Energy Shaping for Systems with Two Degrees of Underactuation

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Applied Mathematics

Waterloo, Ontario, Canada, 2011

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis we are going to study the energy shaping problem on controlled Lagrangian systems with degree of underactuation less than or equal to two. Energy shaping is a method of stabilization by designing a suitable feedback control force on the given controlled Lagrangian system so that the total energy of the feedback equivalent system has a non-degenerate minimum at the equilibrium. The feedback equivalent system can then be stabilized by a further dissipative force. Finding a feedback equivalent system requires solving a system of PDEs. The existence of solutions for this system of PDEs is guaranteed, under some conditions, in the case of one degree of underactuation. Higher degrees of underactuation, however, requires a more careful study on the system of PDEs, and we apply the formal theory of PDEs to achieve this purpose in the case of two degrees of underactuation.

The thesis is divided into four chapters. First, we review the basic notion of energy shaping and state the results for the case of one degree of underactuation. We then devise a general scheme to solve the energy shaping problem with degree of underactuation equal to one, together with some examples to illustrate the general procedure. After that we review the tools from the formal theory of PDEs, as a preparation for solving the problem with two degrees of underactuation. We derive an equivalent involutive system of PDEs from which we can deduce the existence of solutions which suit the energy shaping requirement.

Acknowledgements

First and foremost, I would like to thank my supervisors Professors Dong Eui Chang and George Labahn, who introduced me to the study of control theory and the formal theory of PDEs. I am especially grateful for their financial support during the summer course in 2009 at the Research Institute for Symbolic Computation (RISC), Hagenberg, Austria, which consolidated my understanding of the formal theory.

Special thanks also go to Professors Anthony Bloch, Kirsten Morris, Raymond McLenaghan and Mark Giesbrecht who shared the time to serve as the committee members in my thesis defense, and who provided valuable comments and suggestions to the thesis.

During my four year study in the University of Waterloo, I am happy to have a number of friends who accompanied me in the ups and downs, as well as the winters and summers which I have never come across before. I am fortunate to have Helen Warren as our department secretary, who provided helpful reminders and assistance when completing various forms and procedures towards graduation. And I am fascinated to have those chipmunks and squirrels on campus who brought me and my officemate a lot of joys and colours in my life in Waterloo.

I would also like to thank Sam Chow, my childhood friend who emigrated to Canada when we were 12, and who helped me a lot in adapting to the everyday life in Canada, without which my life as a newcomer here would have become more difficult. I am especially indebted to him for driving me between Waterloo and Toronto a number of times.

Last but not least, I would like to thank my mother and my beloved Mabel Tsang. I cannot forget their love, patience and encouragement during all these years.

Dedicated to My Mother

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List of Symbols

$[ij, l]$	Christoffel symbols of the first kind
$\lrcorner$	Contraction operator
Γ^i_{jk}	Christoffel symbol of the second kind
$\langle \alpha, v \rangle$	Canonical pairing between $\alpha \in T^*Q$ and $v \in TQ$
$\mathcal{EL}$	Euler-Lagrange operator
$\mathcal{E}$	Fibered manifold
$\mathcal{R}_r^{(s)}$	Prolongation of $\mathcal{R}_r$ by s times, then followed by projection back to order r
$\mathcal{R}_r$	System of PDEs of order r
$\mathcal{R}_{r+s}$	Prolongation of $\mathcal{R}$ by r times
$\nabla_v m$	Covariant derivative of mass matrix m (as a $(0, 2)$ tensor) along vector v
∇, $\widehat{\nabla}$	Metric connections associated with mass matrices m and $\widehat{m}$
π_r^{r+s}	Projection map from $J_{r+s}\mathcal{E}$ onto $J_r\mathcal{E}$
E	Energy function of a mechanical system; it is a sum of potential and kinetic energy
F	External force acting on a mechanical system

g	Gravitational constant
G_r	Symbol for the system $\mathcal{R}_r$
$J_r\mathcal{E}$	Jet bundle of $\mathcal{E}$ of order r
L	Lagrangian of a mechanical system
m	Mass matrix of a mechanical system
m^{ij}	(i,j)-entry of the inverse of mass matrix m
m_{ij}	(i,j)-th entry of the mass matrix m
Q	Configuration space of a mechanical system
$S^r(T^*Q)$	Set of all symmetric, order r, tensor products of T^*Q
T^*Q	Cotangent bundle of the configuration space Q
TQ	Tangent bundle of the configuration space Q
u	Control force on a mechanical system
V	Potential energy of a mechanical system
W	Control bundle of a mechanical system
W°	Annihilator of the control bundle W
$\widehat{T}$	A mass matrix-related tensor defined by $\widehat{T} = m\widehat{m}^{-1}m$
(L^ℓ, F^ℓ, W^ℓ)	Linearization of the controlled Lagrangian system (L, F, W)
n_1	Degree of underactuation; $n_1 = n - \dim W$
n	Degree of freedom; $n = \dim TQ$

Chapter 1

Introduction

Energy shaping is a method of stabilizing the equilibrium of a mechanical system by altering (or "shaping") the total energy of the system via a feedback control force. Under the scheme of energy shaping, a feedback control force is designed in a such way that the total energy of the resulting feedback equivalent mechanical system has a non-degenerate minimum at the equilibrium. Using a Lyapunov argument (together with LaSalle invariance principle), the feedback equivalent system (and hence the original given system) can then be asymptotically stabilized by a further dissipative (i.e. energy-consuming) control force.

The idea of energy shaping is thus on one side intuitive from the physical point of view, yet on the other side it also poses challenging mathematical questions. The major difficulty lies in the fact that for an underactuated mechanical system, i.e. one in which we do not have full control, finding a feedback equivalent system is equivalent to finding a solution for the unknown mass matrix and potential energy governed by a set of PDEs, also known as the matching conditions. The focus of this thesis, as a result, is to answer the following question: Under what condition(s) will there be a solution to the matching conditions? and if a solution exists, how can we find it?

The idea of energy shaping has its roots in the work in robot manipulator control [31], in which every joint is controlled independently. Meanwhile, independent study of energy shaping for Euler-Lagrangian systems was also proposed in [17]. The Euler-Lagrangian approach offers an alternative (other than the state-space formalism and transfer function approach) to solving problems in system theory. However, the extensive use of energy shaping in controlled Lagrangian systems has only occurred recently (e.g. [4, 5, 11]). In particular, it is shown in [11] that there is an equivalence between the controlled Lagrangian and controlled Hamiltonian approach, implying that the treatment done on one side is valid for the other side. At the same time, interest was shifted from fully-actuated systems (e.g. [31]) to under-actuated systems (e.g. [2, 8, 16]). In [3], potential shaping is introduced besides the kinetic energy shaping. A more formal treatment of energy shaping on underactuated systems can be found in [1, 2], where λ-matching conditions are stated. It appears that we might have a larger solution set by incorporating various kinds of forces into the system [32]. In addition, energy shaping using gyroscopic forces, i.e. forces which do not dissipate energy in the system, up to degree two is considered in [8] in a general setting and incorporated in the corresponding matching conditions, resulting in a system of quasilinear PDEs for the potential energy and the mass matrix entries. The papers [7, 8] also considered the concept of *local* force shaping, i.e. modifying the external force acting on a given system in a local sense.

In this thesis we are going to follow the general setting as in [8] where gyroscopic forces are considered in the process of energy shaping. The advantage of this approach is twofold: The introduction of gyroscopic force shaping substantially reduces the number of PDEs to be solved (as compared to the original λ approach where no gyroscopic force is considered), and allows a larger set of possible solutions. Then, depending on the degrees of underactuation, we will have a number of PDEs from the matching conditions regarding the potential energy function and the mass matrix of the feedback equivalent system, with more PDEs to be solved when the number of unactuated joints increases. It should be noted that for one degree of underactuation, the energy shapability is related to the controllability of the linearization of the given controlled Lagrangian system [7, 8]. This result is based on the fact the linearization serves as the intial conditions for the matching PDEs. What remains unsolved is then the case where we have more than one unactuated joints. Indeed, there is no satisfactory result in the current literature regarding higher degrees of underactuation.

The case of higher degrees of underactuation is highly nontrivial in the sense that we cannot decide on the existence of solutions directly from the original given system of PDEs. Unlike the case of one degree of underactuation where we only have PDE for potential energy and one for the mass matrix entries, higher degrees of underactuation implies more PDEs and hence we cannot directly copy the argument in the case of one degree of underactuation. Indeed, assuming the system of PDEs is analytic and we are only interested in analytic solutions, the unknown functions should have equal mixed partials regardless of the order of differentiation. Therefore, by equating mixed partials of the unknown functions, we may come across new equations out of the original system of PDEs. These new equations are called integrability conditions or compatibility conditions depending on the nature of the equations themselves. It may appear that the resulting new equations are so restrictive that no common solution exist at all. Hence, we need a systematic approach to find out all these new equations in order to conclude the existence of solution.

In this regard, the formal theory of PDEs offers us a tool for the (local) solvability problem of systems of PDEs. The theory itself, starting roughly from the 1920's, comprises knowledge from differential algebra and differential geometry. Historically there are at least three different directions in this area. One direction (largely due to Cartan) is the study of compatibility conditions using exterior calculus. Another direction considers distinguishing independent and dependent variables in the differentiation of PDEs. This is mainly done by Janet and Riquier who introduced the principal and parametric derivatives, and who suggested a total ordering for the derivatives. Their work was later summarized by Pommaret [23, 24]. Yet there is another direction which is highly abstract, mainly investigated by Spencer [29], Goldschmidt [14] and Quillen [25], who put the whole theory in a more systematic framework with the language of (co)homology and algebraic geometry. It turns out that the involutivity is a crucial property in the study of formal solutions of PDEs, and it is known that under some conditions we can find an equivalent yet involutive system which shares the same set of solutions to the original system of PDEs, hence resolving the issue of integrability/compatibility conditions. [18, 23, 24] There is a substantial list of literatures devoted to this area. [23, 24] offer a comprehensive account of the theory (especially on the Janet's approach) while [27, 28] favor the application of the theory to computer algebra.

This thesis is divided into four chapters. In the first chapter we review the basic notion of energy shaping and state the shapability criteria [8] for systems with one degrees of underactuation. Then in the second chapter we will introduce a general procedure under which one can find the control force that "shapes" and stabilizes a system with one unactuated joint, together with some examples to illustrate the procedure. After that we will introduce the necessary tools from the formal theory of PDEs, including a procedure to find out an equivalent involutive system of PDEs which shares the same set of solutions to the original system of PDEs. In the last chapter we will apply the formal theory to derive workable criteria for energy shapability of systems with two degrees of underactuation. Again, an example is included to demonstrate how these criteria can be checked. The results in this thesis are the first time where the formal theory of PDEs is applied successfully to derive workable criteria for energy shapability compared to [13], and it opens up the possibility of using the formal theory to answer the energy shaping problem in higher degrees of underactuation.

Chapter 2

Energy Shaping for Controlled Lagrangian Systems

In this thesis we will focus on the stabilization of a certain kind of mechanical system, namely controlled Lagrangian systems whose degrees of underactuation are one or two, by the method of energy shaping. In this chapter we will quickly go over these concepts, and review some results from [8] about the energy shaping problem on systems with only one unactuated joint.

2.1 Controlled Lagrangian Systems

We first define a controlled Lagrangian system on a configuration space Q which is a n-dimensional differentiable manifold. The dimension n is sometimes called the degree of freedom for the given system. A (simple) controlled Lagrangian system on the tangent bundle TQ is a triple (L, F, W) with

- the Lagrangian $L(q, \dot{q}) = \frac{1}{2}m(\dot{q}, \dot{q}) - V(q)$ defined on TQ, where m is the symmetric, positive definite, nondegenerate mass matrix, and $V(q)$ is the potential energy of the system.

- F is the external force.

- W is the control bundle, which is a sub-bundle of the cotangent bundle T^*Q.

Thus, the equations of motion for a controlled Lagrangian system have the following form:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F + u,$$

where u is the control force which is onto W.

In this thesis, we are interested in *underactuated* systems: We assume that we do not have full control of the system so that some joints are *unactuated*, or equivalently, the control bundle W is a proper subbundle of T^*Q. From now on, we denote $\dim W$ by n_2 [1] so that the degree of underactuation is $n_1 := n - n_2$.

In most circumstances we will work in local coordinates. In particular, we may express the external force F by $(F_1, \cdots, F_n)$ and the control force u by $(0, \cdots, 0, u_{n_1+1}, \cdots, u_n)$, and thus by the definition of the Lagrangian, the equations of motions can be written in the following form:

$$\begin{aligned} m_{ij}\ddot{q}^j + [jk,i]\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^i} &= F_i + 0, \qquad i = 1, \cdots, n_1 \\ m_{ij}\ddot{q}^j + [jk,i]\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^i} &= F_i + u_i, \qquad i = n_1 + 1, \cdots, n \end{aligned}$$

where it is understood that Einstein summation convention has been adopted, and $[ij,l]$ are the Christoffel symbols of the first kind:

$$[ij,l] = \frac{1}{2}\Big(\frac{\partial m_{il}}{\partial q^j} + \frac{\partial m_{jl}}{\partial q^i} - \frac{\partial m_{ij}}{\partial q^l}\Big)$$

In the later sections, we frequently make use of the canonical pairing between the tangent bundle and cotangent bundle. For any $v = v^i \frac{\partial}{\partial q^i}$ and $w = w_i dq^i$, we define

$$\langle v, w\rangle = v^i w_i.$$

In particular, the pairing of a force (which is TQ-valued) and a velocity vector is a scalar.

[1] Here by n_2 it is understood to be the fiber dimension of the control bundle.

2.1.1 Feedback Equivalent Systems

We now study the stabilization problem for controlled Lagrangian systems. Suppose a given system has an equilibrium point, say $(q, \dot{q}) = (0, 0)$ after a suitable change of coordinates, which is not stable. One may stabilize the system at the equilibrium by applying certain control force u within W. Once a control force is chosen and applied on the given system, the resulting closed loop system will be a controlled Lagrangian system with a (possibly) different mass matrix and potential energy. This gives rise to the concept of feedback equivalent systems:

Definition 2.1.1 *Two controlled Lagrangian systems* (L, F, W) *and* $(\widehat{L}, \widehat{F}, \widehat{W})$*, where*

$$L(q, \dot{q}) = \frac{1}{2} m(\dot{q}, \dot{q}) - V(q) \quad \text{and} \quad \widehat{L}(q, \dot{q}) = \frac{1}{2} \widehat{m}(\dot{q}, \dot{q}) - \widehat{V}(q),$$

are feedback equivalent *if for any control* $u \in W$*, there exists* $\widehat{u} \in \widehat{W}$ *such that the closed loop dynamics are the same, and vice versa.*

Remark: The above definition only qualifies the concept of feedback equivalence without giving any computationally testable criteria. Depending on the information on the mechanical systems (e.g. how do the external forces depend on the velocity), we may derive different *matching conditions* governing the feedback equivalence of two systems.

It is then a direct consequence that two controlled Lagrangian systems (L, F, W) and $(\widehat{L}, \widehat{F}, \widehat{W})$ are feedback equivalent if and only if

ELM1 $m^{-1}W = \widehat{m}^{-1}\widehat{W}$;[2]

ELM2 $\langle \mathcal{EL}(L) - F - m\widehat{m}^{-1}(\mathcal{EL}(\widehat{L}) - \widehat{F}), W^{\circ} \rangle = 0$,[3]

[2] Equivalently it means $m^{-1}u = \widehat{m}^{-1}\widehat{u}$.

[3] The expression $\mathcal{EL}(L) - F - m\widehat{m}^{-1}(\mathcal{EL}(\widehat{L}) - \widehat{F})$ is exactly the elimination of the second order time derivatives from the equations of motion for the two systems, leaving only the control force u. Since W° is the annihilator of the control bundle, the inner product is zero.

where $W^\circ = \{X \in TQ \mid \langle \alpha, X \rangle = 0, \forall \alpha \in W\}$ and $\mathcal{EL} := \frac{d}{dt}\frac{\partial}{\partial \dot{q}^i} - \frac{\partial}{\partial q^i}$ is the Euler-Lagrange operator. Furthermore, using **ELM2**, the control forces u, $\widehat{u}$ that brings the same set of equations of motion for the closed loop systems are related by the following expression:

$$u = \mathcal{EL}(L) - F - m\widehat{m}^{-1}(\mathcal{EL}(\widehat{L}) - \widehat{F}) + m\widehat{m}^{-1}\widehat{u}. \tag{2.1}$$

There are a number of ways to choose such a control force to stabilize a given controlled Lagrangian system, and energy shaping is one of these methods. Generally speaking, under the framework of energy shaping, we try to alter ("shape") the given potential energy and/or the kinetic energy (equivalently changing the mass matrix) by a suitable choice of control force u so that the shaped energy function has a non-degenerate minimum at the equilibrium. Using a Lyapunov stability argument, one then tries to show that we can achieve asymptotic stability at the equilibrium by an additional dissipative force. We will elaborate on this point in the next section.

2.1.2 Energy and Force

Given a controlled Lagrangian system, the energy function E is simply

$$E = \frac{1}{2}m(\dot{q}, \dot{q}) + V(q),$$

and it can be checked that the time derivative of the energy function is equal to $\langle F, \dot{q} \rangle$. Thus, we can treat forces as T^*Q-valued functions defined on TQ, i.e. $F : TQ \to T^*Q$ or we write $F = F(q, \dot{q}) = F_i(q, \dot{q})dq^i$. We can have two types of forces:

1. **Dissipative force** F: For all $(q, \dot{q}) \in TQ$, $\langle F(q, \dot{q}), \dot{q} \rangle \leq 0$.

2. **Gyroscopic force** F; For all $(q, \dot{q}) \in TQ$, $\langle F(q, \dot{q}), \dot{q} \rangle = 0$.

In other words, dissipative forces are those which dissipate energy from a mechanical system, while gyroscopic forces do not change the energy content of the mechanical system at all.

In what follows, we consider only forces which can be decomposed into a sum of homogeneous forces. In local coordinates, a homogeneous force $F = F_i(\dot{q})dq^i$ is a force whose components $F_i(\dot{q})$ are homogeneous polynomial of degree r in $\dot{q}$, for some $r \in \mathbb{N}$. We can identify each homogeneous polynomial of degree r with a symmetric tensor product of degree r. The collection of all these symmetric product constitute a vector space, denoted as $S^r(T^*Q)$.

Definition 2.1.2 *A homogeneous force $F : TQ \to T^*Q$ of degree r on Q is a map defined as follows:*

$$F(v) = \underbrace{v \lrcorner v \lrcorner \cdots v \lrcorner}_{r \text{ times}} \tilde{F}$$

for some section $\tilde{F}$ of $S^r(T^*Q) \otimes T^*Q$ (i.e. symmetric in the first r indices), where $\lrcorner$ denotes the contraction operator, i.e. for any vector $v = v^i \frac{\partial}{\partial q^i}$, and $\tilde{F} = \tilde{F}_{j_1 \cdots j_r} dq^{j_1} \otimes \cdots \otimes dq^{j_r}$,

$$v \lrcorner \tilde{F} = v^i F_{i i_2 \cdots i_r} dq^{i_2} \otimes \cdots \otimes dq^{i_r}$$

With an abuse of notation, we sometimes identify F with $\tilde{F}$ such that we write $F(v, \ldots, v, w) = \langle F(v), w \rangle$ for any $w \in TQ$, where $\langle , \rangle$ is the canonical pairing between T^*Q and TQ.

Remarks Here are some facts regarding homogeneous forces [8]:

1. Homogeneous forces of degree one are linear in velocity, i.e. $F(q, \dot{q}) = K(q)\dot{q}$.

2. Dissipative forces which are linear in velocity are of the form $F(q, \dot{q}) = -D(q)\dot{q}$, where $D(q)$ can be represented as a symmetric positive definite matrix.

3. For any homogeneous force F which is quadratic in velocity, F is dissipative if and only if F is gyroscopic.

4. Any gyroscopic force which is quadratic in velocity can be expressed as

$$F(q, \dot{q}) = C_{ijk}\dot{q}^i\dot{q}^j dq^k,$$

where $C_{ijk} = C_{ijk}(q)$ such that $C_{ijk} + C_{jki} + C_{kij} = 0$ and $C_{ijk} = C_{jik}$.[4]

5. The introduction of gyroscopic force in the process of energy shaping is to provide couplings on the underactuated mechanical system sufficient enough to make stabilization possible. In this sense we can say that the use of gyroscopic forces enlarges the set of shapable mechanical systems.

2.2 Energy Shaping and Matching Conditions

We are now ready to state the matching conditions which govern the energy shapability of a given controlled Lagrangian system. In particular, we will state the results from [8] for the shapability of a controlled Lagrangian system with one degree of underactuation.

2.2.1 Matching Conditions

Suppose we now have two controlled Lagrangian systems (L, F, W) and $(\widehat{L}, \widehat{F}, \widehat{W})$, where $F = F_1 + F_2$ and $\widehat{F} = \widehat{F}_1 + \widehat{F}_2$ are their homogeneous force decompositions up to second degree.[5] We want to find out the matching conditions, i.e. conditions under which these two systems are feedback equivalent to each other. First, from **ELM1**, we know that the feedback equivalence implies

$$m^{-1}W = \widehat{m}^{-1}\widehat{W}.$$

[4]The cyclic property of C_{ijk} is due to the fact that F is gyroscopic; C_{ijk} is symmetric in the first two indices since the force is homogeneous by definition.

[5]In general we can consider forces which are dependent on velocity up to arbitrary degrees. But taking into consider that most forces are general one degree or two (e.g. drag force due to air resistance, Lorenz force for a moving point charge under magnetic field), our setting of using force depending on velocity up to two degree works in most mechanical systems.

Then, from **ELM2**, we also have

$$\langle m\nabla_{\dot q}\dot q + dV - F_1^\flat \dot q - F_2(\dot q,\dot q) - m\widehat{m}^{-1}(\widehat{m}\widehat{\nabla}_{\dot q}\dot q + d\widehat{V} - \widehat{F}_1^\flat \dot q - \widehat{F}_2(\dot q,\dot q)), Z\rangle = 0, \qquad (2.2)$$

where ∇ and $\widehat{\nabla}$ are the metric connections associated with the mass matrices m and $\widehat{m}$ respectively, i.e. in local coordinates,

$$\nabla_X Y = \left(X^j \frac{\partial Y^i}{\partial q^j} + \Gamma^i_{jk} X^j Y^k \right) \frac{\partial}{\partial q^i}, \qquad \forall X = X^i \frac{\partial}{\partial q^i}, Y = Y^i \frac{\partial}{\partial q^i} \in TQ,$$

in which $\Gamma^i_{jk} = m^{ir}[jk, r]$ are the Christoffel symbols of the second kind. The notation $F_1^\flat$ is defined by

$$\langle F_1^\flat X, Y\rangle = F_1(X,Y),$$

for all $X, Y \in TQ$. Now, by collecting terms of equal orders in $\dot q$ in (2.2), we can obtain the following matching conditions:

Theorem 2.2.1 (Matching Conditions [8]) *(L, F, W) and $(\widehat{L}, \widehat{F}, \widehat{W})$ are feedback equivalent systems if and only if the following equations are satisfied:*

$$\begin{aligned}
(dV - m\widehat{m}^{-1} d\widehat{V})|_{W^\circ} &= 0 && (2.3)\\
\widehat{F}_1(X, \widehat{m}^{-1} m Z) &= F_1(X, Z) && (2.4)\\
\widehat{F}_2(X, Y, \widehat{m}^{-1} m Z) &= \widehat{K}(X, Y, \widehat{m}^{-1} m Z) + F_2(X, Y, Z) && (2.5)\\
\widehat{W} &= \widehat{m} m^{-1} W && (2.6)
\end{aligned}$$

for all $X, Y \in TQ$, $Z \in W^\circ$. Here $\widehat{K} \in \Gamma(S^2(T^*Q) \otimes T^*Q)$ is a T^*Q-valued map defined using mass matrices m and $\widehat{m}$ and their associated connections ∇, $\widehat{\nabla}$ by:

$$\widehat{K}(X, Y, Z) = \widehat{m}(\widehat{\nabla}_X Y - \nabla_X Y, Z),$$

for all $X, Y, Z \in TQ$.[6]

[6] It can be easily checked that $\nabla_X Y - \widehat{\nabla}_X Y$ is symmetric in X and Y, hence the map K is well-defined.

In what follows, we will always assume W is integrable, that is, there exists local coordinates $q^1, \ldots, q^n$ so that we can write

$$W^\circ = \text{Span } \left\{ \frac{\partial}{\partial q^\alpha} \;\middle|\; \alpha = 1, \ldots, n_1 \right\}, \quad W = \text{Span } \{dq^a \mid a = n_1+1, \ldots, n\}.$$

With the only exception in the subsequent sections reviewing the notions of formal theory of PDEs, we will consistently use Greek indices which run from 1 to n_1 while Roman alphabetical indices $(i, j, k, \cdots)$ run from 1 to n unless otherwise stated. Now, by some algebraic manipulations [8], the following matching conditions in local coordinates can be obtained:

Theorem 2.2.2 ([8]) *$(L, 0, W)$ is feedback equivalent to $(\widehat{L}, \widehat{F}, \widehat{W})$ with a gyroscopic force $\widehat{F}$ of degree 2 if and only if there exists a non-degenerate mass matrix $\widehat{m}$ and a potential function $\widehat{V}$ such that the following equations are satisfied:*

$$\frac{\partial V}{\partial q^\alpha} - \widehat{m}^{ia} m_{\alpha a} \frac{\partial \widehat{V}}{\partial q^i} = 0, \tag{2.7}$$

$$\widehat{J}_{\alpha\beta\gamma} + \widehat{J}_{\beta\gamma\alpha} + \widehat{J}_{\gamma\alpha\beta} = 0, \tag{2.8}$$

where $\widehat{J}_{\alpha\beta\gamma}$ is defined by [7]

$$\widehat{J}_{\alpha\beta\gamma} = \frac{1}{2} \widehat{m}^{ia} m_{\alpha a} \widehat{m}^{jb} m_{\beta b} \widehat{m}^{kc} m_{\gamma c} \left(\frac{\partial \widehat{m}_{ij}}{\partial q^k} - \Gamma^r_{ki} \widehat{m}_{rj} - \Gamma^r_{jk} \widehat{m}_{ri} \right).$$

Notice that (2.7) is just a direct translation of (2.3) in local coordinates, while (2.8) is done by polarizing (2.5) (c.f. [8]). We sometimes call equations of the type of (2.7) potential matching conditions/potential PDEs, and (2.8) the kinetic matching conditions/kinetic PDEs.

Moreover, one can reduce the number of unknowns to be solved by introducing

$$\widehat{T} = m \widehat{m}^{-1} m,$$

[7] The expression inside the bracket is exactly the (i, j)-th entry of the covariant derivative of mass matrix $\widehat{m}$ (treated as a $(0, 2)$ tensor) along $\frac{\partial}{\partial q^k}$, i.e. $\left(\nabla_{\frac{\partial}{\partial q^k}} \widehat{m} \right)_{ij}$.

for every pair of mass matrices m and $\widehat{m}$. Then, one can check that [8] the matching conditions can be simplified using $\widehat{T}$ instead of $\widehat{m}$:

Theorem 2.2.3 ([8]) *$(L, 0, W)$ is feedback equivalent to $(\widehat{L}, \widehat{F}, \widehat{W})$ with a gyroscopic force $\widehat{F}$ of degree 2 if and only if there exists a non-degenerate mass matrix $\widehat{m}$ and a potential function $\widehat{V}$ such that the following equations are satisfied:*

$$\frac{\partial V}{\partial q^\alpha} - \widehat{T}_{j\alpha} m^{ij} \frac{\partial \widehat{V}}{\partial q^i} = 0 \tag{2.9}$$

$$\widehat{J}_{\alpha\beta\gamma} + \widehat{J}_{\beta\gamma\alpha} + \widehat{J}_{\gamma\alpha\beta} = 0 \tag{2.10}$$

where m_{ij} (resp. m^{ij}) is the (i, j)-entry of m (resp. m^{-1}), $\widehat{T}_{ij} = m_{ia}\widehat{m}^{ab}m_{bj}$ and $\widehat{J}_{\alpha\beta\gamma}$ are defined by

$$\widehat{J}_{\alpha\beta\gamma} = \frac{1}{2}\widehat{T}_{\gamma s} m^{sk} \left(\frac{\partial \widehat{T}_{\alpha\beta}}{\partial q^k} - \Gamma^r_{\beta k}\widehat{T}_{\alpha r} - \Gamma^r_{\alpha k}\widehat{T}_{\beta r} \right).$$

It should be noted that by using $\widehat{T}$, only those entries in the first n_1 rows of $\widehat{T}$ will appear in the PDEs, in contrast to the case where *all* entries of $\widehat{m}$ are used as in Theorem 2.2.3.

Before we move on to general mechanical systems with degree of underactuation one, it is worth mentioning a related result concerning the shapability for a *linear* controlled Lagrangian system. A linear controlled Lagrangian system is a triple $(\overline{L}, \overline{F}, \overline{W})$ such that

$$\overline{L} = \frac{1}{2}M_{ij}\dot{q}^i\dot{q}^j - \frac{1}{2}S_{ij}q^iq^j,$$
$$\overline{F} = A_i\dot{q}^i + B_iq^i,$$

and $\overline{W}$ is a trivial bundle over Q, while M_{ij}, S_{ij}, A_i and B_i are all constant. In a similar fashion, one can also define the linearization of any given controlled Lagrangian system. For any controlled Lagrangian system (L, F, W), its linearized controlled Lagrangian system, denoted as (L^ℓ, F^ℓ, W^ℓ), at the equilibrium $(q, \dot{q}) = (q_e, 0)$ is given by

$$\begin{aligned}
L^\ell &= \frac{1}{2} m_{ij}(q_e)\dot{q}^i \dot{q}^j - \frac{1}{2}\frac{\partial^2 V}{\partial q^i \partial q^j}(q_e)(q^i - q_e^i)(q^j - q_e^j),\\
F^\ell &= \frac{\partial F}{\partial q^i}(q^i - q_e^i) + \frac{\partial F}{\partial \dot{q}^i}(q_e, 0)\dot{q}^i,\\
W^\ell &= W(q_e).
\end{aligned}$$

Without loss of generality, $\frac{\partial V}{\partial q^i}(q_e)$ and $F(q_e, 0)$ are intentionally left out as they should be zero at the equilibrium.

Now, recall that a linear system $\dot{x} = Ax$ is oscillatory if A is diagonalizable and all eigenvalues of A are nonzero and purely imaginary. One can then show that [8] for any second order system $\ddot{x} = Ax$ is oscillatory if and only if A is diagonalizable and has only negative real eigenvalues, and hence one can determine whether a given linear controlled Lagrangian system is oscillatory or not. This in turn is related to the energy shapability of that linear system:

Theorem 2.2.4 ([8]) *A linear controlled Lagrangian system $(L, 0, W)$ is feedback equivalent to a linear controlled Lagrangian system $(\overline{L}, 0, \overline{W})$ with positive definite energy if and only if the uncontrollable dynamics of $(L, 0, W)$, if any, is oscillatory.*

Note that the above theorem is true for any *linear* system with any degree of underactuation. As a result, what is interesting is the case where a given system is *nonlinear* in general.

2.2.2 Systems with One Degree of Underactuation

When a given system $(L, 0, W)$ has only one degree of underactuation, then the matching conditions in Theorem 2.2.3 reduce to 2 PDEs, one for $\widehat{V}$ and one for $\widehat{T}$:

$$\frac{\partial V}{\partial q^1} - \widehat{T}_{j1} m^{ij} \frac{\partial \widehat{V}}{\partial q^j} = 0$$
$$\widehat{T}_{1s} m^{sk} \left(\frac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) = 0.$$

Suppose the linearization $(L^\ell, 0, W^\ell)$ of the given system is controllable, or its uncontrollable part is oscillatory, then by Theorem 2.2.4, it is feedback equivalent to a linear $(\overline{L}^\ell, 0, \overline{W}^\ell)$ where $\overline{L}^\ell = \frac{1}{2}\dot{q}^T \overline{M}\dot{q} - \frac{1}{2}q^T \overline{S} q$, where $\overline{M}, \overline{S} \succ 0$. It can be checked that $\overline{M}$ and $\overline{S}$ are compatible with the two matching conditions for the original nonlinear controlled Lagrangian system, and hence we can use them as initial conditions for this system of PDEs. From the Cauchy-Kowalevski theorem solutions to these 2 PDEs are known to always exist, and the shapability problem can be summarized as follows [8]:

Theorem 2.2.5 ([8]) *Given $(L, 0, W)$ with one degree of underactuation, let $(L^\ell, 0, W^\ell)$ be its linearized system at equilibrium $(q, \dot{q}) = (0, 0)$. Then there exists a feedback equivalent $(\widehat{L}, \widehat{F}, \widehat{W})$ with $\widehat{F}$ gyroscopic of degree 2 and $\widehat{V}$ having a non-degenerate minimum at $(0, 0)$ if and only if the uncontrollable dynamics, if any, of $(L^\ell, 0, W^\ell)$ is oscillatory. In addition if $(L^\ell, 0, W^\ell)$ is controllable, then $(\widehat{L}, \widehat{F}, \widehat{W})$ can be exponentially stabilized by any linear dissipative feedback onto $\widehat{W}$.*

This theorem characterizes the energy shapability of a given system with one degree of underactuation. We have briefly explained the "if" part of the proof of the above theorem while the "only if" argument (which can be done similarly) can be found in [8]. Exponential stability is achieved by the fact that a linear controlled Lagrangian system is controllable if and only if it is (exponentially) stabilizable [8], the feedback-invariant property of controllability and by the use of Lyapunov indirect method. This sort of argument will appear again when we come to the case where the degree of underactuation is two.

Chapter 3

Examples on Energy Shaping with One Degree of Underactuation

In the previous chapter, we briefly reviewed the concept of energy shaping and stated the matching conditions for shaping a given controlled Lagrangian system. In this chapter, we are going to pursue this further by introducing a general strategy for shaping a controlled Lagrangian system with one degree of underactuation [8, 9]. As mentioned in Chapter 1, we only have 1 PDE for the potential energy function $\widehat{V}$ and 1 PDE for the $\widehat{T}$ (which in turn is related to the mass matrix $\widehat{m}$) for the feedback equivalent ("shaped") system $(\widehat{L}, 0, \widehat{W})$. Solving this system of PDEs can be very routine and does not incorporate any ad hoc treatments which are only suitable for some examples, and the number of PDEs to be solved is in general reduced by using gyroscopic force shaping, i.e. energy shaping using gyroscopic force as well.

To put it simply, solving an energy shaping problem involves the following steps:

1. Solve $\widehat{T}$ and $\widehat{V}$ satisfying the system of PDEs that governs feedback equivalence.

2. Find out the corresponding external force on the feedback equivalent system.

3. Choose a suitable dissipative force to asymptotically stabilize the given system.

We will first elaborate the above steps, in particular step 2 where the external force is computed, and demonstrate the procedure by some examples. The contents of this chapter will appear in [20].

3.1 Defining the Gyroscopic Force Terms $\widehat{C}_{ijk}$

Suppose we want to find a feedback equivalent controlled Lagrangian system $(\widehat{L}, \widehat{F}, \widehat{W})$ for a given system $(L, 0, W)$. Recall from Theorem 2.2.2 that in order to achieve this we need to solve for the PDEs for $\widehat{m}$ and $\widehat{V}$. Solving for $\widehat{m}$ and $\widehat{V}$ is not the end of story, though, for we also need to find out $\widehat{F}$ and more importantly, the control forces u and $\widehat{u}$ that provide the feedback equivalence.

To be more precise, we can rephrase the whole problem as follows: given a controlled Lagrangian system $(L, F = 0, W)$ where $L = \frac{1}{2}m_{ij}\dot{q}^i\dot{q}^j - V(q)$ and $W = \text{span}\{dq^{n_1+1}, \cdots dq^n\})$, we want to find a feedback equivalent system $(\widehat{L}, \widehat{F}, \widehat{W})$ with

$$\begin{aligned}
\widehat{L} &= \frac{1}{2}\widehat{m}_{ij}\dot{q}^i\dot{q}^j - \widehat{V}(q),\\
\widehat{F} &= \widehat{C}_{ijk}\dot{q}^i\dot{q}^j dq^k,\\
\widehat{W} &= \widehat{m}m^{-1}W,
\end{aligned}$$

such that $\widehat{m}$ and $\widehat{V}$ are positive definite and the gyroscopic force term satisfying the following:

$$\widehat{C}_{ijk} = \widehat{C}_{jik}; \qquad \widehat{C}_{ijk} + \widehat{C}_{jki} + \widehat{C}_{kij} = 0.$$

To find the gyroscopic force terms $\widehat{C}_{ijk}$, we need to go back to (2.5). Following the idea in [9], we define

$$\widehat{A}_{ijk} := m_{ip}m_{jq}m_{kr}\widehat{m}^{pl}\widehat{m}^{qs}\widehat{m}^{rt}\widehat{C}_{lst} \tag{3.1}$$

$$\widehat{S}_{ijk} := m_{ip}m_{jq}\widehat{m}^{pl}\widehat{m}^{qs}(m_{kr}\widehat{m}^{rt}\widehat{[ls,t]} - [ls,k]). \tag{3.2}$$

Then, the matching condition (2.5) with $F = 0$ is equivalent to

$$\widehat{A}_{ij\alpha} = \widehat{S}_{ij\alpha}.$$

It should be noted that, however, the above equality only holds for $\alpha = 1, \cdots, n_1$, i.e. we only have the information on the gyroscopic force $\widehat{F}$ restricted to $S^2(T^*Q) \otimes \widehat{m}^{-1} m W^0$, not to the whole space $S^2(T^*Q) \otimes T^*Q$. A natural extension of this gyroscopic force to $S^2(T^*Q) \otimes T^*Q$ can be done by making use of the cyclic relation[1]

$$\widehat{A}_{ijk} + \widehat{A}_{jki} + \widehat{A}_{kij} = 0,$$

to write

$$\begin{aligned}
\widehat{A}_{\alpha\beta\gamma} + \widehat{A}_{\beta\gamma\alpha} + \widehat{A}_{\gamma\alpha\beta} &= 0 \\
\widehat{A}_{a\beta\gamma} + \widehat{A}_{\beta\gamma a} + \widehat{A}_{\gamma a\beta} &= 0 \\
\widehat{A}_{ab\gamma} + \widehat{A}_{b\gamma a} + \widehat{A}_{\gamma ab} &= 0 \\
\widehat{A}_{abc} + \widehat{A}_{bca} + \widehat{A}_{cab} &= 0,
\end{aligned}$$

where $a, b, c = n_1 + 1, \cdots n$. As $\widehat{A}_{ij\alpha} = \widehat{S}_{ij\alpha}$, the above four cyclic relations become

$$\begin{aligned}
\widehat{S}_{\alpha\beta\gamma} + \widehat{S}_{\beta\gamma\alpha} + \widehat{S}_{\gamma\alpha\beta} &= 0 \\
\widehat{S}_{a\beta\gamma} + \widehat{A}_{\beta\gamma a} + \widehat{S}_{\gamma a\beta} &= 0 \\
\widehat{S}_{ab\gamma} + \widehat{A}_{b\gamma a} + \widehat{A}_{\gamma ab} &= 0 \\
\widehat{A}_{abc} + \widehat{A}_{bca} + \widehat{A}_{cab} &= 0.
\end{aligned}$$

Notice that the first cyclic relation, namely $\widehat{S}_{\alpha\beta\gamma} + \widehat{S}_{\beta\gamma\alpha} + \widehat{S}_{\gamma\alpha\beta} = 0$, is exactly the PDE for $\widehat{m}$, and hence if we have a solution $\widehat{m}$ for the matching condition, this cyclic relation should be automatically satisfied. Then, the remaining 3 cyclic relations allow us to define $\widehat{A}_{ijk}$ on the whole space (Note that $a, b, c = n_1 + 1, \cdots, n$ in each of the following steps):

[1] $\widehat{A}_{ijk}$ being cyclic is straightforward: Suppose $\frac{\partial}{\partial q^i}, i = 1, \cdots n$ are basis vectors for TQ, then

$$\widehat{A}_{ijk} = \widehat{A}\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}, \frac{\partial}{\partial q^k}\right) = m\widehat{m}^{-1}\widehat{F}\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}, \frac{\partial}{\partial q^k}\right) = \widehat{F}\left(m\widehat{m}^{-1}\frac{\partial}{\partial q^i}, m\widehat{m}^{-1}\frac{\partial}{\partial q^j}, m\widehat{m}^{-1}\frac{\partial}{\partial q^k}\right),$$

which is cyclic.

(a) $\widehat{A}_{ij\alpha} = \widehat{S}_{ij\alpha}$.

(b) Define $\widehat{A}_{\beta\gamma a} = -\widehat{S}_{a\beta\gamma} - \widehat{S}_{\gamma a\beta}$.

(c) Define $\widehat{A}_{\gamma ab} = \widehat{A}_{b\gamma a} = -\frac{1}{2}\widehat{S}_{ab\gamma}$.

(d) Finally, we can choose any $\widehat{A}_{abc}$ such that $\widehat{A}_{abc} + \widehat{A}_{bca} + \widehat{A}_{cab} = 0$. For simplicity, we can simply take $\widehat{A}_{abc} = 0$.

Once $\widehat{A}_{ijk}$ is determined, we can obtain the gyroscopic force terms $\widehat{C}_{ijk}$ by (3.1), or equivalently by,

$$\widehat{C}_{ijk} = \widehat{m}_{xi}\widehat{m}_{yj}\widehat{m}_{zk}m^{xr}m^{ys}m^{zt}\widehat{A}_{rst}. \tag{3.3}$$

The above approach works when $\widehat{m}$ is solved. Suppose on the contrary we solve the $\widehat{T}$ matching condition instead, then we can still make use of the above approach by first computing $\widehat{m}$:

$$\widehat{m} = m\widehat{T}^{-1}m.$$

Notice that $\widehat{m} \succ 0$ if and only if $\widehat{T} \succ 0$. Hence, we also require $\widehat{T} \succ 0$, at least in a neighbourhood of $(q, \dot{q}) = (0, 0)$, when we solve the matching conditions. One advantage of using $\widehat{T}$ instead of $\widehat{m}$ is the reduction of the number of unknowns to be solved in the matching conditions: We have all the entries in the upper triangular part of $\widehat{m}$ in those PDEs, but if we express these PDEs in terms of $\widehat{T}$, only $\widehat{T}_{\alpha k}$, $\alpha = 1, \cdots, n_1$ will appear.

3.2 General Procedure for Energy Shaping Problem

Now we are at the stage to state the general procedure under which one can systematically solve any energy shaping problem with arbitrary degree of underactuation [10]:

S1. Check that the linearization of the given controlled Lagrangian is controllable or its uncontrollable subsystem is oscillatory. If neither holds, then stop; otherwise, proceed to the next step. [2]

S2. Get a solution for $\widehat{V}$ and the (α, i) entries of $\widehat{T}$ which solve the matching PDEs (5.1) and (5.7), keeping in mind that the $n_1 \times n_1$ matrix $[T_{\alpha\beta}]$ is positive definite around $q = 0$ and $\widehat{V}$ has a non-degenerate minimum at 0. In particular, $\widehat{T}_{11}$ should be positive around $q = 0$ when the degree of underactuation n_1 is one.

S3. Choose the rest of the entries $\widehat{T}_{ab}$ of $\widehat{T}$ so that $\widehat{T}$ is positive definite, at least at $q = 0$. In particular, when the degree of freedom n is two, one should choose $\widehat{T}_{22} > (\widehat{T}_{12})^2/\widehat{T}_{11}$.

S4. Obtain the mass matrix $\widehat{m}$ of the feedback equivalent system, through the equation: $\widehat{m} = m\widehat{T}^{-1}m$.

S5. Obtain the gyroscopic force $\widehat{F}$ by computing $\widehat{S}_{ijk}$, $\widehat{A}_{ijk}$ and then $\widehat{C}_{ijk}$ by (3.2), (3.3) and steps (a) – (d) located just above (3.3).

S6. Compute the control bundle $\widehat{W}$, which is given by

$$\widehat{W} = \mathrm{Span} \left\{ \begin{bmatrix} m^{ai}\widehat{m}_{i1} \\ \cdots \\ m^{ai}\widehat{m}_{in} \end{bmatrix} \;\middle|\; a = n_1 + 1, \cdots, n \right\}$$

S7. Choose a dissipative, $\widehat{W}$-valued linear control force $\widehat{u}$. In particular, for systems with degree of underactuation equal to n_1, one may choose

$$\widehat{u} = -K^T D K \dot{q}, \tag{3.4}$$

[2]For one degree of underactuation, linear controllability or the presence of oscillatory uncontrollable subdynamics is necessary by Theorem 2.2.5; For two degree of underactuation (and $n \geq 4$) we will also show that such conditions are required for shaping a nonlinear system.

where D is any symmetric positive definite $(n - n_1) \times (n - n_1)$ matrix and K is the $(n - n_1) \times n$ matrix defined by

$$K = \begin{bmatrix} m^{n_1+1i}\widehat{m}_{i1} & \cdots & m^{n_1+1i}\widehat{m}_{in} \\ \vdots & \ddots & \vdots \\ m^{ni}\widehat{m}_{i1} & \cdots & m^{ni}\widehat{m}_{in} \end{bmatrix}.$$

The above choice of $\widehat{u}$ will guarantee that it is dissipative and onto $\widehat{W}$.

S8. Compute the corresponding control force u:

$$u_a = [jk, a]\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^a} - m_{ar}\widehat{m}^{rs}\left(\widehat{[jk, s]}\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^s} - \widehat{C}_{jks}\dot{q}^j\dot{q}^k - \widehat{u}_s\right) \tag{3.5}$$

where $a = n_1 + 1, \cdots, n$. Note that u_α, where $\alpha = 1, \cdots n_1$ are zero.

It should be emphasized that once we realize the existence of solutions to the matching PDEs, we can apply the above procedure to design the feedback control, irrespective of the degree of underactuation. The main difficulty of solving energy shaping problem lies on the difficulty of proving existence of solutions for the matching conditions.

3.3 Examples

We now present two examples to illustrate the general procedure for solving the energy shaping problem for one degree of underactuation. The first example is the inverted pendulum on a running cart and the second one is the ball and beam problem. More examples can be found in the upcoming paper [20].

3.3.1 Inverted Pendulum on a Cart

In this example, we assume the masses are concentrated at one point while the rod has negligible mass in order to simplify our model. Refer to Figure 1 for the parameters. The configuration space is

$$Q = \left\{ (q^1, q^2) \;\middle|\; q^1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \;,\; q^2 \in \mathbb{R} \right\}$$

The Lagrangian is given by

$$L(q, \dot{q}) = \frac{1}{2} m_1 \ell^2 (\dot{q}^1)^2 + \frac{1}{2}(m_1 + m_2)(\dot{q}^2)^2 + m_1 \ell \dot{q}^1 \dot{q}^2 \cos q^1 - m_1 g \ell \cos q^1,$$

where g is the gravitational constant. Thus the mass matrix is given by

$$m = \begin{bmatrix} m_1 \ell^2 & m_1 \ell \cos q^1 \\ m_1 \ell \cos q^1 & m_1 + m_2 \end{bmatrix}$$

and the potential energy is

$$V(q) = m_1 g \ell \cos q^1,$$

which does not attain a minimum at $q = 0$, and hence the equilibrium point $(q, \dot{q}) = (0, 0)$ is unstable. Thus, we might use energy shaping method to stabilize this system around $(0, 0)$.

The matching conditions are

$$\left(-\frac{m_1 + m_2}{m_1 \ell^2} \widehat{T}_{11} + \frac{\cos q^1}{\ell} \widehat{T}_{12} \right) \left(\frac{\partial \widehat{T}_{11}}{\partial q^1} + \frac{2 m_1 \widehat{T}_{11} \cos q^1 \sin q^1 - 2 \widehat{T}_{12} m_1 \ell \sin q^1}{-(m_1 + m_2) + m_1 \cos^2 q^1} \right) + \left(\frac{\cos q^1}{\ell} \widehat{T}_{11} - \widehat{T}_{12} \right) \frac{\partial \widehat{T}_{11}}{\partial q^2} = 0$$

$$\left(-\frac{m_1 + m_2}{m_1 \ell^2} \widehat{T}_{11} + \frac{\cos q^1}{\ell} \widehat{T}_{12} \right) \frac{\partial \widehat{V}}{\partial q^1} + \left(\frac{\cos q^1}{\ell} \widehat{T}_{11} - \widehat{T}_{12} \right) \frac{\partial \widehat{V}}{\partial q^2} + m_1 g \ell \sin q^1 (-(m_1 + m_2) + m_1 \cos^2 q^1) = 0$$

We now try to obtain closed form solutions for $\widehat{T}$ and $\widehat{V}$. First, we notice that if $\widehat{T}_{11}$ is a constant, we will basically obtain the original given system. Hence, we try the next

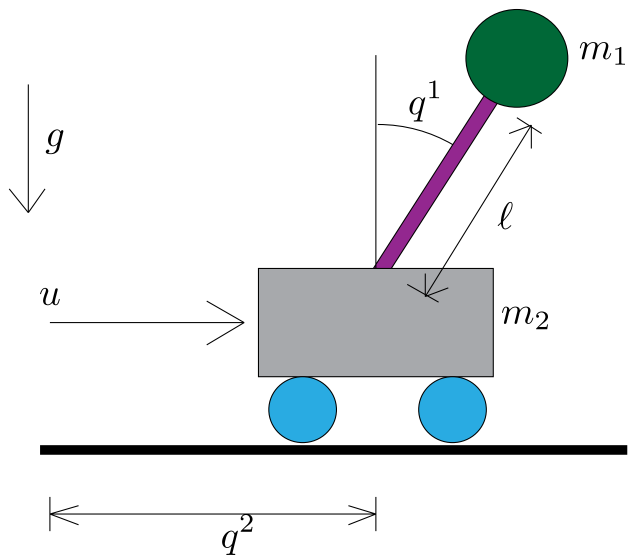

Figure 3.1: An inverted pendulum on a running cart

simplest possible candidate:

$$\widehat{T}_{11} = A_0 + A_1 \cos^2 q^1,$$

where A_0 and $A_1 \neq 0$ are constants to be determined. Putting this *ansatz* into the first matching condition, we can obtain

$$\widehat{T}_{12} = \frac{(m_1 + m_2)(A_0 + A_1 \cos^2 q^1)}{m_1 \ell \cos q^1} \quad \text{or} \quad \widehat{T}_{12} = \frac{A_0 m_1 + A_1(m_1 + m_2)}{m_1 \ell} \cos q^1$$

Notice that the first solution for $\widehat{T}_{12}$ will lead to a potential energy function $\widehat{V}$ whose Hessian is *not* positive definite at $q = 0$, hence we should resort to the second solution of $\widehat{T}_{12}$ and solve the matching condition for $\widehat{V}$ to obtain

$$\widehat{V} = \frac{1}{A_0} m_1^2 g \ell^3 \cos q^1 + f\left(q^2 + \frac{A_1 \ell}{A_0} \sin q^1\right),$$

where $f = f(x)$ is a smooth function yet to be determined. To satisfy the requirement that $\widehat{T}, \widehat{V} \succ 0$ (at least in a neighbourhood of $(0,0)$), we may impose $A_1 > 0 > A_0$, $f(x) = x^2$ and take any $\widehat{T}_{22}$ which makes $\det \widehat{T} > 0$. In particular, suppose we take $A_0 = -\epsilon$, where $\epsilon > 0$, $A_1 = 2$ and

$$\widehat{T}_{22} = \frac{\cos^2 q^1 (2(m_1 + m_2) - \epsilon m_1)^2 + 1}{m_1^2 \ell^2 (2\cos^2 q^1 - \epsilon)},$$

which makes $\det \widehat{T} = \frac{1}{m_1^2 \ell^2}$ for all q. In short, we have the following $\widehat{T}$ matrix and potential energy $\widehat{V}$:

$$\widehat{T} = \begin{bmatrix} 2\cos^2 q^1 & \frac{2(m_1+m_2)-\epsilon m_1}{m_1 \ell} \cos q^1 \\ \frac{2(m_1+m_2)-\epsilon m_1}{m_1 \ell} \cos q^1 & \frac{\cos^2 q^1 (2(m_1+m_2)-\epsilon m_1)^2+1}{m_1^2 \ell^2 (2\cos^2 q^1 - \epsilon)} \end{bmatrix}$$
$$\widehat{V} = \frac{1}{\epsilon} m_1^2 g \ell^3 \cos q^1 - \left(\frac{\epsilon q^2 + 2\ell \sin q^1}{\epsilon} \right)^2,$$

which is defined in a subset $\mathcal{R}_\epsilon$ of Q, where

$$\mathcal{R}_\epsilon = \left(-\cos^{-1} \sqrt{\frac{\epsilon}{2}}, \cos^{-1} \sqrt{\frac{\epsilon}{2}} \right) \times \mathbb{R}.$$

The resulting mass matrix is $\widehat{m} = \begin{bmatrix} \widehat{m}_{11} & \widehat{m}_{12} \\ \widehat{m}_{12} & \widehat{m}_{22} \end{bmatrix}$ where

$$\widehat{m}_{11} = \frac{m_1^2 \ell^4 (4\cos^2 q^1 (m_1 + m_2)^2 + 1 - 8\cos^4 q^1 (m_1^2 + m_1 m_2) + 4 m_1^2 \cos^6 q^1)}{2\cos^2 q^1 - \epsilon}$$
$$m_{12} = \frac{m_1^2 \ell^3 \cos q^1 (-4\epsilon \cos^2 q^1 (m_1^2 + m_1 m_2) + 1 + 2\epsilon (m_1 + m_2)^2 + 2\epsilon m_1^2 \cos^4 q^1)}{2\cos^2 q^1 - \epsilon}$$
$$m_{22} = \frac{m_1^2 \ell^2 (\epsilon^2 m_1^2 \cos^4 q^1 + \cos_1^2 (1 - 2\epsilon^2 m_1^2 - 2\epsilon^2 m_1 m_2) + \epsilon^2 (m_1 + m_2)^2)}{2\cos^2 q^1 - \epsilon}$$

The computation of the gyroscopic force terms is straightforward but too complicated to be stated explicitly, thus we just state the final gyroscopic force $\widehat{F} = [\widehat{F}_1, \widehat{F}_2]^T$ that appears in the equations of motion for the shaped system:

$$\widehat{F}_1 = \frac{m_1^2 \ell^2 \cos q^1 \sin q^1}{(2\cos^2 q^1 - \epsilon)^2} \dot{q}^2 (2\ell \dot{q}^1 + \epsilon \dot{q}^2)(2\epsilon(m_1^2 \cos^2 q^1 - (m_1 + m_2)^2) + 2\epsilon^2 m_1 (m_1 \sin^2 q^1 + m_2) - 1)$$
$$\widehat{F}_2 = -\frac{m_1^2 \ell^2 \cos q^1 \sin q^1}{(2\cos^2 q^1 - \epsilon)^2} \dot{q}^1 (2\ell \dot{q}^1 + \epsilon \dot{q}^2)(2\epsilon(m_1^2 \cos^2 q^1 - (m_1 + m_2)^2) + 2\epsilon^2 m_1 (m_1 \sin^2 q^1 + m_2) - 1)$$

The control bundle $\widehat{W}$ is equal to

$$\widehat{W} = \text{Span}\left\{ \begin{bmatrix} m^{2i} \widehat{m}_{i1} \\ m^{2i} \widehat{m}_{i2} \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} \frac{2\ell}{\epsilon} \cos q^1 \\ 1 \end{bmatrix} \right\}.$$

We now choose a control force $\widehat{u}$ which is $\widehat{W}$-valued, according to step S7. with $D = 1$:

$$\widehat{u} = -\begin{bmatrix} \left(\frac{2\ell}{\epsilon} \cos q^1\right)^2 & \frac{2\ell}{\epsilon} \cos q^1 \\ \frac{2\ell}{\epsilon} \cos q^1 & 1 \end{bmatrix} \begin{bmatrix} \dot{q}^1 \\ \dot{q}^2 \end{bmatrix}$$

One can then compute u by (3.5), which is omitted here due to lack of space. Note that by Theorem 2.2.5, local exponential stability is guaranteed around $(q, \dot{q}) = (0, 0)$. To find out the region of attraction, however, one needs LaSalle invariance principle. We start by choosing $r > 0$ so that

$$\Omega_r = \{(q, \dot{q}) \in \mathcal{R}_\epsilon \times \mathbb{R}^2 \mid \widehat{E}(q, \dot{q}) \leq r\}$$

is compact. Then we define the set

$$\begin{aligned} \mathcal{E} &= \{(q, \dot{q}) \in \Omega_r \mid d\widehat{E}/dt = 0\} \\ &= \{(q, \dot{q}) \in \Omega_r \mid 2\ell \dot{q}^1 \cos q^1 + \epsilon \dot{q}^2 = 0\}. \end{aligned}$$

We note that the total energy function $\widehat{E}$ has a zero time derivative if and only if $\dot{q}^2 = -\left(\frac{2\ell}{\epsilon} \cos q^1\right) \dot{q}^1$, from which we have

$$q^2 = -\frac{2\ell}{\epsilon} \sin q^1 + C, \tag{3.6}$$

where C is a constant. Let $\mathcal{M}$ be the largest invariant subset of $\mathcal{E}$ and consider an arbitrary trajectory $(q(t), \dot{q}(t))$ in $\mathcal{M}$. This trajectory should satisfy the equations of motion of the feedback equivalent system together with (3.6). Substitute (3.6) into those equations of motion, we have

$$\sin q^1 (\dot{q}^1)^2 = \frac{2C(2\cos^2 q^1 - \epsilon) + \frac{m_1 g \ell^2}{2} \sin 2q^1}{\ell^3 m_1^2}, \tag{3.7}$$

$$\ddot{q}^1 = \frac{4C \cos q^1 + m_1 g \ell^2 \sin q^1}{m_1^2 \ell^3}. \tag{3.8}$$

Multiplying (3.8) by $\cos q^1$ and subtracting it from (3.7), one can obtain

$$\sin q^1 (\dot{q}^1)^2 - \cos q^1 \ddot{q}^1 = -\frac{2C\epsilon}{\ell^3 m_1^2}.$$

Then by integration twice with respect to t, we have

$$\sin q^1 = \frac{C\epsilon}{\ell^3 m_1^2} t^2 + C_1 t + C_2,$$

where C_1, C_2 are constant. Now, since $\sin q^1$ is always bounded, the above equation holds only if $C = C_1 = 0$, implying that q^1 must be a constant. As $C = 0$ and $\dot{q}^1 = 0$, (3.8) implies $\sin q^1 = 0$, i.e. $q^1 = 0$ or π. When $q^1 = 0$, so is q^2. In other words, $\mathcal{M} = \{(0,0,0,0)\}$. Hence, by LaSalle invariance principle, every trajectory in Ω_r will appraoch $(0,0,0,0)$ asymptotically. Note that when $\epsilon \to 0^+$, $\mathcal{R}_\epsilon \to (-\pi/2, \pi/2) \times \mathbb{R}$. As a result, we can enlarge the region of attraction by letting $\epsilon \to 0^+$. Since Ω_r is chosen to be compact, we also have exponential stability over Ω_r

3.3.2 The Ball and Beam System

Following the notations as in [15],[3] the ball and beam system has the following Lagrangian:

$$L(q, \dot{q}) = \frac{1}{2}((\dot{q}^1)^2 + (\ell^2 + (q^1)^2)(\dot{q}^2)^2) - g q^1 \sin q^2.$$

[3]With the exception of using ℓ instead of L for the length of the bar, to avoid confusion with the L for the Lagrangian.

See Figure 2 for the parameters. The mass matrix is given by

$$m = \begin{bmatrix} 1 & 0 \\ 0 & \ell^2 + (q^1)^2 \end{bmatrix},$$

and the potential energy is

$$V(q) = gq^1 \sin q^2.$$

Again the equilibrium point $(0, 0)$ is unstable and we apply the energy shaping method. The two matching conditions are

$$(\ell^2 + (q^1)^2)\widehat{T}_{11}\frac{\partial \widehat{T}_{11}}{\partial q^1} + \widehat{T}_{12}\left(\frac{\partial \widehat{T}_{11}}{\partial q^2} - \frac{2q^1\widehat{T}_{12}}{\ell^2 + (q^1)^2}\right) = 0$$

$$(\ell^2 + (q^1)^2)\widehat{T}_{11}\frac{\partial \widehat{V}}{\partial q^1} + \widehat{T}_{12}\frac{\partial \widehat{V}}{\partial q^2} - g(\ell^2 + (q^1)^2)\sin q^2 = 0.$$

Notice that we have a quadratic term for $\widehat{T}_{12}$ in the first matching condition. Hence, we may try an *ansatz* for $\widehat{T}_{12}$ first and then solve for $\widehat{T}_{11}$, assuming $\widehat{T}_{11} = \widehat{T}_{11}(q^1)$ so that Maple can handle it. We thus have the following general solutions:

$$\widehat{T}_{11} = A_0\sqrt{\ell^2 + (q^1)^2}$$

$$\widehat{T}_{12} = \frac{A_0}{\sqrt{2}}(\ell^2 + (q^1)^2).$$

For simplicity, we now take $A_0 = \sqrt{2}$ implying $\widehat{T}_{11} = \sqrt{2(\ell^2 + (q^1)^2)}$ and $\widehat{T}_{12} = \ell^2 + (q^1)^2$. The resulting potential energy, by solving the second matching condition, takes the form

$$\widehat{V} = g(1 - \cos q^2) + f\left(q^2 - \frac{1}{\sqrt{2}}\ln\frac{q^1 + \sqrt{\ell^2 + (q^1)^2}}{\ell}\right).$$

Again, we take $f(x) = x^2$ to ensure that $\widehat{V}$ has a minimum at $q = 0$. The positive definiteness requirement for $\widehat{T}$ is met by taking $\widehat{T}_{22}$ sufficiently large. Here we take

$$\widehat{T}_{22} = \sqrt{2}(\ell^2 + (q^1)^2)^{\frac{3}{2}}.$$

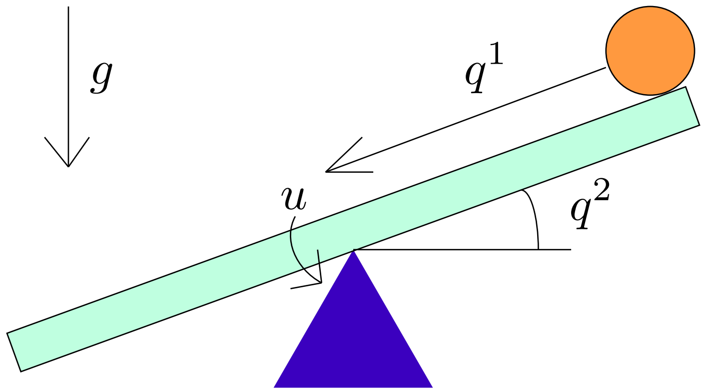

Figure 3.2: The ball and beam system.

Notice that the resulting $\widehat{T}$ is positive definite everywhere. The corresponding mass matrix is

$$\widehat{m} = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{\ell^2+(q^1)^2}} & -1 \\ -1 & \sqrt{2(\ell^2+(q^1)^2)} \end{bmatrix}.$$

With all these at hand, we can calculate the gyroscopic terms. By definition, we have

$$\begin{aligned} \widehat{S}_{111} &= 0 \\ \widehat{S}_{121} = \widehat{S}_{211} &= \frac{1}{\sqrt{2}}\sqrt{\ell^2+(q^1)^2}\,q^1 \\ \widehat{S}_{221} &= (\ell^2+(q^1)^2)q^1, \end{aligned}$$

for all $(i,j) \neq (1,1)$. Hence from the scheme detailed in Section 3.1, the $\widehat{A}_{ijk}$ terms can be chosen as follows:

$$\begin{aligned}\widehat{A}_{111} = \widehat{A}_{222} &= 0\\ \widehat{A}_{112} &= -\sqrt{2}\sqrt{\ell^2+(q^1)^2}q^1\\ \widehat{A}_{121} = \widehat{A}_{211} = \widehat{A}_{221} &= \frac{1}{\sqrt{2}}\sqrt{\ell^2+(q^1)^2}q^1\\ \widehat{A}_{122} = \widehat{A}_{212} &= -\frac{1}{2}\sqrt{\ell^2+(q^1)^2}q^1.\end{aligned}$$

We thus obtain the gyroscopic force terms as follows:

$$\begin{aligned}\widehat{C}_{111} = \widehat{C}_{222} &= 0\\ \widehat{C}_{112} &= -\frac{q^1}{\ell^2+(q^1)^2}\\ \widehat{C}_{121} = \widehat{C}_{211} &= \frac{1}{2(\ell^2+(q^1)^2)}\\ \widehat{C}_{122} = \widehat{C}_{212} &= 0\\ \widehat{C}_{221} &= 0\end{aligned}$$

Combining these gyroscopic force terms together, we can now obtain the expression for the gyroscopic force $\widehat{F} = [\widehat{F}_1, \widehat{F}_2]^T$:

$$\begin{aligned}\widehat{F}_1 &= \frac{q^1\dot{q}^1\dot{q}^2}{\ell^2+(q^1)^2}\\ \widehat{F}_2 &= -\frac{q^1(\dot{q}^1)^2}{\ell^2+(q^1)^2}\end{aligned}$$

Now, for the control force, we first compute the control bundle $\widehat{W}$ is spanned by

$$\begin{bmatrix}-\frac{1}{\sqrt{2(\ell^2+(q^1)^2)}}\\ 1\end{bmatrix}.$$

Hence, we can define the dissipative control force $\widehat{u}$ by

$$\widehat{u} = -k\begin{bmatrix}-\frac{1}{\sqrt{2(\ell^2+(q^1)^2)}}\\ 1\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{2(\ell^2+(q^1)^2)}} & 1\end{bmatrix}\begin{bmatrix}\dot{q}^1\\ \dot{q}^2\end{bmatrix} = k\begin{bmatrix}-\frac{1}{2(\ell^2+(q^1)^2)} & \frac{1}{\sqrt{2(\ell^2+(q^1)^2)}}\\ \frac{1}{\sqrt{2(\ell^2-(q^1)^2)}} & -1\end{bmatrix}\begin{bmatrix}\dot{q}^1\\ \dot{q}^2\end{bmatrix},$$

where k is any positive number. In what follows we take $k = 100$. We can then compute the corresponding control force u using (3.5), which gives

$$u = -\sqrt{\ell^2 + (q^1)^2}\left(50\sqrt{2}\dot{q}^2 - \frac{1}{\sqrt{2}}q^1(\dot{q}^2)^2 + \ln\frac{\ell}{q^1 + \sqrt{\ell^2 + (q^1)^2}} + \sqrt{2}g\sin q^2 + \sqrt{2}q^2\right)$$
$$+ gq^1\cos q^2 + 50\dot{q}^1 + q^1\dot{q}^1\dot{q}^2 - \frac{q^1(\dot{q}^1)^2}{\sqrt{2(\ell^2 + (q^1)^2)}}.$$

Again, we can apply LaSalle invariance principle to estimate the region of attraction for equilibrium $(q, \dot{q}) = (0, 0)$. Here we compare stability performance using our control law with the one using the linear quadratic regulator (LQR) approach. Through the LQR method we simply obtain a control law for the linearization of a given (possibly nonlinear) system, and apply this linear control law to the given system.

For simulation purpose, we take $\ell = 1$ and $g = 9.8$. For the LQR method, we take Q to be the 4×4 diagonal matrix and $R = 1$. The resulting controller parameters are

$$K = [K_1 \quad K_2 \quad K_3 \quad K_4] = [19.6007 \quad -20.1562 \quad 6.3506 \quad -6.3503].$$

For initial states which are close to the equilibrium, both methods can stabilize the system asymptotically, but the energy shaping method can stabilize at a faster rate. For instance, when the ball and beam system starts at $(q^1, q^2, \dot{q}^1, \dot{q}^2) = (0.7, 0.5, 0.3, 0.2)$, the energy shaping method can brings the whole system close to the equilibrium when $t > 40$ while the LQR method cannot reduce the oscillatory behaviour of the system until $t > 50$ (See Figure 3.3). Moreover, the energy shaping method gives a larger region of attraction.

When we switch the initial condition to $(q^1, q^2, \dot{q}^1, \dot{q}^2) = (0.8, 0.5, 0.3, 0.2)$, the system can still be asymptotically stabilized by the energy shaping method, but the ball simply go away from the pivot if LQR controller is employed (Figure 3.4). Notice that since $L = 1$ is the length of the beam, the ball should simply fall off from the beam once $q^1 > 1$.

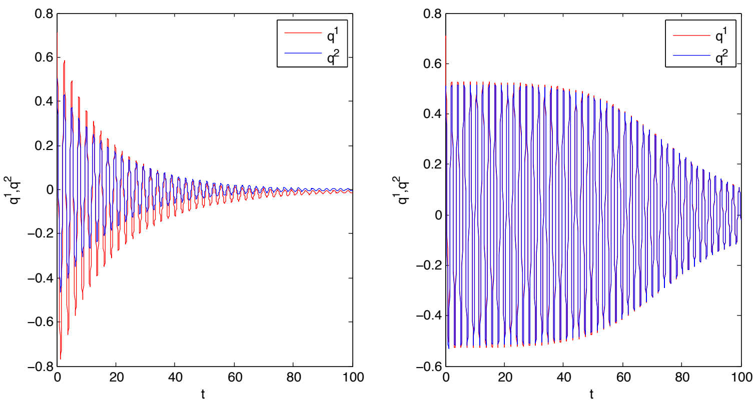

Figure 3.3: Stabilization of ball and beam system. Initial condition: $(q^1, q^2, \dot{q}^1, \dot{q}^2) = (0.7, 0.5, 0.3, 0.2)$. Left: Using energy shaping method; Right: Using LQR controller.

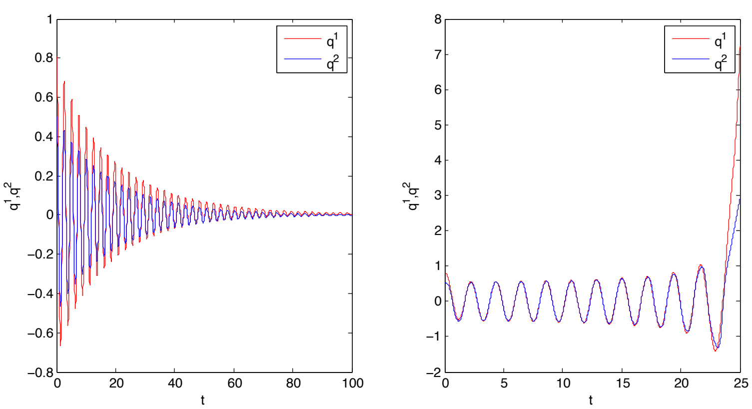

Figure 3.4: Stabilization of ball and beam system. Initial condition: $(q^1, q^2, \dot{q}^1, \dot{q}^2) = (0.8, 0.5, 0.3, 0.2)$. Left: Using energy shaping method; Right: Using LQR controller.

Chapter 4

Formal Theory of PDEs

In previous chapters we focus on the case where the degree of underactuation is one. A natural question is: how to solve the energy shaping problem when we have more joints in which we have no control?

Generally speaking, as the degrees of underactuation increase, the number of matching conditions increase. This implies that we have at least more than one PDEs for $\widehat{V}$, together with many more for those $\widehat{T}$ entries, such that the existence of solutions may not be as obvious as in the case where we only have one unactuated joint. A simple example can illustrate the problem. Consider the following system of two PDEs:

$$\mathcal{R}: \quad \begin{cases} \frac{\partial \widehat{f}}{\partial q^1} = g_1(q^1, q^2) \\ \frac{\partial \widehat{f}}{\partial q^2} = g_2(q^1, q^2) \end{cases}$$

where $\widehat{f} = \widehat{f}(q^1, q^2)$ is the unknown function to be solved and g_1, g_2 are given functions. Then, assuming f to be at least continuously differentiable, we must have $\dfrac{\partial^2 \widehat{f}}{\partial q^1 \partial q^2} = \dfrac{\partial^2 \widehat{f}}{\partial q^2 \partial q^1}$, and this equality of mixed partials, together with the given two PDEs, implies that we actually have one more PDE to satisfy, namely

$$\frac{\partial g_1}{\partial q^2} = \frac{\partial g_2}{\partial q^1}. \tag{4.1}$$

Moreover, it is obvious that not every pair of functions g_1 and g_2 will satisfy (4.1) automatically. In other words, it may happen the original system of PDEs may *not* have a solution, even though it only has two simple PDEs. We call (4.1) the *compatibility condition* for the given system $\mathcal{R}$.

An even more complicated situation arises when we make just small changes on the original system:

$$\overline{\mathcal{R}}: \quad \begin{cases} \frac{\partial \widehat{f}}{\partial q^1} = g_1(\widehat{f}, q^1, q^2) \\ \frac{\partial \widehat{f}}{\partial q^2} = g_2(\widehat{f}, q^1, q^2) \end{cases}$$

This time g_1 and g_2 not only depend on q^1, q^2 but also denote general expressions containing $\widehat{f}$. With the mixed partials at mind, we should notice there should be a "hidden" PDE, i.e.

$$\frac{\partial g_1}{\partial q^2}(\widehat{f}, q^1, q^2) = \frac{\partial g_2}{\partial q^1}(\widehat{f}, q^1, q^2), \tag{4.2}$$

which looks similar to (4.1), but in fact a crucial difference between the two is, with (4.2), we now have an extra equation governing the unknown function $\widehat{f}$. We call (4.1) the *integrability condition*, a "hidden" equation which contains the unknown(s) to be solved.

From this simple illustrative example, it goes without saying that we need a systematic approach to work with a general system of PDEs, and in particular, to find out all integrability (and/or compatibility) conditions, before we can properly handle the energy shaping problem with more unactuated joints. The formal theory of PDEs serves this purpose.

4.1 The Setup of the Formal Theory of PDEs

The formal theory is based on the jet bundle formalism. Consider two manifolds $\mathcal{E}$ and Q with dimensions $n + m$ and n respectively, with $\pi : \mathcal{E} \to Q$ a surjective mapping. We first introduce the following three notions which appear in later discussion:

Fibered manifold: $\mathcal{E}$ is a *fibered manifold* over Q with projection π if there exists coordinate charts (U, Φ) of $\mathcal{E}$ projected onto coordinate charts (V, ϕ) of Q with the following commutative diagram:

$$\begin{tikzcd}
\mathcal{E} \arrow[d, "\pi"] & \supseteq & U \arrow[r, "\Phi"] \arrow[d] & \mathbb{R}^n \times \mathbb{R}^m \arrow[d] \\
Q & \supseteq & V \arrow[r, "\phi"] & \mathbb{R}^n
\end{tikzcd}$$

Morphism: Suppose we have two fibered manifolds $\pi_i : \mathcal{E}_i \to Q_i$ with $i = 1, 2$, a *fibered morphism* $\Phi : \mathcal{E}_1 \to \mathcal{E}_2$ over $\phi : Q_1 \to Q_2$ is a pair of maps (Φ, ϕ) with the following commutative diagram:

$$\begin{tikzcd}
(q, p) \arrow[d, mapsto] & \in & \mathcal{E}_1 \arrow[r, "\Phi"] \arrow[d, "\pi_1"] & \mathcal{E}_2 \arrow[d, "\pi_2"] & \ni & (\phi(q), \Phi(q, p)) \arrow[d, mapsto] \\
q & \in & Q_1 \arrow[r, "\phi"] & Q_2 & \ni & \phi(q)
\end{tikzcd}$$

Fibered submanifold: $\mathcal{R} \to Q$ is a *fibered submanifold* of $\pi : \mathcal{E} \to Q$ if $\mathcal{R}$ is a submanifold of $\mathcal{E}$ and the inclusion map is a morphism.

We then treat the derivatives of the dependent variables with respect to the independent ones as additional, algebraically independent variables. This gives rise to a fibered manifold $\pi : \mathcal{E} \to Q$ with independent variables $q^1, ..., q^n$ as coordinates of the base space Q and the dependent variables $u^1, ..., u^m$ as fiber coordinates. We can then construct the r-th jet bundle $J_r\mathcal{E}$ for $r \geq 1$ in which the fiber coordinates consist of $u^1, ..., u^m$ together with their derivatives up to order r. The canonical projection is denoted as $\pi_r^{r+s} : J_{r+s}\mathcal{E} \to J_r\mathcal{E}$.

Over each bundle we can define a section and its prolongation. A section is a map $\sigma : Q \to \mathcal{E}$ such that $\pi \circ \sigma = \mathrm{id}_Q$. In other words, in local coordinates σ should appear as

$$\sigma : \; q \mapsto (q, f(q)),$$

for some function f. Then the r-th prolongation of a section σ can be done locally by adding derivatives up to order r, i.e.

$$j_r(\sigma) : q \to \left(q, f(q), \frac{\partial^{|\mu|} f(q)}{(\partial q^1)^{\mu_1} \cdots (\partial q^n)^{\mu_n}} \right),$$

where $\mu = (\mu_1, ..., \mu_n), 1 \leq |\mu| = \mu_1 + ... + \mu_n \leq r$. For the sake of convenience, we often express the mixed partials in the following condensed form:

$$p^\alpha_\mu = \frac{\partial^{|\mu|} f(q)}{(\partial q^1)^{\mu_1} \cdots (\partial q^n)^{\mu_n}}.$$

Definition 4.1.1 *A partial differential equation (PDE) of order r is a fibered submanifold $\mathcal{R}_r$ of $J_r\mathcal{E}$. A solution to $\mathcal{R}_r$ is a section σ such that $j_r(\sigma)$ lies in $\mathcal{R}_r$.*

Quite often the PDE is defined as a map $\Phi : J_r\mathcal{E} \to \mathcal{E}'$ where $\Phi = \Phi^\tau(q^i, u^\alpha, p^\alpha_\mu)$ and $\mathcal{E}'$ is another bundle over Q.

4.1.1 Two Basic Operations on Jet Bundles: Prolongations and Projections

Given a system $\mathcal{R}_r$ of PDEs of order r, we can have the following two basic operations:

Prolongation: Imitating the usual chain rule of differentiation, we define the formal derivative $D_i\Phi$ for Φ by

$$D_i\Phi(q^i, u^\alpha, p^\alpha_\mu) = \frac{\partial \Phi^\tau}{\partial q^i} + \sum_\alpha \frac{\partial \Phi^\tau}{\partial u^\alpha} p^\alpha_i + \sum_{\alpha,\mu} \frac{\partial \Phi^\tau}{\partial p^\alpha_\mu} p^\alpha_{\mu+1_i},$$

where $\mu + 1_i = (\mu_1, ..., \mu_{i-1}, \mu_i + 1, \mu_{i+1}, ..., \mu_n)$.

We define the prolongation $\mathcal{R}_{r+1} \subseteq J_{r+1}\mathcal{E}$ for $\mathcal{R}_r$ as the set of PDEs:

$$\mathcal{R}_{r+1} : \qquad \Phi^\tau = 0, \quad D_i\Phi^\tau = 0, \qquad i = 1, ..., n.$$

Notice that the prolonged system $\mathcal{R}_{r+1}$ is *not* necessarily a fibered submanifold.[1] We can generalize the concept of prolongation to define the s-th prolongation $\mathcal{R}_{r+s}$ of $\mathcal{R}_r$ by

$$\mathcal{R}_{r+s} := J_s(\mathcal{R}_r) \cap J_{r+s} \subseteq J_s(J_r\mathcal{E})$$

The intersection means we identify derivatives of the derivatives in $J_r(\mathcal{R}_q)$ [2] with the derivatives of the original u^α, otherwise we must distinguish mixed higher order derivatives, which is not necessary in most circumstances.

When $\mathcal{R}_r = \ker_{f'}\Phi$ [3] for some section $f' : Q \to \mathcal{E}$, we have

$$\mathcal{R}_{r+s} = \ker_{j_s(f')}\rho_s(\Phi)^4$$

This justifies our usage of "prolongation" of a system of PDEs.

Projection: We can also project higher order PDEs into lower order ones. This is done by Gaussian elimination of higher order derivatives by the lower order ones in the equation.

In general, the resulting system of PDEs arising from prolongations of $\mathcal{R}_r$ up to order s followed by projections into $\mathcal{R}_r$, that is, $\pi_r^{r+s}(\mathcal{R}_{r+s})$ is usually denoted as $\mathcal{R}_r^{(s)}$. As mentioned at the beginning of this chapter, prolongation followed by projection does not necessarily retrieve the original system. This fact can be mathematically summarized by the following simple set inclusion:

$$\mathcal{R}_r^{(s)} \subsetneqq \mathcal{R}_r$$

[1] This means that the prolonged system may not have a constant dimension as q varies. We say a system of PDEs is regular if it is a fibered submanifold.

[2] e.g. $u^\alpha_{1,2}$, i.e. differentiate $u^\alpha \in \mathcal{E}$ with respect to x_1 and then differentiate $u^\alpha_1 \in J_1\mathcal{E}$ with respect to x_2

[3] If $\Phi : \mathcal{E}' \to \mathcal{E}$ is a morphism and $f' : Q \to \mathcal{E}$, then $\ker_{f'}\Phi = \{(x, y) \in \mathcal{E}' \mid \Phi(q, p) = f'(q)\}$.

[4] The s-prolongation $\rho_s(\Phi) : J_{r+s}\mathcal{E} \to J_s(\mathcal{E}')$ of Φ is the unique morphism such that $\rho_s(\Phi) \circ j_{r+s} = j_s \circ \mathcal{D}$, where $\mathcal{D} = \Phi \circ j_r$.

Example 1 Consider a system $\mathcal{R}_1$ defined by

$$\mathcal{R}_1 : \begin{cases} u_1 + q^2 u_3 = 0 \\ u_2 = 0 \end{cases},$$

where $u_i,\ i = 1, 2, 3$ stands for partial derivative with respect to q^i. Its first prolongation $\mathcal{R}_2$ is then given by

$$\mathcal{R}_2 : \begin{cases} u_{13} + q^2 u_{33} = 0 \\ u_{12} + u_3 + q^2 u_{23} = 0 \\ u_{11} + q^2 u_{13} = 0 \\ u_{23} = 0 \\ u_{22} = 0 \\ u_{12} = 0 \\ u_1 + q^2 u_3 = 0 \\ u_2 = 0 \end{cases}$$

Hence, besides the defining equations for $\mathcal{R}_1$, $\mathcal{R}_2$ (and hence $\mathcal{R}_1^{(1)}$) has an extra equation $u_3 = 0$, arising from eliminating the second order partials in the second PDE in the system $\mathcal{R}_2$ using the 4th and 6th PDEs from the same system. This extra equation is known as an integrability condition to $\mathcal{R}_1$. It should be noted that integrability conditions cannot be obtained by purely algebraic manipulations on the original system of PDEs; prolongations and projections are required in order to derive integrability conditions. ■

4.1.2 The Concept of Formal Series Solution

We assume that a solution can be expressed in terms of formal power series around a fixed $q_0 \in Q$:

$$u^\alpha(q) = \sum_{|\mu|=0}^{\infty} \frac{a_\mu^\alpha}{\mu!}(q - q_0)^\mu,$$

where $\mu! = \mu_1!...\mu_n!$ and $(q - q_0)^\mu = (q^1 - q_0^1)^{\mu_1}...(q^n - q_0^n)^{\mu_n}$. Substituting this series into a local representation of $\mathcal{R}_r$ and $\mathcal{R}_{r+s}$ yields infinitely many algebraic equations:

$$\begin{array}{rcr}
\mathcal{R}_r : & \Phi^\tau(q_0, a_\mu^\alpha) = 0, & 0 \le |\mu| \le r \\
\mathcal{R}_{r+1} : & (D_i\Phi^\tau)(q_0, a_\mu^\alpha) = 0, & 0 \le |\mu| \le r+1 \\
\mathcal{R}_{r+s} : & (D_\nu\Phi^\tau)(q_0, a_\mu^\alpha) = 0, & 0 \le |\mu| \le r+s, \quad 0 \le |\nu| \le s
\end{array}$$

If $\mathcal{R}_r$ does not generate any integrability conditions, then solving a_μ^α satisfying the above equations will suffice to find a formal solution.

The existence of integrability conditions, however, makes finding the formal solutions termwise rather difficult. This is because in this situation we have to prolong and project a number of times to get all possible integrability conditions of lower orders before we start to determine each coefficient of the formal series. With regard to this, we introduce the idea of formally integrable equations which behave so nicely that the construction of formal solution is made possible.

Definition 4.1.2 *A system $\mathcal{R}_r$ of PDE of order r is* formally integrable *if $\mathcal{R}_{r+s}$ is a fibered manifold for all $s \ge 0$ and $\pi_{r+s}^{r+s+t} : \mathcal{R}_{r+s+t} \to \mathcal{R}_{r+s}$ are epimorphisms for all $s, t \ge 0$.*

The above definition means that for a formally integrable system, we do not further integrability conditions, no matter how many prolongations and projections are carried out on that system.

4.1.3 Symbol

A direct verification of formal integrability as defined in Definition 4.1.2 is difficult computationally, as we have to check infinitely many times whether the projections are epimorphisms. It turns out that, nevertheless, simpler criteria for formal integrability exist so that we can bypass the process of checking epimorphisms infinitely many times. These simpler criteria are partly related to an algebraic property of the highest order derivatives involved in the system, known as *involutivity*. In this section we first construct the symbol for a system of PDEs which consists of the highest order derivatives only, and leave the symbol involutivity and related concepts in subsequent sections.

Usually defining the symbol of a given system is done in a coordinate-free manner, but this approach needs a lot of terminologies and (mainly cohomological) tools which may not be used in actual computations. We therefore avoid this approach and define the symbol using a set of coordinates:

Definition 4.1.3 *The* symbol G_r *of a system* $\mathcal{R}_r$ *is defined to be a family of vector spaces whose local representation is*

$$G_r : \qquad \sum_{|\mu|=r} \frac{\partial \Phi^\tau}{\partial p^\alpha_\mu}(q^i, u^\beta, p^\gamma_\mu) v^\alpha_\mu = 0,$$

where $\tau = 1, \ldots, p;\ \alpha, \beta, \gamma = 1, \ldots, m$ *and* v^α_μ *is the* $|\mu|$*-th vertical differentiation of* u^α [23][5]*, when* $\mathcal{R}_r$ *is locally represented as* $\Phi^\tau(q^i, u^\beta, p^\gamma_\mu) = 0$.

For readers interested in coordinate-free definition of symbol, see [23]. Note that in general the symbol is just a family of vector spaces, not necessarily a *vector bundle* over Q.

[5]Basically the theorem says that a symbol is simply defined by the equations from the original system of PDEs, with lower order terms removed. Since the original unknowns u^α_μ usually do not satisfy the equations defining G_r, we introduce a corresponding notation v^α_μ, which is called the vertical derivative of u^α. In this thesis (and in particular in Chapter 5), we do not distinguish the usual prolongation of a dependent variable (i.e. u^α_μ) with its vertical differentiation (v^α_μ), for if otherwise it might cause confusion by naming too many variables.

By definition, the s-th prolongation $\mathcal{R}_{r+s}$ of a system $\mathcal{R}_r$ of PDEs also has a symbol, denoted as G_{r+s}, which can be easily derived with the knowledge of $\mathcal{R}_r$:

Theorem 4.1.4 ([23]) *The symbol G_{r+s} of $\mathcal{R}_{r+s}$, $s \geq 0$, depends only on G_r by a direct prolongation procedure.*

Proof Express $\mathcal{R}_{r+s}$ as $D_\nu \Phi^\tau = 0$, $0 \leq |\nu| \leq s$, then by formal differentiation of Φ, we have

$$D_i \Phi^\tau = \frac{\partial \Phi^\tau}{\partial p^k_\mu} p^k_{\mu+1_i} + \text{lower order terms}, \qquad |\mu| = r$$
$$D_{ij} \Phi^\tau = \frac{\partial \Phi^\tau}{\partial p^k_\mu} p^k_{\mu+1_i+1_j} + \text{lower order terms}, \qquad |\mu| = r$$

In general, G_{r+s} is thus given by

$$\frac{\partial \Phi^\tau}{\partial p^k_\mu}(q^i, u^\alpha_k, p^\alpha_\mu) v_{\mu+\nu} = 0,$$

where $|\mu| = r$, $|\nu| = s$ and $(q^i, u^\alpha_k, p^\alpha_\mu) \in \mathcal{R}_r$. ■

Example 2 Consider the following system of PDEs of order two:

$$\mathcal{R}_2 : \begin{cases} u_{12} + q^1 u_1 = 0 \\ q^2 u_{11} + u_{22} = 0 \\ u_1 + q^2 u_2 + u = 0 \end{cases}$$

Its symbol is given by

$$G_2 : \begin{cases} U_{12} = 0 \\ q^2 U_{11} + U_{22} = 0 \end{cases}$$

We can prolong the system $\mathcal{R}_2$ to $\mathcal{R}_3$ from which we can derive the symbol G_3:

$$\mathcal{R}_3 : \begin{cases} u_{122} + q^1 u_{12} = 0 \\ u_{112} + q^1 u_{11} + u_1 = 0 \\ q^2 u_{112} + u_{11} + u_{222} = 0 \\ q^2 u_{111} + u_{122} = 0 \\ u_{12} + q^2 u_{22} + u_2 + u_2 = 0 \\ u_{11} + q^2 u_{12} + u_1 = 0 \end{cases} \quad \Rightarrow \quad G_3 : \begin{cases} U_{122} = 0 \\ U_{112} = 0 \\ q^2 U_{112} + U_{222} = 0 \\ q^2 U_{111} + U_{122} = 0 \end{cases}$$

But it is also clear that this symbol G_3 can also be obtained by directly prolonging G_2. ■

The symbol G_r provides a simple criterion to check whether extra integrability condition(s) will occur:

Theorem 4.1.5 ([23, 27]) *If G_{r+1} is a vector bundle, then* $\dim \mathcal{R}_r^{(1)} = \dim \mathcal{R}_{r+1} - \dim G_{r+1}$.

Proof The vector bundle assumption of G_{r+1} is to ensure that our dimension argument works pointwise throughout the base manifold.

Suppose $\mathcal{R}_r$ is locally described as $\Phi^\tau(q^i, u^\alpha, p^\alpha_\mu) = 0$, then

$$\mathcal{R}_{r+1} : \begin{cases} D_i \Phi^\tau(q^i, u^\alpha, p^\alpha_\mu) = 0 \\ \Phi^\tau(q^i, u^\alpha, p^\alpha_\mu) = 0 \end{cases}$$

$$G_{r+1} : \sum_{\alpha, |\mu|=r} \frac{\partial \Phi^\tau}{\partial p^\alpha_\mu} v^\alpha_{\mu+1_i} = 0$$

Hence, the Jacobian of $\mathcal{R}_{r+1}$ is given as follows (arranged in the descending order of derivatives):

$\dfrac{\partial D_i\Phi^\tau}{\partial p^\alpha_\mu}$, $\lvert\mu\rvert = r+1$	$\dfrac{\partial D_i\Phi^\tau}{\partial p^\alpha_\mu}$, $\lvert\mu\rvert \le r$	$\dfrac{\partial D_i\Phi^\tau}{\partial u^\alpha}$
0	$\dfrac{\partial \Phi^\tau}{\partial p^\alpha_\mu}$, $\lvert\mu\rvert \le r$	$\dfrac{\partial \Phi^\tau}{\partial u^\alpha}$

The lower part of blocks is the Jacobian of $\mathcal{R}_r$, hence its rank = codim$\mathcal{R}_r$. For the upper part, notice that the leftmost block is the matrix associated with G_{r+1}. Now, we have 2 possibilities:

Case I G_{r+1} has maximal rank: In this case, $\dim \mathcal{R}_{r+1} = \dim \mathcal{R}_r + \dim G_{r+1}$. Thus, $\dim \mathcal{R}_r^{(1)} = \dim \mathcal{R}_r$.

Case II Otherwise, by row reductions on the upper part of the Jacobian of $\mathcal{R}_{r+1}$ we can obtain some rows with only zeros in the leftmost block. For these rows in the two blocks to the right,

- if it is independent of the rows in the lower part, then we have obtained the integrability conditions;
- otherwise, we get redundant equations which will become identities or compatibility conditions. ■

4.2 Involutive Symbols and Computations

We are now at the stage to explain when will the symbol for a given system of PDEs become *involutive*. Again, we resort to coordinate-dependent approach, though the concept of involutivity is independent of the choice of coordinates. [6]

4.2.1 Involutive Symbols for Solved Systems

We need a specific way of categorizing and prioritizing derivatives. First, we fix a set of local coordinates $q^1, \ldots, q^n$ on Q. In what follows, T^*Q is abbreviated as T^* for simplicity.

Definition 4.2.1 *With local coordinates $q^1, \ldots, q^n$, we can define the following:*

1. *A jet coordinate v^k_μ is said to be class 1 if $\mu_1 \neq 0$. In general, it is of class i if $\mu_1 = \ldots = \mu_{i-1} = 0$ but $\mu_i \neq 0$.*

2. *For $1 \leq i \leq n$, $(S^rT^*)^i$ is defined to the subset of S^rT^* obtained by equating all the class i jet coordinates to zero.*

3. *Given a symbol G_r, we define for any $1 \leq i \leq n$, $(G_r)^i = G_r \cap (S^rT^*)^i \otimes E$, where E is the vertical bundle of $\mathcal{E}$.*[7]

Now, we can solve the linear system defining G_r pointwise in the following manner. We first solve G_r with respect to the maximum number of components of class n, and replace these in the remaining equations. By so doing, only components of class i, where i is at most $n-1$, are left. Then we solve the remaining equations with respect to the maximum number of components of class $n-1$, leaving only components of class i with $i \leq n-2$. We repeat the above steps until we come to class 1 components. We say that the linear system for G_r is *solved*. In each class i equation in its solved form, where $1 \leq i \leq n$, the

[6] The coordinate-free approach is usually done with the concept of Spencer δ-map and its cohomologies. For details, see [23].

[7] Without going into the technical details of vertical bundles, one can treat $S^rT^* \otimes E$ as the vector bundle of r-th order derivatives, and $(S^rT^*)^i \otimes E$ is the vector subbundle of $S^rT^* \otimes E$ consisting of the r-th order derivatives of class i.

component of class i which is a linear combination of other components of class $\leq i$, is called the principal derivative, and the rest of other components are called parametric:

$$\begin{bmatrix} \text{principal} \\ \text{component} \\ \text{of class } i \end{bmatrix} + A(q^i, u^\beta, p^\gamma_\mu) \begin{bmatrix} \text{parametric} \\ \text{components} \\ \text{of class } \leq i \end{bmatrix} = 0.$$

It can be said that the central idea of a solved form is Gaussian elimination, where each of the principal derivatives of class i serves as a pivot for its associated class i equation. With all these at hand, we can then easily determine the size of $(G_r)^i$:

$$\dim(G_r)^i = \dim(S^r T^* \otimes E)^i - (\beta_r^{i+1} + \ldots + \beta_r^i), \qquad 1 \leq i \leq n,$$

where β_r^i is the number of equations of class i.

Theorem 4.2.2 ([23]) *For any fixed local coordinates, we have*

$$\dim G_{r+1} \leq \alpha_r^1 + 2\alpha_r^2 + \ldots + n\alpha_r^n, \tag{4.3}$$

where $\alpha_r^i = \dim(G_r)^{i-1} - \dim(G_r)^i$.

Proof Since by definition, $(G_{r+s})^{i-1} \supseteq (G_{r+s})^i$, $\alpha_{r+s}^i \geq 0$. Telescoping terms and making use of the fact $(G_{r+s})^n = 0$, we have

$$\alpha_{r+s}^i + ... + \alpha_{r+s}^n = \dim(G_{r+s})^{i-1} \tag{4.4}$$

$$\alpha_{r+s}^1 + ... + \alpha_{r+s}^n = \dim\ G_{r+s} \tag{4.5}$$

Meanwhile, it is known that[8]

$$\alpha_{r+s}^i \leq \dim(G_{r+s-1})^{i-1} \tag{4.6}$$

[8]This is usually done by counting dimensions in the following exact sequence:

$$0 \longrightarrow (G_{q+r})^i \longrightarrow (G_{q+r})^{i-1} \xrightarrow{\delta_i} (G_{q+r-1})^{i-1}$$

where δ_i is related to the Spencer δ-map. See [23] for details.

Hence, combining (4.4), (4.5) and (4.6), we have

$$\begin{aligned}
\dim G_{r+s} &\leq \dim G_{r+s-1} + ... + \dim(G_{r+s-1})^i + ... + \dim(G_{r+s-1})^{n-1} \\
&= (\alpha^1_{r+s-1} + ... + \alpha^n_{r+s-1}) + ... + (\alpha^i_{r+s-1} + ... + \alpha^n_{r+s-1}) + ... + \alpha^n_{r+s-1} \\
&= \alpha^1_{r+s-1} + 2\alpha^2_{r+s-1} + ... + n\alpha^n_{r+s-1}
\end{aligned}$$

In particular, when $r = 1$

$$\dim G_{r+s} \leq \alpha^1_q + 2\alpha^2_q + ... + n\alpha^n_q. \qquad \blacksquare$$

We now come to the long-awaited definition of symbol involutivity:

Definition 4.2.3 *The symbol G_r is* involutive *if there exist local coordinates such that the equality holds in* (4.3). *Such local coordinates are called δ-regular.*

4.2.2 Multiplicative Variables

Besides dimension-counting, there is still another method, which resembles the row reduction with pivots, to check the involutiveness of G_r. The central idea comes from the Janet-Riquier theorem [24, 26, 27].

Suppose the the system of PDEs is already in its solved form. For each row of class i, we name $q^1, q^2, \cdots, q^n$ the associated *multiplicative* variables, while all others are called non-multiplicative for that row. Then it turns out that we have yet another equivalent way of checking symbol involutivity:

Theorem 4.2.4 ([23]) *G_r is involutive in the sense of Definition 4.2.3 if and only if we obtain all independent equations of order $r+1$ of the prolongation G_{r+1} by differentiating each equation of G_r with respect to multiplicative variables only (Equivalently, prolongation with respect to non-multiplicative variables does not contribute any new equations).*

Proof Without loss of generality we can assume G_r is already in its solved form. Then, all equations obtained by prolonging each equation in G_r with respect to its multiplicative variables only are independent, as they all have distinct pivots.[9] Since by definition there are β_r^i equations of class i in G_r, we have at least $\sum_{i=1,\dots,n} i\beta_r^i$ independent equations (of order $r+1$) in G_{r+1}. Meanwhile, the number of independent equations in G_{r+1} is rank G_{r+1}, and hence

$$\begin{aligned}\dim G_{r+1} &= mC_{n-1}^{r+n} - \text{rank } G_{r+1} \\ &\leq mC_{n-1}^{r+n} - (\beta_r^1 + 2\beta_r^2 + \dots + n\beta_r^n) \\ &= \alpha_r^1 + 2\alpha_r^2 + \dots + n\alpha_r^n\end{aligned}$$

The last equality is due to a combinatorial argument. Hence, G_r is involutive if and only if we get independent equations of G_{r+1} only by prolonging with respect to multiplicative variables. ■

Example 3 ([23]) Consider the following solved symbol G_2:

$U_{45} - U_{13} = 0$	1	2	3	4	×
$U_{35} - U_{12} = 0$	1	2	3	•	×
$U_{33} - U_{24} = 0$	1	2	3	×	•
$U_{25} - U_{11} = 0$	1	2	•	•	×
$U_{23} - U_{14} = 0$	1	2	•	×	•
$U_{22} - U_{13} = 0$	1	2	•	•	•

The dot board on the right hand side indicates the class of each equation. Numbers $1, 2, 3, \cdots$ denote the multiplicative variables $q^1, q^2, q^3, \cdots$ while × denote the prolongations

[9]Indeed, if u_μ is of class i, then $u_{\mu+1_i}$ with $s \leq i$ is of class $s \leq i$. So, it can only be the prolongation of a component of class s with respect to multiplicative variables of index $\leq s$. This is a contradiction unless $s = i$, in which case we get nothing.

which are not retrievable by prolongations of other multiplicative variables.[10] Alternatively, one can count α_2^i to check if the equality in (4.3) holds (for $r = 2$). In particular, we have the following free variables:

$$\begin{array}{ll} \text{class 1:} & \{v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\} \\ \text{class 2:} & \{v_{24}\} \\ \text{class 3:} & \{v_{34}\} \\ \text{class 4:} & \{v_{44}\} \\ \text{class 5:} & \{v_{55}\} \end{array}$$

Hence, we have $\alpha_2^1 = 5$ and $\alpha_2^i = 1$ for all $i \geq 2$. One can then find that the equality in (4.3) does not hold, as $\sum_{i=1}^5 i\alpha_2^i = 19$ while $\dim G_3 = 13$. Then by Definition 4.2.3, the coordinates are not δ-regular.

Now, adopt the following change of coordinates: $q^1 \mapsto q^5, q^2 \mapsto q^4$ etc. so that G_2 becomes

$$\begin{array}{ll} (4) & U_{55} - U_{14} = 0 \\ (2) & U_{45} - U_{13} = 0 \\ (6) & U_{44} - U_{35} = 0 \\ (1) & U_{35} - U_{12} = 0 \\ (5) & U_{34} - U_{25} = 0 \\ (3) & U_{33} - U_{24} = 0 \end{array} \qquad \boxed{\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & \bullet \\ 1 & 2 & 3 & 4 & \bullet \\ 1 & 2 & 3 & \bullet & \bullet \\ 1 & 2 & 3 & \bullet & \bullet \\ 1 & 2 & 3 & \bullet & \bullet \end{array}}$$

Then one can check that no prolongation with respect to the "dots" leads to new equations. As a result, G_2 is involutive. Notice that although computationally speaking, one needs to find a set of coordinates under which "dot" prolongations give nothing new, symbol involutivity is a concept which is *independent of the choice of coordinates*. Again, one can

[10] [23] does not distinguish prolongations with respect to non-multiplicative variables which lead to new equations, and those which do not. Here we introduce a notation $\times$ simply for illustration.

also show G_2 is involutive by checking the (4.3). This time $\alpha_2^1 = 5$, $\alpha_2^2 = 4$ while the rest are all zero. Hence, the equality in (4.3) holds. ■

4.3 Involutive Systems

We now come back to the original system of PDEs and describe the relation between formal integrability and the involutivity of its symbol. We first define an involutive system of PDE as follows:

Definition 4.3.1 *A system $\mathcal{R}_r \subseteq J_r\mathcal{E}$ of order r on $\mathcal{E}$ is involutive if it is formally integrable and its symbol G_r is involutive.*

The assumption that G_r is involutive in the above definition is necessary because there exists a system $\mathcal{R}_r$ which is formally integrable but its symbol is not involutive:

Example 4 ([27]) Consider the following simple system of PDEs:

$$\mathcal{R}_2 : \begin{cases} u_{11} = 0 \\ u_{22} = 0 \end{cases},$$

where u_{11} and u_{22} are the shorthand for $u_{q^1q^1}$ and $u_{q^2q^2}$ respectively. Since in the symbol G_2, there is only one equation of class 1 and one equation of class 2, we have $\alpha_2^1 = 0$, $\alpha_2^2 = 1$. Meanwhile, $\dim G_3 = 0$ since $u_{111} = u_{112} = u_{122} = u_{222} = 0$.[11] Hence, $\dim G_3 < \alpha_2^1 + 2\alpha_2^2$, meaning that G_2 is not involutive.

Further prolongation gives $\dim G_4 = 0$ as we have $u_{1111} = u_{1112} = u_{1122} = u_{1222} = u_{2222} = 0$ in G_4. Hence, $\dim G_4 = \alpha_3^1 + 2\alpha_2^2$ and G_4 is involutive.

[11]Here without ambiguity, we do not distinguish the "vertical" differentiation (in the vertical bundle) with the usual differentiation.

Since G_3 does not generate any integrability conditions (i.e. extra equations of order ≤ 2), $\mathcal{R}_2$ is formally integrable. Thus, we have a system of equations which is formally integrable but not involutive. $\blacksquare$

Further analysis of the action of prolongations and projections leads to the following important and useful theorem:

Theorem 4.3.2 (Criterion of involutiveness [23, 24]) *Let $\mathcal{R}_r \subseteq J_r\mathcal{E}$ be a system of order r over $\mathcal{E}$ such that $\mathcal{R}_{r+1}$ is a fibered submanifold of $J_{r+1}\mathcal{E}$. If G_r is involutive and if the map $\pi_r^{r+1} : \mathcal{R}_{r+1} \to \mathcal{R}_r$ is an epimorphism, then $\mathcal{R}_r$ is involutive.*[12]

This theorem is useful in the sense that we now have a *finite* test of formal integrability.[13] In order to make use of this theorem, we have to, at the very first step, make sure that G_r is involutive. Involutiveness of symbol, however, is not always true for any given system of PDE, and in case it is indeed not involutive, we still have the following important fact at hand:

Theorem 4.3.3 (Prolongation Theorem [30]) *For the symbol G_r of any system $\mathcal{R}_r$, there exists an integer $\widehat{r}(n, m, r) \geq r$ such that $G_{\widehat{r}}$ is involutive, where $\widehat{r}$ depends on n, the number of independent variables (i.e. $q^1, ..., q^n$), and m, the number of components (i.e. $u^1, ..., u^m$).*

The bound $\widehat{r}$ is rather conservative and grows exponentially [30] as m and n increase. Nevertheless, this theorem ensures theoretically that we can always get an involutive symbol after *finitely many* times of prolongations.

In general, we have a scheme which can generate an involutive system:

[12] If we only assume G_r is 2-acyclic (a looser condition than involutiveness), then $\mathcal{R}_r$ is already formally integrable [23]. Since in real applications it is hard to check 2-acyclicity, we avoid to state the theorem in this way.

[13] "finite" in the sense that by the very definition of formal integrability, we need to check for infinitely many projections $\pi_{q+r}^{q+r+s} : \mathcal{R}_{q+r+s} \to \mathcal{R}_{q+r}$.

Theorem 4.3.4 (Cartan-Kuranishi theorem,[18, 23, 24, 27]) *For every strongly regular system*[14] $\mathcal{R}_r$ *of order* r*, there exist two integers* s *and* t *such that* $\mathcal{R}^{(t)}_{r+s}$ *is involutive and has the same solution space as* $\mathcal{R}_r$*.*

Here is the general procedure for constructing this $\mathcal{R}^{(s)}_{q+r}$:

1. We start from $\mathcal{R}_q$ and compute G_q. If G_q is involutive, then consider whether $\mathcal{R}^{(1)}_q = \mathcal{R}_q$ by checking the dimensions, making use of Theorem 4.1.5 if necessary.
   - If the equality holds, then we are done.
   - Otherwise, replace $\mathcal{R}_q$ by $\mathcal{R}^{(1)}_q$ and start again.

   If G_q is not involutive, then go to 2.

2. If G_q is not involutive, compute G_{q+1}.
   - If G_{q+1} is involutive, go back to 1., by replacing q by $q+1$.
   - If not, prolong again until we get an involutive G_{q+r} for some r. Then, go back to 1. using this symbol G_{q+r}.

Though this procedure in general generates an iterated system

$$(...((((\mathcal{R}_{q+r_1})^{(1)})_{+r_2})^{(1)})...)_{+r_s},$$

it is actually in the form of $\mathcal{R}^{(s)}_{q+r}$ since by the prolongation theorem, G_{q+r} is ultimately involutive[15]. This procedure must stop after finitely many steps by using a Noetherian argument.[23]

As mentioned before, an involutive system allows one to construct a formal series solution. This can be summarized by the famous Cartan-Kähler theorem:

[14] A system $\mathcal{R}_r$ is called strongly regular if $\mathcal{R}^{(t)}_{r+s}$ is a fibered manifold and the symbol $G^{(t)}_{r+s}$ of $\mathcal{R}^{(t)}_{r+s}$ is a vector bundle over Q for all $s, t \geq 0$ [24].

[15] When G_{q+r} is involutive, then G_{q+r+1} is also involutive and $(\mathcal{R}^{(1)}_{q+r})_{+1} = \mathcal{R}^{(1)}_{q+r+1}$. [23]

Theorem 4.3.5 (Cartan-Kähler theorem,[23]) *If $\mathcal{R}_r$ is an involutive and analytic system of order r, then there exists one and only one analytic solution $u^k = f^k(q)$ such that*

1. *$(q_0, \partial_\mu f^k(q_0))$ with $|\mu| \leq r-1$ is a point of $\pi^r_{r-1}(\mathcal{R}_r)$;*
2. *For $i = 1, \cdots, n$, the α^i_r parametric derivatives $\partial_\mu f^k(q)$ of class i are equal for $q^{i+1} = q_0^{i+1}, \cdots, q^n = q_0^n$ given analytic functions of $q^1, \cdots, q^i$.*

It should be noted that analyticity requirement for the given system cannot be dropped. Indeed, Lewy gives a simple example in which the existence of solution fails even if we relax the requirement of analyticity to just being smooth (C^∞). Nevertheless, it poses little trouble in energy shaping because for most of the time we are dealing with systems whose coefficients are all analytic. For details of Lewy's counterexample, consult [19] or [12].

Example 5 ([24]) Consider the following system of PDEs:

$$\mathcal{R}_1 : \quad \begin{cases} y_4 - q^3 y_2 - y = 0 \\ y_3 - q^4 y_1 = 0 \end{cases}$$

Its symbol G_1 is

$$\begin{array}{l} Y_4 - q^3 Y_2 = 0 \\ Y_3 - q^4 Y_1 = 0 \end{array} \qquad \boxed{\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \bullet \end{array}}$$

which is involutive. However, the "dot" prolongation of the second PDE in $\mathcal{R}_1$ leads to a new PDE of order 1, implying that $\mathcal{R}_1^{(1)} \neq \mathcal{R}_1$. Indeed, $\mathcal{R}_1^{(1)}$ is given by

$$\mathcal{R}_1^{(1)} : \quad \begin{cases} y_4 - q^3 y_1 - y = 0 \\ y_3 - q^4 y_1 = 0 \\ y_2 - y_1 = 0 \end{cases}$$

One can then check that $G_1^{(1)}$ is involutive:

$$
\begin{array}{l}
Y_4 - q^3 Y_2 = 0 \\
Y_3 - q^4 Y_1 = 0 \\
Y_2 - Y_1 = 0
\end{array}
\qquad
\boxed{\begin{array}{cccc}
1 & 2 & 3 & 4 \\
1 & 2 & 3 & \bullet \\
1 & 2 & \bullet & \bullet
\end{array}}
$$

Furthermore, we also have $\mathcal{R}_1^{(2)} = \mathcal{R}_1^{(1)}$. Hence, we have an involutive system $\mathcal{R}_1^{(2)}$. ■

Chapter 5

Energy Shaping on Systems with Two Degrees of Underactuation

In the previous chapter we described the set of tools that we will need to solve the PDEs for our energy shaping problem. In this chapter we describe a method for solving the resulting PDEs that occur when we have two degrees of underactuation and at least four degrees of freedom. We first look at the case when the degrees of freedom $n = 4$ without any gyroscopic force on the given system, followed by the case where $n \geq 4$ and finally the case where gyroscopic force is also present in the given system. Basically we derive inequations to be satisfied at $q = 0$, which are conditions under which we can shape the potential and kinetic energy of the given system. In practice, such inequations can be easily checked by working with the linearization of the given system: After linearizing the given mechanical system, one applies energy shaping to get a feedback equivalent linear system which provides the initial conditions for $\widehat{V}$ and $\widehat{T}$ of the nonlinear system.

The results in this chapter is the first time where the formal theory of PDEs has been successfully applied to the energy shaping problem of higher degrees of underactuation to give rise to workable criteria for shapability. Existing results like [13] also used the machinery from the formal theory, but the results therein are limited compared to [7, 8].

This is because [13] can only conclude that a system is shapable if and only if there is a common solution from the kinetic shaping and potential shaping, without mentioning any criteria to check the existence of such common solution. Finding a common solution, however, is at the heart of energy shaping problem which the authors tried to avoid. Furthermore, [13] also avoided the use of quadratic gyroscopic forces in the process of energy shaping, making the set of shapable systems restrictive. Here, we will answer the question of energy shaping for higher degrees of underactuation by properly addressing the common solution issue, and using quadratic gyroscopic forces in the shaping process to help enlarge the set of shapable mechanical systems. The results in this chapter will appear in the submitted paper [21].

5.1 Some Preparatory Work

When the degree of underactuation $n_1 = 2$, the matching conditions (2.9) and (2.10) are equivalent to the following six PDEs, in which two are for $\widehat{V}$ and four for $\widehat{T}$:

$$\widehat{T}_{1s} m^{sk} \frac{\partial \widehat{V}}{\partial q^k} = \frac{\partial V}{\partial q^1}$$

$$\widehat{T}_{2s} m^{sk} \frac{\partial \widehat{V}}{\partial q^k} = \frac{\partial V}{\partial q^2}$$

$$\widehat{T}_{1s} m^{sk} \left(\frac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) = 0$$

$$\widehat{T}_{2s} m^{sk} \left(\frac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) + 2\widehat{T}_{1s} m^{sk} \left(\frac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k} \widehat{T}_{2r} - \Gamma^r_{2k} \widehat{T}_{1r} \right) = 0$$

$$\widehat{T}_{1s} m^{sk} \left(\frac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k} \widehat{T}_{2r} \right) + 2\widehat{T}_{2s} m^{sk} \left(\frac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k} \widehat{T}_{2r} - \Gamma^r_{2k} \widehat{T}_{1r} \right) = 0$$

$$\widehat{T}_{2s} m^{sk} \left(\frac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k} \widehat{T}_{2r} \right) = 0.$$

Notice that the above system of PDEs is quasilinear, and basically only two differential operators appear, namely $\widehat{T}_{1s} m^{sk} \frac{\partial}{\partial q^k}$ and $\widehat{T}_{2s} m^{sk} \frac{\partial}{\partial q^k}$. In what follows we make an assumption on these differential operators and develop a strategy of finding an involutive system of PDEs.

5.1.1 Auxiliary Functions g_1 and g_2

To simplify our argument, we introduce two auxiliary functions g_1 and g_2 so that the above system of PDEs is equivalent to

$$\mathcal{R}_1 : \left\{ \begin{array}{lr} \Phi_1 : & \widehat{T}_{1s} m^{sk} \dfrac{\partial \widehat{V}}{\partial q^k} = \dfrac{\partial V}{\partial q^1} \\ \Phi_2 : & \widehat{T}_{2s} m^{sk} \dfrac{\partial \widehat{V}}{\partial q^k} = \dfrac{\partial V}{\partial q^2} \\ \Phi_3 : & \widehat{T}_{1s} m^{sk} \left(\dfrac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) = 0 \\ \Phi_4 : & \widehat{T}_{2s} m^{sk} \left(\dfrac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) = -2g_1 \\ \Phi_5 : & \widehat{T}_{1s} m^{sk} \left(\dfrac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k} \widehat{T}_{2r} - \Gamma^r_{2k} \widehat{T}_{1r} \right) = g_1 \\ \Phi_6 : & \widehat{T}_{2s} m^{sk} \left(\dfrac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k} \widehat{T}_{2r} - \Gamma^r_{2k} \widehat{T}_{1r} \right) = g_2 \\ \Phi_7 : & \widehat{T}_{1s} m^{sk} \left(\dfrac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k} \widehat{T}_{2r} \right) = -2g_2 \\ \Phi_8 : & \widehat{T}_{2s} m^{sk} \left(\dfrac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k} \widehat{T}_{2r} \right) = 0\ . \end{array} \right.$$

The use of these auxillary functions will become clear as we proceed. In what follows, we define the following differential operators:

$$\begin{aligned}
X_1 &= X_1^k \frac{\partial}{\partial q^k} = \widehat{T}_{1s} m^{sk} \frac{\partial}{\partial q^k} \\
X_2 &= X_2^k \frac{\partial}{\partial q^k} = \widehat{T}_{2s} m^{sk} \frac{\partial}{\partial q^k} \\
X_3 &= X_3^k \frac{\partial}{\partial q^k} = \delta_{3s} m^{sk} \frac{\partial}{\partial q^k} \\
X_4 &= X_4^k \frac{\partial}{\partial q^k} = \delta_{4s} m^{sk} \frac{\partial}{\partial q^k} \ .
\end{aligned}$$

We assume that these four differential operators are linearly independent, say,

$$X_1^3 X_2^4 - X_2^3 X_1^4 \neq 0 \tag{5.1}$$

Remark: In what follows, we denote the differential operators as X_i, $i = 1, \cdots, 4$. When we want to point out the k-th component of those operators, we will write X_i^k.

5.1.2 Involutive Distribution Assumption

To minimize the number of integrability conditions at later stages, we further assume that the distribution spanned by X_1 and X_2 is involutive, that is, the Lie bracket $[X_1, X_2]$ should satisfy

$$[X_1, X_2] = f_1 X_1 + f_2 X_2, \tag{5.2}$$

for some analytic functions f_1 and f_2. Rewriting(5.2) as

$$[X_1, X_2] = f_1 X_1 + f_2 X_2 + 0 \cdot X_3 + 0 \cdot X_4$$

implies that this extra assumption brings about two new equations to the original system of PDEs, namely

$$\begin{aligned}
\det(X_1, X_2, [X_1, X_2], X_4) &= 0 \\
\det(X_1, X_2, X_3, [X_1, X_2]) &= 0.
\end{aligned}$$

The involutive distribution assumption ensures that we will not come across new quasilinear differential operators after each prolongation step. In this way we will have integrability conditions from X_1 and X_2 only, hence reducing the number of possible integrability conditions. It should be noted that this is only an extra assumption on the differential operators; it has nothing to do with the involutivity of the system $\mathcal{R}_1$ and its symbol G_1. With this involutive distribution assumption, we are now looking for a solution set which is smaller than the one proposed by the original problem of energy shaping. We first derive some preliminary results for this assumption on X_1 and X_2.

Lemma 5.1.1 *On the system $\mathcal{R}_1$, the functions f_1 and f_2 in* (5.2) *are purely algebraic expression of $\widehat{T}_{ij}$, g_1 and g_2.*

Proof By Cramer's rule, we know that

$$\begin{aligned} f_1 &= \frac{\det([X_1, X_2], X_2, X_3, X_4)}{\det(X_1, X_2, X_3, X_4)} \\ &= \frac{\det(m)\det([X_1, X_2], X_2, X_3, X_4)}{\det(m)\det(X_1, X_2, X_3, X_4)} \\ &= \frac{\det(Expr_k, \widehat{T}_{2k}, \delta_{3k}, \delta_{4k})}{\det(\widehat{T}_{1k}, \widehat{T}_{2k}, \delta_{3k}, \delta_{4k})} \\ &= \frac{Expr_1\ \widehat{T}_{22} - Expr_2\ \widehat{T}_{12}}{\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2}, \end{aligned}$$

where $Expr_k$, $k = 1, \ldots, 4$ are defined by

$$Expr_k = m_{jk}\left(\widehat{T}_{1s}m^{si}\frac{\partial}{\partial q^i}(\widehat{T}_{2t}m^{tj}) - \widehat{T}_{2t}m^{ti}\frac{\partial}{\partial q^i}(\widehat{T}_{1s}m^{sj})\right).$$

Similarly, we have

$$f_2 = \frac{Expr_2\ \widehat{T}_{11} - Expr_1\ \widehat{T}_{12}}{\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2}.$$

It suffices to obtain an explicit formula for $Expr_k$. In this regard we have

$$\begin{aligned}
Expr_k &= \widehat{T}_{1s}m^{si}\left(m_{jk}\frac{\partial}{\partial q^i}(\widehat{T}_{2t}m^{tj})\right) - \widehat{T}_{2t}m^{ti}\left(m_{jk}\frac{\partial}{\partial q^i}(\widehat{T}_{1s}m^{sj})\right)\\
&= \widehat{T}_{1s}m^{si}\left(\frac{\partial(\widehat{T}_{2t}m^{tj}m_{jk})}{\partial q^i} - \widehat{T}_{2t}m^{tj}\frac{\partial m_{jk}}{\partial q^i}\right) - \widehat{T}_{2t}m^{ti}\left(\frac{\partial(\widehat{T}_{1s}m^{sj}m_{jk})}{\partial q^i} - \widehat{T}_{1s}m^{sj}\frac{\partial m_{jk}}{\partial q^i}\right)\\
&= \widehat{T}_{1s}m^{si}\left(\frac{\partial \widehat{T}_{2k}}{\partial q^i} - \widehat{T}_{2t}m^{tj}\frac{\partial m_{jk}}{\partial q^i}\right) - \widehat{T}_{2t}m^{ti}\left(\frac{\partial \widehat{T}_{1k}}{\partial q^i} - \widehat{T}_{1s}m^{sj}\frac{\partial m_{jk}}{\partial q^i}\right)\\
&= \widehat{T}_{1s}m^{si}\frac{\partial \widehat{T}_{2k}}{\partial q^i} - \widehat{T}_{2t}m^{ti}\frac{\partial \widehat{T}_{1k}}{\partial q^i} - \widehat{T}_{1s}\widehat{T}_{2t}(m^{si}m^{tj} - m^{ti}m^{sj})\frac{\partial m_{jk}}{\partial q^i}\\
&= X_1\widehat{T}_{2k} - X_2\widehat{T}_{1k} - \widehat{T}_{1s}\widehat{T}_{2t}m^{si}m^{tj}\left(\frac{\partial m_{jk}}{\partial q^i} - \frac{\partial m_{ik}}{\partial q^j}\right).
\end{aligned}$$

Using the definition of Christoffel symbols Γ^i_{jk}, we can further simplify the term $I := m^{si}m^{tj}\left(\frac{\partial m_{jk}}{\partial q^i} - \frac{\partial m_{ik}}{\partial q^j}\right)$: Since by definition $\Gamma^t_{ik} = \frac{1}{2}m^{tj}\left(\frac{\partial m_{jk}}{\partial q^i} + \frac{\partial m_{ji}}{\partial q^k} - \frac{\partial m_{ik}}{\partial q^j}\right)$, we have

$$2\Gamma^t_{ik} = m^{tj}\left(\frac{\partial m_{jk}}{\partial q^i} - \frac{\partial m_{ik}}{\partial q^j}\right) + m^{tj}\frac{\partial m_{ji}}{\partial q^k}$$

and thus,

$$S = m^{si}\left(2\Gamma^t_{ik} - m^{tj}\frac{\partial m_{ij}}{\partial q^k}\right). \tag{5.3}$$

Similarly, $\Gamma^s_{kj} = \frac{1}{2}m^{si}\left(\frac{\partial m_{ik}}{\partial q^j} + \frac{\partial m_{ij}}{\partial q^k} - \frac{\partial m_{kj}}{\partial q^i}\right)$ implies that $2\Gamma^s_{kj} = -m^{si}\left(\frac{\partial m_{jk}}{\partial q^i} - \frac{\partial m_{ik}}{\partial q^j}\right) + m^{si}\frac{\partial m_{ij}}{\partial q^k}$ and as a result,

$$S = m^{tj}\left(-2\Gamma^s_{kj} + m^{si}\frac{\partial m_{ij}}{\partial q^k}\right). \tag{5.4}$$

Combining (5.3) and (5.4), we have

$$\begin{aligned}
S &= \frac{1}{2}m^{si}\left(2\Gamma^t_{ik} - m^{tj}\frac{\partial m_{ij}}{\partial q^k}\right) + \frac{1}{2}m^{tj}\left(-2\Gamma^s_{kj} + m^{si}\frac{\partial m_{ij}}{\partial q^k}\right)\\
&= m^{si}\Gamma^t_{ik} - m^{tj}\Gamma^s_{kj}.
\end{aligned}$$

In short, we now have

$$Expr_k = X_1\widehat{T}_{2k} - X_2\widehat{T}_{1k} - \widehat{T}_{1s}\widehat{T}_{2t}(m^{si}\Gamma^t_{ik} - m^{tj}\Gamma^s_{kj}).$$

We can conclude our proof by verifying that $Expr_1$ and $Expr_2$, after elimination of $X_\gamma\widehat{T}_{\alpha\beta}$, are purely algebraic. Such an elimination is possible by using the fact that $\widehat{T}_{ij}$ satisfy the four PDEs (Φ_4, Φ_5, Φ_6, Φ_7). Hence

$$\begin{aligned} Expr_1 &= X_1\widehat{T}_{12} - X_2\widehat{T}_{11} - \widehat{T}_{1s}\widehat{T}_{2t}(m^{si}\Gamma^t_{1i} - m^{tj}\Gamma^s_{1j}) \\ &= [g_1 + \widehat{T}_{1s}m^{si}(\Gamma^t_{1i}\widehat{T}_{2t} + \Gamma^t_{2i}\widehat{T}_{1t})] - [-2g_1 + 2\widehat{T}_{2s}m^{sj}\Gamma^t_{1j}\widehat{T}_{1t}] \\ &\qquad\qquad\qquad\qquad\qquad - \widehat{T}_{1s}m^{si}\Gamma^t_{1i}\widehat{T}_{2t} + \widehat{T}_{2s}m^{sj}\Gamma^t_{1j}\widehat{T}_{1t} \\ &= 3g_1 + \widehat{T}_{1s}m^{si}\Gamma^t_{2i}\widehat{T}_{1t} - \widehat{T}_{2t}m^{ti}\Gamma^s_{1i}\widehat{T}_{1s}. \end{aligned}$$

Similarly, we have

$$Expr_2 = -3g_2 + \widehat{T}_{1s}m^{si}\Gamma^t_{2i}\widehat{T}_{2t} - \widehat{T}_{2s}m^{si}\Gamma^t_{1i}\widehat{T}_{2t}\ .$$

■

With the extra assumption of involutive distribution, we now need to consider the solution for the following system of PDEs:

$$\overline{\mathcal{R}}_1 : \begin{cases} \Phi_1 : & \widehat{T}_{1s} m^{sk} \frac{\partial \widehat{V}}{\partial q^k} = \frac{\partial V}{\partial q^1} \\ \Phi_2 : & \widehat{T}_{2s} m^{sk} \frac{\partial \widehat{V}}{\partial q^k} = \frac{\partial V}{\partial q^2} \\ \Phi_3 : & \widehat{T}_{1s} m^{sk} \left(\frac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) = 0 \\ \Phi_4 : & \widehat{T}_{2s} m^{sk} \left(\frac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) = -2g_1 \\ \Phi_5 : & \widehat{T}_{1s} m^{sk} \left(\frac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k} \widehat{T}_{2r} - \Gamma^r_{2k} \widehat{T}_{1r} \right) = g_1 \\ \Phi_6 : & \widehat{T}_{2s} m^{sk} \left(\frac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k} \widehat{T}_{2r} - \Gamma^r_{2k} \widehat{T}_{1r} \right) = g_2 \\ \Phi_7 : & \widehat{T}_{1s} m^{sk} \left(\frac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k} \widehat{T}_{2r} \right) = -2g_2 \\ \Phi_8 : & \widehat{T}_{2s} m^{sk} \left(\frac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k} \widehat{T}_{2r} \right) = 0 \\ \Phi_9 : & \det(X_1, X_2, [X_1, X_2], X_4) = 0 \\ \Phi_{10} : & \det(X_1, X_2, X_3, [X_1, X_2]) = 0 \ . \end{cases}$$

We first observe that Φ_1 to Φ_8 in $\mathcal{R}_1$ can be grouped into four decoupled pairs (Φ_1 with Φ_2; Φ_3 with Φ_4, etc.), in which the differential operator, either X_1 or X_2, acts on $\widehat{V}$ and $\widehat{T}_{\alpha\beta}$. Such pairs are convenient in terms of symbol involutivity,

Lemma 5.1.2 *The system*

$$\begin{cases} X_1^k \dfrac{\partial \widehat{H}}{\partial q^k} = h_1 \\ X_2^k \dfrac{\partial \widehat{H}}{\partial q^k} = h_2 \end{cases}$$

where $\widehat{H} = \widehat{H}(q)$ is the unknown to be solved, and h_1, h_2 are analytic functions which do not appear in the level of symbol, has an involutive symbol. This system has an integrability condition given by $[X_1, X_2]\widehat{H} = X_1 h_2 - X_2 h_1$.

Proof By Cramer's rule, we can solve $\frac{\partial \widehat{H}}{\partial q^1}$ and $\frac{\partial \widehat{H}}{\partial q^2}$ as follows:

$$
\begin{aligned}
(X_1^3X_2^4 - X_1^4X_2^3)\frac{\partial \widehat{H}}{\partial q^3} &= (X_1^4X_2^\alpha - X_2^4X_1^\alpha)\frac{\partial \widehat{H}}{\partial q^\alpha} + X_2^4h_1 - X_1^4h_2 \\
(X_1^3X_2^4 - X_1^4X_2^3)\frac{\partial \widehat{H}}{\partial q^4} &= (X_2^3X_1^\alpha - X_1^3X_2^\alpha)\frac{\partial \widehat{H}}{\partial q^\alpha} + X_1^3h_2 - X_2^3h_1
\end{aligned}
$$

where α runs from 1 to 2 (or 1 to $n-2$ for general $n \geq 4$). Thus, this system has a symbol given by

$$
\begin{aligned}
(X_1^3X_2^4 - X_1^4X_2^3)\frac{\partial \widehat{H}}{\partial q^3} &= (X_1^4X_2^\alpha - X_2^4X_1^\alpha)\frac{\partial \widehat{H}}{\partial q^\alpha} \\
(X_1^3X_2^4 - X_1^4X_2^3)\frac{\partial \widehat{H}}{\partial q^4} &= (X_2^3X_1^\alpha - X_1^3X_2^\alpha)\frac{\partial \widehat{H}}{\partial q^\alpha}
\end{aligned}
\qquad
\boxed{\begin{array}{cccc} 1 & 2 & 3 & \bullet \\ \\ 1 & 2 & 3 & 4 \end{array}}
$$

The "dot" board is a bookkeeping way of indicating that the first and second equation are of class 4 and 3 respectively. The prolongation of the first equation $(X_1^3X_2^4 - X_1^4X_2^3)\frac{\partial \widehat{H}}{\partial q^3} = (X_1^4X_2^\alpha - X_2^4X_1^\alpha)\frac{\partial \widehat{H}}{\partial q^\alpha}$ with respect to the "dot" (i.e. q^4) is a linear combination of other prolongations with respect to the multiplicative variables. Indeed, this linear combination can be derived from the fact that the Lie bracket $[\overline{X}_1, \overline{X}_2]$ is a differential operator of order 1 only, where

$$
\begin{aligned}
\overline{X}_1 &:= (X_1^3X_2^4 - X_1^4X_2^3)\frac{\partial}{\partial q^3} - (X_1^4X_2^\alpha - X_2^4X_3^\alpha)\frac{\partial}{\partial q^\alpha} = (X_2^4X_1^k - X_1^4X_2^k)\frac{\partial}{\partial q^k} \\
\overline{X}_2 &:= (X_1^3X_2^4 - X_1^4X_2^3)\frac{\partial}{\partial q^4} - (X_2^3X_1^\alpha - X_1^3X_2^\alpha)\frac{\partial}{\partial q^\alpha} = (X_1^3X_2^k - X_2^3X_1^k)\frac{\partial}{\partial q^k} \ .
\end{aligned}
$$

Hence, by Theorem 4.2.4, the symbol for the system of these two PDEs is involutive. Moreover, the integrability condition is

$$
[\overline{X}_1, \overline{X}_2]\widehat{H} = \overline{X}_1(X_1^3h_2 - X_2^3h_1) - \overline{X}_2(X_2^4h_1 - X_1^4h_2) \ .
$$

We now prove that this is the same as $[X_1, X_2]\widehat{H} = X_1 h_2 - X_2 h_1$. We have

$$\begin{aligned}
[\overline{X}_1, \overline{X}_2]\widehat{H} &= \overline{X}_1(X_1^3 X_2^k - X_2^3 X_1^k)\frac{\partial \widehat{H}}{\partial q^k} - \overline{X}_2(X_2^4 X_1^k - X_1^4 X_2^k)\frac{\partial \widehat{H}}{\partial q^k} \\
&= (\overline{X}_1 X_1^3 + \overline{X}_2 X_1^4)\left(X_2^k \frac{\partial \widehat{H}}{\partial q^k}\right) - (\overline{X}_1 X_2^3 + \overline{X}_2 X_2^4)\left(X_1^k \frac{\partial \widehat{H}}{\partial q^k}\right) \\
&\quad + ((X_1^3 \overline{X}_1 + X_1^4 \overline{X}_2) X_2^k - (X_2^3 \overline{X}_1 + X_2^4 \overline{X}_2) X_1^k)\frac{\partial \widehat{H}}{\partial q^k} \; .
\end{aligned}$$

But since $X_i^k \frac{\partial \widehat{H}}{\partial q^k} = h_i$ for $i = 1, 2$, and $X_1^3 \overline{X}_1 + X_1^4 \overline{X}_2 = (X_1^3 X_2^4 - X_1^4 X_2^3) X_1^m \frac{\partial}{\partial q^m}$, $X_2^3 \overline{X}_1 + X_2^4 \overline{X}_2 = (X_1^4 X_2^4 - X_1^4 X_2^3) X_2^m \frac{\partial}{\partial q^m}$, we have

$$[\overline{X}_1, \overline{X}_2]\widehat{H} = (\overline{X}_1 X_1^3 + \overline{X}_2 X_1^4) h_2 - (\overline{X}_1 X_2^3 + \overline{X}_2 X_2^4) h_1 + (X_1^3 X_2^4 - X_1^4 X_2^3)(X_1 X_2^k - X_2 X_1^k)\frac{\partial \widehat{H}}{\partial q^k} \tag{5.5}$$

Meanwhile, similar computation gives

$$\begin{aligned}
&\overline{X}_1(X_1^3 h_2 - X_2^3 h_1) - \overline{X}_2(X_2^4 h_1 - X_1^4 h_2) \\
&= (\overline{X}_1 X_1^3 + \overline{X}_2 X_1^4) h_2 - (\overline{X}_1 X_2^3 + \overline{X}_2 X_2^4) h_1 + (X_1^3 X_2^4 - X_1^4 X_2^3)(X_1 h_2 - X_2 h_1)
\end{aligned} \tag{5.6}$$

Hence, canceling common terms in (5.5) and (5.6), we arrive at $[X_1, X_2]\widehat{H} = X_1 h_2 - X_2 h_1$ as desired. ■

5.2 Getting an Involutive System

We now apply the algorithm from the Cartan-Kuranishi theorem to obtain an equivalent, involutive system of PDEs. We first start by observing that the symbol G_1 is involutive.

Corollary 5.2.1 *The symbol G_1 for the system $\mathcal{R}_1$ (the one defined by Φ_1 to Φ_8 only) is involutive.*

Proof By Lemma 5.1.2, each decoupled pair of PDEs forms an involutive system. Each pair is exclusively for the partials of one of the unknowns: $\widehat{V}$, $\widehat{T}_{11}$, $\widehat{T}_{12}$ or $\widehat{T}_{22}$. Hence, the whole system $\mathcal{R}_1$ defined by these four pairs has an involutive symbol. ■

Lemma 5.1.2 states that we should have one integrability condition for each of $\widehat{V}$, $\widehat{T}_{11}$, $\widehat{T}_{12}$ and $\widehat{T}_{22}$. In particular, we can exploit some properties of the integrability condition for $\widehat{V}$.

Lemma 5.2.2 *The integrability condition for $\widehat{V}$ is purely algebraic in $\mathcal{R}_1$. We can use this equation to define $\widehat{T}_{13}$ algebraically provided that*

$$m^{s3}\frac{\partial^2 V}{\partial q^s \partial q^2} \neq 0. \tag{5.7}$$

Proof By Lemma 5.1.2, the integrability condition for $\widehat{V}$ is given by

$$[X_1, X_2]\widehat{V} = X_1\left(\frac{\partial V}{\partial q^2}\right) - X_2\left(\frac{\partial V}{\partial q^1}\right).$$

Since $[X_1, X_2] = f_1 X_1 + f_2 X_2$, we have

$$f_1\frac{\partial V}{\partial q^1} + f_2\frac{\partial V}{\partial q^2} = X_1\left(\frac{\partial V}{\partial q^2}\right) - X_2\left(\frac{\partial V}{\partial q^1}\right). \tag{5.8}$$

The left hand side of (5.8) is purely algebraic, since we know f_1 and f_2 are purely algebraic from Lemma 5.1.1. The right hand side of (5.8) also does not contain any derivatives of unknown variables, since V is given. Hence, (5.8) is purely algebraic. We now show that this can algebraically define $\widehat{T}_{13}$. First, we note that the left hand side of (5.8) is equal to

$$\begin{aligned}
&\frac{1}{\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2}\left[\frac{\partial V}{\partial q^1}(Expr_1\widehat{T}_{22} - Expr_2\widehat{T}_{12}) + \frac{\partial V}{\partial q^2}(Expr_2\widehat{T}_{11} - Expr_1\widehat{T}_{12})\right] \\
&= \frac{1}{\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2}\left[\left(\frac{\partial V}{\partial q^1}\widehat{T}_{22} - \frac{\partial V}{\partial q^2}\widehat{T}_{12}\right)Expr_1 + \left(\frac{\partial V}{\partial q^2}\widehat{T}_{11} - \frac{\partial V}{\partial q^1}\widehat{T}_{12}\right)Expr_2\right] \\
&= \frac{1}{\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2}\left[\left(\frac{\partial V}{\partial q^1}\widehat{T}_{22} - \frac{\partial V}{\partial q^2}\widehat{T}_{12}\right)(3g_1 + \widehat{T}_{1s}m^{si}\Gamma^t_{2i}\widehat{T}_{1t} - \widehat{T}_{2t}m^{ti}\Gamma^s_{1i}\widehat{T}_{1s})\right. \\
&\qquad\left. + \left(\frac{\partial V}{\partial q^2}\widehat{T}_{11} - \frac{\partial V}{\partial q^1}\widehat{T}_{12}\right)(-3g_2 + \widehat{T}_{1s}m^{si}\Gamma^t_{2i}\widehat{T}_{2t} - \widehat{T}_{2s}m^{si}\Gamma^t_{1i}\widehat{T}_{2t})\right],
\end{aligned}$$

while the right hand side of (5.8) is equal to

$$\widehat{T}_{1s}m^{sk}\frac{\partial^2 V}{\partial q^k \partial q^2} - \widehat{T}_{2s}m^{sk}\frac{\partial^2 V}{\partial q^k \partial q^1}\ .$$

Now notice that g_1 first appears in Φ_4 and Φ_5. If we replace g_1 by

$$g_1 = \overline{g}_1 - \frac{1}{3} m^{3i} \Gamma^3_{2i} (\widehat{T}_{13})^2$$

and trace down the calculations, we conclude that all results obtained so far do not change by such replacement and, in addition, we can remove all quadratic terms of $\widehat{T}_{13}$ in (5.8).

Finally, since we assume $\frac{\partial V}{\partial q^i} = 0$ at $q = 0$ for $i = 1, \ldots, 4$, the left hand side of (5.8) vanishes at $q = 0$. Hence, in order to define $\widehat{T}_{13}$ using (5.8), we require the $\widehat{T}_{13}$ to be non-vanishing on the right hand side of (5.8), that is,

$$m^{s3} \frac{\partial^2 V}{\partial q^s \partial q^2} \neq 0 \qquad \blacksquare$$

Remark : When (5.7) holds, then $\widehat{T}_{13}$ is defined by

$$\widehat{T}_{13} = \frac{(\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2) \left(-\widehat{T}_{1\widehat{s}} m^{\widehat{s}k} \frac{\partial^2 V}{\partial q^k \partial q^2} + \widehat{T}_{2s} m^{sk} \frac{\partial^2 V}{\partial q^k \partial q^1} \right) + P_1}{(\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2) m^{3k} \frac{\partial^2 V}{\partial q^k \partial q^2} - P_2}, \tag{5.9}$$

where $\widehat{s}$ runs through $1, 2$ and 4 only, with P_1, P_2 defined by

$$\begin{aligned} P_1 = & \left(\frac{\partial V}{\partial q^1} \widehat{T}_{22} - \frac{\partial V}{\partial q^2} \widehat{T}_{12} \right) (\widehat{T}_{1\widehat{s}} m^{\widehat{s}i} \Gamma^{\widehat{t}}_{2i} \widehat{T}_{1\widehat{t}} - \widehat{T}_{2t} m^{ti} \Gamma^s_{1i} \widehat{T}_{1s}) \\ & + \left(\frac{\partial V}{\partial q^2} \widehat{T}_{11} - \frac{\partial V}{\partial q^1} \widehat{T}_{12} \right) (-3g_2 + \widehat{T}_{1\widehat{s}} m^{\widehat{s}i} \Gamma^t_{2i} \widehat{T}_{2t} - \widehat{T}_{2s} m^{si} \Gamma^t_{1i} \widehat{T}_{2t}) \\ P_2 = & \left(\frac{\partial V}{\partial q^1} \widehat{T}_{22} - \frac{\partial V}{\partial q^2} \widehat{T}_{12} \right) (\widehat{T}_{1\widehat{s}} m^{\widehat{s}i} \Gamma^3_{2i} + \widehat{T}_{13} m^{3i} \Gamma^{\widehat{t}}_{2i}) \\ & - \left(\frac{\partial V}{\partial q^2} \widehat{T}_{11} - \frac{\partial V}{\partial q^1} \widehat{T}_{12} \right) (m^{3i} \Gamma^t_{2i} \widehat{T}_{2t} - \widehat{T}_{2s} m^{si} \Gamma^3_{1i}), \end{aligned}$$

where $\widehat{s}, \widehat{t}$ runs for 1, 2 and 4 only. Notice that due to the presence of partials of V, both P_1 and P_2 are zero at $q = 0$. We will make use of this fact in later proofs.

We now need to consider the solution for the following system of PDEs:

$$\overline{\mathcal{R}}_1 : \begin{cases} \Phi_1 : & \widehat{T}_{1s} m^{sk} \frac{\partial \widehat{V}}{\partial q^k} = \frac{\partial V}{\partial q^1} \\ \Phi_2 : & \widehat{T}_{2s} m^{sk} \frac{\partial \widehat{V}}{\partial q^k} = \frac{\partial V}{\partial q^2} \\ \Phi_3 : & \widehat{T}_{1s} m^{sk} \left(\frac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) = 0 \\ \Phi_4 : & \widehat{T}_{2s} m^{sk} \left(\frac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) = -2(\overline{g}_1 - \frac{1}{3} m^{3i} \Gamma^3_{2i} (\widehat{T}_{13})^2) \\ \Phi_5 : & \widehat{T}_{1s} m^{sk} \left(\frac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k} \widehat{T}_{2r} - \Gamma^r_{2k} \widehat{T}_{1r} \right) = \overline{g}_1 - \frac{1}{3} m^{3i} \Gamma^3_{2i} (\widehat{T}_{13})^2 \\ \Phi_6 : & \widehat{T}_{2s} m^{sk} \left(\frac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k} \widehat{T}_{2r} - \Gamma^r_{2k} \widehat{T}_{1r} \right) = g_2 \\ \Phi_7 : & \widehat{T}_{1s} m^{sk} \left(\frac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k} \widehat{T}_{2r} \right) = -2g_2 \\ \Phi_8 : & \widehat{T}_{2s} m^{sk} \left(\frac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k} \widehat{T}_{2r} \right) = 0 \\ \Phi_9 : & \det(X_1, X_2, [X_1, X_2], X_4) = 0 \\ \Phi_{10} : & \det(X_1, X_2, X_3, [X_1, X_2]) = 0 \end{cases}$$

where g_1 is replaced by $\overline{g}_1 - \frac{1}{3} m^{3i} \Gamma^3_{2i} (\widehat{T}_{13})^2$ so that $\widehat{T}_{13}$ is well-defined by using the integrability condition for $\widehat{V}$. Here, we do not explicitly eliminate $\widehat{T}_{13}$ for the sake of clarity, but from now on, we should eliminate $\widehat{T}_{13}$ in the system of PDEs whenever it appears.

Lemma 5.2.3 *The symbol $\overline{G}_1$ of $\overline{\mathcal{R}}_1$, after eliminating $\widehat{T}_{13}$ using the integrability condition for $\widehat{V}$, is involutive if*

$$\widehat{T}_{1s} m^{s4} \neq 0 \tag{5.10}$$

$$\widehat{T}_{1s} m^{s4} - \frac{m^{3k} \frac{\partial^2 V}{\partial q^k \partial q^1}}{m^{3s} \frac{\partial^2 V}{\partial q^s \partial q^2}} \widehat{T}_{2t} m^{t4} \neq 0. \tag{5.11}$$

Proof By Corollary 5.2.1, we know that the first eight PDEs (Φ_1 to Φ_8) constitute a system of PDEs with an involutive symbol. We now show that the whole system $\overline{\mathcal{R}}_1$, after eliminating $\widehat{T}_{13}$, has an involutive symbol. This is done by observing that Φ_9 and Φ_{10} can be treated as class 4 equations for $\widehat{T}_{23}$ and $\widehat{T}_{24}$. We first consider Φ_{10}, which is equivalent to $\det(\widehat{T}_{1k}, \widehat{T}_{2k}, \delta_{3k}, Expr_k) = 0$ or, more explicitly,

$$(\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2)(X_1\widehat{T}_{24} - X_2\widehat{T}_{14}) = 0,$$

in the level of symbol $\overline{G}_1$. Thus, this PDE can be used to solve $\frac{\partial \widehat{T}_{24}}{\partial q^4}$ provided that its coefficient in the PDE is nonzero, i.e. if (5.10) holds.

We now come to Φ_9, which is $\det(\widehat{T}_{1k}, \widehat{T}_{2k}, Expr_k, \delta_{4k}) = 0$ or more explicitly,

$$(\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2)(X_1\widehat{T}_{23} - X_2\widehat{T}_{13}) = 0$$

in the level of symbol $\overline{G}_1$. Making use of (5.9) to eliminate $\widehat{T}_{13}$, the above PDE in $\overline{G}_1$ around $q = 0$ is

$$(\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2)\left(X_1\widehat{T}_{23} - X_2\left(\frac{\widehat{T}_{2s}m^{sk}\frac{\partial V}{\partial q^k \partial q^1} - \widehat{T}_{1\widehat{s}}m^{\widehat{s}k}\frac{\partial^2 V}{\partial q^k \partial q^2}}{m^{3s}\frac{\partial^2 V}{\partial q^s \partial q^2}}\right)\right) = 0\ .$$

Hence, Φ_9 can be used to define $\frac{\partial \widehat{T}_{23}}{\partial q^4}$ provided that its coefficient is nonzero, or equivalently, if (5.11) holds. Since Φ_9 and Φ_{10} are both PDEs of class 4 and the rest of the system $\overline{\mathcal{R}}_1$ has an involutive symbol, we can conclude that the symbol $\overline{G}_1$ of the whole system is involutive. ■

Since $\overline{\mathcal{R}}_1$ differs from $\mathcal{R}_1$ by having two extra equations of class 4, the number of integrability conditions in $\overline{\mathcal{R}}_1$ is still four. The one for $\widehat{V}$ has been used to define and eliminate $\widehat{T}_{13}$. Hence, we are left with the integrability conditions for $\widehat{T}_{11}$, $\widehat{T}_{12}$ and $\widehat{T}_{22}$. If we can show that these equations are also of class 4, then we can conclude that $\overline{\mathcal{R}}_1^{(1)}$ is involutive and the whole prolongation-projection algorithm ends.

Lemma 5.2.4 *The integrability conditions for $\widehat{T}_{11}$, $\widehat{T}_{12}$ and $\widehat{T}_{22}$ in their solved forms on the system $\mathcal{R}_1^{(1)}$ are of class 4 if*

$$\widehat{T}_{2s}m^{s4} \neq 0 \tag{5.12}$$

$$\widehat{T}_{1s}m^{s4}\widehat{T}_{1t}m^{tk}\Gamma^4_{2k} \neq \widehat{T}_{2s}m^{s4}\widehat{T}_{1t}m^{tk}\Gamma^4_{1k}. \tag{5.13}$$

Proof We first derive, at the level of symbol, the three integrability conditions explicitly. By Lemma 5.1.2 and the involutive assumption on the differential operators X_1 and X_2, the integrability condition for $\widehat{T}_{11}$ is

$$[X_1, X_2]\widehat{T}_{11} = (f_1 X_1 + f_2 X_2)\widehat{T}_{11} \ .$$

By Lemma 5.1.1, f_1 and f_2 are purely algebraic, and we can eliminate $X_1\widehat{T}_{11}$ and $X_2\widehat{T}_{11}$, as they satisfy Φ_3 and Φ_4, by purely algebraic expressions. Thus the right hand side of the above equation, after such elimination, does not appear at the level of symbol. In other words, we can simply consider the left hand side of the above integrability condition:

$$\begin{aligned}[X_1, X_2]\widehat{T}_{11} &= X_1(X_2\widehat{T}_{11}) - X_2(X_1\widehat{T}_{11}) \\ &= X_1\left(2\widehat{T}_{2s}m^{sk}\Gamma^r_{1k}\widehat{T}_{1r} - 2\overline{g}_1 + \frac{2}{3}m^{3i}\Gamma^3_{2i}(\widehat{T}_{13})^2\right) - X_2(2\widehat{T}_{1s}m^{sk}\Gamma^r_{1k}\widehat{T}_{1r}),\end{aligned}$$

by using Φ_3 and Φ_4.

Now, we observe that

$$\begin{aligned}[X_1, X_2]\widehat{T}_{11} &= 2[X_1(X_2^k(\Gamma^r_{1k}\widehat{T}_{1r})) - X_2(X_1^k(\Gamma^r_{1k}\widehat{T}_{1r})) - X_1\overline{g}_1 + \cdots] \\ &= 2\left[\left(X_1^s\frac{\partial X_2^k}{\partial q^s} - X_2^s\frac{\partial X_1^k}{\partial q^s}\right)\Gamma^r_{1k}\widehat{T}_{1r} + (X_1^sX_2^k - X_2^sX_1^k)\frac{\partial}{\partial q^s}(\Gamma^r_{1k}\widehat{T}_{1r}) - X_1\overline{g}_1 + \cdots\right],\end{aligned}$$

where we omit all the terms that do not contain the $\frac{\partial}{\partial q^4}$ derivatives of $\overline{g}_1$, g_1 or $\widehat{T}_{14}$. The use of shorthand X_k^i (which is by definition $\widehat{T}_{ks}m^{si}$) helps simplify our computations. Notice that the term $X_1^s\frac{\partial X_2^k}{\partial q^s} - X_2^s\frac{\partial X_1^k}{\partial q^s}$ is simply the k-th component of the Lie bracket $[X_1, X_2]$, which is in turn equal to $f_1X_1^k + f_2X_2^k$ by the involutive assumption. Moreover, we know by Lemma 5.1.1 that f_1 and f_2 are purely algebraic after substituting the defining PDEs from $\overline{\mathcal{R}}_1$. Hence the integrability condition for $\widehat{T}_{11}$, at the level of symbol, is simply

$$(X_1^sX_2^k - X_2^sX_1^k)\frac{\partial}{\partial q^s}(\Gamma^{\overline{r}}_{1k}\widehat{T}_{1\overline{r}}) - X_1\overline{g}_1 + \cdots = 0,$$

where $\overline{r}$ runs from 3 to 4, by taking into account $X_1\widehat{T}_{\alpha\beta}$ and $X_2\widehat{T}_{\alpha\beta}$ are zero at the level of symbol. Using the product rule of differentiation, we realize this integrability condition contains derivatives of $\widehat{T}_{14}$ and $\overline{g}_1$:

$$(X_1^s X_2^k - X_2^s X_1^k)\Gamma_{1k}^4 \frac{\partial \widehat{T}_{14}}{\partial q^s} - X_1 \overline{g}_1 + \cdots = 0. \tag{5.14}$$

We do the same trick for $\widehat{T}_{22}$:

$$[X_1, X_2]\widehat{T}_{22} = 2\left[(X_1^s X_2^k - X_2^s X_1^k)\frac{\partial}{\partial q^s}(\Gamma_{2k}^{\overline{r}} \widehat{T}_{2\overline{r}}) + X_2 g_2\right].$$

At first sight, this does not contain any $\widehat{T}_{1\overline{r}}$, but as we know from $\overline{G}_1$, $X_1\widehat{T}_{2\overline{r}} = X_2\widehat{T}_{1\overline{r}}$, we have, at the level of symbol,

$$X_1(\Gamma_{2k}^{\overline{r}}\widehat{T}_{2\overline{r}}) = \Gamma_{2k}^{\overline{r}} X_1 \widehat{T}_{2\overline{r}} = \Gamma_{2k}^{\overline{r}} X_2 \widehat{T}_{1\overline{r}} = X_2(\Gamma_{2k}^{\overline{r}}\widehat{T}_{1\overline{r}}).$$

Hence, the integrability condition for $\widehat{T}_{22}$ should be

$$\begin{aligned} & X_2^s X_2^k \frac{\partial}{\partial q^s}(\Gamma_{2k}^{\overline{r}}\widehat{T}_{1\overline{r}}) + X_2 g_2 + \cdots = 0 \\ \Rightarrow \qquad & X_2^s X_2^k \Gamma_{2k}^4 \frac{\partial \widehat{T}_{14}}{\partial q^s} + X_2 g_2 + \cdots = 0, \end{aligned} \tag{5.15}$$

where we omit again terms that do not contain the derivatives of $\widehat{T}_{14}$.

In a similar fashion one can show that the integrability condition for $\widehat{T}_{12}$ is given by

$$X_2^s X_2^k \frac{\partial}{\partial q^s}(\Gamma_{1k}^{\overline{r}}\widehat{T}_{1\overline{r}}) + (X_1^s X_2^k - X_2^s X_1^k)\frac{\partial}{\partial q^s}(\Gamma_{2k}^{\overline{r}}\widehat{T}_{1\overline{r}}) + X_1 g_2 - X_2 \overline{g}_1 + \cdots = 0,$$

or equivalently,

$$X_2^s X_2^k \Gamma_{1k}^4 \frac{\partial \widehat{T}_{14}}{\partial q^s} + (X_1^s X_2^k - X_2^s X_1^k)\Gamma_{2k}^4 \frac{\partial \widehat{T}_{14}}{\partial q^s} + X_1 g_2 - X_2 \overline{g}_1 + \cdots = 0, \tag{5.16}$$

at the level of the symbol $\overline{G}_1$.

We now show that these PDEs can solve $\dfrac{\partial \overline{g}_1}{\partial q^4}$, $\dfrac{\partial g_2}{\partial q^4}$ and $\dfrac{\partial \widehat{T}_{14}}{\partial q^4}$ respectively provided that (5.12) and (5.13) are satisfied. This is done by computing the determinant of the coefficient matrix of these three derivatives:

$$\begin{vmatrix} -\widehat{T}_{1s}m^{s4} & 0 & \widehat{T}_{2s}m^{sk}\Gamma^4_{1k}\widehat{T}_{1t}m^{t4} - \widehat{T}_{1s}m^{sk}\Gamma^4_{1k}\widehat{T}_{2t}m^{t4} \\ 0 & \widehat{T}_{2s}m^{s4} & \widehat{T}_{2s}m^{sk}\Gamma^4_{2k}\widehat{T}_{2t}m^{t4} \\ -\widehat{T}_{2s}m^{s4} & \widehat{T}_{1s}m^{s4} & \widehat{T}_{2s}m^{sk}\Gamma^4_{2k}\widehat{T}_{1t}m^{t4} + \widehat{T}_{2s}m^{sk}\Gamma^4_{1k}\widehat{T}_{2t}m^{t4} - \widehat{T}_{1s}m^{sk}\Gamma^4_{2k}\widehat{T}_{2t}m^{t4} \end{vmatrix}.$$

We change all the X^i_k back into $\widehat{T}_{ks}m^{si}$ since we want our criteria to be expressed in terms of $\widehat{T}$ and m only. The determinant of this coefficient matrix can be simplified to give $(\widehat{T}_{2s}m^{s4})^2(\widehat{T}_{1s}m^{s4}\widehat{T}_{1t}m^{tk}\Gamma^4_{2k} - \widehat{T}_{2s}m^{s4}\widehat{T}_{1t}m^{tk}\Gamma^4_{1k})$. We can solve the three class 4 derivatives uniquely if and only if the coefficient matrix has a nonzero determinant. This concludes the proof. ■

Remark In the proof we are not concerned about derivatives of unknowns other than $\overline{g}_1$, g_2 and $\widehat{T}_{14}$ though they may appear in the symbol as well. This is valid in the proof as we use the three integrability conditions to define derivatives of $\overline{g}_1$, g_2 and $\widehat{T}_{14}$ only.

We can now summarize our results into the following

Theorem 5.2.5 *If $n = 4$, and if* (5.1), (5.7), (5.10), (5.11), (5.12) *and* (5.13) *hold, at least at $q = 0$, then the system $\overline{\mathcal{R}}^{(1)}_1$ is involutive.*

Proof $\overline{\mathcal{R}}^{(1)}_1$ is defined by Φ_1 to Φ_{10}, together with 4 equations, derived from the integrability conditions for $\widehat{V}$, $\widehat{T}_{11}$, $\widehat{T}_{12}$ and $\widehat{T}_{22}$. The one for $\widehat{V}$, as proved in Lemma 5.2.2, solves $\widehat{T}_{13}$ if (5.7) holds. The resulting system of PDEs, after eliminating $\widehat{T}_{13}$, still has an involutive symbol. The reason for this is two-fold. First, Φ_1 to Φ_{10} constitute a system of PDEs with involutive symbol, as proved in Lemma 5.2.3. Secondly, by Lemma 5.2.4, the extra integrability conditions from $\widehat{T}_{\alpha\beta}$ are of class 4, if (5.12) and (5.13) hold.

Now, by Theorem 4.3.2, if we can show that $\overline{\mathcal{R}}^{(1)}_1 = \pi^2_1((\overline{\mathcal{R}}^{(1)}_1)_{+1})$, then we can conclude that $\overline{\mathcal{R}}^{(1)}_1$ is involutive. But such an equality is true since, with the exception of the integrability condition for $\widehat{V}$, all integrability conditions for $\overline{\mathcal{R}}_1$ are of class 4, and hence we cannot generate further integrability conditions. ■

It should be noted that the above procedure of obtaining an involutive system of PDEs is coordinate-dependent. Here we abide by the choice of coordinates as depicted in [23], [24], where $\frac{\partial}{\partial q^i}$ are classified as class i, and we place higher priority for those derivatives in higher classes. One can choose to prioritize coordinates in several different manners, for example, we can define $\frac{\partial}{\partial q^1}$ as class 4 (i.e. highest priority) etc., and obtain an involutive system with a similar set of inequality constraints. In other words, we have the following.

Theorem 5.2.6 *If $n = 4$, and the following inequalities*

$$X_1^1 X_2^2 - X_1^2 X_2^1 \neq 0 \tag{5.17}$$

$$m^{3s} \frac{\partial^2 V}{\partial q^s \partial q^2} \neq 0 \tag{5.18}$$

$$\widehat{T}_{1s} m^{s1} \neq 0 \tag{5.19}$$

$$\widehat{T}_{1s} m^{s1} - \frac{m^{3k} \frac{\partial^2 V}{\partial q^k \partial q^1}}{m^{3s} \frac{\partial^2 V}{\partial q^s \partial q^2}} \widehat{T}_{2t} m^{t1} \neq 0 \tag{5.20}$$

$$\widehat{T}_{2s} m^{s1} \neq 0 \tag{5.21}$$

$$\widehat{T}_{1s} m^{s1} \widehat{T}_{1t} m^{tk} \Gamma^4_{2k} \neq \widehat{T}_{2s} m^{s1} \widehat{T}_{1t} m^{tk} \Gamma^4_{1k} \tag{5.22}$$

hold (at least at $q = 0$), then the system $\overline{\mathcal{R}}_1^{(1)}$ is involutive.

To conclude this section, we now point out the role of the auxiliary functions g_1 and g_2. When they were first introduced, we brought up some more PDEs than originally proposed, which might be a disadvantage. However, these additional functions help develop the involutive system of PDEs in the following aspects:

1. It reduces the number of prolongations to be done under the Cartan-Kuranishi scheme; Prolonging g_1 and g_2 is basically equivalent to prolonging the original PDEs twice.

2. It allows us to define $\widehat{T}_{13}$ through a purely algebraic equation: Without them it is hard to define $\widehat{T}_{13}$ as the equation involves derivatives of other unknowns.

3. We have two more free dependent variables to work: This is how the proof of Theorem 5.2.4 can be done.

5.3 The Case when $n \geq 4$ and the Shapability Theorem

We first complete our argument of finding an equivalent involutive system for an arbitrary degree of freedom $n \geq 4$. The generalization to the case $n \geq 4$ is in fact rather straightforward. First of all, Φ_1 to Φ_8 remain the same except that the indices $r, s, t, \ldots$ runs from 1 to n instead of 1 to 4. We need n linearly independent differential operators X_i, that is,

$$\begin{aligned} X_1 &= \widehat{T}_{1s} m^{sk} \frac{\partial}{\partial q^k} \\ X_2 &= \widehat{T}_{2s} m^{sk} \frac{\partial}{\partial q^k} \\ X_i &= \delta_{is} m^{sk} \frac{\partial}{\partial q^k}, \qquad i \geq 3 \ . \end{aligned}$$

As before, we can place a further assumption that the differential operators X_1 and X_2 span an involutive distribution, that is, assumption (5.2).[1] The way we choose to define X_i allows f_1 and f_2 in (5.2) to remain purely algebraic, as in Lemma 5.1.1. The only difference for $n > 4$ is the number of extra equations due to this involutivity assumption. Previously when $n = 4$, we have two extra PDEs (Φ_9 and Φ_{10}). When $n > 4$, we would have $n - 2$

[1] Again, what we mean here is to find a solution which satisfies the matching conditions *and* this extra assumption on the differential operators.

extra PDEs:

$$\det(X_1, X_2, [X_1, X_2], X_4, X_5, \ldots, X_{n-1}, X_n) = 0$$
$$\det(X_1, X_2, [X_1, X_2], X_3, X_5, \ldots, X_{n-1}, X_n) = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\det(X_1, X_2, [X_1, X_2], X_3, X_4, \ldots, X_{n-2}, X_{n-1}) = 0 \ .$$

In other words, every time n increases by 1, we have one additional PDE. Nevertheless, we have two more entries in $\widehat{T}$ in the meantime. Indeed, we can assign each of these extra PDEs to solve the class 4 derivatives of $\widehat{T}_{23}, \widehat{T}_{24}, \ldots, \widehat{T}_{2n}$, and still have some free entries in the first row of $\widehat{T}$. Notice that (5.10) and (5.11) will guarantee that we can solve these class n derivatives.

Finally, the proof of Lemma 5.2.4 (i.e. the integrability conditions for $\widehat{T}_{\alpha\beta}$ are all of class n) is essentially the same for $n > 4$. Hence, if we define $\frac{\partial}{\partial q^n}$ as class n derivatives etc., then we will have the following generalization of Theorem 5.2.5.

Theorem 5.3.1 $\overline{\mathcal{R}}_1^{(1)}$ *is involutive if the following holds (at least at $q = 0$)*

$$X_1^{n-1} X_2^n - X_2^{n-1} X_1^n \neq 0 \tag{5.23}$$
$$m^{s3} \frac{\partial^2 V}{\partial q^s \partial q^2} \neq 0 \tag{5.24}$$
$$\widehat{T}_{1s} m^{sn} \neq 0 \tag{5.25}$$
$$\widehat{T}_{1s} m^{sn} - \frac{m^{3k} \frac{\partial^2 V}{\partial q^k \partial q^1}}{m^{3s} \frac{\partial^2 V}{\partial q^s \partial q^2}} \widehat{T}_{2t} m^{tn} \neq 0 \tag{5.26}$$
$$\widehat{T}_{2s} m^{sn} \neq 0 \tag{5.27}$$
$$\widehat{T}_{1s} m^{sn} \widehat{T}_{1t} m^{tk} \Gamma^4_{2k} \neq \widehat{T}_{2s} m^{sn} \widehat{T}_{1t} m^{tk} \Gamma^4_{1k} \ . \tag{5.28}$$

As before, similar conditions can be derived if we prioritize partials in various different manners. In particular, when we rank $\frac{\partial}{\partial q^1}$ as class n derivatives, etc., then we will have the following alternate generalization of Theorem 5.2.5.

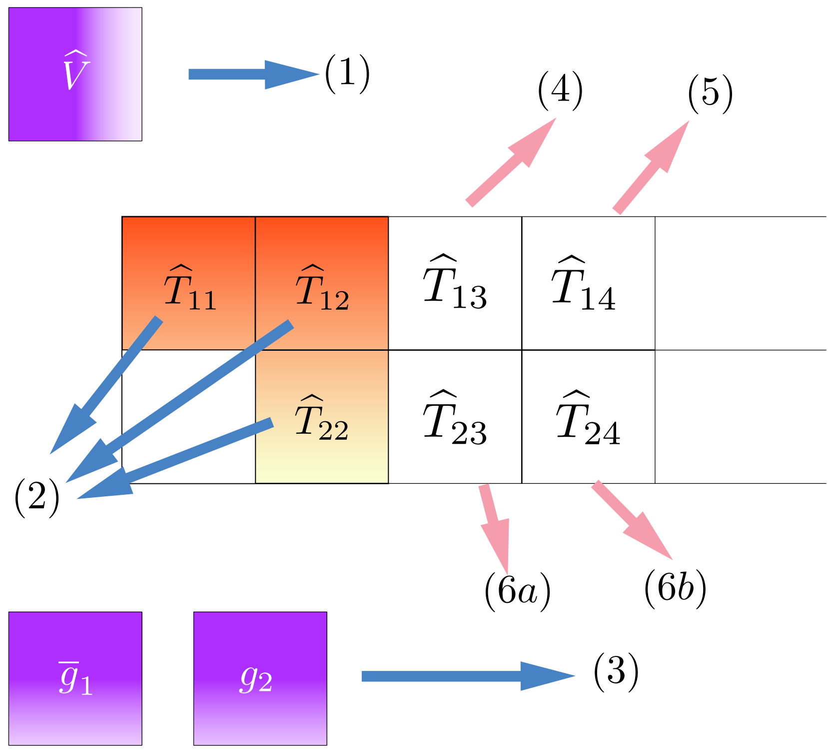

Figure 5.1: The allocation of each entries in the $\widehat{T}$ matrix. Only the first two rows of $\widehat{T}$ appear in the system of PDEs. (1) $\widehat{V}$ defined by the 2 potential PDEs; (2) $\widehat{T}_{\alpha\beta}$ defined by the 6 kinetic PDEs; (3) $\frac{\partial \overline{g}_1}{\partial q^1}, \frac{\partial g_2}{\partial q^1}$ defined by two of the integrability conditions of $\widehat{T}_{\alpha\beta}$; (4) $\widehat{T}_{13}$ algebraically defined by the integrability condition for $\widehat{V}$; (5) $\frac{\partial \widehat{T}_{14}}{\partial q^1}$ defined by one of the integrability condition for $\widehat{T}_{\alpha\beta}$; (6a) $\frac{\partial \widehat{T}_{23}}{\partial q^1}$ defined by $\det(X_1, X_2, [X_1, X_2], X_4, X_5, \cdots, X_n) = 0$; (6b) $\frac{\partial \widehat{T}_{24}}{\partial q^1}$ defined by $\det(X_1, X_2, [X_1, X_2], X_3, X_5, X_6, \cdots, X_n) = 0$. When $n > 4$, we can arbitrarily associate the rest of the determinant equations to $\widehat{T}_{\alpha b}$, where $\alpha = 1, 2$ and $b = 5, 6, 7, \cdots$.

Corollary 5.3.2 $\overline{\mathcal{R}}_1^{(1)}$ *is involutive if* (5.17) *to* (5.22) *hold (at least at* $q = 0$*).*

Since $\overline{\mathcal{R}}_1^{(1)}$ is involutive, it is natural to ask if we have an analytic solution. The answer is affirmative by the following theorem of stabilizability.

Theorem 5.3.3 *Let* $(L, 0, W)$ *be a controlled Lagrangian system with* $n \geq 4$ *degrees of freedom having a linearized system* $(L^\ell, 0, W^\ell)$*. Suppose the uncontrollable dynamics of* $(L^\ell, 0, W^\ell)$*, if any, is oscillatory, and that there exists a linear controlled Lagrangian system* $(\overline{L}, 0, \overline{W})$ *feedback equivalent to* $(L^\ell, 0, W^\ell)$ *such that the inequations* (5.1)*,* (5.7)*,* (5.10)*,* (5.11)*,* (5.12) *and* (5.13) *are satisfed at* $q = 0$.[2]

Then there exists a controlled Lagrangian system $(\widehat{L}, \widehat{F}, \widehat{W})$ *that is feedback equivalent to* $(L, 0, W)$*, with a positive definite mass matrix* $\widehat{m}$*, a gyroscopic force* $\widehat{F}$ *of degree 2, and a potential function* $\widehat{V}$ *having a non-degenerate minimum at* $q = 0$*. In particular, we can obtain a nonlinear controlled Lagrangian system* $(\widehat{L}, \widehat{F}, \widehat{W})$ *whose linearization is equal to* $(\overline{L}, 0, \overline{W})$*. Furthermore, if* $(L^\ell, 0, W^\ell)$ *is controllable, then any linear dissipative feedback force onto* $\widehat{W}$ *exponentially stabilizes the system* $(\widehat{L}, \widehat{F}, \widehat{W})$*.*

Proof We first need to check that defining $\widehat{T}_{13}$ by (5.9) does not bring any extra restriction to the linearized system. Indeed, at $q = 0$, (5.9) reduces to

$$\begin{aligned} & \widehat{T}_{2s} m^{sk} \frac{\partial^2 V}{\partial q^k \partial q^1}\bigg|_{q=0} = \widehat{T}_{1s} m^{sk} \frac{\partial^2 V}{\partial q^k \partial q^1}\bigg|_{q=0} \\ \Rightarrow \qquad & \widehat{T}_{2s} m^{sk} \frac{\partial}{\partial q^k}\left(\widehat{T}_{1t} m^{tl} \frac{\partial \widehat{V}}{\partial q^l}\right)\bigg|_{q=0} = \widehat{T}_{1s} m^{sk} \frac{\partial}{\partial q^k}\left(\widehat{T}_{2t} m^{tl} \frac{\partial \widehat{V}}{\partial q^l}\right)\bigg|_{q=0}. \end{aligned}$$

Since $\frac{\partial \widehat{V}}{\partial q^i}(0) = 0$, the above equation reduces further to

$$\widehat{T}_{2s} m^{sk} \widehat{T}_{1t} m^{tl} \left.\frac{\partial^2 \widehat{V}}{\partial q^k \partial q^l}\right|_{q=0} = \widehat{T}_{1s} m^{sk} \widehat{T}_{2t} m^{tl} \left.\frac{\partial^2 \widehat{V}}{\partial q^l \partial q^k}\right|_{q=0},$$

[2]Here it is understood that $\widehat{T}_{ij}(0)$ are replaced by $\overline{T}_{ij}$, and $\widehat{V}(0)$ by the potential energy of $(\overline{L}, 0, \overline{W})$ in those inequations.

which is obviously true.

Hence, we conclude that there are analytic solutions for $\widehat{T}$ and $\widehat{V}$ once we impose suitable initial conditions. We look for initial conditions from the linearized system $(L^\ell, 0, W^\ell)$ of the given controlled Lagrangian system $(L, 0, W)$. It can be proven (c.f. [8]) that there exists a linear controlled Lagrangian system $(\overline{L}, 0, \overline{W})$ which is feedback equivalent to $(L^\ell, 0, W^\ell)$, and which has a positive definite symmetric mass matrix $\overline{M}$ and a potential energy $\overline{U} = \frac{1}{2}q^T\overline{S}q$, where $\overline{S}$ is positive definite and symmetric, if and only if the uncontrollable dynamics of $(L, 0, W)$, if any, is oscillatory. Then, $\overline{U}$ and the corresponding $\overline{T} = m(0)\overline{M}^{-1}m(0)$ can serve as the initial condition for the PDEs governing the unknown nonlinear $\widehat{V}$ and $\widehat{T}$. Thus, we can now apply the Cartan-Kähler theorem on the first order system to conclude the existence of a solution. Using a continuity argument, we can ensure that the nonlinear solutions $\widehat{m}$ and $\widehat{V}$ to this initial value problem are positive definite (at least locally around $q = 0$).

For exponential stability, it can be proved (cf. [8]) that any linear mechanical system, with positive definite mass matrix m and potential energy V, is controllable if and only if it can be exponentially stabilized by a linear dissipative feedback. Then the Lyapunov linearization method can be used to conclude that the same feedback can exponentially stabilize the given nonlinear system. ■

5.4 Example: Three Linked Carts with Inverted Pendulum

We illustrate the use of the theorems developed in this paper through an example of three linked carts with an inverted pendulum.

For simplicity, we assume point masses for the carts and the inverted pendulum, each with

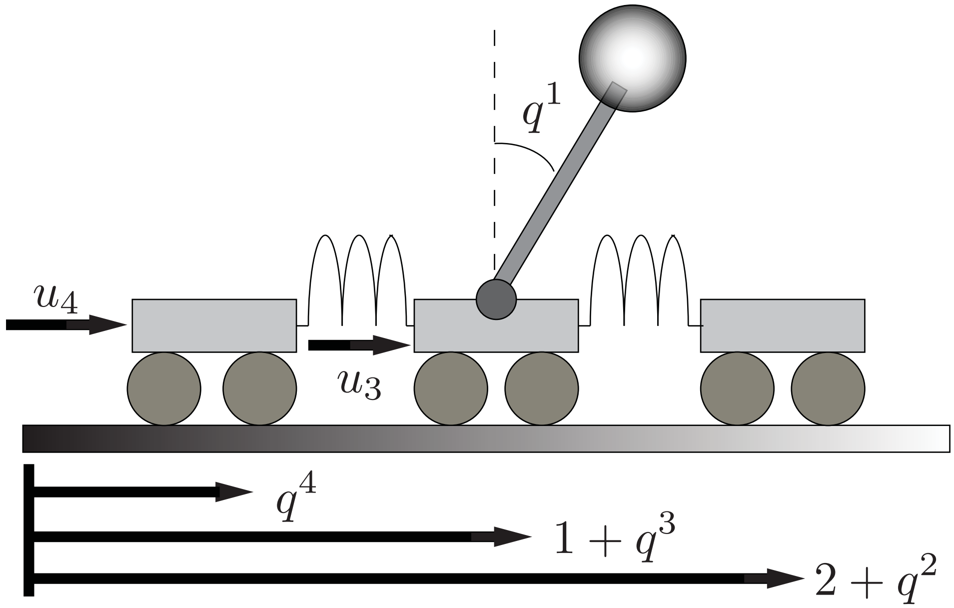

Figure 5.2: Three linked carts with an inverted pendulum.

a mass of 1 kg. The pendulum has a length of 1 m and each spring has a natural length of 1 m. We take $g = 98/10$ ms^{-2}. Note that due to the natural length of the springs, we denote the distance of each cart from the origin as shown in Figure 5.2 so that at equilibrium, $q^i = 0$ for all $i = 1, \cdots, 4$. In this case we can compute the mass matrix

$$m = \begin{bmatrix} 1 & 0 & \cos q^1 & 0 \\ 0 & 1 & 0 & 0 \\ \cos q^1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the potential energy

$$V = \frac{1}{2}\left((1+q^2-q^3)^2 + (1+q^3-q^4)^2\right) + 9.8\cos q^1$$

for the system. The control bundle W is spanned by dq^3 and dq^4. Now, notice that the Christoffel symbols Γ^i_{jk} are zero at $q = 0$. Hence, to ensure that (5.13) is still satisfied (at least at $q = 0$), we do the following change of coordinates: $q^i = z^i$ for $i = 1, 2, 3$ and $q^4 = z^1 z^4 + z^4$. By so doing, only $\Gamma^4_{14} = \Gamma^4_{41}$ are nonzero at $z = 0$. Under the new coordinates,

$$m = \begin{bmatrix} 1 + (z^4)^2 & 0 & \cos(z^1) & z^4(z^1+1) \\ 0 & 1 & 0 & 0 \\ \cos(z^1) & 0 & 2 & 0 \\ z^4(z^1+1) & 0 & 0 & (z^1+1)^2 \end{bmatrix},$$

and the potential energy is

$$V = \frac{1}{2}((1 + z^2 - z^3)^2 + (1 + z^3 - z^1 z^4 - z^4)^2) + 9.8 \cos z^1.$$

We now need to impose suitable initial conditions for $\widehat{T}$ and $\widehat{V}$ in the new coordinates. Following [8], we can set up these initial conditions by considering the linearization of the given system. The linearized system has a mass matrix given by

$$m^\ell = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It can be proved that the linearized system is controllable. A feedback equivalent system $(\overline{L}, 0, \overline{W})$ is given by

$$\overline{T} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 10 & 4 & 1 \\ 3 & 4 & 100 & 0 \\ 1 & 1 & 0 & 100 \end{bmatrix}, \qquad \overline{S} = \begin{bmatrix} \frac{367}{50} & -\frac{7}{25} & \frac{3}{2} & -\frac{1}{5} \\ -\frac{7}{25} & \frac{11}{100} & -\frac{1}{4} & -\frac{1}{10} \\ \frac{3}{2} & -\frac{1}{4} & 1 & 0 \\ -\frac{1}{5} & -\frac{1}{10} & 0 & 1 \end{bmatrix},$$

both of which are positive definite. Furthermore, we can check that $\widehat{T}$ and $\widehat{V}$ satisfy the inequalities (5.1), (5.7), (5.10), (5.11), (5.12) and (5.13) around $z = 0$. Hence, a solution exists by Theorem 5.3.3. We can now incorporate these initial conditions to the system of PDEs, leading to the following solutions

$$\begin{aligned}
\widehat{T}_{11} &= 2\cos^2 z^1 - 1 + 2z^4 + 100(z^4)^2 \\
\widehat{T}_{12} &= 2\cos z^1 + z^4 \\
\widehat{T}_{13} &= 3\cos z^1 \\
\widehat{T}_{14} &= (z^1+1)(100z^4+1) \\
\widehat{T}_{22} &= 10 \\
\widehat{T}_{23} &= 4 \\
\widehat{T}_{24} &= z^1 + 1 \\
\widehat{V} &= (F(z^1,z^2,z^3))^2 + (G(z^1,z^2,z^4))^2 + \frac{4}{25}\cos^2 z^1 - \frac{49}{5}\cos z^1 \\
&\quad - \frac{6}{25} + \frac{1}{50}(10 + 2z^2 - 10z^3)\sin z^1 + \frac{3}{50}(z^2)^2 + \frac{1}{50}(5 - 5z^3)z^2,
\end{aligned}$$

where $F(z^1,z^2,z^3) = \frac{8}{5}\sin z^1 - \frac{z^2}{5} + z^3$ and $G(z^1,z^2,z^4) = -\frac{1}{5}\sin z^1 + \frac{(z^1)^2}{2} - \frac{z^2}{10} + z^1z^4 + z^4$. It is easily checked that $\widehat{V}$ is positive definite at $z = 0$. The same is true for $\widehat{T}$, when we assign $\widehat{T}_{33}$ and $\widehat{T}_{44}$ in such a way that they are 100 when $z = 0$. Hence, we have shaped the energy of the given system, and by its linear controllability, we can conclude that the resulting feedback equivalent system can be asymptotically stabilized by an additional dissipative feedback.

5.5 Energy Shaping on Systems with Gyroscopic Forces

We end this chapter by deriving the corresponding shapability criteria when a given mechanical system has an external gyroscopic forces of degrees two.

The matching conditions for a given controlled Lagrangian system (L, F, W) where F is gyroscopic are given by the following [8]:

$$\widehat{T}_{\alpha s} m^{sk} \frac{\partial \widehat{V}}{\partial q^k} = \frac{\partial V}{\partial q^\alpha}$$
$$\widehat{J}_{\alpha\beta\gamma} + \widehat{J}_{\beta\gamma\alpha} + \widehat{J}_{\gamma\alpha\beta} = 0,$$

where in the presence of gyroscopic forces,

$$\widehat{J}_{\alpha\beta\gamma} = \frac{1}{2} \widehat{T}_{\gamma s} m^{sk} \left(\frac{\partial \widehat{T}_{\alpha\beta}}{\partial q^k} - \Gamma^i_{\alpha k} \widehat{T}_{\beta i} - \Gamma^i_{\beta i} \widehat{T}_{\alpha i} \right) - \widehat{T}_{\alpha r} \widehat{T}_{\beta s} m^{ri} m^{sj} B_{ij\gamma},$$

in which B_{ijk} are the gyroscopic force terms for the external force F acting on the given mechanical system, i.e.

$$F = B_{ijk} \dot{q}^i \dot{q}^j dq^k.$$

To find an equivalent involutive system of PDEs, we can follow the argument as in the case where there is no gyroscopic force term, i.e. by assuming involutive distribution for the vector fields spanned by X_1 ansd X_2 and then introducing auxiliary functions g_1 and g_2, so that we have for $n = 4$, the following system of PDEs:

$$\overline{\mathcal{R}}_1 : \begin{cases} \Phi_1 : & \widehat{T}_{1s} m^{sk} \frac{\partial \widehat{V}}{\partial q^k} = \frac{\partial V}{\partial q^1} \\ \Phi_2 : & \widehat{T}_{2s} m^{sk} \frac{\partial \widehat{V}}{\partial q^k} = \frac{\partial V}{\partial q^2} \\ \Phi_3 : & \widehat{T}_{1s} m^{sk} \left(\frac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) - 2\widehat{T}_{1r} \widehat{T}_{1s} m^{ri} m^{sj} B_{ij1} = 0 \\ \Phi_4 : & \widehat{T}_{2s} m^{sk} \left(\frac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) - 2\widehat{T}_{1r} \widehat{T}_{1s} m^{ri} m^{sj} B_{ij2} = -2g_1 \\ \Phi_5 : & \widehat{T}_{1s} m^{sk} \left(\frac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k} \widehat{T}_{2r} - \Gamma^r_{2k} \widehat{T}_{1r} \right) - 2\widehat{T}_{1r} \widehat{T}_{2s} m^{ri} m^{sj} B_{ij1} = g_1 \\ \Phi_6 : & \widehat{T}_{2s} m^{sk} \left(\frac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k} \widehat{T}_{2r} - \Gamma^r_{2k} \widehat{T}_{1r} \right) - 2\widehat{T}_{1r} \widehat{T}_{2s} m^{ri} m^{sj} B_{ij2} = g_2 \\ \Phi_7 : & \widehat{T}_{1s} m^{sk} \left(\frac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k} \widehat{T}_{2r} \right) - 2\widehat{T}_{2r} \widehat{T}_{2s} m^{ri} m^{sj} B_{ij1} = -2g_2 \\ \Phi_8 : & \widehat{T}_{2s} m^{sk} \left(\frac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k} \widehat{T}_{2r} \right) - 2\widehat{T}_{2r} \widehat{T}_{2s} m^{ri} m^{sj} B_{ij2} = 0 \\ \Phi_9 : & \det(X_1, X_2, [X_1, X_2], X_4) = 0 \\ \Phi_{10} : & \det(X_1, X_2, X_3, [X_1, X_2]) = 0 \ . \end{cases}$$

Then, as in Theorem 5.1.1, if we write $[X_1, X_2] = f_1 X_1 + f_2 X_2$, then it can be proved that both f_1 and f_2 are algebraic after replacing all the derivatives using the defining equations in $\overline{\mathcal{R}}_1$. In this case, the expressions $Expr_1$ and $Expr_2$ in Theorem 5.1.1 become

$$Expr_a = (-1)^{a+1} g_a + \widehat{T}_{ar}\widehat{T}_{2s} m^{sj}(2m^{ri}B_{ij1} - \Gamma^r_{1j}) - \widehat{T}_{ar}\widehat{T}_{1s} m^{sj}(2m^{ri}B_{ij2} - \Gamma^r_{2j}),$$

for $a = 1, 2$.

Again, we can define $\widehat{T}_{13}$ algebraically by the integrability condition for $\widehat{V}$, which reads

$$\frac{1}{\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2}\left[Expr_1\left(\frac{\partial V}{\partial q^1}\widehat{T}_{22} - \frac{\partial V}{\partial q^2}\widehat{T}_{12}\right) + Expr_2\left(\frac{\partial V}{\partial q^1}\widehat{T}_{11} - \frac{\partial V}{\partial q^2}\widehat{T}_{12}\right)\right]$$
$$= \widehat{T}_{1s} m^{sk}\frac{\partial^2 V}{\partial q^k \partial q^2} - \widehat{T}_{2s} m^{sk}\frac{\partial^2 V}{\partial q^k \partial q^1}.$$

In other words, $\widehat{T}_{13}$ is algebraically defined by the integrability condition for $\widehat{V}$ as long as (5.7) holds. We replace g_1 by $\overline{g}_1$ to avoid any quadratic terms of $\widehat{T}_{13}$:

$$g_1 = \overline{g}_1 + \frac{1}{3}(\widehat{T}_{13})^2 m^{3j}(2m^{3i}B_{ij2} - \Gamma^3_{2j}).$$

The determinant equations Φ_9 and Φ_{10} arising from the involutive distribution assumption can define $\dfrac{\partial \widehat{T}_{23}}{\partial q^1}$ and $\dfrac{\partial \widehat{T}_{24}}{\partial q^1}$ provided that (5.10) and (5.11) hold.

However, we need some modifications when we come to the integrability conditions for $\widehat{T}_{\alpha\beta}$.

Let us consider the integrability condition for $\widehat{T}_{11}$. Since $\overline{g}_1$ and g_1 differ by some terms without any $\widehat{T}_{14}$ (and $\widehat{T}_{24}$), it is safe to copy all the steps in the proof of Theorem 5.2.4, only to add the gyroscopic terms accordingly to give

$$(X_1^s X_2^k - X_2^s X_1^k)\frac{\partial}{\partial q^s}(\Gamma^{\overline{r}}_{1k}\widehat{T}_{1\overline{r}}) - X_1\overline{g}_1 + X_1(\widehat{T}_{1r}\widehat{T}_{1s}m^{ri}m^{sj}B_{ij2}) - X_2(\widehat{T}_{1r}\widehat{T}_{1s}m^{ri}m^{sj}B_{ij1}) + \cdots$$
$$= 0$$

at the level of the symbol $\overline{G}_1$. Once again for simplicity we omit all the terms that do not have the derivatives of $\overline{g}_1$ or $\widehat{T}_{14}$. Using the fact that $B_{ijk} = B_{jik}$, one can simplify the above integrability condition:

$$(X_1^s X_2^k - X_2^s X_1^k)\Gamma_{1k}^4 \frac{\partial \widehat{T}_{14}}{\partial q^s} - X_1 \overline{g}_1 + 2\widehat{T}_{1s} m^{4i} m^{sj} B_{ij2} X_1 \widehat{T}_{14}$$
$$-2\widehat{T}_{1s} m^{4i} m^{sj} B_{ij1} X_2 \widehat{T}_{14} + \cdots = 0. \quad (5.29)$$

The integrability conditions for $\widehat{T}_{22}$ and $\widehat{T}_{12}$ can be manipulated in the usual manner, only to notice that $X_1 \widehat{T}_{2\overline{r}} = X_2 \widehat{T}_{1\overline{r}}$:

$$X_2^s X_s^k \Gamma_{2k}^4 \frac{\partial \widehat{T}_{14}}{\partial q^s} + X_2 g_2 + 2\widehat{T}_{2s} m^{4i} m^{sj} B_{ij2} X_2 \widehat{T}_{14} + \cdots = 0 \quad (5.30)$$

$$X_2^s X_2^k \Gamma_{1k}^4 \frac{\partial \widehat{T}_{14}}{\partial q^s} + (X_1^s X_2^k - X_2^s X_1^k)\Gamma_{2k}^4 \frac{\partial \widehat{T}_{14}}{\partial q^s} + X_1 g_2 - X_2 \overline{g}_1$$
$$+2(\widehat{T}_{2s} m^{4i} m^{sj} B_{ij2} X_1 \widehat{T}_{14} + \widehat{T}_{1s} m^{sj} m^{4i} B_{ij2} X_2 \widehat{T}_{14} - \widehat{T}_{2s} m^{4i} m^{sj} B_{ij1} X_1 \widehat{T}_{14}) + \cdots = 0. \quad (5.31)$$

We are now ready to associate each of the above integrability conditions (5.29)–(5.31) to the derivatives of $\overline{g}_1$, g_2 and $\widehat{T}_{14}$. As we know from the example in section 5.4 the Christoffel symbols are usually zero at $q = 0$, we can simply consider the determinant of the coefficient matrix for $\frac{\partial \overline{g}_1}{\partial q^1}, \frac{\partial g_2}{\partial q^1}$ and $\frac{\partial \widehat{T}_{14}}{\partial q^1}$ by ignoring all terms that contain those Christoffel symbols. This leads to the following coefficient matrix:

$$\begin{bmatrix} 2\widehat{T}_{1s} m^{4i} m^{sj} (B_{ij2} \widehat{T}_{1t} m^{t1} - B_{ij1} \widehat{T}_{2t} m^{t1}) & -\widehat{T}_{1x} m^{x1} & 0 \\ 2\widehat{T}_{2s} m^{4i} m^{sj} B_{ij2} \widehat{T}_{1t} m^{t1} & 0 & \widehat{T}_{2y} m^{y1} \\ \widehat{T}_{2s} m^{4i} m^{sj} (B_{ij2} - B_{ij1}) \widehat{T}_{1t} m^{t1} + \widehat{T}_{1s} m^{4i} m^{sj} B_{ij2} \widehat{T}_{2t} m^{t1} & -\widehat{T}_{2x} m^{x1} & \widehat{T}_{1y} m^{y1} \end{bmatrix},$$

whose determinant, denoted as Coe here, is given by

$$Coe := (\widehat{T}_{1s} m^{4i} m^{sj} B_{ij2} + \widehat{T}_{2s} m^{4i} m^{sj} (B_{ij1} - B_{ij2})) \widehat{T}_{1t} m^{t1} (\widehat{T}_{2y} m^{y1})^2$$
$$+ 2\widehat{T}_{2s} m^{4i} m^{sj} B_{ij2} (\widehat{T}_{1t} m^{t1})^3 - 2\widehat{T}_{2s} m^{4i} m^{sj} B_{ij1} (\widehat{T}_{2t} m^{t1})^3.$$

Therefore, we can define the derivatives $\dfrac{\partial \overline{g}_1}{\partial q^1}, \dfrac{\partial g_2}{\partial q^1}$ and $\dfrac{\partial \widehat{T}_{14}}{\partial q^1}$ if and only if

$$Coe \neq 0, \tag{5.32}$$

at $q = 0$. As a result, we can state the theorem of energy shapability when the given system also has gyroscopic force terms:

Theorem 5.5.1 *Let (L, F, W) be a controlled Lagrangian system with $n = 4$ degrees of freedom having a linearized system $(L^\ell, 0, W^\ell)$, and F is the external gyroscopic force acting on the nonlinear system. Suppose the uncontrollable dynamics of $(L^\ell, 0, W^\ell)$, if any, is oscillatory, and that there exists a linear controlled Lagrangian system $(\overline{L}, 0, \overline{W})$ feedback equivalent to $(L^\ell, 0, W^\ell)$ such that the inequations* (5.1)*,* (5.7)*,* (5.10)*,* (5.11) *and* (5.32) *are satisfed at $q = 0$.*

Then there exists a controlled Lagrangian system $(\widehat{L}, \widehat{F}, \widehat{W})$ that is feedback equivalent to (L, F, W), with a positive definite mass matrix $\widehat{m}$, a gyroscopic force $\widehat{F}$ of degree 2, and a potential function $\widehat{V}$ having a non-degenerate minimum at $q = 0$. In particular, we can obtain a nonlinear controlled Lagrangian system $(\widehat{L}, \widehat{F}, \widehat{W})$ whose linearization is equal to $(\overline{L}, 0, \overline{W})$. Furthermore, if $(L^\ell, 0, W^\ell)$ is controllable, then any linear dissipative feedback force onto $\widehat{W}$ exponentially stabilizes the system $(\widehat{L}, \widehat{F}, \widehat{W})$.

Remarks

1. A similar result can be stated for $n \geq 4$ and/or with different choices of coordinates.

2. It should be noted that gyroscopic force terms do not appear in the linearized system. In other words, given the same Euler-Lagrangian, the linearization is the same no matter what external gyroscopic force acts on the system. We have more to say about this property in the epilogue.

Chapter 6

Epilogue

In this thesis we investigated the energy shapability of controlled Lagrangian systems. For systems with at least four degrees of freedom and exactly two degrees of underactuation, we used the formal theory of PDEs to derive the corresponding criteria under which energy shaping is possible. We also illustrated the criteria of energy shapability with a three-cart-one-inverted pendulum example.

Nevertheless, it should be noted that there are still a number of open questions still remaining in the realm of energy shaping. First of all, our approach as described in Chapter 5 cannot be applied directly to the case when the degrees of freedom n is 3 while the degree of underactuation n_1 remains 2. Recall that our success in Chapter 5 lies on the fact that some of the $\widehat{T}$ entries are associated with two PDEs so that we have integrability conditions for these entries, but for the remaining entries of $\widehat{T}$ we can associate each PDE to different entries of $\widehat{T}$, thus avoiding more integrability conditions arising. This does not work in the case when $n = 3$ and $n_1 = 2$. In particular, Lemma 5.2.4 does not work as all the free variables from $\widehat{T}$ have already been exhausted (not to mention the non-existence of $\widehat{T}_{14}$ in a 3×3 matrix). As a result, we have more integrability conditions than before, implying that to obtain an equivalent involutive system one needs to continue the Cartan-Kuranishi argument in which most computations will be far more tedious.

Another natural extension of this problem is the case where the degree of underactuation n_1 goes beyond 2. Generally speaking, we have n_1 PDEs for $\widehat{V}$, but the number of PDEs for $\widehat{T}$ increases faster than the order of n_1 as n_1 increases, e.g. when $n_1 = 2$ we have 4 PDEs for $\widehat{T}$ from the original system of PDEs, but when $n_1 = 3$ and $n_1 = 4$, we have 10 and 20 respectively. Thus, using the formal theory of PDEs to tackle the problem of higher degrees of underactuation becomes a more challenging task.

Yet another question to ask is whether the extra assumption on the distribution spanned by the differential operators $\widehat{T}_{1s} m^{sk} \frac{\partial}{\partial q^k}$ and $\widehat{T}_{2s} m^{sk} \frac{\partial}{\partial q^k}$ can be dropped. Although it seems natural to remove this artificial assumption, the consequence of such removal can be enormous: We have more integrability conditions for $\widehat{V}$, whose coefficients of the leading derivatives are all vanishing at the equilibrium point. We thus need a rather different approach than the one presented in Chapter 5 in order to tackle the regularity issue.

Towards the end of Chapter 5 we discussed the energy shaping problem for a given system with gyroscopic external force F. We mentioned that for the same Euler-Lagrangian L, $(L, 0, W)$ and (L, F, W) share the same linearized system. Thus, we may ask the following natural question: Suppose for each fixed $\epsilon > 0$, we choose a nonzero gyroscopic force $F_\epsilon = F(q, \dot{q}, \epsilon)$ with $F_\epsilon \to 0$ as $\epsilon \to 0$, giving rise to a family of controlled Lagrangian systems (L, F_ϵ, W). Suppose further that after linearization we have a feedback equivalent linear system $(\overline{L}, 0, \overline{W})$[1] satisfying (5.1), (5.7), (5.10), (5.11) and (5.32). Then we know that there exists a feedback equivalent nonlinear system $(\widehat{T}_\epsilon, \widehat{F}_\epsilon, \widehat{W}_\epsilon)$ to (L, F_ϵ, W) for each fixed $\epsilon > 0$. Can we pass the limit $\epsilon \to 0$ to obtain a well-defined solution? And if such limit exists, is it feedback equivalent to the original given system $(L, 0, W)$? Under what condition(s) can we have such a well-defined answer?

Although it may seem as a detour, it helps improve our shapability theorem if we know a

[1]This linearized system does not depend on ϵ, as it should be feedback equivalent to $(L^\ell, 0, W^\ell)$, the linearization of the given system $(L, 0, W)$ (or (L, F_ϵ, W)).

positive answer to the above question with regard to a one-parameter family of controlled Lagrangian systems with varying gyroscopic forces: Since we have a choice to choose the gyroscopic force (at least theoretically speaking), we can bypass the trouble of vanishing Christoffel symbols at $q = 0$, thus avoiding the change of coordinates which appear when we computed the example of three linked carts with one inverted pendulum.

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