# Signaling in Frequency Selective Gaussian Interference Channels

by

Ehsan Ebrahimzadeh Houlasou

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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**Abstract**

Sharing communication resources in wireless communication networks, due to the ever increasing growth in the number of users and the growing demand for higher data rates, appears to be inevitable. Consequently, present wireless communication networks should provide service for a large number of users through a frequency selective and interference limited medium rather than a single band, noise limited channel. In this thesis, we study a Gaussian interference network with orthogonal frequency sub-bands with *slow faded* and *frequency-selective* channel coefficients. The network is decentralized in the sense that there is no central node to assign the frequency sub-bands to the users. Moreover, due to lack of a feedback link between the two ends of any transmitter-receiver pair, all transmitters are unaware of the channel coefficients. Since the channel is assumed to be static during the communication period of interest, the concept of outage probability is employed in order to assess the performance of the network.

In a scenario where all transmitters distribute their available power uniformly across the sub-bands, we investigate the problem of how establishing a nonzero correlation $\rho$ among the Gaussian signals transmitted by each user along different frequency sub-bands can improve the outage probability at each of the receivers. Specifically, we show in a general $k$-user interference channel over $N$ orthogonal frequency sub-bands that , when receivers treat interference as noise, $\rho = 0$ is a point of local extremum for the achievable rate at each receiver, for any realization of channel coefficients. Moreover, in the case of $K = 2$ with arbitrary number of sub-bands, it is verified that there exists a finite level of Signal-to-Noise Ratio (SNR) such that the achievable rate has a local minimum at $\rho = 0$, which is not necessarily the case when $K > 2$.

We then concentrate on a 2-user interference channel over 2 orthogonal frequency sub-bands and characterize the behavior of the outage probability in the high SNR regime. We consider two simple decoding strategies at the receiver. In the first scenario, receivers simply treat interference as noise. In the second scenario, the receivers have the choice either to decode the desired signal treating interference as noise or to decode interference treating the desired signal as noise before decoding the interference free signal. Indeed, in both cases, we first show that the achievable rate is an increasing function of $\rho$ in the high SNR regime, which suggests to repeat the same signal over the sub-bands. This observation, *in a sense*, reflects to the behavior of the outage probability, the scaling behavior of which in the high SNR regime is characterized for the Rayleigh fading scenario.

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Aside from a great academic experience the name of Waterloo is associated with pure existential moments. I would like to gratify all the friends contributed to making such a delightful time and remember the these moments while reading these lines.

Last but definitely not least, I would like to express my appreciations towards my wonderful family for their kind presense which cannot be expressed in words.

## Dedication

This work is dedicated to my beloved family.

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# Chapter 1

# Introduction

## 1.1 Summary of Prior Art

Current wireless communication systems, due to the growing demand for higher data rates, should provide service for a large number of users through a shared time or frequency band. Points to point scenarios with fixed channel characteristics are no longer practical and dealing with multi-user, frequency selective, and interference limited systems is indispensable. Most of the studies on multiuser fading channels assume some knowledge of channel parameters and a certain level of coordination among the users for encoding/decoding and resource allocation. These assumptions vary based on the application of interest.

Tse and Hanly initiated the study of multiuser fading channels in companion papers [1] and [2]. Assuming perfect channel side information (CSI) at both the transmitters and the receiver, they derived the capacity region of fading multiple access channels as well as the optimal resource allocation strategies in two different senses; namely: the *throughput capacity region* and the *delay limited capacity region*.

Throughput capacity corresponds to the ergodic achievable rates through different variations of the channel. In contrast, the delay limited capacity corresponds to the maximum instantaneous data rate that can be maintained through all fading states which is also referred to as *zero outage capacity*. This definition is very strict for practical purposes as an extremely poor fading condition even for a short time duration can dramatically decrease the rate that can be constantly supplied during the whole transmission. In this case, if some transmission outage under severe fading conditions is allowed, this definition can be relaxed and the outage capacity can be defined as the maximum instantaneous information rate during non-outage fading conditions such that the allowed average transmission outage probability is satisfied.

From this viewpoint, the study of fading MAC channels was followed up in [3], where the outage capacity region with non-zero outage for a fading MAC is implicitly obtained and the corresponding optimal power allocation is characterized under the perfect CSI assumption at both the transmitters and the receiver.

Within the context of multi-user fading channels, Li and Goldsmith in [4] and [5], assuming perfect CSI at both the transmitter and the receivers, investigated the three aforementioned types of capacity regions for the fading broadcast channels in different spectrum sharing strategies. Moreover, the optimal resource allocation strategy for each regime is also characterized.

The assumption of perfect CSI at the transmitter side rests upon the presence of a feedback path from each receiver to the corresponding transmitter to send the estimated channel gains. These assumptions, however, fail in many wireless communication scenarios, such as distant pairs of mobile user/base station in a cellular network that operate in the same frequency band.

A better multi-user communication scenario to capture the essential characteristics of a wireless communication system is the fading interference channel. The (Shannon) capacity region of interference channels is not known even for the static two user case. The best known inner bound is the coding scheme proposed by Han and Kobauashi[6]. The capacity region of a Gaussian interference channel (GIC) is characterized to within one bit in [7], by establishing a novel outer bound and show that it is achievable within 1 bit/s/Hz by a Han-Kobayashi (HK) type scheme. Based upon this outer bound, the Diversity Multiplexing Trade off (DMT) of fading interference channels is studied in [8].

Proposed by Zheng and Tse[9] in the context of MIMO point-point channels, DMT can be described a measure for the fundamental trade off between the communication rate and decay rate of the outage probability with SNR, in the high SNR regime. The DMT of the MIMO GIC is considered in [10] and [11]. The DMT of a symmetric GIC with no CSI at the transmitter side is considered in [12], for a a class of achievable schemes known as multilevel superposition coding. The idea of compound interference channel[13] and claimed to be DMT optimal. As a special case of this scenario, The DMT of the HK scheme is then characterized. In [14], the DMT of a fixed-power-split HK is evaluated. Assuming a level of source cooperation and partial CSIT, the DMT of the two user fading GIC is studied in [15]. The DMT of the GIC in the asymmetric setting is studied in [16]. Finally, in [17] the optimal DMT of the two user GIC with Rayleigh fading is characterized.

All the works mentioned thus far primarily deal with the flat fading scenario, while frequency selective interference channels, which are the main focus of this study, have received less attention. The capacity region of a frequency selective Gaussian interference channel is derived in [18] for the strong interference regime, under an average power constraint. First, a frequency selective Gaussian interference channel is modeled as s set of independent parallel memory-less Gaussian interference channels, then the fixed channel results due to [6] and [19] which suggest the use of multiple access codes for coding, the capacity region of a frequency selective Gaussian interference channel is obtained.

For the weak interference, however, successive interference cancellation techniques are no longer efficient. Moreover, sequential cancellation techniques are more prone to errors in estimation of channel gains and are only practical when the number of users is small.

A pragmatic yet mathematically tractable decoding scheme which provides an inner bound for the capacity region of a general interference channel is to treat interference as noise. This scheme is well suitable for the applications where the decoder complexity or latency is a concern. In this scenario, assuming a random Gaussian codebook generation for the encoding, the transmission strategy for maximizing the achievable rate is solely based on power allocation. Depending on the level of cooperation between the users and the knowledge of channel parameters and power constraints various schemes have been proposed in the literature.

In order to find the largest achievable rate region in this scenario, a centralized algorithm is proposed [20] which assumes the presence of a spectrum management center for spectrum allocation to the users. Subsequently, it was shown in [21] that this algorithm can be interpreted as a dual algorithm which leads to a better numerical efficiency for solving the optimization problem.

Within the context of distributed algorithms with no centralized control, Iterative Water Filling Algorithm(IWFA) was proposed in [22] treating the problem as a competitive rate maximization game. The interference channel is modeled as a non-cooperative game where each transmitter is a player who aims at maximizing his own transmission rate. The existence and the uniqueness of the Nash equilibrium in this game is proved in [22]. In IWFA, users maximize their own transmission rates sequentially based on the knowledge of their intended channel gain and the noise plus interference power spectral density which is fed back from their own intended receiver.

To improve the convergence speed of IWFA, simultaneous IWFA was proposed in [23] where users perform their rate maximization strategy simultaneously. An asynchronous version of IWFA, which provably converges globally under certain conditions, was proposed in [24] which provides a unified framework that includes sequential and simultaneous versions.

The issues of efficiency and fairness in this non-cooperative spectrum sharing game are investigated in [25] by a set of self-enforcing spectrum sharing rules within the framework of repeated games. This approach, however, requires the knowledge of all the channel gains and power constraints at each transmitter and is only investigated for the flat fading case.

In [26], studied the spectrum allocation problem in frequency selective interference channels in the framework of cooperative game theory. It is shown that the Nash bargaining solution can be computed in this scenario using convex optimization techniques under the joint TDM/FDM strategies. A survey on both the competitive and cooperative game theoretic techniques over frequency selective interference channels can be found in [27].

## 1.2 Contributions

We study an interference channel of $K$ transmitter-receiver pairs sharing a number of $N$ of frequency sub-bands with static and frequency-selective channel gains. channel gains are realizations of independent random variables with known distributions. There is no central managing node to assign the sub-bands to the users. As such, orthogonal frequency division is not an option. Due to lack of feedback links from receivers to transmitters, no transmitter is aware of the channel gains. By the same token, no transmitter is able to know the occupied sub-bands by the other transmitters. In a scenario where all the transmitters employ Gaussian codebooks and distribute their available power uniformly across the spectrum, we examine the possible advantage of transmitting signals of correlation $\rho \neq 0$ over different sub-bands. Due to lack of knowledge about the channel gains, outage probability is the right tool to assess the performance of the network. The term outage is referred to the event that the achievable rate per user is less than its actual transmission rate. Due to the symmetry, we only consider the achievable rate at receiver 1, denoted by $R_1(\mathsf{H}_1, \rho, \text{snr})$.

In the first part we formulate the problem for a $K$-user interference channel over $N$ frequency sub-bands. Calculating the first and second order derivatives of the achievable rate per user with respect to $\rho$, it is shown that $\rho = 0$ is a point of local extremum (minimum or maximum) for achievable rate per user regardless of the realizations of channel gains and the value of SNR. Necessary and sufficient conditions are developed in terms of the channel gains and the value of SNR so that $\rho = 0$ is a point of minimum for outage capacity per user. In the case of $K = 2$, with arbitrary number of sub-bands, it is established that $\rho = 0$ is not a point of local maximum for achievable rate for infinitely many values of SNR, using a technical measure theoretic lemma. For the general scenario when $k > 2$, simulation results are given to demonstrate that this observations does not hold.

In the second part, we focus on the 2-user scenario, when the transmission is performed over 2 orthogonal frequency sub-bands. First, it is shown that for any realization of the channel coefficients, the achievable rate is a monotonic function of $\rho$. We then show that in the high SNR regime, the achievable rate is in fact an increasing function of $\rho$, which implies to set $\rho = 1$ in order to maximize the achievable rate. Interestingly, the achievable does not saturate with SNR when $\rho = 1$. This scaling behavior helps us to show that, in a sense, we should set $\rho = 1$ in order to minimize the probability of outage $\mathbb{P}(R_1(\mathsf{H}_1, \rho, \text{snr}) < R)$, in the high snr regime.

We then characterize the scaling behavior of the outage probability, when the transmission rate scales with $\log(\text{snr})$; in fact $R = r\log(\text{snr})$. In order to characterize the scaling behavior of the outage probability we assume that the channel gains are circularly symmetric complex Gaussian random variables with variances 1 and $\sigma^2$ for the forward and cross-over gains respectively. This is done by giving tight upper and lower bounds on the probability of outage. Following a random matrix analysis, we show that the outage probability scales like $2\frac{1}{\text{snr}^{1-r}}$.

In the third part, we consider a scenario where the receivers have the choice to either directly decode the desired signal, treating interference as Gaussian noise or to decode and cancel interference before decoding the desired signal. The transmission scheme and the characteristics of the channel are assumed to be the same as before. Following the same approach as the one used for analyzing the previous scenario, we prove that $\rho = 1$ is the optimal correlation coefficient for minimizing the outage probability in the high snr regime. Following similar bounding techniques along with a probabilistic analysis of the same flavor as before, the outage probability in this scenario scales like $\frac{1}{1+\sigma^2}\frac{1}{\text{snr}^{1-r}}$. We finally characterize the scaling of the outage probability in the interference free scenario is also analyzed which is proved to be $8(1-\gamma)\frac{1}{\text{snr}^{2-r}}$, where $\gamma$ is the Euler's constant.

## 1.3 Notations

The sets of natural, real and complex numbers are shown by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. For a complex quantity $a$, its conjugate, real part, imaginary part, and absolute value are represented by $a^*$, $\Re(a)$, $\Im(a)$, and $|a|$, respectively. Random quantities are shown in bold such as $\boldsymbol{x}$ with realization $x$. Vectors are shown by an arrow on top such as $\vec{x}$. Sets are shown by script letters such as $\mathcal{A}$. Sequences are shown by $(x_n)_{n\in\mathbb{N}}$. A diagonal matrix whose diagonal elements are $x_1, \cdots, x_n$ is shown by $\text{diag}(x_1, \cdots, x_n)$. $0_{n\times 1}$ denotes a vector of size $n$ whose all elements are zeros. The $(m, n)$ element of a matrix $C$ is shown by $C_{m,n}$. The transpose, conjugate transpose, inverse and determinant of a matrix $A$ are shown by $A^{\mathrm{t}}$, $A^{\dagger}$, $A^{-1}$ and $\det(A)$, respectively. For any event $\mathcal{E}$, its complement is shown by $\mathcal{E}^c$, its indicator function is shown by $\mathbf{1}(\mathcal{E})$ and its probability is shown by $\mathbb{P}(\mathcal{E})$. For any random variable $\boldsymbol{x}$, $F_{\boldsymbol{x}}(x)$ is the Cumulative Distribution Function (CDF) of $\boldsymbol{x}$, $f_{\boldsymbol{x}}$ is the probability density function of $\boldsymbol{x}$ and the expectation of $\boldsymbol{x}$ is denoted by $\mathbb{E}[\boldsymbol{x}]$. A circularly symmetric complex normal random vector with mean $\vec{m}$ and covariance matrix $C$ is shown by $\mathcal{CN}(\vec{m}, C)$. A uniform random variable over the interval $[a, b]$ is denoted by $\mathcal{U}(a, b)$. Finally, for functions $f(n)$ and $g(n)$, we write $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$ and write $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

# Chapter 2

# Problem Formulation for the general $K$-user Interference Channel

In this chapter, we formulate the problem for a $k$-user interference channel and study the achievable rates at the receivers. We investigate sufficient conditions under which the achievable rate at a given receiver is maximized. It is shown in a frequency selective interference channel with Gaussian signaling over different sub-bands, transmitting correlated signals results in higher achievable rates at the receivers. We finally remark that fining closed form expressions for the achievable rates in this scenario is a hard problem even in the 2-user case and arbitrary number of sub-bands.

## 2.1 System Model

We consider a $K$-user interference channel where the transmission is performed over $N$ sub-bands. The channel gains on different sub-bands are complex valued in general and are assumed to be static along the whole communication period of interest. The received signal on the $n^{th}$ sub-band at the $k^{th}$ receiver is given by

$$\boldsymbol{y}_k^{(n)} = h_{k,k}^{(n)}\boldsymbol{x}_k^{(n)} + \sum_{\substack{l=1\\ l\neq k}}^{K} h_{l,k}^{(n)}\boldsymbol{x}_l^{(n)} + \boldsymbol{z}_k^{(n)}, \tag{2.1}$$

for any $1 \leq k \leq K$ and $1 \leq n \leq N$, where $\boldsymbol{x}_k^{(n)}$ is the transmitted signal over the $n^{th}$ sub-band by the $k^{th}$ user, $h_{l,k}^{(n)}$ is the channel gain from the $l^{th}$ transmitter to the $k^{th}$ receiver over the $n^{th}$ sub-band and $\boldsymbol{z}_k^{(n)} \sim \mathcal{CN}(0,1)$ is the additive noise at the $k^{th}$ receiver over the $n^{th}$ sub-band. Throughout the paper, we denote $\sum_{l=1}^{K} h_{l,k}^{(m)} h_{l,k}^{(n)^*}$ and $\sum_{\substack{l=1 \\ l\neq k}}^{K} h_{l,k}^{(m)} h_{l,k}^{(n)^*}$ by $g_k^{(m,n)}$ and $\widetilde{g}_k^{(m,n)}$, respectively. Each user transmits jointly Gaussian signals across different sub-bands. In fact,

$$\vec{\boldsymbol{x}}_k \triangleq \begin{pmatrix} \boldsymbol{x}_k^{(1)} & \cdots & \boldsymbol{x}_k^{(N)} \end{pmatrix}^{\mathrm{t}} \sim \mathcal{CN}\left(0_{N\times 1}, \frac{\mathrm{snr}}{N} C(\rho)\right), \tag{2.2}$$

where the $N \times N$ correlation matrix $C(rho)$ is given by

$$C(\rho)_{m,n} = 1 + (\rho - 1)\mathbf{1}(m \neq n) \tag{2.3}$$

i.e., for simplicity of presentation, the correlation coefficient of the transmitted signals on any two sub-bands is assumed to be the same value $\rho$. We remark that $C(\rho)$ has only two different eigenvalues $1-\rho$ and $1+(N-1)\rho$ of orders $N-1$ and 1, respectively. This requires $\rho \in [-\frac{1}{N-1}, 1]$ for $C(\rho)$ to be nonnegative-definite. Assuming users treat each other as Gaussian noise, the achievable rate of the $k^{th}$ user is given by

$$R_k(\mathrm{snr}, \mathsf{H}_k, \rho) \triangleq \log \frac{\det \Omega_k}{\det \Gamma_k}, \tag{2.4}$$

where

$$\Omega_k \triangleq \frac{\mathrm{snr}}{N} \sum_{l=1}^{K} H_{l,k} C(\rho) H_{l,k}^{\dagger} + I_N, \tag{2.5}$$

$$\Gamma_k \triangleq \frac{\mathrm{snr}}{N} \sum_{\substack{l=1 \\ l\neq k}}^{K} H_{l,k} C(\rho) H_{l,k}^{\dagger} + I_N, \tag{2.6}$$

$$H_{l,k} \triangleq \mathrm{diag}(h_{l,k}^{(1)}, \cdots, h_{l,k}^{(N)}). \tag{2.7}$$

and

$$\mathsf{H}_k \triangleq (h_{l,k}^{(n)})_{\substack{1\leq l\leq K \\ 1\leq n\leq N}}. \tag{2.8}$$

Throughout this chapter, we think of $\mathsf{H}_k$ as an element of the Euclidean space $\mathbb{C}^{NK}$. Also, for notational simplicity, we write $R_k(\mathrm{snr}, \mathsf{H}_k, \rho)$ as $R_k(\mathsf{H}_k, \rho)$ while remembering its dependence on snr.

**Remark 1** *It is hard to derive closed form expressions for the achievable rates at the receivers for the general setup.This is also the case even for the 2-user scenario with arbitrary number of sub-bands*

For the 2-user case with $N$ independent sub-bands, the achievable rate is given by

$$R_1(\mathrm{snr}, \mathsf{H}_1, \rho) \triangleq \log \frac{\det \Omega}{\det \Gamma}, \tag{2.9}$$

where

$$\Omega = \begin{bmatrix} 1 + \frac{\mathrm{snr}}{N}(|h_{11}^{(1)}|^2 + |h_{21}^{(1)}|^2) & \cdots & \rho\frac{\mathrm{snr}}{N}(h_{11}^{(1)}h_{11}^{(N)^*} + h_{21}^{(1)}h_{21}^{(N)^*}) \\ \vdots & \ddots & \vdots \\ \rho\frac{\mathrm{snr}}{N}(h_{11}^{(N)}h_{11}^{(1)^*} + h_{21}^{(N)}h_{21}^{(1)^*}) & \cdots & 1 + \frac{\mathrm{snr}}{N}(|h_{11}^{(N)}|^2 + |h_{21}^{(N)}|^2) \end{bmatrix}, \tag{2.10}$$

and

$$\Gamma = \begin{bmatrix} 1 + \frac{\mathrm{snr}}{N}|h_{21}^{(1)}|^2 & \cdots & \rho\frac{\mathrm{snr}}{N}h_{21}^{(1)}h_{21}^{(N)^*} \\ \vdots & \ddots & \vdots \\ \rho\frac{\mathrm{snr}}{N}h_{21}^{(N)}h_{21}^{(1)^*} & \cdots & 1 + \frac{\mathrm{snr}}{N}|h_{21}^{(N)}|^2 \end{bmatrix}. \tag{2.11}$$

Finding closed form expressions for the achievable rate, thereby finding optimal values of $\rho$ rests upon calculating the determinant the matrices $\Omega$ and $\Gamma$. The determinant of $\Gamma$ can be calculated by a recursive relation in terms of the dimensions of the matrix. Derivation of this quantity is given in details in Appendix A. It is however hard to derive a closed form expression for $\det \Omega$ for $N > 2$, even in the high snr regime which in turn implies that it is hard to find closed form expressions for the rate.

In this chapter, we study the behavior of the received rate $R_1(\mathsf{H}_1, \rho)$ as a function of $\rho$ for the general case of $k$ user channel with $N$ independent sub-band. In the next chapter, we restrict the focus of study to the 2-user scenario with 2 independent sub-bands.

## 2.2 On the behaviour of of $\rho = 0$

In general, $R_1(\mathsf{H}_1, \rho) : [-\frac{1}{N+1}, 1] \to (0, \infty)$ is not a convex or concave function. Moreover, the value of $\rho$ that maximizes $R_1(\mathsf{H}_1; \cdot)$ depends on snr and $\mathsf{H}_1$. It is shown in Appendix

B that

$$\frac{\partial}{\partial \rho} R_1(\mathsf{H}_1, \rho)$$
$$= \frac{\text{snr}}{N} \sum_{1 \leq m < n \leq N} \left( g_1^{(m,n)} [\Omega_1^{-1}]_{m,n} - \widetilde{g}_1^{(m,n)} [\Gamma_1^{-1}]_{m,n} \right). \tag{2.12}$$

Note that for $\rho = 0$, $[\Omega_1^{-1}]_{m,n} = [\Gamma_1^{-1}]_{m,n} = 0$ for any $m \neq n$. Therefore, $\rho = 0$ is always an answer for $\frac{\partial}{\partial \rho} R_1(\mathsf{H}_1, \rho) = 0$. The second order derivative $\frac{\partial^2}{\partial \rho^2} R_1(\mathsf{H}_1; 0)$ is calculated in Appendix C in (13). For $\rho = 0$ to be a local minimum, we require $\frac{\partial^2}{\partial \rho^2} R_1(\mathsf{H}_1; 0) > 0$, or equivalently,

$$\sum_{1 \leq m < n \leq N} \left( \frac{|\widetilde{g}_1^{(m,n)}|^2}{\left(1 + \frac{\text{snr}}{N} \widetilde{g}_1^{(m,m)}\right) \left(1 + \frac{\text{snr}}{N} \widetilde{g}_1^{(n,n)}\right)} - \frac{|g_1^{(m,n)}|^2}{\left(1 + \frac{\text{snr}}{N} g_1^{(m,m)}\right) \left(1 + \frac{\text{snr}}{N} g_1^{(n,n)}\right)} \right) > 0. \tag{2.13}$$

We now study the behavior of $\rho = 0$ for some special cases of the problem. The first scenario of interest is the $k$-user non-frequency selective interference channel, where for every transmit-receive pair, all the channel gains over different sub-bands are the assumed to be the same. In this case, we show that it does not happen that $\rho = 0$ is a point of local minimum.

**Remark 2** *The frequency selective property for the channels is required here for the condition (2.13) to hold. It can be shown that if the channel gains in this setup are assumed to be static during the whole communication period of interest, then $\rho = 0$ is a point of local maximum for $R_1(\mathsf{H}_1, \rho)$.*

**P**roof: See Appendix D. ∎

The second scenario of interest which sheds light on analyzing orthogonal scenarios is the single user frequency selective channel where it is shown that independent signaling is more efficient.

**Remark 3** *In a single user frequency selective Gaussian channel with described Gaussian signaling over the independent sub bands of a frequency selective channel, $\rho = 0$ is a point of global maximum for $R_1(\mathsf{H}_1, \rho)$.*

Proof: See Appendix E. ■

Next, Let us verify the condition $\frac{\partial^2}{\partial \rho^2} R_1(\mathsf{H}_1; 0) > 0$. Assuming all channel gains are realizations of independent random variables, one can write (2.13) as the event

$$\mathcal{E}(\mathrm{snr}) \triangleq \left\{ \sum_{1 \le m < n \le N} \frac{\boldsymbol{a}_{m,n}}{\mathrm{snr}^2} + \frac{\boldsymbol{b}_{m,n}}{\mathrm{snr}} + \boldsymbol{c}_{m,n} > 0 \right\}, \tag{2.14}$$

where the random variables $\boldsymbol{a}_{m,n}$, $\boldsymbol{b}_{m,n}$ and $\boldsymbol{c}_{m,n}$ are given by

$$\boldsymbol{a}_{m,n} \triangleq |\widetilde{\boldsymbol{g}}_1^{(m,n)}|^2 - |\boldsymbol{g}_1^{(m,n)}|^2, \tag{2.15}$$

$$\begin{aligned} \boldsymbol{b}_{m,n} \;&\triangleq\; \frac{1}{N} |\widetilde{\boldsymbol{g}}_1^{(m,n)}|^2 (\boldsymbol{g}_1^{(m,m)} + \boldsymbol{g}_1^{(n,n)}) \\ &\quad - \frac{1}{N} |\boldsymbol{g}_1^{(m,n)}|^2 (\widetilde{\boldsymbol{g}}_1^{(m,m)} + \widetilde{\boldsymbol{g}}_1^{(n,n)}) \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \boldsymbol{c}_{m,n} \;&\triangleq\; \frac{1}{N^2} \Big( \boldsymbol{g}_1^{(m,m)} \boldsymbol{g}_1^{(n,n)} |\widetilde{\boldsymbol{g}}_1^{(m,n)}|^2 \\ &\qquad - \widetilde{\boldsymbol{g}}_1^{(m,m)} \widetilde{\boldsymbol{g}}_1^{(n,n)} |\boldsymbol{g}_1^{(m,n)}|^2 \Big). \end{aligned} \tag{2.17}$$

Therefore, one needs to verify if $\mathcal{E}(\mathrm{snr})$ is an almost sure event. We show that with probability 1 there exists a finite level of SNR, say $\mathrm{snr}_0$, such that for any $\mathrm{snr} > \mathrm{snr}_0$, the point $\rho = 0$ is a local minimum for $R_1(\mathsf{H}_1, \rho)$. To this purpose, one can prove almost surely the event $\mathcal{E}^c(\mathrm{snr})$ does not occur infinitely often.

A standard approach to verify this is to invoke Borel-Cantelli Lemma [28] and verify there exists an increasing sequence $(\mathrm{snr}_q)_{q \in \mathbb{N}}$ such that $\sum_{q \in \mathbb{N}} \mathbb{P}(\mathcal{E}^c(\mathrm{snr}_q)) < \infty$. This is hard to verify in general. Note, however, that

$$\lim_{\mathrm{snr} \to \infty} \mathcal{E}(\mathrm{snr}) = \lim_{\mathrm{snr} \to \infty} \sum_{1 \le m < n \le N} \frac{\boldsymbol{a}_{m,n}}{\mathrm{snr}^2} + \frac{\boldsymbol{b}_{m,n}}{\mathrm{snr}} + \boldsymbol{c}_{m,n} = \sum_{1 \le m < n \le N} \boldsymbol{c}_{m,n}. \tag{2.18}$$

Therefore, if

$$\mathbb{P}\left(\boldsymbol{c}_{m,n} > 0\right) = 1 \qquad 1 \le m < n \le N, \tag{2.19}$$

then the following measure theoretic lemma guarantees that $\mathcal{E}^c(\mathrm{snr})$ does not occur infinitely often.

**Lemma 1** *Let $(\Omega; \mathcal{S}; P)$ be a probability space and $X_n$ be random variables on $\Omega$ to $\mathbb{R}$. Let $X_n \to X$ almost surely and $X > 0$ almost surely. Then, $\mathbb{P}(X_n < 0 \ \text{ infinitely often}) = 0$.*

**P**roof: See Appendix F. ■

In the following proposition we show that (2.19) always holds if $K = 2$. Then, we remark that this is not necessarily the case when $K > 2$.

**Proposition 1** *Let the channel gains be realizations of continuous random variables. If $K = 2$, the probability that $\rho = 0$ is not a point of local minimum for $R_1(\mathsf{H}_1, \rho)$ for infinitely many values of* snr *is* $0$.

**P**roof: As remarked, wee only need to show that for any $1 \le m, n \le N$, $\mathbb{P}(\boldsymbol{c}_{m,n} > 0) = 1$. We have

$$\boldsymbol{c}_{m,n} = \frac{\boldsymbol{g}_1^{(m,m)} \boldsymbol{g}_1^{(n,n)} |\widetilde{\boldsymbol{g}}_1^{(m,n)}|^2}{N^2} \times \left(1 - \frac{\widetilde{\boldsymbol{g}}_1^{(m,m)} \widetilde{\boldsymbol{g}}_1^{(n,n)} |\boldsymbol{g}_1^{(m,n)}|^2}{\boldsymbol{g}_1^{(m,m)} \boldsymbol{g}_1^{(n,n)} |\widetilde{\boldsymbol{g}}_1^{(m,n)}|^2}\right). \tag{2.20}$$

If $K = 2$,

$$\widetilde{\boldsymbol{g}}_1^{(m,m)} \widetilde{\boldsymbol{g}}_1^{(n,n)} = |\boldsymbol{h}_{2,1}^{(m)}|^2 |\boldsymbol{h}_{2,1}^{(n)}|^2 \tag{2.21}$$

and

$$|\widetilde{\boldsymbol{g}}_1^{(m,n)}|^2 = |\boldsymbol{h}_{2,1}^{(m)} \boldsymbol{h}_{2,1}^{(n)^*}|^2. \tag{2.22}$$

Therefore, $\widetilde{\boldsymbol{g}}_1^{(m,m)} \widetilde{\boldsymbol{g}}_1^{(n,n)} = |\widetilde{\boldsymbol{g}}_1^{(m,n)}|^2$ and (2.20) simplifies to

$$\boldsymbol{c}_{m,n} = \frac{\boldsymbol{g}_1^{(m,m)} \boldsymbol{g}_1^{(n,n)} |\widetilde{\boldsymbol{g}}_1^{(m,n)}|^2}{N^2} \left(1 - \frac{|\boldsymbol{g}_1^{(m,n)}|^2}{\boldsymbol{g}_1^{(m,m)} \boldsymbol{g}_1^{(n,n)}}\right). \tag{2.23}$$

By Cauchy-Schwarz inequality, $\frac{|\boldsymbol{g}_1^{(m,n)}|^2}{\boldsymbol{g}_1^{(m,m)}\boldsymbol{g}_1^{(n,n)}} \leq 1$. Since the channel gains are realizations of continuous random variables, we have $\mathbb{P}(\frac{|\boldsymbol{g}_1^{(m,n)}|^2}{\boldsymbol{g}_1^{(m,m)}\boldsymbol{g}_1^{(n,n)}} < 1) = 1$ and $\mathbb{P}(\boldsymbol{g}_1^{(m,m)}\boldsymbol{g}_1^{(n,n)}|\widetilde{\boldsymbol{g}}_1^{(m,n)}|^2 > 0) = 1$. Therefore, $\mathbb{P}(\boldsymbol{c}_{m,n} > 0) = 1$. ∎

**Remark 4** *The fact that for sufficiently large* snr*, the event* $\mathcal{E}(\mathrm{snr})$ *does occur almost surely can alternatively be proven by dominated convergence theorem*[1] *[28],*

$$\begin{aligned}
\lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}(\mathrm{snr})) &= \lim_{\mathrm{snr}\to\infty} \mathbb{P}\left(\sum_{1\leq m<n\leq N} \frac{\boldsymbol{a}_{m,n}}{\mathrm{snr}^2} + \frac{\boldsymbol{b}_{m,n}}{\mathrm{snr}} + \boldsymbol{c}_{m,n} > 0\right) \\
&= \mathbb{P}\left(\sum_{1\leq m<n\leq N} \boldsymbol{c}_{m,n} > 0\right) = 1. \qquad (2.24)
\end{aligned}$$

For the general $k$-user scenario, when $k > 2$, $\lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}(\mathrm{snr}))$ does not necessarily exist. Fig. 2.1 presents plots of $\mathbb{P}(\mathcal{E}(\mathrm{snr}))$ in terms of snr in dB for the cases $K = 2$ (left plot) and $K = 5$ (right plot). In both cases, $N = 2$ and the channel gains are realizations of independent Rayleigh random variables with PDF $p(h) = he^{-\frac{h^2}{2}}\mathbf{1}(h > 0)$. As proved in Proposition 2, if $K = 2$, then $\lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}(\mathrm{snr})) = 0$. However, it is seen that if $K = 5$, $\lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}(\mathrm{snr}))$ does not exist.

[1]Note that $\mathbb{P}\left(\sum_{1\leq m<n\leq N} \frac{\boldsymbol{a}_{m,n}}{\mathrm{snr}^2} + \frac{\boldsymbol{b}_{m,n}}{\mathrm{snr}} + \boldsymbol{c}_{m,n} > 0\right) = \mathbb{E}\left[\mathbf{1}\left(\sum_{1\leq m<n\leq N} \frac{\boldsymbol{a}_{m,n}}{\mathrm{snr}^2} + \frac{\boldsymbol{b}_{m,n}}{\mathrm{snr}} + \boldsymbol{c}_{m,n} > 0\right)\right]$ and all indicator functions on a probability space are dominated by an integrable function, i.e., the constant function 1.

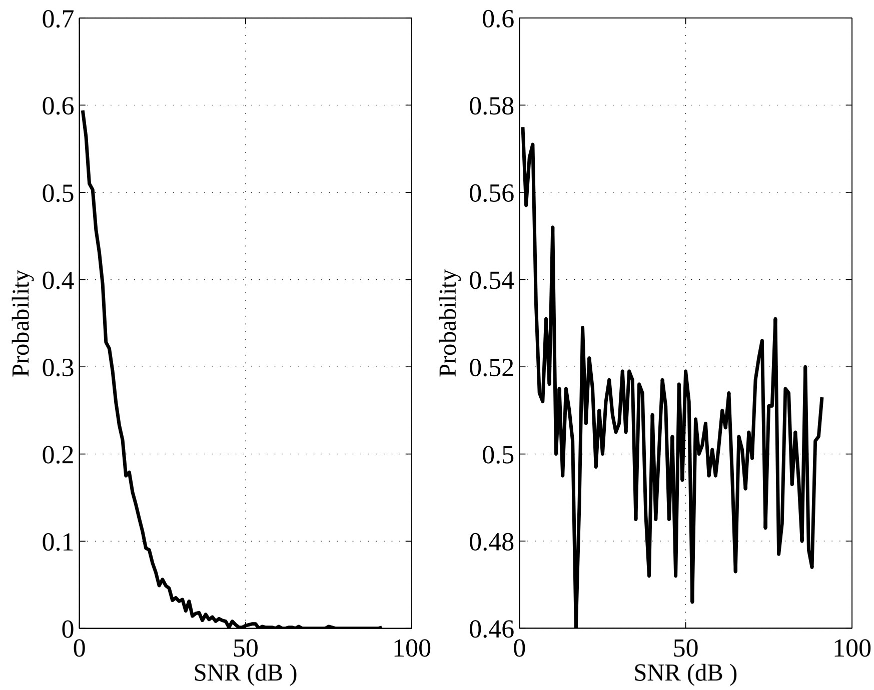


Figure 2.1: Plots of $\lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}(\mathrm{snr}))$ for $K = 2$ (left plot) and $K = 5$ (right plot). In both cases, $N = 2$ and the channel gains are realizations of independent Rayleigh random variables with PDF $p(h) = he^{-\frac{h^2}{2}}\mathbf{1}(h > 0)$.

# Chapter 3

# Results on the 2-user Interference Channel

In the previous chapter, we considered a $k$-user frequency selective Gaussian interference channel where there is no cooperation between the transmitters and there is no feedback link from any of the receivers to any of the transmitters. We studied achievable rates in a scenario where the transmitters put jointly Gaussian signals over different sub-bands and the decoding strategy at the receivers is to treat interference as noise. It is established in the 2-user case that, given any realization of channel gains, higher rates can be achieved when the transmitters send jointly Gaussian signals with non zero correlation coefficients.

In this chapter, we give closed form expressions for the achievable rates in the scenario discussed so far in the 2-user case where the transmission is performed over 2 independent sub bands.

The channel gains over different sub-bands are realizations of independent circularly symmetric complex Gaussian random variables which are assumed to be fixed during the whole communication period of interest. Outage probability, defined as the probability that a predetermined rate cannot be maintained under some fading conditions, is a well-suited measure to assess the performance of the system in this setting. We study the outage probability of the system in the high snr regime and characterize its asymptotic behavior.

It is shown in the asymptotic regime that *in a sense* the best transmission strategy for minimizing the outage probability, is to set $\rho = 1$ which means to repeat the same signal for transmission over the other sub-band. The scaling behavior of the outage probability in this scenario is analyzed.

We also provide simulation results which suggest transmitting jointly Gaussian signals with a correlation coefficient $\rho \neq 1$ for a range of fairly small values of snr, in order to minimize the outage probability.

Finally, we compare this full spread scheme where the transmitters transmit signals over the whole frequency band, with alternative schemes where the transmitters transmit over only one sub-band, with all the transmission power on the adopted sub-band. In the case there is absolutely no cooperation between the users, each user randomly picks a sub-band for the transmission which may result in mutual interference or interference free transmission. The other possibility is that the transmitters cooperatively decide on which sub-band to pick before-hand so as to avoid interference, which is the well-known orthogonal frequency devision scheme.

## 3.1 Full Spread Transmission over 2 independent sub-bands

Consider a 2-user interference channel where the transmission is performed over 2 independent sub-bands. The channel gains on different sub-bands are assumed to be static along the whole communication period of interest. The receivers have the full knowledge of the channel gains while the transmitters are unaware of the specific realization of the these gains and only know their distribution. The received signal on sub-band $i$ at receiver 1 is given by(the expression for the received signal at the other receiver can be written likewise)

$$\boldsymbol{y}_1^{(i)} = \boldsymbol{a}_i \boldsymbol{x}_1^{(i)} + \boldsymbol{b}_i \boldsymbol{x}_2^{(i)} + \boldsymbol{z}_1^{(i)} \qquad 1 \leq i \leq 2, \tag{3.1}$$

where, $\boldsymbol{x}_k^{(i)}$ is the transmitted signal over sub-band $i$ by the transmitters $k$, $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ respectively represent the forward and cross-over channel gains over sub-band $i$ and $\boldsymbol{z}_1^{(i)} \sim \mathcal{CN}(0,1)$ is the circularly symmetric complex Gaussian additive noise at the receiver over sub-band $i$.

All channel gains are assumed to be independent complex Gaussian random variables. Specifically, the forward gains are distributed as $\mathcal{CN}(0,1)$ and the cross-over gains are distributed as $\mathcal{CN}(0,\sigma^2)$. Each user transmits jointly Gaussian signals across different sub-bands. In fact,

$$\vec{\boldsymbol{x}}_k \triangleq \left(\boldsymbol{x}_k^{(1)}, \boldsymbol{x}_k^{(2)}\right)^{\mathrm{t}} \sim \mathcal{CN}\left(\begin{bmatrix}0 & 0\end{bmatrix}^{\mathrm{t}}, \frac{\mathrm{snr}}{2}\begin{bmatrix}1 & \rho \\ \rho^* & 1\end{bmatrix}\right), \quad k = 1, 2. \tag{3.2}$$

Assuming users treat each other as Gaussian noise, the achievable rate at receiver 1 is given by

$$R_1(\mathrm{snr}, \mathsf{H}_1, \rho) \triangleq \log \frac{\det \Omega_1}{\det \Gamma_1}, \tag{3.3}$$

where

$$\Omega_1 \triangleq \begin{bmatrix} 1 + \frac{\mathrm{snr}}{2}(|\boldsymbol{a}_1|^2 + |\boldsymbol{b}_1|^2) & \frac{\mathrm{snr}}{2}\rho(\boldsymbol{a}_1\boldsymbol{a}_2^* + \boldsymbol{b}_1\boldsymbol{b}_2^*) \\ \frac{\mathrm{snr}}{2}\rho^*(\boldsymbol{a}_1^*\boldsymbol{a}_2 + \boldsymbol{b}_1^*\boldsymbol{b}_2) & 1 + \frac{\mathrm{snr}}{2}(|\boldsymbol{a}_2|^2 + |\boldsymbol{b}_2|^2) \end{bmatrix}, \tag{3.4}$$

$$\Gamma_1 \triangleq \begin{bmatrix} 1 + \frac{\mathrm{snr}}{2}|\boldsymbol{b}_1|^2 & \frac{\mathrm{snr}}{2}\rho\boldsymbol{b}_1\boldsymbol{b}_2^* \\ \frac{\mathrm{snr}}{2}\rho^*\boldsymbol{b}_1^*\boldsymbol{b}_2 & 1 + \frac{\mathrm{snr}}{2}|\boldsymbol{b}_2|^2 \end{bmatrix}, \tag{3.5}$$

and

$$\mathsf{H}_1 = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{b}_1 \\ \boldsymbol{a}_2 & \boldsymbol{b}_2 \end{bmatrix}. \tag{3.6}$$

Therefore, $R_1(\mathrm{snr}, \mathsf{H}_1, \rho)$ can be expanded as follows:

$$\begin{aligned} &R_1(\mathsf{H}_1, \rho, \mathrm{snr}) = \\ &\log\Bigg(\frac{(1 + \frac{\mathrm{snr}}{2}(|\boldsymbol{a}_1|^2 + |\boldsymbol{b}_1|^2))(1 + \frac{\mathrm{snr}}{2}(|\boldsymbol{a}_2|^2 + |\boldsymbol{b}_2|^2))}{(1 + \frac{\mathrm{snr}}{2}|\boldsymbol{b}_1|^2)(1 + \frac{\mathrm{snr}}{2}|\boldsymbol{b}_2|^2) - (\frac{\mathrm{snr}}{2})^2|\rho|^2|\boldsymbol{b}_1{\boldsymbol{b}_2}^*|^2} \\ &-\frac{(\frac{\mathrm{snr}}{2})^2|\rho|^2|\boldsymbol{a}_1{\boldsymbol{a}_2}^* + \boldsymbol{b}_1{\boldsymbol{b}_2}^*|^2}{(1 + \frac{\mathrm{snr}}{2}|\boldsymbol{b}_1|^2)(1 + \frac{\mathrm{snr}}{2}|\boldsymbol{b}_2|^2) - (\frac{\mathrm{snr}}{2})^2|\rho|^2|\boldsymbol{b}_1{\boldsymbol{b}_2}^*|^2}\Bigg). \end{aligned} \tag{3.7}$$

As it is observed in (3.7), the achievable rate is only a function of the size of the correlation coefficient $\rho$ which in turn implies that considering only real correlation coefficients is enough for our purpose.

Since the channel gains hardly change over the time scale of the communication, outage probability is indeed a good performance measure in this setup. The outage probability in this scheme can be written as

$$\begin{aligned} &\mathbb{P}(R_1(\mathsf{H}_1, \rho, \mathrm{snr}) < R) = \\ &\mathbb{P}(\log\Bigg(\frac{(1 + \frac{\mathrm{snr}}{2}(|\boldsymbol{a}_1|^2 + |\boldsymbol{b}_1|^2))(1 + \frac{\mathrm{snr}}{2}(|\boldsymbol{a}_2|^2 + |\boldsymbol{b}_2|^2))}{(1 + \frac{\mathrm{snr}}{2}|\boldsymbol{b}_1|^2)(1 + \frac{\mathrm{snr}}{2}|\boldsymbol{b}_2|^2) - (\frac{\rho\mathrm{snr}}{2})^2|\boldsymbol{b}_1{\boldsymbol{b}_2}^*|^2} \\ &\frac{-(\frac{\rho\mathrm{snr}}{2})^2|\boldsymbol{a}_1{\boldsymbol{a}_2}^* + \boldsymbol{b}_1{\boldsymbol{b}_2}^*|^2}{(1 + \frac{\mathrm{snr}}{2}|\boldsymbol{b}_1|^2)(1 + \frac{\mathrm{snr}}{2}|\boldsymbol{b}_2|^2) - (\frac{\rho\mathrm{snr}}{2})^2|\boldsymbol{b}_1{\boldsymbol{b}_2}^*|^2}\Bigg) < R). \end{aligned} \tag{3.8}$$

Our objective is to minimize the outage probability of the system by choosing an appropriate correlation coefficient $\rho$. Once the optimal correlation coefficient is found, we characterize the scaling behavior of the outage probability in the high snr regime.

## 3.2 On the Optimal Correlation Coefficient

In this section we first find the optimal value of $\rho$ which maximizes the achievable rate at the receivers, then show that this correlation coefficient is in a sense optimal in the sense of minimizing the outage probability.

First, notice that given a set of channel gains $\mathcal{H} \subseteq \mathbb{C}^4$, if the achievable rate at the receiver is a monotonic function of $\rho$, so is the outage probability of the system, for any given rate $R$. In particular, if

$$R_1(\mathsf{H}_1, \rho_1, \mathrm{snr}) \leq R_1(\mathsf{H}_1, \rho_2, \mathrm{snr}) \quad \forall \mathsf{H}_1 \in \mathcal{H}$$

then

$$\mathbb{P}(R_1(\mathsf{H}_1, \rho_1, \mathrm{snr}) < R | \mathsf{H}_1 \in \mathcal{H}) \geq \mathbb{P}(R_1(\mathsf{H}_1, \rho_2, \mathrm{snr}) < R | \mathsf{H}_1 \in \mathcal{H}) \tag{3.9}$$

Regardless of the rate $R$. Likewise, if for all realizations of the channel gains in $\mathcal{H}$ the achievable rate is a decreasing function of $\rho$, the outage probability given $\mathcal{H}$ is a decreasing function in terms of $\rho$.

We show for $\rho > 0$ that given a realization of the channel gains, the achievable rate at the receiver is either an increasing or a decreasing function of $\rho$, depending on the value of snr. To this purpose observe that the achievable rate can be expressed as

$$R_1(\mathsf{H}_1, \rho, \mathrm{snr}) = \frac{\boldsymbol{\alpha} - \boldsymbol{\beta}\rho^2}{\boldsymbol{\gamma} - \boldsymbol{\eta}\rho^2}, \tag{3.10}$$

where

$$\begin{aligned}
\boldsymbol{\alpha} &\triangleq (1 + \frac{\mathrm{snr}}{2}(|\boldsymbol{a}_1|^2 + |\boldsymbol{b}_1|^2))(1 + \frac{\mathrm{snr}}{2}(|\boldsymbol{a}_2|^2 + |\boldsymbol{b}_2|^2)) \\
\boldsymbol{\beta} &\triangleq (\frac{\mathrm{snr}}{2})^2 |\boldsymbol{a}_1 \boldsymbol{a}_2{}^* + \boldsymbol{b}_1 \boldsymbol{b}_2{}^*|^2 \\
\boldsymbol{\gamma} &\triangleq (1 + \frac{\mathrm{snr}}{2}|\boldsymbol{b}_1|^2)(1 + \frac{\mathrm{snr}}{2}|\boldsymbol{b}_2|^2) \\
\boldsymbol{\eta} &\triangleq (\frac{\mathrm{snr}}{2})^2 |\boldsymbol{b}_1 \boldsymbol{b}_2{}^*|^2
\end{aligned} \tag{3.11}$$

Calculating $\frac{\partial}{\partial \rho} R_1(\mathsf{H}_1, \rho, \mathrm{snr})$, it turns out that the received rate is an increasing function of $\rho > 0$, if the following random event occurs

$$\mathcal{A}(\mathrm{snr}) \triangleq \{\boldsymbol{\alpha}\boldsymbol{\eta} - \boldsymbol{\beta}\boldsymbol{\gamma} > 0\} \tag{3.12}$$

Therefore, in order to characterize the behavior of the outage probability, it should be determined if the event $\mathcal{A}(\mathrm{snr})$ happens. In this light, we can find the optimal value of $\rho$ and analyze the outage probability for the corresponding value.

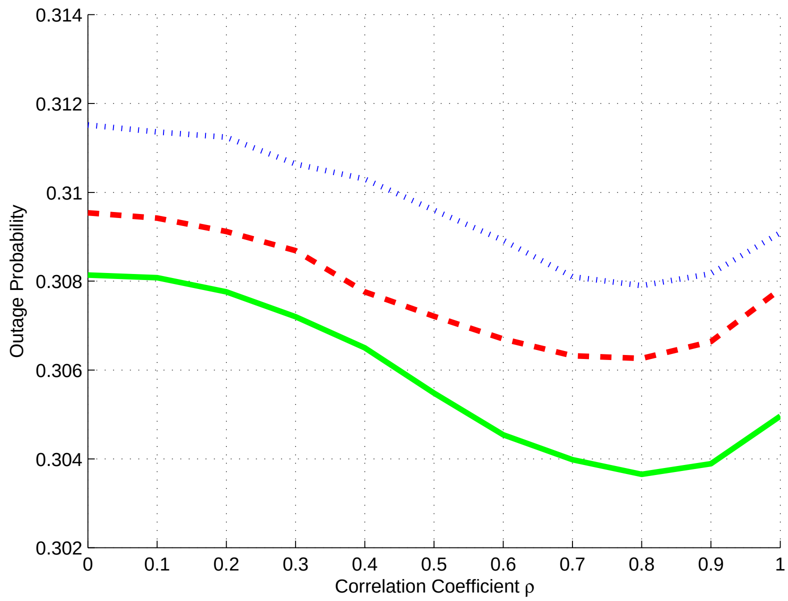


Figure 3.1: Plot of the outage probability, $\mathbb{P}(R_1(\mathsf{H}, \rho, \text{snr}) < R)$, versus $\rho$ in 3 independent Monte Carlo simulation with $10^6$ iterations with channel gains $\boldsymbol{a}_i, \boldsymbol{b}_i \sim \mathcal{CN}(0,1)$, $R = 0.5$ and $\text{snr} = 0$ db.

**Remark 5** *Since the event $\mathcal{A}(\text{snr})$ depends on the channel gains which are indeed random, there exist some values of* snr *for which the outage probability is not a monotonic function of $\rho$.*

In Figures (3.1) and (3.2) the outage probability of the system which is obtained by a Monte Carlo simulation with $10^6$ iterations is given as function of $\rho$. In Figure (3.1), the channel gains over each sub-band are assumed to be complex Gaussian random variables; i. e. $\boldsymbol{a}_i, \boldsymbol{b}_i \sim \mathcal{CN}(0,1)$ for $1 \leq i \leq 2$. The curves are obtained for outage rate threshold $R = 0.5$ and signal to noise ratio $\text{snr} = 0$ db. In Figure (3.2), channel gains over each sub-band are assumed to be uniform random variables; i. e. $\boldsymbol{a}_i, \boldsymbol{b}_i \sim \mathcal{U}(0,1)$. The curves are obtained for outage rate threshold $R = 0.5$ and signal to noise ratio $\text{snr} = 14$ db. As it is observed in these plots, the outage probability is not a monotonic function of $\rho$ and the minimum is achieved for some $0 < \rho < 1$.

We now show that for sufficiently large snr, $\mathcal{A}(\text{snr})$ does occur almost surly, so the achievable rate is an increasing function of $\rho$ and the optimal value for the correlation coefficient is $\rho = 1$.

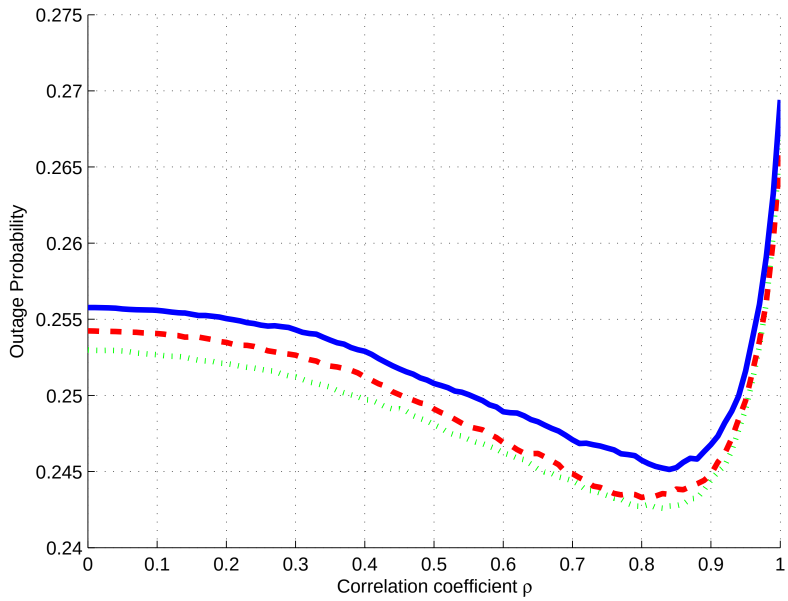


Figure 3.2: Plot of the outage probability, $\mathbb{P}(R_1(\mathsf{H}, \rho, \text{snr}) < R)$, versus $\rho$ in 3 independent Monte Carlo simulation with $10^6$ iterations when the channel gains over different sub-bands are assumed to be independent Uniform random variables $\mathcal{U}(0, 1)$, $R = 0.5$ and snr $= 14$ db.

**Proposition 2**

$$\lim_{\text{snr}\to\infty} \mathbb{P}(R_1(\mathsf{H}_1, \rho, \text{snr}) \leq R_1(\mathsf{H}_1, 1, \text{snr})) = 1 \tag{3.13}$$

**P**roof: We show that $\lim_{\text{snr}\to\infty} \mathbb{P}(\mathcal{A}(\text{snr})) = 1$, which implies what we want. Since $\boldsymbol{\alpha\eta} - \boldsymbol{\beta\gamma} = \sum_{i=1}^{4} c_i \text{snr}^4$ is a polynomial function of snr, in order for $\boldsymbol{\alpha\eta} - \boldsymbol{\beta\gamma}$ to be positive in the high snr regime, we only need to verify if the coefficient of the highest order is positive. This is indeed the case since

$$\begin{aligned} c_4 &= |\boldsymbol{b}_1\boldsymbol{b}_2{}^*|^2 \left[(|\boldsymbol{a}_1|^2 + |\boldsymbol{b}_1|^2)(|\boldsymbol{a}_2|^2 + |\boldsymbol{b}_2|^2) - |\boldsymbol{a}_1\boldsymbol{a}_2{}^* + \boldsymbol{b}_1\boldsymbol{b}_2{}^*|^2\right] \\ &= |\boldsymbol{b}_1\boldsymbol{b}_2{}^*|^2 \left[1 - \frac{|\boldsymbol{a}_1\boldsymbol{a}_2{}^* + \boldsymbol{b}_1\boldsymbol{b}_2{}^*|^2}{(|\boldsymbol{a}_1|^2 + |\boldsymbol{b}_1|^2)(|\boldsymbol{a}_2|^2 + |\boldsymbol{b}_2|^2)}\right], \end{aligned} \tag{3.14}$$

and by Cauchy-Schwarz inequality, the second term is non-negative. Since the channel gains are realizations of continuous random variables, we have

$$\mathbb{P}(|\boldsymbol{b}_1\boldsymbol{b}_2{}^*|^2 \left[1 - \frac{|\boldsymbol{a}_1\boldsymbol{a}_2{}^* + \boldsymbol{b}_1\boldsymbol{b}_2{}^*|^2}{(|\boldsymbol{a}_1|^2 + |\boldsymbol{b}_1|^2)(|\boldsymbol{a}_2|^2 + |\boldsymbol{b}_2|^2)}\right] > 0) = 1 \tag{3.15}$$

which implies that $\mathbb{P}(\lim_{\text{snr}\to\infty} \mathcal{A}(\text{snr})) = 1$. Finally, by the dominated convergence theorem

$$\lim_{\text{snr}\to\infty} \mathbb{P}(\mathcal{A}(\text{snr})) = 1 \tag{3.16}$$

■

Define

$$\mathcal{E}(\text{snr}) = \{\omega \in \Omega | R_1(\mathsf{H}_1, \rho, \text{snr}) \leq R_1(\mathsf{H}_1, 1, \text{snr})\} \tag{3.17}$$

By Proposition 2, for any $\epsilon > 0$, there exists some $N(\epsilon) > 0$ such that for all snr $\geq N(\epsilon)$, $\mathbb{P}(\mathcal{E}(\text{snr})) \geq 1 - \epsilon$.

Now, we show that the monotonicity of the achievable rate in the high snr regime translates, in a sense to be defined, to the monotonicity of the outage probability in terms of $\rho$. For any given rate $R$ and any $\epsilon > 0$, when snr $\geq N(\epsilon)$,we can write

$$\begin{aligned} &\mathbb{P}(R_1(\mathsf{H}_1, \rho, \text{snr}) < R) = \\ &\mathbb{P}(R_1(\mathsf{H}_1, \rho, \text{snr}) < R, \mathcal{E}(\text{snr})) + \mathbb{P}(R_1(\mathsf{H}_1, \rho, \text{snr}) < R, \mathcal{E}^c(\text{snr})) \end{aligned} \tag{3.18}$$

Therefore,

$$\mathbb{P}(R_1(\mathsf{H}_1, \rho, \text{snr}) < R | \mathcal{E}(\text{snr}))(1 - \epsilon) \leq \mathbb{P}(R_1(\mathsf{H}_1, \rho, \text{snr}) < R) \tag{3.19}$$

and

$$\mathbb{P}(R_1(\mathsf{H}_1, \rho, \text{snr}) < R) \leq \mathbb{P}(R_1(\mathsf{H}_1, \rho, \text{snr}) < R | \mathcal{E}(\text{snr})) + \epsilon.$$

which implies that the optimal outage probability is bounded from above by

$$\inf_{\rho} \mathbb{P}(R_1(\mathsf{H}_1, \rho, \text{snr}) < R) \leq \mathbb{P}(R_1(\mathsf{H}_1, 1, \text{snr}) < R | \mathcal{E}(\text{snr})) + \epsilon \tag{3.20}$$

and is bounded from below by

$$\inf_{\rho} \mathbb{P}(R_1(\mathsf{H}_1, \rho, \text{snr}) < R) \geq \mathbb{P}(R_1(\mathsf{H}_1, 1, \text{snr}) < R | \mathcal{E}(\text{snr}))(1 - \epsilon) \tag{3.21}$$

Therefore, in this sense, $\rho = 1$ is the optimal correlation coefficient for the outage probability in the high snr regime.

## 3.3 Asymptotic Analysis of Outage Probability

In this section, we study the outage probability of the system in the high snr regime. Recall that the outage probability of the system for a given rate $R$ can be written as

$$\mathbb{P}(R_1(\mathsf{H}_1,\rho,\mathrm{snr}) < R) =$$
$$\mathbb{P}(\log\left(\frac{(1+\frac{\mathrm{snr}}{2}(|\boldsymbol{a}_1|^2+|\boldsymbol{b}_1|^2))(1+\frac{\mathrm{snr}}{2}(|\boldsymbol{a}_2|^2+|\boldsymbol{b}_2|^2))}{(1+\frac{\mathrm{snr}}{2}|\boldsymbol{b}_1|^2)(1+\frac{\mathrm{snr}}{2}|\boldsymbol{b}_2|^2)-(\frac{\rho\mathrm{snr}}{2})^2|\boldsymbol{b}_1\boldsymbol{b}_2{}^*|^2} \frac{-(\frac{\rho\mathrm{snr}}{2})^2|\boldsymbol{a}_1\boldsymbol{a}_2{}^*+\boldsymbol{b}_1\boldsymbol{b}_2{}^*|^2}{(1+\frac{\mathrm{snr}}{2}|\boldsymbol{b}_1|^2)(1+\frac{\mathrm{snr}}{2}|\boldsymbol{b}_2|^2)-(\frac{\rho\mathrm{snr}}{2})^2|\boldsymbol{b}_1\boldsymbol{b}_2{}^*|^2}\right) < R)$$

As shown earlier in this chapter, in order to minimize the outage probability in the high snr regime, we should set $\rho = 1$. Interestingly, it can be verified in this case that the $\mathrm{snr}^2$ term in the denominator vanishes which means that any given finite rate can be handled with sufficiently large snr. More precisely, when $\rho = 1$, the outage probability can be written as:

$$\begin{aligned}
&\mathbb{P}(R_1(\mathsf{H}_1,1,\mathrm{snr}) < R) = \\
&\mathbb{P}(\frac{(1+\frac{\mathrm{snr}}{2}(|\boldsymbol{a}_1|^2+|\boldsymbol{b}_1|^2))(1+\frac{\mathrm{snr}}{2}(|\boldsymbol{a}_2|^2+|\boldsymbol{b}_2|^2))-(\frac{\mathrm{snr}}{2})^2|\boldsymbol{a}_1\boldsymbol{a}_2{}^*+\boldsymbol{b}_1\boldsymbol{b}_2{}^*|^2}{(1+\frac{\mathrm{snr}}{2}|\boldsymbol{b}_1|^2)(1+\frac{\mathrm{snr}}{2}|\boldsymbol{b}_2|^2)-(\frac{\mathrm{snr}}{2})^2|\boldsymbol{b}_1\boldsymbol{b}_2{}^*|^2} < 2^R) = \\
&\mathbb{P}(\frac{1+\frac{\mathrm{snr}}{2}(|\boldsymbol{a}_1|^2+|\boldsymbol{b}_1|^2+|\boldsymbol{a}_2|^2+|\boldsymbol{b}_2|^2)+(\frac{\mathrm{snr}}{2})^2|\boldsymbol{a}_1\boldsymbol{b}_2-\boldsymbol{b}_1\boldsymbol{a}_2|^2}{1+\frac{\mathrm{snr}}{2}(|\boldsymbol{b}_1|^2+|\boldsymbol{b}_2|^2)} < 2^R) = \\
&\mathbb{P}(\frac{\frac{\mathrm{snr}}{2}(|\boldsymbol{a}_1|^2+|\boldsymbol{a}_2|^2)+(\frac{\mathrm{snr}}{2})^2|\boldsymbol{a}_1\boldsymbol{b}_2-\boldsymbol{b}_1\boldsymbol{a}_2|^2}{1+\frac{\mathrm{snr}}{2}(|\boldsymbol{b}_1|^2+|\boldsymbol{b}_2|^2)} < 2^R-1) \qquad (3.22)
\end{aligned}$$

The aim is to characterize the scaling behavior of the outage probability (3.22) in the high snr regime. Since the achievable rate in this case scales with log(snr), we are interested in investigating the outage probability when $R = r\log(\mathrm{snr})$ and seeking for the highest possible value of $r$ for which the outage probability goes to zero.
Define $n \triangleq \frac{\mathrm{snr}}{2}$. In order for (3.22) to be precisely calculated, the distribution of the random variable

$$\boldsymbol{h}_n \triangleq \frac{(|\boldsymbol{a}_1|^2+|\boldsymbol{a}_2|^2)n+|\boldsymbol{a}_1\boldsymbol{b}_2-\boldsymbol{b}_1\boldsymbol{a}_2|^2n^2}{1+(|\boldsymbol{b}_1|^2+|\boldsymbol{b}_2|^2)n} \qquad (3.23)$$

is needed, which in turn depends on the joint probability distribution of the random variables involved. Define

$$M = \begin{bmatrix} \boldsymbol{a}_1 & -\boldsymbol{a}_2 \\ \frac{1}{\sigma}\boldsymbol{b}_1{}^* & \frac{1}{\sigma}\boldsymbol{b}_2{}^* \end{bmatrix}. \qquad (3.24)$$

Since the columns of the matrix $M$ are independent multivariate normal random variables with covariance matrix $\boldsymbol{I}$, the matrix $A = MM^\dagger$ has Wishart distribution $\mathcal{W}(\boldsymbol{I}, 2)$, given as follows

$$
\begin{aligned}
f_A(B) &= \begin{cases} \frac{1}{\pi}\exp(-\mathrm{tr}(B)) & \text{if } B \text{ is positive definite} \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{\pi}\exp(-(b_{11}+b_{22})) & b_{11}b_{22} > |b_{12}|^2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{3.25}
$$

Given this joint probability density function, our final objective is to characterize the scaling behavior of the following probability in terms of $n$.

$$
\begin{aligned}
&\mathbb{P}(\boldsymbol{h}_n < 2^R - 1) \\
&\mathbb{P}(\frac{\boldsymbol{A}_{11}n + \sigma^2|\boldsymbol{A}_{12}|^2 n^2}{1+\sigma^2\boldsymbol{A}_{22}n} < (2n)^r - 1)
\end{aligned} \tag{3.26}
$$

However, calculating this probability precisely is a hard problem. Alternatively, we give upper and lower bounds on (3.26) in the asymptotic regime and show that these bounds are tight in the asymptotic sense. This gives a characterization of the scaling behavior of the outage probability of the system in the high snr regime.

### 3.3.1 Upper Bound

Corresponding to the sequence of random variables $\{\boldsymbol{h}_n\}_{n=1}^{\infty}$ defined earlier, define the sequence of random variables $\{\hat{\boldsymbol{h}}_n\}_{n=1}^{\infty}$ by only keeping the highest order terms of both the nominator and the denominator of $\boldsymbol{h}_n$ as follows

$$
\hat{\boldsymbol{h}}_n \triangleq \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}} n. \tag{3.27}
$$

Observe that

$$
\boldsymbol{h}_n - \hat{\boldsymbol{h}}_n = \frac{(\boldsymbol{A}_{11}\boldsymbol{A}_{22} - |\boldsymbol{A}_{12}|^2)n}{\boldsymbol{A}_{22}(1+\sigma^2\boldsymbol{A}_{22}n)} = \frac{(|\boldsymbol{a}_1\boldsymbol{b}_1 - \boldsymbol{a}_2\boldsymbol{b}_2|^2)n}{(|\boldsymbol{b}_1|^2+|\boldsymbol{b}_2|^2)(1+n(|\boldsymbol{b}_1|^2+|\boldsymbol{b}_2|^2))} > 0 \tag{3.28}
$$

Therefore, for any given rate $R$,

$$
\mathbb{P}(\boldsymbol{h}_n < R) \leq \mathbb{P}(\hat{\boldsymbol{h}}_n < R), \tag{3.29}
$$

which means

$$\mathbb{P}(R_1(\mathsf{H}_1, 1, \mathrm{snr}) < R) \leq \mathbb{P}(\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}} < \frac{(2n)^r - 1}{n}) \tag{3.30}$$

We now precisely compute the right hand side of (3.30). Let us first compute the joint probability of the off diagonal elements and the second diagonal element of the random matrix $A$. We do so by eliminating the first diagonal element from the joint probability density function which in turn is done by integrating the joint probability density function on this variable.

$$\begin{aligned} f_{\boldsymbol{A}_{22},\Re(\boldsymbol{A}_{12}),\Im(\boldsymbol{A}_{12})}(x,y,z) &= \int f_A(B)\ \mathrm{d}b_{11} = \int_{\frac{y^2+z^2}{x}}^{\infty} \frac{1}{\pi}\exp(-x-t)\mathrm{d}t \\ &= \frac{1}{\pi}\exp(-x - \frac{y^2+z^2}{x}) \end{aligned} \tag{3.31}$$

In order to compute (3.30), we need to analyze the behavior of the probability distribution function of $\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}$ near zero. More strongly, we determine the probability density function of this random variable. We may do so by using the transformation $(x,y,z) \longrightarrow (u,v,w)$, where

$$\begin{aligned} u &:= x, \\ v &:= y, \\ w &:= \frac{y^2+z^2}{x}. \end{aligned} \tag{3.32}$$

It is straightforward that the set of equations (3.32) has real roots $(x_1,y_1,z_1) = (u,v,\sqrt{uw-v^2})$ and $(x_2,y_2,z_2) = (u,v,-\sqrt{uw-v^2})$. Therefore, the joint distribution of the new variables can be written as

$$\begin{aligned} &f_{\boldsymbol{A}_{22},\Re(\boldsymbol{A}_{12}),\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(u,v,w) = \\ &f_{\boldsymbol{A}_{22},\Re(\boldsymbol{A}_{12}),\Im(\boldsymbol{A}_{12})}(x_1,y_1,z_1)|\det(J_1)| + f_{\boldsymbol{A}_{22},\Re(\boldsymbol{A}_{12}),\Im(\boldsymbol{A}_{12})}(x_2,y_2,z_2)|\det(J_2)| \end{aligned} \tag{3.33}$$

where the Jacobian matrices $J_1$ and $J_2$ are given as follows

$$J_1 = \begin{bmatrix} 1 & 0 & \frac{w}{2\sqrt{uw-v^2}} \\ 0 & 1 & \frac{-2v}{2\sqrt{uw-v^2}} \\ 0 & 0 & \frac{u}{2\sqrt{uw-v^2}} \end{bmatrix}, \quad J_2 = \begin{bmatrix} 1 & 0 & \frac{-w}{2\sqrt{uw-v^2}} \\ 0 & 1 & \frac{2v}{2\sqrt{uw-v^2}} \\ 0 & 0 & \frac{-u}{2\sqrt{uw-v^2}} \end{bmatrix}$$

Therefore, one can write

$$\begin{aligned}
f_{\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(w) &= \int_0^\infty \int_{-\sqrt{uw}}^{\sqrt{uw}} f_{\boldsymbol{A}_{22},\Re(\boldsymbol{A}_{12}),\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(u,v,w)\mathrm{d}v\mathrm{d}u \\
&= \int_0^\infty \int_{-\sqrt{uw}}^{\sqrt{uw}} \frac{1}{\pi}\exp(-u-w)\frac{u}{\sqrt{uw-v^2}}\mathrm{d}v\mathrm{d}u \\
&= \frac{1}{\pi}\exp(-w)\int_0^\infty u\exp(-u)\int_{-\sqrt{uw}}^{\sqrt{uw}} \frac{1}{\sqrt{uw-v^2}}\mathrm{d}v\mathrm{d}u \\
&= \frac{1}{\pi}\exp(-w)\int_0^\infty u\exp(-u)\arcsin(\frac{v}{\sqrt{uw}})\Big|_{-\sqrt{uw}}^{\sqrt{uw}} = \exp(-w) \qquad (3.34)
\end{aligned}$$

This observation implies that

$$\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}} \sim \exp(1) \qquad (3.35)$$

Therefore, (3.30) can be computed as follows:

$$\mathbb{P}(\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}} < \frac{(2n)^r-1}{n}) = 1-\exp(-\frac{(2n)^r-1}{n}) \qquad (3.36)$$

This means, as long as $r < 1$, the probability of outage goes to zero. Using Taylor series expansion for the exponential function, the right hand side is asymptotic to

$$2\frac{1}{\mathrm{snr}^{1-r}} + o(\frac{1}{\mathrm{snr}^{1-r}}).$$

This means that the outage probability $\mathbb{P}(R_1(\mathsf{H}_1, 1, \mathrm{snr}) < r\log(\mathrm{snr}))$ at least scales as fast as $2\frac{1}{\mathrm{snr}^{1-r}}$.

### 3.3.2 Lower Bound

Now, we bound $\mathbb{P}(R_1(\mathsf{H}_1, 1, \mathrm{snr}) < r\log(\mathrm{snr}))$ from below and analyze the scaling behavior of the lower bound. It turns out that the lower bound shows the same scaling behavior as the upper bound and in this sense the given bounds are tight.

**Proposition 3** *When* $r > \frac{1}{2}$,

$$\mathbb{P}(R_1(\mathsf{H}_1, 1, \mathrm{snr}) < r\log(\mathrm{snr})) \sim 2\frac{1}{\mathrm{snr}^{1-r}} \qquad (3.37)$$

**Proof:** For any $\epsilon > 0$ and $\ell > 0$, we can write

$$\begin{aligned}
\mathbb{P}(\hat{\boldsymbol{h}}_n < (2n)^r - (2n)^{r-\epsilon} - 1) &= \mathbb{P}(\hat{\boldsymbol{h}}_n - \boldsymbol{h}_n + \boldsymbol{h}_n < (2n)^r - (2n)^{r-\epsilon} - 1) \\
&= \mathbb{P}(\hat{\boldsymbol{h}}_n - \boldsymbol{h}_n + \boldsymbol{h}_n < (2n)^r - (2n)^{r-\epsilon} - 1, |\boldsymbol{h}_n - \hat{\boldsymbol{h}}_n| < (2n)^{r-\epsilon}) \\
&+ \mathbb{P}(\hat{\boldsymbol{h}}_n - \boldsymbol{h}_n + \boldsymbol{h}_n < (2n)^r - (2n)^{r-\epsilon} - 1, |\boldsymbol{h}_n - \hat{\boldsymbol{h}}_n| > (2n)^{r-\epsilon}) \\
&\leq \mathbb{P}(\boldsymbol{h}_n < (2n)^r - 1, |\boldsymbol{h}_n - \hat{\boldsymbol{h}}_n| < (2n)^{r-\epsilon}) + \mathbb{P}(|\boldsymbol{h}_n - \hat{\boldsymbol{h}}_n| > (2n)^{r-\epsilon}) \\
&\overset{(a)}{\leq} \mathbb{P}(\boldsymbol{h}_n < (2n)^r - 1) + \mathbb{P}(\frac{\boldsymbol{A}_{11}}{\sigma^2 \boldsymbol{A}_{22}} > (2n)^{r-\epsilon}) \\
&= \mathbb{P}(\boldsymbol{h}_n < (2n)^r - 1) + \mathbb{P}((\frac{\boldsymbol{A}_{11}}{\sigma^2 \boldsymbol{A}_{22}})^{\ell} > (2n)^{\ell(r-\epsilon)}) \\
&\overset{(b)}{\leq} \mathbb{P}(\boldsymbol{h}_n < (2n)^r - 1) + \frac{\mathbb{E}((\frac{\boldsymbol{A}_{11}}{\sigma^2 \boldsymbol{A}_{22}})^{\ell})}{(2n)^{\ell(r-\epsilon)}} \qquad (3.38)
\end{aligned}$$

where (a) holds since $|\boldsymbol{h}_n - \hat{\boldsymbol{h}}_n| \leq \frac{\boldsymbol{A}_{11}}{\sigma^2 \boldsymbol{A}_{22}}$. As shown earlier in (3.28), since $(\frac{\boldsymbol{A}_{11}}{\sigma^2 \boldsymbol{A}_{22}})^{\ell}$ is a positive random variable,(b) holds due to Markov's inequality.

Note that $\frac{\boldsymbol{A}_{11}}{\sigma^2 \boldsymbol{A}_{22}}$ is the quotient of two independent chi squared distributions with the same degrees of freedom 4. If $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two independent chi squared random variables with $d_1$ and $d_2$ degrees of freedom respectively, then the distribution of the random variable $\frac{\boldsymbol{u}_1/d_1}{\boldsymbol{u}_2/d_2}$ is known as F-distribution with $d_1$ and $d_2$ degrees of freedom, denoted by $F(d_1, d_2)$ [29]. The $k$-th moment of an $F(d_1, d_2)$ distribution exists and is finite only when $2k < d_2$. Therefore, $\frac{\boldsymbol{A}_{11}}{\sigma^2 \boldsymbol{A}_{22}} \sim F(4, 4)$ and its $k$-th moment is finite if $k < 2$. This implies that $\mathbb{E}((\frac{\boldsymbol{A}_{11}}{\sigma^2 \boldsymbol{A}_{22}})^{\ell})$ is constant if $\ell = 1$.

As shown earlier, $\mathbb{P}(\hat{\boldsymbol{h}}_n < (2n)^r - (2n)^{r-\epsilon})$ decays as fast as $\frac{1}{\mathrm{n}^{1-r}}$. Therefore, the last term in the right hand side of (3.38) decays faster than $\frac{1}{n^{1-r}}$ if $1 - r < r - \epsilon$; which implies $r > \frac{1}{2}$ for this to hold.

Therefore, the lower bound to $\mathbb{P}(R_1(\mathsf{H}_1, 1, \mathrm{snr}) < r \log(\mathrm{snr}))$, similar to the upper bound given earlier, scales like $2\frac{1}{\mathrm{snr}^{1-r}}$ when $r > \frac{1}{2}$ and the proof is complete. ∎

## 3.4 Single Sub-band Transmission Schemes

In this section, we compare the full spread scheme analyzed so far with some possible single sub-band transmission schemes in this setup. As for the single sub-band transmission schemes, we consider two scenarios based on the level of cooperation between the users.

### 3.4.1 Non-cooperative scenario

Let us first look into a scenario where there is absolutely no cooperation between the transmitters. That is, none of the transmitters are aware of the choice of the other transmitter's sub-band for transmission. In this scheme, each user chooses one of the sub bands and transmits a Gaussian signal with full power over the chosen sub band, unaware of the choice of the other user. In this case, there are two possible situations.

When both of the users choose the same sub band, say the first sub band, the outage probability of the system given a rate $R$ can be computed as follows

$$\begin{aligned}
\mathbb{P}(R_1(\mathsf{H},\mathrm{snr}) < R) &= \mathbb{P}(\log\left(1+\frac{\mathrm{snr}|\boldsymbol{a}_1|^2}{1+\mathrm{snr}|\boldsymbol{b}_1|^2}\right) < R) \\
&= \mathbb{E}\left[\mathbb{P}\left(|\boldsymbol{a}_1|^2 < \frac{(2^R-1)1+\mathrm{snr}|\boldsymbol{b}_1|^2}{\mathrm{snr}}\right)\Big||\boldsymbol{b}_1|^2\right] \\
&= \int_0^\infty (1-\exp(-\frac{(2^R-1)(1+\mathrm{snr}x)}{\mathrm{snr}})\frac{1}{2\sigma^2}\exp(-\frac{x}{2\sigma^2})\mathrm{d}x \\
&= 1-\exp(-\frac{2^R-1}{\mathrm{snr}})\int_0^\infty \frac{1}{2\sigma^2}\exp(-(2^R-1+\frac{1}{2\sigma^2})x)\mathrm{d}x \\
&= 1-\frac{1}{1+2\sigma^2(2^R-1)}\exp(-\frac{2^R-1}{\mathrm{snr}}) \qquad (3.39)
\end{aligned}$$

As it can be observed, for any finite rate $R$, the probability of outage is saturated and does not decay to zero which poses a major performance loss on the system.

In the case, when the users choose different sub bands, the transmission is performed over orthogonal sub bands; therefore, the outage probability can be computed as follows:

$$\begin{aligned}
\mathbb{P}(R_1(\mathsf{H},\mathrm{snr}) < R) &= \mathbb{P}(\log(1+\mathrm{snr}|\boldsymbol{a}_1|^2) < R) = \mathbb{P}(|\boldsymbol{a}_1|^2 < \frac{2^R-1}{\mathrm{snr}}) \\
&= 1-\exp(-\frac{2^R-1}{\mathrm{snr}}) \qquad (3.40)
\end{aligned}$$

In this case, the achievable rate scales like $\log(\mathrm{snr})$ which suggests that we can handle rates of this order with vanishing probability of outage. Specifically, for $R = r\log(\mathrm{snr})$, the probability of outage scales like $\frac{1}{\mathrm{snr}^{1-r}}$. This means as long as $r < 1$, for the the given rate $R = r\log(\mathrm{snr})$, the probability of outage goes to zero.

Since the choice of sub-bands is made at random by the transmitters, both of the above scenarios happen with equal probabilities; therefore the overall outage probability in this system can be written as

$$\begin{aligned}\mathbb{P}(R_1(\mathsf{H}_1, \mathrm{snr}) < R) &= \frac{1}{2}(1 - \frac{1}{1+\sigma^2(2^R-1)}\exp(-\frac{2^R-1}{\mathrm{snr}})) + \frac{1}{2}(1-\exp(-\frac{2^R-1}{\mathrm{snr}})) \\ &= 1 - \frac{1}{1+\sigma^2(2^R-1)}\exp(-\frac{2^R-1}{\mathrm{snr}}) \qquad (3.41)\end{aligned}$$

### 3.4.2 Orthogonal Scheme

The other single sub-band scenario of interest is the well-known orthogonal frequency division. In this scheme, each sub-band is assigned beforehand to one of the transmitters and the transmission is done interference-free with the full power assigned to the adopted sub-band. The analysis of outage probability in this case is similar to the one given in the latter case of the non-cooperative scenario just described. Therefore the outage probability of the system, given a rate of $R = r\log(\mathrm{snr})$ is

$$\mathbb{P}(R_1(\mathsf{H}, \mathrm{snr}) < r\log(\mathrm{snr})) = 1 - \exp(-\frac{\mathrm{snr}^r - 1}{\mathrm{snr}}) \sim \frac{1}{\mathrm{snr}^{1-r}} \qquad (3.42)$$

# Chapter 4

# How Does Decoding Interference Help?

Next, let us consider a scenario where the receivers have the choice to either directly decode the desired signal, treating interference as Gaussian noise or to decode and cancel interference before decoding the desired signal. Note that for decoding interference, we simply treat the desired signal as noise. Furthermore, notice that in this scheme we still avoid decoder complexity as the adopted interference cancellation technique does nothing more than pretending interference as the desired signal.

Moreover, let us assume that the transmitters have the choice to transmit correlated signals over the sub-bands from a single codebook or to transmit independent signals over the sub-bands using two independent codebooks. In the following, we study the possible scenarios when the transmitters and the receivers have these capabilities.

## 4.1 Single Codebook Transmission Scheme

Suppose that the transmitters generate codewords, to be transmitted over the sub-bands, from a single jointly Gaussian code book. At the receiver side, we consider two possible scenarios for the decoding. In the case that the receiver treats interference as noise, as formulated before, the achievable rate can be written as

$$
\begin{aligned}
&I_{11}(\mathsf{H}_1,\rho,\mathrm{snr}) \triangleq \\
&\log\left(\frac{(1+\frac{\mathrm{snr}}{2}(|\boldsymbol{a}_1|^2+|\boldsymbol{b}_1|^2))(1+\frac{\mathrm{snr}}{2}(|\boldsymbol{a}_2|^2+|\boldsymbol{b}_2|^2))}{(1+\frac{\mathrm{snr}}{2}|\boldsymbol{b}_1|^2)(1+\frac{\mathrm{snr}}{2}|\boldsymbol{b}_2|^2)-(\frac{\mathrm{snr}}{2})^2\rho^2|\boldsymbol{b}_1\boldsymbol{b}_2{}^*|^2}\right.\\
&\left.-\frac{(\frac{\mathrm{snr}}{2})^2\rho^2|\boldsymbol{a}_1\boldsymbol{a}_2{}^*+\boldsymbol{b}_1\boldsymbol{b}_2{}^*|^2}{(1+\frac{\mathrm{snr}}{2}|\boldsymbol{b}_1|^2)(1+\frac{\mathrm{snr}}{2}|\boldsymbol{b}_2|^2)-(\frac{\mathrm{snr}}{2})^2\rho^2|\boldsymbol{b}_1\boldsymbol{b}_2{}^*|^2}\right)
\end{aligned}
\tag{4.1}
$$

In the second case, the receiver first decodes interference considering the desired signal as noise which can be achieved with rate

$$
\begin{aligned}
&I_{12}(\mathsf{H}_1,\rho,\mathrm{snr}) \triangleq \log\left(\frac{(1+\frac{\mathrm{snr}}{2}(|\boldsymbol{a}_1|^2+|\boldsymbol{b}_1|^2))(1+\frac{\mathrm{snr}}{2}(|\boldsymbol{a}_2|^2+|\boldsymbol{b}_2|^2))}{(1+\frac{\mathrm{snr}}{2}|\boldsymbol{a}_1|^2)(1+\frac{\mathrm{snr}}{2}|\boldsymbol{a}_2|^2)-(\frac{\mathrm{snr}}{2})^2\rho^2|\boldsymbol{a}_1\boldsymbol{a}_2{}^*|^2}\right.\\
&\left.-\frac{(\frac{\mathrm{snr}}{2})^2\rho^2|\boldsymbol{a}_1\boldsymbol{a}_2{}^*+\boldsymbol{b}_1\boldsymbol{b}_2{}^*|^2}{(1+\frac{\mathrm{snr}}{2}|\boldsymbol{a}_1|^2)(1+\frac{\mathrm{snr}}{2}|\boldsymbol{a}_2|^2)-(\frac{\mathrm{snr}}{2})^2\rho^2|\boldsymbol{a}_1\boldsymbol{a}_2{}^*|^2}\right).
\end{aligned}
\tag{4.2}
$$

Then the interference free signal is decoded with the rate

$$
I_{13}(\mathsf{H}_1,\rho,\mathrm{snr}) \triangleq \log\left((1+\frac{\mathrm{snr}}{2}|\boldsymbol{a}_1|^2)(1+\frac{\mathrm{snr}}{2}|\boldsymbol{a}_2|^2)-(\frac{\mathrm{snr}}{2})^2\rho^2|\boldsymbol{a}_1\boldsymbol{a}_2{}^*|^2\right).
\tag{4.3}
$$

So, the achievable rate in this case is the lowest achievable rate in these two steps; indeed

$$
\begin{aligned}
&\min\left\{\log\left(\frac{(1+\frac{\mathrm{snr}}{2}(|\boldsymbol{a}_1|^2+|\boldsymbol{b}_1|^2))(1+\frac{\mathrm{snr}}{2}(|\boldsymbol{a}_2|^2+|\boldsymbol{b}_2|^2))}{(1+\frac{\mathrm{snr}}{2}|\boldsymbol{a}_1|^2)(1+\frac{\mathrm{snr}}{2}|\boldsymbol{a}_2|^2)-(\frac{\mathrm{snr}}{2})^2\rho^2|\boldsymbol{a}_1\boldsymbol{a}_2{}^*|^2}\right.\right.\\
&\left.-\frac{(\frac{\mathrm{snr}}{2})^2\rho^2|\boldsymbol{a}_1\boldsymbol{a}_2{}^*+\boldsymbol{b}_1\boldsymbol{b}_2{}^*|^2}{(1+\frac{\mathrm{snr}}{2}|\boldsymbol{a}_1|^2)(1+\frac{\mathrm{snr}}{2}|\boldsymbol{a}_2|^2)-(\frac{\mathrm{snr}}{2})^2\rho^2|\boldsymbol{a}_1\boldsymbol{a}_2{}^*|^2}\right),\\
&\left.\log\left((1+\frac{\mathrm{snr}}{2}|\boldsymbol{a}_1|^2)(1+\frac{\mathrm{snr}}{2}|\boldsymbol{a}_2|^2)-(\frac{\mathrm{snr}}{2})^2\rho^2|\boldsymbol{a}_1\boldsymbol{a}_2{}^*|^2\right)\right\}.
\end{aligned}
\tag{4.4}
$$

Therefore, the achievable rate in this scheme is given by

$$
R_1(\mathsf{H}_1,\rho,\mathrm{snr}) = \max\{I_{11}(\mathsf{H}_1,\rho,\mathrm{snr}),\min\{I_{12}(\mathsf{H}_1,\rho,\mathrm{snr}),I_{13}(\mathsf{H}_1,\rho,\mathrm{snr})\}\}
$$

We claim that in this case $\rho = 1$ maximizes the achievable rate in the high snr regime, almost surely.

**Claim 1**

$$
\lim_{\mathrm{snr}\to\infty}\mathbb{P}(R_1(\mathsf{H}_1,\rho,\mathrm{snr}) \le R_1(\mathsf{H}_1,1,\mathrm{snr})) = 1.
\tag{4.5}
$$

**P**roof: Note that $I_{13}(\mathsf{H}_1, \rho, \mathrm{snr})$ is a decreasing function of $\rho$ regardless of snr. However, as shown earlier in the previous chapter, $I_{11}(\mathsf{H}_1, \rho, \mathrm{snr})$ is an increasing function of $\rho$ in the high snr regime, almost surely. By the same argument, so is the $I_{12}(\mathsf{H}_1, \rho, \mathrm{snr})$, which implies that the $\rho = 1$ maximizes the achievable rate in both cases. That is:

$$\lim_{\mathrm{snr}\to\infty} \mathbb{P}(I_{11}(\mathsf{H}_1, \rho, \mathrm{snr}) \leq I_{11}(\mathsf{H}_1, 1, \mathrm{snr})) = 1, \tag{4.6}$$

and

$$\lim_{\mathrm{snr}\to\infty} \mathbb{P}(I_{12}(\mathsf{H}_1, \rho, \mathrm{snr}) \leq I_{12}(\mathsf{H}_1, 1, \mathrm{snr})) = 1. \tag{4.7}$$

Define

$$I'(\mathsf{H}_1, \rho, \mathrm{snr}) = \max\{I_{11}(\mathsf{H}_1, \rho, \mathrm{snr}), I_{12}(\mathsf{H}_1, \rho, \mathrm{snr})\} \tag{4.8}$$

Define

$$\mathcal{E}_1(\rho, \mathrm{snr}) \triangleq \{\omega \in \Omega : I'(\mathsf{H}_1, \rho, \mathrm{snr}) \leq I'(\mathsf{H}_1, 1, \mathrm{snr})\} \tag{4.9}$$

By (4.6) and (4.7), we have

$$\lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}_1(\rho, \mathrm{snr})) = 1. \tag{4.10}$$

Furthermore, for any $\rho < 1$, note that $I_{13}(\mathsf{H}_1, \rho, \mathrm{snr})$ scales like $O(\mathrm{snr}^2)$ while $I_{12}(\mathsf{H}_1, \rho, \mathrm{snr})$ scales like $O(\mathrm{snr})$. Therefore, for sufficiently large snr, $I_{13}(\mathsf{H}_1, \rho, \mathrm{snr}) \geq I_{12}(\mathsf{H}_1, \rho, \mathrm{snr})$ almost surely. Define

$$\mathcal{E}_2(\rho, \mathrm{snr}) \triangleq \{\omega \in \Omega : R_1(\mathsf{H}_1, \rho, \mathrm{snr}) = I'(\mathsf{H}_1, \rho, \mathrm{snr})\} \tag{4.11}$$

Therefore, for $\rho < 1$ we have

$$\lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}_2(\rho, \mathrm{snr})) = \lim_{\mathrm{snr}\to\infty} \mathbb{P}(I_{13}(\mathsf{H}_1, \rho, \mathrm{snr}) \geq I_{12}(\mathsf{H}_1, \rho, \mathrm{snr})) = 1 \tag{4.12}$$

Note that $R_1(\mathsf{H}_1, \rho, \mathrm{snr})$ and $I'(\mathsf{H}_1, \rho, \mathrm{snr})$ are both continuous functions of $\rho$. Therefore, $\lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}_2(1, \mathrm{snr})) = 1$; which implies that $\mathcal{E}_2(\rho, \mathrm{snr})$ happens almost surely for all $\rho \geq 0$. Define

$$\mathcal{E}_3(\rho, \mathrm{snr}) \triangleq \{\omega \in \Omega : R_1(\mathsf{H}_1, \rho, \mathrm{snr}) \leq R_1(\mathsf{H}_1, 1, \mathrm{snr})\} \tag{4.13}$$

We now show that for any $\rho > 0$, $\lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}_3(\rho, \mathrm{snr})) = 1$.

$$
\begin{aligned}
& \lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}_3(\rho,\mathrm{snr})) \\
= \quad & \lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}_3(\rho,\mathrm{snr}),\mathcal{E}_2(\rho,\mathrm{snr})) + \lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}_3(\rho,\mathrm{snr}),\mathcal{E}_2^c(\rho,\mathrm{snr})) \\
\stackrel{(a)}{=} \quad & \lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}_1(\rho,\mathrm{snr}),\mathcal{E}_2(\rho,\mathrm{snr})) \stackrel{(b)}{=} 1,
\end{aligned} \tag{4.14}
$$

where (a) holds since

$$
\lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}_3(\rho,\mathrm{snr}),\mathcal{E}_2^c(\rho,\mathrm{snr})) \leq \lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}_2^c(\rho,\mathrm{snr})) = 0. \tag{4.15}
$$

Moreover, in order to justify (b), note that by the union bound

$$
\begin{aligned}
& \lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}_1^c(\rho,\mathrm{snr}) \cup \mathcal{E}_2^c(\rho,\mathrm{snr})) \\
\leq \quad & \lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}_2^c(\rho,\mathrm{snr})) + \lim_{\mathrm{snr}\to\infty} \mathbb{P}(\mathcal{E}_1^c(\rho,\mathrm{snr})) = 0.
\end{aligned} \tag{4.16}
$$

Considering the complement event completes the proof.

■

**Remark 6** *It is important to note that in this case, similar to the case of simply treating interference as noise, the achievable rate shows a scaling behavior with* snr *only when* $\rho = 1$. *This drives us to give an alternative proof for the optimality of* $\rho = 1$ *in the high* snr *regime, in the sense of maximizing the achievable rate.*

**P**roof: Since the achievable rate scales with snr when $\rho = 1$, we have

$$
\lim_{\mathrm{snr}\to\infty} \frac{R_1(\mathsf{H}_1,\rho,\mathrm{snr})}{R_1(\mathsf{H}_1,1,\mathrm{snr})} = 0, \tag{4.17}
$$

almost surely. Moreover, it is well-known that almost sure convergence implies convergence in probability[30]. Therefore, for any $\epsilon > 0$,

$$
\lim_{\mathrm{snr}\to\infty} \mathbb{P}(\frac{R_1(\mathsf{H}_1,\rho,\mathrm{snr})}{R_1(\mathsf{H}_1,1,\mathrm{snr})} \geq \epsilon) = 0 \tag{4.18}
$$

Now let $\epsilon = 1$. We have

$$
\begin{aligned}
& \lim_{\mathrm{snr}\to\infty} \mathbb{P}(\frac{R_1(\mathsf{H}_1,\rho,\mathrm{snr})}{R_1(\mathsf{H}_1,1,\mathrm{snr})} \leq 1) \\
& \lim_{\mathrm{snr}\to\infty} \mathbb{P}(R_1(\mathsf{H}_1,\rho,\mathrm{snr}) \leq R_1(\mathsf{H}_1,1,\mathrm{snr})) = 1.
\end{aligned} \tag{4.19}
$$

■

## 4.2 Outage probability Analysis

Following the same argument as the one presented in the case of treating interference as noise, since $\rho = 1$ maximizes the achievable rate in the high snr regime almost surely, it is the optimal choice of correlation coefficients in order to minimize the outage probability in the following sense. For any $\epsilon > 0$

$$\begin{aligned} & \mathbb{P}(R_1(\mathsf{H}_1, 1, \mathrm{snr}) < R|\mathcal{E}(\mathrm{snr}))(1-\epsilon) \leq \\ & \inf_{\rho} \mathbb{P}(R_1(\mathsf{H}_1, \rho, \mathrm{snr}) < R) \\ \leq \quad & \mathbb{P}(R_1(\mathsf{H}_1, 1, \mathrm{snr}) < R|\mathcal{E}(\mathrm{snr})) + \epsilon, \end{aligned} \tag{4.20}$$

where $\mathcal{E}(\mathrm{snr})$ is the event $\{\omega \in \Omega : R_1(\mathsf{H}_1, \rho, \mathrm{snr}) \leq R_1(\mathsf{H}_1, 1, \mathrm{snr})\}$, which was proved that happens almost surely.

Therefore, we analyze the outage probability of the system in this scenario which is given by

$$\begin{aligned} & \mathbb{P}(R_1(\mathsf{H}_1, 1, \mathrm{snr}) < r\log(\mathrm{snr})) = \\ & \mathbb{P}(\max\left\{\min\left\{\sigma^2\frac{\boldsymbol{A}_{22}n + |\boldsymbol{A}_{21}|^2n^2}{1+\boldsymbol{A}_{11}n}, \boldsymbol{A}_{11}n\right\}, \frac{\boldsymbol{A}_{11}n + \sigma^2|\boldsymbol{A}_{21}|^2n^2}{1+\sigma^2\boldsymbol{A}_{22}n}\right\} < (2n)^r - 1() \end{aligned} \tag{4.21}$$

In order to characterize the scaling behavior of the outage probability (4.21), we adopt the same approach of giving tight upper and lower bounds to this probability in the sense of scaling behavior.

### 4.2.1 Upper Bound

Recall the notations $\boldsymbol{h}_n = \frac{\boldsymbol{A}_{11}n+\sigma^2|\boldsymbol{A}_{21}|^2n^2}{1+\sigma^2\boldsymbol{A}_{22}n}$ and $\hat{\boldsymbol{h}}_n = \frac{|\boldsymbol{A}_{21}|^2}{\boldsymbol{A}_{22}}n$. In the same line, define

$$\boldsymbol{h}'_n = \sigma^2\frac{\boldsymbol{A}_{22}n + |\boldsymbol{A}_{21}|^2n^2}{1+\boldsymbol{A}_{11}n}, \tag{4.22}$$

and

$$\hat{\boldsymbol{h}}'_n = \frac{\sigma^2|\boldsymbol{A}_{21}|^2}{\boldsymbol{A}_{11}}n. \tag{4.23}$$

As shown earlier, $\boldsymbol{h}_n(\omega) \geq \hat{\boldsymbol{h}}_n(\omega)$, for any $\omega \in \Omega$. By the same argument

$$\boldsymbol{h}'_n \geq \hat{\boldsymbol{h}}'_n. \tag{4.24}$$

We now bound the outage probability (4.21) from above as follows. Let $\alpha := 2\frac{(2n)^r-1}{n}$.

$$\begin{aligned}
&\mathbb{P}(R_1(\mathsf{H}_1, 1, \mathrm{snr}) < r\log(\mathrm{snr})) = \\
&\mathbb{P}(\max\left\{\min\left\{\sigma^2\frac{\boldsymbol{A}_{22}n + |\boldsymbol{A}_{21}|^2n^2}{1+\boldsymbol{A}_{11}n}, \boldsymbol{A}_{11}n\right\}, \frac{\boldsymbol{A}_{11}n + \sigma^2|\boldsymbol{A}_{21}|^2n^2}{1+\sigma^2\boldsymbol{A}_{22}n}\right\} < (2n)^r - 1) \leq \\
&\mathbb{P}(\max\left\{\min\{\sigma^2\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}}, \boldsymbol{A}_{11}\}, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}\right\} < \alpha) = \\
&\mathbb{P}(\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}} < \frac{\alpha}{\sigma^2}, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}} < \alpha) + \mathbb{P}(\boldsymbol{A}_{11} < \alpha, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}} < \alpha) \\
&-\mathbb{P}(\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}} < \frac{\alpha}{\sigma^2}, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}} < \alpha, \boldsymbol{A}_{11} < \alpha)
\end{aligned} \tag{4.25}$$

In order to compute the probabilities in the right hand side of (4.25), we again take advantage of a transformation of variables, so as to obtain the joint density of the random variables involved, having in hands the joint probability

$$f_{\boldsymbol{A}_{11},\boldsymbol{A}_{22},\Re(\boldsymbol{A}_{12}),\Im(\boldsymbol{A}_{12})}(x,y,z,t) = \frac{1}{\pi}\exp(-x-y)\mathbf{1}_{\{xy>z^2+t^2\}} \tag{4.26}$$

Define:

$$\begin{aligned}
u &:= x, \\
v &:= z, \\
w &:= \frac{z^2+t^2}{x}, \\
s &:= \frac{z^2+t^2}{y}.
\end{aligned} \tag{4.27}$$

After characterizing the real roots of the set of equations (4.27) and their corresponding Jacobian matrices, the joint probability density function of the new variables can be presented as follows:

$$f_{\boldsymbol{A}_{11},\Re(\boldsymbol{A}_{12}),\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}},\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(u,v,w,s) = \frac{1}{\pi}\exp(-u-\frac{uw}{s})\frac{u^2w}{s^2\sqrt{uw-v^2}}\mathbf{1}_{\{u>s\}} \tag{4.28}$$

Eliminating $\Re(\boldsymbol{A}_{12})$, yields

$$f_{\boldsymbol{A}_{11},\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}},\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(u,w,s) = \exp(-u-\frac{uw}{s})\frac{u^2w}{s^2}\mathbf{1}_{\{u>s\}} \tag{4.29}$$

Then

$$\begin{aligned}
f_{\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}},\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(w,s) &= \int_s^\infty \exp\left(-u-\frac{uw}{s}\right)\frac{u^2w}{s^2}\mathrm{d}u \\
&= \frac{ws}{(w+s)^3}\int_{w+s}^\infty u^2e^{-u}\mathrm{d}u \\
&= -\frac{ws}{(w+s)^3}\left[(u^2+2u+2)e^{-u}\right]_{w+s}^\infty \\
&= \frac{ws((w+s)^2+1)}{(w+s)^3}e^{-(w+s)}, \qquad (4.30)
\end{aligned}$$

and

$$\begin{aligned}
f_{\boldsymbol{A}_{11},\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(u,s) &= \int_0^\infty \exp\left(-u-\frac{uw}{s}\right)\frac{u^2w}{s^2}\mathbf{1}_{\{u>s\}}\mathrm{d}w \\
&= e^{-u}\mathbf{1}_{\{u>s\}}\int_0^\infty w\exp(-w)\,\mathrm{d}w \\
&= -e^{-u}\mathbf{1}_{\{u>s\}}\left[e^{-w}(w+1)\right]_0^\infty \\
&= e^{-u}\mathbf{1}_{\{u>s\}}. \qquad (4.31)
\end{aligned}$$

Finding a closed form expression for the probability distribution function $F_{\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}},\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(w,s)$ is a hard problem. As such, since we are interested in bounding the outage probability from above, an upper bound on $F_{\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}},\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(\alpha,\alpha)$ is sufficient for our purpose Later, we show this bound is tight.

$$\begin{aligned}
&\mathbb{P}\left(\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}}<\frac{\alpha}{\sigma^2},\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}<\alpha\right) \\
&= \int_0^{\frac{\alpha}{\sigma^2}}\int_0^\alpha f_{\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}},\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(w,s)\mathrm{d}s\mathrm{d}w \\
&= \int_0^{\frac{\alpha}{\sigma^2}}\int_0^\alpha \frac{ws((w+s)^2+1)}{(w+s)^3}e^{-(w+s)}\mathrm{d}s\mathrm{d}w \\
&\overset{(a)}{\leq} \int_0^{\frac{\alpha}{\sigma^2}}\int_0^\alpha \frac{ws(1+(w+s)^2)}{(w+s)^3}\mathrm{d}s\mathrm{d}w \\
&= \int_0^{\frac{\alpha}{\sigma^2}}\int_0^\alpha \frac{ws}{(w+s)^3}\mathrm{d}s\mathrm{d}w + \int_0^{\frac{\alpha}{\sigma^2}}\int_0^\alpha \frac{ws}{(w+s)}\mathrm{d}s\mathrm{d}w \\
&\overset{(b)}{=} \frac{1}{2(1+\sigma^2)}\alpha + o(\alpha), \qquad (4.32)
\end{aligned}$$

where(a) holds since $e^{-x} \leq 1$, for $x \geq 0$, and (b) follows from the calculations in Appendix G.

It is possible to find $F_{\boldsymbol{A}_{11}, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(u, s)$ in closed form. Indeed,

$$F_{\boldsymbol{A}_{11}, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(u, s) = \begin{cases} 1 - (u+1)e^{-u} & s > u \\ 1 - se^{-u} - e^{-s} & s \leq u \end{cases} . \tag{4.33}$$

Therefore,

$$\mathbb{P}\left(\boldsymbol{A}_{11} < \alpha, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}} < \alpha\right) = F_{\boldsymbol{A}_{11}, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(\alpha, \alpha) = 1 - (\alpha + 1)e^{-\alpha} = \frac{\alpha^2}{2} - \frac{\alpha^3}{3} + \cdots . \tag{4.34}$$

Moreover since

$$\mathbb{P}\left(\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}} < \frac{\alpha}{\sigma^2}, \boldsymbol{A}_{11} < \alpha, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}} < \alpha\right) \leq \mathbb{P}\left(\boldsymbol{A}_{11} < \alpha, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}} < \alpha\right), \tag{4.35}$$

there is no need to calculate this probability, we can simply bound

$$\mathbb{P}\left(\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}} < \frac{\alpha}{\sigma^2}, \boldsymbol{A}_{11} < \alpha, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}} < \alpha\right)$$

from below with 0; as it does not have any effect on the scaling behavior of the upper bound on the outage probability.

Considering (4.43) and (4.34), it turns out that

$$\mathbb{P}(R_1(\mathsf{H}_1, 1, \mathrm{snr}) < r\log(\mathrm{snr})) \leq \frac{1}{(1+\sigma^2)}\frac{1}{\mathrm{snr}^{1-r}} + o(\frac{1}{\mathrm{snr}^{1-r}}) \tag{4.36}$$

### 4.2.2 Lower Bound

In order to characterize the scaling behavior of the outage probability, we bound (4.21) from below and show that the given lower bound meets the upper bound developed earlier in the sense of scaling behavior. First note that

$$\begin{aligned}
&\mathbb{P}(R_1(\mathsf{H}_1, 1, \mathrm{snr}) < r\log(\mathrm{snr})) = \\
&\mathbb{P}(\max\left\{\min\left\{\sigma^2\frac{\boldsymbol{A}_{22}n + |\boldsymbol{A}_{21}|^2 n^2}{1 + \boldsymbol{A}_{11}n}, \boldsymbol{A}_{11}n\right\}, \frac{\boldsymbol{A}_{11}n + \sigma^2|\boldsymbol{A}_{21}|^2 n^2}{1 + \sigma^2\boldsymbol{A}_{22}n}\right\} < (2n)^r - 1) = \\
&\mathbb{P}(\boldsymbol{A}_{11} < \alpha, \boldsymbol{h}_n < (2n)^r - 1) + \mathbb{P}(\boldsymbol{h}'_n < (2n)^r - 1, \boldsymbol{h}_n < (2n)^r - 1, \boldsymbol{A}_{11} \geq \alpha)
\end{aligned} \tag{4.37}$$

We now treat each term in the right hand side of separately and give tight lower bounds for each term using the same techniques as those used for bounding the outage probability from below treated in section (3.3.2) in the case of treating interference as noise. It turns out that outage probability in this scenario behaves as follows.

**Proposition 4** *In the scenario where the receivers have the choice to either directly decode the desired signal, treating interference as Gaussian noise or to decode and cancel interference before decoding the desired signal, we have*

$$\mathbb{P}(R_1(\mathsf{H}_1, 1, \mathrm{snr}) < r\log(\mathrm{snr})) \sim \frac{1}{1+\sigma^2}\frac{1}{\mathrm{snr}^{1-r}} \tag{4.38}$$

*for* $r > \frac{1}{2}$

**P**roof: We bound the probabilities in the right hand side of (4.37). For any $\epsilon > 0$ and $\ell > 0$, we can write

$$\begin{aligned}
& \mathbb{P}(\hat{\boldsymbol{h}}_n < (2n)^r - (2n)^{r-\epsilon} - 1, \boldsymbol{A}_{11} < \alpha) \\
= \; & \mathbb{P}(\hat{\boldsymbol{h}}_n - \boldsymbol{h}_n + \boldsymbol{h}_n < (2n)^r - (2n)^{r-\epsilon} - 1, \boldsymbol{A}_{11} < \alpha) \\
= \; & \mathbb{P}(\hat{\boldsymbol{h}}_n - \boldsymbol{h}_n + \boldsymbol{h}_n < (2n)^r - (2n)^{r-\epsilon} - 1, |\boldsymbol{h}_n - \hat{\boldsymbol{h}}_n| < (2n)^{r-\epsilon}, \boldsymbol{A}_{11} < \alpha) \\
+ \; & \mathbb{P}(\hat{\boldsymbol{h}}_n - \boldsymbol{h}_n + \boldsymbol{h}_n < (2n)^r - (2n)^{r-\epsilon} - 1, |\boldsymbol{h}_n - \hat{\boldsymbol{h}}_n| > (2n)^{r-\epsilon}, \boldsymbol{A}_{11} < \alpha) \\
\leq \; & \mathbb{P}(\boldsymbol{h}_n < (2n)^r - 1, |\boldsymbol{h}_n - \hat{\boldsymbol{h}}_n| < (2n)^{r-\epsilon}, \boldsymbol{A}_{11} < \alpha) + \mathbb{P}(|\boldsymbol{h}_n - \hat{\boldsymbol{h}}_n| > (2n)^{r-\epsilon}) \\
\overset{(a)}{\leq} \; & \mathbb{P}(\boldsymbol{h}_n < (2n)^r - 1, \boldsymbol{A}_{11} < \alpha) + \mathbb{P}(\frac{\boldsymbol{A}_{11}}{\sigma^2 \boldsymbol{A}_{22}} > (2n)^{r-\epsilon}) \\
\overset{(b)}{\leq} \; & \mathbb{P}(\boldsymbol{h}_n < (2n)^r - 1, \boldsymbol{A}_{11} < \alpha) + \frac{\mathbb{E}(\frac{\boldsymbol{A}_{11}}{\sigma^2 \boldsymbol{A}_{22}})}{(2n)^{r-\epsilon}}
\end{aligned} \tag{4.39}$$

where (a) holds since $|\boldsymbol{h}_n - \hat{\boldsymbol{h}}_n| \leq \frac{\boldsymbol{A}_{11}}{\sigma^2 \boldsymbol{A}_{22}}$. Since $\frac{\boldsymbol{A}_{11}}{\sigma^2 \boldsymbol{A}_{22}}$ is a positive random variable, (b) holds due to Markov's inequality.

We now compute $\mathbb{P}(\hat{\boldsymbol{h}}_n < (2n)^r - (2n)^{r-\epsilon} - 1, \boldsymbol{A}_{11} < \alpha)$ based on (4.33). Define $\beta = \frac{(2n)^r - (2n)^{r-\epsilon} - 1}{n}$.

$$\begin{aligned}
\mathbb{P}(\hat{\boldsymbol{h}}_n < (2n)^r - (2n)^{r-\epsilon} - 1, \boldsymbol{A}_{11} < \alpha) \; &= \; F_{\boldsymbol{A}_{11}, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(\alpha, \beta) = 1 - \beta e^{-\alpha} - e^{-\beta} \\
&\geq \; 1 - \beta(1 - \alpha + \frac{\alpha^2}{2}) - (1 - \beta + \frac{\beta^2}{2}) = \alpha\beta - \frac{\beta^2}{2} - \beta\frac{\alpha^2}{2} \\
&= \; \frac{2}{(2n)^{2-2r}} + o(\frac{1}{n^{2-2r}})
\end{aligned} \tag{4.40}$$

We now give a tight lower bound on the second term in (4.37). Applying the same argument twice, as the one used above, for any $\epsilon, \epsilon' > 0$, we can write

$$\begin{aligned}
& \mathbb{P}(\hat{\boldsymbol{h}}_n < (2n)^r - (2n)^{r-\epsilon} - 1, \hat{\boldsymbol{h}}_n' < (2n)^r - (2n)^{r-\epsilon'} - 1, \boldsymbol{A}_{11} \geq \alpha) \\
\leq \ & \mathbb{P}(\boldsymbol{h}_n < (2n)^r - 1, \hat{\boldsymbol{h}}_n' < (2n)^r - (2n)^{r-\epsilon'} - 1, \boldsymbol{A}_{11} \geq \alpha) + \frac{\mathbb{E}(\frac{\boldsymbol{A}_{11}}{\sigma^2 \boldsymbol{A}_{22}})}{(2n)^{r-\epsilon}} \\
\leq \ & \mathbb{P}(\boldsymbol{h}_n < (2n)^r - 1, \boldsymbol{h}_n' < (2n)^r - 1, \boldsymbol{A}_{11} \geq \alpha) + \frac{\mathbb{E}(\frac{\boldsymbol{A}_{11}}{\sigma^2 \boldsymbol{A}_{22}})}{(2n)^{r-\epsilon}} + \frac{\mathbb{E}(\sigma^2 \frac{\boldsymbol{A}_{22}}{\boldsymbol{A}_{11}})}{(2n)^{r-\epsilon'}} \qquad (4.41)
\end{aligned}$$

Furthermore, by the definition we have

$$\begin{aligned}
& \mathbb{P}(\hat{\boldsymbol{h}}_n < n^r - n^{r-\epsilon} - 1, \hat{\boldsymbol{h}}_n' < n^r - n^{r-\epsilon'} - 1, \boldsymbol{A}_{11} \geq \alpha) = \\
& \mathbb{P}(\frac{|\boldsymbol{A}_{21}|^2}{\boldsymbol{A}_{11}} < \frac{\beta}{\sigma^2}, \frac{|\boldsymbol{A}_{21}|^2}{A_2 2} < \beta, A \geq \alpha) = \\
& \mathbb{P}\left(\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}} < \frac{\beta}{\sigma^2}, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}} < \beta\right) - \mathbb{P}\left(\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}} < \frac{\beta}{\sigma^2}, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}} < \beta, \boldsymbol{A}_{11} < \alpha\right) \qquad (4.42)
\end{aligned}$$

The first probability in the right hand side of (4.42) can be bounded from below as follows

$$\begin{aligned}
& \mathbb{P}\left(\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}} < \frac{\alpha}{\sigma^2}, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}} < \alpha\right) \\
& = \int_0^{\frac{\alpha}{\sigma^2}} \int_0^{\alpha} f_{\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}}, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(w, s) \mathrm{d}s \mathrm{d}w \\
& = \int_0^{\frac{\alpha}{\sigma^2}} \int_0^{\alpha} \frac{ws((w+s)^2+1)}{(w+s)^3} e^{-(w+s)} \mathrm{d}s \mathrm{d}w \\
& \overset{(a)}{\geq} \int_0^{\frac{\alpha}{\sigma^2}} \int_0^{\alpha} \frac{ws(1-(w+s))}{(w+s)^3} \mathrm{d}s \mathrm{d}w \\
& = \int_0^{\frac{\alpha}{\sigma^2}} \int_0^{\alpha} \frac{ws}{(w+s)^3} \mathrm{d}s \mathrm{d}w - \int_0^{\frac{\alpha}{\sigma^2}} \int_0^{\alpha} \frac{ws}{(w+s)^2} \mathrm{d}s \mathrm{d}w \\
& \overset{(b)}{=} \frac{1}{2(1+\sigma^2)} \alpha + o(\alpha), \qquad (4.43)
\end{aligned}$$

where(a) holds since $1 - x \leq e^{-x}$, for $x \geq 0$, and (b) follows from the calculations in Appendix G.

As for the second term in (4.42), note that

$$\begin{aligned}\mathbb{P}\left(\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{11}} < \frac{\beta}{\sigma^2}, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}} < \beta, \boldsymbol{A}_{11} < \alpha\right) &\leq \mathbb{P}\left(\frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}} < \beta, \boldsymbol{A}_{11} < \alpha\right) \\ &= F_{\boldsymbol{A}_{11}, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(\alpha, \beta) \end{aligned} \tag{4.44}$$

Since $F_{\boldsymbol{A}_{11}, \frac{|\boldsymbol{A}_{12}|^2}{\boldsymbol{A}_{22}}}(\alpha, \beta)$ scales like $\frac{1}{n^{2-2r}}$, the scaling behavior of the term in the right hand side of (4.44) is dominated by the others.

Moreover, as shown earlier, since $\mathbb{E}(\frac{\boldsymbol{A}_{11}}{\sigma^2 \boldsymbol{A}_{22}})$ and $\mathbb{E}(\sigma^2 \frac{\boldsymbol{A}_{22}}{\boldsymbol{A}_{11}})$ are constant, the last term in the right hand side of (4.39) and the last two terms in the right hand side of (4.41) decay faster than $\frac{1}{n^{1-r}}$ when $r > \frac{1}{2}$ and the proof is complete. ■

## 4.3 Interference-free Scenario

Let us now compare the results developed so far for the scaling behavior of the frequency selective 2-user Gaussian interference channel with 2 sub-bands under different schemes with the scaling behavior of the outage probability in a point to point frequency selective fading channel where there is no interfering source. The outage probability in this scenario, when jointly Gaussian signals with correlation coefficient $\rho$ is transmitted over the sub-bands is given by

$$\begin{aligned}&\mathbb{P}(R_1(\mathsf{H}_1, \rho, \mathrm{snr}) < R) = \\ &\mathbb{P}\left(\log\left((1 + \frac{\mathrm{snr}}{2}|\boldsymbol{a}_1|^2)(1 + \frac{\mathrm{snr}}{2}|\boldsymbol{a}_2|^2) - \rho^2(\frac{\mathrm{snr}}{2})^2|\boldsymbol{a}_1\boldsymbol{a}_2|^2\right) < R\right).\end{aligned} \tag{4.45}$$

By Remark 2, we know that the optimal correlation coefficient in this scenario is $\rho = 0$. Therefore, the outage probability can be written as

$$\mathbb{P}(R_1(\mathsf{H}_1, 0, \mathrm{snr}) < R) = \mathbb{P}\left(\log\left((1 + \frac{\mathrm{snr}}{2}|\boldsymbol{a}_1|^2)(1 + \frac{\mathrm{snr}}{2}|\boldsymbol{a}_2|^2)\right) < R\right) \tag{4.46}$$

In order to characterize the behavior of the outage probability in the high snr regime, given a rate $R = r\log(\mathrm{snr})$, we again give bounds on the outage probability and characterize the scaling behavior of the upper and lower bounds.

### 4.3.1 Upper Bound

As for the upper bound, we neglect 1 in compare to $\frac{\text{snr}}{2}|\boldsymbol{a}_1|^2$ and $\frac{\text{snr}}{2}|\boldsymbol{a}_2|^2$ and obtain

$$\mathbb{P}(R_1(\mathsf{H}_1, 0, \text{snr}) < r\log(\text{snr})) \leq \mathbb{P}\left(|\boldsymbol{a}_1|^2|\boldsymbol{a}_2|^2 < \frac{\text{snr}^r}{(\frac{\text{snr}}{2})^2}\right) \tag{4.47}$$

Note that $|\boldsymbol{a}_1|^2$ and $|\boldsymbol{a}_2|^2$ are two independent exponential random variables. The probability density function of the product of two independent exponential random variables $X$ and $Y$ both of which have have the same parameter 1 is given by[31]

$$f_{XY}(u) = 2K_0(2\sqrt{u}), \quad u > 0 \tag{4.48}$$

where $K_0(z)$ is the modified Bessel function of the second kind.

Define $\eta = \frac{\text{snr}^r}{(\frac{\text{snr}}{2})^2}$. The outage probability (4.47) can be calculated as follows

$$\mathbb{P}\left(|\boldsymbol{a}_1|^2|\boldsymbol{a}_2|^2 < \eta\right) = \int_0^\eta 2K_0(2\sqrt{u})\mathrm{d}u = \int_0^{2\sqrt{\eta}} uK_0(u)\mathrm{d}u \tag{4.49}$$

In order to evaluate the last integral in (4.49) An asymptotic form for this function is given in [29] as follows:

$$K_0(z) = -\left(\ln(\frac{1}{2}z) + \gamma\right) I_0(z) + \frac{\frac{1}{4}z^2}{(1!)^2} + (1+\frac{1}{2})\frac{(\frac{1}{4}z^2)^2}{(2!)^2} + (1+\frac{1}{2}+\frac{1}{3})\frac{(\frac{1}{4}z^2)^3}{(3!)^2} + \cdots \tag{4.50}$$

where

$$I_0(z) = 1 + \frac{\frac{1}{4}z^2}{(1!)^2} + \frac{(\frac{1}{4}z^2)^2}{(2!)^2} + \frac{(\frac{1}{4}z^2)^3}{(3!)^2} + \cdots \tag{4.51}$$

is the modified Bessel function of the first kind and $\gamma$ is Euler's constant.

Since each term in the Laurent series expansion of the function $K_0(z)$ is analytic, the last integral in (4.49) can be readily calculated. Therefore

$$\mathbb{P}\left(|\boldsymbol{a}_1|^2|\boldsymbol{a}_2|^2 < \eta\right) = \int_0^\eta 2K_0(2\sqrt{u})\mathrm{d}u = 2(1-\gamma)\eta + o(\eta) \tag{4.52}$$

Therefore, the probability of outage $\mathbb{P}(R_1(\mathsf{H}_1, 0, \text{snr}) < r\log(\text{snr}))$ in this scenario scales at least as fast as $8(1-\gamma)\frac{1}{\text{snr}^{2-r}}$.

### 4.3.2 Lower Bound

Following the same bounding scheme we have used so far, for any $\epsilon > 0$, we have

$$\begin{aligned}
&\mathbb{P}(R_1(\mathsf{H}_1, 0, \text{snr}) < R) = \\
&\mathbb{P}\left(\log\left((1+\frac{\text{snr}}{2}|\boldsymbol{a}_1|^2)(1+\frac{\text{snr}}{2}|\boldsymbol{a}_2|^2)\right) < r\log(\text{snr})\right) \geq \\
&\mathbb{P}\left(|\boldsymbol{a}_1|^2|\boldsymbol{a}_2|^2(\frac{\text{snr}}{2})^2 < (\text{snr}^r - \text{snr}^{r-\epsilon})\right) - \mathbb{P}(1+\frac{\text{snr}}{2}(|\boldsymbol{a}_1|^2+|\boldsymbol{a}_2|^2) > \text{snr}^{r-\epsilon}) = \\
&\mathbb{P}\left(|\boldsymbol{a}_1|^2|\boldsymbol{a}_2|^2 < \frac{\text{snr}^r - \text{snr}^{r-\epsilon}}{(\frac{\text{snr}}{2})^2}\right) - \mathbb{P}(|\boldsymbol{a}_1|^2+|\boldsymbol{a}_2|^2 > \frac{\text{snr}^{r-\epsilon}-1}{\frac{\text{snr}}{2}})
\end{aligned} \tag{4.53}$$

Since $|\boldsymbol{a}_1|^2$ and $|\boldsymbol{a}_2|^2$ are two independent exponential random variables, $|\boldsymbol{a}_1|^2 + |\boldsymbol{a}_2|^2$ is distributed as a chi-squared random variable with 4 degrees of freedom. Therefore, the second term in the right hand side of (4.53) can be calculated as follows

$$\mathbb{P}(|\boldsymbol{a}_1|^2+|\boldsymbol{a}_2|^2 > \frac{\text{snr}^{r-\epsilon}-1}{\frac{\text{snr}}{2}}) = \int_{\frac{\text{snr}^{r-\epsilon}-1}{\frac{\text{snr}}{2}}}^{\infty} xe^{-x}\mathrm{d}x = (1+\frac{\text{snr}^{r-\epsilon}-1}{\frac{\text{snr}}{2}})e^{-\frac{\text{snr}^{r-\epsilon}-1}{\frac{\text{snr}}{2}}}, \tag{4.54}$$

which exponentially goes to zero when $r > 1$.

Since the first term in the right hand side of (4.53) scales like $8(1-\gamma)\frac{1}{\text{snr}^{2-r}}$, with the same steps taken to prove the upper bound, we have, for $r > 1$, that

$$\mathbb{P}(R_1(\mathsf{H}_1, 0, \text{snr}) < r\log(\text{snr})) \sim 8(1-\gamma)\frac{1}{\text{snr}^{2-r}}. \tag{4.55}$$

### 4.3.3 Analysis of the case $\rho = 1$

Let us then compute the outage probability for the case $\rho = 1$. This is of merit for two reasons; first it gives the characterization of the outage probability of the point to point frequency selective channel in the worst case. Secondly, the maximum achievable rate with vanishing probability over the point to point channel can be verified to be congruous with that of the interference channel in the strong interference regime, which was derived earlier. This comparison does make sense since, in the strong interference regime, the interfering signal can decently be decoded and thrown away.

Let $\rho = 1$ in (4.45). Therefore,

$$\mathbb{P}(R_1(\mathsf{H}_1, 1, \text{snr}) < R) = \mathbb{P}\left(\log\left(1+\frac{\text{snr}}{2}(|\boldsymbol{a}_1|^2+|\boldsymbol{a}_2|^2)\right) < R\right). \tag{4.56}$$

Now, let $R = r\log(\text{snr})$. We characterize the scaling behavior of the outage probability which gives us the maximum value of $r$ for which we have a vanishing outage probability. As remarked earlier, $|\boldsymbol{a}_1|^2 + |\boldsymbol{a}_2|^2$ is distributed as $\chi^2(4)$, which gives

$$\mathbb{P}(R_1(\mathsf{H}_1, 1, \text{snr}) < r\log(\text{snr})) = \mathbb{P}\left((|\boldsymbol{a}_1|^2 + |\boldsymbol{a}_2|^2) < \frac{\text{snr}^r - 1}{\frac{\text{snr}}{2}}\right) =$$
$$\int_0^{\alpha} xe^{-x}\mathrm{d}x = 1 - (1+\alpha)e^{-\alpha} \sim \frac{\alpha^2}{2} \tag{4.57}$$

This means that, the outage probability scales like $\frac{1}{\text{snr}^{2-2r}}$ and the maximum possible value of $r$ so that the probability of outage goes to zero is $r = 1$, which is the same as that of the frequency selective interference channel scenario.

## 4.4 Transmitting Independent Codebooks Over the Sub-bands

In this scheme, each of the transmitters performs the Gaussian signaling over the sub-bands independently with the same signal to noise ratio, $\frac{\text{snr}}{2}$, using two independent codebooks. In this case, there are four possible scenarios at the receivers'side depending on the decoding strategies over different sub-bands. As before, we only consider the achievable rate at receiver 1 which is given in the following case.

When the receiver treats interference as noise on both of the sub-bands, the receiver achieves a rate of

$$I_{11}(\mathsf{H}_1, \text{snr}) = \log\left(1 + \frac{\frac{\text{snr}}{2}|\boldsymbol{a}_1|^2}{1 + \frac{\text{snr}}{2}|\boldsymbol{b}_1|^2}\right) + \log\left(1 + \frac{\frac{\text{snr}}{2}|\boldsymbol{a}_2|^2}{1 + \frac{\text{snr}}{2}|\boldsymbol{b}_2|^2}\right). \tag{4.58}$$

This scheme does not make sense since we ignore the possibility of the diversity that transmitting over independent sub-bands can provide and as it can be seen the achievable rate saturates with snr.

When the receiver decodes interference on both of the sub-bands, the achievable rate at the receiver is

$$\begin{aligned} I_{12}(\mathsf{H}_1, \text{snr}) &= \min\left\{\log\left(1 + \frac{\frac{\text{snr}}{2}|\boldsymbol{b}_1|^2}{1 + \frac{\text{snr}}{2}|\boldsymbol{a}_1|^2}\right), \log\left(1 + \frac{\text{snr}}{2}|\boldsymbol{a}_1|^2\right)\right\} \\ &+ \min\left\{\log\left(1 + \frac{\frac{\text{snr}}{2}|\boldsymbol{b}_2|^2}{1 + \frac{\text{snr}}{2}|\boldsymbol{a}_2|^2}\right), \log\left(1 + \frac{\text{snr}}{2}|\boldsymbol{a}_2|^2\right)\right\}. \end{aligned} \tag{4.59}$$

Over each of the sub-bands, interference is decoded first and is thrown away and then the interference free signal is decoded. Therefore, over each sub-band the achievable rate is the minimum of the rates achieved at each step.

When the receiver decodes interference over the first sub-band and treats interference as noise over the second sub-band, we achieve a rate of

$$\begin{aligned}
I_{13}(\mathsf{H}_1, \mathrm{snr}) &= \min\left\{\log\left(1 + \frac{\frac{\mathrm{snr}}{2}|\boldsymbol{b}_1|^2}{1 + \frac{\mathrm{snr}}{2}|\boldsymbol{a}_1|^2}\right), \log\left(1 + \frac{\mathrm{snr}}{2}|\boldsymbol{a}_1|^2\right)\right\} \\
&+ \log\left(1 + \frac{\frac{\mathrm{snr}}{2}|\boldsymbol{a}_2|^2}{1 + \frac{\mathrm{snr}}{2}|\boldsymbol{b}_2|^2}\right). \qquad (4.60)
\end{aligned}$$

Likewise, When the receiver treats interference as noise over the first sub-band and decodes interference over the second sub-band, the achievable rate at the receiver is

$$\begin{aligned}
I_{14}(\mathsf{H}_1, \mathrm{snr}) &= \log\left(1 + \frac{\frac{\mathrm{snr}}{2}|\boldsymbol{a}_1|^2}{1 + \frac{\mathrm{snr}}{2}|\boldsymbol{b}_1|^2}\right) \\
&+ \min\left\{\log\left(1 + \frac{\frac{\mathrm{snr}}{2}|\boldsymbol{b}_2|^2}{1 + \frac{\mathrm{snr}}{2}|\boldsymbol{a}_2|^2}\right), \log\left(1 + \frac{\mathrm{snr}}{2}|\boldsymbol{a}_2|^2\right)\right\}. \qquad (4.61)
\end{aligned}$$

Therefore, the achievable rate in this scheme is given by

$$R_1(\mathsf{H}_1, \mathrm{snr}) = \max\{I_{11}(\mathsf{H}_1, \mathrm{snr}), I_{12}(\mathsf{H}_1, \mathrm{snr}), I_{13}(\mathsf{H}_1, \mathrm{snr}), I_{14}(\mathsf{H}_1, \mathrm{snr})\} \qquad (4.62)$$

As it can be observed, in none of the these scenarios the achievable rate at the receiver scales with snr in the asymptotic regime. This implies that the achievable rate saturates when snr is increased.

Therefore, transmitting independent codebooks over different sub-bands is inferior to the single code-book transmission scheme in the high snr regime.

# Chapter 5

# Concluding Remarks

Studying frequency selective, interference limited, fading wireless channels, due to the growing demand of higher data rate and band width and ubiquity of wireless service is an essentially important problem. Most of the studies in the context of fading interference channels deal with the flat fading scenario under the assumption of partial or perfect CSI at the transmitter. These assumptions are not valid for many realistic wireless communication scenarios, since there might not be a feedback link from the receiver to the corresponding transmitter for sending channel characteristics.

In this ground, we studied a Gaussian interference channel under the assumption of slow and frequency-selective fading. We considered a non-co-operative scenario where there is no channel state information at the transmitter side and different transmitters cannot cooperate with each other regarding their transmission scheme. In this setting, we analyzed the outage probability in a scenario where all transmitters transmit correlated jointly Gaussian signal with their power uniformly distributed along different sub-bands and as for the decoding scheme receives treat interference as noise.

## 5.1 Summary of Contributions

In the first part, the achievable rate at each of the receivers is formulated for a $k$-user interference channel over $N$ orthogonal frequency sub-bands. It is shown, for any realization of channel coefficients, that the derivative of the achievable rate as a function of $\rho$ in the point $\rho = 0$ is zero. For a non frequency-selective scenario, it is verified that $\rho = 0$ is a point of local maximum for the achievable rate at the receivers. This means that the frequency selective characteristic of the channel is essential for correlated signaling to improve the the achievable rate at the receiver. Furthermore, when there is no source of interference, i. e. in the point to point frequency selective fading channel, it is demonstrated that $\rho = 0$ is the global maximum for the achievable rate at the receiver.

Moreover, in the case of $K = 2$ with arbitrary number of sub-bands, using a technical measure theoretic lemma, it is established that $\rho = 0$ is not a point of local maximum for the achievable rate for infinitely many values of SNR. For the general scenario when $k > 2$, simulation results are provided to show that this observations does not hold.

In the second part, we focused on the 2-user scenario where the transmission is performed over 2 orthogonal frequency sub-bands. First, it is shown by considering the monotonicity of the achievable rate in terms of $\rho$ in the high SNR regime that the optimal scheme in order to maximize the achievable rate is to repeat the same signal over the sub-bands; which means to set $\rho = 1$. It is observed that in this case that the achievable rate scalse with $\log(\mathrm{snr})$. We then showed that $\rho = 1$ is, in a sense, the optimal correlation coefficient in order to minimize the probability of outage $\mathbb{P}(R_1(\mathsf{H}_1, \rho, \mathrm{snr}) < R)$, in the high snr regime. We then characterized the scaling behavior of the outage probability, when $R = r\log(\mathrm{snr})$, in the Rayleigh fading scenario. It turned out that, the outage probability in this case scales like $2\frac{1}{\mathrm{snr}^{1-r}}$. Although having the same scaling exponent, the scaling behavior of the probability of outage in the orthogonal frequency division scheme, which is in fact $\frac{1}{\mathrm{snr}^{1-r}}$, decays with a smaller constant term compared to the transmission scheme we analyzed.

In the third part, maintaining the same assumptions on the channel characteristics and the transmission scheme, we allowed the receivers to have the choice either to decode the desired signal treating interference as noise or to decode interference treating the desired signal as noise before decoding the interference free signal. It turned out in this case, as in the case of treating interference as noise, we should set $\rho = 1$ in order to achieve the minimum outage probability in the high snr regime. The outage probability in this scenario scales like $\frac{1}{1+\sigma^2}\frac{1}{\mathrm{snr}^{1-r}}$, where $\sigma$ is the standard deviation of the cross gains. This implies that this scheme shows a better scaling behavior than the orthogonal frequency division scheme. In the same line, the scaling of the outage probability in the interference free scenario is also analyzed which is proved to be $8(1-\gamma)\frac{1}{\mathrm{snr}^{2-r}}$, where $\gamma$ is the Euler's constant.

## 5.2 Future Work

This study can be extended in a number of ways. In the following we suggest some natural open problems.

1) In the general $K$-user scenario with arbitrary number of sub-bands, the behavior of the achievable rate at the point $\rho$ can be better characterized. In other words, what is the probability that $\rho = 0$ is not a point of local maximum infinitely often?

2) It is a problem of considerable interest to prove for the general scenario if a certain behavior of the achievable rate in terms of $\rho$ can be reflected into the behavior of the outage probability in the strict sense or at least for a highly probable set of realizations of the channel coefficients.

3) How about the scaling behavior of the outage probability in the user case with an arbitrary number of sub-bands?

4) In the present work, the complexity of the receiver is a concern. What is the scaling of the outage probability if the receiver employs more complex strategies for decoding interference?

# APPENDICES

## Appendix A

In order to calculate the determinant of the matrix $\Gamma$, we derive a recursive relation for the determinant of the sequence of matrices define below:

$$\Gamma(n) \triangleq \frac{\text{snr}}{N} H_{2,1} C(\rho) H_{2,1}^{\dagger} + I_n \tag{1}$$

with $H_{2,1} \triangleq \text{diag}(h_{2,1}^{(1)}, \cdots, h_{2,1}^{(n)})$ and $[C(\rho)]_{i,j} = 1 + (\rho - 1)\mathbf{1}(i \neq j)$.
With this notation, it can be easily observed that $\Gamma = \Gamma(N)$. Now we give a recursive relation for $\det(\Gamma(n))$ as follows

$$\begin{aligned}
\det(\Gamma(n)) &= \begin{vmatrix}
1 + \frac{\text{snr}}{N}|h_{2,1}^{(1)}|^2 & \cdots & \rho\frac{\text{snr}}{N}h_{2,1}^{(1)}h_{2,1}^{(n-1)^*} & \rho\frac{\text{snr}}{N}h_{2,1}^{(1)}h_{2,1}^{(n)^*} \\
\rho\frac{\text{snr}}{N}h_{2,1}^{(2)}h_{2,1}^{(1)^*} & \cdots & \rho\frac{\text{snr}}{N}h_{2,1}^{(2)}h_{2,1}^{(n-1)^*} & \rho\frac{\text{snr}}{N}h_{2,1}^{(2)}h_{2,1}^{(n)^*} \\
\vdots & \ddots & \vdots & \vdots \\
\rho\frac{\text{snr}}{N}h_{2,1}^{(n-1)}h_{2,1}^{(1)^*} & \cdots & 1 + \frac{\text{snr}}{N}|h_{2,1}^{(n-1)}|^2 & \rho\frac{\text{snr}}{N}h_{2,1}^{(n-1)}h_{2,1}^{(n)^*} \\
\rho\frac{\text{snr}}{N}h_{2,1}^{(n)}h_{2,1}^{(1)^*} & \cdots & \rho\frac{\text{snr}}{N}h_{2,1}^{(n)}h_{2,1}^{(n-1)^*} & 1 + \frac{\text{snr}}{N}|h_{2,1}^{(n)}|^2
\end{vmatrix} \\
&\overset{(a)}{=} \begin{vmatrix}
1 + \frac{\text{snr}}{N}|h_{2,1}^{(1)}|^2 & \cdots & \rho\frac{\text{snr}}{N}h_{2,1}^{(1)}h_{2,1}^{(n-1)^*} & 0 \\
\rho\frac{\text{snr}}{N}h_{2,1}^{(2)}h_{2,1}^{(1)^*} & \cdots & \rho\frac{\text{snr}}{N}h_{2,1}^{(2)}h_{2,1}^{(n-1)^*} & 0 \\
\vdots & \ddots & \vdots & \vdots \\
\rho\frac{\text{snr}}{N}h_{2,1}^{(n-1)}h_{2,1}^{(1)^*} & \cdots & 1 + \frac{\text{snr}}{N}|h_{2,1}^{(n-1)}|^2 & (\rho - 1)\frac{\text{snr}}{N}h_{2,1}^{(n-1)}h_{2,1}^{(n)^*} - \frac{h_{2,1}^{(n)^*}}{h_{2,1}^{(n-1)^*}} \\
\rho\frac{\text{snr}}{N}h_{2,1}^{(n)}h_{2,1}^{(1)^*} & \cdots & \rho\frac{\text{snr}}{N}h_{2,1}^{(n)}h_{2,1}^{(n-1)^*} & 1 + (1 - \rho)\frac{\text{snr}}{N}|h_{2,1}^{(n)}|^2
\end{vmatrix}
\end{aligned} \tag{2}$$

Where (a) holds since adding a multiple of $n - 1^{th}$ column of $\Gamma(n)$ to the $n^{th}$ column does not change the determinant. Expanding the determinant along the last column, we have

$$\det(\Gamma(n)) = (1 + (1-\rho)\frac{\text{snr}}{N}|h_{2,1}^{(n)}|^2)\det(\Gamma(n-1))$$
$$+((\rho-1)\frac{\text{snr}}{N}h_{2,1}^{(n-1)}h_{2,1}^{(n)^*} - \frac{h_{2,1}^{(n)^*}}{h_{2,1}^{(n-1)^*}})\begin{vmatrix} 1+\frac{\text{snr}}{N}|h_{2,1}^{(1)}|^2 & \cdots & \rho\frac{\text{snr}}{N}h_{2,1}^{(1)}h_{2,1}^{(n-1)^*} \\ \vdots & \ddots & \vdots \\ \rho\frac{\text{snr}}{N}h_{2,1}^{(n-2)}h_{2,1}^{(1)^*} & \cdots & \rho\frac{\text{snr}}{N}h_{2,1}^{(n-2)}h_{2,1}^{(n-1)^*} \\ \rho\frac{\text{snr}}{N}h_{2,1}^{(n)}h_{2,1}^{(1)^*} & \cdots & \rho\frac{\text{snr}}{N}h_{2,1}^{(n)}h_{2,1}^{(n-1)^*} \end{vmatrix} \quad (3)$$

To compute the last determinant in (3), it is enough to deduce the last row with factor $\frac{h_{21}^i}{h_{21}^n}$ from $i^{th}$ row for all $1 \le i \le n-2$ and make the matrix upper triangular. Doing so, we have

$$\det(\Gamma(n)) = (1+(1-\rho)\frac{\text{snr}}{N}|h_{2,1}^{(n)}|^2)\det(\Gamma(n-1)) + \rho\frac{\text{snr}}{N}|h_{2,1}^{(n)}|^2\prod_{i=1}^{n-1} 1+(1-\rho)\frac{\text{snr}}{N}|h_{2,1}^{(n)}|^2 \quad (4)$$

Therefore, noting that $\det(\Gamma(2)) = (1+|h_{2,1}^{(1)}|^2)(1+|h_{2,1}^{(2)}|^2) - |(\rho\frac{\text{snr}}{N})h_{2,1}^{(1)}h_{2,1}^{(2)}|^2$, one can recursively compute $\det(\Gamma(N))$.

# Appendix B

It is well-known [32] that for any square matrix $A$ with positive determinant, $\frac{\partial \log \det A}{\partial [A]_{i,j}} = [A^{-1}]_{i,j}$. Then

$$\begin{aligned}
\frac{\partial}{\partial \rho} R_k(\mathrm{snr}, \mathsf{H}_k, \rho) &= \frac{1}{2} \sum_{k=1}^{K} \frac{\partial}{\partial \rho} \log \frac{\det \Omega_k}{\det \Gamma_k} \\
&= \frac{1}{2} \sum_{k=1}^{K} \left( \frac{\partial}{\partial \rho} \log \det \Omega_k - \frac{\partial}{\partial \rho} \log \det \Gamma_k \right). \qquad (5)
\end{aligned}$$

But,

$$\begin{aligned}
\frac{\partial \log \det \Omega_k}{\partial \rho} &= \sum_{m=1}^{N} \sum_{n=1}^{N} \frac{\partial \log \det \Omega_k}{\partial [\Omega_k]_{m,n}} \frac{\partial [\Omega_k]_{m,n}}{\partial \rho} \\
&= \frac{\mathrm{snr}}{N} \sum_{m=1}^{N} \sum_{n=1}^{N} [\Omega_k^{-1}]_{m,n} g_k^{(m,n)} (1 - \delta_{m,n}) \\
&= \frac{2\mathrm{snr}}{N} \sum_{1 \le m < n \le N} [\Omega_k^{-1}]_{m,n} g_k^{(m,n)} \qquad (6)
\end{aligned}$$

and similarly,

$$\frac{\partial \log \det \Gamma_k}{\partial \rho} = \frac{2\mathrm{snr}}{N} \sum_{1 \le m < n \le N} [\Gamma_k^{-1}]_{m,n} \widetilde{g}_k^{(m,n)}. \qquad (7)$$

This verifies the expression for $\frac{\partial}{\partial \rho} R_1(\mathrm{snr}, \mathsf{H}_1, \rho)$ by noting (6) and (7) and letting $k = 1$.

# Appendix C

As shown in appendix A,

$$\frac{\partial}{\partial\rho}R_k(\mathsf{snr},\mathsf{H}_k,\rho)=\frac{\mathsf{snr}}{N}\sum_{1\le u<v\le U}\Big([\Omega_k^{-1}]_{m,n}\,g_k^{(m,n)}-[\Gamma_k^{-1}]_{m,n}\widetilde{g}_k^{(m,n)}\Big).$$

For any invertible square matrix $A$ whose elements are functions of a parameter $a$, $\frac{\partial A^{-1}}{\partial a}=-A^{-1}\frac{\partial A}{\partial a}A^{-1}$. Hence,

$$\frac{\partial\Omega_k^{-1}}{\partial\rho}=-\Omega_k^{-1}\frac{\partial\Omega_k}{\partial\rho}\Omega_k^{-1},\quad \frac{\partial\Gamma_k^{-1}}{\partial\rho}=-\Gamma_k^{-1}\frac{\partial\Gamma_k}{\partial\rho}\Gamma_k^{-1},\tag{8}$$

for any $1\le k\le K$. However,

$$\begin{aligned}\frac{\partial\Omega_k}{\partial\rho} &= \frac{\partial}{\partial\rho}\left(\frac{\text{snr}}{N}\sum_{l=1}^{K}H_{l,k}C(\rho)H_{l,k}^{\text{t}}+I_N\right)\\ &= \frac{\text{snr}}{N}\sum_{l=1}^{K}H_{l,k}\frac{\partial C(\rho)}{\partial\rho}H_{l,k}^{\text{t}}\\ &= \frac{\text{snr}}{N}\sum_{l=1}^{K}H_{l,k}\left(1_{N\times1}1_{N\times1}^{\text{t}}-I_N\right)H_{l,k}^{\text{t}},\end{aligned}\tag{9}$$

where the last step is by the fact that $\frac{\partial C(\rho)}{\partial\rho}$ is a $N\times N$ matrix such that any element on its main diagonal is 0 and any of its off-diagonal elements is 1, i.e., $\frac{\partial C(\rho)}{\partial\rho}=1_{N\times1}1_{N\times1}^{\text{t}}-I_N$. By the same token,

$$\frac{\partial\Gamma_k}{\partial\rho}=\frac{\text{snr}}{N}\sum_{\substack{l=1\\ l\neq k}}^{K}H_{l,k}\left(1_{N\times1}1_{N\times1}^{\text{t}}-I_N\right)H_{l,k}^{\text{t}}.\tag{10}$$

By (8) and (9),

$$\begin{aligned}\frac{\partial[\Omega_k^{-1}]_{m,n}}{\partial\rho}\bigg|_{\rho=0} &= -\frac{1}{\left[\Omega_k\big|_{\rho=0}\right]_{m,m}\left[\Omega_k\big|_{\rho=0}\right]_{n,n}}\left[\frac{\partial\Omega_k}{\partial\rho}\right]_{m,n}\\ &= -\frac{\frac{\text{snr}}{N}g_k^{(m,n)}(1-\delta_{m,n})}{\left(1+\frac{\text{snr}}{N}g_k^{(m,m)}\right)\left(1+\frac{\text{snr}}{N}g_k^{(n,n)}\right)}.\end{aligned}\tag{11}$$

Similarly, by (8) and (10),

$$\frac{\partial[\Gamma_k^{-1}]_{m,n}}{\partial\rho}\Big|_{\rho=0} = -\frac{\frac{\mathrm{snr}}{N}\widetilde{g}_k^{(m,n)}(1-\delta_{m,n})}{\left(1+\frac{\mathrm{snr}}{N}\widetilde{g}_k^{(m,m)}\right)\left(1+\frac{\mathrm{snr}}{N}\widetilde{g}_k^{(n,n)}\right)}. \tag{12}$$

Finally, (11) and (12) yield

$$\begin{aligned}
&\frac{\partial\frac{\partial}{\partial\rho}R_k(\mathsf{snr},\mathsf{H}_k,\rho)}{\partial\rho}\Big|_{\rho=0} = \\
&\frac{\mathsf{snr}}{N}\sum_{1\le m<n\le N}\left(\frac{\frac{\mathrm{snr}}{N}|\widetilde{g}_k^{(m,n)}|^2}{\left(1+\frac{\mathrm{snr}}{N}\widetilde{g}_k^{(m,m)}\right)\left(1+\frac{\mathrm{snr}}{N}\widetilde{g}_k^{(n,n)}\right)}\right. \\
&\left.-\frac{\frac{\mathrm{snr}}{N}|g_k^{(m,n)}|^2}{\left(1+\frac{\mathrm{snr}}{N}g_k^{(m,m)}\right)\left(1+\frac{\mathrm{snr}}{N}g_k^{(n,n)}\right)}\right).
\end{aligned} \tag{13}$$

Letting $k=1$, the expression for the second order derivative $\frac{\partial^2}{\partial\rho^2}R_1(\mathsf{H}_1;0)$ is derived.

# Appendix D

In this case, for each pair of transmitter and receivers the channel gains over different sub-bands are the same; that is for any $1 \leq l, k \leq K$,

$$h_{l,k}^{(m)} = h_{l,k}^{(n)} \triangleq h_{l,k} \quad 1 \leq m, n \leq N.$$

This implies that

$$\widetilde{g}_1^{(m,n)} = \sum_{\substack{l=1 \\ l \neq k}}^{K} |h_{l,k}|^2 \triangleq \widetilde{g}_1 \quad \text{and} \quad g^{(m,n)} = \sum_{l=1}^{K} |h_{l,k}|^2 \triangleq g_1 \quad 1 \leq m, n \leq N.$$

Therefore,

$$\begin{aligned}
&\frac{\partial^2}{\partial \rho^2} R_1(\mathsf{H}_1; 0) \\
&\sum_{1 \leq m < n \leq N} \left( \frac{|\frac{\text{snr}}{N} \widetilde{g}_1^{(m,n)}|^2}{\left(1 + \frac{\text{snr}}{N} \widetilde{g}_1^{(m,m)}\right)\left(1 + \frac{\text{snr}}{N} \widetilde{g}_1^{(n,n)}\right)} - \frac{|\frac{\text{snr}}{N} g_1^{(m,n)}|^2}{\left(1 + \frac{\text{snr}}{N} g_1^{(m,m)}\right)\left(1 + \frac{\text{snr}}{N} g_1^{(n,n)}\right)} \right) \\
&= \frac{(N-1)(N-2)}{2} \left( \frac{|\frac{\text{snr}}{N} \widetilde{g}_1|^2}{\left(1 + \frac{\text{snr}}{N} \widetilde{g}_1\right)\left(1 + \frac{\text{snr}}{N} \widetilde{g}_1\right)} - \frac{|\frac{\text{snr}}{N} g_1|^2}{\left(1 + \frac{\text{snr}}{N} g_1\right)\left(1 + \frac{\text{snr}}{N} g_1\right)} \right) \\
&= \frac{(N-1)(N-2)}{2} \left( \frac{\frac{\text{snr}}{N} \widetilde{g}_1}{1 + \frac{\text{snr}}{N} \widetilde{g}_1} + \frac{\frac{\text{snr}}{N} g_1}{1 + \frac{\text{snr}}{N} g_1} \right) \left( \frac{\frac{\text{snr}}{N} \widetilde{g}_1}{1 + \frac{\text{snr}}{N} \widetilde{g}_1} - \frac{\frac{\text{snr}}{N} g_1}{1 + \frac{\text{snr}}{N} g_1} \right) \\
&= \frac{(N-1)(N-2)}{2} \left( \frac{\frac{\text{snr}}{N} \widetilde{g}_1}{1 + \frac{\text{snr}}{N} \widetilde{g}_1} + \frac{\frac{\text{snr}}{N} g_1}{1 + \frac{\text{snr}}{N} g_1} \right) \left( \frac{-\frac{\text{snr}}{N} |h_{11}|^2}{(1 + \frac{\text{snr}}{N} \widetilde{g}_1)(1 + \frac{\text{snr}}{N} g_1)} \right) < 0 \qquad (14)
\end{aligned}$$

As the second derivative is negative, the result is implied.

# Appendix E

Denote the channel gain from the transmitter to the receiver over the $n^{th}$ sub band with $h_{(n)}$. In this case, due to absence of interfering signals, the achievable rate at the receiver can be written as follows

$$\begin{aligned}
R_1(\mathsf{H}_1;\rho) &= \frac{N}{2}\log\left(\frac{\text{snr}}{N}\det(\begin{bmatrix} \frac{N}{\text{snr}}+|h_{(1)}|^2 & \rho h_{(1)}h_{(2)} & \cdots & \rho h_{(1)}h_{(N)} \\ \rho h_{(2)}h_{(1)} & \frac{N}{\text{snr}}+|h_{(1)}|^2 & \cdots & \rho h_{(2)}h_{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ \rho h_{(N)}h_{(1)} & \rho h_{(N)}h_{(2)} & \cdots & \frac{N}{\text{snr}}+|h_{(N)}|^2 \end{bmatrix})\right) \\
&\overset{(a)}{\leq} \frac{1}{2}\log\left(\prod_{i=1}^{N-1}(1+\frac{\text{snr}}{N}|h_{(i)}|^2)\right) \qquad (15)
\end{aligned}$$

Where (a) Hadamard's inequality for the determinant of positive definite matrices. In the last inequality, equality occurs when the matrix is diagonal which can be achieved when $\rho = 0$. This completes the proof.

# Appendix F

Define
$$\Omega_1 := \{\omega \in \Omega : X_n(\omega) \to X(\omega)\}$$
and
$$\Omega_2 := \{\omega \in \Omega : X(\omega) > 0\}.$$
Let
$$A := \{\omega \in \Omega : X_n(\omega) \leq 0 \ \text{ infinitely often}\}.$$
Clearly, $A \cap (\Omega_1 \cap \Omega_2) = \emptyset$. Hence, as $\mathbb{P}(\Omega_1 \cap \Omega_2) = 1$, we get $\mathbb{P}(A) = 0$.
The fact that $\mathbb{P}(X_n \leq 0) \to 0$ is a byproduct of the previous observation. One may write
$$A = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{\omega \in \Omega : X_m(\omega) \leq 0\}.$$
Define
$$A_n := \bigcup_{m=n}^{\infty} \{\omega \in \Omega : X_m(\omega) \leq 0\}$$
The events $An$, $n \in \mathbb{N}$ are decreasing, i.e., $A_{n+1} \subset A_n$. Therefore, by "continuity" of $\mathbb{P}(\cdot)$ along nested chains of events,
$$\lim_{n\to\infty} \mathbb{P}(A_n) = 0.$$
Note that $\{\omega \in \Omega : X_n(\omega) \leq 0\} \subset A_n$. Hence,
$$\lim_{n\to\infty} \mathbb{P}(X_n \leq 0) \leq \lim_{n\to\infty} \mathbb{P}(A_n) = 0.$$

# Appendix G

In this appendix, we give bounds to or precisely calculate some definite integrals which appear useful throughout the work.

$$\int_0^b \int_0^a \frac{ws}{(w+s)^3} \mathrm{d}s\mathrm{d}w = \int_0^b w \left[\frac{-(s+\frac{w}{2})}{(w+s)^2}\right]_{s=0}^a \mathrm{d}w =$$
$$\int_0^b (-\frac{aw+\frac{w^2}{2}}{(w+a)^2} + \frac{1}{2})\mathrm{d}w = \left[-\frac{a^2}{2(w+a)}\right]_{w=0}^b = \frac{ab}{2(a+b)} \tag{16}$$

$$\int_0^b \int_0^a \frac{ws}{(w+s)^2} \mathrm{d}s\mathrm{d}w = \int_0^b w \left[\frac{w}{(w+s)} + \ln(w+s)\right]_{s=0}^a \mathrm{d}w =$$
$$\int_0^b -\frac{wa}{w+a}\mathrm{d}w + \int_0^b w\ln(1+\frac{a}{w})\mathrm{d}w = -a\left[w - a\ln(w+a)\right]_{w=0}^b + \int_0^b w\ln(1+\frac{a}{w})\mathrm{d}w =$$
$$-ab + a^2\ln(1+\frac{b}{a}) + \int_0^b w\ln(1+\frac{a}{w})\mathrm{d}w \le -ab + a^2\ln(1+\frac{b}{a}) + \frac{ab}{t_0}\ln(a+t_0), \tag{17}$$

where $t_0$ is the solution of the equation $\ln(1+t) = \frac{t}{1+t}$.

$$\int_0^b \int_0^a \frac{ws}{w+s} \mathrm{d}s\mathrm{d}w = \int_0^b w\left[a - w\ln(w+s)\right]_{s=0}^a \mathrm{d}w =$$
$$\int_0^b aw - w^2\ln(1+\frac{a}{w})\mathrm{d}w \le \int_0^b aw\mathrm{d}w = \frac{ab^2}{2} \tag{18}$$

$$\int_0^a \int_0^b \int_0^a e^{-u-\frac{uw}{s}} \frac{u^2 w}{s^2} \mathbf{1}_{\{u>s\}} \mathrm{d}s\mathrm{d}w\mathrm{d}u = \int_0^a \int_0^b \int_0^{\min\{u,a\}} e^{-u-\frac{uw}{s}} \frac{u^2 w}{s^2} \mathrm{d}s\mathrm{d}w\mathrm{d}u =$$
$$\int_0^a \int_0^b \int_0^u e^{-u-\frac{uw}{s}} \frac{u^2 w}{s^2} \mathrm{d}s\mathrm{d}w\mathrm{d}u = \int_0^a \int_0^b ue^{-u} \int_{\frac{1}{u}}^{\infty} uwe^{-uwt}\mathrm{d}t\mathrm{d}w\mathrm{d}u =$$
$$\int_0^a ue^{-u} \int_0^b e^{-w}\mathrm{d}w\mathrm{d}u = (1-e^{-b})\int_0^a ue^{-u}\mathrm{d}u = (1-e^{-b})(1-(a+1)e^{-a}) \tag{19}$$

# Appendix H

$$
\begin{aligned}
& \mathbb{E}\left(\left(\frac{\boldsymbol{A}_{11}\boldsymbol{A}_{22} - |\boldsymbol{A}_{12}|^2}{\sigma^2 \boldsymbol{A}_{22}^2}\right)^{\ell}\right) \\
= \quad & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \left(\frac{xy - z^2 - t^2}{\sigma^2 y^2}\right)^{\ell} f_{\boldsymbol{A}_{11},\boldsymbol{A}_{22},\Re(\boldsymbol{A}_{12}),\Im(\boldsymbol{A}_{12})}(x,y,z,t)\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t \\
= \quad & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \frac{1}{\pi}\left(\frac{xy - z^2 - t^2}{\sigma^2 y^2}\right)^{\ell} e^{-x-y}\mathbf{1}_{\{xy > z^2 + t^2\}}\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t \\
= \quad & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{\frac{z^2+t^2}{y}}^{\infty} \frac{1}{\pi}\left(\frac{xy - z^2 - t^2}{\sigma^2 y^2}\right)^{\ell} e^{-x-y}\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t \\
= \quad & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{e^{-\frac{z^2+t^2}{y}}e^{-y}}{\pi\sigma^{2\ell}y^{\ell}} \int_{0}^{\infty} x^{\ell}e^{-x}\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t \\
\overset{(a)}{=} \quad & \int_{0}^{\infty} \frac{\ell! e^{-y}}{\pi\sigma^{2\ell}y^{\ell}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{z^2+t^2}{y}}\mathrm{d}z\mathrm{d}t\mathrm{d}y \\
\overset{(b)}{=} \quad & \int_{0}^{\infty} \frac{\ell! e^{-y}}{\pi\sigma^{2\ell}y^{\ell}} (\sqrt{\pi y})^2\mathrm{d}y \\
= \quad & \frac{\ell!}{\sigma^{2\ell}} \int_{0}^{\infty} y^{1-\ell}e^{-y}\mathrm{d}y
\end{aligned} \tag{20}
$$

For an integer $\ell$, the last integral exists if and only if $\ell < 2$.

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