

Pick Interpolation and the Distance Formula

by

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Abstract

The classical interpolation theorem for the open complex unit disk, due to Nevanlinna and Pick in the early 20th century, gives an elegant criterion for the solvability of the problem as an eigenvalue problem. In the 1960s, Sarason reformulated problems of this type firmly in the language of operator theoretic function theory. This thesis will explore connections between interpolation problems on various domains (both single and several complex variables) with the viewpoint that Sarason's work suggests.

In Chapter 1, some essential preliminaries on bounded operators on Hilbert space and the functionals that act on them will be presented, with an eye on the various ways distances can be computed between operators and a certain type of ideal. The various topologies one may define on $\mathcal{B}(\mathcal{H})$ will play a prominent role in this development. Chapter 2 will introduce the concept of a reproducing kernel Hilbert space, and a distinguished operator algebra that we associate to such spaces known as the *multiplier algebra*. The various operator theoretic properties that multiplier algebras enjoy will be presented, with a particular emphasis on their invariant subspace lattices and the connection to distance formulae.

In Chapter 3, the Nevanlinna-Pick problem will be investigated in general for any reproducing kernel Hilbert space, with the basic heuristic for distance formulae being presented. Chapter 4 will treat a large class of reproducing kernel Hilbert spaces associated to measure spaces, where a Pick-like theorem will be established for many members of this class. This approach will closely follow similar results in the literature, including recent treatments by McCullough and Cole-Lewis-Wermer.

Reproducing kernel Hilbert spaces where the analogue of the Nevanlinna-Pick theorem holds are particularly nice. In Chapter 5, the operator theory of these so-called *complete Pick spaces* will be developed, and used to tackle certain interpolation problems where additional constraints are imposed on the solution. A non-commutative view of interpolation will be presented, with the non-commutative analytic Toeplitz algebra of Popescu and Davidson-Pitts playing a prominent role.

It is often useful to consider reproducing kernel Hilbert spaces which arise as natural products of other spaces. The Hardy space of the polydisk is the prime example of this. A general commutative and non-commutative view of such spaces will be presented in Chapter 6, using the left regular representation of higher-rank graphs, first introduced by Kribs-Power. A recent factorization theorem of Bercovici will be applied to these algebras, from which a Pick-type theorem may be deduced. The operator-valued Pick problem for these spaces will also be discussed.

In Chapter 7, the various tools developed in this thesis will be applied to two related problems, known as the Douglas problem and the Toeplitz corona problem.

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For B.B.B.M., N.H. and N.Z.

Contents

1	Introduction and preliminaries	1
1.1	Pick interpolation problems	3
1.2	A primer on dual algebra theory	6
1.2.1	Topologies on $\mathcal{B}(\mathcal{H})$	7
1.2.2	Pre-dual factorization properties	8
1.2.3	The infinite ampliation	10
2	Reproducing kernel Hilbert spaces and their multipliers	13
2.1	Evaluation operators and positive semidefinite kernels	13
2.2	Multipliers and multiplication operators	18
2.3	Full kernels, topologies on $\mathcal{M}(\mathcal{H})$ and reflexivity	20
2.4	Multipliers between spaces	25
3	The distance formula	27
3.1	Invariant subspaces for multiplier algebras	30
3.2	Scalar-valued kernels	34
3.3	Algebras with property $\mathbb{A}_1(1)$	40
3.4	Examples	44
3.4.1	Isometric functional calculus	46
3.5	The infinite ampliation of a multiplier algebra and matrix-valued interpolation	47
4	Reproducing kernel Hilbert spaces associated to measures	51
4.1	Spaces associated to measures	51
4.2	Hardy spaces	54
4.3	Bergman spaces	56
5	Complete Pick kernels	59
5.1	The noncommutative analytic Toeplitz algebra	60
5.2	Invariant subspaces of a complete Pick space	63
5.3	Matrix-valued interpolation II	66
6	Interpolation in product domains	71
6.1	Products of kernels	71
6.2	The polydisk	73
6.3	Left regular representations of other semigroups	74

6.4	Matrix-valued interpolation III	77
7	Douglas-type factorization and the Toeplitz corona problem	81
7.1	Majorization and factorization of multipliers	81
7.2	A more general Douglas problem	84
7.3	Tangential interpolation and a distance formula	85
7.4	Tangential families for complete Pick spaces	89
7.5	The Toeplitz corona problem	91
7.6	Tangential interpolation and the Toeplitz corona problem for other algebras	93
7.6.1	Bergman spaces	94
7.6.2	Products of complete Pick kernels	95
	Appendix - Finite dimensional multiplier algebras	97
A.1	A numerical estimate	97
	Bibliography	98

Chapter 1

Introduction and preliminaries

The modern study of bounded operators on Hilbert space owes much of its success to the classical theory of holomorphic functions. The research areas encapsulated by function theory and operator theory are broad and numerous, and have been an invaluable source of mathematical research for more than half a century. The overarching theme in this thesis has its roots in the seminal work of Arne Beurling [20], where the invariant subspaces of the unilateral shift operator are completely characterized in the language of function theory. Beurling's work demonstrated that function theory can lead to new insight in the theory of operators. The opposite perspective is equally valuable. That is, when can we employ operator theoretic facts to solve problems in function theory? A key motivating example for this viewpoint is Donald Sarason's contribution to the theory of interpolation using Hilbert space operators [62].

The classical Nevanlinna-Pick interpolation theorem gives an elegant criterion for solving interpolation problems on the unit disk: given z_1, \dots, z_n in the complex unit disk \mathbb{D} and complex numbers w_1, \dots, w_n , there is a holomorphic function f on \mathbb{D} which satisfies $f(z_i) = w_i$ and $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| \leq 1$ if and only if the *Pick matrix*

$$\mathbb{P} := \left[\frac{1 - w_i \overline{w_j}}{1 - z_i \overline{z_j}} \right]$$

has nonnegative eigenvalues. This condition is easy to verify, and is remarkable in its sole dependence on the choice of interpolation data. Pick was the first to prove this theorem in 1916, with Nevanlinna in 1919 furnishing a different approach which parametrized the solutions. In light of this historical precedent, we will refer to a general existential interpolation problem as a *Pick* problem. Parametrizing the solutions to a Pick problem (when they exist) will be called a *Nevanlinna* problem. As the title suggests, we will restrict our attention to Pick problems in this thesis.

Let $H^\infty(\mathbb{D})$ denote the Banach algebra of bounded analytic functions on \mathbb{D} , and $H^2(\mathbb{D})$ be the *Hardy space* of holomorphic functions on \mathbb{D} with square summable Taylor coefficients. Functions in $H^\infty(\mathbb{D})$ induce a natural multiplication map on $H^2(\mathbb{D})$, and the operator algebra induced by this representation is generated by the coordinate function z . Sarason's novel approach to the Nevanlinna-Pick theorem essentially starts with the observation that if g is any function in $H^\infty(\mathbb{D})$ such that $g(z_i) = 0$ for $i = 1, \dots, n$ and

f is *any* function (for example, a polynomial) which satisfies $f(z_i) = w_i$, then $f + g$ is also a solution. In other words, if \mathcal{I} is the ideal of functions in H^∞ which vanish at the z_i , then there is a solution to the interpolation problem of norm at most 1 if and only if $\text{dist}(f, \mathcal{I}) := \inf_{g \in \mathcal{I}} \|f - g\|_\infty \leq 1$. Sarason then uses a prototype of the commutant lifting theorem of Sz. Nagy and Foiaş to show that this distance is at most 1 if and only if \mathbb{P} is positive semidefinite.

The Hilbert space $H^2(\mathbb{D})$ belongs to a remarkable class known as reproducing kernel Hilbert spaces. These are Hilbert spaces of complex-valued functions which have the property that point evaluation is continuous with respect to the ambient Hilbert space norm. The common L^2 spaces are typically not reproducing kernel Hilbert space, but many spaces of analytic functions are. Any reproducing kernel Hilbert space admits a *multiplier algebra* of complex-valued functions which pointwise multiply the space back into itself. The multiplier algebra of $H^2(\mathbb{D})$ is $H^\infty(\mathbb{D})$, and so we may regard the classical Pick problem as a problem pertaining to the Hardy space. The Pick problem for *any* reproducing kernel Hilbert space may be formulated precisely, and unfortunately does not have the same elegant solution as the classical theorem in most cases. This thesis will explore many examples of reproducing kernel Hilbert spaces where, even though the analogue of the classical Nevanlinna-Pick theorem does not hold, one still obtains a solvability criterion in terms of the spectrum of matrices depending on the data set. Our approach utilizes tools from operator theory developed in the last 30 years, employing the so-called *dual algebra* techniques of Brown, Apostol, Bercovici, Foiaş, Pearcy and many others.

The remainder of this chapter will consist of a brief overview of the “Pick-type” theorems for other reproducing kernel Hilbert spaces in the literature in Section 1.1, and a short introduction to dual algebra techniques in Section 1.2. Chapter 2 gives a broad introduction to reproducing kernel Hilbert space, as well as their vector-valued analogues. Various operator theoretic properties of multiplier algebras are introduced. While none are surprising or difficult, there seems to be a void in the literature for facts pertaining to general multiplier algebras.

In Chapter 3, a very general approach to the Pick problem will be described using purely operator algebraic techniques based on the results contained in the paper of the author and Davidson [24], which appears in the journal *Integral Equations and Operator Theory*. A sufficient condition for solvability of the Pick problem is stated in the language of dual algebras. These results will be extended to a large class of holomorphic reproducing kernel Hilbert spaces in Chapter 4, where spaces whose norm arises from a measure are examined. The contents of Chapter 4 make up the body of the author’s short paper [37], which has been accepted for publication in *Proceedings of the American Mathematical Society*.

In Chapter 5, the notion of a Complete Pick space is discussed. These are precisely the spaces for which the analogue of the Nevanlinna-Pick theorem (and its matrix-valued versions) hold. These spaces have been totally classified, and their multiplier algebras always unitarily embed into the multiplier algebra of the *Drury-Arveson* space H_d^2 . Drury-Arveson space has recently distinguished itself as the correct multivariable analogue of

Hardy space, and has been the subject of intense research in the last decade due to its connection to important problems in multivariable operator theory. While the Pick problem for Drury-Arveson space is resolved, we will instead examine *constrained* Pick interpolation, where one imposes additional restrictions on the solution function. Many of these problems arise by considering the Pick problem for strict subalgebras of the multiplier algebra. We will employ the tools developed in Chapter 3 to address constrained interpolation problems for complete Pick spaces.

Given multiple reproducing kernel Hilbert spaces, their Hilbert space tensor product naturally forms a reproducing kernel Hilbert space on the product of their domains. Chapter 6 will examine the Pick problem for product space of this type, where each factor is itself a complete Pick space. The simplest case of this is the Hardy space of the polydisk $H^2(\mathbb{D}^d)$, whose Pick problem has only been resolved in the last 15 years. We obtain a sufficient solvability criterion for these spaces by ascending to non-commutative extensions of their multiplier algebras.

Finally, in Chapter 7, concepts related to the Pick problem known as tangential interpolation and the Toeplitz corona problem will be discussed. Dual algebraic methods will again be employed to solve these problems for large classes of multiplier algebras and their subalgebras. The content of this chapter is largely derived from the paper of the author and Raghupathi [38], which has been accepted for publication by the *Indiana University Mathematics Journal*.

1.1 Pick interpolation problems

It is profitable to transfer some of the ideas used in Sarason's proof of the Nevanlinna-Pick theorem to a general Hilbert space setting. A Hilbert space \mathcal{H} of \mathbb{C} -valued functions on X is a *reproducing kernel Hilbert space* if point evaluations are continuous, i.e. the map sending f to $f(x)$ is bounded for every x in X . Consequently, for every $x \in X$, there is a function k_x , called the *reproducing kernel at x* , which satisfies $f(x) = \langle f, k_x \rangle$ for $f \in \mathcal{H}$. There is an associated positive definite kernel on $X \times X$ that is given by $k(x, y) := \langle k_y, k_x \rangle$.

A *multiplier* φ of \mathcal{H} is a function on X with the property that φf is in \mathcal{H} for every f in \mathcal{H} . Each multiplier φ induces a bounded multiplication operator M_φ on \mathcal{H} . The *multiplier algebra* of \mathcal{H} , denoted as $\mathcal{M}(\mathcal{H})$, is the operator algebra consisting of all such M_φ . The adjoints of multiplication operators are characterized by the fundamental identity $M_\varphi^* k_x = \overline{\varphi(x)} k_x$. There is a convenient way to test whether or not a multiplier is contractive. The condition $\|M_\varphi\| \leq 1$ is equivalent to the non-negativity of $\langle (I - M_\varphi M_\varphi^*) h, h \rangle$ for any h in \mathcal{H} . By choosing h to be a finite span of kernel functions, it is elementary to verify that the contractivity of M_φ implies that the Pick matrix $[(1 - \varphi(x_i) \overline{\varphi(x_j)}) k(x_i, x_j)]$ is positive semidefinite. Thus, the positivity of the Pick matrix always follows from the existence of a contractive solution to the given interpolation problem. We say that the multiplier algebra $\mathcal{M}(\mathcal{H})$ has the *Pick property* if this condition is also sufficient.

Unfortunately, there is an abundance of reproducing kernel Hilbert spaces which fail to have the Pick property. The Bergman space of the disk, $L_a^2(\mathbb{D})$, is the Hilbert space of holomorphic functions on \mathbb{D} which are square integrable with respect to planar Lebesgue

measure on \mathbb{D} . Bergman space is easily seen to have the same multipliers as Hardy space, and it follows from basic linear algebra that it cannot have the Pick property (see Section 3.1).

Let Ω be a multiply connected region in \mathbb{C} ; that is, a connected region whose boundary consists of $g + 1$ disjoint, closed analytic Jordan curves (here g is the genus on Ω). In [1], Abrahamse proved the following Pick interpolation theorem for Ω .

Theorem 1.1.1 (Abrahamse). *Suppose Ω is a multiply connected region of genus g , that $z_1, \dots, z_n \in \Omega$ and w_1, \dots, w_n are complex numbers. There is a holomorphic function f on Ω such that $f(z_i) = w_i$ for $i = 1, \dots, n$ and $\sup_{z \in \Omega} |f(z)| \leq 1$ if and only if the matrices*

$$[(1 - w_i \bar{w}_j)k^\alpha(z_i, z_j)]_{i,j=1}^n$$

have nonnegative eigenvalues for every α , where $\{k^\alpha\}_{\alpha \in \mathbb{T}^g}$ is a family of positive semidefinite functions on $\Omega \times \Omega$ indexed by the torus \mathbb{T}^g .

In this setting, positivity of a single Pick matrix was not sufficient to guarantee the existence of a solution. If one fixes the values z_1, \dots, z_n and only allows the target values w_1, \dots, w_n to vary, Federov and Vinnikov demonstrated that for an annular region Ω , only two kernels are required in order to guarantee a solution [32]. As we will see, there are many results in Pick interpolation theory whose solvability criteria are given by the positivity of an infinite family of matrices. Results analogous to the Federov-Vinnikov result are extremely valuable and equally rare.

Cole, Lewis and Wermer [22] approached the Pick problem in substantial generality by considering the problem for any uniform algebra. If X is a compact Hausdorff space, a uniform algebra \mathcal{A} is a subalgebra of $C(X)$ which contains the identity function and separates the points in X . Let $M(\mathcal{A})$ denote the Gelfand spectrum of \mathcal{A} . The *weak* Pick problem for \mathcal{A} seeks solutions to the Pick problem for \mathcal{A} with data $x_1, \dots, x_n \in M(\mathcal{A})$ and $w_1, \dots, w_n \in \mathbb{C}$, where the solution has norm at most $1 + \epsilon$ for fixed $\epsilon > 0$. A necessary and sufficient condition for this is if

$$[(1 - w_i \bar{w}_j)k^\mu(x_i, x_j)]_{i,j=1}^n$$

are positive semidefinite for a certain family of kernels $\{k^\mu\}$. These kernels arise from Borel measures μ on $M(\mathcal{A})$ which satisfy the property that evaluation at the points x_1, \dots, x_n are bounded linear functionals on the space $\overline{\mathcal{A}}^{L^2(\mu)}$. This is an enormous and complicated family of kernels which does not admit an elegant parametrization as in Abrahamse's result. Nonetheless, the generality in which the Cole-Lewis-Wermer approach applies is admirable, and indeed a great source of motivation for the results in this thesis.

Recently, Davidson, Paulsen, Raghupathi and Singh studied the classical Pick problem with the additional constraint that the solution f must satisfy the condition $f'(0) = 0$ [25]. Equivalently, the solution f is required to belong to the subalgebra

$$H_1^\infty := \{f \in H^\infty(\mathbb{D}) : f'(0) = 0\}$$

of $H^\infty(\mathbb{D})$. Beurling's theorem for the shift was used to characterize the invariant subspace

lattice for this algebra, which in turn parametrize a family of kernels k^α on the disk, where α ranges over the unit sphere in \mathbb{C}^2 . The condition

$$[(1 - w_i \overline{w_j})k^\alpha(z, z_j)] \geq 0$$

for each α is a necessary and sufficient criterion for the solvability of the Pick problem for H_1^∞ . Additionally, the authors demonstrated that essentially all α are required if the data points are freely varied. This was the first appearance of a so-called *constrained* Pick interpolation theorem, and we will see that it can be generalized substantially in many settings.

By replacing the complex scalars w_1, \dots, w_n with complex matrices W_1, \dots, W_n , we can also consider matrix-valued interpolation. The classical theorem still holds with the criterion that the matrix

$$\left[\frac{1 - W_i W_j^*}{1 - z_i \overline{z_j}} \right]_{i,j=1}^n,$$

is positive semidefinite, where the solution function is a matrix with entries in H^∞ . A kernel for which the classical theorem holds for all matrix valued functions is called a complete Pick kernel. Results of McCullough [44, 45] and Quiggin [55], building on (unpublished) work by Agler, provide a classification of complete Pick kernels. Agler and McCarthy [3] showed that every space with a complete Pick kernel is equivalent to the restriction of Drury-Arveson to some subset of the unit ball in \mathbb{C}^n .

In [2], Agler and McCarthy also carried out a deep analysis of the Pick problem for the bidisk \mathbb{D}^2 . The solvability condition in their case is a factorization theorem for bounded, holomorphic functions on the bidisk.

Theorem 1.1.2. *Suppose $z_1, \dots, z_n \in \mathbb{D}^2$ and $w_1, \dots, w_n \in \mathbb{C}$. The following are equivalent.*

1. *There is a function $\varphi \in H^\infty(\mathbb{D}^2)$ such that $\|\varphi\|_\infty \leq 1$ and $\varphi(z_i) = w_i$ $i = 1, \dots, n$.*
2. *There are \mathbb{C} -valued positive semidefinite kernels Γ_1 and Γ_2 on $\mathbb{D}^2 \times \mathbb{D}^2$ such that*

$$1 - w_i \overline{w_j} = (1 - z_i^1 \overline{z_j^1})\Gamma_1(z_i, z_j) + (1 - z_i^2 \overline{z_j^2})\Gamma_2(z_i, z_j).$$

Condition (2) in the above theorem is equivalent to the positivity of the Pick matrices

$$[(1 - w_i \overline{w_j})k(z_i, z_j)]$$

for every positive semidefinite kernel k on $\mathbb{D}^2 \times \mathbb{D}^2$ whose associated reproducing kernel Hilbert space has $H^\infty(\mathbb{D}^2)$ as its multiplier algebra.

Lastly, we mention the work of McCullough [46], which was the first to explicitly demonstrate the connection between dual algebra theory and Pick interpolation. While his methods ultimately differ from what we will see here, the basic essence is very close. The idea is that given the ideal of functions in a multiplier algebra that vanish on the

interpolation nodes, \mathcal{I}_E , we wish to find a formula for the distance

$$\text{dist}(f, \mathcal{I}_E)$$

which can be expressed in terms of the data obtained from Pick matrices. A consequence of the Hahn-Banach theorem says that this distance is approximately achieved by $\omega(f)$, where ω is some weak-* continuous functional which annihilates \mathcal{I}_E . By recognizing that the multiplier algebra is an operator algebra, and invoking trace duality, we find that

$$\text{dist}(f, \mathcal{I}_E) = \sup \|(M_f^K)^*|_{\mathcal{M}_K}\|$$

where the supremum is taken over all trace class operators K , M_f^K is a certain multiplication operator on a Hilbert space determined by K , and \mathcal{M}_K is the span of certain kernel functions on K . This distance formula then determines a solvability criterion in terms of a family of Pick matrices indexed by the trace class. McCullough then goes on to show that in concrete examples, this family can be naturally reduced in size in order to get more tractable conditions.

Thus, there are many examples in the literature where a single matricial condition must be replaced with infinitely many. From a computational perspective, this creates an obvious problem. When seeking a solution, it likely will not be the case that the simultaneous positivity of an infinite family of matrices will be verified algorithmically (though it does provide a reasonable approach to verifying that *no* solution exists, for one need only find one kernel for which the associated Pick matrix is not positive semidefinite). Nonetheless, the solvability of the Pick problem is an end in itself. The study of the problem has resulted in a better understanding of many spaces of functions and their multiplier algebras. We shall see in the remainder of this thesis that there is a tremendous amount carryover from other areas of operator theory and function theory to Pick interpolation. Many of the results we will employ reflect decades of intense research in operator theory. This thesis primarily serves to illuminate this connection.

1.2 A primer on dual algebra theory

In this section, we provide a very brief overview of the results from dual algebra theory that we will require in later chapters. The study of dual algebras was largely initiated by Scott Brown [21], and has become a tremendously powerful tool in operator theory, particularly with respect to the invariant subspace problem for large classes of operators. Much of the work in this thesis relies on very deep and difficult results in dual algebra theory, proved over the span of two decades by many individuals.

For elementary topics in operator theory with a specific eye towards topics like reflexivity and weak-* functionals, Conway [23] is an effective source. More advanced topics pertaining to dual algebra theory are covered nearly completely in the manuscript of Bercovici, Foias and Pearcy [18].

1.2.1 Topologies on $\mathcal{B}(\mathcal{H})$

Suppose \mathcal{H} is a complex Hilbert space, and $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on \mathcal{H} . Then $\mathcal{B}(\mathcal{H})$ is a dual Banach space, and its predual $\mathcal{B}(\mathcal{H})_*$ is canonically identified with the ideal of trace class operators $\mathcal{C}_1(\mathcal{H})$. This is accomplished via the bilinear pairing

$$(T, K) := \text{tr}(TK)$$

where $T \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{C}_1(\mathcal{H})$. The corresponding weak-* topology induced on $\mathcal{B}(\mathcal{H})$ is the weakest topology such that the functionals $T \mapsto \text{tr}(TK)$ are continuous for all $K \in \mathcal{C}_1(\mathcal{H})$. Given two vectors $f, g \in \mathcal{H}$, define the rank 1 operator fg^* by $fg^*h := \langle h, g \rangle f$. Any trace-class operator K can be decomposed as $K = \sum_{i=1}^{\infty} f_i g_i^*$ [23, Theorem 18.13], where $\{f_i\}$ and $\{g_i\}$ are square-summable sequences in \mathcal{H} which satisfy $\sum_{i=1}^{\infty} \|f_i\|^2 = \sum_{i=1}^{\infty} \|g_i\|^2 = \|K\|_1$ (here $\|K\|_1$ is the trace norm of K). Conversely, any operator of the form

$$K = \sum_{i=1}^{\infty} f_i g_i^*$$

is easily seen to be trace class. The weak-* continuous functional on $\mathcal{B}(\mathcal{H})$ induced by K is then given by

$$\omega(T) = \text{tr}(TK) = \sum_{i=1}^{\infty} \langle T f_i, g_i \rangle.$$

Consequently, a net $\{T_\lambda\}_{\lambda \in \Lambda}$ of operators in $\mathcal{B}(\mathcal{H})$ converges to T in the weak-* topology if and only if

$$\sum_{i=1}^{\infty} \langle T_\lambda f_i, g_i \rangle \rightarrow \sum_{i=1}^{\infty} \langle T f_i, g_i \rangle$$

for every pair of sequences $\{f_i\}, \{g_i\} \subset \mathcal{H}$ satisfying $\sum_{i=1}^{\infty} \|f_i\|^2 < \infty$ and $\sum_{i=1}^{\infty} \|g_i\|^2 < \infty$.

Given two distinct Hilbert spaces \mathcal{H} and \mathcal{K} , we may also identify a predual structure on the operator space $\mathcal{B}(\mathcal{H}, \mathcal{K})$. This is done by identifying $\mathcal{B}(\mathcal{H}, \mathcal{K})$ with the subspace of $\mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ determined by the map

$$T \mapsto \begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix},$$

where the above operator matrix is written with respect to the external direct sum $\mathcal{H} \oplus \mathcal{K}$. It follows immediately that a net of operators $\{T_\lambda\} \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$ converges to T in the weak-* topology on $\mathcal{B}(\mathcal{H}, \mathcal{K})$ if and only if

$$\sum_{i=1}^{\infty} \langle T_\lambda f_i, g_i \rangle_{\mathcal{K}} \rightarrow \sum_{i=1}^{\infty} \langle T f_i, g_i \rangle_{\mathcal{K}}$$

for every pair of sequences $\{f_i\} \subset \mathcal{H}$ and $\{g_i\} \subset \mathcal{K}$ satisfying $\sum_{i=1}^{\infty} \|f_i\|^2 < \infty$ and $\sum_{i=1}^{\infty} \|g_i\|^2 < \infty$.

The weak operator topology (WOT) on $\mathcal{B}(\mathcal{H})$ is the topology determined by the

convergence criterion

$$T_\lambda \xrightarrow{\text{WOT}} T \text{ if and only if } \langle T_\lambda f, g \rangle \rightarrow \langle T f, g \rangle \text{ for every } f, g \in \mathcal{H}.$$

Similarly, the strong operator topology (SOT) is given by

$$T_\lambda \xrightarrow{\text{SOT}} T \text{ if and only if } \|T_\lambda f\| \rightarrow \|T f\| \text{ for every } f \in \mathcal{H}.$$

The WOT and SOT on $\mathcal{B}(\mathcal{H}, \mathcal{K})$ are defined analogously. The following basic result characterizes the SOT and WOT continuous linear functionals on $\mathcal{B}(\mathcal{H})$ [23, Proposition 8.1].

Proposition 1.2.1. *Suppose ω is a linear function on $\mathcal{B}(\mathcal{H})$. The following are equivalent.*

1. ω is WOT continuous.
2. ω is SOT continuous.
3. There is a natural number n and vectors $f_1, \dots, f_n, g_1, \dots, g_n \in \mathcal{H}$ so that

$$\omega(T) = \sum_{i=1}^n \langle T f_i, g_i \rangle \text{ for all } T \in \mathcal{B}(\mathcal{H}).$$

In particular, a WOT continuous function is automatically weak-* continuous. Given two vectors $f, g \in \mathcal{H}$, the rank 1 operator $f g^*$ induces the functional ω determined by

$$\omega(T) := \text{tr}(T f g^*) = \langle T f, g \rangle.$$

We shall call such an ω a *vector functional*.

1.2.2 Predual factorization properties

Definition 1.2.2. A *dual subspace* \mathcal{S} is a weak-* closed subspace of $\mathcal{B}(\mathcal{H})$. A *dual algebra* is a unital weak-* closed subalgebra (not necessarily self-adjoint) of $\mathcal{B}(\mathcal{H})$.

Given a dual subspace \mathcal{S} , we let \mathcal{S}_\perp denote the *pre-annihilator* of \mathcal{S} :

$$\mathcal{S}_\perp := \{\omega \in \mathcal{B}(\mathcal{H})_* : \omega|_{\mathcal{S}} = 0\}.$$

Of course, \mathcal{S}_\perp can be identified with those trace-class operators K which satisfy $\text{tr}(SK) = 0$ for all $S \in \mathcal{S}$. We may then form the quotient space

$$Q_{\mathcal{S}} := \mathcal{B}(\mathcal{H})_* / \mathcal{S}_\perp,$$

and by an application of the Hahn-Banach theorem, it can be verified that $Q_{\mathcal{S}}^* \cong \mathcal{S}$ via the pairing

$$(S, \omega + \mathcal{S}_\perp) = \omega(S)$$

for $\omega \in \mathcal{B}(\mathcal{H})_*$ and $S \in \mathcal{S}$. Thus \mathcal{S} has a predual, and we let \mathcal{S}_* denote the specific one constructed above (preduals are not necessarily unique).

We are principally interested in dual subspaces whose preduals have strong factorization properties. The following definitions capture the idea that if \mathcal{S} is sufficiently ‘small’, then vector functionals completely exhaust its predual.

Definition 1.2.3. Suppose m and n are natural numbers and that \mathcal{S} is a dual subspace of $\mathcal{B}(\mathcal{H})$. The subspace \mathcal{S} is said to have *property* $\mathbb{A}_{m,n}$ if for every $m \times n$ array $[\omega_{ij}]$ of functionals $\omega_{ij} \in \mathcal{S}_*$, there are vectors $f_1, \dots, f_m, g_1, \dots, g_n \in \mathcal{H}$ so that

$$\omega_{ij}(S) = \langle Sf_i, g_j \rangle$$

for every $1 \leq i \leq m$, $1 \leq j \leq n$ and $S \in \mathcal{S}$. If, in addition, there is a constant $r \geq 1$ such that for every $\epsilon > 0$, the vectors f_i, g_j above may be chosen so that

$$\|f_i\|^2 < r \sum_{j=1}^n \|\omega_{ij}\| + \epsilon, \quad 1 \leq i \leq m,$$

$$\|g_j\|^2 < r \sum_{i=1}^m \|\omega_{ij}\| + \epsilon, \quad 1 \leq j \leq n,$$

we say that \mathcal{S} has *property* $\mathbb{A}_{m,n}(r)$. If $m = n$, we say that \mathcal{S} has property \mathbb{A}_n and property $\mathbb{A}_n(r)$, respectively.

It follows immediately that if \mathcal{S} has property $\mathbb{A}_{m,n}(r)$, then it also has the properties $\mathbb{A}_{m,n}$ and $\mathbb{A}_{m',n'}(r')$ for every $m' < m$, $n' < n$ and $r' < r$. We will also require infinite versions of these properties.

Definition 1.2.4. We say that \mathcal{S} has property \mathbb{A}_{\aleph_0} if for every infinite array $[\omega_{ij}]_{i,j=1}^{\infty}$ of functionals in \mathcal{S}_* there are infinite sequences of vectors $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ such that $\omega_{ij}(S) = \langle Sf_i, g_j \rangle$ for every i, j and $S \in \mathcal{S}$. For $r \geq 1$, we say that \mathcal{S} has property $\mathbb{A}_{\aleph_0}(r)$ if for every infinite array $[\omega_{ij}]_{i,j=1}^{\infty}$ of functionals in \mathcal{S}_* with *summable rows and columns* and every $\epsilon > 0$, the vector sequences $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ may be chosen so that

$$\|f_i\|^2 < r \sum_{j=1}^{\infty} \|\omega_{ij}\| + \epsilon, \quad 1 \leq i \leq \infty,$$

$$\|g_i\|^2 < r \sum_{j=1}^{\infty} \|\omega_{ij}\| + \epsilon, \quad 1 \leq i \leq \infty.$$

For natural numbers m and n , the properties $\mathbb{A}_{\aleph_0,n}$, \mathbb{A}_{m,\aleph_0} , $\mathbb{A}_{\aleph_0,n}(r)$ and $\mathbb{A}_{m,\aleph_0}(r)$ are defined analogously.

Remark 1.2.5. Based on the above definitions alone, it is not clear that property $\mathbb{A}_{\aleph_0}(r)$ implies property \mathbb{A}_{\aleph_0} . This is indeed the case, and its proof can be found in [18, Theorem 3.7].

All of the properties $\mathbb{A}_{m,n}$ and $\mathbb{A}_{m,n}(r)$ are hereditary for weak-* closed subspaces. That is, if \mathcal{S} has one of the properties, then any weak-* closed subspace \mathcal{T} of \mathcal{S} has it as

well. This fact follows immediately by taking a weak-* continuous Hahn-Banach extension of any weak-* continuous functional on \mathcal{T} and extending to all of \mathcal{S} (which can be done with at most an ϵ increase in norm).

Given countable cardinals m and n , we let $M_{m,n}(\mathcal{S})$ denote the operator space of $\mathcal{B}(\mathcal{H}^{(m)}, \mathcal{H}^{(n)})$ consisting of $m \times n$ matrices with entries in \mathcal{S} . If m or n is infinite, then we interpret $M_{m,n}(\mathcal{S})$ as $\mathcal{S} \bar{\otimes} \mathcal{B}(\ell_m^2, \ell_n^2)$, where $\bar{\otimes}$ denotes the *spatial* tensor product. The subspace $M_{m,n}(\mathcal{S})$ is closed in the weak-* topology, and hence has a predual induced by the trace class in $\mathcal{B}(\ell_m^2, \ell_n^2)$. There is a natural identification [5, Section 1] between $M_{m,n}(\mathcal{S})_*$ and $M_{m,n}(\mathcal{S}_*)$, where the latter is simply the collection of formal arrays of weak-* continuous functionals on \mathcal{S} . This identification is explicitly given by the pairing

$$([\omega_{ij}], [S_{ij}]) = \sum_{i,j} \omega_{ij}(S_{ij}).$$

This identification enables us to formulate the following proposition.

Proposition 1.2.6. *Suppose \mathcal{S} is a dual subspace of $\mathcal{B}(\mathcal{H})$ and that m and n are countable cardinals. Then $M_{m,n}(\mathcal{S})$ has property \mathbb{A}_1 if and only if \mathcal{S} has property $\mathbb{A}_{m,n}$. Moreover, if \mathcal{S} has property $\mathbb{A}_{m,n}(r)$ where m and n are finite, then $M_{m,n}(\mathcal{S})$ has property $\mathbb{A}_1(r(mn)^{1/4})$.*

There is a much stronger property defined on dual subspaces which we now introduce.

Definition 1.2.7. Suppose \mathcal{S} is a dual subspace of $\mathcal{B}(\mathcal{H})$. Then \mathcal{S} is said to have *property* $X(0,1)$ if, for each $\omega \in \mathcal{S}_*$, there are sequences of vectors $\{f_i\}, \{g_i\} \subset \mathcal{H}$ so that

1. $\omega(S) = \langle Sf_i, g_i \rangle$ for every i .
2. $\lim_{i \rightarrow \infty} \|f_i\| \|g_i\| = \|\omega\|$.
3. $\lim_{i \rightarrow \infty} \|kf_i^*\| = \lim_{i \rightarrow \infty} \|g_i k^*\| = 0$ for every $k \in \mathcal{H}$.

It immediately follows from the above definition that property $X(0,1)$ implies property $\mathbb{A}_1(1)$. In fact, much more is true. The proof of the following result can be found in [18, Theorem 3.6].

Theorem 1.2.8 (Apostol-Bercovici-Foias-Pearcy). *Suppose \mathcal{S} has property $X(0,1)$. Then \mathcal{S} has property $\mathbb{A}_{\aleph_0}(1)$.*

Lastly, we present a more recent result of Bercovici which provides an elegant sufficient condition for a dual algebra \mathcal{A} to have property $X(0,1)$ [17].

Theorem 1.2.9 (Bercovici). *Suppose \mathcal{A} is a dual algebra such that the commutant \mathcal{A}' contains two isometries with pairwise orthogonal ranges. Then \mathcal{A} has property $X(0,1)$.*

1.2.3 The infinite ampliation

Given a dual algebra \mathcal{A} on a Hilbert space \mathcal{H} and a countable cardinal k , the k -th ampliation $\mathcal{A}^{(k)}$ is an isometric representation of \mathcal{A} on $H^{(k)}$, the direct sum of k copies of \mathcal{H} , with elements $A^{(k)} = A \oplus \cdots \oplus A$, the direct sum of k copies of A . The algebras \mathcal{A} and

$\mathcal{A}^{(k)}$ are obviously isometrically isomorphic, and the preduals \mathcal{A}_* and $\mathcal{A}_*^{(k)}$ are isometrically isomorphic as well. A rank k functional on \mathcal{A} converts to a rank one functional on $\mathcal{A}^{(k)}$ since

$$\sum_{i=1}^k \langle Af_i, g_i \rangle = \langle A^{(k)}f, g \rangle$$

where $f = (f_1, \dots, f_k)$ and $g = (g_1, \dots, g_k)$ in $H^{(k)}$. The precise formulation is given as follows [23, Proposition 21.3].

Proposition 1.2.10. *If $K \in \mathcal{C}_1(\mathcal{H})$ is trace-class with rank at most k , then there are vectors $f, g \in \mathcal{H}^{(k)}$ such that $\|f\|\|g\| = \|K\|_1$ and*

$$\langle T^{(k)}f, g \rangle_{\mathcal{H}^{(k)}} = \text{tr}(TK)$$

for all T in $\mathcal{B}(\mathcal{H})$.

In other words, the infinite ampliation $\mathcal{A}^{(\infty)}$ always has property $\mathbb{A}_1(1)$ and even has property $X(0, 1)$ [18, Proposition 3.9]. In particular, given any dual algebra \mathcal{A} , there is always a completely isometric representation π of \mathcal{A} such that $\pi(\mathcal{A})$ has property $X(0, 1)$. This relatively mundane fact will prove to be extremely useful in Chapter 3.

Chapter 2

Reproducing kernel Hilbert spaces and their multipliers

2.1 Evaluation operators and positive semidefinite kernels

We begin with the fundamental definition for this thesis.

Definition 2.1.1. Suppose \mathcal{L} is any Hilbert space and X a set. A *reproducing kernel Hilbert space* $\mathcal{H} := \mathcal{H}(X, E, \mathcal{L})$ is a Hilbert space of \mathcal{L} -valued functions on X with the property that point evaluation is continuous. In other words, the linear maps

$$E_x : \mathcal{H} \rightarrow \mathcal{L}$$

$$E_x f = f(x)$$

are bounded for every x in X .

It is natural to require that a function f in \mathcal{H} which has the property that $f(x) = 0$ for every x in X be the zero function. This amounts to requiring that the set

$$\text{span}_{x \in X} \text{Ran } E_x^*$$

is norm dense in \mathcal{H} , since $f(x) = 0$ is equivalent to the orthogonality of f and the range of E_x^* . This also implies that the operators E_x are always non-zero. Henceforth, any reproducing kernel Hilbert space \mathcal{H} will be assumed to have this property. As we shall see in the next section, functions in \mathcal{H} of the form $E_x^* u$ are extremely useful for calculations.

Definition 2.1.2. A $\mathcal{B}(\mathcal{L})$ -valued kernel on $X \times X$ is a function $K : X \times X \rightarrow \mathcal{B}(\mathcal{L})$ satisfying

1. $K(x, x) \neq 0$ for every x in X .
2. For every u_1, \dots, u_n in \mathcal{L} we have

$$\sum_{i,j=1}^n \langle K(x_i, x_j) u_j, u_i \rangle_{\mathcal{L}} \geq 0.$$

The following proposition is immediate from the above definition.

Proposition 2.1.3. *Given a reproducing kernel Hilbert space $\mathcal{H}(X, \mathcal{L})$, the function*

$$K(x, y) := E_y E_x^*$$

is a $\mathcal{B}(\mathcal{L})$ -valued kernel function on $X \times X$.

What is more surprising is that there is essentially a converse to the above proposition. That is, given a $\mathcal{B}(\mathcal{L})$ -valued kernel on $X \times X$, there is a straightforward way of constructing a reproducing kernel Hilbert space on X uniquely determined by K . The following theorem is originally due to Moore [49] for scalar-valued kernels.

Theorem 2.1.4. *(Moore) Suppose K is a $\mathcal{B}(\mathcal{L})$ -valued kernel function on $X \times X$. There is a unique reproducing kernel Hilbert space \mathcal{H} of \mathcal{L} -valued functions on X such that $K(x, y) = E_y E_x^*$.*

Proof. Let K be a $\mathcal{B}(\mathcal{L})$ -valued kernel on $X \times X$. For each $x \in X$ and $u \in \mathcal{L}$, define an \mathcal{L} -valued function on X , denoted $E_x^* u$, as follows:

$$(E_x^* u)(y) := K(x, y)u.$$

Let \mathcal{V} denote the formal vector space obtained by taking finite linear combinations of the functions $E_x^* u$. What follows now is a standard Hilbert space construction. First define a sesquilinear form on \mathcal{V} :

$$\left\langle \sum a_i E_{x_i}^* u_i, \sum b_j E_{y_j}^* v_j \right\rangle := \sum a_i \bar{b}_j \langle K(y_j, x_i) u_i, v_j \rangle_{\mathcal{L}}.$$

Let \mathcal{N} denote the set of elements f in \mathcal{V} with the property that $\langle f, f \rangle = 0$. It is routine to verify that \mathcal{N} is a subspace, and that the induced sesquilinear form on \mathcal{V}/\mathcal{N} is both well-defined and a true inner product. Let \mathcal{H} denote the Hilbert space completion of the inner product space \mathcal{V}/\mathcal{N} .

It remains to show that \mathcal{H} is a reproducing kernel Hilbert space on X . As the notation suggests, E_x^* induces a bounded linear operator from \mathcal{L} into \mathcal{H} . To see this, we use the closed graph theorem. If u_n is a sequence of vectors in \mathcal{L} converging to 0 and $E_x^* u_n$ is converging to some equivalence class $[g]$ in \mathcal{H} , we have

$$\|[g]\|_{\mathcal{H}}^2 = \lim \langle E_x^* u_n, E_x^* u_n \rangle_{\mathcal{H}} \stackrel{\text{def}}{=} \lim \langle K(x, x) u_n, u_n \rangle_{\mathcal{L}} = 0.$$

For an equivalence class of functions $[f]$ in \mathcal{H} , we now define the evaluation map as

$$f(x) := E_x[f] \in \mathcal{L},$$

which are bounded by the above calculation. By construction, we have $E_x E_y^* = K(y, x)$, as desired. Note that this also implies $E_x \neq 0$ since $K(x, x) \neq 0$ by assumption.

For the uniqueness claim, suppose Q is another positive $\mathcal{B}(\mathcal{L})$ -valued kernel on $X \times X$ and \mathcal{H}' is the resulting Hilbert space obtained by the above construction. If $Q(y, x)$ factors

as $F_x F_y^*$, it is routine to verify that the map

$$\sum a_i E_x^* u_i \mapsto \sum a_i F_x^* u_i$$

extends to a unitary $U : \mathcal{H} \rightarrow \mathcal{H}'$ which satisfies $Uf(x) = f(x)$ for every $x \in X$. \square

Theorem 2.1.4 together with Proposition 2.1.3 imply the following.

Corollary 2.1.5. *There is a bijective correspondance between \mathcal{L} -valued reproducing kernel Hilbert spaces on X and $\mathcal{B}(\mathcal{L})$ -valued kernels on $X \times X$.*

Also implicit in Theorem 2.1.4 is that every $\mathcal{B}(\mathcal{L})$ -valued kernel admits a factorization in some Hilbert space.

Corollary 2.1.6. *Suppose K is a $\mathcal{B}(\mathcal{L})$ -valued kernel. There is a Hilbert space \mathcal{H} such that, for each $x \in X$, there are operators $E_x \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ so that $K(x, y) = E_y E_x^*$.*

In light of the above results, we can now refer to *the* kernel associated to \mathcal{H} and we write $\mathcal{H} = \mathcal{H}(K)$ when differentiation among spaces is required. Given a reproducing kernel Hilbert space \mathcal{H} , it is often useful to know if a given \mathcal{L} -valued function on the set X actually belongs to \mathcal{H} . If $f, g \in \mathcal{H}$, we use the notation fg^* to refer to the rank 1 operator determined by $h \mapsto \langle h, g \rangle_{\mathcal{H}} f$.

Proposition 2.1.7. *Suppose \mathcal{H} is a reproducing kernel Hilbert space of \mathcal{L} -valued functions on a set X and suppose f is any \mathcal{L} -valued function on X . Then f belongs to \mathcal{H} if and only if there is a constant $C > 0$ so that the operator matrices*

$$\left[C^2 E_{x_i} E_{x_j}^* - f(x_i) f(x_j)^* \right]_{i,j=1}^n$$

are positive semidefinite for every finite subset $\{x_1, \dots, x_n\} \subset X$.

Proof. For a function f in \mathcal{H} , let $C \geq \|f\|^2$. Given $x_1, \dots, x_n \in X$ and u_1, \dots, u_n in \mathcal{L} , we have

$$\left\langle (C^2 I_{\mathcal{H}} - f(x_i) f(x_j)^*) \sum_{i=1}^n E_{x_i}^* u_i, \sum_{j=1}^n E_{x_j}^* u_j \right\rangle_{\mathcal{H}} \geq 0,$$

which is easily seen to be equivalent to the positivity of the desired operator matrix.

Conversely, assume that all such operator matrices are positive and fix a finite subset $F = \{x_1, \dots, x_n\}$. The positivity of the matrix associated to F is equivalent to the inequality

$$C^2 \begin{bmatrix} E_{x_1} \\ \vdots \\ E_{x_n} \end{bmatrix} \begin{bmatrix} E_{x_1}^* & \dots & E_{x_n}^* \end{bmatrix} \geq \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix} \begin{bmatrix} f(x_1)^* & \dots & f(x_n)^* \end{bmatrix}.$$

By the Douglas factorization lemma (see section 7.2), there is an operator $T_F \in \mathcal{B}(\mathcal{H}, \mathbb{C}) \cong \mathcal{H}$ so that $\|T_F\| \leq C$ and

$$T_F \begin{bmatrix} E_{x_1}^* & \dots & E_{x_n}^* \end{bmatrix} = \begin{bmatrix} f(x_1)^* & \dots & f(x_n)^* \end{bmatrix}.$$

Let $g_F \in \mathcal{H}$ be the representing vector for T_F . It follows that $g_F(x_i) = (g_F^* E_{x_i}^*)^* = (f(x_i)^*)^* = f(x_i)$ for each $x_i \in F$. Now let g be any weak cluster point of $\{g_F : F \subset X \text{ finite}\}$ in \mathcal{H} . A moment's thought reveals that for any finite subset F of X and any $x_i \in F$, we have $g_F(x_i) = f(x_i)$. Therefore $f(x) = g(x)$ for every $x \in X$ and $f \in \mathcal{H}$. \square

Remark 2.1.8. The infimum of all constants C in the above theorem is the Hilbert space norm of f .

If \mathcal{H} is a reproducing kernel Hilbert space of \mathbb{C} -valued functions on a set X , we will always use a lower case Roman letter for the kernel function on $X \times X$. Since E_x is now a bounded functional on \mathcal{H} , there is some representing function k_x in \mathcal{H} so that $\langle f, k_x \rangle_{\mathcal{H}} = f(x)$ for any f in \mathcal{H} . Consequently, the kernel function for \mathcal{H} takes the convenient form $k(x, y) = \langle k_y, k_x \rangle_{\mathcal{H}}$. We call k_x the *reproducing kernel at x for \mathcal{H}* . To finish off this section, we will briefly describe a number of important (and a few mundane) examples.

Example 2.1.9. Except in trivial cases (see the next example), L^2 spaces are typically not reproducing kernel Hilbert spaces.

Example 2.1.10. Suppose \mathbb{A} is some index set and $\mathcal{H} = \ell^2(\mathbb{A})$ with the canonical orthonormal basis $\{e_a\}_{a \in \mathbb{A}}$. Then \mathcal{H} is a reproducing kernel Hilbert space on \mathbb{A} , and the reproducing kernel at a is given by $k_a = e_a$. In some sense, \mathcal{H} is an uninteresting example, since it is the Hilbert space direct sum of one-dimensional spaces (which are always trivially reproducing kernel Hilbert spaces).

Example 2.1.11 (Bergman space). Let dA denote the area Lebesgue measure on the complex unit disk \mathbb{D} and define the Bergman space on \mathbb{D} as follows:

$$L_a^2(\mathbb{D}) := \left\{ f \in \text{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty \right\}.$$

Then $L_a^2(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D}, A)$ and is a reproducing kernel Hilbert space on \mathbb{D} whose kernel function is given by $k(z, w) = \frac{1}{(1 - z\bar{w})^2}$. The basic properties of Bergman spaces of arbitrary regions in \mathbb{C}^d will be discussed in Chapter 4.

Example 2.1.12 (Hardy space). Define the Hardy space $H^2(\mathbb{D})$ as the set of holomorphic functions on \mathbb{D} with square summable Taylor coefficients about the point $z = 0$. For $f, g \in H^2$, the inner product is given by

$$\langle f, g \rangle_{H^2} = \sum_{k=1}^{\infty} \hat{f}(k) \overline{\hat{g}(k)},$$

where $\hat{f}(n)$ denotes the n^{th} Taylor coefficient of f about the point $z = 0$. A direct computation shows that the kernel function for $H^2(\mathbb{D})$ is given by

$$\frac{1}{1 - z\bar{w}},$$

the *Szegő kernel*. Hardy spaces of \mathbb{D} and other regions will be described in much more detail in Chapter 4.

Example 2.1.13 (Drury-Arveson space). For $1 \leq d \leq \infty$, let

$$\mathbb{B}_d = \{(z_1, \dots, z_d) \in \mathbb{C}^d : |z_1|^2 + \dots + |z_d|^2 < 1\}$$

denote the open unit ball in \mathbb{C}^d (when $d = \infty$, \mathbb{B}_d is to be interpreted as the unit ball in the Hilbert space $\ell^2(\mathbb{N})$). Drury-Arveson space is a space of holomorphic functions on \mathbb{B}_d whose kernel function is given by

$$\frac{1}{1 - \langle z, w \rangle_{\mathbb{C}^d}}.$$

When $d = 1$, Drury-Arveson space is nothing more than Hardy space $H^2(\mathbb{D})$. These spaces are extremely important as they are, essentially, an exhaustive list of those kernels for which Pick's theorem holds. Chapter 5 will be devoted to the study of Drury-Arveson space.

Example 2.1.14 (Sobolev space). The Sobolev space W_1^2 is the set of absolutely continuous functions f on $[0, 1]$ which satisfy $f(0) = f(1) = 0$ and $f' \in L^2[0, 1]$. We may endow $W_{0,1}^2$ with a sesquilinear form

$$\langle f, g \rangle := \int_0^1 f'(x) \overline{g'(x)} dx.$$

This is easily seen to be a true inner product, since the condition $f(0) = f(1) = 0$ ensures that 0 is the only constant function in W_1^2 . Under this inner product, W_1^2 is complete and point evaluations are bounded. The kernel function for this space is given by

$$k(x, y) = \begin{cases} (1-y)x & : x \leq y \\ (1-x)y & : y \leq x. \end{cases}$$

Example 2.1.15. Define a family of Hilbert function spaces of holomorphic functions $\{H_s\}_{s \in \mathbb{R}}$ on \mathbb{D} by declaring the norm

$$\|f\|_s = \sum_{n=0}^{\infty} (n+1)^{-s} |\hat{f}(n)|^2 < \infty$$

where $\hat{f}(n)$ is the n^{th} Taylor coefficient of f at 0. This induces an inner product given by

$$\langle f, g \rangle = \sum_{n=0}^{\infty} (n+1)^{-s} \hat{f}(n) \overline{\hat{g}(n)}.$$

Powers of z form an orthogonal basis for these spaces. The values $s = -1, 0, 1$ are particularly important, as they give us Bergman space, Hardy space and *Dirichlet* space, respectively.

Example 2.1.16 (Segal-Bargmann space). The Segal-Bargmann space is the Hilbert

space of entire functions on \mathbb{C} with kernel $k(z, w) := e^{\bar{z}w}$.

Example 2.1.17. Suppose $\mathcal{H} = \mathcal{H}(k, X)$ is a scalar-valued reproducing kernel Hilbert space. Then for any auxiliary Hilbert space \mathcal{L} , the kernel function $k(\cdot, \cdot) \otimes I_{\mathcal{L}}$ determines an \mathcal{L} -valued reproducing kernel Hilbert space on X .

Example 2.1.18. Suppose $n \geq 1$ is a natural number and let $X = \{1, 2, \dots, n\}$. For any Hilbert space \mathcal{L} , fix operators T_1, \dots, T_n in $\mathcal{B}(\mathcal{L})$ and define an \mathcal{L} -valued kernel function on $X \times X$ by

$$K(i, j) := T_i T_j^*$$

The associated reproducing kernel Hilbert space is given by $\mathcal{H} = \overline{\text{span}(\text{Ran}(T_i^*) : 1 \leq i \leq n)}^{\mathcal{L}}$. When $\mathcal{L} = \mathbb{C}$, the algebra $\mathcal{M}(\mathcal{H})$ is nothing more than the algebra of diagonal $n \times n$ matrices under the action of a similarity. See Appendix A.

2.2 Multipliers and multiplication operators

Given a reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(K)$, we now seek to define a distinguished operator algebra associated to \mathcal{H} .

Definition 2.2.1. Suppose \mathcal{H} is a reproducing kernel Hilbert space and that Φ is a function from X into $\mathcal{B}(\mathcal{L})$. We say that Φ is a *multiplier of \mathcal{H}* if the pointwise product Φf is in \mathcal{H} whenever f is in \mathcal{H} . The \mathbb{C} -algebra of all such multipliers is denoted $\text{Mult}(\mathcal{H})$. If K is the $\mathcal{L} \times \mathcal{L}$ -valued kernel which determines \mathcal{H} , the notation $\text{Mult}(K)$ will also be used for $\text{Mult}(\mathcal{H})$.

Example 2.2.2. As sets, multiplier algebras are typically easy to compute.

1. $\text{Mult}(\ell^2(\mathbb{A})) = \ell^\infty(\mathbb{A})$.
2. $\text{Mult}(H^2(\mathbb{D})) = \text{Mult}(L_a^2(\mathbb{D})) = H^\infty(\mathbb{D})$, the algebra of bounded holomorphic functions on \mathbb{D} (see Chapter 4).
3. Given $T_1, \dots, T_n \in \mathcal{B}(\mathcal{L})$, let $K(i, j) = T_j T_i^*$. The set $\text{Mult}(K)$ consists of tuples of operators $[A_1, \dots, A_n]$ with $A_j \in \mathcal{B}(\mathcal{H})$ such that the subspaces $\overline{\text{Ran } T_i^*}$ are coinvariant for each A_j .
4. Since scalar multipliers are always bounded in the sup norm (see Proposition 2.3.4), the multipliers of the Segal-Bergman space are the constant functions by Liouville's theorem.
5. For Drury-Arveson space, there are bounded, holomorphic functions on \mathbb{B}_d which are not multipliers, i.e. $\text{Mult}(H_d^2) \subsetneq H^\infty(\mathbb{B}_d)$. The details may be found in Arveson's treatise on this space [9].

We will use capital Greek letters for multipliers in most cases. Lower case Greek letters will be used specifically in the scalar case $\mathcal{L} = \mathbb{C}$. Every multiplier Φ induces a natural multiplication operator, denoted M_Φ whose action is given by $M_\Phi f = \Phi f$. To see

that M_Φ is bounded, we again appeal to the closed graph theorem. If f_n is a sequence of functions in \mathcal{H} converging to 0 and $M_\Phi f_n$ is converging to some g in \mathcal{H} , we have

$$g(x) = E_x g = \lim E_x M_\Phi f_n = \lim \Phi(x) f_n(x) = 0,$$

for any x in X , hence $g = 0$. The algebra of all such multiplication operators will be called the *multiplier algebra of \mathcal{H}* and will be denoted either $\mathcal{M}(\mathcal{H})$ or $\mathcal{M}(K)$, depending on the context. Of course, $\mathcal{M}(K)$ is always unital. Unless otherwise specified, the norm of a multiplier Φ will always be taken to be the operator norm of M_Φ . One of the most elementary and useful properties of multiplication operators is their relationship with evaluation operators, namely:

$$E_x M_\Phi = \Phi(x) E_x.$$

In the important case where $\mathcal{L} = \mathbb{C}$, taking adjoints in the above equation implies that the adjoint of a multiplication operator has an abundance of eigenvectors:

$$M_\varphi^* k_x = \overline{\varphi(x)} k_x.$$

Proposition 2.2.3. *Suppose Φ and Ψ are multipliers of a reproducing kernel Hilbert space \mathcal{H} . Then $M_\Phi = M_\Psi$ if and only if $\Phi(x) = \Psi(x)$ for every $x \in X$*

Proof. Use the above calculation combined with the fact that sums of elements of the form $E_x^* u$ are dense in \mathcal{H} . □

The following result is a very useful characterization of multipliers on \mathcal{H} .

Theorem 2.2.4. *Suppose \mathcal{H} is a reproducing kernel Hilbert space and Φ is a $\mathcal{B}(\mathcal{L})$ -valued function on X . Then Φ is a multiplier of \mathcal{H} if and only if there is a constant $C > 0$ such that the operator matrices*

$$[C^2 K(x_i, x_j) - \Phi(x_i) K(x_i, x_j) \Phi(x_j)^*]_{i,j=1}^n$$

are positive for every finite subset $\{x_1, \dots, x_n\}$ of X .

Proof. Suppose Φ is a multiplier and let $C \geq \|M_\Phi\|$. We then have

$$\langle (C^2 I_H - M_\Phi M_\Phi^*) f, f \rangle_{\mathcal{H}} \geq 0$$

for any f in \mathcal{H} . This inequality holds if and only if it holds on a dense subset of \mathcal{H} , and so without loss of generality we may as well assume that $f = \sum_{i=1}^n E_{x_i}^* u_i$ for any $u_i \in \mathcal{L}$ and any finite subset $\{x_1, \dots, x_n\} \subset X$. By the observations made preceding this proposition,

we have

$$\begin{aligned}
0 &\leq \langle (C^2 I_H - M_\Phi M_\Phi^*) f, f \rangle_{\mathcal{H}} \\
&= \sum_{i,j=1}^n \left\langle (C^2 I_H - M_\Phi M_\Phi^*) E_{x_i}^* u_i, E_{x_j}^* u_j \right\rangle_{\mathcal{H}} \\
&= \sum_{i,j=1}^n \left\langle (C^2 E_{x_i}^* u_i, E_{x_j}^* u_j) \right\rangle_{\mathcal{H}} - \sum_{i,j=1}^n \left\langle M_\Phi^* E_{x_i}^* u_i, M_\Phi^* E_{x_j}^* u_j \right\rangle_{\mathcal{H}} \\
&= \sum_{i,j=1}^n \left\langle (C^2 E_{x_i}^* u_i, E_{x_j}^* u_j) \right\rangle_{\mathcal{H}} - \sum_{i,j=1}^n \left\langle E_{x_i}^* \Phi^*(x_i) u_i, E_{x_j}^* \Phi^*(x_j) u_j \right\rangle_{\mathcal{H}} \\
&= \sum_{i,j=1}^n \left\langle (C^2 K(x_i, x_j) u_i, u_j) \right\rangle_{\mathcal{H}} - \sum_{i,j=1}^n \left\langle K(x_i, x_j) \Phi^*(x_j) u_i, \Phi^*(x_i) u_j \right\rangle_{\mathcal{H}} \\
&= \left\langle (C^2 K(x_i, x_j) - \Phi(x_i) K(x_i, x_j) \Phi(x_j)^*) u_i, u_j \right\rangle_{\mathcal{L}}.
\end{aligned}$$

The last line is precisely equivalent to the positivity of the desired operator matrix. Conversely, suppose Φ is a $\mathcal{B}(\mathcal{L})$ -valued function such that all of the given operator matrices are positive. Reversing the above calculation shows that multiplication by Φ induces a bounded linear map on a dense subset of \mathcal{H} , which we extend continuously to get a multiplication operator on all of \mathcal{H} . \square

Theorem 2.2.4 also gives a very useful characterization of those bounded operators on \mathcal{H} which arise as multiplication operators.

Corollary 2.2.5. *Suppose $T \in \mathcal{B}(\mathcal{H})$ is an operator with the property that for every $x \in X$, there is an operator $T_x \in \mathcal{B}(\mathcal{L})$ such that*

$$E_x T = T_x E_x.$$

Then T is a multiplication operator.

Proof. Suppose $T \in \mathcal{B}(\mathcal{H})$ satisfies the given hypothesis. Define a $\mathcal{B}(\mathcal{L})$ -valued function Φ on X by setting $\Phi(x) := T_x$. We claim that Φ is a multiplier and that $T = M_\Phi$. To see this, the calculation in the proof of Theorem 2.2.4 may be followed, and observe that the positivity $C^2 I_{\mathcal{H}} - T T^*$ is equivalent to the positivity of all operator matrices of the form $[C^2 K(x_i, x_j) - \Phi(x_i) K(x_i, x_j) \Phi(x_j)^*]_{i,j=1}^n$. \square

2.3 Full kernels, topologies on $\mathcal{M}(\mathcal{H})$ and reflexivity

We are primarily interested in studying the various properties of $\mathcal{M}(\mathcal{H})$ as an operator algebra.

Definition 2.3.1. A $\mathcal{B}(\mathcal{L})$ -valued kernel K on $X \times X$ is said to be *full* if the evaluation map E_x is surjective for every $x \in X$. A reproducing kernel Hilbert space $\mathcal{H}(K)$ is called full when K is full.

Example 2.3.2. Suppose k is a scalar-valued kernel for a reproducing kernel Hilbert space \mathcal{H} on X . As long as $k_x \neq 0$ for every $x \in X$, the kernel k is obviously full. If Y is the subset of X for which $k_y = 0$ for each $y \in Y$, then it is perhaps more natural to regard \mathcal{H} as a reproducing kernel Hilbert space over $X \setminus Y$, and under this identification k is a full kernel. Consequently, we always assume that a scalar-valued space is full. It could very well be the case that a closed subspace of a full space may fail to be full. This is a very important issue that will be discussed in the next chapter.

Example 2.3.3. For a full scalar kernel k and any $n \geq 1$, the \mathcal{M}_n -valued kernel $k \otimes I_n$ is also full. If \mathcal{L} is an infinite dimensional Hilbert space, then $k \otimes I_{\mathcal{L}}$ is full if $\mathcal{H}(k)$ contains the constant function 1.

A full reproducing kernel Hilbert space has many interesting operator theoretic properties. The first is a useful lower bound on the operator norm of a multiplier.

Proposition 2.3.4. *Suppose \mathcal{H} is a full reproducing kernel Hilbert space and Φ is a multiplier of \mathcal{H} . Then*

$$\|M_{\Phi}\| \geq \sup_{x \in X} \|\Phi(x)\|.$$

Proof. Rescaling if necessary, assume that $\|M_{\Phi}\| = 1$. It follows that $I_{\mathcal{H}} - M_{\Phi}M_{\Phi}^* \geq 0$ and that

$$0 \leq E_x E_x^* - E_x M_{\Phi} M_{\Phi}^* E_x^* = E_x E_x^* - F(x) E_x E_x^* F(x)^*.$$

By the Douglas factorization lemma (see Section 7.2), for each x there is a contraction $C_x \in B(\mathcal{L})$ such that $C_x E_x = F(x) E_x$. Since E_x is surjective, it follows that $F(x) = C_x$ is a contraction. \square

Remark 2.3.5. If the fullness hypothesis of Proposition 2.3.4 is dropped, the stated conclusion can fail for trivial reasons. To see this, suppose X is a singleton, that $\mathcal{H} = \mathcal{L}$, and that $E_x = P$ is a non-trivial projection in $\mathcal{B}(\mathcal{H})$. If T is any contraction acting on \mathcal{H} such that $\text{Ran } T \subset \ker P$, then $PT = 0$. On the other hand, let \tilde{T} be any operator with $\|T\| > 1$ such that $\tilde{T}P = 0$. It follows that T is a multiplier since $PT = \tilde{T}P$, but $\|T\| < \|\tilde{T}\|$.

Remark 2.3.6. Suppose \mathcal{H} is a scalar-valued reproducing kernel Hilbert space on X and there is a Borel measure μ on X so that \mathcal{H} is a closed subspace of $L^2(X, \mu)$. Then $\|M_{\varphi}\| = \sup_{x \in X} |\varphi(x)|$ for every $M_{\varphi} \in \mathcal{M}(\mathcal{H})$. By Proposition 2.3.4, we always have $\|M_{\varphi}\| \geq \sup_{x \in X} |\varphi(x)|$. Conversely, if $f \in \mathcal{H}$, we have

$$\|\varphi f\|^2 = \int_X |\varphi f|^2 d\mu \leq \|\varphi\|_{\infty}^2 \|f\|^2.$$

In particular, the multiplier norm on Bergman spaces $L_a^2(\mathbb{D})$ is always given by the sup norm on \mathbb{D} .

Theorem 2.3.7. *Suppose \mathcal{H} is a reproducing kernel Hilbert space which is separable and full. The multiplier algebra $\mathcal{M}(\mathcal{H})$ is closed in the weak-* topology. In other words, $\mathcal{M}(\mathcal{H})$ is a dual algebra.*

Proof. By the separability of \mathcal{H} and since $\mathcal{M}(\mathcal{H})$ is convex, it suffices to show that $\mathcal{M}(\mathcal{H})$ is sequentially WOT-closed [23, Proposition 20.3]. Suppose Φ_n is a sequence of multipliers such that M_{Φ_n} is WOT-convergent to an operator $T \in \mathcal{B}(\mathcal{H})$. For each $x \in X$, the sequence of vectors $E_x M_{\Phi_n} f = \Phi_n(x)f(x)$ is converging weakly in \mathcal{L} to $E_x T f$. Define a map $\Phi(x)$ from \mathcal{L} into itself as follows:

$$\Phi(x)(E_x f) := E_x T f.$$

Since E_x is surjective, the map $\Phi(x)$ is defined on all of \mathcal{L} . To see that it is well defined, suppose $E_x f = E_x g$ for $f, g \in H$. Now compute:

$$\begin{aligned} \Phi(x)E_x f &= E_x T f = \text{weak-}\lim_n E_x \Phi_n(x) f \\ &= \text{weak-}\lim_n M_{\Phi_n} E_x f \\ &= \text{weak-}\lim_n M_{\Phi_n} E_x g \\ &= \Phi(x)E_x g. \end{aligned}$$

It is easy to see that $\Phi(x)$ is linear, and its boundedness follows from the estimate:

$$\begin{aligned} \|\Phi(x)E_x f\| &= \|E_x T f\| \leq \limsup_n \|E_x M_{\Phi_n} f\| \\ &= \limsup_n \|\Phi_n(x)E_x f\| \leq \limsup_n \|\Phi_n(x)\| \|E_x f\|. \end{aligned}$$

Finally, we claim that Φ is a multiplier of \mathcal{H} and that M_Φ is the WOT limit of M_{Φ_n} . By construction, we have $E_x T = \Phi(x)E_x$ for every $x \in X$. This implies that Φ is a multiplier by Corollary 2.2.5. \square

One might expect that $\mathcal{M}(\mathcal{H})$ is actually closed in the weak operator topology when \mathcal{H} is full. At the end of this section, we will actually show that $\mathcal{M}(\mathcal{H})$ is reflexive, from which being WOT-closed follows as an easy consequence. Presently, we will describe more properties of the weak-* topology on $\mathcal{M}(\mathcal{H})$.

Proposition 2.3.8. *Suppose \mathcal{H} is a reproducing kernel Hilbert space which is separable and full. If Φ is a multiplier of \mathcal{H} , then the evaluation map*

$$\pi_x : M_\Phi \mapsto \Phi(x)$$

is a weak- to weak-* continuous, contractive homomorphism of $\mathcal{M}(\mathcal{H})$ into $\mathcal{B}(\mathcal{L})$.*

Proof. The claim that π_x is a homomorphism is immediate, and the contractivity of π_x follows from Proposition 2.3.4. For the continuity claim, the separability assumption ensures that we only verify that that π_x is weak-* to WOT sequentially continuous [23, Proposition 20.3]. To this end, suppose that Φ_n is a sequence of multipliers converging weak-* to Φ . We must show that $\Phi_n(x)$ is WOT convergent to $\Phi(x)$ in $\mathcal{B}(\mathcal{L})$. By assumption, we have $E_x M_{\Phi_n} f = \Phi_n(x)f(x)$ converging weakly to $E_x M_\Phi f = \Phi(x)f(x)$. Since E_x is surjective, it must be the case that $\Phi_n(x)u$ converges weakly to $\Phi(x)u$ for every $u \in \mathcal{L}$, from which WOT convergence follows. \square

One of the consequences of Proposition 2.3.8 is that if M_{Φ_x} is a net of multipliers converging weak-* to M_Φ , then $\Phi_x(x)$ converges weak-* in $\mathcal{B}(\mathcal{L})$ to $\Phi(x)$ for every $x \in X$. If one imposes a boundedness assumption, there is a converse to this.

Proposition 2.3.9. *Suppose \mathcal{H} is a separable and full reproducing kernel Hilbert space and suppose $\{M_\Phi\}$ is a net of multipliers such that $\sup_x \|M_{\Phi_x}\| \leq C < \infty$. If, for every $x \in X$, the operators $\Phi_x(x)$ converges weak-* to some $\Phi(x)$ in $\mathcal{B}(\mathcal{L})$, then Φ is a multiplier of \mathcal{H} and M_{Φ_x} converges weak-* to M_Φ .*

Proof. The boundedness claim ensures that there is some M_Φ that is a weak-* cluster point of $\{M_{\Phi_x}\}$. There is a subnet $M_{\Phi_{x'}}$ such that $M_{\Phi_{x'}}$ converges weak-* to M_Ψ . We then have that $E_x M_{\Phi_{x'}} = \Phi_{x'}(x) E_x$ converges weak-* to $\Psi(x) E_x$. By assumption, the subnet $\Phi_{x'}(x) E_x$ also converges to $\Phi(x) E_x$. This implies that $\Phi(x) = \Psi(x)$ for every x , and hence that Φ is actually a multiplier and that $M_\Phi = M_\Psi$ by Corollary 2.2.3.

We are done if we can show that M_{Φ_x} actually converges. It is enough to show that M_{Φ_x} WOT-converges to M_Φ since the net is bounded. If f, g are any functions in \mathcal{H} , we must show that $\langle M_{\Phi_x} f, g \rangle \rightarrow \langle M_\Phi f, g \rangle$, and the boundedness assumption means that we can replace g with a function of the form $\sum_{i=1}^n E_{x_i}^* u_i$. We have

$$\begin{aligned} \langle M_{\Phi_x} f, \sum_{i=1}^n E_{x_i}^* u_i \rangle_{\mathcal{H}} &= \sum_{i=1}^n \langle f, E_{x_i}^* \Phi_x^*(x_i) u_i \rangle_{\mathcal{L}} \\ &\rightarrow \sum_{i=1}^n \langle f, E_{x_i}^* \Phi^*(x_i) u_i \rangle_{\mathcal{L}} \\ &= \langle M_\Phi f, \sum_{i=1}^n E_{x_i}^* u_i \rangle_{\mathcal{H}}, \end{aligned}$$

as desired. \square

For much of this thesis, we will principally be investigating distance formulae for a distinguished ideal in $\mathcal{M}(\mathcal{H})$. Given a finite subset $E = \{x_1, \dots, x_n\} \subset X$, define

$$\mathcal{I}_E := \{M_\Phi \in \mathcal{M}(\mathcal{H}) : \Phi(x_i) = 0, i = 1, \dots, n\}.$$

Proposition 2.3.8 immediately yields the following corollary.

Corollary 2.3.10. *Suppose \mathcal{H} is a reproducing kernel Hilbert space which is separable and full. Then \mathcal{I}_E is a weak-* closed, two-sided ideal in $\mathcal{M}(\mathcal{H})$.*

Proof. We have $\mathcal{I}_E = \bigcap_{i=1}^n \ker \pi_{x_i}$, which is weak-* closed since each π_{x_i} is weak-* to weak-* continuous. \square

Given a WOT closed subspace \mathcal{S} of $\mathcal{B}(\mathcal{H})$, the *reflexive hull* $\text{Ref } \mathcal{S}$ is defined as

$$\text{Ref } \mathcal{S} := \{T \in \mathcal{B}(\mathcal{H}) : Th \in \mathcal{S}[h], h \in \mathcal{H}\}.$$

The space $\text{Ref } \mathcal{S}$ is clearly WOT closed and satisfies $\text{Ref } \text{Ref } \mathcal{S} = \text{Ref } \mathcal{S}$. The subspace \mathcal{S} is said to be *reflexive* if $\mathcal{S} = \text{Ref } \mathcal{S}$. If $\mathcal{S} = \mathcal{A}$ is a WOT closed subalgebra of $\mathcal{B}(\mathcal{H})$,

let $\text{Lat } \mathcal{A}$ denote the lattice of \mathcal{A} -invariant subspaces for \mathcal{A} . It is routine to verify that $\text{Ref } \mathcal{A} = \text{Alg Lat } \mathcal{A} = \{T \in \mathcal{B}(\mathcal{H}) : P_L^\perp T P_L = 0, L \in \text{Lat } \mathcal{A}\}$.

Theorem 2.3.11. *Suppose \mathcal{H} is a full reproducing kernel Hilbert space. The multiplier algebra $\mathcal{M}(\mathcal{H})$ is reflexive.*

Proof. It is equivalent to show that $\mathcal{M}(\mathcal{H})^*$ is reflexive. Suppose $T \in \text{Ref}(\mathcal{M}(\mathcal{H})^*)$. For each $x \in X$ and $u \in \mathcal{L}$ we have

$$TE_x^*u \in \overline{\mathcal{M}(\mathcal{H})^*E_x^*u} \subset \text{Ran } E_x^*.$$

Since E_x is surjective, the range of E_x^* is closed. Consequently, for every $u \in \mathcal{L}$, we may find a $v_x \in \mathcal{L}$ such that $TE_x^*u = E_x^*v_x$. Define a map $\Phi(x)^* : \mathcal{L} \rightarrow \mathcal{L}$ by $\Phi(x)^*u = v_x$. If we can show that $\Phi(x)^*$ is linear and bounded for each x , then by construction we have $E_x T^* = \Phi(x)E_x$, which shows that $T^* = M_\Phi$ is a multiplier of \mathcal{H} by Corollary 2.2.5.

If $u_1, u_2 \in \mathcal{L}$ and $c \in \mathbb{C}$, we have

$$E_x^*F(x)^*(cu_1 + u_2) = T^*E_x^*(cu_1 + u_2) = E_x^*(cF(x)^*u_1 + F(x)^*u_2).$$

Since E_x^* is injective, the above calculation proves that $F(x)^*$ is linear. For boundedness, we yet again appeal to the closed graph theorem. Suppose $u_n \rightarrow 0$ in \mathcal{L} and that $F(x)^*u_n \rightarrow v$ in \mathcal{L} . We have

$$E_x^*v = \lim_n E_x^*F(x)^*u_n = \lim_n TE_x^*u_n = 0.$$

Hence $v = 0$ and the proof is complete. \square

It is worth pointing out that the above proof is drastically simplified when \mathcal{H} is a scalar reproducing kernel Hilbert space. Indeed, if $T^* \in \text{Alg Lat}(\mathcal{M}(\mathcal{H})^*)$, then there must exist some scalar, say $\varphi(x)$, such that $T^*k_x = \overline{\varphi(x)}k_x$. It follows that φ is a multiplier and $T = M_\varphi$. This simple proof was first observed in the 1960s by Sarason [61] where the reflexivity of non-self-adjoint algebras was examined. The earliest reference for reflexivity in the vector-valued setting is a recent paper of Barbian [12], though the above approach leads to a significantly shorter proof. Since reflexive algebras are WOT closed, we immediately get the following corollary.

Corollary 2.3.12. *Suppose \mathcal{H} is a full reproducing kernel Hilbert space. The multiplier algebra $\mathcal{M}(\mathcal{H})$ is closed in the weak operator topology.*

We end this section with a nice application of Corollary 2.3.12 regarding the multipliers of a kernel of the form $K(\cdot, \cdot) \otimes I_{\mathcal{K}}$.

Theorem 2.3.13. *Suppose $\mathcal{H} = \mathcal{H}(K, \mathcal{L}, X)$ is a full reproducing kernel Hilbert space. Then for any separable Hilbert space \mathcal{K} , we have*

$$\mathcal{M}(K \otimes I_{\mathcal{K}}) = \mathcal{M}(K) \overline{\otimes} B(\mathcal{K}).$$

Proof. Any elementary tensor of the form $M_\Phi \otimes A$ in $\mathcal{M}(K) \otimes B(\mathcal{K})$ is obviously a multiplier on $\mathcal{H} \otimes \mathcal{K}$. Since $\mathcal{M}(K \otimes I_{\mathcal{K}})$ is closed in the WOT topology, taking WOT limits of finite linear combinations of elementary tensors implies that $\mathcal{M}(K \otimes I_{\mathcal{K}}) \supset \mathcal{M}(K) \overline{\otimes} B(\mathcal{K})$. For the reverse inclusion, fix an orthonormal basis $\{e_k\}_{k=1}^\infty$ for \mathcal{K} and let P_n denote the orthogonal projection onto the subspace $\text{span}\{e_k : 1 \leq k \leq n\}$. If $M_\Phi \in \mathcal{M}(K \otimes I_{\mathcal{K}})$, we claim that $(I_H \otimes P_n)M_\Phi(I_H \otimes P_n) \in \mathcal{M}(K \otimes I_{\mathcal{K}})$. In fact, if $u \in \mathcal{L}$, $x \in X$ and $\kappa \in \mathcal{K}$, we have

$$\begin{aligned} ((I_H \otimes P_n)M_\Phi^*(I_H \otimes P_n)(E_x^* \otimes I_{\mathcal{K}})(u \otimes \kappa) &= (I_H \otimes P_n)M_\Phi^*(E_x^* \otimes I_{\mathcal{K}})(u \otimes P_n\kappa) \\ &= (I_H \otimes P_n)(E_x^* \otimes I_{\mathcal{K}})\Phi(x)^*(u \otimes P_n\kappa) \\ &= (E_x^* \otimes P_n)\Phi(x)^*(u \otimes P_n\kappa) \\ &= (E_x^* \otimes I_{\mathcal{K}})(I_{\mathcal{L}} \otimes P_n)\Phi(x)^*(I_{\mathcal{L}} \otimes P_n)(u \otimes \kappa), \end{aligned}$$

as desired. Since $\mathcal{M}(K) \overline{\otimes} B(\mathcal{K})$ is WOT closed and $\text{WOT-lim}_{n \rightarrow \infty} P_n M_\Phi P_n = M_\Phi$, we have that $M_\Phi \in \mathcal{M}(K) \overline{\otimes} B(\mathcal{K})$, which completes the proof. \square

2.4 Multipliers between spaces

Given two reproducing kernel Hilbert spaces $\mathcal{H}_1 = \mathcal{H}_1(K_1, K_1, X)$ and $\mathcal{H}_2 = \mathcal{H}_2(K_2, \mathcal{L}_2, X)$ on X , we define the set of multipliers $\text{Mult}(\mathcal{H}_1, \mathcal{H}_2)$ as the $\mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$ -valued functions Φ on X such that

$$\Phi f \in \mathcal{H}_2$$

for all $f \in \mathcal{H}_1$, where $\Phi f(x) := \Phi(x)f(x)$. The *multiplier space* $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is the operator space of all the induced multiplication operators M_Φ where $\Phi \in \text{Mult}(\mathcal{H}_1, \mathcal{H}_2)$. The various characterization of multipliers in $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ follow immediately. Let E^1 and E^2 denote the evaluation operators on \mathcal{H}_1 and \mathcal{H}_2 , respectively.

Theorem 2.4.1. *Suppose Φ is a $\mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$ -valued function on X . Then Φ is a multiplier of \mathcal{H} if and only if there is a constant $C > 0$ such that the operator matrices*

$$[C^2 K_1(x_i, x_j) - \Phi(x_i)K_2(x_i, x_j)\Phi(x_j)^*]_{i,j=1}^n$$

are positive for every finite subset $\{x_1, \dots, x_n\}$ of X . If T is any operator in $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ which, for any $x \in X$ and $f \in \mathcal{H}_1$, satisfies the intertwining relationship

$$E_x^2 T f = T_x f(x)$$

for some operators $T_x \in \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$, then $T = M_\Phi$ is a multiplier where $\Phi(x) = T_x$.

The corresponding reflexivity result is as follows.

Theorem 2.4.2. *Suppose \mathcal{H}_1 and \mathcal{H}_2 are reproducing kernel Hilbert spaces on X and \mathcal{H}_1 is full. Then $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ is a reflexive subspace of $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. In particular, $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ is closed in the induced weak operator topology and is a dual space of operators.*

In later chapters, we will also require the following identification, the proof of which follows from the proof of Theorem 2.3.13.

Theorem 2.4.3. *Suppose $\mathcal{H} = \mathcal{H}(K, \mathcal{L}, X)$ is a full reproducing kernel Hilbert space. Then for any separable Hilbert spaces \mathcal{K}_1 and \mathcal{K}_2 , we have*

$$\mathcal{M}(\mathcal{H} \otimes \mathcal{K}_1, \mathcal{H} \otimes \mathcal{K}_2) = \mathcal{M}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2).$$

Lastly, we will require a strong notion of equivalence between two scalar-valued spaces. Suppose \mathcal{H}_1 and \mathcal{H}_2 are scalar-valued reproducing kernel Hilbert spaces on X with kernels k^1 and k^2 , respectively, and that $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a unitary map. We say that U is a *reproducing kernel Hilbert space isomorphism* if, for every $x \in X$, there is a non-zero scalar c_x such that $U^*k_x^2 = \overline{c_x}k_x^1$. In light of Theorem 2.4.1, this means that U is precisely given by a multiplier $\varphi \in \text{Mult}(\mathcal{H}_1, \mathcal{H}_2)$ with $\varphi(x) = c_x$ for each $x \in X$.

Proposition 2.4.4. *Suppose \mathcal{H}_1 and \mathcal{H}_2 are scalar-valued reproducing kernel Hilbert spaces on a set X and that $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a reproducing kernel Hilbert space isomorphism. Then $\text{Mult}(\mathcal{H}_1) = \text{Mult}(\mathcal{H}_2)$ and the unitary U induces a unitary equivalence between $\mathcal{M}(\mathcal{H}_1)$ and $\mathcal{M}(\mathcal{H}_2)$.*

Proof. A \mathbb{C} -valued function φ on X is in $\mathcal{M}(\mathcal{H}_1)$ if and only if there is a constant $C > 0$ so that

$$\left[C^2 - \varphi(x_i)\overline{\varphi(x_j)}k^1(x_i, x_j) \right]_{i,j=1}^n$$

are positive semidefinite for every finite subset $\{x_1, \dots, x_n\}$ of X . Schur multiplying by the Gramian $[(c_i\overline{c_j})^{-1}]_{i,j=1}^n$ and noting that U is unitary, we have

$$\left[C^2 - \varphi(x_i)\overline{\varphi(x_j)}k^2(x_i, x_j) \right]_{i,j=1}^n \geq 0,$$

and so φ is a multiplier on \mathcal{H}_2 as well. Let M_φ^1 and M_φ^2 denote the multiplication operators for φ on \mathcal{H}_1 and \mathcal{H}_2 , respectively. If $U = M_\psi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$, we have

$$U^*(M_\varphi^1)^*k_x^1 = \overline{\varphi(x)\psi(x)}k_x = (M_\varphi^2)^*U^*k_x,$$

which implies that $M_\varphi^1 U = U M_\varphi^2$. □

Chapter 3

The distance formula

In Chapter 2, we saw that if \mathcal{H} is a reproducing kernel Hilbert space, its multiplier algebra $\mathcal{M}(\mathcal{H})$ is a dual algebra of operators on \mathcal{H} . From this point forward, we shall always assume that \mathcal{H} is full. This assumption is often not necessary, but the reproducing kernel Hilbert spaces we encounter most frequently all enjoy this property. Suppose that E is a finite subset of X and, as before, let \mathcal{I}_E denote the weak-* closed ideal of multipliers in $\mathcal{M}(\mathcal{H})$ that vanish on E . When $E = \{x\}$ is a singleton, we write $\mathcal{I}_E = \mathcal{I}_x$.

If \mathcal{A} is any unital weak-* closed subalgebra of $\mathcal{M}(\mathcal{H})$, we say that \mathcal{A} is a *dual algebra of multipliers of \mathcal{H}* . Given such an \mathcal{A} , one may form the ideal $\mathcal{I}_E^{\mathcal{A}}$ consisting of multipliers in \mathcal{A} which vanish on E . If the context demands it, we will use the notation $\mathcal{I}_E^{\mathcal{A}}$ instead of \mathcal{I}_E . Note that since \mathcal{A} is unital, the ideal \mathcal{I}_E is proper.

Example 3.0.5. Suppose \mathcal{I}_S is the ideal of functions in $\mathcal{M}(\mathcal{H})$ which vanish on a (possibly infinite) subset $S \subset X$. Then $\mathcal{A} := \mathbb{C} + \mathcal{I}_S$ is a dual algebra of multipliers on \mathcal{H} . A multiplication operator M_Φ in $\mathcal{M}(\mathcal{H})$ belongs to \mathcal{A} if and only if $\Phi(x) = \Phi(y)$ for every pair of points x and y in S . Thus, the Pick problem for \mathcal{A} specifies a function that not only satisfies the given interpolation data, but also identifies every point in S .

We are principally interested in calculating the distance between an element in M_Φ in \mathcal{A} and the ideal \mathcal{I}_E :

$$\text{dist}(M_\Phi, \mathcal{I}_E) := \inf_{M_\Psi \in \mathcal{I}_E} \|M_\Phi - M_\Psi\|.$$

Of course, a distance formula like this can be written down for any subset of \mathcal{A} . It means something rather special for this particular choice of ideal. Suppose $E = \{x_1, \dots, x_n\} \subset X$ and $W_1, \dots, W_n \in \mathcal{B}(\mathcal{L})$ are given. If Φ is *any* multiplier in \mathcal{A} such that $\Phi(x_i) = W_i$ for $1 \leq i \leq n$, then $\Phi(x_i) + \Psi(x_i) = W_i$ for $1 \leq i \leq n$. Thus, the smallest possible norm of any multiplier that solves the interpolation problem given by the data sets E and W_1, \dots, W_n is precisely $\text{dist}(M_\Phi, \mathcal{I}_E)$. These seemingly innocuous observations allow us to formulate the most important principle in this thesis.

Theorem 3.0.6 (The distance formula approach to the Pick problem). *Suppose $\mathcal{H} = \mathcal{H}(X, K, \mathcal{L})$ is a full reproducing kernel Hilbert space and the data sets $E = \{x_1, \dots, x_n\} \subset X$ and $W_1, \dots, W_n \in \mathcal{B}(\mathcal{L})$ are given. If \mathcal{A} is a dual algebra of multipliers*

on \mathcal{H} , then there is a contractive multiplier M_Ψ in \mathcal{A} such that $\Psi(x_i) = W_i$ for $1 \leq i \leq n$ if and only if

$$\text{dist}(M_\Phi, \mathcal{I}_E) \leq 1$$

for any Φ satisfying $\Phi(x_i) = W_i$ for $1 \leq i \leq n$.

This reformulation of the Pick problem was strongly suggested by Sarason [62], and was explicitly developed in the groundbreaking work of McCullough [46]. Slight variations of this theme can be found in the work of Cole, Lewis and Wermer [22] where general interpolation problems for uniform algebras were studied. Our approach to interpolation problems in the dual algebra framework will differ from those found in the existing literature. The primary goal is to use the powerful theory of dual algebras, first introduced by Brown in the 1970s [21], to simultaneously handle the Pick problem in many settings.

In order for Theorem 3.0.6 to not be a vacuous statement, it is essential that there is at least one solution (of any norm) which interpolates the data. In the scalar-valued case, for example, this follows readily if \mathcal{A} separates the points in E . Of course, if there are no solutions, the Pick problem has a negative answer for the particular choice of data. Consequently, we will also assume that there is at least one solution. Of course, if the ideal \mathcal{I}_E is zero, Theorem 3.0.6 is a tautology. Fortunately, all the examples we are interested in have arbitrary interpolants and non-trivial ideals.

In light of Theorem 3.0.6, it becomes valuable to get a handle on the quotient norm on $\mathcal{A}/\mathcal{I}_E$. Recall that if Φ is a *contractive* multiplier on \mathcal{H} that satisfies $\Phi(x_i) = W_i$ for $1 \leq i \leq n$, then by Theorem 2.2.4, we have

$$[K(x_i, x_j) - W_i K(x_i, x_j) W_j^*]_{i,j=1}^n \geq 0 \quad (3.1)$$

The Pick problem for $\mathcal{M}(\mathcal{H})$ asks whether or not Inequality 3.1 is also a *sufficient* condition which guarantees the existence of a contractive multiplier which interpolates the given data sets. If we replace the operators W_i with scalars w_i and the kernel K with a scalar kernel k , the above operator matrix becomes the familiar looking Pick matrix

$$[(1 - w_i \overline{w_j})k(x_i, x_j)]_{i,j=1}^n.$$

Definition 3.0.7 (Pick properties). A $\mathcal{B}(\mathcal{L})$ -valued kernel K is said to have the *Pick property* if

$$[K(x_i, x_j) - W_i K(x_i, x_j) W_j^*]_{i,j=1}^n \geq 0$$

is equivalent to the existence of a contractive solution to the interpolation problem with data sets $\{x_1, \dots, x_n\}$ and $\{W_1, \dots, W_n\}$. A scalar-valued kernel k is said to be a *Pick kernel* if it has the Pick property. A scalar-valued kernel k is said to be a *complete Pick kernel* if, for every integer $m \geq 1$, the $\mathcal{M}_m(\mathbb{C})$ -valued kernel $k \otimes I_m$ has the Pick property.

In Chapter 5, kernels with the complete Pick property will be explored. Of course, the Nevanlinna-Pick theorem asserts that the Szegő kernel is a Pick kernel (even a complete Pick kernel). The next proposition shows the natural connection between those kernels which have the Pick property, and the multiplier algebras for which the quotient $\mathcal{M}(\mathcal{H})/\mathcal{I}_E$ admits a natural isometric representation into a subspace of \mathcal{H} .

Proposition 3.0.8. *Suppose $\mathcal{H} = \mathcal{H}(K, X, \mathcal{L})$ is a reproducing kernel Hilbert space and let $E = \{x_1, \dots, x_n\} \subset X$. The kernel K has the Pick property if and only if*

$$\text{dist}(M_\Phi, \mathcal{I}_E) = \|M_\Phi^*|_{\mathfrak{M}}\|$$

where $\mathfrak{M} := \overline{\text{span}}(\text{Ran } E_{x_i}^* : 1 \leq i \leq n)$. In other words, K has the Pick property if and only if the mapping:

$$\mathcal{M}_\Phi + \mathcal{I}_E \mapsto P_{\mathfrak{M}} M_\Phi$$

is an isometric representation of $\mathcal{M}(\mathcal{H})/\mathcal{I}_E$ on \mathfrak{M} .

Proof. First we note that for any $\Phi \in \text{Mult}(\mathcal{H})$, we have $\text{dist}(M_\Phi) \geq \|M_\Phi^*|_{\mathfrak{M}}\|$. To see this, let $M_\Psi \in \mathcal{I}_E$. Then for each $x_i \in E$, we have $E_{x_i} M_\Psi = \Psi(x_i) E_{x_i} = 0$, which implies that $M_\Psi^*|_{\mathfrak{M}} = 0$ and hence that

$$\|M_\Phi - M_\Psi\| \geq \|(M_\Phi^* - M_\Psi^*)|_{\mathfrak{M}}\| = \|M_\Phi^*|_{\mathfrak{M}}\|.$$

Now suppose K has the Pick property and that, normalizing if necessary, we have $\|M_\Phi^*|_{\mathfrak{M}}\| = 1$, which implies the operator inequality

$$P_{\mathfrak{M}} - P_{\mathfrak{M}} M_\Phi M_\Phi^* P_{\mathfrak{M}} \geq 0.$$

By checking this positivity condition against vectors of the form $E_{x_1}^* v_1 + \dots + E_{x_n}^* v_n$, we see that it is equivalent to the positivity of the operator matrix

$$[K(x_i, x_j) - \Phi(x_i) K(x_i, x_j) \Phi(x_j)^*]_{i,j=1}^n \geq 0.$$

Since K has the Pick property, this implies the existence of a contractive multiplier Γ such that $\Gamma(x_i) = \Phi(x_i)$ for $1 \leq i \leq n$. Now $M_\Phi - M_\Gamma$ is in \mathcal{I}_E , hence we have $\text{dist}(M_\Phi, \mathcal{I}_E) \leq \|M_\Phi - (M_\Phi - M_\Gamma)\| = \|M_\Gamma\| \leq 1$, as desired.

Conversely, suppose the distance formula holds and that the given Pick matrix is positive. As we have already seen, this implies that $\text{dist}(M_\Phi, \mathcal{I}_E) = \|M_\Phi^*|_{\mathfrak{M}}\| \leq 1$. By a standard weak- $*$ compactness argument, there is a multiplier $M_\Gamma \in \mathcal{I}_E$ such that $\|M_\Phi - M_\Gamma\| \leq 1$. It follows that $\Phi - \Gamma$ is the desired contractive solution to the interpolation problem. The last statement of the theorem follows from the fact that compression to a co-invariant subspace is a homomorphism. \square

Remark 3.0.9. For a scalar-valued space, it follows immediately that $\mathfrak{M} = \text{span}\{k_x : x \in E\}$, which is at most n -dimensional. More generally, if \mathcal{L} is a Hilbert space of dimension m , then for the kernel $k \otimes I_{\mathcal{L}}$, we have $\mathfrak{M} = \text{span}\{k_x : x \in E\} \otimes \mathbb{C}^m$. These are the most important examples, since in this case the restriction $M_\Phi^*|_{\mathfrak{M}}$ acts on finite dimensional space. In particular, this means that checking the positivity of the Pick matrix can be accomplished algorithmically.

3.1 Invariant subspaces for multiplier algebras

Unfortunately, most kernels do not have the Pick property. The simplest non-example is the Bergman kernel

$$\frac{1}{(1 - z\bar{w})^2}.$$

Since $H^2(\mathbb{D})$ and $L_a^2(\mathbb{D})$ share the same multiplier algebra, namely $H^\infty(\mathbb{D})$, and the Szegő kernel for $H^2(\mathbb{D})$ is a Pick kernel, the Bergman kernel having the Pick property would suggest that

$$\left[\frac{1 - w_i \bar{w}_j}{1 - z_i \bar{z}_j} \right]_{i,j=1}^n \geq 0$$

if and only if

$$\left[\frac{1 - w_i \bar{w}_j}{(1 - z_i \bar{z}_j)^2} \right]_{i,j=1}^n \geq 0,$$

which is not always true.

The distance formula approach to interpolation indicates that, given a kernel K and a finite subset E of X , the subspace \mathfrak{M} must encode enough information in order to isometrically represent $\mathcal{M}(\mathcal{H}) + \mathcal{I}_E$. For the Bergman space, this means there is some $\varphi \in H^\infty(\mathbb{D})$ and E so that

$$\text{dist}(M_\varphi, \mathcal{I}_E) > \|M_\varphi^*|_{\mathfrak{M}}\|.$$

In keeping with the theme of representing the quotient algebra $\mathcal{A}/\mathcal{I}_E$ on subspaces of \mathcal{H} associated to the subset E (i.e. finite dimensional spaces for scalar-valued spaces), it is natural to ask the following question:

Question 3.1.1. Given a dual algebra of multipliers \mathcal{A} on a full reproducing kernel Hilbert space \mathcal{H} , is there a family of unital, contractive representations $\{\pi_\alpha\}_{\alpha \in \mathbb{A}}$ of the quotient $\mathcal{A}/\mathcal{I}_E$ such that $\bigoplus_{\alpha \in \mathbb{A}} \pi_\alpha$ is isometric and the condition $\sup_{\alpha \in \mathbb{A}} \|\pi_\alpha(M_\Phi + \mathcal{I}_E)\| \leq 1$ is equivalent to the positivity of family of Pick matrices?

This approach represents an overarching theme in Pick interpolation theory. This program was carried out for multiply-connected regions by Abrahamse [1], uniform algebras by Cole, Lewis and Wermer [22], the bidisk by Agler and McCarthy [2], for vector-valued multiplier algebras by McCullough [46] and for subalgebras of $H^\infty(\mathbb{D})$ by Davidson et al [25] and Raghupathi [56, 57].

If \mathcal{H} is a full reproducing kernel Hilbert space, the reflexivity of $\mathcal{M}(\mathcal{H})$ demonstrates that this operator algebra has an abundance of invariant subspaces. Given an operator subspace \mathcal{S} of $\mathcal{B}(\mathcal{H})$, an invariant subspace $L \in \text{Lat } \mathcal{S}$ is said to be *cyclic* if it is of the form $L = \mathcal{S}[h] := \overline{\mathcal{S}h}$ for some $h \in \mathcal{H}$ (h is called the *cyclic* vector for L). For reasons we shall see shortly, it is profitable to restrict our attention to cyclic invariant subspaces of $\mathcal{M}(\mathcal{H})$.

Suppose L is an invariant subspace of \mathcal{A} and let P_L denote the orthogonal projection of \mathcal{H} onto L . Define

$$X_L := X \setminus \{x \in X : f(x) = 0 \text{ for every } f \in L\}.$$

Then L is an \mathcal{L} -valued reproducing kernel Hilbert space on X_L since

$$E_x P_L f = E_x f = f(x)$$

for every $x \in X_L$ and $f \in L$. The evaluation operator $E_x P_L : L \rightarrow \mathcal{L}$ induces the $\mathcal{B}(\mathcal{L})$ -valued kernel function $K^L(x, y) := E_x P_L E_y^*$.

Lemma 3.1.2. *If M_Φ is in \mathcal{A} , then Φ is a multiplier for every $L \in \text{Lat}(\mathcal{A})$. The multiplication operator induced by Φ on L is given by $M_\Phi^L := M_\Phi|_L$.*

Proof. To see this, we need only verify the intertwining relationship with evaluation operators. For each $x \in X_L$, we have

$$E_x^L M_\Phi|_L = E_x P_L M_\Phi P_L = E_x M_\Phi P_L = \Phi(x) E_x P_L = \Phi(x) E_x^L. \quad \square$$

Remark 3.1.3. In the scalar-valued case, the reproducing kernel function for L at x is given by $P_L k_x$. The positive semidefinite kernel on $X \times X$ is therefore given by $k^L(x, y) = \langle P_L k_y, k_x \rangle_{\mathcal{H}}$. We also have, by the above lemma, that $(M_\Phi^L)^* k_x^L = \overline{\varphi(x)} k_x^L$.

We conclude this section with a discussion of a natural collection of contractive representations of the quotient $\mathcal{A}/\mathcal{I}_E$. As above, let $L \in \text{Lat} \mathcal{A}$ and recall that the evaluation map for L is given by $E_x^L = E_x P_L$. We define the following two distinguished subspaces of L :

$$\begin{aligned} \mathfrak{M}_L &:= \overline{\text{span}} \{ \text{Ran}(E_{x_i}^L)^* : x_i \in E \} \\ &= L \ominus \left(\bigcap_{x_i \in E} \ker(E_{x_i}^L) \right) \\ &= P_L \mathfrak{M}; \\ \mathfrak{N}_L &:= L \ominus \mathcal{I}_E L. \end{aligned}$$

Note that $\overline{\mathcal{I}_E L}$ is also invariant for \mathcal{A} and is contained in L . Thus \mathfrak{N}_L is a semi-invariant subspace. In particular, $P_{\mathfrak{N}_L} M_\Phi P_L = P_{\mathfrak{N}_L} M_\Phi P_{\mathfrak{N}_L}$ for any $M_\Phi \in \mathcal{A}$ and so compression to \mathfrak{N}_L is a contractive homomorphism of \mathcal{A} into $\mathcal{B}(\mathfrak{N}_L)$. Note that $\mathfrak{M}_L = P_L \mathfrak{M}$ is also semi-invariant for \mathcal{A} since each $\ker(E_{x_i}^L)$ is invariant for \mathcal{A} .

If $f = (E_{x_1}^L)^* u_1 + \cdots + (E_{x_n}^L)^* u_n$ is a typical element in \mathfrak{M}_L , then for any $M_\Psi \in \mathcal{I}_E$ and $g \in L$, we have

$$\begin{aligned} \langle f, \Psi g \rangle_L &= \langle (E_{x_1}^L)^* u_1 + \cdots + (E_{x_n}^L)^* u_n, \Psi g \rangle_L \\ &= \sum_{i=1}^n \langle u_i, \Psi(x_i) g(x_i) \rangle_{\mathcal{L}} = 0, \end{aligned}$$

since $\Psi(x_i) = 0$ for each $x_i \in E$. This shows that \mathfrak{M}_L is orthogonal to $\mathcal{I}_E L$, and hence

that $\mathfrak{M}_{\mathcal{L}} \subset \mathfrak{N}_L$. It follows that the maps

$$\begin{aligned} M_{\Phi} + \mathcal{I}_E &\mapsto P_{\mathfrak{N}_L} M_{\Phi} P_{\mathfrak{N}_L} \\ M_{\Phi} + \mathcal{I}_E &\mapsto P_{\mathfrak{M}_L} M_{\Phi} P_{\mathfrak{M}_L} \end{aligned}$$

are well-defined. The following proposition gives some lower bounds for the norms of these maps, and also shows that we need only consider cyclic invariant subspaces.

Proposition 3.1.4. *Suppose that \mathcal{A} is a dual algebra of multipliers on a full reproducing kernel Hilbert space and $E = \{x_1, \dots, x_n\}$ is a finite subset of X . Then the following distance estimates hold for any $M_{\Phi} \in \mathcal{A}$:*

$$\begin{aligned} \text{dist}(M_{\Phi}, \mathcal{I}_E) &\geq \sup_{L \in \text{Lat}(\mathcal{A})} \|P_{\mathfrak{N}_L} M_{\Phi} P_{\mathfrak{N}_L}\| = \sup_{L \in \text{CycLat}(\mathcal{A})} \|P_{\mathfrak{N}_L} M_{\Phi} P_{\mathfrak{N}_L}\| \\ &\geq \sup_{L \in \text{CycLat}(\mathcal{A})} \|P_{\mathfrak{M}_L} M_{\Phi} P_{\mathfrak{M}_L}\| = \sup_{L \in \text{Lat}(\mathcal{A})} \|P_{\mathfrak{M}_L} M_{\Phi} P_{\mathfrak{M}_L}\|. \end{aligned}$$

Proof. Suppose L is an invariant subspace for \mathcal{A} . For $M_{\Phi} \in \mathcal{A}$ and $M_{\Psi} \in \mathcal{I}_E$, compute

$$\begin{aligned} \|M_{\Phi} - M_{\Psi}\| &\geq \|P_{\mathfrak{N}_L}(M_{\Phi} - M_{\Psi})P_{\mathfrak{N}_L}\| \\ &= \|P_{\mathfrak{N}_L} M_{\Phi} P_{\mathfrak{N}_L}\| \geq \|P_{\mathfrak{M}_L} M_{\Phi} P_{\mathfrak{M}_L}\|. \end{aligned}$$

Taking an infimum over $M_{\Psi} \in \mathcal{I}_E$ and a supremum over $\text{Lat}(\mathcal{A})$, we obtain

$$\text{dist}(f, \mathcal{I}_E) \geq \sup_{L \in \text{Lat}(\mathcal{A})} \|P_{\mathfrak{N}_L} M_{\Phi} P_{\mathfrak{N}_L}\| \geq \sup_{L \in \text{CycLat}(\mathcal{A})} \|P_{\mathfrak{N}_L} M_{\Phi} P_{\mathfrak{N}_L}\|.$$

Since \mathfrak{M}_L is contained in \mathfrak{N}_L , we have

$$\sup_{L \in \text{CycLat}(\mathcal{A})} \|P_{\mathfrak{N}_L} M_{\Phi} P_{\mathfrak{N}_L}\| \geq \sup_{L \in \text{CycLat}(\mathcal{A})} \|P_{\mathfrak{M}_L} M_{\Phi} P_{\mathfrak{M}_L}\|.$$

Now consider an arbitrary element $L \in \text{Lat}(\mathcal{A})$. Then

$$\begin{aligned} \|P_{\mathfrak{N}_L} M_{\Phi} P_{\mathfrak{N}_L}\| &= \|P_{\mathfrak{N}_L} M_{\Phi} P_L\| = \sup_{\|h\|=1, h \in L} \|P_{\mathfrak{N}_L} M_{\Phi} P_L h\| \\ &= \sup_{\|h\|=1, h \in L} \|(P_L - P_{\mathcal{I}_E L}) M_{\Phi} P_{\mathcal{A}[h]} h\| \\ &\leq \sup_{\|h\|=1, h \in L} \|(P_L - P_{\mathcal{I}_E \mathcal{A}[h]}) P_{\mathcal{A}[h]} M_{\Phi} P_{\mathcal{A}[h]} h\| \\ &= \sup_{\|h\|=1, h \in L} \|P_{\mathfrak{N}_{\mathcal{A}[h]}} M_{\Phi} P_{\mathcal{A}[h]} h\| \\ &\leq \sup_{L \in \text{CycLat}(\mathcal{A})} \|P_{\mathfrak{N}_L} M_{\Phi} P_L\| = \sup_{L \in \text{CycLat}(\mathcal{A})} \|P_{\mathfrak{N}_L} M_{\Phi} P_{\mathfrak{N}_L}\|. \end{aligned}$$

Similarly,

$$\begin{aligned}
\|P_{\mathfrak{M}_L} M_\Phi P_{\mathfrak{M}_L}\| &= \|P_{\mathfrak{M}_L} M_\Phi P_L\| = \sup_{\|h\|=1, h \in L} \|P_{\mathfrak{M}_L} M_\Phi P_L h\| \\
&= \sup_{\|h\|=1, h \in L} \|P_{\mathfrak{M}_L} M_\Phi P_{\mathcal{A}[h]} h\| \\
&= \sup_{\|h\|=1, h \in L} \|P_{\mathfrak{M}_L} P_{\mathfrak{N}_{\mathcal{A}[h]}} M_\Phi P_{\mathcal{A}[h]} h\| \\
&= \sup_{\|h\|=1, h \in L} \|(P_{\mathcal{A}[h]} - P_{\cap \ker E_{x_i}^L} P_{\mathcal{A}[h]}) M_\Phi P_{\mathcal{A}[h]} h\| \\
&\leq \sup_{\|h\|=1, h \in L} \|(P_{\mathcal{A}[h]} - P_{\cap \ker E_{x_i}^{\mathcal{A}[h]}} P_{\mathcal{A}[h]}) M_\Phi P_{\mathcal{A}[h]} h\| \\
&= \sup_{\|h\|=1, h \in L} \|P_{\mathfrak{M}_{\mathcal{A}[h]}} M_\Phi P_{\mathcal{A}[h]} h\| \\
&\leq \sup_{L \in \text{CycLat}(\mathcal{A})} \|P_{\mathfrak{M}_L} M_\Phi P_L\| = \sup_{L \in \text{CycLat}(\mathcal{A})} \|P_{\mathfrak{M}_L} M_\Phi P_{\mathfrak{M}_L}\|. \quad \square
\end{aligned}$$

Let $\pi_{\mathfrak{N}}$ and $\pi_{\mathfrak{M}}$ denote representations of $\mathcal{A}/\mathcal{I}_E$ determined by

$$\begin{aligned}
\pi_{\mathfrak{N}}(M_\Phi + \mathcal{I}_E) &= \bigoplus_{L \in \text{CycLat } \mathcal{A}} P_{\mathfrak{N}_L} M_\Phi P_{\mathfrak{N}_L}; \\
\pi_{\mathfrak{M}}(M_\Phi + \mathcal{I}_E) &= \bigoplus_{L \in \text{CycLat } \mathcal{A}} P_{\mathfrak{M}_L} M_\Phi P_{\mathfrak{M}_L}.
\end{aligned}$$

Proposition 3.1.4 says that

$$\left\| \bigoplus_{L \in \text{Lat } \mathcal{A}} P_{\mathfrak{N}_L} M_\Phi P_{\mathfrak{N}_L} \right\| = \|\pi_{\mathfrak{N}}(M_\Phi + \mathcal{I}_E)\| \geq \|\pi_{\mathfrak{M}}(M_\Phi + \mathcal{I}_E)\| = \left\| \bigoplus_{L \in \text{Lat } \mathcal{A}} P_{\mathfrak{M}_L} M_\Phi P_{\mathfrak{M}_L} \right\|.$$

With this language in mind, we can now state the analogues of Pick properties for families of kernels in our present context.

Definition 3.1.5. A collection of invariant subspaces \mathbb{P} in $\text{CycLat } \mathcal{A}$ is said to be a *Pick family* for \mathcal{A} if $\pi_{\mathfrak{N}}$ is an isometric representation. We say that \mathbb{P} is a *strong Pick family* if $\pi_{\mathfrak{M}}$ is isometric.

Of course, if \mathbb{P} is a strong Pick family, then it is certainly a Pick family since both $\pi_{\mathfrak{N}}$ and $\pi_{\mathfrak{M}}$ are contractive. Having a strong Pick family will yield a more obvious analogy to the Pick theorem.

Proposition 3.1.6. *Suppose \mathbb{P} is a strong Pick family for \mathcal{A} , that $E = \{x_1, \dots, x_n\}$ is a finite subset of X and that W_1, \dots, W_n belong to $\mathcal{B}(\mathcal{L})$. The following are equivalent:*

1. *There is a contractive $M_\Phi \in \mathcal{A}$ such that $\Phi(x_i) = W_i$.*
2. *The operator matrix*

$$[K^L(x_i, x_j) - W_i K^L(x_i, x_j) W_j^*]_{i,j=1}^n$$

is positive for every $L \in \mathbb{P}$.

Proof. We have already seen the proof that (1) implies (2) in the proof of Proposition 3.0.8, since M_Φ^L is a contractive multiplication operator for each $L \in \text{CycLat } \mathcal{A}$. Conversely, for any $L \in \mathbb{P}$ we have

$$[K^L(x_i, x_j) - W_i K^L(x_i, x_j) W_j^*]_{i,j=1}^n \geq 0$$

if and only if $\|P_{\mathfrak{M}_L} M_\Phi P_{\mathfrak{M}_L}\| \leq 1$ for any $M_\Phi \in \mathcal{A}$ which satisfies $\Phi(x_i) = W_i$. Since \mathbb{P} is a strong Pick family, this implies that $\text{dist}(M_\Phi, \mathcal{I}_E) \leq 1$. It follows that there is some $M_\Psi \in \mathcal{I}_E$ such that $M_{\Phi-\Psi}$ is a contraction and $\Phi(x_i) - \Psi(x_i) = W_i$ for $1 \leq i \leq n$. \square

If \mathbb{P} is a Pick family, one might hope that there is a nice characterization of solutions to the Pick problem for \mathcal{A} in terms of a family of operator matrices relating to \mathbb{P} . Unfortunately, the only reasonable replacement of statement (2) in Proposition 3.1.6 would be

$$P_{\mathfrak{M}_L} - P_{\mathfrak{M}_L} M_\Phi P_{\mathfrak{M}_L} \geq 0$$

for each $L \in \mathbb{P}$. This expression still contains an apparent dependency on the arbitrary solution Φ . However, if Γ is any other solution, we have $M_\Phi - M_\Gamma \in \mathcal{I}_E$ and so $P_{\mathfrak{M}_L} M_\Phi P_{\mathfrak{M}_L} = P_{\mathfrak{M}_L} M_\Gamma P_{\mathfrak{M}_L}$. Consequently, as long one knows an example of an arbitrary solution in advance, it is conceivable that the above condition is no more abstract than that found in Proposition 3.1.6. They are both, admittedly, very difficult conditions to verify. Fortunately, in the case we are most interested in (scalar-valued interpolation), the difference between strong Pick families and Pick families is hardly an issue.

3.2 Scalar-valued kernels

In this section, we assume that \mathcal{H} is a scalar-valued reproducing kernel Hilbert space with kernel k and that \mathcal{A} is a dual algebra of multipliers on \mathcal{H} . We saw in the last section that cyclic invariant subspaces play a special role in interpolation theory. However, given a data set $E = \{x_1, \dots, x_n\}$ and a function $f \in \mathcal{H}$ which annihilates E , we have $h|_E = 0$ for every $h \in L = \mathcal{A}[f]$. Consequently, $k_{x_i}^L = 0$ for each $x_i \in E$, and hence $\mathfrak{M}^L = \{0\}$. Since we are primarily concerned with the evaluation of *multipliers* rather than functions in the underlying Hilbert space, it is valuable to extend the kernel k^L to more of X . The following lemma shows that, in certain cases, this extension is possible for cyclic invariant subspaces. Moreover, this extended kernel is also an eigenvector of the adjoints of multiplication operators. Recall that $X_L = X \setminus \{x \in X : f(x) = 0 \text{ for every } f \in L\}$.

Lemma 3.2.1. *Suppose \mathcal{A} is a dual algebra of multipliers on \mathcal{H} and set $L = \mathcal{A}[f]$ for some $f \in \mathcal{H}$ and $E = \{x\} \subset X$. There is a function $\tilde{k}^L \in L$ which satisfies the following properties:*

1. *If $x \in X_L$, then $k_x^L = \tilde{k}_x^L$.*
2. *When the subspace $\mathcal{A}[f] \ominus \mathcal{I}_x[f]$ is non-zero, it is spanned by \tilde{k}_x^L .*

3. For every $M_\varphi \in \mathcal{A}$ and $x \in X$ such that $\mathcal{A}[f] \ominus \mathcal{I}_x[f] \neq \{0\}$, we have

$$P_L M_\varphi^* \tilde{k}_x^L = \overline{\varphi(x)} \tilde{k}_x^L,$$

and thus

$$\langle M_\varphi \tilde{k}_x^L, \tilde{k}_x^L \rangle = f(x) \|\tilde{k}_x^L\|^2 \quad \text{for all } x \in X.$$

Proof. Since $L = \mathcal{A}[h]$ is a cyclic subspace and $\dim \mathcal{A}/\mathcal{I}_x = 1$, it follows that $\dim \mathcal{A}[f]/\mathcal{I}_x[f] \leq 1$. If $k_x^L = P_L k_x \neq 0$, then this is an eigenvector for $P_L \mathcal{A}^*$ as we saw in the last section. For $\psi \in \mathcal{I}_x$, we have

$$\langle k_x^L, \psi f \rangle = \langle P_L M_\psi^* k_x^L, f \rangle = \langle \overline{\psi(x)} k_x^L, f \rangle = 0.$$

So $P_L k_x$ belongs to $\mathcal{A}[f] \ominus \mathcal{I}_x[f]$ and we set $\tilde{k}_x^L = k_x^L$.

When $P_L k_x = 0$ and $\dim \mathcal{A}[f]/\mathcal{I}_x[f] = 1$, let \tilde{k}_x^L be any unit vector in $\mathcal{A}[f] \ominus \mathcal{I}_x[f]$. Then for $M_\varphi \in \mathcal{A}$, $M_\varphi - \varphi(x) \mathbf{1}$ lies in \mathcal{I}_x . Hence

$$\langle M_\varphi \tilde{k}_x^L, \tilde{k}_x^L \rangle = \varphi(x) \langle \tilde{k}_x^L, \tilde{k}_x^L \rangle - \langle (\varphi - \varphi(x) \mathbf{1}) \tilde{k}_x^L, \tilde{k}_x^L \rangle = \varphi(x).$$

Also, since $\mathcal{I}_x[h]^\perp \in \text{Lat}(\mathcal{A}^*)$, we have that $P_L M_\varphi^* k_x^L$ belongs to $\mathcal{A}[f] \ominus \mathcal{I}_x[f] = \mathbb{C} k_x^L$. The previous computation shows that

$$P_L M_\varphi^* \tilde{k}_x^L = \overline{f(x)} \tilde{k}_x^L. \quad \square$$

Porism 3.2.2. Suppose $L \in \text{Lat } \mathcal{A}$. Then for any $M_\varphi \in \mathcal{A}$ and $E = \{x\} \subset X$, we have

$$P_{\mathfrak{N}_L} M_\varphi^* P_{\mathfrak{N}_L} = \overline{\varphi(x)} P_{\mathfrak{N}_L}.$$

Proof. Let $\{k_i\}$ be an orthonormal basis for \mathfrak{N}_L . The proof of Lemma 3.2.1 shows that $\varphi(x) = \langle M_\varphi k_i, k_i \rangle$ for each i . Now, if there are constants a_i so that

$$P_{\mathfrak{N}_L} M_\varphi^* k_i = \sum_i a_i k_i,$$

taking inner products against each of the k_i shows that $a_i = \overline{\varphi(x)}$ for each i , as desired. \square

The extended kernel \tilde{k}^L allows us to evaluate the multipliers at a generally much larger subset of X (see Example 3.2.4 below) than just using $P_L k_x$. However, some continuity is lost for evaluation of functions in L . Since we are primarily interested in interpolation questions about the multiplier algebra, evaluation of the multipliers is more important. In order to try and reduce some of the notation we have assembled, we adopt the following convention.

Definition 3.2.3. For any dual algebra \mathcal{A} of multipliers on \mathcal{H} and any cyclic invariant subspace $L \in \text{CycLat}(\mathcal{A})$, let k_x^L denote the extended reproducing kernel on L constructed in Lemma 3.2.1 at the point x .

We will explicitly use the function $P_L k_x$ if we need to differentiate it from k_x^L .

Example 3.2.4. In spaces of analytic functions, it is often possible to fully describe the kernel structure on $L \in \text{CycLat}(\mathcal{A})$. Indeed, suppose that $H = L_a^2(\mathbb{D})$ is the Bergman space, and $\mathcal{A} = H^\infty$. Let $L = H^\infty[h]$ for some non-zero function $h \in L_a^2(\mathbb{D})$. Then

$$X_L = \{x \in \mathbb{D} : P_L k_x \neq 0\} = \{x \in \mathbb{D} : h(x) \neq 0\}.$$

However, since the Bergman space consists of analytic functions, h vanishes only to some finite order on each of its zeros.

It is routine to verify that for each $n \geq 0$ and $x \in \mathbb{D}$, there is a function $k_{x,n} \in L_a^2(\mathbb{D})$ such that

$$\langle h, k_{x,n} \rangle = h^{(n)}(x) \quad \text{for } h \in L_a^2(\mathbb{D}).$$

Suppose that h vanishes at x with multiplicity $r \geq 0$. We claim that $P_L k_{x,r} \neq 0$ and $P_L k_{x,n} = 0$ for $0 \leq n < r$. Indeed, for any $f \in H^\infty$ and $n \leq r$,

$$\begin{aligned} \langle fh, k_{x,n} \rangle &= (fh)^{(r)}(x_i) \\ &= \sum_{j=0}^r \binom{r}{j} f^{(j)}(x) h^{(r-j)}(x) \\ &= \begin{cases} 0 & \text{if } 0 \leq n < r \\ f(x)h^{(r)}(x) & \text{if } n = r. \end{cases} \end{aligned}$$

So $P_L k_{x,n} = 0$ for $0 \leq n < r$. Set $k_x^L = P_L k_{x,r} / \|P_L k_{x,r}\|$. This calculation shows that if $f \in \mathcal{I}_x$, then $\langle fh, k_x^L \rangle = 0$. So k_x^L belongs to $\mathcal{A}[h] \ominus \mathcal{I}_x[h]$. Now for $f, g \in H^\infty$,

$$\begin{aligned} \langle fh, M_g^* k_x^L \rangle &= \langle gfh, k_x^L \rangle \\ &= g(x)f(x)h^{(r)}(x_i) = \langle fh, \overline{g(x)}k_x^L \rangle. \end{aligned}$$

It follows that

$$(M_g|_L)^* k_x^L = P_L M_g^* k_x^L = \overline{g(x)}k_x^L.$$

Thus g is a multiplier for this reproducing kernel.

An identical construction is possible for any space of analytic functions on the unit disk for which extracting a term from the Taylor expansion is a bounded functional.

Remark 3.2.5. The Bergman space is also a good place to illustrate why dealing with cyclic invariant subspaces is preferable. The Bergman shift B (multiplication by z on $L_a^2(\mathbb{D})$) is a universal dilator for strict contractions [5, Chapter 10]. For example, fix a point $x \in \mathbb{D}$. Then B has an invariant subspace L such that $\mathfrak{N}_x = L \ominus \mathcal{I}_x L$ is infinite dimensional. The computations in Lemma 3.2.1 show that $(M_\varphi|_L)^*|_{\mathfrak{N}_x} = \overline{f(x)}I_{\mathfrak{N}_x}$ for any $\varphi \in H^\infty$, and hence there is no canonical choice for k_x^L . On the other hand, we can always identify a kernel structure on any invariant subspace $L \in \text{Lat } \mathcal{A}$ if we allow multiplicity. The subspaces $\mathfrak{N}_x = L \ominus \mathcal{I}_x L$ satisfy $P_{\mathfrak{N}_x} M_\varphi^*|_{\mathfrak{N}_x} = \overline{\varphi(x)}P_{\mathfrak{N}_x}$ for any $M_\varphi \in \mathcal{A}$. So if k is any unit vector in \mathfrak{N}_x , we obtain

$$\langle M_\varphi k, k \rangle = \varphi(x).$$

The spaces $\{\mathfrak{N}_x : x \in X\}$ are linearly independent and together they span L . See the continued discussion later in Remark 3.2.12.

We now make a return to Pick interpolation on some finite subset $E = \{x_1, \dots, x_n\}$ of X by functions in the algebra \mathcal{A} . It could be the case that \mathcal{A} fails to separate certain points in X , and so we impose the natural constraint that E contains at most one representative from any set of points that \mathcal{A} identifies. It follows that the kernels k_{x_i} form a linearly independent set. Indeed, since \mathcal{A} separates these points, we can find elements $p_1, \dots, p_n \in \mathcal{A}$ such that $p_i(x_j) = \delta_{ij}$. Hence if $\sum_{i=1}^n \alpha_i k_{x_i} = 0$, we find that

$$0 = M_{p_i}^* \left(\sum_{i=1}^n \alpha_i k_{x_i} \right) = \alpha_i k_{x_i} \quad \text{for } 1 \leq i \leq n.$$

The quotient algebra $\mathcal{A}/\mathcal{I}_E$ is n -dimensional, and is spanned by the idempotents $\{p_i + \mathfrak{J} : 1 \leq i \leq n\}$. We seek to establish useful formulae for the norm on $\mathcal{A}/\mathcal{I}_E$. These so-called operator algebras of idempotents have been studied by Paulsen [51] (see also Appendix A, where we show that every scalar-valued multiplier algebra on a finite set is naturally identified with a k -idempotent operator algebra).

We will now summarize the relationship between \mathfrak{N}_L and \mathfrak{M}_L when L is cyclic.

Lemma 3.2.6. *Given a finite subset $E \subset X$ on which \mathcal{A} separates points, and $L = \mathcal{A}[h]$ in $\text{CycLat}(\mathcal{A})$, the space \mathfrak{N}_L is a reproducing kernel Hilbert space over E with kernel $\{k_x^L : x \in E\}$; and the non-zero elements of this set form a basis for \mathfrak{N}_L . The subspace \mathfrak{M}_L is contained in \mathfrak{N}_L and is spanned by*

$$\{k_x^L = P_L k_x : x \in E, h(x) \neq 0\},$$

and it is a reproducing kernel Hilbert space over $\{x \in E : h(x) \neq 0\}$.

Proof. For each $x \in E$,

$$k_x^L \in L \ominus \mathcal{I}_x[h] \subset L \ominus \mathcal{I}_E[h] = \mathfrak{N}_L.$$

Let $E_L = \{x \in E : k_x^L \neq 0\}$. Then for $x \in E \setminus E_L$, Lemma 3.2.1 says that $L = \mathcal{A}[h] = \overline{\mathcal{I}_x L}$. It is easy to check that, since \mathcal{A} separates E , we have $\mathcal{I}_{E_L} = \prod_{x \in E \setminus E_L} \mathcal{I}_x$ and hence that $\overline{\mathcal{I}_{E_L} L} = L$. We may factor $\mathcal{I}_E = \mathcal{I}_{E_L} \mathcal{I}_{E \setminus E_L}$ and observe that

$$\overline{\mathcal{I}_E L} = \overline{\mathcal{I}_{E_L} \mathcal{I}_{E \setminus E_L} L} = \overline{\mathcal{I}_{E_L} L}.$$

Now $\dim \mathcal{A}/\mathcal{I}_{E_L} = |E_L|$, so $\dim \mathfrak{N}_L \leq |E_L|$. But \mathfrak{N}_L contains the non-zero vectors k_x^L for $x \in E_L$. For $f \in \mathcal{A}$ and $x \in E_L$,

$$P_{\mathfrak{N}_L} M_\varphi^* k_x^L = P_L P_{\mathcal{I}_E[h]}^\perp M_\varphi^* k_x^L = P_L M_\varphi^* k_x^L = \overline{\varphi(x)} k_x^L.$$

Because \mathcal{A} separates the points of E_L , it follows that these vectors are eigenfunctions for distinct characters of \mathcal{A} , and thus are linearly independent. This set has the same cardinality as $\dim \mathfrak{N}_L$, and therefore it forms a basis.

Now \mathfrak{M}_L is spanned by $\{P_L k_x : x \in E\}$, and it suffices to use the non-zero elements. These coincide with k_x^L on $E_L^0 := \{x \in E : h(x) \neq 0\}$. This is a subset of the basis for \mathfrak{N}_L , and hence \mathfrak{M}_L is a subspace of \mathfrak{N}_L . That \mathfrak{M}_L and \mathfrak{N}_L are reproducing kernel Hilbert space on E_L and E_L^0 , respectively, follows now from the fact that they generated by a kernel structure. \square

The equality $\mathfrak{M}_L = \mathfrak{N}_L$ holds in the following important case. The proof is immediate from the lemma.

Corollary 3.2.7. *Suppose that E is a finite subset of X on which \mathcal{A} separates points, and $L = \mathcal{A}[h]$ in $\text{CycLat}(\mathcal{A})$. If h does not vanish on E , then $\mathfrak{M}_L = \mathfrak{N}_L$.*

The above results indicate that, for scalar-valued Pick interpolation, the difference between strong Pick families and Pick families does not drastically change the type of results we could hope for. For the sake of clarity, we will repeat the precise formulations of the Pick problem for dual algebras of multipliers of a scalar-valued kernel below.

Definition 3.2.8. The collection $\{k^L : L \in \text{CycLat}(\mathcal{A})\}$ is said to be a *Pick family of kernels for \mathcal{A}* if for every finite subset E of X and every $M_\varphi \in \mathcal{A}$,

$$\text{dist}(\varphi, \mathcal{I}_E) = \sup_{L \in \text{CycLat}(\mathcal{A})} \|P_{\mathfrak{N}_L} M_\varphi^*|_{\mathfrak{N}_L}\|.$$

Similarly, $\{k^L : L \in \text{CycLat}(\mathcal{A})\}$ is said to be a *strong Pick family* if

$$\text{dist}(\varphi, \mathcal{I}_E) = \sup_{L \in \text{CycLat}(\mathcal{A})} \|P_{\mathfrak{M}_L} M_\varphi^*|_{\mathfrak{M}_L}\|.$$

If φ is a contractive multiplier which satisfies $\varphi(x_i) = w_i$, Lemma 3.2.1 says that $(M_\varphi^L)^* k_{x_i}^L = \overline{w_i} k_{x_i}^L$ for each $x_i \in E$. This yields the following solution to the Pick problem for \mathcal{A} when a Pick family is present. The proof is contained in the proof of Proposition 3.1.6.

Theorem 3.2.9. *Let \mathcal{A} be a dual algebra of multipliers on a Hilbert space \mathcal{H} . The family $\{k^L : L \in \text{CycLat}(\mathcal{A})\}$ is a Pick family of kernels for \mathcal{A} if and only if the following statement holds:*

Given $E = \{x_1, \dots, x_n\}$ distinct points in X which are separated by \mathcal{A} , and complex scalars w_1, \dots, w_n , there is a multiplier f in the unit ball of \mathcal{A} such that $f(x_i) = w_i$ for $1 \leq i \leq n$ if and only if the Pick matrices

$$\left[(1 - w_i \overline{w_j}) k^L(x_i, x_j) \right]_{n \times n}$$

are positive definite for every $L \in \text{CycLat}(\mathcal{A})$. Similarly, $\{k^L : L \in \text{CycLat}(\mathcal{A})\}$ is a strong Pick family if and only if the solvability of the Pick problem is equivalent to the Pick matrices

$$\left[(1 - w_i \overline{w_j}) \langle P_L k_{x_j}, k_{x_i} \rangle \right]_{n \times n}$$

being positive semidefinite for every $L \in \text{CycLat} \mathcal{A}$.

Even in the scalar case, a strong Pick family still seems somehow more desirable than a Pick family, since the resulting kernel functions are intrinsic to the Hilbert space. This is not always the case, but it does happen in important special cases. The following is one instance where it may occur, and indeed it does so frequently.

Corollary 3.2.10. *Suppose $\{k^{\mathcal{A}[h]} : L \in \text{CycLat } \mathcal{A} \text{ and } h(x_i) \neq 0 \text{ for every } x_i \in E\}$ is a Pick family for \mathcal{A} . Then it is also a strong Pick family.*

While we have been assuming that \mathcal{A} separates the points in the set E , it turns out that we can drop this assumption as long as we have a strong Pick family. In particular, as long as the associated family of Pick matrices are positive semidefinite, it must be the case that solutions of (with no norm constraint) exist.

Proposition 3.2.11. *Suppose \mathcal{A} is a dual algebra of multipliers on \mathcal{H} and that*

$$[(1 - w_i \overline{w_j})k^L(x_i, x_j)] \geq 0$$

for at least one subspace L of the form $L = \mathcal{A}[h]$, where $h(x_i) \neq 0$ for $i = 1, \dots, n$. Then, there is a multiplication operator $M_\varphi \in \mathcal{A}$ such that $\varphi(x_i) = w_i$ for each $i = 1, \dots, n$.

Proof. Suppose $L = \mathcal{A}[h]$ where h does not vanish on E . Note that in this case we have $k_x^L = P_L k_x$ for every $x \in E$. We saw that if \mathcal{A} separates the points in E , then $k_{x_i}^L$ and $k_{x_j}^L$ are linearly independent for any $x_i, x_j \in E$. In fact, the converse to this holds as well. To see this, suppose \mathcal{A} does not separate $x, y \in E$ and find a sequence $\{M_{\psi_n}\} \subset \mathcal{A}$ such that $\psi_n h$ converges to k_z^L for a fixed $z \in X$. Then we have

$$\begin{aligned} \langle k_x^L, k_z^L \rangle &= k_z^L(x) = \lim_{n \rightarrow \infty} \psi_n(x)h(x) = \lim_{n \rightarrow \infty} \psi_n(y)h(x) \\ &= \frac{h(y)}{h(x)} k_z^L(y) = \frac{h(y)}{h(x)} \langle k_y^L, k_z^L \rangle. \end{aligned}$$

Since z was arbitrary, it follows that $\overline{h(x)}k_y^L = \overline{h(y)}k_x^L$, and so k_x^L and k_y^L are linearly dependent.

To prove the proposition, we first assume that \mathcal{A} fails to separate any pair of points in E , so that $\varphi(x_i) = \varphi(x_j)$ for every $x_i \in E$ and $\varphi \in \mathcal{A}$. By the above reasoning, since each $k_{x_i}^L$ is non-zero, we may find non-zero numbers $t_1, \dots, t_n \in \mathbb{C}$ so that $k_x^L := k_{x_1}^L = t_i k_{x_i}^L$. It follows that

$$[(1 - w_i \overline{w_j})k^L(x_i, x_j)] = [(1 - w_i \overline{w_j})t_i \overline{t_j}]k^L(x, x) \geq 0.$$

This implies that $[1 - w_i \overline{w_j}] \geq 0$, and so the vector (w_1, \dots, w_n) is in the range of the matrix $[1]_{i,j}$. It follows that $w_1 = \dots = w_n$, and so the Pick problem in this case reduces to the trivial single point case. For the general case, the subset E may be split into equivalence classes E_1, \dots, E_k where \mathcal{A} identifies all points in each class. If $E_i = \{x_{i_1}, \dots, x_{i_k}\}$, then the above reasoning implies that $w_{i_1} = \dots = w_{i_k}$ for each i . Now to construct a multiplier M_φ in \mathcal{A} that interpolates the data, simply choose functions φ_i so that $\varphi_i|_{E_j} = \delta_{ij}$, and take $\varphi = \sum_{i=1}^k w_{i_k} \varphi_i$. \square

Remark 3.2.12. It is worthwhile to see what happens when dealing with invariant subspaces for \mathcal{A} which are not cyclic. Indeed, there is little additional complication when the subspaces $\mathfrak{N}_x = L \ominus \mathfrak{J}_x L$ have dimension greater than one. These subspaces are at a positive angle to each other (even when they are infinite dimensional) because the restriction of $P_L M_\varphi^*$ to \mathfrak{N}_x is $\overline{\varphi(x)} P_{\mathfrak{N}_x}$ by Porism 3.2.2. When \mathcal{A} separates points x and y , the boundedness of M_φ^* yields a positive angle between eigenspaces. Moreover, the spaces \mathfrak{N}_x for $x \in X$ span L (since the spaces \mathfrak{M}_x span L). We are interested in the norm $\|P_{\mathfrak{N}_L} M_\varphi^*|_{\mathfrak{N}_L}\|$. Since the span of the spaces \mathfrak{N}_x is dense in L , this norm is approximately achieved at a vector $h = \sum h_x$ where this is a finite sum of vectors $h_x \in \mathfrak{N}_x$. Since

$$P_L M_\varphi^* h = \sum \overline{\varphi(x)} h_x,$$

it follows that $K = \text{span}\{h_x : x \in X\}$ is invariant for $P_L M_\varphi^*$ for all $M_\varphi \in \mathcal{A}$. In particular, we obtain that $\|P_{\mathfrak{N}_L} M_\varphi^*|_{\mathfrak{N}_L}\| \leq 1$ if and only if $\|P_K M_\varphi^*|_K\| \leq 1$ for each subspace K of the form just described. This is equivalent to saying $P_K - P_K M_\varphi M_\varphi^*|_K \geq 0$ because of semi-invariance. Because the h_k span K , this occurs if and only if

$$0 \leq \left[\langle (I - M_\varphi M_\varphi^*) h_{x_j}, h_{x_i} \rangle \right] = \left[(1 - \varphi(x_i) \overline{\varphi(x_j)}) \langle h_{x_j}, h_{x_i} \rangle \right].$$

Thus, the norm condition is equivalent to the simultaneous positivity of a family of Pick matrices. Moreover, in the case of \mathcal{I}_E for a finite set $E = \{x_1, \dots, x_n\}$ which is separated by \mathcal{A} , this family of Pick matrices is positive if and only if we have positivity of the operator matrix

$$\left[(1 - \varphi(x_i) \overline{\varphi(x_j)}) P_{\mathfrak{N}_{x_i}} P_{\mathfrak{N}_{x_j}} \right]_{i,j=1}^n.$$

If we can put a bound on the dimension of \mathfrak{N}_L , the above operator matrix acts on finite dimensional space. For example, if $L = \mathcal{A}[f_1, \dots, f_k]$, then $\dim(\mathfrak{N}_L) \leq kn$.

3.3 Algebras with property $\mathbb{A}_1(1)$

Given two operator algebras $\mathcal{A} \subset \mathcal{B} \subset \mathcal{B}(\mathcal{H})$, we say that \mathcal{A} is *relatively reflexive* in \mathcal{B} if $\mathcal{A} = (\text{Alg Lat } \mathcal{A}) \cap \mathcal{B}$. If \mathcal{B} is reflexive, then $\text{Alg Lat } \mathcal{A} \subset \text{Alg Lat } \mathcal{B} = \mathcal{B}$, and so \mathcal{A} is actually reflexive. Turning back to multipliers, suppose \mathcal{A} is a dual algebra of multipliers that is relatively reflexive. The reflexivity of the whole multiplier algebra (c.f. Theorem 2.3.11) implies that \mathcal{A} is actually reflexive.

Since any M_Φ in \mathcal{A} defines the multiplication operator M_Φ^L on every invariant subspace L of \mathcal{A} , we see that

$$\mathcal{A} \subset \bigcap_{L \in \text{Lat } \mathcal{A}} \text{Mult}(L).$$

There is a slight abuse of notation here, since it is really the *multipliers* which induce the multiplication operators in \mathcal{A} that are contained in the above intersection. On the other hand, if Φ is simultaneously a multiplier on each L in $\text{Lat } \mathcal{A}$, then Φ is, in particular, contained in $\text{Mult}(\mathcal{H})$. Moreover, since Φ also satisfies $\Phi L \subset L$, it must be the case that M_Φ leaves each L invariant. Consequently, the reflexivity of \mathcal{A} implies that M_Φ is in \mathcal{A} ,

and so \mathcal{A} is the *largest* algebra of multipliers for the family of subspaces $\text{Lat}(\mathcal{A})$. We can summarize this discussion with the following proposition.

Proposition 3.3.1. *Suppose \mathcal{A} is a dual algebra of multipliers on a full reproducing kernel Hilbert space \mathcal{H} and that \mathcal{A} is relatively reflexive in $\mathcal{M}(\mathcal{H})$. Then we have*

$$\mathcal{A} = \bigcap_{L \in \text{Lat}(\mathcal{A})} \text{Mult}(L) = \bigcap_{L \in \text{CycLat}(\mathcal{A})} \text{Mult}(L).$$

Proof. The first equality is implied by the discussion above. The second follows from the observation that if Φ is in $\bigcap_{L \in \text{CycLat}(\mathcal{A})} \text{Mult}(L)$ and $M \in \text{Lat } \mathcal{A}$, then for any $f \in M$ we have

$$M_\Phi f \in \mathcal{A}[f] \subset M. \quad \square$$

If $\{\mathcal{H}_i\}$ is a family of reproducing kernel Hilbert spaces and \mathcal{A} is an algebra of functions which satisfies

$$\mathcal{A} = \bigcap_i \text{Mult}(\mathcal{H}_i),$$

then \mathcal{A} is called *realizable*. This term is typically used to describe a Banach algebra of functions, but we feel it is reasonable to adopt it for this discussion (see Agler and McCarthy [4, Chapter 13]). Proposition 3.3.1 indicates that a realizable dual algebra of multipliers \mathcal{A} is the largest subalgebra of $\mathcal{M}(\mathcal{H})$ that leaves $\text{Lat}(\mathcal{A})$ invariant. If \mathcal{A} were not realizable, we could replace it with $\text{Alg Lat}(\mathcal{A})$, which *is* realizable (since $\text{Lat Alg Lat } \mathcal{A} = \text{Lat } \mathcal{A}$). An obvious question is prompted by this discussion: which multiplier algebras have the property that every dual subalgebra is reflexive? There is a partial answer to this question, due to Loginov and Shulman [43].

Theorem 3.3.2 (Loginov-Shulman). *Suppose \mathcal{S} is a reflexive subspace of $\mathcal{B}(\mathcal{H})$. Then \mathcal{S} has property \mathbb{A}_1 if and only if every weak-* closed subspace of \mathcal{S} is reflexive.*

This leads to several immediate questions. Which multiplier algebras have property \mathbb{A}_1 , and what additional properties do such algebras have? Since the Pick problem has an inherently quantitative meaning, it is perhaps more profitable to consider property $\mathbb{A}_1(r)$. We begin with a more general discussion of distance formulae, developed in generality by Hadwin and Nordgren [35] along with Kraus and Larson [40].

Given dual algebras \mathcal{A} and \mathcal{B} with $\mathcal{A} \subset \mathcal{B}$, we say that \mathcal{A} is *relatively hyper-reflexive* with respect to \mathcal{B} if there exists a constant $C > 0$ so that

$$\text{dist}(\mathcal{B}, \mathcal{A}) \leq C \sup_{L \in \text{Lat } \mathcal{A}} \|P_L^\perp B P_L\|$$

for every $B \in \mathcal{B}$. It is straight forward to verify that relative hyper-reflexivity implies relative reflexivity, and that the distance estimate

$$\text{dist}(\mathcal{B}, \mathcal{A}) \geq \sup_{L \in \text{Lat } \mathcal{A}} \|P_L^\perp B P_L\|$$

always holds. Using the same techniques as in Proposition 3.1.4, it is also clear that we

can restrict our attention to cyclic invariant subspaces, i.e. that

$$\sup_{L \in \text{Lat } \mathcal{A}} \|P_L^\perp B P_L\| = \sup_{L \in \text{CycLat } \mathcal{A}} \|P_L^\perp B P_L\|.$$

The following theorem relates the concepts of relative hyper-reflexivity and property $\mathbb{A}_1(r)$.

Theorem 3.3.3 (Hadwin-Nordgren, Kraus-Larson). *Suppose \mathcal{B} is a dual subalgebra of $\mathcal{M}(\mathcal{H})$ and has property $\mathbb{A}_1(r)$. Then every WOT-closed unital subalgebra \mathcal{A} of \mathcal{B} is reflexive. Moreover, \mathcal{A} is relatively hyper-reflexive in \mathcal{B} with distance constant at most r .*

If \mathcal{B} has property $\mathbb{A}_1(1)$, we obtain an exact distance formula.

Corollary 3.3.4. *Suppose that \mathcal{B} has property $\mathbb{A}_1(1)$, and let \mathcal{A} be a WOT-closed unital subalgebra. Then*

$$\text{dist}(B, \mathcal{A}) = \sup_{L \in \text{CycLat}(\mathcal{A})} \|P_L^\perp B P_L\| \quad \text{for all } B \in \mathcal{B}.$$

We can use the same methods to obtain a distance formula to any weak-* closed ideal. An argument similar to this one is contained in the proof of [27, Theorem 2.1], where the Pick problem is studied for the noncommutative analytic Toeplitz algebra (see Chapter 5). We first state the following basic consequence of the Hahn-Banach theorem for extending weak-* continuous functionals.

Lemma 3.3.5. *Suppose X is a Banach space and Y is a closed subspace of the dual space X^* . Then for any $\omega \in X^*$, we have the distance formula*

$$\text{dist}(\omega, Y) = \sup\{|\omega(x)| : x \in Y_\perp, \|x\| \leq 1\}.$$

Theorem 3.3.6. *Suppose that \mathcal{A} is a dual algebra on \mathcal{H} with property $\mathbb{A}_1(r)$, and let \mathcal{I} be any weak-* closed ideal of \mathcal{A} . Then we obtain*

$$\sup_{L \in \text{CycLat}(\mathcal{A})} \|P_{L \ominus \mathcal{I}L} A|_{L \ominus \mathcal{I}L}\| \leq \text{dist}(A, \mathcal{I}) \leq r \sup_{L \in \text{CycLat}(\mathcal{A})} \|P_{L \ominus \mathcal{I}L} A|_{L \ominus \mathcal{I}L}\|,$$

for any $A \in \mathcal{A}$.

Proof. By an argument identical to that of Proposition 3.1.4, the first inequality follows immediately. Conversely, given $A \in \mathcal{A}$ and $\varepsilon > 0$, by Lemma 3.3.5 above, choose $\omega \in \mathcal{A}_*$ such that $\omega|_{\mathcal{I}} = 0$, $\|\omega\| < 1 + \varepsilon$ and $|\omega(A)| > \text{dist}(A, \mathcal{I}) - \varepsilon$. Using property $\mathbb{A}_1(r)$, we obtain vectors f and g with $\|f\| = 1$ and $\|g\| < r + \varepsilon$ so that $\omega(A) = \langle Af, g \rangle$. Set $L = \mathcal{A}[f]$ in $\text{CycLat } \mathcal{A}$. Since L is invariant, we can and do replace g by $P_L g$. Moreover, since $\omega|_{\mathcal{I}} = 0$, the function g is orthogonal to $\mathcal{I}[f] = \mathcal{I}L$. Hence, y belongs to $L \ominus \mathcal{I}L$ and so

$$\begin{aligned} \text{dist}(A, \mathcal{I}) &< |\langle Ax, y \rangle| + \varepsilon = |\langle A P_L f, P_{L \ominus \mathcal{I}L}^\perp g \rangle| + \varepsilon \\ &= |\langle P_{L \ominus \mathcal{I}L} A P_L f, g \rangle| + \varepsilon = |\langle P_{L \ominus \mathcal{I}L} A P_{L \ominus \mathcal{I}L} f, g \rangle| + \varepsilon \\ &\leq \|P_{L \ominus \mathcal{I}L} A P_{L \ominus \mathcal{I}L}\| \|f\| \|g\| + \varepsilon \\ &< (r + \varepsilon) \|P_{L \ominus \mathcal{I}L} A|_{L \ominus \mathcal{I}L}\| + \varepsilon. \end{aligned}$$

It follows that the rightmost quantity in the statement of the theorem dominates $\text{dist}(A, \mathcal{I})$. \square

As an immediate consequence, we get a general Pick interpolation theorem for dual algebras of multipliers with property $\mathbb{A}_1(r)$.

Theorem 3.3.7. *Suppose that $\mathcal{H} = \mathcal{H}(K, \mathcal{L}, X)$ is a full reproducing kernel Hilbert space and that $E = \{x_1, \dots, x_n\} \subset X$. Suppose \mathcal{A} is a dual algebra of multipliers on \mathcal{H} with property $\mathbb{A}_1(r)$. Then the following distance estimate holds:*

$$\begin{aligned} \sup\{\|P_{\mathfrak{N}_L} M_{\Phi}^*|_{\mathfrak{N}_L}\| : L \in \text{CycLat}(\mathcal{A})\} &\leq \text{dist}(M_{\Phi}, \mathcal{I}_E) \\ &\leq r \sup\{\|P_{\mathfrak{N}_L} M_{\Phi}^*|_{\mathfrak{N}_L}\| : L \in \text{CycLat}(\mathcal{A})\}. \end{aligned}$$

In particular, if $r = 1$ then $\{K^L : L \in \text{CycLat}(\mathcal{A})\}$ is a Pick family of kernels for \mathcal{A} .

For a scalar-valued space \mathcal{H} , we obtain the above theorem in terms of Pick matrices.

Corollary 3.3.8. *Suppose \mathcal{A} is a dual algebra of multipliers on a reproducing kernel Hilbert space \mathcal{H} on X that has property $\mathbb{A}_1(1)$. Let $\{x_1, \dots, x_n\}$ be distinct points in X separated by \mathcal{A} and let $\{w_1, \dots, w_n\}$ be complex numbers. There is a multiplier f in the unit ball of \mathcal{A} such that $f(x_i) = w_i$ for each i if and only if*

$$\left[(1 - w_i \overline{w_j}) k^L(x_i, x_j) \right] \geq 0 \quad \text{for all } L \in \text{CycLat}(\mathcal{A}).$$

If \mathcal{A} instead has property $\mathbb{A}_1(r)$ for some $r \geq 1$, the positivity of the above matrices implies the existence of a solution of norm at most r .

As we have seen, it is convenient to use the spaces \mathfrak{M}_L instead of \mathfrak{N}_L (i.e. that we are dealing with a *strong* Pick family). In light of Corollary 3.2.7, this is possible for cyclic subspaces $L = \mathcal{A}[h]$ provided that h does not vanish on E . A refinement of the $\mathbb{A}_1(1)$ property can make this possible.

Theorem 3.3.9. *Let \mathcal{A} be a dual algebra of multipliers on \mathcal{H} , and let $E = \{x_1, \dots, x_n\}$ be a finite subset of X which is separated by \mathcal{A} . Suppose that \mathcal{A} has property $\mathbb{A}_1(1)$ with the additional stipulation that each weak-* continuous functional ω on \mathcal{A} with $\|\omega\| < 1$ can be written as $\omega(A) = \langle Af, g \rangle$ with $\|f\| \|g\| < 1$ such that f does not vanish on E . Then there is a multiplier in the unit ball of \mathcal{A} with $\varphi(x_i) = w_i$ for $x_i \in E$ if and only if*

$$\left[(1 - w_i \overline{w_j}) \langle P_L k_{x_j}, k_{x_i} \rangle \right] \geq 0$$

for all cyclic subspaces $L = \mathcal{A}[h]$ where h does not vanish on E . In particular, the collection of kernels

$$\{k^{\mathcal{A}[h]} : h(x_i) \neq 0 \text{ for every } x_i \in E\}$$

is a strong Pick family for \mathcal{A} .

3.4 Examples

A clear benefit of the dual algebra approach to Pick interpolation theory is that the structure of dual algebras and their preduals has received remarkable attention in the last 30 years. Fortunately, multiplier algebras lend themselves readily to predual techniques. In this section, we will describe some well known examples from the literature. The bestiary of multiplier algebras with property $\mathbb{A}_1(1)$ will be greatly expanded in subsequent chapters.

Let $\mathcal{H} = H^2$ be Hardy space, so that $\mathcal{M}(\mathcal{H}) = H^\infty$. It is well known that this algebra has property $\mathbb{A}_1(1)$ [14, Theorem 3.7]. Since this is a fundamental result, we include an elegant and folkloric proof communicated by Hari Bercovici.

Theorem 3.4.1. *The unilateral shift has property $\mathbb{A}_1(1)$.*

Proof. Suppose ω is a weak-* continuous functional on $\mathcal{M}(H^2(\mathbb{D})) \cong H^\infty(\mathbb{D})$. There are sequences of functions $f = (f_i)$ and $g = (g_i)$ in $(H^2)^\infty$ such that

$$\omega(\varphi) = \sum_{i=1}^{\infty} \langle \varphi f_i, g_i \rangle = \langle \varphi^{(\infty)} f, g \rangle, \varphi \in H^\infty(\mathbb{D})$$

and $\sum_{i=1}^{\infty} |f_i|^2 \sum_{i=1}^{\infty} |g_i|^2 = \|\omega\|^2$. Let $L := (H^\infty)^\infty[f]$ and notice that by the von Neumann-Wold decomposition, the algebra $(H^\infty)^\infty|_L$ is unitarily equivalent to H^∞ . Applying this unitary to the function sequences f and g allows us to write ω as a rank 1 functional. \square

Remark 3.4.2. Of course, once we know that every weak-* continuous functional ω on H^∞ can be written as $\omega(\varphi) = \langle \varphi f, g \rangle$, we can apply inner-outer factorization to f and write $f = \gamma F$. It follows that $\omega(\varphi) = \langle \varphi F, M_\gamma^* g \rangle$, and so when applying our interpolation result (Theorem 3.3.9), it suffices to restrict to invariant subspaces generated by *cyclic* vectors. It follows that only one kernel is required in order to solve the Pick problem in our framework, namely the Szegő kernel (as expected!).

Perhaps more importantly, Theorem 3.3.7 provides an interpolation theorem for *any* weak-* closed subalgebra of H^∞ . Generally, we will refer to an interpolation problem concerning a strict subalgebra of $\mathcal{M}(\mathcal{H})$ as a *constrained* interpolation problem. The greatest advantage of the approach that we have taken is that predual factorization properties are inherited by weak-* closed subspaces. In particular, if the full multiplier algebra has property $\mathbb{A}_1(1)$, so does every dual algebra of multipliers on \mathcal{H} . These constrained theorems fashion suitable generalizations of the results seen in Davidson-Paulsen-Raghupathi-Singh [25] and Raghupathi [57].

Returning to H^∞ , since any weak-* linear functional φ on \mathcal{A} extends to a functional on H^∞ with an arbitrarily small increase in norm, it can be factored as $\varphi = [hk^*]$ where h is outer and $\|h\| \|k\| < \|\varphi\| + \varepsilon$. Therefore the more refined Theorem 3.3.9 applies.

Theorem 3.4.3. *Let \mathcal{A} be a weak-* closed unital subalgebra of H^∞ . Let $E = \{z_1, \dots, z_n\}$ be a finite subset of \mathbb{D} which is separated by \mathcal{A} . Then there is a multiplier φ in the unit*

ball of \mathcal{A} with $\varphi(z_i) = w_i$ for $1 \leq i \leq n$ if and only if

$$\left[(1 - w_i \bar{w}_j) \langle P_L k_{z_i}, k_{z_j} \rangle \right] \geq 0$$

for all cyclic subspaces $L = \mathcal{A}[h]$ where h is outer.

In [25], the algebra $H_1^\infty = \{f \in H^\infty : f'(0) = 0\}$ is studied. Beurling's Theorem was used to show that there is a simple parameterization of the cyclic invariant subspaces $H_1^\infty[h]$ for h outer, namely

$$H_{\alpha,\beta}^2 := \text{span}\{\alpha + \beta z, z^2 H^2\} \quad \text{for all } (\alpha, \beta) \text{ with } |\alpha|^2 + |\beta|^2 = 1.$$

It was shown that these provide a Pick family for H_1^∞ , and the consequent interpolation theorem. Raghupathi [57] carries out this program for the class of algebras $H_B^\infty = \mathbb{C}1 + BH^\infty$ where B is a finite Blaschke product.

Example 3.4.4 (Algebras of the form $\mathbb{C} + BH^\infty$). Suppose B is a finite Blaschke product of degree p (including multiplicity), and form the dual algebra of multipliers $\mathcal{A} := \mathbb{C} + BH^\infty$ on H^2 . By Theorem 3.4.3, subspaces of the form $\mathcal{A}[h]$ where h is outer form a strong Pick family for \mathcal{A} . Notice that $BH^\infty[h] = BH^2$, and hence $\mathcal{A}[h] = \text{span}\{h, BH^2\}$. By projecting h onto the $(BH^2)^\perp$, we obtain the decomposition

$$\mathcal{A}[h] = P_{BH^2}^\perp h \oplus BH^2.$$

As we have seen, if B contains zeroes of multiplicity no greater than 1, then BH^2 is spanned by the kernel functions corresponding to each zero. For higher multiplicity zeroes, the kernel functions which evaluate derivatives are also contained in $(BH^2)^\perp$ (cf. Example 3.2.4). In either case, $P_{BH^2}^\perp h$ is contained in a p dimensional subspace. If we let $\{e_1, \dots, e_p\}$ denote an orthonormal basis for $(BH^2)^\perp$, then we may consider Pick matrices generated by invariant subspaces of the form

$$L_a := \mathbb{C}(a_1 e_1 + \dots + a_p e_p) \oplus BH^2, \quad a = (a_1, \dots, a_p) \in \partial \mathbb{B}_p,$$

which are easily seen to be invariant for \mathcal{A} and include all subspaces of the form $\mathcal{A}[h]$ for h outer. Thus, we obtain an explicit parametrization of a strong Pick family for \mathcal{A} by the compact set \mathbb{B}_p . Of course, one might ask if the positivity of every Pick matrix associated to L_a (or at least a dense subset of \mathbb{B}_p) is required to guarantee a solution. In [25], analysis of this type was carried out for the algebra $\mathbb{C} + z^2 H^\infty$ acting on H^2 . The answer is surprising: all subspaces of the form $\mathbb{C}(a + bz) \oplus z^2 H^2$ are required where the monomial $a + bz$ is outer (that is, $|a| < |b|$).

In [56], Raghupathi shows that Abrahamse's interpolation result for multiply connected domains [1] is equivalent to the interpolation problem for certain fixed-point subalgebras of H^∞ . These algebras arise from recognizing that the disk is a universal cover for an arbitrary multiply connected region. Essentially, Raghupathi shows that these algebras have a property not unlike property $\mathbb{A}_1(1)$, and then he shows that one may restrict to outer functions which are so called *character automorphisms* of this fixed point algebra.

3.4.1 Isometric functional calculus

Suppose that T is an absolutely continuous contraction on a Hilbert space \mathcal{H} , and let \mathcal{A}_T be the unital, weak- $*$ -closed algebra generated by T . The Sz.Nagy dilation theory provides a weak- $*$ continuous, contractive homomorphism $\mathbb{F} : H^\infty \rightarrow \mathcal{A}_T$ given by $\mathbb{F}(\varphi) = P_{\mathcal{H}}f(U)|_{\mathcal{H}}$, where U is the minimal unitary dilation of T . If \mathbb{F} is isometric (i.e. $\|\mathbb{F}(\varphi)\| = \|\varphi\|_\infty$ for every $\varphi \in H^\infty$), then \mathbb{F} is also a weak- $*$ homeomorphism. In addition, \mathbb{F} being isometric ensures that the preduals $(\mathcal{A}_T)_*$ and L_1/H_0^1 are isometrically isomorphic. In this case, we say that T has an *isometric functional calculus*. See [18] for relevant details. The following deep result of Bercovici [16] will be used.

Theorem 3.4.5 (Bercovici). *Suppose T is an absolutely continuous contraction on \mathcal{H} and that T has an isometric functional calculus. Then \mathcal{A}_T has property $\mathbb{A}_1(1)$.*

We will use Theorem 3.4.5 to show that a wide class of reproducing kernel Hilbert spaces admit Pick families of kernels for arbitrary dual subalgebras of H^∞ .

Example 3.4.6. Let μ be a finite Borel measure on \mathbb{D} , and let $P^2(\mu)$ denote the closure of the polynomials in $L^2(\mu)$. A *bounded point evaluation* for $P^2(\mu)$ is a point x for which there exists a constant $M > 0$ with $|p(x)| \leq M\|p\|_{P^2(\mu)}$ for every polynomial p . A point x is said to be an *analytic bounded point evaluation* for $P^2(\mu)$ if x is in the interior of the set of bounded point evaluations, and that the map $z \rightarrow f(z)$ is analytic on a neighborhood of x for every $f \in P^2(\mu)$.

It follows that if x is a bounded point evaluation, then there is a kernel function k_x in $P^2(\mu)$ so that $p(x) = \langle p, k_x \rangle$. For an arbitrary $f \in P^2(\mu)$, if we set $f(x) = \langle f, k_x \rangle$, then these values will agree with f a.e. with respect to μ . For both the Hardy space and Bergman space, the set of analytic bounded point evaluation is all of \mathbb{D} . A ground-breaking theorem of Thomson shows that either $P^2(\mu) = L^2(\mu)$ or $P^2(\mu)$ has analytic bounded point evaluations [64].

Let m be Lebesgue area measure on the disk. For $s > 0$, define a weighted area measure on \mathbb{D} by $d\mu_s(z) = (1 - |z|)^{s-1}dm(z)$. The monomials z^n form an orthogonal basis for $P^2(\mu)$. This includes the Bergman space for $s = 1$. For these spaces, every point in \mathbb{D} is an analytic point evaluation.

The following result appears as [4, Theorem 4.6].

Theorem 3.4.7. *Let μ be a measure on \mathbb{D} such that the set of analytic bounded point evaluations of $P^2(\mu)$ contains all of \mathbb{D} . Then $\mathcal{M}(P^2(\mu))$ is isometrically isomorphic and weak- $*$ homeomorphic to H^∞ .*

Corollary 3.3.8 yields the following interpolation result for these spaces.

Theorem 3.4.8. *Let μ be a measure on \mathbb{D} such that the set of analytic bounded point evaluations of $P^2(\mu)$ contains all of \mathbb{D} . Suppose \mathcal{A} is a dual subalgebra of $\mathfrak{M}(P^2(\mu))$. Then \mathcal{A} has a Pick family of kernels.*

Example 3.4.9. In particular, Theorem 3.4.8 provides a Pick condition for Bergman space $A^2 := L_a^2(\mathbb{D})$, whose reproducing kernel $k_x^B = (1 - \bar{x}z)^{-2}$ is not a Pick kernel. The

multiplier algebra of Bergman space has property $\mathbb{A}_1(1)$ as a consequence of much stronger properties, but the subspace lattice of the Bergman shift is immense.

Lastly, we invoke the grandfather of all predual factorization results, namely the theorem of Scott Brown [21], which shows that there is some $r \geq 1$ so that \mathcal{A}_T has property $\mathbb{A}_1(r)$ for any cyclic subnormal operator T on \mathcal{H} . This was later improved to property $\mathbb{A}_1(1)$ for every subnormal operator by Bercovici and Conway [13]. This includes, for instance, the multiplication by z operator on the spaces $R^2(K, \mu)$, the closure of the rational functions in $L^2(K, \mu)$ on some compact subset $K \subset \mathbb{C}$ with Borel measure μ . The analytic bounded point evaluations on $R(K)$ (see above) naturally form a reproducing kernel Hilbert space whose multiplier algebra is generated by M_z . Analogous Pick interpolation results can then be stated on these domains with our terminology.

3.5 The infinite ampliation of a multiplier algebra and matrix-valued interpolation

In this section, we will show that any dual algebra of multipliers \mathcal{A} on a full reproducing kernel Hilbert space \mathcal{H} admits a Pick-type theorem and will use this fact to establish a general matrix-valued interpolation result for scalar-valued kernels. While we cannot hope that every such algebra has property $\mathbb{A}_1(1)$ (see Appendix A), recall that the infinite ampliation $\mathcal{A}^{(\infty)}$ does have it. This observation is essentially the starting point for the distance formulae in Arveson [8], McCullough [46], and Raghupathi-Wick [58].

Given a finite subset $E = \{x_1, \dots, x_n\} \subset X$, we may once again form the ideal \mathcal{I}_E in \mathcal{A} . Of course, we have the following

$$\text{dist}(M_\Phi, \mathcal{I}_E) = \text{dist}(M_\Phi^{(\infty)}, \mathcal{I}_E^{(\infty)}).$$

The algebra $\mathcal{A}^{(\infty)}$ is a dual algebra of multipliers on the reproducing kernel Hilbert space $\mathcal{H} \otimes \ell^2$ in the obvious way. Since it has property $\mathbb{A}_1(1)$, we immediately obtain the following distance formula, valid for any dual algebra of multipliers.

Proposition 3.5.1. *Suppose \mathcal{A} is a dual algebra of multipliers on a full reproducing kernel Hilbert space \mathcal{H} . The distance to the ideal \mathcal{I}_E is given by*

$$\text{dist}(M_\Phi, \mathcal{I}_E) = \sup \left\{ \left\| P_{\mathfrak{N}_L} (M_\Phi^*)^{(\infty)} P_{\mathfrak{N}_L} \right\| : L \in \text{CycLat } \mathcal{A}^{(\infty)} \right\}$$

for any $M_\Phi \in \mathcal{A}$.

Now we assume that k is a scalar-valued kernel. For a natural number $r \geq 2$, we naturally identify (cf. Theorem 2.3.13) the algebras $\mathcal{M}(\mathcal{H} \otimes \ell_r^2)$ and $M_r(\mathcal{M}(\mathcal{H}))$. Moreover, the ideal of functions in this algebra that vanishes on a finite set $E \subset X$ is precisely given by $M_r(\mathcal{I}_E)$. The classical Pick theorem for matrices says that given z_1, \dots, z_n in the disk, and $r \times r$ matrices W_1, \dots, W_n , there is a function Φ in the the unit ball of $\mathcal{M}_r(H^\infty)$ such

that $\Phi(z_i) = W_i$ if and only if the Pick matrix

$$\left[\begin{array}{c} I_r - W_i W_j^* \\ 1 - z_i \bar{z}_j \end{array} \right]_{r \times r}$$

is positive semidefinite. Equivalently, we have

$$\text{dist}(M_\Phi, \mathcal{M}_r(\mathcal{I}_E)) = \|M_\Phi^*|_{\mathfrak{M} \otimes \mathbb{C}^r}\|.$$

In other words, the representation $M_\Phi + \mathcal{I}_E \mapsto P_{\mathfrak{M}} M_\Phi P_{\mathfrak{M}}$ is a complete isometry.

Any dual subalgebra \mathcal{A} of $\mathcal{M}(\mathcal{H})$ determines the dual subalgebra $\mathcal{M}_r(\mathcal{A})$ of $\mathcal{M}_r(\mathcal{M}(\mathcal{H}))$. Suppose that $E = \{x_i : 1 \leq i \leq n\}$ is a finite subset of X separated by \mathcal{A} and \mathcal{I}_E the corresponding ideal. For $M_\Phi = [M_{\varphi_{ij}}] \in M_r(\mathcal{A})$, any subspace of the form $L^{(r)}$ for $L \in \text{Lat}(\mathcal{A})$ is invariant for $\mathcal{M}_r(\mathcal{A})$. Conversely, any invariant subspace of $\mathcal{M}_r(\mathcal{A})$ takes this form.

The subspace $L^{(r)}$ is cyclic if and only if L is r -cyclic, since if f_1, \dots, f_r is a cyclic set for L , then $f := (x = f_1, \dots, f_r)$ is a cyclic vector for $L^{(r)}$ and vice versa. So in general we cannot hope to use only the cyclic subspaces from \mathcal{A} , but rather its finitely generated invariant subspaces. We will have to deal with some multiplicity of the kernels on these spaces. This can be handled as in the discussion in Remark 3.2.12.

Definition 3.5.2. Suppose E is a finite subset of X which is separated by \mathcal{A} . If for any $M_\Phi \in M_r(\mathcal{A})$, we have

$$\text{dist}(M_\Phi, \mathcal{M}_r(\mathfrak{J}_E)) = \sup_{L^{(r)} \in \text{CycLat } M_r(\mathcal{A})} \|(P_{\mathfrak{N}_L} \otimes I_r) M_\Phi (P_{\mathfrak{N}_L} \otimes I_r)\|$$

then we say that $\text{Lat } \mathcal{A}$ is an $r \times r$ Pick family for \mathcal{A} . If this holds for all $r \geq 1$, then we say that $\text{Lat } \mathcal{A}$ is a complete Pick family for \mathcal{A} .

The matrix-valued interpolation theorem that follows from the above definitions can now be stated.

Theorem 3.5.3. Suppose \mathcal{A} is a dual algebra of multipliers on \mathcal{H} that separates points in the finite set $E = \{x_1, \dots, x_n\}$ and let $W_1, \dots, W_n \in M_r$. If $\text{Lat } \mathcal{A}$ is an $r \times r$ Pick family, there is a contractive $M_\Phi \in \mathcal{M}_r(\mathfrak{A})$ with $\Phi(x_i) = W_i$ if and only if the matrices

$$\left[(I_r - W_i W_j^*) \otimes K^L(x_i, x_j) \right]$$

are positive semidefinite for every r -generated $L \in \text{Lat } \mathcal{A}$, where

$$K^L(x_i, x_j) := P_{L \ominus \mathfrak{J}_{x_j} L} P_{L \ominus \mathfrak{J}_{x_i} L}.$$

Proof. Suppose $M_\Psi \in M_r(\mathcal{A})$ is an arbitrary interpolant. Since $\text{Lat } \mathcal{A}$ is an $r \times r$ Pick family, we have

$$\text{dist}(M_\Psi, \mathcal{M}_r(\mathfrak{J}_E)) = \sup_L \|(P_{\mathfrak{N}_L} \otimes I_r) M_\Psi (P_{\mathfrak{N}_L} \otimes I_r)\|,$$

where L ranges over all the r -cyclic invariant subspaces for \mathcal{A} . Arguing as we have previously done several times, if the terms in the above equation are at most 1, we have

$$(P_{\mathfrak{N}_L} \otimes I_r)(I - M_{\Psi} M_{\Psi}^*)(P_{\mathfrak{N}_L} \otimes I_r) \geq 0.$$

As we observed in Remark 3.2.12, \mathfrak{N}_L is spanned by the spaces \mathfrak{N}_{x_i} for $1 \leq i \leq n$. These subspaces are eigenspaces for $(P_L \mathfrak{A}|_L)^*$, and by applying this fact to M_{Ψ} entry-wise, we have

$$(P_L \otimes I_r) M_{\Psi}^* (P_{\mathfrak{N}_{x_i}} \otimes I_r) = P_{\mathfrak{N}_{x_i}} \otimes W_i^*.$$

Therefore the positivity of the operator above is equivalent to the positivity of

$$\begin{aligned} \left[(P_{\mathfrak{N}_{x_i}} \otimes I_r)(I - M_{\Psi} M_{\Psi}^*)(P_{\mathfrak{N}_{x_j}} \otimes I_r) \right] &= \left[(I_r - W_i W_j^*) \otimes P_{\mathfrak{N}_{x_i}} P_{\mathfrak{N}_{x_j}} \right] \\ &= \left[(I_r - W_i W_j^*) \otimes K^L(x_i, x_j) \right] \end{aligned}$$

as desired. \square

Remark 3.5.4. Note that for a subspace of the form $L = \mathcal{A}[f_1, \dots, f_r]$, the subspace $L \ominus \mathfrak{J}_x L$ has dimension at most r . We can therefore consider kernels of the form K^L above as being M_r -valued.

Generally, property $\mathbb{A}_1(1)$ is not inherited by matrix algebras. This failure even burdens our most fundamental example in this subject: the unilateral shift [14, Theorem 3.7]. Another additional complication is that even though many algebras are known to have the property $\mathbb{A}_r(1)$, this property does not generally imply that $M_r(\mathcal{A})$ has property $\mathbb{A}_1(1)$. This is rather unfortunate, since it is precisely the latter condition we need to ensure that a reasonable Pick theorem holds for these algebras. We will see a few examples of reproducing kernel Hilbert spaces in Chapter 6 that enjoy these properties. In the meantime, we must settle for something less.

It may be the case that some finite ampliation of the algebra \mathcal{A} will have $\mathbb{A}_1(1)$. As in Theorem 3.3.7, if $M_r(\mathcal{A})$ has property $\mathbb{A}_1(1)$, then we obtain an exact distance formula which yields a Pick type theorem for these algebras. Proposition 3.5.1 immediately yields the following.

Theorem 3.5.5. *Suppose \mathcal{A} is a dual algebra of multipliers on \mathcal{H} . If $M_r(\mathcal{A})$ has property $\mathbb{A}_1(1)$, then $\text{Lat } \mathcal{A}$ is an $r \times r$ Pick family for \mathcal{A} .*

More generally, if the ampliation $M_r(\mathcal{A}^{(s)})$ has $\mathbb{A}_1(1)$, then $\text{Lat}(\mathcal{A}^{(s)})$ is an $r \times r$ Pick family for \mathcal{A} . In particular, $\text{Lat}(\mathcal{A}^{(\infty)})$ is a complete Pick family for any algebra of multipliers \mathcal{A} .

While it appears that ampliations of matrix algebras over some well known multiplier algebras have $\mathbb{A}_1(1)$, we are unaware of any general results of this kind. Such a result would be interesting. To conclude this chapter, we illustrate Theorem 3.5.5 with matrix-valued constrained interpolation on the disk \mathbb{D} .

We return to the case of subalgebras of H^∞ acting on Hardy space. In [25], for $\mathcal{A} = H_1^\infty := \{f \in H^\infty : f'(0) = 0\}$, it was shown that the distance formula for matrix

interpolation *fails* for \mathcal{A} . In our terminology, $\text{CycLat } M_r(H_1^\infty)$ does not yield a Pick family. Thus we cannot drop the assumption that $\mathcal{M}_r(\mathcal{A})$ has $\mathbb{A}_1(1)$. Indeed, the unilateral shift fails to have even property \mathbb{A}_2 . We will show that with ampliations, a general result can be obtained. The following result should be well known, but we do not have a reference. A version of it appears as Theorem 4 in [62].

Lemma 3.5.6. $\mathcal{M}_r(H^{\infty(r)})$ acting on $(H^2 \otimes \mathbb{C}^r)^{(r)}$ has property $\mathbb{A}_1(1)$.

Proof. The proof will follow very closely to that of H^∞ in Theorem 3.4, but the added indexing will make it highly unpleasant. Form the infinite ampliation $\mathcal{M}_r(\mathcal{M}(H^2)^{(\infty)})$. Then any weak-* continuous functional ω on $\mathcal{M}_r(H^\infty)$ with $\|\omega\| < 1$ can be represented as a rank one functional with vectors f, g in $(H^2 \otimes \mathbb{C}^r)^{(\infty)} \simeq H^{2(\infty)} \otimes \mathbb{C}^r$ satisfying $\|f\| \|g\| < 1$. Write $f = (f_1, \dots, f_r)$ and $g = (g_1, \dots, g_r)$ with f_i and g_i in $H^{2(\infty)}$ so that if $\Phi = [\varphi_{ij}] \in \mathcal{M}_r(H^\infty)$, then

$$\omega(\Phi) = \sum_{i,j=1}^r \langle M_{\varphi_{ij}} f_j, g_i \rangle.$$

Let $M = (H^\infty)^{(\infty)}[f_1, \dots, f_r]$. By the Beurling-Lax-Halmos theorem for shifts of infinite multiplicity [36], $H^{\infty(\infty)}|_M$ is unitarily equivalent to $H^{\infty(s)}$ for some $s \leq r$. Thus, by applying the relevant projections and unitaries (cf. Theorem 3.4), we may assume that f_i and g_j live in $H^{2(r)}$. So this means that f and g are then identified with vectors in $(H^2 \otimes \mathbb{C}^r)^{(r)}$, which satisfy the desired norm constraints, as desired. \square

Applying the above factorization lemma to our terminology, we obtain the following.

Theorem 3.5.7. Suppose \mathcal{A} is a dual subalgebra of H^∞ acting on H^2 . Then

$$\{k^L \otimes I_r : L \in \text{CycLat}(\mathcal{A}^{(r)})\}$$

is an $r \times r$ Pick family of kernels for \mathcal{A} .

For the algebra of vanishing first derivatives at the origin, H_1^∞ , this yields a version of the result of Ball, Bolotnikov and Ter Horst [10]. They express their models as invariant subspaces of $\mathcal{M}_r(H^2)$ (in the Hilbert Schmidt norm) instead of $H^{2(r)} \otimes \mathbb{C}^r$, but this is evidently the same space. It suffices to use subspaces which are cyclic for H^∞ . In much the same manner as [25], they obtain an explicit parameterization of these subspaces. We will return to the subject of matrix-valued interpolation for more general spaces in Chapter 5.

Chapter 4

Reproducing kernel Hilbert spaces associated to measures

In this chapter, we present Pick-type theorems for domains in \mathbb{C}^d . As before, for $d \geq 2$, the open unit ball and polydisk in \mathbb{C}^d will be denoted \mathbb{B}_d and \mathbb{D}^d , respectively. The Hardy space $H^2(\Omega)$ is defined as the closure of the multivariable analytic polynomials in $L^2(\partial\Omega, \theta)$, where $\partial\Omega$ is the *distinguished* boundary of Ω (in particular $\partial\mathbb{D}^d = \mathbb{T}^d$) and θ is Lebesgue measure on $\partial\Omega$. The algebra of bounded analytic functions on Ω will be denoted $H^\infty(\Omega)$. In Theorem 4.2.4 in Section 4.2, a Pick theorem for the polydisk and unit ball is obtained.

For an arbitrary bounded domain $\Omega \subset \mathbb{C}^d$ and Lebesgue measure μ on Ω , the Bergman space $L_a^2(\Omega)$ is defined as those functions which are analytic on Ω and contained in $L^2(\Omega, \mu)$. In Theorem 4.3.5 a Pick result for any bounded domain in \mathbb{C}^d is established using Bergman spaces. As is the case with Theorem 4.2.4, the associated Pick matrices arise from absolutely continuous measures on Ω . These results are also valid for any weak*-closed subalgebra of $H^\infty(\Omega)$, as always.

4.1 Spaces associated to measures

This chapter will concern itself only with reproducing kernel Hilbert spaces of analytic functions. We further assume that \mathcal{H} is endowed with an L^2 norm, i.e. there is some set $\Delta \subset \mathbb{C}^d$ and a Borel measure μ on Δ such that \mathcal{H} is a closed subspace of $L^2(\Delta, \mu)$. The set Δ may play different roles depending on the context. For the Hardy space of the polydisk or unit ball, Δ is taken to be either \mathbb{T}^d or $\partial\mathbb{B}_d$, respectively, and μ is the corresponding Lebesgue measure on these sets. For the Bergman space $L_a^2(\Omega)$, we simply take $\Delta = \Omega$ and μ to be Lebesgue measure on Ω . We also assume that \mathcal{H} contains the constant function 1, so that every multiplier of \mathcal{H} is contained in \mathcal{H} .

Given an algebra of multipliers \mathcal{A} on \mathcal{H} and a measure ν on Δ , let $\mathcal{A}^2(\nu)$ denote the closure of \mathcal{A} in $L^2(\Delta, \nu)$. The measure ν is said to be *dominating* for X (with respect to \mathcal{A}) if $\mathcal{A}^2(\nu)$ is a reproducing kernel Hilbert space on X . We will write $k_x^{\mathcal{A}^2(\nu)}$ for the reproducing kernel on this space, or more simply as k_x^ν when the context is clear. The associated

positive definite kernel function on $X \times X$ will be denoted $k^\nu(x, y) := \langle k_y^\nu, k_x^\nu \rangle_{\mathcal{A}^2(\nu)}$. For any such ν , \mathcal{A} is obviously an algebra of multipliers on $\mathcal{A}^2(\nu)$.

Remark 4.1.1. Our notion of a dominating measure differs slightly from Cole-Lewis-Wermer [22]. In their setting, \mathcal{A} is a uniform algebra and Δ is the maximal ideal space of \mathcal{A} . A measure μ on Δ is said to be *dominating* for a subset Λ of Δ if there is a constant C such that $|\varphi(x)| \leq C\|\varphi\|_{L^2(\mu)}$ for every $x \in \Lambda$. Their theorem solves the so-called *weak* Pick problem for \mathcal{A} : given $\epsilon > 0$, $w_1, \dots, w_n \in \mathbb{C}$ and $x_1, \dots, x_n \in \Delta$, there is a function $\varphi \in \mathcal{A}$ with $\varphi(x_i) = w_i$ for $i = 1, \dots, n$ and $\|\varphi\| < 1 + \epsilon$ if and only if

$$[(1 - w_i \overline{w_j})k^\mu(x_i, x_j)] \geq 0$$

for every measure μ which is dominating for $\{x_1, \dots, x_n\}$.

When \mathcal{H} is contained in an ambient L^2 space, we seek a nicer description of the cyclic subspaces $\mathcal{A}[f]$. The following result shows that cyclic subspaces may naturally be identified with \mathcal{H} under a different norm. Douglas and Sarkar obtain a similar result [30, Lemma 2] in the language of Hilbert modules (though we require slightly more information here). Recall that if $H \subset L^2(\Delta, \mu)$, $\mathcal{A} \subset \mathcal{M}(\mathcal{H})$ and ν is some measure on Δ , then $\mathcal{A}^2(\nu)$ is the closure of \mathcal{A} in $L^2(\Delta, \nu)$.

Theorem 4.1.2. *Suppose \mathcal{H} is a reproducing kernel Hilbert space of analytic functions on some bounded domain Ω . Suppose further that there is a measure space (Δ, μ) such that \mathcal{H} is a closed subspace of $L^2(\Delta, \mu)$. If \mathcal{A} is any dual algebra of multipliers on \mathcal{H} , then for $f \in \mathcal{H}$, the space $\mathcal{A}^2(|f|^2\mu)$ is a reproducing kernel Hilbert space on the set $\Omega_f := \{z \in \Omega : \langle f, k_z^f \rangle_{\mathcal{H}} \neq 0\}$. The reproducing kernel for $\mathcal{A}^2(|f|^2\mu)$ is given by*

$$j^f(z, w) = \frac{k^f(z, w)}{\langle f, k_z^f \rangle_{\mathcal{H}} \langle f, k_w^f \rangle_{\mathcal{H}}}.$$

Moreover, there is a reproducing kernel Hilbert space isomorphism

$$U : \mathcal{A}[f] \rightarrow \mathcal{A}^2(|f|^2\mu).$$

In particular, any measure of the form $|f|^2\mu$ is dominating for Ω_f with respect to \mathcal{A} .

Proof. Define a linear map $V : \mathcal{A} \rightarrow \mathcal{A}[f]$ by $V\varphi = \varphi f$. It is clear by the definition of the norms involved that V extends to a unitary

$$V : \mathcal{A}^2(|f|^2\mu) \rightarrow \mathcal{A}[f].$$

We claim that $U := V^*$ is the required isomorphism. For notational convenience, let $\mathcal{A}^2 := \mathcal{A}^2(|f|^2\mu)$. If $\varphi \in \mathcal{A}$ and $z \in \Omega_f$, we have

$$\langle \varphi, V^* k_z^f \rangle_{\mathcal{A}^2} = \langle \varphi f, k_z^f \rangle_{\mathcal{H}} = \left\langle f, (M_\varphi^f)^* k_z^f \right\rangle_{\mathcal{H}} = \varphi(z) \langle f, k_z^f \rangle_{\mathcal{H}}.$$

By assumption, $\langle f, k_z^f \rangle_{\mathcal{H}} \neq 0$, and so the vector $\overline{\langle f, k_z^f \rangle_{\mathcal{H}}}^{-1} V^* k_z^f$ is the reproducing kernel at the point z for any function in \mathcal{A} . In order to show this for any function $\varphi \in \mathcal{A}^2$,

find $\varphi_n \in \mathcal{A}$ with $\|\varphi_n - \varphi\|_{\mathcal{A}^2}$ tending to 0. Since $\varphi \in L^2(\Delta, |f|^2\mu)$, it follows that $\varphi f \in L^2(\Delta, \mu)$. This in turn implies that the sequence $\{\varphi_n f\}$ is Cauchy in $L^2(\Delta, \mu)$ and that its limit must be φf . The subspace $\mathcal{A}[f]$ is closed, and so $\varphi f \in \mathcal{A}[f]$. Taking an inner product against k_z^f implies that $\varphi_n(z) \rightarrow \varphi(z)$ for any $z \in \Omega_f$. Consequently

$$\begin{aligned} \left\langle \varphi, \frac{V^* k_z^f}{\langle f, k_z^f \rangle_{\mathcal{H}}} \right\rangle_{\mathcal{A}^2} &= \left\langle \varphi f, \frac{k_z^f}{\langle f, k_z^f \rangle_{\mathcal{H}}} \right\rangle_{\mathcal{H}} \\ &= \lim_{n \rightarrow \infty} \left\langle \varphi_n f, \frac{k_z^f}{\langle f, k_z^f \rangle_{\mathcal{H}}} \right\rangle_{\mathcal{H}} = \lim_{n \rightarrow \infty} \varphi_n(z) = \varphi(z). \end{aligned}$$

Now set $j_z^f := (\langle f, k_z^f \rangle_{\mathcal{H}})^{-1} V^* k_z^f$. The above reasoning shows that j_z^f is the reproducing kernel for \mathcal{A}^2 , and

$$j^f(z, w) := \langle j_w^f, j_z^f \rangle_{\mathcal{A}^2} = \frac{\langle V^* k_w^f, V^* k_z^f \rangle_{\mathcal{A}^2}}{\langle f, k_z^f \rangle_{\mathcal{H}} \langle f, k_w^f \rangle_{\mathcal{H}}} = \frac{k^f(z, w)}{\langle f, k_z^f \rangle_{\mathcal{H}} \langle f, k_w^f \rangle_{\mathcal{H}}},$$

which proves the theorem. \square

Corollary 4.1.3. *Suppose \mathcal{A} and \mathcal{H} satisfy the hypotheses of Theorem 4.1.2. Then the matrix*

$$\left[(1 - w_i \bar{w}_j) k^f(z_r, z_s) \right]_{r,s=1}^n$$

is positive semidefinite if and only if

$$\left[(1 - w_i \bar{w}_j) j^f(z_r, z_s) \right]_{r,s=1}^n$$

is positive semidefinite.

Proof. By Theorem 4.1.2, the matrix $[(1 - w_r \bar{w}_s) j^f(z_r, z_s)]$ is the Schur product of $[(1 - w_r \bar{w}_s) k^f(z_r, z_s)]$ and $\left[\left(\langle f, k_{z_r}^f \rangle \overline{\langle f, k_{z_s}^f \rangle} \right)^{-1} \right]$, the latter of which is manifestly positive semidefinite. On the other hand, the matrix $\left[\left(\langle f, k_{z_r}^f \rangle \overline{\langle f, k_{z_s}^f \rangle} \right) \right]$ is also positive semidefinite, and so this Schur multiplication is reversible. \square

We can now summarize the results of this section so far.

Theorem 4.1.4. *Let $(\Delta, \mathcal{B}, \mu)$ be a σ -finite measure space such that \mathcal{H} is both a reproducing kernel Hilbert space over a set X and a closed subspace of $L^2(\Delta, \mu)$. Suppose that \mathcal{A} is a dual algebra of multipliers on \mathcal{H} which has property $\mathbb{A}_1(1)$. Then the following statement holds: given $x_1, \dots, x_n \in X$ and $w_1, \dots, w_n \in \mathbb{C}$, there is a multiplier $\varphi \in \mathcal{A}$ such that $\varphi(x_i) = w_i$ and $\|M_\varphi\| \leq 1$ if and only if the matrix*

$$[(1 - w_i \bar{w}_j) k^\nu(x_i, x_j)]_{i,j=1}^n$$

is positive semidefinite for every measure of the form $\nu = |f|^2\mu$ where $f \in H$.

Proof. Apply Corollary 4.1.3 and note that $k^{|f|^2\mu} = j^f$. \square

4.2 Hardy spaces

Just as is the case with the disk, there are several equivalent ways of defining the Hardy spaces $H^2(\Omega)$ where Ω is \mathbb{B}_d or \mathbb{D}^d . Recall that θ is Lebesgue measure on $\partial\Omega$, where $\partial\mathbb{D}^d$ is taken to mean \mathbb{T}^d . We proceed with the basic theory for the polydisk. The analogous statements also hold for the unit ball. See Krantz [39, Chapter 2] for a detailed treatment of these spaces.

For $1 \leq p < \infty$, the Hardy space $H^p(\mathbb{D}^d)$ is the collection of holomorphic functions f on \mathbb{D}^d which satisfy

$$\|f\|_p^p := \sup_{0 < r < 1} \int_{\partial\Omega} |f_r(t)|^p d\theta(t),$$

where $f_r(t) := f(re^{it_1}, \dots, re^{it_d})$. By taking radial limits, one obtains the boundary function \tilde{f} on \mathbb{T}^d by

$$\tilde{f}(t) = \lim_{r \rightarrow 1} f_r(t).$$

In fact, the analogue of Fatou's theorem for \mathbb{T}^d states that \tilde{f} is the limit almost everywhere along any non-tangential path to \mathbb{T}^d . The function \tilde{f} belongs to $L^p(\mathbb{T}^d, \theta)$ and satisfies $\|\tilde{f}\|_{L^p(\mathbb{T}^d, \theta)} = \|f\|_p$.

Conversely, if F is a function in $L^p(\mathbb{T}^d, \theta)$ with Fourier expansion $\sum_{I \in \mathbb{Z}^d} a_I z^I$ (here I is the standard multi-index), then by integrating F against the Poisson kernel for \mathbb{D}^d , one obtains a Harmonic function f on \mathbb{D}^d . The function f belongs to $H^p(\mathbb{D}^d)$ precisely when $a_I = 0$ for any $I \in \mathbb{Z}^d \setminus \mathbb{Z}_+^d$. Consequently, we may interchange the rolls of f and the boundary function \tilde{f} .

Proposition 4.2.1. *The Hardy spaces $H^2(\mathbb{D}^d)$ and $H^2(\mathbb{B}_d)$ are reproducing kernel Hilbert spaces with kernel functions*

$$k^{H^2(\mathbb{D}^d)}(z, w) = \prod_{k=1}^d \frac{1}{1 - z_k \bar{w}_k} \quad \text{and} \quad k^{H^2(\mathbb{B}_d)}(z, w) = \frac{1}{(1 - \langle z, w \rangle_{\mathbb{C}^d})^d}$$

on \mathbb{D}^d and \mathbb{B}_d , respectively.

When $p = \infty$, we obtain the usual Hardy space $H^\infty(\mathbb{D})$ of bounded, holomorphic functions on \mathbb{D}^d . For $\varphi \in H^\infty$, the functions φ_r converge radially in the weak-* topology to the boundary function $\tilde{\varphi}$ in $L^\infty(\mathbb{T}^d, \theta)$ and we have

$$\|\varphi\|_\infty := \sup_{z \in \mathbb{D}^d} |\varphi(z)| = \text{ess sup}_{z \in \mathbb{T}^d} |\tilde{\varphi}(z)| =: \|\tilde{\varphi}\|_{\text{ess}}.$$

For any $\psi \in L^\infty(\mathbb{T}^d, \theta)$, let N_ψ denote the multiplication operator on $L^2(\mathbb{T}^d, \theta)$ given by $N_\psi f = \psi f$. The operator N_ψ is normal and satisfies $\|N_\psi\| = \|\psi\|_{\text{ess}}$.

Proposition 4.2.2. *Suppose Ω is either \mathbb{D}^d or \mathbb{B}_d . Then $\text{Mult}(H^2(\Omega)) = H^\infty(\Omega)$ and for any $\varphi \in H^\infty(\Omega)$ we have*

$$\|M_\varphi\| = \sup_{z \in \Omega} |\varphi(z)|.$$

Moreover, the algebra $H^\infty(\Omega)$ equipped with the weak-* topology from $L^\infty(\partial\Omega, \theta)$ is weak-* homeomorphic to $\mathcal{M}(H^2(\Omega))$.

Proof. We will prove the statement for $\Omega = \mathbb{D}^d$. Suppose φ is a multiplier of $H^2(\mathbb{D}^d)$. Then, since the constant function 1 is contained in $H^2(\mathbb{D}^d)$, we see that φ also belongs to $H^2(\mathbb{D}^d)$. Since φ is a multiplier, it is automatically bounded, and hence contained in $H^\infty(\mathbb{D}^d)$. Conversely, if $\varphi \in H^\infty(\mathbb{D}^d)$ and $f \in H^2(\mathbb{D}^d)$, it is clear that $\varphi f \in H^2(\mathbb{D}^d)$ by the definition of the norms involved. If $\varphi \in H^\infty(\mathbb{D}^d)$, by identifying φ with its boundary function, we have

$$\|M_\varphi\| \leq \|N_\varphi\| = \|\varphi\|_{\text{ess}} = \sup_{z \in \mathbb{D}^d} |\varphi(z)|.$$

Thus, the map $\Theta : \varphi \mapsto M_\varphi$ is a surjective isometry of $H^\infty(\mathbb{D}^d)$ onto $\mathcal{M}(H^2(\mathbb{D}^d))$. To prove that Θ is a weak-* homeomorphism, we first show that it is weak-* to weak-* continuous. This claim will follow if we can show that Θ is weak-* to WOT sequentially continuous. Suppose $\{\varphi_n\} \subset H^\infty(\mathbb{D}^d)$ is a sequence such that the φ_n converge weak-* to φ . For any $f \in L^1(\mathbb{T}^d, \theta)$, we have

$$\int \varphi_n f d\mu \rightarrow \int \varphi f d\mu.$$

Now if $g, h \in H^2(\mathbb{D}^d)$, we have

$$\langle M_{\varphi_n} g, h \rangle = \int_{\mathbb{T}^d} \varphi_n g \bar{h} d\theta \rightarrow \int_{\mathbb{T}^d} \varphi g \bar{h} d\theta = \langle M_\varphi f, g \rangle$$

since $f\bar{g} \in L^1(\Delta, \mu)$, as desired. Finally, we must show that $\text{Ran } \Theta$ is weak-* closed. By the Krein-Smulian theorem it suffices to show that for any ball B_r of radius $r > 0$ in $\mathcal{B}(H^2(\mathbb{D}^d))$, the set $B_r \cap \text{Ran } \mathcal{M}(H^2(\mathbb{D}^d))$ is weak-* compact. Since Θ is an isometry, $B_r \cap \mathcal{M}(H^2(\mathbb{D}^d))$ is the image of a weak-* compact set in $H^\infty(\mathbb{D}^d)$ by the Banach-Alaoglu theorem. Therefore, by continuity of Θ , this set is compact. \square

We will now invoke the following important factorization result of Bercovici-Westwood [19, Theorem 1].

Theorem 4.2.3 (Bercovici-Westwood). *Suppose Ω is either \mathbb{D}^d or \mathbb{B}_d . Then for any function $h \in L^1(\partial\Omega, \theta)$ and $\varepsilon > 0$, there are functions f and g in $H^2(\Omega)$ such that $\|f\|_2 \|g\|_2 \leq \|h\|_1$ and $\|f - g\bar{h}\|_2 < \varepsilon$. In particular, $\mathcal{M}(H^2(\Omega))$ has property $\mathbb{A}_1(1)$.*

Combining the above result with Theorem 4.1.4 proves the following Pick interpolation theorem for dual algebras of multipliers on the Hardy spaces $H^2(\Omega)$.

Theorem 4.2.4. *Suppose Ω is either \mathbb{D}^d or \mathbb{B}_d and that $z_1, \dots, z_n \in \Omega$ and $w_1, \dots, w_n \in \mathbb{C}$. Let \mathcal{A} be any weak*-closed subalgebra of $H^\infty(\Omega)$. There is a function $\varphi \in \mathcal{A}$ with $\sup_{z \in \Omega} |\varphi(z)| \leq 1$ and $\varphi(z_i) = w_i$ for $i = 1, \dots, n$ if and only if the matrix*

$$[(1 - w_i \bar{w}_j) k^\nu(z_i, z_j)]_{i,j=1}^n \geq 0$$

is positive semidefinite for every measure of the form $\nu = |f|^2 \theta$, where θ is Lebesgue measure on $\partial\Omega$ and $f \in H^2(\Omega)$.

McCullough retrieved Theorem 4.2.4 as stated when $\mathcal{A} = H^\infty(\Omega)$ [46, Theorem 5.12]. In this case, it is shown that the function f may be taken to be bounded and have strictly positive modulus on $\partial\Omega$. Given any $\Omega \subset \mathbb{C}^d$ with a twice continuously differentiable boundary, we may define the Hardy space $H^2(\Omega)$ as the closure of

$$A(\Omega) := \{\varphi \in \text{Hol}(\Omega) : \varphi \text{ extends to be continuous on } \partial\Omega\}$$

in $L^2(\partial\Omega, \theta)$, where θ is Lebesgue measure on $\partial\Omega$. These Hardy spaces are all reproducing kernel Hilbert spaces on Ω [39, Chapter 2], and their multiplier algebras can be isometrically identified with $H^\infty(\Omega)$. Prunaru [54] has recently established a number of conditions which imply that $\mathcal{M}(H^2(\Omega))$ has property $\mathbb{A}_1(1)$. In particular, if $A(\Omega)$ is logmodular, or approximating in modulus on $\partial\Omega$, then $\mathcal{M}(H^2(\Omega))$ has property $\mathbb{A}_1(1)$. Perhaps not surprisingly, these are precisely the conditions that would enable one to apply McCullough's results as well. Outside of the unit ball and polydisk, we are not aware of any additional examples which are known to have these properties.

4.3 Bergman spaces

Recall that for a bounded open domain Ω in \mathbb{C}^d , the Bergman space $L_a^2(\Omega)$ is the set of all holomorphic functions on Ω which are square integrable with respect to volume Lebesgue measure on Ω . By a standard mean value argument, $L_a^2(\Omega)$ is easily seen to be a reproducing kernel Hilbert space on Ω . Moreover, since $L_a^2(\Omega)$ is a closed subspace of $L^2(\Omega, \mu)$, we immediately see that the multiplier norm is the supremum norm for $\mathcal{M}(L_a^2(\Omega))$.

Proposition 4.3.1. *Suppose Ω is a bounded domain in \mathbb{C}^d . Then $\text{Mult}(L_a^2(\Omega)) = H^\infty(\Omega)$ and for every $\varphi \in H^\infty(\Omega)$, we have*

$$\|M_\varphi\| = \sup_{z \in \Omega} |\varphi(z)|.$$

Example 4.3.2. If Ω is any bounded domain in \mathbb{C}^d , the kernel function for the Bergman spaces $L_a^2(\Omega)$ is generally difficult to compute. Some familiar examples are given by

$$k^{L_a^2(\mathbb{D}^d)}(z, w) = \prod_{k=1}^d \frac{1}{(1 - z_i \bar{w}_i)^2} \quad \text{and} \quad k^{L_a^2(\mathbb{B}^d)}(z, w) = \frac{1}{(1 - \langle z, w \rangle_{\mathbb{C}^d})^{d+1}}.$$

In addition to the weak-* topology inherited by $H^\infty(\Omega)$ as a dual algebra, there is an additional weak-* topology that it accrues by being a weak-* closed subalgebra of $L^\infty(\Omega, \mu)$. It turns out that these topologies are identical (see, for example, the introduction to [15]). Thus, predual factorization problems for the dual algebra $\mathcal{M}(L_a^2(\Omega))$ can be rephrased as factorization problems in $L^1(\Omega, \mu)$. This was the program carried out by Bercovici in [15], where he proves that all the algebras $\mathcal{M}(L_a^2(\Omega))$ have property $X(0, 1)$. In fact, a much more general theory of factorization of this type has recently been formulated by Prunaru in [53]. His results apply to any instance where \mathcal{H} is a

reproducing kernel Hilbert space on a measure space (X, \mathcal{B}, μ) and \mathcal{H} is a closed subspace of $L^2(X, \mu)$. In particular, it applies to any Bergman space, but not to Hardy space. Any reproducing kernel Hilbert space which satisfies this hypothesis always has the property that $\|M_\varphi\| = \sup_{x \in X} |\varphi(x)|$, for any multiplier φ .

Theorem 4.3.3 (Prunaru). *Let (X, \mathcal{B}, μ) be a σ -finite measure space such that \mathcal{H} is a reproducing kernel Hilbert space over X and a closed subspace of $L^2(X, \mu)$. Then the multiplier algebra $\mathcal{M}(\mathcal{H})$ has property $\mathbb{A}_1(1)$. Moreover, if $\mathcal{M}(\mathcal{H})$ does not have any 1-dimensional invariant subspaces, then $\mathcal{M}(\mathcal{H})$ has property $X(0, 1)$.*

In the particular case where \mathcal{H} is a Bergman space $L_a^2(\Omega)$, Bercovici [15] proved the above result. An application of Theorem 4.1.4 gives us what we need.

Corollary 4.3.4. *Let (X, \mathcal{B}, μ) be a σ -finite measure space such that \mathcal{H} is both a reproducing kernel Hilbert space over X and a closed subspace of $L^2(X, \mu)$. Suppose \mathcal{A} is a dual algebra of multipliers on \mathcal{H} . If $x_1, \dots, x_n \in X$ and $w_1, \dots, w_n \in \mathbb{C}$, then there is a function $\varphi \in \mathcal{A}$ with $\|\varphi\|_\infty \leq 1$ and $\varphi(x_i) = w_i$ for $i = 1, \dots, n$ if and only if*

$$[(1 - w_i \overline{w_j})k^\nu(x_i, x_j)]_{i,j=1}^n \geq 0$$

for every measure of the form $\nu = |f|^2 \mu$, for $f \in H$.

Theorem 4.3.5 now follows from Corollary 4.3.4 by taking X to be a bounded domain Ω and setting $\mathcal{H} = L_a^2(\Omega)$.

Theorem 4.3.5. *Suppose Ω is a bounded domain in \mathbb{C}^d and that $z_1, \dots, z_n \in \Omega$ and $w_1, \dots, w_n \in \mathbb{C}$. Let \mathcal{A} be any weak*-closed subalgebra of $H^\infty(\Omega)$. There is a function $\varphi \in \mathcal{A}$ with $\sup_{z \in \Omega} |\varphi(z)| \leq 1$ and $\varphi(z_i) = w_i$ for $i = 1, \dots, n$ if and only if the matrix*

$$[(1 - w_i \overline{w_j})k^\nu(z_i, z_j)]_{i,j=1}^n \geq 0$$

is positive semidefinite for every measure of the form $\nu = |f|^2 \mu$, where μ is Lebesgue measure on Ω and $f \in L_a^2(\Omega)$.

For an arbitrary region $\Omega \in \mathbb{C}^d$, and a weak-* closed subalgebra \mathcal{A} of $H^\infty(\Omega)$, the Cole-Lewis-Wermer approach is not easily applied, since there may not be a uniform algebra on $\overline{\Omega}$ which is weak-* dense in \mathcal{A} . Thus, we must generally consider \mathcal{A} as a uniform algebra over its own maximal ideal space, an object enormous in its complexity (even for the disk). Consequently, it is no small task to demonstrate all the measures which are dominating for this set. Theorem 4.3.5 provides a significant simplification of the Cole-Lewis-Wermer approach for the algebra $H^\infty(\Omega)$ (and its subalgebras) by instead considering the reproducing kernels arising from a class of absolutely continuous measures on Ω .

Chapter 5

Complete Pick kernels

Recall that a scalar-valued kernel k is said to be a *complete Pick kernel* if, for every $r \geq 1$, $k \otimes I_r$ is a Pick kernel. In other words, matrix interpolation is determined by the positivity of the Pick matrix for the data. More explicitly, if \mathcal{H} is a Hilbert space with reproducing kernel k on a set X , $E = \{x_1, \dots, x_n\}$ is a finite subset of X , and W_1, \dots, W_n are $r \times r$ matrices, then a necessary condition for there to be an element $\Phi \in \mathcal{M}_r(\mathcal{M}(\mathcal{H})) = \mathcal{M}(\mathcal{H}^{(r)})$ with $\Phi(x_i) = W_i$ and $\|M_\Phi\| \leq 1$ is the positivity of the matrix

$$\left[(I_r - W_i W_j^*) k(x_i, x_j) \right].$$

As we saw in Chapter 2, a kernel k is a complete Pick kernel if this is also a sufficient condition.

Drury-Arveson space H_d^2 , on the complex ball \mathbb{B}_d of \mathbb{C}^d (including $d = \infty$) is a complete Pick space with kernel

$$k(w, z) = \frac{1}{1 - \langle w, z \rangle}.$$

See, for example, Davidson-Pitts [28] for one of many proofs of this fact (the proof of Davidson-Pitts uses the distance formula approach). All irreducible complete Pick kernels were classified by McCullough [44, 45] and Quiggin [55], building on work by Agler (unpublished). Another proof was provided by Agler and McCarthy [3], who noticed the universality of the Drury-Arveson kernel.

A kernel k is said to be *irreducible* if for distinct x and y in X , the functions k_x and k_y are linearly independent and $\langle k_x, k_y \rangle \neq 0$.

Theorem 5.0.6 (McCullough, Quiggin, Agler-McCarthy). *Let k be an irreducible kernel on X . The following are equivalent:*

1. k is a complete Pick kernel.
2. The matrices $\left[1 - \frac{1}{k(x_i, x_j)} \right]_{i,j=1}^n$ are positive semidefinite for every finite subset $\{x_1, \dots, x_n\} \subset X$.
3. For some countable cardinal d , there is an injection $b : X \rightarrow \mathbb{B}_d$ and a nowhere-

vanishing function $\delta : X \rightarrow \mathbb{C}$ such that

$$k(x, y) = \frac{\overline{\delta(x)}\delta(y)}{1 - \langle b(x), b(y) \rangle}.$$

In this case, the map $k_x(y) \mapsto \frac{\delta(x)}{1 - \langle b(y), b(x) \rangle}$ extends to an isometry of \mathcal{H} into H_d^2 .

Examples of other spaces with complete Pick kernels include Hardy and Dirichlet space on the disk, as well as the Sobolev-Besov spaces on \mathbb{B}_d and the Sobolev space W_1^2 . Agler and McCarthy contains a detailed account of complete Pick kernels [4]. In practice, verifying condition (2) in the theorem is the easiest way to test whether or not a kernel is a complete Pick kernel. It immediately says that, for example, the Szegő kernel on the polydisk is not a complete Pick kernel. In theory, however, we very much prefer statement (3). Since the span of kernel functions is always a co-invariant subspace for a dual algebra of multipliers, the reproducing kernel Hilbert space associated to an irreducible complete Pick kernel correspond to co-invariant subspaces of Drury-Arveson space, i.e. $\text{span}\{k_z : z \in b(X)\}$. This viewpoint allows us to borrow some power operator theory from Drury-Arveson space.

5.1 The noncommutative analytic Toeplitz algebra

For Drury-Arveson space H_d^2 , the structure of its multiplier algebra has received significant interest in the last decade, though its introduction into the literature is considerably older. In 1978, Drury [31] provided a generalization of von Neumann's inequality to several variables using the multiplier algebra $\mathcal{M}(H_d^2)$. Namely, given a d -variable polynomial p and a row contraction T , i.e. an operator tuple $[T_1, \dots, T_d]$ which satisfies $\sum_{i=1}^d T_i T_i^* \leq I$ (equivalently, $[T_1, \dots, T_d]$ is a contraction when regarded as an element of $\mathcal{B}(H^{(n)}, \mathcal{H})$), we have

$$\|p(T_1, \dots, T_d)\| \leq \|M_p\|_{\mathcal{M}(H_d^2)}.$$

As a dual algebra, $\mathcal{M}(H_d^2)$ is generated by the coordinate functions M_{z_1}, \dots, M_{z_d} , which together form a row contraction. Arveson [9] showed that the row contraction $M := [M_{z_1}, \dots, M_{z_d}]$ serves as the model for commuting row contractions, in the sense that any other commuting row contraction can be dilated to the direct sum of an ampliation of M and a spherical unitary (a spherical unitary is a d -tuple $[A_1, \dots, A_n]$ of commuting normal operators which satisfies $\sum_{i=1}^d A_i^* A_i = I$). He also demonstrated that the tuple $[M_{z_1}, \dots, M_{z_d}]$ is not jointly subnormal, a consequence of the fact that the H_d^2 norm is not induced by a measure when $d > 1$. In particular, the analysis of the Pick problem in Chapter 4 does not carry through to this setting. Moreover, when $d > 1$, the multiplier norm on H_d^2 is generally larger than the supremum norm over the unit ball \mathbb{B}_d .

Returning to general complete Pick kernels, Theorem 5.0.6 shows that, up to rescaling of kernels, any complete Pick space \mathcal{H} , is given by

$$\mathcal{H} = \overline{\text{span}}\{k_x^d : x \in S\}$$

for some subset S of \mathbb{B}_d . Spans of kernel functions are always co-invariant for multiplication operators. Consequently, every irreducible and complete Pick space corresponds to a co-invariant subspace of $\mathcal{M}(H_d^2)$. It was shown in [6] that all such compressions can be identified with complete quotients of $\mathcal{M}(H_d^2)$, which in turn can be identified with a complete quotient of the *non-commutative analytic Toeplitz algebra* \mathfrak{L}_d .

In order to describe \mathfrak{L}_d , we must introduce some notation. Let \mathbb{F}_d^+ denote the free semigroup on d letters, where d is a countable cardinal. A typical “word” in this semigroup will be denoted w , and the empty word will always be written as 0. Form the Hilbert space $\ell^2(\mathbb{F}_d^+)$, and let $\{e_w : w \in \mathbb{F}_d^+\}$ be its standard orthonormal basis. Now define the left creation operators L_i on $\ell^2(\mathbb{F}_d^+)$ for $1 \leq i \leq d$, whose action on the given basis is determined by $L_i e_w = e_{iw}$. Each L_i is an isometry, and the algebraic relation $L_i^* L_j = \delta_{ij} I$ implies that the L_i have pairwise orthogonal ranges. Equivalently, the tuple $[L_1, \dots, L_d]$ is an isometry from $\ell^2(\mathbb{F}_d^+)^{(d)}$ into $\ell^2(\mathbb{F}_d^+)$.

Definition 5.1.1. The *non-commutative analytic Toeplitz algebra* \mathfrak{L}_d is the unital, WOT closed subalgebra of $\mathcal{B}(\ell^2(\mathbb{F}_d^+))$ generated by the left creation operators L_1, \dots, L_d .

The WOT closed algebra generated by the right regular representation is denoted \mathfrak{R}_d and is defined analogously. A basic result in [26] is that \mathfrak{L}_d and \mathfrak{R}_d are unitarily equivalent and equal to each others’ commutants. Consequently, \mathfrak{L}_d is equal to its own double commutant. One of the most important spatial facts about \mathfrak{L}_d is a Beurling-type decomposition for its invariant subspaces. This was perhaps the first clear evidence that \mathfrak{L}_d is a natural, non-commutative generalization of H^∞ . The following theorem can be found in [7, Theorem 2.3] and [27, Theorem 2.1].

Theorem 5.1.2 (Arias-Popescu, Davidson-Pitts). *Every cyclic invariant subspace for \mathfrak{L}_d is given by the range of an isometry in \mathfrak{R}_d , and this choice is unique up to a scalar. More generally, any invariant subspace M for \mathfrak{L}_d can be decomposed into the direct sum of cyclic subspaces, and M is generated by the wandering subspace*

$$M \ominus \left(\bigoplus_{i=1}^d L_i M \right).$$

We will now briefly describe the connection between invariant subspaces and complete quotients of \mathfrak{L}_d . If \mathcal{I} is a WOT-closed, two-sided ideal of \mathfrak{L}_d with range $M = \overline{\mathcal{I}\ell^2(\mathbb{F}_d^+)}$, then [27] shows that there is a normal, completely isometrically isomorphic map from $\mathfrak{L}_d/\mathcal{I}$ to the compression of \mathfrak{L}_d to M^\perp . Conversely, if M is an invariant subspace of both \mathfrak{L}_d and its commutant, the right regular representation algebra \mathfrak{R}_d , then $\mathcal{I} = \{A \in \mathfrak{L}_d : \text{Ran } A \subset M\}$ is a WOT-closed ideal with range M . In particular, if \mathcal{C} is the commutator ideal, it is shown that $\mathcal{M}(H_d^2) \simeq \mathfrak{L}_d/\mathcal{C}$. Moreover, the compression of both \mathfrak{L}_d and \mathfrak{R}_d to H_d^2 agree with $\mathcal{M}(H_d^2)$ [27, Section 3]. On the other hand, if N is a coinvariant subspace of H_d^2 , then $M = H_d \ominus N$ is invariant for both \mathfrak{L}_d and \mathfrak{R}_d . Consequently, we may view the multiplier algebra of any irreducible complete Pick space as both the co-restriction of \mathfrak{L}_d to a span of kernel functions, or as a complete quotient of \mathfrak{L}_d .

A deep result of Bercovici [17] shows that an algebra of operators has the property $X(0, 1)$ if its commutant contains two isometries with pairwise orthogonal ranges. Consequently, \mathfrak{L}_d and $\mathfrak{L}_d \otimes B(\ell^2)$ both have property $\mathbb{A}_1(1)$ (when $d \geq 2$). If $d' > d$, there is

the canonical embedding of \mathbb{B}_d into $\mathbb{B}_{d'}$, and so there is no loss in assuming that $d \geq 2$. It is essential to use this embedding for Hardy space, for example, since $M(H^2) \otimes B(\ell^2)$ does not have property $\mathbb{A}_1(1)$.

The co-restriction of $\mathcal{M}(H_d^2)$ to a span of kernel functions has property $\mathbb{A}_1(1)$ (see below). The stronger property $X(0, 1)$ is not passed along to these quotients from the parent algebra \mathfrak{L}_d (see Section 5.2). Arias and Popescu [7, Proposition 1.2] first observed that these quotients of \mathfrak{L}_d have property $\mathbb{A}_1(1)$. We will actually prove a stronger statement in Chapter 7, and so we state the following without proof.

Theorem 5.1.3 (Arias–Popescu). *Let \mathcal{I} be any WOT-closed ideal of \mathfrak{L}_d and let $M = \overline{\mathcal{I}\ell^2(\mathbb{F}_d^+)}$. Then $\mathcal{A} = P_M^\perp \mathfrak{L}_d|_{M^\perp}$ has property $\mathbb{A}_1(1)$. Furthermore, if a functional $\omega \in \mathcal{A}_*$ can be factored as $\omega(A) = \langle Au, v \rangle$, then u may be chosen to be a cyclic vector for \mathcal{A} .*

The remarks preceding this theorem show that the multiplier algebra of every complete Pick kernel arise in this way. If \mathcal{A} is a dual algebra of multipliers of an irreducible and complete Pick kernel, then we can extend any functional $\omega \in \mathcal{A}_*$ to a functional on all of $\mathcal{M}(\mathcal{H})$ (with a small increase in norm). Consequently, just as in the case of constrained interpolation for subalgebras of H^∞ , we may factor ω with a cyclic (outer) function for $\mathcal{M}(\mathcal{H})$ as the left-hand factor. This yields the following constrained interpolation result for dual algebras of multipliers of irreducible, complete Pick kernels. Following the analogy with the classical case, we say that a function $h \in H_d^2$ is *outer* if it is cyclic for $\mathcal{M}(H_d^2)$.

Theorem 5.1.4. *Suppose k is an irreducible, complete Pick kernel on X . Then any dual subalgebra \mathcal{A} of multipliers of $\mathcal{H} = \mathcal{H}(k)$ admits a strong Pick family of kernels. More specifically, if $E = \{x_1, \dots, x_n\}$ is a finite subset of X which is separated by \mathcal{A} and w_1, \dots, w_n are scalars, then there is a multiplier f in the unit ball of \mathcal{A} with $f(x_i) = w_i$ for $1 \leq i \leq n$ if and only if*

$$\left[(1 - w_i \overline{w_j}) \langle P_L k_{x_j}, k_{x_i} \rangle \right] \geq 0$$

for all cyclic invariant subspaces $L = \mathcal{A}[h]$ of \mathcal{A} where h is an outer function.

Predual factorization properties behave very nicely with respect to direct sums. Consequently, we are able to get a factorization lemma and Pick-type theorem for an arbitrary complete Pick kernel.

Proposition 5.1.5. *Suppose \mathcal{H} is a reproducing kernel Hilbert space with a complete Pick kernel. Then $\mathcal{M}(\mathcal{H})$ has property $\mathbb{A}_1(1)$ and the additional property that any weak-* continuous functional on $\mathcal{M}(\mathcal{H})$ can be factored as $\omega(M_\varphi) = \langle \varphi h, g \rangle$ where h is a cyclic vector for $\mathcal{M}(\mathcal{H})$.*

Proof. When k is an irreducible complete Pick kernel, the result follows immediately from Theorem 5.1.3. For an arbitrary complete Pick kernel on X , by [3, Lemma 7.2], we can write X as the disjoint union of subsets X_i with k irreducible on each X_i and $\langle k_x, k_y \rangle = 0$ precisely when x and y belong to distinct X_i . Define the mutually orthogonal subspaces $\mathcal{H}_i = \overline{\text{span}}(k_x : x \in X_i)$ so that $\mathcal{H} = \bigoplus_i \mathcal{H}_i$.

For each multiplier $f \in \mathcal{M}(\mathcal{H})$, the subspace \mathcal{H}_i is invariant for both M_φ and M_φ^* . Thus $\mathcal{M}(\mathcal{H}) = \bigoplus_i \mathcal{M}(\mathcal{H}_i)$ and each $\mathcal{M}(\mathcal{H}_i)$ is the multiplier algebra of an irreducible and complete Pick kernel. It now follows from the irreducible case that a weak-* continuous functional on $\mathcal{M}(\mathcal{H}_i)$ has the desired factorization properties. If $\omega \in \mathcal{M}(\mathcal{H})_*$ and $\|\omega\| < 1$, we may write $\omega = \sum_{i=1}^n \omega_i$ for functionals $\omega_i \in \mathcal{M}(\mathcal{H}_i)_*$. Thus, there are cyclic functions h_i in \mathcal{H}_i and functions $g_i \in \mathcal{H}_i$ such that

$$\omega \left(\bigoplus_{i=1}^n M_{\varphi_i} \right) = \sum_{i=1}^n \omega_i(M_{\varphi_i}) = \sum_{i=1}^n \langle \varphi_i h_i, g_i \rangle = \langle \varphi h, g \rangle$$

where $h := (h_1, \dots, h_n)$, $g := (g_1, \dots, g_n)$ and $\|h\| \|g\| < 1$. Since g is easily seen to be cyclic for $\mathcal{M}(\mathcal{H})$, all claims are now proved. \square

We finish this section with a generalization of constrained interpolation problems for algebras of the form $\mathbb{C} + BH^\infty$ as seen in Example 3.4.4, where B is a finite Blaschke product.

Example 5.1.6. Suppose \mathcal{I} is a finite codimension ideal in $\mathcal{M}(\mathcal{H})$, where \mathcal{H} is a complete Pick kernel and $\dim(\mathcal{M}(\mathcal{H})/\mathcal{I}) = k$. Let $\mathcal{A} := \mathbb{C} + \mathcal{I}$, and if h is a cyclic vector for $\mathcal{M}(\mathcal{H})$, set $M := \mathcal{I}[h] = \overline{\mathcal{I}h}$. We have

$$\mathcal{A}[h] = \overline{\text{span}(h, \mathcal{I}[h])} = \overline{\text{span}(h, M)} = P_M^\perp h \oplus M.$$

Since \mathcal{I} is of finite codimension, so is M . Find an orthogonal basis $\{e_1, \dots, e_p\}$ for M^\perp where $p \leq k$. For each cyclic vector h , there are scalars $a_1, \dots, a_p \in \mathbb{C}$ such that $h = a_1 e_1 + \dots + a_p e_p$. Rescaling if necessary, we may assume that $|a_1|^2 + \dots + |a_p|^2 = 1$, i.e. $a := (a_1, \dots, a_p) \in \partial\mathbb{B}_p$. For $a \in \partial\mathbb{B}_p$, let $L_a = \mathbb{C}(a_1 e_1 + \dots + a_p e_p) \oplus M$ denote the invariant subspace associated to a . It follows that the family of cyclic invariant subspaces $\{L_a : a \in \partial\mathbb{B}_p\}$ completely exhausts subspaces of the form $\mathcal{A}[h]$ for h outer, and a similar parametrization is obtained as in Example 3.4.4. For instance, if $H = H_2^2$ and $\mathcal{I} = \langle z_1 z_2 \rangle$ is the ideal generated by the monomial $z_1 z_2$, one is required to check positivity with respect to the invariant subspaces

$$L_{a,b,c} = \mathbb{C}(a + bz_1 + cz_2) \oplus \text{span}(z_1^k z_2^l : k, l \geq 1) \text{ for } (a, b, c) \in \partial\mathbb{B}_3.$$

5.2 Invariant subspaces of a complete Pick space

Let \mathcal{H} be a reproducing kernel Hilbert space whose kernel is irreducible and has the complete Pick property. Up to normalization of its kernel functions, there is a subset S of \mathbb{B}_d such that

$$\mathcal{H} = \overline{\text{span}\{k_x : x \in S\}}.$$

As we have seen, the multiplier algebra $\mathcal{M}(\mathcal{H})$ is a complete quotient of \mathfrak{L}_d , and may be identified with the co-restriction $\mathcal{M}(\mathcal{H}) = P_H \mathfrak{L}_d P_H = P_H \mathfrak{L}_d$. In this section, we will make several observations about the invariant subspace structure of multiplier algebras of complete Pick kernels. First, we will provide a simple proof of the Beurling-type

decomposition of invariant subspaces for multiplier algebras of complete Pick spaces of McCullough-Trent [47]. This will then imply that there is a one-to-one correspondence between invariant subspaces and ideals of $\mathcal{M}(\mathcal{H})$.

Lemma 5.2.1. *Suppose \mathcal{H} is an irreducible and complete Pick space and M is an invariant subspace for $\mathcal{M}(\mathcal{H})$. There are multipliers $\{\varphi_i\}_{i=1}^\infty \subset \mathcal{M}(\mathcal{H})$ such that*

$$M = \overline{\text{span}}\{M_{\varphi_i}\mathcal{H}\}.$$

Moreover, $P_M = \sum_{i=1}^\infty M_{\varphi_i}^* M_{\varphi_i}$.

Proof. The subspace $L := M \oplus \mathcal{H}^\perp$ is in $\text{Lat}(\mathfrak{L}_d)$, and so there are isometries $A_i \in \mathfrak{R}_d$ with pairwise orthogonal ranges such that $L = \bigoplus_{i=1}^\infty \overline{A_i \ell^2(\mathbb{F}_n^+)}$. Compressing back down to \mathcal{H} , we have

$$M = \overline{\text{span}}\{P_{\mathcal{H}}A_i P_{\mathcal{H}} \ell^2(\mathbb{F}_n^+)\}.$$

Let $M_{\varphi_i} := P_{\mathcal{H}}A_i$ denote the corresponding multipliers on \mathcal{H} . Now compute

$$\begin{aligned} \sum_{i=1}^\infty M_{\varphi_i}^* M_{\varphi_i} &= \sum_{i=1}^\infty P_{\mathcal{H}}A_i A_i^* P_{\mathcal{H}} \\ &= P_{\mathcal{H}}P_L P_{\mathcal{H}} \\ &= P_{\mathcal{H}}(P_M + P_{\mathcal{H}}^\perp)P_{\mathcal{H}} \\ &= P_M. \end{aligned} \quad \square$$

When \mathcal{H} is an irreducible and complete Pick space of analytic functions on the ball \mathbb{B}_d , a much stronger result is due to Green, Richter and Sundberg, where they show that the sequence of multipliers which generates an invariant subspace for $\mathcal{M}(\mathcal{H})$ has a very special form (namely, the sequence is *inner* in the sense that its radial limits to the boundary of \mathbb{B}_d have absolute value 1 almost everywhere [33]).

Theorem 5.2.2. *Let $\text{Id}(\mathcal{M}(\mathcal{H}))$ denote the lattice of weak-* closed ideals of $\mathcal{M}(\mathcal{H})$. Define the map $\alpha : \text{Id}(\mathcal{M}(\mathcal{H})) \rightarrow \text{Lat}(\mathcal{M}(\mathcal{H}))$ by $\alpha(\mathcal{I}) = \overline{\mathcal{I}1}$. Then α is a complete lattice isomorphism whose inverse β is given by*

$$\beta(M) = \{J \in \mathcal{M}(\mathcal{H}) : J1 \in M\}.$$

Proof. The structure of this proof follows very closely to that of \mathcal{L}_d (where the corresponding statement is true [27]). For any $\mathcal{I} \in \text{Id}(\mathcal{M}(\mathcal{H}))$ and $M \in \text{Lat}(\mathcal{M}(\mathcal{H}))$, it is easily verified that $\alpha(\mathcal{I}) \in \text{Lat}(\mathcal{M}(\mathcal{H}))$ and $\beta(M) \in \text{Id}(\mathcal{M}(\mathcal{H}))$. We first show that $\beta\alpha$ is the identity map.

If \mathcal{I} is an ideal in $\mathcal{M}(\mathcal{H})$, then it is clear that $\mathcal{I} \subset \beta \circ \alpha(\mathcal{I})$. For the reverse inclusion, first note that for any function h cyclic for $\mathcal{M}(\mathcal{H})$ (in particular, the constant function 1), we have

$$\alpha(\mathcal{I}) = \overline{\mathcal{I}f} = \overline{\mathcal{I}h}.$$

Let ω be a weak-* continuous functional that annihilates \mathcal{I} . By Theorem 5.1.3, there are

functions $h, k \in H$ with h cyclic for $\mathcal{M}(\mathcal{H})$ such that

$$\omega(M_\varphi) = \langle M_\varphi h, k \rangle,$$

for any $M_\varphi \in \mathcal{M}(\mathcal{H})$. It follows that

$$k \perp \overline{\mathcal{I}h} = \overline{\mathcal{I}1} = \alpha(\mathcal{I})$$

and so φ annihilates $\beta \circ \alpha(\mathcal{I})$ as well. Thus $\mathcal{I} = \beta \circ \alpha(\mathcal{I})$.

Next we show that $\alpha \circ \beta(M) = M$ for any $M \in \text{Lat}(\mathcal{M}(\mathcal{H}))$. It is clear that $\alpha \circ \beta(M) \subset M$. On the other hand, apply the previous lemma to M and find multipliers $\{g_i\}$ such that $M = \overline{\text{span}\{M_{g_i}\mathcal{H}\}}$. Each g_i is clearly in $\beta(M)$, and so $M \subset \overline{\beta(M)\mathcal{H}} = \overline{\beta(M)1} = \alpha \circ \beta(M)$, as desired. This shows that α is a bijection. Verifying that it is a complete lattice isomorphism is straightforward and we omit it. \square

If k is a complete Pick kernel, one might expect that constrained Pick interpolation results exist on the matrix level. As we saw in Section 4.5, Ball, Bolotnikov and ter Horst tackled this problem for the algebra H_1^∞ [10], where they demonstrate that a Pick interpolation theorem does hold, but it is significantly more complicated than the scalar-valued case. Even though the unilateral shift fails to have property $\mathbb{A}_n(1)$ for $n > 1$, one might expect that Drury-Arveson space in at least two variables might overcome this problem. There is strong evidence to suggest that this is not the case. We first require the following result of Bercovici [15].

Theorem 5.2.3 (Bercovici). *Suppose \mathcal{A} is a dual algebra with property \mathbb{A}_{\aleph_0} . Then there are invariant subspaces M, N for \mathcal{A} such that $M \cap N = \{0\}$.*

Proposition 5.2.4. *The algebra $\mathcal{M}(H_d^2)$ does not have property \mathbb{A}_{\aleph_0} .*

Proof. We show that two non-trivial invariant subspaces of $\mathcal{M}(H_d^2)$ have a non-trivial intersection and then apply Bercovici's theorem above. Indeed, if $M \in \text{Lat} \mathcal{M}(H_d^2)$, then $N = M + (H_d^2)^\perp$ is invariant for \mathfrak{L}_d . Find isometries $A_i \in \mathfrak{A}_d$ with pairwise orthogonal ranges so that $N = \oplus_i A_i \ell^2(\mathbb{F}_d^+)$. As above, each compression $P_{H_d^2} R_i|_{H_d^2}$ is given by multiplier M_{φ_i} in $\mathcal{M}(H_d^2)$. Thus

$$\begin{aligned} M &= P_{H_d^2} N = \sum_i P_{H_d^2} A_i \ell^2(\mathbb{F}_d^+) \\ &= \sum_i P_{H_d^2} A_i P_{H_d^2} \ell^2(\mathbb{F}_d^+) = \sum_i M_{\varphi_i} H_d^2. \end{aligned}$$

In particular, every invariant subspace M contains the range of a non-zero multiplier M_φ . Hence given two invariant subspaces M and N in H_d^2 , we can find non-zero multipliers φ and ψ with $\text{Ran } M_\varphi \subset M$ and $\text{Ran } M_\psi \subset N$. So $M \cap N$ contains $\text{Ran } M_{\varphi\psi}$. \square

This leads to the following question.

Question 5.2.5. Does $\mathcal{M}(H_d^2)$ have property $\mathbb{A}_2(r)$ for any r , or even property \mathbb{A}_2 ?

5.3 Matrix-valued interpolation II

Suppose k is a scalar-valued kernel such that $\mathcal{M}(k)$ has property $\mathbb{A}_1(1)$. In Section 3.5, we saw that even if $M_k(\mathcal{M}(k))$ fails to have property $\mathbb{A}_1(1)$, it still may be possible to solve the Pick problem by introducing multiplicity. In the present section, we will adopt a more versatile technique for solving constrained matrix-valued Pick problems for complete Pick kernels by passing into the non-commutative world.

The Pick interpolation for the non-commutative analytic Toeplitz algebra \mathfrak{L}_d was solved independently by Arias and Popescu [7] and Davidson-Pitts [28]. For a fixed countable cardinal d , let $k = \frac{1}{1-\langle z, w \rangle}$ denote the Drury-Arveson kernel. As we mentioned earlier, the reproducing kernel k_z naturally resides in $\mathcal{H}_d := \ell^2(\mathbb{F}_d^+)$. The following result of Davidson-Pitts [26, Theorem 2.6] formalizes the connection between \mathfrak{L}_d and Drury-Arveson space. Let L_1, \dots, L_d denote the left-creation operators in \mathfrak{L}_d . For a word $w \in \mathbb{F}_d^+$ and $z = (z_1, \dots, z_d) \in \mathbb{B}_d$, the complex number $w(z)$ is formed by instantiating every occurrence of the letter i in w with z_i . For example, $121(z, w) = zwz$.

Theorem 5.3.1 (Davidson-Pitts). *The eigenvectors for \mathfrak{L}_d^* are given by the functions k_z for every $z \in \mathbb{B}_d$ which satisfy*

$$k_z = \sum_{w \in \mathbb{F}_d^+} w(z) e_w$$

and

$$L_w^* k_z = \overline{w(z)} k_z.$$

The normalized kernel functions $k_z / \|k_z\|$ have the property that every WOT continuous, multiplicative linear functional ω on \mathfrak{L}_d is given by

$$\omega(A) = \|k_z\|^{-2} \langle A k_z, k_z \rangle$$

for $A \in \mathfrak{L}_d$. If $\hat{A}(z)$ is the complex number which satisfies $A^* k_z = \overline{\hat{A}(z)} k_z$, then the map $A \mapsto \hat{A}$ is a unital, completely contractive weak-* to weak-* continuous epimorphism of \mathfrak{L}_d onto $\mathcal{M}(H_d^2)$.

The (complete) Pick interpolation theorem for \mathfrak{L}_d follows from the distance formula for WOT closed two-sided ideals in \mathfrak{L}_d , the proof of which follows from the fact that $\mathfrak{L}_D \otimes \mathcal{B}(\mathcal{H})$ has property $\mathbb{A}_1(1)$ and that, like the unilateral shift, \mathfrak{L}_d has a very rigid invariant subspace structure.

Theorem 5.3.2 (Davidson-Pitts). *Suppose \mathfrak{J} is a two-sided, WOT closed ideal in \mathfrak{L}_d and set $M = \text{Ran } \mathfrak{J}$. Then the representation*

$$A + \mathfrak{J} \mapsto P_M^\perp A P_M^\perp = P_M^\perp A$$

is completely isometric.

By taking $\mathcal{I} = \mathcal{I}_E = \{A \in \mathfrak{L}_d : \hat{A}(z_i) = 0 \text{ for } z_i \in E\}$, the complete Pick theorem for \mathfrak{L}_d follows immediately.

Theorem 5.3.3 (Arias-Popescu, Davidson-Pitts). *Suppose $z_1, \dots, z_n \in \mathbb{B}_d$ and $W_1, \dots, W_n \in M_k$. There is a contractive $A \in M_k(\mathfrak{L}_d)$ such that $\hat{A}(z_i) = W_i$ for $i = 1, \dots, n$ if and only if*

$$\left[\frac{1 - W_i W_j^*}{1 - \langle z_i, z_j \rangle} \right]_{i,j=1}^n$$

is positive.

Now suppose \mathcal{A} is a dual subalgebra of \mathfrak{L}_d , and consider the ideals \mathcal{I}_E in \mathcal{A} now for any finite subset $E \subset \mathbb{B}_d$. If L is any invariant subspace for \mathcal{A} and $A \in \mathcal{A}$, we have

$$P_L A^* P_L k_z = \overline{\hat{A}(z)} P_L k_z,$$

just as in the commutative case. Moreover, if $P_L k_z = 0$, but the subspace $\mathcal{A}[h] \ominus \mathcal{J}_z[h]$ is non-zero, then it is spanned by an extended kernel function which we called k_z^L (see Lemma 3.2.1). This extended kernel also enjoys the identity

$$(A|_L)^* k_z^L = \overline{\hat{A}(z)} k_z^L.$$

Thus, everything appears to work just the same when considering constrained interpolation problems for \mathfrak{L}_d . Since \mathfrak{L}_d enjoys the extremely strong property that $\mathfrak{L}_d \otimes \mathcal{B}(\mathcal{H})$ has property $\mathbb{A}_1(1)$, we immediately get the strongest possible Pick interpolation results as a consequence of the matrix-valued interpolation result found in Theorem 3.5.3. For $A = [A_{ij}] \in M_r(\mathcal{A})$, let \hat{A} denote the M_r -valued function given by $\hat{A}(z) := [\hat{A}_{ij}(z)]$.

Theorem 5.3.4. *Suppose $z_1, \dots, z_n \in \mathbb{B}_d$ and $W_1, \dots, W_n \in M_r$ and that \mathcal{A} is a dual subalgebra of \mathfrak{L}_D . There is a contractive $A \in M_r(\mathcal{A})$ such that $\hat{A}(z_i) = W_i$ for $i = 1, \dots, n$ if and only if*

$$[(1 - W_i W_j^*) \otimes K^L(z_i, z_j)] \geq 0$$

for every r -generated $L \in \text{Lat } \mathcal{A}$, where $K^L(z_i, z_j) = P_{L \ominus \mathcal{I}_{z_j} L} P_{L \ominus \mathcal{I}_{z_i} L}$ is an M_r -valued positive semidefinite kernel.

Turning back now to the commutative case, suppose that $k(w, z) = \frac{1}{1 - \langle z, w \rangle}$ is the Drury-Arveson kernel on \mathbb{B}_d . If $A \in \mathfrak{L}_d$ is any operator which satisfies $\hat{A}(z_i) = w_i$, then $\varphi(z_i) = w_i$ where $\varphi \in \mathcal{M}(H_d^2)$ is the compression of A to Drury-Arveson space. Consequently, if we compress the solution to the constrained matrix-valued interpolation problem above, we get a solution (of potentially smaller norm) in the commutative setting. Let q denote the compression map $A \mapsto P_{H_d^2} A$ from \mathfrak{L}_d onto $\mathcal{M}(H_d^2)$. Note that if \mathcal{A} is a dual algebra of multipliers on H_d^2 , then $q^{-1}(\mathcal{A})$ is a dual algebra in \mathfrak{L}_d .

Theorem 5.3.5. *Suppose $z_1, \dots, z_n \in \mathbb{B}_d$ and $W_1, \dots, W_n \in M_r$ and that \mathcal{A} is a dual algebra of multipliers on H_d^2 . There is a contractive $M_\Phi \in M_r(\mathcal{A})$ such that $\Phi(z_i) = W_i$ for $i = 1, \dots, n$ if*

$$[(1 - W_i W_j^*) \otimes K^L(z_i, z_j)] \geq 0$$

for every r -generated $L \in \text{Lat } \tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}} = q^{-1}(\mathcal{A})$ and $K^L(z_i, z_j) = P_{L \ominus \mathcal{J}_{z_j} L} P_{L \ominus \mathcal{J}_{z_i} L}$.

By restricting the multipliers of Drury-Arveson space to any subspace of the form $\overline{\text{span}}\{k_z : z \in S \subset \mathbb{B}_d\}$, Theorem 6.4.1 also holds for any complete Pick space. It has the disadvantage of being not intrinsic to the underlying Hilbert space, instead requiring one to look at representations on full Fock space. Nonetheless, it is far more general than Theorem 3.5.7, which was valid only for subalgebras of H^∞ . Note that the matrix positivity condition in Theorem 6.4.1 is a sufficient condition, and it is possible that it is strictly stronger than the existence of a contractive solution to the Pick problem (since, in general, the norm on \mathfrak{L}_d is larger than the norm on $\mathcal{M}(H_d^2)$).

Example 5.3.6. Returning to the theme of Example 3.4.4 and Example 5.1.6, suppose $d > 1$ and \mathcal{I} is a WOT closed ideal in \mathfrak{L}_d such that $\dim(\mathfrak{L}_d/\mathcal{I}) = p < \infty$. We wish to consider the constrained $r \times r$ matrix-valued Pick problem on the dual algebra $\mathcal{A} := \mathbb{C} + \mathcal{I}$. For any r -tuple of vectors (x_1, \dots, x_r) in $\ell^2(\mathbb{F}_d^+)$, the invariant subspace decomposition for \mathfrak{L}_d [26, Theorem 2.1] implies that there are isometries $B_1, \dots, B_r \in \mathfrak{K}_d$ such that

$$\begin{aligned} L &:= \mathcal{A}[x_1, \dots, x_r] = \text{span}(x_1, \dots, x_r, \mathcal{I}[x_1, \dots, x_r]) \\ &= P_{\mathcal{I}[x_1, \dots, x_r]}^\perp(\text{span}(x_1, \dots, x_r)) \oplus \left(\bigoplus_{i=1}^r B_i \ell^2(\mathbb{F}_d^+)\right). \end{aligned}$$

For notational convenience, set $M := P_{\mathcal{I}[x_1, \dots, x_r]}^\perp(\text{span}(x_1, \dots, x_r))$. For $z \in \mathbb{B}_d$, let $\mathcal{I}_z^{\mathcal{A}}$ denote the ideal in \mathcal{A} of operators A such that $\hat{A}(z) = 0$. It follows that

$$\mathcal{I}_z^{\mathcal{A}} = \{B - \hat{B}(z) : B \in \mathcal{I}\}.$$

If we let \mathcal{I}_z denote the ideal in \mathfrak{L}_d of operators A such that $\hat{A}(z) = 0$, then $\mathcal{I}\mathcal{I}_z = \mathcal{I}_z\mathcal{I}$ is a WOT closed, two-sided ideal in \mathfrak{L}_d , and clearly $\mathcal{I}_z\mathcal{I} \subset \mathcal{I}_z^{\mathcal{A}}$. This implies, in particular, that M is orthogonal to the range of $\mathcal{I}_z\mathcal{I}$. Now compute

$$\begin{aligned} L \ominus \mathcal{I}_z^{\mathcal{A}}L &\subset L \ominus \mathcal{I}_z\mathcal{I}L \\ &= M \oplus \left(\left(\bigoplus B_i \ell^2(\mathbb{F}_d^+)\right) \ominus \mathcal{I}_z \left[\left(\bigoplus B_i \ell^2(\mathbb{F}_d^+)\right)\right]\right) \\ &= M \oplus \left(\left(\bigoplus B_i \ell^2(\mathbb{F}_d^+)\right) \ominus \left(\bigoplus B_i \mathcal{I}_z \ell^2(\mathbb{F}_d^+)\right)\right) \\ &= M \oplus \left(\bigoplus B_i (\ell^2(\mathbb{F}_d^+) \ominus \mathcal{I}_z \ell^2(\mathbb{F}_d^+))\right) \\ &= M \oplus \mathbb{C} \sum_{i=1}^r B_i k_z, \end{aligned}$$

where the third line follows from the fact that \mathcal{I}_z commutes with \mathfrak{K}_d . Note that the left hand side of the above inclusion has dimension at most pr , and the right hand side has dimension at most $pr + 1$. Notice that for $z, w \in \mathbb{B}_d$, we have

$$\left\langle \sum_{i=1}^r B_i k_z, \sum_{j=1}^r B_j k_w \right\rangle = \sum_{i,j} \langle B_j^* B_i k_z, k_w \rangle = rk(w, z),$$

and hence there is no dependence on the B_i . Thus, a sufficient criterion for solving the Pick problem can be realized by replacing subspaces of the form $L \ominus \mathcal{I}_z^{\mathcal{A}}L$ by the larger spaces $M \oplus \mathbb{C} \sum_{i=1}^r B_i k_z$ in the Pick matrices of Theorem 5.3.4, the latter of which are

parametrized by pr -dimensional subspaces containing the complement of the range of \mathcal{I} .

Chapter 6

Interpolation in product domains

In this chapter, we will investigate the Pick problem on domains of the form $X = X_1 \times \cdots \times X_n$. If a positive definite kernel on X can be factored as kernels on each of the X_i , then the corresponding reproducing kernel Hilbert space on X can be identified with the tensor product of the reproducing kernel Hilbert spaces of the factors. This is a natural generalization of the Pick problem for the polydisk seen in Chapter 4. We will be particularly interested in the full matrix-valued generalizations of the Pick problem in this setting, and once again will employ techniques in non-commutative operator algebras.

6.1 Products of kernels

For an arbitrary reproducing kernel Hilbert space on a set X , the Pick problem is not tractable, and so we cannot expect to have general results for products of kernels. However, if each product is itself a complete Pick kernel (as is the case with the polydisk), we can say much more. Recall that if \mathcal{S} and \mathcal{L} are subspaces of $\mathcal{B}(\mathcal{H})$, we denote the spatial tensor product of \mathcal{S} and \mathcal{L} as $\mathcal{S} \overline{\otimes} \mathcal{L}$.

Proposition 6.1.1. *Suppose $m \geq 1$ and that $\mathcal{H}_1, \dots, \mathcal{H}_m$ are scalar-valued reproducing kernel Hilbert spaces on sets X_1, \dots, X_m , with kernels k^1, \dots, k^m , respectively. Define*

$$k := \prod_{i=1}^m k^i; \quad X = \prod_{i=1}^m X_i.$$

Then $\mathcal{H} := \mathcal{H}(k)$ is isomorphic as a reproducing kernel Hilbert space to $\bigotimes_{i=1}^m \mathcal{H}(k^i)$. The unitary that implements this isomorphism induces a unitary equivalence between $\mathcal{M}(\mathcal{H})$ and $\mathcal{M}(\bigotimes_{i=1}^m \mathcal{H}_i)$. If, in addition, $\mathcal{M}(\mathcal{H})$ and each $\mathcal{M}(\mathcal{H}_i)$ are maximal abelian, we have $\mathcal{M}(\bigotimes_{i=1}^m \mathcal{H}_i) = \overline{\bigotimes_{i=1}^m \mathcal{M}(\mathcal{H}_i)}$.

Proof. Given elements $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ in X , the reproducing kernel at x is the function $k_x \in \mathcal{H}$ determined by

$$k_x(y) = k^1(x_1, y_1) \cdots k^m(x_m, y_m) = \langle k_{x_1}^1 \otimes \cdots \otimes k_{x_m}^m, k_{y_1}^1 \otimes \cdots \otimes k_{y_m}^m \rangle.$$

Define a linear map U on reproducing kernels in \mathcal{H} into $\otimes_{i=1}^m \mathcal{H}(k^i)$ as follows:

$$U \sum_{j=1}^n a_j k_{x^j} = \sum_{j=1}^n a_j (\otimes_{i=1}^m k_{x^j}^i)$$

where $x^1, \dots, x^n \in X$. Since U maps a dense set to a dense set, and kernels to kernels, the claims about U will be satisfied if we can show that U is an isometry. Compute:

$$\begin{aligned} \left\| U \sum_{j=1}^n a_j k_{x^j} \right\|^2 &= \left\langle \sum_{j=1}^n a_j \otimes_{i=1}^m k_{x^j}^i, \sum_{l=1}^n a_l \otimes_{i=1}^m k_{x^l}^i \right\rangle \\ &= \sum_{j,l=1}^n a_j \bar{a}_l \prod_{i=1}^m k^i(x_l^i, x_j^i) \\ &= \sum_{j,l=1}^n a_j \bar{a}_l k(x_l, x_j) \\ &= \left\| \sum_{j=1}^n a_j k_{x^j} \right\|^2. \end{aligned}$$

Now, since U is a reproducing kernel Hilbert space isomorphism, the intertwining relationship follows immediately. To show that $\mathcal{M}(\otimes_{i=1}^m \mathcal{H}_i) \supseteq \overline{\otimes_{i=1}^m} \mathcal{M}(\mathcal{H}_i)$, note that if $\varphi_1, \dots, \varphi_m$ are multipliers on $\mathcal{H}_1, \dots, \mathcal{H}_m$, respectively, then

$$\left(\otimes_{i=1}^m M_{\varphi_i} \right)^* k_x = \prod_{i=1}^m \overline{\varphi_i(x_i)} k_{x_i},$$

which implies that $M_{\varphi_1} \otimes \dots \otimes M_{\varphi_m}$ is in $\mathcal{M}(\mathcal{H})$. By taking weak-* limits of sums of elementary tensors, this implies that $\overline{\otimes_{i=1}^m} \mathcal{M}(\mathcal{H}_i) \subseteq \mathcal{M}(\otimes_{i=1}^m \mathcal{H}_i)$. If $\mathcal{M}(\mathcal{H})$ and each $\mathcal{M}(\mathcal{H}_i)$ is maximal abelian, the reverse inclusion follows readily since

$$\begin{aligned} \overline{\otimes_{i=1}^m} \mathcal{M}(\mathcal{H}_i) &= \overline{\otimes_{i=1}^m} \mathcal{M}(\mathcal{H}_i)' = \left(\overline{\otimes_{i=1}^m} \mathcal{M}(\mathcal{H}_i) \right)' \\ &\supseteq \left(\mathcal{M} \left(\otimes_{i=1}^m \mathcal{H}_i \right) \right)' = \mathcal{M}(\mathcal{H}). \quad \square \end{aligned}$$

Proposition 6.1.2. *Suppose \mathcal{H} is a scalar-valued reproducing kernel Hilbert space such that $\mathcal{M}(\mathcal{H})$ has a cyclic vector h . Then $\mathcal{M}(\mathcal{H})$ is maximal abelian.*

Proof. Suppose T is an operator such that $T \in \mathcal{M}(\mathcal{H})'$ and let h be a cyclic vector in \mathcal{H} . Define a function ψ on X by setting $\psi(x) = \frac{Th(x)}{h(x)}$. We claim that ψ is a multiplier of \mathcal{H} and that $T = M_\psi$. Let f be any function in \mathcal{H} and φ_n a sequence of multipliers such that

$\lim_{n \rightarrow \infty} \varphi_n h = f$. For any $x \in X$, we have

$$\begin{aligned} \psi(x)f(x) &= \lim_{n \rightarrow \infty} \psi(x)\varphi_n(x)h(x) = \lim_{n \rightarrow \infty} Th(x)\varphi_n(x) = \lim_{n \rightarrow \infty} \langle M_{\varphi_n}Th, k_x \rangle_{\mathcal{H}} \\ &= \lim_{n \rightarrow \infty} \langle TM_{\varphi_n}h, k_x \rangle_{\mathcal{H}} = \lim_{n \rightarrow \infty} T\varphi_n(x)h(x) = Tf(x). \end{aligned}$$

This shows that ψf is in \mathcal{H} for any f , which implies ψ is a multiplier. \square

Corollary 6.1.3. *Suppose k^1, \dots, k^m are complete Pick kernels. Then*

$$\mathcal{M} \left(\bigotimes_{i=1}^m \mathcal{H}(k^i) \right) = \overline{\bigotimes_{i=1}^m} \mathcal{M}(k^i).$$

Proof. By Proposition 6.1.1 and Proposition 6.1.2, it is enough to show that each $\mathcal{M}(k_i)$ has a cyclic vector, for then $\mathcal{M} \left(\bigotimes_{i=1}^m \mathcal{H}(k^i) \right)$ has one as well. Without loss of generality, we may assume that there are countable cardinals d_1, \dots, d_m and subsets S_1, \dots, S_m of the balls $\mathbb{B}_{d_1}, \dots, \mathbb{B}_{d_m}$, respectively, so that $\mathcal{H}_i = \overline{\text{span}}\{k_s^i : s \in S_i\}$ for $1 \leq i \leq m$. The constant functions $1|_{S_i}$ are easily seen to be cyclic for $\mathcal{M}(\mathcal{H}_i)$. \square

6.2 The polydisk

The first and simplest product kernel one might think of is the product of Szegő kernels for the disk. Even in this case, the Pick problem is difficult and only recently solved. For $d > 1$, let \mathbb{D}^d denote the polydisk in d variables and write any element in \mathbb{D}^d as $z = (z^1, \dots, z^d)$. Recall that the Hardy space $H^2(\mathbb{D}^d)$ is a reproducing kernel Hilbert space on \mathbb{D}^d with kernel

$$k(w, z) = \prod_{j=1}^d \frac{1}{1 - z^j w^j},$$

and whose multiplier algebra is $H^\infty(\mathbb{D}^d)$. A positive definite matrix-valued kernel $K : \mathbb{D}^d \times \mathbb{D}^d \rightarrow M_r$ is said to be *admissible* if

$$(1 - z_i^s z_j^s)K(z_i, z_j) \geq 0$$

for $s = 1, \dots, d$. The following is the analogue of the Pick interpolation theorem for the case $d = 2$. It is due to Agler and McCarthy [2], building on earlier unpublished work of Agler.

Theorem 6.2.1. *Suppose $z_1, \dots, z_n \in \mathbb{D}^2$ and $W_1, \dots, W_n \in M_r$. The following are equivalent.*

1. *There is a function $\Phi \in M_r(H^\infty(\mathbb{D}^2))$ such that $\|\Phi\|_\infty \leq 1$ and $\Phi(z_i) = W_i$ $i = 1, \dots, n$.*
2. *The operator matrix*

$$[(1 - W_i W_j^*) \otimes K(z_i, z_j)]$$

is positive for every admissible kernel K .

3. There are M_r -valued kernels Γ_1 and Γ_2 on $\mathbb{D}^2 \times \mathbb{D}^2$ such that

$$1 - W_i W_j^* = (1 - z_i^1 \bar{z}_j^1) \Gamma_1(z_i, z_j) + (1 - z_i^2 \bar{z}_j^2) \Gamma_2(z_i, z_j)$$

The statement that (3) implies (1) makes use of the so-called realization formula for the bidisk, which in turn relies on Ando's theorem. The well-known examples of Parrott and Varopolous show that Ando's theorem fails in dimension 3 or greater (see, for example, Paulsen [50, Chapter 5]), which makes the case $d > 2$ more complicated. For bounded analytic functions on \mathbb{D}^d , define the Schur-Agler norm as

$$\|\varphi\|_d := \sup\{\|\varphi(rT_1, \dots, rT_d)\| : T_1, \dots, T_d \text{ are commuting contractions, } 0 < r < 1\}.$$

The Schur-Agler norm can be strictly larger than the sup norm. The Agler-McCarthy theorem for the bidisk holds for the polydisk as well, with the exception that the solution to the interpolation problem will be a contraction with respect to the Schur-Agler norm.

Since the sup norm is much better understood than the Schur-Agler norm, one might ask whether or not there is a 'natural' sufficient condition which implies the existence of an interpolant with $\|\varphi\|_\infty \leq 1$ on \mathbb{D}^d . The following curious condition was recently proved to be necessary [34].

Theorem 6.2.2 (Grinshpan-Kaliuzhnyi-Vinnikov-Woerdeman). *Suppose there is a function $\varphi \in H^\infty(\mathbb{D}^d)$ such that $\|\varphi\|_\infty \leq 1$. Then for every pair of natural numbers p, q such that $1 \leq p < q \leq d$, there are positive semidefinite matrices Γ^p and Γ^q such that*

$$(1 - \varphi(z_i) \overline{\varphi(z_j)}) = \prod_{r \neq q} (1 - z_i^r \bar{z}_j^r) \Gamma^q(z_i, z_j) + \prod_{r \neq p} (1 - z_i^r \bar{z}_j^r) \Gamma^p(z_i, z_j). \quad (6.1)$$

Our immediate goal is to prove a Pick interpolation theorem for products of complete Pick kernels. There is such a result, due to Tomerlin [65], which is very much analogous to the Agler-McCarthy theorem, where the solution is a contraction with respect to a Schur-Agler-type norm for that class. Our approach is very different, and will take us on a sojourn back into the non-commutative world.

6.3 Left regular representations of other semigroups

For the remainder of this chapter, let $p \geq 2$ be a natural number and fix countable cardinals d_1, \dots, d_p . Define the following semigroup:

$$\mathcal{S} = \langle e^i : 1 \leq i \leq p, 1 \leq j \leq d_i, e_i^j e_k^j = e_k^j e_i^j \text{ for } i \neq j \rangle.$$

In other words, \mathcal{S} is the semigroup on $d_1 \dots d_p$ letters determined by the the commutation relations $e_i^j e_k^j = e_k^j e_i^j$ for $i \neq j$. For a fixed index i , the generators $e_1^i, \dots, e_{d_i}^i$ are free. Now form the Hilbert space $\ell^2(\mathcal{S})$ and list its canonical orthonormal basis as

$$\{\xi_{e_{w_1}^1 \dots e_{w_p}^p} : w_i \in \mathbb{F}_{d_i}^+ \text{ for } 1 \leq i \leq p\}.$$

Definition 6.3.1. The \mathcal{S} -analytic Toeplitz algebra $\mathfrak{L}_{\mathcal{S}}$ is the unital, WOT closed subalgebra of $\mathcal{B}(\ell^2(\mathcal{S}))$ generated by the left regular representation of \mathcal{S} . Namely,

$$\mathfrak{L}_{\mathcal{S}} = \overline{\text{Alg}}^{\text{WOT}} \{I, L_j^i : 1 \leq i \leq p, 1 \leq j \leq d_i\},$$

where $L_j^i \xi_{e_{w_1}^1 \dots e_{w_p}^p} = \xi_{e_j^i e_{w_1}^1 \dots e_{w_p}^p}$. The right regular representation on \mathcal{S} is defined analogously, and is denoted $\mathfrak{R}_{\mathcal{S}}$.

These algebras are examples of a more general class of algebras associated to higher rank graphs, introduced by Kribs and Power [41] as simultaneous generalizations of \mathfrak{L}_d and $H^\infty(\mathbb{D}^d)$. The graph associated to the semigroup \mathcal{S} above is a single vertex graph with p colours and d_i edges for each colour i . The C^* algebras associated to higher rank graphs were introduced earlier by Kumjian and Pask [42]. Each d_i -tuple of operators $[L_1^i, \dots, L_{d_i}^i]$ is a row isometry (that is, the isometries L_j^i have pairwise orthogonal ranges) which individually generate the left regular representation on the free semigroups $\mathbb{F}_{d_i}^+$ (with infinite multiplicity). One immediately observes that for $i \neq j$, we have the commutation relations $L_k^i L_l^j = L_l^j L_k^i$, which forces the following tensor product decomposition of $\mathfrak{L}_{\mathcal{S}}$.

Proposition 6.3.2. *Suppose \mathcal{S} is the semigroup given above. The algebra $\mathfrak{L}_{\mathcal{S}}$ is unitarily equivalent to the spatial tensor product $\mathfrak{L}_{d_1} \overline{\otimes} \dots \overline{\otimes} \mathfrak{L}_{d_p}$.*

Proof. Let $\{e_w : w \in \mathbb{F}_{d_i}^+\}$ denote the standard orthonormal basis for $\ell^2(\mathbb{F}_{d_i}^+)$. Define a unitary $U : \ell^2(\mathcal{S}) \rightarrow \otimes_{i=1}^p \ell^2(\mathbb{F}_{d_i}^+)$ by

$$U \xi_{e_{w_1}^1 \dots e_{w_p}^p} = e_{w_1} \otimes \dots \otimes e_{w_p}.$$

It is apparent that U induces the required equivalence. \square

Proposition 6.3.3 (Kribs-Power). *The commutant of $\mathfrak{L}_{\mathcal{S}}$ is $\mathfrak{R}_{\mathcal{S}}$ and $(\mathfrak{L}_{\mathcal{S}})'' = \mathfrak{R}_{\mathcal{S}}$.*

Note that if $d_1 = \dots = d_p = 1$, then we may identify $\mathfrak{L}_{\mathcal{S}}$ with the analytic Toeplitz operators on $H^2(\mathbb{D}^p)$, i.e. the multiplier algebra $\mathcal{M}(H^2(\mathbb{D}^p)) = H^\infty(\mathbb{D}^p)$. We will return to this special case later, but for now we need to assume that at least one of the d_i is at least 2.

Theorem 6.3.4. *Suppose d_1, \dots, d_p are countable cardinals with $d_i \geq 2$ for some $1 \leq i \leq p$ and suppose \mathcal{L} is a separable Hilbert space. Then $\mathfrak{L}_{\mathcal{S}} \overline{\otimes} \mathcal{B}(\mathcal{L})$ has property $X(0, 1)$, and therefore property $\mathbb{A}_1(1)$.*

Proof. By Bercovici's theorem [17], it suffices to show that $(\mathfrak{L}_{\mathcal{S}} \otimes \mathcal{B}(\mathcal{L}))'$ contains two isometries with pairwise orthogonal ranges. Since $d_i \geq 2$ for some i , the isometries $R_1^i \otimes I_{\mathcal{L}}$ and $R_2^i \otimes I_{\mathcal{L}}$ satisfy this requirement. \square

Let k^{d_1}, \dots, k^{d_p} denote the Drury-Arveson kernels on the sets $\mathbb{B}_{d_1}, \dots, \mathbb{B}_{d_p}$, respectively. By Corollary 6.1.3, the kernel $k := \prod_{i=1}^p k_{d_i}$ is the kernel function for the reproducing kernel Hilbert space $\mathcal{H} := \oplus_{i=1}^d H_{d_i}^2$, which acts on the set $\mathbb{B} := \prod_{i=1}^p \mathbb{B}_{d_i}$. Moreover, the

multiplier algebra $\mathcal{M}(\mathcal{H})$ can be identified with the spatial tensor product $\overline{\otimes}_{i=1}^p \mathcal{M}(H_{d_i}^2)$. By the identification made in Proposition 6.3.2, we regard \mathcal{H} as a subspace of $\ell^2(\mathcal{S})$. Some parts of the following result appear in Kribs-Power [41]. For the sake of convenience, we include simplified proofs presently.

Theorem 6.3.5. *For every $z = (z_1, \dots, z_p) \in \mathbb{B}$, the reproducing kernel k_z is an eigenvector for $\mathfrak{L}_{\mathcal{S}}^*$. The normalized kernel functions $k_z/\|k_z\|$ have the property that every WOT continuous, multiplicative linear functional ω on \mathfrak{L}_d is given by*

$$\omega(A) = \|k_z\|^{-2} \langle Ak_z, k_z \rangle$$

for $A \in \mathfrak{L}_{\mathcal{S}}$. If $\hat{A}(z)$ is the complex number which satisfies $A^*k_z = \overline{\hat{A}(z)}k_z$, then the map $A \mapsto \hat{A}$ is a unital, completely contractive WOT-to-WOT continuous epimorphism of $\mathfrak{L}_{\mathcal{S}}$ onto $\mathcal{M}(\mathcal{H})$.

Proof. Let $A_1 \otimes \dots \otimes A_p$ be an elementary tensor in $\mathfrak{L}_{\mathcal{S}}$, with $A_i \in \mathfrak{L}_i$ for each $i = 1, \dots, p$. For $z \in \mathbb{B}$, we have

$$(A_1 \otimes \dots \otimes A_p)^*k_z = \overline{\hat{A}(z_1) \dots \hat{A}(z_p)}k_z.$$

For an arbitrary element $A \in \mathfrak{L}_{\mathcal{S}}$, taking WOT limits of finite linear combinations of elementary tensors, we see that $A^*k_z \in \mathbb{C}k_z$. Let $\hat{A}(z)$ denote the corresponding eigenvalue. If ω is a WOT continuous homomorphism for $\mathfrak{L}_{\mathcal{S}}$, then the slice map $\omega_i : \mathfrak{L}_{d_i} \rightarrow \mathbb{C}$ given by

$$\omega_i(A_i) := \omega(I \otimes \dots \otimes I \otimes A_i \otimes I \otimes \dots \otimes I)$$

is a WOT continuous homomorphism on \mathfrak{L}_{d_i} . Therefore there is a point $z_i \in \mathbb{B}_{d_i}$ such that $\omega_i = \omega_{z_i}$, where $\omega_{z_i}(A_i) = \|k_{z_i}\|^{-2} \langle Ak_{z_i}, k_{z_i} \rangle$ for each $A_i \in \mathfrak{L}_{d_i}$. Repeating this argument for each i , we see that there is a point $z = (z_1, \dots, z_p) \in \mathbb{B}$ so that

$$\begin{aligned} \omega(A_1 \otimes \dots \otimes A_p) &= \prod_{i=1}^p \omega_i(A_i) \\ &= \prod_{i=1}^p \|k_{z_i}\|^{-2} \langle A_i k_{z_i}, k_{z_i} \rangle \\ &= \|k_z\|^{-2} \langle A_1 \otimes \dots \otimes A_p k_z, k_z \rangle. \end{aligned}$$

By again taking limits of linear combinations, we find that $\omega = \omega_z$ on all of $\mathfrak{L}_{\mathcal{S}}$.

We now show that \hat{A} defines a multiplier on \mathcal{H} . Let $z_1, \dots, z_n \in \mathbb{B}$ and notice that since $A^*k_{z_i} = \overline{\hat{A}(z_i)}k_{z_i}$, we have

$$\left[(\|A\|^2 - \hat{A}(z_i)\overline{\hat{A}(z_j)})k(z_i, z_j) \right]_{i,j=1}^n \geq 0.$$

This shows that $\hat{A} \in \mathcal{M}(\mathcal{H})$. The rest of the claims about the map $A \mapsto \hat{A}$ follow if we can show that \hat{A} is just the compression A to \mathcal{H} , i.e. that $P_{\mathcal{H}}A = P_{\mathcal{H}}AP_{\mathcal{H}} = M_{\varphi}$ where

$\varphi(z) = \hat{A}(z)$. If $A_1 \otimes \cdots \otimes A_p$ is an elementary tensor, we have

$$\begin{aligned}
(A_1 \widehat{\otimes} \cdots \widehat{\otimes} A_p)(z) &= \langle A_1 \otimes \cdots \otimes A_p k_z, k_z \rangle \\
&= \prod_{i=1}^p \langle A_i k_{z_i}, k_{z_i} \rangle \\
&= \prod_{i=1}^p \langle P_{H_{d_i}^2} A_i k_{z_i}, k_{z_i} \rangle \\
&= (P_{H_{d_1}^2} \otimes \cdots \otimes P_{H_{d_p}^2})(A_1 \otimes \cdots \otimes A_p)(z) \\
&= P_{\mathcal{H}}(A_1 \otimes \cdots \otimes A_p)(z).
\end{aligned}$$

Since compression to a closed subspace is continuous in the WOT topology, the result now follows. \square

Now suppose \mathcal{A} is a dual subalgebra of $\mathfrak{L}_{\mathcal{S}}$, and form the ideal \mathcal{I}_E in \mathcal{A} for any finite subset $E \subset \mathbb{B}$:

$$\mathcal{I}_E := \{A \in \mathcal{A} : \hat{A}(z_i) = 0, \text{ for every } z_i \in E\}.$$

If L is any invariant subspace for \mathcal{A} and $A \in \mathcal{A}$, we have

$$P_L A^* P_L k_z = \overline{\hat{A}(z)} P_L k_z.$$

More generally, if $P_L k_z = 0$, but the subspace $\mathcal{A}[h] \ominus \mathcal{J}_z[h]$ is non-zero, then it is spanned by the extended kernel function k_z^L , which of course satisfies the equation

$$(A|_L)^* k_z^L = \overline{\hat{A}(z)} k_z^L.$$

Since $\mathcal{A} \otimes \mathcal{B}(\mathcal{L})$ has property $\mathbb{A}_1(1)$ for any Hilbert space \mathcal{L} , we immediately get the following matrix-valued Pick interpolation theorem. In light of the above observations, the proof is identical to the analogous result for \mathfrak{L}_d (cf. Theorem 5.3.4). As before, for $A = [A_{ij}] \in M_r(\mathcal{A})$, let \hat{A} denote the M_r -valued function given by $\hat{A}(z) := [\widehat{A_{ij}}(z)]$.

Theorem 6.3.6. *Suppose $z_1, \dots, z_n \in \mathbb{B}$ and $W_1, \dots, W_n \in M_r$ and that \mathcal{A} is a dual subalgebra of $\mathfrak{L}_{\mathcal{S}}$. There is a contractive $A \in M_r(\mathcal{A})$ such that $\hat{A}(z_i) = W_i$ for $i = 1, \dots, n$ if and only if*

$$[(1 - W_i W_j^*) \otimes K^L(z_i, z_j)] \geq 0$$

for every r -generated $L \in \text{Lat } \mathcal{A}$, where $K^L(z_i, z_j) = P_{L \ominus \mathcal{I}_{z_j} L} P_{L \ominus \mathcal{I}_{z_i} L}$ is an M_r -valued positive semidefinite kernel.

6.4 Matrix-valued interpolation III

Turning back now to the commutative case, once again fix countable cardinals d_1, \dots, d_p with $d_i \geq 2$ for at least one i . Define the positive definite kernel $k(z, w) = \prod_{i=1}^p k^{d_i}(z_i, w_i)$ for $z = (z_1, \dots, z_p), w = (w_1, \dots, w_p) \in \mathbb{B} = \mathbb{B}_{d_1} \times \cdots \times \mathbb{B}_{d_p}$. If $A \in \mathfrak{L}_{\mathcal{S}}$, then \hat{A} determines

a multiplier on $\mathcal{H}(k) = \otimes_{i=1}^p H_{d_i}^2$. If E is a finite subset of \mathbb{B} , then the existence of a contractive $A \in \mathfrak{L}_{\mathcal{S}}$ such that $\hat{A}(z_i) = W_i$ implies the existence of a contractive multiplier $M_{\Phi} \in M_r(\mathcal{M}(\mathcal{H}))$ by taking $\Phi = (P_{\mathcal{H}} \otimes I_r)A(P_{\mathcal{H}} \otimes I_r)$ (in other words, $\Phi(z) = \hat{A}(z)$).

Theorem 6.4.1. *Suppose $z_1, \dots, z_n \in \mathbb{B}$ and $W_1, \dots, W_n \in M_r$ and that \mathcal{A} is a dual algebra of multipliers on \mathcal{H} . There is a contractive $M_{\Phi} \in M_r(\mathcal{A})$ such that $\Phi(z_i) = W_i$ for $i = 1, \dots, n$ if*

$$[(1 - W_i W_j^*) \otimes K^L(z_i, z_j)] \geq 0$$

for every r -generated $L \in \text{Lat } \tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}} = q^{-1}(\mathcal{A})$ and $K^L(z_i, z_j) = P_{L \ominus \mathfrak{J}_{z_j} L} P_{L \ominus \mathfrak{J}_{z_i} L}$.

This immediately leads one to speculate if the above positivity condition is necessary as well. If it were the case that $M_r(\mathcal{M}(k))$ had property $\mathbb{A}_1(1)$, then we would obtain a necessary and sufficient condition, where the kernels K^L arise from subspaces of \mathcal{H} rather than $\ell^2(\mathcal{S})$. However, even for a single Drury-Arveson kernel k^d , we know it is not the case that $M_n(\mathcal{M}(H_d^2))$ has property $\mathbb{A}_1(1)$. In general, it is not true that the restriction of an algebra with property $\mathbb{A}_1(1)$ also has property $\mathbb{A}_1(1)$ (if this were true, one would have an easy proof that every singly generated algebra has property $\mathbb{A}_1(1)$, which is still open). Therefore, the above theorem likely represents the best we could hope for using a dual algebra approach to Pick interpolation. Nonetheless, it gives a sufficient condition for a solution in terms of the *multiplier* norm, and not the generally larger Schur-Agler norm obtained by Tomerlin. We currently do not know of any relationship between the operator norm on $\mathfrak{L}_{\mathcal{S}}$ and the Schur-Agler norm on $\mathcal{M}(\mathcal{H})$ (see below).

The various tools present when studying \mathfrak{L}_d (inner-outer factorization and the Beurling theorem for invariant subspaces for instance) are no longer present for $\mathfrak{L}_{\mathcal{S}}$, and their failure is well-known even for $H^\infty(\mathbb{D}^2)$ (see Rudin [59]). Therefore we cannot hope to relate the cyclic subspaces of $\mathfrak{L}_{\mathcal{S}}$ to those in $\mathcal{M}(\mathcal{H})$. Recall that the complete distance formula for a two-sided WOT closed ideals \mathcal{J} of \mathfrak{L}_d is

$$\text{dist}(A, \mathcal{J}) = \|P_M A\|$$

where $M = \mathcal{J}[e_0]$ is the range of \mathcal{J} . In particular, when $\mathcal{J} = \mathcal{C}$ is the WOT closure of the commutator ideal, we get

$$\text{dist}(A, \mathcal{C}) = \|P_{H_d^2} A\| = \|\hat{A}\|_{\mathcal{M}(H_d^2)}.$$

Equivalently, if \mathcal{A} belongs to \mathfrak{L}_d , the *minimal* norm of a multiplier of H_d^2 which is the compression of \mathcal{A} to Drury-Arveson space is precisely $\text{dist}(A, \mathcal{J})$. The proof of these facts are entirely reliant on the correspondence between invariant subspaces and ideal for \mathfrak{L}_d .

Let \mathcal{C} now denote the WOT closure of the commutator ideal in $\mathfrak{L}_{\mathcal{S}}$. For any $h \in \ell^2(\mathcal{S})$, $z \in \mathbb{B}$ and $A, B \in \mathfrak{L}_{\mathcal{S}}$, we have

$$\langle (AB - BA)h, k_z \rangle = 0,$$

which implies that \mathcal{H} is contained in $\text{Ran } \mathcal{C}^\perp$. If φ is a multiplier of \mathcal{H} , then $\text{dist}(A + \mathcal{C})$ is an upper bound for the norm of M_φ , where A is any operator in $\mathfrak{L}_{\mathcal{S}}$ such that $\hat{A} = \varphi$.

Consequently, we have

$$\text{dist}(A, \mathcal{C}) \geq \|P_{\text{Ran } \mathcal{C}}^\perp A\| \geq \|P_{\mathcal{H}} A\| = \|M_\varphi\|.$$

Since \mathfrak{L}_S has property $\mathbb{A}_1(1)$, we can actually get an exact formula for the distance to the commutator ideal.

Corollary 6.4.2. *If \mathfrak{C} is the WOT closure of the commutator ideal of \mathfrak{L}_S , then for any A in \mathfrak{L}_S we have*

$$\text{dist}(A, \mathfrak{C}) = \sup_{h \in \ell^2(S)} \|P_{\mathfrak{C}[h]}^\perp A\| = \sup_{h \in \ell^2(S)} \|P_{\mathfrak{C}[h]}^\perp M_\varphi\|$$

where $\varphi = \hat{A}$.

In particular, if A is any contractive operator belonging to \mathfrak{L}_S which solves the Pick problem, i.e. that $\hat{A}(z_i) = w_i$ for $i = 1, \dots, n$, then there is a contractive multiplier M_φ on \mathcal{H} such that

$$\|M_\varphi\| \leq \sup_{h \in \ell^2(S)} \|P_{\mathfrak{C}[h]}^\perp A\| \tag{6.2}$$

and $\varphi(z_i) = w_i$. Analogous statements hold for operators in $M_r(\mathfrak{L}_S)$.

Question 6.4.3. When is inequality in 6.2 an equality? How does the quantity $\sup_{h \in \ell^2(S)} \|P_{\mathfrak{C}[h]}^\perp M_\varphi\|$ compare to the Schur-Agler norm for φ ?

If $k = k^1 \dots k^p$ is a product of irreducible Pick kernels on the product set $X = X_1 \times \dots \times X_p$, then by passing through to the injective embeddings $X_i \rightarrow \mathbb{B}_{d_i}$ for some cardinal d_i , we may assume that

$$\mathcal{H}(k) = \overline{\text{span}}\{k_{z_1}^{d_1} \otimes \dots \otimes k_{z_p}^{d_p} : (z_1, \dots, z_p) \in X_1 \times \dots \times X_p\}.$$

The analogue of Theorem 6.4.1 for products of irreducible Pick kernels follows immediately.

Example 6.4.4. In order to apply our techniques to the polydisk \mathbb{D}^d , we embed it into the set $\mathbb{B}_2 \times \mathbb{D}^{d-1}$ and identify $H^2(\mathbb{D}^d)$ with a subspace of $H_2^2 \otimes H^2(\mathbb{D}^{d-1})$ via the map $k_{z_1} \dots k_{z_d} \mapsto k_{(z_1, 0)}^2 \dots k_{z_d}$. This map is a reproducing kernel Hilbert space isomorphism which intertwines multipliers, and so we do obtain a sufficient condition for the solvability of the matrix-valued Pick interpolation theorem for $H^\infty(\mathbb{D}^d)$. Moreover, our result gives a more explicit description of the operator-valued kernels K^L , instead of relying on all possible admissible kernels.

As a final note, the results of this section carry through to products of arbitrary Pick kernels by considering the decomposition of the associated multiplier algebras (see Lemma 5.2.1). The details are essentially the same.

Chapter 7

Douglas-type factorization and the Toeplitz corona problem

This chapter focuses on the extension of some recently obtained results in tangential interpolation to a large class of algebras of operators acting on reproducing kernel Hilbert spaces. In particular, these results apply to arbitrary weakly-closed algebras of multipliers on Drury-Arveson space and all other complete Pick spaces. A consequence of these results is an operator corona theorem (often called a Toeplitz corona theorem) for these algebras. This work was motivated by results of Raghupathi and Wick [58] where subalgebras of $H^\infty(\mathbb{D})$ are the main focus. The application of this result to the study of interpolating sequences is the subject of work currently in progress.

Our approach is based on reformulating the interpolation problem as a distance problem and computing a distance formula. A Toeplitz corona theorem is deduced from this formula. There are other approaches to the tangential interpolation problem and Toeplitz corona problem. Schubert [63] approached the problem through the commutant lifting theorem of Sz.-Nagy and Foias and also obtained the best bounds for the solution. Commutant lifting also appears in the (unpublished) work of Helton in this area. A commutant lifting theorem for the non-commutative analytic Toeplitz algebra \mathfrak{L}_d was proved by Popescu [52], from which a Toeplitz corona theorem follows in the same manner as Schubert.

7.1 Majorization and factorization of multipliers

Definition 7.1.1. An operator algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is said to have the *Douglas property* if the following are equivalent for operators A and B in \mathcal{A} :

1. $A^*A \geq B^*B$.
2. There is contraction $C \in \mathcal{A}$ such that $CA = B$.

By switching the roles of the above operators A and B and their adjoints, we immediately see that the statement:

$$AA^* \geq BB^* \text{ if and only if there is contraction } C \in \mathcal{A} \text{ such that } AC = B$$

is equivalent to \mathcal{A}^* having the Douglas property. When studying multiplier algebras, operators of the form AA^* have proven useful in calculations, and so we are primarily interested in the cases where the adjoint of an algebra has the Douglas property. In [29], Douglas proved that $\mathcal{B}(\mathcal{H})$ has the Douglas property.

Example 7.1.2. The C^* -algebra $C([0, 1])$ does not have the Douglas property. Consider $f(x) = |x - 1/2|$ and $g(x) = |x - 1/2|\chi_{[1/2, 1]}$. There is no continuous function h so that $fh = g$.

The following alternative proof of the Douglas lemma shows that any von Neumann algebra has the Douglas property.

Theorem 7.1.3. *If \mathcal{A} is a von Neumann algebra, then \mathcal{A} has the Douglas property.*

Proof. Suppose $A, B \in \mathcal{A}$ and $A^*A \geq B^*B$. It suffices to show that $W^*(A, B)$ has the Douglas property. Let $A = UH$ and $B = VK$ denote the polar decompositions of A and B , respectively, where U and V are partial isometries and H and K are positive. By assumption, we have $H^2 \geq K^2$.

Suppose, for the moment, that we can find a contraction D so that $DH = K$. Write $C := VDU^*$ and compute

$$CA = VDU^*UH = VDH = VK = B.$$

The second equality follows from the fact that U^*U is the projection onto $\text{Ran } A = \text{Ran } H$. Since C is a contraction contained in $W^*(A, B)$, the desired result is obtained assuming such a D exists. We can therefore assume, without loss of generality, that A and B are both positive and $A^2 \geq B^2$.

First assume that A is invertible. Write $C := A^{-1}B$ so that $CA = B$ and notice that C is a contraction since $C^*C = B(A^{-1})^2B \leq B(B^{-1})^2B = I$. In the general case, fix $\varepsilon > 0$. Then

$$(A + \varepsilon I)^2 = A^2 + 2\varepsilon A + \varepsilon^2 I \geq B^2 + 2\varepsilon B + \varepsilon^2 I = (B + \varepsilon I)^2$$

since the square root function is operator monotone. By the above argument, there is a contraction $C_\varepsilon \in W^*(A, B)$ so that $C_\varepsilon(A + \varepsilon I) = B + \varepsilon I$. Any weak* cluster point C of $\{C_\varepsilon\}_{\varepsilon > 0}$ in the unit ball of \mathcal{A} will satisfy $CA = B$. \square

In a recent preprint of McCullough and Trent [48], it is shown that $\mathcal{M}(k)^* \otimes \mathcal{B}(\mathcal{H})$ has the Douglas property if and only if k is a complete Pick kernel. As one might expect, if k is just a Pick kernel, then $\mathcal{M}(k)^*$ has the Douglas property.

Proposition 7.1.4. *Suppose k is a scalar Pick kernel. Then $\mathcal{M}(k)^*$ has the Douglas property.*

Proof. Suppose M_φ and M_ψ are in $\mathcal{M}(k)$ and satisfy $M_\varphi M_\varphi^* \geq M_\psi M_\psi^*$. This operator inequality holds if and only if the matrices

$$\left[\left(\varphi(x_i) \overline{\varphi(x_j)} - \psi(x_i) \overline{\psi(x_j)} \right) \langle k_{x_i}, k_{x_j} \rangle \right]$$

are positive semidefinite for every finite set of points $\{x_1, \dots, x_n\} \subset X$. Now suppose the x_i are chosen so that $f(x_i) \neq 0$ for each i . By Schur multiplying the above matrix by the Gramian $\left[\frac{1}{\varphi(x_i)\overline{\varphi(x_j)}}\right]$, we see that

$$\left[\left(1 - \frac{\psi(x_i)\overline{\psi(x_j)}}{\varphi(x_i)\overline{\varphi(x_j)}} \right) \langle k_{x_i}, k_{x_j} \rangle \right] \geq 0.$$

By the Pick property, there is a contractive multiplier M_γ so that $\gamma(x_i) = \psi(x_i)/\varphi(x_i)$. Enduring this process over all finite subsets of X for which φ does not vanish and taking a weak-* cluster point will yield a contractive multiplier with the desired properties. \square

The proof of the McCullough-Trent result relies entirely on the characterization of complete Pick kernels outlined in Chapter 5. The absence of such a characterization for Pick kernels makes the converse to the above proposition a seemingly more challenging problem.

Question 7.1.5. Is the converse of Proposition 7.1.4 true?

Example 7.1.6. $\mathcal{M}(H^2) = H^\infty$ does not have the Douglas property. Let ω be a singular inner function. Then

$$I = M_z^* M_z \geq M_\omega^* M_\omega = I.$$

Now suppose h is any meromorphic function satisfying $h(z)z = \omega(z)$. Then, since ω does not vanish on \mathbb{D} , the function h must have a non-removable singularity at 0 and so cannot be in H^∞ .

Recall that \mathbb{F}_d^+ is the free semigroup on d letters and \mathfrak{L}_d the weakly closed algebra generated by the left regular representation on \mathbb{F}_d^+ .

Proposition 7.1.7. *For every natural number $d \geq 1$, the non-commutative analytic Toeplitz algebra \mathfrak{L}_d does not have the Douglas property.*

Proof. The case $d = 1$ is handled in Example 7.1.6. For $d \geq 2$, observe that $L_1^* L_1 \geq L_2^* L_2$. Let e_0 denote the orthonormal basis element corresponding to the empty word. A standard result in [26] is that an operator $C \in \mathfrak{L}_d$ is 0 if $Ce_0 = 0$. If there is a $C \in \mathfrak{L}_d$ such that $CL_1 = L_2$, we have

$$R_2 e_0 = L_2 e_0 = CL_1 e_0 = CR_1 e_0 = R_1 C e_0$$

and so $R_1 C e_0 = R_2 e_0$. Multiply by R_1^* to get $C e_0 = 0$, and hence that $C = 0$. \square

The non-commutative analytic Toeplitz algebra does, however, satisfy a weaker version of the Douglas lemma. Just as in the commutative case, any operator A in \mathfrak{L}_d may be factored as $A = VF$ where V is an isometry in \mathfrak{K}_d and F is an operator in \mathfrak{L}_d with dense range.

Proposition 7.1.8. *Suppose $A, B \in \mathfrak{L}_d$ satisfy $A^* A \geq B^* B$ where A is outer (has dense range). Then there is a contraction $C \in \mathfrak{L}_d$ so that $CA = B$.*

Proof. Factor $B = VF$ where V is inner and F is outer. Then $A^*A \geq B^*B = F^*F$. By the Douglas lemma, find a contraction $C \in \mathcal{B}(\mathcal{H})$ so that $CA = F$. If R is an operator in \mathfrak{K}_d , we have

$$0 = RF - FR = RCA - CAR = RCA - CRA.$$

Since A has dense range, this implies $RC - CR = 0$, i.e. that $C \in \mathfrak{L}_d$. \square

As is the case with H^∞ , examining the adjoint \mathfrak{L}_d^* is a much more natural environment for studying the Douglas property.

Proposition 7.1.9. \mathfrak{L}_d^* has the Douglas property.

Proof. Using the Pick interpolation theorem for \mathfrak{L}_d of Davidson-Pitts [28], the proof of Proposition 7.1.4 may be repeated verbatim. The following shorter proof is more elementary. If A, B are in \mathfrak{L}_d and $AA^* \geq BB^*$, find a contraction $C \in \mathcal{B}(\ell^2(\mathbb{F}_d^+))$ so that $AC = B$. For $R \in \mathfrak{K}_d$ we have

$$0 = BR - RB = ACR - RAC = A(CR - RC).$$

Since every element in \mathfrak{L}_d is injective we have $CR - RC = 0$ and so $C \in \mathfrak{L}_d$. \square

7.2 A more general Douglas problem

Douglas's factorization theorem is not restricted to operators from \mathcal{H} into itself. Namely, given Hilbert spaces $\mathcal{H}, \mathcal{H}_1$ and \mathcal{H}_2 and operators $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H})$ which satisfy $AA^* \geq BB^*$, there is a contraction $C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ so that $AC = B$. Now suppose $\mathcal{H} = \mathcal{H}(X, K, \mathcal{L})$, $\mathcal{H}_1 = \mathcal{H}_1(X, K, \mathcal{L}_1)$ and $\mathcal{H}_2 = \mathcal{H}_2(X, K_2, \mathcal{L}_2)$ are full reproducing kernel Hilbert spaces on X . Given $\Phi \in \text{Mult}(\mathcal{H}_1, \mathcal{H})$ and $\Psi \in \text{Mult}(\mathcal{H}_2, \mathcal{H})$, when does the operator inequality $M_\Phi M_\Phi^* \geq M_\Psi M_\Psi^*$ imply the existence of a contractive multiplication operator $M_\Gamma \in \mathcal{M}(\mathcal{H}_2, \mathcal{H}_1)$ such that $M_\Psi M_\Gamma = M_\Phi$? One might be inclined to say that the operator space $\mathcal{M}(\mathcal{H}_2, \mathcal{H}_1)$ has the Douglas property if this question has an affirmative answer. We will avoid this, for the time being, in order to keep naming conventions as streamlined as possible.

The operator inequality $M_\Phi M_\Phi^* \geq M_\Psi M_\Psi^*$ implies that the operator matrices

$$[\Phi(x_i)K_1(x_i, x_j)\Phi(x_j)^* - \Psi(x_i)K_2(x_i, x_j)\Psi(x_j)^*]_{i,j=1}^n$$

are positive for every finite subset $\{x_1, \dots, x_n\}$. This looks suspiciously like a Pick matrix. One immediately wonders if there is a connection between the Douglas problem and Pick interpolation, and more specifically if the positivity of the above operator matrices is ever a sufficient condition. In order to disguise the Douglas problem as an interpolation problem, we must specify three data sets instead of two.

Suppose $E = \{x_1, \dots, x_n\}$, $A_1, \dots, A_n \in \mathcal{B}(\mathcal{L}_1, \mathcal{L})$ and $B_1, \dots, B_n \in \mathcal{B}(\mathcal{L}_2, \mathcal{L})$. The (left) tangential interpolation problem asks if the condition

$$[B_i K_1(x_i, x_j) B_i^* - A_i K_2(x_i, x_j) A_i^*]_{i,j=1}^n \geq 0$$

is equivalent to the existence of a contractive multiplication operator $M_\Gamma \in \mathcal{M}(\mathcal{H}_2, \mathcal{H}_1)$ so that $A_i \Gamma(x_i) = B_i$. If the left tangential interpolation problem has an affirmative solution, then we claim that the Douglas problem has one as well. Indeed, the condition $M_\Phi M_\Phi^* \geq M_\Psi M_\Psi^*$ implies that the tangential matrix above is positive for the data set $A_i = \Psi(x_i)$ and $B_i = \Phi(x_i)$ for $1 \leq i \leq n$. If we label a solution to the tangential interpolation problem for this data set as Γ_E , then any weak-* cluster point of $\{\Gamma_E\}_{E \subset X \text{ finite}}$ will solve the Douglas problem.

Of course, the overarching theme of this thesis is that a single Pick-type condition is rarely sufficient to solve an interpolation problem. Consequently, we will instead turn towards families of matrices which might result in tangential interpolation problems. Just as in the Pick problem, the invariant subspaces of a dual algebra of multipliers play a central role. It is possible to form the notion of a Pick family for the multiplier space $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ determined by subspaces of the form $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)[f]$, but problems arise immediately. Instead, we will focus on a class of tangential interpolation problems that tie in thematically with the constrained interpolation problems studied in Chapter 5.

To conclude this short section, we state the following theorem. The proof follows immediately from the proof of Theorem 7.1.4.

Theorem 7.2.1. *Suppose \mathcal{A} is a dual algebra of multipliers on a scalar-valued reproducing kernel Hilbert space \mathcal{H} . If $\text{CycLat } \mathcal{A}$ is a Pick family for \mathcal{A} , then the following statements are equivalent:*

1. $M_\psi^L (M_\psi^L)^* \geq M_\varphi^L (M_\varphi^L)^*$ for each $L \in \text{CycLat } \mathcal{A}$;
2. There is a contractive $M_\gamma \in \mathcal{A}$ such that $\gamma\psi = \varphi$.

Corollary 7.2.2. *Any dual algebra of multipliers on the Bergman spaces $L_a^2(\Omega)$, the complete Pick spaces, and the Hardy spaces $H^2(\mathbb{D}^d)$ and $H^2(\mathbb{B}_d)$ satisfy the statement of Theorem 7.2.1.*

7.3 Tangential interpolation and a distance formula

Suppose k is a scalar-valued kernel and that $\mathcal{H} = \mathcal{H}(k)$ is a reproducing kernel Hilbert space on X . Fix a countable cardinal k and set $\ell^2 := \ell_k^2$. Then $\mathcal{H} \otimes \ell^2$ is a reproducing kernel Hilbert space of ℓ^2 -valued functions on X , and the spaces of multiplication operators $\mathcal{M}(\mathcal{H}, \mathcal{H} \otimes \ell^2)$ and $\mathcal{M}(\mathcal{H} \otimes \ell^2, \mathcal{H})$ can be identified with the column space $\mathcal{C}(\mathcal{M}(\mathcal{H})) := \mathcal{M}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \ell^2)$ and row space $\mathcal{R}(\mathcal{M}(\mathcal{H})) := \mathcal{M}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H} \otimes \ell^2, \mathcal{H})$, respectively by Theorem 2.4.3.

Given a dual algebra of multipliers \mathcal{A} on \mathcal{H} , column space over \mathcal{A} will be denoted $\mathcal{C}(\mathcal{A})$. By the above reasoning, this is the set of all operators in $\mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \ell^2)$ with entries from \mathcal{A} . A typical multiplication operator in $\mathcal{C}(\mathcal{A})$ will be denoted $\Phi = [\varphi_1, \varphi_2, \dots]^T \in \mathcal{C}(\mathcal{A})$. For $L \in \text{Lat } \mathcal{A}$, $M_\Phi^L = [M_{\varphi_1}^L, M_{\varphi_2}^L, \dots]^T$ will denote the multiplier from L into $L \otimes \ell^2$. Let k_x^L denote the extended kernel function for L described in Lemma 3.2.1. The fundamental relationship between M_Φ^L and k_x^L is given by

$$M_\Phi^* (k_x^L \otimes u) = (\Phi(x)^* u) k_x^L.$$

Multiplication operators in the row space $R(\mathcal{A})$ are given by $M_\Phi = [M_{\varphi_1}, M_{\varphi_2}, \dots]$, and they satisfy

$$M_\Phi^* k_x^L = k_x^L \otimes \Phi(x)^*.$$

At the risk of abusing terminology, the tangential interpolation problem specifies n points $x_1, \dots, x_n \in X$, $w_1, \dots, w_n \in \mathbb{C}$ and vectors $v_1, \dots, v_n \in \ell^2$. We say that k has the *tangential Pick property* if

$$[(\langle v_j, v_i \rangle - w_i \bar{w}_j) k(x_i, x_j)] \geq 0$$

is equivalent to the existence of a contractive $M_\Phi \in \mathcal{C}(\mathcal{M}(\mathcal{H}))$ such that $\Phi(x_i)^* v_i = \bar{w}_i$ for $1 \leq i \leq n$. We always assume that all the v_i are non-zero, for if some $v_i = 0$, a solution exists if and only if $w_i = 0$ (every contractive M_Φ is a solution in this case). That the given matricial condition is necessary is easy, and it works for any invariant subspace of \mathcal{A} , as expected.

Lemma 7.3.1. *Suppose $M_\Phi \in \mathcal{C}(\mathcal{A})$ is a contraction which satisfies $\Phi(x_i)^* v_i = \bar{w}_i$ for $1 \leq i \leq n$. Then the matrix*

$$[(\langle v_j, v_i \rangle - w_i \bar{w}_j) k^L(x_i, x_j)]$$

is positive semidefinite for every $L \in \text{Lat } \mathcal{A}$.

Proof. We have $I_{\mathcal{H} \otimes \ell^2} - M_\Phi M_\Phi^* \geq 0$. Let $h = \sum_{i=1}^n a_i k_{x_i}^L \otimes v_i$ and compute

$$\begin{aligned} 0 &\leq \langle (I_{\mathcal{H} \otimes \ell^2} - M_\Phi M_\Phi^*) h, h \rangle \\ &= \sum_{i,j=1}^n a_i \bar{a}_j \left(\langle k_j^L, k_i^L \rangle_{\mathcal{H}} \langle v_j, v_i \rangle_{\ell^2} - \langle M_\Phi^* k_{x_j}^L \otimes v_j, M_\Phi^* k_{x_i}^L \otimes v_i \rangle_{\mathcal{H} \otimes \ell^2} \right) \\ &= \sum_{i,j=1}^n a_i \bar{a}_j \left(\langle k_j^L, k_i^L \rangle_{\mathcal{H}} \langle v_j, v_i \rangle_{\ell^2} - \langle k_{x_j}^L \otimes \Phi(x_j)^* v_j, k_{x_i}^L \otimes \Phi(x_i)^* v_i \rangle_{\mathcal{H} \otimes \ell^2} \right) \\ &= \sum_{i,j=1}^n a_i \bar{a}_j \left(k^L(x_i, x_j) \langle v_j, v_i \rangle_{\ell^2} - \langle k_{x_j}^L \bar{w}_j, k_{x_i}^L \bar{w}_i \rangle_{\mathcal{H}} \right) \\ &= \sum_{i,j=1}^n a_i \bar{a}_j (\langle v_j, v_i \rangle - w_i \bar{w}_j) k^L(x_i, x_j). \quad \square \end{aligned}$$

As is the case with the Pick problem, the objective is to minimize $\|M_\Phi\|_{\mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \ell^2)}$ over all functions Φ such that $\Phi(x_i)^* v_i = \bar{w}_i$. Given the finite set $\{x_1, \dots, x_n\} \subset X$, we denote $\mathfrak{J} := \mathfrak{J}_E = \{\Psi \in \mathcal{C}(\mathcal{A}) : \Psi(x_i)^* v_i = 0, i = 1, \dots, n\}$. The column space $\mathcal{C}(\mathcal{A})$ is naturally a weak-* closed (in $\mathcal{B}(\mathcal{H} \otimes \ell^2, \mathcal{H})$) \mathcal{A} -bimodule and \mathfrak{J} is weak-* closed submodule of $\mathcal{C}(\mathcal{A})$. An application of standard duality arguments shows us that this optimization problem is equivalent to computing the distance of M_Φ from the submodule \mathfrak{J} . We first establish an easy distance estimate that always holds.

Lemma 7.3.2. *Suppose \mathcal{H} is a RKHS on a set X and that \mathcal{A} is a dual algebra of multipliers on \mathcal{H} . If $x_1, \dots, x_n \in X$ and $v_1, \dots, v_n \in \ell^2$, then the distance from $M_\Phi \in \mathcal{C}(\mathcal{A})$*

to $\mathfrak{J} = \{\Psi \in \mathcal{C}(\mathcal{A}) : G(x_i)^*v_i = 0, i = 1, \dots, n\}$ has the lower bound

$$\text{dist}(M_\Phi, \mathfrak{J}) \geq \sup_{L \in \text{Lat}(\mathcal{A})} \|P_L M_\Phi^*|_{\overline{\mathcal{C}(\mathcal{A})L} \ominus \mathfrak{J}L}\| = \sup_{L \in \text{CycLat}(\mathcal{A})} \|P_L M_\Phi^*|_{\overline{\mathcal{C}(\mathcal{A})L} \ominus \mathfrak{J}L}\|.$$

Proof. For each $L \in \text{Lat}(\mathcal{A})$, $M_\Phi \in \mathcal{C}(\mathcal{A})$ and $M_\Psi \in \mathfrak{J}$ we have

$$\begin{aligned} \|M_\Phi - M_\Psi\|_{\mathcal{C}(\mathcal{A})} &\geq \|P_{\overline{\mathcal{C}(\mathcal{A})L} \ominus \mathfrak{J}L}(M_\Phi - M_\Psi)P_L\| \\ &= \|P_{\overline{\mathcal{C}(\mathcal{A})L} \ominus \mathfrak{J}L}M_\Phi P_L\| \\ &= \|P_L M_\Phi^*|_{\overline{\mathcal{C}(\mathcal{A})L} \ominus \mathfrak{J}L}\|. \end{aligned}$$

The equality in the statement of the theorem is the same as Proposition 3.1.4 \square

Given a subspace $L \in \text{Lat}(\mathcal{A})$, we let $\mathfrak{K}_L = (L \otimes \ell^2) \ominus \overline{\mathfrak{J}L} \subseteq L \otimes \ell^2$. Under certain assumptions on separation of points (see Lemma 7.3.5) it is the case that \mathfrak{K}_L is the span of the functions $\{k_{x_1}^L \otimes v_1, \dots, k_{x_n}^L \otimes v_n\}$. We denote this span as \mathfrak{M}_L . A simple computation shows that the operator $M_\Phi^*|_{\mathfrak{M}_L}$ satisfying $\Phi(x_i)^*v_i = \overline{w_i}$ is a contraction if and only if the $n \times n$ matrix

$$[(\langle v_j, v_i \rangle - w_i \overline{w_j})K^L(x_i, x_j)]$$

is positive semidefinite.

Definition 7.3.3. We will call a family of subspaces $\mathfrak{T} \subset \text{Lat}(\mathcal{A})$ a *tangential Pick family* if for every choice of points $x_1, \dots, x_n \in X$, $w_1, \dots, w_n \in \mathbb{C}$ and $v_1, \dots, v_n \in \ell^2$, we have

$$\text{dist}(M_\Phi, \mathfrak{J}) = \sup_{L \in \mathfrak{T}} \|P_L M_\Phi^*|_{\mathfrak{K}_L}\|.$$

The collection \mathfrak{T} is a *strong tangential Pick family* if

$$\text{dist}(M_\Phi, \mathfrak{J}) = \sup_{L \in \mathfrak{T}} \|P_L M_\Phi^*|_{\mathfrak{M}_L}\|.$$

In light of the above observations and definitions, we have the following: if \mathcal{A} is a dual algebra of multipliers on \mathcal{H} with a strong tangential Pick family \mathfrak{T} and we are given points $x_1, \dots, x_n \in X$, $w_1, \dots, w_n \in \mathbb{C}$ and $v_1, \dots, v_n \in \ell^2$, then there exists a multiplication operator $M_\Phi \in \mathcal{C}(\mathcal{A})$ such that $\|M_\Phi\| \leq 1$ and $\Phi(x_i)^*v_i = \overline{w_i}$ for $i = 1, \dots, n$ if and only if $[(\langle v_j, v_i \rangle - w_i \overline{w_j})k^L(x_i, x_j)] \geq 0$ for all $L \in \mathfrak{T}$.

We will show that if the column space $\mathcal{C}(\mathcal{A})$ has property $\mathbb{A}_1(1)$, then $\text{CycLat}(\mathcal{A})$ is a tangential Pick family for \mathcal{A} . In view of the comments above, this gives rise to a family of matrix positivity conditions that are equivalent to the solvability of the tangential interpolation problem. As is the case with Pick families, the strong families are desirable but in practice we principally deal with the generic variety. Just as in the Pick problem, there are work-arounds that enable us to state the solution to the tangential interpolation problem as a purely matricial condition.

It is entirely possible when dealing with subalgebras that there may be no functions that satisfy the condition $\Phi(x_i)^*v_i = \overline{w_i}$. Therefore, the final claim that a solution of required norm actually exists depends on the existence of at least one multiplication

operator $M_\Phi \in \mathcal{C}(\mathcal{A})$ such that $\Phi(x_i)^*v_i = \overline{w_i}$ (here there is no norm constraint on Φ). The proof of the existence of such a function depends in a crucial way on the fact that there is a function $h \in \mathcal{H}$ that does not vanish at the points x_1, \dots, x_n . The proof is essentially the same as that of Proposition 3.2.11 (albeit more notationally cumbersome), and we omit it.

Proposition 7.3.4. *Suppose \mathcal{A} is a dual algebra of multipliers on \mathcal{H} and that*

$$[(\langle v_j, v_i \rangle - w_i \overline{w_j})k^L(x_i, x_j)] \geq 0$$

for at least one subspace L of the form $L = \mathcal{A}[h]$, where $h(x_i) \neq 0$. Then, there is a multiplication operator $M_\Phi \in \mathcal{C}(\mathcal{A})$ such that $\Phi(x_i)^*v_i = w_i$ for each $i = 1, \dots, n$.

Note that Proposition 7.3.4 is immediate if \mathcal{A} separates points in X . As mentioned earlier we can explicitly write down a basis for the space $\mathcal{C}(\mathcal{A})[h] \ominus \mathfrak{J}[h]$ under the assumption that the function h does not vanish at any of the points x_1, \dots, x_n . As before, let $\mathfrak{K}_L = \mathcal{C}(\mathcal{A})[h] \ominus \mathfrak{J}[h]$ and let $\mathfrak{M}_L = \text{span}\{k_{x_i}^L \otimes v_i, i = 1 \dots n\}$. It is always the case that $\mathfrak{M}_L \subset \mathfrak{K}_L$.

Lemma 7.3.5. *Suppose \mathcal{H} is a reproducing kernel Hilbert space and that \mathcal{A} is a weak-* closed subalgebra of $\mathcal{M}(\mathcal{H})$ and write $L = \mathcal{A}[h]$ for $h \in H$. If h does not vanish on any of the x_i (so that $k_{x_i}^L \neq 0$), then $\mathfrak{M}_L = \mathfrak{K}_L$.*

Proof. For the non-trivial inequality $\mathfrak{K}_L \subset \mathfrak{M}_L$, we use a dimension argument. Let $\varphi_i : \mathcal{C}(\mathcal{A}) \rightarrow \mathbb{C}$ denote the functional $F \mapsto \langle F(x_i), v_i \rangle$ and note that $\mathfrak{J} = \bigcap_{i=1}^n \ker(\varphi_i)$ has codimension at most n . Thus \mathfrak{K}_L is at most n -dimensional. Since h does not vanish on the x_i , the vectors $\{P_L k_{x_i}\}$ are linearly independent if the algebra \mathcal{A} separates the x_i . In this case, \mathfrak{M}_L is spanned by n linearly independent vectors and we are done.

If \mathcal{A} does not separate the x_i , then we can partition the set $\{x_1, \dots, x_n\}$ into equivalence classes X_1, \dots, X_p , where \mathcal{A} identifies points in every X_j . Let \mathfrak{J}_j denote the set of multipliers $M_\Psi \in \mathcal{C}(\mathcal{A})$ such that $\Psi(x_i)^*v_i = 0$ for $x_i \in X_j$. It suffices to prove that

$$\mathcal{C}(\mathcal{A})[h] \ominus \mathfrak{J}_j[h] = \text{span}\{P_L k_i \otimes v_i : x_i \in X_j\}$$

for each j . We have $\mathfrak{J}_j = \bigcap_{x_i \in X_j} \ker(\varphi_i)$, and since any F in $\mathcal{C}(\mathcal{A})$ only takes on a single value on X_j , the codimension of \mathfrak{J}_j is at most $m := \dim(\text{span}\{v_i\})$. On the other hand, since $P_L k_{x_i} \neq 0$, the right hand side always has dimension at least m . Since the right hand side is contained in the left, the proof is complete. \square

We are now in a position to prove the main factorization theorem.

Theorem 7.3.6. *Let \mathcal{H} be a reproducing kernel Hilbert space and let \mathcal{A} be a weak-* closed subalgebra of $\mathcal{M}(\mathcal{H})$. Suppose that the column space $\mathcal{C}(\mathcal{A})$, regarded as a weak-* closed subspace of $\mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \ell^2)$, has property $\mathbb{A}_1(1)$. Then the distance to the subspace \mathfrak{J} is given by*

$$\text{dist}(M_\Phi, \mathfrak{J}) = \sup_{L \in \text{CycLat}(\mathcal{A})} \|P_L M_\Phi^*|_{\mathfrak{K}_L}\|.$$

Suppose further that a functional φ in the predual $\mathcal{C}(\mathcal{A})_*$ can be factored as $\varphi(F) = \langle Fh, K \rangle$ where h does not vanish on any of the points x_i . Then the distance formula above can be improved to

$$\text{dist}(M_\Phi, \mathfrak{J}) = \sup\{\|P_L M_\Phi^*|_{\mathfrak{M}_L}\| : L = \mathcal{A}[h] \in \text{CycLat } \mathcal{A}, h(x_i) \neq 0\}.$$

In other words, $\{\mathcal{A}[h] : h(x_i) \neq 0 \text{ for } i = 1, \dots, n\}$ is a strong tangential Pick family for \mathcal{A} .

Proof. Suppose $M_\Phi \in \mathcal{C}(\mathcal{A})$. There is a contractive weak- $*$ -continuous linear functional ω on $\mathcal{C}(\mathcal{A})$ such that $\text{dist}(F, \mathfrak{J}) < |\varphi(F)| + \epsilon$ and $\varphi(\mathfrak{J}) = \{0\}$. Since $\mathcal{C}(\mathcal{A})$ has property $\mathbb{A}_1(1)$, fix $\epsilon > 0$ and find vectors g in \mathcal{H} and $H = (h_i)_i$ in $\mathcal{H} \otimes \ell^2$ with $\|g\|\|H\| < (1 + \epsilon)$ such that

$$\omega(M_\Phi) = \langle M_\Phi g, H \rangle.$$

Let $L = \mathcal{A}[g]$ and replace H with $(P_L \otimes I)H$. Since $\varphi(\mathfrak{J}) = 0$, we see that $\mathfrak{J}[g]$ is orthogonal to H . It follows that $H \in \mathfrak{K}_L$ and so we have

$$\text{dist}(M_\Phi, \mathfrak{J}) < |\langle M_\Phi g, H \rangle| + \epsilon = |\langle P_{\mathfrak{K}_L} M_\Phi g, H \rangle| + \epsilon < \|P_L M_\Phi^*|_{\mathfrak{K}_L}\|(1 + \epsilon) + \epsilon.$$

It follows that $\text{dist}(M_\Phi, \mathfrak{J}) \leq \sup_{L \in \mathbb{T}} \|P_L M_\Phi^*|_{\mathfrak{K}_L}\|$. The reverse inequality was already shown. If, additionally, the function h does not vanish on any of the x_i , then we may restrict the above supremum to those functions. In this case Lemma 7.3.5 implies $\mathfrak{K}_L = \mathfrak{M}_L$, and so the second statement follows. \square

7.4 Tangential families for complete Pick spaces

In this section we establish that for any dual algebra of multipliers \mathcal{A} of a complete Pick kernel, the collection of subspaces $\text{CycLat } \mathcal{A}$ is a strong tangential Pick family for \mathcal{A} . Recall (cf. Chapter 5) that the multiplier algebra of any irreducible complete Pick kernel is unitarily equivalent to a co-restriction of the non-commutative analytic Toeplitz algebra \mathfrak{L}_d for some countable cardinal d . Moreover, this multiplier algebra may be identified as the complete quotient of \mathfrak{L}_d by a WOT closed ideal. The following theorem shows that the canonical representation of any quotient of \mathfrak{L}_d onto the orthogonal complement of the range of \mathcal{I} has the required predual factorization property. We first require a lemma which follows immediately from Bercovici's theorem on pairwise orthogonal isometries [17].

Lemma 7.4.1. *Suppose $d \geq 2$ is a natural number. The column space $\mathcal{C}(\mathfrak{L}_d) \subset \mathcal{B}(\ell^2(\mathbb{F}_d^+), \ell^2(\mathbb{F}_d^+) \otimes \ell^2)$ has property $\mathbb{A}_1(1)$.*

Proof. Following our convention, we may regard $\mathcal{C}(\mathfrak{L}_d)$ as a weak- $*$ closed subspace of $\mathfrak{L}_d \otimes \mathcal{B}(\ell^2) \subset \mathcal{B}(\mathcal{H} \oplus (\mathcal{H} \otimes \ell^2))$, which is precisely the weak- $*$ topology that $\mathcal{C}(\mathfrak{L}_d)$ inherits. The commutant of $\mathfrak{L}_d \otimes \mathcal{B}(\ell^2)$ contains two isometries with pairwise orthogonal ranges, names $R_1 \otimes I$ and $R_2 \otimes I$. By Bercovici's theorem, $\mathfrak{L}_d \otimes \mathcal{B}(\ell^2)$ has property $X(0, 1)$ and therefore property $\mathbb{A}_1(1)$ as well. Since property $\mathbb{A}_1(1)$ is hereditary, it follows that $\mathcal{C}(\mathfrak{L}_d)$ has it as well. \square

Theorem 7.4.2. *Suppose \mathcal{I} is a weak-* closed ideal in \mathfrak{L}_d and let $M = (\text{Ran } \mathcal{I})^\perp$. Let \mathfrak{A} denote the quotient algebra $\mathfrak{L}_n/\mathcal{I}$ and form the column space $\mathcal{C}(\mathfrak{A})$. Then $\mathcal{C}(\mathfrak{A})$, regarded as a weak-* closed subspace of $\mathcal{B}(M, M \otimes \ell^2)$, has property $\mathbb{A}_1(1)$. Moreover, any weak-* continuous functional φ with $\|\varphi\| < 1$ may be factored as $\varphi(A) = \langle Au, V \rangle$ where u is cyclic for \mathfrak{A} , $V \in M \otimes \ell^2$ and $\|u\|\|V\| < 1$.*

Proof. Suppose $\omega \in \mathcal{C}(\mathcal{A})_*$ is of norm at most 1 and let $\epsilon > 0$. Let $Q : \mathcal{C}(\mathfrak{L}_d) \rightarrow \mathcal{C}(\mathcal{A})$ be given by $Q((B_i)) = (q(B_i))$, where $q : \mathfrak{L}_d \rightarrow \mathcal{A}$ is the canonical quotient map. Then $\omega \circ Q$ is a weak-* continuous functional on $\mathcal{C}(\mathfrak{L}_d)$, which is a weak-* closed subspace of $\mathfrak{L}_d \otimes B(\ell^2)$. Property $\mathbb{A}_1(1)$ is hereditary for weak-* closed subspaces, and so for any $\epsilon > 0$ there are vectors $x \in H_d$ and $Y = (y_i) \in H_n \otimes \ell^2$ so that $\omega \circ Q([A_i]) = \langle (A_i)x, Y \rangle$ and $\|x\|\|Y\| < 1 + \epsilon$.

As in the above discussion, let $R \in \mathfrak{K}_d$ and u be a cyclic vector for \mathfrak{L}_d so that $x = Ru$. Let $V = (R^* \otimes I)Y$ and observe that $\langle Au, V \rangle = \langle Ax, Y \rangle$ for any $A \in \mathcal{C}(\mathfrak{L}_d)$. We also have $\mathcal{C}(\mathcal{I})u \perp V$ and

$$\overline{\mathcal{C}(\mathcal{I})u} = \overline{\mathcal{C}(\mathcal{I}\mathfrak{L}_d)u} = \overline{\mathcal{I}H_n} \otimes \ell^2 = M^\perp \otimes \ell^2.$$

It follows that $V \in M \otimes \ell^2$, which is a co-invariant subspace of $\mathfrak{L}_d \otimes B(\ell^2)$.

For $A \in \mathcal{C}(\mathcal{A})$, find a $B \in \mathcal{C}(\mathfrak{L}_d)$ such that $Q(B) = A$. We have

$$\begin{aligned} \omega(A) &= \omega \circ Q(B) = \langle Bu, V \rangle \\ &= \langle Bu, (P_M \otimes I)V \rangle = \langle (P_M \otimes I)Bu, V \rangle \\ &= \langle (P_M \otimes I)B(P_M \otimes I)u, V \rangle = \langle A(P_M u), V \rangle. \end{aligned}$$

The property $\mathbb{A}_1(1)$ now follows. Note that the vector u is cyclic for \mathfrak{L}_d . Therefore the vector $P_M u$ is cyclic for \mathcal{A} since

$$\overline{\mathcal{A}P_M u} = \overline{P_M \mathcal{A}P_M u} = \overline{P_M \mathfrak{L}_d P_M u} = \overline{P_M \mathfrak{L}_d u} = M. \quad \square$$

Applying the above theorem to the case where M is the closed span of kernel functions and using the classification of irreducible complete Pick spaces, we obtain the following corollary.

Corollary 7.4.3. *Suppose \mathcal{H} is a reproducing kernel Hilbert space with a complete Pick kernel. Then $\mathcal{C}(\mathcal{M}(\mathcal{H}))$ has property $\mathbb{A}_1(1)$ and the additional property that any weak-* continuous functional on $\mathcal{C}(\mathcal{M}(\mathcal{H}))$ can be factored as $\varphi(A) = \langle Ag, H \rangle$ where g is a cyclic vector for $\mathcal{M}(\mathcal{H})$.*

Proof. When k is an irreducible complete Pick kernel, the result follows immediately from Theorem 7.4.2. For an arbitrary complete Pick kernel on X , the proof follows identically to that of Proposition 5.1.5. Namely, we can decompose \mathcal{H} as the direct sum of spaces \mathcal{H}_i with irreducible complete Pick kernels, and $\mathcal{M}(\mathcal{H}) = \oplus_i \mathcal{M}(\mathcal{H}_i)$. A moment's thought reveals that $\mathcal{C}(\mathcal{M}(\mathcal{H})) = \bigoplus_i \mathcal{C}(\mathcal{M}(\mathcal{H}_i))$, and the proof now follows *mutatis mutandis* from the proof of Proposition 5.1.5. \square

This result also extends to weak-* closed subalgebra of multipliers on a complete Pick space.

Corollary 7.4.4. *Suppose \mathcal{H} is a reproducing kernel Hilbert space with a complete Pick kernel and that \mathcal{A} is a weak-* closed subalgebra of $\mathcal{M}(\mathcal{H})$. Then $\mathcal{C}(\mathcal{A})$ has property $\mathbb{A}_1(1)$, and every weak-* continuous functional can be factored as $\varphi(A) = \langle Ah, K \rangle$ where h is a cyclic vector for the full multiplier algebra $\mathcal{M}(\mathcal{H})$.*

We can now apply Theorem 7.3.6 to constrained tangential interpolation for a complete Pick kernel.

Corollary 7.4.5. *Suppose \mathcal{H} is a reproducing kernel Hilbert space with a complete Pick kernel and that \mathcal{A} is a dual algebra of multipliers on \mathcal{H} . Then*

$$\mathfrak{T} := \{k^L : L = \mathcal{A}[h], h \in \mathcal{H} \text{ cyclic for } \mathcal{M}(\mathcal{H})\}$$

is a tangential Pick family for \mathcal{A} . Equivalently, the distance formula from $\mathcal{C}(\mathcal{A})$ to \mathfrak{J}

$$\text{dist}(M_\Phi, \mathfrak{J}) = \sup\{\|P_L M_\Phi^*|_{M_L}\| : L \in \mathfrak{T}\}$$

holds.

7.5 The Toeplitz corona problem

We will now apply the tangential interpolation results of the previous section to obtain a Toeplitz corona theorem. Suppose \mathcal{A} is a dual algebra of multipliers on a reproducing kernel Hilbert space \mathcal{H} . Given functions $\varphi_1, \dots, \varphi_n$ in \mathcal{A} , the Toeplitz corona problem asks that if there is a $\delta > 0$ such that

$$\sum_{i=1}^n M_{\varphi_i} M_{\varphi_i}^* \geq \delta^2 I, \tag{7.1}$$

is it possible to find functions ψ_1, \dots, ψ_n in \mathcal{A} such that

$$\varphi_1 \psi_1 + \dots + \varphi_n \psi_n = 1?$$

In other words, the row operator $[M_{\varphi_1}, \dots, M_{\varphi_n}]$ has a right inverse in $\mathcal{B}(\mathcal{H}, \mathcal{H}^{(n)})$, but is there a right inverse with entries in \mathcal{A} ? One typically requires some type of norm control on the ψ_i as well. By considering the case where $n = 1$, the constant δ^{-1} is easily seen to be the optimal operator norm for the column $[M_{\psi_1}, \dots, M_{\psi_n}]^T$. Using our notation, given a multiplier $M_\Phi \in \mathcal{M}(\mathcal{H}^{(n)}, \mathcal{H})$ with $M_\Phi M_\Phi^* \geq \delta^2 I$, is there a multiplier $M_\Psi \in \mathcal{M}(\mathcal{H}, \mathcal{H}^{(n)})$ such that $\Phi(x)\Psi(x) = 1$ for every $x \in X$ and $\|M_\Psi\|_{\mathcal{M}(\mathcal{H}, \mathcal{H}^{(n)})} \leq \delta^{-1}$? The astute reader will no doubt observe that this is nothing but the Douglas problem for the multiplication operators $[M_{\varphi_1}, \dots, M_{\varphi_n}] \in \mathcal{R}(\mathcal{A})$ and $\delta \in \mathbb{C}$.

For the algebra H^∞ acting on Hardy space, this question was answered affirmatively by Arveson in [8] using his famous distance formula for nest algebras, albeit without optimal norm control. Under the hypothesis that each φ_i was contractive, he showed that the functions g_i could be chosen such that $\|g_i\|_\infty \leq 4n\delta^{-3}$. Schubert ([63]) obtained the result with optimal constants using the commutant lifting theorem of Sz. Nagy and Foias.

This program was carried out in substantial generality by Ball, Trent and Vinnikov [11] for multiplier algebras of complete Pick spaces. Popescu [52] proved a Toeplitz corona theorem for the noncommutative analytic Toeplitz algebra \mathfrak{L}_d using his commutant lifting theorem for row isometries. Finally, Trent and Wick [66] obtained Toeplitz corona theorems for the polydisk and unit ball in terms of families of kernels. If it is the case that the column spaces over $H^\infty(\mathbb{B}_d)$ and $H^\infty(\mathbb{D}^d)$ have property $\mathbb{A}_1(1)$, we would obtain the Trent-Wick theorem as stated. However, we do not know if this is the case.

Toeplitz corona theorems have been used in solving the classical corona problem in a variety of settings. Recently, Ryles and Trent [60] have used operator corona theorems to resolve classical corona problems for subalgebras of H^∞ of the form $\mathbb{C} + BH^\infty$, where B is an inner function. Their approach is rather different from ours. While we obtain a very general Toeplitz corona theorem, their results provide interesting estimates in the classical setting.

If the Toeplitz corona problem has an affirmative solution for the full multiplier algebra $\mathcal{M}(\mathcal{H})$, then for $\varphi_1, \dots, \varphi_n \in \mathcal{A}$, the condition $\sum_{i=1}^n M_{\varphi_i} M_{\varphi_i}^* \geq \delta^2 I$ certainly implies the existence of the solutions ψ_1, \dots, ψ_n in $\mathcal{M}(\mathcal{H})$. However, in order to require that these functions belong to a weak-* closed subalgebra \mathcal{A} , a stronger set of assumptions on the φ_i is generally required. The following result, appearing as (a somewhat weaker statement) Proposition 4.2 in [58] says that if \mathfrak{T} is a strong tangential Pick family for \mathcal{A} and that $M_\Phi^L (M_\Phi^L)^* \geq \delta^2 I$ for every $L \in \mathfrak{T}$, then there are solutions ψ_1, \dots, ψ_n in \mathcal{A} .

Proposition 7.5.1. *Suppose \mathcal{H} is a scalar-valued reproducing kernel Hilbert space on X , that Y is a subset of X and that \mathfrak{T} is a strong tangential Pick family for \mathcal{A} . If $M_\Phi \in \mathcal{R}(\mathcal{A})$ and $M_\gamma \in \mathcal{A}$ such that*

$$M_\Phi^L (M_\Phi^L)^* \geq M_\gamma M_\gamma^*$$

for each $L \in \mathcal{L}$, then there is a multiplier $M_\Psi \in \mathcal{C}(\mathcal{A})$ with $\|M_\Psi\| \leq 1$ such that $\Phi(y)\Psi(y) = \gamma(y)$ for every $y \in Y$.

Proof. For any finite set of points $E = \{x_1, \dots, x_k\} \subset Y$, the positivity condition implies that

$$\left[\left(\langle \Phi(x_j)^*, \Phi(x_i)^* \rangle - \gamma(x_i) \overline{\gamma(x_j)} \right) k^L(x_i, x_j) \right] \geq 0$$

for every $L \in \mathcal{L}$. For $1 \leq i \leq k$, set $v_i = F(x_i)^*$ and $w_i = \gamma(x_i)$ so that the above matrix is of the form in Theorem 7.3.6. Since \mathfrak{T} is a tangential Pick family, for each E there is a corresponding contractive $M_{\Psi_E} \in \mathcal{C}(\mathcal{A})$ such that $\overline{\gamma(x_i)} = \langle \Phi(x_i), \Psi_E(x_i)^* \rangle$ for $i = 1, \dots, k$. By taking adjoints, we have $\Phi(x_i)\Psi_E(x_i) = \gamma$. Direct the collection of finite subsets of Y by inclusion, and apply a standard weak* approximation argument to find there is a contractive multiplier $M_\Psi \in R(\mathcal{A})$ such that $\Psi|_E = \Psi_E$ for every finite $E \subset Y$. It follows that $\Phi(y)\Psi(y) = \gamma(y)$ for every $y \in Y$. \square

Taking $X = Y$ and $\omega(x) = \delta$ for every x in the above proposition yields a solution Ψ which satisfies $\Phi(x)\Psi(x) = \delta$, and hence $\Psi\delta^{-1}$ is the multiplier of optimal norm which solves the Toeplitz corona problem. In other words, in order to solve the Toeplitz corona problem for an arbitrary subalgebra \mathcal{A} , one requires that, in particular, the row multiplier M_Φ^L has a right inverse in $\mathcal{B}(L, L^{(n)})$ for every $L \in \mathfrak{T}$. We can now use Corollary 7.4.5 to

solve the Toeplitz corona problem for subalgebras of $\mathcal{M}(\mathcal{H})$, where \mathcal{H} is a complete Pick space.

Theorem 7.5.2. *Suppose \mathcal{H} is a reproducing kernel Hilbert space with a complete Pick kernel and \mathcal{A} is a dual algebra of multipliers on \mathcal{H} . If $\Phi \in R(\mathcal{A})$ and $\delta > 0$ such that*

$$M_{\Phi}^L (M_{\Phi}^L)^* \geq \delta^2 I_L \quad (7.2)$$

for every $L = \mathcal{A}[h]$ where h is a cyclic vector for $\mathcal{M}(\mathcal{H})$, then there is a $M_{\Psi} \in C(\mathcal{A})$ such that $\Phi(x)\Psi(x) = 1$ for every $x \in X$ and $\|M_{\Psi}\| \leq \delta^{-1}$.

Note that Theorem 7.5.2 works just as well for infinitely many φ_i . The scalar-valued version of the Ball-Trent-Vinnikov result [11] is recovered as a special case of Theorem 7.5.2 when $\mathcal{A} = \mathcal{M}(\mathcal{H})$.

Example 7.5.3. As is the case with Pick interpolation, the positivity criterion in Theorem 7.5.2 may be simplified for “large” subalgebras of $\mathcal{M}(\mathcal{H})$. Form the subalgebra $\mathcal{A} := \mathbb{C} + \mathcal{I}$ for a finite codimensional ideal \mathcal{I} in $\mathcal{M}(\mathcal{H})$. By Example 5.1.6, subspaces of the form

$$L_a = \mathbb{C}(a_1 e_1 + \dots + a_p e_p) \oplus, \quad a = (a_1, \dots, a_p) \in \partial \mathbb{B}_p$$

exhaust all subspaces of the form $\mathcal{A}[h]$ where h is outer. Thus, the hypothesis in Theorem 7.5.2 may be replaced by

$$M_{\Phi}^{L_a} (M_{\Phi}^{L_a})^* \geq \delta^2 I_{L_a}, \quad a \in \partial \mathbb{B}_p.$$

7.6 Tangential interpolation and the Toeplitz corona problem for other algebras

The final results in the last two sections rely on a refinement of property $\mathbb{A}_1(1)$ for the column space $\mathcal{C}(\mathcal{A})$ where a cyclic vector may be chosen in the factorization of the given functional (or at least a function that does not vanish on the interpolation nodes). So far, any attempt to carry over this refinement of property $\mathbb{A}_1(1)$ has been unsuccessful. In the case of Pick interpolation in Chapter 3, this difficulty was overcome by considering the extended kernel k_x^L for $x \in X$ and $L \in \text{CycLat } \mathcal{A}$, where k_x^L is a unit vector in the space $L \ominus \mathcal{I}_x L = \mathcal{A}[h] \ominus \mathcal{I}_x[h]$ and \mathcal{I}_x is the ideal in \mathcal{A} of functions that vanish at x .

Our goal in this section is to show that the space \mathfrak{K}_L is actually spanned by $k_{x_i}^L \otimes v_i$. We do need to (re)impose the condition that \mathcal{A} separates points in E (and for the Toeplitz corona theorem we will actually require \mathcal{A} to separate points on all of X).

Proposition 7.6.1. *Let \mathcal{A} be a dual algebra of multipliers on \mathcal{H} which separates points in the set E and $L = \mathcal{A}[h]$. If k_x^L is the extended kernel function for L described in Lemma 3.2.1, then*

$$\text{span}_{x_i \in E} \{k_{x_i}^L \otimes v_i\} = \mathfrak{K}_L.$$

Remark 7.6.2. If each of the $k_{x_i}^L$ are non-zero (this occurs in many spaces of analytic functions; see Example 3.2.4), then the separating assumption of \mathcal{A} implies that they are linearly independent. A simple dimension argument then proves the proposition.

Proof of Proposition 7.6.1. We proceed by induction on the size of the set $E = \{x_1, \dots, x_k\}$. If $n = 1$ the fact that $v_1 \neq 0$ implies that $\mathcal{I}_x = \mathcal{J}_x$, and so there is nothing to prove, since either the space $\mathfrak{N}_L = \mathfrak{K}_L$ is $\{0\}$, it is spanned by the extended kernel $k_{x_1}^L$. Now suppose $|E| = k$ and the equality

$$\text{span}_{x_i \in E_0} \{k_{x_i}^L \otimes v_i\} = \mathcal{C}(\mathcal{A})[h] \ominus \mathcal{J}_{E_0}[h]$$

holds for $E_0 = \{x_1, \dots, x_{k-1}\} \subsetneq E$. By the same dimension argument we have seen several times now, it suffices to show that if $k_{x_k}^L = 0$ then

$$\mathcal{C}(\mathcal{A})[h] \ominus \mathcal{J}_E[h] = \text{span}_{x_i \in E_0} \{k_{x_i}^L \otimes v_i\}.$$

Instead, we prove that $\mathcal{J}_E[h] = \mathcal{C}(\mathcal{A})[h] \ominus \text{span}_{x_i \in E_0} \{k_{x_i}^L \otimes v_i\}$. The left hand side is contained in the right as we have seen. For the reverse inclusion, suppose $g = (g_1, \dots, g_n) \in \mathcal{C}(\mathcal{A})[h] \ominus \text{span}_{x_i \in E_0} \{k_{x_i}^L \otimes v_i\}$. By the inductive hypothesis, g already belongs to \mathcal{J}_{E_0} . Since $k_{x_k}^L = 0$, we have $\mathcal{A}[h] = \mathcal{I}_x[h]$, which implies that, for $1 \leq i \leq k$, there are sequences φ_m^i in \mathcal{I}_x such that $g_i = \lim_{m \rightarrow \infty} \varphi_m^i h$. But then $(\varphi_m^1(x_k), \dots, \varphi_m^n(x_k))^* v_k = (0, \dots, 0)^* v_k = 0$, and so $g \in \mathcal{J}_{x_k}[h]$. Since \mathcal{A} separates points in E , it is straightforward to verify that $\mathcal{J}_{x_k}[h] \cap \mathcal{J}_{E_0}[h] = \mathcal{J}_E[h]$, and hence that $g \in \mathcal{J}_E[h]$. \square

This immediately yields tangential interpolation results for algebras of multipliers \mathcal{A} such that $\mathcal{C}(\mathcal{A})$ has property $\mathbb{A}_1(1)$, but not necessarily the refined version.

7.6.1 Bergman spaces

In Chapter 4, we saw that for any bounded, open region $\Omega \subset \mathbb{C}^d$, the multiplier algebra of the Bergman space has property $X(0, 1)$. By Proposition 1.2.8, this implies that $\mathcal{M}(L_a^2(\Omega))$ has property $\mathbb{A}_{\mathbb{N}_0}(1)$, which in turn implies that $\mathcal{C}_k(\mathcal{M}(H_\Omega^2))$ has property $\mathbb{A}_1(\sqrt{k})$ by Proposition 1.2.6. Here $\mathcal{C}_k(\mathcal{M}(L_a^2(\Omega)))$ denotes the column space of length k for $\mathcal{M}(L_a^2(\Omega))$. The finite row space $\mathcal{R}_k(\mathcal{M}(L_a^2(\Omega)))$ is defined analogously. The corresponding tangential interpolation theorem and Toeplitz corona theorems are as follows.

Theorem 7.6.3. *Suppose k is a natural number and Ω a bounded open domain in \mathbb{C}^k . Then for any dual algebra of multipliers \mathcal{A} on $L_a^2(\Omega)$*

$$\text{dist}(M_\Phi, \mathcal{J}) \leq \sqrt{k} \sup_{L \in \text{CycLat } \mathcal{A}} \|P_L M_\Phi^* |_{\mathcal{K}_L}\|$$

for any $M_\Phi \in \mathcal{C}_k(\mathcal{A})$.

Theorem 7.6.4. *Suppose that \mathcal{A} is a dual algebra of multipliers on $L_a^2(\Omega)$. If $M_\Phi \in \mathcal{R}_k(\mathcal{A})$ and $\delta > 0$ such that*

$$M_\Phi^L (M_\Phi^L)^* \geq \delta^2 I_L \tag{7.3}$$

for every $L \in \text{CycLat } \mathcal{A}$, then there is a $M_\Psi \in \mathcal{C}_k(\mathcal{A})$ such that $\Phi(x)\Psi(x) = 1$ for every $x \in X$ and $\|M_\Psi\| \leq \sqrt{k}\delta^{-1}$.

7.6.2 Products of complete Pick kernels

We may also apply the results of this section to the non-commutative \mathcal{S} -Teoplitz algebra $\mathfrak{L}_{\mathcal{S}}$ described in Chapter 6. Recall that by Bercovici's theorem, the algebra $\mathfrak{L}_{\mathcal{S}} \overline{\otimes} \mathcal{B}(\mathcal{L})$ has property $X(0, 1)$ for any auxiliary Hilbert space \mathcal{L} . In particular, the column space $\mathcal{C}(\mathfrak{L}_{\mathcal{S}})$ has property $\mathbb{A}_1(1)$. Suppose \mathcal{A} is a dual subalgebra of $\mathfrak{L}_{\mathcal{S}} = \mathfrak{L}_{d_1} \overline{\otimes} \dots \overline{\otimes} \mathfrak{L}_{d_p}$. Given finitely many points $z_1, \dots, z_n \in \mathbb{B} := \mathbb{B}_{d_1} \times \dots \times \mathbb{B}_{d_p}$ and vectors $v_1, \dots, v_n \in \ell^2$, let \mathcal{J} denote the weak-* closed submodule of $\mathcal{C}(\mathcal{A})$ which satisfies $\hat{A}(z_i)^* v_i = 0$ for $A \in \mathcal{C}(\mathcal{A})$, where $\hat{A}(z) = [\hat{A}_1(z), \hat{A}_2(z), \dots]^T$. For $L = \mathcal{A}[h]$, we again let $\mathcal{K}_L = \mathcal{C}(\mathcal{A})[h] \ominus \mathcal{J}[h]$ for any $h \in \ell^2(\mathcal{S})$. Just as in the commutative case, we have $P_L A^* (k_z^L \otimes v) = \langle \hat{A}(z), v_i \rangle k_z^L$. The next two results follow immediately from the discussion in this chapter.

Theorem 7.6.5. *If \mathcal{A} is any dual subalgebra of $\mathfrak{L}_{\mathcal{S}}$ and \mathcal{J} is as above, then*

$$\text{dist}(A, \mathcal{J}) = \sup_{L \in \text{CycLat } \mathcal{A}} \|P_L M_{\Phi}^*|_{\mathcal{K}_L}\|$$

for any $A \in \mathcal{C}(\mathcal{A})$.

Theorem 7.6.6. *Suppose that \mathcal{A} is a dual algebra in $\mathfrak{L}_{\mathcal{S}}$. If $A \in \mathcal{R}(\mathcal{A})$ and $\delta > 0$ such that*

$$AP_L A^* \geq \delta^2 I_L \tag{7.4}$$

for every $L \in \text{CycLat } \mathcal{A}$, then there is a $B \in \mathcal{C}(\mathcal{A})$ such that $AB = I$ and $\|B\| \leq \delta^{-1}$.

Appendix A

Finite dimensional multiplier algebras

In this appendix, we present a complete description of finite dimensional multiplier algebras. In addition, we will provide numerical evidence that there is a finite dimensional multiplier algebra \mathcal{A} with the property that $\text{CycLat}(\mathcal{A})$ does not yield a Pick family of kernels for \mathcal{A} . In particular, this also suggests that \mathcal{A} does not have property $\mathbb{A}_1(1)$. It does, however, have property $\mathbb{A}_1(r)$ for some $r > 1$.

A.1 A numerical estimate

Suppose $X = \{x_1, \dots, x_N\}$ is a finite set and $k : X \times X \rightarrow \mathbb{C}$ is an irreducible kernel. Let y_1, \dots, y_N be vectors in \mathbb{C}^N such that $k(x_i, x_j) = \langle y_j, y_i \rangle$, and let $\{x_1, \dots, x_N\}$ be a dual basis for the y_i . The space $\mathcal{H} = \mathbb{C}^N$ may be regarded as a reproducing kernel Hilbert space over X , with reproducing kernel at x_i given by y_i . The multiplier algebra $\mathcal{M}(\mathcal{H})$ is an N -idempotent operator algebra spanned by the rank one idempotents $p_i = x_i y_i^*$.

If $\{e_i\}$ is the canonical orthonormal basis for \mathbb{C}^N , then one readily sees that $\mathcal{M}(\mathcal{H})$ is similar to the diagonal algebra \mathfrak{D}_N via the similarity S defined by $S e_i = x_i$. Since \mathfrak{D}_N evidently has property $\mathbb{A}_1(1)$, it follows from elementary results on dual algebras that $\mathcal{M}(\mathcal{H})$ has $\mathbb{A}_1(r)$ for some $r \geq 1$. If k is irreducible and a complete Pick kernel, then Theorem 5.1.3 shows that $\mathcal{M}(\mathcal{H})$ has $\mathbb{A}_1(1)$. However, there are many kernels k that cannot be embedded in Drury-Arveson space in this way. We expect that many of these algebras fail to have $\mathbb{A}_1(1)$ and that the distance formula fails in such cases.

Since \mathcal{A} is similar to the diagonal algebra \mathfrak{D}_N , the invariant subspaces are spanned by some subset of $\{x_1, \dots, x_N\}$. Denote them by $L_\sigma = \text{span}\{x_i : i \in \sigma\}$. For $E \subset \{1, \dots, N\}$, the ideal $\mathcal{I} = \mathcal{I}_E = \text{span}\{p_i : i \notin E\}$. Then $\mathcal{I} L_\sigma = L_{\sigma \setminus E}$, and $N_\sigma := N_{L_\sigma} = L_\sigma \ominus L_{\sigma \setminus E}$. The distance formula is obtained as the maximum of compressions to these subspaces—so we need only consider the maximal ones. These arise from $\sigma \supset E$.

For trivial reasons, the distance formula is always satisfied when $N = 2$ and $N = 3$. There is strong numerical evidence to suggest that the formula does hold for $N = 4$, though we do not have a proof. In the following 5-dimensional example, *Wolfram Mathematica 7* was used to find a similarity S such that the distance formula fails.

Example A.1.1. Define the similarity

$$S = \begin{bmatrix} 3 & 1 & 1 & 0 & -1 \\ 0 & 1 & -2 & -1 & 0 \\ -1 & 0 & -1 & 1 & -1 \\ -1 & 1 & 2 & 1 & -1 \\ 1 & 1 & 3 & 1 & -2 \end{bmatrix}.$$

Let $p_i = x_i y_i^*$ for $1 \leq i \leq 5$ be the idempotents which span the algebra $\mathcal{A} := \mathcal{M}(\mathcal{H})$. Let $E = \{1, 2, 3\}$, and form $\mathcal{I} = \mathcal{I}_E = \text{span}\{p_4, p_5\}$.

Consider the element $A = -2p_1 - 3p_2 + 7p_3$. We are interested in comparing $\max_{\sigma} \|P_{N_{\sigma}} A P_{N_{\sigma}}\|$ with $\text{dist}(A, \mathcal{I})$. As noted above, it suffices to use maximal N_{σ} 's formed by the cyclic subspaces that do not vanish on E , namely

$$\begin{aligned} N_{\{123\}} &= \text{span}\{x_1, x_2, x_3\}, \\ N_{\{1234\}} &= \text{span}\{x_1, x_2, x_3, x_4\} \ominus \mathbb{C}x_4, \\ N_{\{1235\}} &= \text{span}\{x_1, x_2, x_3, x_5\} \ominus \mathbb{C}x_5, \text{ and} \\ N_{\{12345\}} &= \text{span}\{x_4, x_5\}^{\perp} = \text{span}\{y_1, y_2, y_3\}. \end{aligned}$$

For notational convenience, set $P_{\sigma} := P_{N_{\sigma}}$. The values of $\|P_{\sigma} A P_{\sigma}\|$ were computed and rounded to four decimal places:

$$\begin{aligned} \|P_{123} A P_{123}\| &= 9.0096, \\ \|P_{1234} A P_{1234}\| &= 10.1306, \\ \|P_{1235} A P_{1235}\| &= 7.4595, \\ \|P_{12345} A P_{12345}\| &= 10.6632. \end{aligned}$$

The Mathematica nonlinear global optimization tool *NMinimize-RandomSearch* was employed with 250 search points, and the following quantity was obtained:

$$\text{dist}(A, \mathcal{I}) \approx 11.9346.$$

Similar results appeared for many different elements of \mathcal{A} , which indicate that $\text{CycLat}(\mathcal{A})$ does not yield a Pick family for \mathcal{A} . Consequently, it must also fail to have $\mathbb{A}_1(1)$. We currently have no example of a dual algebra of multipliers on any \mathcal{H} that fails to have $\mathbb{A}_1(r)$ for every $r \geq 1$, or even fails to have \mathbb{A}_1 .

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