information about these uncertain parameters. We assume that for each uncertain parameter, that there are four bounds: a loose upper and lower bound, called the *outer bounds*, and a tight upper and lower bound, called the *inner bounds*. The outer bounds are guaranteed to contain the uncertain parameter under all circumstances, but the inner bounds only contain the known parameter under particular, but known, circumstances. It is possible for one of the bounds to be tight and one to be loose, e.g., the parameter may be known to lie between the inner lower bound and the outer upper bound. For a given k, we assume that k bounds are outer bounds, one bound lies between its inner and outer bound, if and only if k is nonintegral, and all others are outer bounds; we refer to a combination of k such bounds as a *bound set*.

In the design phase, we do not know which bounds will be inner bounds, we know only k. However, at implementation, the bound set is known. Applying the robust optimization method under a model of cardinality constrained uncertainty proposed in [41], to the linear programming-based interval observer design method of [12], we design the observer such that the maximum of the aforementioned norms over all bound sets defined by k is minimized, i.e., a single optimal interval observer is designed for an entire class of systems. Since we know the bound set at implementation, we can choose appropriate initial conditions and dynamics for the interval observer — without redesigning — such that the observer is an interval observer.

The benefit of optimizing over smaller bound sets is that it yields a lower cost, thereby improving performance of the interval observer. If only the initial conditions are uncertain, using a less-conservative model of uncertainty reduces the ℓ_1-norm of the difference between the upper and lower estimates. If the system's dynamics are uncertain, using a less-conservative model reduces the ℓ_1-norm of the steady state supremum of the difference between the upper and lower estimates. A benefit of using the robust framework of [41] is that only a single optimization must be performed for a class of perturbations, rather than for each combination of perturbations within that class. As is characterized later in this thesis, this circumvents the issue of combinatorial explosion of bound sets.

1.5 Notation and Terminology

Given a matrix $M \in \mathbb{R}^{m \times n}$, the notation M_{ij} denotes the element of M in the ith row and the jth column, and M_j denotes the jth column of M. Given a vector $v \in \mathbb{R}^n$, v_j denotes the jth element of v. Given a vector $v \in \mathbb{R}^n$, $\mathbf{diag}(v) \in \mathbb{R}^{n \times n}$ is the matrix whose (i,i)th element is v_i, and all off-diagonal elements are 0. Given a matrix $M \in \mathbb{R}^{m \times n}$, let $M^\top \in \mathbb{R}^{n \times m}$ be its transpose, and for $M \in \mathbb{R}^{n \times n}$, let $M^{-\top}$ be the transpose of its inverse; for real matrices, $(M^\top)^{-1} = (M^{-1})^\top$. Given two vectors $v_1 \in \mathbb{R}^m$ and $v_2 \in \mathbb{R}^n$, define the column vector $\mathbf{col}(v_1, v_2) \coloneqq \begin{bmatrix} v_1^\top & v_2^\top \end{bmatrix}^\top \in \mathbb{R}^{m+n}$; the **col** function extends to an arbitrary number of arguments. When applied to a matrix $M \in \mathbb{R}^{m \times n}$, $\mathbf{col}(M) \coloneqq \mathbf{col}(M_1, \dots, M_n) \in \mathbb{R}^{mn}$. The vector $\mathbf{1}_n \in \mathbb{R}^n$ is the column vector of 1s. The vector $\mathbf{0}_n \in \mathbb{R}^n$ is the column vector of 0s, and the matrix $\mathbf{0}_{m \times n} \in \mathbb{R}^{m \times n}$ is the matrix of 0s. The matrix $I \in \mathbb{R}^{n \times n}$ is the identity matrix of contextually appropriate dimensions.

The *Kronecker delta* is defined as

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$$

The *binomial coefficient* is defined as

$$\binom{n}{k} \coloneqq \frac{n!}{k!(n-k)!},$$

and is pronounced "n choose k". The *multinomial coefficient* is defined as

$$\binom{n}{k_1, \dots, k_m} \coloneqq \frac{n!}{\prod_{i=1}^m k_i!},$$

and is pronounced "n choose $k_1, \dots, k_m$". Given a scalar $k \in \mathbb{R}$, its floor $\lfloor k \rfloor \in \mathbb{Z}$ is the value of k rounded down to the nearest integer, and its ceiling $\lceil k \rceil \in \mathbb{Z}$ is the value of k rounded up to the nearest integer.

A matrix $M \in \mathbb{R}^{n \times n}$ is said to be *Hurwitz* if the real part of each of its eigenvalues is negative. A matrix $M \in \mathbb{R}^{n \times n}$ is said to be *Metzler* if all its off-diagonal elements are nonnegative, i.e., $M_{ij} \geq 0$ for all $i \neq j$, $i, j \in \{1, \dots, n\}$.

Given a matrix $M \in \mathbb{R}^{n \times n}$ and a locally Lipschitz function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, the dynamical system $\dot{x}(t) = Mx(t) + g(x,t)$ is said to be *positive* if $x(0) \geq 0$ implies $x(t) \geq 0$ for all $t \geq 0$. This is true if M is Metzler and $g(x,t) \geq 0$ for all $t \geq 0$.

When applied to vectors or matrices, the relations $>, <, \geq, \leq$ are taken elementwise. Given matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$, the relation $A \succeq B$ is defined as $A \geq B$, with $A_{ij} > B_{ij}$ for at least one pair (i, j); the relation $A \preceq B$ is defined as $A \leq B$, with $A_{ij} < B_{ij}$ for at least one pair (i, j). Given a scalar $c \in \mathbb{R}$, vector $v \in \mathbb{R}^n$, or matrix $M \in \mathbb{R}^{m\times n}$, the operator $|\cdot|$ is the elementwise absolute value. Given a set S, its cardinality is denoted by $|S|$. Denote by $\mathcal{P}(S)$ the power set of S, i.e., $\mathcal{P}(S) \coloneqq \{S' \mid S' \subseteq S\}$, and by $\mathcal{P}_k(S)$ the set $\{S' \mid S' \subseteq S, |S'| \leq k\} \subseteq \mathcal{P}(S)$; for convenience, for $k \leq 0$, we define $\mathcal{P}_k(S) \coloneqq \varnothing$.

1.6 Contributions

The main contributions of this thesis are:

1. An interval observer design procedure for systems whose only uncertain parameters are their initial conditions.

2. An interval observer design procedure for systems with uncertain dynamical parameters and initial conditions.

3. A method of associating multiple distinct perturbations with individual parameters in a robust cardinality constrained linear program of the form proposed in [41].

1.7 Outline

In Chapter 2, we provide an overview of the robust optimization method of [41]. In Chapter 3, we provide an overview the linear programming-based interval observer design method of [12]. In Chapter 4, we delineate the proposed robust formulation of the interval observer in the case where only the initial conditions of the plant are uncertain. In Chapter 5, we delineate the proposed robust formulation of the interval observer in the case where the dynamics of the plant are characterized by uncertain parameters, in addition to uncertain initial conditions. In Chapters 4 and 5, the performance of the proposed observers is characterized. In Chapter 6, we conclude the thesis and identify areas of future research.

Chapter 2

Cardinality Constrained Robust Optimization

In [41], a method is proposed for protecting against varying numbers of perturbed coefficients, given an LP of the form

Problem 1:

$$\begin{aligned} \textit{minimize: } & c^\top q \\ \textit{subject to: } & Eq \leq b \\ & l \leq q \leq u, \end{aligned}$$

where $E \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, *and* $q, l, u, c \in \mathbb{R}^n$.

All uncertainty is assumed to be in the constraint coefficient matrix E and cost coefficient vector c. The vectors b, l, and u are assumed to be known. Uncertainty in b can be modelled by augmenting q and E [41, Section 2].

The set J_0 contains the indices of the uncertain cost coefficients, and the set J_i, $i \in \{1, \ldots, m\}$ contains the indices of the uncertain coefficients in the ith constraint. If $j \in J_0$, then the jth element of the cost vector lies in the interval $[c_j, c_j + d_j]$, where the cost perturbation term $d_j \geq 0$ for all j. Similarly, if $j \in J_i$, $i \in \{1, \ldots, m\}$, then the jth coefficient of the ith constraint lies in the interval $[E_{ij} - \widehat{E}_{ij}, E_{ij} + \widehat{E}_{ij}]$, where the constraint coefficient perturbation term $\widehat{E}_{ij} \geq 0$ for all i, j. The values of d_j and $\widehat{E}_{ij}$ are known for all i, j. The *protection levels* $\Gamma_0 \in \mathbb{Z}_{\geq 0}$ and $\Gamma_i \in \mathbb{R}_{\geq 0}$, $i \in \{1, \ldots, m\}$, specify the number of perturbations to be protected against in the cost and ith constraints, respectively. The vector $\Gamma \coloneqq \mathbf{col}(\Gamma_0, \Gamma_1, \ldots, \Gamma_m)$ specifies only the cardinalities of the sets of protected coefficients, it does not specify individual coefficients to be protected.

Example 2.0.1. Consider an LP with the form of Problem 1, with the uncertain cost coefficient vector and constraint coefficient matrix

$$\begin{bmatrix}1\\1\end{bmatrix} \leq c \leq \begin{bmatrix}1\\1.2\end{bmatrix} \qquad \begin{bmatrix}0.9 & 2\\-1 & 2\end{bmatrix} \leq E \leq \begin{bmatrix}1.1 & 2\\-0.5 & 3\end{bmatrix}.$$

Given the range of possible values of the cost and constraint coefficients, we take the nominal coefficients to be

$$c = \begin{bmatrix}1\\1\end{bmatrix} \qquad E = \begin{bmatrix}1 & 2\\-0.75 & 2.5\end{bmatrix},$$

and the coefficient perturbations to be

$$d = \begin{bmatrix}0\\0.2\end{bmatrix} \qquad \widehat{E} = \begin{bmatrix}0.1 & 0\\0.25 & 0.5\end{bmatrix}.$$

A coefficient is uncertain if its perturbation is nonzero. For example, $d_1 = 0$, so it is not uncertain, whereas $d_2 = 0.2$, so it is uncertain. The sets of indices of uncertain cost and constraint coefficients are

$$\begin{aligned}J_0 &= \{2\}\\ J_1 &= \{1\}\\ J_2 &= \{1,2\}.\end{aligned}$$

--- △

The robust formulation of Problem 1, as developed in [41], is given in Problem 2.

Problem 2:

$$\begin{aligned}&\textit{minimize:}\, c^\top q + \max_{\{S_0 | S_0 \subseteq J_0, |S_0| \leq \Gamma_0\}} \left\{ \sum_{j \in S_0} d_j |q_j| \right\}\\ &\textit{subject to:} \sum_{j=1}^{n} E_{ij} q_j + \max_{\{S_i \cup \{t_i\} | S_i \subseteq J_i, |S_i| \leq \lfloor \Gamma_i \rfloor, t_i \in J_i \setminus S_i\}} \left\{ \sum_{j \in S_i} \widehat{E}_{ij} |q_j| + \big(\Gamma_i - \lfloor \Gamma_i \rfloor\big) \widehat{E}_{it_i} |q_{t_i}| \right\} \leq b_i\\ &\qquad l \leq q \leq u, \qquad\qquad i \in \{1, \dots, m\}\end{aligned}$$

The cost of Problem 1 has been augmented with

$$\Omega_0(q, \Gamma_0) := \max_{\{S_0 | S_0 \subseteq J_0, |S_0| \leq \Gamma_0\}} \left\{ \sum_{j \in S_0} d_j |q_j| \right\}. \tag{2.1}$$

The set $S_0 \subseteq J_0$ in (2.1) contains the indices of perturbed cost coefficients. Maximizing over S_0 identifies the set of coefficients which, when perturbed, result in the greatest cost for a given solution q. By augmenting the cost with Ω_0, minimizing the cost yields an optimal solution q^* that minimizes the maximum cost over the class of perturbed cost functions defined by Γ_0.

Similarly, the constraints $Eq \leq b$ of Problem 1 have been augmented with

$$\Omega_i(q, \Gamma_i) := \max_{\{S_i \cup \{t_i\} | S_i \subseteq J_i, |S_i| \leq \lfloor \Gamma_i \rfloor, t_i \in J_i \setminus S_i\}} \left\{ \sum_{j \in S_i} \widehat{E}_{ij} |q_j| + \big(\Gamma_i - \lfloor \Gamma_i \rfloor\big) \widehat{E}_{it_i} |q_{t_i}| \right\}, \tag{2.2}$$

where $i \in \{1, \ldots, m\}$. If $q_j > 0$, then E_{ij} will be perturbed in the positive direction, if $q_j < 0$, then E_{ij} will be perturbed in the negative direction. Notice that Ω_i is nonnegative for $i \in \{0, \ldots, m\}$. The set of indices $S_i \cup \{t_i\}$ identifies the $\lfloor \Gamma_i \rfloor$ constraint coefficients E_{ij} to be perturbed by $\widehat{E}_{ij}$, $j \in S_i$, and the constraint coefficient E_{it_i} to be perturbed by $\big(\Gamma_i - \lfloor \Gamma_i \rfloor\big) \widehat{E}_{it_i}$, that maximizes the increase in $\sum_j E_{ij} q_j$. This has the following implication for feasibility.

Proposition 2.0.1. *Any feasible solution to Problem 2 for given* Γ_i, $i \in \{1, \ldots, m\}$, *is also a feasible solution for any* $\Gamma_i' \leq \Gamma_i$, $i \in \{1, \ldots, m\}$.

Proof. By construction, a constraint perturbation (2.2) increases the left-hand side of constraints of the form $\sum_j E_{ij} q_j \leq b_i$. Therefore, as Γ_i increases, thereby increasing the magnitude of $\Omega_i(q, \Gamma_i)$, solutions that satisfy constraint i with insufficient slack become infeasible. Conversely, as Γ_i decreases, the left-hand side of $\sum_j E_{ij} q_j \leq b_i$ decreases, thereby expanding the set of feasible solutions. Consequently, for fixed E, $\widehat{E}$, b, l, u, the set of feasible solutions to Problem 2 using the protection levels $\Gamma_i' \leq \Gamma_i$, $i \in \{1, \ldots, m\}$ is a superset of the feasible solutions to Problem 2 using the constraint protection levels Γ_i, $i \in \{1, \ldots, m\}$. $\square$

Note that Problem 1 is equivalent to Problem 2 when $\Omega_i = 0$, $i \in \{0, \ldots, m\}$, which is achieved by setting $\Gamma_i = 0$ for all i, as this yields $S_i = \varnothing$ for all i and $\big(\Gamma_i - \lfloor \Gamma_i \rfloor\big) = 0$, $i \in \{1, \ldots, m\}$.

Example 2.0.2. Consider the LP described in Example 2.0.1. Suppose the constraint protection levels are $\Gamma_1 = 0$ and $\Gamma_2 = 1$. The robust solution is deterministically feasible for the perturbed constraint matrices

$$\begin{bmatrix} 1 & 2 \\ -0.75 & 2.5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ -1 & 2.5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ -0.75 & 3 \end{bmatrix},$$

since 0 elements in the first row have been perturbed and at most 1 element in the second row has been perturbed. However, the robust solution will not necessarily be feasible for the perturbed constraint matrices

$$\begin{bmatrix} 1.1 & 2 \\ -0.75 & 2.5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}.$$

In the first matrix, $1 > \Gamma_1$ element has been perturbed in the first row, and in the second matrix, $2 > \Gamma_2$ elements have been perturbed in the second row. Setting $\Gamma_1 = |J_1| = 1$ and $\Gamma_2 = |J_2| = 2$ would ensure that the robust solution is feasible for all possible perturbed coefficient matrices.

△

Problem 2 has the following equivalent linear formulation [41, Theorem 1].

Problem 3:

$$\begin{aligned}
\textit{minimize:}\ & c^\top q + \zeta_0 \Gamma_0 + \sum_{j \in J_0} p_{0j} & \\
\textit{subject to:}\ & \sum_j E_{ij} q_j + \zeta_i \Gamma_i + \sum_{j \in J_i} p_{ij} \le b_i & \forall\, i \ne 0 \\
& \zeta_0 + p_{0j} \ge d_j y_j & \forall\, j \in J_0 \\
& \zeta_i + p_{ij} \ge \widehat{E}_{ij} y_j & \forall\, i \ne 0, j \in J_i \\
& p_{ij} \ge 0 & \forall\, i, j \in J_i \\
& y_j \ge 0 & \forall\, j \\
& \zeta_i \ge 0 & \forall\, i \\
& -y_j \le q_j \le y_j & \forall\, j \\
& l_j \le q_j \le u_j & \forall\, j,
\end{aligned}$$

where, $q_j, \zeta_0, p_{0j}, \zeta_i, p_{ij}, y_j \in \mathbb{R}$, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$, are decision variables.

Optimizing over the new decision variables ζ_0 and p_{0j}, and ζ_i and p_{ij}, $i \ne 0$, is equivalent to selecting the sets S_0 and S_i, $i \ne 0$, respectively, in Problem 2 [41, Proof of Theorem 1]. The vector of new decision variables y in Problem 3, and its associated constraints, are the linear equivalent of taking the absolute values of the elements of q in the perturbation terms (2.1) and (2.2).

When exclusively uncertain cost coefficients are considered, the variables ζ_i and p_{ij} for $i \ne 0$ may be eliminated, as they pertain only to uncertainty of constraint coefficients. For indices associated with nonnegative decision variables, i.e., $j \in \{j' \mid l_{j'} \geq 0\}$, there is no need to take the absolute value of q_j when computing the perturbation terms (2.1), (2.2). Since the vector of decision variables y and its associated constraints are the linear equivalent of the absolute value operation, we may eliminate y_j and the constraints $y_j \geq 0$, and replace y_j with q_j in the constraints $\zeta_0 + p_{0j} \geq dy_j$ and $\zeta_i + p_{ij} \geq \widehat{E}y_j$, for $j \in \{j' \mid l_{j'} \geq 0\}$.

2.1 Cardinalities

One of the advantages of the robust formulation of [41], is that a single optimization is needed for a class of perturbations. In this section, we characterize the cardinalities of admissible perturbation sets.

Proposition 2.1.1. *For a fixed J_0 and Γ_0, the number of combinations of indices of perturbed cost coefficients S_0 is*

$$\sum_{k=0}^{\Gamma_0} \binom{|J_0|}{k}.$$

Proof. The set of possible S_0 contains all S_0 that contain k indices of perturbed coefficients, $k \in \{0, \dots, \Gamma_0\}$. For a fixed k, the number of combinations is $|J_0|$ choose k. Therefore, the number of possible S_0 is the sum of $|J_0|$ choose k, $k \in \{0, \dots, \Gamma_0\}$. □

Proposition 2.1.2. *For fixed J_i and Γ_i, $i \in \{1, \dots, m\}$, the number of combinations of indices of perturbed constraint coefficients is*

$$\prod_{i=1}^{m} \left\{ \sum_{k=0}^{\lfloor \Gamma_i \rfloor} \binom{|J_i|}{k} + \left(1 - \delta_{\lceil \Gamma_i \rceil \lfloor \Gamma_i \rfloor}\right) \binom{|J_i|}{\lfloor \Gamma_i \rfloor, 1} \right\}.$$

Proof. For a fixed i, the set of possible constraint perturbations contains all S_i that contain $\lfloor k \rfloor$ indices of maximally perturbed coefficients, $k \in \{0, \dots, \lfloor \Gamma_i \rfloor\}$. For a fixed k, the number of combinations of strictly maximal perturbations is $|J_i|$ choose $\lfloor k \rfloor$. Therefore, the number of possible S_i is the sum of J_i choose k, $k \in \{0, \dots, \lfloor \Gamma_i \rfloor\}$.

By construction of the constraint perturbation term (2.2), no coefficient will be partially perturbed unless the maximum number of coefficients have been maximally perturbed, i.e., $\lfloor \Gamma_i \rfloor$ coefficients have been maximally perturbed, and at most one coefficient will be partially perturbed. There are therefore $|J_i|$ choose $\lfloor k \rfloor, 1$ combinations of $\lceil \Gamma_i \rceil$ perturbations for nonintegral Γ_i. If $\lceil \Gamma_i \rceil = \lfloor \Gamma_i \rfloor$, i.e., for integral Γ_i, there are no such combinations.

The number of index combinations for an individual constraint is the sum of the number of combinations of indices of strictly maximal perturbations and the number of combinations of indices of $\lceil \Gamma_i \rceil$ perturbations. By the product rule of counting, the number of combinations across all constraints is equal to the product of the numbers of index combinations over all m constraints. □

Example 2.1.1. Consider the following nominal coefficients and perturbations,

$$c = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad d = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad \widehat{E} = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

which gives us $J_0 = \{1, 2\}$ and $J_1 = \{1, 2\}$. We take $\Gamma_0 = \Gamma_1 = 2$.

By Proposition 2.1.1, there are four combinations of the indices of cost perturbations,

$$S_0 \in \big\{\varnothing, \{1\}, \{2\}, \{1, 2\}\big\},$$

hence, the perturbed cost vector can take any value in the set

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}.$$

By Proposition 2.1.2, there are four combinations of the indices of constraint perturbations,

$$S_1 \in \big\{\varnothing, \{1\}, \{2\}, \{1, 2\}\big\}.$$

However, unlike the cost coefficient perturbations, the constraint coefficient perturbation corresponding to E_{1j}, $j \in S_1$, may be positive or negative, depending on the sign of the decision variable q_j. There are therefore two perturbed coefficient values we must consider for each $j \in S_1$. The perturbed constraint coefficient matrix can take any value in the set

$$\left\{ \begin{bmatrix} 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \end{bmatrix} \right\}.$$

We see that even for $n = 2$, $m = 1$, there is a relatively large number of potential instances of an uncertain problem. In this example, there are four possible cost vectors, and nine possible constraint matrices, yielding a total of 36 problem instances.

--- △

Example 2.1.1 illustrates the potential computational benefit of the robust formulation of [41]: a single solution is generated, which is optimal over the entire class of perturbed problem instances defined by Γ, in the sense that the highest cost over this class for this fixed solution is minimized.

To illustrate the rapid growth in the number of problem instances to consider as n increases and Γ is varied, we list the number of admissible cost vectors and constraint matrices in Table 2.1. We assume that $|J_i| = n$, for all i, and that all Γ_i, $i \neq 0$, are equal.

Table 2.1: Number of combinations of indices of cost and constraint coefficients for various problem dimensions and protection levels.

<table>
<tr><th>n</th><th>m</th><th>Γ_0</th><th>Γ_i</th><th>$\left\vert\{\text{Possible } S_0\}\right\vert$</th><th>$\left\vert\left\{\text{Possible } \bigcup_{i\neq 0} S_i \cup \{t_i\}\right\}\right\vert$</th></tr>
<tr><td>2</td><td>2</td><td rowspan="3">2</td><td rowspan="2">2</td><td rowspan="3">4</td><td>16</td></tr>
<tr><td rowspan="2">2</td><td rowspan="2">3</td><td>64</td></tr>
<tr><td>1.5</td><td>125</td></tr>
<tr><td>3</td><td>3</td><td rowspan="3">3</td><td rowspan="2">3</td><td rowspan="3">8</td><td>512</td></tr>
<tr><td rowspan="2">3</td><td rowspan="2">4</td><td>4096</td></tr>
<tr><td>2.5</td><td>10000</td></tr>
<tr><td>4</td><td>4</td><td rowspan="3">4</td><td rowspan="2">4</td><td rowspan="3">16</td><td>65536</td></tr>
<tr><td rowspan="2">4</td><td rowspan="2">5</td><td>1048576</td></tr>
<tr><td>3.5</td><td>2476099</td></tr>
<tr><td>5</td><td>5</td><td rowspan="3">5</td><td rowspan="2">5</td><td rowspan="3">32</td><td>33554432</td></tr>
<tr><td rowspan="2">5</td><td rowspan="2">6</td><td>1073741824</td></tr>
<tr><td>4.5</td><td>2176782336</td></tr>
</table>

2.2 Equality Constraints

The LPs used to design the interval observers in [12] contain equality constraints, which are incompatible with the stipulated form of constraints in Problem 1. We characterize the effect of the protection process of [41] on strict equality constraints with uncertain coefficients.

Proposition 2.2.1. *Given a problem of the form of Problem 2, if there is an equality constraint with at least one uncertain coefficient and a nonzero protection level, i.e.,* $J_i \neq \varnothing$ *and* $\Gamma_i > 0$*, where constraint* i *is an equality constraint, then Problem 2 is infeasible.*

Proof. An equality constraint can be formulated as two inequality constraints,

$$\sum_j E_{ij} q_j = b_i \implies \begin{cases} \sum_j E_{ij} q_j \leq b_i \\ \sum_j E_{ij} q_j \geq b_i \end{cases} \implies \begin{cases} \sum_j E_{ij} q_j \leq b_i \\ -\sum_j E_{ij} q_j \leq -b_i. \end{cases} \tag{2.3}$$

Denoting the index of the second constraint by i', the robust formulation of (2.3) is

$$\sum_j E_{ij} q_j + \Omega_i(q, \Gamma_i) \leq b_i$$
$$-\sum_j E_{ij} q_j + \Omega_{i'}(q, \Gamma_{i'}) \leq -b_i.$$

Summing these constraints, we have

$$\Omega_i(q, \Gamma_i) + \Omega_{i'}(q, \Gamma_{i'}) \leq 0. \tag{2.4}$$

Since $\Omega_i \geq 0$, with equality holding if and only if $\Gamma_i = 0$, inequality (2.4) is satisfied if and only if $\Gamma_i = \Gamma_{i'} = 0$. Therefore, equality constraints with uncertain coefficients render Problem 2 infeasible for any nonzero protection level. □

This issue is addressed in our treatment of interval observers.

2.3 Multiple Distinct Perturbations to Individual Cost Coefficients

Problem 2 protects against Γ_0 cost coefficient perturbations. However, this method is based on the assumption that the perturbing of a particular cost coefficient is binary, i.e., it is either maximally perturbed, or it is not perturbed. As we shall discuss in our treatment of interval observers, there can be multiple possible perturbations for a particular cost coefficient, each having a distinct physical interpretation. We propose a method of modifying Problem 2 such that individual cost coefficients may be protected against multiple distinct perturbations. We present this process for the case where there are two distinct perturbations, but this process can be extended to an arbitrary number of distinct perturbations.

Define the vector of dummy variables $q' \in \mathbb{R}^n$ to be a copy of the decision variable vector q. Define $d^0, d' \in \mathbb{R}^n_{\geq 0}$ to be the vectors of cost coefficient perturbations associated with q and q', respectively, in the augmented problem. Define the stacked decision variable vector $Q \coloneqq \mathbf{col}(q, q') \in \mathbb{R}^{2n}$, and the stacked perturbation vector $D \coloneqq \mathbf{col}(d^0, d') \in \mathbb{R}^{2n}_{\geq 0}$. We stipulate that

$$d^0 + d' = d, \tag{2.5}$$

which implies that, for $j \in \{1, \ldots, n\}$, $D_j + D_{j+n} = d_j$, and impose the constraint

$$q' = q, \tag{2.6}$$

which implies $Q_{j+n} = Q_j$.

Replacing q with Q, and d with D in the cost perturbation term (2.1), yields the objective

$$\text{minimize: } c^\top q + \max_{\{S_0 | S_0 \subseteq J_0, |S_0| \leq \Gamma_0\}} \left\{ \sum_{j \in S_0} D_j Q_j \right\}.$$

Note that introducing the decision variable q' implicitly extends the cost vector c to $c' \coloneqq \mathbf{col}(c, \mathbf{0}_n)$, i.e., the dummy variables q' are nominally unweighted, but their associated cost coefficients can be perturbed. This change in the cost coefficient vector changes the set of indices of uncertain cost coefficients from $J_0 \subseteq \{1, \ldots, n\}$ to $J_0' \subseteq \{1, \ldots, 2n\}$.

Proposition 2.3.1. *For a fixed $k \in J_0$, $\big\{k, (k+n)\big\} \cap J_0' \neq \varnothing$, perturbing the cost coefficients c_k' and c_{k+n}' by D_k and D_{k+n}, respectively, is equivalent to perturbing the cost coefficient c_k by d_k in Problem 2.*

Proof. Both Problem 2 and the augmented problem have an unperturbed cost of $c^\top q$. We therefore need only show that the perturbations are equal, i.e.,

$$d_k q_k = D_k Q_k + D_{k+n} Q_{k+n}.$$

By the definitions of D and Q, and applying (2.5) and (2.6), we have

$$\begin{aligned} D_k Q_k + D_{k+n} Q_{k+n} &= d_k^0 q_k + d_k' q_k' \\ &= (d_k^0 + d_k') q_k \\ &= d_k q_k. \end{aligned}$$

□

Proposition 2.3.2. *If $\Gamma_0 = \Gamma_0' = 0$, or $\Gamma_0 = |J_0|$, $\Gamma_0' = |J_0'|$, then the augmented problem yields the same optimal solution as Problem 2.*

Proof. Since the cost perturbation term (2.1) is 0 for a protection level $\Gamma_0 = \Gamma_0' = 0$, the costs of Problem 2 and the augmented problem are $c^\top q$, proving the claim for $\Gamma_0 = \Gamma_0' = 0$.

For $\Gamma_0 = |J_0|$, $\Gamma_0' = |J_0'|$, all uncertain cost coefficients are perturbed. The cost of Problem 2 is perturbed by

$$\sum_{j \in J_0} d_j |q_j|, \tag{2.7}$$

and the cost of the augmented problem is perturbed by

$$\sum_{j \in J_0'} D_j |Q_j|. \tag{2.8}$$

Applying Proposition 2.3.1 for all $k \in J_0$, we have that (2.7) and (2.8) are equal, thereby proving the claim. □

Chapter 3

Interval Observers

We propose a method of interval observer design, based on that of [12], for systems of the form

$$\begin{aligned} \dot{x} &= Ax + \xi(x,t) \\ y &= Cx, \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$ is uncertain but bounded, the nonlinear function $\xi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is uncertain but bounded, $y \in \mathbb{R}^p$ is the output, and $C \in \mathbb{R}^{p\times n}$. We make the following assumptions regarding the boundedness of A and ξ.

Assumption 1. Given a system of the form (3.1), there exist known matrices $\underline{A}$, $\overline{A} \in \mathbb{R}^{n\times n}$, such that $\underline{A} \leq A \leq \overline{A}$.

Assumption 2. Given a system of the form (3.1), there exist known constants $\underline{\xi}$, $\overline{\xi} \in \mathbb{R}^n$, such that $\underline{\xi} \leq \xi(x,t) \leq \overline{\xi}$, for all $t \geq 0$ and all $x \in \mathbb{R}^n$.

An *interval observer* comprises an *upper observer*, whose estimates $\hat{x}^u$ bound the true states x from above, and a *lower observer*, whose estimates $\hat{x}^l$ bound the true states from below:

$$\text{for all } t \geq 0,\ \hat{x}^l(t) \leq x(t) \leq \hat{x}^u(t). \tag{3.2}$$

We refer to (3.2) as the *interval property*, and for it to be satisfied, the upper estimation error $e^u \coloneqq \hat{x}^u - x$ and the lower estimation error $e^l \coloneqq x - \hat{x}^l$ must satisfy positivity.

Summing e^u and e^l, we obtain $(\hat{x}^u - x) + (x - \hat{x}^l) = \hat{x}^u - \hat{x}^l$, i.e., the difference between the upper and lower state estimates. This value is called the *interval error* and we denote it by $e \coloneqq \hat{x}^u - \hat{x}^l$. If $e(t) \geq 0$ for all $t \geq 0$, then $\hat{x}^u(t) \geq \hat{x}^l(t)$ for all $t \geq 0$, i.e., the positivity of the interval error e ensures that $\hat{x}^u$ is always above $\hat{x}^l$. Positivity of the interval error is a necessary, but not sufficient, condition to satisfy the interval property. The observer pair must also be initialized such that $\hat{x}^u(0)$ and $\hat{x}^l(0)$ satisfy (3.2). We therefore make the following assumption.

Assumption 3. Given a system of the form (3.1), there exist known constants $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$, such that $\underline{x}_0 \leq x(0) \leq \overline{x}_0$.

If the evolution of e is governed by positive dynamics, Assumption 3 means we can always set $\hat{x}^{u,l}(0) := \big(\hat{x}^u(0), \hat{x}^l(0)\big)$ such that the pair $\hat{x}^{u,l} := (\hat{x}^u, \hat{x}^l)$ satisfies the interval property with respect to the true state x. An interval observer is said to be *convergent* if $\lim_{t\to\infty} e(t)$ exists and is bounded, and is said to be *asymptotically convergent* if $\lim_{t\to\infty} e(t) = 0$.

A special case of system (3.1) of interest is when there is no model uncertainty,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \xi(y,t) \\ y(t) &= Cx(t), \end{aligned} \tag{3.3}$$

i.e., $\underline{A} = \overline{A}$ and ξ is only a function of the output and time, i.e., $\xi(x,t) \equiv \xi(y,t)$. We call this the *known model (KM) case*; only the initial conditions $x(0)$ of (3.3) are uncertain. In the general case of (3.1), called the *uncertain model (UM) case*, the initial conditions $x(0)$, state matrix A, and the instantaneous values of the nonlinearity ξ are uncertain, but bounded. In both the KM and UM cases, we assume existence and uniqueness of solutions for $x(t)$ for all $t \geq 0$.

3.1 Known Model Interval Observers

A *known model interval observer (KMIO)* is constructed for systems of the form (3.3), and has the dynamics [12]

$$\begin{aligned} \dot{\hat{x}}^u &= A\hat{x}^u + \xi(y,t) + L(y - C\hat{x}^u) \\ \dot{\hat{x}}^l &= A\hat{x}^l + \xi(y,t) + L(y - C\hat{x}^l), \end{aligned} \tag{3.4}$$

initialized at

$$\begin{aligned} \hat{x}^u(0) &= \overline{x}_0 \\ \hat{x}^l(0) &= \underline{x}_0, \end{aligned} \tag{3.5}$$

where $L \in \mathbb{R}^{n\times p}$ is the observer gain matrix. An asymptotically convergent interval observer of the form (3.4) exists if and only if there exist $\lambda \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ that satisfy the following constraints [12, Theorem 3.1]:

$$\begin{aligned} (A - LC)^\top \lambda &< \mathbf{0}_n \\ (A - LC)^\top + \beta I &\geq \mathbf{0}_{n\times n} \\ \lambda &> \mathbf{0}_n. \end{aligned} \tag{3.6}$$

The first and second constraints ensure that $(A-LC)$ is Hurwitz and Metzler, respectively. The quantity βI is added in the Metzler constraint because only the off-diagonal elements of a matrix must be nonnegative to satisfy the Metzler property.

Linear programming is used to generate optimal L matrices for the interval observer dynamics (3.4). But there are three issues we must address first: 1) a linear cost function must be identified; 2) the constraints (3.6) contain strict inequalities, which are not permitted in linear programming, as the feasible set of an LP must be closed, to ensure that the extrema are well-defined; 3) the constraints are not linear in L.

If the constraints (3.6) are satisfied, the ℓ_1-norm of the interval error is [12, Section II.B]

$$\|e\|_1 \coloneqq \left\| \int_0^\infty e(t)dt \right\|_1 = -\big(\hat{x}^u(0) - \hat{x}^l(0)\big)^\top (A - LC)^{-\top}\mathbf{1}_n. \tag{3.7}$$

The function (3.7) is not linear in L, however, modifying the Hurwitz constraints in (3.6) to be strictly equal to $-\mathbf{1}_n$, instead of strictly less than $\mathbf{0}_n$, yields

$$(A - LC)^\top \lambda = -\mathbf{1}_n.$$

Solving for λ and substituting into (3.7) yields the function

$$\|e\|_1 = \big(\hat{x}^u(0) - \hat{x}^l(0)\big)^\top \lambda, \tag{3.8}$$

which is linear in λ.

Introducing the matrix $Z \in \mathbb{R}^{p\times n}$, we define

$$L \coloneqq \mathbf{diag}(\lambda)^{-1} Z^\top. \tag{3.9}$$

Substituting this definition of L into the modified version of (3.6) yields the constraints

$$\begin{aligned} A^\top \lambda - C^\top Z \mathbf{1}_n &= -\mathbf{1}_n \\ A^\top \mathbf{diag}(\lambda) - C^\top Z + \beta I &\geq \mathbf{0}_{n\times n}, \end{aligned}$$

which are linear in λ, Z, and β.

Lastly, at implementation, the constraint $\lambda > \mathbf{0}_n$ is replaced with $\lambda \geq \varepsilon \mathbf{1}_n$, where $\varepsilon \in \mathbb{R}_{>0}$ is an arbitrarily small constant.

System (3.4) with initial conditions (3.5), is an asymptotically convergent interval observer for (3.3), if and only if Problem 4 is feasible. The gain matrix L, defined in (3.9), is computed using an optimal solution to Problem 4:

Problem 4:

$$\begin{aligned} \textit{minimize:}\ & \big(\hat{x}^u(0) - \hat{x}^l(0)\big)^\top \lambda \\ \textit{subject to:}\ & A^\top \lambda - C^\top Z \mathbf{1}_n = -\mathbf{1}_n \\ & A^\top \mathbf{diag}(\lambda) - C^\top Z + \beta I \geq \mathbf{0}_{n\times n} \\ & \lambda > \mathbf{0}_n. \end{aligned}$$

3.2 Uncertain Model Interval Observers

An *uncertain model interval observer (UMIO)* is constructed for systems of the general form (3.1), and has the dynamics [12]

$$\begin{aligned} \dot{\hat{x}}^u &= \overline{A}\hat{x}^u + L(y - C\hat{x}^u) - (\overline{A} - \underline{A})\phi(\hat{x}^u) + \overline{\xi} \\ \dot{\hat{x}}^l &= \overline{A}\hat{x}^l + L(y - C\hat{x}^l) - (\overline{A} - \underline{A})\psi(\hat{x}^l) + \underline{\xi} \\ \hat{x}^u(0) &= \overline{x}_0 \\ \hat{x}^l(0) &= \underline{x}_0, \end{aligned} \tag{3.10}$$

where $\psi : \mathbb{R}^n \to \mathbb{R}^n_{\geq 0}$ and $\phi : \mathbb{R}^n \to \mathbb{R}^n_{\leq 0}$ are defined as

$$\begin{aligned} \psi(x) &= \frac{1}{2}\big(x + |x|\big) \\ \phi(x) &= \frac{1}{2}\big(x - |x|\big). \end{aligned}$$

The function ψ retains the positive elements of x and maps the negative elements to 0, and ϕ retains the negative elements of x and maps the positive elements to 0.

A convergent interval observer of the form (3.10) exists if and only if there exist $\lambda \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ that satisfy the following constraints [12, Theorem 4.2]:

$$\begin{aligned} (\overline{A} - LC)^\top \lambda &< \mathbf{0}_n \\ (\underline{A} - LC)^\top + \beta I &\geq \mathbf{0}_{n\times n} \\ \lambda &> \mathbf{0}_n. \end{aligned}$$

The first and second constraints ensure that $\big(A-LC\big)$ is Hurwitz for all $A \leq \overline{A}$ and Metzler for all $A \geq \underline{A}$, respectively.

We make the following assumption, which is needed to optimize a particular objective; it is not necessary or sufficient for the existence of the interval observers we discuss.

Assumption 4. Given a system of the form (3.1) that satisfies Assumptions 1 and 2, there exists a constant $\varkappa$[1] $\in \mathbb{R}^n_{>0}$ such that $\varkappa \geq |\sup_{t\geq 0} x(t)|$.

Invoking Assumption 4, the ℓ_1-norm of the steady state supremum of the interval error is bounded above by [12, Theorem 4.2]

$$\|\bar{e}\|_1 \coloneqq \limsup_{t\to\infty} \|e(t)\|_1 \leq -\big[2(\overline{A}-\underline{A})\varkappa + \overline{\xi} - \underline{\xi}\big]^\top (\overline{A}-LC)^{-\top}\mathbf{1}_n. \tag{3.11}$$

Following an analogous process to that of Section 3.1, we obtain the linear cost function

$$\|\bar{e}\|_1 \leq \big[2(\overline{A}-\underline{A})\varkappa + \overline{\xi} - \underline{\xi}\big]^\top \lambda, \tag{3.12}$$

and the linear constraints in Problem 5. System (3.10) is a convergent interval observer for (3.1), if and only if Problem 5 is feasible. The gain matrix L, as defined in (3.9), is computed using an optimal solution to Problem 5:

Problem 5:

$$\begin{aligned}
\textit{minimize:}\ & \big[2(\overline{A}-\underline{A})\varkappa + \overline{\xi} - \underline{\xi}\big]^\top \lambda \\
\textit{subject to:}\ & \overline{A}^\top \lambda - C^\top Z \mathbf{1}_n = -\mathbf{1}_n \\
& \underline{A}^\top \mathbf{diag}(\lambda) - C^\top Z + \beta I \geq \mathbf{0}_{n\times n} \\
& \lambda > \mathbf{0}_n
\end{aligned}$$

[1]Pronounced "kappa".

Chapter 4

Robust Known Model Interval Observers

Problem 4 is used to generate an L that minimizes the ℓ_1-norm of the interval error of the KMIO (3.4). By (3.8), $\|e\|_1$ is directly proportional to the difference between the initial conditions of the upper and lower observers. Therefore, initializing the upper and lower observers closer together effects a smaller $\|e\|_1$. We assume that under particular operating conditions, tighter upper and lower initial conditions, $\hat{x}^u(0)$ and $\hat{x}^l(0)$, respectively, may be chosen such that $\hat{x}^l(0) \leq x(0) \leq \hat{x}^u(0)$, which is required to satisfy the interval property (3.2). By casting Problem 4 in the framework of Problem 2, we develop a tunably *robust known model interval observer (RKMIO)*.

4.1 Robust Formulation of the Known Model Interval Observer Problem

Define the constants $\underline{x}_0^{\uparrow}$, $\overline{x}_0^{\downarrow} \in \mathbb{R}^n$ such that $\underline{x}_0 \leq \underline{x}_0^{\uparrow} \leq \overline{x}_0^{\downarrow} \leq \overline{x}_0$ and $(\overline{x}_0^{\downarrow} - \underline{x}_0^{\uparrow}) \preceq (\overline{x}_0 - \underline{x}_0)$. As discussed, using these tighter initial conditions, i.e., setting $\hat{x}^{u,l}(0) = (\underline{x}_0^{\uparrow}, \overline{x}_0^{\downarrow})$, causes an elementwise reduction in the cost vector, which reduces the attainable optimal $\|e\|_1$. Define the notation

$$\Delta\overline{x}_0 \coloneqq \overline{x}_0 - \overline{x}_0^{\downarrow} \geq 0$$
$$\Delta\underline{x}_0 \coloneqq \underline{x}_0^{\uparrow} - \underline{x}_0 \geq 0.$$

We wish to treat perturbations of the upper and lower bounds of the initial conditions separately, allowing for a larger class of uncertainty than merely perturbations in the size of the interval. Using the method developed in Section 2.3, set $Q \coloneqq \mathbf{col}(\lambda, \lambda')$ and $D \coloneqq \mathbf{col}(\Delta\overline{x}_0, \Delta\underline{x}_0)$. We do not duplicate the unweighted decision variables Z or β.

Perturbations of the cost coefficients corresponding to λ are interpreted as upward perturbations of $\overline{x}_0^{\downarrow}$, and perturbations of the cost coefficients corresponding to λ' are interpreted as downward perturbations of $\underline{x}_0^{\uparrow}$. For consistency of notation with Problem 2, we retain the symbol d to represent the redefined cost coefficient perturbation vector D, and for clarity, we use the symbol Λ instead of Q. We also forgo the $'$ notation. Lastly, since $\lambda > \mathbf{0}_n$, we omit the absolute value operation in the cost perturbation term (2.1). We propose the following robust formulation of Problem 4.

Problem 6:

$$\begin{aligned} \textit{minimize: } & (\overline{x}_0^{\downarrow} - \underline{x}_0^{\uparrow})^\top \lambda + \max_{\{S_0 | S_0 \subseteq J_0, |S_0| \le \Gamma_0\}} \left\{ \sum_{j \in S_0} d_j \Lambda_j \right\} \\ \textit{subject to: } & A^\top \lambda - C^\top Z \mathbf{1}_n = -\mathbf{1}_n \\ & A^\top \mathbf{diag}(\lambda) - C^\top Z + \beta I \ge \mathbf{0}_{n \times n} \\ & \lambda > \mathbf{0}_n \\ & \lambda' = \lambda, \end{aligned}$$

Remark 4.1.1 (Computational Complexity of the Robust Known Model Problem). The original KM problem, Problem 4, has $n + pn + 1$ decision variables. Introducing the dummy variables λ' increases the number of decision variables to $2n + pn + 1$. The linear robust formulation, Problem 3, is used to implement the robust KM problem, Problem 6, which introduces the decision variables ζ_0 and p_{0j} for all $j \in J_0$. There is therefore a total of $\big(2n + pn + 2 + |J_0|\big)$ decision variables, where $|J_0| \le 2n$, and we assume $p \le n$.

There exist algorithms that can solve LPs in $O\big(\frac{N^3}{\ln N} U\big)$ time in the worst case [42], where N is the number of decision variables and U is the bit length of the binary encoding of the vectors c, b, and matrix E. Therefore, Problem 6 can be solved in $O\left(\frac{(pn)^3}{\ln n} U\right)$ time.

Perturbing all elements of $(\overline{x}_0^{\downarrow} - \underline{x}_0^{\uparrow})$ by setting $\Gamma_0 = |J_0|$, recovers the cost vector $(\overline{x}_0 - \underline{x}_0)$ of Problem 4. Further, there is exactly one possible perturbed cost vector for $\Gamma_0 = |J_0|$, therefore, we must verify that the optimal costs of Problems 4 and 6 are equal.

Proposition 4.1.2. *If $\Gamma_0 = |J_0|$, then Problem 6 and Problem 4 are equivalent.*

Proof. Setting $\Gamma_0 = |J_0|$ results in Problem 6 optimizing over all uncertain cost coefficients, i.e., $S_0 = J_0$.

$$\begin{aligned}&\big(\overline{x}_0^{\downarrow} - \underline{x}_0^{\uparrow}\big)^\top \lambda + \Delta\overline{x}_0^\top \lambda + \Delta\underline{x}_0^\top \lambda' \\ =&\big(\overline{x}_0^{\downarrow} + \Delta\overline{x}_0\big)^\top \lambda - \big(\underline{x}_0^{\uparrow} - \Delta\underline{x}_0\big)^\top \lambda \\ =&\big(\overline{x}_0 - \underline{x}_0\big)^\top \lambda\end{aligned}$$

Since the cost function and constraints are equivalent, the problems are equivalent. □

Given the pairs $(\overline{x}_0^{\downarrow}, \underline{x}_0^{\uparrow})$, $(\overline{x}_0, \underline{x}_0) \in \mathbb{R}^n \times \mathbb{R}^n$, a set of indices of uncertain cost coefficients J_0, and a set of indices of perturbed cost coefficients S_0, we generate a pair of initial conditions $\hat{x}^{u,l}(0) \in \mathbb{R}^n \times \mathbb{R}^n$ via the mapping

$$\begin{aligned}\Psi : \mathcal{P}_{\Gamma_0}(J_0) &\to \mathbb{R}^n \times \mathbb{R}^n \\ S_0 &\mapsto \big(\hat{x}^u(0), \hat{x}^l(0)\big),\end{aligned} \tag{4.1}$$

where

$$\hat{x}_i^u(0) = \begin{cases} \overline{x}_{0i} & \text{if } i \leq n,\ i \in S_0 \\ \overline{x}_{0i}^{\downarrow} & \text{otherwise,} \end{cases}$$

$$\hat{x}_i^l(0) = \begin{cases} \underline{x}_{0i} & \text{if } i \leq n,\ (i+n) \in S_0 \\ \underline{x}_{0i}^{\uparrow} & \text{otherwise.} \end{cases}$$

A pair of initial conditions $\big(\hat{x}^u(0), \hat{x}^l(0)\big) \in \mathbb{R}^n \times \mathbb{R}^n$ defines a set in $\mathbb{R}^n$ by

$$\big\{x \in \mathbb{R}^n : \hat{x}^l(0) \leq x \leq \hat{x}^u(0)\big\}. \tag{4.2}$$

A generic 2-dimensional example of these sets is illustrated in Figure 4.1. The possible values of $x_1(0)$ and $x_2(0)$ are on the horizontal and vertical axes, respectively. The outermost box encapsulates the interval defined by $(\overline{x}_0, \underline{x}_0)$ guaranteed to contain $x(0)$, and the solid inner box denotes the inner interval defined by $(\overline{x}_0^{\downarrow}, \underline{x}_0^{\uparrow})$. The boxes extending from the solid inner box represent individual perturbations. For example, the rightmost box represents the perturbation of $\overline{x}_{01}^{\downarrow}$ to $\overline{x}_{01}$. If both elements are perturbed in the positive, or negative, direction, the set (4.2) includes the white regions bordered by both boxes representing individual perturbations. For example, if $\overline{x}_{01}^{\downarrow}$ is perturbed to $\overline{x}_{01}$ and $\overline{x}_{02}^{\downarrow}$ is perturbed to $\overline{x}_{02}$, i.e., $(\overline{x}_0^{\downarrow}, \underline{x}_0^{\uparrow})$ is perturbed to $(\overline{x}_0, \underline{x}_0^{\uparrow})$, the set (4.2) comprises the solid inner box, the rightmost box, the topmost box, and the top-right white box. To satisfy the interval property (3.2), the observer's initial conditions $\hat{x}^{u,l}(0)$ must be chosen such that the set (4.2) contains all possible $x(0)$.

Further, for a fixed cost protection level Γ_0, we generate the set of initial conditions effected by perturbing at most Γ_0 elementwise boundaries of the interval (4.2) defined by

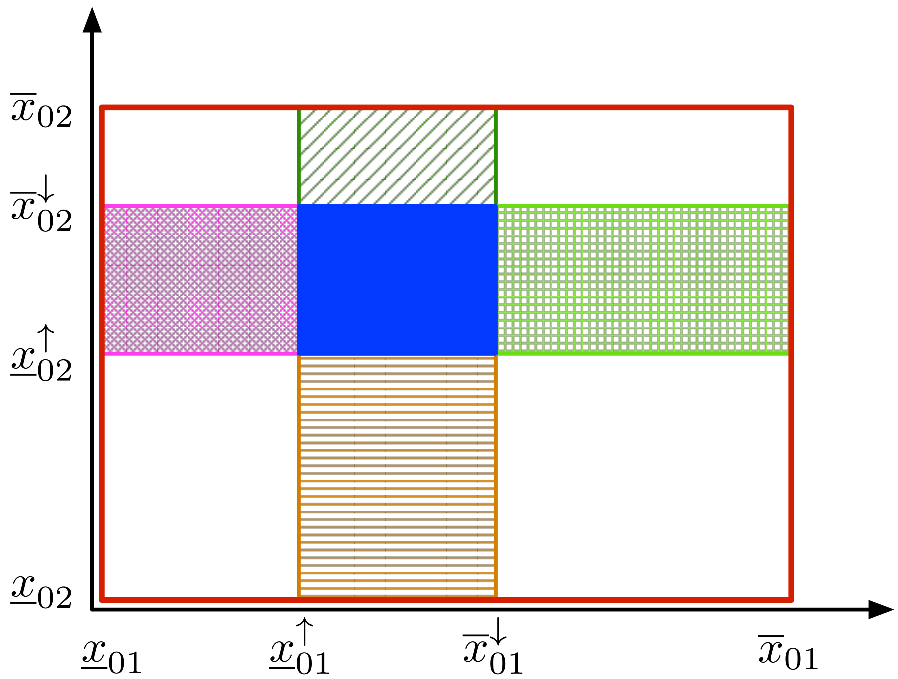

Figure 4.1: A 2-dimensional example of the intervals defined by the pair $(\overline{x}_0^{\downarrow}, \underline{x}_0^{\uparrow}), (\overline{x}_0, \underline{x}_0) \in \mathbb{R}^2 \times \mathbb{R}^2$.

the pair $(\underline{x}_0^{\uparrow}, \overline{x}_0^{\downarrow})$, via the mapping

$$\begin{aligned} \mathcal{X} : \mathcal{P}_{\Gamma_0}(J_0) \times \mathbb{Z}_{\geq 0} &\to \mathbb{R}^n \times \mathbb{R}^n \\ (J_0, \Gamma_0) &\mapsto \left\{ \hat{x}^{u,l}(0) \mid \big(\exists\, S_0 \in \mathcal{P}_{\Gamma_0}(J_0)\big)\big(\Psi(S_0) = \hat{x}^{u,l}(0)\big) \right\}. \end{aligned}$$

Problem 6 generates an L that minimizes the maximum $\|e\|_1$ of the interval observer (4.3), over all initial conditions $\hat{x}^{u,l}(0) \in \mathcal{X}(J_0, \Gamma_0)$ for a fixed Γ_0.

4.2 Optimality

Since Problems 4 and 6 have the same constraints, they admit the same set of feasible observer gain matrices $\mathcal{F}_{KM}^L \subset \mathbb{R}^{n \times p}$. Define the set $\mathcal{L}_{RKM}(J_0, \Gamma_0) \subset \mathcal{F}_{KM}^L$ to be the set of all matrices L constructed using optimal solutions to the robust KM problem, Problem 6, for a given J_0 and Γ_0. Similarly, define the set $\mathcal{L}_{KM} \subset \mathcal{F}_{KM}^L$ to be the set of all matrices

L constructed using optimal solutions to the original KM problem, Problem 4, using the cost vector induced by the initial conditions $(\overline{x}_0, \underline{x}_0)$.

We propose the following RKMIO,

$$\begin{aligned} \dot{\hat{x}}^l &= A\hat{x}^l + \xi(y,t) + L(y - C\hat{x}^l) \\ \dot{\hat{x}}^u &= A\hat{x}^u + \xi(y,t) + L(y - C\hat{x}^u) \\ \hat{x}^{u,l}(0) &\in \mathcal{X}(J_0, \Gamma_0), \end{aligned} \tag{4.3}$$

where $L \in \mathcal{L}_{RKM}(J_0, \Gamma_0)$.

Define the mapping

$$\begin{aligned} e_{\ell_1} : \mathbb{R}^{n\times p} \times \mathcal{X}(J_0, |J_0|) &\to \mathbb{R}_{\geq 0} \\ \big(L, \hat{x}^{u,l}(0)\big) &\mapsto -\big(\hat{x}^u(0) - \hat{x}^l(0)\big)^\top (A - LC)^{-\top}\mathbf{1}_n. \end{aligned} \tag{4.4}$$

By (3.7), the mapping (4.4) gives the ℓ_1-norm of an interval observer with dynamics (3.4), initialized at $\hat{x}^{u,l}(0)$.

Theorem 4.2.1. *Given a system of the form (3.3), constants $\underline{x}_0 \leq \underline{x}_0^\uparrow \leq \overline{x}_0^\downarrow \leq \overline{x}_0$, such that $\underline{x}_0 \preceq \overline{x}_0$ and $\underline{x}_0 \leq x(0) \leq \overline{x}_0$, indices of uncertain cost coefficients J_0, and cost protection level $\Gamma_0 < |J_0|$, the proposed RKMIO (4.3) effects a smaller $\|e\|_1$ than the KMIO (3.4) initialized at $(\overline{x}_0, \underline{x}_0)$, and a maximum $\|e\|_1$ over the set of initial conditions $\mathcal{X}(J_0, \Gamma_0)$ no greater than that of (3.4), i.e.,*

$$\max_{\hat{x}^{u,l}(0)\in\mathcal{X}(J_0,\Gamma_0)} e_{\ell_1}\big(L_{RKM}, \hat{x}^{u,l}(0)\big) \leq \max_{\hat{x}^{u,l}(0)\in\mathcal{X}(J_0,\Gamma_0)} e_{\ell_1}\big(L_{KM}, \hat{x}^{u,l}(0)\big) < e_{\ell_1}\big(L_{KM}, (\overline{x}_0, \underline{x}_0)\big). \tag{4.5}$$

Proof. By (3.7), for a fixed L, choosing initial conditions $\tilde{\hat{x}}_0^{u,l} := (\tilde{\hat{x}}_0^u, \tilde{\hat{x}}_0^l)$ such that $(\tilde{\hat{x}}_0^u - \tilde{\hat{x}}_0^l) \preceq \big(\hat{x}^u(0) - \hat{x}^l(0)\big)$, effects a smaller $\|e\|_1$. Since $(\overline{x}_0, \underline{x}_0) \notin \mathcal{X}(J_0, \Gamma_0)$ for any $\Gamma_0 < |J_0|$, we have

$$\big(\forall\, \Gamma_0 < |J_0|\big)\big(\tilde{\hat{x}}_0^{u,l} \in \mathcal{X}(J_0, \Gamma_0)\big)\big(\hat{x}^{u,l}(0) = (\overline{x}_0, \underline{x}_0)\big) \implies \Big(e_{\ell_1}(L, \tilde{\hat{x}}_0^{u,l}) < e_{\ell_1}\big(L, (\overline{x}_0, \underline{x}_0)\big)\Big). \tag{4.6}$$

Therefore, initializing the KMIO (3.4) at $\hat{x}^{u,l}(0) \in \mathcal{X}(J_0, \Gamma_0)$, $\Gamma_0 < |J_0|$, effects a smaller $\|e\|_1$ than initializing at $(\overline{x}_0, \underline{x}_0)$.

By construction, the optimal cost of Problem 6 is equal to

$$\max_{\hat{x}^{u,l}(0)\in\mathcal{X}(J_0,\Gamma_0)} e_{\ell_1}\big(L, \hat{x}^{u,l}(0)\big),$$

i.e., the maximum $\|e\|_1$ over all initial conditions $\hat{x}^{u,l}(0) \in \mathcal{X}(J_0, \Gamma_0)$. Since Problems 4 and 6 admit the same set of feasible L matrices, by the optimality of $L \in \mathcal{L}_{RKM}$, there exists no $L \in \mathcal{L}_{KM}$ that effects a smaller maximum $\|e\|_1$ over the set of initial conditions $\mathcal{X}(J_0, \Gamma_0)$. Combining this with (4.6), we verify (4.5). □

Theorem 4.2.1 signifies that the $\|e\|_1$ of the RKMIO (4.3) for any $\Gamma_0 < |J_0|$, is strictly less than that of the KMIO (3.4) initialized at $(\overline{x}_0, \underline{x}_0)$, and no greater than that of the KMIO initialized in $\mathcal{X}(J_0, \Gamma_0)$.

4.3 Implementation

In this section, we delineate and illustrate the design process of the proposed RKMIO (4.3).

1. Identify $\overline{x}_0, \underline{x}_0, \overline{x}_0^{\downarrow}, \underline{x}_0^{\uparrow}$.
2. Set Γ_0 to the smallest value such that each possible initial condition $x(0)$ is contained in at least one of the sets (4.2) defined by an initial conditions pair $\hat{x}^{u,l}(0) \in \mathcal{X}(J_0, \Gamma_0)$.
3. Using $c := (\overline{x}_0^{\downarrow} - \underline{x}_0^{\uparrow})$ and $d := \mathbf{col}\big((\overline{x}_0 - \overline{x}_0^{\downarrow}), (\underline{x}_0^{\uparrow} - \underline{x}_0)\big)$, solve Problem 6 using the constraint $\lambda \geq \varepsilon \mathbf{1}_n$, where $\varepsilon \in \mathbb{R}_{>0}$ is an arbitrarily small constant.
4. Using the optimal λ and Z, construct the observer gain matrix $L := \mathbf{diag}(\lambda)^{-1} Z^\top$.
5. Initialize the RKMIO (4.3) in the elementwise smallest interval (4.2) induced by $\hat{x}^{u,l}(0) \in \mathcal{X}(J_0, \Gamma_0)$ such that $\hat{x}^l(0) \leq x(0) \leq \hat{x}^u(0)$, for all possible $x(0)$.

The single resultant RKMIO with gain matrix L, is optimal in the sense that the maximum $\|e\|_1$ over all initial conditions $\hat{x}^{u,l}(0) \in \mathcal{X}(J_0, \Gamma_0)$ is minimized, and is optimal in this sense for any initial conditions in $\mathcal{X}(J_0, \Gamma_0)$. By stipulating that $\hat{x}^{u,l}(0)$ be chosen such that the set (4.2) contains $x(0)$, we ensure the interval property (3.2); by setting Γ_0 as small as is possible while ensuring the interval property, we minimize the attainable optimum cost.

For the class of systems defined by a fixed Γ_0, we perform only a single optimization, i.e., design a single interval observer, whereas the original KMIO (3.4) of [12], is optimal only for a single $\hat{x}^{u,l}(0)$, and would need to be optimized for each set of initial conditions in $\mathcal{X}(J_0, \Gamma_0)$. This is advantageous in situations where the initial conditions of the plant are guaranteed to lie within some range, but under certain circumstances, this range can be refined. For example, as discussed in the Introduction, when performing many similar experiments, but deliberately varying the initial conditions in specific iterations. Or when reinitializing an autonomous robot after it has sustained damage, and its current states are uncertain.

Example 4.3.1. To illustrate the proposed approach, we construct and implement an RKMIO for the system (1.1), whose state space model is

$$A = \begin{bmatrix} -3 & -1 \\ 1 & -4 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -1 \end{bmatrix}.$$

We take the constants $\overline{x}_0, \underline{x}_0, \overline{x}_0^{\downarrow}, \underline{x}_0^{\uparrow}$ to be

$$\overline{x}_0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \overline{x}_0^{\downarrow} = \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix} \quad \underline{x}_0^{\uparrow} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \underline{x}_0 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \tag{4.7}$$

which define the intervals (4.2) illustrated in Figure 4.2, and the cost and perturbation vectors

$$\begin{aligned} c &= (\overline{x}_0^{\downarrow} - \underline{x}_0^{\uparrow}) & d &= \mathbf{col}\big((\overline{x}_0 - \overline{x}_0^{\downarrow}), (\underline{x}_0^{\uparrow} - \underline{x}_0)\big) \\ &= \begin{bmatrix} 1.5 & 0.5 \end{bmatrix}^\top, & &= \begin{bmatrix} 0.5 & 1.5 & 2 & 1 \end{bmatrix}^\top. \end{aligned}$$

The indices of uncertain cost coefficients are those corresponding to nonzero elements of d,

$$J_0 = \{1, 2, 3, 4\}.$$

Setting $\varepsilon = 10^{-6}$, we construct Problem 6 in the linear robust formulation of Problem 3

$$\begin{aligned} \text{minimize: } & c^\top \lambda + \zeta_0 \Gamma_0 + \sum_{j \in J_0} p_{0j} \\ \text{subject to: } & A^\top \lambda - C^\top Z \mathbf{1}_2 = -\mathbf{1}_2 \\ & A^\top \mathbf{diag}(\lambda) - C^\top Z + \beta I \geq 0 \\ & \zeta_0 + p_{0j} \geq d_j \Lambda_j & \forall\, j \in J_0 \\ & p_{0j} \geq 0 & \forall\, j \in J_0 \\ & \zeta_0 \geq 0 \\ & \lambda \geq 10^{-6} \mathbf{1}_2 \\ & \lambda' = \lambda. \end{aligned}$$

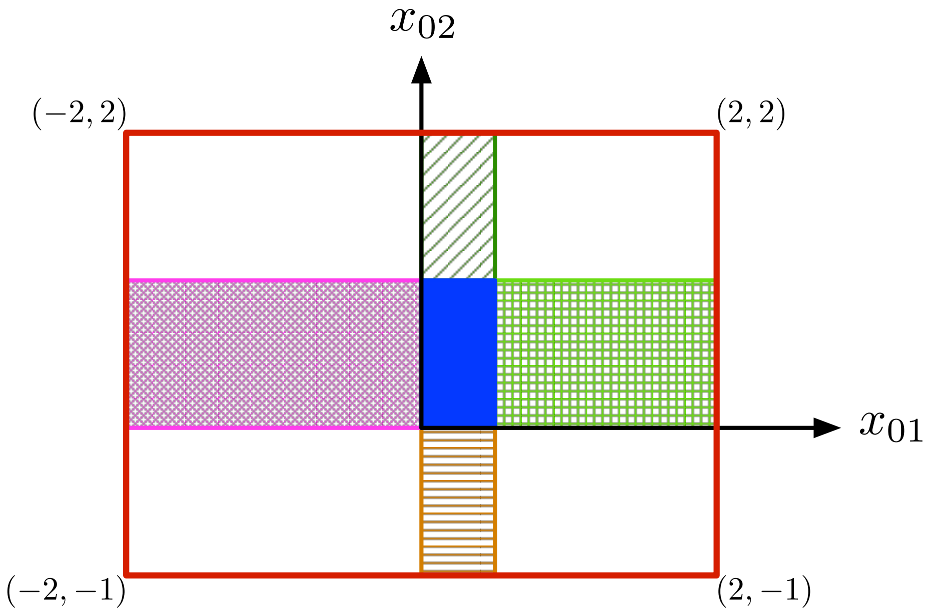

Figure 4.2: Initial conditions intervals defined by (4.7) in Example 4.3.1.

The KMIO of [12] is initialized at $\hat{x}^{u,l}(0) = (\overline{x}_0, \underline{x}_0)$. We compare the maximum $\|e\|_1$ over the set of initial conditions $\mathcal{X}(J_0, \Gamma_0)$ of the proposed RKMIO (4.3), to the $\|e\|_1$ effected by the KMIO of [12]. The maximum $\|e\|_1$ of the RKMIO over the initial conditions $\mathcal{X}(J_0, \Gamma_0)$, the matrix L, and the percent reduction in $\|e\|_1$, i.e.,

$$100\left(1 - \frac{\max_{\hat{x}^{u,l}(0)\in\mathcal{X}(J_0,\Gamma_0)} e_{\ell_1}\left(L_{RKM}, \hat{x}^{u,l}(0)\right)}{e_{\ell_1}\left(L_{KM}, (\overline{x}_0, \underline{x}_0)\right)}\right),$$

are presented in Table 4.1 for $\Gamma_0 \in \{1, 2, 3, 4\}$. Recall that for $\Gamma_0 = |J_0| = 4$, the KMIO and RKMIO are equivalent. Plots of the trial $\Gamma_0 = 1$ are presented in Figures 4.3 and 4.4.

The data in Table 4.1 show that the reduction in $\|e\|_1$ is potentially significant, ranging from 9.00% to 48.0%. However, a reduction in $\|e\|_1$ would also occur by initializing the KMIO of [12] using tighter initial conditions. For comparison, we identify the greatest $\|e\|_1$ effected by initializing the KMIO of [12] in the sets $\mathcal{X}(J_0, \Gamma_0)$, $\Gamma_0 \in \{1, 2, 3\}$, and compute the percent reduction in the maximum $\|e\|_1$ effected by using the RKMIO instead of the KMIO, i.e.,

$$100\left(1 - \frac{\max_{\hat{x}^{u,l}(0)\in\mathcal{X}(J_0,\Gamma_0)} e_{\ell_1}\left(L_{RKM}, \hat{x}^{u,l}(0)\right)}{\max_{\hat{x}^{u,l}(0)\in\mathcal{X}(J_0,\Gamma_0)} e_{\ell_1}\left(L_{KM}, \hat{x}^{u,l}(0)\right)}\right).$$

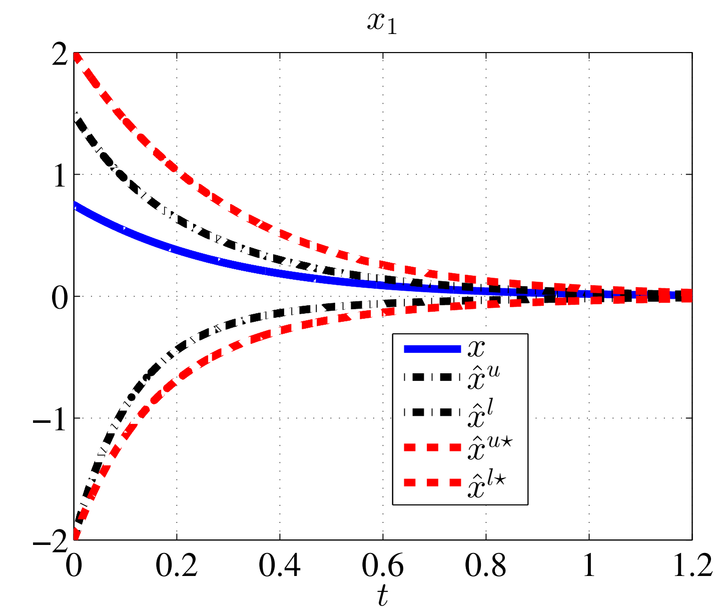

Figure 4.3: The estimates of x_1 of the proposed RKMIO $\hat{x}$ and the original KMIO $\hat{x}^{\star}$.

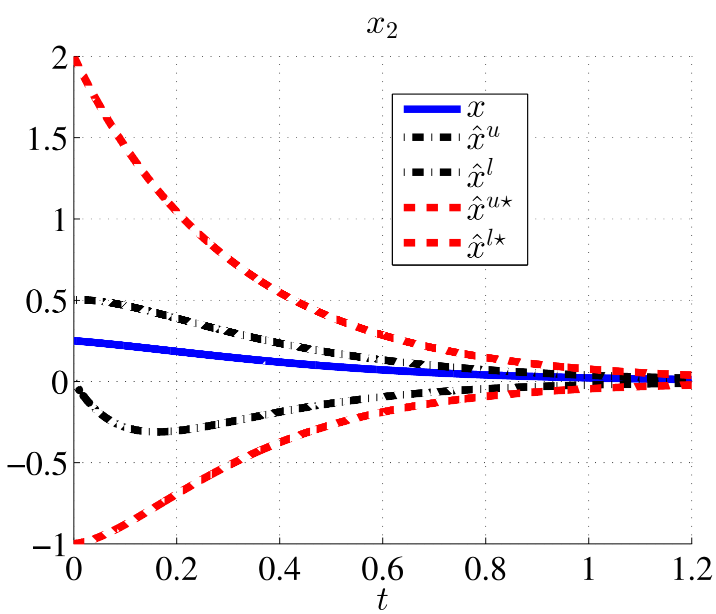

Figure 4.4: The estimates of x_2 of the proposed RKMIO $\hat{x}$ and the original KMIO $\hat{x}^{\star}$.

Table 4.1: Solutions for various Γ_0.

Γ_0	$\|e\|_1$	L	% Reduction
1	1.04	$[\ 4.13 \quad -1.35\ ]^\top$	48.0
2	1.50	$[\ 5.13 \quad -0.564\ ]^\top$	25.0
3	1.82	$[\ 2.37 \quad -4.23\ ]^\top$	9.00
4	2.00	$[\ 3.10 \quad -1.11\ ]^\top$	

Table 4.2: Worst-case comparisons.

Γ_0	$\|e\|_1$	$\hat{x}^{u,l}(0)$	% Reduction
1	1.10	$[\ 1.5 \quad 0.5\ ]^\top \times [\ -2 \quad 0\ ]^\top$	5.45
2	1.56	$[\ 1.5 \quad 2\ ]^\top \times [\ -2 \quad 0\ ]^\top$	3.85
3	1.86	$[\ 1.5 \quad 2\ ]^\top \times [\ -2 \quad -1\ ]^\top$	2.15

We see in Table 4.2 that the reduction in the maximum $\|e\|_1$ over the set of initial conditions $\mathcal{X}(J_0, \Gamma_0)$ is modest, ranging from 2.15% to 5.45%. As expected, the reduction is inversely proportional to Γ_0. In Figures 4.5 and 4.6, we see that even though the RKMIO effects a smaller $\|e\|_1$, its instantaneous upper and lower estimates are not necessarily better than those of the KMIO; the RKMIO's upper estimate $\hat{x}^u$ is above the KMIO's upper estimate $\hat{x}^{u\star}$ for the entire simulation, but this is offset by the improvement in the RKMIO's lower estimate $\hat{x}^l$.

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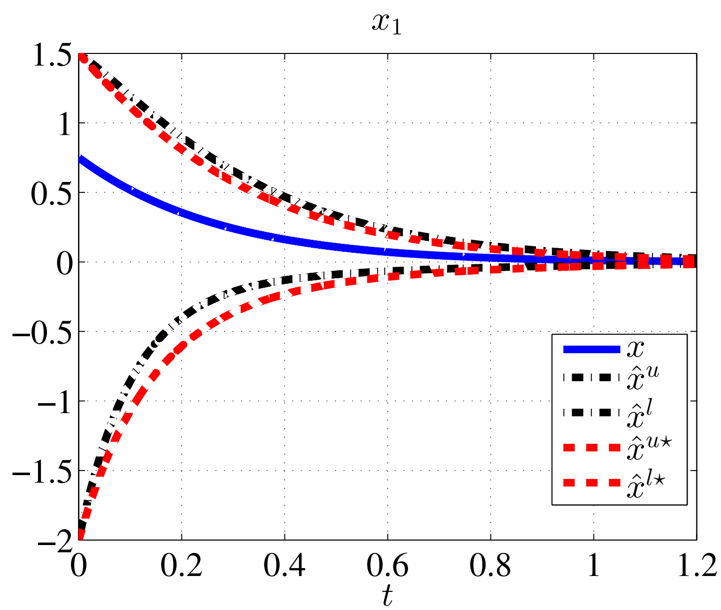

Figure 4.5: The estimates of x_1 of the proposed RKMIO $\hat{x}$ and the original KMIO $\hat{x}^\star$ using the worst-case KMIO initial conditions for $\Gamma_0 = 2$.

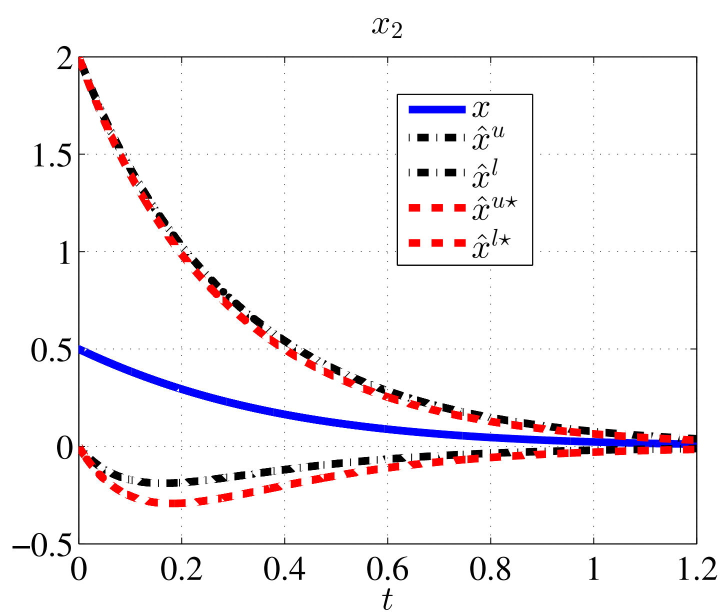

Figure 4.6: The estimates of x_2 of the proposed RKMIO $\hat{x}$ and the original KMIO $\hat{x}^\star$ using the worst-case KMIO initial conditions for $\Gamma_0 = 2$.

4.4 Experimental Validation of Approach

We conduct a Monte Carlo analysis to characterize the reduction in $\|e\|_1$ effected by using the proposed RKMIO (4.3), instead of the original KMIO (3.4) of [12] initialized at $(\overline{x}_0, \underline{x}_0)$.

The parameters $\overline{x}_0, \underline{x}_0, \overline{x}_0^{\downarrow}, \underline{x}_0^{\uparrow} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{p\times n}$ are generated as uniform random variables, such that

$$\begin{aligned} \|\overline{x}_0 - \underline{x}_0\|_1 &= 1 \\ \overline{x}_0^{\downarrow} - \underline{x}_0^{\uparrow} &\geq \frac{2}{n^3}\mathbf{1}_n, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} A_{ij} &\in [0,1] & i,j \in \{1,\dots,n\}, i \neq j \\ A_{ii} &\in \big[-(n-1), 0\big] & i \in \{1,\dots,n\} \\ C &\in [0,1]^{p\times n}. \end{aligned} \tag{4.9}$$

The constraints (4.8) provide consistency in the size of the initial condition intervals (4.2) across trials. The ranges defined for the elements of A in (4.9) effect a bias toward positivity and stability.

Ten thousand trials are conducted for each combination of $n \in \{1,\dots,5\}$ and $p, \Gamma_0 \in \{1,\dots,n-1\}$. In each trial, Problems 6 and 4 are solved for the same parameters. The absolute differences between the ℓ_1-norms of the interval errors of the proposed and the original KMIOs are recorded, as well as the relative difference,

$$1 - \frac{\max_{\hat{x}^{u,l}(0)\in\mathcal{X}(J_0,\Gamma_0)} e_{\ell_1}\big(L_{RKM}, \hat{x}^{u,l}(0)\big)}{e_{\ell_1}\big(L_{KM}, (\overline{x}_0, \underline{x}_0)\big)}.$$

The arithmetic means μ and standard deviations σ of these values are presented in the following tables.

Table 4.3: Empirical results for $n = 2$.

$n=2$		Absolute		Relative	
p	Γ_0	μ	σ	μ	σ
1	1	0.16	0.23	0.11	0.059

In Tables 4.3, 4.4, 4.5, and 4.6, we see that the greatest mean reductions in $\|e\|_1$ for each n are approximately $11\% - 29\%$. As expected, the reduction in $\|e\|_1$ is inversely proportional to Γ_0, as Problems 4 and 6 are equivalent for $\Gamma_0 = n$, and $\Gamma_0 = 0$ does not perturb the initial conditions from $(\overline{x}_0^{\downarrow}, \underline{x}_0^{\uparrow})$. The data suggest that the reduction in $\|e\|_1$

Table 4.4: Empirical results for $n = 3$.

$n = 3$		Absolute		Relative	
p	Γ_0	μ	σ	μ	σ
1	1	0.68	6.6	0.21	0.086
	2	0.30	3.2	0.11	0.060
2	1	0.12	0.30	0.17	0.091
	2	0.040	0.043	0.068	0.055

Table 4.5: Empirical results for $n = 4$.

$n = 4$		Absolute		Relative	
p	Γ_0	μ	σ	μ	σ
1	1	1.5	20	0.26	0.092
	2	0.72	8.6	0.16	0.072
	3	0.41	4.5	0.095	0.052
2	1	0.32	1.9	0.25	0.094
	2	0.14	0.29	0.14	0.071
	3	0.090	0.89	0.086	0.052
3	1	0.080	0.69	0.20	0.10
	2	0.029	0.033	0.084	0.066
	3	0.011	0.016	0.033	0.035

correlates positively with n and negatively with p. These relationships make the proposed RKMIO (4.3) increasingly attractive as the number of states increases, and as the number of outputs decreases, i.e., as the system becomes more complex and as less information is available. The negative correlation with p can be interpreted as the robust formulation compensating for the reduction in the number of measurements. A practical implication is that the robust formulation can be used to justify using fewer sensors. However, for many combinations of n, p, and Γ_0, the standard deviation is greater than the mean. This suggests that the reduction is highly dependent upon the specific plant being observed.

Table 4.6: Empirical results for $n = 5$.

$n = 5$		Absolute		Relative	
p	Γ_0	μ	σ	μ	σ
1	1	1.9	14	0.29	0.093
	2	1.2	9.8	0.19	0.076
	3	0.74	6.0	0.13	0.060
	4	0.51	5.3	0.085	0.046
2	1	0.68	7.8	0.29	0.093
	2	0.45	11	0.19	0.076
	3	0.28	6.6	0.13	0.059
	4	0.16	3.8	0.081	0.045
3	1	0.17	0.22	0.28	0.095
	2	0.10	0.097	0.18	0.079
	3	0.063	0.058	0.11	0.060
	4	0.038	0.035	0.068	0.045
4	1	0.049	0.046	0.22	0.11
	2	0.022	0.024	0.099	0.073
	3	0.0092	0.013	0.041	0.042
	4	0.0026	0.0068	0.010	0.022

Chapter 5

Robust Uncertain Model Interval Observers

Problem 5 is used to generate an L that minimizes the upper bound on the ℓ_1-norm of the steady state supremum of the interval error. The cost of Problem 5 is proportional to the difference between the elementwise upper and lower bounds on the state matrix (3.12). By using tighter upper and lower bounds on the state matrix, we reduce the upper bound on $\|\bar{e}\|_1$. We assume that under particular operating conditions, the upper and lower state matrices $A^{u,l} := (A^u, A^l) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ may be chosen such that $A^l \leq A \leq A^u$. By casting Problem 5 in the framework of Problem 2, we develop a tunably *robust uncertain model interval observer (RUMIO)*.

5.1 Robust Formulation of the Uncertain Model Interval Observer Problem

Define the matrices $\underline{A}^{\uparrow}, \overline{A}^{\downarrow} \in \mathbb{R}^{n\times n}$ such that $\underline{A} \leq \underline{A}^{\uparrow} \leq \overline{A}^{\downarrow} \leq \overline{A}$ and $(\overline{A}^{\downarrow} - \underline{A}^{\uparrow}) \preceq (\overline{A} - \underline{A})$. As discussed, using $A^{u,l}$ such that $\underline{A} \leq A^l \leq \underline{A}^{\uparrow}$ and $\overline{A}^{\downarrow} \leq A^u \leq \overline{A}$, causes an elementwise reduction in the cost vector, which reduces the attainable optimal upper bound on $\|\bar{e}\|_1$. Recall that to satisfy the interval property, $A^{u,l}$ must also satisfy $A^l \leq A \leq A^u$. Define the notation

$$\Delta\overline{A} := \overline{A} - \overline{A}^{\downarrow} \geq \mathbf{0}_{n\times n}$$
$$\Delta\underline{A} := \underline{A}^{\uparrow} - \underline{A} \geq \mathbf{0}_{n\times n}.$$

To treat perturbations of the upper and lower bounds of the state matrix separately, thereby allowing for a larger class of uncertainty than merely perturbations in the width of the interval, we apply the method developed in Section 2.3. We set $Q \coloneqq \mathbf{col}(\overline{\lambda}, \underline{\lambda})$ and $D \coloneqq \mathbf{col}(2\Delta\overline{A}\varkappa, 2\Delta\underline{A}\varkappa)$. We do not duplicate the unweighted decision variables Z or β.

We wish to allow for perturbations to individual elements of the state matrices $\overline{A}^{\downarrow}$ and $\underline{A}^{\uparrow}$. This requires further duplication of the decision variable λ, such that there is exactly one decision variable corresponding to each element of $\overline{A}^{\downarrow}$ and $\underline{A}^{\uparrow}$. We introduce the dummy variables $\overline{\lambda}^{(i)}, \underline{\lambda}^{(i)} \in \mathbb{R}^n, i \in \{1, \dots, n\}$, where the superscript (i) is used rather than repeated $'$ notation. We redefine the stacked decision vector Q as

$$Q' \coloneqq \mathbf{col}\left(\overline{\lambda}^{(1)}, \dots, \overline{\lambda}^{(n)}, \underline{\lambda}^{(1)}, \dots, \underline{\lambda}^{(n)}\right) \in \mathbb{R}^{2n^2},$$

and the cost coefficient perturbation vector as

$$D' \coloneqq \mathbf{col}\big(2\Delta\overline{A}\mathbf{diag}(\varkappa), 2\Delta\underline{A}\mathbf{diag}(\varkappa)\big) \in \mathbb{R}^{2n^2}. \tag{5.1}$$

Introducing the decision variables $\overline{\lambda}^{(i)}$ and $\underline{\lambda}^{(i)}$ changes the sets of indices of uncertain coefficients from $J_i \subseteq \{1, \dots, 2n\}$ to $J_i' \subseteq \{1, \dots, 2n^2\}$, $i \in \{0, \dots, n^2 + n\}$. Note that the maximum cardinality of J_i' is equal to the maximum cardinality of J_i, $i \in \{1, \dots, n^2 + n\}$, as the number of decision variables appearing in each constraint remains the same. Since perturbations to the cost and constraint coefficients have the same physical interpretation, we ensure that the same number of coefficients are perturbed in the cost as in the constraints, by stipulating that the cost protection level be equal to the sum of the ceilings of the constraint protection levels, i.e.,

$$\Gamma_0 = \sum_{i \neq 0} \lceil \Gamma_i \rceil. \tag{5.2}$$

Perturbations to the coefficients corresponding to $\overline{\lambda}$ are interpreted as upward perturbations of $\overline{A}^{\downarrow}$, and perturbations to the coefficients corresponding to $\underline{\lambda}$ are interpreted as downward perturbations of $\underline{A}^{\uparrow}$. For consistency of notation with Problem 2, we retain the symbol d to represent the redefined cost coefficient perturbation vector D', and for clarity, we use the symbol Λ instead of Q'; we also forgo the $'$ notation. Lastly, since $\lambda > \mathbf{0}_n$, we omit the absolute values from the cost and constraint perturbation terms (2.1), (2.2).

By Proposition 2.2.1, an equality constraint with uncertain coefficients cannot be made robust. We therefore propose the following robust formulation of Problem 5, where the equality relation in the Hurwitz constraint is instead a nonstrict inequality.

Problem 7:

$$\begin{aligned}
\textit{minimize: } & \big[2(\overline{A}^{\downarrow} - \underline{A}^{\uparrow})\varkappa + \overline{\xi} - \underline{\xi}\big]^\top \lambda + \max_{\{S_0 | S_0 \subseteq J_0, |S_0| \le \Gamma_0\}} \left\{ \sum_{j \in S_0} d_j \Lambda_j \right\} \\
\textit{subject to: } & \overline{A}_i^{\downarrow\top} \overline{\lambda}^{(i)} - C_i^\top Z \mathbf{1}_n \\
& \quad + \max_{\{S_i \cup \{t_i\} | S_i \subseteq J_i, |S_i| \le \lfloor \Gamma_i \rfloor, t_i \in J_i \setminus S_i\}} \left\{ \sum_{k \in S_i} \Delta \overline{A}_{ki} \overline{\lambda}_k^{(i)} + \big(\Gamma_i - \lfloor \Gamma_i \rfloor\big) \Delta \overline{A}_{t_i i} \overline{\lambda}_{t_i}^{(i)} \right\} \le -1 \\
& - (\underline{A}_{ji}^{\uparrow} \underline{\lambda}_j^{(i)} - C_i^\top Z_j + \delta_{ij}\beta) \\
& \quad + \max_{\{S_r \cup \{t_r\} | S_r \subseteq J_r, |S_r| \le \lfloor \Gamma_r \rfloor, t_r \in J_r \setminus S_r\}} \left\{ \sum_{k \in S_r} \Delta \underline{A}_{ki} \underline{\lambda}_k^{(i)} + \big(\Gamma_r - \lfloor \Gamma_r \rfloor\big) \Delta \underline{A}_{t_r i} \underline{\lambda}_{t_r}^{(i)} \right\} \le 0 \\
& \lambda > \mathbf{0}_n \\
& \overline{\lambda}^{(i)} = \lambda \\
& \underline{\lambda}^{(i)} = \lambda \qquad\qquad i, j \in \{1, \ldots, n\},\ r = ni + j.
\end{aligned}$$

Remark 5.1.1 (Computational Complexity of the Robust Uncertain Model Problem). The original UM problem, Problem 5, has $n + pn + 1$ decision variables and $m = n^2 + n$ constraints. Introducing the dummy variables $\overline{\lambda}^{(i)}$ and $\underline{\lambda}^{(i)}$, $i \in \{1, \ldots, n\}$, increases the number of decision variables to $2n^2 + pn + 1$. The linear robust formulation, Problem 3, is used to implement the robust UM problem, Problem 7, which introduces the decision variables ζ_i, and p_{ij} for all $i \in \{0, \ldots, m\}$, $j \in J_i$. There is therefore a total of $N = \big(3n^2 + pn + n + 1 + \sum_{i=0}^{m} |J_i|\big)$ decision variables, where $|J_0| \le 2n^2$, $|J_i| \le n$ for the n Hurwitz constraints, $|J_i| \le 1$ for the n^2 Metzler constraints, and we assume $p \le n$.

There exist algorithms that can solve LPs in $O\big(\frac{N^3}{\ln N} U\big)$ time in the worst case [42], where U is the bit length of the binary encoding of the vectors c, b, and matrix E. Therefore, Problem 7 can be solved in $O\left(\frac{n^6}{\ln n} U\right)$ time.

Denote by $\mathbf{\Gamma} \coloneqq \mathbf{col}(\Gamma_1, \ldots, \Gamma_{n^2+n})$ the vector of constraint protection levels. Denote by $J \coloneqq \{J_0, \ldots, J_{n^2+n}\}$ the set of sets of indices of uncertain coefficients, and by $|J| \coloneqq \mathbf{col}\big(|J_0|, \ldots, |J_{n^2+n}|\big)$ the vector of cardinalities of the members of J. Denote by $\mathbf{J} \coloneqq \{J_1, \ldots, J_{n^2+n}\}$ the set of indices of uncertain constraint coefficients, and by $|\mathbf{J}| \coloneqq \mathbf{col}\big(|J_1|, \ldots, |J_{n^2+n}|\big)$ the vector of cardinalities of the members of $\mathbf{J}$.

Perturbing all coefficients by setting $\Gamma = |J|$ recovers the cost and constraint coefficients of Problem 5. We verify that the optimal costs of Problems 7 and 5 are equal.

Definition 5.1.2. Given an instance of Problem 7 and an instance of Problem 5, if the optimal costs of both problems are equal, and if there exists an injective mapping from an optimal triple (Λ, Z, β) of Problem 7 to an optimal triple (λ', Z', β') of Problem 5, we say that Problem 7 *recovers* Problem 5.

Proposition 5.1.3. *If $\Gamma = |J|$ and Problem 5 is feasible, then Problem 7 recovers Problem 5.*

To prove Proposition 5.1.3, we first prove that if $\Gamma = |J|$, then Problem 7 has the same cost function and constraints as Problem 5, except the Hurwitz constraint is a nonstrict inequality. Using several intermediary results, we prove that if $\Gamma = |J|$ and Problem 5 is feasible, then the Hurwitz inequality constraint in Problem 7 is always satisfied with equality, thereby rendering Problem 7 and 5 equivalent.

Lemma 5.1.4. *If $\Gamma = |J|$, then Problem 7 has the same cost function and constraints as Problem 5, except the Hurwitz constraint of Problem 7 is a nonstrict inequality.*

Proof. Setting $\Gamma_0 = |J_0|$ results in Problem 7 optimizing over all uncertain cost coefficients, i.e., $S_0 = J_0$,

$$\begin{aligned}
&\left[2(\overline{A}^{\downarrow} - \underline{A}^{\uparrow})\varkappa + \overline{\xi} - \underline{\xi}\right]^\top \lambda + \sum_{j\in J_0}\left\{[2\varkappa_j \Delta\overline{A}_j^\top]\overline{\lambda}^{(j)}\right\} + \sum_{j\in J_0}\left\{[2\varkappa_j \Delta\underline{A}_j^\top]\underline{\lambda}^{(j)}\right\}\\
=&\left[2(\overline{A}^{\downarrow} - \underline{A}^{\uparrow})\varkappa + \overline{\xi} - \underline{\xi}\right]^\top \lambda + \sum_{j\in J_0}\left\{[2\varkappa_j \Delta\overline{A}_j^\top]\lambda\right\} + \sum_{j\in J_0}\left\{[2\varkappa_j \Delta\underline{A}_j^\top]\lambda\right\}\\
=&\left[2(\overline{A}^{\downarrow} - \underline{A}^{\uparrow})\varkappa + \overline{\xi} - \underline{\xi}\right]^\top \lambda + 2\sum_{j\in J_0}\left\{\varkappa_j (\Delta\overline{A}_j + \Delta\underline{A}_j)^\top\right\}\lambda\\
=&\left[2(\overline{A}^{\downarrow} - \underline{A}^{\uparrow})\varkappa + \overline{\xi} - \underline{\xi}\right]^\top \lambda + \left[2(\Delta\overline{A}_j + \Delta\underline{A}_j)\varkappa\right]^\top \lambda\\
=&\left[2(\overline{A} - \underline{A})\varkappa + \overline{\xi} - \underline{\xi}\right]^\top \lambda
\end{aligned}$$

The cost function of Problem 7 for $\Gamma_0 = |J_0|$ is therefore equivalent to the cost function of Problem 5.

Setting $\mathbf{\Gamma} = |\mathbf{J}|$ results in Problem 7 satisfying feasibility over all admissible constraint coefficient perturbations, i.e., $S_i = J_i$, $i \in \{1, \ldots, n^2 + n\}$. We first examine the Hurwitz constraint:

$$\begin{aligned}
\overline{A}_i^{\downarrow\top}\overline{\lambda}^{(i)} - C_i^\top Z\mathbf{1}_n &\leq -1 \qquad i \in \{1, \ldots, n\}\\
\Rightarrow \quad \overline{A}_i^{\downarrow\top}\lambda - C_i^\top Z\mathbf{1}_n &\leq -1 \qquad i \in \{1, \ldots, n\}\\
\Rightarrow \quad \overline{A}^\top\lambda - C^\top Z\mathbf{1}_n &\leq -\mathbf{1}_n,
\end{aligned}$$

which has the same coefficients as the Hurwitz constraint of Problem 5, but is a nonstrict inequality.

We next examine the Metzler constraint:

$$\begin{array}{rrcll} & -(\underline{A}^{\uparrow}_{ji}\underline{\lambda}^{(i)}_j - C_i^\top Z_j + \delta_{ij}\beta) + \Delta\underline{A}_{ji}\underline{\lambda}^{(i)}_j & \leq & 0 & i,j \in \{1,\ldots,n\} \\ \Rightarrow & -(\underline{A}^{\uparrow}_{ji}\lambda_j - C_i^\top Z_j + \delta_{ij}\beta) + \Delta\underline{A}_{ji}\lambda_j & \leq & 0 & i,j \in \{1,\ldots,n\} \\ \Rightarrow & \underline{A}_{ji}\lambda_j - C_i^\top Z_j + \delta_{ij}\beta & \geq & 0 & i,j \in \{1,\ldots,n\} \\ \Rightarrow & \underline{A}^\top\lambda - C^\top Z + \beta I & \geq & \mathbf{0}_{n\times n}, & \end{array}$$

which is equivalent to the Metzler constraint of Problem 5. □

Lemma 5.1.5. *Using a nonstrict inequality relation in the Hurwitz constraint of Problem 7 instead of equality, as in Problem 5, results in*

$$\limsup_{t\to\infty} \|e(t)\|_1 \leq \left[2(\overline{A} - \underline{A})\varkappa + \overline{\xi} - \underline{\xi}\right]^\top \lambda^0 \leq \left[2(\overline{A} - \underline{A})\varkappa + \overline{\xi} - \underline{\xi}\right]^\top \lambda, \tag{5.3}$$

where λ^0 is the λ in the original equality constraint.

Proof. Consider the two equations

$$\lambda = -(\overline{A} - LC)^{-\top}\mathbf{1}_n$$
$$\lambda' = -(\overline{A} - LC)^{-\top}\alpha\mathbf{1}_n,$$

where $\alpha := \mathbf{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n), \alpha_i \geq 1, i \in \{1, \ldots, n\}$. Since $(\overline{A} - LC)^\top$ is Hurwitz and Metzler by construction, we have that $-(\overline{A} - LC)^{-\top}$ is nonnegative [43], and since $\alpha\mathbf{1}_n \geq \mathbf{1}_n$, we have $\lambda' \geq \lambda$, with equality holding if and only if $\alpha = I$. Since α is arbitrary, it follows that $\lambda = -(\overline{A} - LC)^{-\top}\alpha\mathbf{1}_n$ is equivalent to $\lambda \leq -(A - LC)^{-\top}\mathbf{1}_n$. Replacing λ with λ' in the expression for the upper bound on $\|\bar{e}\|_1$ (3.12) yields (5.3). □

Next, we prove that all feasible solutions to Problem 5 are feasible solutions to problem Problem 7, and, consequently, if Problem 5 is feasible, then any optimal solution to Problem 7 will satisfy its Hurwitz constraint with equality.

Corollary 5.1.6. *The set of feasible solutions to Problem 7 is a superset of the feasible solutions to Problem 5.*

Proof. By Lemma 5.1.4, if $\Gamma = |J|$, then Problem 5 can be viewed as an instance of Problem 7, where the Hurwitz constraint is a nonstrict inequality, instead of an equality. If a constraint $\sum_j E_{ij}q_j = b_i$ is feasible, then the constraint $\sum_j E_{ij}q_j \leq b_i$ can be feasibly satisfied with equality. Therefore, if $\Gamma = |J|$ and Problem 5 is feasible, then Problem 7 is also feasible, and its Hurwitz constraint can be feasibly satisfied with equality. Therefore, by Proposition 2.0.1, the set of feasible solutions to Problem 7 is a superset of the feasible solutions to Problem 5 for any $\mathbf{\Gamma} \preceq |\mathbf{J}|$. □

Lemma 5.1.7. *If Problem 5 is feasible, then the Hurwitz constraint in Problem 7 is always satisfied with equality.*

Proof. By Corollary 5.1.6, if Problem 5 is feasible, then satisfying the Hurwitz constraint in Problem 7 is feasible. By Lemma 5.1.5, satisfying the Hurwitz constraint with inequality will effect a greater cost than when satisfied with equality. Therefore, satisfying the Hurwitz constraint with equality is always optimal. □

We can now prove Proposition 5.1.3.

Proof of Proposition 5.1.3. By Lemma 5.1.4, if $\Gamma = |J|$, then Problem 7 is equivalent to Problem 5, where the Hurwitz constraint is a nonstrict inequality instead of an equality. By Lemma 5.1.7, the Hurwitz constraint is always satisfied with equality. Therefore, the two problems have the same optimal cost. It follows from the analysis in the proof of Lemma 5.1.4 and the definition of Λ, that the isomorphism $(\Lambda, Z, \beta) \mapsto \big(\mathbf{col}(\Lambda_1, \ldots, \Lambda_n), Z, \beta\big)$ satisfies Definition 5.1.2. □

Problem 2 allows for positive and negative perturbations to the constraint coefficients, but due to the form of Problem 7, each type of constraint, i.e., Hurwitz and Metzler, and therefore the elements of $\overline{A}^{\downarrow}$ and $\underline{A}^{\uparrow}$, respectively, will only be perturbed in one direction.

Proposition 5.1.8. *In Problem 7, perturbations to the elements of $\overline{A}^{\downarrow}$ will be strictly positive, and perturbations to the elements of $\underline{A}^{\uparrow}$ will be strictly negative.*

Proof. Since λ is positive, all constraint coefficient perturbations will be in the positive direction. In the Hurwitz constraints, whose coefficients are the elements of $\overline{A}^{\downarrow}$, a perturbed coefficient of λ has the value $(\overline{A}^{\downarrow}_{ij} + \Delta\underline{A}_{ij}) = \overline{A}_{ij}$.

Notice that in the Metzler constraint, we have reversed the direction of the inequality to comply with the form of Problem 3. In the Metzler constraints, whose coefficients are the elements of $\underline{A}^{\uparrow}$, a perturbed coefficient of λ has the value $(-\underline{A}^{\uparrow}_{ij} + \Delta\underline{A}_{ij}) = -\underline{A}_{ij}$. Reverting the direction of the inequality to that of Problem 5, the perturbed constraint coefficient of λ is $\underline{A}_{ij}$. □

Define the set of all admissible sets of indices of uncertain constraint coefficients

$$\mathcal{J} \coloneqq \bigcup_{k=1}^{n^2+n} \{\mathcal{P}_{\Gamma_k}(J_k)\} = \Big\{\{\mathcal{P}_{\Gamma_1}(J_1)\}, \ldots, \{\mathcal{P}_{\Gamma_{n^2+n}}(J_{n^2+n})\}\Big\}.$$

Given the pairs $(\overline{A}^{\downarrow}, \underline{A}^{\uparrow})$, $(\overline{A}, \underline{A}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$, a set of sets of indices of uncertain constraint coefficients $\mathbf{J}$, and a set of sets of indices of perturbed constraint coefficients $\mathbf{S} \coloneqq \big\{\{S_1 \cup \{t_1\} \subseteq J_1\}, \ldots, \{S_{n^2+n} \cup \{t_{n^2+n}\} \subseteq J_{n^2+n}\}\big\}$, we generate a pair $(A^u, A^l) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ via the mapping

$$\begin{aligned} \Xi : \mathcal{J} &\to \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \\ \mathbf{S} &\mapsto (A^u, A^l) \end{aligned} \tag{5.4}$$

where

$$A^u_{ji} = \begin{cases} \overline{A}_{ji} & \text{if } i \le n,\ j \in S_i \\ \overline{A}^{\downarrow}_{ji} + \big(\lfloor \Gamma_i \rfloor - \Gamma_i\big)\Delta\overline{A}_{ji} & \text{if } i \le n,\ j \in \{t_i\} \\ \overline{A}^{\downarrow}_{ji} & \text{otherwise,} \end{cases}$$

$$A^l_{ji} = \begin{cases} \underline{A}_{ji} & \text{if } i, j \le n,\ j \in S_{ni+j} \\ \underline{A}^{\uparrow}_{ji} - \Gamma_i \Delta\underline{A}_{ji} & \text{if } i, j \le n,\ j \in \{t_{ni+j}\} \\ \underline{A}^{\uparrow}_{ji} & \text{otherwise.} \end{cases}$$

A pair $A^{u,l}$ defines a set in $\mathbb{R}^{n\times n}$ by

$$\{A \in \mathbb{R}^{n\times n} : A^l \le A \le A^u\}. \tag{5.5}$$

A generic example of these sets for $n = 2$, illustrated as contiguous columns constituting a polygon, is presented in Figure 5.1. Each column represents the possible values of an element of A. The entire region represents the interval defined by $(\overline{A}, \underline{A})$, and the solid inner polygon represents the interval defined by $(\overline{A}^{\downarrow}, \underline{A}^{\uparrow})$. The upper and lower boundaries on the entire region represent the elements of $\overline{A}$ and $\underline{A}$, respectively, corresponding to the element represented by each column. Similarly, the upper and lower boundaries of the solid inner polygon represent the elements of $\overline{A}^{\downarrow}$ and $\underline{A}^{\uparrow}$, respectively, corresponding to the element represented by each column. The boxes extending from the inner solid polygon in each column represent perturbations to a single element of $\overline{A}^{\downarrow}$ or $\underline{A}^{\uparrow}$. For example, the bottom-left box, in the A_{11} column, represents the perturbation of $\underline{A}^{\uparrow}_{11}$ to $\underline{A}_{11}$. To satisfy the interval property (3.2), the state matrices used to construct the interval observer $A^{u,l}$ must be chosen such that the set (5.5) contains all possible A.

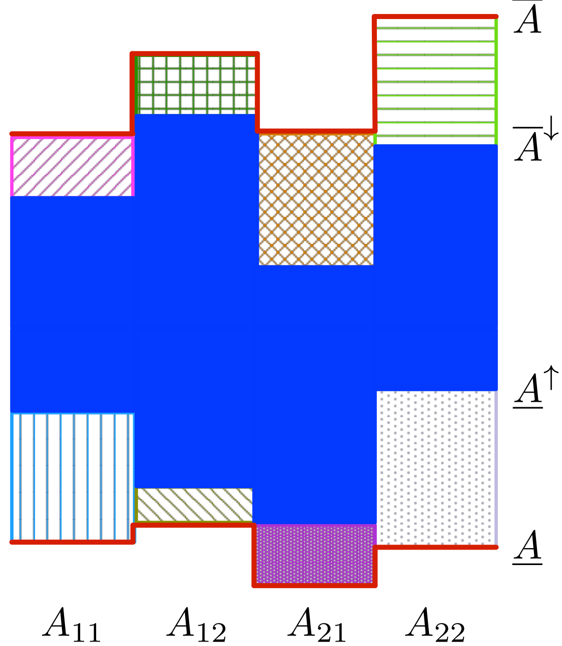

Figure 5.1: An example of the intervals defined by the pairs $(\overline{A}^{\downarrow}, \underline{A}^{\uparrow}), (\overline{A}, \underline{A}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ for $n = 2$.

Further, given constraint protection levels $\mathbf{\Gamma}$, using (5.4) we define the mapping

$$\begin{aligned} \mathcal{A} : \mathcal{J} \times \mathbb{R}^{n^2+n}_{\geq 0} &\to \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \\ (\mathbf{J}, \mathbf{\Gamma}) &\mapsto \left\{ A^{u,l} \mid (\exists\, \mathbf{S} \in \mathcal{J}) \left(\Xi(\mathbf{S}) = A^{u,l} \right) \right\}. \end{aligned} \tag{5.6}$$

The set (5.6) comprises all pairs of upper and lower state matrices effected by perturbing no more than $\mathbf{\Gamma}$ elements of the boundaries of the interval induced by the pair $(\underline{A}^{\uparrow}, \overline{A}^{\downarrow})$ as defined by (5.5). Problem 7 generates an L that minimizes the maximum upper bound on $\|\bar{e}\|_1$ over all state matrix pairs $A^{u,l} \in \mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$ for a fixed $\mathbf{\Gamma}$.

5.2 Optimality

Define $\mathcal{F}^L_{UM}, \mathcal{F}^L_{RUM}(J, \Gamma) \subset \mathbb{R}^{n\times p}$ to be the sets of all feasible observer gain matrices for the original UM problem, Problem 5, and the proposed robust UM problem, Problem 7, respectively. By Corollary 5.1.6, any feasible solution to Problem 7 is also a feasible solution to Problem 5, i.e., $\mathcal{F}^L_{RUM}(J, \Gamma) \supseteq \mathcal{F}^L_{UM}$. Define the set $\mathcal{L}_{RUM}(J, \Gamma) \subset \mathcal{F}^L_{RUM}(J, \Gamma)$ to be the set of all matrices L constructed using optimal solutions to Problem 7 for a given J and Γ. Similarly, define the set $\mathcal{L}_{UM} \subset \mathcal{F}_{UM}^{L}$ to be the set of all matrices L constructed using optimal solutions to Problem 5.

We propose the following *robust uncertain model interval observer (RUMIO)*, whose gain matrix (3.9) is constructed using an optimal solution to Problem 7 for a given J and Γ.

$$\begin{aligned} \dot{\hat{x}}^l &= A^u\hat{x}^l + L(y - C\hat{x}^l) - (A^u - A^l)\psi(\hat{x}^l) + \underline{\xi} \\ \dot{\hat{x}}^u &= A^u\hat{x}^u + L(y - C\hat{x}^u) - (A^u - A^l)\phi(\hat{x}^u) + \overline{\xi} \\ \hat{x}^{u,l}(0) &= (\overline{x}_0, \underline{x}_0) \\ A^{u,l} &\in \mathcal{A}(\mathbf{J}, \mathbf{\Gamma}). \end{aligned} \tag{5.7}$$

Define the mapping

$$\begin{aligned} \bar{e}_{\ell_1} : \mathbb{R}^{n\times p} \times \mathcal{A}\big(\mathbf{J}, |\mathbf{J}|\big) &\to \mathbb{R}_{\geq 0} \\ (L, A^{u,l}) &\mapsto -\big[2(A^u - A^l)\varkappa + \overline{\xi} - \underline{\xi}\big]^\top (A^u - LC)^{-\top}\mathbf{1}_n. \end{aligned} \tag{5.8}$$

By (3.11), the mapping (5.8) upper bounds $\|\bar{e}\|_1$ of an interval observer with dynamics (5.7), constructed with the state matrix pair $A^{u,l}$.

Proposition 5.2.1. *The optimal cost of Problem 7 is no less than the tightest bound (3.12) on $\|\bar{e}\|_1$ of the RUMIO (5.7).*

Proof. Given the optimal state matrix pair (A^u, A^l) constructed using the mapping (5.4), by construction of (3.12), the optimal cost is equal to the tightest bound (3.12) if and only if the optimal perturbed cost vector is equal to

$$\big[2(A^u - A^l)\varkappa + \overline{\xi} - \underline{\xi}\big]. \tag{5.9}$$

Since we stipulate in (5.2) that $\Gamma_0 = \sum_{i\neq 0}\lceil\Gamma_i\rceil$, exactly as many cost coefficients will be perturbed as constraint coefficients, so (5.9) is always a feasible perturbed cost vector. By construction, Problem 7 perturbs $\big[2(\overline{A}^{\downarrow} - \underline{A}^{\uparrow})\varkappa + \overline{\xi} - \underline{\xi}\big]$ such that the maximum cost is minimized. Therefore, the optimal cost cannot be less than the bound (3.12), as this would violate optimality. However, if (5.9) does not effect the greatest maximum cost, then (5.9) will not be the optimal perturbed cost vector. □

Theorem 5.2.2. *Given a system of the form (3.1), with matrices $\underline{A} \leq \underline{A}^{\uparrow} \leq \overline{A}^{\downarrow} \leq \overline{A}$, such that $\underline{A} \preceq \overline{A}$ and $\underline{A} \leq A \leq \overline{A}$, indices of uncertain cost coefficients J_0, indices of uncertain constraint coefficients $\mathbf{J}$, cost protection level $\Gamma_0 < |J_0|$, and constraint protection* *levels* $\mathbf{\Gamma} \preceq |\mathbf{J}|$*, the proposed RUMIO* (5.7) *effects a smaller upper bound on* $\|\bar{e}\|_1$ *than the UMIO* (3.10)*, and a maximum upper bound on* $\|\bar{e}\|_1$ *over the set of state matrices* $\mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$ *no greater than that if it were constructed instead using* $L_{UM} \in \mathcal{L}_{UM}$*, i.e.,*

$$\max_{A^{u,l} \in \mathcal{A}(\mathbf{J},\mathbf{\Gamma})} \bar{e}_{\ell_1}(L_{RUM}, A^{u,l}) \leq \max_{A^{u,l} \in \mathcal{A}(\mathbf{J},\mathbf{\Gamma})} \bar{e}_{\ell_1}\big(L_{UM}, A^{u,l}\big) < \bar{e}_{\ell_1}\big(L_{UM}, (\overline{A}, \underline{A})\big). \tag{5.10}$$

Proof. By (3.11), for a fixed L, choosing any state matrices $\widetilde{A}^{u,l} := (\widetilde{A}^u, \widetilde{A}^l)$ such that $(\widetilde{A}^u - \widetilde{A}^l) \preceq (A^u - A^l)$, necessarily reduces the upper bound on $\|\bar{e}\|_1$. Since $(\overline{A}, \underline{A}) \notin \mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$ for any $\mathbf{\Gamma} \preceq |\mathbf{J}|$, we have

$$\big(\forall\, \Gamma_0 < |J_0|\big)\big(\forall\, \mathbf{\Gamma} \preceq |\mathbf{J}|\big)\big(\widetilde{A}^{u,l} \in \mathcal{A}(\mathbf{J}, \mathbf{\Gamma})\big) \implies \Big(\bar{e}_{\ell_1}(L, \widetilde{A}^{u,l}) < \bar{e}_{\ell_1}(L, (\overline{A}, \underline{A}))\Big). \tag{5.11}$$

Therefore, constructing the observer (3.10) with state matrices $A^{u,l} \in \mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$, $\mathbf{\Gamma} \preceq |\mathbf{J}|$, instead of $(\overline{A}, \underline{A})$, effects a smaller upper bound on $\|\bar{e}\|_1$.

By Proposition 5.2.1, the optimal cost of Problem 7 is no less than

$$\max_{A^{u,l} \in \mathcal{A}(\mathbf{J},\mathbf{\Gamma})} \bar{e}_{\ell_1}(L, A^{u,l}),$$

i.e., the maximum upper bound on the ℓ_1-norm of the steady state supremum of the interval error over all state matrix pairs $A^{u,l} \in \mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$. By Corollary 5.1.6, the set of feasible L matrices admitted by Problem 7 is a superset of the feasible L matrices admitted by Problem 5, and by the optimality of $L \in \mathcal{L}_{RUM}$, we have that there exists no $L \in \mathcal{L}_{UM}$ that effects a smaller maximum upper bound on $\|\bar{e}\|_1$ over the set of state matrix pairs $\mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$. Combining this with (5.11), we verify (5.10). □

5.3 Implementation

In this section, we delineate and illustrate the design process of the proposed RUMIO (5.7).

1. Identify $\overline{x}_0$, $\underline{x}_0$, $\overline{A}$, $\underline{A}$, $\overline{A}^{\downarrow}$, $\underline{A}^{\uparrow}$, $\varkappa$.
2. Set $\mathbf{\Gamma}$ to the elementwise smallest value such that each possible state matrix A is contained in at least one of the sets (5.5) defined by a state matrix pair $A^{u,l} \in \mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$.
3. Using $c := \big[2(\overline{A}^{\downarrow} - \underline{A}^{\uparrow})\varkappa + \overline{\xi} - \underline{\xi}\big]$ and d as defined in (5.1), solve Problem 7 using the constraint $\lambda \geq \varepsilon \mathbf{1}_n$, where $\varepsilon \in \mathbb{R}_{>0}$ is an arbitrarily small constant.

4. Using the optimal λ and Z, construct the observer gain matrix $L \coloneqq \mathbf{diag}(\lambda)^{-1}Z^\top$.

5. Construct the RUMIO (5.7) with the pair $A^{u,l} \in \mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$ that induces the elementwise smallest interval (5.5) such that $A^l \leq A \leq A^u$, for all possible A.

The single resultant RUMIO with gain matrix L, is optimal in the sense that the maximum $\|\bar{e}\|_1$ over all state matrix pairs $A^{u,l} \in \mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$ is minimized, and is optimal in this sense for any state matrix pair in $\mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$. By stipulating that $A^{u,l}$ be chosen such that the set (5.5) contains A, we ensure the interval property (3.2); by setting $\mathbf{\Gamma}$ as small as is possible while ensuring the interval property, we minimize the attainable optimum cost.

For the class of systems defined by a fixed $\mathbf{\Gamma}$, we perform only a single optimization, i.e., design a single interval observer. This is advantageous in situations where the dynamical parameters of the plant are guaranteed to lie within some range, but under certain circumstances, this range can be refined. For example, in wireless networks where the dynamics, i.e., the load, changes more frequently than optimization can be performed, but the load can be identified [44]. A RUMIO (5.7) could be designed for each $\mathbf{\Gamma}$, and as the load changes, the corresponding L and $A^{u,l}$ would become active. Only a handful of such interval observers would need to be designed, *a priori*, but would be optimal for potentially billions of possible dynamics, as characterized in Proposition 2.1.2. The original UMIO (3.10) of [12], is optimal only for a single $A^{u,l}$, and would need to be optimized for each state matrix pair in $\mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$.

Example 5.3.1. To illustrate the proposed approach, we construct and implement a RUMIO for a system based on that in [12, Section IV.C],

$$A = \begin{bmatrix} -2.5 & 0.2 & 1 \\ 0.1 & -0.5 & 1 \\ 0 & 0.3 & -0.8 \end{bmatrix} \qquad \xi = 0.5\sin(x_1^2)\mathbf{1}_3$$
$$C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}.$$

We take the constants $\overline{x}_0$, $\underline{x}_0$, $\overline{\xi}$, $\underline{\xi}$, $\varkappa \in \mathbb{R}^3$ to be

$$\overline{x}_0 = \begin{bmatrix} -0.5 \\ 2.5 \\ 1.5 \end{bmatrix} \quad \underline{x}_0 = \begin{bmatrix} -1.5 \\ 1.5 \\ 0.5 \end{bmatrix} \quad \overline{\xi} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \underline{\xi} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \quad \varkappa = \begin{bmatrix} 1.7 \\ 6 \\ 2.6 \end{bmatrix},$$

and the matrices $\overline{A}, \underline{A}, \overline{A}^{\downarrow}, \underline{A}^{\uparrow} \in \mathbb{R}^{3\times 3}$ to be

$$\overline{A} = \begin{bmatrix} -2.48 & 0.22 & 1.02 \\ 0.12 & -0.48 & 1.02 \\ 0 & 0.32 & -0.78 \end{bmatrix} \quad \overline{A}^{\downarrow} = \begin{bmatrix} -2.49 & 0.21 & 1.01 \\ 0.11 & -0.49 & 1.01 \\ 0 & 0.31 & -0.79 \end{bmatrix}$$
$$\underline{A}^{\uparrow} = \begin{bmatrix} -2.51 & 0.19 & 0.99 \\ 0.09 & -0.51 & 0.99 \\ 0 & 0.29 & -0.81 \end{bmatrix} \quad \underline{A} = \begin{bmatrix} -2.52 & 0.18 & 0.98 \\ 0.08 & -0.52 & 0.98 \\ 0 & 0.28 & -0.82 \end{bmatrix},$$

which define the cost and perturbation vectors[1]

$$\begin{aligned} c &= 2(\overline{A}^{\downarrow} - \underline{A}^{\uparrow})\varkappa + \overline{\xi} - \underline{\xi} \\ &= \begin{bmatrix} 2.412 & 2.412 & 2.344 \end{bmatrix}^\top \end{aligned} \qquad d = \mathbf{col}\big(2\Delta\overline{A}\mathbf{diag}(\varkappa), 2\Delta\underline{A}\mathbf{diag}(\varkappa)\big).$$

The indices of uncertain cost coefficients are those corresponding to nonzero elements of d,

$$J_0 = \{1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18\},$$

and the indices of uncertain constraint coefficients are those corresponding to nonzero elements of the constraint coefficient perturbation matrix[2],

$$\widehat{E} := \begin{bmatrix} \Delta\overline{A}_1^\top & \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 9} \\ \mathbf{0}_{1\times 3} & \Delta\overline{A}_2^\top & \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 9} \\ \mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} & \Delta\overline{A}_3^\top & \mathbf{0}_{1\times 9} \\ \mathbf{0}_{9\times 3} & \mathbf{0}_{9\times 3} & \mathbf{0}_{9\times 3} & \mathbf{diag}\big(\mathbf{col}(\Delta\underline{A})\big) \end{bmatrix},$$

which yields

$$\begin{aligned} J_1 &= \{1, 2\} \\ J_2 &= \{4, 5, 6\} \\ J_3 &= \{7, 8, 9\} \\ J_6 &= \varnothing \\ J_i &= \{6 + i\} \quad i \in \{4, \ldots, 12\} \setminus \{6\}. \end{aligned}$$

[1] Refer to Appendix A.1 for the computed d.

[2] Refer to Appendix A.2 for the computed $\widehat{E}$.

Setting $\varepsilon = 10^{-6}$, we construct Problem 7 in the linear robust formulation of Problem 3:

$$\begin{aligned}
\text{minimize: } & \left[2(\overline{A}^{\downarrow} - \underline{A}^{\uparrow})\varkappa + \overline{\xi} - \underline{\xi}\right]^{\top} \lambda + \zeta_0 \Gamma_0 + \sum_{j \in J_0} p_{0j} \\
\text{subject to: } & \overline{A}_i^{\downarrow\top} \overline{\lambda}^{(i)} - C_i^{\top} Z \mathbf{1}_3 + \zeta_i \Gamma_i + \sum_{j \in J_i} p_{ij} \leq -1 && \forall\, i \neq 0 \\
& -(\underline{A}_{ji}^{\uparrow} \underline{\lambda}_j^{(i)} - C_i^{\top} Z_j + \delta_{ij}\beta) + \zeta_i \Gamma_r + \sum_{j \in J_r} p_{rj} \leq 0 && \forall\, i, j \neq 0,\ r = 3i + j \\
& \zeta_0 + p_{0j} \geq d_j \Lambda_j && \forall\, j \in J_0 \\
& \zeta_i + p_{ij} \geq \widehat{E}_{ij} \Lambda_j && \forall\, i \neq 0, j \in J_i \\
& p_{ij} \geq 0 && \forall\, i, j \in J_i \\
& \zeta_i \geq 0 && \forall\, i \\
& \lambda \geq 10^{-6} \mathbf{1}_3 \\
& \overline{\lambda}^{(i)} = \lambda && \forall\, i \neq 0 \\
& \underline{\lambda}^{(i)} = \lambda && \forall\, i \neq 0.
\end{aligned}$$

The UMIO of [12] is constructed with $A^{u,l} = (\overline{A}, \underline{A})$. We compare the maximum upper bound on $\|\bar{e}\|_1$ over the set of state matrix pairs $\mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$ effected by the proposed RU-MIO (5.7), i.e., the cost of Problem 7, to that effected by the original UMIO (3.10). The maximum upper bound on $\|\bar{e}\|_1$ over the state matrix pairs $\mathcal{A}(\mathbf{J}, \mathbf{\Gamma})$, the matrix L, and the percent reduction in the upper bound on $\|\bar{e}\|_1$ are presented in Table 5.1 for various $\mathbf{\Gamma}$. For ease of exposition, define the scalar parameters $\Gamma_H, \Gamma_M \in \mathbb{Z}_{\geq 0}$. The protection levels for the Hurwitz constraints, i.e., Γ_i, $i \in \{1, 2, 3\}$, are all set to $\min(\Gamma_H, |J_i|)$, e.g., $\Gamma_H = 3$ effects $\Gamma_1 = 3$, $\Gamma_2 = 3$, and $\Gamma_3 = 2$; the protection levels for the Metzler constraints, i.e., Γ_i, $i \in \{4, \ldots, 12\}$, are all set to $\min(\Gamma_M, |J_i|)$, e.g., $\Gamma_M = 1$ effects $\Gamma_i = 1$, $i \in \{4, \ldots, 12\} \setminus \{6\}$, and $\Gamma_6 = 0$. Recall that $\Gamma = |J|$ yields the same solution as Problem 5. Plots of the trial $\Gamma_H, \Gamma_M = 1$ are presented in Figures 5.2, 5.3, and 5.4.

The reduction in the upper bound on $\|\bar{e}\|_1$ ranges from modest to significant. The lowest reduction is 0.613% and the greatest reduction is 22.6%. The reduction in $\|\bar{e}\|_1$ is similarly variable, ranging from 0.673% to 21.5%, and the data suggest that it is correlated with the reduction in the upper bound on $\|\bar{e}\|_1$. The reduction in $\|\bar{e}\|_1$ seems roughly proportional to the reduction in cost.

A reduction in the upper bound on $\|\bar{e}\|_1$ would also occur by constructing the UMIO of [12] with tighter upper and lower state matrices. For comparison, we identify the greatest upper bound on $\|\bar{e}\|_1$ effected by constructing the UMIO of [12] with $A^{u,l}$ effected

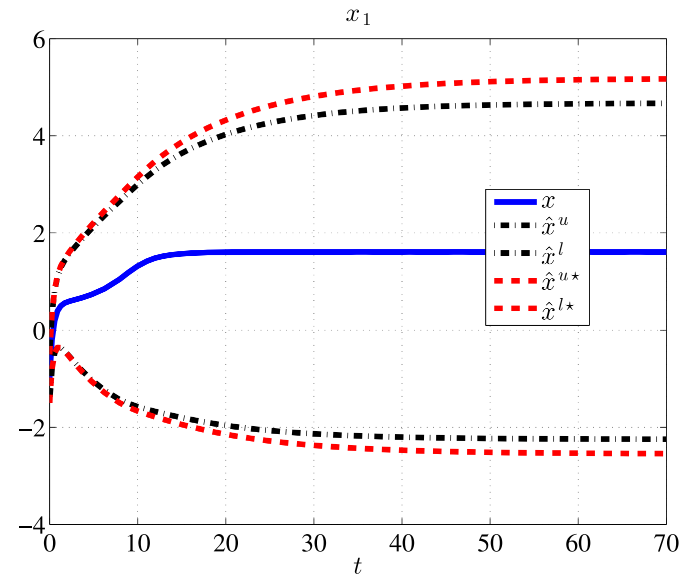

Figure 5.2: The estimates of x_1 of the proposed RUMIO $\hat{x}$ and the original UMIO $\hat{x}^{\star}$.

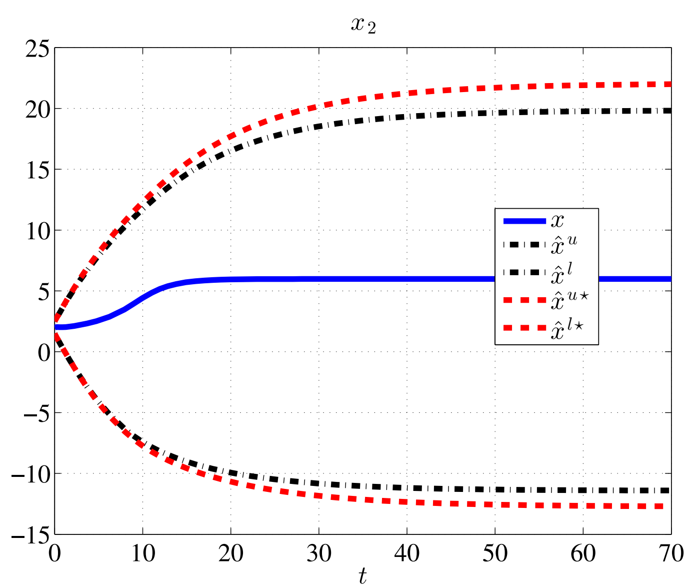

Figure 5.3: The estimates of x_2 of the proposed RUMIO $\hat{x}$ and the original UMIO $\hat{x}^{\star}$.

Table 5.1: Solutions for various Γ.

Γ_H	Γ_M	Cost	$\lVert \bar{e} \rVert_1$	L	% Cost Reduction	% $\lVert \bar{e} \rVert_1$ Reduction
0	1	63.1	46.6	$[\ 0.18 \quad 0.08 \quad 0\ ]^\top$	22.6	21.5
1	0	63.6	49.7	$[\ 0.19 \quad 0.09 \quad 0\ ]^\top$	22.0	16.3
	1	73.4	53.6	$[\ 0.18 \quad 0.08 \quad 0\ ]^\top$	9.94	9.76
2	0	72.5	54.3	$[\ 0.19 \quad 0.09 \quad 0\ ]^\top$	11.0	8.59
	1	81.0	59.0	$[\ 0.18 \quad 0.08 \quad 0\ ]^\top$	0.613	0.673
3	0	73.9	54.6	$[\ 0.19 \quad 0.09 \quad 0\ ]^\top$	2.70	8.08
	1	81.5	59.4	$[\ 0.18 \quad 0.08 \quad 0\ ]^\top$		

by perturbing Γ_0 bounds of $(\overline{A}^{\downarrow}, \underline{A}^{\uparrow})$, as this is the class of systems which Problem 7 minimizes over, as discussed in Proposition 5.2.1. We compute the percent reduction in the maximum upper bound on $\lVert \bar{e} \rVert_1$ effected by using the RUMIO instead of the UMIO.

Table 5.2: Worst-case comparisons.

Γ_H	Γ_M	Γ_0	Cost	$\lVert \bar{e} \rVert_1$	A^u	A^l	% Cost Reduction	% $\lVert \bar{e} \rVert_1$ Reduction
0	1	8	80.4	58.2	$\begin{bmatrix} -2.49 & 0.21 & 1.01 \\ 0.11 & -0.48 & 1.02 \\ 0 & 0.32 & -0.78 \end{bmatrix}$	$\begin{bmatrix} -2.51 & 0.19 & 0.99 \\ 0.09 & -0.52 & 0.98 \\ 0 & 0.28 & -0.82 \end{bmatrix}$	21.5	19.9
3	0						8.08	6.19
1	0	3	76.0	50.6	$\begin{bmatrix} -2.49 & 0.21 & 1.01 \\ 0.11 & -0.49 & 1.01 \\ 0 & 0.32 & -0.79 \end{bmatrix}$	$\begin{bmatrix} -2.51 & 0.19 & 0.99 \\ 0.09 & -0.52 & 0.99 \\ 0 & 0.28 & -0.81 \end{bmatrix}$	16.3	1.78
	1	11	81.3	59.2	$\begin{bmatrix} -2.49 & 0.22 & 1.01 \\ 0.12 & -0.48 & 1.02 \\ 0 & 0.32 & -0.78 \end{bmatrix}$	$\begin{bmatrix} -2.51 & 0.19 & 0.99 \\ 0.08 & -0.52 & 0.98 \\ 0 & 0.28 & -0.82 \end{bmatrix}$	9.72	9.46
2	0	6	79.2	56.3	$\begin{bmatrix} -2.49 & 0.21 & 1.01 \\ 0.11 & -0.48 & 1.01 \\ 0 & 0.32 & -0.78 \end{bmatrix}$	$\begin{bmatrix} -2.51 & 0.19 & 0.99 \\ 0.09 & -0.52 & 0.99 \\ 0 & 0.28 & -0.82 \end{bmatrix}$	8.46	3.55
	1	14	81.4	59.3	$\begin{bmatrix} -2.49 & 0.22 & 1.02 \\ 0.12 & -0.48 & 1.02 \\ 0 & 0.32 & -0.78 \end{bmatrix}$	$\begin{bmatrix} -2.51 & 0.18 & 0.98 \\ 0.08 & -0.52 & 0.98 \\ 0 & 0.28 & -0.82 \end{bmatrix}$	0.491	0.506

We see in Table 5.2 that the reduction in the maximum upper bound on $\lVert \bar{e} \rVert_1$ is modest to significant, ranging from 0.491% to 21.5%. We see a similar trend in the reduction of the actual $\lVert \bar{e} \rVert_1$, ranging from 0.506% to 19.9%. In Figures 5.5, 5.6, and 5.7, we see that, unlike the RKMIO in Example 4.3.1, the RUMIO in the trial $\Gamma_H = 1$, $\Gamma_M = 0$, $\Gamma_0 = 3$, does generate better estimates than the UMIO for all t.

$\triangle$

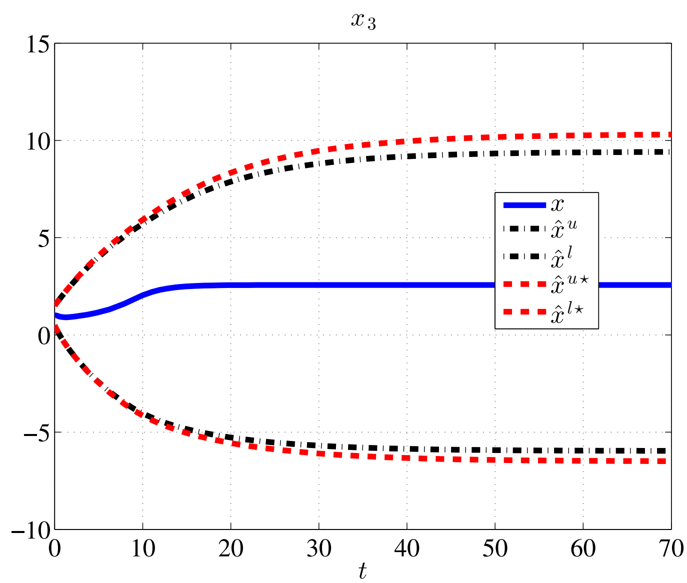

Figure 5.4: The estimates of x_3 of the proposed RUMIO $\hat{x}$ and the original UMIO $\hat{x}^{\star}$.

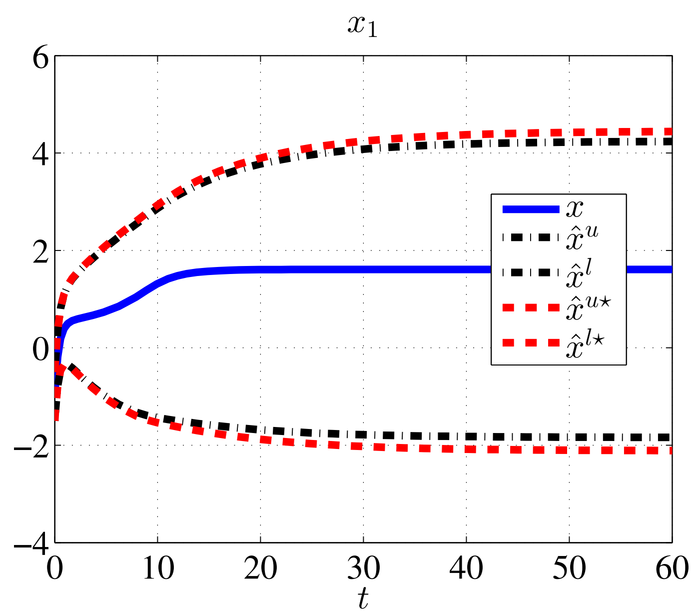

Figure 5.5: The estimates of x_1 of the proposed RUMIO $\hat{x}$ and the original UMIO $\hat{x}^{\star}$ using the worst-case UMIO initial conditions for $\Gamma_H = 1$, $\Gamma_M = 0$, $\Gamma_0 = 3$.

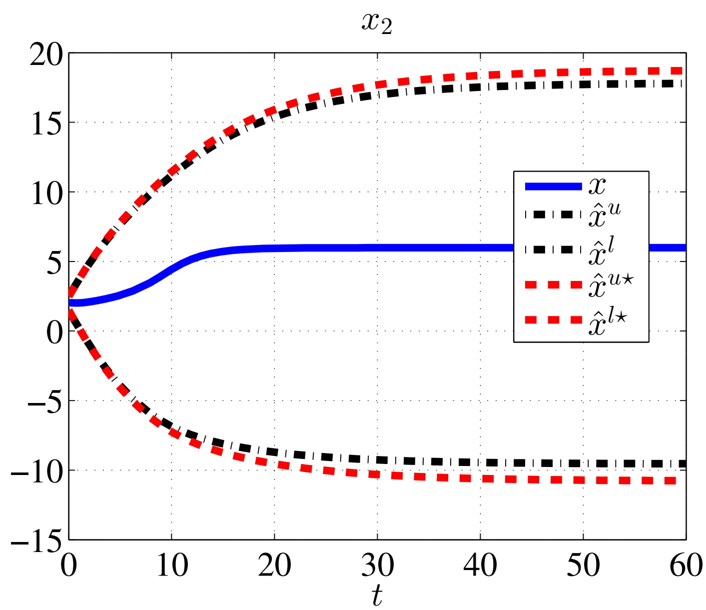

Figure 5.6: The estimates of x_2 of the proposed RUMIO $\hat{x}$ and the original UMIO $\hat{x}^\star$ using the worst-case UMIO initial conditions for $\Gamma_H = 1$, $\Gamma_M = 0$, $\Gamma_0 = 3$.

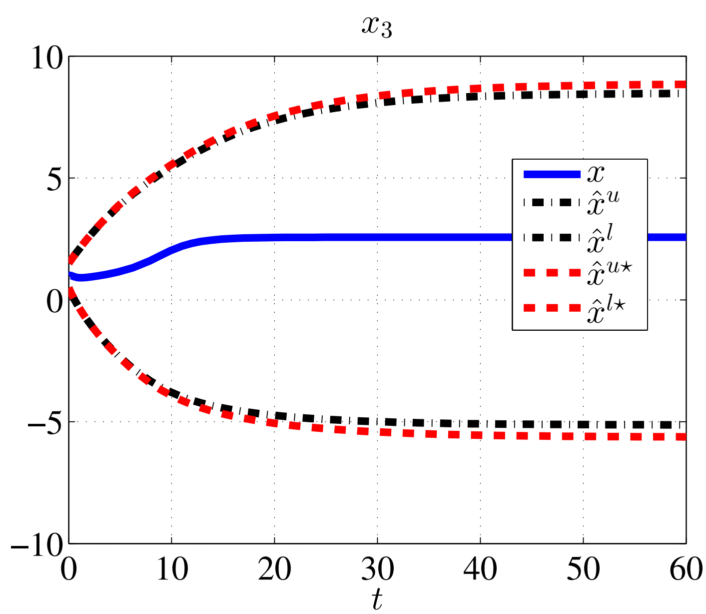

Figure 5.7: The estimates of x_3 of the proposed RUMIO $\hat{x}$ and the original UMIO $\hat{x}^\star$ using the worst-case UMIO initial conditions for $\Gamma_H = 1$, $\Gamma_M = 0$, $\Gamma_0 = 3$.

5.4 Experimental Validation of Approach

We conduct a Monte Carlo analysis to characterize the reduction in the upper bound on $\|\bar{e}\|_1$ effected by using the proposed RUMIO (5.7), instead of the UMIO (3.10) of [12].

The initial conditions $x(0) \in \mathbb{R}^n$ are generated as uniform random variables on the interval $x(0) \in [-n, n]^n$, and the constants $\overline{x}_0, \underline{x}_0 \in \mathbb{R}^n$ are defined as

$$\begin{aligned}\overline{x}_0 &= x(0) + \frac{1}{2n}\mathbf{1}_n \\ \underline{x}_0 &= x(0) - \frac{1}{2n}\mathbf{1}_n,\end{aligned}$$

which guarantees $\|\overline{x}_0 - \underline{x}_0\|_1 = 1$, providing consistency in the width of the initial condition intervals (4.2) across trials.

The elements of the matrices $\overline{A}, \underline{A}, \overline{A}^{\downarrow}, \underline{A}^{\uparrow} \in \mathbb{R}^{n \times n}$ are seeded by the uniform random variables on the intervals

$$\begin{aligned}&\widetilde{A}_{ij} \in [0, 1] && i, j \in \{1, \ldots, n\}, i \neq j \\ &\widetilde{A}_{ii} \in \big[-2n, -n\big] && i \in \{1, \ldots, n\}.\end{aligned} \tag{5.12}$$

The seed matrix $\widetilde{A} \in \mathbb{R}^{n \times n}$ is modified to generate the state matrices such that

$$\begin{aligned}&\|\overline{A} - \underline{A}\|_1 = 1 \\ &\overline{A}^{\downarrow} - \underline{A}^{\uparrow} \geq \frac{2}{n^3}\mathbf{1}_{n \times n}.\end{aligned} \tag{5.13}$$

The ranges defined for the elements of $\widetilde{A}$ in (5.12) ensure the Hurwitz property, and although we do not simulate a specific dynamical system, we make the assumption that ξ is such that the solutions of $\dot{x} = Ax + \xi(x, t)$ are bounded, which is sufficient to satisfy Assumption 4. The constraints (5.13) provide consistency in the sizes of the state matrix intervals (5.5) across trials. The matrix $C \in \mathbb{R}^{p \times n}$ is generated as a uniform random variable on the interval

$$C \in [0, 1]^{p \times n}.$$

We take the constants $\overline{\xi}, \underline{\xi}, \varkappa \in \mathbb{R}^n$ to be

$$\begin{aligned}\overline{\xi} &= 0.1\mathbf{1}_n \\ \underline{\xi} &= -0.1\mathbf{1}_n \\ \varkappa &= 2\max(|\overline{x}_0|, |\underline{x}_0|),\end{aligned}$$

where max is taken elementwise.

Define the constant $\Gamma^\star \in \mathbb{Z}_{\geq 0}$. The protection levels for the Hurwitz constraints are set to $\Gamma_i = \min\bigl(\Gamma^\star, |J_i|\bigr)$, $i \in \{1, \ldots, n\}$, and a randomly populated set $\mathcal{I} \subseteq \{n+1, \ldots, n^2+n\}$ of cardinality $n\Gamma^\star$, contains the indices of the Metzler constraints that have their protection levels set to 1, i.e.,

$$
\begin{aligned}
&\mathcal{I} \subseteq \{n+1, \ldots, n^2+n\} \\
&|\mathcal{I}| = n\Gamma^\star \\
&\Gamma_i = \begin{cases} 1 & \text{if } i \in \mathcal{I} \\ 0 & \text{if } i \in \{n+1, \ldots, n^2+n\} \setminus \mathcal{I}. \end{cases}
\end{aligned}
$$

This causes $n\Gamma^\star$ elements of $\overline{A}^{\downarrow}$, and $n\Gamma^\star$ elements of $\underline{A}^{\uparrow}$, to be perturbed in each trial. The cost protection level is set to $\Gamma_0 = \sum_{i \neq 0} \Gamma_i$.

Five thousand trials are conducted for each combination of $n \in \{1, \ldots, 5\}$ and $p, \Gamma^\star \in \{1, \ldots, n-1\}$. In each trial, Problems 7 and 5 are solved for the same parameters, and the absolute and relative differences between their costs are recorded. The arithmetic means μ and standard deviations σ of these values are presented in the following tables.

Table 5.3: Empirical results for $n = 2$.

$n=2$		Absolute		Relative	
p	$\Gamma^\star$	μ	σ	μ	σ
1	1	0.54	12	0.076	0.062

Table 5.4: Empirical results for $n = 3$.

$n=3$		Absolute		Relative	
p	$\Gamma^\star$	μ	σ	μ	σ
1	1	1.2	4.3	0.15	0.080
	2	1.1	29	0.045	0.061
2	1	0.32	0.30	0.079	0.047
	2	0.053	0.098	0.012	0.018

In Tables 5.3, 5.4, 5.5, and 5.6, we see that the greatest mean reductions in the upper bound on $\|\bar{e}\|_1$ for each n are approximately 7.6% – 22%. As expected, the reduction in the upper bound on $\|\bar{e}\|_1$ is inversely proportional to $\Gamma^\star$, as $\Gamma^\star = n$ recovers the original problem, Problem 5, and $\Gamma^\star = 0$ does not perturb the state matrices from $(\overline{A}^{\downarrow}, \underline{A}^{\uparrow})$. As in the KM case, the data suggest that the reduction in the upper bound on $\|\bar{e}\|_1$ correlates

Table 5.5: Empirical results for $n = 4$.

$n = 4$		Absolute		Relative	
p	$\Gamma^\star$	μ	σ	μ	σ
1	1	2.5	25	0.19	0.083
	2	1.0	9.0	0.082	0.066
	3	0.51	5.6	0.029	0.053
2	1	1.1	0.68	0.16	0.056
	2	0.42	0.31	0.060	0.032
	3	0.12	0.13	0.017	0.015
3	1	0.29	0.29	0.076	0.043
	2	0.053	0.085	0.012	0.014
	3	0.021	0.042	0.0051	0.0081

Table 5.6: Empirical results for $n = 5$.

$n = 5$		Absolute		Relative	
p	$\Gamma^\star$	μ	σ	μ	σ
1	1	4.1	58	0.22	0.080
	2	1.5	5.6	0.11	0.066
	3	0.77	6.5	0.051	0.054
	4	0.51	11	0.019	0.045
2	1	1.8	0.83	0.20	0.051
	2	0.86	0.48	0.095	0.034
	3	0.37	0.33	0.040	0.020
	4	0.12	0.13	0.013	0.011
3	1	1.2	0.71	0.17	0.056
	2	0.50	0.37	0.068	0.035
	3	0.19	0.18	0.025	0.018
	4	0.053	0.066	0.0071	0.0075
4	1	0.28	0.28	0.074	0.043
	2	0.048	0.083	0.011	0.013
	3	0.021	0.036	0.0052	0.0062
	4	0.010	0.023	0.0025	0.0044

positively with n and negatively with p, making the proposed RUMIO (5.7) increasingly attractive as the number of states increases, and as the number of outputs decreases, i.e., as the system becomes more complex and less information is available. The negative correlation with p can be interpreted as the robust formulation compensating for the reduction in the number of measurements. A practical implication is that the robust formulation can be used to justify using fewer sensors. As in the KM case, for many combinations of n, p, and $\Gamma^{\star}$, the standard deviation is greater than the mean, which suggests that the reduction is highly dependent upon the specific plant being observed.

Chapter 6

Conclusions and Future Work

6.1 Conclusions

In this thesis, we applied the robust optimization method of [41] to the linear programming-based interval observer design procedure of [12]. In Chapter 2 we discussed the cardinality constrained robust optimization method of [41]. We extended the theory of [41] to allow for multiple distinct perturbations to individual cost coefficients. In Chapter 3 we delineated the interval observer design procedures of [12]. In Chapter 4 we applied the robust optimization method, with the multiple-perturbations-extension developed in Chapter 2, to the KMIO design procedure of [12]. We proved that the ℓ_1-norm of the interval error of the proposed RKMIO is strictly less than that of the original KMIO, albeit by an indeterminate amount. We empirically characterized the reduction in the ℓ_1-norm of the interval error using a Monte Carlo analysis. In Chapter 5 we applied the robust optimization method of [41], with the multiple-perturbations-extension developed in Chapter 2, to the UMIO design procedure of [12]. We proved that the ℓ_1-norm of the steady state supremum of the interval error of the proposed RUMIO is strictly less than that of the original UMIO, albeit by an indeterminate amount. We empirically characterized the reduction in the upper bound on the ℓ_1-norm of the steady state supremum of the interval error using a Monte Carlo analysis.

The Monte Carlo analyses in Chapters 4 and 5 suggest that the cost reduction effected by the proposed robust formulations correlate positively with the number of states, and negatively with the number of outputs. The proposed RKMIO and RUMIO are therefore increasingly attractive as the plant is more complex, and as as fewer measurements are available. A practical implication is that the robust formulations compensate for the lack of measurements, thereby justifying the use of fewer sensors. However, the standard deviations of the reduction in cost effected by using the robust formulations is relatively high. This suggests that the reduction is highly dependent upon the specific plant being observed.

6.2 Future Work

An important goal of future work is to identify analytic lower bounds on the reduction in the ℓ_1-norm of the interval error, and upper bound on the ℓ_1-norm of the steady state supremum of the interval error, when using the proposed RKMIO and RUMIO, respectively, over the KMIO and UMIO of [12]. Also, an objective function should be identified that enables simultaneous optimization of transient and steady state performance. Identifying such an objective function could allow for the unification of the KM and UM cases, as the constraints in both cases are already equivalent. As discussed in Chapter 5, the cost of the proposed UM problem is conservative, as the cost and constraint perturbations may be mismatched; future research should identify a method of coupling the perturbing of coefficients that correspond to the same dynamical parameters. The proposed robust design procedures should also be extended to uncertainty in the outputs. The RKMIO and RUMIO design procedures are described in terms of the underlying LPs; it would be useful to develop a framework that allows the process to be described in terms of the dynamical system being observed. It would also be useful to identify the class of systems for which these interval observers can be constructed, in terms of more intuitive dynamical properties, rather than linear constraints.

APPENDICES

Appendix A

Robust Unknown Model Interval Observer Example Constants

A.1 Cost Coefficient Perturbations Vector

$$d = [0.034 \quad 0.034 \quad 0 \quad 0.12 \quad 0.12 \quad 0.12 \quad 0.052 \quad 0.052 \quad 0.052 \quad \cdots \\ 0.034 \quad 0.034 \quad 0 \quad 0.12 \quad 0.12 \quad 0.12 \quad 0.052 \quad 0.052 \quad 0.052]^\top$$

A.2 Constraint Coefficient Perturbations Matrix

$$\widehat{E} =$$

$$\begin{bmatrix}
0.01 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.01 & 0.01 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.01 & 0.01 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.01 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.01 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.01 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.01 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.01 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.01
\end{bmatrix}$$

References

[1] David L. Ma and Richard D. Braatz. Robust identification and control of batch processes. *Computers & Chemical Engineering*, 27(8-9):1175–1184, September 2003.

[2] Eric Walter and Hélène Piet-Lahanier. Estimation of parameter bounds from bounded-error data: a survey. *Mathematics and Computers in Simulation*, 32(5-6):449–468, December 1990.

[3] Koren Ward and Alexander Zelinsky. An exploratory robot controller which adapts to unknown environments and damaged sensors. In Alexander Zelinsky, editor, *Field and Service Robotics*, pages 456–463. Springer London, 1998.

[4] Hod Lipson and Josh Bongard. An exploration-estimation algorithm for synthesis and analysis of engineering systems using minimal physical testing. In *ASME Design Automation Conference*, pages 1087–1093, Salt Lake City, USA, 2004. ASME.

[5] Frédéric Mazenc and Olivier Bernard. Interval observers for linear time-invariant systems with disturbances. *Automatica*, 47(1), January 2011.

[6] Fred C. Schweppe. Recursive state estimation: Unknown but bounded errors and system inputs. *IEEE Transactions on Automatic Control*, 13(1):22–28, February 1968.

[7] Dimitri P. Bertsekas and Ian B. Rhodes. Recursive state estimation for a set-membership description of uncertainty. *IEEE Transactions on Automatic Control*, 16(2):117–128, April 1971.

[8] L. Chisci, A. Garulli, and G. Zappa. Recursive state bounding by parallelotopes. *Automatica*, 32(7):1049–1055, July 1996.

[9] T. Alamo, J.M. Bravo, and E.F. Camacho. Guaranteed state estimation by zonotopes. *Automatica*, 41(6):1035–1043, June 2005.

[10] Mohamed Zakaria Hadj-Sadok and Jean-Luc Gouzé. Bounds estimations for uncertain models of wastewater treatment. In *Proceedings of the IEEE International Conference on Control Applications*, volume 1, pages 336–340, Trieste, Italy, 1998.

[11] Marcelo Moisan and Olivier Bernard. Interval observers for non monotone systems application to bioprocess models. In Pavel Zítek, editor, *Proceedings of the IFAC World Congress*, pages 43–48, Prague, Czech Republic, July 2005.

[12] Mustapha Ait Rami, C.H. Cheng, and C. de Prada. Tight robust interval observers: An LP approach. In *IEEE Conference on Decision and Control*, pages 2967–2972, Cancun, Mexico, 2008.

[13] Mustapha Ait Rami, Fernando Tadeo, and Uwe Helmke. Positive observers for linear positive systems, and their implications. *International Journal of Control*, 84(4), April 2011.

[14] Mustapha Ait Rami, M. Schönlein, and Jens Jordan. Estimation of Linear Positive Systems with Unknown Time-Varying Delays. *European Journal of Control*, pages 1–22, 2013.

[15] M. Bolajraf, Mustapha Ait Rami, and Fernando Tadeo. Robust interval observer with uncertainties in the output. In *Mediterranean Conference on Control and Automation*, pages 52–57, Marrakech, Morocco, June 2010.

[16] Zhan Shu, James Lam, Huijun Gao, Baozhu Du, and Ligang Wu. Positive Observers and Dynamic Output-Feedback Controllers for Interval Positive Linear Systems. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 55(10), November 2008.

[17] V. Alcaraz-González, A. Genovesi, J. Harmand, V. González-Alvarez, A. Rapaport, and J.P. Steyer. Robust Exponential Nonlinear Observers for a Class of Lumped Models Useful in Chemical and Biochemical Engineering - Application to a Wastewater Treatment Process. In *International Workshop on Application of Interval Analysis to Systems and Control*, Girona, Spain, 1999.

[18] Jean-Luc Gouzé, A. Rapaport, and Mohamed Zakaria Hadj-Sadok. Interval observers for uncertain biological systems. *Ecological Modelling*, 133(1-2), August 2000.

[19] Matthieu Fruchard, Olivier Bernard, and Jean-Luc Gouzé. Interval observers with guaranteed confidence levels application to the activated sludge process. In Luis Basañez and Juan A de la Puente, editors, *Proceedings of the IFAC World Congress*, pages 1376–1376, Barcelona, Spain, July 2002.

[20] V. Alcaraz-González, J.P. Steyer, J. Harmand, A. Rapaport, V. González-Alvarez, and C. Pelayo-Ortiz. Application of a Robust Interval Observer to an Anaerobic Digestion Process. *Developments in Chemical Engineering and Mineral Processing*, 13(3-4):267–278, May 2008.

[21] Guillaume Goffaux, Alain Vande Wouwer, and Olivier Bernard. Improving continuous-discrete interval observers with application to microalgae-based bioprocesses. *Journal of Process Control*, 19(7):1182–1190, July 2009.

[22] Luc Jaulin. *Interval Robotics*. Brest, France, 2012.

[23] Jordi Meseguer, Vicenç Puig, and Teresa Escobet. Fault Diagnosis Using a Timed Discrete-Event Approach Based on Interval Observers: Application to Sewer Networks. *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans*, 40(5):900–916, September 2010.

[24] Guillaume Goffaux, Alain Vande Wouwer, and Marcel Remy. Vehicle Positioning by a Confidence Interval Observer - Application to an Autonomous Underwater Vehicle. In *IEEE Intelligent Vehicles Symposium*, number 5, pages 309–314, Istanbul, Turkey, June 2007. IEEE.

[25] A. Rapaport and J. Harmand. Robust regulation of a class of partially observed nonlinear continuous bioreactors. *Journal of Process Control*, 12(2):291–302, February 2002.

[26] Denis Efimov, Tarek Raissi, and Ali Zolghadri. Control of Nonlinear and LPV Systems: Interval Observer-Based Framework. *IEEE Transactions on Automatic Control*, 58(3), March 2013.

[27] Dimitris Bertsimas, David B. Brown, and Constantine Caramanis. Theory and Applications of Robust Optimization. *SIAM Review*, 53(3):464–501, January 2011.

[28] Aharon Ben-Tal and Arkadi Nemirovski. Robust solutions of Linear Programming problems contaminated with uncertain data. *Mathematical Programming*, 88(3), September 2000.

[29] Aharon Ben-Tal and Arkadi Nemirovski. Robust Convex Optimization. *Mathematics of Operations Research*, 23(4):769–805, November 1998.

[30] Aharon Ben-Tal and Arkadi Nemirovski. Robust solutions of uncertain linear programs. *Operations Research Letters*, 25(1):1–13, August 1999.

[31] Dimitris Bertsimas, Dessislava Pachamanova, and Melvyn Sim. Robust linear optimization under general norms. *Operations Research Letters*, 32(6), November 2004.

[32] A.L. Soyster. Technical Note–Convex Programming with Set-Inclusive Constraints and Applications to Inexact Linear Programming. *Operations Research*, 21(5):1154–1157, September 1973.

[33] Dimitris Bertsimas and Melvyn Sim. The Price of Robustness. *Operations Research*, 52(1):35–53, January 2004.

[34] Aharon Ben-Tal and Arkadi Nemirovski. Robust optimization - methodology and applications. *Mathematical Programming*, 92(3):453–480, May 2002.

[35] Aharon Ben-Tal and Arkadi Nemirovski. Robust Truss Topology Design via Semidefinite Programming. *SIAM Journal on Optimization*, 7(4):991–1016, November 1997.

[36] Dimitris Bertsimas and David B. Brown. Constrained Stochastic LQC: A Tractable Approach. *IEEE Transactions on Automatic Control*, 52(10):1826–1841, October 2007.

[37] Stephen P. Boyd, Seung-Jean Kim, Dinesh D. Patil, and Mark A. Horowitz. Digital Circuit Optimization via Geometric Programming. *Operations Research*, 53(6):899–932, November 2005.

[38] Dinesh D. Patil, Sunghee Yun, Seung-Jean Kim, Alvin Cheung, Mark A. Horowitz, and Stephen P. Boyd. A New Method for Design of Robust Digital Circuits. In *International Symposium on Quality of Electronic Design*, pages 676–681, 2005.

[39] Kan-Lin Hsiung, Seung-Jean Kim, and Stephen P. Boyd. Power control in lognormal fading wireless channels with uptime probability specifications via robust geometric programming. In *Proceedings of the American Control Conference*, pages 3955–3959, Portland, OR, 2005.

[40] Miguel Sousa Lobo and Stephen P. Boyd. The worst-case risk of a portfolio. 2000.

[41] Dimitris Bertsimas and Melvyn Sim. Robust discrete optimization and network flows. *Mathematical Programming*, 98(1-3):49–71, September 2003.

[42] K.M. Anstreicher. Linear Programming in O([n3/ln n]L) Operations. *SIAM Journal on Optimization*, 9(4):803–812, January 1999.

[43] Kenneth Joseph Arrow. *Studies in resource allocation processes*. Cambridge University Press, Cambridge, 1977.

[44] Sungho Yun and Constantine Caramanis. System level optimization in wireless networks with uncertain customer arrival rates. *2008 46th Annual Allerton Conference on Communication, Control, and Computing*, pages 1022–1029, September 2008.

[45] Michael Grant and Stephen P. Boyd. CVX: Matlab software for disciplined convex programming, version 2.0 beta. http://cvxr.com/cvx, September 2012.

[46] Michael Grant and Stephen P. Boyd. Graph implementations for nonsmooth convex programs. In V. Blondel, S. Boyd, and H. Kimura, editors, *Recent Advances in Learning and Control*, Lecture Notes in Control and Information Sciences, pages 95–110. Springer-Verlag Limited, 2008. http://stanford.edu/~boyd/graph_dcp.html.