Financial Risk Management of Guaranteed Minimum Income Benefits Embedded in Variable Annuities

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Claymore James Marshall

Abstract

A guaranteed minimum income benefit (GMIB) is a long-dated option that can be embedded in a deferred variable annuity. The GMIB is attractive because, for policyholders who plan to annuitize, it offers protection against poor market performance during the accumulation phase, and adverse interest rate experience at annuitization. The GMIB also provides an upside equity guarantee that resembles the benefit provided by a lookback option.

We price the GMIB, and determine the fair fee rate that should be charged. Due to the long dated nature of the option, conventional hedging methods, such as delta hedging, will only be partially successful. Therefore, we are motivated to find alternative hedging methods which are practicable for long-dated options. First, we measure the effectiveness of static hedging strategies for the GMIB. Static hedging portfolios are constructed based on minimizing the Conditional Tail Expectation of the hedging loss distribution, or minimizing the mean squared hedging loss. Next, we measure the performance of semi-static hedging strategies for the GMIB. We present a practical method for testing semi-static strategies applied to long term options, which employs nested Monte Carlo simulations and standard optimization methods. The semi-static strategies involve periodically rebalancing the hedging portfolio at certain time intervals during the accumulation phase, such that, at the option maturity date, the hedging portfolio payoff is equal to or exceeds the option value, subject to an acceptable level of risk. While we focus on the GMIB as a case study, the methods we utilize are extendable to other types of long-dated options with similar features.

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To my parents

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Chapter 1

Introduction

This thesis investigates the valuation and financial risk management of the guaranteed minimum income benefit (GMIB). A GMIB is an option that can be embedded in a deferred variable annuity. This option usually has a term to expiry of at least 10 years. The GMIB is attractive because, for policyholders who plan to annuitize, it offers protection against poor market performance during the accumulation phase, and adverse interest rate experience at annuitization. The GMIB also provides an upside equity guarantee that resembles the benefit provided by a lookback option, which allows policyholders to benefit from strong market performance during the accumulation phase, which subsequently deteriorates before maturity. Furthermore, in the case where a GMIB is embedded in a life annuity, the GMIB helps protect against individual longevity risk by guaranteeing a minimum annuity payment rate at annuitization (which may be higher than the fair annuity payment rate at annuitization, if mortality improves significantly more than expected during the accumulation phase). These features make the GMIB an interesting option to price, and a very challenging option to hedge.

In the context of this thesis, financial risk management involves understanding the risks associated with selling the GMIB, how the risks can be mitigated or controlled, and devising feasible hedging strategies for the GMIB. While this thesis focuses on the GMIB as a case study, the valuation and financial risk management methods we use are extendable to other types of long term options with similar features. One of the aims of this thesis is to be practitioner friendly. The methods are dependent on Monte Carlo simulation, which is flexible, and widely used in practice by practitioners for analyzing complex financial problems. Understanding the risks involved in providing a complex financial guarantee, and how they can be managed, are the cornerstones to financial risk management. Hopefully, some readers will be able to adopt the methods we employ as templates for their own financial problems, making appropriate adjustments as necessary on a case by case basis. We present decompositions of the risks of the GMIB, quantitatively and visually, in order to understand how the risks affect the value of the option and the performance of the hedging strategies. Analogous decompositions may be possible for other long term options.

The structure of Chapter 1 is as follows. In Section 1.1, we discuss the range of popular variable annuity options offered in the U.S. variable annuity market. Section 1.2 discusses the motivation for this thesis. In Section 1.3, we define the maturity value of a variable option with an embedded GMIB option, and discuss the assumptions adopted in order to make the pricing and risk management of the GMIB a manageable task. To give the reader an intuitive understanding of how the GMIB operates, in Section 1.4, we show how the GMIB maturity value behaves for five distinct plausible hypothetical scenarios. Section 1.5 presents the contributions of this thesis concisely. Section 1.6 outlines the structure of this thesis.

1.1 Variable annuity options

With a deferred variable annuity policy, the policyholder pays a large upfront (annuity) premium to the insurance company, which is then invested in the financial markets for many years (the accumulation phase). During the accumulation phase the policyholder may make partial withdrawals or pay further premiums. The accumulation phase ends when the policyholder decides to either receive the balance of their investment account as a lump sum benefit, or annuitize their investment to provide retirement income. We refer to the time point at which this occurs, which is random and depends on the policyholder's behavior, as the maturity date.

A deferred variable annuity (frequently simply referred to as a "variable annuity", and thus we henceforth omit "deferred") is one form of investment before retirement. There are tax incentives available in the U.S. if retirement income is received from a variable annuity. In particular, investment gains are tax deferred until the funds are withdrawn or annuity payments are received. In the late 1990s, U.S. insurance companies started selling variable annuities with options that could be embedded for additional fees. These options offered guaranteed living benefits and/or guaranteed death benefits. They were designed to increase the attractiveness of variable annuities to potential buyers; variable annuities are sold, not bought. These options have proven to be very popular, as demonstrated by the increase in demand for variable annuities since they were first introduced. In Figure 1.1, the left panel displays U.S. variable annuity assets by year, and the right panel displays U.S. variable annuity sales by year.[1] Both assets and sales increased significantly from 2002 to 2007. However, both assets and sales slumped when the global financial crisis struck in late 2007.

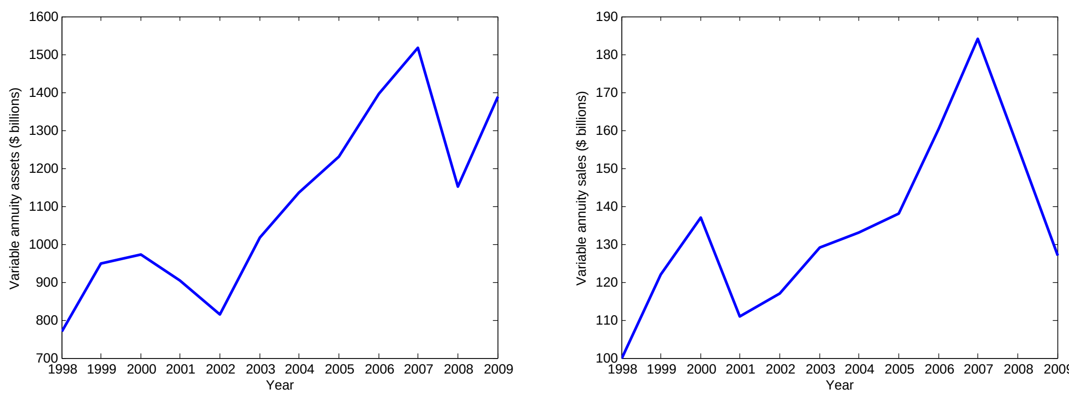

Figure 1.1: *The left (right) panel displays U.S. variable annuity assets (sales), measured in $ billions, for the period 1998-2009.*

Common variable annuity options in the U.S. market today include (different insurers may use different names for each option):

- The **Guaranteed Minimum Death Benefit (GMDB).** Most variable annuity issuers guarantee that if the policyholder dies during the accumulation phase, the

[1]Data sources: "Sales of fixed and variable annuities", Insurance Information Institute, URL: www.iii.org/media/facts/statsbyissue/annuities , and "Responding to the variable annuity crisis", McKinsey and Company, URL: www.mckinsey.com/clientservice/financialservices/pdf/Responding_to_the_Variable_Annuity_Crisis.pdf

policyholder's beneficiaries will receive the total premiums invested (less any withdrawals). The GMDB increases the value of the death benefit. A common death benefit may be the greater of the investment account value at the date of death and either of the following benefits (sometimes the GMDB may include both of the following benefits):

1. The value of the premium(s) invested at a rate of r per year ($r = 5\%$ is common) until either the date of death or the policyholder reaches some threshold age x (x is typically somewhere between 80 and 90); or
2. The highest investment account value over all of the policy anniversaries up until either the date of death or the policyholder reaches some threshold age x.

- The **Guaranteed Minimum Maturity Benefit (GMMB).** This option provides a guaranteed minimum lump sum at the end of the accumulation phase. The exact form of the GMMB varies among issuers. Typically the minimum maturity date must be at least 10 years. One version of the GMMB guarantees that the minimum lump sum received by the policyholder is the greater of either:

  1. A return of the initial investment; or
  2. The maximum investment account value on prespecified policy anniversaries during the accumulation phase.

- The **Guaranteed Minimum Withdrawal Benefit (GMWB).** This option is designed for policyholders who need to make periodic withdrawals (for example, monthly or annually) from their investment, but want a guarantee that they can withdraw at a minimum the total value of the premiums they invested, regardless of poor market performance. Depending on the particular insurer, the policyholder can typically withdraw a maximum of 5-10% of the total value of the premiums they invested, per year without penalty. The policyholder is guaranteed to be able to withdraw the total value of their premiums. After the guaranteed total amount (of the premiums) has been withdrawn, any further withdrawals depend on the remaining balance of the investment account.

- The **Guaranteed Lifetime Withdrawal Benefit (GLWB).** This option is relatively new, introduced to the market a few years ago. It is similar to the GMWB except that the guaranteed withdrawals are paid for life, regardless of whether the investment account value reaches 0, and withdrawal payment rates are lower. The withdrawals cease when the policyholder dies. The guaranteed annual withdrawal amount is typically 4-7% of the total withdrawal base. The value of the total withdrawal base is determined at the time of the first withdrawal. The total withdrawal base is usually the greater of:

1. The value of the premium(s) accumulated at a compound rate of r per year (r is usually between 4-6%) for a maximum of 10 years, with the compounding period stopping on the date of the first withdrawal (if the first withdrawal date is less than 10 years from inception); or
2. The highest investment account value over all of the policy anniversaries between time 0 and the first withdrawal date.

- The **Guaranteed Minimum Income Benefit (GMIB).** The benefits provided by this option are described in detail in Section 1.3.

Variable annuity options transfer the risks associated with investing in the financial markets from the policyholder to the insurance company. The insurance company does not charge an upfront premium for these options, as is traditionally the case for options traded in financial markets. Rather, periodic fees are charged by the insurance company for each option included in a variable annuity policy, during the accumulation phase. In practice, the insurance company makes a profit at the expiry date of the option if the value of the benefits provided by the option are less than the accumulated value of the fees earned (including returns on fee cash flows reinvested by the insurer). The URL www.annuityFYI.com provides a frequently updated list of the most competitive variable annuity options sold by U.S. insurance companies, describing in detail the exact benefits provided by the options of each of the sellers, and the fee rates charged.

Publicly available data on variable annuity options in the U.S. is difficult to obtain. However, the consulting firm Milliman periodically publishes surveys on the dynamics of the U.S. guaranteed living benefit market. Table 1.1 summarizes the relative popularity of the guaranteed living benefit options based on the 2008 Milliman survey (the table is reproduced from Saip (2009)). Twenty one insurance companies participated in the

2008 survey, 13 of whom were ranked in the top 20 based on new variable annuity sales (according to Morningstar, Inc.'s The VARDS Report), representing over 41% of total variable annuity sales for the first half of 2008. The first column of Table 1.1 displays the approximate percentage of insurers that offered each option in the first half of 2008. The remaining columns show the average proportion of policyholders electing each option when the insurer offers the option. Since 2005, the GLWB is the only option which has increased in popularity in each year. The popularity of GMIB declined between 2004 to 2008, although several insurers reported maintaining strong election rates.

		Option election rate				
Option	**Option offered by insurer in 2008**	2004	2005	2006	2007	First half 2008
GMAB with GMWB or GLWB	30%	58%	63%	63%	37%	28%
GMIB	70%	49%	42%	36%	28%	25%
GLWB	90%	51%	21%	38%	47%	57%
GMWB	70%	25%	19%	23%	19%	14%
Combination GMWB/GLWB	95%	N/A	58%	51%	41%	37%
GMAB	85%	21%	16%	12%	11%	10%

Table 1.1: *Average proportion of U.S. variable annuity buyers electing each guaranteed living benefit option when the insurer offers the option.*

There are some practical details related to variable annuity investments that are worth mentioning. Each insurance company selling these variable annuity options has specific rules and restrictions on how the policyholder may split the balance of their investment into sub-accounts, related to different fund managers with different risk/return profiles and investment styles. The specifics of these restrictions can be found in variable annuity prospectuses.[2] Invariably these rules are designed to help limit the insurer's exposure to the risks associated with the variable annuity options. In particular, they limit the overall volatility of the investment account returns (which affects the values of the variable annuity options). Notable rules include:

- Insurance companies typically require that at least $x\%$ of the investment account balance be invested in fixed income portfolios (for example, two large players in the U.S. market set a minimum of 30%). This is often not in the best interests of the policyholder. In the case of the GMIB, we show that it is in the policyholder's

[2] Examples include (as at November 2010) the AXA Equitable (U.S.) Accumulator Series Annuity Prospectus, and the Metlife Class A Variable Annuity Prospectus.

interest to maximize the account volatility if they want to maximize the option value (see Section 2.4.1).

- The range of fund managers the policyholder can choose to invest with is chosen by the insurance company. Some of the fund managers are in-house to the insurer, while others are external managers. Outsourcing of funds management avoids the possibility of perverse incentives for the insurer acting as fund manager.

- One representative major insurance company's prospectus specifically states that at any time it has the right to limit or terminate the policyholder's contributions and allocations to any of the fund managers, and to limit the number of fund managers the policyholder may select. In effect, these rules mean that the insurer has additional control over the investment account volatility.

Although the impact of these rules on the variable annuity option values is difficult to quantify, it is important to be aware of their existence. We approximate the real life situation of investing in many sub-accounts by simply modeling the overall policyholder's investment account balance.

The insurer also typically gives itself certain rights in the variable annuity contract regarding additional premiums. There may also be maximum limits on premium contributions (e.g. $1.5 million), and after the initial premium is invested, the insurer can refuse to accept any additional (typically smaller) premiums at later dates. These rights may protect the insurer in certain situations. For example, if returns on most assets are poor for an extended period of time, the policyholder may have the incentive to invest further premiums to take advantage of the guarantees provided by the variable annuity options, at the expense of the insurer (in particular, the guaranteed return component of the GMIB option, which we describe later in this chapter, could be exploited).

1.2 Motivation for this thesis

As the U.S. variable annuity market is highly competitive, each company tries to differentiate itself from its competitors by making its guarantees appear more attractive, and often more valuable. Furthermore, each company does not want to charge unmarketable fee rates for their guarantees. As a consequence, insurance companies may not be properly pricing the guarantees, or accurately assessing the risks involved with the guarantees. It can be particularly dangerous if the insurance company offers seemingly "cheap" teaser guarantees to attract more business, without actually considering proper risk management of the guarantee, particularly in the product design phase. Waiting until the guarantee approaches maturity deep in-the-money, before implementing an effective hedging program for the guarantee, can be very costly and financially dangerous.

History has demonstrated that the consequences of an insurance company mismanaging guarantees that it offers with its products can be fatal. The failure of Equitable Life Assurance Society in 2000, the oldest mutual insurer in the U.K., provides a very sobering example. For many decades, Equitable Life sold retirement savings products which included guaranteed annuitization options (GAOs). GAOs guaranteed the policyholder that they could convert their retirement investment into an annuity at a guaranteed minimum payment rate. When interest rates are high, GAOs are out-of-the-money. It is when interest rates are low that GAOs become valuable. GAOs were popular in retirement savings products issued in the 1970s and 1980s. During that period long term interest rates reached very high levels (ranging between 8-17%). Actuaries valuing the GAOs apparently believed that interest rates would remain high, in which case the GAOs would not be exercised. However, in the 1990s, with inflation now hovering at low levels, interest rates fell to historically low levels of 4-6%. The GAOs were now deep in the money, and policyholders who had been saving for decades were starting to annuitize their investments. The liabilities generated by the GAOs were significantly underestimated and inadequately reserved for, which ultimately led to the downfall of Equitable Life. U.S. insurance companies selling variable annuity options would be wise to learn from the mistakes of Equitable Life and other U.K. life insurance companies.

At the end of the fourth quarter of 2009, U.S. variable annuity assets totalled $1.4 trillion, and in 2009 total sales were $127 billion. According to the 2008 Milliman Survey of the U.S. guaranteed living benefit market, 96% of variable annuities offered include some form of guaranteed living benefit option. Clearly, understanding the risks associated with variable annuity guarantees is of significant financial importance. There is growing field of research on the valuation and hedging of variable annuity options. Ledlie et al. (2008) provide a detailed overview of the variable annuity options offered in each the major markets around the world. Several authors have priced the popular GMWB ((Chen et al., 2008), (Chen and Forsyth, 2008), (Dai et al., 2008)), some suggesting that insurance companies may be underpricing this guarantee (Milevsky and Salisbury, 2006). Liu (2010) finds that basic forms of GMWBs are priced correctly by the market, and explores the effectiveness of semi-static hedging strategies for GMWBs. Liu finds that semi-static hedging strategies are at least as effective, if not better, than a delta-hedging strategy. Bauer et al. (2008) propose a universal pricing framework for guaranteed minimum benefits in variable annuities, presenting numerical results for the GMxBs (GMDB, GMAB, GMWB and GMIB), all based on a model in which the investment account is modeled as a geometric Brownian motion. Benhamou and Gauthier (2009) employ stochastic interest rate and stochastic (equity) volatility models to price GMxBs, finding that the fair fee rates for GMxBs are higher in their models than those obtained using simpler constant volatility models. Literature on the pricing of GMDBs includes Milevsky and Posner (2001), Ulm (2010) and Belanger et al. (2009). Piscopo and Haberman (2011) prices the GLWB in a no-arbitrage model, with a particular emphasis on measuring the sensitivity of the option to mortality risk.

Currently, there does not seem to be much in the academic literature on the valuation and risk management of GMIBs. The GAO in the U.K. has some features that are similar to a GMIB; namely a guaranteed minimum annuity payment rate. Boyle and Hardy (2003) obtain an analytical pricing formula for the GAO using a one factor interest rate model. They also discuss in detail the feasibility of hedging the three major types of risks in GAOs, namely equity, interest rate and mortality risks, which are the same risk factors for the GMIB. Pelsser (2003) derives a formula for the value of a GAO as a portfolio of long-dated receiver swaptions. The swaptions are shown to be effective instruments for managing the interest rate risk of GAOs. However, as pointed out by Boyle and Hardy (2003), the swaption pricing approach does not deal with the equity or longevity risks. Wilkie et al. (2003) and Hardy (2003) investigate the actuarial approach for the risk management of GAOs, which involves measuring the quantile and conditional tail expectation reserves required, based on projections of the real world GAO liability distributions. They also explore the feasibility of dynamically hedging the GAO. In the case of dynamic hedging, Wilkie et al. stress that it is important to empirically test how well the hedging strategy might perform, taking into account transaction costs and model risks. Furthermore, Wilkie et al. argue that contingency reserves need to be set up to allow for all probable hedging errors. Ballotta and Haberman (2003) obtain analytical formulae for the price of a GAO using the one-factor Heath-Jarrow-Morton interest rate term structure model. Ballotta and Haberman (2006) extend the model of Ballotta and Haberman (2003) by incorporating a stochastic mortality model. They find that the inclusion of stochastic mortality may actually lead to a reduction in the price of the GAO. Ruowei (2007) prices the GAO using the Vasicek and CIR (Cox et al., 1985) (one factor) interest rate models, estimating each model's parameters using maximum-likelihood estimation applied to historical U.S. interest rate data. Van Haastrecht et al. (2009) obtain closed-form formulas for pricing GAOs using a stochastic equity volatility model. They show that GAO prices are much higher using a stochastic volatility model in comparison to using a constant volatility model, particularly for GAOs with out of the money strike prices.

Although GAOs resemble GMIBs through the guaranteed minimum annuity payment rate, the equity risk associated with GMIBs is much more complicated than with GAOs. Specifically, the GMIB is based on annuitizing an amount of funds equal to the benefit base (defined shortly), whereas the GAO is based on (the much simpler situation of) annuitizing the value of the policyholder's investment at maturity. Therefore, findings on GAOs cannot be directly applied to GMIBs. In particular, GAOs are very sensitive to interest rate risk, but as we show in Chapter 3, for GMIBs the equity risk dominates the interest rate risk.

This thesis focuses on the valuation and financial risk management of the GMIB, from the point of view of the insurance company selling the option. Bauer et al. (2008) provide numerical results relating to the valuation of GMIBs, but the depth of the results is limited; there are many other questions that can be asked and answered about the valuation of GMIBs. We are not aware of any comprehensive quantitative research on hedging GMIBs. Identifying effective methods for hedging GMIBs is at least as important as pricing GMIBs, and is a much more challenging task. It is hoped that this thesis will provide some useful insights about the GMIB option.

1.3 The GMIB maturity value

We refer to the issuance date of a variable annuity with an embedded GMIB option as time 0. We refer to the time point at which the policyholder decides to annuitize or receive a lump sum benefit as the maturity date, or time T. We assume T is a positive integer (the maturity date is on a policy anniversary). This is a reasonable assumption as some insurance companies state that the policyholder may only choose to exercise their GMIB option within the 30 days following each policy anniversary (a penalty fee may occur if they wish to annuitize at other points within the year). The insurance company requires that the policyholder invest their funds for a certain period of time, called the *waiting period*, before they can exercise the GMIB option (annuitizing beforehand forfeits the GMIB option). The most common waiting period is 10 years. The insurance company recovers the cost of the GMIB by deducting fees periodically from the policyholder's investment account. We assume the fees are paid annually on each policy anniversary, which is a common practice for the GMIB option.

In this section, we first define the generalized form of the maturity value of a variable annuity with an embedded GMIB option. Next, we define a special case of the maturity value. The special case is the maturity value of a 20 year term certain annuity with an embedded GMIB option. It is a function of the key financial variables – equity returns and interest rates – but not mortality. We use the special case to price the GMIB in Chapter 2. Then in Chapters 3 and 4, we use the special case as the ultimate hedging target for the hedging strategies we explore.

1.3.1 Generalized form of the GMIB maturity value

At maturity a variable annuity with an embedded GMIB option gives the policyholder the following choices:

1. Annuitize the accumulated value of the investment account at annuity payment rates offered by the insurer at maturity.

2. Take the accumulated value of the investment account as a lump sum. The policyholder might buy an immediate annuity from a different insurer with this sum, if

the annuity payment rates offered are more favorable.

3. Annuitize a guaranteed amount of funds at a guaranteed payment rate of g per year.

It is usually optimal to choose Choice 1 or Choice 2 when investment performance is strong during the accumulation phase, or interest rates are high at maturity. It is usually optimal to choose Choice 3 when investment performance is poor during the accumulation phase and/or interest rates are low at maturity.[3] Choices 1 and 2 have the same financial value. They correspond to the policyholder receiving the investment account value at maturity.

The benefits provided by the GMIB option vary slightly between insurance companies, but the core benefits are essentially the same for each company. Based on publicly available information about GMIBs sold by major insurance companies in the U.S. market (www.annuityFYI.com), the generalized form of the maturity value of a variable annuity with an embedded GMIB option is

$$Z(T) = \max\left\{ B(T) g \sum_{j=T}^{\infty} \varrho(j) P(T,j) p(j), A(T) \right\}, \tag{1.1}$$

where:

- $A(T)$ is the value of the policyholder's investment account at time T, after deducting the option fee for the T-th policy year.
- $B(T)$ is the value at time T of the *benefit base* of the GMIB, defined as

  $$B(T) = \max\{A(0)(1+r_g)^T, \max_{n=1,2,\ldots,T} A(n-)\} \tag{1.2}$$

  where $A(0)$ is equal to the policyholder's initially investment at time 0 (the annuity premium), $A(n-)$ is the value of the investment account on the n-th policy anniver-

[3] The guaranteed payment rate g provides a minimum payment rate that protects against low interest rates at maturity. Low interest rates at time T mean the market values of annuities at time T will be higher, and hence the immediate annuity payment rates (which, if fairly priced, are roughly equal to the inverses of the market values of annuities with annual payments of \$1 per year) offered by insurers will be lower.

sary, just before the fee for the n-th policy year is deducted, and r_g is a guaranteed annual rate, which is typically set somewhere in the range of 4-6% per annum.

- g is the guaranteed annual annuity payment rate at time j, specified by the insurance company at time 0. It is set conservatively by the insurer with respect to future mortality and interest rate assumptions.

- $\varrho(j)$ is an inflation adjustment factor applied to g in year $j > T$. Generally $\varrho(j) = 1$ for all j. However, the underlying annuity may allow payments to increase with by say $x\%$ per year to help adjust for inflation, in which case $\varrho(j) > 1$ for $j > T$.

- $P(T, j)$ is the price at time T of a zero coupon bond maturing at time $j > T$ with unit face value. Note that $P(T, T) = 1$. The term structure of interest rates at time T is described by the function $\{P(T, j), j > T\}$, which is assumed to be known at time T.

- $p(j)$ is the probability that an annuity payment is made at time j. For example, if the variable annuity is an M year term certain annuity and a for life annuity thereafter, then $\{p(j) = 1, j = T, T + 1, \ldots, T + M - 1\}$, and $\{p(j), j \geqslant T + M\}$ will depend on future mortality/longevity assumptions.

The values of $\{f(j)\}, \{P(T, j)\}, \{p(j)\}$, all depend on what kind of variable annuity is elected by the policyholder at time 0. The variable annuity may be term certain, a life annuity, 5 year term certain with payments contingent on survival in each year thereafter, joint life and last survivor with 10 year certain, et cetera.

Equation (1.2) is also a slight simplification of a typical benefit base. Inevitably, the insurer imposes age limits on accumulation of the guaranteed benefits. A more formal expression for the benefit base is

$$B(T) = \max\{A(0)(1 + r_g)^{\min(T,y-x)}, \max_{n=1,2,\ldots,\min(T,z-x)} A(n-)\}$$

where x is the age of the policyholder at inception, and $y, z > x$ are specific policyholder ages set arbitrarily by the issuer at the inception of the contract. The actual benefit age limits for a representative insurer are $z = 80$ and $y = 90$. Henceforth, we do not concern ourselves with the age limits because we price and hedge the GMIB option over a decade, and it seems likely that most policyholders will annuitize before their 80th birthday (if the annuity is one of their main sources of retirement income). However, allowing for the benefit age limits is important if there are reasons for believing that policyholders will delay annuitization for an extended period of time.

Equations (1.1) and (1.2) combined illustrate the attractiveness of a GMIB to variable annuity buyers. If investment returns are strong during the accumulation phase, then the policyholder is likely to annuitize the sum $A(T)$ at annuity payment rates offered by the insurer at maturity. Recall that the guaranteed payment rate g is set conservatively, so the annuity payment rate(s) available at time T are likely to be more favorable. Alternatively, the policyholder may want to receive the lump sum $A(T)$. If investment returns are poor during the accumulation phase, then the policyholder is able to convert a guaranteed minimum amount of funds – the benefit base $B(T)$ – into an annuity with payments of $B(T)g$ per year. The benefit base provides a guaranteed return of r_g per year on the inital investment. As of 2010, the most competitive GMIB sellers are offering a guaranteed return of 5% per year. The benefit base also provides the right to receive the maximum value of the investment account on any previous policy anniversary, giving the policyholder the opportunity to lock in gains when investment returns are strong during the accumulation phase. However, these gains will be slightly penalized by the conservative value of g set by the insurance company at time 0 (g is not set equal to its fair value at time 0, based on future interest rate and mortality assumptions used at time 0).

Equation (1.1) makes the following assumptions:

- The policyholder pays a single premium at time 0 and does not make any cash withdrawals before time T. In practice, the policyholder is usually able to pay additional premiums (usually up to a total investment limit of say $2 million), which may be covered by the GMIB option after these funds have been invested for at least the waiting periods beginning from the point at which they are invested. The policyholder is also usually able to make withdrawals of up to a maximum pre-specified percentage of the benefit base each year without paying withdrawal penalty charges (any withdrawals also reduce the value of the benefit base). As noted in

Ledlie et al. (2008), the majority of variable annuity business in the U.S. is by single premium. Therefore assuming there is one single premium seems reasonable.

- In the past (before 2009), to attract large annuity premiums from policyholders, some variable annuity sellers credited a bonus of say 1-5% on the initial premium π to the investment account at time 0. Specifically, this means $A(0) = b\pi$, where b is in the range of 1.01-1.05. We simply assume $b = 1$.

- The policyholder does not lapse or die before time T. In the context of this thesis, by lapsing we mean that the policyholder cancels their variable annuity policy at some point during the accumulation phase, and receives back the value of their investment, subject to any penalty fees charged by the insurer. If a policyholder does lapse during the waiting period, then the GMIB option is forfeited, and the insurance company keeps all the GMIB fees earned. Assuming no possibility of lapsing is a strong assumption, partly because the policyholder may need to withdraw their invested funds for some unforeseen reason (which may or may not be related to the prevailing economic conditions) at some point during the waiting period which is usually 10 years. Allowing for lapses will reduce the GMIB price. However, lapses are notoriously difficult to predict, as there is little data, and because they tend to be correlated to the prevailing economic conditions. Notwithstanding, in Section 2.6, we explore the impact of a constant annual lapse rate on the GMIB price.

- Some companies include "step-up" options in their GMIB contracts. The step-up option allows the policyholder to adjust upwards the guaranteed return component of the benefit base, on a prespecified set of policy anniversaries, if investment performance is strong. For example, suppose the GMIB contract grants the policyholder a step-up option at time m. If $A(m) > A(0)(1 + r_g)^m$, then the policyholder can step-up the benefit base at time m, such that at time T

$$B(T) = \max\{A(m)(1 + r_g)^{T-m}, \max_{n=1,2,\ldots,T} A(n-)\}.$$

However, there are drawbacks to exercising the step-up options. First, the waiting period is restarted. Second, some insurers give themselves the right to revise the GMIB option fee upwards to the fee rate applicable to new policies at the time of exercise, up to a prespecified maximum rate (the maximum fee rate for one representative insurer is 1.5% per year). It is noted that usually no step-up options are offered once the policyholder reaches age 80. The additional complexity of the step-up option is not considered in this thesis, but such a feature only further increases the value of the GMIB option.

- We ignore the impact of ongoing expenses associated with the variable annuity policy.

- Annuity payments are often monthly, or quarterly, but we assume payments occur annually in advance. Varying the payment assumption will not lead to a significant difference in the GMIB price.

	2004	2005	2006	2007	First half 2008
Average % of in-force GMIB policies beyond the waiting period	0%	4%	19%	23%	22%
Average % of in-force GMIB policies beyond the waiting period where the GMIB was in-the-money	N/A	72%	64%	72%	65%
Average % of in-force GMIB policies beyond the waiting period that began income payments in the following calendar period	N/A	4%	6%	5%	2%

Table 1.2: *U.S. industry statistics for GMIBs in-force beyond the minimum waiting period.*

Saip (2009) reports some U.S. industry statistics for GMIBs in-force beyond the minimum waiting period, based on the 2008 Milliman survey of guaranteed living benefit options, which are reproduced in Table 1.2. Recall that these statistics are based on the experience of 21 U.S. insurers (see Section 1.1 for the survey details). The first year for which there were variable annuity policies with GMIBs that had been in-force for more than the minimum waiting period was 2005. As shown in Table 1.2, the average percentage of in-force policies beyond the waiting period grew from 3.7% in 2005 to 21.7% in the first half of 2008. For in-force policies beyond the waiting period in each year between 2005 and 2008, the GMIB option is in-the-money at least 60% of the time, on average. The percentage of in-force policies beyond the waiting period that annuitized in the following calendar period, as shown in the bottom portion of Table 1.2, is quite small. The low annuitization rates suggest that:

- Most policyholders are not experiencing personal cash flow pressures during the waiting period and therefore do not need to annuitize as soon as the waiting period expires. Moreover, many policyholders may take the view that they will not annuitize until they actually need the annuity income stream. In other words, they are drawing down on other savings and selling significant personal assets, before turning to their variable annuity for income in their older ages.

- Many policyholders may be relatively young when they buy variable annuity policies with a GMIB, and the accumulation phase may be quite long. For example, a 40 year old who buys a variable annuity is unlikely to annuitize until they are at least age 65.

The low annuitization rates also indicate that the insurer is exposed to significant model risk related to policyholder exercise behavior. The time of exercise of the GMIB option is at the policyholder's discretion. The accumulation phase could end up being say 20-30 years. The insurer must ensure that it has sufficient funds to meet annuity payments going forward when the policyholder chooses to annuitize. From the insurer's perspective, the "random" exercise time could turn into a big problem in some situations. For example, consider an insurer which has a large group of policies with GMIBs, and say 80% of the policyholders decide to annuitize in a particular year due to an economic crisis. If the insurer has not adequately hedged the GMIBs and they are in-the-money, it exposed to the risk of cash flow shortages in the short term, as it must meet the annuity payments in each year henceforth, and the risk of bankruptcy in the long term if the GMIBs are deep in-the-money. For the more popular GMWB and GLWBs, policyholder behavior is much more easily modeled and projected. This is because the insurer can influence policyholder behavior by imposing penalty charges if the policyholder wants to withdraw more than the maximum guaranteed annual withdrawal amount in each year. The unknown exercise date of the GMIB is one reason why the GMIB is less popular among variable annuity sellers today. With the GMWB and GLWBs, the insurer has a much better idea about policyholder behavior.

1.3.2 GMIB maturity value using a term certain annuity

In Chapters 2, 3 and 4, we study the GMIB assuming that, if the option is exercised, the type of the underlying annuity is a 20 year term certain annuity with annual payments in advance. The justification for using a 20 year term certain annuity is discussed in Section 2.2. In short, this assumption is adopted in order to simplify the analysis without detracting much from the practical usefulness of the results. However, we are not suggesting that longevity/mortality risk is an unimportant consideration in the valuation of GMIBs. Many policyholders will choose a life-related annuity, and so measuring longevity risk is an important part of the overall picture. Modeling mortality risk is outside of the scope of this thesis, but it is a potentially fruitful research topic in itself.

The maturity value of a variable annuity with an embedded GMIB option given by equation (1.1) simplifies to

$$Y(T) = \max\left\{B(T)g\ddot{a}_{\overline{20}|}(T), A(T)\right\}, \tag{1.3}$$

where $B(T)$ is still defined by equation (1.2), and $\ddot{a}_{\overline{20}|}(T)$ is the market value of a 20 year term certain annuity at time T. In addition to the assumptions listed in Section 1.3.1 for the generalized maturity value, we adopt another assumption relating to the maturity date T. The GMIB cannot be exercised until the policy has been in-force for longer than the waiting period. In most cases the waiting period is 10 years from inception. Beyond the waiting period, the policyholder may be able to exercise the GMIB at any time, or there may be restrictions such as exercise is allowed within the 30 days following each policy anniversary; the option is American or Bermudan. The maturity date $T \geqslant 10$ is a random variable that is dependent on policyholder behavior which may or may not be influenced by the prevailing economic conditions. Moreover, there is very little data available on GMIB exercise behavior. As shown in Table 1.2, data is only available from 2005 onwards. Boyle and Hardy (2003) assume there are 10 years to maturity when valuing the guaranteed annuity options, which have features that are similar to GMIBs. In this thesis, we assume the maturity date is fixed at the 10-th policy anniversary, $T = 10$.[4] It is noted that in Section 2.4.4, we briefly explore the sensitivity of the GMIB price to different values of T.

Several important, but perhaps not so obvious, points regarding the GMIB option include:

- The value of g, which is set by the insurance company at the outset, has a large influence on the value of the GMIB.

[4]In passing we note that the investigation of the optimal maturity date, in terms of maximizing the the GMIB maturity value with respect to the financial variables, could be an interesting topic for future research.

- The benefit base $B(T)$ is only used for calculating the guaranteed minimum annuity payments, and cannot be withdrawn as a cash lump sum.
- At time T, even if $B(T) > A(T)$, the policyholder may still be better off taking the lump sum $A(T)$ rather than exercising the GMIB option, because insurance companies explicitly state in their variable annuity prospectuses that they set g conservatively with respect to future mortality and interest rate assumptions. In other words, it would be expected that $g\ddot{a}_{\overline{20}|}(T) < 1$ at time T.

The maturity value of a variable annuity with an embedded GMIB is the maximum of three components:

$$Y(T) = \max(X_1, X_2, X_3) \tag{1.4}$$

where

$$X_1 = \max_{n=1,\dots,T} A(n-)g\ddot{a}_{\overline{20}|}(T),\ X_2 = A(0)(1+r_g)^T g\ddot{a}_{\overline{20}|}(T),\ X_3 = A(T). \tag{1.5}$$

Throughout this thesis we refer to $Y(T)$ as the *GMIB maturity value* (although strictly speaking $Y(T)$ is the value of a variable annuity with an embedded GMIB option at time T), X_1 as the *lookback component*, X_2 as the *guaranteed return component* and X_3 as the *investment account component*. When we say that a particular component is exercised, we mean that it has the highest value among all three components at maturity, and it is optimal for the policyholder to receive the benefit provided by this component at maturity.

The lookback and guaranteed return components appear in slightly modified forms in some of the other variable annuity options discussed in Section 1.1. Therefore, our findings in Chapters 2, 3 and 4, on the pricing and hedging of these components may provide useful information for the financial risk management of other variable annuity options which include similar types of components. It is noted that the payoff of a GAO forms a subset of the event defined by X_1. The GAO payoff is given by $A(T-)g\ddot{a}_{\overline{20}|}(T)$ (ignoring the complication of whether the fee deduction at time T should be allowed for).

As previously mentioned, in Chapter 2, we price the GMIB option based on equation

(1.3). In Chapter 3, we measure the effectiveness of static hedges, where we attempt to hedge the value as given by equation (1.3). In Chapter 4, we use equation (1.3) in the development of the hedging targets for the semi-static hedging strategies. Although equation (1.3) is a special case of the value of a variable annuity with an embedded GMIB, the results we illustrate using this equation provide useful insights into the valuation and risk management of the GMIB with respect to the financial variables that drive it. These insights will still apply when the underlying annuity is life-related (more complex).

1.4 Hypothetical scenarios

As noted in Section 1.3.2, our focus is on valuing and hedging the GMIB option ignoring longevity risk. Therefore the factors driving the GMIB value are investment account (equity) returns during the accumulation phase, and the term structure of interest rates at maturity. In this section, we show how the GMIB maturity value behaves for five distinct plausible hypothetical scenarios. We do this in order to give the reader an intuitive feeling for how the GMIB maturity value varies with equity returns and interest rates.

Figure 1.2 displays the assumed evolution of the investment account during the accumulation phase for each scenario, and Table 1.3 shows the numerical results, using the notation of equations (1.4) and (1.5). To keep things simple, we assume the term structure of interest rates is a flat curve at maturity. We let r denote the annually compounded interest rate at maturity, which varies by scenario (r drives the value of $\ddot{a}_{\overline{20|}}(T)$). In all of these hypothetical scenarios, we assume $g = 6.5\%$ and $r_g = 5\%$. Scenarios 1 and 2 use the actual evolution of the total returns for the S&P 500 index over the decade starting January 1 2000 and ending 1 January 2010, but assume different interest rates of 5% and 10% respectively. Scenario 3 assumes there is a bull market that persists to maturity, with the interest rate set at the plausible level of 7% to curb any inflationary pressures. Scenario 4 assumes there is an equity bubble which bursts at the end of year 3, and thereafter a bear market persists for many years. To stimulate growth, the interest rate is set at a very low level of 2%. Scenario 5 describes the situation where equity markets remain relatively stable, with interest rates set at 5%.

In scenarios 1 and 2 the guaranteed return component X_2 is exercised due to poor equity performance. These scenarios show how sensitive the lookback and guaranteed return components, X_1 and X_2, are to changes in the level of interest rates at maturity. In Scenario 3 it is optimal for the policyholder to receive the proceeds of the investment account (X_3 is exercised). The GMIB option is not exercised. Scenario 4 illustrates a situation where the GMIB option is very valuable to the policyholder. This scenario reflects a "perfect storm" for the GMIB liability. The GMIB locks in the gains of bull market, the stock market crashes and does not recover by the maturity date, and interest rates are very low at maturity. The lookback component is exercised. Scenario 4 loosely reflects the situation for GMIBs sold a few years before the global financial crisis. Scenario 5 demonstrates the point discussed in Section 1.3.1 that $B(T) > A(T)$ does not always imply that the GMIB option is exercised. In Scenario 5 it is optimal to exercise the investment account component since $B(T)g\ddot{a}_{\overline{20|}}(T) < A(T)$, when $g = 6.5\%$.

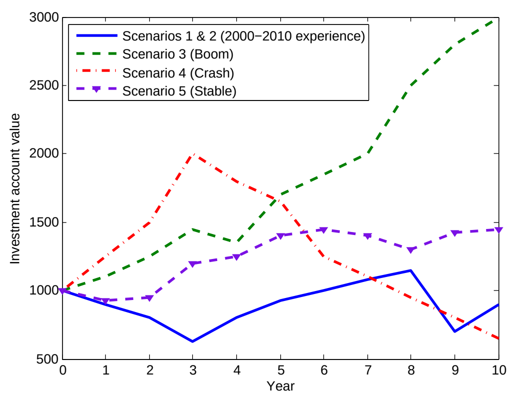

Figure 1.2: *Hypothetical scenarios for the evolution of the investment account during the accumulation phase.*

$\max_{n=1,\ldots,T} A(n-)$	$A(T)$	$B(T)$	**int. rate** r	X_1	X_2	X_3	$Y(T)$
1150	900	1629	5%	978	1385	900	1385
1150	900	1629	10%	700	992	900	992
3000	3000	3000	7%	2210	1200	3000	3000
2000	650	2000	2%	2168	1766	650	2168
1450	1450	1629	5%	1233	1385	1450	1450

Table 1.3: *Analysis of the GMIB maturity value $Y(T)$ for the hypothetical scenarios, assuming $g = 6.5\%$ and $r_g = 5\%$.*

1.5 Contributions of this thesis

This thesis makes the following contributions:

1. We present a valuation method for the GMIB. Furthermore, an analysis of the GMIB option design is conducted.

2. We present a method for constructing static hedging strategies for a GMIB. A hedging portfolio is set up at time 0 and held until maturity, with the aim of minimizing the loss to the insurer at maturity.

3. We present a method for designing and testing a semi-static hedging strategy, where the hedging portfolio is rebalanced at particular time intervals.

We illustrate the results of each method under reasonable model (and parameter) assumptions. This thesis aims to be practical in nature. The results elucidate the importance of careful policy design for complex options with very long expiration dates. We focus on the GMIB option as a particular case study. However, practitioners designing and selling long-dated options may find the methods presented in this thesis useful for the financial risk management of the options they are dealing with.

1.6 Outline of this thesis

In Chapter 2, we value the GMIB, and determine the fair fee rate for the option. The factors influencing the value of the GMIB are investment account (equity) returns, interest rates and mortality. We focus on the sensitivity of the GMIB value to the financial variables. Mortality is not incorporated into the valuation. We present a comprehensive sensitivity analysis of the model employed. We decompose the value of the GMIB at the maturity date, which is rather complicated, to analyze what drives the value of a GMIB. Our approach offers a simple but effective way for insurers to measure the value of the GMIBs they offer, and provides some insights into the risk management of GMIBs and other guarantees that provide similar payoffs. Our model suggests that the fee rates charged by insurance companies for the GMIB option may be too low.

In Chapter 3, we measure the effectiveness of static hedging strategies for the GMIB. Using Monte Carlo simulation, the effectiveness of a static hedging strategy is measured by the empirical hedging loss distribution, where each hedging loss is defined as the difference between the GMIB payoff and the hedging portfolio payoff at the maturity date. Hedging portfolios are constructed by minimizing the Conditional-Tail-Expectation (CTE) of the hedging loss distribution, or minimizing the mean squared hedging loss (MSHL). The positions in the hedging portfolio instruments are determined at the outset from solving either portfolio optimization problem, and are held fixed until the maturity date.

The methods presented in Chapter 3 provide a template for how an insurance company can develop static hedging strategies for groups of variable annuity policies which include GMIBs. Our results suggest which instruments are most important to achieve the best results. Based on the (benchmark) models and assumptions adopted, the performance of the static hedge for the GMIB is imperfect at best. The hedging portfolios do not adequately simultaneously hedge the upside and downside equity guarantees provided by the GMIB. We backtest the performance of static hedging strategies for the period 1997 to 2011. We demonstrate that if the design of the GMIB is simplified, then the static hedges are more effective.

In Chapter 4, we investigate the performance of semi-static hedging strategies for the GMIB option. Semi-static strategies involve periodically rebalancing the hedging portfolio at certain time points during the accumulation phase, such that, at the option maturity date, the hedging portfolio payoff is equal to or exceeds the option value, subject to an acceptable level of risk. We present a practical method for implementing semi-static strategies for the GMIB option, which employs nested Monte Carlo simulations and standard optimization methods. It is noted that this method is versatile, and can be applied to other types of long-dated options. Several examples, illustrating the effectiveness of semi-static hedging strategies for the GMIB, are presented. The performances of the semi-static and static hedging strategies are compared.

Chapter 5 presents our conclusions. We end with suggestions for future research.

Chapter 2

Valuation of a Guaranteed Minimum Income Benefit

2.1 Introduction

As mentioned in Section 1.2, there seems to be little in the academic literature on pricing or hedging GMIBs. Bauer et al. (2008) propose a universal pricing framework for guaranteed minimum benefits in variable annuities, presenting numerical results for GMxBs (including the GMIB), based on a model in which the investment account is modeled as a geometric Brownian motion. In this chapter, we consider the valuation of the GMIB in more detail, and focus on the design elements of GMIBs. The model we use for our valuations is an extension of that in Bauer et al. (2008), as we allow interest rates to follow a random process. Given that the accumulation phase must exceed a decade as part of the contract requirements for the GMIB, the stochastic feature of interest rates has a stronger impact on the GMIB option price. Therefore, incorporating a stochastic interest rate model seems worthwhile. Bauer et al. also assume the fee rate charged for the GMIB option is a percentage of the investment account, but the GMIB fee structure for most U.S. insurers is usually a fixed percentage of the benefit base, rather than the investment account (the fee structure is defined later by equation (2.2)). In constrast to Bauer et al., in our valuations we adopt the fee structure commonly used in practice. Assuming the same fee rate is applied to the investment account and the benefit base, the fees charged based on the benefit base are always at least as large as those based on the investment account. It is noted that Bauer et al. present results on the impact of the inclusion of a GMDB with a GMIB, whereas we consider the valuation of the GMIB in isolation. We present a comprehensive sensitivity analysis of our model parameters. We decompose the maturity value of a variable annuity with an embedded GMIB option, and measure the contributions to the total value from the lookback and guaranteed return components (see equation (1.5)). We conclude that GMIBs appear to be underpriced by insurance companies, which agrees with the existing GMIB pricing results of Bauer et al. It seems that the fair fee rates we obtain are higher than those reported by Bauer et al. However, Bauer et al. present the fair fee rates that should be charged for each of the individual benefit components provided by the GMIB but not for the GMIB as a whole, whereas we calculate the fair fee rates for the GMIB as a whole.

In this chapter we value the GMIB, and determine the fair fee rate that should be charged, based on plausible model assumptions. The value of the GMIB is affected by investment account returns, interest rates and mortality. We focus on the sensitivity of the GMIB price to the financial variables. Mortality is not incorporated into the valuation. We present a comprehensive sensitivity analysis of the model employed. The numerical results presented provide a benchmark for GMIB valuations that use more sophisticated models and complex assumptions. Since the GMIB maturity value is rather complicated, we price the individual components of the GMIB to analyze what drives the GMIB price. The techniques we use to value the GMIB are simple, but they are effective at generating meaningful information for insurers selling GMIBs (such as whether the fee rates they are charging for the GMIB in practice make sense in a highly simplified model of reality). Furthermore, the techniques act as a guide as to things that can be done by insurance companies when they are valuing and monitoring the risks associated with other complex options with similar benefits.

The structure of Chapter 2 is as follows. Section 2.2 describes the models we use to price the GMIB. In Section 2.3, we illustrate the results of this model. Specifically, we calculate the fair fee rates for GMIBs with different contract parameters. In Section 2.4, we provide a sensitivity analysis of all of the model parameters. In Section 2.5, the GMIB price is decomposed, facilitating an understanding of the drivers of its value. In particular, we measure the contributions of the lookback and guaranteed return components to the total

GMIB price. Section 2.6 explores the impact of lapses on the GMIB price. In Section 2.7, we measure the differences in the GMIB price when a continuous fee rate, charged as a percentage of the investment account, is adopted, rather than charging discrete annual fees that are a percentage of the benefit base. In Section 2.8, we illustrate the impact on the GMIB fair fee rate when underlying variable annuity charges are included. Section 2.9 discusses an efficient Monte Carlo simulation method for valuing the GMIB. A control variate for the GMIB is also provided. Section 2.10 gives a summary of the history of GMIBs sold in the U.S. by the major competitive sellers over the past few years. It seems that the U.S. industry underpriced this GMIB, and impact of the global financial crisis led to a wide scale reassessment of the benefits provided by GMIB, and the fee rates charged for the option. Concluding remarks are given in Section 2.11.

2.2 The valuation model

This section discusses the model used to price the GMIB. The GMIB maturity value that we price is given by equation (1.3). The valuation depends on the key financial factors but mortality factors are not incorporated. If we introduced a life related annuity, the theory for pricing becomes complicated. Currently, there are a few very actively traded financial instruments which could be used to hedge the longevity risk associated with a GMIB associated with a life annuity; q-forwards are simple capital market instruments that might provide a basic hedge against mortality risk. However, it seems highly unlikely that we would be able to construct a replicating portfolio in practice which consists of liquid securities. Nevertheless, longevity risk, which is a non-diversifiable risk, is an important consideration for life annuities. The mortality assumptions employed would be a key driver of the value of a GMIB associated with a life annuity.

One justification for assuming that the underlying annuity is a 20 year term certain annuity is that at age 65 the life expectancies for males and females in the 2005 U.S. period life table are 16.7 and 19.5 years respectively (Social Security Online, 2009), and age 65 is a likely retirement age for many variable annuity policyholders. Buying a 20 year term certain annuity will cover the expected number of payments that a retiree at age 65 will need for the remainder of their life. Another justification is that the term certain annuity is also actually one of the choices of annuity type that a variable annuity policyholder may choose, where the term may be of 20 to 30 years.

We value the GMIB using the well known no-arbitrage (risk-neutral) pricing approach. The risk-neutral valuation approach for equity-linked insurance contracts was first presented in the pioneering work of Boyle and Schwartz (1977). Under this approach, there exists a self-financing replicating portfolio which generates a payoff at maturity which exactly matches the GMIB maturity value, and we are calculating the price of the replicating portfolio at time 0.

2.2.1 Models for the financial variables

The policyholder's investment account is the most important financial variable that must be modeled. In reality, the policyholder has a choice of splitting their investment among several fund managers with different risk/reward profiles. As discussed in Section 1.1, there are often restrictions on the percentage that can be invested with riskier asset classes (for example, a maximum of 70% of the premium can be invested in equities and the remaining must be in fixed interest). For simplicity, we assume the policyholder has requested the insurer to invest their annuity premium in a managed portfolio that offers returns perfectly matching the returns of a major stock index. Henceforth, we refer to this portfolio as the stock. Changes in the value of the stock are modeled under the risk-neutral probability measure, which we denote by Q, by the stochastic differential equation (SDE)

$$\mathrm{d}S(t) = r(t)S(t)\mathrm{d}t + \sigma_S S(t)\mathrm{d}W_S^Q(t) \tag{2.1}$$

where $S(t)$ is the stock value at time t, $r(t)$ is the short rate at time t, $\sigma_S > 0$ is the (annualized) instantaneous volatility of the stock and $\{W_S^Q(t), t \in [0, T]\}$ is a standard Brownian motion under Q.[1]

Further notation is introduced in order to define the policyholder's investment account process, $\{A(t), t \in [0, T]\}$. The insurance company does not receive an option premium

[1] Some readers might be tempted to refer to the stochastic process defined by equation (2.1) as a geometric Brownian motion (GBM), but strictly speaking it is not GBM because part of the definition of GBM is that the (annualized) instantaneous expected return is constant for all t.

from the policyholder at the outset for providing the GMIB option. Rather, the insurer deducts a fee from the policyholder's investment account on each policy anniversary. We assume $A(0) = S(0) = \pi$, where π is the value of the single annuity premium invested by the policyholder at time 0. After the first fee is deducted, the stock value is always greater than the investment account value. The size of the fee charged at the end of policy year n is

$$f(n) = \min\{cB(n), A(n-)\} \qquad n = 1, 2, \ldots, T, \tag{2.2}$$

where:

- $A(n-)$ is the value of the investment account on the n-th policy anniversary, just before the fee for the n-th policy year is deducted;
- $c > 0$ is the annual fee rate charged by the insurance company for the GMIB;
- $B(n)$ is the value of the benefit base on the n-th policy anniversary, calculated as

$$B(n) = \max\{A(0)(1 + r_g)^n, \max_{m=1,2,\ldots,n} A(m-)\}.$$

Note that $f(n) \geqslant cA(n-)$ if $f(n) > A(n-)$ for $n = 1, 2, \ldots, T$; $cA(n-)$ is the size of the annual fee charged when it is computed as a percentage of the investment account, which is a common fee structure for many investment products. As of the middle of 2010, many of the competitive GMIB sellers are charging fee rates somewhere between 0.8-1% (www.annuityFYI.com).

Define the stock accumulation factor over the time interval $[n - 1, n)$ as

$$\begin{aligned} R(n) &= \frac{S(n)}{S(n-1)} \\ &= e^{\int_{n-1}^{n} r(s)\mathrm{d}s - \sigma_S^2/2 + \sigma_S (W_S^Q(n) - W_S^Q(n-1))}, \qquad n = 1, 2, \ldots, T. \end{aligned} \tag{2.3}$$

Using this notation,

$$A(n-) = A(n-1)R(n). \tag{2.4}$$

The value of the investment account on the n-th policy anniversary after the annual fee is deducted is given by

$$\begin{aligned} A(n) &= A(n-1)R(n) - f(n) \\ &= A(n-) - f(n) \qquad n = 1, 2, \ldots, T. \end{aligned} \tag{2.5}$$

Also, for $t \in (n-1, n)$, $n = 1, 2, \ldots, T$,

$$A(t-) = A(t) = A(n-1)e^{\int_{n-1}^{t}(r(s)-\sigma_S^2/2)\mathrm{d}s+\sigma_S(W_S^Q(t)-W_S^Q(n-1))}.$$

When pricing any option with a long maturity date (such as 10 years for the GMIB), the stochastic feature of interest rates has a stronger impact on the option price. If the maturity date is long, it is advisable that a stochastic interest rate model be employed instead of assuming deterministic interest rates. We use the Hull-White model for modelling the term structure of interest rates (Hull and White, 1990, 1994). This model is also known as the extended Vasicek model, from the Vasicek (1977) model. Namely, the instantaneous short rate is modeled under the risk-neutral probability measure Q by the SDE

$$\mathrm{d}r(t) = a\{\Theta(t)/a - r(t)\}\mathrm{d}t + \sigma_r \mathrm{d}W_r^Q(t) \tag{2.6}$$

where $a > 0$ is a constant, $\Theta(t)$ is a deterministic function of time that is chosen such that the model term structure matches the market term structure at the start of the projection, $\sigma_r > 0$ is the (annualized) instantaneous volatility of the short rate and $\{W_r^Q(t), t \in [0, T]\}$ is a standard Brownian motion under Q that may be correlated with $\{W_S^Q(t), t \in [0, T]\}$.

Define ρ as the linear correlation coefficient between $\{W_S^Q(t), t \in [0, T]\}$ and $\{W_r^Q(t), t \in [0, T]\}$ such that

$$\mathrm{Cov}^Q(\mathrm{d}W_S^Q(t), \mathrm{d}W_r^Q(u)) = \begin{cases} \rho \ \mathrm{d}t & \text{if } t = u, \\ 0 & \text{if } t \neq u, \end{cases} \tag{2.7}$$

where Cov^Q denotes the covariance under Q. When $\rho \neq 0$, the SDEs of the stock and the short rate can be expressed in terms of two independent standard Brownian motions

$\{W_r^Q(t)\}$ and $\{\widetilde{W}^Q(t)\}$ under Q, using a Cholesky decomposition (Glasserman, 2004):

$$\mathrm{d}r(t) = \{\Theta(t) - ar(t)\}\mathrm{d}t + \sigma_r \mathrm{d}W_r^Q(t) \tag{2.8}$$

$$\mathrm{d}S(t) = r(t)S(t)\mathrm{d}t + \sigma_S S(t)\{\rho \mathrm{d}W_r^Q(t) + (1-\rho^2)^{1/2}\mathrm{d}\widetilde{W}^Q(t)\} \tag{2.9}$$

where

$$\mathrm{d}W_S^Q(t) = \rho \mathrm{d}W_r^Q(t) + (1-\rho^2)^{1/2}\mathrm{d}\widetilde{W}^Q(t).$$

In the valuations presented in this chapter, we simply assume $\rho = 0$ unless otherwise stated. The actual correlation between these two processes is difficult to estimate accurately in practice, and changes for different periods of data.

The function $\Theta(t)$ in equation (2.6) depends on σ_r and a (Brigo and Mercurio, 2006). It is defined as

$$\Theta(t) = \frac{\partial f^M(0,t)}{\partial t} + af^M(0,t) + \frac{\sigma_r^2}{2a}(1 - e^{-2at}), \tag{2.10}$$

where $f^M(0,t)$ is the market instantaneous forward rate at time 0 for the maturity t. By definition,

$$f^M(0,t) = -\frac{\partial \log(P^M(0,t))}{\partial t},$$

where $P^M(0,t)$ is the market price at time 0 of a zero coupon bond with face value of \$1 and maturity date t.

Clearly, more sophisticated models for the underlying processes are available, such as stock price processes that allow for jumps or stochastic volatility and multi-factor interest rate processes. However, the main motivation for our choices is that the models we employ are well understood benchmarks, and their use allows us to isolate and focus on the influence of the contract features rather than idiosyncrasies of the assumed processes. Moreover, note that we only require the investment account values on each policy anniversary. In other words, we only need to model the annual returns. Assuming the annual returns are normally distributed is not unreasonable. One of the well-known stylized facts of empirical stock return data is aggregational normality (Cont, 2001). As we increase the length of the intervals over which stock returns are calculated, the empirical distributions of the returns tend to appear more normally distributed. Furthermore, the shape of the return distribution changes at different time scales. Well known features of stock returns observed over consecutive time intervals of a month, week, day, hour, minute, such as stochastic volatility, large jumps, and volatility clustering, are largely "washed away" in annual return data. Therefore, in a sense, there is not a strong incentive to use a more complicated model for the investment account, such as Heston-type models (Heston, 1993) which are popular equity return models among practitioners.

2.2.2 Pricing equation for the GMIB

Let $E^Q[\cdot]$ indicate that an expectation is computed under a risk-neutral probability measure Q. Furthermore, let $\mathcal{F}_t$ denote all of the information available at time t. More formally,

$$\boldsymbol{F}_{[0,T]} = \{\mathcal{F}_t, 0 \leqslant t \leqslant T\}$$

is the filtration generated by the stock and short rate processes from time 0 to time T. In the Hull-White model the price of a zero coupon bond at time t, maturing at time T with unit face value has an analytical formula, conditional on the value of $r(t)$. It is given by the formula (Brigo and Mercurio, 2006)

$$P(t,T) = \mathcal{A}(t,T)e^{-\mathcal{B}(t,T)r(t)}, \tag{2.11}$$

where

$$\mathcal{B}(t,T) = \frac{1}{a}[1 - e^{-a(T-t)}],$$

$$\mathcal{A}(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\left\{\mathcal{B}(t,T)f^M(0,t) - \frac{\sigma_r^2}{4a}(1 - e^{-2at})\mathcal{B}(t,T)^2\right\}.$$

Therefore, the value at time 0 of the GMIB maturity value can be calculated as

$$\begin{aligned} V(c) &= E^Q\left[e^{-\int_0^T r(t)\mathrm{d}t}\max\{B(T)g\ddot{a}_{\overline{20}|}(T), A(T)\}\right] \\ &= E^Q\left[e^{-\int_0^T r(t)\mathrm{d}t}\max\{B(T)g\sum_{j=T}^{T+19}P(T,j), A(T)\}\right]. \end{aligned} \tag{2.12}$$

For ease of exposition we henceforth refer to $V(c)$ as the *GMIB price* (although strictly speaking $V(c)$ is the price of a variable annuity with an embedded GMIB option). If the issuer wants to hedge the GMIB by investing in a replicating portfolio, the fee rate $c = c^*$ is fair if

$$V(c^*) = \pi, \tag{2.13}$$

where π is the single annuity premium invested at time 0.

Using the model specified in Section 2.2.1, the market is complete if we can trade in the stock and at least one bond at all times. If the market is complete, the risk neutral measure Q is unique. Later, in Section 3.2 we discuss the conditions on the measure changes between the risk-neutral and the real-world (objective) probability measures for the stock and interest rate processes, that must be satisfied for the market to be complete.

In understanding equation (2.13), it is useful to recall the justification for why the arbitrage-free GMIB price is equal to π when c is equal to the fair fee rate. From derivative pricing theory (Björk (2004), Hull (2008), Joshi (2008), Musiela and Rutkowski (2004)), it is known that, in a complete market, the GMIB maturity value can be replicated by investing in a portfolio consisting of the stock and zero coupon bonds costing π dollars at time 0, and then rebalancing this portfolio dynamically in a self-financing way until time T. More succinctly, the payoff of the derivative can be reproduced exactly by investing π dollars at time 0 and following a pre-defined replicating strategy. We emphasize that this replicating strategy is distinct from the concept of the insurer physically investing the policyholder's premium in the stock index portfolio at the outset, and then physically periodically withdrawing fees from the policyholder's investment account.

If the equality given by equation (2.12) does not hold, then the insurer has either made a profit or loss at time 0. Let $H(c)$ denote the *cost of hedging*, which is the excess amount of funds needed by the insurer to hedge the GMIB maturity value, when the fee rate charged is c:

$$H(c) = V(c) - \pi. \tag{2.14}$$

Equation (2.14) is the pricing equation for a variable annuity with an embedded GMIB. If $H(c)$ is positive (negative), the insurance company is undercharging (overcharging) the policyholder for the GMIB.

2.2.3 Other fee charges in practice

It is important to note that administrative and investment management fees associated with the underlying variable annuity contract are not incorporated into our valuations. The actual size of these fees (in total) can be somewhere between 0.5-3% of the policyholder's investment account per year, during the accumulation phase. While the impact of these fees on the GMIB price is not negligible, the actual size of these fees varies with insurance company, the policyholder's choice of investment managers, and mortality assumptions (which we have not incorporated). Making allowances for fees related to the underlying variable annuity contract is rather subjective, and we do not allow for them in our model. Our interest is in determining a fair fee rate for the GMIB option. In other words, we want to calculate the "pure" fair fee rate that should be charged for the benefits provided by the GMIB option. It is noted that incorporating the underlying fees into our model can be easily done, if the sizes of these fees are known with reasonable certainty. How these additional fees will affect the GMIB option is not entirely clear. The value of the lookback and investment account components will be reduced by the additional fees. However, the guaranteed return component is more likely to be exercised in the real world. We briefly explore the impact of additional underlying contract fee charges on the fair fee rates in Section 2.8.

2.2.4 Alternative view of the pricing equation

We can decompose the GMIB price $V(c)$ in order to gain an alternative view of equation (2.14). Equation (2.12) can be expanded to

$$V(c) = E^Q\left[e^{-\int_0^T r(t)\mathrm{d}t}\max\left\{B(T)g\ddot{a}_{\overline{20|}}(T) - A(T), 0\ \right\}\right] + E^Q\left[e^{-\int_0^T r(t)\mathrm{d}t}A(T)\right]. \quad (2.15)$$

The first term on the right hand side of equation (2.15) equals the total value of the benefits provided by the GMIB option. The second term on the right hand side of equation (2.15) is the risk-neutral expected present value of the maturity value of the investment account. We proceed by decomposing the second term. Recall that $f(n) = \min\{cB(n), A(n-)\}$ denotes the size of the fee deducted on the n-th policy anniversary (if c is sufficiently large, $f(n) = 0$ is possible). The following recursive relationship holds for $t = 2, 3, \ldots, T$:

$$\begin{aligned}
A(t) &= A(t-1)R(t) - f(t) \\
&= [A(t-2)R(t-1) - f(t-1)]R(t) - f(t) \\
&= [[A(t-3)R(t-2) - f(t-2)]R(t-1)R(t) - f(t-1)R(t) - f(t) \\
&\quad \vdots \\
&= A(0)\prod_{n=1}^{t} R(n) - \left[\sum_{n=1}^{t-1} f(n)[\prod_{i=n+1}^{t} R(i)] + f(t)\right]
\end{aligned}$$

where $R(n)$ is defined by equation (2.3). Since $A(0) = S(0)$,

$$A(t) = S(t) - \left[\sum_{n=1}^{t-1} f(n)[\prod_{i=n+1}^{t} R(i)] + f(t)\right]. \quad (2.16)$$

Combining equations (2.15) and (2.16) yields

$$
\begin{aligned}
V(c) = {} & E^Q\left[e^{-\int_0^T r(t)\mathrm{d}t}\max\left\{B(T)g\ddot{a}_{\overline{20|}}(T) - A(T), 0\ \right\}\right] + E^Q\left[e^{-\int_0^T r(t)\mathrm{d}t}S(T)\right] \\
& - E^Q\left[e^{-\int_0^T r(t)\mathrm{d}t}\left\{\sum_{n=1}^{T-1} f(n)[\prod_{i=n+1}^{T} R_i] + f(T)\right\}\right] \\
= {} & E^Q\left[e^{-\int_0^T r(t)\mathrm{d}t}\max\left\{B(T)g\ddot{a}_{\overline{20|}}(T) - A(T), 0\ \right\}\right] + S(0) \\
& - E^Q\left[e^{-\int_0^T r(t)\mathrm{d}t}\left\{\sum_{n=1}^{T-1} f(n)[\prod_{i=n+1}^{T} R_i] + f(T)\right\}\right]. \qquad (2.17)
\end{aligned}
$$

We have used the result that

$$
E^Q\left[e^{-\int_t^T r(s)\mathrm{d}s}S(T)\big|\mathcal{F}_t\right] = S(t) \qquad (2.18)
$$

in equation (2.17). Equation (2.18) can be shown as follows. Applying Ito's Lemma to $f(S(t)) = \log(S(t))$ and integrating yields

$$
S(T) = S(t)e^{\int_t^T r(s)\mathrm{d}s - \sigma_S^2(T-t)/2 + \sigma_S(W_S^Q(T) - W_S^Q(t))}.
$$

Therefore,

$$
\begin{aligned}
& E^Q\left[e^{-\int_t^T r(s)\mathrm{d}s}S(T)\big|\mathcal{F}_t\right] \\
& = E^Q\left[S(t)e^{-\int_t^T r(s)\mathrm{d}s + \int_t^T r(s)\mathrm{d}s - \sigma_S^2(T-t)/2 + \sigma_S(W_S^Q(T) - W_S^Q(t))}\big|\mathcal{F}_t\right] \\
& = S(t)E^Q\left[e^{-\sigma_S^2(T-t)/2 + \sigma_S(W_S^Q(T) - W_S^Q(t))}\big|\mathcal{F}_t\right]. \qquad (2.19)
\end{aligned}
$$

Now $W_S^Q(T) - W_S^Q(t)\big|\mathcal{F}_t \sim N(0, T-t)$, and thus by knowing the mean of the lognormal distribution,

$$
E^Q\left[e^{-\sigma_S^2(T-t)/2 + \sigma_S(W_S^Q(T) - W_S^Q(t))}\big|\mathcal{F}_t\right] = e^{-\sigma_S^2(T-t)/2} \cdot e^{\sigma_S^2(T-t)/2} = 1. \qquad (2.20)
$$

Thus equation (2.19) simplifies to equation (2.18).

Rearranging equation (2.17) yields

$$V(c) - S(0) = E^Q \left[e^{-\int_0^T r(t)\mathrm{d}t} \max\left\{ B(T) g \ddot{a}_{\overline{20}|}(T) - A(T), 0 \ \right\} \right]$$
$$- E^Q \left[\sum_{n=1}^{T-1} e^{-\int_0^T r(t)\mathrm{d}t} f(n) [\prod_{i=n+1}^{T} R(i)] + e^{-\int_0^T r(t)\mathrm{d}t} f(T) \right]$$

Using the fact that

$$\prod_{i=n+1}^{T} R(i) = e^{\int_n^T r(s)\mathrm{d}s - \sigma_S^2 (T-n)/2 + \sigma_S (W_S^Q(T) - W_S^Q(n))}$$

we obtain

$$\begin{aligned}
&V(c) - S(0) \\
&= E^Q \left[e^{-\int_0^T r(t)\mathrm{d}t} \max\left\{ B(T) g \ddot{a}_{\overline{20}|}(T) - A(T), 0 \ \right\} \right] \\
&- E^Q \left[\sum_{n=1}^{T-1} e^{-\int_0^n r(t)\mathrm{d}t} f(n) e^{-\sigma_S^2 (T-n)/2 + \sigma_S (W_S^Q(T) - W_S^Q(n))} + e^{-\int_0^T r(t)\mathrm{d}t} f(T) \right] \\
&= E^Q \left[e^{-\int_0^T r(t)\mathrm{d}t} \max\left\{ B(T) g \ddot{a}_{\overline{20}|}(T) - A(T), 0 \ \right\} \right] \\
&- E^Q \left[\sum_{n=1}^{T-1} e^{-\int_0^n r(t)\mathrm{d}t} f(n) \right] \cdot E^Q \left[e^{-\sigma_S^2 (T-n)/2 + \sigma_S (W_S^Q(T) - W_S^Q(n))} \right] - E^Q \left[e^{-\int_0^T r(t)\mathrm{d}t} f(T) \right].
\end{aligned} \tag{2.21}$$

The last step follows from the fact that for $n > 0$, $e^{-\int_0^n r(t)\mathrm{d}t} f(n)$ is independent of $W_S^Q(T) - W_S^Q(n)$. Furthermore, using equation (2.20),

$$E^Q \left[e^{-\sigma_S^2 (T-n)/2 + \sigma_S (W_S^Q(T) - W_S^Q(n))} \right] = 1.$$

Using the fact that $\pi = S(0)$, equation (2.21) can be rearranged into the form of the pricing equation given by equation (2.14):

$$H(c) = V(c) - \pi = G(c) - F(c) \tag{2.22}$$

where

$$G(c) = E^{Q}\left[e^{-\int_0^T r(t)\mathrm{d}t}\max\left\{B(T,c)g\ddot{a}_{\overline{20|}}(T) - A(T,c), 0\ \right\}\right], \tag{2.23}$$

$$F(c) = E^{Q}\left[\sum_{n=1}^{T} e^{-\int_0^n r(t)\mathrm{d}t} f(n,c)\right]. \tag{2.24}$$

For clarity of exposition, we have explicitly identified all terms in equations (2.23) and (2.24) that are functions of the fee rate c. The function $G(c)$ is equal to the expected present value under Q (EPV^Q) of the benefits provided by the GMIB option. Loosely speaking, the GMIB benefits resemble an equity put option on the investment account with a random strike price (and a random maturity date T, which we have assumed is a constant in order to simplify the GMIB valuation). The function $F(c)$ is equal to the EPV^Q of the fees paid for the GMIB option during the accumulation phase. The GMIB pricing equation (2.22) says that the price of a variable annuity with the GMIB option minus the initial annuity premium, is equal to the EPV^Q of the benefits provided by the GMIB minus the EPV^Q of the fees paid. We emphasize that the EPV^Q of the benefits provided by the GMIB option requires the policyholder to annuitize their investment at maturity (and thus receive a stream of income over 20 years), rather than receiving a lump sum benefit at maturity. (However, in theory the policyholder could sell the retirement income stream provided by the GMIB to another party in exchange for a lump sum.) When the GMIB is priced fairly $H(c) = 0$, and $G(c) = F(c)$.

2.3 Valuation results

In this section, we illustrate how the GMIB price varies as a function of the fee rate. The fair fee rate is determined for a realistic set of values of the parameter g. Due to the complexity of the GMIB, we use Monte Carlo simulation to value the GMIB. All of the estimates we compute in this chapter are based on 10^5 scenarios, unless stated otherwise.

The following benchmark parameter assumptions are used, unless indicated otherwise: $T = 10$, $r_g = 5\%$, $A(0) = S(0) = \pi = 1000$, $\sigma_S = 20\%$, $a = 0.35$, $\sigma_r = 1.5\%$ $\rho = 0$, and $\Theta(t)$ depends on a linear approximation of the shape of the U.S. zero coupon bond yield curve halfway through 2008 (the curve is displayed later in Figure 2.7 as the one labeled "Benchmark"). We now briefly explain our choices of parameter values for the stock and short rate models. Our value of π was chosen for neatness in illustrating the results. If we changed the value of the premium from π to $m\pi$, for some $m > 0$, leaving all other parameters unchanged, then the GMIB price will change from $V(c)$ to $mV(c)$. However, the fair fee rate will not change. Setting $\sigma_S = 20\%$ is a common assumption. We set $\sigma_r = 1.5\%$ because this value corresponds roughly to the volatility of the cash rate set by the U.S. Federal Reserve Bank for the past 10-20 years. We set $a = 0.35$ as this value for the speed of reversion is broadly comparable with speed of reversion estimates obtained from several one-factor continuous time short-rate models fitted to U.K. and U.S. data over several decades (for example, see Nowman (1997) and Yu and Phillips (2001)). Boyle and Hardy (2003) also use the Hull-White model to value guaranteed annuity options, and they also assume $a = 0.35$. We set $\rho = 0$ because, as previously mentioned, the actual correlation between these two processes is difficult to estimate accurately in practice. The yield curve we use to calibrate $\Theta(t)$ is one that we believe is representative of a common upward sloping yield curve in a stable, low inflation, economic environment.

2.3.1 Choice of g

The lookback and guaranteed return components are proportional to the guaranteed payment rate g. Therefore, the GMIB price is highly sensitive to the value of g. The insurance company sets the value of g they are prepared to offer at the time the contract is sold. The value of g must be competitive and should be equitable. Its magnitude also depends on the type of annuity, selected by the policyholder at time 0, for which the GMIB offers income protection. For a 20 year term certain annuity, g is likely to be in the range 5-10%. The justification for this range is as follows: if g is set fairly then it should be approximately equal to the inverse of the value of a 20 year term certain annuity. The value of the annuity depends on the assumed interest rate term structure over a 20 year period.

Table 2.1 displays values of 20 year term certain annuities with annual payments made in advance, for various annually compounded interest rates, assuming the interest rate term structure remains flat and constant. In actuarial notation, these are values of $\ddot{a}_{\overline{20}|}$. The estimates of g in the table are equal to $1/\ddot{a}_{\overline{20|}}$. The table shows that values of g between 5% and 10%, correspond to the range of flat interest rate term structures between 0% and 10%. If interest rates do not exceed say 9% over the long term (plausible based on recent history in the U.S.), then a competitive/equitable value for g seems to be somewhere between 5% to 8%. In making our choice of g for the valuations, we keep in mind that insurance companies explicitly state, in their variable annuity prospectuses, that they set g conservatively with respect to future mortality and interest rate assumptions. It is difficult to say what the exact value of g should be. Therefore, in this chapter we present numerical results typically for g of 5.5%, 6.5%, 7.5% and 8.5%, which correspond to cheap through to expensive valuation assumptions, from the insurance company's perspective. It is noted that, based on the benchmark parameter assumptions, $E^Q[\ddot{a}_{\overline{20|}}(T)] = 12.78$. Therefore, a reasonable upper bound for the appropriate value of g is given by $g = 1/\ddot{a}_{\overline{20|}} < 1/12.78 = 7.82\%$. The author considers values of g between 6.5% and 7.5% to be equitable for the policyholder, based on historical U.S. interest rate levels over the past decade. The range also balances the interests of the insurer, who wants to offer a competitive value of g, but also wants some conservatism in setting g.

Constant interest rate r	0%	0.5%	1%	1.5%	2%	2.5%	3%	4%	5%	6%	7%	8%	9%
$\ddot{a}_{\overline{20|}}$ **using** r	20	19.1	18.2	17.4	16.7	16.0	15.3	14.1	13.1	12.2	11.3	10.6	10
$g = 1/\ddot{a}_{\overline{20|}}$	5%	5.2%	5.5%	5.7%	6%	6.3%	6.5%	7.1%	7.6%	8.2%	8.8%	9.4%	10.1%

Table 2.1: *Fair value of g for a 20 year term certain annuity with annual payments in advance.*

2.3.2 Fair fee rates

Figure 2.1 illustrates the relationship between the GMIB price $V(c)$ and the fee rate c for a realistic range of values of g. The standard errors of the GMIB price estimates in Figure 2.1 lie in the range 0.4-2.4. Setting $g < 5\%$ is unlikely to be competitive or equitable. If $g > 10\%$, then the insurer is offering very generous benefits; no insurer is likely to offer a rate so high. Each curve corresponds to the GMIB price for a particular g, and the fee rate at the intersecting point of a curve with the horizontal dotted line corresponds to the fair fee rate for the curve. For any given fee rate, the vertical distance between a curve and the horizontal dotted line corresponds to $H(c)$, the cost of hedging. When any of the curves lie below the horizontal dotted line, the cost of hedging is negative, which can be thought of as profit for the insurer (in an ideal world where the option can be hedged perfectly). For $g > 7\%$, it is not possible for the insurance company to break even at any fee rate (based on the benchmark parameter assumptions); hedging the GMIB using a replicating portfolio requires the insurer to obtain funds from elsewhere. For $g > 7\%$, fair fee rates do not exist because the guaranteed return component, $A(0)(1+r_g)^T g\ddot{a}_{\overline{20}|}(T)$, is very valuable at maturity, and this component does not decrease in value as the fee rate increases – it is independent of the fee rate.

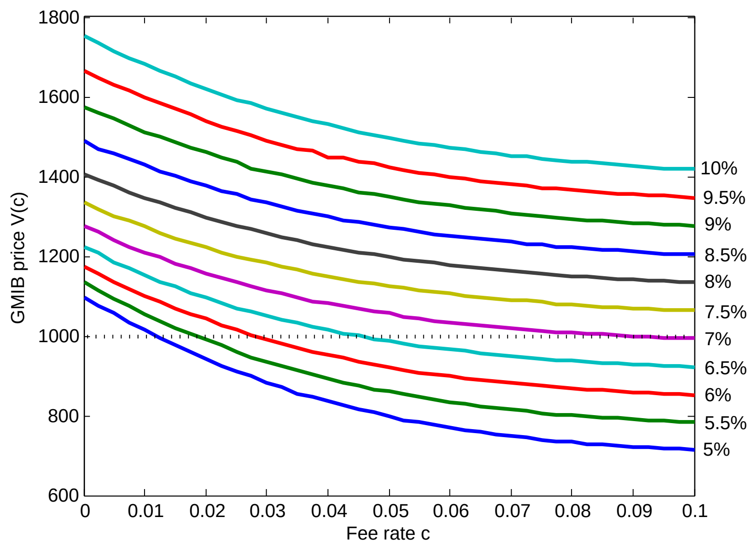

Figure 2.1: *GMIB price $V(c)$ as a function of the fee rate c. Each curve corresponds to a particular value of g. For the curves that intersect with the horizontal dotted line, the fee rate at the intersecting point corresponds to the fair fee rate.*

Using Figure 2.1, we can roughly approximate the fair fee rate for each g to within a few basis points. However, it is always preferable to obtain statistical estimates from the output of a Monte Carlo simulation whenever possible. Since the GMIB price cannot be computed analytically, it is not immediately obvious how we would calculate an estimate of the fair fee rate, and the standard error of this estimate, for a given value of g. As it turns out, an estimate of the fair fee rate and its standard error are easily computed by recognizing that the GMIB price can be estimated once we have simulated N observations from the random vector

$$\widetilde{W} = \left(\int_0^T r(t)\mathrm{d}t, r(T), R(n)\ n = 1, 2, \ldots, T \right) \Big| \mathcal{F}_0,$$

where $R(n)$ is defined by equation (2.3). None of the random variables in $\widetilde{W}$ depend on the fee rate c. Given N observations of $\widetilde{W}$ and any value of $c > 0$, we can compute N values of

$$\phi(\widetilde{W}, c) = e^{-\int_0^T r(t)\mathrm{d}t}\left[\max\{B(T)g\ddot{a}_{\overline{20}|}(T), A(T)\}\right]$$

using equations (2.2), (2.4) and(2.5). Let $\phi_n(c), n = 1, \ldots, N$ denote the N observations of $\phi(c)$. For a given c, we can estimate the GMIB price from the usual Monte Carlo estimator

$$\widehat{V}(c) = \frac{1}{N}\sum_{n=1}^{N}\phi_n(c). \tag{2.25}$$

When g is not so large that a fair fee rate does not exist (the actual value of g at which a fair fee rate no longer exists will depend on the all of the parameter assumptions), there exists a true fair fee rate c such that

$$E^Q[\phi(\widetilde{W}, c)] = \pi.$$

Using Monte Carlo simulation, we know that

$$\frac{1}{N}\sum_{n=1}^{N}\phi_n(c) \to E^Q[\phi(\widetilde{W}, c)], \text{ as } N \to \infty.$$

Therefore, for sufficiently large N, we can approximately estimate the fair fee rate by solving the equation

$$\frac{1}{N}\sum_{n=1}^{N}\phi_n(c) - \pi = 0 \tag{2.26}$$

with respect to c. The approximation improves as N increases. Because equation (2.25) is a continuous function with respect to c, equation (2.26) can be solved numerically. For example, it is easily solved in MATLAB using the “fsolve” function. If a solution cannot be found, then it probably means a fair fee rate does not exist for the parameter assumptions chosen. It is noted the Delta Method could also be used to obtain the standard deviation of the fair fee rate (Casella and Berger, 2001).

To obtain the standard deviation of the fair fee rate, we solve equation (2.26) for J independent Monte Carlo simulations. Let c_j denote the fair fee rate obtained from solving equation (2.26) for the j-th simulation. A more stable estimate of the fair fee rate can be calculated as $\bar{c} = \frac{1}{J}\sum_{j=1}^{J} c_j$. The standard error of the fair fee rate is calculated as

$$\hat{\sigma}_c = \frac{1}{J^{1/2}} \left\{ \frac{1}{J-1} \sum_{j=1}^{J} \left(c_j - \bar{c}^2 \right) \right\}^{1/2} .$$

Table 2.2 reports the fair fee rates and their standard errors for values of g for which a fair fee rate exists, based on $J = 100$ Monte Carlo simulations and $N = 10^5$ scenarios within each simulation. The standard errors of the fair fee rates increase with g.

g	5%	5.5%	6%	6.5%	7%
Fair fee rate	1.19%	1.82%	2.80%	4.50%	8.84%
Std error of fair fee rate	0.0023%	0.0024%	0.0027%	0.0032%	0.0073%

Table 2.2: *Fair fee rates and their standard errors for values of g for which a fair fee rate exists.*

As of the middle of 2010, competitive insurance companies are charging fees of 0.8-1% for GMIB options (www.annuityFYI.com). Our simple model suggests that insurance companies may be underpricing GMIBs for equitable values of g. However, our valuations have ignored policyholder lapse assumptions. The policyholder must wait at least 10 years from inception if they wish to annuitize using the GMIB option. It seems probable that some policyholders would lapse before the minimum maturity date, in which case the fair fee rate would be reduced. Section 2.6 examines the issue of lapses in further detail. Furthermore, as mentioned in Section 2.2.3, our analysis has not allowed for fees relating to the underlying variable annuity contract, which can be somewhere between 0.5-3% of the investment account per year. Allowing for these fees would also reduce the fair fee rate for the GMIB option. Section 2.8 briefly explores the impact on the fair fee rate from allowing for these underlying variable annuity contract fees.

The fair fee rates we have obtained for a GMIB seem to be slightly higher than the fair fee rates reported in Bauer et al. (2008) for GMIBs. However, Bauer et al. (2008) value each of the benefits provided by a GMIB in isolation and determine the fair fee rates for each individual benefit separately but not for the GMIB as a whole. They also use a different fee structure.

2.3.3 Valuing the benefits and the fees separately

Based on the alternative view of the pricing equation discussed in Section 2.2.4, it is possible to measure $G(c)$, the EPV^Q of the benefits provided by the GMIB, and $F(c)$, the EPV^Q of the fees paid for the option, as functions of the fee rate c. Figure 2.2 depicts the cost of hedging $H(c) = V(c) - \pi$, $G(c)$ and $F(c)$ and as functions of the fee rate c, for particular values of g. The standard errors of the $H(c)$, $G(c)$ and $F(c)$ estimates lie in the ranges 0.4-2.4, 0.5-1.1 and 0.0-1.7 respectively. Note that $F(c)$ is independent of g. This figure highlights a tricky issue with the design of the GMIB. Both $G(c)$ and $F(c)$ increase with the fee rate c. Why $G(c)$ increases with c is worth explaining in more detail. As c increases, $B(T)$ and $A(T)$ both decrease. Therefore, since $G(c)$ increases with c because the difference $B(T)g\ddot{a}_{\overline{20|}}(T) - A(T)$ increases, it must be the case that $B(T)$ decreases at a slower rate than $A(T)$ with respect to c. The GMIB design would be much less risky if the benefits provided by the GMIB option did not increase so sharply as a function of the fee rate. Ideally, it would be better, from the point of view of controlling risk, if the benefits provided by a variable annuity option were insensitive to the fee rate (or a monotone decreasing function of the fee rate).

2.4 Sensitivity analysis

In this section we perform a sensitivity analysis of the parameter values in our models. The benchmark parameter values are listed at the start of Section 2.3.

2.4.1 Stock volatility

The left panel of Figure 2.3 displays the relationship between $V(0)$, the GMIB price when the fee rate is zero, and the stock volatility σ_S, for various of g. The right panel of Figure 2.3 shows the GMIB price as a function of the stock volatility when the fair fee rate is charged, for values of $g < 7\%$ (fair fee rates do not exist for $g > 7\%$, and the fair fee

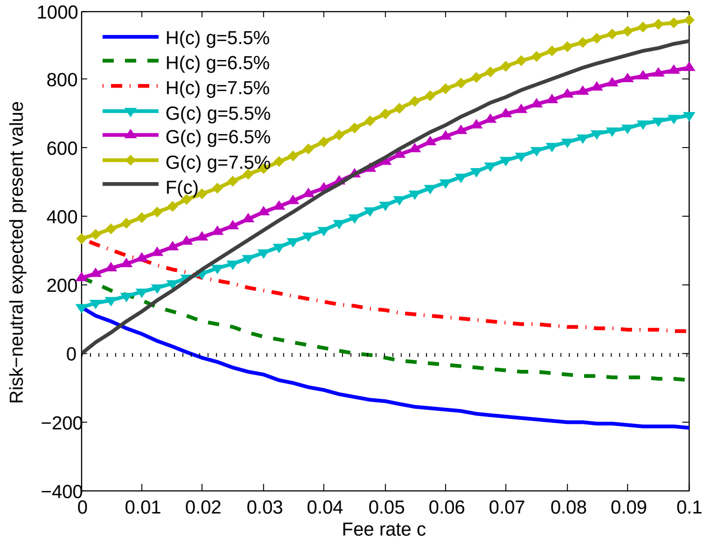

Figure 2.2: *$G(c)$, $F(c)$ and $H(c) = G(c) - F(c)$ as functions of the fee rate c, for particular values of g.*

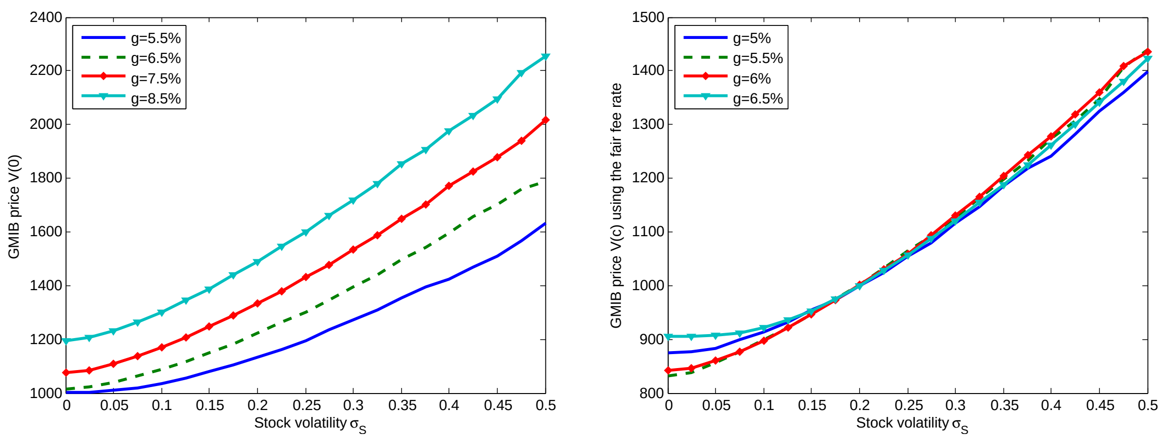

Figure 2.3: *The left panel displays the GMIB price without fee charges $V(0)$ as a function of stock volatility σ_S. The right panel displays the GMIB price as a function of stock volatility when the fair fee rate is charged. Each curve corresponds to a particular value of g.*

rate for $g = 7\%$ is too high to ever realistically be charged). For example, when $g = 6\%$, the fair fee rate is 1.85% and the corresponding curve plots the GMIB price using this fee rate as a function of the stock volatility. The standard errors of the GMIB price estimates lie in the range 0.2-9.9, and increase as the volatility increases. As expected, the GMIB price is a monotonically increasing function of σ_S. A higher volatility gives greater probability to larger values of $B(T)$ and $A(T)$ at time T. In the left panel, starting at a conservative volatility of 10%, each 5% increase in volatility leads to an increase in $V(0)$ of about 4-8%, where the percentage increases in $V(0)$ are gradually increasing. In practice, the policyholder must decide how their money is invested. The insurer provides the policyholder with a range of fund managers with different risk/return profiles (e.g. growth, capital stable, fixed interest). Subject to the insurer charging the same fee rate for a given set of fund managers, the policyholder should allocate their money to fund managers with the highest volatilities, if they want to maximize the value of their GMIB option.

2.4.2 Impact of interest rate assumptions

The Hull-White model is employed for modeling interest rates. In this section we explore the sensitivity of the GMIB price to the parameter values in the Hull-White model. It is shown that the yield curve assumptions have a significant influence on the GMIB price.

Interest rate volatility

The left panel of Figure 2.4 displays the GMIB price without fee charges $V(0)$ plotted against interest rate volatility σ_r for various values of g. The right panel of Figure 2.4 displays the GMIB price as a function of interest rate volatility when the fair fee rate is charged, for lower values of g where the fair fee rate exists. The standard errors of the GMIB price estimates lie in the range 0.9-5.2. The GMIB price is a monotonically increasing function of σ_r. The GMIB maturity value is sensitive to the level of the short rate at time T through $\ddot{a}_{\overline{20|}}(T)$, and thus it makes sense that the value of the GMIB option increases as σ_r increases. Note that a higher interest rate volatility leads to greater variability in the discounting factor, and in the drift term of the stock SDE given by equation (2.1). In the left panel, each 1% increase in interest rate volatility leads to an increase in $V(0)$ of about 0.2-1.5% (for σ_r in the range 0-5%), where the percentage increases in $V(0)$ are gradually increasing. Changes in the interest rate volatility have a much smaller influence on the GMIB price compared to changes in the stock volatility. For interest rate volatilities of less than 2%, which are arguably realistic values for the past decade, the GMIB price remains fairly constant. The fact that the GMIB price is relatively insensitive to the interest rate volatility assumption is consistent with the results of Boyle and Hardy (2003). They find that the guaranteed annuity option is also relatively insensitive to the interest rate volatility assumption for long periods.

The GMIB option is partially an interest rate option through the value of $\ddot{a}_{\overline{20|}}(T)$. In our model, the uncertainty of interest rates is encapsulated through the parameter σ_r. The top panels of Figure 2.5 exhibit the distribution of $\ddot{a}_{\overline{20|}}(T) = \ddot{a}_{\overline{20|}}(10)$ for $\sigma_r = 0.5\%$ and $\sigma_r = 1.5\%$. Since $\ddot{a}_{\overline{20|}}(10)$ is a function of $r(T) = r(10)$, the bottom panels display the distribution of $r(10)$ for $\sigma_r = 0.5\%$ and $\sigma_r = 1.5\%$. The probability of $r(10) < 0$ is approximately 0% for $\sigma_r = 0.5\%$, and 0.14% for $\sigma_r = 1.5\%$. Figure 2.5 is primarily presented to give the reader a feel for the magnitude of $\ddot{a}_{\overline{20|}}(T)$ based on our model assumptions. This figure also gives a sense of how interest rate risk is captured in our valuation model.

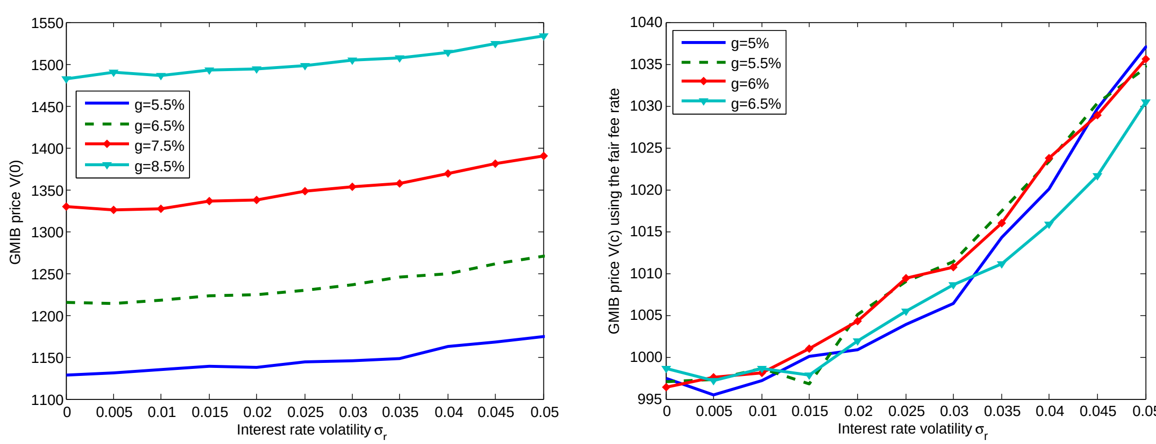

Figure 2.4: *The left panel displays the GMIB price without fee charges $V(0)$ as a function of interest rate volatility σ_r. The right panel displays the GMIB price as a function of interest rate volatility when the fair fee rate is charged. Each curve corresponds to a particular value of g.*

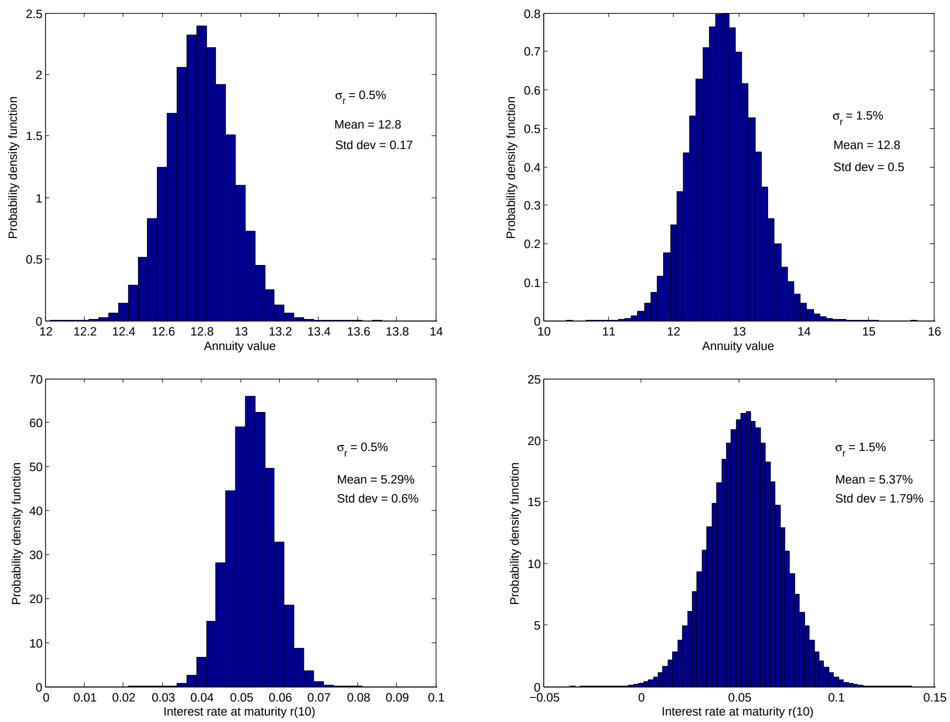

Figure 2.5: *The top panels display the distribution of the 20 year term certain annuity for* $\sigma_r = 0.5\%$ *and* $\sigma_r = 1.5\%$*. The bottom panels display the distribution of* $r(10)$ *for* $\sigma_r = 0.5\%$ *and* $\sigma_r = 1.5\%$*.*

Speed of reversion

The left panel of Figure 2.6 shows the relationship between the GMIB price without fees $V(0)$ and the speed of reversion a in the SDE given by equation (2.6), for the benchmark values of g. The right panel of Figure 2.6 displays the GMIB price as a function of a when the fair fee rate is charged, for lower values of g where the fair fee rate exists (note the range of the y-axis is narrow in the right panel, so the estimation errors are conspicuous). The standard errors of the GMIB price estimates lie in the range 1.7-2.6, and decrease as a increases. We note that in equation (2.12) the term $\ddot{a}_{\overline{20}|}(T) = \sum_{j=T}^{T+19} p(T,j)$ is a function of a. For our calibration of the yield curve, captured in $\Theta(t)$, the GMIB price is relatively insensitive to the value of a for $a > 0.25$. The GMIB price increases as $a \to 0$ because the probability of $r(T) < 0$ increases to a level that has a noticeable impact on the GMIB price. If $r(T) < 0$, the discounting factor tends to be larger in magnitude, and $\ddot{a}_{\overline{20}|}(T) > 20$, which drives up the discounted GMIB maturity value. The probability of $r(T) < 0$ is close to 0 for all $a > 0.25$ (i.e. negligible), but as $a \to 0$ the probability increases sharply to a few percentage points.

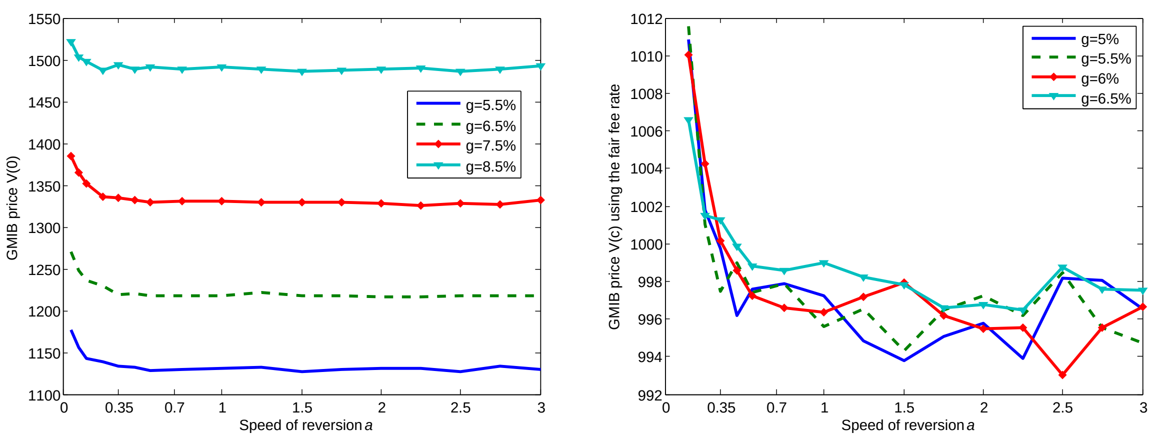

Figure 2.6: *The left panel displays the GMIB price without fee charges $V(0)$ as a function of the speed of reversion a in the Hull-White model. The right panel displays the GMIB price as a function of a when the fair fee rate is charged. Each curve corresponds to a particular value of g.*

Underlying yield curve shape

Figure 2.7 displays five different zero coupon bond yield curves that are used for testing the sensitivity of the GMIB price to the underlying yield curve shape. The yield curve shape affects $\Theta(t)$ in equation (2.6). The curve labeled "Benchmark" is the curve applied to all of the valuations presented in this chapter unless stated otherwise (it is a linear approximation of the shape of the U.S. zero coupon bond yield curve halfway through 2008). The curves labeled "3% Shift" and "6% Shift" are parallel upward shifts of the Benchmark curve, where the sizes of the shifts are 3% and 6% respectively. The shifted curves could occur in practice under different economic conditions to the present (e.g. when inflation rates are high). The curve labeled "Change in Convexity" represents a change in convexity of the Benchmark curve. The shape of the Change in Convexity curve is convex rather than concave as for the Benchmark curve, but to facilitate a comparison with the Benchmark curve the level of the Change in Convexity curve is roughly the same as the Benchmark curve at the short and long maturity dates. The curve labeled "Inverse" captures the shape of an inverted yield curve, where the level of this curve is kept close to the Benchmark curve at the shorter maturity dates. It is noted that the prominent features of each yield curve are deliberately concentrated in the first 10 years of the term structure.

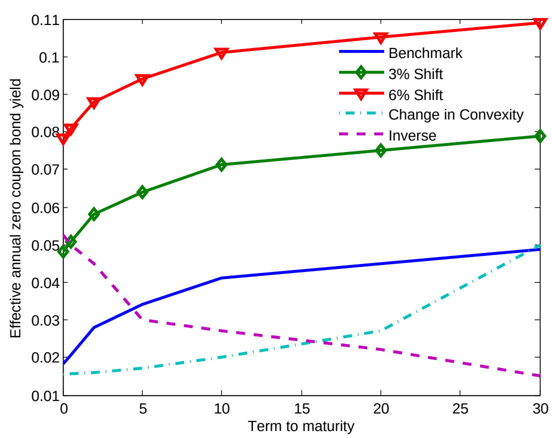

Figure 2.7: *A set of zero coupon bond yield curves used for testing the sensitivity of the GMIB price to the underlying assumed yield curve. Figure 2.8 shows the corresponding GMIB prices.*

The panels in Figure 2.8 display the GMIB price as a function of the fee rate for each of the yield curves shown in Figure 2.7, for $g = 6.5\%$ and $g = 7.5\%$. Each curve corresponds to the GMIB price for a given yield curve. The standard errors of the GMIB price estimates lie in the range 0.5-3.2. All else being constant, as the level of the yield curve increases the GMIB price decreases. In Figure 2.8, the 3% Shift curve is uniformly lower than the Benchmark curve, and similarly the 6% Shift curve is uniformly lower than the 3% Shift

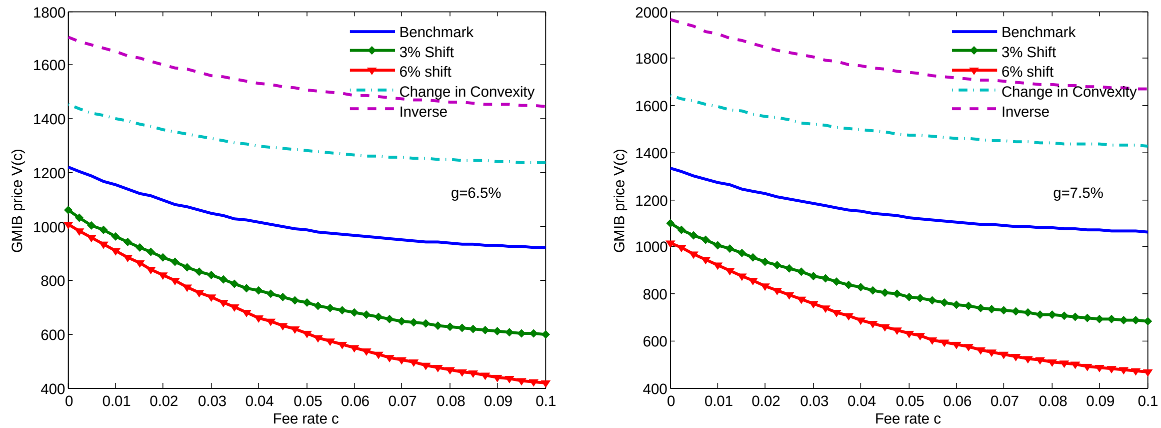

Figure 2.8: *GMIB price $V(c)$ as a function of the fee rate c, assuming $g = 6.5\%$ ($g = 7.5\%$) for each curve in the left (right) panel. Each curve plots the GMIB price using the corresponding yield curve displayed in Figure 2.7.*

curve. This observation makes sense, but it is important to realize that there are several factors affecting the GMIB price in opposite directions when the yield curve is shifted:

(1) The short rate reverts to $\Theta(t)/a$, and $\Theta(t)$ is larger for all t when the yield curve is shifted upwards. Thus the discounting factor will be larger, reducing the GMIB price.

(2) A higher yield curve reduces $\ddot{a}_{\overline{20}|}(T)$, scaling down the values of the lookback and guaranteed return components. Clearly, this also reduces the GMIB price.

(3) The drift coefficient in the SDE of the stock depends on the short rate, and the short rate will tend to follow higher paths during the accumulation phase when $\Theta(t)$ is larger for all t. Hence the investment account will also tend to follow higher paths during the accumulation phase, increasing the GMIB price.

Figure 2.8 shows that the effects of (1) and (2) overwhelm the effect of (3). Figure 2.9, discussed shortly, also shows that as interest rates increase, the GMIB price decreases.

By comparing the Change in Convexity and Benchmark curves in Figure 2.8, it is clear that the convexity of the yield curve has a significant impact on the GMIB price. Figure 2.7 shows that the Change in Convexity curve is lower than the Benchmark curve for all but the longest maturities; this is the reason why in Figure 2.8 the GMIB prices related to the Change in Convexity curve are higher than GMIB prices related to the Benchmark curve. The Inverse curve in Figure 2.8 is the highest of all the curves, demonstrating that the level of the long end of the yield curve significantly affects the GMIB price. Notice that in Figure 2.7, the yields at the long end of the Inverse curve are the lowest among all the curves. Hence, if the long end of a yield curve decreases, the GMIB price will increase sharply. This occurs because when the long end of the curve falls, the bond prices with maturity dates beyond time T increase, and the value of $\ddot{a}_{\overline{20|}}(T)$ increases in turn.

Impact of constant interest rates

It is clear that assumptions for the underlying yield curve shape have a large influence on the GMIB price, largely due to the long time until expiry of the GMIB option. Before moving on to other issues, we consider the impact on the GMIB price from removing the complication of stochastic interest rates. The panels in Figure 2.9 display the GMIB price $V(c)$ as a function of the fee rate c, for $g = 6.5\%$ and $g = 7.5\%$. Each curve assumes the term structure of interest rates is flat and constant through time at a particular continuously compounded annual rate r; r takes the values 2%, 3%, 4%, 5%, 6% and 7%. The standard errors of the GMIB price estimates lie in the range 0.3-2.1. Clearly, when the term structure is flat and shifted upwards, the GMIB price decreases. As the interest rate r increases, the discounting factor and the annuity value both decrease in value, reducing the GMIB price. The outcome of effects (1), (2) and (3) is again demonstrated in Figure 2.9, where a higher interest rate r corresponds to shifting up the yield curve structure. For a given fee rate c, each 1% increase in the interest rate r leads to a decrease in the GMIB price $V(c)$ of about 6-11%.

The results in Figure 2.9 can be loosely compared to those of Bauer et al. (2008) since the short rate is deterministic, although Bauer et al. use different values for T and σ_S, and set $r_g = 6\%$ ($r_g = 6\%$ used to be a common guaranteed rate offered by the major GMIB issuers until the global financial crisis struck – this point is discussed further in Section 2.10). Bauer et al. (2008) report that, using $r = 4\%$, no fair fee rate exists for a GMIB when g is above a certain value[2]. In the left panel of Figure 2.9, where $g = 6.5\%$, the fair

[2]Bauer et al. (2008) actually show there is no fair fee rate for what we call the guaranteed return

fee rate for $r = 4\%$ does not exist, which agrees with the results of Bauer et al. (2008).

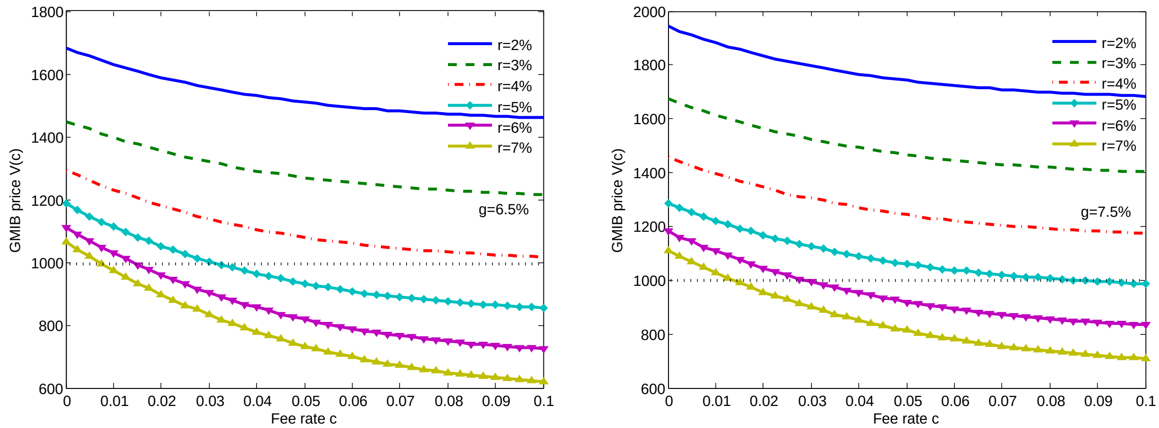

Figure 2.9: *Relationship between GMIB price $V(c)$ and the fee rate c for various constant continuously compounded annual interest rates r, assuming $g = 6.5\%$ for each curve in the left panel and $g = 7.5\%$ for each curve in the right panel. Each curve corresponds to a particular value of r.*

2.4.3 Correlation between the underlying processes

The results presented thus far have assumed that the short rate and stock processes evolve independently over time. This section considers the impact on the GMIB price when these processes are correlated. Recall that we have defined ρ as the linear correlation coefficient between $\{W_S^Q(t), t \in [0, T]\}$ and $\{W_r^Q(t), t \in [0, T]\}$ such that

$$\mathrm{Cov}^Q(\mathrm{d}W_S^Q(t), \mathrm{d}W_r^Q(u)) = \begin{cases} \rho \ \mathrm{d}t & \text{if } t = u, \\ 0 & \text{if } t \neq u. \end{cases}$$

The panels in Figure 2.10 compare the GMIB price $V(c)$ as a function of the fee rate c using various values of ρ, for $g = 6.5\%$ and $g = 7.5\%$. The standard errors of the GMIB price estimates lie in the range 0.6-1.9. The GMIB price is a monotone increasing function of ρ. It is difficult to give a clear explanation for this observed behavior because, regardless of whether the correlation is positive or negative, there are always multiple effects which influence the GMIB price in opposite directions. Consider when $\rho < 0$. In this case, the short rate tends to increase when the stock price decreases. However, when

component of the GMIB, not the entire GMIB contract, but the implications are the same.

the short rate increases the drift term of the stock SDE (see equation (2.1)) also increases. Hence the overall change in the value of the stock is not obvious when $\rho < 0$. However, a partial explanation for the observed behavior is suggested: If $\rho > 0$, the Brownian motion components in the SDEs of both processes tend to move in the same direction, and in the stock SDE there is a magnifying effect on the movement of the stock price since the drift term also moves in the same direction as the random term driven by the Brownian motion. This compounding effect causes the overall volatility of the stock to be slightly higher, and in Section 2.4.1 we have already shown that the GMIB price is sensitive to the stock volatility. If $\rho < 0$, the opposite effect occurs.

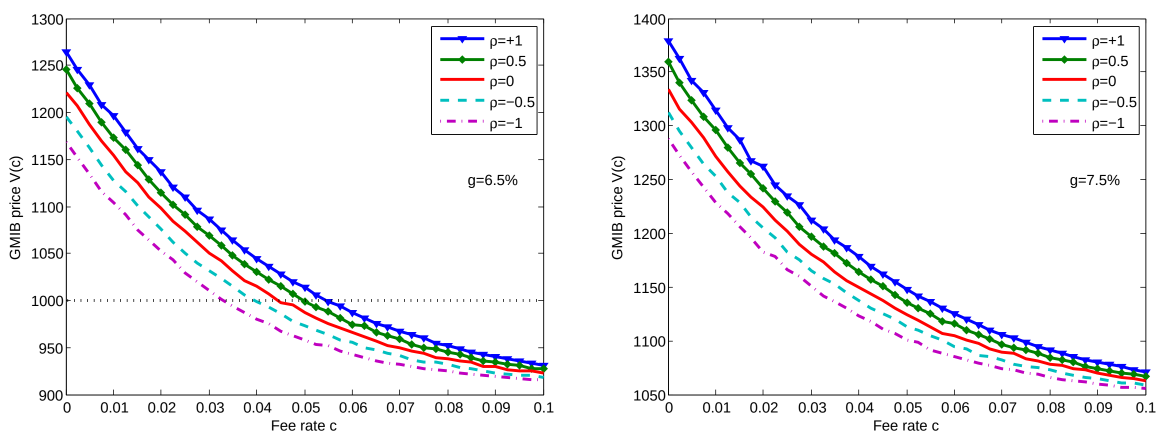

Figure 2.10: *Relationship between GMIB price $V(c)$ and the fee rate c for various values of ρ, assuming $g = 6.5\%$ ($g = 7.5\%$) for each curve in the left (right) panel. Each curve corresponds to a particular value of ρ.*

2.4.4 Varying the GMIB contract parameters

Guaranteed payment rate

Each GMIB seller must decide what guaranteed annual rate r_g it will offer for the guaranteed return component. Typically r_g is set somewhere between 4-6%, though 5% is currently very common. We have assumed $r_g = 5\%$ in our valuation assumptions. In this section we measure the change in the GMIB price from varying r_g. The left panel of Figure 2.11 displays the GMIB price without fees $V(0)$ as a function of r_g for various values of g. The right panel of Figure 2.11 displays the GMIB price as a function of r_g

when the fair fee rate is charged, for lower values of g where the fair fee rate exists. The standard errors of the GMIB price estimates lie in the range 1.7-2.2. The GMIB price increases monotonically with r_g. In the left panel, each 0.5% increase in the guaranteed rate of return increases $V(0)$ by 1-2% if g is 5.5%, 1.5-3% if g is 6.5% or 7.5%, and 2-3.5% if g is 8.5%, where the percentage increases in $V(0)$ are gradually increasing.

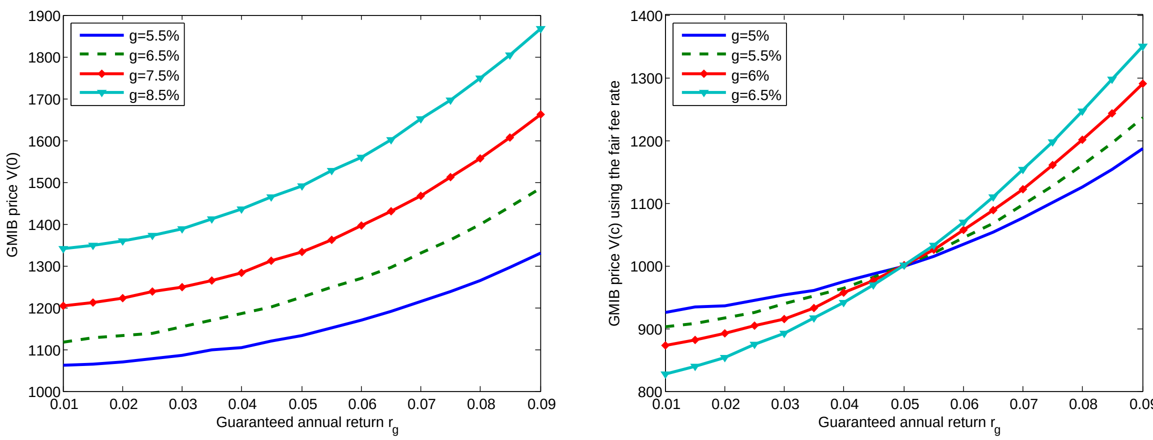

Figure 2.11: *The left panel displays the GMIB price without fees $V(0)$ as a function of the guaranteed annual return r_g. The right panel displays the GMIB price as a function of r_g when the fair fee rate is charged. Each curve corresponds to a particular value of g.*

Maturity date

In this thesis we make the simplifying assumption that the maturity date is $T = 10$ years. In reality the policyholder is able to exercise their GMIB option at any time after the waiting period has elapsed (although there may be restrictions on when they can annuitize without incurring penalty charges, such as within 30 days of each policy anniversary). The panels in Figure 2.12 plot the GMIB price $V(c)$ as a function of the fee rate c for $T = 10, 20, 30$, for $g = 6.5\%$ and $g = 7.5\%$. The standard errors of the GMIB price estimates when $T = 10, 20, 30$ lie in the ranges 0.7-1.8, 0.5-3.3 and 0.6-4.9 respectively. The zero coupon bond yield curve used in the short rate model is still the Benchmark curve up to 30 years, and then from 30 to 50 years the zero coupon bond yield curve is assumed to increase linearly, very gradually, from 4.88% to 5% per year. As T increases, the benefit base $B(T)$ increases (increasing the GMIB price), but the discounting factor decreases in value (decreasing the GMIB price), and a larger number of (annual) fee deductions from the investment account are made (decreasing the GMIB price). Figure 2.12 indicates that the GMIB price increases as T increases. The price increases are driven by the larger benefit base values $B(T)$. The guaranteed return component of the benefit base is very valuable as T increases, particularly when the fee rate charged exceeds say 1% per year.

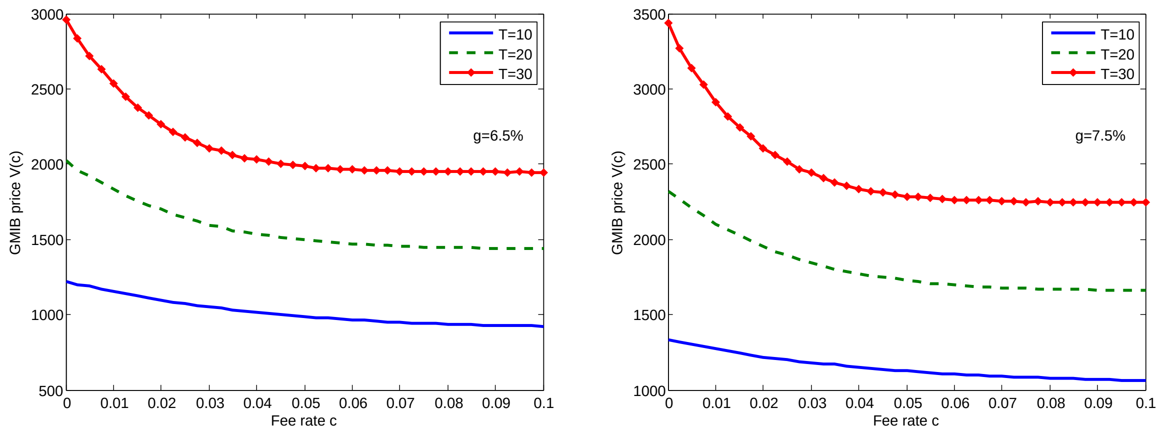

Figure 2.12: *Relationship between GMIB price $V(c)$ and the fee rate c for $T = 10, 20, 30$ assuming $g = 6.5\%$ ($g = 7.5\%$) in the left (right) panel. Each curve corresponds to a particular maturity date T.*

2.5 Decomposing the GMIB price

This section explores why the GMIB appears to be quite valuable. The GMIB price is decomposed, facilitating an understanding of the drivers of its value. This analysis also provides useful information for risk management purposes.

2.5.1 Contributions of each component to the GMIB price

Our goal is to measure the contributions of the maximum and guaranteed return components to the total GMIB price. This concept is important from hedging, risk management and (future) product design perspectives. Recall that we can define the GMIB maturity value as the maximum of three components:

$$X_1 = \max_{n=1,\ldots,T} A(n-) g \ddot{a}_{\overline{20|}}(T),\ X_2 = A(0)(1+r_g)^T g \ddot{a}_{\overline{20|}}(T),\ X_3 = A(T).$$

Let

$$Y_i = e^{-\int_0^T r(t)\mathrm{d}t} X_i, \qquad i = 1, 2, 3.$$

Define the indicator random variable

$$1_A = \begin{cases} 1 & \text{if event } A \text{ occurs,} \\ 0 & \text{if event } A \text{ does not occur.} \end{cases}$$

The contribution of each component to the GMIB price $V(c)$ can be obtained by re-expressing the GMIB price as the sum of three terms:

$$V(c) = E^Q[Y_1 1_{[X_1 > X_2, X_3]}] + E^Q[Y_2 1_{[X_2 > X_1, X_3]}] + E^Q[Y_3 1_{[X_3 > X_1, X_2]}], \tag{2.27}$$

It is noted that in Equation (2.27), the events $X_i = X_j, i \neq j, i, j = 1, 2, 3$ have probability zero and are ignored.

Define

$$y_1 = E^Q[Y_1 1_{[X_1 > X_2, X_3]}],\ y_2 = E^Q[Y_2 1_{[X_2 > X_1, X_3]}]\ddot{a}_{\overline{20|}}(T)],\ y_3 = E^Q[Y_3 1_{[X_3 > X_1, X_2]}]. \tag{2.28}$$

In words, y_1 is the contribution from the lookback component, y_2 is the contribution from the guaranteed return component, y_3 is the contribution from the investment account component. The panels in Figure 2.13 display $y_i,\ i = 1, 2, 3$ as functions of the fee rate for g of 5.5%, 6.5%, 7.5% and 8.5%. The standard errors of the y_i estimates lie in the range 0.5-4.2. In each panel, the sum of the values of the three curves for any given fee rate equals the GMIB price at that fee rate. As the fee rate increases, the sum of the three curves for each particular g must decrease to a lower bound, as seen in Figure 2.1.

In the top left panel of Figure 2.13, when $g = 5.5\%$, it is clear that most of the GMIB price comes from the contribution from the investment account component y_3 for fee rates below 2%, and from the contribution of the guaranteed return component y_2 for fee rates above 2%. For low fee rates, the contributions of the guaranteed return and lookback components are each worth less than the contribution of the investment account component, suggesting the GMIB option is not very valuable when $g = 5.5\%$. However, for high (but unrealistic/unmarketable) fee rates exceeding 5%, the guaranteed return component y_2 of the GMIB becomes valuable because of "fee drag" (most of the funds in the investment account are eaten up by high fees). When $g = 5.5\%$, the fair fee rate is 1.82%. At this fair fee rate, the investment account component y_3 contributes the most to the GMIB price. However, $g = 5.5\%$ is fairly conservative and in practice g is likely to be higher. The top left panel, which displays the contributions for a more equitable g of 6.5%, indicates that for fee rates above 0.5%, the guaranteed return component y_2 contributes the most to the GMIB price. At the fair fee rate of 4.65%, y_2 is worth 70% of the GMIB price, while the lookback component y_1 is worth 16% of the GMIB price. The bottom panels illustrate that as g gets larger, the lookback component y_1 contributes more to the GMIB price than the investment account y_3 for any fee rate. As shown in the bottom right panel, when g is sufficiently large the investment account component y_3 has negligible value while the lookback component y_1 becomes very valuable.

A number of important observations are made from Figure 2.13:

- It is clear that the guaranteed return component y_2 is the dominant contribution to the GMIB price for average/equitable values of g. However, when g is sufficiently large the lookback component y_1 becomes at least as valuable as the guaranteed return component y_2 at lower fee rates. The investment account component y_3 is also valuable for lower values of g.

- As g increases, the contribution of the investment account component y_3 to the GMIB price decreases sharply. This occurs because as g increases, the values of the lookback component (X_1) and guaranteed return component (X_2) are scaled up, and thus both components are more likely to be worth more relative to the value of the investment account component.

- As the fee rate increases, the contribution of the guaranteed return component y_2 increases while the contribution of the lookback component y_1 decreases. This is expected since the guaranteed return component is independent of the fee rate, while the lookback component is a decreasing function of the fee rate.

- The lookback component y_1 is less sensitive to the fee rate than the investment account component y_3, indicating that increasing the fee rate reduces the GMIB price primarily through reducing the contribution from the investment account component y_3, rather than the contribution from the lookback component y_1.

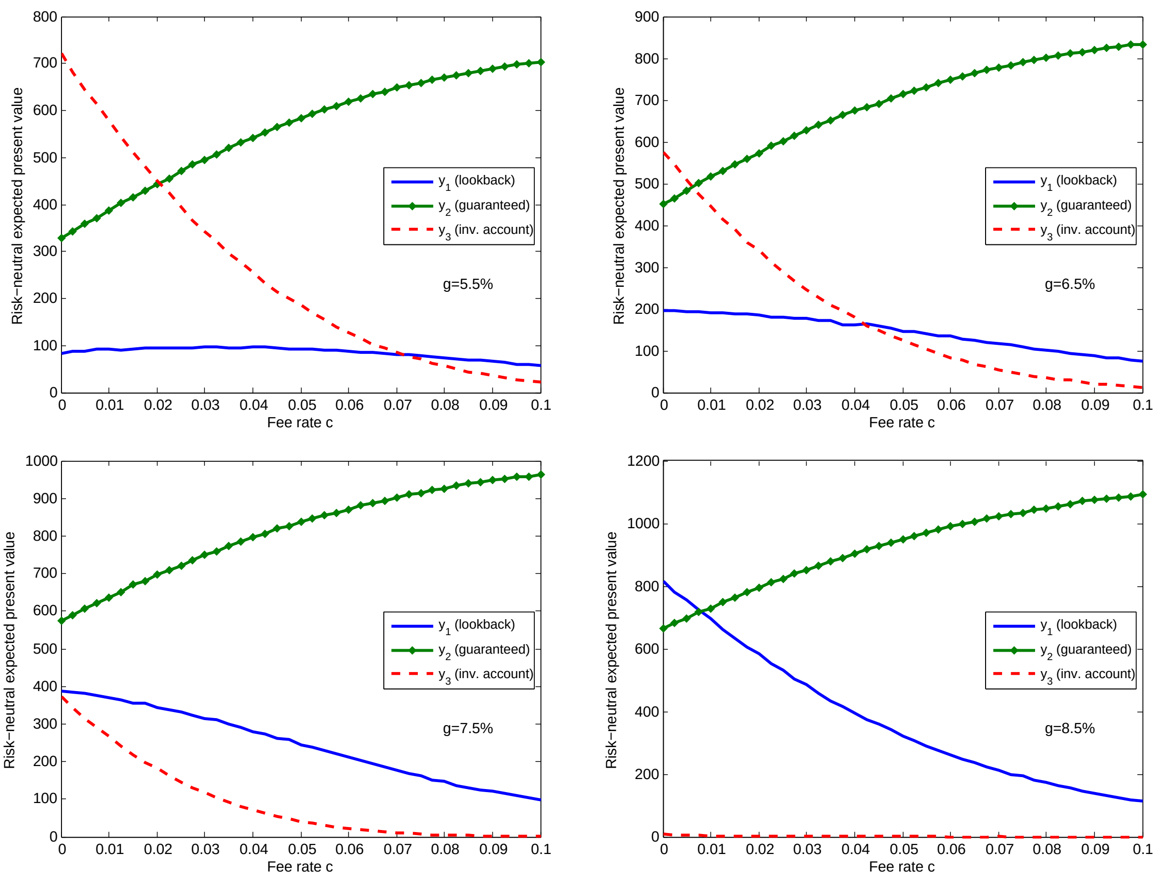

Figure 2.13: *Each panel displays the contributions to the GMIB price from y_i, $i = 1, 2, 3$, (the maximum component, guaranteed return component and investment account component respectively) as functions of the fee rate for a particular value of g. The top left (right) panel displays the contributions for $g = 5.5\%$ ($g = 6.5\%$), and the bottom left (right) panel displays the contributions for $g = 7.5\%$ ($g = 8.5\%$).*

A different but closely related perspective as to the drivers of the GMIB price is obtained by valuing the three components in isolation. The panels in Figure 2.14 display $E^Q[Y_1]$ (EPV^Q of the lookback component), $E^Q[Y_2]$ (EPV^Q of the guaranteed return component) and $E^Q[Y_3]$ (EPV^Q of the investment account component) as functions of the fee rate for g of 5.5%, 6.5%, 7.5% and 8.5%. The standard errors of the $E^Q[Y_i]$ estimates lie in the range 0.7-2.1. It is noted that the $E^Q[Y_3]$ curve is the same in each panel as it does not depend on g. Each panel illustrates that when the lookback and guaranteed return components are valued in isolation, the latter is more valuable except for very low fee rates. The value of the lookback component decreases as the fee rate increases, but the rate of decrease becomes smaller as the fee rate increases. Clearly, the observations drawn from Figure 2.13 are reinforced by Figure 2.14. For average values of g, the guaranteed return component has the highest value, when the three components are valued in isolation.

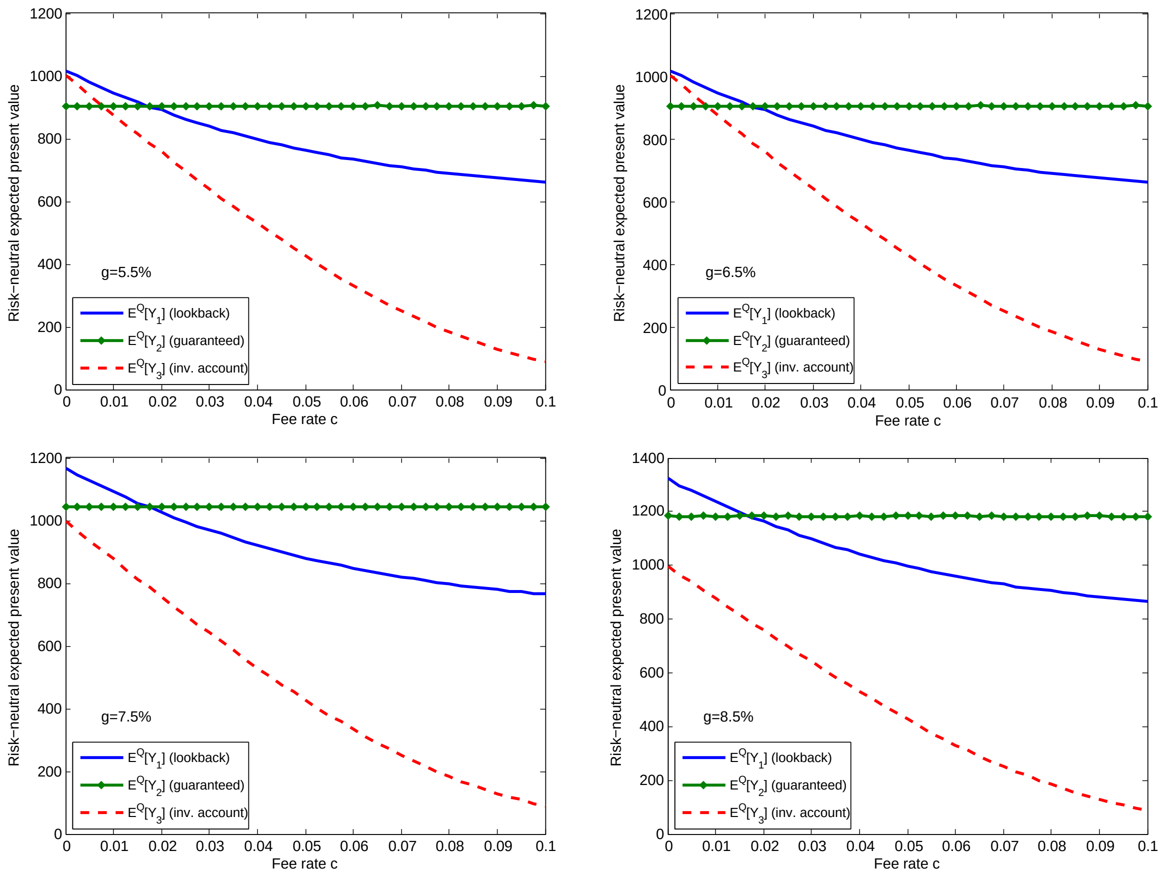

Figure 2.14: *The values of $E^Q[Y_i]$ $i = 1, 2, 3$, (the lookback component, guaranteed return component and investment account component respectively) as functions of the fee rate. The top left (right) panel displays the values for $g = 5.5\%$ ($g = 6.5\%$), and the bottom left (right) panel displays the values for $g = 7.5\%$ ($g = 8.5\%$).*

2.5.2 Valuing simplified GMIBs

Suppose an insurance company selling GMIBs wanted to offer simpler guarantees for their variable annuities. Specifically, suppose the GMIB maturity value is simplified so that it consisted of the maximum of the investment account component and either the lookback component or the guaranteed return component, but not both. It is useful to know how much difference there is between the values of these simpler guarantees and the total GMIB price. The panels in Figure 2.15 display

$$z_1 = E^Q\left[e^{-\int_0^T r(t)\mathrm{d}t}\max\{X_1, X_3\}\right],$$
$$z_2 = E^Q\left[e^{-\int_0^T r(t)\mathrm{d}t}\max\{X_2, X_3\}\right],$$

and for comparison purposes the GMIB price $V(c)$, as functions of the fee rate for $g = 6.5\%$ and $g = 7.5\%$. The standard errors of the z_i estimates lie in the range 0.5-2.8. In words, z_1 is the value of a "lookback only variable annuity option", and z_2 is the value of a "guaranteed return only variable annuity option". A striking observation is that z_2 is closer to $V(c)$ than one might expect. However, $V(c)$ is substantially larger than z_1. This suggests the lookback component does not contribute much to the GMIB price in excess of the guaranteed return component, which is also supported by Figure 2.13. Obviously, the inclusion of the lookback component increases the appeal of a GMIB to variable annuity buyers. It might be argued that variable annuity buyers perceive the lookback component to be quite valuable, but in fact this guarantee contributes little to the value of a GMIB which already includes a guaranteed return component. Nevertheless, in spite of the lookback component appearing to be a cheap benefit for the insurance company to provide, in terms of price, it has the potential to be a very large liability at time T when it is in-the-money. Specifically, if the stock increases sharply in a volatile manner during the accumulation phase, and then sharply declines before time T, the lookback component will be very valuable relative to the other components, and thus should not be ignored when considering hedging strategies for the GMIB. Put another way, a small contribution to the overall price does not imply the risk associated with the lookback component is also negligible.

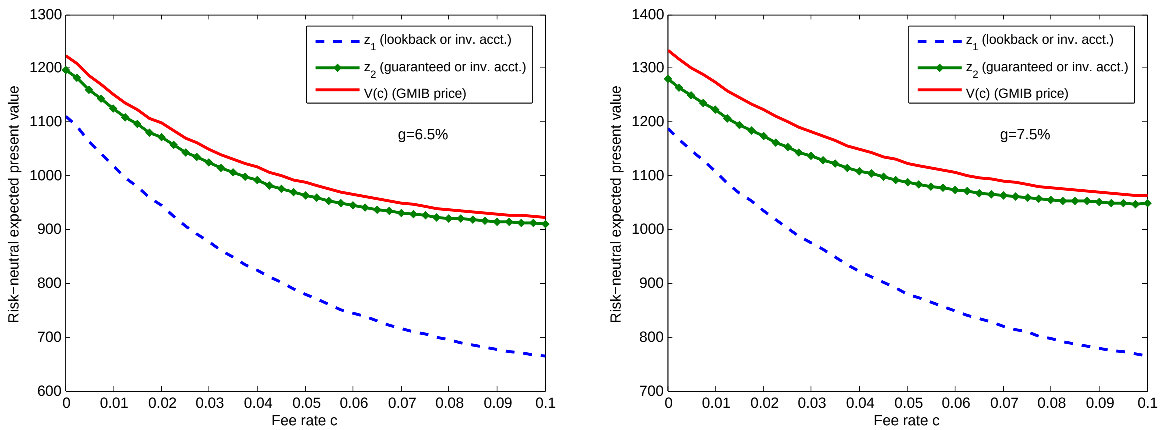

Figure 2.15: *The left (right) panel displays z_1 (lookback or investment account), z_2 (guaranteed return or investment account), and the GMIB price as functions of the fee rate for $g = 6.5\%$ ($g = 7.5\%$).*

2.5.3 Upfront fair fee

The GMIB seller earns the equivalent of an option premium by charging annual fees during the accumulation phase. A simpler, but probably somewhat less marketable alternative, would be to charge one upfront fee, with no fees paid thereafter. This section determines the magnitude of such a fee. The magnitude of the fee gives another measure of the value of the GMIB, and unlike the annual fee payments approach, a fair upfront fee can be calculated for $g > 7\%$. Let φ denote the upfront fee rate charged as a percentage of the annuity premium. Recall that π is the policyholder's annuity premium. The insurance company receives a fee of $\pi\varphi$ at the outset, and invests $\pi(1-\varphi)$ for the policyholder. Define the function $V(0;\varphi)$, which is identical in form to the function $V(c)$ given by equation (2.12), except that $c = 0$ and $A(0) = S(0) = \pi(1-\varphi)$. The benefit base is now

$$B(T) = \max\{\pi(1-\varphi)(1+r_g)^T, \max_{n=1,2,\ldots,T} A(n)\}.$$

The upfront fee rate $\varphi = \varphi^*$ is fair if $\pi = V(0;\varphi^*)$. By charging φ^* the insurance company has the exact amount of funds needed to construct the replicating portfolio for the GMIB. Table 2.3 presents the upfront fair fee rate φ^* and its standard error for various values of g. We can calculate the upfront fair fee and its standard error following the same technique that was used to obtain the fair fee rate and its standard error (except we are now solving for ψ instead of c). For an equitable value of g in the range of 6.5-7.5%, the fair upfront fee φ^* is about 18-25% of the annuity premium π. Variable annuity buyers may find it far less appealing to pay an upfront fee of this magnitude compared to the alternative of paying smaller annual fees during the accumulation phase.

g	5%	5.5%	6%	6.5%	7%	7.5%	8%	8.5%	9%	9.5%	10%
Upfront fair fee rate	9.09%	11.94%	14.91%	18.19%	21.54%	25.05%	28.88%	32.95%	36.65%	40.00%	42.97%
Std error of fair fee rate	0.07%	0.06%	0.07%	0.05%	0.05%	0.05%	0.04%	0.04%	0.04%	0.04%	0.04%

Table 2.3: *The upfront fair fee rate and its standard error for various values of g.*

2.6 Impact of lapses

Lapses have been ignored in the valuations thus far. Allowing for lapses will reduce the GMIB price. It is difficult to say what an appropriate set of withdrawal rate assumptions during the accumulation phase should be. We consider the change in the GMIB price from simply assuming a *constant* annual lapse rate. Let p denote the probability that the policyholder lapses over a given policy year. The GMIB price allowing for lapses is

$$V^L(c) = V(c)(1-p)^T + \sum_{n=1}^{T} E^Q\left[e^{-\int_0^n r(t)\mathrm{d}t} A(n-)\right](1-p)^{n-1}p \qquad (2.29)$$

where $V(c)$ is still given by equation (2.12). In equation (2.29) we assume that if the policyholder lapses during the n-th policy year, they receive the value of the investment account at the end of year n, just before the annual fee for the GMIB option is deducted.

The panels in Figure 2.16 display the GMIB price $V^L(c)$ as a function of the fee rate c using various values of p, for $g = 6.5\%$ and $g = 7.5\%$. The standard errors of the $V^L(c)$ estimates lie in the range 0.5-1.9. The curves labeled $p = 0\%$ are identical to the curves for g of 6.5% and 7.5% presented in Figure 2.1. The left panel shows that the fair fee rate drops significantly even for small lapse rates. For example, a conservative lapse rate of 2.5% p.a. reduces the fair fee rate to about 3.15%. The right panel demonstrates that, while there is no fair fee rate when $g = 7.5\%$ and $p = 0$, if a small constant lapse rate is introduced then it is possible for a fair fee rate to exist when $g = 7.5\%$. These observations show that the fair fee rate is highly sensitive to lapse assumptions. Therefore, if policyholder lapse behavior can be reliably measured, it should be incorporated into the valuations of GMIBs. A complicating factor is that lapses depend on economic conditions in ways that may not be clearly understood, given the short history of these products. It is noted that assuming an annual lapse rate as high as 10% still does not reduce the fair fee rate for $g = 6.5\%$ to a level that lies in the range of fee rates being charged by competitive GMIB sellers (0.8-1%).

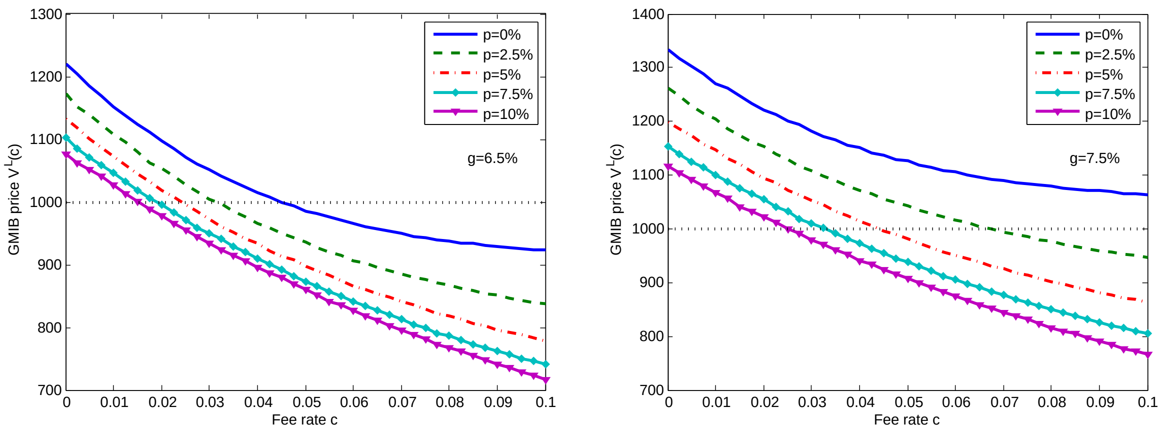

Figure 2.16: *Relationship between GMIB price $V^L(c)$ and the fee rate c for various constant lapse rates p, assuming $g = 6.5\%$ ($g = 7.5\%$) for each curve in the left (right) panel. Each curve corresponds to a particular value of p.*

2.7 Continuous versus discrete fee structure

In our model we have assumed that fee payments are made at discrete time intervals of one year. Each fee is a fixed percentage of the benefit base at the time of payment. Recall that the size of the fee charged at the end of policy year n is

$$f(n) = \min\{A(n-), cB(n)\} \qquad n = 1, 2, \ldots, T.$$

This fee structure is consistent with how fees for the GMIB are calculated by many insurance companies.[3] This fee structure is somewhat unique. For many investment products, fees are charged as a percentage of the underlying asset account value (usually at discrete time intervals such as quarterly or yearly). It is common practice for academic researchers

[3]For example, see the variable annuity prospectus of AXA Equitable Life Insurance Company. URL: www.axa-equitable.com/annuities/accumulator/product-fact-sheets.cfm

valuing options associated with investment products to assume fees are deducted continuously from the asset account at a rate of c per year. This assumption often allows for tractable analytical results, and in general simplifies the implementation of the models. For example, Bauer et al. (2008) used a continuous fee structure to price the GMIB. In most cases, the continuous fee approximation is appropriate for pricing, and leads to results that are close to those obtained assuming discrete fee payments. However, for the GMIB, the fees are calculated as a percentage of the benefit base, and therefore the model more closely resembles reality if the discrete fee structure is adopted. Nevertheless, it is of interest to measure the impact on the GMIB price and fair fee rate from using a continuous fee structure, where fees are deducted from the investment account at a continuous rate of c per year. In this situation, the SDE of the investment account process under Q can be explicitly written as

$$\mathrm{d}A(t) = (r(t) - c)A(t)\mathrm{d}t + \sigma_s A(t)\mathrm{d}W_S^Q(t). \tag{2.30}$$

The fee rate c is analogous to a continuous dividend yield on a stock. It is noted that the stock and short rate SDEs are unchanged.

Under the continuous fee structure, the following recursive relationship holds:

$$\begin{aligned} A(T) &= A(T-1)R(T) - A(T-1)(1-e^{-c})R(T) \\ &\vdots \\ &= A(0)\prod_{n=1}^{T} R(n) - \sum_{n=0}^{T-1} A(n)(1-e^{-c})[\prod_{i=n+1}^{T} R(i)] \end{aligned} \tag{2.31}$$

where $R(n)$ is defined by equation (2.3). Substituting equation (2.31) into equation (2.15) and following the steps presented in section 2.2.4 we obtain

$$V(c) - \pi = G(c) - F(c) \tag{2.32}$$

where

$$F(c) = E^Q \left[\sum_{n=0}^{T-1} A(n)(1 - e^{-c})[\prod_{i=n+1}^{T} R(i)] \right], \tag{2.33}$$

and $G(c)$ has the same form as equation (2.23), except that the sampling of the maximum in the benefit base changes slightly such that

$$B(T) = \max\{\max_{n=1,2,\ldots,T} A(n), A(0)(1 + r_g)^T\}. \tag{2.34}$$

In Figure 2.17, the left panel compares the GMIB price $V(c)$ as a function of the fee rate c under the continuous and discrete fee structures, for the cases $g = 5.5\%, 6.5\%, 7.5\%$. The standard errors of the $V(c)$ estimates under the discrete and continuous fee structures lie in the ranges 0.5-1.9 and 0.5-2.0 respectively. The GMIB price is marginally higher under the continuous fee structure, but the difference shrinks to a negligible amount as g increases. For the curves that intersect with the horizontal dotted line, the fee rate at the intersecting point corresponds to the fair fee rate. The fair fee rates are slightly lower under the discrete fee structure, but the differences are relatively small, lying somewhere between 0-0.5%.

The right panel of Figure 2.17 displays the EPV^Q of the benefits provided by the GMIB option $G(c)$, and the EPV^Q of the fees paid $F(c)$, as functions of the fee rate c under the continuous and discrete fee structures, for the cases $g = 5.5\%, 6.5\%, 7.5\%$. The standard errors of the $G(c)$ and $F(c)$ estimates under the discrete (continuous) fee structures lie in the ranges 0.6-1.1 (0.6-0.9) and 0.0-1.7 (0.0-1.4) respectively. The expected GMIB benefits and fees earned are significantly higher under the discrete fee structure. Despite the differences in the $G(c)$ and $F(c)$ values under the two fee structures, the fair fee rates, corresponding to the points where $G(c) = F(c)$ (for $g = 5.5\%, 6.5\%$), are relatively close.

2.8 Allowing for other fee charges in practice

We have not allowed for various fee charges related to the underlying variable annuity contract in the results we have presented. The charges cover administrative expenses,

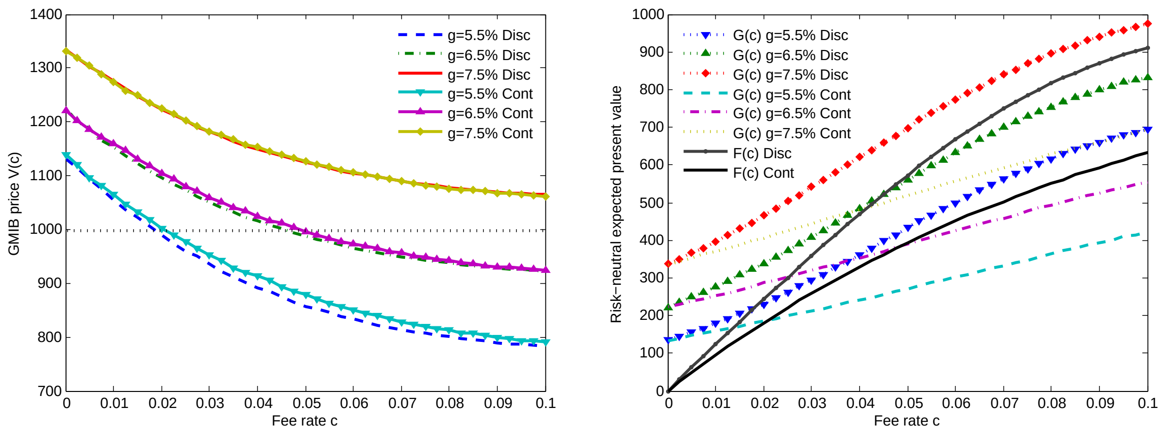

Figure 2.17: *The left panel compares the GMIB price $V(c)$ under the continuous and discrete fee structures as a function of the fee rate c, for $g = 5.5\%, 6.5\%, 7.5\%$. The right panel compares the EPV^Q of the GMIB benefits $G(c)$ and the EPV^Q of the fees paid $F(c)$ as functions of the fee rate c under the continuous and discrete fee structures, for $g = 5.5\%, 6.5\%, 7.5\%$.*

expenses related to mortality risk, portfolio management fees, cost of capital, expense uncertainty loadings and of course variable annuity profit loadings. Moreover, part of the charges may be for covering the cost of implicit guarantees provided to the policyholder, such as a put option expiring at time T, with a strike price equal to the value of the initial investment (i.e. compulsory protection against a loss of principal). There may also be additional fee charges if other types of variable annuity options are included in the variable annuity contract (e.g. the guaranteed minimum death benefit). Including these charges will reduce the investment account value, and this will influence the value of the GMIB option. The total value of these fees is likely to be between 0.5-3% per year. These charges are deducted from the investment account on a periodic basis (perhaps daily). We now illustrate the impact on the GMIB fair fee rate when these variable annuity charges are allowed for.

We assume the variable annuity fees are deducted continuously at a rate of q per year. The only adjustment to the valuation model is in the investment account accumulation factor in each time interval $[n-1, n)$. The investment account equations defined in Section

2.2.1 become

$$A(n-) = A(n-1)R(n)e^{\int_{n-1}^{n} q\mathrm{d}s} - f(n)$$
$$= A(n-1)R(n)e^{-q},$$

$$A(n) = A(n-1)R(n)e^{-q} - f(n) \qquad n = 1, 2 \ldots, T,$$

and for $t \in (n-1, n)$, $n = 1, 2, \ldots, T$,

$$A(t-) = A(t) = A(n-1)e^{\int_{n-1}^{t}(r(s)-q-\sigma_S^2/2)\mathrm{d}s+\sigma_S(W_S^Q(t)-W_S^Q(n-1))}.$$

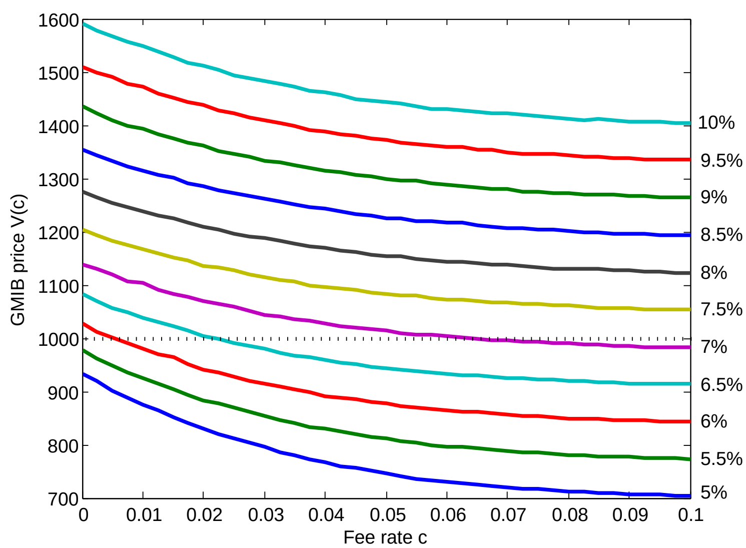

Figure 2.18: *GMIB price $V(c)$ as a function of the (GMIB) fee rate c allowing for variable annuity fees of $q = 2.5\%$. Each curve corresponds to a particular value of g. For the curves that intersect with the horizontal dotted line, the fee rate at the intersecting point corresponds to the fair fee rate.*

We assume a relatively high variable annuity fee rate of $q = 2.5\%$ in the following results. Figure 2.18 depicts the GMIB price $V(c)$ as a function of the fee rate c allowing for variable annuity fees of $q = 2.5\%$. The fair fee rates are lower because the fee drag reduces X_1 and X_3 (while X_2 is unaffected). It turns out that for $g = 5\%$ and $g = 5.5\%$, the GMIB price is always less than the premium; the insurer can offer the GMIB at $c = 0$ and still make a profit. The GMIB fair fee rates (with standard errors) for g of 6%, 6.5% and 7% are

0.54% (0.0028%), 2.18% (0.0035%) and 6.35% (0.0066%). The variable annuity fee rate of $q = 2.5\%$ plus the adjusted GMIB fair fee rate exceed the GMIB fair fee rate when no variable annuity fees are allowed for by 0.24%, 0.18% and 0.01% for g of 6%, 6.5% and 7%. These differences make sense in light of the results in Section 2.7, which showed that the GMIB fair fee rates under both the discrete and continuous fee structures were fairly close to each other. As a rule of thumb, the GMIB fair fee rate calculated when no variable annuity fees are allowed for, is approximately equal to variable annuity fee rate q plus the adjusted GMIB fair fee rate. Figure 2.18 suggests that the GMIB is not underpriced if the underlying variable annuity charges are sufficiently high.

2.9 Monte Carlo simulation of the GMIB price

2.9.1 An efficient simulation method

A straightforward method for simulating the GMIB price involves discretization of the SDEs given by equations (2.1) and (2.6) using the Euler approximation (McLeish, 2005), (Glasserman, 2004). However, this method entails discretization errors, and may be time consuming if the number of time steps used is large. We now describe a Monte Carlo simulation method for pricing the GMIB that is efficient, in the sense that it does not involve any discretization errors and is much faster.

Recall that

$$\boldsymbol{F}_{[0,T]} = \{\mathcal{F}_t, 0 \leqslant t \leqslant T\} \tag{2.35}$$

is the filtration generated by the stock and short rate processes from time 0 to time T. In order to compute the GMIB price using Monte Carlo simulation, we must be able to sample from the joint distribution of

$$\widetilde{\chi} = \left(\int_0^T r(t)\mathrm{d}t, \max_{n=1,2,\ldots,T} A(n-), A(T), r(T) \right) \Big| \mathcal{F}_0.$$

Note that we require the value of $r(T)$ since $\ddot{a}_{\overline{20|}}(T)|\mathcal{F}_T$ is a function of $r(T)$ in the Hull-White model. Now, we can sample from $\widetilde{\chi}$ directly if we are able to sample from the joint distribution of

$$\widetilde{\mathcal{W}} = \left(\int_0^T r(t)\mathrm{d}t, r(T), S(n)\ n = 1, 2, \ldots, T \right) \Big| \mathcal{F}_0.$$

Sampling from $\widetilde{\mathcal{W}}$ is not straightforward as there is dependence between the random variables in the random vector. However, we can sample from $\widetilde{\mathcal{W}}$ by sampling sequentially from $\widetilde{M}_1, \widetilde{M}_2, \ldots, \widetilde{M}_T$, where

$$\widetilde{M}_n = \left(\int_{n-1}^n r(t)\mathrm{d}t, r(n), S(n) \right) \Big| \mathcal{F}_{n-1} \qquad n = 1, 2, \ldots, T.$$

We now outline how we can sample from $\widetilde{M}_n$. First, equations (2.1) and (2.6) can be solved over the time interval $[v, w]$ to give $S(w)|\mathcal{F}_v$ and $r(w)|\mathcal{F}_v$. We use the SDEs defined by equations (2.8) and (2.9) in what follows. Applying Ito's Lemma to $f(S(t)) = \log(S(t))$ and integrating yields

$$S(w) = S(v) \exp \left(\int_v^w r(t)\mathrm{d}t - \frac{\sigma_S^2}{2}(w - v) + \sigma_S \rho \int_v^w \mathrm{d}W_r^Q(t) + \sigma_S (1 - \rho^2)^{1/2} \int_v^w \mathrm{d}\widetilde{W}^Q(t) \right). \tag{2.36}$$

Multiplying equation (2.8) on both sides by the integrating factor e^{at} and rearranging leads to

$$e^{at}\mathrm{d}r(t) + e^{at}ar(t)\mathrm{d}t = e^{at}\Theta(t)\mathrm{d}t + e^{at}\sigma_r \mathrm{d}W_r^Q(t). \tag{2.37}$$

The left hand side of equation (2.37) is equal to

$$e^{at}\mathrm{d}r(t) + \mathrm{d}(e^{at})r(t) = \mathrm{d}(e^{at}r(t))$$

using the stochastic calculus product rule. Thus, integrating equation (2.37) yields

$$r(w) = r(v)e^{-a(w-v)} + \alpha(w) - \alpha(v)e^{-a(w-v)} + \sigma_r \int_v^w e^{-a(w-t)}\mathrm{d}W_r^Q(t) \tag{2.38}$$

where

$$\alpha(t) = f^M(0, t) + \frac{\sigma_r^2}{2a^2}(1 - e^{-at})^2.$$

Equation (2.38) indicates that $r(w)|\mathcal{F}_v$ is a normal random variable.

It can be shown that (Brigo and Mercurio, 2006)

$$\int_v^w r(t)\mathrm{d}t = B(v,w)[r(v)-\alpha(v)] + \log\left(\frac{P^M(0,w)}{P^M(0,v)}\right) + \frac{1}{2}[V(0,w)-V(0,v)] + \frac{\sigma_r}{a}\int_{t=v}^{w}(1-e^{-a(w-t)})\mathrm{d}W_r^Q(t) \tag{2.39}$$

where

$$B(v,w) = \frac{1}{a}\left(1-e^{-a(w-v)}\right),$$

$$V(v,w) = \frac{\sigma_r^2}{a^2}\left(w - v + \frac{2}{a}e^{-a(w-v)} - \frac{1}{2a}e^{-2a(w-v)} - \frac{3}{2a}\right). \tag{2.40}$$

Thus, we see that $\int_v^w r(t)\mathrm{d}t|\mathcal{F}_v$ is a normal random variable. This in turn implies $S(w)|\mathcal{F}_v$ is a lognormal random variable. From equation (2.36), we see that $S(w)|\mathcal{F}_v$ depends on the random quantities $\int_v^w r(t)\mathrm{d}t|\mathcal{F}_v$, $\int_v^w \mathrm{d}W_r^Q(t)|\mathcal{F}_v$ and $\int_v^w \mathrm{d}\widetilde{W}^Q(t)|\mathcal{F}_v$, which are all normally distributed. Therefore, sampling from $\widetilde{M}_n$ can be obtained by sampling from the following multivariate normal random vector:

$$\widetilde{G}_n = \left(\int_{n-1}^{n} r(t)\mathrm{d}t, r(n), \int_{n-1}^{n}\mathrm{d}W_r^Q(t), \int_{n-1}^{n}\mathrm{d}\widetilde{W}^Q(t)\right)\Big|\mathcal{F}_{n-1} \qquad n = 1,2,\ldots,T. \tag{2.41}$$

In order to simulate from $\widetilde{G}_n$, we require the mean and variance of each random variable, and the covariances between all of the random variables. Over the time interval $[v,w]$, the means, variances and covariances are

$$\begin{aligned}
\mu^{(1)}_{v,w} &= E^Q\left[\int_v^w r(t)\mathrm{d}t|\mathcal{F}_v\right] \\
&= B(v,w)[r(v)-\alpha(v)] + \log\left(\frac{P^M(0,v)}{P^M(0,w)}\right) + \frac{1}{2}[V(0,w)-V(0,v)], \\
\mu^{(2)}_{v,w} &= E^Q\left[r(w)|\mathcal{F}_v\right] = r(v)e^{-a(w-v)} + \alpha(w) - \alpha(v)e^{-a(w-v)}, \\
\mu^{(3)}_{v,w} &= E^Q\left[\int_v^w \mathrm{d}W_r^Q(t)|\mathcal{F}_v\right] = 0, \\
\mu^{(4)}_{v,w} &= E^Q\left[\int_v^w \mathrm{d}\widetilde{W}^Q(t)|\mathcal{F}_v\right] = 0,
\end{aligned}$$

$$\begin{aligned}
\Sigma^{(11)}_{v,w} &= \mathrm{Var}^Q\left[\int_v^w r(t)\mathrm{d}t|\mathcal{F}_v\right] = V(v,w), \\
\Sigma^{(22)}_{v,w} &= \mathrm{Var}^Q\left[r(w)|\mathcal{F}_v\right] = \frac{\sigma_r^2}{2a}\left(1-e^{-2a(w-v)}\right), \\
\Sigma^{(33)}_{v,w} &= \mathrm{Var}^Q\left[\int_v^w \mathrm{d}W_r^Q(t)|\mathcal{F}_v\right] = w-v, \\
\Sigma^{(44)}_{v,w} &= \mathrm{Var}^Q\left[\int_v^w \mathrm{d}\widetilde{W}^Q(t)|\mathcal{F}_v\right] = w-v,
\end{aligned}$$

$$\begin{aligned}
\Sigma^{(12)}_{v,w} &= \mathrm{Cov}^Q\left[\int_v^w r(t)\mathrm{d}t, r(v)|\mathcal{F}_v\right] = \frac{\sigma_r^2}{2}B(v,w)^2, \\
\Sigma^{(13)}_{v,w} &= \mathrm{Cov}^Q\left[\int_v^w r(t)\mathrm{d}t, \int_v^w \mathrm{d}W_r^Q(t)|\mathcal{F}_v\right] = \frac{\sigma_r}{a}\left[(w-v)-B(v,w)\right], \\
\Sigma^{(14)}_{v,w} &= \mathrm{Cov}^Q\left[\int_v^w r(t)\mathrm{d}t, \int_v^w \mathrm{d}\widetilde{W}^Q(t)|\mathcal{F}_v\right] = 0, \\
\Sigma^{(23)}_{v,w} &= \mathrm{Cov}^Q\left[r(w), \int_v^w \mathrm{d}W_r^Q(t)|\mathcal{F}_v\right] = \sigma_r B(v,w), \\
\Sigma^{(24)}_{v,w} &= \mathrm{Cov}^Q\left[r(w), \int_v^w \mathrm{d}\widetilde{W}^Q(t)|\mathcal{F}_v\right] = 0, \\
\Sigma^{(34)}_{v,w} &= \mathrm{Cov}^Q\left[\int_v^w \mathrm{d}W_r^Q(t), \int_v^w \mathrm{d}\widetilde{W}^Q(t)|\mathcal{F}_v\right] = 0.
\end{aligned}$$

Hence $\widetilde{G}_n$ has a multivariate normal distribution, $\widetilde{G}_n \sim N(\boldsymbol{\mu}_{n-1,n}, \boldsymbol{\Sigma}_{n-1,n})$, with mean vector

$$\boldsymbol{\mu}_{n-1,n} = [\mu^{(1)}_{n-1,n}, \mu^{(2)}_{n-1,n}, \mu^{(3)}_{n-1,n}, \mu^{(4)}_{n-1,n}]'$$

and covariance matrix

$$\boldsymbol{\Sigma}_{n-1,n} = \begin{bmatrix} \Sigma^{(11)}_{n-1,n} & \Sigma^{(12)}_{n-1,n} & \Sigma^{(13)}_{n-1,n} & \Sigma^{(14)}_{n-1,n} \\ \Sigma^{(12)}_{n-1,n} & \Sigma^{(22)}_{n-1,n} & \Sigma^{(23)}_{n-1,n} & \Sigma^{(24)}_{n-1,n} \\ \Sigma^{(13)}_{n-1,n} & \Sigma^{(23)}_{n-1,n} & \Sigma^{(33)}_{n-1,n} & \Sigma^{(34)}_{n-1,n} \\ \Sigma^{(14)}_{n-1,n} & \Sigma^{(24)}_{n-1,n} & \Sigma^{(34)}_{n-1,n} & \Sigma^{(44)}_{n-1,n} \end{bmatrix}.$$

We can directly compute $A(n-)$ and $A(n)$ if we have realizations from $\widetilde{G}_1, \widetilde{G}_2, \ldots, \widetilde{G}_n$. To compute the GMIB price, we sample in sequence from $\widetilde{G}_1, \widetilde{G}_2, \ldots, \widetilde{G}_T$ for each scenario.

2.9.2 A control variate for variance reduction

Variance reduction techniques can be used to reduce the standard errors of the estimates of option prices obtained by Monte Carlo simulation. In other words, it is possible to obtain estimates with the same standard errors, using a much smaller number of scenarios (and hence simulation time may be considerably reduced). One such technique involves the use of a control variate in simulation (Boyle et al., 1997).

We now outline the logic for deriving a control variate for the GMIB. The control variate should generate a value that is close as possible to the estimate of $V(c)$, while still having an analytical formula. First of all, an analytical formula will require us to approximate the discrete annual payments of $cB(n)$ by a continuously paid fee applied to the investment account, at an annual rate of c. Let the investment account process for the control variate satisfy the following SDE under Q:

$$\mathrm{d}A_c(t) = (r(t) - c)A_c(t)\mathrm{d}t + \sigma_s A_c(t)\mathrm{d}W^Q_S(t), \tag{2.42}$$

where $A_c(0) = A(0) = \pi$. The short rate is still assumed to satisfy the SDE given by

$$\mathrm{d}r(t) = \{\Theta(t) - ar(t)\}\mathrm{d}t + \sigma_r \mathrm{d}W^Q_r(t). \tag{2.43}$$

Let ρ denote the linear correlation coefficient between $\{W^Q_S(t)\}$ and $\{W^Q_r(t)\}$.

Assuming fees are paid continuously, the GMIB maturity value is still a complicated function of $r(T)$, $\max_{n=1,2,\ldots,T} A(n)$, and $A(T)$. We need to simplify the function at maturity. Figure 2.15 indicates that if the lookback component is removed from the GMIB price $V(c)$, the value of the now simplified GMIB, z_2, is only marginally smaller. On the other hand, if the guaranteed return component is removed from $V(c)$, the value of the now simplified GMIB, z_1, is considerably smaller than z_2. Therefore, a potentially useful control variate involves removing the lookback component. The maturity value of a variable annuity with the guaranteed return option is

$$\begin{aligned}\mathcal{Y}(T) &= \max\{A_c(0)(1+r_g)^T g\ddot{a}_{\overline{20}|}(T), A_c(T)\} \\ &= \max\{A_c(0)(1+r_g)^T g\ddot{a}_{\overline{20}|}(T) - A_c(T), 0\} + A_c(T) \end{aligned} \tag{2.44}$$

In the Hull-White model, $\ddot{a}_{\overline{20}|}(T) = \sum_{i=T}^{T+20-1} P(T,i)$ is a function of $r(T)$ through the $P(T,i)$ for $i > T$. To obtain a control variate, we replace $\ddot{a}_{\overline{20}|}(T)$ by an estimate of $E^Q\left[\ddot{a}_{\overline{20}|}(T)|\mathcal{F}_0\right] = E^Q\left[\ddot{a}_{\overline{20}|}(T)\right]$ which can obtained from a separate (prior) Monte Carlo simulation (this simulation takes a few seconds at most). Alternatively, we can compute $E^Q\left[\ddot{a}_{\overline{20}|}(T)\right]$ using numerical integration, since it is an integral with respect to a normal probability density function. Equation (2.44) simplifies to

$$\mathcal{Y}_{cv}(T) = \max\{A_c(0)(1+r_g)^T gE^Q\left[\ddot{a}_{\overline{20}|}(T)\right] - A_c(T), 0\} + A_c(T), \tag{2.45}$$

which is the payoff of a European put option on the investment account value at maturity with a strike price of

$$\mathcal{K} = A_c(0)(1+r_g)^T gaE^Q\left[\ddot{a}_{\overline{20}|}(T)\right],$$

plus the investment account value at maturity. Given that $E^Q[e^{-\int_0^T r(t)\mathrm{d}t}A_c(T)] = A_c(0)e^{-cT}$, the expected present value under Q of the expression in equation (2.45) is evaluated analytically as

$$E^Q[e^{-\int_0^T r(t)\mathrm{d}t}\mathcal{Y}_{cv}(T)] = \mathcal{P}(T,\mathcal{K}) + A_c(0)e^{-cT}, \tag{2.46}$$

where $\mathcal{P}(T,\mathcal{K})$ is the formula for the price at time 0 of the European put option with strike price $\mathcal{K}$ and maturity date T, in a model where the stock follows equation (2.42), and the short rate follows equation (2.43) (Brigo and Mercurio, 2006). Specifically,

$$\mathcal{P}(T,\mathcal{K}) = \mathcal{K}P(0,T)\Phi\left(-\frac{\log\left\{\frac{A_c(0)}{\mathcal{K}P(0,T)}\right\} - cT - \frac{1}{2}v^2(0,T)}{v(0,T)}\right)$$
$$- A_c(0)e^{-cT}\Phi\left(-\frac{\log\left\{\frac{A_c(0)}{\mathcal{K}P(0,T)}\right\} - cT + \frac{1}{2}v^2(0,T)}{v(0,T)}\right)$$

where

$$v^2(0,T) = V(0,T) + \sigma_S^2 T + 2\frac{\rho\sigma_r\sigma_S}{a}\left(T - \frac{1}{a}(1-e^{-aT})\right)$$

and $V(0,T)$ is defined by equation (2.40).

Now we discuss how the control variate is implemented in the Monte Carlo simulation. Recall from Section 2.9.1 that $\widetilde{G}_n \sim N(\boldsymbol{\mu}_{n-1,n}, \boldsymbol{\Sigma}_{n-1,n})$. This 4-dimensional multivariate normal random variable is generated by simulating from a 4-dimensional standard uniform random vector $\boldsymbol{U} = [U_1, U_2, U_3, U_4]'$, where $U_1, \ldots, U_4$ are independent and uniformly distributed on [0,1]. In order to simulate from $\widetilde{G}_n$, we first simulate $Z_i = \Phi^{-1}(U_i) \sim N(0,1)$, where Φ^{-1} is the inverse of the standard normal cumulative distribution function, obtaining a standard normal random vector $\boldsymbol{Z} = [Z_1, Z_2, Z_3, Z_4]'$. Next we calculate the Cholesky square root matrix $\boldsymbol{C}_n$ for which $\boldsymbol{C}_n\boldsymbol{C}_n' = \boldsymbol{\Sigma}_{n-1,n}$ (Glasserman, 2004). Then we are able to calculate $\widetilde{G}_n = \boldsymbol{\mu}_{n-1,n} + \boldsymbol{C}_n\boldsymbol{Z}$.

In order to simulate the present value of the GMIB maturity value for a particular scenario m, we must sample from $\widetilde{G}_1, \widetilde{G}_2, \ldots, \widetilde{G}_T$. Therefore, we require a sample from the $4 \times T$ dimensional matrix

$$\overrightarrow{\boldsymbol{U}} = (\boldsymbol{U}_1, \boldsymbol{U}_2, \ldots, \boldsymbol{U}_T) \tag{2.47}$$

where $\boldsymbol{U}_n$ denotes the 4-dimensional standard uniform random vector that generates $\widetilde{G}_n$.

Define

$$f(\overrightarrow{\boldsymbol{U}}) = e^{-\int_0^T r(t)\mathrm{d}t}\left[\max\{B(T)g\ddot{a}_{\overline{20}|}(T), A(T)\}\right]$$

and

$$f_{cv}(\overrightarrow{\boldsymbol{U}}) = e^{-\int_0^T r(t)\mathrm{d}t}\mathcal{Y}_{cv}(T), \tag{2.48}$$

where $\mathcal{Y}_{cv}(T)$ is given by equation (2.45).

Let M denote the number of scenarios in the Monte Carlo simulation. Define

$$\overrightarrow{\boldsymbol{U}}_m = (\boldsymbol{U}_{1,m}, \boldsymbol{U}_{2,m}, \ldots, \boldsymbol{U}_{T,m}) \qquad m = 1, 2, \ldots, M$$

as the realization of $\overrightarrow{\boldsymbol{U}}$ for the m-th scenario. The standard (crude) Monte Carlo estimator of the GMIB price is

$$\hat{\theta}_0 = \frac{1}{M}\sum_{m=1}^{M} f(\overrightarrow{\boldsymbol{U}}_m).$$

An unbiased estimator using the control variate is

$$\hat{\theta}_1 = E^Q[f_{cv}(\overrightarrow{\boldsymbol{U}})] + \frac{1}{M}\sum_{m=1}^{M}\left(f(\overrightarrow{\boldsymbol{U}}_m) - f_{cv}(\overrightarrow{\boldsymbol{U}}_m)\right),$$

where $E^Q[f_{cv}(\overrightarrow{\boldsymbol{U}})]$ is given by equation (2.46). If the control variate is effective, the variance of this estimator will be significantly smaller than the variance of the standard Monte Carlo estimator. The closer the values of $f(\overrightarrow{\boldsymbol{U}}_m)$ and $f_{cv}(\overrightarrow{\boldsymbol{U}}_m)$ for each m, the more efficient the estimator $\hat{\theta}_1$ will be in terms of minimizing the standard error for a fixed M.

It is possible to find a linear function of $f_{cv}(\overrightarrow{\boldsymbol{U}})$ that is a better control variate than $f_{cv}(\overrightarrow{\boldsymbol{U}})$ itself. For sufficiently large M, an improved unbiased estimator using the control variate is (Glasserman, 2004)

$$\hat{\theta}_2 = \hat{\beta} E^Q[f_{cv}(\overrightarrow{\boldsymbol{U}})] + \frac{1}{M}\sum_{m=1}^{M}\Big(f(\overrightarrow{\boldsymbol{U}}_m) - \hat{\beta} f_{cv}(\overrightarrow{\boldsymbol{U}}_m)\Big)$$

where

$$\hat{\beta} = \frac{\sum_{m=1}^{M}\Big[\Big(f(\overrightarrow{\boldsymbol{U}}_m) - \frac{1}{M}\sum_{m=1}^{M} f(\overrightarrow{\boldsymbol{U}}_m)\Big)\Big(f_{cv}(\overrightarrow{\boldsymbol{U}}_m) - \frac{1}{M}\sum_{m=1}^{M} f_{cv}(\overrightarrow{\boldsymbol{U}}_m)\Big)\Big]}{\sum_{m=1}^{M}\Big(f_{cv}(\overrightarrow{\boldsymbol{U}}_m) - \frac{1}{M}\sum_{m=1}^{M} f_{cv}(\overrightarrow{\boldsymbol{U}}_m)\Big)^2}. \qquad (2.49)$$

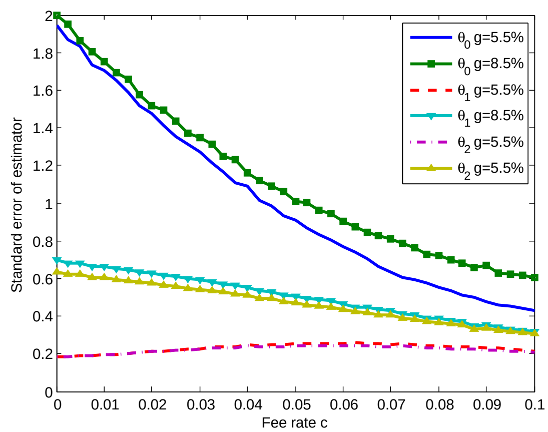

Figure 2.19: *Standard errors of estimators $\hat{\theta}_0$ (standard Monte Carlo estimator), $\hat{\theta}_1$ (control variate estimator), $\hat{\theta}_2$ (improved control variate estimator) as functions of the fee rate c for the cases $g = 5.5\%, 8.5\%$. In each case, $\hat{\theta}_1$ and $\hat{\theta}_2$ are close. Each simulation is based on $M = 10^5$ scenarios.*

Figure 2.19 displays (estimates of) the standard errors of the unbiased GMIB price estimators $\hat{\theta}_0$, $\hat{\theta}_1$ and $\hat{\theta}_2$ as functions of the fee rate c. All standard errors are computed using $M = 10^5$ scenarios. The standard errors of the GMIB price are higher when g is higher. For any fee rate, the standard error of $\hat{\theta}_2$ is marginally smaller than the standard error for $\hat{\theta}_1$. The standard error of $\hat{\theta}_2$ is 45-90% smaller than the standard error of $\hat{\theta}_0$ across the fee rate range 0-10%.

As noted in McLeish (2005), when we compare two different Monte Carlo estimators (of the same expectation/option price), the ratio of the variances of the estimators corresponding to a fixed number of function evaluations can be interpreted as roughly the ratio of computational time required for a predetermined accuracy. Explicitly, the efficiency gain of the control variate estimator $\hat{\theta}_j$ is defined as the variance of the crude Monte Carlo estimator, $\hat{\theta}_0$ divided by the variance of $\hat{\theta}_j$, where both estimators are calculated using the same number of scenarios (function evaluations/observations). An efficiency gain of x indicates that the control variate estimator only requires M/x scenarios to achieve the same variance as the crude Monte Carlo estimator using M scenarios. The efficiency gains of $\hat{\theta}_2$ over $\hat{\theta}_0$ range from 4 to 115, depending on the fee rate. The efficiency gains decrease as the fee rate increases. Clearly, incorporating the control variate into the simulations is well worth the extra effort.

Finally, we note that further reductions in the standard errors of $\hat{\theta}_0$, $\hat{\theta}_1$ and $\hat{\theta}_2$ might be achieved by using an appropriate low-discrepancy sequence (Joy et al., 1996), (Tan and Boyle, 2000). In fact, the author has tested using the low discrepancy sequence suggested by Lemieux and Faure (2009) in conjunction with the control variate estimators, and it was found that additional, albeit small, efficiency gains were made.

2.10 History of the GMIB in the U.S. since 2007

In the past decade there seems to have been widespread underpricing of GMIBs in the U.S. market. A universal pricing correction was observed in the first half of 2009. All major GMIB issuers in the U.S. increased the fees for GMIB option, and reduced the benefits offered by the the GMIB option.[4] Before the correction in 2009, it was very common for guaranteed return component of the benefit base to guarantee 6% per year on the premiums invested. Moreover, the fee rates being charged were about 0.4-0.6% per year – no competitive GMIB seller charged a fee rate close to 1% per year or higher. In the first half of 2009, with the full impact of the global financial crisis unfolding, significant changes were made to the GMIB option (on new policies sold). The guaranteed rate on premiums was almost universally reduced to 5% by the major and most competitive GMIB issuers. At the same time as reducing the GMIB benefits, the fee rates for the GMIB were increased to about 0.9-1% per year, across all competitive issuers. Moreover, restrictions

[4]The following observations were noted by the author of this thesis, who has been monitoring www.annuityFYI.com since the start of 2007.

on the percentage of funds invested with riskier fund managers became standard (for further details see Section 1.1). It seems likely that these adjustments were partly due to issuers realizing that they were exposed to significantly higher liabilities than expected.

It is interesting to note one particular case. In 2007, ING U.S.A. offered a highly competitive GMIB (highly recommended by www.annuityFYI.com at the time), which provided a very generous benefit base relative to its competitors. ING U.S.A. guaranteed an annual rate of return on premiums invested of 7% per year, and allowed the policyholder to lock in the highest investment account value at the end of each quarter (most other issuers only allowed the policyholder to lock in gains on each policy anniversary). It charged a fee rate of 0.75% per year for this option. Our analysis suggests this fee rate may have been too low. In 2009, ING U.S.A. ceased selling the GMIB option altogether. Today ING U.S.A. still sells the other variable annuity options, including the increasingly popular GLWB.

In fact, it appears that most GMIB sellers got the pricing of the GMIB very wrong. Only when the market started to collapse did the issuers take actions to correct the benefits and pricing of the GMIB. On the other hand:

- Insurers may have been aware that the fee rates being charged were too low, but due to competitive pressures, were unable to charge a higher fee rate without reducing their market share of new variable annuity business;
- The GMIB may have been a loss leader product for some companies, if the overall variable annuity charges were profitable;
- Perhaps the fees being earned were profitable, but only while there was not a sharp and sustained drop in equity returns.

The false sense of security that comes from following the majority, assuming that they have got the pricing right (without verifying for yourself), may have played a role.

2.11 Concluding remarks

In this chapter, we proposed a pricing equation for the GMIB, which allows us to determine the fair fee rate for the option. The GMIB was valued using straightforward benchmark models, avoiding complex models with idiosyncracies. It has been shown that interest rate assumptions have a significant influence on the GMIB price. Taken at face value, the model suggests that, based on reasonable parameter assumptions, the fee rates being charged by insurance companies for GMIBs (currently about 0.8-1% per year) may be too low. Specifically, the fee rates being charged are lower than what is needed to dynamically hedge the GMIB. However, we caution that our analysis has not allowed for lapses and underlying variable annuity fee charges. Allowing for these factors reduces the fair fee rate. Sections 2.6 and 2.8 illustrate how these factors affect the fair fee rate. Of the two guarantees provided by the GMIB, namely the guaranteed return and lookback components, the guaranteed return component is the most valuable in terms of pricing. Moreover, the lookback component seems to be a relatively cheap guarantee for the insurance company when paired with the guaranteed return component, and potential buyers may perceive the value of this guarantee to be much higher than what has been calculated in this chapter.

Assumptions made to simplify the analysis included no policy lapses, no cash withdrawals or additional premiums, no lapses, and no underlying variable annuity fees. It is also assumed that the maturity date is on the 10th policy anniversary. Varying each of these assumptions will lead to changes in the fair fee rates and GMIB prices we have presented. In practice insurance companies may be making profits from selling GMIB options partly because of policyholder lapses – the minimum 10 year accumulation phases may end up being too long for some cash-tight policyholders. Section 2.6 showed that the GMIB price is highly sensitive to lapse behavior.

Features of GMIBs that may be worth exploring include assessing the value of step-up options (the step up option was briefly discussed in Section 1.3.1; it gives the policyholder the right to reset the value of the guaranteed return component at specific points during the accumulation phase), measuring the cost of mortality improvement for GMIBs associated with life annuities, and exploring the optimal time, in a purely financial sense, for exercising the GMIB beyond the 10-th policy anniversary. The challenging issue of hedging GMIBs is an important one that needs investigating, but to date does not seem to have been touched on much in the literature. Investigating the effectiveness of possible static hedges is a first step.

Chapter 3

Measuring the Effectiveness of Static Hedging Strategies for a Guaranteed Minimum Income Benefit

3.1 Introduction

The standard assumptions for pricing derivatives include (Hull (2008), Joshi (2008)):

- The ability to rebalance the replicating portfolio on a continuous basis;
- No transaction costs incurred on any trades; and
- The ability to correctly model the underlying asset price dynamics.

These assumptions must hold for a complete market model. In an ideal world where the market is complete, there exists a risk-free delta hedging strategy, such that the GMIB maturity value is equal to the payoff of a replicating portfolio. The annuity premium is invested according to a pre-defined replicating strategy, in such a way that no additional funds are needed for the replicating portfolio payoff to match the GMIB liability at maturity. In Chapter 2, we priced the GMIB using a complete market model. The prices we obtained assume that the standard assumptions listed above are fulfilled in reality. The prices correspond to the costs of employing delta hedging strategies. However, in practice, there are significant difficulties with delta hedging the GMIB option, including the following issues:

- The term to expiry is very long, with a minimum of 10 years. Delta hedging requires frequent rebalancing of the replicating portfolio. The transaction costs will not be negligible if the hedge is to be reasonably accurate from inception to the expiration date. Furthermore, continuous rebalancing is a requirement if the hedge is to be risk-free. In reality, rebalancing is only possible at discrete time intervals.

- The asset price dynamics cannot be modeled precisely; we regularly see sudden large market movements that cannot be accurately predicted by models. When the underlying asset prices feature jumps and/or stochastic volatility, any delta hedge will only be partially successful, in general.

- Rebalancing a large portfolio, (at the same time as other insurers are doing the same thing, perhaps) may cause large price movements in the market, making rebalancing more expensive than anticipated.

- Delta hedging is model dependent. In particular, the calculations of the Greeks (Delta, Gamma, Vega, Rho, etc.) depend on the choices of the stock and interest rate models. If the models do not reasonably approximate movements in stock prices and interest rates in reality, then the hedge may not work as projected.

- There is systemic longevity risk associated with GMIBs embedded in life annuities. Finding liquid financial instruments to adequately hedge the longevity risk may be a difficult or impossible task. In this chapter, we do not model mortality/longevity risk, but rather assume a term certain annuity is selected at maturity. However, we note that longevity risk is a key driver of the value of GMIBs associated with life-contingent annuities. Hedging the longevity risk of GMIBs is a research topic in itself.

An alternative to delta hedging is a static hedging strategy. A static hedging strategy involves selecting an appropriate combination of different financial instruments at inception. This portfolio is then held to the maturity date without rebalancing. The instruments are chosen to generate a combined payoff at maturity that matches the payoff of the

GMIB as closely as possible. The static hedging strategy avoids or mitigates the difficulties involved with implementing delta hedging strategies. Carr et al. (1998) and Derman et al. (2000) provide examples of static hedging strategies applied to exotic option payoffs.

A minimally dynamic approach that extends the concept of the static hedge is the semi-static strategy, under which the hedge portfolio is rebalanced once (say) before the expiration date of the payoff being hedged. Carr and Wu (2004) and Liu (2010) have examined the effectiveness of semi-static hedging strategies for standard options and guaranteed minimum withdrawal benefits respectively. In this chapter, we measure the effectiveness of static hedging strategies for a guaranteed minimum income benefit. In Chapter 4, we will explore the effectiveness of semi-static hedging strategies.

Define the *hedging loss* as the difference between the GMIB value and the hedging portfolio payoff at maturity. Using Monte Carlo simulation, the effectiveness of a static hedging strategy is measured by the empirical hedging loss distribution. Several assumptions are required to develop appropriate static hedging strategies:

1. The range of hedging instruments assumed to be available needs to be specified. We explore the use of a range of standard financial instruments that should all be available in practice, either on exchanges or over-the-counter: the underlying stock index portfolio, zero coupon bonds and long-dated European options with various strike prices. We also explore the addition of lookback options to mitigate one feature of the GMIB maturity value.

2. Given a budget constraint, the optimal positions in each of the hedging instruments must be determined based on minimizing some specified objective function, and there are many possible objective functions to choose from. We illustrate the hedging effectiveness of two distinct objective functions. First, in Section 3.6, we analyze the optimal portfolios obtained from minimizing the Conditional Tail Expectation (CTE) (also known as Conditional Value at Risk, Tail Value at Risk or Expected Shortfall) of the hedging loss distribution. Second, in Section 3.7, we explore the results based on minimizing the mean squared hedging loss (MSHL).

The methods presented provide a template for how an insurance company can develop static hedging strategies for groups of variable annuity policies which include GMIBs.

Our results suggest which instruments are most important to achieve the best results. However, we also illustrate that, based on the (benchmark) models and assumptions described, the performance of the static hedge for the GMIB is imperfect at best. Based on the GMIB option fee rates currently being charged in practice, the hedging portfolios do not adequately simultaneously hedge the upside and downside equity guarantees provided by the GMIB. Changing the model assumptions and parameter values will produce different hedging loss statistics to those reported in this chapter, but the underlying message, that the static hedge approach may not adequately mitigate the risks, is unlikely to change.

The structure of Chapter 3 is as follows. In Section 3.2, we discuss the models employed for the financial variables. In Section 3.3, we describe the method for constructing a static hedging portfolio. We discuss the hedging loss statistics, which measure the performance of a static hedging strategy. Useful decompositions of the hedging loss statistics are proposed, which assist in the risk analysis of a strategy. The CTE and MSHL minimization problems are defined. Section 3.4 lists the benchmark parameter assumptions we adopt for illustrating most of the results in this chapter. Section 3.5 illustrates the effectiveness of a simple hedging portfolio, in which the entire initial investment is invested in a stock index portfolio from time 0 to maturity. The hedging loss statistics for this portfolio act as a benchmark for the hedging loss statistics obtained using more sophisticated static hedging portfolios. Sections 3.6 and 3.7 present the hedging loss statistics for various portfolios, containing different instruments, obtained from minimizing the CTE and MSHL. Section 3.8 shows that hedging the interest rate risk associated with the GMIB option is secondary to hedging the equity risk. Section 3.9 explores the effectiveness of static hedging portfolios when the design of the GMIB option is simplified. In Section 3.10, we illustrate how the performance of the static hedging portfolios change if the fair fee rate for the GMIB option is charged. In Section 3.11, we investigate how a static hedging strategy would have performed if it had been in place for a GMIB issued in each year from 1997 to 2011. This is a challenging backtest, given the nature of the financial market crises experienced during the period. In Section 3.12, we measure the impact on the hedging loss distribution when significant loadings are added to the prices of options included in the hedging portfolios. In Section 3.13, we present a sensitivity analysis of key parameters. Section 3.14 covers some practical risks associated with static hedging strategies, which we have not allowed for in our results. Section 3.15 presents concluding remarks.

3.2 Models for the financial variables

We assume the policyholder has requested the insurer to invest their annuity premium in a managed portfolio that offers returns perfectly matching the returns of a major stock index. Furthermore, it is assumed that options on this stock index are traded. Henceforth, we refer to this managed portfolio as simply the stock. Under the real-world (objective) probability measure, which we denote by P, changes in the value of the stock are modeled as a geometric Brownian motion. Namely,

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma_S S(t)\mathrm{d}W_S^P(t) \tag{3.1}$$

where μ is the (annualized) instantaneous expected return, $\sigma_S > 0$ is the (annualized) instantaneous volatility of the stock, and $\{W_S^P(t), t \in [0,T]\}$ is a standard Brownian motion under P. We assume that the short rate satisfies the Hull-White model under both the P measure, and the risk-neutral measure Q. Explicitly, the short rate under P is assumed to evolve according to the SDE given by

$$\mathrm{d}r(t) = \{\breve{\Theta}(t) - \breve{a}r(t)\}\mathrm{d}t + \sigma_r \mathrm{d}W_r^P(t), \tag{3.2}$$

where $\breve{a} > 0$ is a constant that measures the speed of mean reversion, $\breve{\Theta}(t)$ is a deterministic function of time, $\sigma_r > 0$ is the (annualized) instantaneous volatility of the short rate and $\{W_r^P(t), t \in [0,T]\}$ is a standard Brownian motion under P. The stock and short rate processes may be dependent on each other. The parameter ρ denotes the linear correlation coefficient between $\{W_S^P(t), t \in [0,T]\}$ and $\{W_r^P(t), t \in [0,T]\}$.

Under P, the policyholder's investment account still satisfies equation (2.5), but all of the random variables are computed based on equations (3.1) and (3.2).

We now want to determine the SDEs of the stock and short rate processes under a risk neutral measure, denoted by Q. Using a Cholesky decomposition,

$$\mathrm{d}W_S^P(t) = (1-\rho^2)^{1/2}\mathrm{d}Z_{(1)}^P(t) + \rho\mathrm{d}Z_{(2)}^P(t) \tag{3.3}$$

$$\mathrm{d}W_r^P(t) = \mathrm{d}Z_{(2)}^P(t) \tag{3.4}$$

where $\{Z_{(1)}^P(t)\}$ and $\{Z_{(2)}^P(t)\}$ are independent standard Brownian motions under P. Using equations (3.3) and (3.4),

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma_S S(t)\{(1-\rho^2)^{1/2}\mathrm{d}Z_{(1)}^P(t) + \rho\mathrm{d}Z_{(2)}^P(t)\} \tag{3.5}$$

$$\mathrm{d}r(t) = \{\breve{\Theta}(t) - \breve{a}r(t)\}\mathrm{d}t + \sigma_r\mathrm{d}Z_{(2)}^P(t). \tag{3.6}$$

The Girsanov Theorem tells us that the process $\{Z_{(i)}(t), Z_{(i)}(0) = 0\}, i = 1, 2$, defined by

$$\mathrm{d}Z_{(i)}^Q(t) = \mathrm{d}Z_{(i)}^P(t) - \lambda_{(i)}(t)\mathrm{d}t, \tag{3.7}$$

where $\lambda_{(i)}(t)$ is some real function (satisfying certain technical conditions (Musiela and Rutkowski, 2004)), is a standard Brownian motion under Q. The market is arbitrage free if there exists at least one risk-neutral measure. For the measure Q to be a risk-neutral measure, the discounted price processes of all assets must be martingales. Under the measure Q, the short rate satisfies the SDE

$$\mathrm{d}r(t) = \{\breve{\Theta}(t) - \breve{a}r(t) + \lambda_{(2)}(t)\sigma_r\}\mathrm{d}t + \sigma_r\mathrm{d}Z_{(2)}^Q(t). \tag{3.8}$$

Equation (3.8) follows from combining equations (3.6) and (3.7).

Under the measure Q, the stock satisfies the SDE

$$\begin{aligned}\mathrm{d}S(t) = {} & \left(\mu + \lambda_{(1)}(t)\sigma_S\sqrt{1-\rho^2} + \lambda_{(2)}(t)\sigma_S\rho\right) S(t)\mathrm{d}t \\ & + \sigma_S S(t)\{\sqrt{1-\rho^2}\mathrm{d}Z_{(1)}^Q(t) + \rho\mathrm{d}Z_{(2)}^Q(t)\}.\end{aligned} \tag{3.9}$$

Equation (3.9) follows from combining equations (3.5) and (3.7). For Q to be a risk-neutral measure, we require

$$\mu + \lambda_{(1)}(t)\sigma_S\sqrt{1-\rho^2} + \lambda_{(2)}(t)\sigma_S\rho = r(t). \tag{3.10}$$

Now, the Hull-White model is actually defined under a risk-neutral measure Q', not the real-world measure P. Under Q', the short rate, by definition, satisfies the SDE

$$\mathrm{d}r(t) = \{\Theta(t) - ar(t)\}\mathrm{d}t + \sigma_r \mathrm{d}Z_{(2)}^{Q'}(t), \tag{3.11}$$

where $a > 0$, $\Theta(t)$ is a deterministic function of time that is chosen such that the model term structure matches the market term structure at the start of the projection, and $\{Z_{(2)}^{Q'}(t)\}$ is a standard Brownian motion under Q'. Equation (3.8) can expressed in the form of equation (3.11) if we set

$$\lambda_{(2)}(t) = \frac{(\Theta(t) - ar(t)) - (\breve{\Theta}(t) - \breve{a}r(t))}{\sigma_r}. \tag{3.12}$$

In fact, if we assume that the short rate process satisfies the Hull-White model under P, and that at least one zero coupon bond is traded, then $\lambda_{(2)}(t)$ must satisfy equation (3.12), otherwise there will be arbitrage opportunities.

We now explain why there would be arbitrage opportunities if $\lambda_{(2)}(t)$ does not satisfy equation (3.12) for all t. In the Hull-White model the price at time t of a zero coupon bond maturing at time T has the form $P(t,T) = F(r(t), t, T)$, where F is a smooth function with respect to the three chosen arguments. Under these conditions Ito's Lemma can be used to determine the dynamics of $P(t,T)$. Let P_t^T be shorthand notation for $P(t,T)$. For Q to be a risk-neutral measure, every discounted zero bond price process must be a martingale, otherwise there are arbitrage opportunities. Let $\{M(t) = e^{-\int_0^t r(s)\mathrm{d}s} P_t^T\}$ denote the discounted price process for the zero coupon bond maturing at time T. This process is a martingale if the drift term of the SDE for $\{M(t)\}$ is zero. Now, the short rate follows equation (3.8) under the measure Q. Using Ito's Lemma, it can be shown that

$$\begin{aligned} \mathrm{d}M(t) = e^{-\int_0^t r(s)\mathrm{d}s} \left[\frac{\partial P_t^T}{\partial t} + \{\breve{\Theta}(t) - \breve{a}r(t) + \lambda_{(2)}(t)\sigma_r\} \frac{\partial P_t^T}{\partial r} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 P_t^T}{\partial r^2} - r(t)P_t^T \right] \mathrm{d}t \\ + e^{-\int_0^t r(s)\mathrm{d}s} \sigma_r \frac{\partial P_t^T}{\partial r} \mathrm{d}Z_{(2)}^{Q}(t). \end{aligned} \tag{3.13}$$

In the Hull-White model, defined by equation (3.11), the fundamental PDE for the zero coupon bond maturing at time T is (Musiela and Rutkowski, 2004)

$$\frac{\partial P_t^T}{\partial t} + \{\Theta(t) - ar(t)\}\frac{\partial P_t^T}{\partial r} + \frac{1}{2}\sigma_r^2\frac{\partial^2 P_t^T}{\partial r^2} - r(t)P_t^T = 0, \tag{3.14}$$

with boundary condition $P_T^T = 1$. In an arbitrage free bond market, P_t^T must satisfy this PDE. Substituting equation (3.14) into equation (3.13) yields

$$\mathrm{d}M(t) = \Big((\breve{\Theta}(t) - \breve{a}r(t)) - (\Theta(t) - ar(t)) + \lambda_{(2)}(t)\sigma_r\Big)\frac{\partial P_t^T}{\partial r}\mathrm{d}t + \sigma_r\frac{\partial P_t^T}{\partial r}\mathrm{d}Z_{(2)}^Q(t).$$

Clearly, $\{M(t)\}$ is a martingale if and only if $\lambda_{(2)}(t)$ satisfies equation (3.12).

Given equation (3.12), we can rearrange equation (3.10) to obtain a unique solution for $\lambda_{(1)}(t)$. In summary, assuming that short rate process satisfies the Hull-White model under P, and that a zero coupon bond can be traded, then the risk-neutral measure Q is unique. Hence, the market is also complete.

In this thesis, we assume that the Hull-White model parameters are identical under measures P and Q, implying $\lambda_{(2)}(t) = 0$. When $\lambda_{(2)}(t) = 0$, equation (3.10) can be rearranged to show that, for all t,

$$\lambda_{(1)}(t) = \frac{-(\mu - r(t))}{\sigma_S\sqrt{1-\rho^2}}.$$

In our setting, the stock process satisfies the following SDE under Q:

$$\mathrm{d}S(t) = r(t)S(t)\mathrm{d}t + \sigma_S S(t)\mathrm{d}W_S^Q(t), \tag{3.15}$$

where $\{W_S^Q(t) = \{\sqrt{1-\rho^2}Z_{(1)}^Q(t) + \rho Z_{(2)}^Q(t)\}$ is a standard Brownian motion under Q. Furthermore, under Q the short rate process satisfies the SDE given by

$$\mathrm{d}r(t) = \{\Theta(t) - ar(t)\}\mathrm{d}t + \sigma_r\mathrm{d}W_r^Q(t) \tag{3.16}$$

where $\{W_r^Q(t) = Z_{(2)}^Q(t)\}$ is a standard Brownian motion under Q.

Analytical formulas exist for zero coupon bond prices in the Hull-White model. The price at time t of a zero coupon bond maturing at time T, $P(t,T)$, is given by equation (2.11). Options in the hedging portfolio are valued at time 0 using analytical formulas where possible, otherwise we use Monte Carlo simulation. Analytical formulas for the European put and call options exist when the stock and the short rate are assumed to evolve according to equations (3.15) and (3.16). In order to present the analytical formulas in their full form, let us temporarily extend the stock SDE given by equation (3.15) to allow for a continuous divided yield at rate y per year. The SDE of the stock price under Q becomes

$$\mathrm{d}S(t) = (r(t) - y)S(t)\mathrm{d}t + \sigma_S S(t)\mathrm{d}W_S^Q(t).$$

The price at time t of a European call/put option written on the stock with maturity date T and strike price K is given by (Brigo and Mercurio, 2006)

$$\begin{aligned}\mathcal{O}(t,T,\mathcal{K}) = \psi S(t) e^{-y(T-t)}\Phi\left(\psi\frac{\log\left\{\frac{S(t)}{KP(t,T)}\right\} - y(T-t) + \frac{1}{2}v^2(t,T)}{v(t,T)}\right) \\ - \psi K P(t,T)\Phi\left(\psi\frac{\log\left\{\frac{S(t)}{KP(t,T)}\right\} - y(T-t) - \frac{1}{2}v^2(t,T)}{v(t,T)}\right)\end{aligned} \tag{3.17}$$

where $\psi = 1$ corresponds to a call, and $\psi = -1$ corresponds to a put. In equation (3.17)

$$v^2(t,T) = V(t,T) + \sigma_S^2(T-t) + 2\frac{\rho\sigma_r\sigma_S}{a}\left(T - t - \frac{1}{a}(1 - e^{-a(T-t)})\right)$$

and $V(t,T)$ is defined by equation (2.40). We continue to assume $y = 0$ for simplicity, but it is trivial to adjust the model in order to obtain results with $y > 0$. It is emphasized that the call and put options are valued as functions of the stock, not the policyholder's investment account (which is periodically reduced by fees).

It is noted that more sophisticated models for the dynamics of the stock (which allow for stochastic volatility and jumps), and the term structure of interest rates, could be easily be implemented using the approach in this chapter. We choose a simple model without any idiosyncrasies to set a benchmark. If the static hedge does not work well using a simple model, it will do an even worse job in reality.

3.3 Implementing the static hedging strategies

Recall that we are hedging the GMIB maturity value, defined by equation (1.3), which is

$$Y(T) = \max\left\{B(T)g\ddot{a}_{\overline{20|}}(T), A(T)\right\},$$

where

$$B(T) = \max\{A(0)(1+r_g)^T, \max_{n=1,2,\ldots,T} A(n-)\}.$$

Now,

$$\ddot{a}_{\overline{20|}}(T) = \sum_{j=T}^{T+19} P(T,j) \tag{3.18}$$

where $P(T,j)$ is defined by equation (2.11) in the Hull-White model. At time T, $\ddot{a}_{\overline{20|}}(T)$ is a deterministic function of $r(T)$. Recall that the assumptions include ignoring longevity risk and policy lapses (see Section 1.3).

Let:

- α denote the confidence level of the CTE.
- N denote the number of scenarios.
- K denote the number of hedging instruments.
- $\boldsymbol{y} = [y_1, y_2, \ldots, y_N]'$ denote the vector of simulated GMIB maturity values. The n-th component y_n is the GMIB maturity value for the n-th scenario.
- $\boldsymbol{x} = [x(1), x(2), \ldots, x(K)]'$ denote the vector of hedging instrument positions. The k-th component $x(k)$ is the number of units of hedging instrument k held long in the portfolio.

- $\boldsymbol{z}_n = [z_n(1), z_n(2), \ldots, z_n(K)]'$ denote the vector of simulated hedging instrument payoffs for the nth scenario. The k-th component $z_n(k)$ is the payoff of the k-th instrument for the n-th scenario. Furthermore, let $\boldsymbol{Z} = [\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_N]'$ denote the $N \times K$ market payoff matrix containing the instrument payoffs for all N scenarios.

- $\boldsymbol{\phi} = [\phi(1), \phi(2), \ldots, \phi(K)]'$ denote the vector of hedging instrument prices at time 0. The k-th component $\phi(k)$ is the price of the k-th instrument.

- $\boldsymbol{c} = [c(1), c(2), \ldots, c(K)]'$ denote the vector of transaction costs. The k-th component $c(k)$ is the transaction cost per unit of instrument k bought/sold.

- $\boldsymbol{u} = [u(1), u(2), \ldots, u(K)]'$ denote a vector of real numbers introduced to solve the optimization problems.

We can measure the effectiveness of a given hedging portfolio by analyzing the distribution of the difference between the GMIB maturity value and the hedging portfolio payoff, which we refer to as the hedging loss distribution. We can sample from the hedging loss distribution using the following algorithm:

(1) Compute the hedging instrument prices $\boldsymbol{\phi}$. All option prices are computed using the Q-measure models.

(2) Simulate y_n and $\boldsymbol{z}_n$ for $n = 1, \ldots, N$, using the P-measure models. It should always be checked that the securities market model generated by $\boldsymbol{\phi}$ and $\boldsymbol{Z}$ is arbitrage free. Certain instruments might introduce arbitrage opportunities. Later, in Section 4.2.5, we define a test which should be performed at this step to ensure that arbitrage opportunities do not exist. As it turns out, none of the combinations of hedging instruments that we consider in this chapter permit arbitrage opportunities.

(3) Solve the optimization problem to obtain the vector of optimal instrument positions, which we denote by $\boldsymbol{x} = \hat{\boldsymbol{x}}$.

(4) Calculate the hedging loss observations (henceforth referred to as simply the hedging losses), defined as

$$e_n = y_n - \boldsymbol{z}_n^{'} \hat{\boldsymbol{x}}, \qquad n = 1, \ldots, N. \tag{3.19}$$

In Step (4), when N is not large, to avoid biases in the results, a different set of realizations of $\{y_n, \boldsymbol{z}_n\}_{n=1}^N$, not used to obtain $\hat{\boldsymbol{x}}$, should be used when computing $\{e_n\}_{n=1}^N$. Nevertheless, when N is sufficiently large, the output of Step (2) can be used in Step (4) without introducing any significant bias. As we demonstrate in Section 3.13.1, using $N \geqslant 10{,}000$ is sufficient for producing stable results with low sampling error bias. In Sections 3.5, 3.6 and 3.7, we use the same set of realizations of $\{y_n, \boldsymbol{z}_n\}_{n=1}^N$ for each set of results that we illustrate. This allows us to compare the portfolio statistics for each portfolio using the same underlying "sampling variability". In particular, we can "rank" the CTE values and the MSHL$^{1/2}$ values for different portfolios, without worrying about sampling variability that arises from using different sets of realizations of $\{y_n, \boldsymbol{z}_n\}_{n=1}^N$.

If the hedging portfolio payoff closely matches the GMIB maturity value, then the hedging losses should be small in absolute magnitude. When $e_n > 0$ ($e_n < 0$), the insurance company experiences a loss (profit) at time T. It is emphasized that if the insurer utilizes static hedges for the GMIB, then the premium is not physically invested in the stock, but rather in a combination of hedging instruments. In the remainder of this section, we discuss the universe of instruments, the hedging loss statistics that we use to measure the effectiveness of a particular portfolio, and define the CTE and MSHL optimization problems.

3.3.1 Universe of instruments

In order to implement a hedging portfolio using an optimization problem, the universe of available hedging instruments must first be defined. To be practical, the instruments should be available in practice. We assume the following financial instruments can be bought or sold in any quantity:

- A stock index portfolio (stock) that offers returns perfectly matching the returns of the broad stock index in which the policyholder is invested. Without loss of generality, we assume that the value of one unit of the stock at time 0 is $\pi = 1000$; this allows us to illustrate the results more neatly.[1]

[1] If the initial value of the stock is set equal to $aS(0)$ for $a > 0$, then the optimal positions in the stock and the options, shown in the tables in this chapter, will be scaled by the constant $1/a$.

- Zero coupon bonds (ZCBs) with maturity dates ranging between 10 and 29 years. This range corresponds to the range of annuity payment dates. We refer to a zero coupon bond with a maturity date of T years starting from time 0 as ZCB(T). Without loss of generality, we assume that each bond has a face value of $\pi = 1000$ per unit.

- European put options on the stock index, with expiration dates of $T = 10$ years and judiciously chosen strike prices. We refer to a put option with strike price K as Put(K).

- European (annually sampled) lookback call options at a particular strike price K, which we refer to as LBC(K), and European (annually sampled) lookback put options, which we refer to as LBP. We define the payoffs of these instruments in Section 3.6.4.

Put and lookback options with 10 year exercise dates are uncommon, and their presence on public exchanges is limited. However, historically these types of options have been actively traded over-the-counter; common option writers include investment banks. The motivations for using these options to hedge the GMIB are discussed in Sections 3.5 and 3.6.

3.3.2 The hedging loss statistics

The effectiveness of an optimal strategy $\hat{\boldsymbol{x}}$ is measured by the hedging loss distribution that it generates. The statistics (some being standard risk measures) we compute to describe the hedging loss distribution include the sample mean, sample standard deviation, 1%-percentile, median, Value at Risk (99%-percentile) and the Conditional Tail Expectation. To facilitate comparisons between the results for portfolios minimized using different objective functions, we also compute an estimate of the square root of the mean squared hedging loss ($\text{MSHL}^{1/2}$) for each example, which is calculated as

$$\widehat{\text{MSHL}}^{1/2} = \left(\frac{1}{N} \sum_{n=1}^{N} e_n^2 \right)^{1/2}. \tag{3.20}$$

The $\text{MSHL}^{1/2}$ is a measure of how closely the hedging portfolio payoff matches the GMIB maturity value.

The Value at Risk (VaR) and Conditional Tail Expectation (CTE) form our measures of (tail) risk, for a given static hedging strategy. They are estimated as follows. Let

$$e_{(1)}, e_{(2)}, \ldots, e_{(N)}$$

denote the ordered hedging losses, sorted in ascending order. In other words, $e_{(n)}$ is the n-th smallest hedging loss. The estimate of the VaR at a confidence level of $\alpha \in (0,1)$ is given by $\widehat{\text{VaR}}(\alpha) = e_{(N\alpha)}$, provided $N\alpha$ is an integer. The estimate of the CTE at a confidence level of $\alpha \in (0,1)$ is

$$\widehat{\text{CTE}}(\alpha) = \sum_{n=N\alpha+1}^{N} \frac{e_{(n)}}{N(1-\alpha)}. \tag{3.21}$$

VaR is a used extensively used in the finance industry, particularly in the measurement of trading risk over fixed time horizons (Hull, 2009). However, the CTE is becoming the preferred risk measure in the insurance industry, particularly for setting liability provisions (American Academy of Actuaries, 2005). Wirch and Hardy (1999) study the properties of VaR and CTE in the context of equity-linked guarantees in insurance contracts. Unlike VaR, the CTE is a coherent risk measure in the sense of Artzner et al. (1999). Furthermore, the optimization problems for minimizing the CTE are easier to implement than for VaR, when they are scenario-based; VaR optimization problems are non-convex, and may have many local minima (Gaivoronski and Pflug, 2005). Partly for these reasons, we focus on the CTE as our central measure of risk.

Confidence intervals for the statistics

Whenever Monte Carlo simulation is used to compute statistics of interest, the uncertainty of the estimates, due to sampling variability, should be quantified. The uncertainty of an estimate is often quantified by the standard error of the estimate, or by a confidence interval. We now outline how we can calculate measures of uncertainty for each of our estimates. Let:

- Φ^{-1} denote the inverse of the standard normal cumulative distribution function;
- ε_n be the random variable denoting the hedging loss for the n-th scenario;
- $\bar{e}$ and $\hat{\sigma}_e^2$ denote the estimates of the mean and variance of ε_n;
- $\bar{\omega}$ and $\hat{\sigma}_\omega^2$ denote the estimates of the mean and variance of ε_n^2.

Applying the central limit theorem, an approximate 100β percent confidence interval for $E[\varepsilon_n]$ is given by

$$\bar{e} \pm \Phi^{-1}\left(\frac{1+\beta}{2}\right)\frac{\hat{\sigma}_e}{\sqrt{N}}.$$

By applying the Delta Method (Casella and Berger, 2001) to $f(\hat{\sigma}_e^2) = (\hat{\sigma}_e^2)^{1/2}$, we can obtain an approximate 100β percent confidence interval for $Var[\varepsilon_n]^{1/2}$. It is given by

$$\hat{\sigma}_e \pm \Phi^{-1}\left(\frac{1+\beta}{2}\right)\frac{(\hat{\sigma}_U^2)^{1/2}}{2\hat{\sigma}_e\sqrt{N}}$$

where $\hat{\sigma}_U^2$ is the estimate of the variance of $(\varepsilon_n - E(\varepsilon_n))^2$.

Because $\{\varepsilon_n\}_{n=1}^N$ are independent and identically distributed, so are $\{\varepsilon_n^2\}_{n=1}^N$. Hence $\frac{1}{N}\sum_{n=1}^N \varepsilon_n^2$ converges to a normal distribution as N increases. Therefore, we can also apply the Delta Method to $f(\bar{\omega}) = (\bar{\omega})^{1/2}$ to obtain an approximate 100β percent confidence interval for $\text{MSHL}^{1/2}$. It is given by

$$\bar{\omega}^{1/2} \pm \Phi^{-1}\left(\frac{1+\beta}{2}\right)\frac{(\hat{\sigma}_\omega^2)^{1/2}}{2\bar{\omega}^{1/2}\sqrt{N}}.$$

In the special case where the hedging loss mean is close to 0, $\bar{\omega}^{1/2}$ and $\hat{\sigma}_e$ will be close in value, and the corresponding confidence intervals will be very similar.

Let $e_{(1)}, e_{(2)}, \ldots, e_{(N)}$ denote the ordered hedging losses from smallest to largest. Thus, $e_{(n)}$ is the n-th smallest hedging loss. A nonparametric (approximate) 100β percent confidence interval for the γ-quantile hedging loss, estimated as $e_{(N\gamma)}$ assuming $N\gamma$ is an integer, is given by (Hardy, 2003)

$$\left(e_{(N\gamma-\theta)}, e_{(N\gamma+\theta)}\right)$$

where

$$\theta = \Phi^{-1}\left(\frac{1+\beta}{2}\right)\sqrt{N\gamma(1-\gamma)}.$$

It is usual to round θ to the nearest integer. As this confidence interval is based on the normal approximation to the binomial distribution, we should ensure that $N\gamma > 30$ and $N(1-\gamma) > 30$.

The standard error of the CTE estimator is difficult to determine, for reasons outlined in (Hardy, 2003). Recall that the CTE is the mean hedging loss, given that the hedging loss exceeds the α-quantile of the underlying hedging loss distribution. If the α-quantile of the underlying hedging loss distribution was known with certainty, the standard error of the CTE estimate would be the sample standard deviation of the observations in excess of the α-quantile, divided by the number of observations in excess of the α-quantile. However, in practice the α-quantile is unknown. Using simulation to estimate the α-quantile introduces a second source of uncertainty in the CTE estimate. If the second source of uncertainty is ignored, then a biased low estimate of the standard error of the CTE estimate is given by

$$\tilde{\sigma} = \left(\frac{\hat{\sigma}^2(e_{(n)} : n > N\alpha)}{N(1-\alpha)}\right)^{1/2},$$

where $\hat{\sigma}^2(e_{(n)} : n > N\alpha)$ denotes the sample variance of the hedging loss observations in excess of $e_{(N\alpha)}$, $\{e_{(n)}\}_{n=N\alpha+1}^{N}$. A more accurate estimate of the standard error can be obtained by repeated Monte Carlo simulation. Repeating the (entire) simulation of a strategy many times provides a range of CTE estimates. The standard deviation of the sample of CTE estimates (one CTE estimate per entire simulation) provides an estimate of the standard error of the CTE estimator. This approach is not always feasible if one entire simulation takes many hours. In particular, in Chapter 4, where computation time is a critical issue, repeated Monte Carlo simulation of a semi-static hedging strategy is not a viable option.

Manistre and Hancock (2005) develop an estimator for the variance of the CTE estimator that is valid as the sample size approaches infinity. They also present empirical results that suggest the asymptotic CTE variance estimator is a good approximation to the true variance for finite sample sizes, even for high confidence (alpha) levels. The asymptotic CTE variance estimate is

$$\hat{\sigma}^2(\widehat{\mathrm{CTE}}(\alpha)) = \frac{\hat{\sigma}^2(e_{(n)} : n > N\alpha)}{N(1-\alpha)} + \frac{\alpha}{N(1-\alpha)} \left(\widehat{\mathrm{CTE}}(\alpha) - e_{(N\alpha)}\right)^2 .$$

This estimate is intuitively appealing as it simplifies to the variance of a mean estimator when $\alpha = 0$. An approximate 100β percent confidence interval for $\mathrm{CTE}(\alpha)$ is given by

$$\widehat{\mathrm{CTE}}(\alpha) \pm \Phi^{-1}\left(\frac{1+\beta}{2}\right) \sqrt{\hat{\sigma}^2(\widehat{\mathrm{CTE}}(\alpha))}. \tag{3.22}$$

We use expression (3.22) in the calculation of confidence intervals for reported CTE estimates.

It is noted that the confidence intervals presented here do not capture all sources of uncertainty associated with the hedging loss statistics of a given static hedging strategy. The confidence intervals are calculated conditional on the vector of optimal instrument positions $\hat{\boldsymbol{x}}$. But $\hat{\boldsymbol{x}}$ is a random variable that depends on the selection of scenarios used in the optimization problem. The random variability associated with $\hat{\boldsymbol{x}}$ can be measured by utilizing repeated Monte Carlo simulations. In Section 3.13.1, we illustrate the sensitivity of the optimal value of $\hat{\boldsymbol{x}}$ for different selections of scenarios and values of N. As it turns out, the variability of $\hat{\boldsymbol{x}}$ is reasonably small if N is sufficiently large ($N = 20000$ seems appropriate). Therefore, although the confidence intervals we report for each example in the following sections do not account for the variability of $\hat{\boldsymbol{x}}$, the intervals are still reasonably accurate.

Useful decompositions of the statistics

Recall that the GMIB maturity value is the maximum of three components:

$$Y(T) = \max(X_1, X_2, X_3)$$

where

$$X_1 = \max_{n=1,\ldots,T} A(n-)g\ddot{a}_{\overline{20}|}(T),\ X_2 = A(0)(1+r_g)^T g\ddot{a}_{\overline{20}|}(T),\ X_3 = A(T).$$

We refer to X_1 as the *lookback component*, X_2 as the *guaranteed return component* and X_3 as the *investment account component*. When we say that a particular component is exercised (for a particular real-world scenario), we mean that it is the most valuable component among all three components. To assist in the risk analysis of a particular strategy, useful information may be obtained by decomposing some of the hedging loss statistics. In particular, insights into what instruments effectively hedge the GMIB are gained by decomposing the mean and CTE(99%) into contributions related to when each of the three components are exercised. Here we outline some informative decompositions, which we will calculate for our examples.

Define the indicator function

$$1_{[A](n)} = \begin{cases} 1 & \text{if event } A \text{ occurs for the } n\text{-th scenario} \\ 0 & \text{if event } A \text{ does not occur for the } n\text{-th scenario.} \end{cases} \tag{3.23}$$

The mean hedging loss estimate $\bar{e}$ can be decomposed as the sum of the contributions from each of the components of the GMIB and the hedging instruments:

$$
\begin{aligned}
\bar{e} &= \frac{1}{N}\sum_{n=1}^{N} e_n \\
&= \frac{1}{N}\sum_{n=1}^{N}\left[(y_n - \sum_{k=1}^{K} z_n(k)x(k))(1_{[X_1>X_2,X_3](n)} + 1_{[X_2>X_1,X_3](n)} + 1_{[X_3>X_1,X_2](n)})\right] \\
&= \bar{y}_1^M + \bar{y}_2^M + \bar{y}_3^M - \sum_{k=1}^{K}(\theta_1^M(k) + \theta_2^M(k) + \theta_3^M(k)) \qquad (3.24)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{y}_1^M &= \frac{1}{N}\sum_{n=1}^{N} y_n 1_{[X_1>X_2,X_3](n)}, \\
\bar{y}_2^M &= \frac{1}{N}\sum_{n=1}^{N} y_n 1_{[X_2>X_1,X_3](n)}, \\
\bar{y}_3^M &= \frac{1}{N}\sum_{n=1}^{N} y_n 1_{[X_3>X_1,X_2](n)},
\end{aligned}
$$

$$
\begin{aligned}
\theta_1^M(k) &= \frac{1}{N}\sum_{n=1}^{N} z_n(k)x(k)1_{[X_1>X_2,X_3](n)}, \qquad k = 1,\dots,K, \\
\theta_2^M(k) &= \frac{1}{N}\sum_{n=1}^{N} z_n(k)x(k)1_{[X_2>X_1,X_3](n)}, \qquad k = 1,\dots,K, \\
\theta_3^M(k) &= \frac{1}{N}\sum_{n=1}^{N} z_n(k)x(k)1_{[X_3>X_1,X_2](n)}, \qquad k = 1,\dots,K.
\end{aligned}
$$

The larger the value of $\bar{y}_j^M, j = 1,2,3$, the more that component X_j contributes to $\bar{e}$ for the given hedging portfolio. The term $\theta_j^M(k), k = 1,\dots,K, j = 1,2,3$, is a measure of the effectiveness of the k-th hedging instrument at offsetting the GMIB maturity value when X_j is exercised. The larger the value of $\theta_j^M(k)$, the more effective instrument k is at reducing $\bar{e}$.

Let $y_{(n)}$ and $z_{(n)}(k), k = 1, \ldots, K$, denote the GMIB maturity value and hedging instrument payoffs corresponding to the n-th smallest simulated hedging loss $e_{(n)}$, for $n = N\alpha + 1, \ldots, N$. The CTE estimate can be decomposed in a similar manner to the mean hedging loss:

$$\mathrm{CTE}(\alpha) = \bar{y}_1^C + \bar{y}_2^C + \bar{y}_3^C - \sum_{k=1}^{K} (\theta_1^C(k) + \theta_2^C(k) + \theta_3^C(k)) \tag{3.25}$$

where

$$\bar{y}_1^C = \frac{1}{N(1-\alpha)} \sum_{n=N\alpha+1}^{N} y_{(n)} 1_{[X_1 > X_2, X_3](n)},$$
$$\bar{y}_2^C = \frac{1}{N(1-\alpha)} \sum_{n=N\alpha+1}^{N} y_{(n)} 1_{[X_2 > X_1, X_3](n)},$$
$$\bar{y}_3^C = \frac{1}{N(1-\alpha)} \sum_{n=N\alpha+1}^{N} y_{(n)} 1_{[X_3 > X_1, X_2](n)},$$

$$\theta_1^C(k) = \frac{1}{N(1-\alpha)} \sum_{n=N\alpha+1}^{N} z_{(n)}(k) x(k) 1_{[X_1 > X_2, X_3](n)}, \qquad k = 1, \ldots, K,$$
$$\theta_2^C(k) = \frac{1}{N(1-\alpha)} \sum_{n=N\alpha+1}^{N} z_{(n)}(k) x(k) 1_{[X_2 > X_1, X_3](n)}, \qquad k = 1, \ldots, K,$$
$$\theta_3^C(k) = \frac{1}{N(1-\alpha)} \sum_{n=N\alpha+1}^{N} z_{(n)}(k) x(k) 1_{[X_3 > X_1, X_2](n)}, \qquad k = 1, \ldots, K.$$

The larger the value of $\bar{y}_j^C, j = 1, 2, 3$, the more that component X_j contributes to $\mathrm{CTE}(\alpha)$ for the given hedging portfolio. The term $\theta_j^C(k), k = 1, \ldots, K, j = 1, 2, 3$, is a measure of the effectiveness of the k-th hedging instrument at reducing $\mathrm{CTE}(\alpha)$ when X_j is exercised. A large value for $\theta_j^C(k)$ may suggest instrument k is important for reducing $\mathrm{CTE}(\alpha)$. However, it is important to realize that the relationship is not straightforward; even if instrument k produces a small value for $\theta_j^C(k)$, it may still be important for reducing $\mathrm{CTE}(\alpha)$. The $\theta_j^C(k)$ should not be interpreted in isolation, but in conjunction with the $\theta_j^M(k)$. A concrete example demonstrating this point is given in Section 3.6.4.

We can also decompose the standard deviation estimate into contributions from each of the three components. The sample variance $\hat{\sigma}_e^2$ can be expressed in the following form:

$$\begin{aligned}\hat{\sigma}_e^2 &= \frac{1}{N-1}\sum_{n=1}^{N}(e_n-\overline{e})^2 \\ &= \frac{1}{N-1}\sum_{n=1}^{N}(e_n-\overline{e})^2(1_{[X_1>X_2,X_3](n)}+1_{[X_2>X_1,X_3](n)}+1_{[X_3>X_1,X_2](n)}). \qquad (3.26)\end{aligned}$$

Dividing equation (3.26) by the sample standard deviation, which is $\sqrt{\hat{\sigma}_e^2}=\hat{\sigma}_e>0$, we obtain the sample standard deviation as the sum of three terms related to the three components:

$$\hat{\sigma}_e = SD_{X_1}+SD_{X_2}+SD_{X_3}$$

where

$$\begin{aligned}SD_{X_1} &= \frac{1}{\sqrt{\hat{\sigma}_e^2}(N-1)}\sum_{n=1}^{N}(e_n-\overline{e})^2 1_{[X_1>X_2,X_3](n)}, \\ SD_{X_2} &= \frac{1}{\sqrt{\hat{\sigma}_e^2}(N-1)}\sum_{n=1}^{N}(e_n-\overline{e})^2 1_{[X_2>X_1,X_3](n)}, \\ SD_{X_3} &= \frac{1}{\sqrt{\hat{\sigma}_e^2}(N-1)}\sum_{n=1}^{N}(e_n-\overline{e})^2 1_{[X_3>X_1,X_2](n)}.\end{aligned}$$

The term $SD_{X_j}, j=1,2,3$ measures the contribution of component X_j to the standard deviation for the given hedging portfolio. It is possible to further decompose the standard deviation by expanding the squared differences, but we choose to omit further decompositions.

The $\text{MSHL}^{1/2}$ estimate, given by equation (3.20), can also be decomposed in a similar manner to the standard deviation estimate:

$$\widehat{\text{MSHL}}^{1/2} = \sqrt{MS}_{X_1}+\sqrt{MS}_{X_2}+\sqrt{MS}_{X_3}$$

where

$$\sqrt{MS}_{X_1} = \frac{1}{\widehat{\mathrm{MSHL}}^{1/2} N} \sum_{n=1}^{N} e_n^2 1_{[X_1 > X_2, X_3](n)},$$
$$\sqrt{MS}_{X_2} = \frac{1}{\widehat{\mathrm{MSHL}}^{1/2} N} \sum_{n=1}^{N} e_n^2 1_{[X_2 > X_1, X_3](n)},$$
$$\sqrt{MS}_{X_3} = \frac{1}{\widehat{\mathrm{MSHL}}^{1/2} N} \sum_{n=1}^{N} e_n^2 1_{[X_3 > X_1, X_2](n)}.$$

The term $\sqrt{MS}_{X_1}, j = 1, 2, 3$ measures the contribution of component X_j to the $\mathrm{MSHL}^{1/2}$ for the given hedging portfolio.

3.3.3 CTE minimization problem

Rockafellar and Uryasev (2000) present a method for minimizing the Conditional Tail Expectation (CTE) of a portfolio's loss distribution. Alexander et al. (2006) extend the CTE minimization problem by allowing for a cost function that penalizes large positions in any of the financial instruments in the portfolio. Alexander et al. find that, by including a cost function, it is possible to construct an optimal portfolio with significantly fewer financial instruments, without any significant deterioration in the CTE. The cost function prevents the optimal portfolio from including unrealistically large positive or negative instrument positions. For our problem, this cost function can be interpreted as the transaction costs involved in constructing the hedging portfolio. Specifically, there is a transaction cost for each long or short position in an instrument which is proportional to the number of units in the position.

It can be shown that the optimization problem for minimizing the CTE(α) of the hedging loss distribution is equivalent to a (convex) constrained piecewise linear minimization problem (Alexander et al., 2006):

$$\min_{(\delta, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^K} \left(\delta + \frac{1}{N(1-\alpha)} \sum_{n=1}^{N} \max\left\{ y_n - \boldsymbol{z}_n' \boldsymbol{x} - \delta, 0 \right\} + \sum_{k=1}^{K} c(k)|x(k)| \right)$$

$$\text{subject to:} \quad \boldsymbol{\phi}'\boldsymbol{x} + \sum_{k=1}^{K} c(k)|x(k)| \leqslant \pi.$$

We note that the solution to this optimization problem jointly minimizes the CTE and the transaction costs. Therefore, strictly speaking we are simultaneously minimizing the CTE and the transaction costs involved in constructing the portfolio (for conciseness, we do not mention the transaction costs). The reason transaction costs are included in the objective function is to ensure that the optimization problem is stable when the optimizer searches for the solution. If we do not include the transaction costs in the objective function, we find that occasionally no reasonable solution can be obtained (this issue applies much more so in the optimization problems defined in Chapter 4). The budget constraint says that the cost of constructing the hedging portfolio at time 0, allowing for transaction costs, must not exceed the invested annuity premium $\pi = S(0)$. We allow short selling of the hedging instruments.

Let $\boldsymbol{v} = [v_1, v_2, \ldots, v_N]'$ be a vector of real numbers. The solution to the CTE minimization problem is obtained by solving an equivalent linear programming program of the following form:

$$\min_{(\delta, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{u}) \in \mathbb{R} \times \mathbb{R}^K \times \mathbb{R}^N \times \mathbb{R}^K} \left(\delta + \frac{1}{N(1-\alpha)} \sum_{n=1}^{N} v_n + \sum_{k=1}^{K} c(k)u(k) \right)$$

subject to:

$$\begin{aligned}
&\boldsymbol{\phi}'\boldsymbol{x} + \sum_{k=1}^{K} c(k)u(k) \leqslant \pi, \\
&v_n \geqslant y_n - \boldsymbol{z}_n'\boldsymbol{x} - \delta, \qquad v_n \geqslant 0, \qquad && n = 1, \ldots, N, \\
&u(k) - x(k) \geqslant 0, \qquad u(k) + x(k) \geqslant 0, \qquad && k = 1 \ldots, K.
\end{aligned}$$

We can add a further constraint to the CTE minimization problem. We can choose to minimize the CTE subject to the mean hedging loss being equal to some real number R.

The additional constraint has the form

$$\bar{y} - \bar{\boldsymbol{z}}'\boldsymbol{x} = R, \tag{3.27}$$

where $\bar{y} = 1/N\sum_{n=1}^{N} y_n$ and $\bar{\boldsymbol{z}} = [\bar{z}(1), \bar{z}(2), \ldots, \bar{z}(K)]'$ is a vector containing the mean payoffs of each hedging instrument, where $\bar{z}(k) = 1/N\sum_{n=1}^{N} z_n(k)$. Setting $R = 0$ will produce optimal portfolios with mean hedging losses of 0, implying that the insurer will break-even on average. By not including this constraint we are obtaining the minimum CTE among all possible values of R.

Due to the nature of this optimization problem, and our parameter assumptions, which are listed in Section 3.4, at the optimal solution we always find

$$\boldsymbol{\phi}'\hat{\boldsymbol{x}} + \sum_{k=1}^{K} c(k)|\hat{x}(k)| = \pi.$$

It is noted that if the transaction cost assumptions were increased significantly (to unreasonable levels), this equality may not hold.

3.3.4 MSHL minimization problem

This optimization problem minimizes the mean squared hedging loss (MSHL), allowing for transaction costs and the budget constraint. It is a quadratic programming problem of the following form:

$$\min_{(\boldsymbol{x},\boldsymbol{u})\in\mathbb{R}^K\times\mathbb{R}^K} \left(\frac{1}{N}\sum_{n=1}^{N} \left(y_n - \boldsymbol{z}_n'\boldsymbol{x}\right)^2 + \sum_{k=1}^{K} c(k)u(k) \right)$$

subject to:

$$\boldsymbol{\phi}'\boldsymbol{x} + \sum_{k=1}^{K} c(k)u(k) \leqslant \pi,$$

$$u(k) - x(k) \geqslant 0, \qquad u(k) + x(k) \geqslant 0, \qquad\qquad k = 1\ldots, K.$$

Just like the CTE minimization problem, the solution to the MSHL optimization problem jointly minimizes the MSHL and the transaction costs. Therefore, strictly speaking we are simultaneously minimizing the MSHL and the transaction costs involved in constructing the portfolio (for conciseness, we do not mention the transaction costs). It is also possible to include the mean hedging loss constraint, given by equation (3.27), in the MSHL minimization problem.

Unlike the CTE minimization problem, equality in the budget constraint may not hold when π is sufficiently large. We define the portfolio cost as the optimal portfolio value plus the transaction costs involved in constructing the portfolio:

$$\hat{\psi} = \boldsymbol{\phi}'\hat{\boldsymbol{x}} + \sum_{k=1}^{K} c(k)|\hat{x}(k)|. \tag{3.28}$$

Define the excess funds as

$$\hat{\xi} = \pi - \hat{\psi}.$$

If $\hat{\xi} > 0$ for a particular portfolio, the hedging loss observations, calculated using equation (3.19), should be adjusted. It is reasonable to assume that the excess funds should be invested in the risk free asset, ZCB(10). Suppose the b-th instrument in the hedging portfolio corresponds to ZCB(10). Then $\phi(b)$ denotes the price of ZCB(10), and $c(b)$ denotes the corresponding transaction cost per unit of ZCB(10). The number of units of ZCB(10) bought with the excess funds is given by

$$\hat{x}_{\xi} = \hat{\xi}/(\phi(b) + c(b)) \geqslant 0. \tag{3.29}$$

At the optimal solution,

$$\boldsymbol{\phi}'\hat{\boldsymbol{x}} + \sum_{k=1}^{K} c(k)|\hat{x}(k)| + \hat{x}_{\xi}(\phi(b) + c(b)) = \pi.$$

The adjusted hedging losses are then calculated as

$$e_n = y_n - \boldsymbol{z}_n^{'}\hat{\boldsymbol{x}} - \hat{x}_\xi \pi, \qquad n = 1, \ldots, N, \tag{3.30}$$

where $\phi(b)$ denotes the price of ZCB(10), which has a face value of π dollars.

The CTE and MSHL minimization problems are readily solved using a numerical programming environment such as MATLAB. For an introduction to optimization methods, with applications in MATLAB, see Brandimarte (2006). It is noted that the objective function of the MSHL minimization problem can be written in matrix notation as follows (the objective functions for quadratic optimization problems must be expressed using matrices if the they are to be solved in MATLAB):

$$\begin{aligned}
&\left(\frac{1}{N}\sum_{n=1}^{N}(y_n - \boldsymbol{z}_n'\boldsymbol{x})^2 + \sum_{k=1}^{K}c(k)u(k)\right) \\
&= \frac{1}{N}(\boldsymbol{y} - \boldsymbol{Z}\boldsymbol{x})'(\boldsymbol{y} - \boldsymbol{Z}\boldsymbol{x}) + \boldsymbol{c}'\boldsymbol{u} \\
&= \frac{1}{N}\boldsymbol{y}'\boldsymbol{y} - \frac{2}{N}\boldsymbol{y}'\boldsymbol{Z}\boldsymbol{x} + \frac{1}{N}\boldsymbol{x}'\boldsymbol{Z}'\boldsymbol{Z}\boldsymbol{x} + \boldsymbol{c}'\boldsymbol{u}.
\end{aligned}$$

3.4 Benchmark parameter assumptions

The following benchmark parameter assumptions are used in this chapter, unless indicated otherwise: N = 20,000, $\pi = S(0) = A(0) = 1000$, $T = 10$, $r_g = 5\%$, $\mu = 9\%$, $\sigma_s = 20\%$, $\breve{a} = a = 0.35$, $\sigma_r = 1.5\%$, $\rho = 0$ and $\breve{\Theta}(t) = \Theta(t)$ depends on the zero coupon yield shape labeled "Benchmark" in Figure 2.7. The parameter values for the Q-measure models are identical to those used in Chapter 2. We use the same parameter values of α, and $\Theta(t)$ for both the P and Q measure models of the short rate. For the P-measure model of the stock, we must pick a value for μ. Merton (1980) explained the difficulties involved with estimating mean returns. It is difficult to justify any particular percentage return for μ, but invariably the value of μ will have an impact on the shape of the hedging loss distribution. Since 9-11% is often publicly cited as the long term average total (annual) return for U.S. equity markets, we set $\mu = 9\%$ (giving an expected annual return of $e^\mu - 1 = 9.42\%$).

We set $\alpha = 99\%$ because we are concerned with minimizing the risk of extreme hedging losses. The results of a sensitivity test for α, presented in Section 3.13.2, suggest that $\alpha = 99\%$ is a good choice.

For the GMIB contract parameters we assume $g = 6.5\%$ and $c = 1\%$. We consider $g = 6.5\%$ to be an equitable guaranteed payment rate based on our assumptions (see Section 2.3.1 for the reasoning). We set the fee rate equal to $c = 1\%$, as this is currently one of the highest fee rates being charged in practice by many major U.S. insurance companies. However, we stress that $c = 1\%$ is well below the fair fee rate of 4.5%, calculated using the valuation model of Chapter 2. Therefore, in the examples we present, we are modeling an insurer that has underpriced the GMIB, with respect to the model of Chapter 2. In a perfect world where the assumptions of the pricing model are fulfilled, the insurer will experience a loss when $c = 1\%$, if it follows a delta hedging strategy. We are interested in exploring whether the insurer can construct a static hedge that, using a representative industry fee rate, offers a reasonable likelihood of making a profit at maturity, while managing the downside risk.

The objective function and the budget constraint of each optimization problem allow for transaction costs. The transaction costs for each instrument are assumed to be proportional to the amount bought or sold. If instrument k is the stock or an option, we set $c(k) = 0.5\%\phi(k)$. If instrument k is a bond, we set $c(k) = 0.1\%\phi(k)$. By including transaction costs, the optimal solutions do not include excessively large positive/negative instrument positions. Without the transaction costs, the optimal solutions may yield $x(k) \to \pm\infty$ for any k. Such solutions have little practical meaning, other than indicating arbitrage opportunities may exist in the model.[2] Another obvious reason for including transaction costs is that the optimization problems more closely reflect reality.

Let p_{X_1}, p_{X_2} and p_{X_3} denote the real-world probabilities of exercising the lookback, guaranteed return and investment account components, respectively. We can calculate these

[2]Another more direct way to prevent excessively large positions in any instrument is to include upper and lower limits on the instrument positions in the optimization problem constraints. But it may be difficult to determine what limits are reasonable.

probabilities by simulating under the P-measure models. Using the benchmark parameter values, we find that $p_{X_1} = 20\%$, $p_{X_2} = 27\%$ and $p_{X_3} = 53\%$. The GMIB option is exercised about 47% of the time. We note that the p_{X_i} depend on the contract parameter assumptions, g and c, but they are independent of the actual hedging portfolio.

As part of the benchmark parameter assumptions, all portfolios assume implied option price volatilities of 20%. For completeness, Table 3.17 in Section 3.12 displays, in the row corresponding to $\sigma_i = 20\%$, the prices of all of the hedging instruments included in at least one optimized portfolio example in this chapter.

3.5 Hedging with the stock only

Before using portfolio optimization methods to construct static hedging portfolios, we first consider the effectiveness of a basic, easy to implement, naive static hedging portfolio. This is to invest the policyholder's premium, π, in the stock at time 0 and hold this long position until maturity. The hedging loss statistics for this hedging portfolio form a set of benchmark values, which can be compared with the hedging loss statistics of more sophisticated hedging portfolios presented in the following sections. In order to be consistent with the examples in the following sections, we assume transaction costs of 0.5% per unit invested in the stock.

The top panel of Figure 3.1 displays the probability density function of the hedging loss distribution obtained from using the naive hedging portfolio. The mean hedging loss is -45, indicating that the insurance company will on average make a small profit from implementing this hedge. However, the right tail indicates that occasionally the hedging loss will be very large.

The middle panel of Figure 3.1 offers another perspective on how to hedge the GMIB effectively. The markers in the middle panel plot the GMIB maturity values y_n from each scenario as functions of the stock value at time T, $S(T)$. The 'diamond', '$\times$' and '+' markers correspond to when the lookback, guaranteed return and investment account components are exercised, respectively. We refer to the 'diamond', '$\times$' and '+' markers

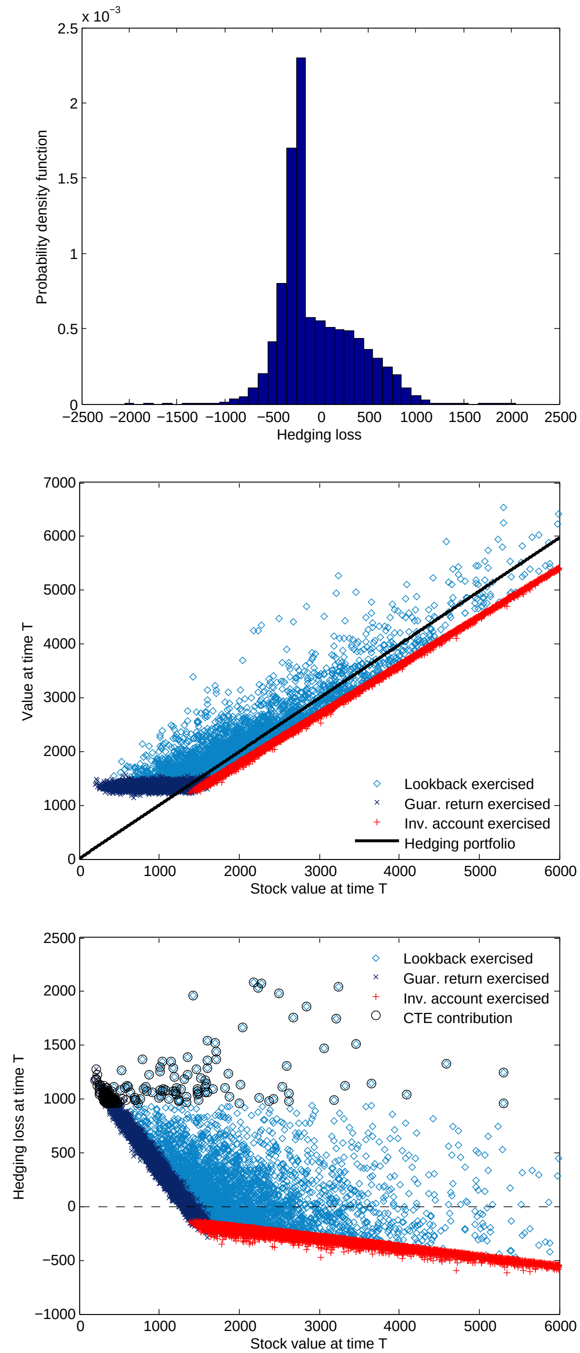

Figure 3.1: *The top panel displays the hedging loss distribution for the stock only portfolio. The middle panel shows the simulated GMIB maturity values y_n and the value of the hedging portfolio as functions of the stock value at time T, $S(T)$. The bottom panel shows the simulated hedging losses e_n as functions of $S(T)$. The y_n and e_n are individually marked according to which component is exercised.*

as the lookback, guaranteed return and investment account markers. The solid line in the middle panel plots the value of the long position in the stock at maturity. When a marker lies above (below) the solid line, the insurance company makes a loss (gain) at time T for that scenario.

In Figure 3.1, when $S(T)$ is between 0 and $A(0)(1+r_g)^T \approx 1630$, the guaranteed return component X_2 generates a floor for the GMIB maturity values, as shown by the dense region of guaranteed return markers. Note that $S(T) < A(0)(1+r_g)^T$ does not imply that the guaranteed return component is always exercised; when the stock price is low at time T, the guaranteed return component is exercised if $A(T) < A(0)(1+r_g)^T g\ddot{a}_{\overline{20}|}(T)$. The investment account markers cluster in a region that increases linearly with $S(T)$. Notice that all of the investment account markers lie below the solid line, which is because $S(T) > A(T)$ (when $c > 0$). The investment account evolves in the same way as the stock index, but is slightly lower in value due to the discrete fee payments. The investment account markers correspond to the scenarios where annuitizing using the GMIB option is not as valuable as receiving the lump sum benefit $A(T)$. The lookback component X_1 is exercised in scenarios where the account value $\{A(t)\}_{t=0}^{T}$ is very high on any particular policy anniversary during the accumulation phase. Some of the hedging losses corresponding to when X_1 is exercised are very large, as shown by the outlying lookback markers. If interest rates are low at maturity (such that $g\ddot{a}_{\overline{20}|}(T)$ is close to or exceeds 1), the lookback and guaranteed return components are more likely to be exercised.

The bottom panel of Figure 3.1 displays the hedging losses e_n, individually marked according to which component is exercised, as functions of $S(T)$. This panel is useful for highlighting the weaknesses of the hedging portfolio; if most of the largest hedging losses are generated by a particular component, then this suggests the hedging portfolio is unable to hedge that component effectively. The markers which correspond to hedging losses included in the CTE calculation are highlighted with black circles around them. The majority of circled markers are lookback markers. None of the investment account markers are circled, indicating that the CTE calculation never includes hedging losses corresponding to when the investment account component is exercised.

It is noted that the horizontal axis is restricted to 0-6,000 in the middle and bottom panels of Figure 3.1, in order to improve their readability. However, there are a small number of scenarios, not shown in the panels, for which $S(T)$ ranges between 6,000-25,000, and the corresponding GMIB maturity value ranges between 5,000-10,000. The middle and bottom panels illustrate clearly that the stock only portfolio performs poorly in two situations:

- If $S(T) < A(0)(1+r_g)^T$, the guaranteed return component provides a floor for the GMIB maturity value. When the guaranteed return component is exercised, the lower the value of $S(T)$, the larger the hedging loss.
- If the stock index increases sharply in a volatile manner during the accumulation phase, and then sharply declines before time T, the lookback component can generate a GMIB maturity value that is significantly larger than the hedging portfolio payoff at time T. The bottom panel shows that the lookback component generates a small number of very large positive hedging losses, that are much larger than any of the hedging losses generated from the guaranteed return component. The lookback component is driving the CTE value.

Port. Cost	$\mathbf{MSHL}^{1/2} =$	$\sqrt{\mathbf{MS}}_{X_1}$	$+\sqrt{\mathbf{MS}}_{X_2}$	$+\sqrt{\mathbf{MS}}_{X_3}$.	**1%-quantile**	**Median**	**VaR(99%)**
1000	388	69	144	176	-771	-186	956
	$(384, 392)$				$(-801, -748)$	$(-189, -184)$	$(936, 976)$
Excess funds	**Std. Deviation =**	$\mathbf{SD}_{X_1}$	$+\ \mathbf{SD}_{X_2}$	$+\ \mathbf{SD}_{X_3}$.		**Mean**	**CTE(99%)**
0	385	73	156	156		-48	1121
	$(380, 390)$					$(-53, -43)$	$(1083, 1159)$
Instrument (k)	$\hat{x}(k)$	$\hat{w}(k)$	**Mean/CTE Contributions**			$i = M$	$i = C$
GMIB			$\bar{y}_1^i$ (Lookback)			433	1416
			$\bar{y}_2^i$ (Guar. return)			369	725
			$\bar{y}_3^i$ (Inv. account)			1585	0
			$\boldsymbol{A}^i = \bar{y}_1^i + \bar{y}_2^i + \bar{y}_3^i$ (GMIB total)			2387	2141
Stock	0.995	1.000	$-\theta_1^i(1)$ (Lookback)			-400	-835
(1)			$-\theta_2^i(1)$ (Guar. return)			-277	-185
			$-\theta_3^i(1)$ (Inv. account)			-1757	0
			$\boldsymbol{B}^i = -\sum_{k=1}^K \sum_{j=1}^3 \theta_j^i(k)$ (Hedge port. total)			-2435	-1020
			$\boldsymbol{C}^i = \bar{y}_1^i - \sum_{k=1}^K \theta_1^i(k)$ (Lookback total)			33	581
			$\boldsymbol{D}^i = \bar{y}_2^i - \sum_{k=1}^K \theta_2^i(k)$ (Guar. return total)			92	540
			$\boldsymbol{E}^i = \bar{y}_3^i - \sum_{k=1}^K \theta_3^i(k)$ (Inv. account total)			-172	0
Mean $= \boldsymbol{A}^M + \boldsymbol{B}^M = \boldsymbol{C}^M + \boldsymbol{D}^M + \boldsymbol{E}^M$				**CTE(99%)** $= \boldsymbol{A}^C + \boldsymbol{B}^C = \boldsymbol{C}^C + \boldsymbol{D}^C + \boldsymbol{E}^C$			

Table 3.1: *Hedging loss statistics for the stock only portfolio.*

In this chapter, for each hedging portfolio we consider, we present a table of statistics describing the (empirical) hedging loss distribution. The following paragraphs explain how to interpret the numbers in each of these tables, by way of example for the current hedging portfolio.

The top section of Table 3.1 displays the hedging loss statistics, and the decompositions of the standard deviation and $\text{MSHL}^{1/2}$. The numbers in brackets below each statistic display the corresponding 95% confidence intervals. How these confidence intervals are computed was discussed in Section 3.3.2. The mean hedging loss is -48, meaning that a small hedging profit is expected. However, the standard deviation is 385, indicating a large amount of uncertainty in the realized value of the hedging loss. The $\text{MSHL}^{1/2}$, which is a measure of how closely the hedging portfolio payoff matches the GMIB maturity value, is 388. The standard deviation and $\text{MSHL}^{1/2}$ decompositions indicate that the guaranteed return and investment account components contribute the most to these two statistics. This is partly because these two components are exercised more often than the lookback component (see the real world exercise probabilities, p_{X_i}, which are given at the start of this section). The VaR and CTE at a confidence level of 99% are 956 and 1121 respectively. Thus, even in a simplified model of reality where the stock price process is modeled as a geometric Brownian motion, at least 1% of the time the hedging loss (after spending all of the annuity premium to buy the stock only hedging portfolio) is at least 956, which is almost as much as the entire initial investment, $\pi = 1000$. This may be unacceptable risk for some insurance companies. The portfolio cost, $\hat{\phi}$, displayed in the top left corner, is computed using equation (3.28).

The lower left section of Table 3.1 shows the optimal number of units in each hedging instrument k. The quantity $\hat{w}(k) = \hat{x}(k)\phi(k)/(\hat{\boldsymbol{x}}'\boldsymbol{\phi})$ is the proportion of the total portfolio value invested in instrument k. The lower right section of Table 3.1 displays the decompositions of the mean and CTE(99%). The mean and CTE are decomposed into the contributions from each of the components, and the contributions from the hedging instruments (just the stock in this particular portfolio) when each component is exercised. How the numbers in the mean and CTE columns are obtained was discussed in Section 3.3.2. The lowest section of the table summarizes the effectiveness of the hedging instruments at hedging each of the components. It is informative to compare the values of $\boldsymbol{C}^i, \boldsymbol{D}^i, \boldsymbol{E}^i, i = C, M$, for the different hedging portfolios presented in the following sections.

The decompositions of the mean and the CTE provide useful information about the effectiveness of the instruments in the hedging portfolio. For example, in the case of the mean decomposition, $\boldsymbol{E}^M = -172$ (a negative contribution to the mean hedging loss), suggesting that the stock is an effective hedge for the investment account component. However, $\boldsymbol{C}^M = 33$ and $\boldsymbol{D}^M = 92$, indicating that the stock provides a poor hedge when the lookback or guaranteed return components are exercised. Similar calculations based on the CTE decomposition indicate that both the guaranteed return and lookback components drive the value of the CTE, and that the stock is not effective at hedging either component.

3.6 Portfolios minimizing the CTE

In this section, we measure the effectiveness of several hedging portfolios with different combinations of instruments. In all of the examples, the optimal positions in the hedging instruments are obtained from solving the CTE minimization problem. By finding the portfolio that minimizes the CTE, we are obtaining the portfolio which minimizes the risk of unacceptably large hedging losses.

Recall that we use $N = 20{,}000$ scenarios. Each time a new set of scenarios is simulated, the hedging loss statistics will change slightly. However, we find that the results are reasonably stable for $N \geqslant 10,000$ scenarios. To give the reader a feel for the stability of the results, a sensitivity test of the hedging loss statistics is reported in Section 3.13.1.

3.6.1 Portfolio C1: stock and ZCB(10)

Consider a portfolio consisting of the stock and 10 year zero coupon bond (ZCB(10)) with face value of $S(0)$ per unit at time T (it is a risk-free asset). We refer to the optimized portfolio as Portfolio C1 (PC1). The optimal portfolio consists of $\hat{x}(1) = 0.943$ units of the stock index and $\hat{x}(2) = 0.078$ units in the bond. Expressed another way, $\hat{w}(1) = 94.8\%$ of the total portfolio value is in the stock, and $\hat{w}(2) = 5.2\%$ of the total portfolio value is in the bond. Compared to the stock only portfolio, PC2 generates a slightly lower CTE at the cost of a slightly higher mean. The VaR, standard deviation and $\text{MSHL}^{1/2}$ are smaller. On the other hand, the potential for hedging profits has been reduced, as seen by the increase in the 1%-quantile. The left panel of Figure 3.2 displays the hedging loss distribution. Comparing the right panels of Figures 3.1 and 3.2, we see that the locations of the markers are somewhat similar.

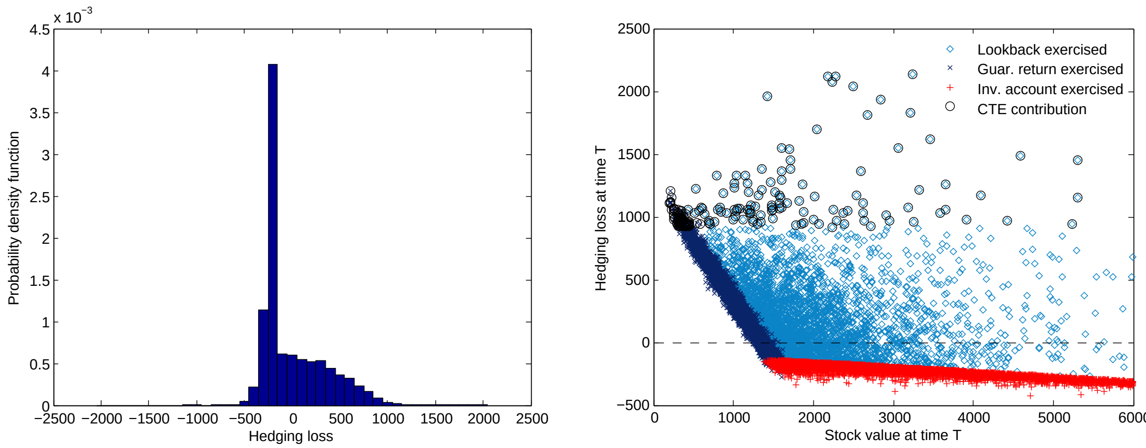

Figure 3.2: *The left panel displays the hedging loss distribution for Portfolio C1. The right panel shows the simulated hedging losses e_n as functions of the stock index value at time T, $S(T)$. The e_n are individually marked according to which component is exercised.*

Table 3.2 presents the hedging loss statistics for PC1. An explanation of the numbers in the table was given in Section 3.5; the only difference is that Table 3.2 includes the decompositions for multiple instruments in the hedging portfolio. Compared to the stock only hedging portfolio, PC1 generates a similar mean decomposition, but the CTE decomposition is somewhat different. The smaller stock position increases the lookback component contribution to the CTE (see $\bar{y}_1^C$ and $\boldsymbol{C}^C$), and the long ZCB(10) position causes the guaranteed return component contribution, $\bar{y}_2^C$, to decrease. Sensitivity tests indicate that if we increase g or r_g, then the optimal position in the stock increases. In particular, if $g \geqslant 7\%$, a leveraged postion in the stock is needed to minimize the CTE.

3.6.2 Portfolio C2: Put$(0.8S(0))$, stock and ZCB(10)

The middle panel of Figure 3.1 suggests that a put option with a strike price of around $0.5S(0)$-$2S(0)$, exercisable at time T, may help reduce the guaranteed return component

Port. Cost	**MSHL**$^{1/2}$ **=**	$\sqrt{\mathbf{MS}}_{X_1}$	$+\sqrt{\mathbf{MS}}_{X_2}$	$+\sqrt{\mathbf{MS}}_{X_3}$.	**1%-quantile**	**Median**	**VaR(99%)**
1000	326	84	150	93	-420	-167	921
	$(322, 330)$				$(-429, -410)$	$(-169, -166)$	$(906, 943)$
Excess funds	**Std. Deviation =**	$\mathbf{SD}_{X_1}$	$+\ \mathbf{SD}_{X_2}$	$+\ \mathbf{SD}_{X_3}$.		**Mean**	**CTE(99%)**
0	326	84	150	93		1	1107
	$(322, 330)$					$(-4, 5)$	$(1064, 1150)$
Instrument (k)	$\hat{x}(k)$	$\hat{w}(k)$	**Mean/CTE Contributions**			$i = M$	$i = C$
GMIB			$\bar{y}_1^i$ (Lookback)			433	2048
			$\bar{y}_2^i$ (Guar. return)			369	561
			$\bar{y}_3^i$ (Inv. account)			1585	0
			$\boldsymbol{A}^i = \bar{y}_1^i + \bar{y}_2^i + \bar{y}_3^i$ (GMIB total)			2387	2610
Stock (1)	0.943	0.948	$-\theta_1^i(1)$ (Lookback)			-379	-1294
			$-\theta_2^i(1)$ (Guar. return)			-263	-130
			$-\theta_3^i(1)$ (Inv. account)			-1666	0
ZCB(10) (2)	0.078	0.052	$-\theta_1^i(2)$ (Lookback)			-15	-46
			$-\theta_2^i(2)$ (Guar. return)			-21	-32
			$-\theta_3^i(2)$ (Inv. account)			-42	0
			$\boldsymbol{B}^i = -\sum_{k=1}^{K}\sum_{j=1}^{3}\theta_j^i(k)$ (Hedge port. total)			-2386	-1503
			$\boldsymbol{C}^i = \bar{y}_1^i - \sum_{k=1}^{K}\theta_1^i(k)$ (Lookback total)			38	707
			$\boldsymbol{D}^i = \bar{y}_2^i - \sum_{k=1}^{K}\theta_2^i(k)$ (Guar. return total)			85	399
			$\boldsymbol{E}^i = \bar{y}_3^i - \sum_{k=1}^{K}\theta_3^i(k)$ (Inv. account total)			-122	0
Mean $= \boldsymbol{A}^M + \boldsymbol{B}^M = \boldsymbol{C}^M + \boldsymbol{D}^M + \boldsymbol{E}^M$				**CTE(99%)** $= \boldsymbol{A}^C + \boldsymbol{B}^C = \boldsymbol{C}^C + \boldsymbol{D}^C + \boldsymbol{E}^C$			

Table 3.2: *Hedging loss statistics for Portfolio C1.*

contribution to the CTE. We have explored adding puts with strike prices of $0.1nS(0)$, $n = 5, 6, \ldots, 25$ to the hedging portfolio. We found that, by trial and error (i.e. running the optimization problem many times, for different combinations of instruments), the smallest CTE is obtained from the addition of a put with a strike price of $0.8S(0)$ (results based on other strikes are not shown), which we abbreviate by Put$(0.8S(0))$. We also considered simultaneously including two or more puts with different strikes to the hedging portfolio, but found there was no noticeable improvement over the addition of just Put$(0.8S(0))$. A quick procedure for finding the optimal put strike price, in terms of minimizing the CTE, involves initially including many puts with different strikes in the portfolio and identifying which ones have the largest (long) positions. The puts with the largest positions are then each individually included in an existing portfolio consisting of the stock and ZCB(10). The optimal strike price corresponds to the put which helps generate the smallest possible CTE.

Here we report the results obtained from using a portfolio which includes the instruments in PC1 and Put$(0.8S(0))$. We refer to the optimized portfolio as Portfolio C2 (PC2). In Figure 3.3, the left panel displays the hedging loss distribution for PC2. In the right panel we see that there is a kink in the region of guaranteed return markers, which is caused

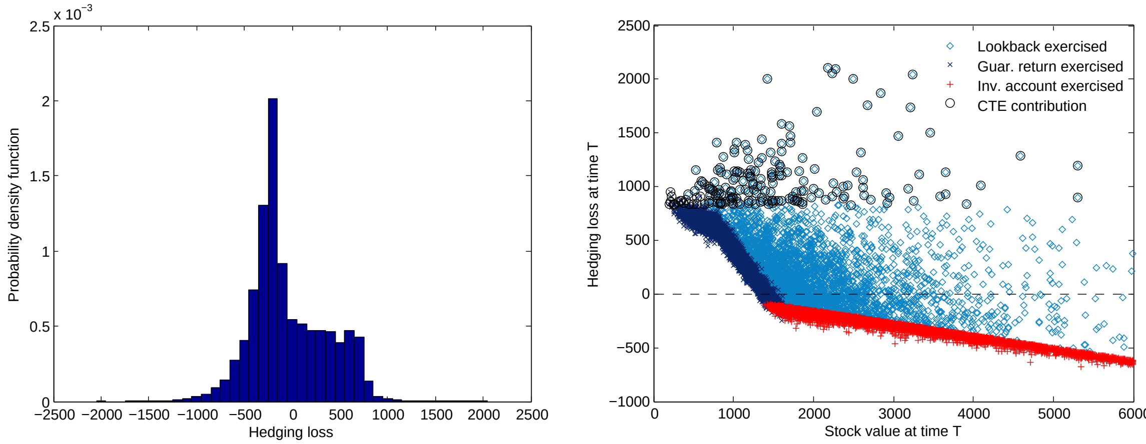

Figure 3.3: *The left panel displays the hedging loss distribution for Portfolio C2. The right panel shows the simulated hedging losses e_n as functions of $S(T)$. The e_n are individually marked according to which component is exercised.*

Port. Cost	$\mathbf{MSHL}^{1/2}$ =	$\sqrt{\mathbf{MS}_{X_1}}$	+ $\sqrt{\mathbf{MS}_{X_2}}$	+ $\sqrt{\mathbf{MS}_{X_3}}$.	**1%-quantile**	**Median**	**VaR(99%)**
1000	403	73	134	196	-910	-155	832
	$(399, 407)$				$(-947, -880)$	$(-158, -152)$	$(815, 852)$
Excess funds	**Std. Deviation** =	$\mathbf{SD}_{X_1}$	+ $\mathbf{SD}_{X_2}$	+ $\mathbf{SD}_{X_3}$.		**Mean**	**CTE(99%)**
0	401	77	144	179		-41	1056
	$(396, 405)$					$(-46, -35)$	$(1007, 1104)$

Instrument (k)	$\hat{x}(k)$	$\hat{w}(k)$	**Mean/CTE Contributions**	$i = M$	$i = C$
GMIB			$\bar{y}_1^i$ (Lookback)	433	2409
			$\bar{y}_2^i$ (Guar. return)	369	152
			$\bar{y}_3^i$ (Inv. account)	1585	0
			$\boldsymbol{A}^i = \bar{y}_1^i + \bar{y}_2^i + \bar{y}_3^i$ (GMIB total)	2387	2560
Put($0.8S(0)$)	0.682	0.027	$-\theta_1^i(1)$ (Lookback)	0	-9
(1)			$-\theta_2^i(1)$ (Guar. return)	-9	-32
			$-\theta_3^i(1)$ (Inv. account)	0	0
Stock (2)	1.021	1.027	$-\theta_1^i(2)$ (Lookback)	-411	-1506
			$-\theta_2^i(2)$ (Guar. return)	-284	-37
			$-\theta_3^i(2)$ (Inv. account)	-1804	0
ZCB(10) (3)	-0.080	-0.054	$-\theta_1^i(3)$ (Lookback)	16	71
			$-\theta_2^i(3)$ (Guar. return)	22	8
			$-\theta_3^i(3)$ (Inv. account)	42	0
			$\boldsymbol{B}^i = -\sum_{k=1}^K \sum_{j=1}^3 \theta_j^i(k)$ (Hedge port. total)	-2428	-1505
			$\boldsymbol{C}^i = \bar{y}_1^i - \sum_{k=1}^K \theta_1^i(k)$ (Lookback total)	38	965
			$\boldsymbol{D}^i = \bar{y}_2^i - \sum_{k=1}^K \theta_2^i(k)$ (Guar. return total)	97	90
			$\boldsymbol{E}^i = \bar{y}_3^i - \sum_{k=1}^K \theta_3^i(k)$ (Inv. account total)	-176	0
Mean $= \boldsymbol{A}^M + \boldsymbol{B}^M = \boldsymbol{C}^M + \boldsymbol{D}^M + \boldsymbol{E}^M$			**CTE(99%)** $= \boldsymbol{A}^C + \boldsymbol{B}^C = \boldsymbol{C}^C + \boldsymbol{D}^C + \boldsymbol{E}^C$		

Table 3.3: *Hedging loss statistics for Portfolio C2.*

by the inclusion of Put($0.8S(0)$). Table 3.3 displays the hedging loss statistics for PC2. Compared to PC1, PC2 generates a smaller VaR and CTE. Unlike PC1, PC2 is expected to generate a small profit. However, the standard deviation of PC2 is 23% larger, implying greater uncertainty in the realized hedging loss. Comparing the CTE decompositions in Tables 3.2 and 3.3, we see that the inclusion of the Put($0.8S(0)$) markedly reduces the guaranteed return total $\boldsymbol{D}^C$, but the lookback total $\boldsymbol{C}^C$ increases. The increase in the $\boldsymbol{C}^C$ figure is also partly due to the short position in ZCB(10). In the right panel of Figure 3.3, it is clear that the majority of circled markers are lookback markers. The Put($0.8S(0)$) reduces all of the hedging losses generated when $S(T) < 0.8S(0)$, most of which are generated from the guaranteed return component being exercised. The CTE is still large because the lookback component is not being hedged effectively.

Before moving on, we note that adding call options with various strike prices to a portfolio already including put options, does not improve the hedging loss statistics. This is because the put-call parity relationship holds approximately. The relationship is not exact because of the presence of transaction costs in the optimization problems.

3.6.3 Portfolio C3: Put($0.8S(0)$), stock and multiple ZCBs

The GMIB payoff is a function of the short rate at maturity. Therefore, including interest rate sensitive hedging instruments may improve the static hedges. Consider a portfolio which includes the instruments in PC2 and zero coupon bonds with maturity dates of $T_i = 10, 15, 20, 29$. We refer to the optimized portfolio as Portfolio C3 (PC3). The maturity dates of the bonds lie in the range of the GMIB annuity payment dates. In our model, ZCB(10) is a risk-free asset, but any ZCB with maturity date $T_i > 10$ is a function of $r(T)$ at time $T = 10$. Thus, any improvement in the hedging loss statistics would suggest that the additional ZCBs with different maturities, are hedging against the interest rate risk component of the GMIB.

Table 3.4 exhibits the hedging loss statistics for PC3. The optimal positions in ZCB(15) and ZCB(20) are both 0, and thus are excluded from Table 3.4 for conciseness. The optimal portfolio includes a long position of 3.338 units in ZCB(29), and a short position of -1.35 units in ZCB(10). The reason why only one ZCB with $T_i > 10$ is necessary, is because the ZCBs with $T_i > 10$ are all comonotonic with respect to $r(T)$. The sensitivity of the ZCBs to $r(T)$ increases with T_i. The longest dated bond, ZCB(29), which offers the greatest sensitivity to $r(T)$, always appears to be included in the optimal portfolio.[3] The positions in Put($0.8S(0)$) and the stock are close to those for PC2. Therefore, the increase in short position in ZCB(10) is used primarily to fund the long position in the interest sensitive instrument, ZCB(29).

We have experimented with adding different combinations of ZCBs, expiring at different times, to hedging portfolios already including the instruments of PC2. Sensitivity tests indicate that adding a ZCB with maturity $T_i > 10$ to a portfolio which already includes ZCB(10), will help reduce the CTE slightly. For the current example, we find that including ZCB(29) generates the smallest CTE, and no other distinct ZCBs are needed to reduce the CTE any further. In all of our experiments with different ZCB combinations, the same pattern emerges; a non-zero position in the longest dated bond exists, and a non-zero position in the shortest dated option (ZCB(10)) may also exist, but all other ZCBs with intermediate maturity dates have optimal positions of 0.

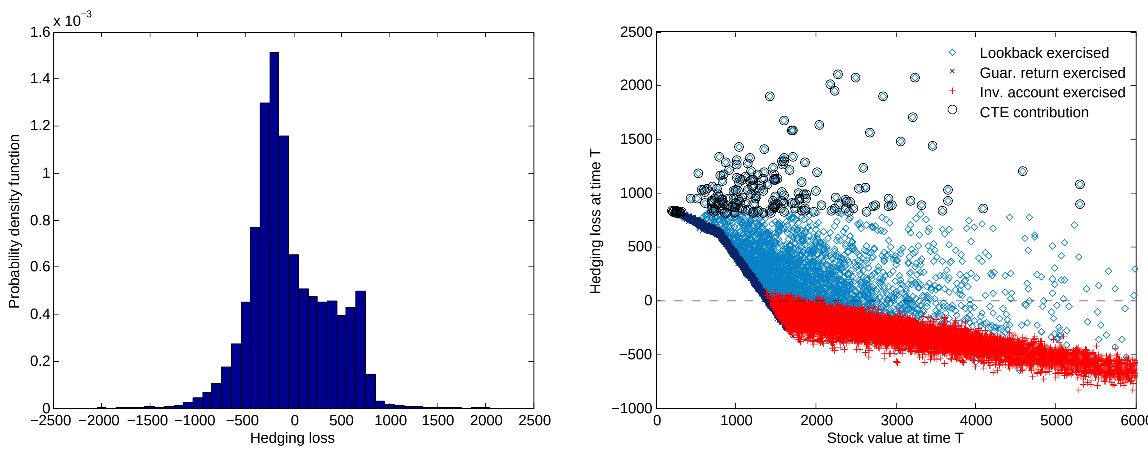

Figure 3.4: *The left panel displays the hedging loss distribution for Portfolio C3. The right panel shows the simulated hedging losses e_n as functions of the stock index value at time T, $S(T)$. The e_n are individually marked according to which component is exercised.*

PC3 offers a small improvement over PC2. The mean, 1%-quantile, VaR and CTE are

[3] If a ZCB with maturity date $T_i > 29$ was added to the portfolio, then the optimal portfolio would include a long position in this instrument, and the position in ZCB(29) would then be 0.

Port. Cost	$\mathbf{MSHL}^{1/2} =$	$\sqrt{\mathbf{MS}}_{X_1}$	$+\sqrt{\mathbf{MS}}_{X_2}$	$+\sqrt{\mathbf{MS}}_{X_3}$.	**1%-quantile**	**Median**	**VaR(99%)**
1000	414	71	135	208	-960	-134	815
	$(409, 418)$				$(-997, -938)$	$(-138, -129)$	$(803, 832)$
Excess funds	**Std. Deviation =**	$\mathbf{SD}_{X_1}$	$+\ \mathbf{SD}_{X_2}$	$+\ \mathbf{SD}_{X_3}$.		**Mean**	**CTE(99%)**
0	411	75	146	191		-43	1039
	$(407, 416)$					$(-49, -38)$	$(991, 1087)$
Instrument (k)	$\hat{x}(k)$	$\hat{w}(k)$	**Mean/CTE Contributions**			$i = M$	$i = C$
GMIB			$\bar{y}_1^i$ (Lookback)			433	2427
			$\bar{y}_2^i$ (Guar. return)			369	114
			$\bar{y}_3^i$ (Inv. account)			1585	0
			$\boldsymbol{A}^i = \bar{y}_1^i + \bar{y}_2^i + \bar{y}_3^i$ (GMIB total)			2387	2542
Put$(0.8S(0))$	0.664	0.026	$-\theta_1^i(1)$ (Lookback)			0	-9
(1)			$-\theta_2^i(1)$ (Guar. return)			-8	-31
			$-\theta_3^i(1)$ (Inv. account)			0	0
Stock (2)	1.028	1.036	$-\theta_1^i(2)$ (Lookback)			-414	-1528
			$-\theta_2^i(2)$ (Guar. return)			-286	-22
			$-\theta_3^i(2)$ (Inv. account)			-1816	0
ZCB(10) (3)	-1.350	-0.908	$-\theta_1^i(3)$ (Lookback)			264	1236
			$-\theta_2^i(3)$ (Guar. return)			367	115
			$-\theta_3^i(3)$ (Inv. account)			719	0
ZCB(29) (4)	3.338	0.846	$-\theta_1^i(4)$ (Lookback)			-247	-1157
			$-\theta_2^i(4)$ (Guar. return)			-343	-106
			$-\theta_3^i(4)$ (Inv. account)			-666	0
			$\boldsymbol{B}^i = -\sum_{k=1}^{K}\sum_{j=1}^{3}\theta_j^i(k)$ (Hedge port. total)			-2430	-1503
			$\boldsymbol{C}^i = \bar{y}_1^i - \sum_{k=1}^{K}\theta_1^i(k)$ (Lookback total)			36	969
			$\boldsymbol{D}^i = \bar{y}_2^i - \sum_{k=1}^{K}\theta_2^i(k)$ (Guar. return total)			99	70
			$\boldsymbol{E}^i = \bar{y}_3^i - \sum_{k=1}^{K}\theta_3^i(k)$ (Inv. account total)			-178	0
Mean $= \boldsymbol{A}^M + \boldsymbol{B}^M = \boldsymbol{C}^M + \boldsymbol{D}^M + \boldsymbol{E}^M$					**CTE(99%)** $= \boldsymbol{A}^C + \boldsymbol{B}^C = \boldsymbol{C}^C + \boldsymbol{D}^C + \boldsymbol{E}^C$		

Table 3.4: *Hedging loss statistics for Portfolio C3.*

slightly smaller, but the standard deviation and $\text{MSHL}^{1/2}$ are slightly larger. Comparing the left panels of Figure 3.4 and 3.3, we see that the general shapes of the hedging loss distributions for PC2 and PC3 are similar. However, there are distinct differences between the right panels of Figures 3.4 and 3.3 in terms of the locations of the different types of markers. There is remarkably less variability in the locations of the guaranteed return markers for PC3. This observation makes sense because the only source of variability in the guaranteed return component is through $r(T)$. ZCB(29) is partially hedging the guaranteed return component. Meanwhile, the variability in the locations of the investment account markers has increased considerably; this observation is also supported by the standard deviation decomposition. This occurs for two reasons. Firstly, the inclusion of the interest rate sensitive instrument increases the variability of the hedging portfolio payoff. Secondly, the investment account component payoff is independent of $r(T)$. Comparing Tables 3.3 and 3.4, we see that PC3 has a slightly larger $\boldsymbol{C}^C$ value, and a smaller $\boldsymbol{D}^C$ value. The reduction in the CTE is mainly derived from reducing the largest hedging losses generated by the guaranteed return component.

3.6.4 Portfolios C4A, C4B: lookback and put options, stock and ZCBs

All of the portfolios we have considered so far have difficulties with hedging the lookback component. Lookback options, which are traded over-the-counter in practice, may be useful for hedging the lookback component of the GMIB. The payoff of a European lookback put option with expiration date T is (Björk, 2004)

$$\nu_1(T) = \max\left(\max_{0 \leqslant t \leqslant T} S(t) - S(T), 0\right).$$

The payoff of a forward European lookback call option with expiration date T is

$$\nu_2(T) = \max\left(\max_{0 \leqslant t \leqslant T} S(t) - K, 0\right).$$

Unfortunately, lookback options are expensive. However, it does not seem unreasonable that an insurer could arrange to buy, over-the-counter, modified versions of conventional lookback options which sample on an annual basis, reflecting the features of the lookback component, rather than on a continuous basis. These modified options would be considerably cheaper to buy, and less risky for the option writers. Specifically, suppose an insurer can buy European annually sampled lookback put options and forward lookback call options on the stock, maturing in $T = 10$ years, with payoffs given by

$$\upsilon_P(T) = \max\left(\max_{n=0,1,\ldots,T} S(n) - S(T), 0\right)$$

and

$$\upsilon_C(T) = \max\left(\max_{n=0,1,\ldots,T} S(n) - K, 0\right),$$

respectively. Henceforth, we refer to the annually sampled lookback put as LBP, and the annually sampled forward lookback call option with strike price K as LBC(K). The largest difficulty with finding writers for these options is the long term to expiry. Further practical issues related to these options are discussed in Section 3.14. No analytical formulas exist for these annually sampled lookback options. Their prices, under the model in Section 3.2, must be obtained using simulation.

In the following two examples, the portfolio includes the stock, ZCB(10), ZCB(29), LBP, LBC(K_L) and Put(K_P), where K_L and K_P are appropriately chosen to minimize the CTE. The optimal strike prices for the options are found heuristically. They can be identified by rerunning the optimization problem many times for different strike prices, and finding which strike prices produce the lowest CTE. Sensitivity tests indicate that including two or more LBCs in the portfolio is unnecessary; the CTE will not noticeably shrink any further if one LBC, with an optimal strike price, is already in the portfolio. We note that if ZCBs with maturity dates $11, \ldots, 28$, are included in the portfolio, they will all have optimal positions of 0 (see Section 3.6.3).

In the first example, the hedging portfolio is optimized to obtain the smallest possible CTE, allowing for all possible instrument combinations, just as for the previous examples. It turns out that the CTE is minimized by including LBC($1.6S(0)$) and Put($1.1S(0)$). We refer to the optimized portfolio as Portfolio C4A (PC4A). PC4A yields drastic reductions in the CTE, but at a cost of generating a mean loss of 178 and a median loss of 210.

Obviously, a hedging strategy is not particularly appealing if a material loss is expected. Therefore, in the second example, the hedging portfolio is designed to minimize the CTE, subject to a mean hedging loss constraint of 0. It turns out that the CTE is minimized, subject to a mean constraint of 0, by including LBC($1.6S(0)$) and Put($1.3S(0)$). We refer to the optimized break-even portfolio as Portfolio C4B (PC4B). The higher CTE obtained with PC4B reflects the trade-off between minimizing the CTE and achieving a mean hedging loss of 0. The optimal strike price of $1.6S(0)$ for the LBC, for both PC4A and PC4B, may be partially explained by the fact that $S(0)(1 + r_g)^T \approx 1.63S(0)$. This optimal strike price is also a result of assuming $g = 6.5\%$. If g is increased (decreased), the optimal strike price will increase (decrease).

PC4A: optimizing without a mean constraint

The left panel of Figure 3.5 shows the hedging loss distribution for PC4A.[4] It is informative to compare the left panels of Figures 3.4 and 3.5. Comparing PC4A to PC3, we see that both the left and right tails have thinned significantly, and the median and the mode have increased. Table 3.5 shows that there are huge reductions in the standard deviation, MSHL$^{1/2}$, VaR and CTE. However, there is a trade off in that the mean and median hedging losses are positive and large. But the VaR and CTE for PC4A are much smaller than for any of the previous portfolios. Furthermore, both risk measures are close in magnitude, indicating that the (right) tail risk has been significantly ameliorated. It seems that LBC($1.6S(0)$) is very effective at hedging the lookback component, and the LBP is not needed in the portfolio.

The right panel of Figure 3.5 shows drastic changes in the locations of the different types of markers, compared to the previous portfolios. Contrary to the previous portfolios, the majority of the investment account markers correspond to positive hedging losses. Moreover, the CTE decomposition includes, for the first time, a positive contribution from

[4] In our experience, optimizers which use large-scale algorithms to solve linear/quadratic programming problems have difficulties in successfully solving the problems in this section. The optimization problems may be unbounded, using a large-scale algorithm. The difficulties arise from the inclusion of the lookback options. In MATLAB, the CTE minimization problem can be solved using the built in function *linprog*, which by default uses a large-scale algorithm. To obtain portfolios like PC4A and PC4B in MATLAB, it may be necessary to switch to the simplex algorithm in the options to the function *linprog*. The only disadvantage with using the simplex algorithm is that it takes longer to find the solution.

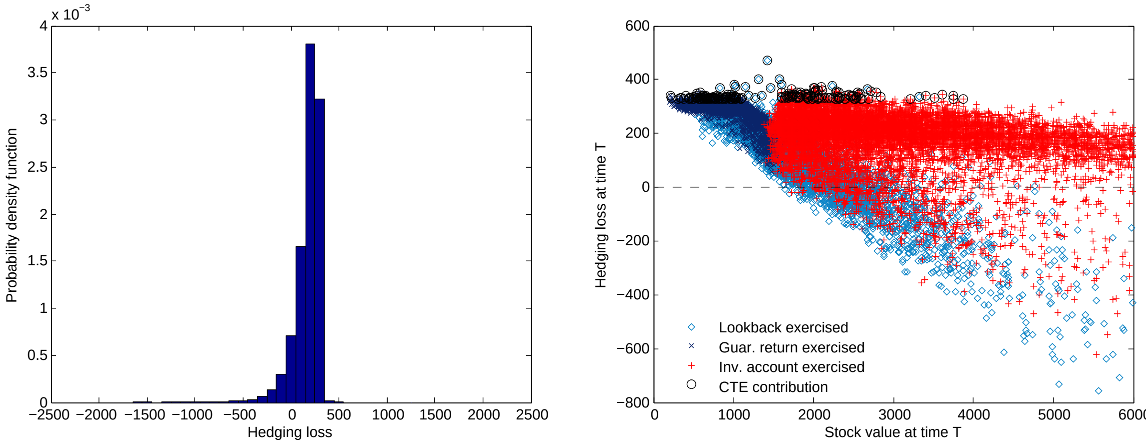

Figure 3.5: *The left panel displays the hedging loss distribution for Portfolio C4A. The right panel shows the simulated hedging losses e_n as functions of the stock index value at time T, $S(T)$. The e_n are individually marked according to which component is exercised.*

Port. Cost	**MSHL$^{1/2}$ =**	$\sqrt{\mathbf{MS}}_{X_1}$	+ $\sqrt{\mathbf{MS}}_{X_2}$	+ $\sqrt{\mathbf{MS}}_{X_3}$.	**1%-quantile**	**Median**	**VaR(99%)**
1000	224	28	95	101	-314	210	327
	$(222, 225)$				$(-337, -293)$	$(209, 212)$	$(326, 329)$
Excess funds	**Std. Deviation =**	$\mathbf{SD}_{X_1}$	+ $\mathbf{SD}_{X_2}$	+ $\mathbf{SD}_{X_3}$.		**Mean**	**CTE(99%)**
0	135	31	58	46		178	339
	$(132, 139)$					$(176, 180)$	$(336, 342)$
Instrument (k)	$\hat{x}(k)$	$\hat{w}(k)$	**Mean/CTE Contributions**			$i = M$	$i = C$
GMIB			$\bar{y}_1^i$ (Lookback)			433	260
			$\bar{y}_2^i$ (Guar. return)			369	670
			$\bar{y}_3^i$ (Inv. account)			1585	895
			$\boldsymbol{A}^i = \bar{y}_1^i + \bar{y}_2^i + \bar{y}_3^i$ (GMIB total)			2387	1825
LBC$(1.6S(0))$ (1)	0.704	0.208	$-\theta_1^i(1)$ (Lookback)			-169	-106
			$-\theta_2^i(1)$ (Guar. return)			-1	0
			$-\theta_3^i(1)$ (Inv. account)			-667	-196
LBP (2)	0.020	0.004	$-\theta_1^i(2)$ (Lookback)			-3	-3
			$-\theta_2^i(2)$ (Guar. return)			-1	-3
			$-\theta_3^i(2)$ (Inv. account)			-1	0
Put$(1.1S(0))$ (3)	0.200	0.022	$-\theta_1^i(3)$ (Lookback)			0	-1
			$-\theta_2^i(3)$ (Guar. return)			-9	-34
			$-\theta_3^i(3)$ (Inv. account)			0	0
Stock (4)	0.228	0.229	$-\theta_1^i(4)$ (Lookback)			-92	-32
			$-\theta_2^i(4)$ (Guar. return)			-63	-75
			$-\theta_3^i(4)$ (Inv. account)			-402	-227
ZCB(10) (5)	0.000	0.000	$-\theta_1^i(5)$ (Lookback)			0	0
			$-\theta_2^i(5)$ (Guar. return)			0	0
			$-\theta_3^i(5)$ (Inv. account)			0	0
ZCB(29) (6)	2.127	0.538	$-\theta_1^i(6)$ (Lookback)			-158	-83
			$-\theta_2^i(6)$ (Guar. return)			-218	-407
			$-\theta_3^i(6)$ (Inv. account)			-424	-319
			$\boldsymbol{B}^i = -\sum_{k=1}^{K}\sum_{j=1}^{3}\theta_j^i(k)$ (Hedge port. total)			-2209	-1486
			$\boldsymbol{C}^i = \bar{y}_1^i - \sum_{k=1}^{K}\theta_1^i(k)$ (Lookback total)			12	35
			$\boldsymbol{D}^i = \bar{y}_2^i - \sum_{k=1}^{K}\theta_2^i(k)$ (Guar. return total)			75	151
			$\boldsymbol{E}^i = \bar{y}_3^i - \sum_{k=1}^{K}\theta_3^i(k)$ (Inv. account total)			91	153
Mean $= \boldsymbol{A}^M + \boldsymbol{B}^M = \boldsymbol{C}^M + \boldsymbol{D}^M + \boldsymbol{E}^M$				**CTE(99%)** $= \boldsymbol{A}^C + \boldsymbol{B}^C = \boldsymbol{C}^C + \boldsymbol{D}^C + \boldsymbol{E}^C$			

Table 3.5: *Hedging loss statistics for Portfolio C4A.*

the investment account component, $\bar{y}_3^C$. A much smaller proportion of circled markers are lookback markers. The lookback component no longer drives the CTE value; the LBC($1.6S(0)$) seems to be very effective at hedging this component. It seems that all three components can be hedged reasonably well using the available instruments, because there are no longer any large hedging loss outliers. It appears that reducing the CTE further requires more funds to be added to the budget constraint. Recall that the fee rate being charged is also below the fair fee rate obtained from the valuation model of Chapter 2. If a higher fee rate is charged, then the results will improve. In Section 3.10 we investigate how the results change when the fair fee rate, obtained from the model of Chapter 2, is charged.

PC4B: optimizing with a mean constraint of 0

The CTE minimization problem now includes the constraint given by equation (3.27), where $R = 0$. The left panel of Figure 3.6 shows that the hedging loss distribution for PC4B has a much thicker left tail, compared to PC4A. Table 3.6 shows the mean is now 0, but the CTE has increased by 19% to 404. This is the trade-off between minimizing the CTE and achieving a mean of 0. The standard deviation and VaR are also higher. The lookback component contributes the most to the standard deviation, which is partly explained by the fact that $p_{X_3} = 53\%$, and by looking at the spread of the lookback markers in the right panel of Figure 3.6. Significant changes in the optimal instrument positions include larger positions in the LBC($1.6S(0)$), the put option, stock and ZCB(29), which are funded by a short position in ZCB(10). The mean decomposition indicates that the LBC($1.6S(0)$), stock and ZCB(29) are the instruments primarily responsible for shifting the mean toward 0. Comparing the right panels of Figures 3.5 and 3.6, we see that the outlying hedging profits (below the dashed line) obtained with PC4B are significantly larger than the outlying profits obtained with PC4A.

Note that the $\theta_j^C(1), j = 1, 2, 3$, which are supposed to be measures of the effectiveness of LBC($1.6S(0)$) at reducing the CTE, are very small. If the CTE decomposition was considered in isolation, these measures might be interpreted by the reader as saying that LBC($1.6S(0)$) is not effective at minimizing the CTE. However, the LBC($1.6S(0)$) is vital for reducing the CTE and the mean, but in this case this instrument is primarily used

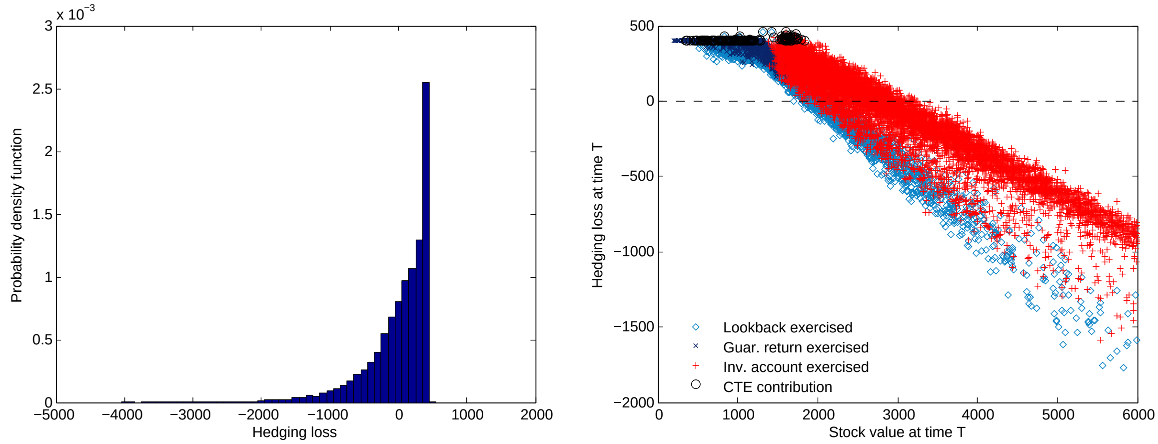

Figure 3.6: *The left panel displays the hedging loss distribution for Portfolio C4B. The right panel shows the simulated hedging losses e_n as functions of the stock index value at time T, $S(T)$. The e_n are individually marked according to which component is exercised.*

Port. Cost	**MSHL$^{1/2}$ =**	$\sqrt{\mathbf{MS}}_{X_1}$	$+\sqrt{\mathbf{MS}}_{X_2}$	$+\sqrt{\mathbf{MS}}_{X_3}$**.**	**1%-quantile**	**Median**	**VaR(99%)**
1000	476	69	84	323	-1748	141	399
	$(464, 488)$				$(-1795, -1681)$	$(134, 148)$	$(399, 399)$
Excess funds	**Std. Deviation =**	$\mathbf{SD}_{X_1}$	$+\ \mathbf{SD}_{X_2}$	$+\ \mathbf{SD}_{X_3}$**.**		**Mean**	**CTE(99%)**
0	476	69	84	323		0	404
	$(464, 488)$					$(-7, 7)$	$(402, 405)$
Instrument (k)	$\hat{x}(k)$	$\hat{w}(k)$	**Mean/CTE Contributions**			$i = M$	$i = C$
GMIB			$\bar{y}_1^i$ (Lookback)			433	101
			$\bar{y}_2^i$ (Guar. return)			369	1044
			$\bar{y}_3^i$ (Inv. account)			1585	314
			$\boldsymbol{A}^i = \bar{y}_1^i + \bar{y}_2^i + \bar{y}_3^i$ (GMIB total)			2387	1460
LBC$(1.6S(0))$ (1)	0.770	0.228	$-\theta_1^i(1)$ (Lookback)			-185	-35
			$-\theta_2^i(1)$ (Guar. return)			-1	0
			$-\theta_3^i(1)$ (Inv. account)			-729	-9
LBP (2)	0.000	0.000	$-\theta_1^i(2)$ (Lookback)			0	0
			$-\theta_2^i(2)$ (Guar. return)			0	0
			$-\theta_3^i(2)$ (Inv. account)			0	0
Put$(1.3S(0))$ (3)	0.419	0.074	$-\theta_1^i(3)$ (Lookback)			-3	-3
			$-\theta_2^i(3)$ (Guar. return)			-35	-146
			$-\theta_3^i(3)$ (Inv. account)			0	0
Stock (4)	0.419	0.422	$-\theta_1^i(4)$ (Lookback)			-169	-21
			$-\theta_2^i(4)$ (Guar. return)			-117	-257
			$-\theta_3^i(4)$ (Inv. account)			-740	-148
ZCB(10) (5)	-0.619	-0.416	$-\theta_1^i(5)$ (Lookback)			121	28
			$-\theta_2^i(5)$ (Guar. return)			168	458
			$-\theta_3^i(5)$ (Inv. account)			329	133
ZCB(29) (6)	2.733	0.693	$-\theta_1^i(6)$ (Lookback)			-203	-50
			$-\theta_2^i(6)$ (Guar. return)			-280	-804
			$-\theta_3^i(6)$ (Inv. account)			-545	-200
			$\boldsymbol{B}^i = -\sum_{k=1}^{K}\sum_{j=1}^{3}\theta_j^i(k)$ (Hedge port. total)			-2387	-1056
			$\boldsymbol{C}^i = \bar{y}_1^i - \sum_{k=1}^{K}\theta_1^i(k)$ (Lookback total)			-4	19
			$\boldsymbol{D}^i = \bar{y}_2^i - \sum_{k=1}^{K}\theta_2^i(k)$ (Guar. return total)			104	295
			$\boldsymbol{E}^i = \bar{y}_3^i - \sum_{k=1}^{K}\theta_3^i(k)$ (Inv. account total)			-100	89
Mean $= \boldsymbol{A}^M + \boldsymbol{B}^M = \boldsymbol{C}^M + \boldsymbol{D}^M + \boldsymbol{E}^M$					**CTE(99%)** $= \boldsymbol{A}^C + \boldsymbol{B}^C = \boldsymbol{C}^C + \boldsymbol{D}^C + \boldsymbol{E}^C$		

Table 3.6: *Hedging loss statistics for Portfolio C4B.*

in the optimization problem to meet the constraint of a mean of 0, rather than focusing only on minimizing the CTE, as demonstrated by PC4A. It happens to be the case that the largest hedging losses for PC4B are effectively hedged by ZCB(29), but these hedging losses would not be the largest ones if LBC($1.6S(0)$) was not included in the portfolio. Therefore, it is important to realize that the $\theta_j^C(i)$ should not be interpreted in isolation, but in conjunction with the $\theta_j^M(i)$.

Before continuing, we note that it is possible to construct an efficient frontier mapping the minimized CTE as a function of the mean hedging loss. That is, we could determine the minimum CTE for a range of plausible values of the mean hedging loss. The insurer could then decide what level of risk (as measured by the CTE) it is willing to accept, and identify the level of expected profit/loss it will make for accepting that level of risk. However, it must be remembered that the minimum CTE for a given level of expected return varies with the combinations of instruments in the portfolio. At different levels of expected hedging loss, the optimal portfolio will most likely consist of different instruments, and in particular, different optimal strike prices. For example, consider PC4A and PC4B, which use different optimal put option strike prices. The optimal instruments are found by trial and error for each level of expected return. Constructing an efficient frontier may be time consuming.

3.7 Portfolios minimizing the MSHL

In this section, we measure the effectiveness of static hedging portfolios obtained from solving the MSHL minimization problem. By finding the portfolio that minimizes the MSHL, we are obtaining a portfolio which severely penalizes against both very large positive and negative hedging losses. The payoff of the hedging portfolio is matched as closely as possible to the GMIB maturity value over as many scenarios as possible. Any excess funds are invested in the risk free asset until maturity. As it turns out, in our examples, there are never any excess funds. However, if for example the fair fee rate was charged, then there would most likely be positive excess funds.

3.7.1 Portfolio M1: Put($1.6S(0)$), stock and ZCBs

A benchmark portfolio for minimizing the MSHL consists of the stock, ZCB(10), ZCB(29) and a put option. We have analyzed adding puts with strike prices of $0.1nS(0), n = 5, 6, \ldots, 25$ to the hedging portfolio. We found that the smallest MSHL is obtained from the addition of Put($1.6S(0)$). Note that $S(0)(1 + r_g)^T \approx 1.63S(0)$, which may partially explain why the strike price $1.6S(0)$ is optimal. We refer to the optimized portfolio as Portfolio M1 (PM1). The left panel of Figure 3.7 indicates that the hedging loss distribution has a thick right tail. In the right panel we see that the locations of the markers are noticeably different compared to those obtained from minimizing the CTE. All of the circled markers are lookback markers. Table 3.7 displays the hedging loss statistics for PM1. Compared to PC3, which includes similar instruments, PM1 generates a $\text{MSHL}^{1/2}$ which is 39% smaller, a standard deviation which is roughly 48% smaller, a CTE that is about 14% larger, and a mean hedging loss of 136 instead of a profit.

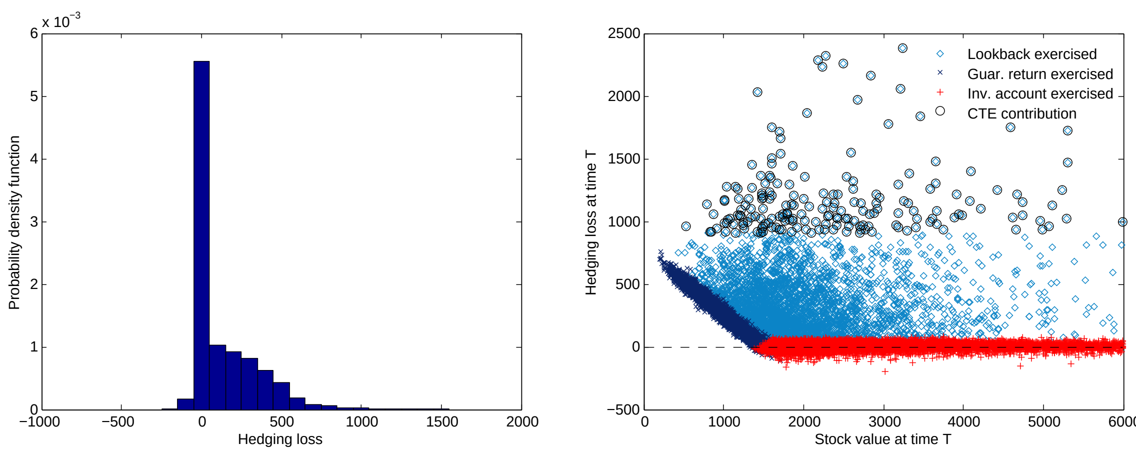

Figure 3.7: *The left panel displays the hedging loss distribution for Portfolio M1. The right panel shows the simulated hedging losses e_n as functions of the stock value at time T, $S(T)$. The e_n are individually marked according to which component is exercised.*

3.7.2 Portfolios M2A, M2B: lookback and put options, stock and ZCBs

In the following two examples, the portfolio includes the stock, ZCB(10), ZCB(29), LBP, LBC(K_L) and Put(K_P), where K_L and K_P are appropriately chosen to minimize the

Port. Cost	$\mathbf{MSHL}^{1/2}=$	$\sqrt{\mathbf{MS}}_{X_1}$	$+\sqrt{\mathbf{MS}}_{X_2}$	$+\sqrt{\mathbf{MS}}_{X_3}$.	**1%-quantile**	**Median**	**VaR(99%)**
1000	252	157	93	2	-61	33	907
	$(246, 257)$				$(-64, -58)$	$(32, 34)$	$(857, 947)$
Excess funds	**Std. Deviation =**	$\mathbf{SD}_{X_1}$	$+\ \mathbf{SD}_{X_2}$	$+\ \mathbf{SD}_{X_3}$.		**Mean**	**CTE(99%)**
0	212	144	68	0		136	1187
	$(207, 217)$					$(133, 139)$	$(1130, 1244)$

Instrument (k)	$\hat{x}(k)$	$\hat{w}(k)$	**Mean/CTE Contributions**	$i = M$	$i = C$
GMIB			$\bar{y}_1^i$ (Lookback)	433	3657
			$\bar{y}_2^i$ (Guar. return)	369	0
			$\bar{y}_3^i$ (Inv. account)	1585	0
			$\boldsymbol{A}^i = \bar{y}_1^i + \bar{y}_2^i + \bar{y}_3^i$ (GMIB total)	2387	3657
Put$(1.6S(0))$ (1)	0.376	0.112	$-\theta_1^i(1)$ (Lookback)	-8	-35
			$-\theta_2^i(1)$ (Guar. return)	-59	0
			$-\theta_3^i(1)$ (Inv. account)	0	0
Stock (2)	0.907	0.912	$-\theta_1^i(2)$ (Lookback)	-365	-2467
			$-\theta_2^i(2)$ (Guar. return)	-253	0
			$-\theta_3^i(2)$ (Inv. account)	-1602	0
ZCB(10) (3)	-0.404	-0.271	$-\theta_1^i(3)$ (Lookback)	79	404
			$-\theta_2^i(3)$ (Guar. return)	110	0
			$-\theta_3^i(3)$ (Inv. account)	215	0
ZCB(29) (4)	0.977	0.247	$-\theta_1^i(4)$ (Lookback)	-72	-372
			$-\theta_2^i(4)$ (Guar. return)	-100	0
			$-\theta_3^i(4)$ (Inv. account)	-195	0
			$\boldsymbol{B}^i = -\sum_{k=1}^{K}\sum_{j=1}^{3}\theta_j^i(k)$ (Hedge port. total)	-2251	-2470
			$\boldsymbol{C}^i = \bar{y}_1^i - \sum_{k=1}^{K}\theta_1^i(k)$ (Lookback total)	67	1187
			$\boldsymbol{D}^i = \bar{y}_2^i - \sum_{k=1}^{K}\theta_2^i(k)$ (Guar. return total)	67	0
			$\boldsymbol{E}^i = \bar{y}_3^i - \sum_{k=1}^{K}\theta_3^i(k)$ (Inv. account total)	3	0

Mean $= \boldsymbol{A}^M + \boldsymbol{B}^M = \boldsymbol{C}^M + \boldsymbol{D}^M + \boldsymbol{E}^M$	**CTE(99%)** $= \boldsymbol{A}^C + \boldsymbol{B}^C = \boldsymbol{C}^C + \boldsymbol{D}^C + \boldsymbol{E}^C$

Table 3.7: *Hedging loss statistics for Portfolio M1.*

MSHL. As it turns out, in both portfolios the MSHL is minimized when LPC($1.6S(0)$) and Put($1.6S(0)$) are included (found by trial and error). Just like in Section 3.6.4, we first present the results from using Portfolio M2A (PM2A), which achieves the smallest possible MSHL$^{1/2}$ without a mean constraint. Then we display the results from using Portfolio M2B (PM2B), which minimizes MSHL$^{1/2}$ subject to a mean constraint of 0.

PM2A: optimizing without a mean constraint

The left panel of Figure 3.8 displays the hedging loss distribution for PM2A. The right tail has thinned with the addition of the LBC($1.6S(0)$) and LBP. The right panel indicates that guaranteed return component is driving the CTE value. Comparing the hedging loss statistics of PM2A, shown in Table 3.8, with those of PC4A, we see that PM2A has a significantly larger standard deviation, VaR and CTE, but a smaller mean and MSHL$^{1/2}$.

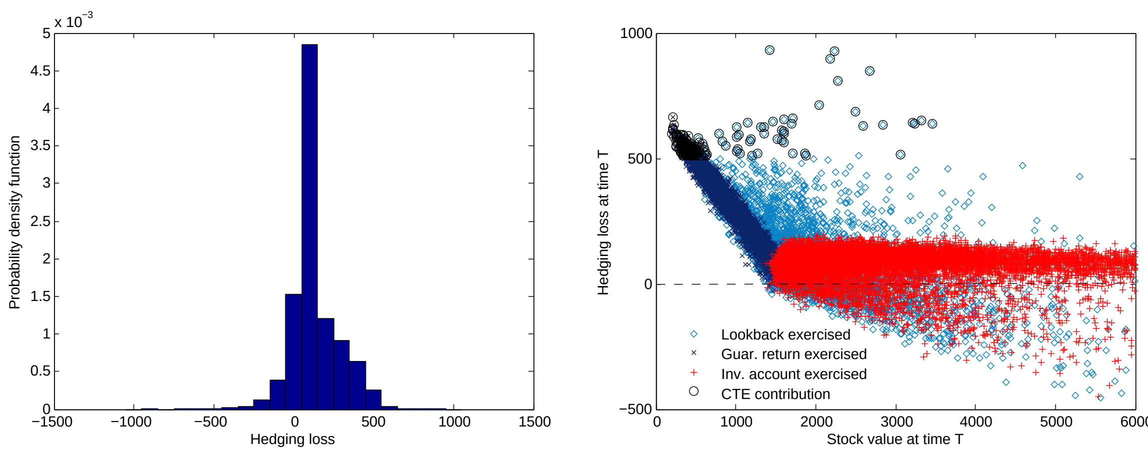

Figure 3.8: *The left panel displays the hedging loss distribution for Portfolio M2A. The right panel shows the simulated hedging losses e_n as functions of the stock index value at time T, $S(T)$. The e_n are individually marked according to which component is exercised.*

PM2B: optimizing with a mean constraint of 0

The MSHL minimization problem now includes the constraint given by equation (3.27), where $R = 0$. The left panel of Figure 3.9 indicates the hedging loss distribution for PM2B has a thicker right tail, compared to PM2A. Table 3.9 shows that the MSHL$^{1/2}$ has increased by 45%, and the standard deviation has doubled. The VaR and CTE have also

Port. Cost	**MSHL$^{1/2}$ =**	$\sqrt{\text{MS}}_{X_1}$	$+\sqrt{\text{MS}}_{X_2}$	$+\sqrt{\text{MS}}_{X_3}$.	**1%-quantile**	**Median**	**VaR(99%)**
1000	193	37	128	28	-205	106	510
	$(191, 195)$				$(-217, -192)$	$(105, 107)$	$(504, 518)$
Excess funds	**Std. Deviation =**	**SD**$_{X_1}$	**+ SD**$_{X_2}$	**+ SD**$_{X_3}$.		**Mean**	**CTE(99%)**
0	141	32	106	3		132	562
	$(139, 143)$					$(130, 134)$	$(551, 574)$
Instrument (k)	$\hat{x}(k)$	$\hat{w}(k)$	**Mean/CTE Contributions**			$i = M$	$i = C$
GMIB			$\bar{y}_1^i$ (Lookback)			433	743
			$\bar{y}_2^i$ (Guar. return)			369	1040
			$\bar{y}_3^i$ (Inv. account)			1585	0
			$\boldsymbol{A}^i = \bar{y}_1^i + \bar{y}_2^i + \bar{y}_3^i$ (GMIB total)			2387	1783
LBC($1.6S(0)$) (1)	0.427	0.126	$-\theta_1^i(1)$ (Lookback)			-103	-219
			$-\theta_2^i(1)$ (Guar. return)			-1	0
			$-\theta_3^i(1)$ (Inv. account)			-405	0
LBP (2)	0.064	0.014	$-\theta_1^i(2)$ (Lookback)			-10	-33
			$-\theta_2^i(2)$ (Guar. return)			-5	-31
			$-\theta_3^i(2)$ (Inv. account)			-2	0
Put($1.6S(0)$) (3)	0.055	0.016	$-\theta_1^i(3)$ (Lookback)			-1	-4
			$-\theta_2^i(3)$ (Guar. return)			-9	-50
			$-\theta_3^i(3)$ (Inv. account)			0	0
Stock (4)	0.487	0.489	$-\theta_1^i(4)$ (Lookback)			-196	-196
			$-\theta_2^i(4)$ (Guar. return)			-136	-142
			$-\theta_3^i(4)$ (Inv. account)			-861	0
ZCB(10) (5)	0.026	0.018	$-\theta_1^i(5)$ (Lookback)			-5	-7
			$-\theta_2^i(5)$ (Guar. return)			-7	-20
			$-\theta_3^i(5)$ (Inv. account)			-14	0
ZCB(29) (6)	1.332	0.337	$-\theta_1^i(6)$ (Lookback)			-99	-130
			$-\theta_2^i(6)$ (Guar. return)			-137	-388
			$-\theta_3^i(6)$ (Inv. account)			-266	0
			$\boldsymbol{B}^i = -\sum_{k=1}^{K}\sum_{j=1}^{3}\theta_j^i(k)$ (Hedge port. total)			-2255	-1221
			$\boldsymbol{C}^i = \bar{y}_1^i - \sum_{k=1}^{K}\theta_1^i(k)$ (Lookback total)			20	154
			$\boldsymbol{D}^i = \bar{y}_2^i - \sum_{k=1}^{K}\theta_2^i(k)$ (Guar. return total)			75	408
			$\boldsymbol{E}^i = \bar{y}_3^i - \sum_{k=1}^{K}\theta_3^i(k)$ (Inv. account total)			38	0
Mean = $\boldsymbol{A}^M + \boldsymbol{B}^M = \boldsymbol{C}^M + \boldsymbol{D}^M + \boldsymbol{E}^M$					**CTE(99%)** = $\boldsymbol{A}^C + \boldsymbol{B}^C = \boldsymbol{C}^C + \boldsymbol{D}^C + \boldsymbol{E}^C$		

Table 3.8: *Hedging loss statistics for Portfolio M2A.*

significantly increased. This is the trade-off between minimizing the MSHL and obtaining a mean of 0. The optimal instrument positions have changed for all of the instruments. In the right panel of Figure 3.9, all of the circled markers are guaranteed return markers. It seems that the MSHL is minimized, the CTE still remains relatively large because the guaranteed return component is not hedged effectively.

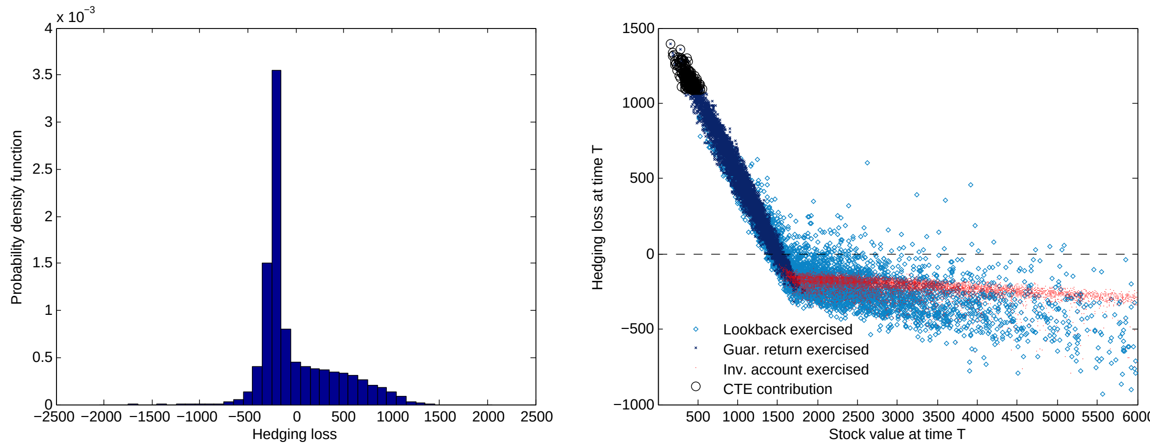

Figure 3.9: *The left panel displays the hedging loss distribution for Portfolio M2B. The right panel shows the simulated hedging losses e_n as functions of the stock index value at time T, $S(T)$. The e_n are individually marked according to which component is exercised.*

It appears that portfolios obtained from minimizing the MSHL have a much higher tail risk than the portfolios obtained from minimizing the CTE. Comparing PC4B to PM2B, PC4B is much more desirable as the hedging loss distribution does not exhibit a thick right tail, whereas PM2B does. We therefore conclude that static hedging portfolios constructed by minimizing the CTE are preferable to portfolios constructed by minimizing the MSHL.

3.8 Interest rate risk

Some readers may naturally think of the GMIB option as an interest rate option, as it is related to the annuity payment rate at maturity (which is a function of the interest rate term structure). In all of the portfolios we looked at thus far, we have largely focussed on the equity risk. The reason for this is because the equity risk associated with the GMIB option dominates the interest rate risk. To see why the interest rate risk is secondary to

Port. Cost	**MSHL$^{1/2}$ =**	$\sqrt{\textbf{MS}}_{X_1}$	$+\sqrt{\textbf{MS}}_{X_2}$	$+\sqrt{\textbf{MS}}_{X_3}$.	**1%-quantile**	**Median**	**VaR(99%)**
1000	280	38	184	58	-462	-102	813
	$(277, 283)$				$(-480, -448)$	$(-104, -101)$	$(800, 832)$
Excess funds	**Std. Deviation =**	$\textbf{SD}_{X_1}$	$+\ \textbf{SD}_{X_2}$	$+\ \textbf{SD}_{X_3}$.		**Mean**	**CTE(99%)**
0	280	38	184	58		0	888
	$(277, 283)$					$(-4, 4)$	$(875, 901)$
Instrument (k)	$\hat{x}(k)$	$\hat{w}(k)$	**Mean/CTE Contributions**			$i = M$	$i = C$
GMIB			$\bar{y}_1^i$ (Lookback)			433	0
			$\bar{y}_2^i$ (Guar. return)			369	1370
			$\bar{y}_3^i$ (Inv. account)			1585	0
			$\boldsymbol{A}^i = \bar{y}_1^i + \bar{y}_2^i + \bar{y}_3^i$ (GMIB total)			2387	1370
LBC$(1.6S(0))$ (1)	0.316	0.093	$-\theta_1^i(1)$ (Lookback)			-76	0
			$-\theta_2^i(1)$ (Guar. return)			-1	0
			$-\theta_3^i(1)$ (Inv. account)			-299	0
LBP (2)	0.186	0.041	$-\theta_1^i(2)$ (Lookback)			-28	0
			$-\theta_2^i(2)$ (Guar. return)			-14	-119
			$-\theta_3^i(2)$ (Inv. account)			-6	0
Put$(1.6S(0))$ (3)	-0.337	-0.100	$-\theta_1^i(3)$ (Lookback)			7	0
			$-\theta_2^i(3)$ (Guar. return)			53	406
			$-\theta_3^i(3)$ (Inv. account)			0	0
Stock (4)	0.616	0.619	$-\theta_1^i(4)$ (Lookback)			-248	0
			$-\theta_2^i(4)$ (Guar. return)			-171	-245
			$-\theta_3^i(4)$ (Inv. account)			-1087	0
ZCB(10) (5)	0.166	0.111	$-\theta_1^i(5)$ (Lookback)			-32	0
			$-\theta_2^i(5)$ (Guar. return)			-45	-166
			$-\theta_3^i(5)$ (Inv. account)			-88	0
ZCB(29) (6)	0.934	0.236	$-\theta_1^i(6)$ (Lookback)			-69	0
			$-\theta_2^i(6)$ (Guar. return)			-96	-357
			$-\theta_3^i(6)$ (Inv. account)			-186	0
			$\boldsymbol{B}^i = -\sum_{k=1}^{K}\sum_{j=1}^{3}\theta_j^i(k)$ (Hedge port. total)			-2387	-482
			$\boldsymbol{C}^i = \bar{y}_1^i - \sum_{k=1}^{K}\theta_1^i(k)$ (Lookback total)			-13	0
			$\boldsymbol{D}^i = \bar{y}_2^i - \sum_{k=1}^{K}\theta_2^i(k)$ (Guar. return total)			94	888
			$\boldsymbol{E}^i = \bar{y}_3^i - \sum_{k=1}^{K}\theta_3^i(k)$ (Inv. account total)			-82	0
Mean $= \boldsymbol{A}^M + \boldsymbol{B}^M = \boldsymbol{C}^M + \boldsymbol{D}^M + \boldsymbol{E}^M$					**CTE(99%)** $= \boldsymbol{A}^C + \boldsymbol{B}^C = \boldsymbol{C}^C + \boldsymbol{D}^C + \boldsymbol{E}^C$		

Table 3.9: *Hedging loss statistics for Portfolio M2B.*

the equity risk, consider Figure 3.10. The left panel plots the hedging losses as a function of the short rate at maturity, $r(T)$, for PC3. Similarly, the right panel shows PC4B. In both cases, there is no discernable pattern between $r(T)$ and the different types of markers. All three types of markers are spread across all plausible values of the short rate at maturity, $r(T)$. The majority of the markers are clustered around $E^P[r(T)] = 5.35\%$. Large outlying hedging losses are driven by the behavior of equity returns, not the behavior of the short rate.

Although not documented here, we have tested including standard interest rate related options within Hull-White model, such as caps and floors. They do not help reduce the CTE by any noticeable amount. However, we have not considered the use of hybrid equity-interest rate type options. For example, knock out put options, where the knock out feature is related to some aspect of the interest rate term structure (such as the cash rate rising above a certain level), may be useful as they are cheaper than standard put options. The GMIB option is generally less valuable when interest rates increase, all else being equal. Therefore, knock out features may be related to increases in interest rates. If hybrid options were considered, it is advisable to use a more sophisticated interest rate model (for a survey of standard interest rate models, see Brigo and Mercurio (2006)).

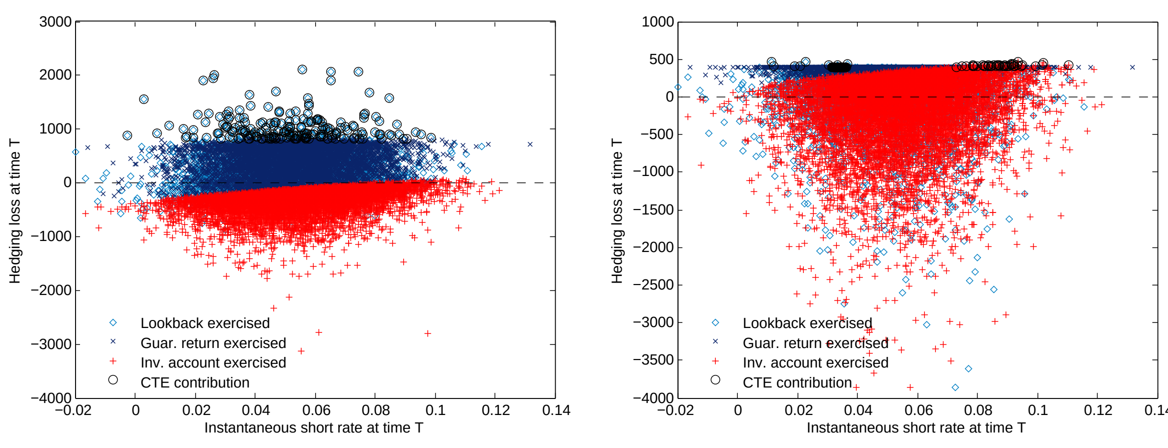

Figure 3.10: *The right (left) panel shows the simulated hedging losses e_n for PC3 (PC4B) as functions of the short rate at maturity, $r(T)$. The e_n are individually marked according to which component is exercised.*

3.9 Hedging simplified GMIBs

The GMIB option has a complex benefit structure. It provides a downside equity guarantee (guaranteed return component) and an upside equity guarantee (lookback component). In this section, we demonstrate that if the benefit structure of the GMIB is simplified, then it is easier to hedge with a static portfolio. We illustrate the hedging loss distributions for two simplified versions of the GMIB. The first version, referred to as the *embedded lookback option*, has a maturity value given by $Y_l(T) = \max(X_1, A(0)g\ddot{a}_{\overline{20}|}(T), X_3)$. The second version, referred to as the *embedded guaranteed return option*, has a maturity value given by $Y_{gr}(T) = \max(X_2, X_3)$. The hedging portfolios are designed to minimize the CTE, subject to a mean constraint of 0. The hedging instruments are those of PC4B. To be consistent with the previous examples, the same fee structure is used; note that the fee rate of 1% is now being applied to less valuable options. The hedging loss statistics in this section can be referenced against the statistics for PC4B.

3.9.1 Hedging the embedded lookback option

This simplified GMIB does not contain the guaranteed return component. We recognize that this simplified option may be unattractive to many policyholders because the (implicit) downside equity guarantee will be significantly lower if there is a downward trend in the stock over the accumulation phase. However, it is of interest to see how well the upward equity guarantee can be hedged with a static portfolio, when the hedging portfolio does not have to be concerned about allocating resources to hedge the guaranteed return component.

We refer to the optimized portfolio as Portfolio E1 (PE1). In Figure 3.11, the left panel indicates the hedging loss distribution has a left tail, but no right tail. All of the hedging loss statistics for PE1, shown in Table 3.10, are smaller than the corresponding statistics for PC4B. In particular, the CTE has decreased by 64%. The locations of the different types of markers in the right panel are similar to those in the corresponding right panel for PC4B (Figure 3.6). There are no large hedging loss outliers, and the VaR and CTE are close in value, which suggests that the embedded lookback option is hedged well using the available instruments. It seems that when $X_2 = 0$, the LBP effectively hedges any large GMIB maturity values generated by the lookback component, and LBC($1.6S(0)$) is not needed for hedging this component. The reverse situation occurs for PC4B; in PC4B a large position in LBC($1.6S(0)$) is held, while the LBP position is 0.

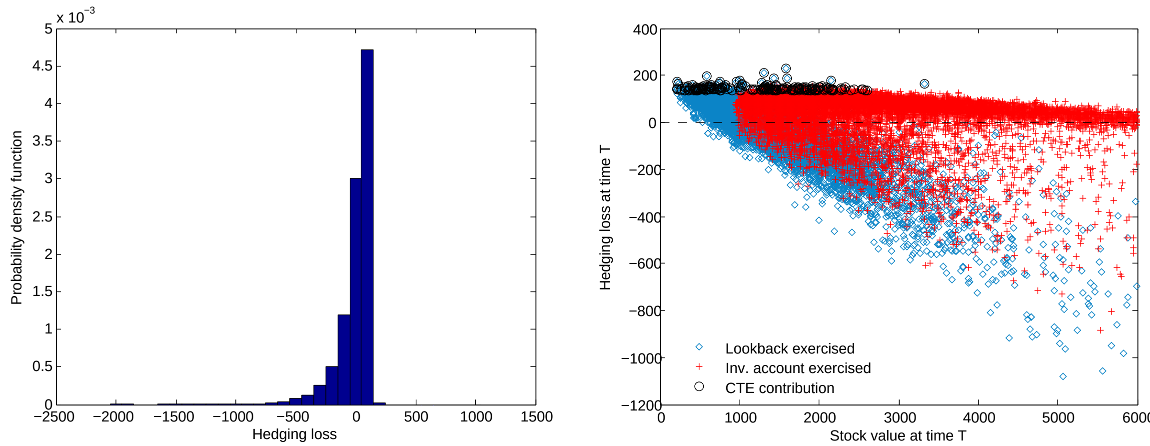

Figure 3.11: *The left panel displays the hedging loss distribution for Portfolio E1. The right panel shows the simulated hedging losses e_n as functions of the stock index value at time T, $S(T)$. The e_n are individually marked according to which component is exercised.*

3.9.2 Hedging the embedded guaranteed return option

This simplified GMIB does not contain the lookback component. We refer to the optimized portfolio as Portfolio E2 (PE2). The left panel of Figure 3.12 shows that the hedging loss distribution for PE2 has a left tail, but no right tail. Table 3.11 displays the hedging loss statistics. All of the hedging loss statistics are lower than the corresponding statistics for PC4B, but not by as much as those for PE1. In the right panel of Figure 3.12, the majority of circled markers are guaranteed return markers. Similarly to the hedging loss distribution for PE1, there are no large hedging loss outliers, and the VaR and CTE are close in value, suggesting that the embedded guaranteed return option is hedged reasonably well using the available instruments. However, the CTE is still relatively large. It appears that more funds are needed to reduce the CTE further.

The value of the CTE is dependent on the value of the guaranteed return component X_2, which is extremely sensitive to the assumed ZCB yield curve structure. Now, X_2 decreases

Port. Cost	$\mathbf{MSHL}^{1/2} =$	$\sqrt{\mathbf{MS}}_{X_1}$	$+\sqrt{\mathbf{MS}}_{X_2}$	$+\sqrt{\mathbf{MS}}_{X_3}$.	**1%-quantile**	**Median**	**VaR(99%)**
1000	140	82	0	58	-534	44	136
	$(135, 144)$				$(-559, -513)$	$(43, 46)$	$(135, 137)$
Excess funds	**Std. Deviation =**	$\mathbf{SD}_{X_1}$	$+ \mathbf{SD}_{X_2}$	$+ \mathbf{SD}_{X_3}$.		**Mean**	**CTE(99%)**
0	140	82	0	58		0	146
	$(135, 144)$					$(-2, 2)$	$(144, 148)$
Instrument (k)	$\hat{x}(k)$	$\hat{w}(k)$	**Mean/CTE Contributions**			$i = M$	$i = C$
GMIB			$\bar{y}_1^i$ (Lookback)			626	639
			$\bar{y}_2^i$ (Guar. return)			0	0
			$\bar{y}_3^i$ (Inv. account)			1681	791
			$\boldsymbol{A}^i = \bar{y}_1^i + \bar{y}_2^i + \bar{y}_3^i$ (GMIB total)			2308	1430
LBC($1.6S(0)$) (1)	-0.054	-0.016	$-\theta_1^i(1)$ (Lookback)			13	6
			$-\theta_2^i(1)$ (Guar. return)			0	0
			$-\theta_3^i(1)$ (Inv. account)			51	7
LBP (2)	0.848	0.187	$-\theta_1^i(2)$ (Lookback)			-189	-304
			$-\theta_2^i(2)$ (Guar. return)			0	0
			$-\theta_3^i(2)$ (Inv. account)			-31	0
Put($1.3S(0)$) (3)	0.080	0.014	$-\theta_1^i(3)$ (Lookback)			-7	-25
			$-\theta_2^i(3)$ (Guar. return)			0	0
			$-\theta_3^i(3)$ (Inv. account)			0	0
Stock (4)	0.983	0.990	$-\theta_1^i(4)$ (Lookback)			-560	-357
			$-\theta_2^i(4)$ (Guar. return)			0	0
			$-\theta_3^i(4)$ (Inv. account)			-1845	-870
ZCB(10) (5)	-0.553	-0.372	$-\theta_1^i(5)$ (Lookback)			212	277
			$-\theta_2^i(5)$ (Guar. return)			0	0
			$-\theta_3^i(5)$ (Inv. account)			341	277
ZCB(29) (6)	0.777	0.197	$-\theta_1^i(6)$ (Lookback)			-113	-161
			$-\theta_2^i(6)$ (Guar. return)			0	0
			$-\theta_3^i(6)$ (Inv. account)			-179	-133
			$\boldsymbol{B}^i = -\sum_{k=1}^K \sum_{j=1}^3 \theta_j^i(k)$ (Hedge port. total)			-2308	-1285
			$\boldsymbol{C}^i = \bar{y}_1^i - \sum_{k=1}^K \theta_1^i(k)$ (Lookback total)			-18	75
			$\boldsymbol{D}^i = \bar{y}_2^i - \sum_{k=1}^K \theta_2^i(k)$ (Guar. return total)			0	0
			$\boldsymbol{E}^i = \bar{y}_3^i - \sum_{k=1}^K \theta_3^i(k)$ (Inv. account total)			18	71
Mean $= \boldsymbol{A}^M + \boldsymbol{B}^M = \boldsymbol{C}^M + \boldsymbol{D}^M + \boldsymbol{E}^M$					**CTE(99%)** $= \boldsymbol{A}^C + \boldsymbol{B}^C = \boldsymbol{C}^C + \boldsymbol{D}^C + \boldsymbol{E}^C$		

Table 3.10: *Hedging loss statistics for Portfolio E1.*

as the yield curve shifts upwards. In all of our examples, we have assumed the ZCB yield curve structure follows the Benchmark curve in Figure 2.7 (in the current economic environment, this curve is very plausible). If the fitted ZCB curve was changed to, say, the ZCB curve in 2007, which is higher (see Figure 3.15), then the prices of ZCB(10) and ZCB(29) would be cheaper; more money can be invested in the bond instruments. The maturity value of the embedded guaranteed return option will also decrease (we assume interest rates and the stock returns are independent under P). The CTE would decrease. In fact, using the 2007 curve, the CTE is actually negative (about -200). Profits are highly likely within the model, since the ZCB yields are about 5.25-5.5% p.a. across all maturities while the guaranteed return rate r_g is 5% per annum. The positions in ZCB(10) and ZCB(29) increase, while the other instrument positions remain about the same. From Table 3.11 we see that the LBP is not useful for the guaranteed return option. However, the position in LBC($1.6S(0)$) is 0.592 units, which suggests that it contributes significantly to hedging large embedded guaranteed return option maturity values. The spreads between the guaranteed rate r_g and the 10 and 29 year ZCB yields heavily influence the location of the hedging loss distribution, and in particular, the value of the CTE.

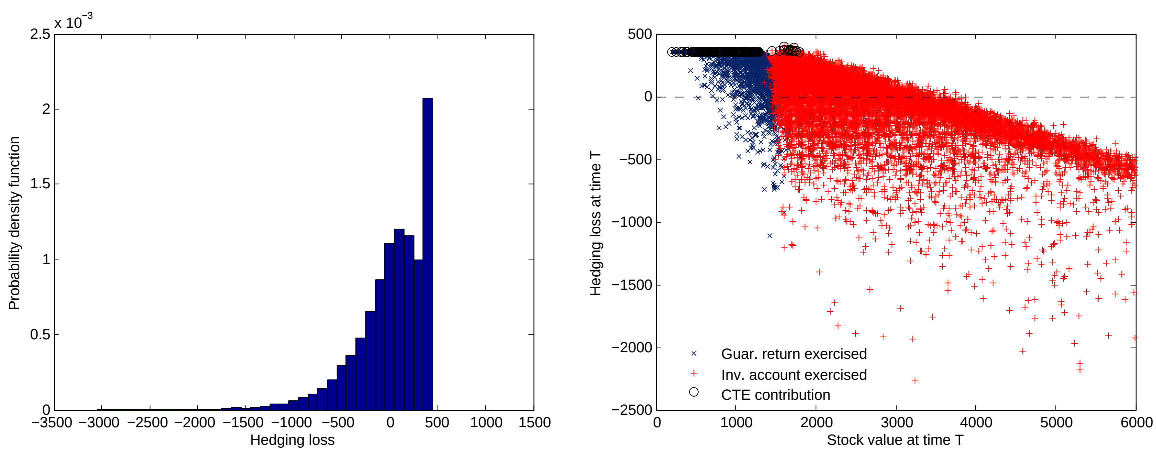

Figure 3.12: *The left panel displays the hedging loss distribution for Portfolio E2. The right panel shows the simulated hedging losses e_n as functions of the stock index value at time T, $S(T)$. The e_n are individually marked according to which component is exercised.*

The results in this section send a message to insurers regarding option policy design. If the GMIBs had simpler benefit structures, by providing a downside guarantee (guaranteed return component) or an upside guarantee (lookback component), but not both, then

Port. Cost	**MSHL$^{1/2}$ =**	$\sqrt{\textbf{MS}}_{X_1}$	$+\sqrt{\textbf{MS}}_{X_2}$	$+\sqrt{\textbf{MS}}_{X_3}$.	**1%-quantile**	**Median**	**VaR(99%)**
1000	382	0	91	292	-1308	87	359
	$(373, 391)$				$(-1372, -1262)$	$(82, 93)$	$(359, 359)$
Excess funds	**Std. Deviation =**	$\textbf{SD}_{X_1}$	$+\ \textbf{SD}_{X_2}$	$+\ \textbf{SD}_{X_3}$.		**Mean**	**CTE(99%)**
0	382	0	91	292		0	360
	$(373, 391)$					$(-5, 5)$	$(359, 361)$

Instrument (k)	$\hat{x}(k)$	$\hat{w}(k)$	**Mean/CTE Contributions**	$i = M$	$i = C$
GMIB			$\bar{y}_1^i$ (Lookback)	0	0
			$\bar{y}_2^i$ (Guar. return)	452	1274
			$\bar{y}_3^i$ (Inv. account)	1876	104
			$\boldsymbol{A}^i = \bar{y}_1^i + \bar{y}_2^i + \bar{y}_3^i$ (GMIB total)	2328	1378
LBC(1.6$S(0)$) (1)	0.592	0.175	$-\theta_1^i(1)$ (Lookback)	0	0
			$-\theta_2^i(1)$ (Guar. return)	-17	0
			$-\theta_3^i(1)$ (Inv. account)	-687	-3
LBP (2)	0.000	0.000	$-\theta_1^i(2)$ (Lookback)	0	0
			$-\theta_2^i(2)$ (Guar. return)	0	0
			$-\theta_3^i(2)$ (Inv. account)	0	0
Put(1.3$S(0)$) (3)	0.510	0.089	$-\theta_1^i(3)$ (Lookback)	0	0
			$-\theta_2^i(3)$ (Guar. return)	-45	-186
			$-\theta_3^i(3)$ (Inv. account)	0	0
Stock (4)	0.510	0.513	$-\theta_1^i(4)$ (Lookback)	0	0
			$-\theta_2^i(4)$ (Guar. return)	-181	-431
			$-\theta_3^i(4)$ (Inv. account)	-1067	-59
ZCB(10) (5)	-0.703	-0.472	$-\theta_1^i(5)$ (Lookback)	0	0
			$-\theta_2^i(5)$ (Guar. return)	234	653
			$-\theta_3^i(5)$ (Inv. account)	469	49
ZCB(29) (6)	2.745	0.695	$-\theta_1^i(6)$ (Lookback)	0	0
			$-\theta_2^i(6)$ (Guar. return)	-345	-977
			$-\theta_3^i(6)$ (Inv. account)	-688	-64
			$\boldsymbol{B}^i = -\sum_{k=1}^{K}\sum_{j=1}^{3}\theta_j^i(k)$ (Hedge port. total)	-2328	-1018
			$\boldsymbol{C}^i = \bar{y}_1^i - \sum_{k=1}^{K}\theta_1^i(k)$ (Lookback total)	0	0
			$\boldsymbol{D}^i = \bar{y}_2^i - \sum_{k=1}^{K}\theta_2^i(k)$ (Guar. return total)	97	334
			$\boldsymbol{E}^i = \bar{y}_3^i - \sum_{k=1}^{K}\theta_3^i(k)$ (Inv. account total)	-97	26
Mean $= \boldsymbol{A}^M + \boldsymbol{B}^M = \boldsymbol{C}^M + \boldsymbol{D}^M + \boldsymbol{E}^M$			**CTE(99%)** $= \boldsymbol{A}^C + \boldsymbol{B}^C = \boldsymbol{C}^C + \boldsymbol{D}^C + \boldsymbol{E}^C$		

Table 3.11: *Hedging loss statistics for Portfolio E2.*

static hedges are more effective. The right tails are cut off. However, we see that the guaranteed return component is very valuable when interest rates are low. In particular, this component is difficult to hedge in the current economic environment, with a budget constraint of $\pi = 1000$. Hopefully insurance companies, when designing long-dated embedded options in their products, carefully take into consideration the difficulties involved with developing static-type hedges for the options, particularly when they must simultaneously hedge equity risks at both ends of the spectrum. Effective risk management starts at the product design phase.

3.10 Charging the fair fee rate

In all of the results illustrated thus far, we have assumed that the fee rate for the GMIB option is $c = 1\%$ per year. However, using the valuation model in Chapter 2, we found that the fair fee rate is 4.5% when $g = 6.5\%$. The notion of a fair fee rate for a static hedge is slightly different to that using the no-arbitrage pricing model in Chapter 2. In Chapter 2, the fair fee rate was defined as the rate that should be charged in a perfect world where the standard assumptions of option pricing theory hold. Using the fair fee rate, the insurer will break even. In practice, those assumptions do not hold. The results of Chapter 2 should be considered as a benchmark, but do not provide the final answer for making decisions in practice. A fair fee rate for a static hedging strategy is a fee rate that the insurer is comfortable with charging, which depends on its risk and expected loss/profit preferences. Other considerations also come into play, such as marketing considerations (e.g. a lower fee rate will attract more annuity business). Given that there is a range of fee rates which might be considered fair by the insurer, we use the fair fee rate of Chapter 2 as our benchmark "fair" fee rate in this section (and in Chapter 4).

We now illustrate how the performances of the static hedging portfolios change if the fair fee rate is charged. As we discussed in Chapter 2, it is not clear whether insurers are charging a fair fee rate for the GMIB option; allowing for lapses or additional underlying variable annuity contract fees reduces the fair fee rate by noticeable amounts. When the fair rate changes, the real-world probabilities of exercising the lookback, guaranteed return and investment account components change. Table 3.12 shows the real-world exercise probabilities for the benchmark case, where $g = 6.5\%$ and $c = 1\%$, and for the fair fee case, where $g = 6.5\%$ and $c = 4.5\%$.

	g	c	p_{X_1}	p_{X_2}	p_{X_3}
Benchmark case	6.5%	1%	0.20	0.27	0.53
Fair fee case	6.5%	4.5%	0.23	0.49	0.28

Table 3.12: *Real-world probabilities of the lookback, guaranteed return and investment account components being exercised for different GMIB contract parameter values.*

We consider two portfolios. Both portfolios are designed to minimize the CTE, without a mean constraint. The first portfolio we consider includes a put option, the stock, and ZCBs. The instruments in this portfolio are of the same type as PC3. We refer to the optimized portfolio as Portfolio F1 (PF1). Table 3.13 displays the hedging loss statistics and the optimal hedging instruments (including optimal strike prices) for PF1. Comparing the results in this table to the results for PC3 (Table 3.4) gives a sense of how the static hedge changes when the fair fee is charged. The mean is -382, so there is no obvious need to introduce a mean constraint, if the insurer measures risk by the CTE. Introducing a mean constraint different to -382 will lead to a larger minimized CTE. If the fair fee rate is charged, the insurer can expect to make a profit. However, the standard deviation and MSHL$^{1/2}$ are large. Both statistics are driven by the investment account component, which generates large hedging profits that are consistently far away from the mean hedging loss. In the left panel of Figure 3.13, we see that the distribution has several local maxima, with noticeable left and right tails. In the right panel, the majority of circled markers are clearly lookback markers. Clearly, even when the fair fee rate is charged, it is still difficult to hedge the lookback component without a lookback option.

The instruments in the second portfolio are of the same type as PC4A/PC4B. We refer to the optimized portfolio as Portfolio F2 (PF2). Table 3.14 displays the hedging loss statistics and the optimal hedging instruments (including optimal strike prices) for PF2. Again we see the same story as for PC4A and PC4B. The LBC is very effective at hedging the lookback component, which when not hedged, produces large CTE values. Given that there is no mean constraint, PF2 can be directly compared to PC4A. We see that when the fair fee rate is charged a profit of 122 is expected, when the objective is to simply to minimize the CTE. Furthermore, the 1%-quantile is much smaller than for PC4A.

Port. Cost	**MSHL**$^{1/2}$ =	$\sqrt{\textbf{MS}}_{X_1}$	+ $\sqrt{\textbf{MS}}_{X_2}$	+$\sqrt{\textbf{MS}}_{X_3}$.	**1%-quantile**	**Median**	**VaR(99%)**
1000	707	101	101	505	-1838	-358	478
	$(700, 715)$				$(-1893, -1792)$	$(-371, -343)$	$(478, 490)$
Excess funds	**Std. Deviation =**	**SD**$_{X_1}$	+ **SD**$_{X_2}$	+ **SD**$_{X_3}$.		**Mean**	**CTE(99%)**
0	595	69	117	409		-382	646
	$(589, 601)$					$(-391, -374)$	$(610, 683)$
Instrument (k)	$\hat{x}(k)$	$\hat{w}(k)$	**Mean/CTE Contributions**			$i = M$	$i = C$
GMIB			$\bar{y}_1^i$ (Lookback)			471	2033
			$\bar{y}_2^i$ (Guar. return)			664	119
			$\bar{y}_3^i$ (Inv. account)			715	0
			$\boldsymbol{A}^i = \bar{y}_1^i + \bar{y}_2^i + \bar{y}_3^i$ (GMIB total)			1850	2152
Put($0.8S(0)$) (1)	0.803	0.032	$-\theta_1^i(1)$ (Lookback)			0	-12
			$-\theta_2^i(1)$ (Guar. return)			-10	-35
			$-\theta_3^i(1)$ (Inv. account)			0	0
Stock (2)	0.807	0.812	$-\theta_1^i(2)$ (Lookback)			-495	-1202
			$-\theta_2^i(2)$ (Guar. return)			-545	-19
			$-\theta_3^i(2)$ (Inv. account)			-951	0
ZCB(10) (3)	-0.794	-0.533	$-\theta_1^i(3)$ (Lookback)			186	726
			$-\theta_2^i(3)$ (Guar. return)			390	67
			$-\theta_3^i(3)$ (Inv. account)			218	0
ZCB(29) (4)	2.724	0.690	$-\theta_1^i(4)$ (Lookback)			-241	-940
			$-\theta_2^i(4)$ (Guar. return)			-503	-91
			$-\theta_3^i(4)$ (Inv. account)			-280	0
			$\boldsymbol{B}^i = -\sum_{k=1}^K \sum_{j=1}^3 \theta_j^i(k)$ (Hedge port. total)			-2232	-1506
			$\boldsymbol{C}^i = \bar{y}_1^i - \sum_{k=1}^K \theta_1^i(k)$ (Lookback total)			-80	605
			$\boldsymbol{D}^i = \bar{y}_2^i - \sum_{k=1}^K \theta_2^i(k)$ (Guar. return total)			-5	41
			$\boldsymbol{E}^i = \bar{y}_3^i - \sum_{k=1}^K \theta_3^i(k)$ (Inv. account total)			-298	0
Mean $= \boldsymbol{A}^M + \boldsymbol{B}^M = \boldsymbol{C}^M + \boldsymbol{D}^M + \boldsymbol{E}^M$				**CTE(99%)** $= \boldsymbol{A}^C + \boldsymbol{B}^C = \boldsymbol{C}^C + \boldsymbol{D}^C + \boldsymbol{E}^C$			

Table 3.13: *Hedging loss statistics for Portfolio F1.*

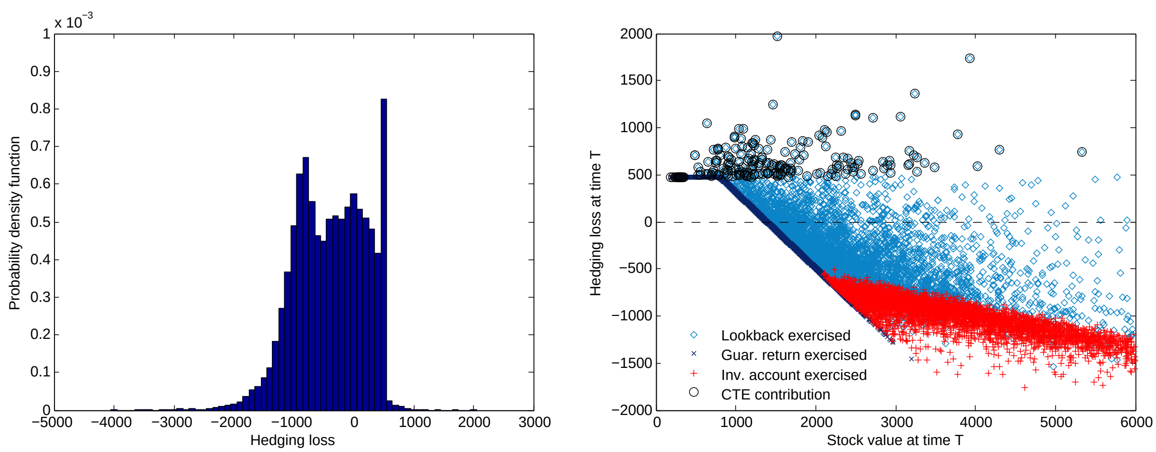

Figure 3.13: *Hedging loss statistics for Portfolio F1.*

Compared to PF1, PF2 generates a smaller standard deviation and a smaller MSHL$^{1/2}$. Overall the hedging loss distribution for PF2 would appear to be more desirable than the distribution for PF1, in terms of risk and reward, for most insurers.

Port. Cost	**MSHL**$^{1/2}$ =	$\sqrt{\mathbf{MS}}_{X_1}$	+ $\sqrt{\mathbf{MS}}_{X_2}$	+$\sqrt{\mathbf{MS}}_{X_3}$.	**1%-quantile**	**Median**	**VaR(99%)**
1000	245	82	50	113	-600	-148	137
	$(242, 247)$				$(-619, -585)$	$(-153, -143)$	$(137, 137)$
Excess funds	**Std. Deviation** =	**SD**$_{X_1}$	+ **SD**$_{X_2}$	+ **SD**$_{X_3}$.		**Mean**	**CTE(99%)**
0	212	64	65	83		-122	149
	$(210, 213)$					$(-125, -119)$	$(145, 154)$
Instrument (k)	$\hat{x}(k)$	$\hat{w}(k)$		**Mean/CTE Contributions**		$i = M$	$i = C$
GMIB				$\bar{y}_1^i$ (Lookback)		471	681
				$\bar{y}_2^i$ (Guar. return)		664	953
				$\bar{y}_3^i$ (Inv. account)		715	0
				$\boldsymbol{A}^i = \bar{y}_1^i + \bar{y}_2^i + \bar{y}_3^i$ (GMIB total)		1850	1634
LBC($1.6S(0)$)	0.623	0.183		$-\theta_1^i(1)$ (Lookback)		-237	-264
(1)				$-\theta_2^i(1)$ (Guar. return)		-53	0
				$-\theta_3^i(1)$ (Inv. account)		-464	0
LBP (2)	0.012	0.003		$-\theta_1^i(2)$ (Lookback)		-2	-3
				$-\theta_2^i(2)$ (Guar. return)		-1	-1
				$-\theta_3^i(2)$ (Inv. account)		0	0
Put($0.8S(0)$)	-0.017	-0.001		$-\theta_1^i(3)$ (Lookback)		0	0
(3)				$-\theta_2^i(3)$ (Guar. return)		0	1
				$-\theta_3^i(3)$ (Inv. account)		0	0
Stock (4)	0.000	0.000		$-\theta_1^i(4)$ (Lookback)		0	0
				$-\theta_2^i(4)$ (Guar. return)		0	0
				$-\theta_3^i(4)$ (Inv. account)		0	0
ZCB(10) (5)	0.180	0.120		$-\theta_1^i(5)$ (Lookback)		-42	-52
				$-\theta_2^i(5)$ (Guar. return)		-88	-128
				$-\theta_3^i(5)$ (Inv. account)		-49	0
ZCB(29) (6)	2.754	0.695		$-\theta_1^i(6)$ (Lookback)		-243	-310
				$-\theta_2^i(6)$ (Guar. return)		-509	-728
				$-\theta_3^i(6)$ (Inv. account)		-283	0
				$\boldsymbol{B}^i = -\sum_{k=1}^{K}\sum_{j=1}^{3}\theta_j^i(k)$ (Hedge port. total)		-1972	-1484
				$\boldsymbol{C}^i = \bar{y}_1^i - \sum_{k=1}^{K}\theta_1^i(k)$ (Lookback total)		-54	52
				$\boldsymbol{D}^i = \bar{y}_2^i - \sum_{k=1}^{K}\theta_2^i(k)$ (Guar. return total)		13	97
				$\boldsymbol{E}^i = \bar{y}_3^i - \sum_{k=1}^{K}\theta_3^i(k)$ (Inv. account total)		-82	0
Mean = $\boldsymbol{A}^M + \boldsymbol{B}^M = \boldsymbol{C}^M + \boldsymbol{D}^M + \boldsymbol{E}^M$				**CTE(99%)** = $\boldsymbol{A}^C + \boldsymbol{B}^C = \boldsymbol{C}^C + \boldsymbol{D}^C + \boldsymbol{E}^C$			

Table 3.14: *Hedging loss statistics for Portfolio F2.*

3.11 Backtesting the static hedging strategy

In this section, we backtest the performance of a static hedging portfolio which includes lookback options. The first GMIB option was introduced in the U.S. variable annuity market in 1996-1997. GMIBs typically have a waiting period of 10 years before exercise is possible. We assume the GMIB is exercised on the 10-th policy anniversary in this backtest. We are able to determine the actual hedging loss/profit of the static strategies

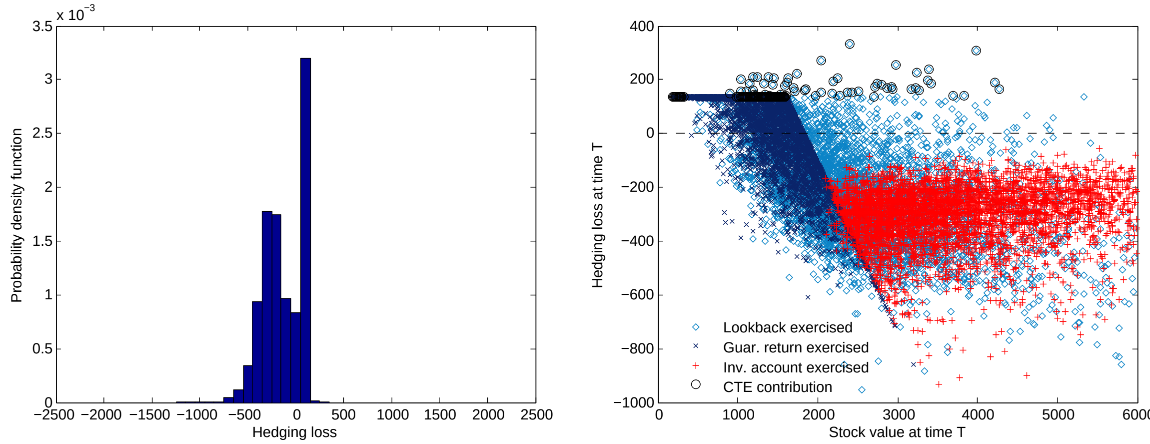

Figure 3.14: *Hedging loss statistics for Portfolio F2.*

for GMIBs issued between 1997 and 2001. For GMIBs issued in 2002 and beyond, we can measure the performance of the static strategy at the start of 2011, based on the hedging portfolio and liability values at that time.

We calculate static hedging portfolios for GMIBs issued at the start of each year. The fee rate is set at 1% for all issue years. The available hedging instruments include the LBC(1.7$S(0)$), LBP, Put(1.3$S(0)$), stock, ZCB(10) and ZCB(29). Each portfolio is optimized to minimize the CTE(99%) subject to a mean constraint of 0. Most insurers are likely to want the hedging strategy to at least break-even on average, even if this results in a slightly higher CTE. The portfolios are optimized at the start of each issue year based on the observed ZCB yield curve at that time (the one-factor interest rate model is fitted to the prevailing ZCB yield curve). Over the period 1997-2011, a broad spectrum of ZCB yield curve shapes existed in the U.S. Figure 3.15 displays a selection of historical yield curves (all calculated at the start of the calendar year), capturing the range of shapes observed between 1997-2011. Of particular noteworthiness is the yield curve shape at the start of 2011. This curve is an input in determining the hedging portfolio and liability values as at 2011. All other model parameters are set equal to the benchmark parameter assumptions. Hence, the optimal instrument positions for each issue year will vary primarily because of different ZCB yield curve assumptions (the other source of variation comes from the selection of the scenarios in the optimization problem). Table 3.15 displays the optimal portfolio instrument positions for each issue year.

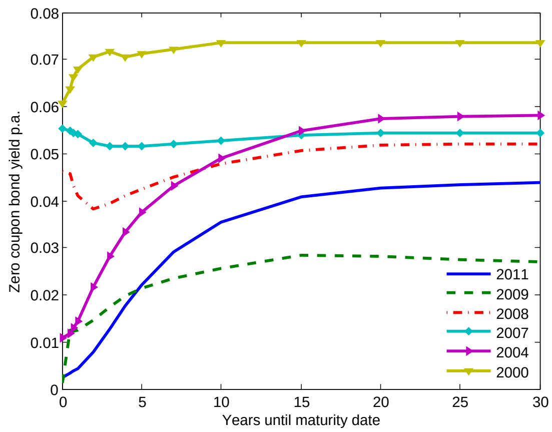

Figure 3.15: *U.S. zero coupon bond yield curves for a selection of calendar years.*

Time 0	**Time** T	**LBC**$(1.6S(0))$	**LBP**	**Put**$(1.3S(0))$	**Stock**	**ZCB(10)**	**ZCB(29)**
1997	2007	0.59	0.05	0.28	0.38	0.13	1.72
1998	2008	0.57	0.06	0.32	0.40	0.15	1.52
1999	2009	0.71	0.01	0.36	0.38	-0.31	2.43
2000	2010	0.40	0.16	0.36	0.51	0.47	0.13
2001	2011	0.63	0.03	0.29	0.35	0.26	1.39
2002	2012	0.62	0.03	0.32	0.37	0.17	1.54
2003	2013	0.60	0.05	0.37	0.46	-0.27	2.22
2004	2014	0.58	0.07	0.37	0.45	-0.06	1.72
2005	2015	0.69	0.00	0.36	0.39	-0.15	2.06
2006	2016	0.76	0.00	0.40	0.40	-0.57	2.75
2007	2017	0.71	0.00	0.41	0.41	-0.49	2.69
2008	2018	0.74	0.00	0.42	0.42	-0.56	2.73
2009	2019	1.52	0.00	0.00	0.00	-0.49	2.18
2010	2020	0.79	0.00	0.40	0.40	-0.60	2.66
2011	2021	0.87	0.00	0.33	0.33	-0.55	2.61

Table 3.15: *Optimal hedging instrument positions at time 0 for GMIBs issued from 1997 to 2011.*

In this backtest, the investment returns of the policyholder's investment account are assumed to match the returns on the S&P 500 Total Return Index. Furthermore, we assume that 10-year put and lookback options on the S&P 500 Total Return Index are available. Figure 3.16 displays the evolution of the S&P 500 Total Return Index over the period of interest. The circles on the curve denote the GMIB issue dates and policy anniversaries. The sharp drops in the index correspond to the dot-com bubble crash (starting in 2000), and the credit crunch (starting in 2007).

For GMIBs issued in 1997-2001, the hedging losses are known. For GMIBs issued in 2002

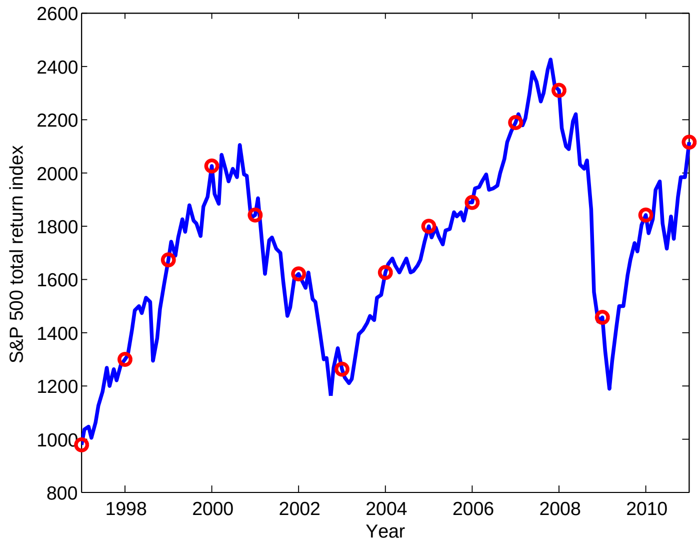

Figure 3.16: *Evolution of the S&P 500 Total Return Index from the start of 1997 to the end of 2011.*

and beyond, we simulate the hedging loss distributions as at 1 January 2011 conditional on past experience. This gives us a forecast of how the hedging strategy is expected to perform when the option is exercised. For a GMIB issued at the start of year $s \leqslant 2011$, the hedging losses, valued at 1 Jan 2011, are calculated as

$$e_{n,(2011)} = y_{n,(2011)} - \boldsymbol{z}'_{n,(2011)}\boldsymbol{x}_s, \qquad n = 1, \ldots, N,$$

where:

- $y_{n,(2011)}$ is the GMIB maturity value for the n-th scenario, valued at 1 Jan 2011 conditional on historical experience.
- $\boldsymbol{z}_{n,(2011)}$ is the vector of hedging instrument payoffs in year T for the n-th scenario, valued at 1 Jan 2011 conditional on historical experience.
- $\boldsymbol{x}_s$ is the vector of the optimal instrument positions set at the start of year s.

The top portion of Table 3.16 displays the actual hedging losses for GMIBs issued in 2001 or earlier. The bottom portion of the table shows the hedging loss statistics at 1 Jan 2011 for GMIBs issued in the last decade. For each issue year, the value of the policyholder's investment account as at 2011 is shown. Furthermore, the real-world probabilities of exercising the lookback, guaranteed return and investment account components, conditional on actual experience up to 2011, are reported for each issue year.

Studying Table 3.16 in conjunction with Figure 3.16 helps explain the hedging loss statistics. It is difficult to make any general statements based on Table 3.16. There are a number of factors influencing the performance of each hedging strategy including the ZCB yield curve shape in the issue year, the prevailing interest rates at the maturity date, and the peaks and troughs in the index over time. Notwithstanding, we make the following observations:

- The static hedge occasionally produces large losses and profits, as seen for issue years 1997 and 1999.
- The real world probabilities for each issue year make sense when one looks at Figure 3.16.
- In all issue years except 2009, the right tail risks have been hedged effectively. The sharp appreciation of the index since 2009 has generated the large tail risk measures for GMIBs issued in 2009.
- The static hedges tend to produce negative mean hedging losses when the index rises sharply over the accumulation phase, as seen for issue years 2003, 2009, and 2010. However, this is not always the case, as seen for issue year 1997, where the index did rise significantly over the accumulation phase.

Overall, the static hedges perform reasonably well, allowing for the fact that the GMIB option is underpriced. A variety of economic conditions were experienced between 1997-2011, and the static hedges appear to be robust in the majority of circumstances. Charging the fair fee will improve the performance of the static strategies in this backtest. However, we do not explore the impact of charging the fair fee rate, because it will vary with each issue year due to different ZCB yield curve assumptions. For an indication of how the yield curve shape impacts the fair fee rate, see Section 2.4.2.

Time 0	Time T	MSHL$^{1/2}$	1%-quantile	Median	VaR(99%)	Std dev	Mean	CTE(99%)	$A(2011)$	p_{X_1}	p_{X_2}	p_{X_3}
1997	2007						381			0	0	1
1998	2008						127			0	0	1
1999	2009						-452			0	1	0
2000	2010						50			0	1	0
2001	2011						-71			0	1	0
2002	2012	95	-100	95	148	63	71	154	1169	0.00	0.83	0.17
2003	2013	109	-334	-35	94	98	-47	101	1531	0.41	0.00	0.59
2004	2014	94	-233	54	147	89	32	156	1201	0.07	0.47	0.46
2005	2015	131	-358	102	175	120	52	184	1094	0.07	0.48	0.45
2006	2016	220	-574	197	279	187	116	305	1053	0.09	0.46	0.45
2007	2017	185	-571	166	201	173	67	228	916	0.08	0.53	0.39
2008	2018	229	-746	179	225	218	70	250	877	0.09	0.51	0.40
2009	2019	1127	-4190	166	1003	1119	-134	1018	1426	0.29	0.11	0.60
2010	2020	475	-1810	66	318	468	-80	345	1139	0.23	0.22	0.55
2011	2021	498	-1872	161	388	498	-6	418	1000	0.21	0.28	0.51

Table 3.16: *Hedging loss statistics at the start of 2011 for GMIBs issued at the start of each year from 1997 to 2011.*

3.12 Impact of increasing the option prices

There are potential difficulties with implementing a static hedge of the type presented in this chapter. Unfortunately, there are unlikely to be many natural sellers of 10 year put options. That is, few investors are likely to naturally gain from being short on a broad stock index over a decade. Furthermore, any party willing to write/sell lookback options of the types we have considered (possibly an investment bank), is going to charge significant loadings for the risks involved with hedging their own short positions. Therefore, it is interesting to explore how much the static hedge deteriorates when significant loadings are added to the option prices.

The implied volatilities of the options in all the strategies presented thus far have been = 20%. This section assesses the impact on the hedging loss distribution when the the implied volatilities, denoted by σ_i, for all of the options are increased to 25% or 30%. It is emphasized that we are not changing the stock volatility parameter $\sigma_S = 20\%$ used to simulate the paths of the stock under P. Only the option prices are increased. Table 3.17 displays the hedging instrument prices for different option implied volatilities. The lookback option prices are computed using Monte Carlo simulation, based on 200,000 scenarios. The standard errors for the lookback option prices are shown in brackets. Analytical formulas exist for the put options and the ZCBs.

Tables 3.18, 3.19, 3.20 and 3.21 illustrate the impact of different implied volatilities, σ_i, on various portfolios which minimize the CTE. In these tables, the results for $\sigma_i = 20\%$ correspond to PC3, PF1, PF2 and PC4B, respectively.[5] The hedging loss statistics in Tables 3.18, 3.19 and 3.20 display common characteristics as σ_i increases. The 1% quantile, median, mean, VaR(99%) and CTE(99%) all increase as σ_i increases. There do not appear to be any consistent patterns in the structure of the optimal portfolio positions as σ_i increases. The portfolios in Table 3.21 are designed to minimize the CTE and satisfy a mean constraint of 0, so the results show different behavior as σ_i increases. The tail risk measures increase rapidly, but there do not appear to be any other obvious patterns.

In passing, it is noted that a more realistic version of this analysis may use higher implied volatilities for the lookback options compared to the plain vanilla put options.

Implied vol.	LBC(1.6S(0))	LBP	Put(0.8S(0))	Put(1.1S(0))	Put(1.3S(0))	Stock	ZCB(10)	ZCB(29)
$\sigma_i = 20\%$	293.7 (1.3)	219.4 (0.5)	39.4	109.0	174.4	1000	667.8	251.8
$\sigma_i = 25\%$	392.6 (1.8)	317.2 (0.7)	68.5	154.7	227.8	1000	667.8	251.8
$\sigma_i = 30\%$	503.3 (2.6)	420.9 (0.9)	100.6	200.5	280.1	1000	667.8	251.8

Table 3.17: *Hedging instrument prices for different implied volatilities. The benchmark assumption is $\sigma_i = 20\%$.*

Implied vol	**MSHL**$^{1/2}$	**1%-quantile**	**Median**	**VaR(99%)**	**Std Dev**	**Mean**	**CTE(99%)**
$\sigma_i = 20\%$	477	-1283	-157	794	470	-77	1011
	$(461, 492)$	$(-1415, -1187)$	$(-166, -147)$	$(782, 826)$	$(455, 485)$	$(-90, -64)$	$(921, 1101)$
$\sigma_i = 25\%$	462	-1201	-125	839	460	-41	1046
	$(447, 476)$	$(-1320, -1104)$	$(-134, -116)$	$(828, 869)$	$(446, 474)$	$(-54, -28)$	$(959, 1132)$
$\sigma_i = 30\%$	342	-552	-119	898	342	18	1072
	$(334, 350)$	$(-610, -526)$	$(-125, -113)$	$(866, 925)$	$(334, 350)$	$(9, 28)$	$(997, 1148)$

		Put(0.8S(0))	Stock	ZCB(10)	ZCB(29)
$\sigma_i = 20\%$	$\hat{x}(k)$	0.819	1.067	-0.846	1.825
	$\hat{w}(k)$	0.032	1.074	-0.569	0.463
$\sigma_i = 25\%$	$\hat{x}(k)$	0.718	1.059	-0.979	2.142
	$\hat{w}(k)$	0.050	1.066	-0.658	0.543
$\sigma_i = 30\%$	$\hat{x}(k)$	0.254	0.965	-0.679	1.815
	$\hat{w}(k)$	0.026	0.970	-0.456	0.460

Table 3.18: *Hedging loss statistics and optimal instrument positions for portfolios including 10-year put options, in the case where the fee rate is 1%. The results for $\sigma_i = 20\%$ correspond to PC3.*

[5] The results for PC3, PF1, PF2 and PC4B are based on $N = 5{,}000$. The hedging loss statistics are slightly different to those shown in earlier tables. The different statistics arise from two sources: sampling error from different selections of scenarios, and slightly different optimal instrument positions.

Implied vol	MSHL$^{1/2}$	1%-quantile	Median	VaR(99%)	Std Dev	Mean	CTE(99%)
$\sigma_i = 20\%$	589	-1334	-331	424	491	-325	609
	$(579, 599)$	$(-1393, -1293)$	$(-357, -307)$	$(424, 429)$	$(483, 498)$	$(-339, -311)$	$(522, 695)$
$\sigma_i = 25\%$	555	-1232	-294	450	477	-284	641
	$(546, 564)$	$(-1283, -1207)$	$(-319, -272)$	$(450, 468)$	$(470, 484)$	$(-297, -271)$	$(553, 729)$
$\sigma_i = 30\%$	490	-1009	-252	483	431	-232	673
	$(483, 497)$	$(-1054, -984)$	$(-275, -235)$	$(476, 499)$	$(425, 437)$	$(-244, -220)$	$(586, 761)$

		Put(0.8S(0))	Stock	ZCB(10)	ZCB(29)
$\sigma_i = 20\%$	$\hat{x}(k)$	0.739	0.739	-0.700	2.758
	$\hat{w}(k)$	0.029	0.743	-0.470	0.698
$\sigma_i = 25\%$	$\hat{x}(k)$	0.723	0.728	-0.727	2.790
	$\hat{w}(k)$	0.050	0.732	-0.488	0.706
$\sigma_i = 30\%$	$\hat{x}(k)$	0.590	0.691	-0.743	2.940
	$\hat{w}(k)$	0.060	0.695	-0.499	0.744

Table 3.19: *Hedging loss statistics and optimal instrument positions for portfolios including 10-year put options, in the case where the fair fee rate is charged. The results for $\sigma_i = 20\%$ correspond to PF1.*

Implied vol.	MSHL$^{1/2}$	1%-quantile	Median	VaR(99%)	Std Dev	Mean	CTE(99%)
$\sigma_i = 20\%$	245	-600	-148	137	212	-122	149
	$(242, 248)$	$(-619, -585)$	$(-153, -143)$	$(137, 137)$	$(210, 214)$	$(-125, -119)$	$(145, 154)$
$\sigma_i = 25\%$	206	-463	-41	229	205	-20	246
	$(203, 209)$	$(-486, -449)$	$(-51, -27)$	$(229, 229)$	$(202, 208)$	$(-26, -15)$	$(235, 257)$
$\sigma_i = 30\%$	221	-358	61	330	205	82	350
	$(218, 224)$	$(-383, -344)$	$(51, 76)$	$(330, 330)$	$(202, 208)$	$(77, 88)$	$(338, 363)$

Implied vol.		LBC(1.6S(0))	LBP	Put(0.8S(0))	Stock	ZCB(10)	ZCB(29)
$\sigma_i = 20\%$	$\hat{x}(k)$	0.623	0.012	-0.017	0.000	0.180	2.754
	$\hat{w}(k)$	0.183	0.003	-0.001	0.000	0.120	0.695
$\sigma_i = 25\%$	$\hat{x}(k)$	0.616	0.020	-0.029	0.000	0.087	2.755
	$\hat{w}(k)$	0.242	0.006	-0.002	0.000	0.058	0.695
$\sigma_i = 30\%$	$\hat{x}(k)$	0.616	0.011	-0.017	0.000	-0.013	2.753
	$\hat{w}(k)$	0.311	0.005	-0.002	0.000	-0.009	0.695

Table 3.20: *Hedging loss statistics and optimal instrument positions for portfolios including the lookback options, in the case where the fair fee rate is charged. The results for $\sigma_i = 20\%$ correspond to PF2.*

Implied vol.	MSHL$^{1/2}$	1%-quantile	Median	VaR(99%)	Std Dev	Mean	CTE(99%)
$\sigma_i = 20\%$	482	-1807	147	397	482	0	397
	$(457, 507)$	$(-2029, -1658)$	$(134, 159)$	$(397, 397)$	$(457, 507)$	$(-13, 13)$	$(397, 398)$
$\sigma_i = 25\%$	826	-3157	257	666	826	0	669
	$(782, 869)$	$(-3391, -2817)$	$(233, 279)$	$(666, 666)$	$(782, 869)$	$(-23, 23)$	$(666, 671)$
$\sigma_i = 30\%$	796	-2848	173	951	796	0	999
	$(758, 834)$	$(-3178, -2550)$	$(154, 193)$	$(943, 965)$	$(758, 834)$	$(-22, 22)$	$(980, 1018)$

	Instrument (k)	LBC(1.6S(0))	LBP	Put(1.3S(0))	Stock	ZCB(10)	ZCB(29)
$\sigma_i = 20\%$	$\hat{x}(k)$	0.789	0.000	0.404	0.404	-0.603	2.746
	$\hat{w}(k)$	0.233	0.000	0.071	0.406	-0.404	0.695
$\sigma_i = 25\%$	$\hat{x}(k)$	0.733	0.000	0.680	0.680	-1.232	2.751
	$\hat{w}(k)$	0.290	0.000	0.156	0.685	-0.829	0.698
$\sigma_i = 30\%$	$\hat{x}(k)$	0.467	-0.074	0.508	0.898	-1.379	2.655
	$\hat{w}(k)$	0.237	-0.032	0.143	0.906	-0.928	0.674

Table 3.21: *Hedging loss statistics and optimal instrument positions for portfolios including the lookback options, in the case where the fee rate is 1% and a mean constraint of 0 is included in the optimization problem. The results for $\sigma_i = 20\%$ correspond to PC4B.*

3.13 Sensitivity Analysis

This section is presented to give the reader a sense of the stability of the results illustrated in this chapter.

3.13.1 Stability of the optimal hedging portfolios

The number of scenarios N is the key parameter that drives the stability of the results of a given strategy. Here we report the sensitivity of the hedging loss distributions for PC3 and PC4B using 1,000, 10,000 and 20,000 scenarios. Since the distributions depend on the variability of the optimal instrument positions, we also report the sensitivity of the optimal instrument positions. We measure the stability of the results by using repeated Monte Carlo simulations. For 20 independent Monte Carlo simulations, each consisting of N scenarios, we record the key hedging loss statistics and the optimal instrument positions. Then we calculate the mean and standard deviation of the 20 repeated simulation estimates of the key statistics and optimal instrument positions. It is noted that in each independent Monte Carlo simulation, the set of scenarios used to compute the optimal instrument positions is different to the set of scenarios used to calculate the hedging loss statistics; recall that this is done to remove any bias in the hedging loss distribution.

Tables 3.22 and 3.23 display the mean and variance of the key statistics and optimal instrument positions for PC3 and PC4A, based on different values of N. The hedging loss statistics and optimal instrument positions of PC3 and PC4B shown in Tables 3.4 and 3.6 respectively, are within a reasonable distance of the reported mean statistics in Tables 3.22 and 3.23. The hedging loss statistics and the optimal instrument positions exhibit much lower variability in PC4B. The lookback call options play a major role in stabilizing the hedging loss distribution.

For both PC3 and PC4B, the statistics appear to be relatively stable for $N \geqslant 10{,}000$. We conclude that the statements we have made in the analysis of each portfolio in this chapter are robust to the selection of the scenarios.

		MSHL$^{1/2}$	1%-quantile	Median	VaR(99%)	Std Dev	Mean	CTE(99%)
$N = 1{,}000$	Mean	508	-1344	-155	867	495	-83	1096
	Std dev	252	1093	61	61	229	133	45
$N = 10{,}000$	Mean	512	-1428	-161	834	502	-96	1067
	Std dev	91	388	19	21	83	48	24
$N = 20{,}000$	Mean	506	-1415	-159	831	497	-94	1066
	Std dev	42	197	10	15	39	30	28

		Put($0.8S(0)$)	Stock	ZCB(10)	ZCB(29)
$N = 1{,}000$	Mean	0.75	1.07	-1.13	2.58
	Std dev	0.38	0.15	0.80	1.94
$N = 10{,}000$	Mean	0.80	1.09	-1.21	2.71
	Std dev	0.12	0.06	0.20	0.51
$N = 20{,}000$	Mean	0.80	1.09	-1.26	2.85
	Std dev	0.06	0.03	0.15	0.38

Table 3.22: *Mean and variance of optimal hedging instrument positions and hedging loss statistics for PC3, obtained using 20 independent Monte Carlo simulations.*

		MSHL$^{1/2}$	1%-quantile	Median	VaR(99%)	Std Dev	Mean	CTE(99%)
$N = 1{,}000$	Mean	464	-1710	143	387	464	7	395
	Std dev	28	119	5	7	28	13	7
$N = 10{,}000$	Mean	477	-1769	140	391	477	0	397
	Std dev	10	41	4	3	10	6	2
$N = 20{,}000$	Mean	475	-1769	140	390	475	0	397
	Std dev	6	37	4	2	6	4	2

		LBC($1.6S(0)$)	LBP	Put($1.3S(0)$)	Stock	ZCB(10)	ZCB(29)
$N = 1{,}000$	Mean	0.77	0.00	0.40	0.40	-0.59	2.73
	Std dev	0.03	0.00	0.03	0.03	0.05	0.01
$N = 10{,}000$	Mean	0.77	0.00	0.41	0.41	-0.60	2.74
	Std dev	0.01	0.00	0.01	0.01	0.01	0.00
$N = 20{,}000$	Mean	0.77	0.00	0.41	0.41	-0.60	2.74
	Std dev	0.01	0.00	0.01	0.01	0.01	0.00

Table 3.23: *Mean and variance of optimal hedging instrument positions and hedging loss statistics for PC4B, obtained using 20 independent Monte Carlo simulations.*

3.13.2 Changing the confidence level in CTE minimization problems

Tables 3.24 and 3.25 display the hedging loss statistics for PC3 and PC4B using different CTE confidence levels α in the CTE minimization problem. To better understand how the choice of α influences the right tail risk, we report the CTEs at 90%, 95% and 99% confidence levels for all cases. For both portfolios, $\alpha = 0.99$ seems to offer favorable results, in terms of minimizing the risk of extremely large hedging losses. Setting α too low increases the risk of larger hedging losses when extremely adverse scenarios with low probability occur.

α	$\text{MSHL}^{1/2}$	1%-quantile	Median	VaR(99%)	Std Dev	Mean	CTE(90%)	CTE(95%)	CTE(99%)
0.99	414	-960	-134	815	411	-43	719	794	1039
0.95	283	-239	-49	845	273	76	636	715	1108
0.90	274	-171	59	891	238	136	618	734	1187
0.85	276	-143	70	907	234	147	621	748	1206
0.80	275	-127	63	902	233	145	625	752	1205

Table 3.24: *Hedging loss statistics for PC3 using different CTE confidence levels.*

α	$\text{MSHL}^{1/2}$	1%-quantile	Median	VaR(99%)	Std Dev	Mean	CTE(90%)	CTE(95%)	CTE(99%)
0.99	470	-1766	139	387	471	0	387	387	391
0.95	474	-1771	142	386	474	0	387	387	393
0.90	476	-1771	145	386	476	0	386	388	396
0.85	482	-1793	150	400	482	0	390	396	409
0.80	475	-1786	150	428	475	0	400	415	443

Table 3.25: *Hedging loss statistics for PC4B using different CTE confidence levels.*

3.14 Practical risks with using a static hedging strategy

Risks associated with the use of a static hedging strategy, which we have not modeled, include basis risk and counter-party risk. These risks are difficult to measure and model reliably. However, it is important to be aware of their existence.

In our context, basis risk occurs because the returns on the stock index that we are modeling are unlikely to coincide with the returns of the policyholder's investment account.

We have assumed that the returns are identical. As mentioned in Section 1.1, the policyholder may split the balance of their investment into sub-accounts related to different fund managers. If the returns of the stock index process tend to differ considerably from the returns of the investment account, the static hedge may not work as intended. To further complicate the problem, each individual policyholder may split their investment differently. Therefore, developing a static hedge for a portfolio of GMIBs, where each policy experiences different investment account returns, is a rather complex task. Basis risk is obviously difficult to model. Basis risk can be reduced if the policyholder is forced to invest at least $x\%$ (e.g. $x = 50$) of their premium with managed funds that have a track record of producing returns which are strongly positively correlated with the returns on the stock index.

Counter-party risk arises because the hedging portfolio may include one or more long positions in derivatives with long terms to expiry. Given that the term to expiry is 10 years, the possibility that the option seller fails to meet its obligations at maturity is not negligible. History has repeatedly shown that, over a decade, market conditions may change significantly. The seller could go bankrupt, or a severe downgrade in its credit rating over the period may occur, for a variety of reasons. Consider the hypothetical example where an insurance company bought, say, a 10 year lookback call option from the investment bank Lehman Brothers, as part of a static hedge for a GMIB. The hedge would most likely fail miserably. The counter-party risk in the examples we have considered, is much higher than in the situation where say one year options are used in a hedging portfolio which is rebalanced annually. To mitigate counter-party risk, the insurer should carefully investigate the long term creditworthiness of any party involved in an over-the-counter transaction. In the case where the option is traded on an exchange, which may be a possibility for long-dated plain vanilla options such as 10 year put options, counter-party risk is significantly reduced because of margin calls (but as exchange based contracts are standardized, the basis risk might be higher).

3.15 Concluding remarks

This chapter has measured the effectiveness of various static hedging strategies for the GMIB. Under the assumption that the GMIB is underpriced (the fee rate being set at 1%, one of the highest rates currently charged), the performance of static hedging strategies for the GMIB is imperfect at best. The hedging portfolios do not adequately simultaneously hedge the upside and downside equity guarantees (the lookback and guaranteed return components respectively). We discovered that a particular hedging instrument, the lookback call option, is very effective at hedging the GMIB, particularly when the lookback component is exercised. The results from Portfolios C4A and C4B, in Section 3.6.4, demonstrate this point clearly. Without the lookback call option, the lookback component is responsible for generating a very large CTE for the hedging loss distribution. We then demonstrated that when the fair fee rate is charged, the static hedging strategies produce much more favorable hedging loss distributions. However, significant tail risk still exists even when the fair fee rate is charged, unless the lookback call option is included in the hedging portfolio.

We backtested the performance of static hedging strategies for GMIBs issued in each year from 1997 to 2011, under the assumption that the GMIB is underpriced. Overall, most of the static hedges generated, or are expected to generate, small hedging losses.

The following modeling issues will decrease the effectiveness of static hedging strategies:

1. We assumed the stock follows a geometric Brownian motion process, which is well known to produce thin-tailed equity return distributions (Hardy, 2003). In practice, equity return distributions exhibit much fatter negative return tails. Using a fatter-tailed model for the stock is likely to produce hedging loss distributions with thicker right tails (the likelihood of larger losses increases).

2. The implied volatilities of options with terms to expiry of 10 years may be higher than what we have assumed (i.e. option prices will be higher) due to the higher inherent risks of long-dated options – particularly for the lookback options. This means smaller instrument positions for the same budget constraint, which in turn will shift the hedging loss distribution to the right (higher hedging losses). Given

the importance of this issue, in Section 3.12 we investigated the impact of higher implied volatilities for the options.

Lapse assumptions have also not been taken into consideration. Allowing for the possibility of the policyholder lapsing during the accumulation phase will reduce the cost of a static hedging strategy. Static hedges may be more profitable. If the policyholder lapses, they forfeit the GMIB option, and they may, say, only be entitled to receive the value of their investment account when they lapse. The hedging portfolio could be sold at that point, most likely for a profit as the liability is reduced, or held until a later date if it seems optimal to do so. The potential profits from lapses are difficult to measure because policyholders are more likely to lapse when the GMIB option value is low. If a policyholder does lapse, there is also the issue of liquidity risk. If the hedging portfolio is to be liquidated, it may be difficult for an insurer to sell any long-dated options it holds at reasonable prices.

We have explored the addition of up-and-out put options in the hedging portfolio, in order to improve the hedging strategies in situations where the stock performs poorly during the accumulation phase, and the stock value at maturity is low. These options are cheaper than put options, and thus more downside protection can be included in the portfolio, provided that the stock stays below the knock-out barrier during the term to expiry of these options. We have found that the addition of such options does not offer any material improvement for the static hedging strategies.

Hybrid option instruments, which have payoffs that are functions of the stock process and interest rates, may be useful for static hedging strategies. For example, a hybrid knock-in put option which has a positive payoff only if interest rates are below some predetermined level at maturity, may be cheap and useful for hedging the guaranteed return component when it is deep-in-the-money (the guaranteed return component has a higher payoff when interest rates are low at maturity). We have not investigated the usefulness of such instruments.

There is an interesting point to be made about the pricing and hedging of GMIBs (that may not be well known to actuaries). In Chapter 2, we showed that the value of a simplified GMIB, consisting of just the guaranteed return and investment account components (we referred to this guarantee as the embedded guaranteed return option in Section 3.9.2), increases only slightly when the lookback component is also included to “complete ” the GMIB option. However, this chapter has shown that when the GMIB is hedged with a static portfolio, the lookback component contributes significantly to the risk of large hedging losses – particularly if a lookback call option is not included in the hedging portfolio. In other words, a small contribution to the price of a guarantee does not necessarily imply a small contribution to the risks involved in hedging the guarantee (the risks may increase substantially).

Chapter 4

An Investigation of Periodic Rebalancing Hedging Strategies for a Guaranteed Minimum Income Benefit

4.1 Introduction

In Chapter 3, we discussed in detail the shortcomings of delta-hedging strategies for options with long maturity dates. We are motivated to find alternative hedging methods which work well in practice for long-dated options, but which do not rely heavily on the standard assumptions of option pricing theory. The effectiveness of static hedges for the GMIB option was illustrated in Chapter 3. In this chapter, we investigate the performance of semi-static hedging strategies for the GMIB option.

In general, the goal of a semi-static hedging strategy for a long-dated option is to construct a hedging portfolio that is rebalanced at particular time points during the accumulation phase, such that, at the option maturity date, the hedging portfolio payoff is equal to or exceeds the option value, subject to an acceptable level of risk. We refer to the time interval between two rebalancing points as the hedging horizon. The choice of the length of each hedging horizon depends on several factors including:

- The trade-off between the transaction costs involved in rebalancing, and the risk of large positive deviations between the option price and the portfolio value at the end of each hedging horizon;
- The expiration dates of options that can be included in the hedging portfolio.

A semi-static hedging strategy should be self financing: no additional external funds should be required at any rebalancing point to reduce shortfalls between the option price and the hedging portfolio value. Furthermore, any reasonable semi-static hedging strategy is designed subject to some measure of risk. There are no universally accepted risk measures associated with semi-static hedging strategies. In this chapter, we consider trading strategies based on minimizing the mean square hedging error (MSHE) in each horizon, and minimizing the conditional-tail-expectation of the hedging error distribution in each horizon.

The investigation is done from the perspective of an insurer who has sold a variable annuity with an embedded GMIB option. The insurer wishes to hedge the option from inception to maturity, using just the annuity premium. We illustrate the effectiveness of semi-static hedging strategies through several examples. Many of the examples are based on rebalancing the hedging portfolio in such a way that the hedging portfolio value matches the GMIB price at the end of each hedging horizon, as closely as possible. By matching the GMIB price at the end of a hedging horizon, the insurer could, at least in theory, sell the GMIB to another party at that time without incurring a loss (by transferring the funds provided by the hedging portfolio to the buyer of the GMIB liability).

Long computation times are often a major challenge with implementing semi-static hedging strategies. The method we use requires nested simulations, which in general can be very computationally expensive (time consuming). As a first stepping stone to tackling the complex problem of developing, and forecasting the performance of, semi-static hedging strategies, we work within the model framework presented in Section 3.2. Using this model framework enables us to exploit the speed of the efficient simulation method discussed in Section 2.9.1, which is vital for completing the nested simulations in reasonable time frames.[1] Using the efficient simulation method, we can produce fast simulations of the hedging loss distribution at maturity, based on our choice of semi-static hedging strategy. The use of more complicated SDEs for the stock and the interest rate, which must be discretized in order to simulate trajectories, leads to problems in respect of computation time. Because nested simulations are necessary, the generation of a sufficient number of scenarios for making meaningful inferences could take a very long time (e.g. simulating 1,000 scenarios may take days or longer).

The structure of Chapter 4 is as follows. The remainder of Section 4.1 outlines the method we use to test semi-static hedging strategies for the GMIB, and discusses the contribution of this chapter to the literature on hedging methods. Section 4.2 describes the steps involved in implementing the method. Although the method is described specifically for the case of the GMIB, the steps involved can easily be adjusted to accommodate for any long-dated option. In Section 4.3, we deal with numerical stability issues related to the implementation of the method. Section 4.4 lists the benchmark parameter assumptions we adopt for illustrating most of the results in this chapter. For each semi-static strategy that we investigate, we report the key statistics summarizing the hedging loss distribution in a table. We also present detailed tables summarizing the behavior of certain semi-static strategies. Section 4.5 explains how to the interpret the numbers in the tables spread throughout the latter part of this chapter. Before investigating semi-static strategies for the GMIB option, we first test the strategies for a simple derivative, the 10 year call option, in Section 4.6. The purpose of investigating the performance of the semi-static strategies for the 10-year call option is to build some intuition as to how the semi-static strategies behave for different hedging horizon lengths and transaction cost assumptions, and various choices of allowable hedging instruments. Section 4.7 describes the hedging strategy types that we consider for the GMIB, which are used in the examples presented in the remainder of the chapter. Sections 4.8 and 4.9 illustrate the performance of MSHE minimization and CTE minimization hedging strategies for the GMIB. Section 4.10 investigates the change in performance of the semi-static strategies when the hedging target is changed from the GMIB price to the real-world expected present value of the benefits provided

[1] For the author, during the writing of this chapter, a Monte Carlo simulation which is complete in a "reasonable time frame" is one that is done in less than 24 hours. All simulations were performed in MATLAB on a 64-bit laptop with the following specifications: Intel i7-2600 (4 cores) CPU @2Ghz, 8GB RAM.

by the GMIB. In Section 4.11, we assess the impact of model risk for the semi-static strategies. We assume the real-world stock returns follow a 2-state regime switching lognormal process, rather than the benchmark lognormal return process. This section gives some indication of the robustness of the semi-static strategies. Section 4.13 gives an indication of the stability of the results obtained by using the semi-static hedging method. Section 4.14 provides a summary of the results in the chapter.

4.1.1 Preliminary notation

In this chapter, we present equations that combine simulated paths of the stock, short rate and short rate accumulation factor processes under both a risk-neutral probability measure, denoted by Q, and the real-world probability measure, denoted by P. Therefore, for clarity of exposition, we mark all random variables simulated under the risk-neutral measure with tildes above their symbols. For brevity, we collectively refer to the short rate and short rate accumulation factor processes as the interest rate processes.

Let us partition the interval from time 0 to time T into I hedging horizons/intervals. Let t_i denote the i-th rebalancing time point such that

$$0 = t_0 < t_1 < \ldots < t_{I-1} < t_I = T.$$

We always assume equally spaced hedging horizons; $t_i - t_{i-1} = \tau, i = 1, \ldots, I$, for some constant τ, which implies $I = T/\tau$.

Define

$$\boldsymbol{S}(t_i) = [S(t_1), S(t_2), \ldots, S(t_i)]',$$

and

$$\boldsymbol{D}(t_i) = [e^{\int_{t_0}^{t_1} r(s)\mathrm{d}s}, e^{\int_{t_1}^{t_2} r(s)\mathrm{d}s}, \ldots, e^{\int_{t_{i-1}}^{t_i} r(s)\mathrm{d}s}]', \qquad i = 1, 2, \ldots, I.$$

The GMIB price at time t_i is

$$V(t_i) = V\left(t_i, \boldsymbol{S}(t_i), \boldsymbol{D}(t_i), r(t_i)\right) = E^Q\left[e^{-\int_{t_i}^{T}\widetilde{r}(s)\mathrm{d}s}\max\{\widetilde{B}(T)g\widetilde{\ddot{a}}_{\overline{20}|}(T), \widetilde{A}(T)\}\Big|\mathcal{F}_{t_i}\right], \quad (4.1)$$

where $\mathcal{F}_{t_i}$ denotes all information available at time t_i. In equation (4.1), we must know the value of $r(t_i)$ in order to simulate the value of $\widetilde{r}(T)$, and hence evaluate $\widetilde{\ddot{a}}_{\overline{20}|}(T)$.

Note that we can express the GMIB price at time $t_i < T$ in an alternative form:

$$\begin{aligned} V(t_i) = {} & E^Q\left[\exp\{-\int_{t_i}^{T}\widetilde{r}(s)\mathrm{d}s\}\max\{\widetilde{B}(T)g\widetilde{\ddot{a}}_{\overline{20}|}(T) - \widetilde{A}(T), 0\}\Big|\mathcal{F}_{t_i}\right] + S(t_i) \\ & - \sum_{n=1}^{\lfloor t_i \rfloor} e^{\int_n^{t_i} r(s)\mathrm{d}s - \sigma_S^2(t_i - n)/2 + \sigma_S(W_S^P(t_i) - W_S^P(n))} f(n) \\ & - E^Q\left[e^{-\int_{t_i}^{T}\widetilde{r}(s)\mathrm{d}s}\left(\sum_{n=\lfloor t_i \rfloor+1}^{T-1}[\prod_{i=n+1}^{T}\widetilde{R}(i)]\widetilde{f}(n) + \widetilde{f}(T)\right)\Bigg|\mathcal{F}_{t_i}\right], \end{aligned}$$

where:

- $\lfloor t_i \rfloor$ denotes the largest integer less than or equal to t_i;
- $f(n)$ is defined by equation (2.2);
- $R(i)$ is defined by equation (2.3).

At maturity,

$$V(T) = \max\{B(T)g\ddot{a}_{\overline{20}|}(T), A(T)\}. \quad (4.2)$$

4.1.2 How the semi-static hedging strategy is implemented

The implementation of a semi-static hedging strategy depends on whether the goal of the strategy is to either:

(1) Generate a hedging portfolio payoff at maturity which matches (or exceeds) the GMIB maturity value, subject to an acceptable level of risk;

(2) Generate a hedging portfolio value which matches (or exceeds) the GMIB price at the end of each hedging horizon, subject to an acceptable level of risk.

Goal (1) would be chosen if the insurer intends to hold the GMIB liability to maturity, and they are only concerned with the risk associated with the final realized hedging loss (profit). Goal (2) would be chosen by an insurer who may want to transfer the GMIB liability to another party at some future time point before maturity. Goal (2) might also be necessary for reporting and solvency purposes. As a special case, both goals coincide over the last hedging horizon. The two distinct goals are mentioned because, prior to the last rebalancing point before maturity, a trading strategy designed to meet Goal (2) may not be the best strategy for meeting Goal (1). The reason for the distinction between the goals is that the choice of goal may affect the choice of the hedging target for each horizon, and the criteria for rebalancing the hedging portfolio in an optimal way.

The semi-static hedging strategy is implemented as follows. Suppose we are at the start of the i-th hedging horizon, time t_{i-1}. We have a short position in a variable annuity with an embedded GMIB option, and we wish to hedge the liability using a budget constraint of $b(t_{i-1})$ dollars. We must choose a vector of hedging instrument positions at time t_{i-1}, denoted by $\boldsymbol{x}(t_{i-1})$, in such a way that we maximize the chance of meeting either Goal (1) or (2). Let

- $Y(t_i)|\mathcal{F}_{t_{i-1}}$ denote the *hedging target* over the i-th hedging horizon. The aim of the hedging strategy is to produce a hedging portfolio value at time t_i which meets the hedging target. A natural candidate for the hedging target is the GMIB price at time t_i, given by equation (4.1).
- $\boldsymbol{Z}(t_i)|\mathcal{F}_{t_{i-1}}$ denote the vector of hedging instrument payoffs at time t_i.
- $Y(t_i) - \boldsymbol{x}(t_{i-1})'\boldsymbol{Z}(t_i)|\mathcal{F}_{t_{i-1}}$ denote the *hedging error* for the i-th horizon.

Criteria must be defined for choosing the instrument positions $\boldsymbol{x}(t_{i-1})$. There are many possibilities. In this chapter, we illustrate the results from choosing $\boldsymbol{x}(t_{i-1}), i = 1, \ldots, I$ by minimizing the following objective functions:

1. The mean square hedging error (MSHE) of $Y(t_i) - \boldsymbol{x}(t_{i-1})'\boldsymbol{Z}(t_i)|\mathcal{F}_{t_{i-1}}$ over each horizon. This objective function is a natural choice for minimizing the difference between the hedging target and the hedging portfolio payoff over each horizon.

2. The Conditional Tail Expectation (CTE) of $Y(t_i) - \boldsymbol{x}(t_{i-1})'\boldsymbol{Z}(t_i)|\mathcal{F}_{t_{i-1}}$ over each horizon. Strategies obtained from minimizing this objective function are designed to minimize the likelihood of large positive hedging errors.

Minimizing the chosen objective function produces an optimal trading strategy for horizon i, denoted by $\hat{\boldsymbol{x}}(t_{i-1})$, set in place at the start of the horizon.

In the last hedging horizon, the hedging target is always set equal to the GMIB maturity value, given by equation (4.2). We refer to the hedging *error* distribution at time $t_I = T$ as the hedging *loss* distribution. A *hedging loss (profit)* is realized if the GMIB maturity value exceeds (is less than) the hedging portfolio payoff at time T. For Goal (1), the successfulness of a particular strategy is measured by the shape of the hedging loss distribution. Monte Carlo simulation is used to sample from the hedging loss distribution. Let $e_j, j = 1, 2, \ldots, J$ denote observations from the hedging loss distribution at time T. To be clear, these hedging loss observations (which we refer to as the "hedging losses") are the realized losses (profits) at time T from implementing the semi-static hedging strategy over I hedging horizons, between time 0 and time T, for J distinct scenarios.

If an insurer aims to meet Goal (2), then, in the context of this chapter, it should minimize the MSHE using the GMIB price as the hedging target. However, if an insurer plans to meet Goal (1), then there is flexibility in the choice of hedging target and objective function. In this chapter, all of our hedging loss results are presented assuming that the insurer has Goal (1) in mind; we present comprehensive results of each strategy as at the maturity date. However, some useful information for meeting Goal (2) is also presented. In particular, for many examples, we show the mean hedging error, and the hedging error standard deviation, for each hedging horizon between time 0 and time T. Further information related to Goal (2), such as the shapes of the hedging error distributions at particular rebalancing points of interest, is easily obtained using the method described here.

As we illustrate in our results, the choice of hedging target materially impacts on the hedging loss distribution at time T. Goal (1) could be met by using any appropriate hedging target, which does not unreasonably increase the risk of large hedging losses. We consider two hedging targets for meeting Goal (1):

- The GMIB price calculated using option pricing theory. We calculate the GMIB price using the model in Chapter 2.
- The expected present value of the benefits provided by the GMIB under the real-world measure P. That is we are valuing the GMIB liability under P instead of Q. We refer to this hedging target as the *P-valuation target*. In Section 4.10, we show how the hedging loss distribution changes when the hedging target is the *P-valuation target* instead of the GMIB price. The reader is reminded that in perfect world where the market is complete, trading the GMIB liability at a price equal to the real-world expectation of the future benefits (instead of pricing using the risk-neutral measure) will generate an arbitrage opportunity.

4.1.3 Contribution to the literature on hedging methods

Standard option hedging strategies are based on frequently rebalancing the hedging portfolio with respect to movements in one or more of the Greeks of the option (delta-hedging being the simplest case). There is a broad field of literature on the properties associated with alternative hedging strategies. These alternative strategies are designed to deal with hedging options in incomplete markets. In incomplete markets, the intrinsic risk of an option cannot be fully hedged. Each alternative hedging strategy is based on some form of risk-minimization criterion. Some of these strategies are now outlined.

Quantile hedging involves constructing a hedging strategy which maximizes the probability of a successful hedge under the real-world measure P, given a constraint on the required cost (Föllmer and Leukert, 1999). The MSHE minimization strategy we analyze is related to the intertwined subjects of mean-variance hedging and quadratic hedging methods. These methods often involve minimizing the squared difference between the terminal payoffs of a derivative and a self-financing trading strategy (i.e. quadratic risk minimization). Pioneering studies on quadratic hedging criteria for pricing and hedging general contingent claims include Föllmer and Sondermann (1986), Föllmer and Schweizer (1989), Duffie and Richardson (1991), Schweizer (1992) and Schäl (1994). Schweizer (2001) provides an general overview of quadratic hedging approaches, with further references to earlier work. Today, the literature on quadratic hedging methods is vast, and it continues to proliferate. For more recent developments on quadratic hedging methods see, for example, Černý and Kallsen (2007, 2009). As an alternative to hedging methods based on quadratic risk minimization, Coleman et al. (2003) investigate hedging strategies obtained by piecewise linear risk minimization.

Some hedging strategies have been investigated for variable annuity options. Coleman et al. (2006) analyze local risk minimization (quadratic risk minimization) hedging strategies for a variable annuity with an embedded GMDB, allowing for equity jump risk (using Merton's jump diffusion model (Merton, 1976)) and interest rate risk (using the Vasicek model (Vasicek, 1977)). They investigate hedging with the underlying stock, and hedging with standard options. Hedging with standard options performs better than annual or monthly hedging with the underlying stock, particularly when equity jump risk is allowed for. Furthermore, they show that ignoring stochastic interest rate risk in the calculations of hedging strategies may generate large hedging errors, and the benefits of hedging with options over hedging with just the underlying stock may be lost. It is shown that it is possible to reduce interest rate risk by including a stochastic interest rate model in the calculations of hedging strategies. In a related paper on hedging variable annuities with embedded GMDBs, Coleman et al. (2007) find that when implied volatility risk is modeled, local risk minimization hedging strategies which use standard options still tend to be more effective at reducing risk, compared to strategies which use the underlying stock. Liu (2010) measures the performance of semi-static strategies for the GMWB. However, Liu restricts the performance measurement to one hedging horizon. In this chapter, we measure the performance of semi-static strategies for the GMIB, based on accumulating the horizon by horizon hedging profits/losses over the entire accumulation phase. This approach is more complex to implement, and requires much more computation time.

There seems to be a dearth of literature, accessible to the majority of practitioners, on the computational implementation of hedging strategies of the semi-static type. The main contribution of this chapter is to provide a bridge between the complex mathematical theory of hedging strategies based on rebalancing at discrete fixed time points, and the actual implementation of such strategies. We use the GMIB option as a case study. We show the results obtained from using semi-static strategies for the GMIB. Our target audience includes risk managers responsible for the hedging programs of long-dated complex financial guarantees.

4.2 The method

In this section, we describe the method used to implement the semi-static hedging strategies for the GMIB. We emphasize that this method is flexible and can be extended to any long-dated option. However, if the option has features which make pricing difficult or time consuming, such as an American option, this approach may become very time consuming. The exact amount of computation time is problem specific.

4.2.1 Simulating the hedging loss distribution

The j-th hedging loss e_j is calculated using the following algorithm, which we refer to as the *hedging loss simulation algorithm* (HLS algorithm).

Starting at $i = 1$:

(1) The information available at time t_{i-1} includes the realized values of the random vector

$$\Omega(t_{i-1}) = (\boldsymbol{S}(t_{i-1}), \boldsymbol{D}(t_{i-1}), r(t_{i-1}))\,.$$

Given $\Omega(t_{i-1}) = \omega(t_{i-1})$, simulate (under P) N *sub-scenarios* representing possible paths of the stock and interest rate processes from time t_{i-1} to time t_i. Let $\{\omega_n(t_i|t_{i-1})\}_{n=1}^{N}$ denote the realized values of

$$\Omega(t_i|t_{i-1}) = \Omega(t_i)|\mathcal{F}_{t_{i-1}} = (\boldsymbol{S}(t_i), \boldsymbol{D}(t_i), r(t_i))\,|\mathcal{F}_{t_{i-1}}$$

for each sub-scenario.

(2) Simulate an observation of the hedging target $Y(t_i)|\mathcal{F}_{t_{i-1}}$ for each $\omega_n(t_i|t_{i-1}), n = 1, \ldots, N$. Let $\{y_n(t_i)\}_{n=1}^N$ denote the hedging target values for each of the N sub-scenarios of the i-th hedging horizon. Using the GMIB price or the P-valuation target as the hedging target requires the use of nested simulations. Section 4.2.2 discusses the details of how the hedging target values are calculated.

(3) Calculate the prices of the instruments in the hedging portfolio at time t_{i-1}.

(4) Calculate the hedging instrument payoffs (prices) at time t_i for each sub-scenario. It is emphasized that the instrument payoffs are functions of $\{\omega_n(t_i|t_{i-1})\}_{n=1}^N$.

(5) Check that the single-period market model described by the combination of the set of simulated instrument payoffs, and the set of instrument prices, does not permit arbitrage opportunities. Section 4.2.5 describes how to test for arbitrage within the model.

(6) Determine the hedging portfolio $\boldsymbol{x}(t_{i-1}) = \hat{\boldsymbol{x}}(t_{i-1})$, based on the hedging targets and hedging instrument payoffs for the N sub-scenarios, which minimizes the chosen objective function over the current hedging horizon. Section 4.2.3 describes the process for obtaining the optimal portfolio positions. The cost of the optimal hedging portfolio constructed at time t_{i-1}, denoted by $\hat{b}(t_{i-1})$, cannot exceed the funds available at time t_{i-1}, denoted by $b(t_{i-1})$. If $\hat{b}(t_{i-1}) < b(t_{i-1})$, then the excess funds $\hat{\xi}(t_{i-1}) = b(t_{i-1}) - \hat{b}(t_{i-1})$ are invested in zero coupon bonds (risk-free assets) over the interval $[t_{i-1}, t_i]$. In the case of the GMIB option, initially $b(0) = S(0) = \pi$, where π is the policyholder's premium.

(7) Simulate under P one trajectory for each random process, denoted by $\Omega(t_i|t_{i-1}) = \omega_A(t_i|t_{i-1})$ which represents the actual (realized) movement of the process over the interval $[t_{i-1}, t_i]$. Note that this step is independent of the simulated sub-scenarios.

(8) Using the output of Step (7), compute the actual hedging target, denoted by $y_A(t_i)$, and the actual total hedging portfolio payoff (which includes any excess funds invested in zero coupon bonds), denoted by $\psi_A(t_i)$. We explicitly define $\psi_A(t_i)$ in Section 4.2.3 after further notation is defined. The *actual total hedging target error*

at time t_i is defined as

$$\eta_A(t_i) = y_A(t_i) - \psi_A(t_i).$$

The actual total hedging target error is not a realized loss for the insurer unless either:

- $t_i = T$, and $V(T) - \psi_A(T) < 0$; or
- $t_i < T$, and the insurer transfers the liability to another party when $V(t_i) - \psi_A(t_i) < 0$

(9) Set $b(t_i) = \psi_A(t_i)$.

(10) Repeat Steps (1) to (9) for $i = 2, 3, \ldots, I$.

(11) Evaluate $e_j = V(T) - \psi_A(T)$.

The HLS algorithm is used to compute e_j for $j = 1, \ldots, J$. Strictly speaking the notation in the above steps should include j subscripts as the variables are different for each simulated scenario j. However, in order to keep the notation clean, we only include the j subscripts in the notation when we define equations that relate to observations of variables from more than one particular scenario j.

4.2.2 Calculating the hedging target values

This section provides the details of Step (2) of the HLS algorithm described in Section 4.1.2. Recall that at the start of the i-th horizon, the method involves estimating the hedging target for each of the N simulated sub-scenarios. To be clear, the n-th hedging target value (estimate) is calculated conditional on one possible path of the underlying random processes over the horizon, given by $\omega_n(t_i|t_{i-1})$. Each hedging target value is calculated using the following algorithm, which we refer to as the *conditional hedging target simulation algorithm* (CHTS algorithm). We describe the CHTS algorithm for the natural case where the hedging target is the GMIB price. By using the CHTS algorithm within the HLS algorithm, we have constructed a nested Monte Carlo simulation.

The steps involved in generating $\{y_n(t_i)\}_{n=1}^{N}$ for the i-th horizon (of the j-th scenario), when $i < I$, are as follows, starting at $n = 1$:

(1) Given $\omega_n(t_i|t_{i-1})$, simulate (under Q) M *conditional-paths* of the stock and interest rate processes from time t_i to time T. Define

$$W(n) = e^{-\int_{t_i}^{T} \widetilde{r}(s)\mathrm{d}s} \max\{\widetilde{B}(T)g\widetilde{\ddot{a}}_{\overline{20}|}(T), \widetilde{A}(T)\}\Big| \left(\Omega(t_i|t_{i-1}) = \omega_n(t_i|t_{i-1})\right). \qquad (4.3)$$

Let $\{W_m(n)\}_{m=1}^{M}$ denote the observations of $W(n)$ based on the M conditional-paths.

(2) The hedging target value for the n-th sub-scenario is given by

$$y_n(t_i) = \frac{1}{M}\sum_{m=1}^{M} W_m(n), \qquad n = 1, \ldots, N. \qquad (4.4)$$

(3) Repeat Steps (1) to (2) for $n = 2, 3, \ldots, N$.

At the last horizon, when $i = I$, the hedging target value for the n-th scenario is defined as $y_n(T) = V_n(T)$, where $V_n(T)$ is the observed value of the random variable given by equation (4.2). When $i = I$, no nested simulations are needed.

In the case where the hedging target is the P-valuation target (as mentioned in Section 4.1.2), the only change to the CHTS algorithm is in Step (1). The random processes are simulated under P and equation (4.3) is replaced by

$$W(n) = e^{-\int_{t_i}^{T} r(s)\mathrm{d}s} \max\{B(T)g\ddot{a}_{\overline{20}|}(T), A(T)\}\Big| \left(\Omega(t_i|t_{i-1}) = \omega_n(t_i|t_{i-1})\right). \qquad (4.5)$$

The description of the CHTS algorithm applies to the GMIB option, but it equally applies to any option which is valued using simulation. We make the obvious comment that if the hedging target observations for a particular option can be calculated using numerical integration (or analytically), then it is usually computationally more efficient to use such an approach, instead of introducing a nested simulation.

4.2.3 The optimization problems

At the start of the i-th hedging horizon, the optimal hedging portfolio $\boldsymbol{x}(t_{i-1})$ must be constructed, given the budget $b(t_{i-1}) > 0$. We illustrate the effectiveness of the two optimization problems for constructing semi-static hedging strategies:

1. Minimizing the mean squared hedging error (MSHE) over each hedging horizon;
2. Minimizing the conditional-tail-expectation (CTE) of the hedging error distribution for each hedging horizon.

We assume proportional transaction costs are incurred when instruments are traded.

Further notation

Let

- α be the confidence level of the CTE associated with the CTE minimization problem.
- N denote the number of hedging target observations for each hedging horizon.
- $y_n(t_i)$ denote the hedging target observation for the n-th sub-scenario of the i-th hedging horizon.
- K_i denote the number of instruments included in the hedging portfolio for the i-th hedging horizon.
- $\boldsymbol{x}(t_{i-1}) = [x(1, t_{i-1}), x(2, t_{i-1}), \ldots, x(K_i, t_{i-1})]'$ denote the vector of hedging instrument positions specified at time t_{i-1}. The k-th component $x(k, t_{i-1})$ is the number of units of hedging instrument k held long over the i-th horizon.
- $\boldsymbol{c}(t_{i-1}) = [c(1, t_{i-1}), c(2, t_{i-1}), \ldots, c(K_i, t_{i-1})]'$ denote the vector of hedging instrument transaction costs. The k-th component $c(k, t_{i-1})$ is the transaction cost per unit of instrument k bought/sold at time t_i.
- $\boldsymbol{u}(t_{i-1}) = [u(1, t_{i-1}), u(2, t_{i-1}), \ldots, u(K_i, t_{i-1})]'$ denote a vector of real numbers introduced to solve the optimization problems.

- $\boldsymbol{z}_n(t_i) = [z_n(1,t_i), z_n(2,t_i), \ldots, z_n(K_i,t_i)]'$ denote the vector of simulated hedging instrument payoffs for the n-th sub-scenario over the i-th hedging horizon.

- $\boldsymbol{\phi}(t_{i-1}) = [\phi(1,t_{i-1}), \phi(2,t_{i-1}), \ldots, \phi(K_i,t_{i-1})]'$ denote the vector of hedging instrument prices at time t_{i-1}.

- $x_L(k)$ and $x_U(k)$ denote the lower and upper limits on the position for hedging instrument k.

Instruments and transaction costs

The stock and zero coupon bonds expiring at the end of each hedging horizon are included as instruments in all hedging portfolios. Henceforth, we assign the instrument indexes s and b as unique references to the stock and the zero coupon bond (risk-free bond) respectively. Over the i-th horizon we set

$$\phi(s,t_{i-1}) = S(t_{i-1}), \qquad z_n(s,t_i) = S_n(t_i),$$
$$\phi(b,t_{i-1}) = \pi P(t_{i-1},t_i), \qquad z_n(b,t_i) = \pi,$$

where $P(t_{i-1},t_i)$ is defined by equation (2.11).

Other instruments we consider in certain portfolios include:

- European call and put options tradable at each rebalancing point, which expire precisely at the end of the current hedging horizon (the term to expiry is τ years). The strike prices of these instruments are set as functions of $S(t_{i-1})$. We refer to these instruments as τ-year options.

- European call and put options which expire at the maturity date of the GMIB option, time T. The strike prices of these instruments are set at time 0. We refer to these instruments as T-year options. In the i-th horizon, the set of T-year option payoffs (given by $\{z_n(k,t_i)\}_{n=1}^{N}$, where k is a T-year option) are the prices of the option at time t_i for each sub-scenario.

Section 4.3.2 discusses further details relating to the inclusion of option instruments in hedging portfolios.

Additional notation is necessary to clearly describe the transaction costs for a semi-static strategy, because some instrument positions set up at time t_{i-1} do not expire at time t_i. Each hedging instrument belongs to one of two mutually exclusive sets. The *Rebalance Set*, denoted by $\mathcal{R}$, includes all instruments for which active (non-zero) positions set up at time t_{i-2} still exist at time t_{i-1}. Changes to the active positions will incur transaction costs. The stock and T-year options belong to $\mathcal{R}$. The *Buy Set*, denoted by $\mathcal{B}$, includes all instruments for which active positions at time t_{i-1} must be bought/sold. The instrument positions expire at time t_i. The zero coupon bond and the τ-year options belong to $\mathcal{R}$. By default, at time t_0 all instruments belong to the Buy Set. The transaction costs incurred in obtaining the optimal portfolio position at the start of the i-th hedging horizon are given by

$$\theta_{TC}(i) = \sum_{k \in \mathcal{B}} c(k, t_{i-1})|x(k, t_{i-1})| + \sum_{k \in \mathcal{R}} c(k, t_{i-1})|x(k, t_{i-1}) - \hat{x}(k, t_{i-2})|. \tag{4.6}$$

Optimization problem definitions

The two optimization problems are similar to the optimization problems defined in Chapter 3. Slight adjustments to the transaction cost constraints are made to ensure that equation (4.6) is satisfied at the optimal solution. The MSHE minimization problem for the i-th hedging horizon is

$$\min_{(\boldsymbol{x}, \boldsymbol{u}) \in \mathbb{R}^{K_i} \times \mathbb{R}^{K_i}} \left(\sum_{n=1}^{N} \left(y_n(t_i) - \boldsymbol{z}_n(t_i)'\boldsymbol{x}(t_{i-1})\right)^2 + \boldsymbol{c}(t_{i-1})'\boldsymbol{u}(t_{i-1}) \right)$$

subject to:

$\boldsymbol{\phi}(t_{i-1})'\boldsymbol{x}(t_{i-1}) + \boldsymbol{c}(t_{i-1})'\boldsymbol{u}(t_{i-1}) \leqslant b(t_{i-1}),$

$u(k,t_{i-1}) - x(k,t_{i-1}) \geqslant 0, \qquad u(k,t_{i-1}) + x(k,t_{i-1}) \geqslant 0, \text{ if } k \in \mathcal{B},$

$u(k,t_{i-1}) - x(k,t_{i-1}) + \hat{x}(k,t_{i-2}) \geqslant 0, \qquad u(k,t_{i-1}) + x(k,t_{i-1}) - \hat{x}(k,t_{i-2}) \geqslant 0, \text{ if } k \in \mathcal{R},$

$x_L(k) \leqslant x(k,t_{i-1}) \leqslant x_U(k), \qquad k = 1,\dots,K_i.$

Let CTE(α) denote the Conditional Tail Expectation of the hedging loss distribution at a confidence level of $\alpha \in (0,1)$. Let $\boldsymbol{v}$ denote an $N \times 1$ vector of real numbers. The CTE(α) minimization problem for the i-th horizon is

$$\min_{(\delta,\boldsymbol{x},\boldsymbol{v},\boldsymbol{u})\in\mathbb{R}\times\mathbb{R}^{K_i}\times\mathbb{R}^N\times\mathbb{R}^{K_i}} \left(\delta + \frac{1}{N(1-\alpha)}\sum_{n=1}^{N} v_n(t_i) + \boldsymbol{c}(t_{i-1})'\boldsymbol{u}(t_{i-1})\right)$$

subject to:

$\boldsymbol{\phi}(t_{i-1})'\boldsymbol{x}(t_{i-1}) + \boldsymbol{c}(t_{i-1})'\boldsymbol{u}(t_{i-1}) \leqslant b(t_{i-1}),$

$v_n(t_i) \geqslant y_n(t_i) - \boldsymbol{z}_n(t_i)'\boldsymbol{x}(t_{i-1}) - \delta, \qquad v_n(t_i) \geqslant 0, \qquad n = 1,\dots,N,$

$u(k,t_{i-1}) - x(k,t_{i-1}) \geqslant 0, \qquad u(k,t_{i-1}) + x(k,t_{i-1}) \geqslant 0, \text{ if } k \in \mathcal{B},$

$u(k,t_{i-1}) - x(k,t_{i-1}) + \hat{x}(k,t_{i-2}) \geqslant 0, \qquad u(k,t_{i-1}) + x(k,t_{i-1}) - \hat{x}(k,t_{i-2}) \geqslant 0, \text{ if } k \in \mathcal{R},$

$x_L(k) \leqslant x(k,t_{i-1}) \leqslant x_U(k), \qquad k = 1,\dots,K_i.$

4.2.4 The total hedging portfolio payoff

At the end of the i-th hedging horizon, the actual hedging portfolio payoff can be calculated. Recall that if the optimal hedging portfolio cost, $\hat{b}(t_{i-1})$, is less than the funds available at time t_{i-1}, $b(t_{i-1})$, then the excess funds, denoted by $\hat{\xi}(t_{i-1}) = b(t_{i-1}) - \hat{b}(t_{i-1})$ are invested in zero coupon bonds until time t_i. Investing excess funds in bonds gives a cushion against possible hedging losses in subsequent hedging horizons. Let

$$\hat{x}_\xi(t_{i-1}) = \frac{\hat{\xi}(t_{i-1})}{\phi(b, t_{i-1}) + c(b, t_{i-1})}$$

denote the number of zero coupon bonds bought at time t_{i-1}, using the excess funds. The MSHE minimization problem may permit $\hat{\xi}(t_{i-1}) > 0$, but the CTE minimization problem, because of the nature of the objective function (and our use of reasonable transaction cost rate assumptions), always uses all funds available such that $\hat{\xi}(t_{i-1}) = 0$.

Once the actual trajectory for each random process is simulated (under P), the actual total hedging portfolio payoff at time t_i is calculated as

$$\psi_A(t_i) = \hat{\boldsymbol{x}}(t_{i-1})'\boldsymbol{z}_A(t_i) + \hat{x}_\xi(t_{i-1}) z_A(b, t_i), \tag{4.7}$$

where the notation $\boldsymbol{z}_A(t_i)$ represents the actual vector of hedging instrument payoffs at time t_i. This payoff is used to construct the hedging portfolio for the $(i+1)$-th horizon. To be clear, the total amount of funds invested in the zero coupon bonds at time t_{i-1} is $(\hat{x}(b, t_{i-1}) + \hat{x}_\xi(t_{i-1}))\pi P(t_{i-1}, t_i)$.

4.2.5 Testing for arbitrage

The simulations of the hedging instrument payoffs should not permit arbitrage opportunities in any given hedging horizon. The fundamental theorem of asset pricing states that a single-period securities market model is arbitrage free if and only if there exists a state price vector (Panjer et al., 1998). In the current context, a state price vector is a strictly positive vector

$$\boldsymbol{\psi}(t_i) = [\psi_1(t_i), \psi_2(t_i), \ldots, \psi_N(t_i)]'$$

such that

$$\boldsymbol{\phi}(t_{i-1}) = \boldsymbol{Z}(t_i)'\boldsymbol{\psi}(t_i),$$

where $\boldsymbol{Z}(t_i) = [\boldsymbol{z}_1(t_i), \ldots, \boldsymbol{z}_N(t_i)]'$ is the $N \times K_i$ matrix containing the instrument payoffs for all N scenarios over the i-th horizon.

To test whether the securities market model for the current hedging horizon has a state price vector, we solve the following linear programming problem:

$$\max_{(c,\boldsymbol{\psi}(t_i))\in\mathbb{R}\times\mathbb{R}^N} C$$

$$\begin{aligned}
&\text{subject to:}\\
&\boldsymbol{\phi}(t_{i-1}) = \boldsymbol{Z}(t_i)'\boldsymbol{\psi}(t_i),\\
&\psi_n(t_i) \geqslant C, \qquad n = 1, \ldots, N.
\end{aligned}$$

If a solution exists such that C is strictly positive, then the model is arbitrage free. If $C \leqslant 0$, or a solution does not exist, then arbitrage opportunities exist. If the model is not arbitrage free, then certain securities within the model should be removed such that the model becomes arbitrage free. Section 4.3.3 discusses this issue further. In particular, arbitrage opportunities are more likely to exist if N is small and several deep out-of-the-money options are included in the hedging portfolio.

4.3 Issues surrounding implementing the method

Here we deal with numerical stability issues related to the implementation of the method. This section can be skipped by readers who are not concerned with implementation issues.

4.3.1 Speeding up the simulations

The computation time for each scenario j depends on the number of sub-scenarios N (generated for each hedging horizon), and the number of simulated conditional paths M used in the calculation of the hedging target value for each sub-scenario. The choices for M and N are discretionary. Choosing larger values for each parameter will improve the accuracy of the optimal hedging portfolio positions chosen over each hedging horizon.

As N increases, the number of hedging target values increases. We obtain a more accurate description of the distribution of the hedging target in each hedging horizon. As a result, the output of the optimization problem in a given horizon is an optimal hedging portfolio which more accurately meets the hedging objective in that horizon, provided sufficient funds are available, particularly with respect to the tails of the hedging target distribution. Hence the actual total hedging portfolio payoff is more likely to match the actual hedging target, particularly when the actual hedging target is an outlier. Furthermore, a larger N means that the single-period market model (for each horizon) is more likely to be arbitrage free when deep out-of-the-money options are included in the single-period market model.

As M increases, the variances of the hedging target estimates decrease. For example, if the hedging target is the GMIB price, then the standard error of the estimate of the GMIB price decreases. In other words, the hedging target values used in the optimization problems are more accurately estimated.

Unfortunately, a larger value for either N or M significantly increases the computation time of each scenario. However, if we can reduce the standard errors of the N hedging target estimators using a variance reduction technique, we can reduce M for a predetermined level of accuracy in the hedging target estimators. In other words, we will be able to speed up the simulations. For example, we may want to set M such that at least $x\%$ (e.g. 95%) of the N hedging target estimates (for any given horizon) have standard errors which are less than some upper bound ϵ^*. It is noted that while it would be nice to choose M such that all of the standard errors are less than some small ϵ^*, this is not practical unless we can set M very large (which we cannot, as it will be too time consuming). A variance reduction technique may greatly assist in achieving this accuracy objective. A clear way to express this idea is through a simple example. Suppose in a given horizon the N hedging target estimates range between 900 and 1300, and the standard errors of these estimates range from 5 to 50, for at least $0.95N$ of the target estimates, using $M = 250$ scenarios. Less than $0.05N$ of the target estimates may have standard errors that are larger than 50. And suppose we want the standard errors of the hedging target estimates to be less than $\epsilon^* = 25$ at least 95% of the time. One way to approximately achieve this result is to set $M = 1000$ (to halve the standard error of an estimator based on n observations, we need $4n$ observations). Another possible solution is to use an effective variance reduction technique, in which case it may be possible to achieve $\epsilon^* = 25$ (at least 95% of the time) using say $M \leqslant 250$.

We can only test whether $x\%$ of the hedging target estimates have standard errors less than ϵ^* by trial and error, so we cannot always be sure that the level of accuracy we desire will always be met. But for practical purposes, we simply want to be confident that the accuracy objective is satisfied in most hedging horizons. We now briefly explain why the standard errors of the hedging target estimates vary in different situations. The standard error of the n-th hedging target estimate for the i-th horizon depends on $\omega_n(t_i|t_{i-1})$. The standard errors of the hedging target estimates increase in the cases where, projecting to maturity, X_1 (lookback component) or X_3 (investment account component) have high exercise probabilities. These cases tend to occur when the stock price has risen sharply up to the current horizon. The standard errors of the hedging target estimates are smaller in the cases where, projecting to maturity, X_2 (guaranteed return component) has a high exercise probability. This case tends to occur when the stock price has fallen sharply up to the current horizon. All else being equal, the GMIB maturity value is less variable when X_2 is deep in-the-money. This is because X_2 depends on just one random variable at time T, $r(T)$, and is not influenced by the stock price path.

Fortunately, we have an effective variance reduction technique for when the hedging target is the GMIB price. The standard errors of the hedging target estimates can be reduced significantly by using the control variate we proposed in Section 2.9.2. Adjustments to the notation in Section 2.9.2 are needed in order to precisely define the control variate estimator of the hedging target for the n-th sub-scenario of the i-th horizon.

Define

$$\overrightarrow{\boldsymbol{U}}(t_i) = (\boldsymbol{U}_i, \boldsymbol{U}_{i+1}, \ldots, \boldsymbol{U}_I),$$

where $\boldsymbol{U}_i$ denotes the 4-dimensional standard uniform random vector that generates the movements in the stock and interest rate processes between time t_{i-1} and time t_i, using the efficient simulation method discussed in Section 2.9.1. $\overrightarrow{\boldsymbol{U}}(t_i)$ is analogous to expression (2.47), allowing for adjustments in notation. $\overrightarrow{\boldsymbol{U}}(t_i)$ contains all of the uniform random variables used to evaluate the hedging target at time t_i.

At the start of the i-th horizon, time t_{i-1}, the method involves simulating N sub-scenarios which represent possible movements of the stock and interest rate processes over the interval $[t_{i-1}, t_i]$. We use the notation $x'_n(t_i)$ to denote the realization of some random variable $X'(t_i)$ for the n-th sub-scenario, conditional on the information known at time t_{i-1}.

For the n-th sub-scenario of the i-th horizon, we define the following random variables at time t_i:

$$f_n(\overrightarrow{\boldsymbol{U}}) = W(n) = e^{-\int_{t_i}^{T}\widetilde{r}(t)\mathrm{d}t}\left[\max\{\widetilde{B}(T)g\ddot{a}_{\overline{20|}}(T), \widetilde{A}(T)\}\right]\Big|\left(\Omega(t_i|t_{i-1}) = \omega_n(t_i|t_{i-1})\right), \tag{4.8}$$

and

$$f_{cv,n}(\overrightarrow{\boldsymbol{U}}) = e^{-\int_{t_i}^{T}\widetilde{r}(t)\mathrm{d}t}\mathcal{Y}_{cv}(n, t_i, T)\Big|\left(\Omega(t_i|t_{i-1}) = \omega_n(t_i|t_{i-1})\right), \tag{4.9}$$

where $\mathcal{Y}_{cv}(n, t_i, T)$ is given by

$$\mathcal{Y}_{cv}(n, t_i, T) = \max\{\widetilde{A}_c(t_i)(1+r_g)^T g E^Q\left[\tilde{\ddot{a}}_{\overline{20|}}(T)|r_n(t_i)\right] - \widetilde{A}_c(T), 0\} + \widetilde{A}_c(T). \tag{4.10}$$

In equation (4.10), $\{\widetilde{A}_c(s)\}_{s=t_i}^{T}$ satisfies the SDE given by equation (2.42). The initial starting point for $\{\widetilde{A}_c(s)\}_{s=t_i}^{T}$ is $\widetilde{A}_c(t_i) = A_n(t_i)$.

Now, $E^Q\left[f_{cv,n}(\overrightarrow{\boldsymbol{U}})\right]$ has an analytical formula that is analogous to equation (2.46):

$$E^Q\left[f_{cv,n}(\overrightarrow{\boldsymbol{U}})\right] = \mathcal{P}_n(t_i, T, \mathcal{K}_n(t_i)) + A_n(t_i)e^{-c(T-t_i)}$$

where $\mathcal{P}_n(t_i, T, \mathcal{K}_n(t_i))$ is the formula for the price at time t_i of a European put option expiring at time T with strike price given by

$$\mathcal{K}_n(t_i) = A(0)(1+r_g)^T g E^Q \left[\widetilde{\ddot{a}}_{\overline{20|}}(T) | r_n(t_i) \right].$$

The analytical formula for $\mathcal{P}_n(t_{i+1}, T, \mathcal{K}_n(t_{i+1}))$ is given by equation (3.17).

Define

$$\overrightarrow{\boldsymbol{U}}_m(t_i) = (\boldsymbol{U}_{i,m}, \boldsymbol{U}_{i+1,m}, \ldots, \boldsymbol{U}_{I,m}) \qquad m = 1, 2, \ldots, M,$$

as the realization of $\overrightarrow{\boldsymbol{U}}(t_i)$ which generates the m-th set of conditional-paths of the stock and interest rate processes from time t_i to time T. An unbiased estimate of the hedging target for the n-th sub-scenario of the i-th horizon, allowing for the control variate, is given by

$$y_n(t_i) = \hat{\beta} E^Q[f_{cv,n}(\overrightarrow{\boldsymbol{U}})] + \frac{1}{M} \sum_{m=1}^{M} \left(f_n(\overrightarrow{\boldsymbol{U}}_m) - \hat{\beta} f_{cv,n}(\overrightarrow{\boldsymbol{U}}_m) \right) \tag{4.11}$$

where $\hat{\beta}$ is given by equation (2.49). The expressions $f(\overrightarrow{\boldsymbol{U}}_m)$ and $f_{cv}(\overrightarrow{\boldsymbol{U}}_m)$ in equation (2.49) are replaced by expressions (4.8) and (4.9). Equation (4.11) replaces equation (4.4) in Step (2) of the CHTS algorithm.

We find that, based on $M = 200$, this control variate estimator provides average efficiency gains that range between between 4-50 (for each sub-scenario of each horizon)[2]. The efficiency gains of some hedging target estimators exceed 100. Hence it is certainly worthwhile to use the control variate estimator.

The control variate estimator also provides significant efficiency gains when the hedging targets are P-valuation targets instead of GMIB prices. To use the control variate estimator in the case where the hedging targets are P-valuation targets, only one change to

[2]Efficiency gains are defined in Section 2.9.2

equation (4.11) is needed. The expression $f_n(\overrightarrow{\boldsymbol{U}})$, given by equation (4.8), is replaced by

$$f_n(\overrightarrow{\boldsymbol{U}}) = e^{-\int_{t_i}^{T} r(s)\mathrm{d}s} \max\{B(T)g\ddot{a}_{\overline{20}|}(T), A(T)\}\Big| \left(\Omega(t_i|t_{i-1}) = \omega_n(t_i|t_{i-1})\right).$$

4.3.2 Comments on the range of hedging instruments

The range of hedging instruments that a semi-static strategy may use is an important issue. As already mentioned, the universe of instruments we consider includes the stock, zero coupon bonds, τ-year options and T-year options. Our motivation for using strategies with τ-year options is that they are practical, and, depending on the underlying, liquid instruments. For example, exchange traded options on the S&P500 Index are natural instruments for hedging GMIBs that are directly linked to, or are highly correlated with the returns of the S&P500 Index. Our motivation for using strategies with T-year options is based on the principle of matching asset and liability cashflows as closely as possible, in order to reduce risk. Of course, trading in T-year options may be problematic in practice. The liquidity of these long-dated instruments is questionable (making rebalancing complicated), and it is likely that the implied volatilities of these options will be higher than the 20% implied volatility assumption that we use in our examples. Furthermore, it may not be possible to trade T-year options across a wide range of strike prices. Despite the fact that strategies using T-year options may be difficult to implement in practice, we still investigate the performances of such strategies for insights.

For the τ-year options, the put-call parity equation holds at the end of each hedging horizon. The equation is not exact, however, because of the presence of transaction costs. This equation introduces approximate linear dependence between the options, stock and the zero coupon bond. When linear dependence is present, it can be difficult to interpret the economic meaning of the optimal instrument positions. Furthermore, the numerical optimization process often fails. Therefore, we avoid including τ-year call and put options with identical strike prices. Because put-call parity holds approximately, there is no material benefit from including puts and calls with identical strikes in hedging portfolios.

In the model framework of Section 3.2, the prices of zero coupon bonds, τ-year and T-year vanilla options can be calculated using analytical formulas at each rebalancing time point. If more sophisticated models are employed for either the stock or interest rates, then analytical formulas may not exist for the relevant option prices. If the option prices cannot be obtained using numerical integration, it may be necessary to add another set of Monte Carlo simulations, in each hedging horizon, to compute the option prices. Furthermore, if the options do not expire at the end of the hedging horizon, then another layer of nested Monte Carlo simulations may be needed to compute the instrument payoffs, which are the option prices at the end of the horizon, for every sub-scenario. These extra simulation layers will significantly increase the computation time of each scenario. These complications partly explain why we have used simple benchmark models for the stock and interest rate processes.

4.3.3 Avoiding arbitrage in each hedging horizon

Section 4.2.5 described a test for arbitrage that should be conducted at the start of each hedging horizon. If the test reveals that arbitrage opportunities exist for the set of sub-scenarios of horizon i, then the single-period market model should be adjusted to remove these arbitrage opportunities. One of the most direct ways to remove arbitrage opportunities is to remove the instruments which are generating the arbitrage opportunities from the model. In our examples, it is fairly easy to identify the instruments which should be removed; specifically, deep in-the-money and deep out-of-the-money options. In this section, we discuss an adjustment to the method for situations where arbitrage opportunities exist.

Each hedging strategy we investigate is constructed from a given universe of hedging instruments. The universe of instruments must yield an arbitrage-free single-period market model. We refer to the *available universe* of hedging instruments as all of the instruments that can be potentially included in a hedging portfolio at a particular rebalancing point. Each strategy type we illustrate depends on an available universe of hedging instruments. In each hedging horizon, a subset of instruments from the available universe which generates an arbitrage free single-period market model is referred to as a *permissible universe* of instruments.

When the available universe consists of just the stock and a risk-free bond, the available universe is identical to the permissible universe, for reasonable stock price parameter values (e.g. non-zero volatility). However, when the universe includes deep out-of-the-money options, arbitrage opportunities may exist for a particular horizon. We now describe one way to obtain a permissible universe of instruments when the available universe permits arbitrage. There are two relevant cases. The first corresponds to the case where the available universe includes τ-year options. The second corresponds to the case where the available universe includes T-year options.

If T-year options are used, we set the strike prices of these options at time 0. Specifically, we choose the strike prices for a set of T-year options, as K_T equally spaced values between appropriate upper and lower end points, X_L and X_U. Namely, the strike price for the k-th consecutive T-year option strike price is given by

$$X(k) = \begin{cases} X_L & \text{if } k = 1, \\ X(k-1) + \Delta & \text{if } k = 2, \dots, K_T - 1, \\ X_U & \text{if } k = K_T. \end{cases}$$

Suppose we are at the start of the i-th horizon. Let $S_n(t_{i-1}, t_i)$ denote the value of the stock at time t_i for the n-th sub-scenario, given the value of $S(t_{i-1})$. Step (1) of the HLS algorithm includes simulating the sample $\{S_n(t_{i-1}, t_i)\}_{n=1}^N$. Let $S_{(\alpha)}(t_{i-1}, t_i)$ denote the α-quantile of the sample $\{S_n(t_{i-1}, t_i)\}_{n=1}^N$. The stock quantiles are used to determine reasonable upper and lower limits to the option strike prices for the i-th horizon. Let $\vec{\alpha}(p)$ denote the p-th component of the following 1×7 vector:

$$\vec{\alpha} = [0.005, 0.01, 0.025, 0.05, 0.1, 0.2, 0.4].$$

Let $\mathcal{X}_i$ denote the set of instruments included in the single-period market model for horizon i. The stock and the risk-free bond are included in $\mathcal{X}_i$ by default. To determine a permissible universe for a given hedging strategy we use the following algorithm, starting at $p = 1$:

(1) Set the stock price confidence level corresponding to the lower end point of the range of feasible strike prices, $\alpha_p = \vec{\alpha}(p)$. (The confidence level corresponding go

the upper end point of the range of feasible strike prices is then set equal to $(1-\alpha_p)$.

(2) Screen the strike prices of the options in the available universe for feasible initial conditions.

- For τ-year option strategies: If instrument k is a τ-year option with strike price $X(k)$, then $k \in \mathcal{X}_i$ if

$$S_{(\alpha_p)}(t_{i-1}, t_i) < X(k) < S_{(1-\alpha_p)}(t_{i-1}, t_i) \tag{4.12}$$

- For T-year option strategies: At the start of each hedging horizon, if instrument k is a T-year option with strike price $X(k)$, then $k \in \mathcal{X}_i$ if

$$\begin{aligned} S_{(\alpha_p)}(t_{i-1}, t_i) - \Delta\delta_c < X(k) < S_{(1-\alpha_p)}(t_{i-1}, t_i) + \Delta\delta_p, \qquad & \text{if } i < I \\ S_{(\alpha_p)}(t_{i-1}, t_i) < X(k) < S_{(1-\alpha_p)}(t_{i-1}, t_i) \qquad & \text{if } i = I, \end{aligned} \tag{4.13}$$

where $\delta_c = 1$ if k is a call ($\delta_c = 0$ if k is a put), and $\delta_p = 1$ if k is a put ($\delta_p = 0$ if k is a call).

(3) Test the market model generated by $\mathcal{X}_i$ for arbitrage. If the market model is arbitrage free, then $\mathcal{X}_i$ is the permissible universe. If arbitrage exists and $p < 7$, then $p = p + 1$ and go back to Step (1). Else if arbitrage exists and $p = 7$, then the permissible universe consists of just the stock and the risk-free bond.

As α_p increases in the above algorithm, fewer out-of-the-money options will end up in the permissible universe. We note that our specific choice of feasible initial conditions is not unique; many variations are possible. Also, the feasible initial conditions in Step (2) do not guarantee that the market model is arbitrage-free – Step (3) is necessary. The market model may still permit arbitrage because we have simulated a finite number of sub-scenarios for the instrument payoffs. In particular, arbitrage is most likely to exist in the case where deep out-of-the money options (which already satisfy the feasible initial conditions) are included in the market model. However, as the number of sub-scenarios N increases, arbitrage opportunities are less likely to exist in the cases where $\mathcal{X}_i$ contains deep out-of-the money options. Unfortunately, N cannot be set too large if computation time is an issue.

The reasoning for equation (4.12) in the case of τ-year options is as follows. If instrument k is a put option with $X(k) < S_{(\alpha_p)}(t_{i-1}, t_i)$, then for any hedging horizon there is a material probability that this instrument will produce a payoff of 0 for every sub-scenario (i.e. arbitrage exists). If $X(k) > S_{(1-\alpha_p)}(t_{i-1}, t_i)$ then arbitrage opportunities may exist between instrument k, the stock and other deep in-the-money puts. In particular, deep in-the-money puts are likely to produce non-zero payoffs for all sub-scenarios, and there may be linear dependence between the payoffs of these puts (and the stock) which produces arbitrage opportunities.[3] When instrument k is a call option, the reverse arguments apply for equation (4.12).

The reasoning for equation (4.13) in the case of T-year options is similar to the reasoning for τ-year options. The main difference is that the strike price boundaries can be wider without introducing arbitrage opportunities. If instrument k is a put option with $X(k) < S_{(\alpha_p)}(t_{i-1}, t_i)$, then its price at time t_{i-1} will be negligible (tending towards 0, using a price based on option pricing theory), which may cause numerical problems when solving the optimization problems. The test for arbitrage often fails without the lower strike boundary constraint. If $X(k) > S_{(1-\alpha_p)}(t_{i-1}, t_i) + \Delta$, approximate linear dependence may arise between the payoffs (prices at time t_i) of deep in-the-money put options and the stock, leading to arbitrage opportunities.[4] At the last hedging horizon, the reasoning for the τ-year options applies. The reverse arguments apply when instrument k is a call option.

We emphasize to less informed readers that setting reasonable feasible initial conditions is very important. It is usually not possible to numerically solve the optimization problems when the hedging portfolio includes options which do not satisfy the feasible initial conditions.

[3] MATLAB reports that, for the no-arbitrage optimization problem (defined in Section 4.2.5), the optimizaton problem constraints are overly stringent, and thus no feasible starting point can be found.

[4] MATLAB reports the optimizaton problem constraints are overly stringent.

4.4 Benchmark parameter assumptions

In the remainder of this chapter, we will illustrate the performance of a variety of semi-static hedging strategies for the GMIB option. The benchmark parameter values we use are the same as the benchmark values adopted in Chapters 2 and 3. Unless stated otherwise, $\pi = b(0) = A(0) = S(0) = 1000$, $T = 10$, $r_g = 5\%$, $\sigma_S = 20\%$, $\breve{a} = a = 0.35$, $\sigma_r = 1.5\%$ $\rho = 0$, and $\breve{\Theta}(t) = \Theta(t)$ depends on a linear approximation of the shape of the U.S. zero coupon bond yield curve halfway through 2008 (the curve is displayed in Figure 2.7 as the one labeled "Benchmark"). The transaction cost rate for the zero coupon bond is set to $c(b, t_i) = 0.05\%\phi(b, t_i)$.[5] For any other instrument $k, k \neq b$ we set $c(k, t_i) = 0.5\%\phi(k, t_i)$. A lower transaction cost rate is adopted for the zero coupon bond as, in practice, transaction costs for risk-free assets tend to be lower. We do not set constraints on the limits for the stock and zero coupon bond positions. For any option instruments, no short selling is allowed ($x_L(k) = 0$ if instrument k is an option) but there are no upper limit constraints. However, in Section 4.8.3 we briefly explore how the semi-static strategies are affected when option short selling is permitted.

The number of scenarios we use is a trade off between stability of the hedging loss distribution and computation time. Unless stated otherwise, $J = 1000$ and $N = 200$. Because we use a control variate in estimating the hedging targets, we set $M = 200$. These values for N and M appear to produce relatively stable results, while not producing excessively large computation times for simulating each scenario. If a control variate is not used, we advise setting $M \geqslant 1000$. We show in Section 4.13 that our choice for N is sufficiently large to produce stable results. If computation time is less of an issue, increasing J is always desirable.

4.5 Understanding the format of the results

For each hedging strategy that we investigate, we report the key statistics summarizing the hedging loss distribution in a table. In many cases we also present tables summarizing the

[5] Including a transaction cost constraint for the zero coupon bond, even if it is very small, helps stabilize the optimization problem when we search for a solution.

behavior of the hedging strategy for particular hedging horizons during the accumulation phase. In this section, we explain how to interpret the numbers in the tables. We explain by example, referring to Tables 4.2 and 4.3 in Section 4.6, to assist the reader in quickly grasping the meanings of the numbers. Tables in later sections follow the same format as Tables 4.2 and 4.3. We note that in Tables 4.2 and 4.3, the liability being hedged is a 10-year European call option, not the GMIB option, but the numbers have the same interpretations.

4.5.1 Tables describing the hedging loss distribution

Table 4.2 reports the hedging loss statistics describing the hedging loss distribution at time T, for different types of hedging strategies. Each strategy type involves a different combination of hedging instruments. For each strategy type, the hedging loss statistics are compared for different rebalancing frequencies. Furthermore, for each strategy type, we measure the impact of transaction costs by comparing the results of the case where the benchmark transaction costs are included to the case where the transaction costs are negligible (where we set $c(k, t_{i-1}) = 0.01\%\phi(k, t_{i-1})$ for all k). We present the results for the negligible transaction cost case in order to more clearly understand the relationship between rebalancing frequency and the hedging loss statistics, with particular emphasis on the tail risk measures. When transaction costs are allowed for, the relationship between rebalancing frequency and the hedging loss statistics may become blurry.

For each strategy and rebalancing frequency combination of interest, we describe the key features of the resulting hedging loss distribution. Table 4.2 shows the estimates of the square root of the mean squared hedging loss ($\text{MSHL}^{1/2}$), 5%-quantile hedging loss, median hedging loss, Value at Risk at a 95% confidence level (i.e. 95%-quantile hedging loss), standard deviation, mean, CTE(95%) and CTE(99%). The numbers in brackets below each estimate are the corresponding 95% confidence intervals for the estimate. The confidence intervals are calculated using the formulas presented in Section 3.3.2, where N in the formulas now corresponds to J. The CTE(99%) provides a conservative measure of tail risk (the high confidence level accommodates for model risk), but unfortunately it is subject to considerable sampling error as it is only based on $0.01J$ hedging loss observations. Given that we use $J = 1000$ (due to time constraints), our CTE(99%) estimates are based on just 10 observations. Therefore we also report the CTE(95%) estimate, which is a more stable estimate.

When the hedging loss statistics for multiple rebalancing frequencies are shown, the hedging losses for each frequency are produced using common random numbers, for the actual stock and interest rate processes, for each scenario j. By this we mean that, for each scenario j, the hedging loss observations for each rebalancing frequency are produced using the same actual stock and short rate paths. Specifically, we simulate the actual paths of the stock and interest rate processes at time intervals corresponding to the highest rebalancing frequency (which is quarterly in our examples, except for one case). This way we have all the relevant variates for the lower rebalancing frequency calculations. For a particular scenario j, the differences between the three rebalancing frequencies in terms of simulated random variables, allowing for common random numbers, come from different simulated sub-scenarios over each horizon. The output of Steps (1) to (6) and (8) to (10) in the HLS algorithm will be different, even with common random numbers. Using common random numbers reduces the influence of sampling errors when comparing the the hedging loss statistics for the different rebalancing frequencies. It is particularly important to use common random numbers when the total number of scenarios generated, J, is small.

4.5.2 Tables describing the behavior of a strategy

Table 4.3 is designed to give the reader a feel for the expected behavior of a particular strategy over the accumulation phase. It can be explained precisely by introducing further notation. Let $r_A^{(j)}$ denote the *actual* realization of some random variable R for the j-th scenario. The random variable we are observing from, although not explicitly defined, will be clear from the notation.

The actual total hedging target error for the i-th hedging horizon of the j-th scenario is given by

$$\begin{aligned}\theta_{TE}^{(j)}(i) &= y_A^{(j)}(t_i) - [\boldsymbol{z}_A^{(j)}(t_i)'\hat{\boldsymbol{x}}^{(j)}(t_{i-1}) + \hat{x}_\xi^{(j)}(t_{i-1}) z_A^{(j)}(b, t_i)] \\ &= y_A^{(j)}(t_i) - \psi_A^{(j)}(t_i), \qquad i = 1, \dots, I-1.\end{aligned}$$

At the final hedging horizon,

$$\theta_{TE}^{(j)}(I) = V_A^{(j)}(T) - \psi_A^{(j)}(T) = e_j,$$

where $V_A^{(j)}(T)$ is the maturity value of the liability (given by equation (4.2) in the case of the GMIB). For $t_i < T$, the actual hedging target error $\theta_{TE}^{(j)}(i)$ measures the amount of mismatch between the actual hedging target and the actual total hedging portfolio payoff for the i-th horizon. This statistic is particularly important to monitor if the insurer wants to meet Goal (2), as outlined in Section 4.1.2, and the hedging target is the liability price. In Table 4.3, the "Mean total target error" and "Std total target error" for the i-th horizon correspond to the mean and standard deviation of $\{\theta_{TE}^{(j)}(i)\}_{j=1}^{J}$. Note that the values for horizon I are the same as the mean and standard deviation in Table 4.2 for the corresponding strategy and rebalancing frequency combination.

For each hedging horizon, it is useful to know how well the objective function is minimized over each hedging horizon. Let

$$\gamma_n^{(j)}(t_i) = y_n^{(j)}(t_i) - \boldsymbol{z}_n^{(j)}(t_i)'\hat{\boldsymbol{x}}^{(j)}(t_{i-1}), \qquad n = 1, \ldots, N,$$

denote the n-th hedging error observation for the i-th hedging horizon of the j-th scenario. In the case of a MSHE minimization strategy, we define the square root of the minimized MSHE (objective function) for the i-th hedging horizon of the j-th scenario as

$$\theta_{MS}^{(j)}(i) = \left(\frac{1}{N} \sum_{n=1}^{N} (\gamma_n^{(j)}(t_i))^2 \right)^{1/2}.$$

In Table 4.3, the "Mean min obj. (MSHE$^{1/2}$)" and "Std min obj. (MSHL$^{1/2}$)" for the i-th horizon correspond to the mean and standard deviation of $\{\theta_{MS}^{(j)}(i)\}_{j=1}^{J}$. The Mean min obj. MSHL$^{1/2}$ is a measure of the average degree of mismatch between the optimized hedging portfolio payoff (excluding excess funds) and the hedging target at the end of each horizon.

Let

$$\gamma^{(j)}_{(1)}(t_i), \gamma^{(j)}_{(2)}(t_i), \ldots, \gamma^{(j)}_{(N)}(t_i)$$

denote the ordered hedging error observations, sorted in ascending order. In the case of a CTE minimization strategy, we define the minimized CTE (objective function) for the i-th hedging horizon of the j-th scenario as

$$\theta^{(j)}_{CTE}(i) = \frac{1}{N(1-\alpha)} \sum_{n=N\alpha+1}^{N} \gamma^{(j)}_{(n)}(t_i).$$

In forthcoming tables related to CTE minimization strategies, "Mean min obj. (CTE)" and "Std min obj. (CTE)" for the i-th horizon correspond to the mean and standard deviation of $\{\theta^{(j)}_{CTE}(i)\}_{j=1}^{J}$. The Mean min obj. (CTE) indicates how well the optimized hedging portfolio can minimize the CTE hedging error distribution at the end of each horizon, on average.

In the case of a MSHE minimization strategy, it may be possible to minimize the MSHE for the i-th horizon using a portfolio that costs less than the amount of funds available. The actual excess funds for the i-th hedging horizon of the j-th scenario are given by

$$\theta^{(j)}_{EF}(i) = \hat{\xi}^{(j)}(t_{i-1}) = (\phi^{(j)}(k, t_{i-1}) + c^{(j)}(k, t_{i-1}))\hat{x}^{(j)}_{\xi}(t_{i-1}), \qquad i = 1, \ldots, I.$$

The excess funds are invested in zero coupon bonds. The excess funds provide security against unfavorable movements between the hedging target and the portfolio value over future hedging horizons. In Table 4.3, the "Mean excess funds" and "Std excess funds" for the i-th horizon correspond to the mean and standard deviation of $\{\theta^{(j)}_{EF}(i)\}_{j=1}^{J}$.

The actual total transaction costs for the i-th hedging horizon of the j-th scenario are given by

$$\theta_{TC}^{(j)}(i) = \sum_{k \in \mathcal{B}} c^{(j)}(k, t_{i-1}) |\hat{x}^{(j)}(k, t_{i-1})| + \sum_{k \in \mathcal{R}} c^{(j)}(k, t_{i-1}) |\hat{x}^{(j)}(k, t_{i-1}) - \hat{x}^{(j)}(k, t_{i-2})| \qquad i = 1, \ldots, I.$$

In Table 4.3, the "Mean trans. costs" and "Std trans. costs" for the i-th horizon correspond to the mean and standard deviation of $\{\theta_{TC}^{(j)}(i)\}_{j=1}^{J}$. The transaction costs at time 0 are often larger than subsequent rebalancing points because the portfolio is constructed from scratch.

In Table 4.3, the section titled "**Mean** $\hat{\boldsymbol{x}}(t_{i-1})$" displays the means of the optimal positions in each of the hedging instruments, for the given hedging horizons. Specifically, for instrument k, the cell for the i-th horizon contains the mean of $\{\hat{x}^{(j)}(k, t_{i-1})\}_{j=1}^{J}$. Similarly, the section titled "**Std Dev** $\hat{\boldsymbol{x}}(t_{i-1})$" displays the standard deviations of the optimal positions in each of the hedging instruments, for the given hedging horizons.

4.6 Semi static hedging of a 10-year call option

Before investigating semi-static hedging strategies for the GMIB option, we first test the MSHE minimization strategies for a simple derivative. The liability is a European call option with a 10 year expiry date. It has a strike price of $X = 2000$. The initial budget for the hedging portfolio is set equal to the price of the option (the option price is determined using option pricing theory). We consider the hedging strategies using two different models:

(1) In a simplified model where interest rates are deterministic. The continuously compounded interest rate is set equal to r per year for all maturities, and is assumed to be constant over time. In this case we are operating in the standard Black-Scholes framework.

(2) Using the model adopted in this thesis for pricing the GMIB option; the short rate satisfies the SDE given by equation (3.2).

A semi-static hedging strategy is characterized by the following factors:

- The type of objective function being optimized in each horizon;
- The type of hedging target;
- The available hedging instruments at each rebalancing point.

To aid our exposition, we organize strategies into types based on these factors. For each strategy type, we present the results for different rebalancing frequencies. We consider the effectiveness of two strategy types for the 10-year call option:

- **Strategy 1**: This type minimizes the MSHE. The hedging target is the call option price. (At horizon I the hedging target is the option payoff.) The hedging portfolio consists of the stock and the risk-free bond. These are the instruments used in a delta-hedging strategy. This strategy should resemble the delta hedging strategy as the rebalancing frequency increases.

- **Strategy 2**: This type minimizes the MSHE. The hedging target is the call option price. The hedging portfolio consists of the stock, risk-free bond and up to eight τ-year call options. The τ-year call option strike prices available at time t_{i-1} are chosen as evenly spaced values between $S_{(0.025)}(t_{i-1}, t_i)$ and $S_{(0.975)}(t_{i-1}, t_i)$ (this notation was defined in Section (4.3.3). Table 4.1 displays the strike prices for various rebalancing frequencies, measured in units of $S(t_{i-1})$.

Rebalancing frequency	*Strike prices measured in units of* $S(t_{i-1})$							
Annual	0.72	0.85	0.97	1.09	1.22	1.34	1.46	1.59
Half-Yearly	0.78	0.87	0.95	1.03	1.12	1.20	1.28	1.37
Quarterly	0.84	0.89	0.95	1.01	1.07	1.12	1.18	1.24
Monthly	0.90	0.93	0.96	1.00	1.03	1.06	1.09	1.13

Table 4.1: *The range of τ-year call option strike prices available at the i-th horizon for different rebalancing frequencies.*

We do not claim that Strategy 2 is the best type of strategy for hedging the 10-year call option. Many variations of Strategy 2 are possible, such as using a different group of strike prices. The optimal available universe of hedging instruments is unknown. However, the selection of options in Strategy 2 should be adequate for the hedging portfolio payoff distribution to closely match the hedging target distribution over each horizon, provided sufficient funds are available.

The purpose of considering the simplified examples in this section is to build some intuition as to how the strategies behave for different hedging horizon lengths and transaction cost assumptions, and various choices of allowable hedging instruments. In the case of more complex options, such as the GMIB, the relationships may not be so clear. Also, by comparing the results based on deterministic and stochastic interest rate models, we obtain insight into how interest rate risk affects the quality of the hedging strategies.

4.6.1 Assuming constant interest rates

In this simplified model the stock price process is a geometric Brownian motion under Q:

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma_S S(t)\mathrm{d}\widetilde{W}(t),$$

where $\{\widetilde{W}(t), t \in [0, T]\}$ is a standard Brownian motion under Q. An advantage of working in this simple setting is that at each rebalancing point, nested simulations are not necessary to compute the hedging target values. The hedging target values can be computed analytically using the Black-Scholes call option formula. Thus, the computation time for each scenario is significantly reduced. We use parameter assumptions that are similar to those for the GMIB. We set $S(0) = 1000$, $\sigma_S = 20\%$, $\mu = 9\%$, $T = 10$, $r = 4\%$. The price at time 0 of a 10 year call option with a strike price of $X = 2000$ is 149.50. Therefore the initial budget is $b(0) = 149.50$.

Table 4.2 displays the hedging loss distribution statistics obtained using Strategies 1 and 2 for the Black-Scholes 10-year call option. We report the results for the monthly rebalancing frequency as a one-off case. This is because the common random number simulations of each scenario, allowing for the monthly rebalancing case, only takes a few seconds. Unfortunately, for all of the subsequent examples, reporting the common random number simulations with the monthly rebalancing frequency, for 1000 scenarios, is too time consuming.

Strategy 1
Negligible transaction costs

Rebal freq	MSHL$^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Annual	130	-234	-31	170	126	-34	283	469
	$(122, 138)$	$(-251, -220)$	$(-40, -19)$	$(142, 199)$	$(117, 134)$	$(-41, -26)$	$(239, 327)$	$(376, 562)$
Half-yearly	91	-164	-15	123	89	-19	204	305
	$(86, 96)$	$(-186, -150)$	$(-22, -8)$	$(111, 153)$	$(83, 94)$	$(-24, -13)$	$(175, 232)$	$(254, 355)$
Quarterly	66	-126	-9	90	65	-12	151	228
	$(62, 70)$	$(-136, -112)$	$(-13, -5)$	$(81, 122)$	$(61, 69)$	$(-16, -8)$	$(129, 172)$	$(191, 266)$
Monthly	39	-68	-5	54	39	-5	102	166
	$(36, 43)$	$(-75, -61)$	$(-7, -3)$	$(47, 77)$	$(36, 43)$	$(-7, -2)$	$(84, 119)$	$(130, 203)$

Strategy 1
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})$ $k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$

Rebal freq	MSHL$^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Annual	130	-230	-13	184	128	-19	303	492
	$(120, 139)$	$(-259, -217)$	$(-20, -6)$	$(167, 217)$	$(118, 138)$	$(-27, -11)$	$(254, 351)$	$(358, 625)$
Half-yearly	94	-151	3	149	94	1	243	406
	$(86, 102)$	$(-165, -133)$	$(-2, 7)$	$(133, 184)$	$(86, 102)$	$(-5, 7)$	$(205, 282)$	$(308, 503)$
Quarterly	74	-87	8	131	72	14	225	374
	$(66, 82)$	$(-99, -77)$	$(5, 11)$	$(110, 165)$	$(65, 80)$	$(9, 18)$	$(188, 262)$	$(287, 462)$
Monthly	53	-25	18	111	45	27	169	261
	$(48, 58)$	$(-31, -19)$	$(15, 20)$	$(100, 130)$	$(41, 50)$	$(24, 30)$	$(147, 191)$	$(236, 286)$

Strategy 2
Negligible transaction costs

Rebal freq	MSHL$^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Annual	6	-9	0	7	6	-1	13	25
	$(6, 7)$	$(-12, -7)$	$(-1, 0)$	$(6, 8)$	$(6, 7)$	$(-1, 0)$	$(10, 15)$	$(17, 32)$
Half-yearly	6	-6	0	6	6	0	15	35
	$(5, 8)$	$(-7, -5)$	$(0, 0)$	$(5, 8)$	$(5, 8)$	$(-1, 0)$	$(10, 19)$	$(17, 54)$
Quarterly	5	-4	0	4	5	0	12	32
	$(3, 7)$	$(-4, -3)$	$(0, 0)$	$(3, 6)$	$(3, 7)$	$(0, 0)$	$(7, 17)$	$(14, 51)$
Monthly	2	-2	0	4	2	0	8	14
	$(2, 3)$	$(-2, -1)$	$(0, 0)$	$(3, 5)$	$(2, 3)$	$(0, 1)$	$(6, 9)$	$(7, 21)$

Strategy 2
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})$ $k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$

Rebal freq	MSHL$^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Annual	18	-5	9	33	15	11	50	80
	$(17, 20)$	$(-8, -1)$	$(8, 9)$	$(31, 36)$	$(13, 17)$	$(10, 12)$	$(43, 57)$	$(64, 97)$
Half-yearly	25	-8	11	44	21	13	64	93
	$(21, 29)$	$(-13, -3)$	$(10, 12)$	$(42, 49)$	$(16, 26)$	$(12, 15)$	$(56, 71)$	$(77, 109)$
Quarterly	39	-11	14	74	33	21	108	164
	$(35, 42)$	$(-20, -6)$	$(13, 16)$	$(69, 85)$	$(29, 37)$	$(19, 23)$	$(95, 122)$	$(128, 200)$
Monthly	88	-16	22	195	75	45	283	425
	$(79, 96)$	$(-24, -11)$	$(20, 26)$	$(161, 220)$	$(67, 83)$	$(40, 50)$	$(250, 317)$	$(350, 501)$

Table 4.2: *Hedging loss distribution statistics derived from Strategies 1 and 2 for the Black-Scholes 10-year call option.*

Comments on Strategy 1 with negligible transaction costs:

- On average small profits are expected, but the average profit tends toward 0 as the rebalancing frequency increases. The median also tends toward 0 as rebalancing frequency increases, as expected.
- The right tail risk is substantial, for annual rebalancing, as indicated by the VaR and CTE risk measures. However, these risk measures decrease rapidly as the rebalancing frequency increases.
- The potential for large profits reduces as the rebalancing frequency increases, as indicated by the 5%-quantiles.
- Overall, as the rebalancing frequency increases, the spread of the hedging loss distribution narrows.

All of the above observations are inevitable if transaction costs are negligible.

Comments on Strategy 1 allowing for the benchmark transaction costs:

- The impact of additional transaction costs for higher rebalancing frequencies is clearly demonstrated. Annual rebalancing produces a mean hedging profit of 19, but higher rebalancing frequencies generate mean hedging losses.
- Even with transaction costs incurred, increasing the rebalancing frequency reduces the right tail risk. However, compared to Strategy 1 with negligible transaction costs, the right tail risk is higher at each rebalancing frequency.

Comments on Strategy 2 with negligible transaction costs:

- Strategy 2 is expected to break-even at each rebalancing frequency. Furthermore, compared to Strategy 1, the standard deviation and $\text{MSHL}^{1/2}$ are vastly smaller. These observations suggest that Strategy 2 provides a very good fit to the hedging target most of the time.
- The tails of the distribution are much smaller, compared to Strategy 1.

- Comparing the annual and half-yearly results, we see that increasing the rebalancing frequency does not necessarily reduce the right tail risk. Furthermore, we see that the half-yearly and quarterly right tail risk measures are similar. These differences are not consequences of sampling error, because the same observations arise from re-running the simulations for another 1000 scenarios. Hence, we suggest that when τ-year options are including in the hedging portfolio, increasing the rebalancing frequency does not necessarily reduce the right tail risk. This behavior will present itself again for the much more complex case of the GMIB option.

Comments on Strategy 2 allowing for the benchmark transaction costs:

- Unlike Strategy 1, the negative impact of transaction costs is much more pronounced. For example, the tail risk measures of the monthly rebalancing case are more than double the corresponding measures for the quarterly rebalancing case, because of transaction costs.

- The transaction costs cause the standard deviation of the hedging loss distribution to increase, as the rebalancing frequency increases.

Table 4.3 illustrates the behavior of Strategy 1 over the accumulation phase, using annual rebalancing and allowing for the benchmark transaction costs, for hedging the Black-Scholes 10-year call option. For Strategy 1, the mean total target error decreases over each horizon. The optimal hedging strategy involves borrowing funds to invest in the stock; the hedging strategy is similar to the delta hedging strategy for a call option. As previously mentioned, the purpose of this type of table is to provide the reader with an indication of how a strategy performs over each hedging horizon. Due to space constraints, we only illustrate the annual rebalancing case.

Table 4.4 illustrates the behavior of Strategy 2, using annual rebalancing. Unlike Strategy 1, the mean total target error increases over time. On average, funds are now borrowed to invest in a combination of the stock and all of the available call options.

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	2	1	-1	-4	-8	-8	-11	-13	-16	-19
Std total target error	20	30	38	48	58	72	82	89	104	128
Mean min obj. (MSHE$^{1/2}$)	19	26	30	34	38	43	49	53	57	67
Std min obj. (MSHE$^{1/2}$)	3	13	17	21	26	29	40	44	48	64
Mean excess funds	0	7	11	16	22	28	32	38	42	48
Std excess funds	0	6	13	19	25	33	40	49	55	64
Mean transaction costs	2	1	1	1	1	1	1	1	1	1
Std transaction costs	0	1	1	1	1	1	1	1	1	1
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.49	0.50	0.50	0.51	0.52	0.54	0.54	0.54	0.55	0.56
Risk-free bond	-0.34	-0.40	-0.46	-0.52	-0.60	-0.68	-0.76	-0.84	-0.93	-1.05
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.02	0.12	0.18	0.21	0.25	0.28	0.31	0.34	0.37	0.41
Risk-free bond	0.02	0.16	0.24	0.31	0.39	0.46	0.54	0.62	0.71	0.82

Table 4.3: *Behavior of Strategy 1 for hedging the Black-Scholes 10-year call option, using annual rebalancing and allowing for the benchmark transaction costs.*

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	1	2	2	3	4	5	6	8	9	11
Std total target error	1	3	4	5	6	8	8	9	11	15
Mean min obj. (MSHE$^{1/2}$)	1	2	3	4	5	6	7	8	10	14
Std min obj. (MSHE$^{1/2}$)	0	1	3	4	5	6	8	8	9	12
Mean excess funds	0	0	0	0	0	0	0	0	0	0
Std excess funds	0	1	1	1	1	1	1	1	2	2
Mean transaction costs	1	1	1	1	1	1	1	1	2	2
Std transaction costs	0	1	1	1	1	1	2	2	2	3
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.07	0.11	0.13	0.14	0.16	0.17	0.18	0.18	0.19	0.19
Risk-free bond	-0.02	-0.04	-0.07	-0.09	-0.13	-0.16	-0.20	-0.24	-0.29	-0.35
Call($0.72S(t_{i-1})$)	0.20	0.17	0.16	0.15	0.14	0.13	0.13	0.12	0.12	0.11
Call($0.85S(t_{i-1})$)	0.06	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.08	0.09
Call($0.97S(t_{i-1})$)	0.08	0.08	0.09	0.09	0.08	0.08	0.08	0.08	0.08	0.08
Call($1.09S(t_{i-1})$)	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.08
Call($1.22S(t_{i-1})$)	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.06	0.06
Call($1.34S(t_{i-1})$)	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.05
Call($1.46S(t_{i-1})$)	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04
Call($1.59S(t_{i-1})$)	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.06	0.07
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.05	0.06	0.08	0.11	0.14	0.17	0.19	0.23	0.26	0.32
Risk-free bond	0.03	0.05	0.06	0.10	0.15	0.20	0.26	0.33	0.42	0.57
Call($0.72S(t_{i-1})$)	0.05	0.10	0.12	0.14	0.15	0.16	0.16	0.17	0.19	0.21
Call($0.85S(t_{i-1})$)	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.11	0.24
Call($0.97S(t_{i-1})$)	0.01	0.02	0.02	0.03	0.04	0.05	0.06	0.07	0.10	0.22
Call($1.09S(t_{i-1})$)	0.01	0.01	0.02	0.02	0.03	0.04	0.05	0.06	0.08	0.22
Call($1.22S(t_{i-1})$)	0.01	0.01	0.02	0.02	0.03	0.03	0.04	0.06	0.08	0.19
Call($1.34S(t_{i-1})$)	0.01	0.01	0.02	0.02	0.03	0.04	0.04	0.05	0.07	0.19
Call($1.46S(t_{i-1})$)	0.02	0.02	0.02	0.03	0.03	0.03	0.04	0.05	0.06	0.16
Call($1.59S(t_{i-1})$)	0.04	0.04	0.05	0.06	0.06	0.07	0.07	0.09	0.10	0.21

Table 4.4: *Behavior of Strategy 2 for hedging the Black-Scholes 10-year call option, using annual rebalancing and allowing for the benchmark transaction costs.*

4.6.2 Allowing for the one-factor short rate model

The purpose of this section is to obtain a sense of the change in performance of the hedging strategies when a second source of uncertainty is introduced. The hedging target values can be computed analytically using the formula given by equation (3.17). We use the same parameter assumptions as for the GMIB option, listed in Section 4.4, except for the initial budget $b(0)$. Under this model, the price of a 10-year European call option with a strike price of $X = 2000$ is 153.57. Thus, the initial budget is $b(0) = 153.57$. Table 4.5 displays the hedging loss distribution statistics for Strategies 1 and 2.

Comments on Strategy 1:

- In the case of negligible transaction costs, the comments made for the constant interest rate case still apply.
- For each rebalancing frequency, the tail risk measures are higher compared to the corresponding constant interest rate cases. Similarly, the 5%-quantiles are lower. The tails are wider because of the additional interest rate uncertainty.

Comments on Strategy 2:

- Annual rebalancing generates the lowest tail risk measures, regardless of transaction costs. Thus, increasing the rebalancing frequency, when the hedging portfolio includes τ-year options, increases the risk of extreme hedging losses.
- Compared to the results of the corresponding constant interest rate cases, the standard deviation and tail risk measures are much higher.

The hedging loss distribution for Strategy 2 changes markedly when interest rates are stochastic. Its tails become thicker and wider. This is an important observation. It suggests that a robust analysis of the performance of semi-static hedging strategies for a long-dated option should allow for stochastic interest rates. Any results obtained from a model with deterministic interest rates may not be adequately capturing the risk associated with a particular hedging strategy.

Rebal freq	MSHL$^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Strategy 1								
Negligible transaction costs								
Annual	152	-268	-28	221	149	-30	336	507
	$(144, 161)$	$(-294, -249)$	$(-39, -15)$	$(182, 258)$	$(140, 158)$	$(-39, -20)$	$(292, 380)$	$(403, 610)$
Half-yearly	122	-190	-14	209	122	-9	302	439
	$(115, 129)$	$(-204, -184)$	$(-19, -5)$	$(182, 231)$	$(115, 129)$	$(-16, -1)$	$(267, 336)$	$(393, 484)$
Quarterly	108	-167	-1	181	108	-1	267	438
	$(100, 116)$	$(-184, -158)$	$(-6, 4)$	$(157, 207)$	$(100, 116)$	$(-8, 6)$	$(229, 304)$	$(323, 554)$
Strategy 1								
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})$ $k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$								
Annual	154	-262	-16	231	153	-18	356	577
	$(145, 163)$	$(-285, -240)$	$(-28, -7)$	$(205, 268)$	$(143, 163)$	$(-28, -9)$	$(307, 406)$	$(446, 709)$
Half-yearly	124	-189	4	208	124	5	318	501
	$(114, 135)$	$(-202, -173)$	$(-3, 9)$	$(193, 241)$	$(114, 135)$	$(-3, 13)$	$(272, 364)$	$(359, 643)$
Quarterly	113	-131	15	229	110	27	317	452
	$(106, 121)$	$(-152, -120)$	$(7, 21)$	$(212, 254)$	$(103, 117)$	$(21, 34)$	$(284, 351)$	$(399, 505)$
Strategy 2								
Negligible transaction costs								
Annual	73	-113	-5	111	73	-2	188	337
	$(66, 80)$	$(-127, -103)$	$(-8, -2)$	$(97, 131)$	$(66, 80)$	$(-7, 2)$	$(155, 221)$	$(246, 428)$
Half-yearly	80	-122	-5	132	80	-1	221	364
	$(73, 87)$	$(-138, -110)$	$(-9, -2)$	$(107, 166)$	$(73, 87)$	$(-6, 4)$	$(187, 255)$	$(292, 437)$
Quarterly	87	-126	-7	148	87	-1	241	415
	$(76, 97)$	$(-135, -112)$	$(-10, -4)$	$(123, 170)$	$(76, 97)$	$(-6, 5)$	$(199, 283)$	$(280, 549)$
Strategy 2								
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})$ $k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$								
Annual	83	-95	5	150	82	16	242	411
	$(74, 92)$	$(-111, -83)$	$(2, 7)$	$(133, 175)$	$(73, 90)$	$(11, 21)$	$(204, 281)$	$(300, 522)$
Half-yearly	96	-96	8	197	93	23	294	478
	$(86, 106)$	$(-112, -86)$	$(5, 11)$	$(167, 221)$	$(83, 103)$	$(17, 29)$	$(252, 335)$	$(369, 588)$
Quarterly	111	-86	13	223	106	35	350	607
	$(98, 125)$	$(-107, -73)$	$(9, 17)$	$(188, 268)$	$(93, 119)$	$(29, 42)$	$(295, 405)$	$(422, 792)$

Table 4.5: *Hedging loss distribution statistics derived from Strategies 1 and 2 for the 10-year call option (under the one-factor interest rate model).*

Table 4.6 shows the behavior of Strategy 1, using annual rebalancing, for hedging the 10-year call option under the one-factor interest rate model. The results are similar to those for the constant interest rate case in Table 4.3. In particular, the expected instrument positions over each hedging horizon are almost identical. The standard deviations of the total target error, minimized MSHE and excess funds are higher, for most horizons, under the one-factor interest rate model.

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	-2	-3	-4	-6	-6	-10	-11	-17	-17	-18
Std total target error	29	40	52	65	76	88	102	115	130	153
Mean min obj. (MSHE$^{1/2}$)	23	30	35	42	48	53	59	65	70	79
Std min obj. (MSHE$^{1/2}$)	3	13	20	28	34	40	47	54	61	78
Mean excess funds	1	10	16	21	28	32	39	45	52	59
Std excess funds	1	12	19	27	34	41	49	59	68	78
Mean transaction costs	3	1	1	1	1	1	1	1	1	2
Std transaction costs	0	1	1	1	1	1	1	1	1	2
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.50	0.52	0.53	0.55	0.57	0.57	0.57	0.58	0.57	0.58
Risk-free bond	-0.36	-0.43	-0.50	-0.58	-0.66	-0.75	-0.83	-0.93	-1.01	-1.14
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.02	0.12	0.17	0.21	0.24	0.28	0.30	0.33	0.36	0.41
Risk-free bond	0.02	0.16	0.24	0.31	0.39	0.47	0.54	0.62	0.72	0.84

Table 4.6: *Behavior of Strategy 1 for hedging the 10-year call option (assuming stochastic interest rates), using annual rebalancing and allowing for the benchmark transaction costs.*

Table 4.7 illustrates the behavior of Strategy 2, using annual rebalancing. The total target errors, minimized objective function values and excess funds all tend to be higher and more variable than in the constant interest rate case.

4.7 Hedging strategy types for the GMIB

We consider the effectiveness of three strategy types for the GMIB:

- **Strategy 1**: The hedging portfolio consists of the stock and the zero coupon bond.
- **Strategy 2**: The hedging portfolio consists of the stock, zero coupon bond, and τ-year call and put options. The put and call option strike prices available at time t_{i-1} are chosen as equally spaced values in the intervals $[S_{(0.05)}(t_{i-1}, t_i), S(t_{i-1})]$ and $[S(t_{i-1}), S_{(0.95)}(t_{i-1}, t_i)]$, respectively. Table 4.8 displays the strike prices for various rebalancing frequencies, measured in units of $S(t_{i-1})$.

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	1	2	2	3	5	4	9	9	13	16
Std total target error	22	28	35	39	49	55	63	72	79	82
Mean min obj. ($\text{MSHE}^{1/2}$)	13	16	21	24	28	32	34	36	37	37
Std min obj. ($\text{MSHE}^{1/2}$)	1	8	14	18	24	30	36	42	52	63
Mean excess funds	0	4	7	9	11	13	15	16	18	18
Std excess funds	0	8	12	15	19	23	28	31	34	35
Mean transaction costs	1	1	1	1	1	1	1	2	2	2
Std transaction costs	0	1	1	1	1	2	2	2	2	3
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.10	0.14	0.17	0.18	0.19	0.20	0.21	0.21	0.21	0.20
Risk-free bond	-0.03	-0.07	-0.11	-0.14	-0.17	-0.21	-0.26	-0.30	-0.35	-0.39
Call($0.72S(t_{i-1})$)	0.19	0.16	0.15	0.14	0.13	0.13	0.12	0.12	0.12	0.12
Call($0.85S(t_{i-1})$)	0.06	0.07	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08
Call($0.97S(t_{i-1})$)	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.09	0.10
Call($1.09S(t_{i-1})$)	0.08	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.08	0.08
Call($1.22S(t_{i-1})$)	0.06	0.07	0.06	0.06	0.06	0.06	0.06	0.06	0.07	0.07
Call($1.34S(t_{i-1})$)	0.06	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.06
Call($1.46S(t_{i-1})$)	0.05	0.04	0.05	0.05	0.05	0.05	0.04	0.04	0.04	0.05
Call($1.59S(t_{i-1})$)	0.09	0.08	0.08	0.08	0.07	0.07	0.07	0.07	0.06	0.06
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.06	0.09	0.12	0.15	0.18	0.21	0.24	0.26	0.29	0.35
Risk-free bond	0.04	0.07	0.11	0.17	0.23	0.29	0.37	0.43	0.54	0.69
Call($0.72S(t_{i-1})$)	0.08	0.14	0.16	0.16	0.17	0.18	0.18	0.19	0.20	0.26
Call($0.85S(t_{i-1})$)	0.04	0.06	0.07	0.07	0.08	0.08	0.09	0.10	0.12	0.22
Call($0.97S(t_{i-1})$)	0.04	0.05	0.06	0.06	0.06	0.07	0.07	0.08	0.11	0.24
Call($1.09S(t_{i-1})$)	0.05	0.05	0.05	0.05	0.06	0.06	0.07	0.08	0.10	0.22
Call($1.22S(t_{i-1})$)	0.05	0.06	0.05	0.06	0.06	0.06	0.07	0.07	0.09	0.20
Call($1.34S(t_{i-1})$)	0.06	0.06	0.06	0.06	0.06	0.06	0.07	0.07	0.08	0.19
Call($1.46S(t_{i-1})$)	0.07	0.06	0.07	0.06	0.07	0.07	0.07	0.07	0.08	0.19
Call($1.59S(t_{i-1})$)	0.11	0.11	0.10	0.11	0.10	0.10	0.10	0.10	0.12	0.19

Table 4.7: *Behavior of Strategy 2 for hedging the 10-year call option (assuming stochastic interest rates), using annual rebalancing and allowing for the benchmark transaction costs.*

- **Strategy 3**: The hedging portfolio consists of the stock, zero coupon bond and T-year call and put options. The available strike prices for the T-year call and put options, which are determined at time 0, are displayed in Table 4.8, measured in units of $S(0)$. At time 0, the probability of the future stock price at time T being between 0.8 and 6.4 is 89.4%.

τ-year option strike prices *Measured in units of $S(t_{i-1})$*								
Rebalancing frequency	**Puts**				**Calls**			
Annual	0.77	0.85	0.92	1.00	1.12	1.25	1.37	1.49
Half-Yearly	0.82	0.88	0.94	1.00	1.08	1.15	1.23	1.31
Quarterly	0.86	0.91	0.95	1.00	1.05	1.10	1.15	1.20

T-year option strike prices *Measured in units of $S(0)$*							
Puts				**Calls**			
0.80	1.60	2.40	3.20	4.00	4.80	5.60	6.40

Table 4.8: *The range of τ-year and T-year option strike prices available at the i-th horizon for different rebalancing frequencies.*

With Strategy 2, we choose to use out-of-the-money and at-the-money options. We do not claim that the instruments we have chosen will yield the best results. We do not have a formal criterion for choosing the optimal strike prices. The put options are designed to help keep the hedging portfolio value in line with the hedging targets which occur when the stock price jumps downwards. The call options are designed to help in a similar way when the stock price jumps upwards. Developing a criterion for choosing the optimal available universe of hedging instruments, for the semi-static hedging strategy of a long-dated option, is, perhaps, an avenue for future research.

We compare the hedging loss distributions for each semi-static strategy type to the hedging loss distributions obtained from two static hedging strategies. Both static strategies are designed to minimize the CTE(99%). The first static strategy corresponds to Portfolio F1 (shown in Section 3.10, for the case of stochastic interest rates); it includes a Put($0.8S(0)$), the stock, ZCB(10) and ZCB(29). The second static strategy corresponds to Portfolio F2 (also shown in Section 3.10); it includes LBC($1.6S(0)$), LBP, Put($0.8S(0)$), the stock, ZCB(10) and ZCB(29). Henceforth, these strategies are referred to as the *static put strategy* and the *static lookback strategy*. In practice, the static put strategy should be relatively straightforward to implement (the put is out of the money). However, it may not be possible to implement the optimal static lookback strategy without substantial additional funds, because the lookback options may have much higher implied volatilities than the 20% implied volatility assumption we have assumed (i.e. the lookbacks may sell at much higher prices). Recall that Section 3.12 has explored the decrease in effectiveness of the static strategies from higher than expected option prices. Thus, even though the static put strategy is less favorable than the static lookback strategy in terms of risk, we show the results for the static put strategy because it is more practicable than the static lookback strategy. To allow for consistent comparisons with the semi-static results, we use the common random numbers in each scenario to obtain the hedging losses from the static put and lookback strategies. Given that the number of scenarios is $J = 1000$, the hedging loss statistics for the static strategies are expected to moderately vary with each example; our best estimates of the statistics for the static put and lookback strategies are given in Section 3.10, because the estimates there are based on 20,000.[6]

For each strategy that we illustrate, the highest rebalancing frequency considered is quarterly. This is partly due to balancing the trade off between transaction costs and reducing tail risk, and partly due to time constraints. A strategy can easily be implemented at a higher rebalancing frequency if desired. However, as illustrated in the following results, higher transaction costs do significantly erode the quality of a hedging strategy.

4.8 Using MSHE minimization hedging strategies

In this section, we investigate the performance of MSHE minimization hedging strategies for the GMIB, using the benchmark parameter assumptions. MSHE minimization strategies are intuitively appealing. In each hedging horizon, the hedging portfolio payoff is matched closely to the locations where the hedging targets occur with high probabilities. For all of the strategies, the hedging target is the GMIB price. In Section 4.8.1, we test hedging strategies in the simplified setting where interest rates are constant. In Section 4.8.2, we test the strategies under the one-factor interest rate model. Comparing the results allows us to isolate the impact of any additional complications which arise when

[6] As a side issue, the reader can gain a sense of the sensitivity of the hedging loss statistics for the static strategies, for small sample sizes, by comparing the results for each relevant example in this chapter.

interest rates are stochastic. Section 4.8.3 briefly explores how the results are affected by permitting option short selling. In Section 4.8.4, we illustrate through figures how Strategy 2 behaves under different scenarios for the stock and interest rate processes. These figures are designed to give the reader another perspective on how the semi-static strategy works.

4.8.1 Assuming constant interest rates

Section 2.4.2 displayed the fair fee rates for the GMIB for constant parallel yield curves. In this section we assume $r = 5\%$, for which the fair fee rate is $c = 3\%$ (which we adopt here). We note that the static hedging strategies (which up until now we have not used in a constant interest rate model) include ZCB(10), but ZCB(29) is redundant. Tables 4.9 and 4.10 display the results from applying Strategies 1, 2 and 3.

Comments on Strategy 1:

- With negligible transaction costs, the mean and median are negative, and both tend toward 0 as the rebalancing frequency increases. However, if we account for transaction costs, the mean and median are positive for the half-yearly and quarterly cases.
- Allowing for transaction costs significantly increases the VaR(95%) and CTE risk measures.
- The static lookback strategy appears to yield the most desirable hedging loss distribution.

Comments on Strategy 2:

- The hedging portfolio matches the GMIB liability remarkably closely, as indicated by the MSHL$^{1/2}$. Introducing the options vastly reduces the tail risk measures, compared to Strategy 1. However, the sharp reduction in risk comes at cost. A small loss is expected.

Strategy 1
Negligible transaction costs

Rebal freq	MSHL$^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Annual	148	-260	-20	209	146	-26	322	533
	$(139, 157)$	$(-284, -248)$	$(-28, -10)$	$(182, 238)$	$(137, 155)$	$(-35, -17)$	$(277, 368)$	$(439, 628)$
Half-yearly	110	-188	-8	154	109	-13	252	388
	$(102, 117)$	$(-208, -176)$	$(-14, -2)$	$(133, 188)$	$(102, 116)$	$(-19, -6)$	$(216, 288)$	$(318, 458)$
Quarterly	76	-132	-4	102	75	-9	169	270
	$(71, 81)$	$(-143, -117)$	$(-8, 1)$	$(94, 121)$	$(70, 81)$	$(-13, -4)$	$(144, 194)$	$(225, 315)$
Static w/	561	-1000	-247	533	511	-234	611	824
10-yr put	$(538, 585)$	$(-1045, -942)$	$(-301, -191)$	$(523, 548)$	$(490, 531)$	$(-265, -202)$	$(560, 663)$	$(592, 1057)$
Static w/	156	-301	-49	129	151	-39	130	132
lookback	$(148, 164)$	$(-331, -275)$	$(-59, -37)$	$(129, 129)$	$(144, 158)$	$(-49, -30)$	$(129, 131)$	$(126, 139)$

Strategy 1
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})\ k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$

Rebal freq	MSHL$^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Annual	154	-252	-7	260	154	-7	374	535
	$(145, 162)$	$(-266, -234)$	$(-15, 1)$	$(228, 307)$	$(145, 162)$	$(-16, 3)$	$(334, 415)$	$(463, 608)$
Half-yearly	111	-157	10	200	111	10	295	453
	$(103, 119)$	$(-175, -145)$	$(4, 16)$	$(176, 230)$	$(103, 118)$	$(3, 17)$	$(259, 331)$	$(378, 529)$
Quarterly	87	-98	27	165	82	30	241	325
	$(82, 93)$	$(-107, -88)$	$(22, 32)$	$(151, 196)$	$(77, 87)$	$(25, 35)$	$(215, 267)$	$(280, 371)$
Static w/	586	-1044	-334	530	519	-272	594	758
10-yr put	$(561, 611)$	$(-1141, -977)$	$(-389, -283)$	$(516, 540)$	$(498, 540)$	$(-304, -240)$	$(557, 631)$	$(603, 912)$
Static w/	158	-311	-68	129	149	-52	129	129
lookback	$(150, 166)$	$(-328, -301)$	$(-77, -56)$	$(129, 129)$	$(143, 156)$	$(-61, -42)$	$(129, 129)$	$(129, 129)$

Strategy 2
Negligible transaction costs

Rebal freq	MSHL$^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Annual	13	-16	3	18	12	2	30	59
	$(11, 14)$	$(-20, -14)$	$(2, 3)$	$(16, 21)$	$(11, 14)$	$(2, 3)$	$(24, 36)$	$(38, 80)$
Half-yearly	13	-14	2	20	12	3	35	61
	$(11, 14)$	$(-17, -13)$	$(2, 3)$	$(18, 24)$	$(11, 14)$	$(2, 3)$	$(29, 42)$	$(47, 76)$
Quarterly	15	-18	2	23	15	3	41	85
	$(13, 17)$	$(-21, -15)$	$(1, 2)$	$(20, 26)$	$(12, 17)$	$(2, 3)$	$(32, 50)$	$(56, 115)$
Static w/	588	-1041	-303	536	532	-250	638	953
10-yr put	$(558, 617)$	$(-1103, -988)$	$(-367, -240)$	$(523, 550)$	$(506, 559)$	$(-283, -217)$	$(573, 702)$	$(661, 1245)$
Static w/	157	-312	-42	129	152	-39	131	140
lookback	$(149, 166)$	$(-343, -290)$	$(-59, -29)$	$(129, 129)$	$(145, 160)$	$(-48, -30)$	$(127, 135)$	$(119, 161)$

Strategy 2
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})\ k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$

Rebal freq	MSHL$^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Annual	24	-4	14	45	17	17	65	99
	$(22, 26)$	$(-8, -1)$	$(13, 15)$	$(42, 51)$	$(15, 19)$	$(16, 18)$	$(56, 73)$	$(72, 127)$
Half-yearly	30	3	17	63	20	23	85	121
	$(28, 32)$	$(2, 4)$	$(16, 18)$	$(59, 70)$	$(18, 22)$	$(21, 24)$	$(77, 94)$	$(108, 133)$
Quarterly	39	5	24	77	25	30	106	148
	$(36, 41)$	$(4, 6)$	$(22, 25)$	$(71, 87)$	$(23, 27)$	$(28, 31)$	$(95, 117)$	$(121, 175)$
Static w/	561	-1004	-334	521	491	-272	591	756
10-yr put	$(540, 581)$	$(-1070, -933)$	$(-407, -268)$	$(512, 535)$	$(473, 509)$	$(-302, -242)$	$(541, 640)$	$(530, 982)$
Static w/	167	-330	-67	129	157	-58	129	129
lookback	$(158, 176)$	$(-356, -305)$	$(-77, -56)$	$(129, 129)$	$(149, 164)$	$(-67, -48)$	$(129, 129)$	$(129, 129)$

Table 4.9: *Hedging loss distribution statistics derived from Strategies 1 and 2 for the GMIB, assuming constant interest rates.*

- When transaction costs are included, the 5%-quantiles are roughly around 0. This indicates that at least 95% of the time Strategy 2 will yield a small hedging loss. Furthermore, the VaR and CTE risk measures increase significantly for each rebalancing frequency, when transaction costs are included.

- Regardless of transaction costs, it seems that the tail risk measures increase with rebalancing frequency. Recall that in the case of hedging the 10-year call option, where we assumed constant interest rates and negligible transaction costs, we found that increasing the rebalancing frequency of strategies involving options did not reduce the right tail risk.

- When transaction costs are included, annual rebalancing yields better results than half-yearly or quarterly rebalancing.

- The annual rebalancing strategy produces smaller tail risk measures than the static lookback strategy, even when transaction costs are included.

Strategy 3 Negligible transaction costs								
Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	44	-52	5	48	43	2	80	156
	$(32, 55)$	$(-62, -46)$	$(4, 6)$	$(42, 55)$	$(32, 55)$	$(-1, 5)$	$(60, 100)$	$(70, 243)$
Half-yearly	36	-44	4	45	36	3	74	129
	$(27, 45)$	$(-50, -36)$	$(3, 5)$	$(41, 54)$	$(27, 45)$	$(0, 5)$	$(63, 86)$	$(101, 156)$
Quarterly	28	-29	4	35	28	3	64	117
	$(22, 33)$	$(-34, -25)$	$(3, 4)$	$(32, 42)$	$(22, 33)$	$(2, 5)$	$(52, 76)$	$(86, 149)$
Static w/	577	-1024	-325	535	510	-271	623	883
10-yr put	$(553, 602)$	$(-1097, -959)$	$(-371, -278)$	$(521, 547)$	$(489, 532)$	$(-302, -239)$	$(571, 675)$	$(658, 1107)$
Static w/	155	-300	-66	129	146	-51	130	133
lookback	$(147, 163)$	$(-317, -286)$	$(-73, -51)$	$(129, 129)$	$(140, 153)$	$(-60, -42)$	$(128, 132)$	$(125, 142)$

Strategy 3 Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})$ $k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$								
Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	64	-42	22	75	60	22	142	341
	$(34, 94)$	$(-46, -36)$	$(20, 24)$	$(68, 83)$	$(29, 91)$	$(18, 26)$	$(85, 200)$	$(69, 613)$
Half-yearly	54	-30	21	75	49	24	124	255
	$(29, 80)$	$(-36, -25)$	$(20, 24)$	$(69, 83)$	$(21, 76)$	$(21, 27)$	$(78, 169)$	$(34, 476)$
Quarterly	54	-14	23	90	46	29	140	280
	$(38, 70)$	$(-22, -12)$	$(22, 25)$	$(83, 98)$	$(28, 64)$	$(26, 32)$	$(102, 178)$	$(101, 459)$
Static w/	575	-990	-329	520	507	-271	602	800
10-yr put	$(551, 599)$	$(-1082, -939)$	$(-393, -280)$	$(510, 540)$	$(487, 528)$	$(-303, -240)$	$(563, 641)$	$(664, 937)$
Static w/	171	-319	-62	129	162	-55	129	129
lookback	$(157, 185)$	$(-353, -304)$	$(-73, -51)$	$(129, 129)$	$(149, 175)$	$(-66, -45)$	$(129, 129)$	$(129, 129)$

Table 4.10: *Hedging loss distribution statistics derived from Strategy 3 for the GMIB, assuming constant interest rates.*

Comments on Strategy 3:

- The tail risk measures are lower than the corresponding values for Strategy 1, but higher than the corresponding values for Strategy 2.
- Compared to Strategy 2, there is a higher likelihood of making a small profit, as indicated by the 5%-quantiles.

Some readers may initially find it surprising that, in the case of Strategy 2 with negligible transaction costs, increasing the rebalancing frequency leads to a deterioration in performance. A strategy which has more opportunities to rebalance, in a setting where transaction costs are negligible, is not expected to be worse off than a strategy with fewer opportunities to rebalance.[7] If rebalancing less often produces better results, the more flexible strategy could choose to rebalance only at the time points of the less flexible strategy. However, under the method we use, this type of forward decision making is not taken into consideration. In the method we use, at each rebalancing point a new optimization problem is solved, and these optimization problems are myopic. The problems are solved with respect to movements over the current hedging horizon. The optimal solutions are not forward horizon looking. In Section 4.8.4, we provide some reasoning for why increasing the rebalancing frequency, for semi-static hedging strategies using options, may lead to a deterioration in performance.

Tables 4.11, 4.12 and 4.13 display the behavior of Strategies 1, 2 and 3 respectively, using annual rebalancing. Strategies 2 and 3 minimize the objective function, the MSHE, much more effectively than Strategy 1. On average, Strategy 1 accumulates excess funds over time. Strategies 2 and 3 do not offer any excess funds; all available funds are invested in the options in order to minimize the objective function. With Strategy 3, a significant proportion of the funds at each rebalancing point are invested in the 10-year put options with strike prices of $1.6S(0)$ and $2.4S(0)$. These strike prices are in the vicinity of the GMIB maturity value, in the case where there is a high probability of the guaranteed return component being exercised.

[7] In making this statement we are ignoring the complication (in our implementation) that by changing the rebalancing frequency from, say, annually to half-yearly, the term to expiry of the τ-year options changes from one year to half a year.

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	5	6	5	3	3	1	-2	-3	-3	-7
Std total target error	37	45	53	61	69	80	90	111	132	154
Mean min obj. ($MSHE^{1/2}$)	22	28	33	39	44	51	57	67	79	102
Std min obj. ($MSHE^{1/2}$)	3	15	21	26	31	34	42	51	68	89
Mean excess funds	0	5	10	14	18	23	29	35	43	52
Std excess funds	0	5	11	17	23	29	36	44	55	68
Mean transaction costs	3	1	1	1	1	1	1	1	1	2
Std transaction costs	0	0	0	0	1	1	1	1	1	1
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.55	0.56	0.55	0.53	0.50	0.48	0.44	0.41	0.38	0.35
Risk-free bond	0.4472	0.4273	0.42	0.45	0.49	0.53	0.60	0.67	0.75	0.84
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.02	0.13	0.18	0.20	0.22	0.23	0.23	0.24	0.24	0.26
Risk-free bond	0.02	0.16	0.22	0.25	0.27	0.29	0.31	0.34	0.38	0.47

Table 4.11: *Behavior of Strategy 1 for hedging the GMIB, using annual rebalancing and allowing for the benchmark transaction costs (assuming interest rates are constant).*

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	5	7	7	8	7	10	12	13	15	17
Std total target error	29	30	33	35	34	32	31	30	25	17
Mean min obj. ($MSHE^{1/2}$)	9	10	11	12	13	14	15	16	16	19
Std min obj. ($MSHE^{1/2}$)	1	3	4	5	7	8	9	11	13	13
Mean excess funds	0	0	0	0	0	0	0	0	0	0
Std excess funds	0	0	1	1	2	2	2	2	1	1
Mean transaction costs	3	1	1	1	1	1	1	1	2	2
Std transaction costs	0	1	1	1	1	1	1	1	2	3
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.48	0.51	0.50	0.49	0.46	0.44	0.40	0.37	0.33	0.30
Risk-free bond	0.51	0.48	0.47	0.48	0.52	0.56	0.63	0.69	0.78	0.85
Put($0.77S(t_{i-1})$)	0.08	0.08	0.09	0.08	0.09	0.08	0.08	0.07	0.06	0.01
Put($0.85S(t_{i-1})$)	0.06	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.06	0.08
Put($0.92S(t_{i-1})$)	0.07	0.05	0.05	0.05	0.04	0.04	0.04	0.04	0.05	0.15
Put($1.00S(t_{i-1})$)	0.01	0.07	0.08	0.10	0.10	0.10	0.11	0.12	0.12	0.07
Call($1.12S(t_{i-1})$)	0.11	0.07	0.06	0.05	0.05	0.05	0.05	0.05	0.05	0.07
Call($1.25S(t_{i-1})$)	0.04	0.07	0.06	0.06	0.06	0.06	0.06	0.06	0.06	0.05
Call($1.37S(t_{i-1})$)	0.04	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.03	0.04
Call($1.49S(t_{i-1})$)	0.13	0.08	0.07	0.07	0.07	0.07	0.06	0.06	0.06	0.06
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.02	0.12	0.18	0.22	0.25	0.26	0.27	0.28	0.28	0.33
Risk-free bond	0.02	0.14	0.22	0.28	0.32	0.36	0.39	0.44	0.53	0.76
Put($0.77S(t_{i-1})$)	0.08	0.08	0.08	0.08	0.09	0.09	0.09	0.08	0.08	0.04
Put($0.85S(t_{i-1})$)	0.06	0.06	0.05	0.05	0.05	0.06	0.06	0.06	0.07	0.14
Put($0.92S(t_{i-1})$)	0.04	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.06	0.22
Put($1.00S(t_{i-1})$)	0.02	0.07	0.07	0.06	0.07	0.07	0.08	0.09	0.10	0.17
Call($1.12S(t_{i-1})$)	0.03	0.05	0.06	0.06	0.06	0.06	0.07	0.07	0.08	0.16
Call($1.25S(t_{i-1})$)	0.04	0.05	0.05	0.05	0.06	0.06	0.07	0.07	0.08	0.15
Call($1.37S(t_{i-1})$)	0.05	0.06	0.06	0.05	0.05	0.05	0.06	0.06	0.06	0.14
Call($1.49S(t_{i-1})$)	0.08	0.08	0.08	0.08	0.09	0.09	0.09	0.09	0.10	0.15

Table 4.12: *Behavior of Strategy 2 for hedging the GMIB, using annual rebalancing and allowing for the benchmark transaction costs (assuming interest rates are constant).*

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	7	9	10	9	12	11	12	14	18	22
Std total target error	31	31	32	34	33	36	36	33	31	60
Mean min obj. (MSHE$^{1/2}$)	11	13	13	14	15	17	19	21	23	38
Std min obj. (MSHE$^{1/2}$)	1	4	5	6	8	10	12	15	20	36
Mean excess funds	0	0	0	0	1	1	1	1	1	1
Std excess funds	0	0	1	1	2	3	4	5	7	5
Mean transaction costs	5	1	1	1	1	1	1	2	2	4
Std transaction costs	1	1	1	0	1	1	4	7	6	9
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.80	0.85	0.85	0.84	0.82	0.79	0.75	0.70	0.66	0.49
Risk-free bond	0.05	-0.07	-0.10	-0.11	-0.10	-0.07	-0.02	0.09	0.21	0.62
Put$(0.80S(0),T)$	1.70	0.78	0.29	0.09	0.01	0.00	0.00	0.00	0.00	0.01
Put$(1.60S(0),T)$	0.45	0.81	0.98	1.07	1.06	0.92	0.71	0.50	0.34	0.22
Put$(2.40S(0),T)$	0.00	0.00	0.01	0.04	0.10	0.21	0.29	0.34	0.35	0.29
Put$(3.20S(0),T)$	0.00	0.00	0.00	0.00	0.01	0.03	0.07	0.11	0.14	0.11
Call$(4.00S(0),T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.04
Call$(4.80S(0),T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.02
Call$(5.60S(0),T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.01
Call$(6.40S(0),T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.02
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.07	0.09	0.10	0.11	0.11	0.13	0.17	0.23	0.27	0.35
Risk-free bond	0.12	0.16	0.17	0.17	0.19	0.26	0.46	0.83	1.22	1.36
Put$(0.80S(0),T)$	0.84	0.76	0.47	0.23	0.07	0.03	0.01	0.01	0.03	0.05
Put$(1.60S(0),T)$	0.34	0.35	0.32	0.40	0.47	0.53	0.51	0.43	0.37	0.38
Put$(2.40S(0),T)$	0.00	0.02	0.09	0.21	0.33	0.44	0.42	0.37	0.30	0.32
Put$(3.20S(0),T)$	0.00	0.00	0.00	0.04	0.11	0.20	0.32	0.36	0.34	0.22
Call$(4.00S(0),T)$	0.00	0.00	0.00	0.00	0.00	0.01	0.06	0.08	0.06	0.14
Call$(4.80S(0),T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.04	0.08	0.10	0.11
Call$(5.60S(0),T)$	0.00	0.00	0.00	0.00	0.02	0.02	0.05	0.06	0.08	0.08
Call$(6.40S(0),T)$	0.00	0.00	0.00	0.00	0.00	0.03	0.04	0.10	0.17	0.14

Table 4.13: *Behavior of Strategy 3 for hedging the GMIB, using annual rebalancing and allowing for the benchmark transaction costs (assuming interest rates are constant).*

4.8.2 Allowing for the one-factor short rate model (the benchmark example)

The results illustrated in this section form the benchmark for comparing the performance of semi-static hedging strategies for the GMIB. We provide a comprehensive analysis. These results can be compared to those of other strategy types in the following sections, such as strategies based on minimizing the CTE. Tables 4.14 and 4.15 display the results from applying Strategies 1, 2 and 3.

Comments on Strategy 1:

- The tail risk measures tend to be higher, and the mean hedging losses tend to be lower, compared to the corresponding results for the constant interest rate cases. These observations reflect the additional interest rate uncertainty.
- When transaction costs are included, the mean and median hedging losses from half-yearly and quarterly rebalancing are negative, unlike the corresponding results for the constant interest rate case.

Comments on Strategy 2:

- Strategy 2 exhibits less risk than Strategy 1, but this comes at a cost of a lower mean hedging profit.
- The tail risk measures and the $\text{MSHL}^{1/2}$ are significantly larger than the corresponding results for the constant interest rate cases.

Comments on Strategy 3:

- The $\text{MSHL}^{1/2}$ is much lower, compared to Strategies 1 and 2.
- The tail risk measures are much lower than for Strategies 1 and 2. In particular, the VaR(95%) for Strategy 3 is considerably smaller than for Strategies 1 and 2.

Strategy 1
Negligible transaction costs

Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	189	-351	-60	234	179	-61	356	549
	$(180, 198)$	$(-371, -335)$	$(-76, -47)$	$(204, 269)$	$(170, 188)$	$(-72, -50)$	$(310, 402)$	$(467, 632)$
Half-yearly	161	-301	-46	196	151	-54	269	372
	$(153, 168)$	$(-326, -282)$	$(-64, -36)$	$(169, 221)$	$(144, 159)$	$(-63, -45)$	$(242, 295)$	$(319, 424)$
Quarterly	147	-280	-40	168	140	-46	263	416
	$(140, 155)$	$(-311, -258)$	$(-52, -25)$	$(152, 194)$	$(132, 148)$	$(-55, -38)$	$(226, 300)$	$(358, 475)$
Static w/	715	-1353	-371	477	597	-394	531	740
10-yr put (PF1)	$(683, 746)$	$(-1431, -1311)$	$(-435, -325)$	$(476, 477)$	$(573, 620)$	$(-431, -357)$	$(492, 570)$	$(577, 903)$
Static w/	254	-458	-156	136	217	-131	139	148
lookback (PF2)	$(242, 265)$	$(-498, -448)$	$(-180, -130)$	$(136, 136)$	$(208, 226)$	$(-144, -118)$	$(136, 142)$	$(134, 162)$

Strategy 1
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})\ k \neq b,\ c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$

Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	183	-339	-55	233	175	-55	376	610
	$(173, 194)$	$(-360, -315)$	$(-64, -42)$	$(191, 269)$	$(164, 186)$	$(-66, -44)$	$(320, 432)$	$(473, 747)$
Half-yearly	159	-280	-36	207	155	-37	325	565
	$(148, 171)$	$(-311, -257)$	$(-45, -23)$	$(183, 242)$	$(143, 168)$	$(-47, -28)$	$(269, 381)$	$(375, 756)$
Quarterly	146	-234	-14	210	145	-15	333	610
	$(133, 158)$	$(-258, -222)$	$(-21, -2)$	$(191, 238)$	$(132, 158)$	$(-24, -6)$	$(274, 392)$	$(427, 794)$
Static w/	714	-1250	-372	477	601	-385	512	652
10-yr put (PF1)	$(680, 747)$	$(-1420, -1207)$	$(-430, -302)$	$(477, 477)$	$(575, 627)$	$(-422, -348)$	$(482, 543)$	$(499, 806)$
Static w/	245	-474	-150	136	213	-122	138	146
lookback (PF2)	$(236, 255)$	$(-487, -449)$	$(-174, -121)$	$(136, 136)$	$(206, 219)$	$(-135, -109)$	$(135, 141)$	$(131, 161)$

Strategy 2
Negligible transaction costs

Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	116	-216	-29	151	112	-31	228	344
	$(110, 122)$	$(-234, -205)$	$(-36, -24)$	$(137, 178)$	$(106, 118)$	$(-38, -24)$	$(199, 257)$	$(274, 415)$
Half-yearly	124	-230	-32	150	119	-35	244	406
	$(118, 131)$	$(-249, -215)$	$(-39, -24)$	$(136, 172)$	$(112, 127)$	$(-42, -27)$	$(207, 280)$	$(319, 494)$
Quarterly	126	-245	-35	150	120	-38	234	369
	$(119, 133)$	$(-266, -225)$	$(-45, -25)$	$(132, 171)$	$(113, 127)$	$(-46, -31)$	$(201, 267)$	$(282, 456)$
Static w/	701	-1265	-392	477	582	-391	492	549
10-yr put (PF1)	$(669, 733)$	$(-1408, -1217)$	$(-456, -324)$	$(476, 477)$	$(557, 607)$	$(-427, -355)$	$(480, 503)$	$(490, 608)$
Static w/	243	-445	-156	136	209	-126	140	153
lookback (PF2)	$(234, 252)$	$(-470, -419)$	$(-175, -129)$	$(136, 136)$	$(202, 215)$	$(-139, -113)$	$(136, 144)$	$(134, 172)$

Strategy 2
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})\ k \neq b,\ c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$

Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	110	-212	-18	164	108	-21	231	332
	$(104, 116)$	$(-225, -189)$	$(-25, -11)$	$(142, 186)$	$(103, 114)$	$(-27, -14)$	$(206, 256)$	$(292, 373)$
Half-yearly	113	-207	-18	165	112	-19	242	370
	$(107, 120)$	$(-230, -192)$	$(-24, -10)$	$(154, 185)$	$(106, 118)$	$(-26, -12)$	$(213, 272)$	$(306, 434)$
Quarterly	115	-203	-3	182	115	-7	271	384
	$(109, 122)$	$(-223, -190)$	$(-11, 3)$	$(166, 218)$	$(109, 122)$	$(-14, 0)$	$(240, 302)$	$(336, 433)$
Static w/	706	-1281	-407	476	573	-414	507	628
10-yr put (PF1)	$(675, 738)$	$(-1351, -1214)$	$(-470, -345)$	$(450, 477)$	$(548, 597)$	$(-449, -378)$	$(477, 538)$	$(473, 783)$
Static w/	250	-462	-172	136	211	-135	139	148
lookback (PF2)	$(241, 259)$	$(-474, -442)$	$(-196, -149)$	$(136, 136)$	$(205, 217)$	$(-148, -122)$	$(135, 143)$	$(127, 169)$

Table 4.14: *Hedging loss distribution statistics derived from Strategies 1 and 2 for the GMIB (benchmark example).*

Rebal freq	MSHL$^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Strategy 3 — Negligible transaction costs								
Annual	73	-94	-1	89	73	-3	145	284
	(61, 85)	(−106, −86)	(−5, 2)	(79, 102)	(61, 85)	(−8, 1)	(115, 175)	(165, 402)
Half-yearly	71	-88	-3	75	71	-6	124	223
	(60, 83)	(−105, −74)	(−7, 1)	(70, 87)	(59, 83)	(−11, −2)	(101, 147)	(144, 302)
Quarterly	68	-97	-4	65	67	-8	127	285
	(57, 78)	(−107, −88)	(−7, −1)	(59, 72)	(57, 78)	(−12, −3)	(92, 161)	(150, 419)
Static w/	731	-1411	-344	477	620	-388	536	766
10-yr put (PF1)	(694, 768)	(−1589, −1291)	(−421, −277)	(477, 477)	(590, 650)	(−426, −349)	(470, 603)	(440, 1092)
Static w/	240	-448	-153	136	208	-120	142	163
lookback (PF2)	(231, 249)	(−469, −426)	(−178, −112)	(136, 136)	(202, 215)	(−132, −107)	(136, 147)	(136, 191)
Strategy 3 — Transaction costs: $c(k,t_{i-1}) = 0.5\%\phi(k,t_{i-1})$ $k \neq b$, $c(b,t_{i-1}) = 0.05\%\phi(b,t_{i-1})$								
Rebal freq	MSHL$^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Annual	70	-71	20	116	67	20	177	305
	(62, 79)	(−93, −62)	(16, 22)	(103, 126)	(59, 76)	(16, 24)	(148, 206)	(207, 404)
Half-yearly	73	-56	23	125	66	30	195	354
	(61, 84)	(−63, −48)	(21, 27)	(115, 139)	(55, 78)	(26, 34)	(157, 232)	(207, 502)
Quarterly	71	-30	29	135	60	38	203	351
	(63, 79)	(−37, −25)	(26, 33)	(120, 149)	(52, 68)	(34, 42)	(171, 235)	(256, 446)
Static w/	714	-1250	-372	477	601	-385	512	652
10-yr put (PF1)	(680, 747)	(−1420, −1207)	(−430, −302)	(477, 477)	(575, 627)	(−422, −348)	(482, 543)	(499, 806)
Static w/	245	-474	-150	136	213	-122	138	146
lookback (PF2)	(236, 255)	(−487, −449)	(−174, −121)	(136, 136)	(206, 219)	(−135, −109)	(135, 141)	(131, 161)

Table 4.15: *Hedging loss distribution statistics derived from Strategy 3 for the GMIB (benchmark example).*

The results clearly illustrate that using a stochastic interest rate model can significantly alter the performance of a semi-static hedging strategy for a long-dated option. This is an important consideration for future research related to semi-static hedging strategies. The literature on hedging methods for derivatives often assumes that interest rates are constant. The results here show that it is advisable to use a model which captures the stochastic nature of interest rates, even for options driven by equity risk (such as the GMIB option).

Additional useful information about the hedging strategies can be obtained from appropriate figures. Figure 4.1 displays the hedging loss distributions for Strategies 1, 2 and 3, using annual rebalancing and allowing for transaction costs (all on the same scale, for ease of comparison). The shapes of the distributions are intuitive. As more tail risk is hedged, the mean hedging loss increases.

In Figure 4.2, the left panels illustrate the relationship between the hedging losses and the corresponding stock prices at maturity, $S(T)$. We note that the horizontal axis in each panel is restricted to 0-6000, but occasionally there are hedging losses for $S(T) > 6000$.

The right panels display the relationship between the hedging losses and the maximum stock price, sampled on each policy anniversary, $\max_{n=1,\ldots,T} S(n)$. The latter relationship is explored because it is related to the value of the lookback component of the GMIB. For Strategy 1, the largest losses are generated by the lookback component. For Strategy 2, the largest losses are attributed to the guaranteed return component, and to a lesser extent, the lookback component. For Strategy 3, the lookback component is responsible for producing a few very large hedging loss outliers. All of the semi-static strategies are effective at minimizing the hedging losses generated by the investment account component.

Tables 4.14 and 4.15 show that all of the semi-static strategies are less risky than the static put strategy. However, Strategies 1 and 2 are unable to reduce the right tail risk to levels obtained using the static lookback strategy. Strategy 3 with annual rebalancing is able to produce a smaller VaR(95%) than the static lookback case. However, as indicated by the CTE measures, in rare circumstances Strategy 3 produces large hedging losses. The static lookback strategy does a remarkable job of literally cutting off the right tail.

In passing, we note that Strategy 3 and the static strategies may not be practicable. The implied volatilities (prices) of the T-year vanilla and lookback options may be higher than expected. On the other hand, Strategies 1 and 2 can easily be implemented in practice; τ-year options (for $\tau \leqslant 1$) on stock indices are likely to be actively traded securities. But Strategy 2 offers much lower tail risk than Strategy 1.

Tables 4.16, 4.17 and 4.18 display the behavior of Strategies 1, 2 and 3 respectively, using annual rebalancing and allowing for transaction costs. Comparing these tables to the corresponding constant interest rate tables (Tables 4.11, 4.12 and 4.13) illustrates the impact of stochastic interest rates on the strategies. Overall, the standard deviations of the total target errors, minimized objective function values and excess funds are higher with variable interest rates. For Strategies 1 and 2, less funds are invested in the stock, on average, with variable interest rates. The negative mean total target errors for Strategy 2 appear to be generated partly by the mean excess funds. The behavior of Strategy 3 is consistent with its behavior in the corresponding constant interest rate case.

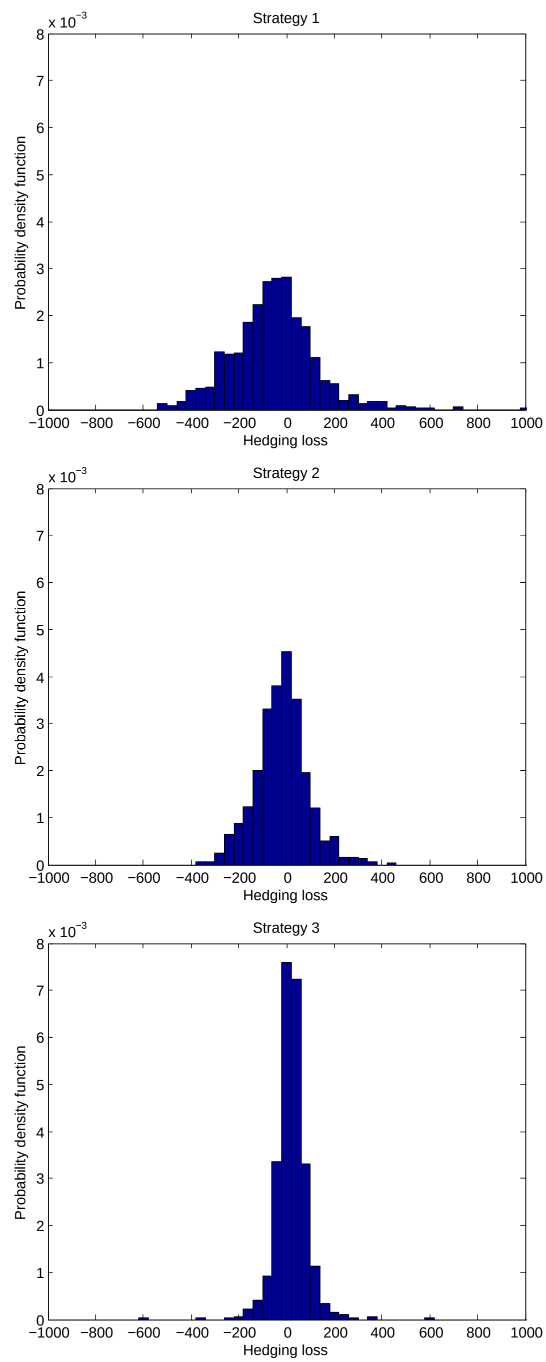

Figure 4.1: *The hedging loss distributions derived from Strategies 1, 2 and 3, using annual rebalancing and allowing for transaction costs (benchmark example).*

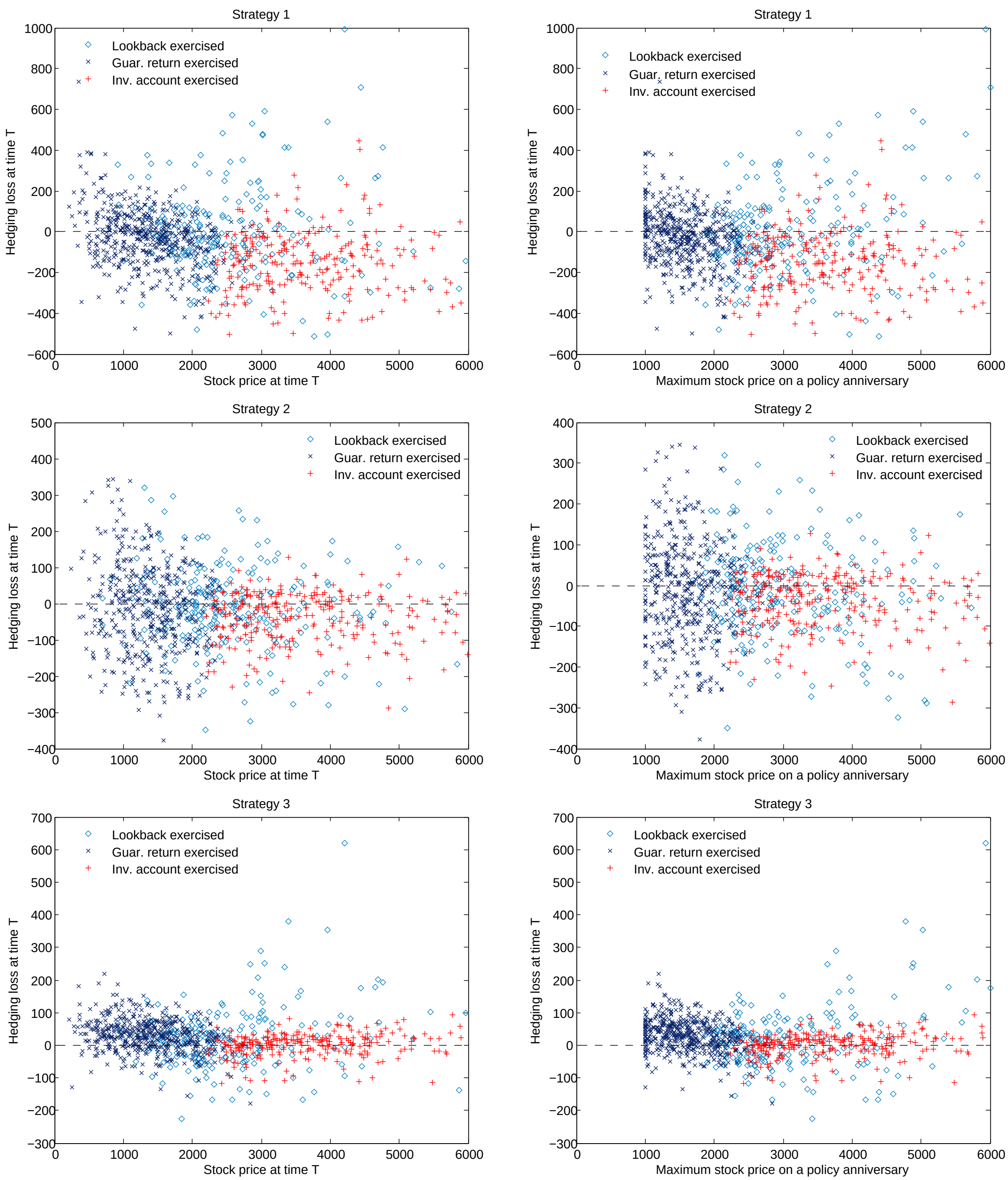

Figure 4.2: *Hedging losses e_n as functions of the stock price at time T, $S(T)$, and as functions of the maximum stock price on a policy anniversary, $\max_{n=1,\dots,T} S(n)$, derived from Strategies 1, 2 and 3, based on annual rebalancing and the benchmark transaction costs (benchmark example).*

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	0	-2	-6	-12	-17	-23	-32	-39	-44	-55
Std total target error	40	52	65	73	86	99	112	127	148	175
Mean min obj. ($\text{MSHE}^{1/2}$)	33	41	45	51	55	60	67	72	81	96
Std min obj. ($\text{MSHE}^{1/2}$)	3	13	21	26	28	34	42	52	61	77
Mean excess funds	1	13	21	28	35	44	51	61	71	83
Std excess funds	1	17	26	34	41	50	61	70	80	94
Mean transaction costs	2	1	1	1	1	1	1	1	1	1
Std transaction costs	0	1	0	1	1	1	1	1	1	1
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.39	0.41	0.41	0.40	0.38	0.35	0.32	0.29	0.26	0.23
Risk-free bond	0.62	0.57	0.57	0.58	0.60	0.65	0.72	0.80	0.91	1.01
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.02	0.13	0.16	0.18	0.20	0.21	0.21	0.21	0.21	0.22
Risk-free bond	0.02	0.17	0.22	0.25	0.28	0.31	0.32	0.34	0.38	0.46

Table 4.16: *Behavior of Strategy 1 for hedging the GMIB, using annual rebalancing and allowing for the benchmark transaction costs (benchmark example).*

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	2	-2	-2	-6	-10	-11	-12	-16	-18	-21
Std total target error	34	44	49	55	63	75	82	91	100	108
Mean min obj. ($\text{MSHE}^{1/2}$)	26	31	34	37	39	42	46	48	52	55
Std min obj. ($\text{MSHE}^{1/2}$)	1	10	15	18	21	24	31	35	40	47
Mean excess funds	0	8	14	17	22	27	31	36	41	46
Std excess funds	1	13	20	24	30	35	42	49	55	62
Mean transaction costs	2	1	1	1	1	1	1	1	2	2
Std transaction costs	0	1	1	1	1	1	1	1	1	3
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.34	0.37	0.38	0.37	0.35	0.33	0.31	0.28	0.25	0.22
Risk-free bond	0.67	0.61	0.59	0.59	0.62	0.66	0.72	0.78	0.89	0.98
Put($0.77S(t_{i-1})$)	0.10	0.10	0.10	0.09	0.09	0.09	0.08	0.07	0.07	0.04
Put($0.85S(t_{i-1})$)	0.05	0.05	0.05	0.05	0.04	0.04	0.04	0.05	0.04	0.04
Put($0.92S(t_{i-1})$)	0.05	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.10
Put($1.00S(t_{i-1})$)	0.04	0.05	0.06	0.07	0.07	0.07	0.08	0.08	0.08	0.06
Call($1.12S(t_{i-1})$)	0.07	0.07	0.07	0.06	0.06	0.06	0.05	0.05	0.06	0.05
Call($1.25S(t_{i-1})$)	0.06	0.07	0.06	0.05	0.05	0.05	0.05	0.04	0.04	0.05
Call($1.37S(t_{i-1})$)	0.05	0.06	0.05	0.04	0.04	0.04	0.04	0.03	0.04	0.04
Call($1.49S(t_{i-1})$)	0.18	0.11	0.09	0.08	0.07	0.07	0.06	0.06	0.05	0.05
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.04	0.13	0.18	0.20	0.21	0.22	0.22	0.23	0.22	0.26
Risk-free bond	0.04	0.17	0.24	0.28	0.32	0.35	0.39	0.43	0.49	0.74
Put($0.77S(t_{i-1})$)	0.14	0.15	0.14	0.13	0.13	0.14	0.13	0.12	0.12	0.10
Put($0.85S(t_{i-1})$)	0.08	0.08	0.08	0.08	0.08	0.08	0.07	0.08	0.08	0.10
Put($0.92S(t_{i-1})$)	0.07	0.07	0.06	0.06	0.07	0.07	0.07	0.06	0.07	0.18
Put($1.00S(t_{i-1})$)	0.06	0.06	0.07	0.07	0.08	0.08	0.08	0.09	0.09	0.13
Call($1.12S(t_{i-1})$)	0.06	0.06	0.07	0.07	0.06	0.07	0.07	0.07	0.07	0.13
Call($1.25S(t_{i-1})$)	0.06	0.07	0.07	0.07	0.07	0.07	0.06	0.06	0.07	0.13
Call($1.37S(t_{i-1})$)	0.07	0.08	0.08	0.07	0.07	0.07	0.07	0.06	0.07	0.10
Call($1.49S(t_{i-1})$)	0.11	0.11	0.10	0.10	0.09	0.09	0.09	0.09	0.09	0.13

Table 4.17: *Behavior of Strategy 2 for hedging the GMIB, using annual rebalancing and allowing for the benchmark transaction costs (benchmark example).*

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	0	-1	-3	-5	-12	-14	-20	-25	-33	-36
Std total target error	35	47	57	63	72	79	89	99	106	133
Mean min obj. (MSHE$^{1/2}$)	26	30	35	37	40	43	46	48	50	51
Std min obj. (MSHE$^{1/2}$)	1	9	16	23	24	28	30	36	40	44
Mean excess funds	1	10	16	21	26	31	37	43	51	57
Std excess funds	1	15	22	29	35	42	50	57	67	73
Mean transaction costs	0	0	0	0	0	0	0	0	0	0
Std transaction costs	0	0	0	0	0	0	0	0	0	0
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.34	0.37	0.37	0.36	0.33	0.31	0.29	0.25	0.23	0.21
Risk-free bond	0.66	0.62	0.60	0.61	0.65	0.69	0.75	0.84	0.94	1.00
Put($0.77S(t_{i-1})$)	0.07	-0.05	-0.04	-0.17	-0.09	-0.18	-0.11	-0.17	-0.32	-0.23
Put($0.85S(t_{i-1})$)	0.03	0.09	0.07	0.12	0.06	0.10	0.06	0.11	0.13	0.13
Put($0.92S(t_{i-1})$)	0.07	0.03	0.02	0.02	0.04	0.02	0.03	-0.01	-0.03	0.03
Put($1.00S(t_{i-1})$)	0.04	0.05	0.07	0.07	0.05	0.06	0.06	0.07	0.09	0.08
Call($1.12S(t_{i-1})$)	0.06	0.07	0.06	0.06	0.07	0.06	0.05	0.07	0.05	0.03
Call($1.25S(t_{i-1})$)	0.07	0.07	0.06	0.06	0.04	0.05	0.05	0.03	0.03	0.05
Call($1.37S(t_{i-1})$)	0.01	0.05	0.06	0.05	0.04	0.03	0.03	0.04	0.05	0.03
Call($1.49S(t_{i-1})$)	0.22	0.10	0.06	0.06	0.05	0.06	0.04	0.01	0.02	0.02
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.08	0.16	0.21	0.25	0.25	0.27	0.25	0.26	0.27	0.34
Risk-free bond	0.08	0.19	0.26	0.31	0.34	0.39	0.42	0.47	0.53	0.80
Put($0.77S(t_{i-1})$)	0.38	0.70	0.94	1.42	1.19	2.34	1.12	1.71	3.47	1.62
Put($0.85S(t_{i-1})$)	0.36	0.51	0.83	0.88	0.82	1.16	0.88	1.10	1.53	1.11
Put($0.92S(t_{i-1})$)	0.30	0.37	0.60	0.57	0.52	0.75	0.64	0.64	0.88	0.87
Put($1.00S(t_{i-1})$)	0.20	0.23	0.31	0.34	0.32	0.46	0.39	0.35	0.41	0.56
Call($1.12S(t_{i-1})$)	0.14	0.16	0.20	0.24	0.21	0.29	0.24	0.25	0.27	0.34
Call($1.25S(t_{i-1})$)	0.15	0.17	0.21	0.25	0.22	0.29	0.29	0.30	0.36	0.32
Call($1.37S(t_{i-1})$)	0.18	0.21	0.23	0.35	0.28	0.34	0.35	0.37	0.43	0.40
Call($1.49S(t_{i-1})$)	0.17	0.22	0.24	0.32	0.27	0.31	0.36	0.53	0.45	0.49

Table 4.18: *Behavior of Strategy 3 for hedging the GMIB, using annual rebalancing and allowing for the benchmark transaction costs (benchmark example).*

4.8.3 Permitting short selling of options

As part of the benchmark parameter assumptions, we do not permit short selling of options. Our justification for this restriction is because short selling increases the risks associated with trading strategies. Nevertheless, for completeness, we briefly explore what happens if short selling is allowed for Strategy 2 (i.e. no constraint is made for $x_L(k), k = 1, \ldots, K_i$).

Table 4.19 displays the hedging loss statistics for Strategy 2, when short selling is permitted. To measure the impact of short selling, these statistics should be compared to the corresponding results for Strategy 2 when short selling is not permitted, in Table 4.14. It is clear that short selling markedly increases the CTE measures, but interestingly, the VaR measures tend to be slightly lower. The mean hedging losses and 5% quantiles also tend to be slightly lower, as a result of selling deep out-of-the money options. As expected, the standard deviations are significantly higher. A very small number of the hedging losses exceeded 1000; losses of this magnitude were not seen when short selling was not permitted.

Table 4.20 displays the behavior of Strategy 2 when short selling is permitted, using annual rebalancing and allowed for transaction costs. On average, the optimal strategy involves a short position in the deepest out-of-the-money put option. Thus, when the stock price crashes during the accumulation phase, the hedging losses are more likely to be larger.

Permitting short selling of options does not help improve the hedging loss distribution, in the sense of minimizing the risk of very large losses which occur with small probability. As expected, short selling substantially increases the risk of large hedging losses, as measured by the increases in the CTE.

Strategy 2 (with short selling) Negligible transaction costs								
Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	138	-229	-31	136	133	-36	269	593
	$(117, 159)$	$(-250, -213)$	$(-39, -24)$	$(112, 154)$	$(111, 156)$	$(-45, -28)$	$(190, 347)$	$(263, 922)$
Half-yearly	174	-244	-36	139	170	-37	382	1012
	$(137, 212)$	$(-271, -226)$	$(-43, -27)$	$(119, 178)$	$(131, 210)$	$(-48, -26)$	$(247, 517)$	$(467, 1558)$
Quarterly	153	-247	-40	120	145	-47	318	755
	$(136, 169)$	$(-274, -228)$	$(-47, -33)$	$(103, 165)$	$(127, 164)$	$(-56, -38)$	$(229, 407)$	$(503, 1007)$
Static w/	711	-1293	-404	476	585	-406	512	651
10-yr put (PC3)	$(677, 746)$	$(-1405, -1237)$	$(-465, -344)$	$(476, 477)$	$(557, 612)$	$(-442, -369)$	$(478, 545)$	$(481, 820)$
Static w/	252	-463	-159	136	216	-129	144	171
lookback (PC4B)	$(241, 262)$	$(-494, -438)$	$(-180, -129)$	$(136, 136)$	$(209, 223)$	$(-142, -116)$	$(137, 150)$	$(136, 207)$

Strategy 2 (with short selling) Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})$ $k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$								
Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	128	-222	-20	142	124	-25	271	442
	$(119, 136)$	$(-244, -187)$	$(-26, -12)$	$(125, 156)$	$(115, 133)$	$(-32, -18)$	$(228, 314)$	$(319, 566)$
Half-yearly	134	-212	-15	165	132	-21	310	665
	$(117, 151)$	$(-233, -186)$	$(-24, -11)$	$(135, 183)$	$(114, 150)$	$(-29, -13)$	$(236, 383)$	$(396, 933)$
Quarterly	134	-195	-12	177	133	-12	346	677
	$(118, 150)$	$(-220, -180)$	$(-21, -6)$	$(145, 212)$	$(117, 150)$	$(-20, -4)$	$(272, 420)$	$(455, 899)$
Static w/	730	-1403	-394	477	606	-407	501	594
10-yr put (PC3)	$(694, 766)$	$(-1501, -1326)$	$(-460, -330)$	$(476, 477)$	$(578, 634)$	$(-445, -370)$	$(482, 520)$	$(498, 689)$
Static w/	243	-427	-156	136	209	-125	141	158
lookback (PC4B)	$(234, 252)$	$(-453, -417)$	$(-176, -137)$	$(136, 137)$	$(202, 215)$	$(-138, -112)$	$(135, 147)$	$(129, 188)$

Table 4.19: *Hedging loss distribution statistics derived from Strategy 2 for the GMIB, when option short selling is allowed.*

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	4	1	1	-1	-5	-8	-12	-16	-20	-25
Std total target error	36	44	55	62	72	82	90	108	113	124
Mean min obj. (MSHE$^{1/2}$)	26	31	35	39	42	45	49	51	53	56
Std min obj. (MSHE$^{1/2}$)	1	10	15	23	25	32	37	45	63	62
Mean excess funds	0	9	13	17	22	27	32	37	43	48
Std excess funds	1	15	20	27	32	39	46	53	60	67
Mean transaction costs	2	1	1	1	1	1	1	2	2	2
Std transaction costs	0	1	1	1	1	1	1	1	1	2
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.33	0.37	0.37	0.38	0.36	0.34	0.31	0.28	0.24	0.23
Risk-free bond	0.67	0.62	0.60	0.60	0.62	0.66	0.72	0.80	0.90	0.96
Put($0.77S(t_{i-1})$)	0.03	-0.04	-0.11	-0.13	-0.20	-0.14	-0.06	-0.12	-0.19	-0.33
Put($0.85S(t_{i-1})$)	0.05	0.05	0.10	0.08	0.11	0.07	0.01	0.06	0.00	0.09
Put($0.92S(t_{i-1})$)	0.07	0.06	0.02	0.01	0.00	0.01	0.02	0.00	0.07	0.10
Put($1.00S(t_{i-1})$)	0.02	0.04	0.07	0.08	0.09	0.09	0.09	0.10	0.07	0.06
Call($1.12S(t_{i-1})$)	0.08	0.07	0.06	0.04	0.04	0.03	0.03	0.02	0.03	0.02
Call($1.25S(t_{i-1})$)	0.07	0.06	0.07	0.06	0.05	0.05	0.04	0.07	0.05	0.06
Call($1.37S(t_{i-1})$)	0.01	0.06	0.04	0.03	0.04	0.05	0.05	0.03	0.03	0.02
Call($1.49S(t_{i-1})$)	0.21	0.09	0.08	0.07	0.05	0.03	0.03	0.02	0.02	0.00
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.07	0.13	0.17	0.20	0.21	0.23	0.23	0.23	0.24	0.27
Risk-free bond	0.08	0.16	0.23	0.27	0.31	0.35	0.38	0.41	0.48	0.71
Put($0.77S(t_{i-1})$)	0.39	0.69	0.90	1.15	1.46	1.47	1.56	1.62	3.31	1.85
Put($0.85S(t_{i-1})$)	0.36	0.45	0.50	0.79	0.68	0.74	1.04	0.95	1.60	1.04
Put($0.92S(t_{i-1})$)	0.27	0.30	0.34	0.50	0.42	0.47	0.52	0.55	0.93	0.85
Put($1.00S(t_{i-1})$)	0.18	0.20	0.21	0.22	0.25	0.29	0.29	0.29	0.49	0.39
Call($1.12S(t_{i-1})$)	0.13	0.14	0.16	0.18	0.17	0.22	0.20	0.21	0.21	0.25
Call($1.25S(t_{i-1})$)	0.14	0.16	0.18	0.22	0.22	0.28	0.27	0.26	0.26	0.32
Call($1.37S(t_{i-1})$)	0.17	0.21	0.23	0.26	0.29	0.33	0.34	0.34	0.42	0.40
Call($1.49S(t_{i-1})$)	0.17	0.21	0.24	0.28	0.34	0.37	0.39	0.45	0.67	0.57

Table 4.20: *Behavior of Strategy 2 for hedging the GMIB when option short selling is permitted, using annual rebalancing and allowing for the benchmark transaction costs.*

4.8.4 Examples of simulated scenarios

In this section, we present a set of figures that are designed to give the reader some insight into how the hedging target distribution evolves under different scenarios for the stock and interest rate processes. We also give some reasoning for why, using Strategy 2, increasing the rebalancing frequency may lead to a deterioration in the hedging loss distribution.

Figures 4.3, 4.4 and 4.5 display the behavior of Strategy 2, using annual rebalancing and allowing for transaction costs, for three different scenarios, labeled A, B and C. In Scenario A, the stock price rises sharply during the accumulation phase, but crashes just before maturity, and the lookback component is exercised (the hedge roughly breaks even at maturity). In Scenario B, the stock price trends downwards, and the guaranteed return component is exercised (a loss occurs at maturity). In Scenario C, there is a persistent rise in the stock price, and thus the investment account component is exercised (a profit occurs at maturity). In each figure, the left panels show the optimal hedging portfolio payoff and the hedging targets as functions of the end of horizon stock price. The right panels display the hedging targets and the actual portfolio payoff as functions of the end of horizon short rate.

For comparison, Figure 4.6 displays the behavior of Strategy 2 for one particular scenario in the case where interest rates are constant. In this particular scenario the investment account component is exercised. Comparing the panels of Figure 4.6 and those for Scenarios A, B and C, provides a feel for how stochastic interest rates influence semi-static strategies. We see that when interest rates are constant, the hedging targets are bunched much more closely together, and the hedging portfolio is able to more closely match the hedging target distribution across the range of plausible stock price values.

Our results have shown that when the GMIB option is hedged using Strategy 2, increasing the rebalancing frequency does not improve the results. In fact, increasing the rebalancing frequency appears to lead to a deterioration in the results. This occurs even when transaction costs are assumed to be negligible. When we hedge with the τ-year options, part of the budget in each horizon is usually spent on buying out-of-the-money options, to reduce the hedging errors in the tails of the hedging target distribution. This can be seen in the left panels of Figures 4.3, 4.4 and 4.5. However, most of the time, the actual hedging target is not among the hedging target values in the tails. The money spent on buying tail risk protection could otherwise be part of the excess funds invested in risk-free bonds, increasing the cushion against hedging error shortfalls in future hedging horizons. For example, in Figure 4.5, consider the left panels for the first and second hedging horizons; the put option with a strike of 770 expires worthless at time 1. When the rebalancing frequency increases from, say, annually to half-yearly, the tail risk protection is adjusted twice as often. There are more opportunities for hedging the outlying hedging target values, and the cost of buying the deep out-of-the-money eats away at the excess funds. It seems that hedging tail risk with deep out-of-the-money options, at a rebalancing frequency higher than annually, is counterproductive. The same reasoning is applied to Strategy 3.

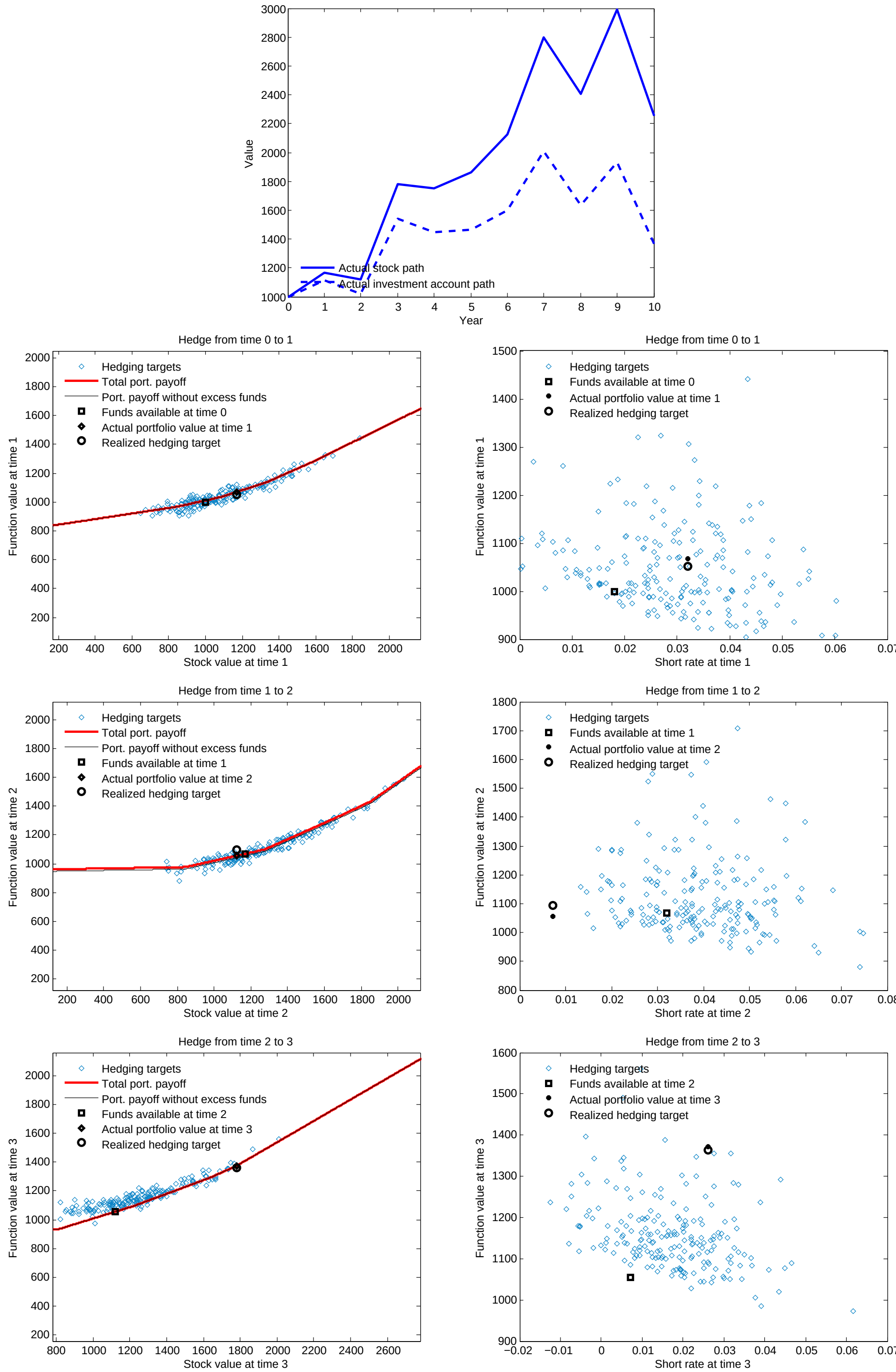

Figure 4.3: *Evolution of Strategy 2 for Scenario A, where the Lookback component X_1 exercised.*

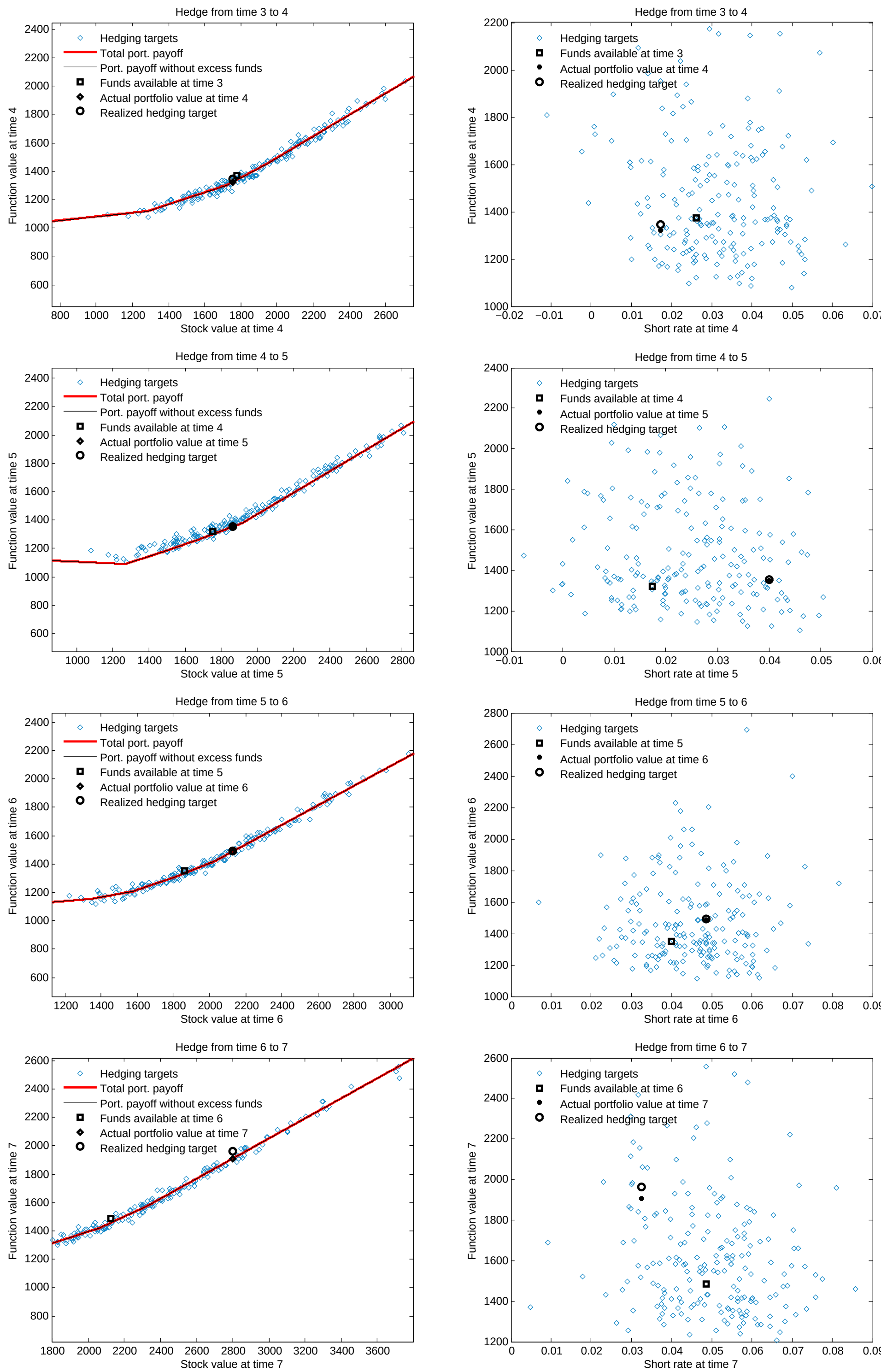

Figure 4.3 (Continued): *Evolution of Strategy 2 for Scenario A, where the Lookback component* X_1 *exercised.*

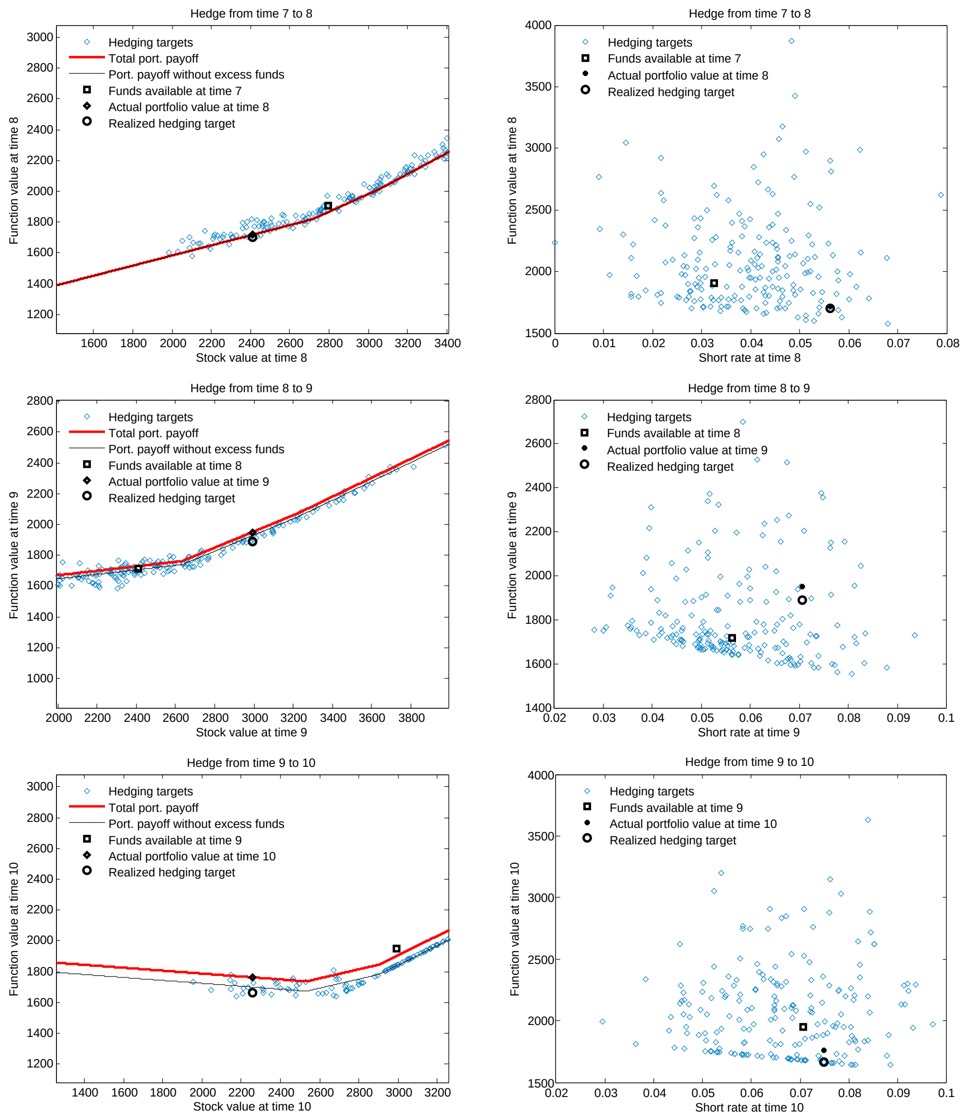

Figure 4.3 (Continued): *Evolution of Strategy 2 for Scenario A, where the Lookback component X_1 exercised.*

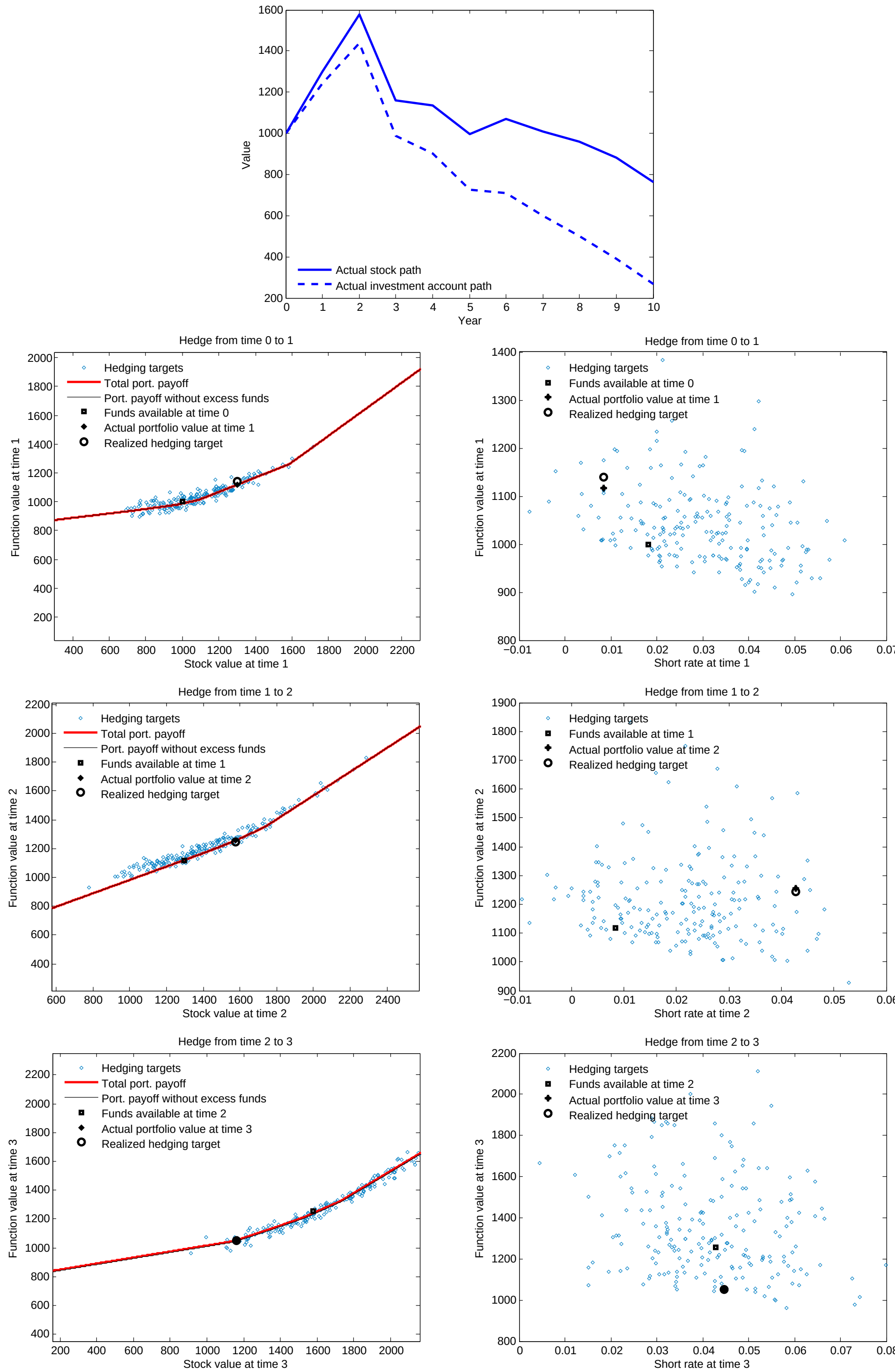

Figure 4.4: *Evolution of Strategy 2 for Scenario B, where the Guaranteed return component X_2 exercised.*

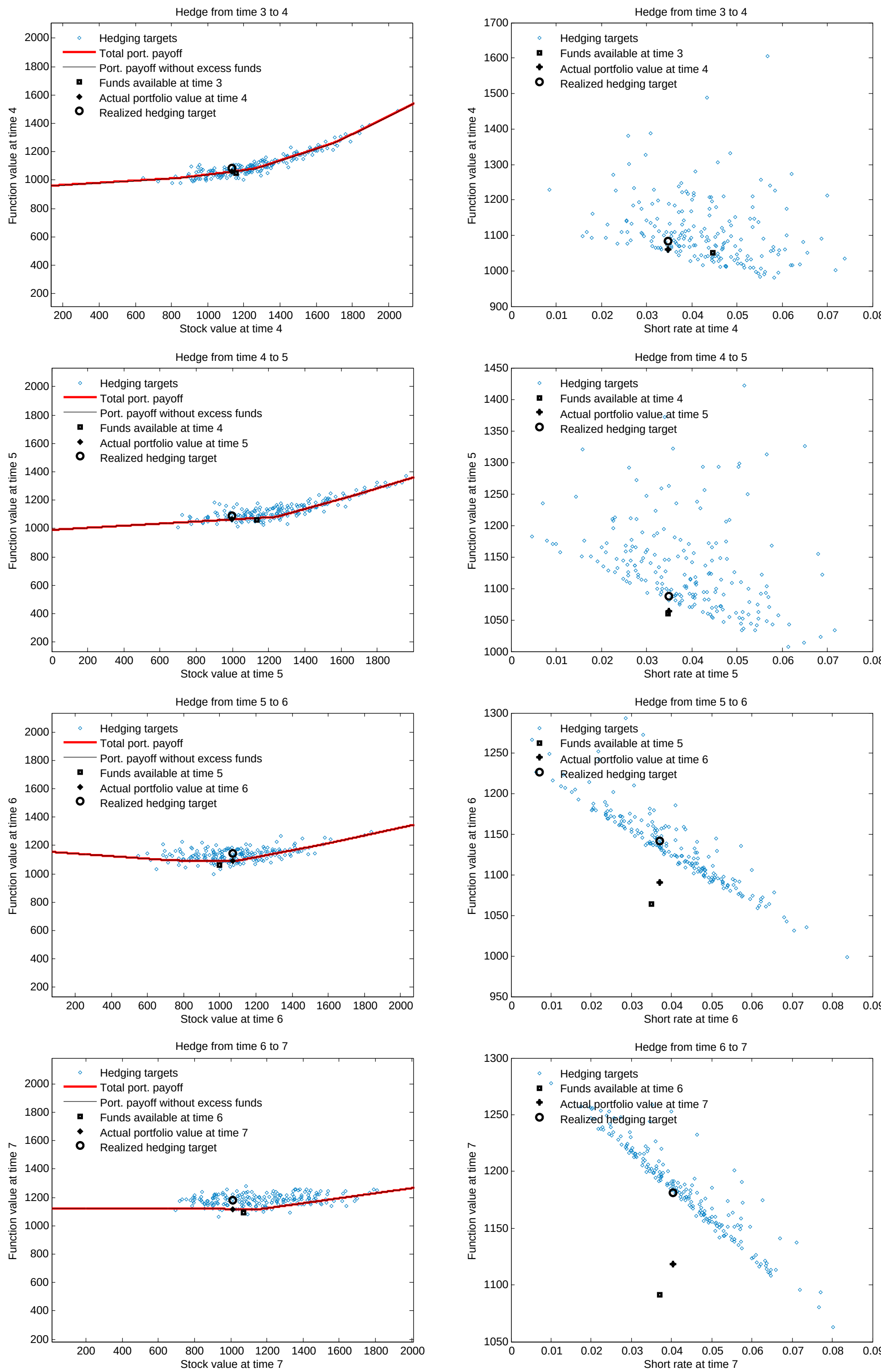

Figure 4.4 (Continued): *Evolution of Strategy 2 for Scenario B, where the Guaranteed return component X_2 exercised.*

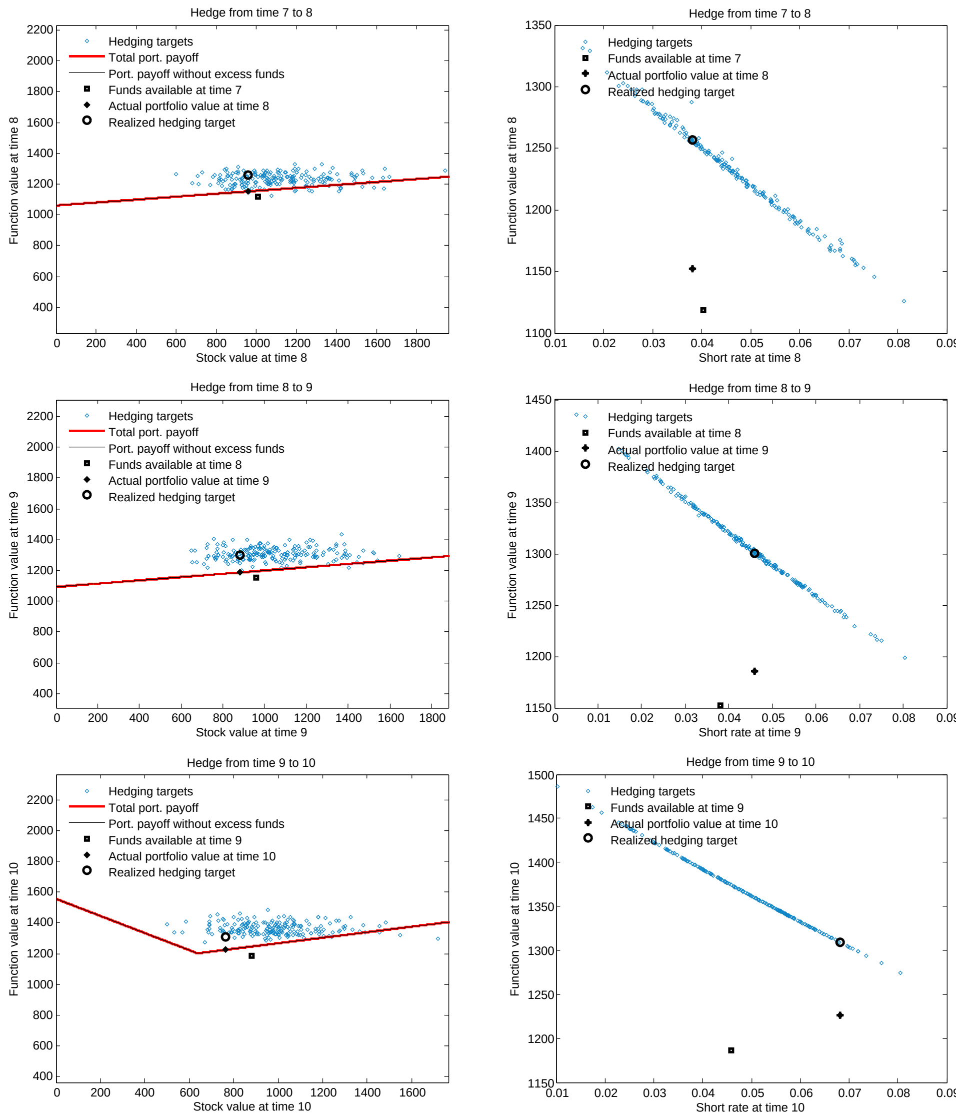

Figure 4.4 (Continued): *Evolution of Strategy 2 for Scenario B, where the Guaranteed return component X_2 exercised.*

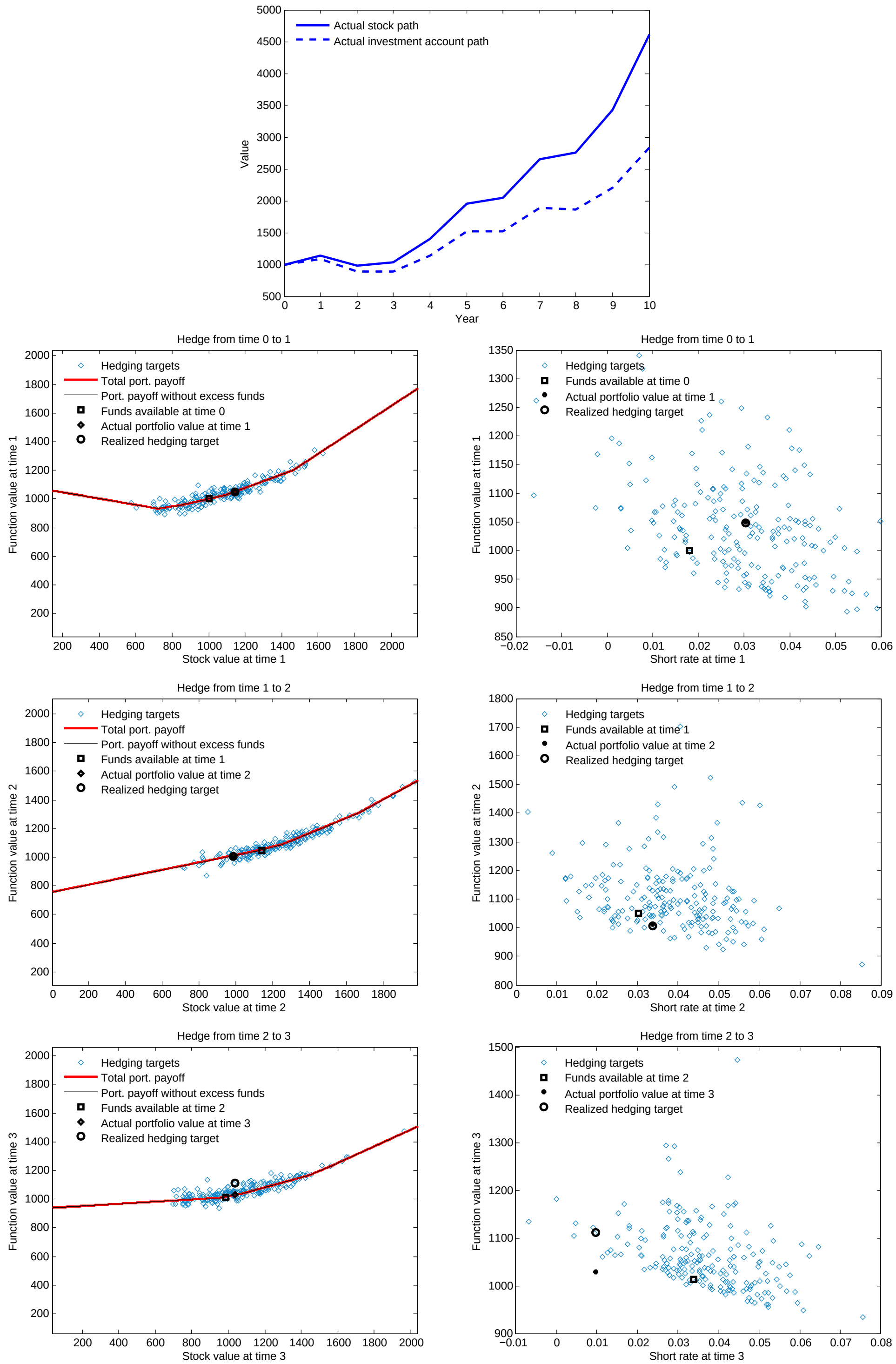

Figure 4.5: *Evolution of Strategy 2 for Scenario C, where the Investment account component X_3 exercised.*

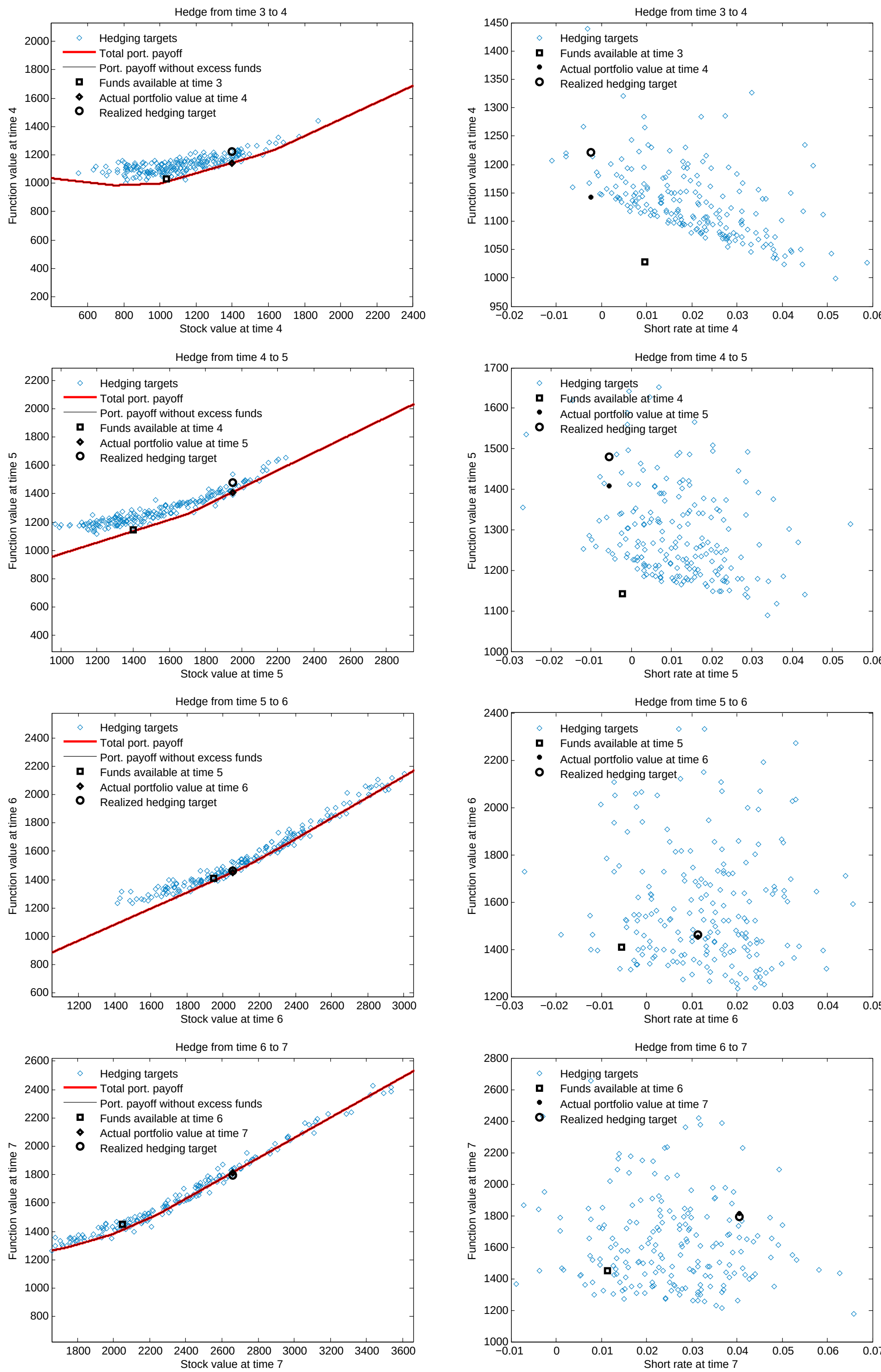

Figure 4.5 (Continued): *Evolution of Strategy 2 for Scenario C, where the Investment account component* X_3 *exercised.*

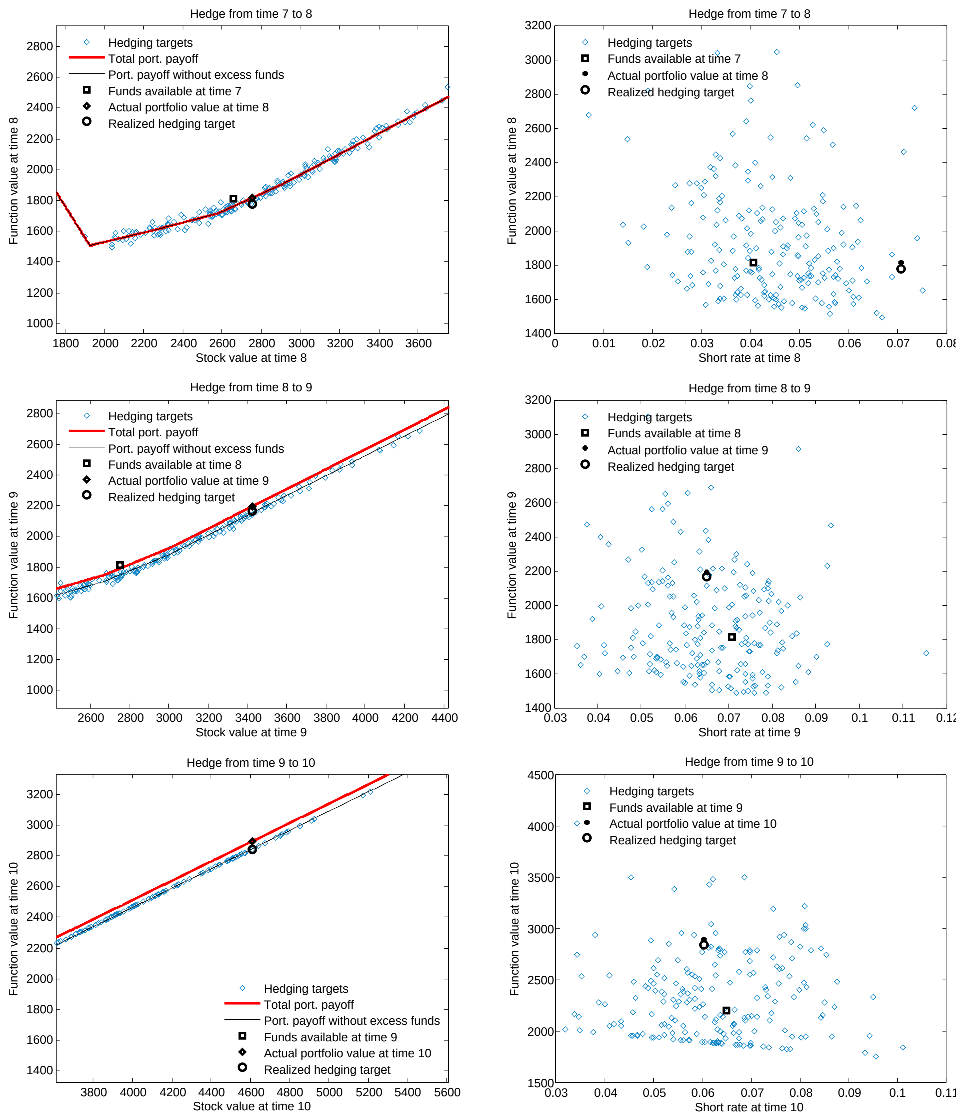

Figure 4.5 (Continued): *Evolution of Strategy 2 for Scenario C, where the Investment account component X_3 exercised.*

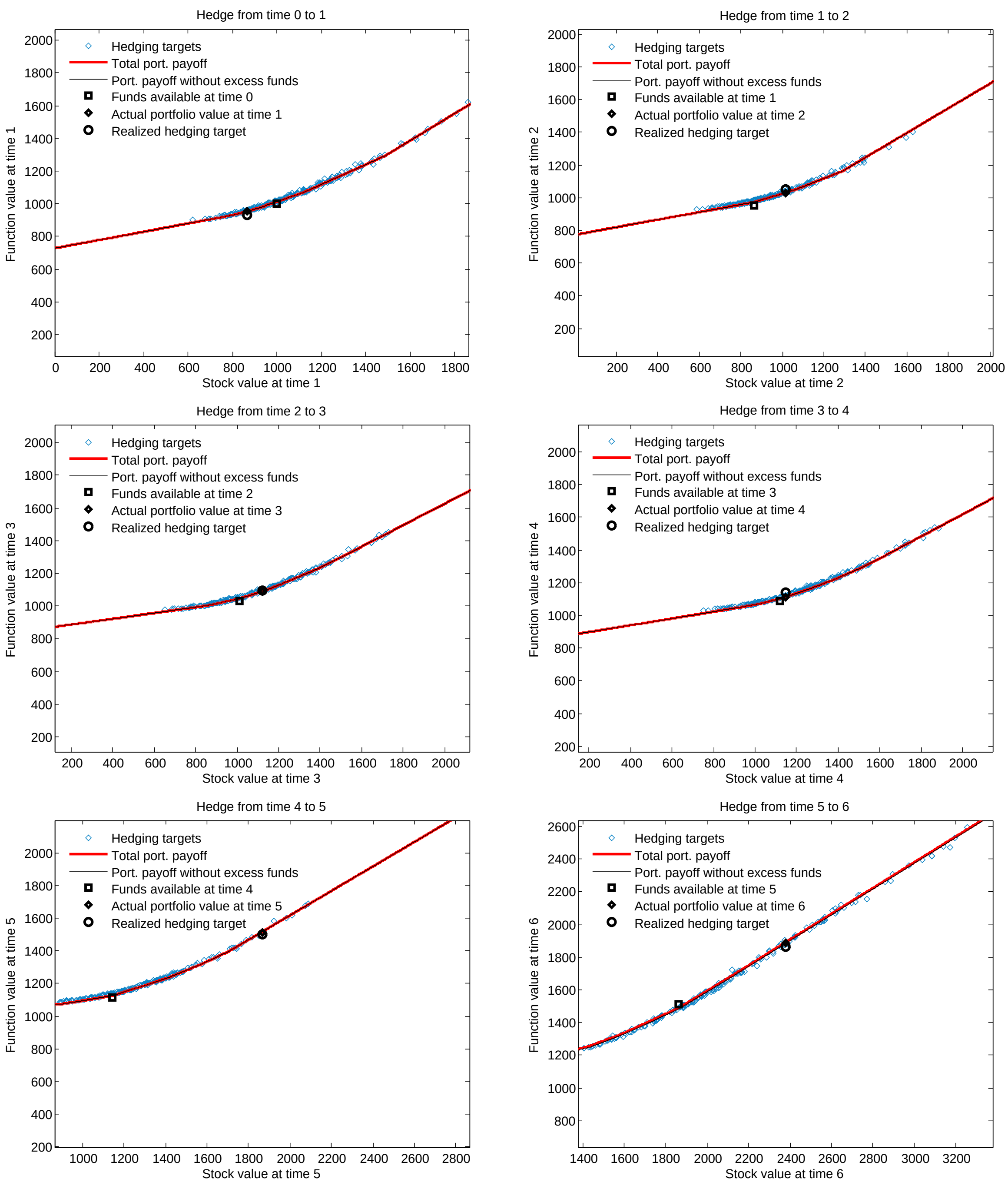

Figure 4.6: *Evolution of Strategy 2 for one particular scenario, in the case where interest rates are constant.*

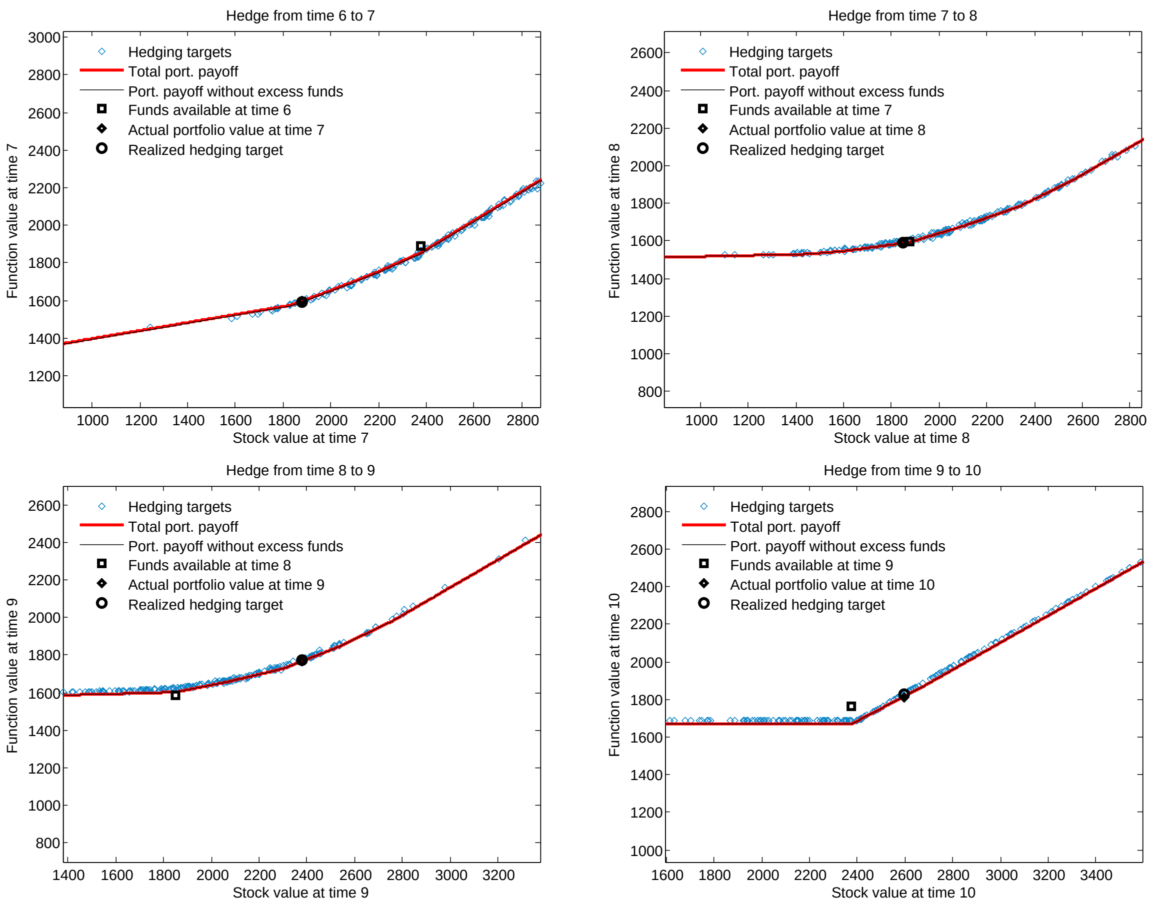

Figure 4.6 (Continued): *Evolution of Strategy 2 for one particular scenario, in the case where interest rates are constant.*

4.9 CTE minimization strategies

The examples shown so far have all been based on minimizing the MSHE in each hedging horizon. In this we illustrate the results from using strategies designed to minimize the CTE of the hedging error distribution in each horizon. The hedging target is still the GMIB price. If we use a CTE minimization objective function, we must specify the desired confidence level, α, for the CTE. Using different values for α produces slightly different results. Based on a preliminary set of simulations at different confidence levels, we set the confidence level at the value which approximately provides the lowest VaR(95%) and CTE(95%), subject to a reasonable hedging loss mean. Of the confidence levels we tested (ranging from 0.5 to 0.99), we found that $\alpha = 0.85$ yielded the most favorable results for Strategies 2 and 3. Strategy 1 produced similar results for any $\alpha \geqslant 0.75$ (it is difficult to accurately hedge the tail risk with just the stock). It is noted that setting $\alpha \geqslant 0.95$ for Strategies 2 and 3 seems to produce slightly higher tail risk measures, after taking into consideration sampling errors. This may be partly because a relatively small sample of $N = 200$ observations is used in minimizing the CTE objective function, and thus very high confidence levels lead to minimized objective functions that are very sensitive to the small number of observations in the right tail.

Tables 4.21 and 4.22 display the hedging loss results from applying Strategies 1, 2 and 3. Strategy 1 is not well-suited to minimize the CTE, as the stock is not a natural instrument for hedging the tail risk. However, the results for Strategy 1 are shown for completeness. The results in the tables suggest that semi-static strategies for the GMIB, based on CTE minimization, produce reasonable hedging loss distributions.

It is informative to compare the differences between the results of the CTE minimization strategies and MSHE minimization strategies in Section 4.8.2. For Strategies 1 and 2, the tail risk measures are higher for the CTE minimization strategies. The means also tend to be slightly higher for the CTE minimization strategies. For Strategy 3, the means and CTE measures are lower for the CTE minimization strategies, but the VaR measures are comparable. We see that the $\text{MSHL}^{1/2}$ estimates for the MSHE minimization strategies tend to be lower than the corresponding $\text{MSHL}^{1/2}$ estimates for the CTE minimization strategies, as expected.

Tables 4.23, 4.24 and 4.25 display the behavior of Strategies 1, 2 and 3 respectively, based on minimizing the CTE objective function, using annual rebalancing and allowing for transaction costs. The behavior of Strategy 1 is slightly different for the MSHE and CTE minimization cases. For Strategy 2, the CTE minimization case produces smaller stock positions compared to the MSHE minimization case. For Strategy 3, the CTE minimization case uses higher stock positions, compared to the MSHE case. Furthermore, the positions in each of the T-year put options follow similar patterns in the CTE and MSHE minimization cases.

Strategy 1
Negligible transaction costs

Rebal freq	$\text{MSHL}^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Annual	198	-356	-57	244	189	-57	390	622
	(188, 208)	(−370, −340)	(−68, −44)	(208, 291)	(178, 200)	(−69, −46)	(334, 447)	(487, 756)
Half-yearly	168	-309	-33	217	163	-40	323	503
	(159, 177)	(−334, −286)	(−49, −27)	(195, 243)	(154, 172)	(−50, −30)	(280, 366)	(391, 615)
Quarterly	156	-276	-25	233	154	-27	301	419
	(149, 164)	(−301, −255)	(−37, −12)	(207, 250)	(146, 161)	(−37, −18)	(273, 328)	(344, 495)
Static w/	698	-1295	-339	477	588	-377	521	693
10-yr put (PF1)	(665, 731)	(−1400, −1216)	(−390, −278)	(476, 477)	(562, 613)	(−414, −341)	(489, 553)	(548, 837)
Static w/	238	-428	-139	136	209	-114	137	141
lookback (PF2)	(228, 247)	(−454, −414)	(−166, −96)	(136, 136)	(202, 215)	(−127, −101)	(136, 139)	(135, 147)

Strategy 1
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})\ k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$

Rebal freq	$\text{MSHL}^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Annual	199	-352	-41	285	195	-42	398	607
	(189, 210)	(−383, −340)	(−56, −22)	(228, 316)	(185, 205)	(−54, −30)	(350, 446)	(487, 727)
Half-yearly	167	-280	-14	258	166	-14	345	469
	(159, 175)	(−313, −257)	(−24, −1)	(233, 290)	(158, 174)	(−25, −4)	(314, 377)	(409, 528)
Quarterly	155	-248	11	264	155	10	348	464
	(148, 163)	(−263, −224)	(−5, 20)	(242, 292)	(148, 163)	(0, 19)	(318, 379)	(418, 510)
Static w/	685	-1248	-338	477	585	-356	539	780
10-yr put (PF1)	(656, 713)	(−1337, −1208)	(−400, −265)	(477, 477)	(564, 606)	(−393, −320)	(497, 581)	(617, 942)
Static w/	244	-450	-139	136	213	-121	140	153
lookback (PF2)	(235, 253)	(−470, −436)	(−167, −112)	(136, 136)	(207, 219)	(−134, −108)	(136, 144)	(131, 175)

Strategy 2
Negligible transaction costs

Rebal freq	$\text{MSHL}^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Annual	127	-244	-18	178	125	-20	229	307
	(121, 133)	(−272, −219)	(−28, −4)	(165, 202)	(120, 131)	(−28, −12)	(210, 248)	(264, 349)
Half-yearly	137	-258	-21	191	135	-20	248	334
	(130, 143)	(−271, −241)	(−32, −10)	(176, 208)	(129, 142)	(−29, −12)	(227, 270)	(294, 374)
Quarterly	143	-264	-10	203	142	-17	264	342
	(136, 149)	(−285, −244)	(−23, −3)	(186, 231)	(135, 148)	(−25, −8)	(243, 285)	(301, 383)
Static w/	679	-1271	-367	477	573	-366	524	711
10-yr put (PF1)	(649, 709)	(−1312, −1191)	(−417, −306)	(477, 477)	(549, 596)	(−401, −330)	(480, 568)	(486, 937)
Static w/	241	-454	-144	136	211	-117	138	146
lookback (PF2)	(231, 251)	(−482, −431)	(−162, −118)	(136, 136)	(204, 218)	(−130, −104)	(136, 141)	(134, 158)

Strategy 2
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})\ k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$

Rebal freq	$\text{MSHL}^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Annual	124	-213	-11	188	124	-8	246	341
	(118, 131)	(−229, −199)	(−20, −4)	(176, 205)	(118, 130)	(−16, −1)	(223, 269)	(284, 398)
Half-yearly	132	-209	-5	214	132	-2	274	370
	(126, 139)	(−238, −195)	(−12, 3)	(196, 238)	(126, 139)	(−10, 6)	(252, 297)	(329, 411)
Quarterly	139	-208	11	243	138	13	305	414
	(132, 146)	(−235, −184)	(2, 23)	(223, 266)	(131, 145)	(4, 21)	(280, 330)	(350, 478)
Static w/	705	-1371	-377	477	594	-381	522	698
10-yr put (PF1)	(675, 736)	(−1430, −1279)	(−420, −311)	(476, 477)	(571, 616)	(−418, −344)	(488, 555)	(532, 864)
Static w/	249	-470	-140	136	218	-120	140	155
lookback (PF2)	(238, 259)	(−496, −454)	(−158, −110)	(136, 136)	(211, 226)	(−133, −106)	(136, 144)	(134, 177)

Table 4.21: *Hedging loss distribution statistics derived from Strategies 1 and 2, based on minimizing the CTE, for hedging the GMIB.*

Rebal freq	$\text{MSHL}^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Strategy 3								
Negligible transaction costs								
Annual	78	-163	-1	76	77	-15	102	155
	$(70, 86)$	$(-188, -128)$	$(-6, 2)$	$(71, 82)$	$(69, 85)$	$(-20, -10)$	$(89, 115)$	$(109, 201)$
Half-yearly	80	-141	-6	70	79	-16	110	206
	$(70, 91)$	$(-175, -120)$	$(-9, -1)$	$(66, 76)$	$(69, 89)$	$(-21, -11)$	$(85, 135)$	$(97, 314)$
Quarterly	69	-119	-6	76	68	-12	108	163
	$(60, 78)$	$(-146, -109)$	$(-9, -3)$	$(70, 87)$	$(60, 76)$	$(-16, -8)$	$(95, 121)$	$(124, 202)$
Static w/	714	-1330	-360	477	606	-379	560	875
10-yr put (PF1)	$(684, 745)$	$(-1443, -1263)$	$(-408, -322)$	$(477, 477)$	$(582, 629)$	$(-417, -342)$	$(494, 626)$	$(590, 1160)$
Static w/	239	-442	-153	136	208	-118	139	151
lookback (PF2)	$(230, 248)$	$(-464, -426)$	$(-169, -116)$	$(136, 136)$	$(202, 214)$	$(-131, -105)$	$(134, 144)$	$(125, 176)$

Rebal freq	$\text{MSHL}^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Strategy 3								
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})$ $k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$								
Annual	88	-106	26	119	86	15	152	205
	$(73, 102)$	$(-151, -90)$	$(21, 30)$	$(109, 131)$	$(71, 101)$	$(10, 21)$	$(138, 166)$	$(163, 248)$
Half-yearly	74	-97	26	120	71	21	159	219
	$(69, 79)$	$(-122, -83)$	$(21, 30)$	$(113, 132)$	$(65, 76)$	$(17, 25)$	$(144, 173)$	$(197, 241)$
Quarterly	71	-79	23	133	67	23	169	229
	$(67, 76)$	$(-96, -69)$	$(18, 28)$	$(124, 145)$	$(63, 72)$	$(19, 27)$	$(153, 185)$	$(179, 279)$
Static w/	703	-1274	-407	476	576	-403	509	636
10-yr put (PF1)	$(667, 739)$	$(-1388, -1207)$	$(-462, -347)$	$(476, 477)$	$(546, 607)$	$(-439, -367)$	$(480, 538)$	$(490, 782)$
Static w/	242	-438	-151	136	207	-126	140	155
lookback (PF2)	$(232, 251)$	$(-465, -413)$	$(-175, -131)$	$(136, 136)$	$(200, 214)$	$(-139, -113)$	$(133, 147)$	$(118, 191)$

Table 4.22: *Hedging loss distribution statistics derived from Strategy 3, based on minimizing the CTE, for hedging the GMIB.*

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	-2	-7	-9	-11	-16	-19	-24	-29	-37	-42
Std total target error	41	56	69	83	97	109	129	147	164	195
Mean min obj. (CTE(85%))	81	84	84	88	93	98	105	113	118	135
Std min obj. (CTE(85%))	12	38	55	70	87	104	124	144	163	188
Mean transaction costs	2	1	1	1	1	1	1	1	1	1
Std transaction costs	0	0	0	0	1	1	1	1	1	1
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.41	0.42	0.41	0.39	0.37	0.34	0.31	0.28	0.24	0.22
Risk-free bond	0.60	0.58	0.59	0.61	0.64	0.71	0.79	0.88	1.00	1.10
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.04	0.14	0.18	0.20	0.21	0.22	0.22	0.21	0.20	0.20
Risk-free bond	0.04	0.17	0.22	0.25	0.27	0.29	0.31	0.32	0.34	0.37

Table 4.23: *Behavior of Strategy 1 for hedging the GMIB, using annual rebalancing and allowing for the benchmark transaction costs (based on CTE minimization).*

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	2	1	-1	-2	-2	-3	-6	-6	-7	-8
Std total target error	34	46	54	64	72	83	93	104	113	124
Mean min obj. (CTE(85%))	42	43	45	45	46	48	49	48	49	44
Std min obj. (CTE(85%))	4	29	42	51	62	72	84	96	108	120
Mean transaction costs	2	1	1	1	1	1	1	1	1	2
Std transaction costs	0	1	1	1	1	1	1	1	1	2
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.31	0.35	0.36	0.35	0.34	0.31	0.28	0.25	0.22	0.20
Risk-free bond	0.70	0.65	0.64	0.65	0.67	0.72	0.79	0.88	0.98	1.03
Put($0.77S(t_{i-1})$)	0.10	0.09	0.09	0.09	0.09	0.08	0.08	0.06	0.06	0.03
Put($0.85S(t_{i-1})$)	0.04	0.05	0.04	0.05	0.04	0.04	0.04	0.04	0.04	0.02
Put($0.92S(t_{i-1})$)	0.02	0.04	0.04	0.05	0.04	0.04	0.04	0.04	0.03	0.10
Put($1.00S(t_{i-1})$)	0.00	0.04	0.05	0.06	0.07	0.07	0.08	0.08	0.09	0.08
Call($1.12S(t_{i-1})$)	0.12	0.09	0.08	0.06	0.06	0.05	0.05	0.05	0.04	0.04
Call($1.25S(t_{i-1})$)	0.05	0.05	0.05	0.05	0.04	0.04	0.04	0.04	0.05	0.04
Call($1.37S(t_{i-1})$)	0.05	0.07	0.05	0.05	0.05	0.05	0.04	0.04	0.04	0.03
Call($1.49S(t_{i-1})$)	0.15	0.10	0.08	0.08	0.07	0.07	0.07	0.06	0.06	0.05
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.04	0.11	0.16	0.19	0.21	0.22	0.22	0.22	0.22	0.25
Risk-free bond	0.04	0.14	0.21	0.26	0.30	0.35	0.38	0.42	0.49	0.72
Put($0.77S(t_{i-1})$)	0.15	0.16	0.15	0.15	0.15	0.14	0.13	0.12	0.11	0.09
Put($0.85S(t_{i-1})$)	0.07	0.09	0.09	0.09	0.09	0.08	0.08	0.08	0.08	0.07
Put($0.92S(t_{i-1})$)	0.05	0.07	0.07	0.08	0.07	0.07	0.07	0.07	0.07	0.19
Put($1.00S(t_{i-1})$)	0.01	0.06	0.08	0.08	0.09	0.09	0.10	0.10	0.11	0.14
Call($1.12S(t_{i-1})$)	0.08	0.09	0.09	0.08	0.08	0.07	0.07	0.08	0.07	0.10
Call($1.25S(t_{i-1})$)	0.08	0.08	0.07	0.08	0.07	0.07	0.07	0.07	0.08	0.12
Call($1.37S(t_{i-1})$)	0.09	0.09	0.09	0.09	0.09	0.08	0.07	0.08	0.07	0.11
Call($1.49S(t_{i-1})$)	0.13	0.13	0.12	0.12	0.11	0.11	0.11	0.11	0.10	0.12

Table 4.24: *Behavior of Strategy 2 for hedging the GMIB, using annual rebalancing and allowing for the benchmark transaction costs (based on CTE minimization).*

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	3	4	3	2	4	5	4	5	10	15
Std total target error	23	27	30	30	37	39	46	48	63	86
Mean min obj. (CTE(85%))	17	18	18	18	19	19	21	24	35	78
Std min obj. (CTE(85%))	1	6	9	13	17	25	35	38	52	80
Mean transaction costs	5	1	0	1	1	2	2	3	5	5
Std transaction costs	0	0	0	0	1	1	1	2	4	4
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.77	0.80	0.80	0.81	0.80	0.81	0.83	0.92	1.16	0.70
Risk-free bond	-0.06	-0.14	-0.18	-0.22	-0.27	-0.35	-0.47	-0.68	-1.22	0.01
Put$(0.80S(0), T)$	0.28	0.20	0.02	0.00	0.01	0.01	0.00	0.02	0.04	0.05
Put$(1.60S(0), T)$	0.92	1.05	1.17	1.19	1.00	0.69	0.38	0.23	0.25	0.16
Put$(2.40S(0), T)$	0.01	0.02	0.04	0.09	0.20	0.35	0.48	0.50	0.55	0.34
Put$(3.20S(0), T)$	0.00	0.00	0.00	0.02	0.06	0.13	0.21	0.33	0.48	0.19
Call$(4.00S(0), T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.03
Call$(4.80S(0), T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02
Call$(5.60S(0), T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01
Call$(6.40S(0), T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.01
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.02	0.05	0.05	0.06	0.08	0.10	0.12	0.25	0.57	0.57
Risk-free bond	0.03	0.09	0.12	0.13	0.15	0.17	0.20	0.39	1.09	1.51
Put$(0.80S(0), T)$	0.30	0.39	0.13	0.06	0.11	0.12	0.07	0.24	0.38	0.31
Put$(1.60S(0), T)$	0.08	0.17	0.29	0.39	0.48	0.52	0.47	0.47	0.66	0.44
Put$(2.40S(0), T)$	0.03	0.03	0.13	0.21	0.26	0.27	0.31	0.37	0.53	0.45
Put$(3.20S(0), T)$	0.00	0.00	0.02	0.08	0.21	0.35	0.48	0.56	0.65	0.29
Call$(4.00S(0), T)$	0.00	0.00	0.01	0.00	0.01	0.01	0.01	0.01	0.00	0.11
Call$(4.80S(0), T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.09
Call$(5.60S(0), T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.04	0.06
Call$(6.40S(0), T)$	0.00	0.00	0.00	0.00	0.01	0.02	0.02	0.04	0.09	0.08

Table 4.25: *Behavior of Strategy 3 for hedging the GMIB, using annual rebalancing and allowing for the benchmark transaction costs (based on CTE minimization).*

4.10 Using P-valuation hedging targets

Up to this point we have assumed that the hedging target is the GMIB price, calculated using option pricing theory. In this section, we investigate the effect of changing the hedging target to the P-valuation target. All of the results we illustrate are based on strategies minimizing the MSHE. The purpose of the results presented in this section is to show how changing the hedging target (to some thing meaningful other than the GMIB price) may produce hedging loss distributions with very different risk and profit profiles.

Recall that the P-valuation target is the expected present value of the benefits provided by the GMIB under the real-world measure. This hedging target might be used by an insurer who wants to meet Goal (1); in other words, the insurer does not plan to trade the liability before maturity. Using the P-valuation target will change the shape of the hedging loss distribution. In the case of the GMIB option, the P-valuation target is usually higher than the GMIB price, because, loosely speaking, the expected present values of the lookback and investment account components are higher under P than under Q (because most of the time $\mu > r(t)$ in the drift component of the stock SDE), while the expected present value of the guaranteed return component is unchanged (we assumed the market price of interest rate risk is 0). As it turns out, using the P-valuation target will decrease the mean hedging loss, but increase the tail risk measures.

Tables 4.26 and 4.27 display the results from applying Strategies 1, 2 and 3, using the P-valuation target. Large hedging profits are expected for each strategy, but the tail risk measures are considerably larger than any of the tail risk measures for the strategies shown in the previous sections. Furthermore, the results for Strategies 1 and 2 are very similar under the same transaction cost assumptions. The tail risk measures for Strategy 3 are lower than for Strategies 1 and 2, but still considerably higher than the corresponding tail risk measures for Strategy 3 when the hedging target is the GMIB price.

Tables 4.28, 4.29 and 4.30 display the behavior of Strategies 1, 2 and 3, respectively, using the P-valuation target, annual rebalancing and allowing for transaction costs. It is informative to compare these tables with the corresponding tables where the hedging target is the GMIB price (Tables 4.16, 4.17 and 4.18). For all of the strategies, the average positions in the stock (over time) are higher when the P-valuation target is used.

Rebal freq	$MSHL^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Strategy 1								
Negligible transaction costs								
Annual	472	-940	-239	289	388	-270	470	728
	$(448, 497)$	$(-1019, -861)$	$(-267, -216)$	$(253, 326)$	$(367, 409)$	$(-294, -246)$	$(403, 537)$	$(598, 858)$
Half-yearly	484	-961	-249	276	393	-284	484	748
	$(458, 511)$	$(-1024, -911)$	$(-277, -222)$	$(221, 342)$	$(369, 416)$	$(-308, -260)$	$(409, 559)$	$(607, 888)$
Quarterly	497	-987	-262	282	402	-292	488	879
	$(468, 525)$	$(-1061, -938)$	$(-287, -233)$	$(239, 335)$	$(376, 428)$	$(-317, -267)$	$(403, 572)$	$(684, 1073)$
Static w/	698	-1321	-337	477	596	-364	572	936
10-yr put (PF1)	$(663, 732)$	$(-1436, -1231)$	$(-402, -266)$	$(476, 477)$	$(567, 625)$	$(-400, -327)$	$(503, 642)$	$(620, 1251)$
Static w/	243	-460	-133	136	214	-116	140	155
lookback (PF2)	$(234, 253)$	$(-478, -435)$	$(-155, -102)$	$(136, 136)$	$(207, 221)$	$(-129, -102)$	$(134, 147)$	$(122, 189)$

Rebal freq	$MSHL^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Strategy 1								
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})\ k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$								
Annual	470	-914	-237	307	394	-258	561	939
	$(448, 492)$	$(-980, -879)$	$(-266, -191)$	$(251, 362)$	$(374, 413)$	$(-282, -233)$	$(465, 657)$	$(830, 1048)$
Half-yearly	480	-936	-223	322	404	-259	577	1068
	$(456, 505)$	$(-1008, -876)$	$(-257, -190)$	$(265, 389)$	$(380, 428)$	$(-284, -234)$	$(464, 690)$	$(745, 1391)$
Quarterly	483	-929	-210	319	412	-252	625	1220
	$(459, 506)$	$(-1014, -879)$	$(-253, -178)$	$(275, 365)$	$(388, 436)$	$(-277, -226)$	$(497, 753)$	$(940, 1500)$
Static w/	714	-1250	-372	477	601	-385	512	652
10-yr put (PF1)	$(680, 747)$	$(-1420, -1207)$	$(-430, -302)$	$(477, 477)$	$(575, 627)$	$(-422, -348)$	$(482, 543)$	$(499, 806)$
Static w/	245	-474	-150	136	213	-122	138	146
lookback (PF2)	$(236, 255)$	$(-487, -449)$	$(-174, -121)$	$(136, 136)$	$(206, 219)$	$(-135, -109)$	$(135, 141)$	$(131, 161)$

Rebal freq	$MSHL^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Strategy 2								
Negligible transaction costs								
Annual	463	-894	-247	262	379	-266	476	896
	$(437, 488)$	$(-979, -818)$	$(-274, -219)$	$(222, 306)$	$(355, 402)$	$(-289, -242)$	$(387, 566)$	$(692, 1099)$
Half-yearly	507	-991	-264	278	412	-296	497	898
	$(478, 535)$	$(-1087, -904)$	$(-292, -232)$	$(225, 329)$	$(386, 437)$	$(-321, -270)$	$(409, 585)$	$(719, 1077)$
Quarterly	536	-1019	-281	278	435	-313	559	1106
	$(506, 566)$	$(-1133, -986)$	$(-311, -249)$	$(225, 352)$	$(406, 464)$	$(-340, -286)$	$(439, 680)$	$(752, 1460)$
Static w/	719	-1334	-385	477	599	-398	573	952
10-yr put (PF1)	$(687, 751)$	$(-1403, -1259)$	$(-451, -324)$	$(476, 477)$	$(573, 625)$	$(-435, -361)$	$(491, 655)$	$(570, 1335)$
Static w/	244	-447	-153	136	209	-126	136	137
lookback (PF2)	$(234, 253)$	$(-476, -425)$	$(-176, -126)$	$(136, 136)$	$(202, 216)$	$(-139, -113)$	$(136, 137)$	$(137, 137)$

Rebal freq	$MSHL^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Strategy 2								
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})\ k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$								
Annual	428	-822	-196	293	362	-229	548	945
	$(408, 448)$	$(-871, -784)$	$(-224, -170)$	$(231, 358)$	$(343, 380)$	$(-251, -206)$	$(451, 645)$	$(829, 1062)$
Half-yearly	450	-868	-205	305	379	-243	540	959
	$(429, 472)$	$(-938, -825)$	$(-234, -177)$	$(263, 366)$	$(359, 399)$	$(-267, -220)$	$(445, 636)$	$(756, 1163)$
Quarterly	474	-902	-215	340	407	-242	657	1296
	$(450, 498)$	$(-948, -866)$	$(-245, -174)$	$(264, 430)$	$(382, 433)$	$(-268, -217)$	$(521, 792)$	$(901, 1692)$
Static w/	714	-1250	-372	477	601	-385	512	652
10-yr put (PF1)	$(680, 747)$	$(-1420, -1207)$	$(-430, -302)$	$(477, 477)$	$(575, 627)$	$(-422, -348)$	$(482, 543)$	$(499, 806)$
Static w/	245	-474	-150	136	213	-122	138	146
lookback (PF2)	$(236, 255)$	$(-487, -449)$	$(-174, -121)$	$(136, 136)$	$(206, 219)$	$(-135, -109)$	$(135, 141)$	$(131, 161)$

Table 4.26: *Hedging loss distribution statistics derived from Strategies 1 and 2, using P-valuation targets, for hedging the GMIB.*

Strategy 3								
Negligible transaction costs								
Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	448	-909	-204	261	373	-250	380	558
	$(419, 478)$	$(-1005, -819)$	$(-230, -178)$	$(220, 311)$	$(347, 398)$	$(-273, -227)$	$(336, 424)$	$(477, 639)$
Half-yearly	482	-1009	-209	251	399	-270	407	689
	$(452, 512)$	$(-1125, -934)$	$(-237, -174)$	$(210, 295)$	$(374, 425)$	$(-295, -246)$	$(339, 474)$	$(480, 897)$
Quarterly	506	-1049	-215	263	422	-279	517	1030
	$(475, 537)$	$(-1163, -950)$	$(-244, -189)$	$(212, 323)$	$(392, 453)$	$(-305, -253)$	$(395, 640)$	$(619, 1440)$
Static w/	713	-1366	-412	477	585	-409	502	602
10-yr put (PF1)	$(683, 744)$	$(-1467, -1265)$	$(-458, -365)$	$(476, 477)$	$(562, 607)$	$(-445, -373)$	$(482, 522)$	$(504, 699)$
Static w/	247	-458	-158	136	209	-133	140	152
lookback (PF2)	$(238, 257)$	$(-490, -445)$	$(-178, -138)$	$(136, 136)$	$(202, 216)$	$(-146, -120)$	$(135, 144)$	$(129, 176)$

Strategy 3								
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})\ k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$								
Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	386	-761	-155	294	336	-190	414	565
	$(362, 410)$	$(-862, -713)$	$(-187, -133)$	$(269, 336)$	$(316, 357)$	$(-211, -169)$	$(371, 457)$	$(486, 645)$
Half-yearly	393	-807	-139	298	347	-186	437	650
	$(370, 416)$	$(-897, -779)$	$(-161, -110)$	$(266, 347)$	$(327, 366)$	$(-207, -164)$	$(385, 489)$	$(536, 763)$
Quarterly	382	-809	-72	341	350	-152	488	743
	$(357, 406)$	$(-906, -723)$	$(-101, -51)$	$(296, 377)$	$(329, 372)$	$(-174, -130)$	$(431, 546)$	$(647, 838)$
Static w/	698	-1326	-329	476	593	-367	546	821
10-yr put (PF1)	$(664, 731)$	$(-1454, -1274)$	$(-396, -254)$	$(476, 477)$	$(567, 619)$	$(-404, -331)$	$(490, 602)$	$(537, 1105)$
Static w/	234	-414	-140	136	205	-112	139	151
lookback (PF2)	$(225, 244)$	$(-443, -399)$	$(-168, -111)$	$(136, 136)$	$(198, 212)$	$(-125, -100)$	$(136, 143)$	$(134, 168)$

Table 4.27: *Hedging loss distribution statistics derived from Strategy 3, using P-valuation targets, for hedging the GMIB.*

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	173	120	69	16	-34	-81	-132	-175	-214	-258
Std total target error	83	105	132	157	189	225	258	294	336	394
Mean min obj. (MSHE$^{1/2}$)	186	141	114	104	98	96	97	97	100	114
Std min obj. (MSHE$^{1/2}$)	5	63	74	84	83	88	98	102	109	117
Mean excess funds	0	2	15	44	83	126	166	209	248	289
Std excess funds	0	9	33	67	103	139	173	210	249	292
Mean transaction costs	5	1	1	1	1	1	1	1	1	2
Std transaction costs	0	1	1	1	1	1	1	1	1	1
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	1.03	0.91	0.78	0.68	0.59	0.51	0.44	0.38	0.31	0.26
Risk-free bond	-0.04	0.09	0.23	0.32	0.40	0.49	0.60	0.71	0.85	0.99
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.05	0.09	0.13	0.17	0.20	0.23	0.28	0.31	0.30	0.33
Risk-free bond	0.05	0.10	0.13	0.17	0.23	0.28	0.32	0.37	0.43	0.52

Table 4.28: *Behavior of Strategy 1 for hedging the GMIB, using annual rebalancing and allowing for the benchmark transaction costs (using the P-valuation target).*

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	173	122	72	21	-27	-72	-118	-158	-193	-229
Std total target error	82	104	130	154	183	218	248	279	317	362
Mean min obj. (MSHE$^{1/2}$)	185	141	113	100	92	88	86	82	77	77
Std min obj. (MSHE$^{1/2}$)	5	61	72	80	82	87	96	100	107	118
Mean excess funds	0	2	15	42	78	117	154	191	227	262
Std excess funds	0	11	35	69	102	136	168	202	235	273
Mean transaction costs	5	1	1	1	1	1	1	2	2	2
Std transaction costs	0	1	1	1	1	1	1	1	2	2
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	1.01	0.92	0.80	0.68	0.58	0.49	0.42	0.35	0.29	0.24
Risk-free bond	-0.02	0.07	0.20	0.30	0.40	0.49	0.60	0.72	0.85	0.96
Put($0.77S(t_{i-1})$)	0.16	0.15	0.16	0.15	0.14	0.14	0.12	0.11	0.09	0.05
Put($0.85S(t_{i-1})$)	0.02	0.07	0.06	0.07	0.06	0.06	0.06	0.06	0.05	0.04
Put($0.92S(t_{i-1})$)	0.01	0.03	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.11
Put($1.00S(t_{i-1})$)	0.00	0.06	0.08	0.08	0.07	0.07	0.07	0.07	0.07	0.06
Call($1.12S(t_{i-1})$)	0.05	0.01	0.01	0.03	0.04	0.05	0.06	0.06	0.06	0.05
Call($1.25S(t_{i-1})$)	0.02	0.01	0.02	0.03	0.04	0.04	0.04	0.04	0.04	0.05
Call($1.37S(t_{i-1})$)	0.02	0.02	0.02	0.03	0.04	0.04	0.04	0.03	0.03	0.04
Call($1.49S(t_{i-1})$)	0.04	0.05	0.07	0.06	0.07	0.06	0.07	0.07	0.06	0.05
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.09	0.09	0.13	0.16	0.21	0.25	0.28	0.29	0.28	0.35
Risk-free bond	0.09	0.10	0.13	0.18	0.25	0.32	0.38	0.44	0.53	0.74
Put($0.77S(t_{i-1})$)	0.34	0.33	0.28	0.24	0.26	0.29	0.23	0.27	0.18	0.19
Put($0.85S(t_{i-1})$)	0.10	0.19	0.16	0.14	0.12	0.10	0.13	0.20	0.12	0.11
Put($0.92S(t_{i-1})$)	0.05	0.10	0.11	0.10	0.09	0.09	0.09	0.08	0.10	0.19
Put($1.00S(t_{i-1})$)	0.00	0.12	0.12	0.11	0.10	0.09	0.08	0.08	0.09	0.13
Call($1.12S(t_{i-1})$)	0.10	0.03	0.03	0.05	0.07	0.07	0.07	0.09	0.10	0.14
Call($1.25S(t_{i-1})$)	0.06	0.03	0.05	0.06	0.06	0.06	0.06	0.07	0.08	0.13
Call($1.37S(t_{i-1})$)	0.07	0.06	0.06	0.07	0.07	0.07	0.07	0.06	0.07	0.12
Call($1.49S(t_{i-1})$)	0.09	0.10	0.11	0.11	0.11	0.10	0.10	0.10	0.10	0.12

Table 4.29: *Behavior of Strategy 2 for hedging the GMIB, using annual rebalancing and allowing for the benchmark transaction costs (using the P-valuation target).*

Horizon i	1	2	3	4	5	6	7	8	9	10
Mean total target error	172	120	71	26	-18	-55	-96	-130	-165	-190
Std total target error	80	97	111	137	165	192	226	260	299	336
Mean min obj. (MSHE$^{1/2}$)	185	131	98	80	71	66	67	66	67	80
Std min obj. (MSHE$^{1/2}$)	4	67	79	81	85	85	90	93	95	98
Mean excess funds	0	1	12	35	67	101	135	168	198	224
Std excess funds	0	8	30	61	95	127	158	191	227	260
Mean transaction costs	7	2	2	2	2	2	2	3	4	7
Std transaction costs	2	2	2	2	2	2	2	2	4	7
Mean $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	1.20	1.26	1.23	1.16	1.07	1.00	0.94	0.90	0.93	0.43
Risk-free bond	-0.39	-0.71	-0.84	-0.85	-0.80	-0.75	-0.69	-0.64	-0.72	0.66
Put$(0.80S(0),T)$	0.24	0.33	0.26	0.20	0.12	0.05	0.02	0.01	0.02	0.04
Put$(1.60S(0),T)$	0.02	0.10	0.16	0.23	0.30	0.34	0.32	0.34	0.27	0.09
Put$(2.40S(0),T)$	0.23	0.39	0.37	0.32	0.23	0.18	0.19	0.25	0.38	0.24
Put$(3.20S(0),T)$	0.00	0.10	0.22	0.28	0.33	0.37	0.39	0.39	0.40	0.12
Call$(4.00S(0),T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.04
Call$(4.80S(0),T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02
Call$(5.60S(0),T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01
Call$(6.40S(0),T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.02
Std Dev $\hat{\boldsymbol{x}}(t_{i-1})$										
Stock	0.14	0.13	0.12	0.12	0.13	0.14	0.13	0.17	0.36	0.35
Risk-free bond	0.30	0.34	0.36	0.37	0.38	0.38	0.38	0.52	1.13	1.37
Put$(0.80S(0),T)$	0.65	0.72	0.57	0.43	0.31	0.20	0.11	0.08	0.22	0.17
Put$(1.60S(0),T)$	0.10	0.23	0.29	0.34	0.36	0.34	0.33	0.36	0.44	0.34
Put$(2.40S(0),T)$	0.22	0.33	0.37	0.37	0.33	0.30	0.31	0.35	0.41	0.28
Put$(3.20S(0),T)$	0.00	0.25	0.35	0.38	0.38	0.39	0.42	0.47	0.48	0.21
Call$(4.00S(0),T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.12
Call$(4.80S(0),T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.02	0.09
Call$(5.60S(0),T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.05	0.07
Call$(6.40S(0),T)$	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.04	0.09	0.14

Table 4.30: *Behavior of Strategy 3 for hedging the GMIB, using annual rebalancing and allowing for the benchmark transaction costs (using the P-valuation target).*

4.11 Assessing the impact of model risk

There is a large amount of statistical evidence showing that log stock returns are not independent, identically distributed (IID) normal random variables (i.e. stock prices do not satisfy the geometric Brownian motion process). Stock returns exhibit stochastic volatility and jumps. Given that equity risk dominates the GMIB option, it is of great interest to assess the impact of stock return model risk. In this section, we measure the impact on the performance of the semi-static strategies if the actual stock return distribution under P does not satisfy the IID normal assumption. If the strategies perform very differently under a different equity price model, then the semi-static strategies may not be robust in practice.

An alternative model for stock price returns, which has gained some acceptance as a better model for stock returns at low and high frequencies, is the regime switching lognormal (RSLN) model. The RSLN model assumes the stock return process, under P, lies in one of K_s states over a given time interval. Let $E_{[t,t+\tau)}$ be a random variable denoting the state in the interval $[t, t+\tau)$. Under the RSLN model

$$\log\left(\frac{S(t+\tau)}{S(t)}\right) \Big| \left(E_{[t,t+\tau)} = k\right) \sim N(u_k, \sigma_k^2), \qquad k = 1, \dots, K_s.$$

The K_s-state Markov chain transition matrix $\boldsymbol{P}$ contains the probabilities of moving between states. The component in the i-th row at the j-th column corresponds to

$$p_{ij} = P[E_{[t,t+\tau)} = j | E_{[t-\tau,t)} = i] \qquad i, j = 1, \dots, K_s.$$

Hardy (2001) finds the 2-state regime switching lognormal (RSLN2) model provides a reasonable fit to S&P 500 monthly log total return data.

We assess the impact of model risk by assuming the *actual* stock return distribution under P satisfies the RSLN2 model ($K_s = 2$). An advantage of using a lognormal regime switching model is that it is computationally fast, at simulating returns between rebalancing time points, in the existing semi-static model framework. We fit the RSLN2 model to monthly time intervals ($\tau = 1/12$). Hence the parameters of the model,

$\Theta = \{\mu_1, \mu_2, \sigma_1, \sigma_2, p_{1,2}, p_{2,1}\}$, are measured on a monthly basis. At time 0, the probability of being in State 1 is set equal to the long-run 2-state Markov chain equilibrium probability given by $\pi_1 = p_{21}/(p_{12}+p_{21})$ (For State 2, $\pi_2 = 1-\pi_1$). Methods for estimating the parameters include maximum likelihood estimation (Hardy, 2001) and Bayesian Markov Chain Monte Carlo estimation (Hartman and Heaton, 2011). Table 4.31 displays the parameter sets we use to assess the impact of model risk. State 1 corresponds to a positive, strong equity market, while State 2 corresponds to a negative, weak equity market. Parameter set A corresponds to the values obtained by Hardy (2001). These RLSN2 parameter values are broadly consistent with the parameter estimates in Hartman and Heaton (2011). Parameter set B is a variation of set A, which allows for a higher transition probability of 10% from the strong to the weak market state. Parameter set C allows for much more adverse outcomes than sets A and B. For comparison purposes, we also show the benchmark parameter values, used in all previous examples of this chapter, expressed on a monthly basis. Table 4.32 displays the key statistics of the stock price distribution at the end of year 5 and year 10 under each parameter set (assuming the stock price at time 0 is 1000). These statistics were obtained using simulation, based on 10^6 scenarios. This table is provided to give the reader a sense of the stock price distribution features for each parameter set.

Parameter Set	μ_1	μ_2	σ_1	σ_2	p_{12}	p_{21}
A	0.0126	-0.0185	0.035	0.0748	0.0398	0.3798
B	0.0126	-0.0185	0.035	0.0748	0.1	0.4
C	0.0126	-0.0185	0.035	0.09	0.1	0.2
Benchmark Assumptions	0.0058	0	0.0577	0	0	0

Table 4.31: *Different parameter set assumptions for the RSLN2 model.*

Parameter Set	**Mean**	**Std dev**	**1%-quantile**	**5%-quantile**	**Median**	**95%-quantile**	**99%-quantile**
Distribution at end of year 5							
A	1888	638	748	995	1807	3049	3755
B	1580	616	550	751	1489	2717	3449
C	1305	677	290	454	1181	2573	3445
Benchmark Assumptions	1569	739	501	679	1419	2963	4027
Distribution at end of year 10							
A	3566	1753	962	1404	3228	6869	9259
B	2494	1429	550	842	2181	5203	7336
C	1704	1331	197	360	1351	4233	6528
Benchmark Assumptions	2462	1729	461	711	2015	5705	8776

Table 4.32: *The stock price distribution under different parameter set assumptions.*

Adjusting the implementation of the method for testing semi-static strategies, to allow for a different actual stock return distribution model, is straightforward. The only change to the HLS algorithm occurs in Step (7); the stock returns are now simulated using the RLSN2 model. The hedging targets are still simulated using the existing models (geometric Brownian motion for the stock). This situation reflects reality, where the modeler (user of the results) does not know the true underlying distributions of the stock returns, but simulates the returns using, in our case, a normal distribution. Here, the "true" underlying distribution is the RLSN2 model. We do not adjust the strike prices of the options because the actual stock return distribution under P over each hedging horizon has changed; the modeler does not know what the actual return distribution is.

Tables 4.33, 4.34 and 4.35 illustrate the performances of Strategies 1, 2 and 3 under each of the parameter assumption sets for the RSLN2 model. All of the results in this section are based on strategies minimizing the MSHE, where the hedging target is the GMIB price, $b(0) = \pi$, and transaction costs are included. For each strategy type, the results under parameter set A are the most favorable, while the results under set C are the least favorable. This observation suggests that semi-static hedging strategies for the GMIB tend to perform better when equity returns are on average higher. Interestingly, Strategy 1 performs poorly under set C, but the relative performance of Strategy 2 under set C is much better. Strategy 3 is the best performer at minimizing the tail risk under all of the parameter sets, but this comes at a cost; the mean hedging losses are all positive for Strategy 3.

Each table also shows how the static hedging strategies behave under the RSLN2 model. Note that in Chapter 3, we did not test the static strategies for their robustness against model misspecification. Hence, the results in this section supplement the findings in Chapter 3. The static lookback strategy appears to be remarkably robust under each model, in terms of controlling the risk of extreme losses, although the mean hedging profit decreases when equity returns are on average lower. In contrast, the tail risk of the static put strategy is quite variable under different return distributions. It is noted that by comparing the hedging loss statistics in each of Tables 4.33, 4.34 and 4.35, for a particular static strategy and a certain parameter set, the reader can gain a sense of the sensitivity of the results of that static strategy under that parameter set (for statistics based on 1000 scenarios).

Overall, it appears that the Strategies 2 and 3, and the static lookback strategy, are robust against model misspecification. The results give credibility to the use of the method in this chapter for testing semi-static strategies for the GMIB.

The way we have tested model misspecification in this section is an important risk management technique. Any hedging strategy for a long-dated option should be tested for its robustness against model misspecification, using an approach similar to what we have described here.

Strategy 1, using parameter set A

Transaction costs: $c(k,t_{i-1}) = 0.5\%\phi(k,t_{i-1})\ k \neq b,\ c(b,t_{i-1}) = 0.05\%\phi(b,t_{i-1})$

Rebal freq	MSHL$^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Annual	245	-413	-204	59	148	-195	149	280
	$(238,252)$	$(-426,-404)$	$(-216,-196)$	$(34,98)$	$(141,156)$	$(-204,-186)$	$(115,183)$	$(193,368)$
Half-yearly	225	-389	-180	68	142	-175	149	279
	$(218,232)$	$(-413,-373)$	$(-193,-173)$	$(49,99)$	$(135,149)$	$(-184,-166)$	$(117,181)$	$(202,355)$
Quarterly	206	-361	-163	56	132	-159	130	247
	$(199,213)$	$(-377,-343)$	$(-171,-153)$	$(48,79)$	$(126,139)$	$(-167,-150)$	$(101,160)$	$(168,326)$
Static w/	914	-1487	-837	19	444	-799	252	460
10-yr put (PF1)	$(889,939)$	$(-1533,-1410)$	$(-858,-809)$	$(-31,127)$	$(419,468)$	$(-826,-771)$	$(177,326)$	$(419,501)$
Static w/	248	-410	-219	131	138	-206	135	137
lookback (PF2)	$(241,255)$	$(-427,-386)$	$(-225,-214)$	$(98,135)$	$(130,146)$	$(-215,-198)$	$(134,136)$	$(137,137)$

Strategy 1, using parameter set B

Transaction costs: $c(k,t_{i-1}) = 0.5\%\phi(k,t_{i-1})\ k \neq b,\ c(b,t_{i-1}) = 0.05\%\phi(b,t_{i-1})$

Rebal freq	MSHL$^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Annual	207	-385	-122	177	173	-115	276	463
	$(199,215)$	$(-398,-373)$	$(-133,-111)$	$(159,205)$	$(164,181)$	$(-125,-104)$	$(236,316)$	$(357,568)$
Half-yearly	182	-346	-97	168	155	-95	256	421
	$(174,189)$	$(-357,-335)$	$(-108,-86)$	$(142,189)$	$(147,162)$	$(-105,-86)$	$(220,292)$	$(346,496)$
Quarterly	162	-309	-83	148	142	-79	238	390
	$(156,169)$	$(-330,-294)$	$(-97,-70)$	$(135,173)$	$(135,149)$	$(-88,-71)$	$(203,274)$	$(306,475)$
Static w/	672	-1170	-504	405	505	-444	486	566
10-yr put (PF1)	$(649,695)$	$(-1222,-1116)$	$(-563,-445)$	$(366,475)$	$(488,522)$	$(-476,-413)$	$(456,515)$	$(467,664)$
Static w/	232	-393	-182	136	188	-136	137	137
lookback (PF2)	$(225,239)$	$(-423,-382)$	$(-199,-165)$	$(136,136)$	$(183,194)$	$(-148,-124)$	$(136,137)$	$(137,137)$

Strategy 1, using parameter set C

Transaction costs: $c(k,t_{i-1}) = 0.5\%\phi(k,t_{i-1})\ k \neq b,\ c(b,t_{i-1}) = 0.05\%\phi(b,t_{i-1})$

Rebal freq	MSHL$^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Annual	223	-289	7	396	222	25	624	1053
	$(203,243)$	$(-310,-266)$	$(-8,19)$	$(348,455)$	$(203,241)$	$(11,38)$	$(530,718)$	$(807,1298)$
Half-yearly	212	-248	15	369	210	32	599	1060
	$(187,238)$	$(-278,-239)$	$(2,31)$	$(314,434)$	$(186,234)$	$(19,45)$	$(494,704)$	$(702,1418)$
Quarterly	234	-245	25	376	229	44	643	1219
	$(172,296)$	$(-270,-224)$	$(14,39)$	$(321,436)$	$(168,291)$	$(30,59)$	$(487,799)$	$(572,1866)$
Static w/	553	-1059	107	478	551	-41	621	1009
10-yr put (PF1)	$(529,576)$	$(-1115,-1010)$	$(64,144)$	$(478,479)$	$(530,573)$	$(-75,-6)$	$(544,698)$	$(801,1217)$
Static w/	201	-406	73	137	198	-33	145	178
lookback (PF2)	$(192,210)$	$(-422,-384)$	$(25,99)$	$(136,137)$	$(191,206)$	$(-46,-21)$	$(135,155)$	$(125,232)$

Table 4.33: *Hedging loss statistics derived from Strategy 1 for the GMIB, under different parameter assumptions for the RSLN2 model.*

Strategy 2, using parameter set A

Transaction costs: $c(k,t_{i-1}) = 0.5\%\phi(k,t_{i-1})\ k \neq b,\ c(b,t_{i-1}) = 0.05\%\phi(b,t_{i-1})$

Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	89	-165	-9	105	86	-21	159	251
	$(83, 94)$	$(-180, -152)$	$(-16, -4)$	$(97, 124)$	$(81, 91)$	$(-26, -15)$	$(138, 180)$	$(204, 298)$
Half-yearly	90	-166	-7	112	88	-16	174	287
	$(84, 95)$	$(-184, -153)$	$(-12, -2)$	$(104, 125)$	$(83, 94)$	$(-21, -11)$	$(149, 199)$	$(230, 345)$
Quarterly	94	-157	3	141	94	-6	218	331
	$(88, 101)$	$(-179, -148)$	$(-2, 7)$	$(124, 176)$	$(88, 101)$	$(-12, 0)$	$(190, 247)$	$(272, 390)$
Static w/	908	-1440	-850	26	441	-794	216	502
10-yr put (PF1)	$(883, 933)$	$(-1509, -1378)$	$(-872, -824)$	$(-31, 93)$	$(417, 465)$	$(-821, -767)$	$(146, 286)$	$(389, 615)$
Static w/	256	-418	-229	105	141	-214	133	137
lookback (PF2)	$(249, 263)$	$(-445, -397)$	$(-240, -223)$	$(69, 132)$	$(134, 149)$	$(-222, -205)$	$(125, 140)$	$(136, 137)$

Strategy 2, using parameter set B

Transaction costs: $c(k,t_{i-1}) = 0.5\%\phi(k,t_{i-1})\ k \neq b,\ c(b,t_{i-1}) = 0.05\%\phi(b,t_{i-1})$

Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	104	-178	-16	152	103	-18	225	342
	$(98, 111)$	$(-208, -162)$	$(-25, -12)$	$(132, 169)$	$(96, 110)$	$(-24, -12)$	$(196, 254)$	$(282, 401)$
Half-yearly	108	-187	-11	150	107	-13	240	361
	$(101, 114)$	$(-212, -174)$	$(-18, -4)$	$(135, 180)$	$(100, 114)$	$(-20, -6)$	$(207, 272)$	$(310, 413)$
Quarterly	109	-179	-5	188	109	-1	262	379
	$(103, 116)$	$(-214, -155)$	$(-12, 4)$	$(166, 211)$	$(103, 116)$	$(-8, 5)$	$(235, 290)$	$(346, 412)$
Static w/	697	-1199	-540	428	522	-462	475	498
10-yr put (PF1)	$(670, 725)$	$(-1291, -1144)$	$(-594, -476)$	$(396, 476)$	$(500, 545)$	$(-495, -430)$	$(460, 489)$	$(467, 529)$
Static w/	230	-395	-175	136	189	-132	136	137
lookback (PF2)	$(220, 241)$	$(-411, -377)$	$(-190, -155)$	$(136, 136)$	$(180, 198)$	$(-143, -120)$	$(136, 137)$	$(137, 137)$

Strategy 2, using parameter set C

Transaction costs: $c(k,t_{i-1}) = 0.5\%\phi(k,t_{i-1})\ k \neq b,\ c(b,t_{i-1}) = 0.05\%\phi(b,t_{i-1})$

Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	134	-231	-9	203	133	-10	300	484
	$(123, 145)$	$(-248, -205)$	$(-16, -1)$	$(180, 227)$	$(122, 144)$	$(-19, -2)$	$(254, 346)$	$(324, 644)$
Half-yearly	148	-230	-7	231	148	-7	361	622
	$(137, 160)$	$(-254, -213)$	$(-16, 3)$	$(201, 263)$	$(137, 160)$	$(-16, 3)$	$(306, 417)$	$(459, 784)$
Quarterly	151	-235	3	242	151	2	386	668
	$(137, 164)$	$(-263, -211)$	$(-5, 10)$	$(208, 294)$	$(137, 164)$	$(-7, 11)$	$(324, 449)$	$(467, 869)$
Static w/	562	-1074	106	478	559	-62	591	875
10-yr put (PF1)	$(538, 587)$	$(-1174, -1010)$	$(41, 146)$	$(478, 479)$	$(537, 581)$	$(-96, -27)$	$(533, 649)$	$(695, 1056)$
Static w/	210	-406	49	137	205	-44	137	137
lookback (PF2)	$(199, 220)$	$(-438, -384)$	$(22, 84)$	$(136, 137)$	$(197, 213)$	$(-57, -32)$	$(137, 137)$	$(137, 137)$

Table 4.34: *Hedging loss statistics derived from Strategy 2 for the GMIB, under different parameter assumptions for the RSLN2 model.*

Rebal freq	$\text{MSHL}^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Strategy 3, using parameter set A								
Transaction costs: $c(k,t_{i-1}) = 0.5\%\phi(k,t_{i-1})\ k \neq b,\ c(b,t_{i-1}) = 0.05\%\phi(b,t_{i-1})$								
Annual	53	-62	14	87	51	13	138	263
	$(47, 59)$	$(-71, -55)$	$(11, 16)$	$(79, 98)$	$(45, 57)$	$(10, 16)$	$(113, 163)$	$(174, 351)$
Half-yearly	51	-45	19	94	46	21	133	216
	$(46, 55)$	$(-53, -37)$	$(17, 21)$	$(88, 101)$	$(41, 51)$	$(18, 24)$	$(115, 151)$	$(160, 273)$
Quarterly	58	-29	25	119	49	31	156	221
	$(53, 64)$	$(-37, -23)$	$(23, 27)$	$(109, 131)$	$(43, 55)$	$(28, 34)$	$(141, 170)$	$(179, 263)$
Static w/	931	-1499	-853	48	451	-815	265	514
10-yr put (PF1)	$(905, 957)$	$(-1574, -1434)$	$(-872, -834)$	$(-36, 140)$	$(424, 478)$	$(-843, -787)$	$(190, 339)$	$(423, 605)$
Static w/	254	-409	-227	132	142	-210	135	137
lookback (PF2)	$(244, 264)$	$(-424, -399)$	$(-235, -219)$	$(99, 134)$	$(130, 155)$	$(-219, -201)$	$(134, 136)$	$(136, 137)$
Strategy 3, using parameter set B								
Transaction costs: $c(k,t_{i-1}) = 0.5\%\phi(k,t_{i-1})\ k \neq b,\ c(b,t_{i-1}) = 0.05\%\phi(b,t_{i-1})$								
Rebal freq	**$\text{MSHL}^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	53	-57	17	105	49	20	139	192
	$(50, 56)$	$(-66, -49)$	$(14, 19)$	$(95, 117)$	$(46, 52)$	$(17, 23)$	$(126, 152)$	$(162, 222)$
Half-yearly	55	-35	23	111	46	29	150	213
	$(51, 58)$	$(-46, -30)$	$(21, 26)$	$(102, 124)$	$(43, 50)$	$(26, 32)$	$(135, 165)$	$(180, 246)$
Quarterly	62	-21	31	129	48	39	165	233
	$(58, 66)$	$(-27, -15)$	$(28, 33)$	$(120, 141)$	$(44, 52)$	$(36, 42)$	$(149, 181)$	$(183, 282)$
Static w/	684	-1190	-500	473	515	-450	480	493
10-yr put (PF1)	$(658, 710)$	$(-1271, -1126)$	$(-562, -434)$	$(426, 476)$	$(495, 535)$	$(-482, -418)$	$(474, 486)$	$(464, 522)$
Static w/	230	-409	-166	136	190	-129	136	137
lookback (PF2)	$(222, 238)$	$(-421, -395)$	$(-186, -147)$	$(136, 136)$	$(184, 195)$	$(-141, -118)$	$(136, 137)$	$(136, 137)$
Strategy 3, using parameter set C								
Transaction costs: $c(k,t_{i-1}) = 0.5\%\phi(k,t_{i-1})\ k \neq b,\ c(b,t_{i-1}) = 0.05\%\phi(b,t_{i-1})$								
Rebal freq	**$\text{MSHL}^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	66	-76	23	117	63	21	152	219
	$(61, 72)$	$(-85, -64)$	$(19, 26)$	$(109, 126)$	$(57, 69)$	$(17, 25)$	$(137, 166)$	$(175, 263)$
Half-yearly	64	-46	27	127	56	32	173	261
	$(58, 70)$	$(-57, -39)$	$(24, 30)$	$(115, 140)$	$(50, 62)$	$(28, 35)$	$(151, 196)$	$(177, 344)$
Quarterly	70	-21	36	133	55	43	195	317
	$(64, 76)$	$(-30, -14)$	$(33, 39)$	$(119, 147)$	$(49, 61)$	$(40, 47)$	$(168, 222)$	$(237, 397)$
Static w/	570	-1086	122	479	567	-58	589	830
10-yr put (PF1)	$(543, 597)$	$(-1136, -1031)$	$(59, 158)$	$(478, 479)$	$(542, 592)$	$(-93, -23)$	$(539, 639)$	$(713, 947)$
Static w/	198	-387	66	137	196	-31	137	138
lookback (PF2)	$(189, 207)$	$(-412, -366)$	$(40, 103)$	$(136, 137)$	$(188, 203)$	$(-43, -19)$	$(137, 137)$	$(137, 138)$

Table 4.35: *Hedging loss statistics derived from Strategy 3 for the GMIB, under different parameter assumptions for the RSLN2 model.*

4.12 Hedging effectiveness when the fee rate is low

The GMIB price is a monotone decreasing function of the fee rate. Furthermore, as illustrated in Figure 2.1, the GMIB price approaches a lower boundary as the fee rate grows, because of the guaranteed return component. The performance of a semi-static (or static) strategy will improve as the fee rate increases, because the GMIB maturity value will be smaller, on average. But the marginal improvement of a semi-static strategy will decrease as the fee rate gets larger, because the guaranteed return component is independent of the fee rate. The results shown thus far in this chapter are based on the fee rate being 4.5%, which corresponds to the fair fee rate under the pricing model of Chapter 2. In this section, we provide an indication of the deterioration in the performance of the semi-static strategies from using a fee rate of 1%; this fee rate corresponds to a representative industry fee rate.

Tables 4.36 and 4.37 display the results from applying Strategies 1, 2 and 3, when the fee rate is 1%. The strategies are based on a MSHE minimization objective, and the hedging target is the GMIB price. Note that the static hedging strategies now correspond to Portfolios C3 and C4B from Chapter 3. It is informative to compare the results in Tables 4.36 and 4.37 with the corresponding results for when the fee rate is 4.5% (Tables 4.14 and 4.15). By reducing the fee rate, all of the hedging loss statistics for the semi-static strategies increase. In particular, the tail risk measures are much higher. When the fee rate is 1%, Strategy 2 appears to offer the lowest tail risk measures. This is in contrast to the results based on a fee rate of 4.5%, where Strategy 3 offered the lowest tail risk measures.

For completeness, Tables 4.38 and 4.39 show the corresponding results when the P-valuation targets are used. The major differences in the hedging loss distributions, compared to the cases where the GMIB price is the hedging target, are that the mean hedging losses are lower, but the tail risk measures are much higher. Lower means and higher tail risk measures were also seen in the results in the case where the fee rate is 4.5%, in Section 4.10.

In this chapter, we have used the fair fee rate $c = 4.5\%$ as the benchmark fee rate. But in Chapter 3, the benchmark fee rate was $c = 1\%$. We briefly explain the reasoning for using the fair fee rate in Chapter 4. When $c = 1\%$, there are not enough funds at time 0 to construct a hedging portfolio payoff that can adequately match the hedging targets at the end of the first hedging horizon. Thus, appropriate option positions are unaffordable, at least in the earlier hedging horizons. The semi-static strategy is at a disadvantage from the start. A favorable hedging result can only occur if the stock price increases at various stages during the accumulation phase, in such a way that at least one rebalancing point, sufficient funds are available to construct a hedging portfolio that has a payoff, at the end of the next hedging horizon, that adequately matches the hedging targets. In other words, under the assumption of $c = 1\%$, it is not possible to see the true potential of a semi-static strategy. In contrast, when $c = 4.5\%$, at time 0 the hedging portfolio has enough funds to reasonably match the range of hedging targets (when the hedging target is set equal to the GMIB price) at the end of the first horizon. This point is best understood by inspecting, in Figures 4.3, 4.4 and 4.5, the panels displaying the hedging targets as functions of the stock price for time 0 to time 1. Loosely speaking, the playing field is roughly even at inception, when the fee rate is set around 4.5%. If the semi-static strategy does not perform well, then it is not because that the strategy has an unreasonably low amount of funds at time 0 to adequately complete the task.

Rebal freq	MSHL$^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Strategy 1								
Negligible transaction costs								
Annual	204	-158	51	421	187	82	576	833
	$(190, 218)$	$(-189, -146)$	$(37, 62)$	$(375, 461)$	$(175, 199)$	$(71, 94)$	$(516, 637)$	$(729, 938)$
Half-yearly	184	-136	59	392	162	86	527	741
	$(171, 196)$	$(-154, -120)$	$(52, 70)$	$(352, 415)$	$(152, 173)$	$(76, 96)$	$(475, 578)$	$(638, 844)$
Quarterly	179	-108	62	366	154	91	519	787
	$(165, 193)$	$(-120, -92)$	$(55, 70)$	$(336, 409)$	$(142, 167)$	$(82, 101)$	$(458, 581)$	$(660, 915)$
Static w/	441	-698	-128	679	439	-43	819	1131
10-yr put (PC3)	$(404, 478)$	$(-750, -624)$	$(-146, -99)$	$(647, 708)$	$(403, 475)$	$(-71, -16)$	$(755, 884)$	$(939, 1322)$
Static w/	574	-1030	162	399	574	-12	400	406
lookback (PC4B)	$(457, 692)$	$(-1174, -892)$	$(129, 190)$	$(399, 399)$	$(458, 691)$	$(-48, 23)$	$(398, 402)$	$(396, 416)$
Strategy 1								
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})\ k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$								
Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	227	-164	64	427	206	95	646	1088
	$(207, 247)$	$(-198, -143)$	$(50, 77)$	$(402, 491)$	$(187, 224)$	$(83, 108)$	$(554, 739)$	$(838, 1339)$
Half-yearly	205	-112	68	403	177	102	595	942
	$(185, 224)$	$(-131, -103)$	$(59, 79)$	$(378, 448)$	$(159, 196)$	$(91, 113)$	$(513, 676)$	$(704, 1180)$
Quarterly	209	-67	81	424	170	121	631	983
	$(189, 228)$	$(-77, -58)$	$(73, 89)$	$(383, 463)$	$(152, 188)$	$(110, 131)$	$(547, 715)$	$(803, 1164)$
Static w/	425	-684	-153	682	419	-72	768	946
10-yr put (PC4)	$(406, 444)$	$(-774, -634)$	$(-178, -130)$	$(658, 704)$	$(400, 439)$	$(-98, -46)$	$(726, 809)$	$(789, 1103)$
Static w/	510	-1083	104	399	509	-37	400	402
lookback (PC4B)	$(464, 555)$	$(-1312, -930)$	$(70, 138)$	$(399, 399)$	$(464, 553)$	$(-68, -5)$	$(399, 400)$	$(398, 406)$
Strategy 2								
Negligible transaction costs								
Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	179	-43	71	364	144	107	544	846
	$(163, 195)$	$(-58, -35)$	$(64, 80)$	$(342, 428)$	$(129, 158)$	$(98, 116)$	$(474, 615)$	$(672, 1019)$
Half-yearly	183	-49	72	382	149	105	562	913
	$(165, 200)$	$(-67, -41)$	$(64, 78)$	$(352, 417)$	$(133, 165)$	$(96, 114)$	$(488, 637)$	$(740, 1087)$
Quarterly	193	-61	61	436	163	104	614	933
	$(175, 212)$	$(-88, -54)$	$(55, 70)$	$(380, 460)$	$(146, 180)$	$(94, 114)$	$(542, 687)$	$(759, 1108)$
Static w/	440	-659	-139	700	438	-45	825	1098
10-yr put (PC4)	$(417, 463)$	$(-736, -594)$	$(-155, -117)$	$(666, 718)$	$(415, 461)$	$(-72, -18)$	$(768, 882)$	$(955, 1241)$
Static w/	540	-963	139	399	540	-24	399	399
lookback (PC4B)	$(478, 603)$	$(-1181, -819)$	$(110, 172)$	$(399, 399)$	$(479, 601)$	$(-57, 10)$	$(399, 399)$	$(398, 400)$
Strategy 2								
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})\ k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$								
Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	185	-32	77	396	147	113	557	860
	$(169, 201)$	$(-46, -26)$	$(74, 82)$	$(359, 437)$	$(131, 162)$	$(104, 122)$	$(489, 624)$	$(684, 1036)$
Half-yearly	196	-33	75	417	159	116	606	984
	$(177, 216)$	$(-45, -25)$	$(68, 82)$	$(367, 457)$	$(140, 177)$	$(106, 125)$	$(525, 687)$	$(769, 1199)$
Quarterly	204	-41	72	427	167	116	645	1060
	$(183, 225)$	$(-56, -33)$	$(65, 80)$	$(383, 495)$	$(148, 187)$	$(106, 127)$	$(556, 734)$	$(829, 1291)$
Static w/	396	-562	-130	664	394	-39	778	1054
10-yr put (PC4)	$(377, 414)$	$(-615, -529)$	$(-152, -106)$	$(649, 690)$	$(375, 413)$	$(-64, -15)$	$(716, 841)$	$(802, 1305)$
Static w/	439	-812	136	399	439	13	399	401
lookback (PC4B)	$(401, 478)$	$(-945, -714)$	$(109, 167)$	$(399, 399)$	$(400, 479)$	$(-14, 40)$	$(399, 400)$	$(398, 405)$

Table 4.36: *Hedging loss distribution statistics derived from Strategies 1 and 2 for the GMIB (when the fee rate is 1%, and hedging target is the GMIB price).*

Strategy 3								
Negligible transaction costs								
Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	189	-39	81	392	147	119	548	851
	$(173, 204)$	$(-59, -25)$	$(74, 91)$	$(367, 424)$	$(132, 161)$	$(110, 128)$	$(482, 613)$	$(681, 1020)$
Half-yearly	186	-33	77	403	145	116	538	784
	$(172, 200)$	$(-49, -24)$	$(71, 85)$	$(362, 435)$	$(133, 158)$	$(107, 125)$	$(483, 593)$	$(642, 925)$
Quarterly	204	-28	71	401	167	118	638	1098
	$(181, 227)$	$(-40, -20)$	$(63, 78)$	$(367, 437)$	$(144, 189)$	$(108, 129)$	$(538, 739)$	$(851, 1345)$
Static w/	435	-657	-156	691	430	-67	809	1087
10-yr put (PC4)	$(413, 456)$	$(-751, -604)$	$(-175, -133)$	$(665, 708)$	$(408, 451)$	$(-93, -40)$	$(750, 867)$	$(869, 1305)$
Static w/	516	-974	106	399	515	-39	399	400
lookback (PC4B)	$(464, 569)$	$(-1146, -844)$	$(76, 140)$	$(399, 399)$	$(465, 566)$	$(-71, -7)$	$(399, 399)$	$(398, 401)$

Strategy 3								
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})$ $k \neq b$, $c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$								
Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	205	-16	91	415	154	135	611	901
	$(188, 221)$	$(-40, -3)$	$(86, 101)$	$(384, 455)$	$(139, 169)$	$(125, 144)$	$(537, 685)$	$(792, 1010)$
Half-yearly	206	-2	95	410	153	138	615	969
	$(187, 225)$	$(-17, 3)$	$(88, 100)$	$(371, 474)$	$(134, 172)$	$(128, 147)$	$(533, 698)$	$(737, 1201)$
Quarterly	220	-1	103	401	166	145	673	1198
	$(197, 244)$	$(-9, 9)$	$(93, 109)$	$(356, 439)$	$(142, 190)$	$(135, 155)$	$(561, 784)$	$(966, 1429)$
Static w/	421	-595	-140	697	419	-45	801	1025
10-yr put (PC3)	$(401, 441)$	$(-727, -555)$	$(-158, -113)$	$(671, 717)$	$(399, 439)$	$(-71, -19)$	$(751, 850)$	$(849, 1201)$
Static w/	492	-898	132	399	492	-7	401	408
lookback (PC4B)	$(441, 543)$	$(-1149, -737)$	$(100, 161)$	$(399, 399)$	$(442, 543)$	$(-38, 23)$	$(398, 403)$	$(396, 421)$

Table 4.37: *Hedging loss distribution statistics derived from Strategy 3 for the GMIB (when the fee rate is 1%, and hedging target is the GMIB price).*

Rebal freq	MSHL$^{1/2}$	5%-quantile	Median	VaR(95%)	Std dev	Mean	CTE(95%)	CTE(99%)
Strategy 1								
Negligible transaction costs								
Annual	703	-1264	-276	812	656	-255	1412	2276
	$(666, 741)$	$(-1326, -1224)$	$(-317, -211)$	$(718, 921)$	$(613, 698)$	$(-296, -215)$	$(1188, 1637)$	$(1825, 2727)$
Half-yearly	728	-1326	-289	756	670	-285	1391	2412
	$(680, 775)$	$(-1432, -1233)$	$(-336, -245)$	$(662, 929)$	$(615, 725)$	$(-327, -244)$	$(1128, 1654)$	$(1613, 3210)$
Quarterly	788	-1369	-339	783	728	-303	1621	3239
	$(729, 847)$	$(-1452, -1299)$	$(-366, -292)$	$(678, 993)$	$(657, 799)$	$(-348, -258)$	$(1267, 1974)$	$(2301, 4177)$
Static w/	421	-595	-140	697	419	-45	801	1025
10-yr put (PC3)	$(401, 441)$	$(-727, -555)$	$(-158, -113)$	$(671, 717)$	$(399, 439)$	$(-71, -19)$	$(751, 850)$	$(849, 1201)$
Static w/	492	-898	132	399	492	-7	401	408
lookback (PC4B)	$(441, 543)$	$(-1149, -737)$	$(100, 161)$	$(399, 399)$	$(442, 543)$	$(-38, 23)$	$(398, 403)$	$(396, 421)$
Strategy 1								
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})\ k \neq b,\ c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$								
Annual	665	-1235	-256	820	613	-258	1201	1815
	$(630, 699)$	$(-1314, -1154)$	$(-292, -219)$	$(674, 927)$	$(577, 648)$	$(-296, -220)$	$(1056, 1347)$	$(1496, 2135)$
Half-yearly	686	-1270	-261	845	632	-267	1244	1904
	$(650, 721)$	$(-1371, -1166)$	$(-311, -219)$	$(680, 956)$	$(594, 669)$	$(-306, -228)$	$(1085, 1404)$	$(1539, 2269)$
Quarterly	706	-1307	-272	871	654	-269	1382	2221
	$(669, 744)$	$(-1401, -1180)$	$(-310, -243)$	$(693, 1049)$	$(612, 695)$	$(-309, -228)$	$(1190, 1574)$	$(1855, 2586)$
Static w/	395	-603	-135	674	393	-41	770	971
10-yr put (PC3)	$(380, 411)$	$(-645, -555)$	$(-150, -113)$	$(658, 697)$	$(377, 410)$	$(-66, -17)$	$(725, 816)$	$(812, 1130)$
Static w/	428	-833	158	399	428	24	399	399
lookback (PC4B)	$(399, 458)$	$(-997, -743)$	$(127, 185)$	$(399, 399)$	$(398, 458)$	$(-2, 51)$	$(399, 399)$	$(399, 399)$
Strategy 2								
Negligible transaction costs								
Annual	640	-1215	-241	755	592	-244	1150	1924
	$(605, 675)$	$(-1293, -1132)$	$(-300, -213)$	$(648, 855)$	$(554, 630)$	$(-280, -207)$	$(978, 1322)$	$(1443, 2404)$
Half-yearly	720	-1303	-302	800	663	-281	1342	2404
	$(652, 789)$	$(-1415, -1208)$	$(-335, -270)$	$(631, 927)$	$(584, 743)$	$(-322, -240)$	$(1053, 1631)$	$(1264, 3544)$
Quarterly	768	-1324	-328	948	711	-291	1582	2841
	$(703, 833)$	$(-1461, -1257)$	$(-360, -284)$	$(813, 1086)$	$(635, 788)$	$(-335, -247)$	$(1277, 1888)$	$(1740, 3943)$
Static w/	416	-639	-147	689	413	-55	775	942
10-yr put (PC3)	$(398, 434)$	$(-687, -592)$	$(-173, -127)$	$(672, 720)$	$(394, 431)$	$(-80, -29)$	$(739, 811)$	$(831, 1053)$
Static w/	486	-933	125	399	486	-15	399	400
lookback (PC4B)	$(435, 538)$	$(-1059, -858)$	$(97, 150)$	$(399, 399)$	$(435, 537)$	$(-45, 15)$	$(399, 399)$	$(399, 401)$
Strategy 2								
Transaction costs: $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})\ k \neq b,\ c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$								
Annual	675	-1164	-221	904	643	-208	1466	2341
	$(638, 713)$	$(-1232, -1109)$	$(-257, -182)$	$(758, 1073)$	$(601, 685)$	$(-248, -168)$	$(1252, 1680)$	$(1990, 2693)$
Half-yearly	696	-1224	-249	853	656	-234	1405	2330
	$(656, 737)$	$(-1306, -1152)$	$(-290, -196)$	$(746, 1011)$	$(610, 702)$	$(-275, -194)$	$(1180, 1629)$	$(1742, 2918)$
Quarterly	770	-1267	-262	970	735	-230	1722	3133
	$(714, 826)$	$(-1361, -1189)$	$(-316, -222)$	$(877, 1209)$	$(671, 799)$	$(-276, -185)$	$(1408, 2037)$	$(2238, 4029)$
Static w/	421	-595	-140	697	419	-45	801	1025
10-yr put (PC3)	$(401, 441)$	$(-727, -555)$	$(-158, -113)$	$(671, 717)$	$(399, 439)$	$(-71, -19)$	$(751, 850)$	$(849, 1201)$
Static w/	492	-898	132	399	492	-7	401	408
lookback (PC4B)	$(441, 543)$	$(-1149, -737)$	$(100, 161)$	$(399, 399)$	$(442, 543)$	$(-38, 23)$	$(398, 403)$	$(396, 421)$

Table 4.38: *Hedging loss distribution statistics derived from Strategies 1 and 2 for the GMIB (when the fee rate is 1%, and P-valuation hedging targets are used).*

Strategy 3 Negligible transaction costs								
Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	662	-1202	-210	826	627	-212	1225	1784
	$(626, 698)$	$(-1313, -1110)$	$(-246, -175)$	$(708, 953)$	$(591, 663)$	$(-251, -173)$	$(1081, 1369)$	$(1618, 1950)$
Half-yearly	725	-1305	-242	944	689	-228	1466	2285
	$(685, 765)$	$(-1432, -1222)$	$(-276, -210)$	$(766, 1119)$	$(646, 731)$	$(-271, -185)$	$(1273, 1660)$	$(2069, 2501)$
Quarterly	773	-1346	-275	905	727	-263	1585	2951
	$(725, 822)$	$(-1540, -1261)$	$(-323, -235)$	$(760, 1072)$	$(673, 782)$	$(-308, -218)$	$(1306, 1865)$	$(2348, 3555)$
Static w/	432	-693	-132	694	429	-50	810	1083
10-yr put (PC3)	$(408, 455)$	$(-744, -617)$	$(-151, -108)$	$(672, 726)$	$(406, 452)$	$(-77, -23)$	$(757, 863)$	$(923, 1243)$
Static w/	528	-1035	141	399	528	-24	400	402
lookback (PC4B)	$(462, 594)$	$(-1134, -888)$	$(108, 164)$	$(399, 399)$	$(463, 593)$	$(-57, 8)$	$(399, 400)$	$(398, 406)$

Strategy 3 **Transaction costs:** $c(k, t_{i-1}) = 0.5\%\phi(k, t_{i-1})\ k \neq b,\ c(b, t_{i-1}) = 0.05\%\phi(b, t_{i-1})$								
Rebal freq	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
Annual	640	-1085	-183	854	616	-175	1226	1826
	$(603, 677)$	$(-1200, -1047)$	$(-217, -142)$	$(727, 955)$	$(579, 653)$	$(-213, -137)$	$(1084, 1367)$	$(1603, 2050)$
Half-yearly	683	-1194	-181	928	660	-178	1357	1919
	$(644, 723)$	$(-1339, -1111)$	$(-226, -143)$	$(852, 1079)$	$(621, 700)$	$(-218, -137)$	$(1206, 1507)$	$(1763, 2076)$
Quarterly	704	-1190	-142	1102	691	-136	1571	2205
	$(662, 745)$	$(-1266, -1113)$	$(-174, -86)$	$(914, 1245)$	$(648, 733)$	$(-179, -93)$	$(1401, 1740)$	$(1913, 2496)$
Static w/	442	-608	-131	692	441	-36	826	1112
10-yr put (PC3)	$(421, 464)$	$(-739, -573)$	$(-154, -114)$	$(669, 721)$	$(419, 463)$	$(-63, -9)$	$(766, 886)$	$(970, 1254)$
Static w/	527	-1009	164	399	527	-8	399	401
lookback (PC4B)	$(471, 582)$	$(-1190, -857)$	$(124, 195)$	$(399, 399)$	$(472, 582)$	$(-41, 25)$	$(399, 400)$	$(398, 403)$

Table 4.39: *Hedging loss distribution statistics derived from Strategy 3 for the GMIB (when the fee rate is 1%, and P-valuation hedging targets are used).*

4.13 Stability of the semi-static hedging method

The purpose of this section is to provide the reader with a sense of the stability of the results of each *scenario* generated by the semi-static hedging method. Recall that each scenario involves nested simulations, and optimizing the hedging portfolio based on the nested simulation output, for each hedging horizon; variability in the results arises from several sources. The optimization hedging portfolios for each hedging horizon will be more stable as N, the number of sub-scenarios, increases. Furthermore, the hedging target estimate for each sub-scenario will have a lower standard error as M increases.

For a specific semi-static strategy, we can test the stability of the results of each scenario by running common Monte Carlo simulations (for fixed N and M). Each common simulation is based on the same strategy type and rebalancing frequency, and common random numbers are used for generating the actual stock and interest rate processes. Specifically, Step (7) of the HLS algorithm is the same for each common simulation, but all of the other steps in the algorithm will have different output. In theory, the results of each common simulation should give identical results as N and M approach infinity. However, as already mentioned, there is a balance between accuracy and computation time in using the semi-static hedging method. Thus, here we show how the results vary for different values of N. Numerical results suggest that, if the control variate discussed in Section 4.3.1 is used, then, when N is sufficiently large, M does not need to be set any higher than 200 to produce relatively stable results. Hence, we do not show how the results vary with M, because the value of N has a much larger influence on the stability of the results.

Table 4.40 displays the results of three common Monte Carlo simulations of the hedging loss distribution for Strategy 2, using annual rebalancing and negligible transaction costs, for the cases where $N = 100$, $N = 200$ and $N = 300$. Comparing the hedging loss statistics of each simulation, for a specific value of N, gives an indication as to how stable the results are. Differences between the hedging losses of the three common simulations arise from different optimal instrument positions in each hedging horizon (because the set of hedging target values vary), which in turn affect the portfolio payoffs and budget constraints of subsequent hedging horizons.

Figure 4.7 provides another perspective on the stability of the results of each scenario. The top, middle and bottom panels, display the hedging losses for 25 scenarios, generated by the three common simulations, for $N = 100$, $N = 200$ and $N = 300$, respectively. It is noted that the scenarios are independent of each other. As expected, the hedging losses for each simulation tend to be closer in value as N increases.

Overall, the results for $N = 200$ and $N = 300$ are fairly close to each other, while the results for $N = 100$ are noticeably more variable. This gives us confidence that $N = 200$ is sufficient for reasonably accurate results. Because the computation time of each scenario using $N = 300$ is significantly more than for $N = 200$, we have opted for using $N = 200$ in the examples presented in this chapter. (Although not shown here, Strategy 3 also shows fairly stable results for $N \geqslant 200$.)

				$N = 100$				
Common simulation number	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
#1	130	-225	-28	160	126	-34	247	439
	(121, 139)	(−257, −211)	(−34, −20)	(143, 180)	(116, 135)	(−42, −26)	(205, 290)	(282, 596)
#2	129	-240	-28	144	124	-35	231	425
	(120, 139)	(−267, −220)	(−35, −21)	(132, 170)	(114, 134)	(−43, −28)	(187, 275)	(264, 585)
#3	127	-225	-28	149	123	-34	241	417
	(118, 136)	(−243, −214)	(−37, −20)	(134, 170)	(113, 132)	(−42, −27)	(200, 283)	(279, 555)
				$N = 200$				
Common simulation number	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
#1	116	-212	-26	139	111	-31	203	288
	(109, 122)	(−247, −192)	(−35, −18)	(124, 159)	(105, 118)	(−38, −25)	(180, 226)	(258, 318)
#2	117	-218	-22	147	113	-32	205	293
	(110, 125)	(−250, −196)	(−30, −15)	(129, 168)	(106, 120)	(−39, −25)	(184, 227)	(253, 334)
#3	117	-219	-25	137	113	-34	211	314
	(110, 124)	(−252, −193)	(−32, −20)	(123, 166)	(106, 120)	(−41, −27)	(184, 238)	(274, 355)
				$N = 300$				
Common simulation number	**MSHL$^{1/2}$**	**5%-quantile**	**Median**	**VaR(95%)**	**Std dev**	**Mean**	**CTE(95%)**	**CTE(99%)**
#1	120	-237	-26	138	114	-36	206	334
	(113, 127)	(−246, −215)	(−33, −19)	(126, 161)	(107, 122)	(−43, −29)	(176, 236)	(237, 431)
#2	122	-235	-25	144	117	-35	229	387
	(114, 129)	(−249, −216)	(−34, −18)	(126, 165)	(109, 124)	(−42, −28)	(193, 264)	(295, 479)
#3	119	-233	-25	142	114	-36	210	343
	(112, 127)	(−252, −215)	(−33, −20)	(123, 159)	(107, 121)	(−43, −29)	(179, 241)	(244, 442)

Table 4.40: *Results of three common simulations of the hedging loss distribution for Strategy 2, using annual rebalancing and negligible transaction costs, for the cases where $N = 100, 200, 300$ ($M = 200$ in all of the simulations). In each simulation, common random numbers are used for the actual values of the stock and interest rate variables.*

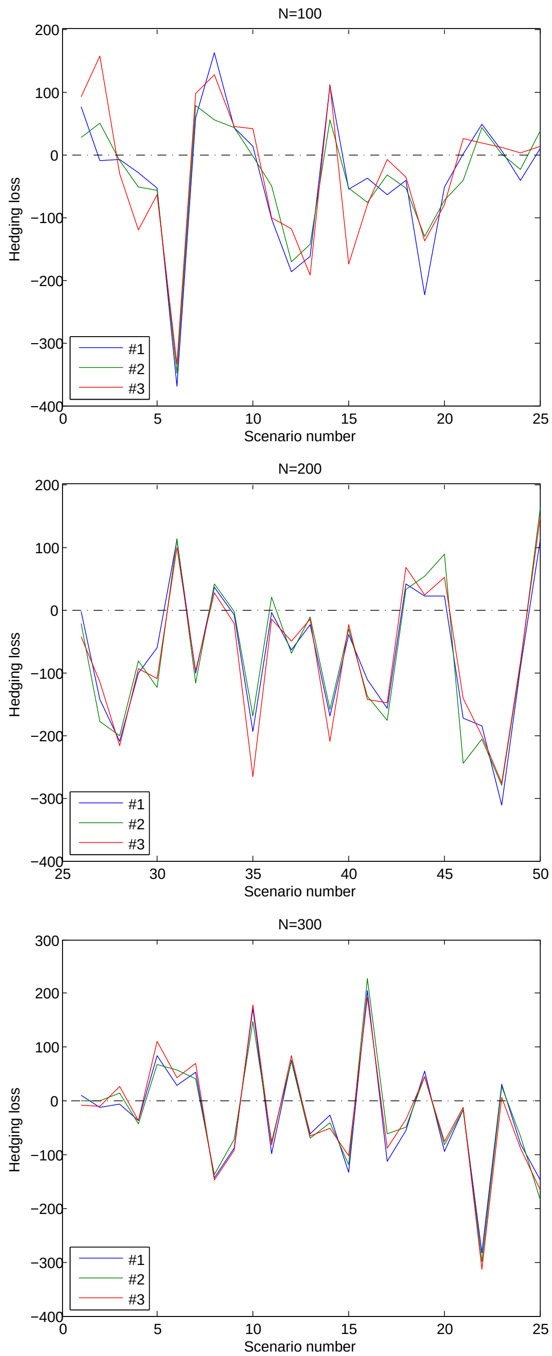

Figure 4.7: *The hedging losses of 25 scenarios for Strategy 2, based on annual rebalancing and negligible transaction costs, generated by each of the three common simulations, for the cases where* $N = 100, 200, 300$. *The scenarios are independent of each other.*

4.14 Summary of the results and concluding remarks

This chapter has measured the performance of several types of semi-static hedging strategies for the GMIB option. Three types of semi-static strategies were considered for the GMIB (Section 4.7). Strategy 1 involves hedging with just the stock (and bonds). Under Strategy 2, the hedging portfolio may include options which expire at the end of each horizon (which we referred to as τ-year options). Under Strategy 3, the hedging portfolio may include T-year options, which expire at the maturity date of the GMIB, time T.

In Section 4.8, we demonstrated that the results of any semi-static strategy are much more favorable in a model where interest rates are constant, compared to a model in which interest rates are stochastic. The results are more favorable in the sense that the hedging portfolios more closely match the GMIB liability over time, and the tail risk measures are smaller. This observation suggests that investigations of the effectiveness of semi-static strategies for long-dated options should employ stochastic interest rate models.

If the hedging target is the GMIB price, Strategies 1 and 2 perform best using a MSHE minimization objective (Section 4.8.2), while Strategy 3 performs best using a CTE minimization objective (Section 4.9). Changing the hedging target from the GMIB price to the P-valuation target markedly changes the hedging loss distribution (Section 4.10). Using a P-valuation target produces a hedging loss distribution with a much higher mean hedging profit (i.e. lower mean loss), and much higher tail risk measures. Overall, setting the hedging target equal to the GMIB price gives a better trade-off between risk and expected loss.

The semi-static strategies appear to be robust against model risk (Section 4.11). We found the semi-static, and static lookback, strategies performed well if the modeler implemented the strategies assuming the actual stock return distribution was normally distributed, when in fact the true stock return distribution satisfied the RSLN2 model. However, the semi-static strategies are far less effective if the fee rate charged for the GMIB option is 1%, rather than the fair fee rate of 4.5% (Section 4.12).

Comparing the performances of the semi-static and static hedging strategies for the GMIB, we find that the static lookback strategy outperforms the semi-static strategies in most situations. The T-year lookback options are able to cut-off the right tail risk. However, as previously mentioned in Chapter 3, in practice, implementing the static lookback strategy will be problematic. The implied volatilities of the lookback options are likely to be much higher than the 20% implied volatility assumption we adopted. Moreover, it may be difficult to find a seller for a lookback option with such a long term to expiry. Therefore, we conclude that the semi-static strategies offer viable, practical alternative hedging strategies for the GMIB.

We argue that Strategy 2, using annual rebalancing, with the MSHE minimization objective and the hedging target set equal to the GMIB price, is the most effective, and practical, semi-static hedging strategy for the GMIB option. However, Strategy 2 is by no means perfect. Although Strategy 2 is feasible in practice, it may not work well in periods where financial markets are highly turbulent. Option implied volatilities rise sharply when market conditions are unfavorable, particularly for out-of-the-money put options. Higher option costs will reduce the effectiveness of Strategy 2 at mitigating the tail risk of the hedging loss distribution.

We now end with some more general comments. Semi-static hedging strategies mitigate many of the problems associated with dynamic hedging strategies (such as delta hedging). Dynamic hedging strategies usually assume continuous time trading is possible. In practice, rebalancing of the hedging portfolio can only be achieved at discrete time points, and in volatile markets being unable to trade continuously increases the risk of hedge slippage. Furthermore, transaction costs incurred by frequent rebalancing increases hedge slippage. Large jumps in the prices of the underlying securities may also increase the likelihood of large hedging errors for dynamic hedging strategies. A semi-static hedging strategy can be implemented with a fixed, small number of rebalancing time points. Transaction costs can be limited. Temporary asset price fluctuations between rebalancing time points do not impact on the hedging portfolio. The risks surrounding extreme liquidity events are also lessened.

Note that none of the strategies we have considered include interest sensitive instruments. This is an issue that may be worth exploring further. We have some preliminary findings. Although not shown here, we have investigated including an instrument corresponding to a bank account which earns a floating rate of interest. The payoff of the bank account at the end of each horizon varies mildly with changes in the short rate over the horizon. Our investigations suggest that none of the strategies considered here noticeably benefit from the inclusion of a bank account instrument. Similarly, we have tested the inclusion of a long term zero coupon bond which matures at time T, or slightly afterwards. Again no material differences in the hedging loss distributions were observed for strategies with and without the long term zero coupon bond. As noted in Chapter 3, it seems that hedging the equity risk of the GMIB is paramount.

The semi-static hedging method, proposed in this chapter, is a versatile hedging method for long-dated options. It can be used to hedge all kinds of long-dated options. However, one considerable drawback of this method is the considerable computation times involved in generating a sufficient number of scenarios, for making reliable inferences. But, given the rapid pace at which computing power is increasing, the computation time barrier will become less of an issue over time.

Chapter 5

Conclusions and Potential Research Directions

5.1 Summary of key findings

In Chapter 2, a pricing equation for the GMIB option was derived, which allowed us to determine the fair fee rate that should be charged for the GMIB, based on plausible parameter assumptions. At first glance, it seems that the GMIB option may be underpriced by insurers. That is, the fee rates being charged by insurers may be too low. However, when we account for policy lapses and/or make allowances for underlying variable annuity fees, the fair fee rate reduces to levels that are more comparable with the fee rates currently being charged by insurers. A decomposition of the GMIB price was presented, which enabled us to determine the drivers of its value. We found that the guaranteed return component provides the largest contribution to the GMIB price, and that the lookback component does not contribute much to the value of the GMIB option when the guaranteed return component is already included.

In Chapter 3, we comprehensively assessed the performance of static hedging strategies designed to minimize the CTE of the hedging loss distribution at maturity. For standard static hedging portfolios including long-dated put options, the CTE value is driven by large hedging losses related to the lookback component. Even though the lookback component does not necessarily contribute much to the GMIB price, it is the dominant factor driving the tail risk of static hedging strategies, if left unhedged. We showed that the addition of a lookback call option to the hedging portfolio was crucial for hedging the lookback component, and hence minimizing the CTE of the hedging loss distribution. The CTE minimization strategies were compared to static hedging strategies designed to minimize the mean squared hedging loss at maturity. We found that static hedging strategies optimized to minimize the CTE generated hedging loss distributions that were much more desirable, particularly because they addressed the tail risk. It was shown that hedging the equity risk of the GMIB option was much more important than hedging the interest rate risk.

In many of the examples we illustrated, we set the fee rate equal to 1%, in line with fee rates currently being charged in practice for the GMIB option. This fee rate is far below the fair fee rate, with respect to the model of Chapter 2. The performance of each static hedging strategy was imperfect at best, when a fee rate of 1% was used. We then demonstrated that when the fair fee rate is charged, the results significantly improved. However, significant tail risk still exists even when the fair fee rate is charged, unless the lookback call option is included in the hedging portfolio.

In Chapter 4, we investigated whether semi-static hedging strategies could outperform static hedging strategies, in terms of reducing risk, as measured by the VaR and CTE. Key observations were:

(1) Semi-static hedging strategies which use τ-year options (options which have a term to expiry of one hedging horizon) or T-year options show considerably less tail risk than semi-static hedging strategies using just the stock.

(2) The semi-static strategies appear to be robust against model risk. We found the semi-static and static lookback strategies performed well if the modeler implemented the strategies assuming the actual stock return distribution was normally distributed, when in fact the true stock return distribution satisfied the 2-state lognormal regime switching model.

(3) For the semi-static strategies to be effective, the fair fee rate needed to be charged.

If a low fee rate of 1% was charged, the semi-static strategies did not perform too well.

Overall, we found that a static hedging strategy using lookback options outperforms the semi-static strategies, in most situations. However, in practice, T-year lookback option prices are likely to be much higher than the prices we obtained in using our model. Thus, the static strategy using lookback options is unlikely to be as effective in practice, as the insurer will not have enough funds to buy the optimal lookback positions (unless they obtain funds from elsewhere). We concluded that the semi-static strategies using τ-year options offer viable, practical hedging strategies for the GMIB. Section 4.14 comments on the advantages of using semi-static hedging methods, as opposed to dynamic hedging methods (e.g. delta hedging), for long-dated options.

This thesis has investigated the financial risk management of the GMIB option as a case study. However, the methods presented in this thesis can be applied to other complex long-dated options. Our results should also be instructive in determining what to investigate when pricing and hedging other long-dated options. Specifically, the following issues should be considered:

- For an embedded option that can be valued at a certain point in time (European options in particular), which is offered in exchange for periodic fees (specified by some fee rate), we can develop a pricing equation for the option, that is a function of the fee rate. Using the pricing equation, fair fee rates can be determined. Often, alternative views of the pricing equation can be derived, as demonstrated in Sections 2.2.2 and 2.2.4. Analyzing different forms of the pricing equation may provide additional insights into what drives the option value.
- The prices of options with complex payoffs can be decomposed in a similar manner to the approach outlined in Section 2.5.
- It is important to explore the impact of policyholder lapses, as they may significantly affect the price of the option (consider Section 2.6). It is noted that lapses are not usually investigated in the (academic) literature on pricing embedded options in long term insurance products.

- If static hedging strategies are investigated, decompositions of the hedging loss statistics helps identify what hedging instruments are useful. Appropriately constructed figures, such as those shown in Chapter 3, may also help in identifying and understanding the risks. The figures can be instructive in determining whether the right kind of hedging instruments are included in the hedging portfolio, in order to adequately mitigate the tail risks. In some ways, the figures can be just as useful as the numbers. Figures are certainly more helpful and effective than numbers when attempting to explain to an audience the risks involved with hedging an option.

- The semi-static hedging method, proposed in Chapter 4, is a versatile hedging method for long-dated options. It can be used to hedge all kinds of long-dated options. The results obtained from using this method appear to be promising. The choice of hedging target, and the choice of the objective function that is to be optimized in each hedging horizon, are important issues to consider when using this method. Unfortunately, a significant drawback of this method is the considerable computation times involved in generating a sufficient number of scenarios, for making reliable inferences.

5.2 Comments on the GMIB option design

The exercise time of the GMIB option is random, after the 10 year waiting period expires. Valuing the GMIB as an American or Bermudan option is a challenging task. Monte Carlo simulation methods for pricing American options could be used to value the GMIB (Longstaff and Schwartz, 2001). But the problem is that the exercise date for each policyholder will depend on the personal circumstances of the policyholder (e.g. ill-health in old age, death of spouse, family issues), which may or may not be influenced by the prevailing economic conditions. The policyholder is less likely to be concerned about when it is optimal to exercise the GMIB, from the point of view of maximizing the financial value of the option, after locking up their funds for at least 10 years, particularly if the optimal strategy involves waiting another couple of years.

If insurers simplified the GMIB by say, restricting the exercise times to every 5-th policy anniversary after the waiting period expires (with penalties for exercising at other times), then the valuation and risk management of the GMIB becomes a much more manageable task. The unattractiveness of these restrictions could be mitigated by charging lower fee rates. An alternative approach is to give the policyholder the choice, at inception, of when they would like to set the exercise date of the GMIB. They might for example pick the date at which they plan to retire.

As noted in Section 2.3.3, the GMIB would be less risky if the benefits provided by the option did not increase sharply as a function of the fee rate charged for the option. From the point of view of controlling risk, an option should be designed such that the benefits provided by the option do not increase as the fee rate (which is supposed to cover the benefits provided by the option) increases. The benefits provided by the option should be insensitive to the fee rate, or a monotone decreasing function of the fee rate. One way to circumvent this problem is to charge an upfront fee at time 0 for the option, instead of annual fees, as discussed in Section 2.5.3. But obviously an upfront fee may be unattractive to the policyholder, if it is a significant proportion of their initial investment.

5.3 Future research directions

Future research directions that may be fruitful include:

- **Testing the static and semi-static hedging strategies using more complex equity price and interest rate models.** In particular, it is worthwhile testing semi-static hedging strategies using a stochastic volatility model for equity prices. Given that option prices in such a model also depend on volatility levels, an additional factor, volatility, must be allowed for in the construction of hedging portfolios. Additional instruments, which are sensitive to volatility, may need to be added to the hedging portfolios. Examples include equity options that have expiry dates which are further than the next portfolio rebalancing time point; at the next rebalancing time point, the option prices will be a function of stock and volatility levels. Liu (2010) has investigated this concept over a single-period hedging horizon.
- **Developing a method for pricing the GMIB in a generalized setting where the policyholder can choose to annuitize (with the option) on any future**

policy anniversary date, once the waiting period has expired. It would be interesting to see the optimal exercise strategy, which will depend on the realized path of the investment account, and the term structure of interest rates that exists on each policy anniversary. The findings may help existing GMIB policyholders in deciding whether they should annuitize with the GMIB option on their policy anniversary in the current calendar year, or hold off and wait say another 5 years to maximize their potential stream of annuity payments. Note that the optimal exercise strategy will be based purely on the financial value of the option, and will not take into account individual policyholder behavior, which is difficult, or impossible, to capture in a model.

- **Testing the effectiveness of stratified sampling techniques in the semi-static hedging method.** In Chapter 4, we set the hedging target values in each hedging horizon based on the simulated values of the stock and interest rate processes over the hedging horizon. It was important to have simulated a few outlying hedging target values (as a function of the stock price; see Figures 4.3, 4.4 and 4.5), in order to reduce the (potential) hedging errors in the tails of the hedging target distribution. The optimal hedging portfolios might be improved by using a stratified sampling technique (in each horizon) which simulates an appropriate number of hedging target values in the tails of the hedging target distribution.

- **Applying the semi-static hedging method to other types of long-dated options.** Given that the GLWB is becoming increasingly popular, there is likely to be considerable interest in illustrations of the performance of semi-static hedging strategies for the GLWB.

- **Devising semi-static hedging methods that explicitly incorporate dynamic decision rules for rebalancing the hedging portfolio.** For example, within a given hedging horizon, if the stock increases or decreases by more than $x\%$, then the portfolio is immediately reassessed and rebalanced if necessary.

- **Studying the longevity risk associated with the GMIB.** Like the guaranteed annuity option, longevity risk is a key driver of the value of the GMIB. Longevity risk is also significant for the GLWB. Investigating hedging strategies for the GMIB/GLWB, that also somehow hedge the longevity risk, perhaps only partially at best, is an area certainly worth exploring. A stochastic mortality model will be needed (for a review of such models, see Cairns et al. (2008)). Given that the Lee–Carter model (Lee and Carter, 1992) is considered a benchmark model for mortality projection, the use of a Bayesian form of the Lee–Carter model (Czado et al. (2005), Pedroza (2006)), which adequately models all sources of uncertainty, may be one starting point for incorporating longevity risk.

- **Measuring how lapse assumptions impact on hedging strategies for the GMIB (and other variable annuity options).** In the literature, there has not been much investigation of lapse assumptions in the pricing or hedging of variable annuity options. However, as we have demonstrated in Chapter 2, accounting for lapses can materially change the results.

- **Pricing the step-up options associated with some GMIB options.** These step-up options were briefly discussed in Section 1.3.1.

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