Even Cycle and Even Cut Matroids

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis we consider two classes of binary matroids, even cycle matroids and even cut matroids. They are a generalization of graphic and cographic matroids respectively. We focus on two main problems for these classes of matroids. We first consider the Isomorphism Problem, that is the relation between two representations of the same matroid. A representation of an even cycle matroid is a pair formed by a graph together with a special set of edges of the graph. Such a pair is called a signed graph. A representation for an even cut matroid is a pair formed by a graph together with a special set of vertices of the graph. Such a pair is called a graft. We show that two signed graphs representing the same even cycle matroid relate to two grafts representing the same even cut matroid. We then present two classes of signed graphs and we solve the Isomorphism Problem for these two classes. We conjecture that any two representations of the same even cycle matroid are either in one of these two classes, or are related by a local modification of a known operation, or form a sporadic example. The second problem we consider is finding the excluded minors for these classes of matroids. A difficulty when looking for excluded minors for these classes arises from the fact that in general the matroids may have an arbitrarily large number of representations. We define degenerate even cycle and even cut matroids. We show that a 3-connected even cycle matroid containing a 3-connected non-degenerate minor has, up to a simple equivalence relation, at most twice as many representations as the minor. We strengthen this result for a particular class of non-degenerate even cycle matroids. We also prove analogous results for even cut matroids.

Acknowledgements

I am sincerely thankful to my supervisor, Dr Bertrand Guenin, for his tremendous contributions both to this thesis and to my training as a mathematician. It is a pleasure to work with someone who conducts research with such enthusiasm and perseverance.

I would also like to thank the faculty members and graduate students of the Department of Combinatorics and Optimization for their role in providing a friendly and stimulating environment in which to conduct research.

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Notation

$[n]$	Set $\{1,\dots,n\}$
$\bar{A}$	Complement of set A
$A-B$	Set of elements of A not in B
$A \triangle B$	Symmetric difference of sets A and B
$\text{cycle}(G)$	Set of cycles or cycle matroid of G, page 1
$\text{cut}(G)$	Set of cuts or cut matroid of G, page 3
$\text{ecycle}(G,\Sigma)$	Set of even cycles or even cycle matroid of (G,Σ), page 4
$\text{ecut}(G,T)$	Set of even cuts or even cut matroid of (G,T), page 5
$\text{ecycle}(M,\Sigma)$	Even cycles of the signed matroid (M,Σ), page 24
$\delta_G(U)$	Cut induced by U, page 3
$V_{odd}(H)$	Set of vertices of odd degree in H, page 11
$\text{loop}(G)$	Set of loops of G, page 6
$\text{pin}(G,T)$	Set of pins of (G,T), page 23
$\mathscr{B}_G(X)$	Boundary of X in G, page 2
$\mathscr{I}_G(X)$	Interior of X in G, page 23
$\text{W}_{\text{flip}}[G,\mathbb{S}]$	Graph obtained from G by performing Whitney-flips on $\mathbb{S}$, page 50

$\mathbb{S} \odot \mathbb{S}'$	Concatenation of sequences $\mathbb{S}$ and $\mathbb{S}'$, page 63
$\mathrm{Cat}(G, \mathbb{S})$	Caterpillar of G with respect to $\mathbb{S}$, page 70
$\lambda_M(X)$	Connectivity function of M, page 22
M^*	Dual of the matroid M
$M \setminus e$	Element deletion, page 21
M/e	Element contraction, page 21
$M\|_X$	Restriction of M to X, page 24
$M_1 \oplus_1 M_2$	1-sum of matroids M_1 and M_2, page 91
$M_1 \oplus_2 M_2$	2-sum of matroids M_1 and M_2, page 93

Chapter 1

Introduction

1.1 The graphic and cographic cases: two problems

Let G be a graph. For a set $X \subseteq E(G)$, we write $V_G(X)$ to refer to the set of vertices incident with an edge of X and $G[X]$ for the subgraph with vertex set $V_G(X)$ and edge set X. A subset C of edges of G is a *cycle* if $G[C]$ is a graph where every vertex has even degree. An inclusion-wise minimal non-empty cycle is a *circuit*. We denote by $\text{cycle}(G)$ the set of all cycles of G. A *cycle* for a binary matroid M is the symmetric difference of circuits of M. Since the cycles of G correspond to the cycles of the *cycle matroid* of G, we identify $\text{cycle}(G)$ with that matroid and say that G is a *representation* of that matroid. The classes of matroids considered in this work all arise from graphs. Hence, when referring to a representation of a matroid we will always mean a graphic representation of the matroid. When referring to a matrix representing a matroid over some field (which will usually be the binary field), we will refer to that matrix as the *matrix representation* of the matroid.

Cycle matroids are also referred to as graphic matroids. An example of a cycle matroid is given in Figure 1.1. On the left we have the matrix representation of the matroid over the binary field. Columns 1 to 10 represent elements 1 to 10 (in this order). Elements $1,2,3,4,5,6$ form a basis of the matroid; the element 7 forms a fundamental circuit with 1 and 3. On the right we have a graph representation of the matroid. The matrix representation is the incidence matrix of the graph. Note that the basis $\{1,2,3,4,5,6\}$ corresponds to a spanning tree in the graph. Edges $1,3,7$ form a circuit and edges $2,3,4,5,6,8,10$ form a cycle of the graph (hence of the cycle matroid).

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}
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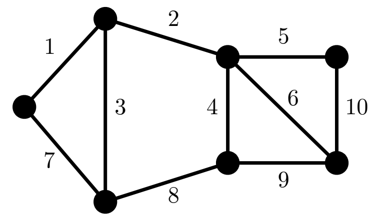

Figure 1.1: Example of a cycle matroid.

We may ask when two graphs represent the same cycle matroid. We define an operation on graphs which preserves cycles as follows. Given sets A, B we denote by $A - B$ the set $\{a \in A : a \notin B\}$. Given a set of edges X of G, we define the *boundary* of X in G as $\mathscr{B}_G(X) = V_G(X) \cap V_G(\bar{X})$, where $\bar{X} = E(G) - X$. Consider a graph G and let $X \subseteq E(G)$. Suppose that $\mathscr{B}_G(X) = \{u_1, u_2\}$ for some $u_1, u_2 \in V(G)$. Let G' be obtained by identifying vertices u_1, u_2 of $G[X]$ with vertices u_2, u_1 of $G[\bar{X}]$ respectively. Then G' is obtained from G by a *Whitney-flip* on X. We will also use the term Whitney-flip for the operation consisting of identifying two vertices from distinct components, or the operation consisting of partitioning the graph into components each of which is a block of G. An example of two graphs related by Whitney-flips is given in Figure 1.2. In this example the set X is given by edges $5, 6, 9, 10$.

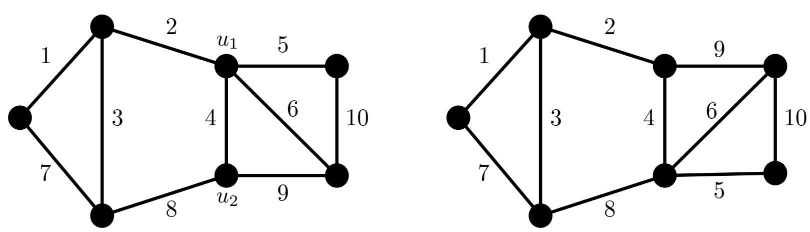

Figure 1.2: Example of a Whitney-flip.

It is easy to see that Whitney-flips preserve cycles. Hence, two graphs related by a sequence of Whitney-flips have the same cycles; in particular they are representations of the same cycle matroid. In [38] Whitney proved that the converse also holds.

Theorem 1.1 (Whitney '33)**.** *Two graphs represent the same cycle matroid if and only if they are related by Whitney-flips.*

In light of Theorem 1.1, we define two graphs to be *equivalent* if one can be obtained from the other by a sequence of Whitney-flips.

Now we introduce another basic class of binary matroids. Given a set of vertices U, we denote by $\delta_G(U)$ the *cut* induced by U, that is $\delta_G(U) := \{(u,v) \in E(G) : u \in U, v \notin U\}$. An inclusion-wise minimal cut is a *bond*. We denote by $\text{cut}(G)$ the set of all cuts of G. Since the cuts of G correspond to the cycles of the *cut matroid* of G, we identify $\text{cut}(G)$ with that matroid and say that G is a *representation* of that matroid. For example, in the graph in Figure 1.1 edges $2,8$ form a bond of the graph, hence a circuit of the cut matroid. Edges $1,3,5,8,10$ form a cut of the graph, hence a cycle of the cut matroid. Cut matroids are also referred to as cographic matroids, as they are duals of graphic matroids. In fact, for every graph G every cycle has an even intersection with every cut, hence the matroid $\text{cycle}(G)$ is the dual of $\text{cut}(G)$. Therefore Theorem 1.1 may be restated as follows.

Theorem 1.2 (Whitney '33)**.** *Two graphs represent the same cut matroid if and only if they are related by Whitney-flips.*

An excluded minor for a minor closed class of matroids is a matroid M which is not in the class, but such that every proper minor of M is in the class. The class of cycle matroids is a minor closed class; in [34] Tutte found the excluded minors for this class. The matroids in the following theorem are defined in Appendix B.

Theorem 1.3 (Tutte '59)**.** *Let M be a binary matroid. Then M is a cycle matroid if and only if M has no F_7, F_7^*, $M(K_5)^*$ or $M(K_{3,3})^*$ minor.*

Theorem 1.1 and Theorem 1.3 provide solutions to two problems for the classes of cycle and cut matroids. The first one is the problem of determining when two graphs represent the same matroid. We refer to this problem as the *Isomorphism Problem*. The *Excluded Minor Problem* is the problem of finding all the excluded minors for a minor closed class of matroids. Hence Theorem 1.3 provides an answer to the Excluded Minor Problem for cycle matroids and, by duality, cut matroids. Note that in general a class of matroids may have an infinite set of excluded minors. This happens, for example, for real representable matroids (see [18]). However, this is not the case for binary matroids, as recently proved by Geelen, Gerards and Whittle [11].

1.2 Thesis overview

The first class of matroids that we consider in this work is a generalization of the class of cycle matroids which arises from signed graphs. A *signed graph* is a pair (G, Σ) where G is a graph and $\Sigma \subseteq E(G)$. We call Σ a *signature* of G. A subset $B \subseteq E(G)$ is Σ-*even* (respectively Σ-*odd*) if $|B \cap \Sigma|$ is even (respectively odd). When there is no ambiguity we omit the prefix Σ when referring to Σ-even and Σ-odd sets. Given a signed graph (G, Σ), we denote by $\text{ecycle}(G, \Sigma)$ the set of all even cycles of (G, Σ). It can be verified that $\text{ecycle}(G, \Sigma)$ is the set of cycles of a binary matroid, which we call the *even cycle matroid*. We identify $\text{ecycle}(G, \Sigma)$ with that matroid and say that (G, Σ) is a *representation* of that matroid. Note that, if Σ is empty, all the cycles of (G, Σ) are even, hence $\text{ecycle}(G, \Sigma)$ is a cycle matroid. Hence the class of even cycle matroids contains the class of cycle matroids.

An example of an even cycle matroid is given in Figure 1.3. On the left we can see the matrix representation of the matroid. Columns 1 to 10 represent elements 1 to 10 (in this order). Elements $1, 2, 3, 4, 5, 6, 7$ form a basis of the matroid. The fundamental circuits for elements $8, 9, 10$ are $\{1, 2, 4, 7, 8\}$, $\{1, 3, 4, 6, 7, 9\}$ and $\{5, 6, 10\}$ respectively. On the right we have a signed graph representation of the matroid. The bold edges form the signature (we use this convention throughout this work). Edges $\{2, 3, 4, 8\}$ form an odd circuit of the signed graph, hence not a circuit of the matroid. Sets $\{5, 6, 10\}$ and $\{1, 3, 4, 6, 7, 9\}$ are both even cycles of the signed graph, hence cycles of the even cycle matroid. Note that the basis $\{1, 2, 3, 4, 5, 6, 7\}$ corresponds to a spanning tree in the graph plus an edge forming an odd circuit with the tree.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

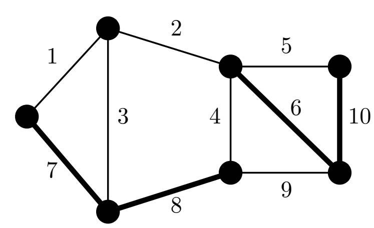

Figure 1.3: Even cycle matroid. Bold edges are odd.

The second class of matroids we consider is a generalization of the class of cut matroids. A *graft* is a pair (G, T) where G is a graph, $T \subseteq V(G)$ and $|T|$ is even. The vertices in T

are the *terminals* of the graft. A cut $\delta(U)$ is *T-even* (respectively *T-odd*) if $|T \cap U|$ is even (respectively odd). When there is no ambiguity we omit the prefix T when referring to T-even and T-odd cuts. We denote by $\text{ecut}(G,T)$ the set of all even cuts of (G,T). It can be verified that $\text{ecut}(G,T)$ is the set of cycles of a binary matroid, which we call the *even cut matroid* represented by (G,T). We identify $\text{ecut}(G,T)$ with that matroid and say that (G,T) is a *representation* of that matroid. Note that, if T is empty, all the cuts of (G,T) are even, hence $\text{ecut}(G,T)$ is a cut matroid.

An example of the matrix representation and the graft representation of an even cut matroid is given in Figure 1.4, where the white vertices of the graph are the terminals (we use this convention throughout this work). The set of edges $\{1,2,6\}$ forms an odd cut of the graft, hence not a cycle of the even cut matroid. On the other hand, the sets $\{2,3,8\}$ and $\{1,2,3,4,6,10\}$ form even cuts, hence cycles of the matroid. Some basic properties of even cycle and even cut matroids are discussed in Chapter 2.

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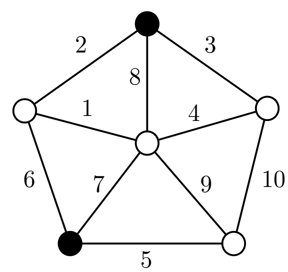


Figure 1.4: Even cut matroid. White vertices are terminals.

1.2.1 Problem 1: isomorphism

In the first part of this dissertation we focus on the following problem.

Isomorphism Problem for even cycles: What is the relation between two representations of the same even cycle matroid?

The Isomorphism Problem has been solved for even cycle matroids which are graphic, by Shih (in his doctoral disseration, see [30]) and independently by Gerards, Lovász, Schrijver, Seymour, Truemper (see [13]). We report the second result here, while Shih's result, which describes the structure of the graphs more precisely, is presented in Chapter 4.

Theorem 1.4. *Let* (G,Σ) *and* (G',Σ') *be signed graphs. Suppose that* $\mathrm{ecycle}(G,\Sigma) = \mathrm{ecycle}(G',\Sigma')$ *and that this matroid is a cycle matroid. Then* (G,Σ) *and* (G',Σ') *are related by a sequence of Whitney-flips, signature exchanges, and Lovász-flips.*

We need to define the terms "signature exchange" and "Lovász-flip". Given signed graphs (G,Σ) and (G',Σ'), where G and G' are equivalent, we say that Σ' is obtained from Σ by a *signature exchange* if $\Sigma \triangle \Sigma'$ is a cut of G (where $\triangle$ denotes symmetric difference). Every set Σ' which may be obtained from Σ by a signature exchange is a *signature* of (G,Σ).

Given a graph G we denote by $\mathrm{loop}(G)$ the set of all loops of G. Let (G,Σ) be a signed graph. A vertex s is a *blocking vertex* of (G,Σ) if every odd circuit of $(G,\Sigma) \setminus \mathrm{loop}(G)$ uses s. A pair of vertices s,t is a *blocking pair* if every odd circuit of $(G,\Sigma) \setminus \mathrm{loop}(G)$ uses at least one of s,t. Note that s is a blocking vertex (respectively s,t is a blocking pair) of (G,Σ) if and only if there exists a signature Σ' of (G,Σ) such that $\Sigma' \subseteq \delta(s) \cup \mathrm{loop}(G)$ (respectively $\Sigma' \subseteq \delta(s) \cup \delta(t) \cup \mathrm{loop}(G)$).

Consider a signed graph (G,Σ) and vertices $v_1, v_2 \in V(G)$ where $\Sigma \subseteq \delta_G(v_1) \cup \delta_G(v_2) \cup \mathrm{loop}(G)$. So v_1, v_2 is a blocking pair of (G,Σ). We can construct a signed graph (G',Σ) from (G,Σ) by replacing the endpoints x,y of every odd edge e with new endpoints x',y' as follows:

(a) if $x = v_1$ and $y = v_2$, then $x' = y'$ (i.e. e becomes a loop);

(b) if $x = y$ (i.e. e is a loop), then $x' = v_1$ and $y' = v_2$;

(c) if $x = v_1$ and $y \neq v_1, v_2$, then $x' = v_2$ and $y' = y$;

(d) if $x = v_2$ and $y \neq v_1, v_2$, then $x' = v_1$ and $y' = y$.

Then we say that (G',Σ) is obtained from (G,Σ) by a *Lovász-flip* on v_1, v_2. In Section 3.4 we show that Lovász-flips preserve even cycles. An example of two signed graphs related by a Lovász-flip is given in Figure 1.5, where the white vertices represent the blocking pairs.

Suppose that (G_1,Σ_1) and (G_2,Σ_2) are signed graphs where G_1 and G_2 are equivalent and Σ_2 is obtained from Σ_1 by a signature exchange. Then we say that (G_1,Σ_1) and (G_2,Σ_2) are *equivalent* signed graphs. Let $D := \Sigma_1 \triangle \Sigma_2$. As D is a cut of G_1 (and G_2), for every cycle C of G_1, $|D \cap C|$ is even. Hence $|C \cap \Sigma_1|$ is even if and only if $|C \cap \Sigma_2|$ is even. It

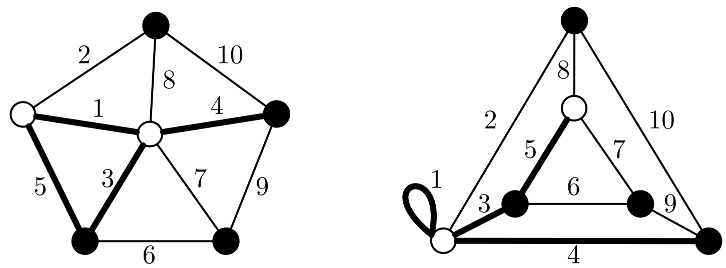


Figure 1.5: Signed graphs related by a Lovász-flip. Bold edges are odd.

follows that equivalent signed graphs represent the same even cycle matroid. Now suppose that for two signed graphs (G_1,Σ_1), (G_2,Σ_2) we have $\text{ecycle}(G_1,\Sigma_1) = \text{ecycle}(G_2,\Sigma_2)$ and $\text{cycle}(G_1) = \text{cycle}(G_2)$. Then, by Theorem 1.1, G_1 and G_2 are equivalent. A cycle of G_1 is Σ_1-even if and only if it is Σ_2-even and is Σ_1-odd if and only if it is Σ_2-odd. Hence, Σ_2 is a signature of G_1 and Σ_1 is a signature of G_2. It follows that (G_1,Σ_1) and (G_2,Σ_2) are equivalent. We conclude that, if G_1 and G_2 are equivalent graphs and $\text{ecycle}(G_1,\Sigma_1) = \text{ecycle}(G_2,\Sigma_2)$ for some signatures Σ_1 and Σ_2, then (G_1,Σ_1) and (G_2,Σ_2) are equivalent. Thus the Isomorphism Problem is easily solved for signed graphs having equivalent underlying graphs. Therefore we focus on the Isomorphism Problem for the case that the two graphs are inequivalent. We say that two graphs G_1 and G_2 are *siblings* if G_1 and G_2 are inequivalent and, for some signatures Σ_1 and Σ_2, we have $\text{ecycle}(G_1,\Sigma_1) = \text{ecycle}(G_2,\Sigma_2)$. We extend this terminology to the signed graphs and say that (G_1,Σ_1) and (G_2,Σ_2) are *siblings*. We call the pair Σ_1,Σ_2 the *matching signature pair* for G_1, G_2. In Chapter 3 we prove that, given siblings G_1,G_2, their matching signature pair is unique, up to signature exchange.

The other Isomorphism Problem we consider is the following.

Isomorphism Problem for even cuts: What is the relation between two representations of the same even cut matroid?

In Chapter 3 we show how the Isomorphism Problem for even cycles relates to the Isomorphism Problem for even cuts. In particular we show that, if two graphs G_1,G_2 are siblings, then there exist sets of terminals T_1 and T_2 such that $\text{ecut}(G_1,T_1) = \text{ecut}(G_2,T_2)$. In this case we also say that (G_1,T_1) and (G_2,T_2) are *siblings* and we call the pair T_1,T_2 the *matching terminal pair* for G_1, G_2. We show that the converse is also true, that is, if two grafts (G_1,T_1) and (G_2,T_2) are siblings, then there exist signatures Σ_1 and Σ_2 such that

(G_1,Σ_1) and (G_2,Σ_2) are siblings. We also show that the matching signature pair can be obtained from the matching terminal pair and vice-versa. An example of two siblings is given in Figure 1.6. The two graphs are not equivalent as, for example, the edge 1 is a loop in the graph on the left and not a loop in the graph on the right. Given the signatures (edges in bold), we have two signed graphs with the same even cycles. The terminals (white vertices) determine two grafts with the same even cuts.

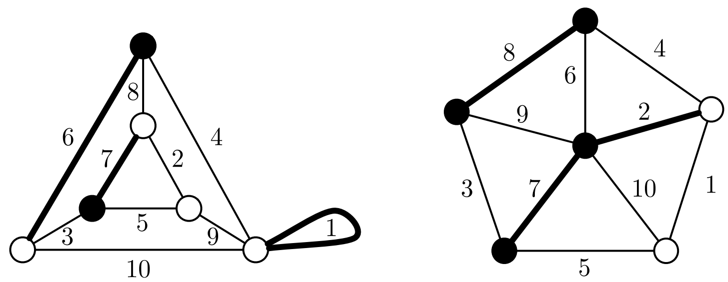


Figure 1.6: Siblings. Bold edges are odd, white vertices are terminals.

We focus on the Isomorphism Problem for even cycles: in Chapter 4 we present two classes of siblings and we characterize all the operations relating two siblings in the same class, thus solving the Isomorphism Problem for these classes. We conjecture that, up to Whitney-flips, signature exchanges, Lovász-flips and some reductions, every pair of siblings is either contained in one of these two classes, or is a modification of an operation for graphic matroids, or forms a sporadic example. We discuss this conjecture in more details in Section 4.4.

An example of two siblings in the first class is given in Figure 1.7, where dotted lines represent vertices that are identified (we use this convention throughout this work). A signature for both graphs is $\alpha_1 \triangle \alpha_2$ (corresponding to the edges in darker grey in the figure).

An example of two siblings in the second class is given in Figure 1.8. Note that, even though the underlying graphs are isomorphic, there is no isomorphism between the two graphs which preserves the edge labels. These two signed graphs are related by a shuffle, an operation defined in Chapter 4.

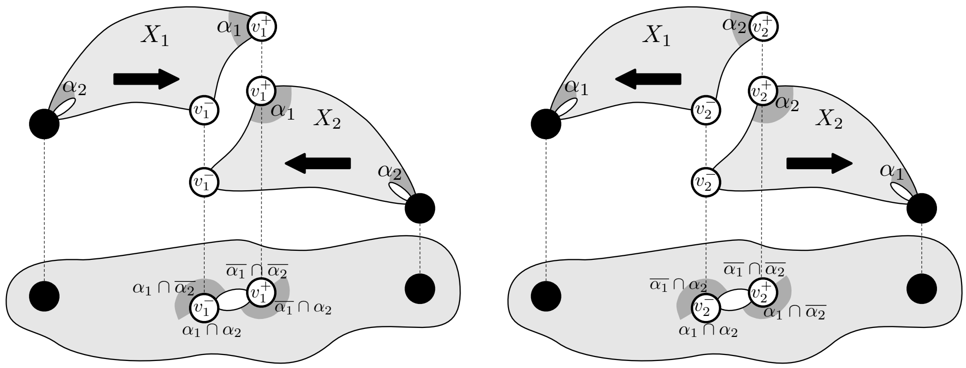


Figure 1.7: Siblings in the first class.

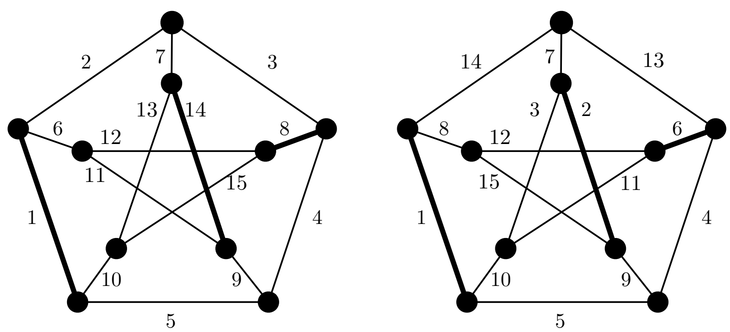


Figure 1.8: Siblings in the second class. Bold edges are odd.

1.2.2 Problem 2: bounding the number of representations

In the previous section we presented operations which relate signed graphs representing the same even cycle matroid. With such operations we obtain signed graphs which are not equivalent; thus an even cycle matroid may have inequivalent representations. The situation may be quite complicated; in fact in general there is no bound on the number of inequivalent representations that an even cycle matroid may have.

We say that a signed graph (G,Σ) is *degenerate* if some signed graph (G',Σ'), equivalent to (G,Σ), has a blocking pair. An even cycle matroid M is *degenerate* if some representation (G,Σ) of M is degenerate; it is non-degenerate otherwise. Note that an even cycle matroid may have both degenerate and non-degenerate representations. Degenerate even cycle matroids may have an arbitrary number of inequivalent representations. As an example, consider the construction in Figure 1.9. Each of the graphs $G_1,\ldots,G_4$ may be any graph. As an example we chose G_1 to be the graph with edges $1,2,3,4,5,6$ given in the figure. The arrows indicate how each piece is flipped. The odd edges, in both graphs, are $1,2,3$. Note that, for every $i \in [4]$, the two vertices in $V_{G_i} \cap V_{G_{i+1}}$ form a blocking pair and it is possible to obtain the signed graph on the right from the signed graph on the left by signature exchanges and Lovász-flips on these blocking pairs. In general we may have an arbitrary number of graphs $G_1, \ldots, G_k$ and we may flip any subset of them. Thus a degenerate even cycle matroid may have an exponential number of pairwise inequivalent representations. We give a more precise description of this operation in Chapter 3.

We do not give a characterization of siblings with blocking pairs here. However, in a paper in preparation (see [16]) we characterize the structure of signed graphs with two distinct blocking pairs.

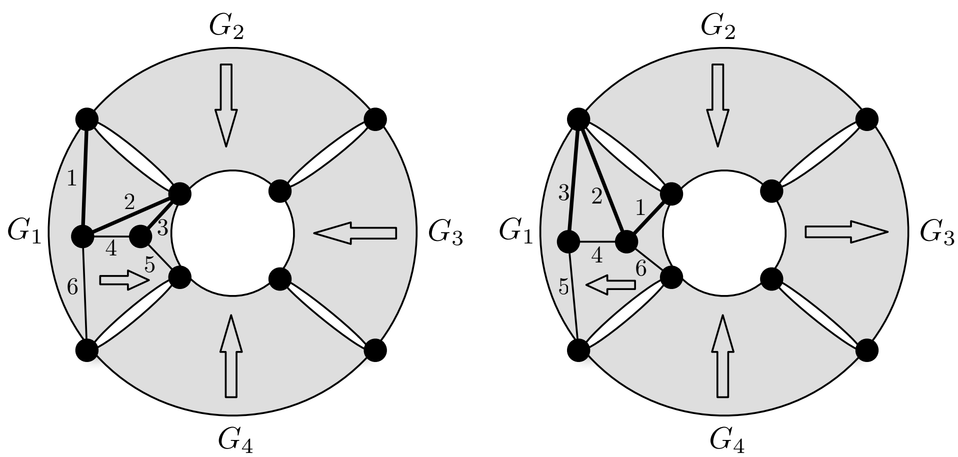


Figure 1.9: Inequivalent siblings.

In Chapter 8 we show that every 3-connected even cycle matroid containing a fixed 3-connected non-degenerate even cycle matroid as a minor has, up to equivalence, a bounded number of representations (where the bound depends on the minor). More specifically, we prove the following.

Theorem. *Let M be a* 3*-connected even cycle matroid which contains as a minor a non-degenerate* 3*-connected matroid N. Then the number of equivalence classes of the representations of M is at most twice the number of equivalence classes of the representations of N.*

The above result is an easy corollary of a stronger result, namely Theorem 8.1, which is proved in Chapter 8.

An example of an even cycle matroid which is non-degenerate is given by the matroid R_{10}, which was introduced by Hoffman in [17] and plays a central role in Seymour's decomposition of regular matroids [28] (see also [1]). A matrix representation of R_{10} is given in Appendix B. R_{10} has six representations as an even cycle matroid, all isomorphic to the signed graph $(K_5, E(K_5))$. This signed graph does not have a blocking pair, as the removal of any two vertices leaves an odd triangle. Hence R_{10} is a non-degenerate even cycle matroid and the theorem above implies that every 3-connected even cycle matroid containing R_{10} as a minor has, up to equivalence, at most 12 representations. In fact, R_{10} has another property, stronger than being non-degenerate. We discuss this property in Chapter 8 and prove a result which implies that every connected even cycle matroid containing R_{10} as a minor has, up to equivalence, at most six representations.

A similar situation occurs for even cut matroids. Given a graph H, we denote by $V_{odd}(H)$ the set of vertices of H of odd degree. Given a graft (G,T) we say that $J \subseteq E(G)$ is a *T-join* of G if $T = V_{odd}(G[J])$. Note that, if J is a T-join of G, a cut C of G is T-even if and only if $|C \cap J|$ is even. We say that two grafts (G_1, T_1) and (G_2, T_2) are *equivalent* if G_1 and G_2 are equivalent and a T_1-join of G_1 is a T_2-join of G_2. As G_1 and G_2 are equivalent, $\text{cut}(G_1) = \text{cut}(G_2)$. Moreover, for $i = 1,2$, a cut C of G_i is T_i-even if and only if $|C \cap J|$ is even. It follows that equivalent grafts represent the same even cut matroid. The converse is also true: suppose that G_1 and G_2 are equivalent graphs and there exist sets of terminals T_1 for G_1 and T_2 for G_2 such that $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$. Let J be a T_1-join of G_1. As (G_1, T_1) and (G_2, T_2) have the even cuts, J is also a T_2-join of G_2. Hence the grafts (G_1, T_1) and (G_2, T_2) are equivalent. We conclude that, given equivalent graphs G_1 and G_2, for two sets of terminals T_1 for G_1 and T_2 for G_2, we have $\text{ecut}(G_1, T_1) = \text{ecut}(G_2, T_2)$ if and only if (G_1, T_1) and (G_2, T_2) are equivalent. The example in Figure 1.6 shows that an even cut matroid may have inequivalent representations.

In general, an even cut matroid may have an arbitrary number of inequivalent representations. By a *path* P of a graph G we mean a set of edges of G such that $G[P]$ is a path in the usual sense. We say that a graft (G,T) has a *covering path* if $|T| \leq 2$ and has a *covering pair* if $|T| \leq 4$. This terminology comes from the fact that if (G,T) has a covering path (respectively a covering pair) then there exists a path P (respectively disjoint paths P, P') of G such that P (respectively $P \cup P'$) is a T-join of G. We say that a graft (G,T) is *degenerate* if some graft (G', T') equivalent to (G,T) has a covering pair. An even cut matroid M is *degenerate* if some representation (G,T) of M is degenerate; it is non-degenerate otherwise.

Note that an even cut matroid may have both degenerate and non-degenerate representations. There is no bound on the number of inequivalent representations that a degenerate even cut matroid may have. An example is given in Figure 1.10, where white vertices are terminals and dotted lines denote vertices that are identified. $G_1, \ldots, G_4$ may be any set of graphs; the arrows indicate how every piece is flipped. In general we may have an arbitrary number of graphs $G_1, \ldots, G_k$ and we may flip any subset of them.

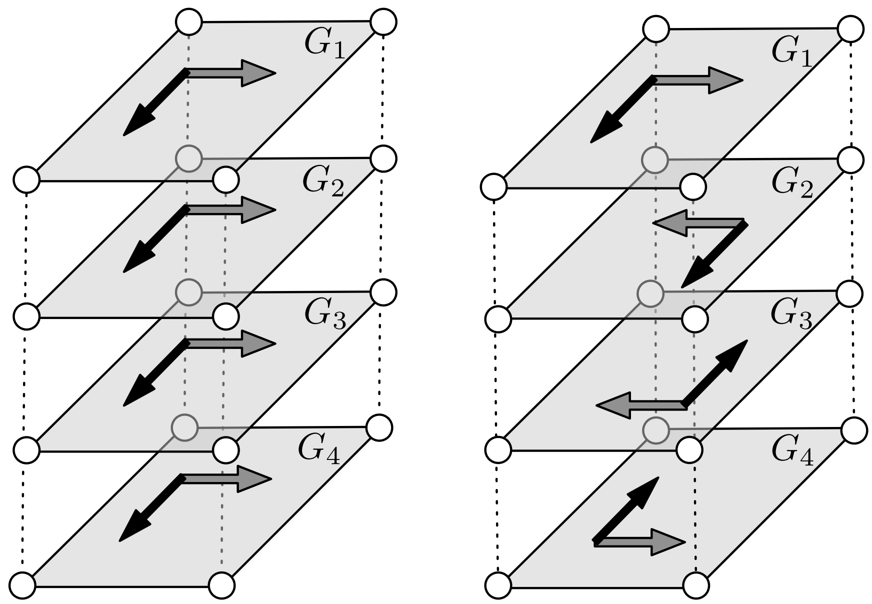


Figure 1.10: Inequivalent representations of an even cut matroid. White vertices are terminals, dotted lines denote vertices that are identified.

In Chapter 9 we show that every 3-connected even cut matroid containing a fixed 3-connected non-degenerate even cut matroid as a minor has, up to equivalence, a bounded number of inequivalent representations. More precisely, we show the following.

Theorem. *Let M be a* 3*-connected even cut matroid which contains as a minor a* 3*-connected matroid N which is non-degenerate. Then the number of equivalence classes of the representations of M is at most twice the number of equivalence classes of the representations of N.*

The above result is an easy corollary of a stronger result, namely Theorem 9.1, which is proved in Chapter 9.

The matroid R_{10} is also an even cut matroid. R_{10} has, up to equivalence, 10 representations as an even cut matroid, which are all isomorphic to the graft in Figure 1.11. Hence R_{10} is a non-degenerate even cut matroid and the theorem above implies that every 3-connected even cut matroid containing R_{10} as a minor has, up to equivalence, at most 20 representations. R_{10} has a stronger property than being non-degenerate. We discuss this property in Chapter 9 and we prove a result which implies that every connected even cut matroid containing R_{10} as a minor has, up to equivalence, at most 10 representations.

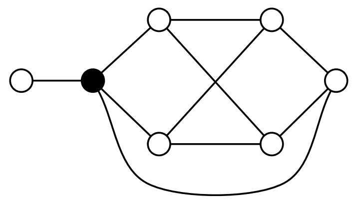


Figure 1.11: Graft representation of R_{10}. White vertices are terminals.

1.2.3 Problem 3: excluded minors

When working with other classes of matroids a first attempt to find excluded minors usually involves proving results about the connectivity of such excluded minors. For example, the proofs of the excluded minors for graphic [34], ternary [2, 27] and quaternary [9] matroids all rely on the fact that every excluded minor for these classes is 3-connected. This is not the case for the excluded minors for even cycle matroids. For example, any matroid obtained by a 2-sum of a copy of R_{10} and a minimally non-graphic matroid is an excluded minor for even cycle matroids which is not 3-connected. An explanation of why these are excluded minors is given in Chapter 7.

Another difficulty when looking for excluded minors for even cycle and even cut matroids arises from the fact that they may have many inequivalent representations. We give an idea of why theorems bounding the number of representations, as the ones in the previous section, may help to find excluded minors. Given two even cycle matroids M and N, where N is a minor of M, we say that a representation (H,Γ) of N *extends* to M if there exists a representation (G,Σ) of M such that (H,Γ) is a minor of (G,Σ) (we define minors for signed graphs in the next chapter). Suppose $\mathscr{F}$ is the set of signed graphs equivalent to (H,Γ). Then we say that $\mathscr{F}$ extends to M if some signed graph in $\mathscr{F}$ extends to M. In

Chapter 8 we prove a stronger result than the theorem stated in the previous section. In fact we show the following.

Theorem. *Let M be a 3-connected even cycle matroid which contains as a minor a non-degenerate 3-connected matroid N. Then every equivalence class of representations of N extends to at most two equivalence classes of representations of M.*

Let N be an even cycle matroid which is a minor of a matroid M. Let $\mathscr{F}$ be an equivalence class of representations of N. Suppose we can show that, if $\mathscr{F}$ does not extend to M, then there exists a matroid M' such that:

(i) N is a minor of M';

(ii) M' is a minor of M;

(iii) $\mathscr{F}$ does not extend to M';

(iv) the size of M' is bounded by a function of N.

Now suppose M is a binary excluded minor for the class of even cycle matroids which contains a minor N, where N is a non-degenerate even cycle matroid with k inequivalent representations. Then no representation of N extends to M. Let $\mathscr{F}$ be an equivalence class of representations of N and let M' be a matroid with the properties above. M' may still be an even cycle matroid, but M' has at most $2k-2$ inequivalent representations, by the theorem above and by the fact that $\mathscr{F}$ does not extend to M. Thus we may repeat the same reasoning with M' and M instead of N and M. After at most $2k$ steps we will obtain a matroid $\hat{M}$ such that N is a minor of $\hat{M}$, no representation of N extends to $\hat{M}$ and the size of $\hat{M}$ only depends on N. This would show that every excluded minor for the class of even cycle matroids containing N as a minor has bounded size. Moreover, a precise characterization of the matroid M' with the properties (i)-(iv) above may lead to an algorithm to find such excluded minors.

In Chapter 2 we introduce basic properties of even cycle and even cut matroids. In Chapter 3 we consider the relation between two signed graphs representing the same even cycle matroid and two grafts representing the same even cut matroid. Chapter 4 contains results which provide a partial answer to the Isomorphism Problem for even cycle matroids; such results are proved in Chapter 6. Chapter 5 contains results on 2-separations and Whitney-flips which are used in subsequent chapters. In Chapter 7 we discuss the problem of finding the excluded minors for the classes of even cycle and even cut matroids; we discuss stabilizer-type theorems, which are proved in Chapters 8 and 9. The final Chapter 10 contains open problems and discussion on future work.

The results in Chapter 3 and Section 5.1 and an early version of the results in Chapters 4 and 8 (with respective proofs) are joint work with Paul Wollan.

1.3 Related results

In this section we survey recent results about the Isomorphism Problem and the Excluded Minor problem for other classes of matroids arising from graphs. We start by introducing a very general class of matroids arising from biased graphs and then present results for two special subclasses of these matroids. A *theta graph* is a graph formed by two circuits intersecting exactly in a path with at least one edge. A set $\mathbb{B}$ of circuits in a graph is *linear* if for every $C_1, C_2 \in \mathbb{B}$ forming a theta graph, the third circuit in $C_1 \cup C_2$ is also in $\mathbb{B}$. A *biased graph* is a pair $(G, \mathbb{B})$, where G is a graph and $\mathbb{B}$ is a linear set of circuits of G. The circuits in $\mathbb{B}$ are called *balanced*. Biased graphs were introduced by Zaslavsky (see [40] and [41]). A family of matroids arising from biased graphs is the family of *frame matroids*. The frame matroid represented by a biased graph $(G, \mathbb{B})$ has as ground set the set of edges of the graph. The circuits of the matroid are the sets of edges of one of the following four types: balanced circuits; two disjoint unbalanced circuits together with a minimal path connecting them; two unbalanced circuits sharing exactly one vertex; a theta graph with all circuits unbalanced. Frame matroids include a wide variety of matroids, for example Dowling matroids [6]. We present results about two special classes of frame matroids.

Frame matroids arising from a biased graph $(G, \emptyset)$ (that is, all circuits are unbalanced) are called *bicircular matroids*. Bicircular matroids were first introduced by Simões-Pereira [31]. The Isomorphism Problem for bicircular matroids has been widely studied and a complete characterization of when two graphs represent the same bicircular matroid is known (see [37], [4] and [19]). The operations relating two graphs representing the same bicircular matroid are relatively simple and they act locally on the graph. Recently Goddyn and DeVos (see [5]) announced that they have found the excluded minors for this class. The main part of their proof consists in showing that every excluded minor for the class of bicircular matroids has at most nine elements. This proof uses the above mentioned results about representations of bicircular matroids.

A second important class of frame matroids arises from signed graphs. Let (G,Σ) be a signed graph and $\mathbb{B}$ be the set of even circuits of (G,Σ). Then $(G,\mathbb{B})$ is a biased graph. The frame matroid represented by $(G,\mathbb{B})$ is a *signed-graphic matroid*. Signed-graphic matroids are in general very complicated objects, but there has been recent progress on regular and near-regular signed-graphic matroids. A matroid is *regular* if it is representable over every field and *near-regular* if it is representable over every field, except possibly the binary field. There are 31 regular excluded minors for signed graphic matroids, as recently proved in [24] by Slilaty et al. All but two of these excluded minors are excluded minors for projective-planar graphs. Work on the isomorphism problem for this class has been conducted by Pendavingh and Van Zwam [23], who studied a recognition algorithm for near-regular signed-graphic matroids. They introduced three operations which relate representations of the same near-regular signed-graphic matroid in the case in which the signed graph is cylindrical. We will not define this term here; we just remark that for the recognition algorithm it is sufficient to consider the cylindrical case.

Another general class of matroids arising from biased graphs is the class of lift matroids (also defined in [41]). The circuits of the *lift matroid* represented by the biased graph $(G,\mathbb{B})$ are the sets of edges of one of the following three types: balanced circuits; two unbalanced circuits sharing at most one vertex; a theta graph with all circuits unbalanced. Even cycle matroids are a basic class of lift matroids. In fact, we already noted that, given a signed graph (G,Σ) and the set $\mathbb{B}$ of even circuits of (G,Σ), $(G,\mathbb{B})$ is a biased graph. Moreover, given any two odd circuits C_1,C_2 of (G,Σ) which intersect exactly in a path, the third circuit in $C_1 \cup C_2$ is even. Thus the lift matroid represented by $(G,\mathbb{B})$ does not contain any theta graph with all circuits unbalanced. Hence the circuits of the lift matroid are exactly the circuits of ecycle(G,Σ). Note that even cycle matroids are different from the signed-graphic matroids defined above. In fact, two vertex-disjoint odd circuits in a signed graph (G,Σ) form a circuit of ecycle(G,Σ), but not a circuit of the signed-graphic matroid represented by (G,Σ). Little is known about the Isomorphism Problem and the Excluded Minor Problem for the class of lift matroids.

1.4 Motivation

Even cycle and even cut matroids arise naturally in the literature. The class of even cycle matroids is the smallest minor closed class of matroids which properly contains all single-element co-extensions of cycle matroids. The class of even cut matroids is the smallest minor closed class of matroids which properly contains all single-element co-extensions of cut matroids. Hence these classes are the first natural generalization of cycle and cut matroids. Even cycle and even cut matroids and their duals also seem to be good candidates to be the building blocks for the class of binary matroids without an $AG(3,2)$ minor.

Signed graphs have been fruitfully used to find shorter proofs of important results. A first example is the proof of Theorem 1.3 given by Gerards in [12]; this proof is much shorter than the original one and relies mainly on graph theoretical results. Signed graphs have also been used by Geelen and Gerards (see [8]) to give an alternative proof of Seymour's decomposition of regular matroids.

Our original motivation for working with these classes of matroids was a conjecture by Seymour about flows in matroids. Given a graph G, two vertices $s,t \in V(G)$ and a vector $w \in \mathbb{R}_+^{E(G)}$, consider the following problems:

$$\begin{array}{lll} \min & w^T x & \\ \text{s.t.} & x(P) \geq 1, \qquad \forall\ (s,t)\text{-path } P & \qquad \text{(IP)} \\ & x \in \{0,1\}^{E(G)} & \end{array}$$

$$\begin{array}{lll} \max & \mathbb{1}^T y & \\ \text{s.t.} & \sum(y_P : e \in P, P\ (s,t)\text{-path}) \leq w_e, \qquad \forall e \in E(G) & \qquad \text{(D)} \\ & y \geq 0 & \end{array}$$

Note that (D) is the dual of the LP relaxation of (IP). A solution to (IP) can be interpreted as a minimum (s,t)-cut, while a solution to (D) gives a fractional maximal st-flow (for undirected graphs). By the Max-Flow Min-Cut Theorem of Ford and Fulkerson (see [7]), for all $w \in \mathbb{R}_+^{E(G)}$ the optimal value of (IP) is equal to the optimal value of (D). We can generalize the concept of minimum cut and maximum flow to binary matroids. Given a matroid M and $f \in E(M)$, a set of the form $C - \{f\}$, where C is a circuit of M using f, is called an *f-path*. We can define the analogue of (IP) and (D) in terms of f-paths.

Let M be a matroid, $f \in E(M)$ and $w \in \mathbb{R}_+^{E(M)-\{f\}}$. Consider

$$\begin{array}{lll} \min & w^T x & \\ \text{s.t.} & x(P) \geq 1, \quad \forall\, f\text{-path } P & \text{(IP')} \\ & x \in \{0,1\}^{E(M)-\{f\}} & \end{array}$$

$$\begin{array}{lll} \max & \mathbb{1}^T y & \\ \text{s.t.} & \sum(y_P : e \in P, P\ f\text{-path}) \leq w_e, \quad \forall e \in E(M)-\{f\} & \text{(D')} \\ & y \geq 0 & \end{array}$$

We say that M is f-*flowing* if, for all $w \in \mathbb{R}_+^{E(M)-\{f\}}$, the optimal values of (IP') and (D') are the same. M is 1-*flowing* if it is f-flowing for all $f \in E(M)$. An example of a matroid that is not 1-flowing is $U_{2,4}$. As being 1-flowing is closed under minors, it follows that non-binary matroids are not 1-flowing. Seymour (see [26]) conjectured the following.

Conjecture 1.5 (Seymour 1977)**.** *A binary matroid M is* 1*-flowing if and only if it contains no $AG(3,2)$, T_{11} or T_{11}^* minor.*

The matroids in Conjecture 1.5 are defined in Appendix B. In [26] Seymour solved the analogous problem of determining when (D') and its dual both have integer solutions for all integral vectors w and all elements e. We will not state this result here, as the precise statement would require a few definitions. A consequence of this result is that, for a binary matroid M and a fixed element $e \in E(M)$, (D') and its dual both have integer solutions for all integral vectors w if and only if M has no F_7^* minor using the element e.

Guenin showed that Seymour's conjecture holds for even cycle and even cut matroids (see [14]). Hence finding the excluded minors for even cycle and even cut matroids would be a first step toward solving Seymour's Conjecture for general binary matroids.

Chapter 2

Preliminaries

In this chapter we present some basic properties of even cycle and even cut matroids. In particular we specify what the matrix representation, the bases and co-cycles are, we illustrate how minor operations on the matroids correspond to minor operations on the representations and we present some simple results about connectivity. In the second section we relate degenerate signed graphs and grafts. We assume that the reader is familiar with the basics of matroid theory. Our terminology generally follows that of Oxley [21]. Unless otherwise specified, we will only consider binary matroids in the rest of this work. Thus the reader should substitute the term “binary matroid” every time “matroid” appears in this text.

2.1 Basic properties

2.1.1 Matrix-representations

Even cycle and even cut matroids are binary matroids: we now explain how to obtain their matrix representation from a signed graph or a graft representation. Let (G,Σ) be a signed graph. Let $A(G)$ be the incidence matrix of G, i.e. the columns of $A(G)$ are indexed by the edges of G, the rows of $A(G)$ are indexed by the vertices of G and entry (v,e) of $A(G)$ is 1 if vertex v is incident to edge e in G and 0 otherwise. Let S be the transpose of the characteristic vector of Σ; hence S is a row vector indexed by $E(G)$ and S_e is 1 if $e \in \Sigma$ and 0 otherwise. Let A be the binary matrix obtained from $A(G)$ by adding row S. Let $M(A)$

be the binary matroid represented by A. Let C be a cycle of $M(A)$. Then C intersects every cut of G and Σ with even parity. The sets that intersect every cut of G with even parity are exactly the cycles of G. Thus $M(A) = \text{ecycle}(G,\Sigma)$. Note that in constructing A we may replace $A(G)$ with any binary matrix whose rows span the cut space of G.

Let (G,T) be a graft and J a T-join of G. Let $\hat{A}(G)$ be a binary matrix whose rows span the cycle space of G. Let $\hat{S}$ be the transpose of the incidence vector of J; hence $\hat{S}$ is a row vector indexed by $E(G)$ and $\hat{S}_e$ is 1 if $e \in J$ and 0 otherwise. Construct a matrix $\hat{A}$ from $\hat{A}(G)$ by adding row $\hat{S}$. Let $M(\hat{A})$ be the binary matroid represented by $\hat{A}$. Let C be a cycle of $M(\hat{A})$. Then C intersects every cycle of G and J with even parity. The sets that intersect every cycle of G with even parity are exactly the cuts of G. Moreover, a cut intersects J with even parity if and only if it is T-even. Thus $M(\hat{A}) = \text{ecut}(G,T)$.

2.1.2 Bases and co-cycles

Consider a signed graph (G,Σ). What is a basis for $\text{ecycle}(G,\Sigma)$? A set $F \subseteq E(G)$ is dependent if and only if it contains an even cycle. As we consider graphs up to equivalence, and identifying two vertices in distinct components of a graph is a Whitney-flip, we may assume without loss of generality that G is connected. If (G,Σ) does not contain any odd cycles, then $\text{ecycle}(G,\Sigma) = \text{cycle}(G)$ and a basis is just formed by a spanning tree. If (G,Σ) contains at least one odd cycle, every basis for $\text{ecycle}(G,\Sigma)$ is formed by a spanning tree B together with an edge $f \in \bar{B}$ forming an odd cycle with edges in B.

The co-cycle space of $\text{ecycle}(G,\Sigma)$ is the space spanned by the rows of A, where A is the binary matrix representation of $\text{ecycle}(G,\Sigma)$. From the construction in the previous section we have the following.

Remark 2.1. *The co-cycles of* $\text{ecycle}(G,\Sigma)$ *are the cuts of* G *and the signatures of* (G,Σ).

Consider a graft (G,T). What is a basis for $\text{ecut}(G,T)$? A set $F \subseteq E(G)$ is dependent if and only if it contains an even cut. Hence, if (G,T) does not contain any odd cut then $\text{ecut}(G,T) = \text{cut}(G)$ and a basis is just formed by the complement of a spanning tree. If (G,T) contains at least one odd cut, every basis for $\text{ecut}(G,T)$ is formed by the complement $\bar{B}$ of a spanning tree B together with an edge $f \in B$ forming an odd cut with edges in $\bar{B}$. The co-cycle space of $\text{ecut}(G,T)$ is the space spanned by the rows of $\hat{A}$, where $\hat{A}$ is the binary matrix representation of $\mathrm{ecut}(G,T)$. Note that the symmetric difference of a cycle and a T-join is a T-join. From the construction in the previous section we have the following.

Remark 2.2. *The co-cycles of* $\mathrm{ecut}(G,T)$ *are the cycles of* G *and the* T*-joins of* (G,T).

2.1.3 Minors

Let M be a matroid and e an element of M. We denote by $M\setminus e$ the matroid obtained from M by deleting e and by M/e the matroid obtained from M by contracting e.

Remark 2.3. *Let* M *be a matroid and* $e\in E(M)$.

(1) The cycles of $M\setminus e$ *are the cycles of* M *not using* e.

(2) The cycles of M/e *are the cycles of* M *not using* e *and the cycles of* M *using* e*, with the element* e *removed.*

For any two disjoint subsets C,D of $E(M)$, we denote by $M/C\setminus D$ the matroid obtained from M by contracting the elements in C and deleting the elements in D. This is well defined, as minor operations commute. Given a graph G and C,D disjoint subsets of $E(G)$ we denote by $G/C\setminus D$ the graph obtained from G by contracting C and deleting D. We ignore isolated vertices in graphs.

Even cycle matroids

In this section we define minor operations for signed graphs. Let (G,Σ) be a signed graph and let $e\in E(G)$. Then $(G,\Sigma)\setminus e$ is defined as $(G\setminus e,\Sigma-\{e\})$. This definition implies that the even cycles of $(G,\Sigma)\setminus e$ are the even cycles of (G,Σ) not using e. We define $(G,\Sigma)/e$ as $(G\setminus e,\emptyset)$ if e is an odd loop of (G,Σ) and as $(G\setminus e,\Sigma)$ if e is an even loop of (G,Σ); otherwise $(G,\Sigma)/e$ is equal to $(G/e,\Sigma')$, where Σ' is any signature of (G,Σ) which does not contain e. By definition, the even cycles of $(G,\Sigma)/e$ are either even cycles of (G,Σ) not using e or even cycles of (G,Σ) using e, with the element e removed. By Remark 2.3 we conclude that

$$\mathrm{ecycle}(G,\Sigma)/C\setminus D=\mathrm{ecycle}\big((G,\Sigma)/C\setminus D\big).$$

Given two signed graphs $(G,\Sigma),(H,\Gamma)$, we say that (H,Γ) is a *minor* of (G,Σ), denoted $(H,\Gamma)\leq(G,\Sigma)$, if $(H,\Gamma)=(G,\Sigma)/C\setminus D$ for some disjoint sets $D,C\subseteq E(G)$.

Even cut matroids

In this section we define minor operations for grafts. Let (G,T) be a graft and let $e \in E(G)$. Then $(G,T)\setminus e$ is defined as $(G\setminus e, T')$, where $T' = \emptyset$ if e is an odd bridge of (G,T) and $T' = T$ otherwise. Note that the even cuts of $(G,T)\setminus e$ are either even cuts of (G,T) not using e or even cuts of (G,T) with the element e removed. $(G,T)/e$ is equal to $(G/e, T')$, where T' is defined as follows. Let u, v be the ends of e in G and let w be the vertex obtained by contracting e. If $x \neq w$, then $x \in T'$ if and only if $x \in T$; $w \in T'$ if and only if $|\{u,v\} \cap T| = 1$. Note that the cuts of $(G,T)/e$ are the cuts of (G,T) not using the element e. Moreover a cut in $(G,T)/e$ is even if and only if it is even in (G,T). By Remark 2.3 we conclude that

$$\text{ecut}(G,T)/C\setminus D = \text{ecut}\big((G,T)/D\setminus C\big).$$

Given two grafts $(G,T),(H,R)$, we say that (H,R) is a *minor* of (G,T), denoted $(H,R) \leq (G,T)$, if $(H,R) = (G,T)\setminus D/C$ for some disjoint sets $D, C \subseteq E(G)$.

2.2 Checking for isomorphism

It is easy to check whether two signed graphs (G_1,Σ_1) and (G_2,Σ_2) are siblings, that is, checking whether $\text{ecycle}(G_1,\Sigma_1) = \text{ecycle}(G_2,\Sigma_2)$. Let F be a set of edges forming a spanning tree of G_1. If (G_1,Σ_1) is bipartite, let $B := F$; otherwise, let f be an edge in $\bar{F}$ forming a Σ_1-odd cycle in G_1 with edges in F and let $B := F \cup \{f\}$. Then B is a basis of $\text{ecycle}(G_1,\Sigma_1)$. For every $e \in \bar{B}$, there is a unique subset C_e of B such that $C_e \cup \{e\}$ is an even cycle of (G_1,Σ_1) (these are the fundamental circuits of $\text{ecycle}(G_1,\Sigma_1)$ with respect to B). To check whether $\text{ecycle}(G_1,\Sigma_1) = \text{ecycle}(G_2,\Sigma_2)$, it suffices to check that B is a basis of $\text{ecycle}(G_2,\Sigma_2)$ and that, for every $e \in \bar{B}$, C_e is an even cycle in (G_2,Σ_2).

2.3 Connectivity

Let M be a matroid with rank function r. Given $X \subseteq E(M)$ we define $\lambda_M(X)$, the *connectivity function* of M, to be equal to $r(X) + r(\bar{X}) - r(E(M)) + 1$. The set X is a *k-separation* of M if $\min\{|X|,|\bar{X}|\} \geq k$ and $\lambda_M(X) = k$. M is *k-connected* if it has no r-separations for any $r < k$. Let G be a graph and let $X \subseteq E(G)$. The set X is a *k-separation* of G if

$\min\{|X|,|\bar{X}|\} \geq k$, $|\mathscr{B}_G(X)| = k$ and both $G[X]$ and $G[\bar{X}]$ are connected. Note that with this definition two parallel edges of G form a 2-separation of G. A graph G is *k-connected* if it has no r-separations for any $r < k$. We relate graph connectivity with connectivity for even cycle and even cut matroids. The proof of the next two results is given in the more general setting of signed matroids.

2.3.1 Even cycle matroids

Recall that we denote by $\text{loop}(G)$ the set of loops of G. A signed graph (G,Σ) is *bipartite* if G has no Σ-odd cycle. Equivalently, (G,Σ) is bipartite if Σ is a cut of G.

Proposition 2.4. *Suppose that* $\text{ecycle}(G,\Sigma)$ *is* 3*-connected. Then:*

(1) $|\text{loop}(G)| \leq 1$ *and if* $e \in \text{loop}(G)$ *then* $e \in \Sigma$*;*

(2) $G \setminus \text{loop}(G)$ *is* 2*-connected;*

(3) if G *has a* 2*-separation* X *then* $(G[X],\Sigma \cap X)$ *and* $(G[\bar{X}],\Sigma \cap \bar{X})$ *are both non-bipartite.*

2.3.2 Even cut matroids

Given a separation X of G, we define the *interior* of X in G to be $\mathscr{I}_G(X) = V_G(X) - \mathscr{B}_G(X)$. Given a graft (G,T), we say that an edge e of G is a *pin* if e is an odd bridge of G incident to a vertex of degree one, which we call the *head* of the pin. Hence the head of a pin is a terminal. We denote by $\text{pin}(G,T)$ the set of pins of (G,T).

Proposition 2.5. *Suppose that* $\text{ecut}(G,T)$ *is* 3*-connected. Then:*

(1) $|\text{pin}(G,T)| \leq 1$*;*

(2) $G/\text{pin}(G,T)$ *is* 2*-connected;*

(3) if G *has a* 2*-separation* X *then* $T \cap \mathscr{I}_G(X)$ *and* $T \cap \mathscr{I}_G(\bar{X})$ *are both non-empty.*

2.3.3 Signed matroids

Recall that we only consider binary matroids in this work. A pair (M,Σ) where M is a matroid and $\Sigma \subseteq E(M)$ is a *signed matroid*. A set $C \subseteq E(M)$ is Σ-even if $|C \cap \Sigma|$ is even. The set of all cycles of M that are Σ-even forms the set of cycles of a matroid which we denote by $\mathrm{ecycle}(M,\Sigma)$. A signed matroid (M,Σ) is *bipartite* if all cycles of M are even. We denote by $M|_X$ the restriction of M to the set X, i.e. the matroid $M \setminus \bar{X}$.

We denote by $\mathrm{loop}(M)$ the set of loops (i.e. one-element circuits) of the matroid M. We may generalize Proposition 2.4 and Proposition 2.5 to the following result.

Proposition 2.6. *Suppose that* $\mathrm{ecycle}(M,\Sigma)$ *is* 3*-connected. Then*

(1) $|\mathrm{loop}(M)| \leq 1$ *and if* $e \in \mathrm{loop}(M)$ *then* $e \in \Sigma$*;*

(2) $M \setminus \mathrm{loop}(M)$ *is* 2*-connected;*

(3) if M *has a* 2*-separation* X *then* $(M|_X, \Sigma \cap X)$ *and* $(M|_{\bar{X}}, \Sigma \cap \bar{X})$ *are both non-bipartite.*

Before proving Proposition 2.6, we show how it implies the two results for even cycle and even cut matroids. Let (G,Σ) be a signed graph and let $M := \mathrm{cycle}(G)$. Then $\mathrm{ecycle}(M,\Sigma) = \mathrm{ecycle}(G,\Sigma)$ and Proposition 2.4 follows directly from Proposition 2.6.

Let (G,T) be a graft and $M := \mathrm{cut}(G)$. Let J be a T-join of G. Then an even cycle of (M,J) is a cut of G which intersects J with even parity. Hence $\mathrm{ecycle}(M,J) = \mathrm{ecut}(G,T)$. Note that loops of M are bridges of G and an even bridge of G is a loop of $\mathrm{ecycle}(M,J)$. Moreover, if X is a k-separation of G, then X is a k-separation of $\mathrm{cycle}(G)$, hence a k-separation of M (because $\mathrm{cycle}(G)$ is the dual of M). For a set $X \subseteq E(G)$, $M|_X = \mathrm{cut}(G/\bar{X})$. Hence the cuts of $M|_X$ are the cuts of $G/\bar{X}$. It follows that $(M|_X, J \cap X)$ is bipartite if and only if every cut of $G/\bar{X}$ is J-even. This happens if and only if $T \cap \mathscr{I}_G(X)$ is empty. Hence Proposition 2.5 follows from Proposition 2.6.

To prove Proposition 2.6, we require a definition and a preliminary result. Let (M,Σ) be a signed matroid and $X \subseteq E(M)$. We say that X is a *k-(i,j)-separation* of (M,Σ), where $i,j \in \{0,1\}$, if the following hold:

(a) X is a k-separation of M;

(b) $i = 0$ when $(M|_X, \Sigma \cap X)$ is bipartite and $i = 1$ otherwise;

(c) $j = 0$ when $(M|_{\bar{X}}, \Sigma \cap \bar{X})$ is bipartite and $j = 1$ otherwise.

Lemma 2.7. *Let* (M,Σ) *be a non-bipartite signed matroid and* $M_S :=$ ecycle(M,Σ). *For every* k*-*(i,j)*-separation* X *of* (M,Σ)*, we have* $\lambda_{M_S}(X) = k+i+j-1$.

Proof. Let r be the rank function of M and r_S be the rank function of M_S. As (M,Σ) is non-bipartite, a basis for M_S consists of a basis B for M plus an element $e \in \bar{B}$ such that the fundamental circuit of e in M is Σ-odd. Hence $r_S(M_S) = r(M)+1$. Similarly, if $(M|_X, \Sigma \cap X)$ (respectively $(M|_{\bar{X}}, \Sigma \cap \bar{X})$) is non-bipartite, then the rank of X (respectively $\bar{X}$) in M_S is one more that in M, otherwise the rank of X (respectively $\bar{X}$) is the same in both matroids. Thus $r_S(X) = r(X)+i$ and $r_S(\bar{X}) = r(\bar{X})+j$. Hence

$$\begin{aligned}\lambda_{M_S}(X) &= r_S(X)+r_S(\bar{X})-r_S(M_S)+1\\ &= r(X)+i+r(\bar{X})+j-r(M)-1+1\\ &= \lambda_M(X)+i+j-1.\end{aligned}$$

□

Proof of Proposition 2.6. Let $M_S :=$ ecycle(M,Σ). As M_S is 3-connected, it has no loops, no co-loops and no parallel elements. We may assume that (M,Σ) is non-bipartite, for otherwise $M_S = M$ and M is 3-connected. **(1)** Let e be a loop of M. Then $e \in \Sigma$ for otherwise e would be a loop of M_S. There do not exist distinct loops e, f of M, for otherwise $\{e, f\}$ would be a circuit of M_S and e, f would be in parallel in M_S. **(2)** Suppose that X is a 1-(i,j)-separation of (M,Σ). By Lemma 2.7, $\lambda_{M_S}(X) = 1+i+j-1 \leq 2$. As M_S is 3-connected, X is not a 2-separation; hence either $|X| = 1$ or $|\bar{X}| = 1$. The single element in X (or $\bar{X}$) is not a co-loop of M, for otherwise it is a co-loop of M_S. Hence X or $\bar{X}$ is a loop of M. **(3)** Suppose that X is a 2-(i,j)-separation of (M,Σ). As M_S is 3-connected, $\lambda_{M_S}(X) \geq 3$. By Lemma 2.7, $2+i+j-1 \geq 3$, hence $i = j = 1$. □

2.4 Constructing even cuts from even cycles and vice versa

2.4.1 Matroids that are both even cycle and even cut

We give a simple construction that produces matroids that are both even cycle and even cut matroids. Let (G,Σ) be a signed graph such that G is planar. Let G^* be the planar dual of G. Then every edge of G corresponds to an edge of G^* and $\text{cycle}(G) = \text{cut}(G^*)$. Now define $T = V_{odd}(G^*[\Sigma])$. By this definition, Σ is a T-join of G^*. Hence C is a Σ-even cycle of G if and only if C is a T-even cut of G^*. It follows that $\text{ecycle}(G,\Sigma) = \text{ecut}(G^*,T)$.

Note that there are matroids which are both even cycle and even cut matroids and do not arise from this construction. An example is given by the matroid R_{10}. As discussed in Section 1.2.2, R_{10} is both an even cycle and an even cut matroid. All representations of R_{10} as an even cycle matroid are isomorphic to the signed graph $(K_5, E(K_5))$, which is clearly non-planar. All representations of R_{10} as an even cut matroid are isomorphic to the graft in Figure 1.11. The algorithm we used to find these representations is given in Appendix A.

2.4.2 Folding and unfolding

In this section we define an operation that relates signed graphs with blocking pairs to grafts with covering pairs. For our purpose the position of the loops is immaterial. Thus we will assume that all loops form distinct components of the graph.

Consider a graph H with a vertex v and $\alpha \subseteq \delta_H(v) \cup \text{loop}(H)$. We say that G is obtained from H by *splitting* v into v_1, v_2 according to α if $V(G) = V(H) - \{v\} \cup \{v_1, v_2\}$ and for every $e = (u,w) \in E(H)$:

(a) if $e \notin \alpha \cup \delta_H(v)$, then $e = (u,w)$ in G;

(b) if $e \in \text{loop}(H) \cap \alpha$, then $e = (v_1, v_2)$ in G;

(c) if $e \in \delta_H(v) \cap \alpha$ and $w = v$ then $e = (u, v_1)$ in G;

(d) if $e \in \delta_H(v) - \alpha$ and $w = v$ then $e = (u, v_2)$ in G.

Consider a signed graph (H,Γ) where $\Gamma \subseteq \delta_H(s) \cup \delta_H(t) \cup \text{loop}(H)$ for two distinct vertices s,t of H. Choose $\alpha, \beta \subseteq E(H)$, where $\alpha \Delta \beta = \Gamma$, $\alpha \subseteq \delta(s) \cup \text{loop}(H)$, $\beta \subseteq \delta(t) \cup \text{loop}(H)$, and $\alpha \cap \beta \cap \text{loop}(H) = \emptyset$. Construct a graft (G,T) as follows:

(a) split s into s_1, s_2 according to α;

(b) split t into t_1, t_2 according to β;

(c) set $T = \{s_1, s_2, t_1, t_2\}$.

Then (G,T) is obtained by *unfolding* (H,Γ) according to vertices s,t and signature Γ (or according to vertices s,t and α,β). Note that the resulting graft (G,T) depends on the choice of α,β, not only on Γ. Finally, we say that (H,Γ) is obtained by *folding* the graft (G,T) with the pairing s_1,s_2 and t_1,t_2. We denote by M^* the dual of a matroid M.

Remark 2.8. *Let (H,Γ) be a signed graph with $\Gamma \subseteq \delta(s) \cup \delta(t) \cup \text{loop}(H)$ and let (G,T) be a graft obtained by unfolding (H,Γ) according to s,t and Γ. Then:*

(1) a set of edges is an even cycle of (H,Γ) if and only if it is a cycle or a T-join of G;

(2) $\text{ecycle}(H,\Gamma) = \text{ecut}(G,T)^*$.

Proof. Suppose we choose α and β as in the definition of unfolding. Suppose that C is an even cycle of (H,Γ). For every $v \in V(H) - \{s,t\}$, $|\delta_H(v) \cap C| = |\delta_G(v) \cap C|$, which is even. For $i = 1,2$ define $d(s,i) = |C \cap \delta_G(s_i)|$ and $d(t,i) = |C \cap \delta_G(t_i)|$. Since C is a cycle $d(s,1), d(s,2)$ have the same parity and so do $d(t,1), d(t,2)$. Note that $\alpha = \delta_G(s_1)$, $\beta = \delta_G(t_1)$ and $\Gamma = \alpha \Delta \beta$. Thus, as $|C \cap \Gamma|$ is even, $d(s,1)$ and $d(t,1)$ have the same parity. Thus $d(s,1)$, $d(s,2)$, $d(t,1)$, $d(t,2)$ are either all even or all odd. In the former case C is a cycle of G, in the later case it is a T-join of G. The converse is similar. Finally, (2) follows from (1) and Remark 2.2. □

In particular, it follows by Remark 2.8 that if M is an even cycle matroid represented by a signed graph with a blocking pair, then M is also the dual of an even cut matroid. Vice versa, if M is an even cut matroid represented by a graft with a covering pair, then M is the dual of an even cycle matroid. Note that not all matroids which are both an even cycle and the dual of even cut matroid arise from this construction. An example is given, once again, by the matroid R_{10}, which is both an even cycle and an even cut matroid and is self-dual.

2.4.3 Unbounded number of representations

Theorem 1.1 states that any two representations of the same cycle matroid are equivalent. In light of this result, it is natural to ask whether we can bound the number of inequivalent representations that even cycle and even cut matroids may have. Unfortunately this is not the case, as the following two examples illustrate.

Consider a signed graph (H,Γ) with $\Gamma \subseteq \delta(s) \cup \delta(t) \cup \text{loop}(H)$ and let (G,T) be a graft obtained by unfolding (H,Γ) according to s,t and Γ. Let (G',T') be a graft which is equivalent to (G,T), where $|T'| = 4$. Let (H',Γ') be obtained by folding (G',T') according to some arbitrary pairing of the vertices of T'. Then by Remark 2.8(2),

$$\text{ecycle}(H,\Gamma) = \text{ecut}(G,T)^* = \text{ecut}(G',T')^* = \text{ecycle}(H',\Gamma').$$

This construction gives rise to the example in Figure 2.1. Suppose for instance that we

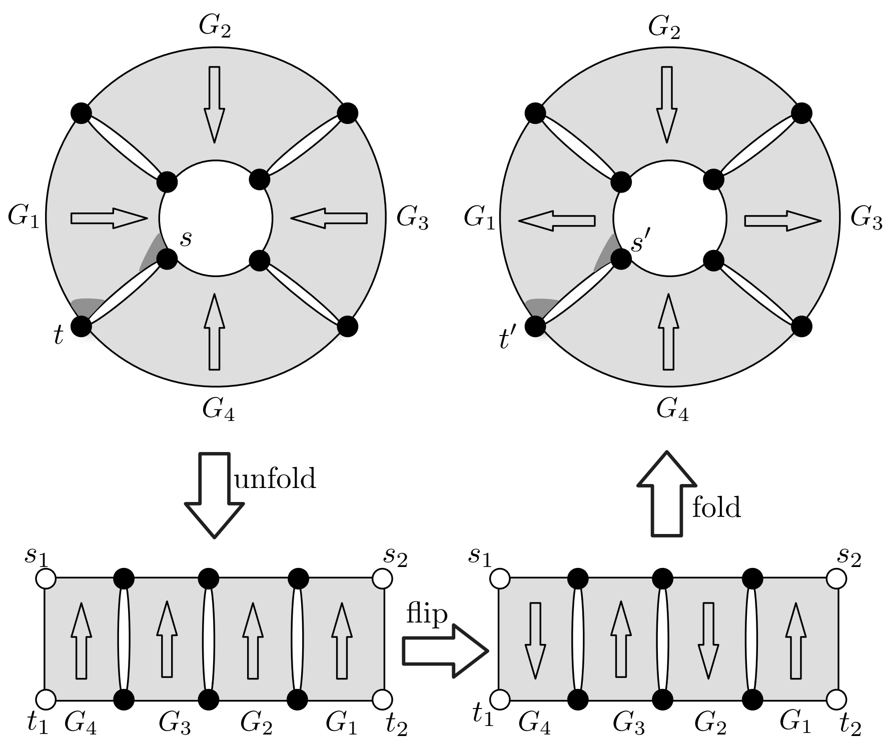


Figure 2.1: Inequivalent signed graphs.

choose G to be the graph with vertex set $\{v_1,\ldots,v_k\} \cup \{v'_1,\ldots,v'_k\}$ and edges $\{(v_i,v_{i+1}),$ $(v'_i,v'_{i+1}),(v_i,v'_{i+1}),(v'_i,v_{i+1})\}$ for all $i \in [k-1]$. Let $T = \{v_1,v'_1,v_k,v'_k\}$ and let G' be any graph obtained from G by a Whitney-flip on vertices v_i,v'_i for some $i \in \{2,\ldots,k-1\}$. Then $T' = T$ and $(H,\Gamma),(H',\Gamma')$ are inequivalent representations. We conclude that an even cycle matroid may have an arbitrary number of inequivalent representations.

We consider an analogous construction for grafts with a covering pair. Let (G,T) be a graft with $|T| = 4$. Let (H,Γ) be a signed graph obtained by folding (G,T) with some pairing of the vertices in T. Let (H',Γ') be a signed graph equivalent to (H,Γ) and having a blocking pair. Let (G',T') be obtained by some unfolding of (H',Γ') according to the blocking pair. Then by Remark 2.8(2),

$$\text{ecut}(G,T) = \text{ecycle}(H,\Gamma)^* = \text{ecycle}(H',\Gamma')^* = \text{ecut}(G',T').$$

This construction gives rise to the example in Figure 2.2. In general, consider any graft

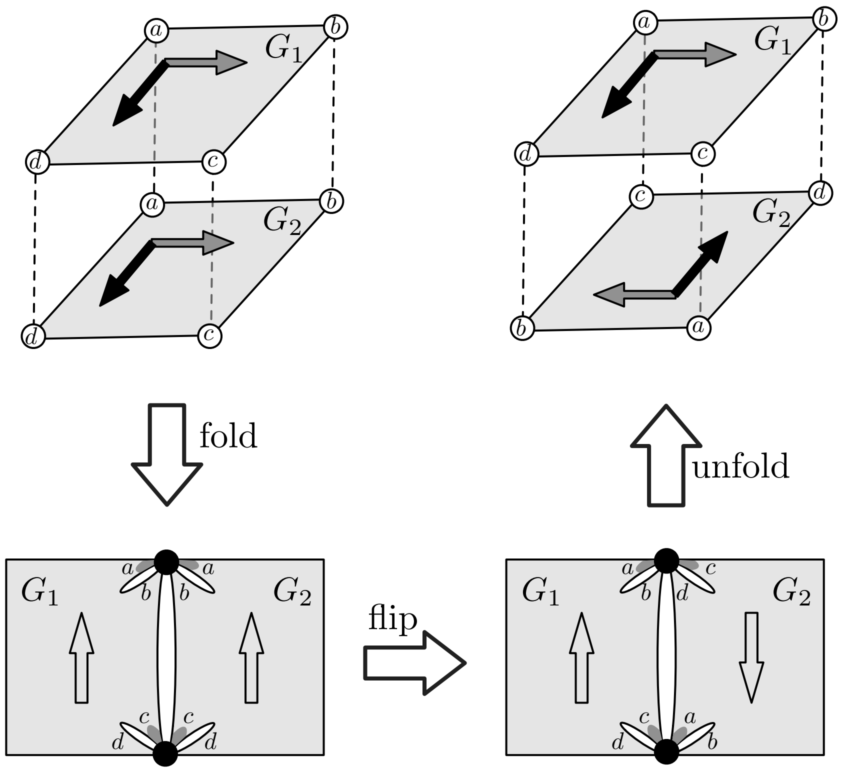


Figure 2.2: Inequivalent grafts.

(G,T) with $|T| = 4$ such that $E(G)$ can be partitioned into sets $X_1,\dots,X_k$ with the properties that $\mathscr{B}_G(X_i) = T$ and $G[X_i]$ is connected for every $i \in [k]$. Let (H,Γ) be obtained from (G,T) by folding. Let H' be any graph obtained from H by a Whitney-flip on X_i, for some $i \in [k-1]$. Then $\mathscr{B}_{H'}(X_1)$ is a blocking pair of (H',Γ'), where $\Gamma' = \Gamma$. It follows that (H,T) and (H',T') are inequivalent representations of the same even cut matroid. We conclude that an even cut matroid may have an arbitrary number of inequivalent representations.

2.5 Lifts and projections

Let N and M be matroids where $E(N) = E(M)$. We say that N is a *lift* of M if for some matroid M', where $E(M') = E(M) \cup \{\Omega\}$, $M = M'/\Omega$ and $N = M' \setminus \Omega$. If N is a lift of M then M is a *projection* of N. Lifts and projections were introduced in [10]. Every even cycle matroid M is a lift of a cycle matroid; indeed, for any representation (G,Σ) of M we may construct (G',Σ') by adding an odd loop Ω. Then $\text{ecycle}(G',\Sigma')/\Omega$ is a cycle matroid. Every even cut matroid is a lift of a cut matroid. In fact, suppose $M = \text{ecut}(G,T)$ and (G',T') is obtained from (G,T) by adding an odd bridge Ω. Then $\text{ecut}(G',T')/\Omega = \text{cut}(G' \setminus \Omega)$ is a cut matroid. The following result shows that degenerate even cycle matroids are projections of cycle matroids.

Remark 2.9. *Let (H,Γ) be a signed graph.*

(1) If (H,Γ) has a blocking vertex, then $\text{ecycle}(H,\Gamma)$ *is a cycle matroid.*

(2) If (H,Γ) has a blocking pair, then $\text{ecycle}(H,\Gamma)$ *is a projection of a cycle matroid.*

Proof. **(1)** Suppose that $\Gamma \subseteq \delta_H(s) \cup \text{loop}(H)$ for some vertex s of H. Let G be obtained from H by splitting s according to Γ. Then $\text{cycle}(G) = \text{ecycle}(H,\Gamma)$. **(2)** Suppose that $\Gamma \subseteq \delta_H(s) \cup \delta_H(t) \cup \text{loop}(H)$ for a pair of vertices s,t of H. Let G be obtained from H by splitting s into s_1,s_2 according to $\delta_H(s) \cap \Gamma$ and by adding an edge $\Omega = (s_1,s_2)$. Let $M' = \text{ecycle}(G,\Gamma)$. Then by construction $(G,\Gamma)/\Omega = (H,\Gamma)$, hence $M'/\Omega = M$. Moreover, $\text{ecycle}(G,\Gamma) \setminus \Omega = M' \setminus \Omega$ is a cycle matroid, as t is a blocking vertex of $(G,\Gamma) \setminus \Omega$. □

Next we show that degenerate even cut matroids are projections of cut matroids.

Remark 2.10. *Let (G,T) be a graft.*

1. *If $|T| = 2$, then* $\text{ecut}(G,T)$ *is a cut matroid.*
2. *If $|T| = 4$, then* $\text{ecut}(G,T)$ *is a projection of a cut matroid.*

Proof. **(1)** Suppose that (G,T) is a graft with $T = \{u,v\}$. Let H be obtained from G by identifying u and v. Then $\text{cut}(H) = \text{ecut}(G,T)$. **(2)** Suppose that (G,T) is a graft with $T = \{a,b,c,d\}$. Let $M := \text{ecut}(G,T)$. Let H be obtained from G by adding an edge Ω

with endpoints a and b. Let $M' := \mathrm{ecut}(H,T)$. Then, by construction, $(H,T) \setminus \Omega = (G,T)$, hence $M'/\Omega = M$. Let $N := M' \setminus \Omega$. By construction, $N = \mathrm{ecut}((H,T)/\Omega)$. As the graft $(H,T)/\Omega$ has exactly two terminals, by (1) N is a cut matroid and M is a projection of N. □

Chapter 3

Pairing isomorphism problems

In this chapter we study the relation between even cycle and even cut matroids. We present results relating pairs of signed graph siblings to pairs of graft siblings. These results are proved in the more general setting of signed matroids.

3.1 Results

The main result of this chapter shows how the Isomorphism Problems for even cycle and even cut matroids are related.

Theorem 3.1. *Let G_1 and G_2 be graphs such that* $\mathrm{cycle}(G_1) \neq \mathrm{cycle}(G_2)$.

(1) Suppose there exists a pair $\Sigma_1, \Sigma_2 \subseteq E(G_1)$ such that $\mathrm{ecycle}(G_1,\Sigma_1) = \mathrm{ecycle}(G_2,\Sigma_2)$. *For $i = 1,2$, if (G_i,Σ_i) is bipartite define $C_i := \emptyset$, otherwise let C_i be a Σ_i-odd cycle of G_i. Let $T_i := V_{odd}(G_i[C_{3-i}])$. Then* $\mathrm{ecut}(G_1,T_1) = \mathrm{ecut}(G_2,T_2)$.

(2) Suppose there exists a pair $T_1 \subseteq V(G_1), T_2 \subseteq V(G_2)$ (where $|T_1|,|T_2|$ are even) such that $\mathrm{ecut}(G_1,T_1) = \mathrm{ecut}(G_2,T_2)$. *For $i = 1,2$, if $T_i = \emptyset$ let $\Sigma_{3-i} = \emptyset$, otherwise let $t_i \in T_i$ and $\Sigma_{3-i} := \delta_{G_i}(t_i)$. Then* $\mathrm{ecycle}\ (G_1,\Sigma_1) = \mathrm{ecycle}(G_2,\Sigma_2)$.

We illustrate this result with an example. Consider the signed graphs (G_i,Σ_i), for $i = 1,2,3$, in Figure 3.1. The signed graph (G_2,Σ_2) is obtained from (G_1,Σ_1) by a Lovász-flip on vertices b, f; (G_3,Σ_3) is obtained from (G_2,Σ_2) first by a signature exchange $\Sigma_3 :=$

$\Sigma_2 \triangle \delta_{G_2}(b)$, then by moving loop 9 to vertex a (this is a Whitney-flip) and finally by performing a Lovász-flip on vertices a, f. As Lovász-flips, Whitney-flips and signature exchanges preserve even cycles, $\text{ecycle}(G_1, \Sigma_1) = \text{ecycle}(G_3, \Sigma_3)$.

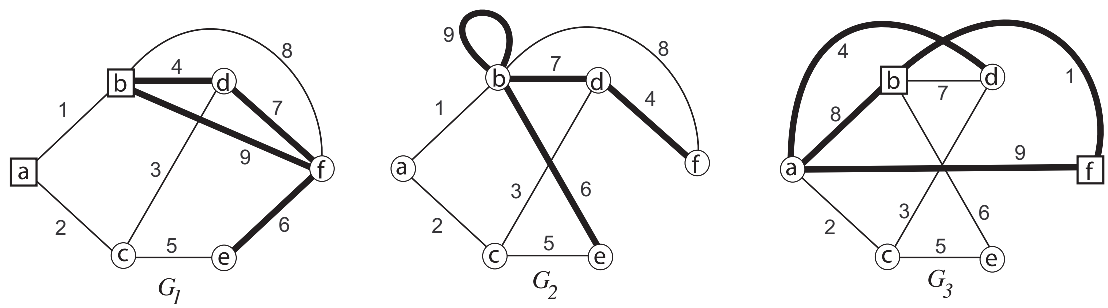


Figure 3.1: Bold edges of G_i are in Σ_i, square vertices of G_1, G_3 are T_1, T_3.

In the same figure, consider the grafts (G_1, T_1) and (G_3, T_3) where $T_1 = \{a, b\}$ and $T_3 = \{b, f\}$. These grafts are obtained using the construction in Theorem 3.1(1). Pick an odd cycle $\{4, 7, 9\}$ of (G_1, Σ_1) and let T_3 be the set of vertices of odd degree in $G_3[\{4, 7, 9\}]$. Pick an odd cycle $\{1, 8, 9\}$ of (G_3, Σ_3) and let T_1 be the set of vertices of odd degree in $G_1[\{1, 8, 9\}]$. Then, $\text{ecut}(G_1, T_1) = \text{ecut}(G_3, T_3)$. We can also consider the reverse construction, as in Theorem 3.1(2). Pick $a \in T_1$, then $\delta_{G_1}(a) = \{1, 2\}$. Now $\{1, 2\} \triangle \Sigma_3$ is a cut of G_3, hence $\{1, 2\}$ is a signature of (G_3, Σ_3). Similarly, pick $b \in T_3$, then $\delta_{G_3}(b) = \{1, 6, 7, 8\}$ is a signature of (G_1, Σ_1).

Recall the definition of matching signature pairs and matching terminal pairs given in Section 1.2.1. In Section 3.2 we show that, given siblings G_1, G_2, there exists exactly one matching signature pair (up to signature exchange) and exactly one matching terminal pair. Note that for uniqueness the condition that $\text{cycle}(G_1) \neq \text{cycle}(G_2)$ is necessary, as otherwise any pairs $\Sigma_1 = \Sigma_2$ will yield the same even cycles.

Suppose that we can solve the Isomorphism Problem for even cycle matroids. Does Theorem 3.1 then provide us with a solution to the Isomorphism Problem for even cut matroids? Consider $(G, T), (G', T')$ such that $\text{ecut}(G, T) = \text{ecut}(G', T')$. Theorem 3.1 implies that there exists Σ, Σ' such that $\text{ecycle}(G, \Sigma) = \text{ecycle}(G', \Sigma')$. Suppose that we can transform (G, Σ) into (G', Σ') by a sequence of operations that preserve even cycles at each step. Can we transform (G, T) into (G', T') by a sequence of operations that preserve even cuts at each step? We have a sequence of signed graphs (G_i, Σ_i) for $i = 1, \ldots, n$ which have all the same even cycles and where $(G,\Sigma) = (G_1,\Sigma_1)$ and $(G',\Sigma') = (G_n,\Sigma_n)$. Can we find $T_1,\ldots,T_n$ such that $\text{ecut}(G_i,T_i) = \text{ecut}(G_j,T_j)$ for all $i,j \in [n]$? The example in Figure 3.1 shows that this is not always the case. The graphs G_1 and G_3 determine T_1 and T_3 uniquely. But it is not possible to find a set T_2 such that $\text{ecut}(G_1,T_1) = \text{ecut}(G_2,T_2)$, because the edge 9 is a loop in G_2 but is contained in the T_1-even cut $\{6,7,8,9\}$ of G_1.

This leads to the following definition: a set of graphs $\{G_1,\ldots,G_n\}$ is *harmonious* if for all $i,j \in [n], i \neq j$, $\text{cycle}(G_i) \neq \text{cycle}(G_j)$ and there exist $\Sigma_1,\ldots,\Sigma_n$ and $T_1,\ldots,T_n$ such that $\text{ecycle}(G_i,\Sigma_i) = \text{ecycle}(G_j,\Sigma_j)$ and $\text{ecut}(G_i,T_i) = \text{ecut}(G_j,T_j)$ for all $i,j \in [n]$. For instance the set $\{G_1,G_2,G_3\}$ in Figure 3.1 is not harmonious. In fact, no large set of graphs is harmonious.

Theorem 3.2. *Suppose that $\{G_1,\ldots,G_n\}$ is a harmonious set of graphs. Then $n \leq 3$.*

The bound of 3 is best possible. A construction that yields a harmonious set of 3 graphs $\{G_1,G_2,G_3\}$ is as follows: let (G_1,Σ_1) be any signed graph with vertices u,v where $\Sigma_1 \subseteq \delta_{G_1}(u) \cup \delta_{G_1}(v)$. Let (G_2,Σ_2) be obtained from (G_1,Σ_1) by a Lovász-flip on u,v, and let (G_3,Σ_3) be obtained from $(G_1,\Sigma_1 \triangle \delta_{G_1}(u))$ by a Lovász-flip on u,v. Finally, let $T_1 = \{u,v\}$ and for $i = 2,3$, let T_i be the vertices in G_i corresponding to u,v.

Theorem 3.1 and 3.2 are proved in the next section in the more general context of signed matroids.

In Chapter 4 we define two special classes of siblings. We show that, for every pair of siblings (G_1,Σ_1), (G_2,Σ_2) in one of the two classes, there exist equivalent signed graphs (G_1',Σ_1') and (G_2',Σ_2') respectively such that (G_1',Σ_1') and (G_2',Σ_2') are related by exactly one of a set of operations that we define. Thus the Isomorphism Problem is solved for these classes of even cycle matroids and, by the results in this chapter, also for the corresponding even cut matroids. This is in contrast with the discussion above about the three signed graphs in Figure 3.1.

3.2 Generalization to signed matroids

In this section we will generalize to matroids the concepts introduced in the previous section. Given a signed matroid (M,Σ), we say that Σ' is a *signature* of (M,Σ) if $\text{ecycle}(M,\Sigma) = \text{ecycle}(M,\Sigma')$. It can be readily checked that Σ' is a signature of (M,Σ) if and only if

$\Sigma' = \Sigma \triangle D$ for some co-cycle D of M. The operation that consists of replacing a signature of a signed matroid by another signature is called *signature exchange*. When $M = \text{cycle}(G)$ for some graph G, then $\text{ecycle}(M,\Sigma) = \text{ecycle}(G,\Sigma)$ and the aforementioned definitions for signed matroids correspond to the definitions for signed graphs.

3.2.1 Pairs

Let $(M_1,\Sigma_1),(M_2,\Sigma_2)$ be signed matroids such that $\text{ecycle}(M_1,\Sigma_1) = \text{ecycle}(M_2,\Sigma_2)$. A cycle (respectively co-cycle) of M_1 is *preserved* if it is a cycle (respectively co-cycle) of M_2. A signature of (M_1,Σ_1) is *preserved* if it is a signature of (M_2,Σ_2). The main result of this section is the following.

Theorem 3.3. *Suppose that* $\text{ecycle}(M_1,\Sigma_1) = \text{ecycle}(M_2,\Sigma_2)$. *Then there exists* $\Gamma_1,\Gamma_2 \subseteq E(M_1)$ *such that* $\text{ecycle}(M_1^*,\Gamma_1) = \text{ecycle}(M_2^*,\Gamma_2)$ *and, for* $i = 1,2$, *the* Γ_i*-even co-cycles of* M_i *are exactly the preserved co-cycles of* M_i. *Moreover, if* (M_i,Σ_i) *is bipartite, then so is* (M_{3-i}^*,Γ_{3-i}).

The proof requires a number of preliminaries. Given a signed matroid (M,Σ), the co-cycles of $\text{ecycle}(M,\Sigma)$ are the sets that intersect every Σ-even cycle of M with even cardinality. Thus we have the following.

Remark 3.4. *The co-cycles of* $\text{ecycle}(M,\Sigma)$ *are the co-cycles of* M *and the signatures of* (M,Σ),

which in turns implies the following.

Remark 3.5. *Suppose that* $\text{ecycle}(M_1,\Sigma_1) = \text{ecycle}(M_2,\Sigma_2)$.

(1) If B *is a non-preserved co-cycle of* M_1, *then* B *is a signature of* (M_2,Σ_2).

(2) If B *is a non-preserved signature of* (M_1,Σ_1), *then* B *is a co-cycle of* M_2.

Proof. For both (1) and (2), Remark 3.4 implies that B is a co-cycle of $\text{ecycle}(M_1,\Sigma_1)$, hence B is a co-cycle of $\text{ecycle}(M_2,\Sigma_2)$. Remark 3.4 implies that B is either a co-cycle of M_2 or a signature of (M_2,Σ_2). For (1), B is not a co-cycle of M_2. For (2), B is not a signature of (M_2,Σ_2). □

Lemma 3.6. *Suppose that* $\text{ecycle}(M_1,\Sigma_1) = \text{ecycle}(M_2,\Sigma_2)$*, for signed matroids* (M_1,Σ_1)*,* (M_2,Σ_2)*. For* $i = 1,2$*, there exists* $\Gamma_i \subseteq E(M_i)$ *such that, for every co-cycle* D *of* M_i*,* D *is preserved if and only if it is* Γ_i*-even. Moreover, if* (M_{3-i},Σ_{3-i}) *is bipartite, then* $\Gamma_i = \emptyset$*.*

Proof. Fix $i \in \{1,2\}$. Let B be a co-basis of M_i. For every $e \notin B$, let D_e denote the unique co-circuit in $B \cup \{e\}$ (these are the fundamental co-circuits of M_i). Then we let $e \in \Gamma_i$ if and only if D_e is non-preserved. Consider now an arbitrary co-cycle D of M_i. D may be expressed as the symmetric difference of a set of distinct fundamental co-circuits D_e, where, say, s of these are non-preserved. By construction, $|D \cap \Gamma_i| = s$. By Remark 3.5(1), non-preserved co-cycles of M_i are signatures of (M_{3-i},Σ_{3-i}). Moreover, the symmetric difference of an even (respectively odd) number of signatures of (M_{3-i},Σ_{3-i}) is a co-cycle of M_{3-i} (respectively a signature of (M_{3-i},Σ_{3-i})). It follows that D is a co-cycle of M_{3-i} when s is even and is a signature of (M_{3-i},Σ_{3-i}) when s is odd. If (M_{3-i},Σ_{3-i}) is non-bipartite then signatures of (M_{3-i},Σ_{3-i}) are not co-cycles of M_{3-i} and the result follows. If (M_{3-i},Σ_{3-i}) is bipartite then every co-cycle of M_i is preserved. As a consequence, $\Gamma_i = \emptyset$ and the result follows as well. □

Proof of Theorem 3.3. Lemma 3.6 implies that, for $i = 1,2$, there exists $\Gamma_i \subseteq E(M_1)$ such that the preserved co-cycles of M_i are exactly the Γ_i-even co-cycles of M_i. Therefore $\text{ecycle}(M_1^*,\Gamma_1) = \text{ecycle}(M_2^*,\Gamma_2)$. Again by Lemma 3.6, if (M_i,Σ_i) is bipartite, then $\Gamma_{3-i} = \emptyset$, so (M_{3-i}^*,Γ_{3-i}) is bipartite. □

3.2.2 Uniqueness

The main observation in this section is the following.

Proposition 3.7. *Suppose that* (M_1,Σ_1) *and* (M_2,Σ_2) *are signed matroids such that* $M_1 \neq M_2$ *and* $\text{ecycle}(M_1,\Sigma_1) = \text{ecycle}(M_2,\Sigma_2)$*. For* $i = 1,2$*, the* Σ_i*-even cycles of* M_i *are exactly the preserved cycles of* M_i*. In particular,* Σ_1 *and* Σ_2 *are unique up to signature exchanges.*

Proposition 3.7 follows directly from the following remark.

Remark 3.8. *Suppose that* $\text{ecycle}(M_1,\Sigma_1) = \text{ecycle}(M_2,\Sigma_2)$*. If* C *is a* Σ_1*-odd cycle of* M_1 *which is preserved, then* $\text{cycle}(M_1) = \text{cycle}(M_2)$*.*

Proof. Every odd cycle C' of (M_1,Σ_1) is of the form $C' := C \triangle B$, where B is an even cycle of (M_1,Σ_1). As B is an even cycle of (M_2,Σ_2), C' is a cycle of (M_2,Σ_2). Hence, $\text{cycle}(M_1) \subseteq \text{cycle}(M_2)$. As C is a cycle of M_1 and M_2 and C is Σ_1-odd, C is also a preserved Σ_2-odd cycle of M_2. By symmetry, the reverse inclusion holds as well. □

3.2.3 Odd cycles and signatures

Remark 3.9. *Suppose that* $\text{ecycle}(M_1,\Sigma_1) = \text{ecycle}(M_2,\Sigma_2)$*, for signed matroids* (M_1,Σ_1)*,* (M_2,Σ_2)*, where* $M_1 \neq M_2$*. If* (M_1,Σ_1) *is bipartite, let* $\Sigma := \emptyset$*. Otherwise there exists a non-preserved co-cycle* D *of* M_2*; let* $\Sigma := D$*. Then* Σ *is a signature of* (M_1,Σ_1)*.*

Proof. We may assume that (M_1,Σ_1) is non-bipartite. By Theorem 3.3, there exists Γ_1,Γ_2 such that, for $i = 1,2$, the Γ_i-even co-cycles of M_i are exactly the preserved co-cycles of M_i. If every co-cycle of M_2 is preserved, then (M_2^*,Γ_2) is bipartite. It follows, from Theorem 3.3 applied to (M_1^*,Γ_1) and (M_2^*,Γ_2), and from Proposition 3.7, that (M_1,Σ_1) is bipartite, a contradiction. Hence, some co-cycle D of M_2 is non-preserved. The result then follows by Remark 3.5(1). □

The signature Σ of (M_1,Σ_1) in Remark 3.9 is called an M_2*-standard signature*. When there is no ambiguity we omit the prefix M_2.

Theorem 3.10. *Let* (M_1,Σ_1)*,* (M_2,Σ_2) *be signed matroids such that* $M_1 \neq M_2$ *and let* $\Gamma_1 \subseteq E(M_1)$*,* $\Gamma_2 \subseteq E(M_2)$*. Assume that* $\text{ecycle}(M_1,\Sigma_1) = \text{ecycle}(M_2,\Sigma_2)$ *and* $\text{ecycle}(M_1^*,\Gamma_1) = \text{ecycle}(M_2^*,\Gamma_2)$*. If, for* $i = 1,2$*,* Σ_i *is an* M_{3-i}*-standard signature, then for any* $D \subseteq E(M_1)$ *the following hold:*

(1) Suppose that (M_1,Σ_1) *is non-bipartite. Then*
D is a Σ_1-odd cycle of M_1 if and only if D is a Σ_2-even signature of (M_2^*,Γ_2).

(2) Suppose that $(M_1,\Sigma_1),(M_2,\Sigma_2)$ *are non-bipartite. Then*
D is a Σ_1-odd signature of (M_1^*,Γ_1) if and only if D is a Σ_2-odd signature of (M_2^*,Γ_2).

Proof. We begin with the proof of (1). Let D be a Σ_1-odd cycle of M_1. Remark 3.8 implies that D is non-preserved. Remark 3.5(1) implies that D is a signature of (M_2^*,Γ_2). If $\Sigma_2 = \emptyset$, then D is trivially Σ_2-even. Otherwise, as Σ_2 is a standard signature, Σ_2 is a co-cycle of M_1.

Since M_1 is a binary matroid, cycles and co-cycles have an even intersection, hence D is Σ_2-even. Conversely, let D be a Σ_2-even signature of (M_2^*, Γ_2). As (M_1, Σ_1) is non-bipartite, there exists a Σ_1-odd cycle C of M_1. By the first part of the proof, C is a Σ_2-even signature of (M_2^*, Γ_2). Therefore $C \triangle D$ is a Σ_2-even cycle of M_2, hence a Σ_1-even cycle of M_1. Thus D is a Σ_1-odd cycle of M_1. We now proceed with the proof of (2). Let D be a Σ_1-odd signature of (M_1^*, Γ_1). Moreover, let C be a Σ_1-odd cycle of M_1. Then $D \triangle C$ is a Σ_1-even signature of (M_1^*, Γ_1). By part (1) and symmetry between M_1 and M_2, $D \triangle C$ is a Σ_2-odd cycle of M_2. Also, by part (1), C is a Σ_2-even signature of (M_2^*, Γ_2). Hence $D = (D \triangle C) \triangle C$ is a Σ_2-odd signature of (M_2^*, Γ_2). Hence every Σ_1-odd signature of (M_1^*, Γ_1) is a Σ_2-odd signature of (M_2^*, Γ_2). The other inclusion follows by symmetry between M_1 and M_2. □

3.2.4 Harmonious sets

A set of matroids $\{M_1, \ldots, M_n\}$ is *harmonious* if $M_i \neq M_j$, for all distinct $i, j \in [n]$, and there exist signatures $\Sigma_1, \ldots, \Sigma_n$ and $\Gamma_1, \ldots, \Gamma_n$ such that $\text{ecycle}(M_i, \Sigma_i) = \text{ecycle}(M_j, \Sigma_j)$ and $\text{ecycle}(M_i^*, \Gamma_i) = \text{ecycle}(M_j^*, \Gamma_j)$, for all $i, j \in [n]$. An example of three matroids forming a harmonious set was given at the end of Section 3.1.

Theorem 3.11. *Suppose that $\{M_1, \ldots, M_n\}$ is a harmonious set of matroids. Then $n \leq 3$.*

Proof. Suppose for a contradiction that there exists a harmonious set $\{M_1, \ldots, M_4\}$. Note that, by Proposition 3.7, $\Sigma_1, \ldots, \Sigma_4, \Gamma_1, \ldots, \Gamma_4$ are unique up to resigning. First suppose that (M_k, Σ_k) is bipartite for some $k \in [4]$. Then, by Theorem 3.3, (M_i^*, Γ_i) is bipartite for every $i \in [4] - \{k\}$. Hence, for $i, j \in [4] - \{k\}$, $i \neq j$, the matroids M_i, M_j have the same co-cycles, hence $M_i = M_j$, a contradiction. Therefore, for every $i \in [4]$, (M_i, Σ_i) is non-bipartite and by duality (M_i^*, Γ_i) is non-bipartite as well. By Theorem 3.3, a co-cycle C of M_4 is non-preserved if and only if it is Γ_4-odd. We fix C to be an odd co-cycle of (M_4, Γ_4), and conclude that C is non-preserved for M_i, for all $i \in [3]$. By definition, C is an M_4-standard signature for (M_i, Σ_i), for all $i \in [3]$.

For every $i \in [3]$, let C_i be a C-odd signature of (M_i^*, Γ_i). Note that such signatures exist because (M_i, Σ_i) is non-bipartite, hence an odd circuit of (M_i, Σ_i) can be added to the signature of (M_i^*, Γ_i) to change its parity. By Theorem 3.10(2), C_i is a signature of (M_4^*, Γ_4) for every $i \in [3]$. The symmetric difference of two signatures of (M_4^*, Γ_4) is a cycle of M_4. Moreover, for some $j, k \in [3]$, $j \neq k$, C_j and C_k have the same parity with respect to Σ_4.

Hence $D := C_j \triangle C_k$ is a Σ_4-even cycle of M_4, so D is a Σ_i-even cycle of M_i for every $i \in [4]$. Therefore $C_j = D \triangle C_k$ is a C-odd signature of both $(M_j^*, \Gamma_j), (M_k^*, \Gamma_k)$. Now let C' be a Σ_4-odd cycle of M_4. By Theorem 3.10(1), C' is a C-even signature of (M_j^*, Γ_j) and (M_k^*, Γ_k). Therefore $C_j \triangle C'$ is a C-odd cycle of both M_j and M_k. Hence, by Remark 3.8, $M_j = M_k$, a contradiction. □

3.3 Applications to signed graphs and grafts

In this section we show how the results for signed matroids apply to signed graphs and grafts.

Remark 3.12. *Let (G,T) be a graft, let Γ be a T-join of G and let $M = cut(G)$.*

(1) A cut of G is T-even if and only if it is Γ-even. In particular, $\text{ecut}(G,T) = \text{ecycle}(M,\Gamma)$.

(2) A set of edges is a T-join of G if and only if it is a signature of (M,Γ).

Proof of Theorem 3.1. We begin with the proof of (1). We omit the cases when (G_1,Σ_1) or (G_2,Σ_2) is bipartite. For $i = 1,2$, let $M_i := \text{cycle}(G_i)$. By Theorem 3.3 there exists Γ_1, Γ_2 such that $\text{ecycle}(M_1^*, \Gamma_1) = \text{ecycle}(M_2^*, \Gamma_2)$. Since B_i is an odd cycle of (M_i, Σ_i) it is non-preserved. It follows from Remark 3.5(1) that B_i is a signature of $(M_{3-i}^*, \Gamma_{3-i})$. Hence, $\text{ecycle}(M_1^*, B_2) = \text{ecycle}(M_2^*, B_1)$. Let T_i be the vertices of odd degree in $G_i[B_{3-i}]$. Remark 3.12(1) implies that $\text{ecut}(G_1,T_1) = \text{ecut}(G_2,T_2)$.

We proceed with the proof of (2). We omit the cases when $T_1 = \emptyset$ or $T_2 = \emptyset$. For $i = 1,2$ let $M_i = \text{cut}(G_i)$ and let Γ_i be a T_i-join of G_i. Remark 3.12(1) implies that $\text{ecycle}(M_1,\Gamma_1) = \text{ecycle}(M_2,\Gamma_2)$. By Theorem 3.3 there exist $\tilde{\Sigma}_1, \tilde{\Sigma}_2$ such that $\text{ecycle}(M_1^*, \tilde{\Sigma}_1) = \text{ecycle}(M_2^*, \tilde{\Sigma}_2)$. As $\Sigma_i = \delta_{G_{3-i}}(t_{3-i})$ is a T_{3-i}-odd cut of G_{3-i}, by Remark 3.12(1), Σ_i is a Γ_{3-i}-odd cycle of (M_{3-i}, Γ_{3-i}). It follows from Remark 3.5(1) that Σ_i is a signature of $(M_i^*, \tilde{\Sigma}_i)$. We conclude that $\text{ecycle}(G_1,\Sigma_1) = \text{ecycle}(M_1^*,\Sigma_1) = \text{ecycle}(M_2^*,\Sigma_2) = \text{ecycle}(G_2,\Sigma_2)$. □

Let G_1 and G_2 be inequivalent graphs. Suppose that $\text{ecycle}(G_1,\Sigma_1) = \text{ecycle}(G_2,\Sigma_2)$ and $\text{ecut}(G_1,T_1) = \text{ecut}(G_2,T_2)$. If (G_1,Σ_1) is bipartite, let $\Sigma := \emptyset$. Otherwise, by Remark 3.9, there exists a T_2-odd cut D of (G_2,T_2); let $\Sigma := D$. Then Σ is a standard signature of (G_1,Σ_1). Given a signature $\tilde{\Sigma}_i$ of (G_i,Σ_i), $\Sigma_i \triangle \tilde{\Sigma}_i$ is a cut D of G_i. We say that $\tilde{\Sigma}_i$ is *T_i-even* (respectively *T_i-odd*) if D is a T_i-even (respectively T_i-odd) cut.

Proposition 3.13. *Let G_1 and G_2 be inequivalent graphs. Suppose that* $\text{ecycle}(G_1,\Sigma_1) = \text{ecycle}(G_2,\Sigma_2)$ *and* $\text{ecut}(G_1,T_1) = \text{ecut}(G_2,T_2)$. *If Σ_1,Σ_2 are standard signatures then the following hold.*

(1) Suppose that (G_1,Σ_1) is non-bipartite. Then
D is a Σ_1-odd cycle of G_1 if and only if D is a Σ_2-even T_2-join of G_2;

(2) Suppose that $(G_1,\Sigma_1),(G_2,\Sigma_2)$ are non-bipartite. Then
D is a Σ_1-odd T_1-join of G_1 if and only if D is a Σ_2-odd T_2-join of G_2;

(3) Suppose that $T_1 \neq \emptyset$. Then
D is a T_1-odd cut of G_1 if and only if D is T_2-even signature of (G_2,Σ_2);

(4) Suppose that $T_1,T_2 \neq \emptyset$. Then
D is a T_1-odd signature of (G_1,Σ_1) if and only if D is T_2-odd signature of (G_2,Σ_2).

We illustrate Proposition 3.13 on the example in Figure 3.1. We have that $\Sigma_1' := \delta_{G_3}(f) = \{1,9\}$ is a standard signature of (G_1,Σ_1) and $\Sigma_3' := \delta_{G_1}(a) = \{1,2\}$ is a standard signature of (G_3,Σ_3). Then the odd cycle $\{4,7,9\}$ of (G_1,Σ_1') is a Σ_3'-even T_3-join of G_3. The set $\{1\}$ is a Σ_1'-odd T_1-join of G_1 and a Σ_3'-odd T_3-join of G_3. Moreover $\{1,3,5\} = \delta_{G_1}(\{a,c\})$ is a T_1-odd cut of G_1. As $\{1,3,5\}\triangle\Sigma_3' = \{2,3,5\} = \delta_{G_3}(c)$, $\{1,3,5\}$ is a T_3-even signature of (G_3,Σ_3'). Finally, $\{2,9\}$ is a T_1-odd signature of (G_1,Σ_1') which is also a T_3-odd signature of (G_3,Σ_3').

Proof of Proposition 3.13. We prove parts (1) and (3) only, as statements (2) and (4) follow similarly from Theorem 3.10(2). We begin with the proof of (1). For $i = 1,2$, let $M_i := \text{cycle}(G_i)$. Clearly, D is a cycle of G_1 if and only if D is a cycle of M_1. Let Γ_2 be a T_2-join of G_2. Remark 3.12(2) implies that D is a T_2-join of G_2 if and only if D is a signature of (M_2^*,Γ_2). The result now follows from Theorem 3.10(1). We proceed with the proof of (3). For $i = 1,2$, let $M_i := \text{cut}(G_i)$ and let Γ_i be a T_i-join of G_i. Remark 3.12(1) implies that D is a T_1-odd cut of G_1 if and only if D is a Γ_1-odd cycle of M_1. Since Σ_2 is a standard signature of (M_2^*,Σ_2), Σ_2 is a Γ_1-odd cycle of M_1. It follows from Theorem 3.10(1) that Σ_2 is Γ_2-even. D is a T_2-even signature of (G_2,Σ_2) if and only if D is a signature of (M_2^*,Σ_2) such that $\Sigma_2\triangle D$ is T_2-even. Equivalently, by Remark 3.12(1), $\Sigma_2\triangle D$ is Γ_2-even. As Σ_2 is Γ_2-even, this occurs if and only if D is Γ_2-even. The result now follows from Theorem 3.10(1). □

3.4 A matroid operation

Consider a graft (H,T) with $|T|=4$. Let $T=\{t_1,t_2,t_3,t_4\}$. Suppose that H has a 2-separation X such that $t_1,t_2 \in V_H(X)$ and $t_3,t_4 \in V_H(\bar{X})$. Construct a graph H' from $H[X]$ and $H[\bar{X}]$ by identifying vertex t_1 with t_3 and identifying vertex t_2 with t_4. Let C be a circuit of H where both $C \cap X$ and $C \cap \bar{X}$ are non-empty. Define $T' := V_{odd}(H'[C])$. Then we say that (H',T') is obtained from (H,T) by a *simple shift* on X with pairing t_1,t_3 and t_2,t_4. In this section we show how Whitney-flips, Lovázs-flips and simple shifts all arise from the same matroid construction. We require the following observation.

Lemma 3.14. *Let M be a matroid and let a,b,c,d denote distinct elements of M. Suppose that $\{a,b,c,d\}$ is both a cycle and a co-cycle of M. Then $M/\{a,b\} \setminus \{c,d\} = M \setminus \{a,b\}/\{c,d\}$.*

Proof. Let $M_1 := M/\{a,b\} \setminus \{c,d\}$ and let $M_2 := M \setminus \{a,b\}/\{c,d\}$. We want to show that the cycles of M_1 are exactly the cycles of M_2. By symmetry between M_1 and M_2, it suffices to show that every cycle of M_1 is a cycle of M_2. Let C be any cycle of M_1. Then there exists a cycle D of M such that $C \subseteq D \subseteq C \cup \{a,b\}$. Since $\{a,b,c,d\}$ is a co-cycle of M and M is binary, $|D \cap \{a,b,c,d\}|$ is even. Hence, none of a,b are in D or both of a,b are in D. In the former case, $D=C$ and C is cycle of M_2 as required. In the latter case, $D = C \cup \{a,b\}$. Since $\{a,b,c,d\}$ is a cycle of M, $D \triangle \{a,b,c,d\} = C \cup \{c,d\}$ is a cycle of M. It follows that C is cycle of M_2. □

Consider a graph G which consists of components $G[X_1], G[X_2]$ for some partition X_1, X_2 of $E(G)$. For $i=1,2$, pick vertices $s_i, t_i \in G[X_i]$. Denote by C the set of edges $\{a,b,c,d\}$ where $a=(s_1,t_1), b=(s_2,t_2), c=(s_1,t_2), d=(s_2,t_1)$. Let H be the graph obtained from G by adding the edges in C. Since C is a circuit and a cut of H, it is a cycle and a co-cycle of cycle(H). Lemma 3.14 implies that cycle$(H) \setminus \{a,b\}/\{c,d\} =$ cycle$(H)/\{a,b\} \setminus \{c,d\}$. It follows that cycle$(H \setminus \{a,b\}/\{c,d\}) =$ cycle$(H/\{a,b\} \setminus \{c,d\})$. It can now be easily verified that $H \setminus \{a,b\}/\{c,d\}$ and $H/\{a,b\} \setminus \{c,d\}$ are related by a Whitney-flip and that any two graphs related by a single Whitney-flip can be obtained in that way. In particular, graphs related by Whitney-flips have the same set of cycles.

Consider a graph G. Pick vertices s_1,t_1,s_2,t_2 of G. Denote by C the set of edges $\{a,b,c,d\}$ where $a=(s_1,t_1), b=(s_2,t_2), c=(s_1,t_2), d=(s_2,t_1)$. Let H be the graph obtained from G by adding edges in C. Since C is an even cycle of the signed graph (H,C), it is a cycle of ecycle(H,C). Since C is a signature of (H,C), by Remark 2.1 it is a co-cycle of ecycle(H,C). Lemma 3.14 implies that ecycle$(H,C)\setminus\{a,b\}/\{c,d\} =$ ecycle$(H,C)/\{a,b\}\setminus\{c,d\}$. It follows that ecycle$\big((H,C)\setminus\{a,b\}/\{c,d\}\big) =$ ecycle$\big((H,C)$ $/\{a,b\}\setminus\{c,d\}\big)$. It can now be easily verified that $(H,C)\setminus\{a,b\}/\{c,d\}$ and $(H,C)/\{a,b\}\setminus$ $\{c,d\}$ are related by a Lovász-flip (and possibly signature exchanges) and that any two signed graphs related by a single Lovász-flip can be obtained in that way. In particular, graphs related by Lováz-flips have the same set of even cycles.

Consider a graph G which consists of components $G[X_1],G[X_2]$ for some partition X_1,X_2 of $E(G)$. For $i=1,2$, pick vertices $s_i,t_i,u_i,v_i \in V(G[X_i])$ (where these vertices are not necessarily all distinct). Denote by C the set of edges $\{a,b,c,d\}$ where $a=(s_1,s_2), b=(t_1,t_2), c=(u_1,u_2), d=(v_1,v_2)$. Let H be the graph obtained from G by adding the edges in C. Let $T := \{s_1,s_2,t_1,t_2,u_1,u_2,v_1,v_2\}$. Since C is an even cut of (H,T), it is a cycle of ecut(H,T). Moreover, C is a T-join of H. It follows from Remark 2.2 that C is a co-cycle of ecut(H,T). Lemma 3.14 implies that ecut$(H,T)\setminus\{a,b\}/\{c,d\} =$ ecut(H,T) $/\{a,b\}\setminus\{c,d\}$. It follows that ecut$\big((H,T)/\{a,b\}\setminus\{c,d\}\big) =$ ecut$\big((H,T)\setminus\{a,b\}/\{c,d\}\big)$. Hence, the two grafts $(H,T)\setminus\{a,b\}/\{c,d\}$ and $(H,T)/\{a,b\}\setminus\{c,d\}$ have the same even cuts. It can now be easily verified that $(H,T)\setminus\{a,b\}/\{c,d\}$ and $(H,T)/\{a,b\}\setminus\{c,d\}$ are related by a simple shift.

Chapter 4

Even cycle isomorphism

In this chapter we provide a partial answer to the Isomorphism Problem for even cycle matroids. First we present a result by Shih that solves the Isomorphism Problem for even cycle matroids which are graphic. We show, as a direct consequence of the results in Chapter 3, that this also solves the Isomorphism Problem for even cut matroids which are cographic. In Sections 4.2 and 4.3, we introduce two classes of even cycle siblings: Shih siblings and quad siblings. For each one of these classes, we provide a list of operations and we show that any two siblings in the class are related by Whitney-flips and exactly one of these operations, thus solving the Isomorphism Problem for these two classes; these results are presented in Secions 4.5 and 4.6. In Section 4.4 we present a conjecture for the Isomorphism Problem for even cycle matroids.

4.1 The graphic and cographic case

In this section we consider the Isomorphism Problem for graphic even cycle matroids. Suppose that for a signed graph (H,Γ), ecycle(H,Γ) is a graphic matroid. Hence there exists a graph G such that ecycle$(H,\Gamma) =$ cycle(G). If (H,Γ) does not contain any odd cycles, then cycle$(H) =$ cycle(G), the two graphs are equivalent and the Isomorphism Problem is solved. Thus we assume that (H,Γ) contains an odd cycle C. Every odd cycle of H can be generated by C and a basis for the even cycles of H. Thus cycle(G) is a subspace of cycle(H) and dim(cycle$(G)) =$ dim(cycle$(H)) - 1$. Moreover, if we know the structure of G and H, then we can determine the signature Γ by Theorem 3.1, as the signature pair is unique in this case. Therefore the following result (proved by Shih in his doctoral dissertation, see [30]) provides an answer to the Isomorphism Problem for graphic even cycle matroids.

Theorem 4.1. *Suppose G,H are graphs such that* $\text{cycle}(G)$ *is a subspace of* $\text{cycle}(H)$ *and* $dim(\text{cycle}(G)) = dim(\text{cycle}(H)) - 1$. *Then there exist graphs G',H', equivalent to G,H respectively, such that one of the following holds.*

(1) H' is obtained from G' by identifying two distinct vertices.

(2) There exist graphs $G_1,\ldots,G_4$ (not necessarily all non-empty) and distinct vertices $x_i,y_i,z_i \in V(G_i)$ such that G' is obtained by identifying x_i,y_{3-i},z_{2+i} to a vertex w_i, for $i = 1,\ldots,4$ (where the indices are modulo 4). Moreover, H' is obtained by identifying x_1,x_2,x_3,x_4 to a vertex x, identifying y_1,y_2,y_3,y_4 to a vertex y and identifying z_1,z_2,z_3,z_4 to a vertex z.

(3) There exist graphs $G_1,\ldots,G_k$, with $k \geq 3$, and distinct vertices $x_i,y_i,z_i \in V(G_i)$ for $i = 1,\ldots,k$, such that G' is obtained by identifying $z_1,\ldots,z_k$ to a vertex z and for $i = 1,\ldots,k$ identifying y_{i-1} and x_i to a vertex w_i (where the indices are modulo k). Moreover, H' is obtained by identifying y_{i-1}, z_i, x_{i+1} to a vertex w'_i, for $i = 1,\ldots,k$ (where the indices are modulo k).

An example of outcome (2) is given in Figure 4.1, where dotted lines represent vertices that are identified. G' is the graph on the left and H' the graph on the right. Let P_1 be a (y,z)-path in G_1 and P_2 be a (y,z) path in G_2. Then $P_1 \cup P_2$ is a cycle of H' and not a cycle of G'. Let $T := V_{odd}(G'[P_1 \cup P_2]) = \{w_1,w_2,w_3,w_4\}$. By Theorem 3.1, $\text{ecut}(G',T) = \text{cut}(H')$ and we may choose $\Gamma := \delta_{G'}(w_1)$ (shaded in the figure).

An example of outcome (3) is given in Figure 4.2, where the graph on the left is G' and the one on the right is H'. In this example we chose G_1 to be the graph with edges $1,2,3$ as in the figure. The arrows indicate how each piece is flipped. We may choose $\Gamma := \delta_{G'}(w_1)$ (shaded in the figure).

Note that Theorem 4.1 also answers the Isomorphism Problem for even cut matroids in the case that the even cut matroid represented by a graft (G,T) is cographic. In fact, by Theorem 3.1, we have $\text{cycle}(G) = \text{ecycle}(H,\Gamma)$ if and only if $\text{ecut}(G,T) = \text{cut}(H)$, for some set of terminals T of G.

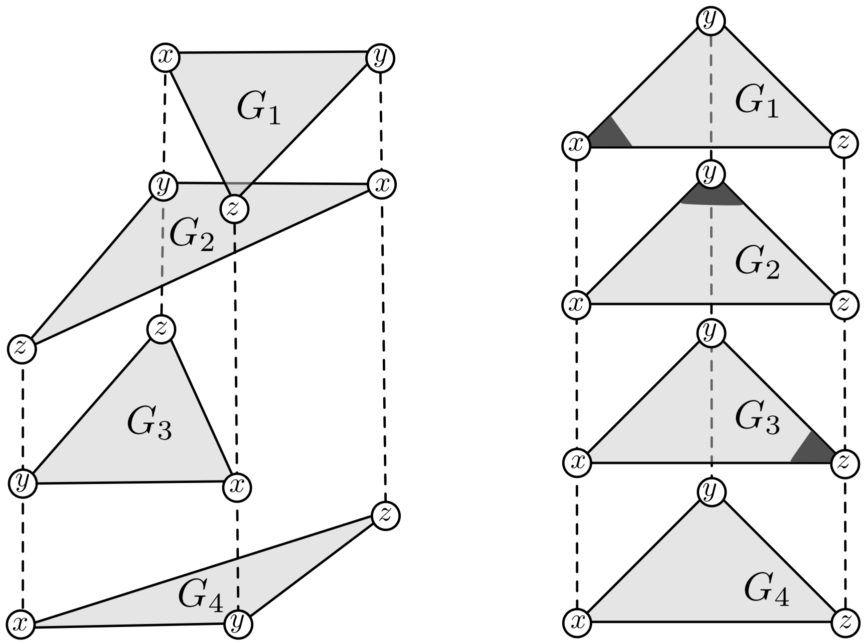


Figure 4.1: Shih operation 2.

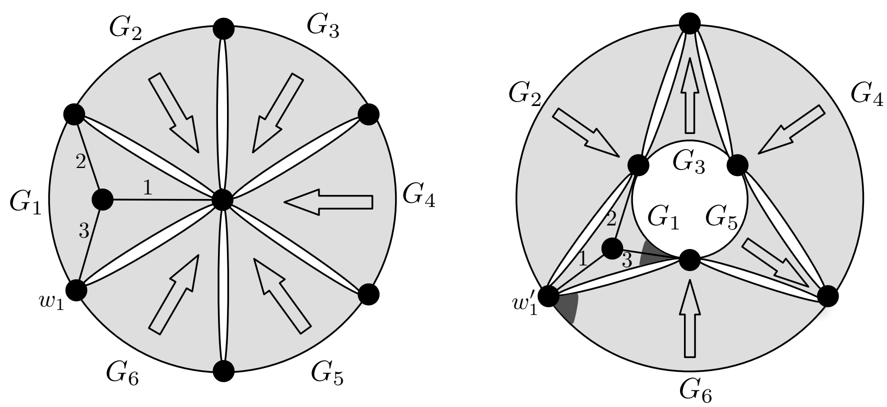


Figure 4.2: Shih operation 3.

As the Isomorphism Problem is solved for graphic matroids, we will mostly consider non-graphic matroids in this chapter. Moreover, the case in which the graphs are equivalent is trivial, hence we will only consider the Isomorphism Problem for representations that are not equivalent.

4.2 The class of Shih siblings

Let signed graphs (G_1,Σ_1),(G_2,Σ_2) be siblings and let T_1,T_2 be the matching terminal pair. If $|T_1| = 2$ or $|T_2| = 2$, we say that (G_1,Σ_1),(G_2,Σ_2) are *Shih siblings*.

Suppose $|T_2| = 2$ and let H_2 be the graph obtained from G_2 by identifying the two vertices in T_2. Then $\text{ecut}(G_2,T_2) = \text{cut}(H_2)$. It follows that $\text{ecut}(G_1,T_1) = \text{cut}(H_2)$. Therefore Theorem 4.1 gives a characterization of Shih siblings. For example, the graphs G_1 and H_2 may be as in Figure 4.2 and we may obtain G_2 from the graph on the right by splitting a vertex (for example, w'_1) into vertices v^+ and v^-. Then, up to resigning, $\Sigma_1 = \delta_{G_2}(v^+)$ and Σ_2 is still $\delta_{G_1}(w_1)$.

Note that Theorem 4.1 completely characterizes the structure of G_1 and H_2 in cases (2) and (3) and G_2 is obtained from H_2 by simply splitting any vertex. Moreover, the matching signature pair is uniquely determined, by the results in Chapter 3. However, if $|T_1| = |T_2| = 2$, case (1) of the theorem occurs. What Theorem 4.1 states in this case is that there exist equivalent graphs H_1,H_2 such that, for $i = 1,2$, H_i is obtained from G_i by identifying two vertices. Hence Theorem 4.1 does not characterize the structure of the graphs in this case. Therefore we treat this type of siblings separately from the other Shih siblings and we provide an explicit characterization of them. Let signed graphs (G_1,Σ_1),(G_2,Σ_2) be siblings and let T_1,T_2 be the matching terminal pair, where $|T_1| = |T_2| = 2$. For $i = 1,2$, let H_i be obtained from G_i by identifying the two vertices in T_i. Then $\text{cut}(H_1) = \text{ecut}(G_1,T_1) = \text{ecut}(G_2,T_2) = \text{cut}(H_2)$. By Theorem 1.2, H_1,H_2 are equivalent. This justifies the following definition.

Consider a pair of equivalent graphs H_1 and H_2. Suppose that, for $i = 1,2$, we have $\alpha_i \subseteq \delta_{H_i}(v_i) \cup \text{loop}(H_i)$ for some $v_i \in V(H_i)$. Then, for $i = 1,2$, let G_i be obtained from H_i by splitting v_i into v_i^-,v_i^+ according to α_i and let $T_i := \{v_i^-,v_i^+\}$. Since H_1 and H_2 are equivalent, $\text{cut}(H_1) = \text{cut}(H_2)$. Thus

$$\text{ecut}(G_1,T_1) = \text{cut}(H_1) = \text{cut}(H_2) = \text{ecut}(G_2,T_2).$$

In particular, if G_1,G_2 are not equivalent, $(G_1,T_1),(G_2,T_2)$ are siblings. Let Σ_1,Σ_2 be the matching signature pair for G_1,G_2. If $(G_1,\Sigma_1),(G_2,\Sigma_2)$ are inequivalent we say that the tuple $\mathbb{T} = (H_1,v_1,\alpha_1,H_2,v_2,\alpha_2)$ is a *split-template* and that (G_1,Σ_1),(G_2,Σ_2) (respectively $(G_1,T_1),(G_2,T_2)$) are *split siblings* which *arise* from $\mathbb{T}$. Split siblings are a special type of Shih siblings, namely the type arising from outcome (1) in Theorem 4.1. An explicit characterization of split siblings representing a 3-connected matroid is given in Section 4.5.

4.3 The class of quad siblings

Let (H_1,Γ_1) and (H_2,Γ_2) be a pair of equivalent signed graphs. Suppose that, for $i=1,2$, $\Gamma_i \subseteq \delta_{H_i}(v_i) \cup \delta_{H_i}(w_i) \cup \mathrm{loop}(H_i)$ for some $v_i, w_i \in V(H_i)$. Then, for $i=1,2$, let (G_i,T_i) be the graft obtained by unfolding (H_i,Γ_i) according to v_i,w_i and α_i,β_i (where $\Gamma_i = \alpha_i \Delta \beta_i$). It follows from Remark 2.8(2) that

$$\mathrm{ecut}(G_1,T_1) = \mathrm{ecycle}(H_1,\Gamma_1)^* = \mathrm{ecycle}(H_2,\Gamma_2)^* = \mathrm{ecut}(G_2,T_2).$$

In particular, if G_1,G_2 are not equivalent, then $(G_1,T_1),(G_2,T_2)$ are siblings. Let Σ_1,Σ_2 be the matching signature pair for G_1,G_2. If G_1,G_2 are not equivalent, we say that the tuple $\mathbb{T} = (H_1,v_1,w_1,\alpha_1,\beta_1,H_2,v_2,w_2,\alpha_2,\beta_2)$ is a *quad-template* and that (G_1,Σ_1),(G_2,Σ_2) (respectively (G_1,T_1),(G_2,T_2)) are *quad siblings* which *arise* from $\mathbb{T}$. An explicit characterization of quad siblings representing a 3-connected non-graphic matroid is given in Section 4.6.

4.4 Isomorphism Conjecture

In this section we present a conjecture about the relation between signed graphs siblings. We are not very precise in the definitions of the outcomes of the conjecture. However, these outcomes arose in a sketch of the proof of this conjecture with some connectivity hypothesis.

Conjecture 4.2. *Suppose* (G_1,Σ_1) *and* (G_2,Σ_2) *are siblings and* $\mathrm{ecycle}(G_1,\Sigma_1)$ *is non-graphic. Then there exist signed graphs* (G_1',Σ_1'), (G_2',Σ_2') *such that, for* $i=1,2$, (G_i',Σ_i') *is obtained from* (G_i,Σ_i) *by a sequence of Whitney-flips, Lovász-flips and signature exchanges and one of the following occurs:*

(1) $(G_1',\Sigma_1') = (G_2',\Sigma_2')$;

(2) (G_1',Σ_1') *and* (G_2',Σ_2') *are either Shih siblings or quad siblings;*

(3) (G'_1, Σ'_1) *and* (G'_2, Σ'_2) *may be reduced;*

(4) (G'_1, Σ'_1) *and* (G'_2, Σ'_2) *belong to a sporadic set of examples;*

(5) (G'_1, Σ'_1) *and* (G'_2, Σ'_2) *are obtain by a local modification of one of the operations in Shih's Theorem.*

The reductions in part (3) are similar to, and include, the reductions described in Sections 4.2 and 4.3. The small set of examples in part (4) arise from a construction like the one in Figure 4.3.

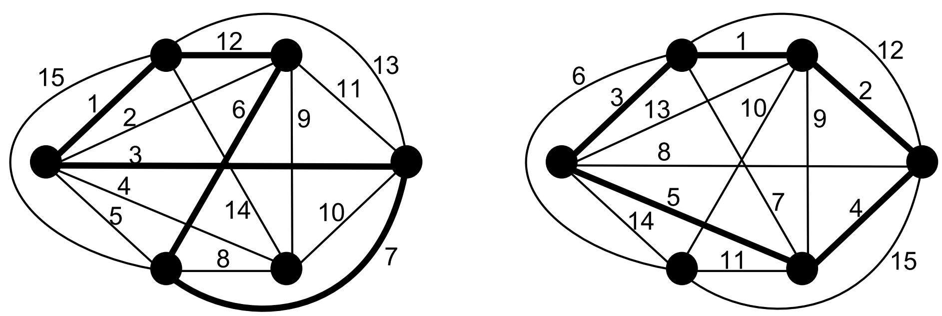


Figure 4.3: Sporadic example. Bold edges are odd.

Outcome (5) is constructed as follows. Let G be a graph and (H,Γ) be a signed graph such that cycle$(G) =$ ecycle(H,Γ). Suppose e, f, g are edges forming an odd triangle in (H,Γ). Let v_{ef} be the vertex in H incident to e and f; define v_{fg} and v_{eg} similarly. Construct a graph H' by adding a new vertex v and three new edges $\bar{e}, \bar{f}, \bar{g}$ to H as follows: $\{\bar{e}, \bar{f}, \bar{g}\}$ form a triad in H' incident to the new vertex v. The other end of $\bar{e}$ (respectively $\bar{f}, \bar{g}$) in H' is v_{fg} (respectively v_{eg}, v_{ef}). Now construct a graph G' from G by adding edges $\bar{e}, \bar{f}, \bar{g}$, where $\bar{e}$ is parallel to e, $\bar{f}$ is parallel to f and $\bar{g}$ is parallel to g. Then ecut$(H', \{v, v_{ef}, v_{eg}, v_{fg}\}) =$ ecut(G', T'), where $T' := V_{odd}(G[\{e, f, g\}])$. Hence the graphs G' and H' are siblings. An example of this construction is given in Figure 4.4.

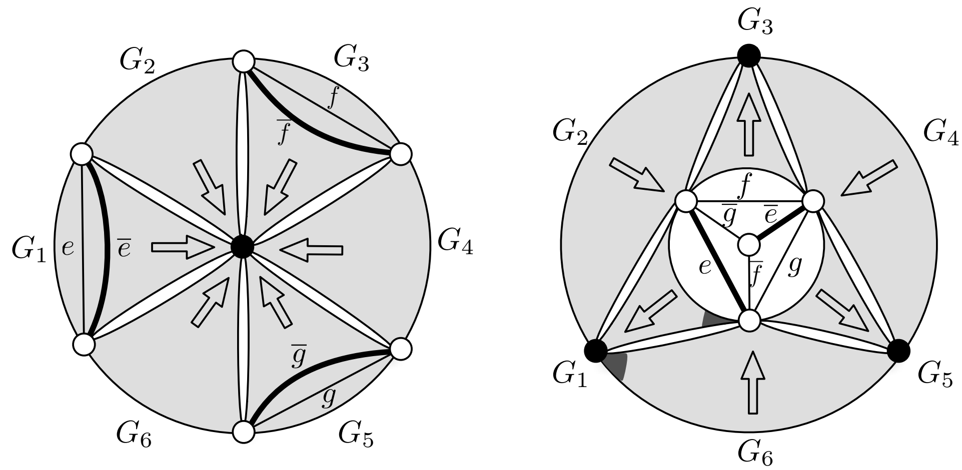


Figure 4.4: Modification of Shih's operation. Bold and shaded edges are odd, white vertices are terminals.

4.5 Isomorphism for Shih siblings

Let signed graphs (G_1,Σ_1),(G_2,Σ_2) be Shih siblings and let T_1,T_2 be the matching terminal pair. Suppose $|T_2| = 2$, and let H_2 be the graph obtained from G_2 by identifying the two vertices in T_2. Then $\text{ecut}(G_1,T_1) = \text{ecut}(G_2,T_2) = \text{cut}(H_2)$ and some graphs G_1' and H_2', equivalent to G_1 and H_2 respectively, satisfy one of the outcomes of Theorem 4.1. Outcomes (2) and (3) completely characterize the structure of G_1 and H_2. The aim of this section is to provide a structural characterization of outcome (1). Recall that if outcome (1) occurs, then (G_1,Σ_1),(G_2,Σ_2) are split siblings. The proof of the following result is given in Chapter 6.

Theorem 4.3. *Let M be a 3-connected even cycle matroid. If (G_1,Σ_1) and (G_2,Σ_2) are representations of M which are split siblings, then they are either:*

(1) simple siblings, or

(2) nova siblings, or

(3) reducible.

We say that (G_1,Σ_1) and (G_2,Σ_2) are simple (respectively nova) *siblings* if, for $i = 1,2$, there exists (G_i',Σ_i') equivalent to (G_i,Σ_i) such that (G_1',Σ_1') and (G_2',Σ_2') are simple (respectively nova) twins. It remains to define the terms "simple twins", "nova twins" and "reducible". We need some preliminary definitions.

By a *sequence* $(X_1,\dots,X_k)$ we mean a family of sets $\{X_1,\dots,X_k\}$ where X_i precedes X_j when $i<j$. We say that $\mathbb{S}=(X_1,\dots,X_k)$ is a *w-sequence* of G if, for all $i\in[k]$, X_i is a 2-separation of the graph obtained from G by performing Whitney-flips on $X_1,\dots,X_{i-1}$ (in this order). We denote by $\mathrm{W}_{\mathrm{flip}}[G,\mathbb{S}]$ the graph obtained from G by performing Whitney-flips on $X_1,\dots,X_k$ (in this order). For our purpose the position of loops is irrelevant. Hence we will assume that loops form distinct components of the graph. Therefore, if G,G' are equivalent graphs that are 2-connected, except for possible loops, then $G'=\mathrm{W}_{\mathrm{flip}}[G,\mathbb{S}]$ for some w-sequence $\mathbb{S}$ of G.

A family $\mathbb{S}=\{X_1,\dots,X_k\}$ of sets of edges of a graph G is a *w-star* if

(a) $X_i\cap X_j=\emptyset$, for all $i,j\in[k]$, where $i\neq j$;

(b) there exist distinct $z,v_1,\dots,v_k\in V(G)$ such that $\mathscr{B}_G(X_i)=\{z,v_i\}$, for all $i\in[k]$;

(c) no edge with ends z,v_i is in X_i, for all $i\in[k]$.

Vertex z is the *center* of the w-star $\mathbb{S}$.

Consider a split-template $(H_1,v_1,\alpha_1,H_2,v_2,\alpha_2)$. If H_1,H_2 are 2-connected, except for possible loops, we have that $H_2=\mathrm{W}_{\mathrm{flip}}[H_1,\mathbb{S}]$ for some w-sequence $\mathbb{S}$. In this case we slightly abuse terminology and say that $(H_1,v_1,\ \alpha_1,\ H_2,\ v_2,\ \alpha_2,\mathbb{S})$ is a split-template. (This is only well defined for the case where H_1,H_2 are 2-connected up to loops).

Remark 4.4. *Let $\mathbb{T}=(H_1,v_1,\alpha_1,H_2,v_2,\alpha_2)$ be a split-template and let (G_1,Σ_1), (G_2,Σ_2) be split siblings that arise from $\mathbb{T}$. Then, up to signature exchange, we have $\Sigma_1=\Sigma_2=\alpha_1\triangle\alpha_2$.*

Proof. For $i=1,2$, vertex v_i of H_i gets split into vertices v_i^-,v_i^+ of G_i. By construction, $\alpha_i=\delta_{G_i}(v_i^-)$, for $i=1,2$. As $v_1^-\in T_1$, Theorem 3.1 implies that α_1 is a signature of (G_2,Σ_2). As α_2 is a cut of G_2, $\alpha_1\triangle\alpha_2$ is a signature of (G_2,Σ_2). By symmetry, $\alpha_1\triangle\alpha_2$ is also a signature of (G_1,Σ_1). □

4.5.1 Simple twins

Consider a split-template $\mathbb{T} = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2, \mathbb{S})$. If $\mathbb{S} = \emptyset$, i.e. $H_1 = H_2$, then $\mathbb{T}$ is *simple* and $(G_1, \Sigma_1), (G_2, \Sigma_2)$ arising from $\mathbb{T}$ are *simple twins*. By Remark 4.4, we may assume that $\Sigma_1 = \Sigma_2 = \alpha_1 \triangle \alpha_2$. Suppose that vertex v_1 of H_1 gets split into vertices v_1^-, v_1^+ of G_1. Then $\alpha_1 \subseteq \delta_{G_1}(v_1^-)$ and $\alpha_2 \subseteq \delta_{G_1}(v_2)$. Hence, v_1^- and v_2 form a blocking pair of (G_1, Σ_1). Thus we have the following.

Remark 4.5. *Simple twins have blocking pairs.*

It can easily be verified that two simple twins are related by Lovász-flips.

4.5.2 Nova twins

Let (G, Σ) be a signed graph with distinct vertices s_1 and s_2. For $i = 1, 2$, let C_i denote a circuit of H_i using s_i and avoiding s_{3-i}. Suppose that C_1 and C_2 are either vertex disjoint or that C_1 and C_2 intersect exactly in a path. In the former case let P denote a path with ends $u_i \in V_G(C_i) - \{s_i\}$, for $i = 1, 2$, such that $V_G(P) \cap \big(V_G(C_1) \cup V_G(C_2)\big) = \{u_1, u_2\}$. In the latter case, define P to be the empty set. We say that the triple (C_1, C_2, P) form $\{s_1, s_2\}$-*handcuffs*. We say that $X \subseteq G$ is a *handcuff-separation* if X is a 2-separation of G and there exist $\{s_1, s_2\}$-handcuffs of $(G[X], \Sigma \cap X)$, where s_1, s_2 are the vertices in $\mathscr{B}_G(X)$.

A split-template $\mathbb{T} = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2, \mathbb{S})$ is *nova* if, for $i = 1, 2$:

(N1) $\mathbb{S}$ is a w-star of H_i with center v_i, and

(N2) all $X' \subseteq X \in \mathbb{S}$ with $\mathscr{B}_{H_i}(X') = \mathscr{B}_{H_i}(X)$ are handcuff-separations of $(H_i, \alpha_1 \triangle \alpha_2)$.

We say that $(G_1, \Sigma_1), (G_2, \Sigma_2)$ arising from $\mathbb{T}$ are *nova twins*. We could have defined nova twins omitting condition (N2). This would yield a weaker version of Theorem 4.3. However, the stronger version is needed for the stabilizer theorem for even cycle matroids discussed in Chapter 8.

4.5.3 Reduction

Consider grafts (G_1, T_1) and (G_2, T_2) where, for $i = 1, 2$, T_i consists of vertices v_i^-, v_i^+. We write $(G_1, T_1) \oplus (G_2, T_2)$ to indicate the graft (G, T) where G is obtained from G_1 and G_2

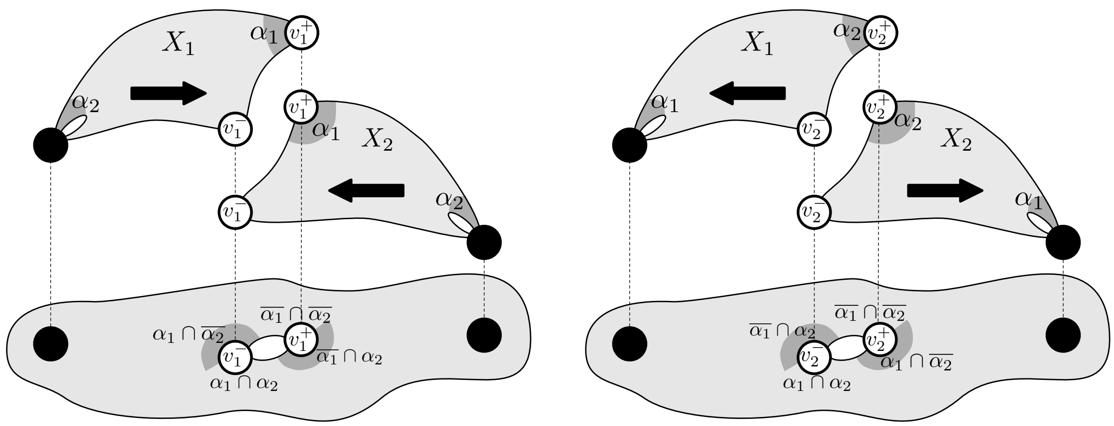


Figure 4.5: Example of nova twins with $|\mathbb{S}| = 2$.

by identifying vertex v_1^- with v_2^- and by identifying vertex v_1^+ with vertex v_2^+. Denote by v^- (respectively v^+) the vertex in G corresponding to v_1^-, v_2^- (respectively v_2^-, v_2^+) and let $T = \{v^-, v^+\}$. Note that (G,T) is defined uniquely from (G_1,T_1) and (G_2,T_2) up to a possible Whitney-flip on $E(G_1)$.

Consider split siblings $(G_1,\Sigma_1),(G_2,\Sigma_2)$ and let T_1,T_2 be the matching terminal pair. Suppose that there exists $X \subseteq E(G_1)$ such that $\mathscr{B}_{G_1}(X) = T_1$. For $i = 1,2$, let H_i be obtained from G_i by identifying the vertices in T_i to a single vertex v_i. Then $H_1[X]$ is a block of H_1 attached to vertex v_1. As $(G_1,T_1),(G_2,T_2)$ are split siblings, $H_2[X]$ is also a block of H_2 attached to v_2. It follows that $\mathscr{B}_{G_2}(X) = T_2$. For $i = 1,2$, define $G_i' := G_i[X]$ and $G_i'' := G_i[\bar{X}]$. Let T_i' and T_i'' denote the vertices corresponding to T_i in G_i' and G_i'' respectively. Then, for $i = 1,2$, $(G_i,T_i) = (G_i',T_i') \oplus (G_i'',T_i'')$. Observe that $(G_1',T_1'),(G_2',T_2')$ are split siblings and so are $(G_1'',T_1''),(G_2'',T_2'')$. We say in that case that $(G_1,\Sigma_1),(G_2,\Sigma_2)$ are *reducible*.

4.6 Isomorphism for quad siblings

The main result of this section is the following.

Theorem 4.6. *Let M be a 3-connected non-graphic even cycle matroid. If $(G_1,\Sigma_1),(G_2,\Sigma_2)$ are representations of M which are quad siblings, then they are either:*

(1) shuffle siblings,

(2) tilt siblings,

(3) twist siblings,

(4) widget siblings,

(5) gadget siblings, or

(6) Δ-reducible.

The proof of Theorem 4.6 is in Chapter 6.

We say that $(G_1,\Sigma_1),(G_2,\Sigma_2)$ are *shuffle* (respectively *tilt*, *twist*, *widget*, *gadget*) *siblings* if, for $i=1,2$, there exists (G'_i,Σ'_i) equivalent to (G_i,Σ_i) such that $(G'_1,\Sigma'_1),(G'_2,\Sigma'_2)$ are shuffle (respectively tilt, twist, widget, gadget) twins. The terms "shuffle twins", "tilt twins", "twist twins", "widget twins", "gadget twins" and "Δ-reducible" are defined in the next sections.

4.6.1 Shuffle twins

Consider a graph G and let $\{a,b,c,d\} \subseteq V(G)$. Suppose that $E(G)$ can be partitioned into sets $X_1,\dots,X_4$ (not necessarily all non-empty) such that, for all $i \in [4]$, $\mathscr{B}_G(X_i) \subseteq \{a,b,c,d\}$. For all $i \in [4]$, denote by a_i (respectively b_i,c_i,d_i) the copy of vertex a (respectively b,c,d) of $G[X_i]$. Then construct G' by:

- identifying vertices a_1,b_2,c_3,d_4 to a vertex a';
- identifying vertices b_1,a_2,d_3,c_4 to a vertex b';
- identifying vertices c_1,d_2,a_3,b_4 to a vertex c';
- identifying vertices d_1,c_2,b_3,a_4 to a vertex d'.

We say that G and G' are *shuffle twins*. We will show that they are siblings with matching terminal pair $\{a,b,c,d\}$ and $\{a',b',c',d'\}$. Shuffle twins were introduced by Norine and Thomas [20].

Let H (respectively H') be obtained by folding $(G,\{a,b,c,d\})$ (respectively $(G',\{a',b',c',d'\})$) with the pairing a,b and c,d (respectively a',b' and c',d'). Let $\alpha := \delta_G(a)$, $\beta :=$

$\delta_G(c)$, $\alpha' := \delta_{G'}(a')$ and $\beta' := \delta_{G'}(c')$. Then $(H_1, \alpha\triangle\beta)$ and $(H_2, \alpha'\triangle\beta')$ are equivalent, hence G and G' are quad siblings with matching terminal pair $\{a,b,c,d\}$ and $\{a',b',c',d'\}$.

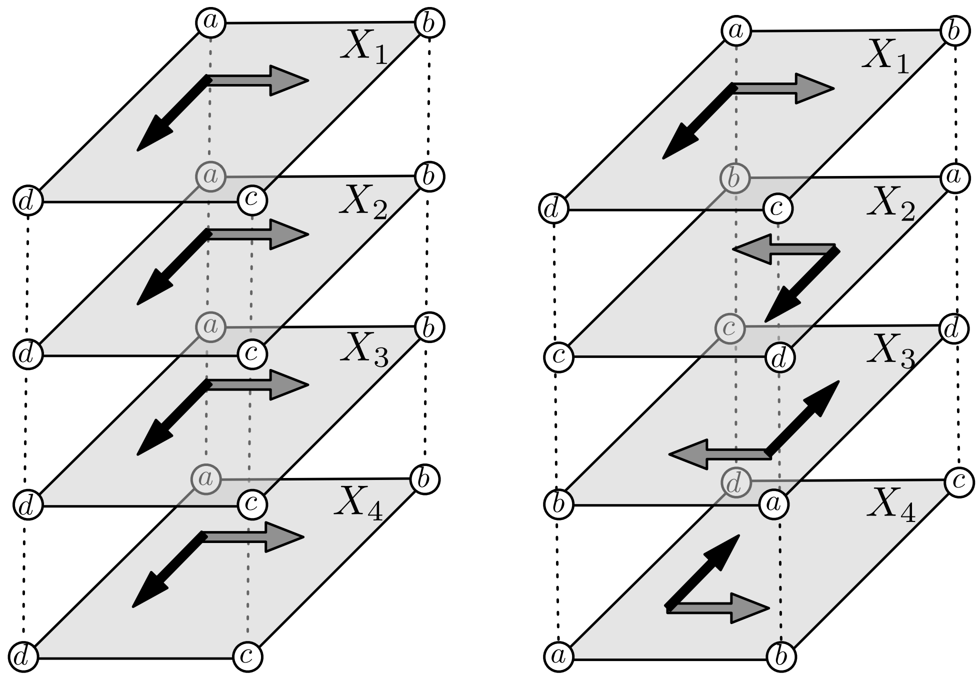


Figure 4.6: Shuffle twins.

4.6.2 Tilt twins

Consider a graph G with distinct edges $e, f, g, h \in E(G)$ and distinct vertices a_1, a_2, b_1, b_2, c, d. Suppose e, f have ends a_1, a_2 and g, h have ends b_1, b_2. Suppose we can partition $E(G)$ into $X_1, X_2, \{e, f, g, h\}$, such that $V_G(X_1) \cap V_G(X_2) = \{c, d\}$ and $a_1, b_1 \in V_G(X_1)$, $a_2, b_2 \in V_G(X_2)$. For all $i \in [2]$, denote by c_i (respectively d_i) the copy of vertex c (respectively d) in $G[X_i]$. Construct G' from $G[X_1], G[X_2]$ by:

- identifying vertices a_1 and a_2;
- identifying vertices b_1 and b_2;

- joining c_1, c_2 with edges e, g;
- joining d_1, d_2 with edges f, h.

We say that G and G' are *tilt twins*. In general, we say that G, G' are tilt twins even if not all edges e, f, g, h in the above construction are present. Tilt twins were introduced by Gerards [13] .

Let H (respectively H') be obtained by folding $(G, \{a_1, a_2, b_1, b_2\})$ (respectively $(G', \{c_1, c_2, d_1, d_2\})$) with the pairing a_1, a_2 and b_1, b_2 (respectively c_1, c_2 and d_1, d_2). Let $\alpha := \delta_G(a_1)$, $\beta := \delta_G(b_1)$, $\alpha' := \delta_{G'}(c_1)$ and $\beta' := \delta_{G'}(d_1)$. Then $(H_1, \alpha \triangle \beta)$ and $(H_2, \alpha' \triangle \beta')$ are equivalent, hence G and G' are quad siblings with matching terminal pair $\{a_1, a_2, b_1, b_2\}$ and $\{c_1, c_2, d_1, d_2\}$.

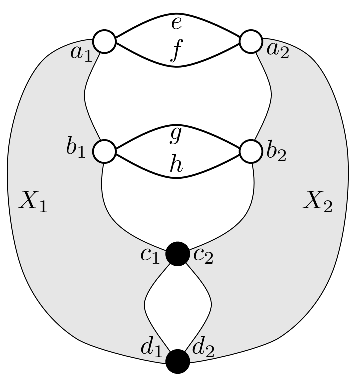


Figure 4.7: Tilt twins.

4.6.3 Twist twins

Consider a graph G with distinct edges e, f, g, h and distinct vertices a_1, a_2, b, c, d. Suppose e, f have ends a_1, a_2 and g, h have ends b, c. Suppose we can partition $E(G)$ into $X_1, X_2, \{e, f, g, h\}$ such that $V_G(X_1) \cap V_G(X_2) = \{b, c, d\}$ and $a_1 \in V(X_1)$, $a_2 \in V(X_2)$. For all $i \in [2]$ let b_i (respectively c_i, d_i) denote the copy of vertex b (respectively c, d) in $G[X_i]$. Construct G' from $G[X_1], G[X_2]$ by:

- identifying vertices a_1 and a_2;

- identifying vertices b_1 and c_2, calling the resulting vertex $\tilde{b}$;
- identifying vertices c_1 and b_2, calling the resulting vertex $\tilde{c}$;
- joining $\tilde{b}, \tilde{c}$ with edges e, g;
- joining d_1, d_2 with edges f, h.

We say that G and G' are *twist twins*. In general, we say that G, G' are twist twins even if not all edges e, f, g, h in the above construction are present.

Let H (respectively H') be obtained by folding $(G, \{a_1, a_2, b, c\})$ (respectively $(G', \{\tilde{b}, \tilde{c}, d_1, d_2\})$) with the pairing a_1, a_2 and b, c (respectively $\tilde{b}, \tilde{c}$ and d_1, d_2). Let $\alpha := \delta_G(a_1)$, $\beta := \delta_G(b)$, $\alpha' := \delta_{G'}(\tilde{b})$ and $\beta' := \delta_{G'}(d_1)$. Then $(H_1, \alpha \triangle \beta)$ and $(H_2, \alpha' \triangle \beta')$ are equivalent, hence G and G' are quad siblings with matching terminal pair $\{a_1, a_2, b, c\}$ and $\{\tilde{b}, \tilde{c}, d_1, d_2\}$.

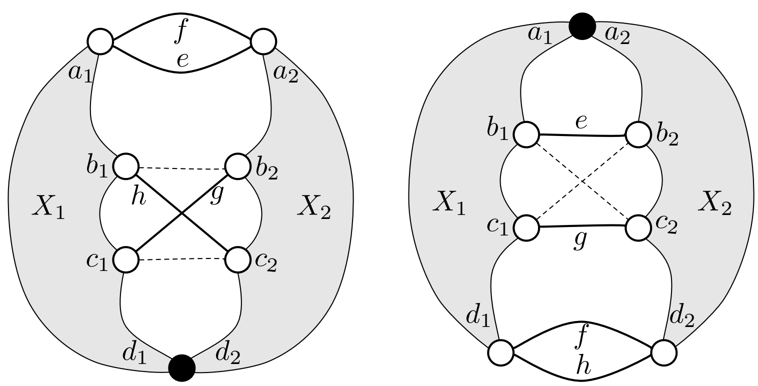


Figure 4.8: Twist twins.

4.6.4 Widget twins

Consider a graph H_1 with distinct edges $a, b, c, d, e, f, \ell_1, \ell_2, \ell_3, \ell_4$ and distinct vertices v_1, z_1, w_1, w_2. Suppose a, b have ends v_1, w_2; c, d have ends z_1, w_2; e, f have ends v_1, w_1 and $\mathrm{loop}(H_1) = \{\ell_1, \ell_2, \ell_3, \ell_4\}$. Suppose we can partition $E(H_1)$ into $X, \{a, b, c, d, e, f\}, \mathrm{loop}(H_1)$ such that $\delta_{H_1}(w_2) = \{a, b, c, d\}$ and $\mathscr{B}_{H_1}(X) = \{v_1, z_1, w_1\}$. Let $H_2 = \mathrm{W}_{\text{flip}}[H_1, \{a, b, c, d\}]$. Let the vertices in H_2 which are not in $\mathscr{B}_{H_2}(\{a, b, c, d\})$ be labeled as in H_1. Let $v_2 \in$

$V(H_2)$ be the endpoint of c distinct from w_2. Let $\gamma \subseteq \delta_{H_1}(v_1) \cap X$. Define $\alpha_1 := \gamma \cup \{a, e, \ell_1, \ell_2\}$; $\beta_1 := \{e, f, \ell_3, \ell_4\}$; $\alpha_2 := \gamma \cup \{f, c, \ell_1, \ell_3\}$ and $\beta_2 := \{a, c, \ell_2, \ell_4\}$. Let $\mathbb{T} = (H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2)$. Note that $\mathbb{T}$ is a quad-template. Let (G_1, Σ_1) and (G_2, Σ_2) be the quad siblings arising from $\mathbb{T}$. We say that G_1 and G_2 are *widget twins*.

4.6.5 Gadget twins

Consider a graph H_1 with distinct edges $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, \ell_1, \ell_2, \ell_3, \ell_4$ and distinct vertices v_1, z_1, u_1, w_1, w_2. Suppose a_i, b_i have ends v_1, w_i, for $i = 1, 2$; c_1, d_1 have ends $z_1 w_1$; c_2, d_2 have ends $u_1 w_2$ and $\text{loop}(H_1) = \{\ell_1, \ell_2, \ell_3, \ell_4\}$. Suppose we can partition $E(H_1)$ into sets $X, \{a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2\}, \text{loop}(H_1)$ such that $\delta_{H_1}(w_i) = \{a_i, b_i, c_i, d_i\}$, for $i = 1, 2$, and $\mathscr{B}_{H_1}(X) = \{v_1, z_1, u_1\}$. Let $H_2 = \mathrm{W}_{\text{flip}}[H_1, (\{a_1, b_1, c_1, d_1\}, \{a_2, b_2, c_2, d_2\})]$. Let the vertices in H_2 which are not in $\mathscr{B}_{H_2}(\{a_i, b_i, c_i, d_i\})$ be labeled as in H_1. Let $v_2 \in V(H_2)$ be the endpoint of c_1 distinct from w_1. Let $\gamma \subseteq \delta_{H_1}(v_1) \cap X$. Define $\alpha_1 := \gamma \cup \{a_1, a_2, \ell_1, \ell_2\}$, $\beta_1 := \{a_1, c_1, \ell_3, \ell_4\}$, $\alpha_2 = \gamma \cup \{c_1, c_2, \ell_1, \ell_3\}$ and $\beta_2 := \{a_2, c_2, \ell_2, \ell_4\}$. Let $\mathbb{T} = (H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2)$. Note that $\mathbb{T}$ is a quad-template. Let (G_1, Σ_1) and (G_2, Σ_2) be the quad siblings arising from $\mathbb{T}$. We say that G_1 and G_2 are *gadget twins*.

4.6.6 Δ-reduction

Consider siblings $(G_1, \Sigma_1), (G_2, \Sigma_2)$ and suppose that edges $\{e_1, e_2, e_3\}$ form a triangle of both G_1 and G_2 and (after possibly resigning) $\{e_1, e_2, e_3\} \cap \Sigma_i = \emptyset$, for $i = 1, 2$. Let H be a graph with distinct vertices v_{12}, v_{13}, v_{23}. For $i = 1, 2$, let G'_i be the graph obtained from G_i by (for all distinct $j, k \in [3]$) identifying the vertex of G_i incident to both e_j, e_k with the vertex v_{jk} of H, and by then deleting the edges e_1, e_2, e_3. We say that (G'_1, Σ_1) and (G'_2, Σ_2) are obtained by a Δ-*substitution* from (G_1, Σ_1) and (G_2, Σ_2) and that (G_1, Σ_1) and (G_2, Σ_2) are obtained by a Δ-*reduction* from (G'_1, Σ_1) and (G'_2, Σ_2). By possibly omitting some of the edges of the triangle, we will make sure to not create parallel edges of the same parity when applying a Δ-reduction. Note that in this case (G'_1, Σ_1) and (G'_2, Σ_2) are also siblings.

We say that siblings (G_1, Σ_1), (G_2, Σ_2) are Δ-*irreducible* if no Δ-reduction is possible in (G_1, Σ_1), (G_2, Σ_2), otherwise we say that the siblings are Δ-*reducible*. We mainly consider Δ-reductions to simplify the definitions of the various types of quad siblings. For example, suppose (G_1, Σ_1) and (G_2, Σ_2) are tilt twins, with the same notation as in the definition of tilt twins in Section 4.6.2. Suppose that G_1 contains edges e_1, e_2 with ends a_1, c and a_2, c respectively. Then $\{e, e_1, e_2\}$ is an even triangle of both G_1 and G_2 and such a triangle may be substituted by any graph H.

Chapter 5

Whitney-flips

In this chapter we provide results about equivalent graphs and grafts which will be used in the subsequent chapters. The results in Section 5.1 are used in Chapter 6 (to prove the theorems stated in Chapter 4) and in Chapter 8. The results in Sections 5.2 and 5.6 are used to prove the results in Chapter 9. A difficulty when dealing with Whitney-flips comes from crossing 2-separations. We show how, in the cases we are interested in, we can reduce to considering only Whitney-flips on non-crossing separations. Throughout this chapter graphs are 2-connected. However, the notions of w-sequences, the operation W_{flip} and the results in this chapter extend naturally to the class of graphs that are 2-connected except for possible loops.

5.1 Whitney-flips avoiding vertices

Recall the definitions of w-sequence and w-star given in Section 4.5. We say that two sets X,Y are *crossing* if all of $X\cap Y, X-Y, Y-X$ and $\overline{X}\cap\overline{Y}$ are non-empty. A family of sets (or sequence) $\mathbb{S}$ is *non-crossing* if X,Y are non-crossing for every $X,Y\in\mathbb{S}$.

Remark 5.1. *Let G be a graph and let $\mathbb{S}=(X_1,\dots,X_k)$ be a non-crossing w-sequence for G. Then for any permutation $i_1,\dots,i_k$ of $1,\dots,k$, $\mathbb{S}'=(X_{i_1},\dots,X_{i_k})$ is a w-sequence and $\mathrm{W}_{\mathit{flip}}[G,\mathbb{S}]=\mathrm{W}_{\mathit{flip}}[G,\mathbb{S}']$.*

In light of the previous remark, given a non crossing w-sequence $(X_1,\dots,X_k)$, we call the family $\mathbb{S}:=\{X_1,\dots,X_k\}$ a w-sequence and the notation $\mathrm{W}_{\text{flip}}[G,\mathbb{S}]$ is well defined.

We can now state the first of the two main technical results of this section.

Proposition 5.2. *Let G, G' be 2-connected equivalent graphs and let $Z \subseteq V(G)$, where $|Z| \leq 2$. There exist a w-sequence $\mathbb{S}_1$ of G and a graph H with a non-crossing w-sequence $\mathbb{S}_2$ such that:*

(1) $H = \mathrm{W}_{flip}[G, \mathbb{S}_1]$, where $Z \cap \mathscr{B}_G(X) = \emptyset$ for all $X \in \mathbb{S}_1$; and

(2) $G' = \mathrm{W}_{flip}[H, \mathbb{S}_2]$, where $Z \cap \mathscr{B}_G(X) \neq \emptyset$ for all $X \in \mathbb{S}_2$.

Note that we cannot replace $|Z| \leq 2$ by $|Z| \leq k$ for any $k > 2$ in the previous proposition, as the following example illustrates. Suppose that G consists of edges e_1, e_2, e_3, e_4, e_5 that form a circuit with edges appearing in that order. Let G' be the graph obtained from G by re-arranging the edges to form a circuit with edges appearing in order e_1, e_3, e_5, e_2, e_4. Suppose that Z consists of 3 consecutive vertices of the circuit in G. Then every 2-separation of G contains a vertex of Z but there is no non-crossing w-sequence $\mathbb{S}$ for which $G' = \mathrm{W}_{\text{flip}}[G, \mathbb{S}]$. The other result in this section is the following.

Proposition 5.3. *Consider 2-connected equivalent graphs G, G' and let $z \in V(G), z' \in V(G')$. There exist w-sequences $\mathbb{L}$ of G, $\mathbb{L}'$ of G' and graphs H and H' such that:*

(1) $H = \mathrm{W}_{flip}[G, \mathbb{L}]$, where $z \notin \mathscr{B}_G(X)$ for all $X \in \mathbb{L}$;

(2) $H' = \mathrm{W}_{flip}[G', \mathbb{L}']$, where $z' \notin \mathscr{B}_{G'}(X)$ for all $X \in \mathbb{L}'$; and

(3) $H' = \mathrm{W}_{flip}[H, \mathbb{S}]$,

where $\mathbb{S}$ is a w-star of H with center z and a w-star of H' with center z'.

Recall that w-stars were defined in Section 4.5. The proofs of Propositions 5.2 and 5.3 are postponed until Section 5.4.

5.2 Whitney-flips preserving paths

A sequence $(X_1, \ldots, X_k)$ is *nested* if $X_i \subset X_{i+1}$, for $i = 1, \ldots, k-1$. In particular, nested sequences are non-crossing. Let G be a graph and P a path in G. We say that a Whitney-flip on a 2-separation X *preserves* P if P is a path of $\mathrm{W}_{\text{flip}}[G, X]$. Note that this occurs if and only if the ends of P are both in $V_G(X)$ or both in $V_G(\bar{X})$. Similarly, we say that a w-sequence $\mathbb{S}$ of G *preserves* P if P is a path in $\mathrm{W}_{\text{flip}}[G,\mathbb{S}]$. The main result of this section is the following.

Proposition 5.4. *Let G and G' be equivalent graphs and let P be a path in G. Then there exists a graph H such that:*

(1) $H = \mathrm{W}_{\text{flip}}[G,\mathbb{S}_1]$, for some w-sequence $\mathbb{S}_1$ which preserves P, and

(2) $G' = \mathrm{W}_{\text{flip}}[H,\mathbb{S}_2]$, for some nested w-sequence $\mathbb{S}_2$, where no $X \in \mathbb{S}_2$ preserves P.

The next section contains results needed to prove Propositions 5.2, 5.3 and 5.4. The proofs follow in Sections 5.4 and 5.5. Section 5.6 provides two results about Whitney-flips on grafts which will be used in Chapter 9.

5.3 Flowers

For a graph H, we say that a partition $\mathbb{F} = \{B_1,\dots,B_t\}$ of $E(H)$, with $t \geq 2$, is a *flower* if there exist distinct $u_1,\dots,u_t \in V(H)$ such that (after possibly relabeling $B_1,\dots,B_t$),

(a) $H[B_i]$ is connected, for every $i \in [t]$, and

(b) $\mathscr{B}_H(B_i) = \{u_i,u_{i+1}\}$, for every $i \in [t]$ (where $t+1=1$).

For $i \in [t]$, B_i (or $H[B_i]$) is a *petal* with *attachments* u_i,u_{i+1}. We say that the flower is *maximal* if no petal has a cut-vertex separating its attachments. Maximal flowers correspond to generalized circuits as introduced by Tutte in [36]. The term flower was introduced to describe crossing 3-separations in matroids (see [22]).

Given two partitions $\mathbb{F}_1,\mathbb{F}_2$ of the same set, we say that $\mathbb{F}_1$ is a *refinement* of $\mathbb{F}_2$ if every set in $\mathbb{F}_2$ is the union of sets in $\mathbb{F}_1$. Note that, for every flower $\mathbb{F}$, there is a maximal flower that is a refinement of $\mathbb{F}$. Let $\mathbb{S}_1,\mathbb{S}_2$ be families of sets over the same ground set. We say that $\mathbb{S}_1,\mathbb{S}_2$ are *independent* if for every $X \in \mathbb{S}_1$ and $Y \in \mathbb{S}_2$, X and Y do not cross. This definition extends to sequences of sets. Thus we can talk about pairs of independent w-sequences and pairs of independent flowers.

For a graph H, we say that a partition $\mathbb{F} = \{B_1,\dots,B_t\}$ of $E(H)$, with $t \geq 2$, is a *leaflet* if there exist distinct $u_1,u_2 \in V(H)$ such that:

(a) $H[B_i]$ is connected for every $i \in [t]$, and

(b) $\mathscr{B}_H(B_i) = \{u_1, u_2\}$ for every $i \in [t]$.

Remark 5.5. *Let G be a 2-connected graph and let X, Y be 2-separations of G that cross. Then $\mathbb{F} := \{X \cap Y, X - Y, Y - X, \bar{X} \cap \bar{Y}\}$ is either a flower or a leaflet.*

Let $\mathbb{F}$ be a flower of G. We say that a 2-separation X of G, where X is the union of petals of $\mathbb{F}$, is a 2-separation of $\mathbb{F}$. The following lemma characterizes pairs of independent flowers.

Lemma 5.6. *Let $\mathbb{F}_1, \mathbb{F}_2$ be distinct maximal flowers of G. The following are equivalent.*

(1) $\mathbb{F}_1, \mathbb{F}_2$ are independent.

(2) The set of all 2-separations of $\mathbb{F}_1$ is independent from the set of all 2-separations of $\mathbb{F}_2$.

(3) There exist petals B_1 of $\mathbb{F}_1$ and B_2 of $\mathbb{F}_2$ such that $\bar{B}_1 \subset B_2$ and $\bar{B}_2 \subset B_1$.

(4) There is no leaflet $\{B_1, B_2, B_3, B_4\}$ with $\mathbb{F}_1 = \{B_1 \cup B_2, B_3 \cup B_4\}$ and $\mathbb{F}_2 = \{B_1 \cup B_3, B_2 \cup B_4\}$.

Proof. It is easy to see that (3) $\Rightarrow$ (2) and that (2) $\Rightarrow$ (1). Let us show that (1) $\Rightarrow$ (3).

Claim 1. *For $i = 1, 2$, no petal of $\mathbb{F}_i$ can be partitioned into a set $\mathbb{S}$ of petals of $\mathbb{F}_{3-i}$ with $|\mathbb{S}| > 1$.*

Proof. Suppose for a contradiction that $B_i \in \mathbb{F}_i$ can be partitioned into a set $\mathbb{S}$ of petals of $\mathbb{F}_{3-i}$, where $|\mathbb{S}| > 1$. Let $\mathbb{F}' := \mathbb{S} \cup \{\bar{B}_i\}$. Then $\mathbb{F}_{3-i}$ is a refinement of $\mathbb{F}'$, hence $\mathbb{F}'$ is a flower. It follows that the sets in $\mathbb{S}$ are petals of $\mathbb{F}_i$, a contradiction as $\mathbb{F}_i$ is maximal. ◇

It follows from the claim that there exists a petal $B_1 \in \mathbb{F}_1$ that is not included in any petal of $\mathbb{F}_2$ and that there exists a petal $B_2 \in \mathbb{F}_2$ such that $B_2 \cap B_1$ and $B_2 - B_1$ are non-empty. As $B_1 - B_2$ is non-empty and B_1, B_2 do not cross, by (1) we must have that $B_1 \cup B_2 = E(G)$, i.e. (3) holds. Let us show that (1) $\Leftrightarrow$ (4). Clearly, if (4) does not hold then neither does (1). Suppose (1) does not hold, i.e. some petals $X \in \mathbb{F}_1$ and $Y \in \mathbb{F}_2$ cross. Let $\mathbb{F} = \{X \cap Y, X - Y, Y - X, \bar{X} \cap \bar{Y}\}$. Remark 5.5 implies that $\mathbb{F}$ is either a flower or a leaflet. The former case contradicts the fact that $\mathbb{F}_1$ is maximal, and the latter case shows that (4) does not hold. □

Given sequences $\mathbb{S} = (S_1, \ldots, S_k)$ and $\mathbb{S}' = (S'_1, \ldots, S'_r)$ we denote by $\mathbb{S} \odot \mathbb{S}'$ the concatenated sequence $(S_1, \ldots, S_k, S'_1, \ldots, S'_r)$. Consider a flower $\mathbb{F}$ of G and let $\mathbb{S}$ be a w-sequence $(X_1, \ldots, X_k)$ such that, for every $i \in [k]$, X_i is a 2-separation of the flower $\mathbb{F}$ in the graph $\mathrm{W}_{\text{flip}}[G, (X_1, \ldots, X_{i-1})]$. We then say that $\mathbb{S}$ is a w-sequence for the flower $\mathbb{F}$ of G.

Remark 5.7. *Let $\mathbb{S}$ be a w-sequence of G and suppose that $\mathbb{S} = \mathbb{S}_1 \odot \mathbb{S}_2$ for some independent sequences $\mathbb{S}_1, \mathbb{S}_2$. Let $\mathbb{S}'$ be obtained from $\mathbb{S}$ by rearranging the order of sets in $\mathbb{S}$ such that, for $i = 1,2$ and every $X, Y \in \mathbb{S}_i$, if X precedes Y in $\mathbb{S}_i$ it does so in $\mathbb{S}'$ as well. Then $\mathbb{S}'$ is a w-sequence of G and $\mathrm{W}_{\text{flip}}[G, \mathbb{S}] = \mathrm{W}_{\text{flip}}[G, \mathbb{S}']$. In particular, if $\mathbb{F}_1, \mathbb{F}_2$ are independent flowers and, for $i = 1,2$, $\mathbb{S}_i$ is a w-sequence for flower $\mathbb{F}_i$, then $\mathrm{W}_{\text{flip}}[G, \mathbb{S}_1 \odot \mathbb{S}_2] = \mathrm{W}_{\text{flip}}[G, \mathbb{S}_2 \odot \mathbb{S}_1]$.*

Lemma 5.8. *Let G and H be equivalent 2-connected graphs. Then there exists a set of maximal independent flowers $\mathbb{F}_1, \ldots, \mathbb{F}_k$ and there exists, for each $i \in [k]$, a w-sequence $\mathbb{S}_i$ of $\mathbb{F}_i$ such that*

$$H = \mathrm{W}_{\text{flip}}[G, \mathbb{S}_1 \odot \ldots \odot \mathbb{S}_k].$$

Proof. Since G and H are equivalent and 2-connected, there exists a w-sequence $\mathbb{S}$ of G for which $H = \mathrm{W}_{\text{flip}}[G, \mathbb{S}]$. Let us proceed by induction on the cardinality ℓ of $\mathbb{S}$. Let X be the last set in $\mathbb{S}$ and let $\mathbb{S}'$ be the sequence for which $\mathbb{S} = \mathbb{S}' \odot (X)$. Let $\mathbb{F}'$ be the maximal flower that refines $\{X, \bar{X}\}$. If $\ell = 1$, then $\mathbb{F}'$ and (X) are the required flower and corresponding sequence. Otherwise, by induction, there exists a set of maximal independent flowers $\mathbb{F}_1, \ldots, \mathbb{F}_r$ and there exists, for each $i \in [r]$, a w-sequence $\mathbb{S}_i$ of $\mathbb{F}_i$ such that $H = \mathrm{W}_{\text{flip}}[G, \mathbb{S}_1 \odot \ldots \odot \mathbb{S}_r \odot (X)]$. Suppose $\mathbb{F}' = \mathbb{F}_i$ for some $i \in [r]$. Because of Remark 5.7, we may assume that $\mathbb{F}' = \mathbb{F}_r$. Then $\mathbb{F}_1, \ldots, \mathbb{F}_r$ and $\mathbb{S}_1, \ldots, \mathbb{S}_r \odot (X)$ are the required flowers and corresponding w-sequences. Thus we may assume that $\mathbb{F}'$ is distinct from $\mathbb{F}_i$ for all $i \in [r]$. Suppose that $\mathbb{F}'$ is independent from $\mathbb{F}_1, \ldots, \mathbb{F}_r$. Then $\mathbb{F}_1, \ldots, \mathbb{F}_r, \mathbb{F}'$ and $\mathbb{S}_1, \ldots, \mathbb{S}_r, (X)$ are the required flowers and corresponding w-sequences. Hence, we may assume that for some $i \in [k]$, $\mathbb{F}'$ and $\mathbb{F}_i$ are not independent. Because of Remark 5.7, we may assume that $\mathbb{F}'$ and $\mathbb{F}_r$ are not independent. It follows from Lemma 5.6 that there exists a leaflet $\{B_1, B_2, B_3, B_4\}$ of $H' := \mathrm{W}_{\text{flip}}[G, \mathbb{S}_1 \odot \ldots \odot \mathbb{S}_{r-1}]$, where $\mathbb{F}_r = \{B_1 \cup B_2, B_3 \cup B_4\}$ and $\mathbb{F}' = \{B_1 \cup B_3, B_2 \cup B_4\}$. Hence $\mathbb{S}_r = (B_1 \cup B_2)$ and $X = B_1 \cup B_3$. It follows that $\mathrm{W}_{\text{flip}}[H', \mathbb{S}_r \odot (X)] = \mathrm{W}_{\text{flip}}[H', (B_2 \cup B_3)]$. Then $\mathbb{F}_1, \ldots, \mathbb{F}_r$ and $\mathbb{S}_1, \ldots, \mathbb{S}_{r-1}, (B_2 \cup B_3)$ are the required flowers and corresponding w-sequences. □

Let G be a graph and let $\mathbb{F} = \{B_1, \ldots, B_t\}$ be a flower of G. If $H = \mathrm{W}_{\text{flip}}[G, (B_i)]$ for some $i \in [t]$, then we say that H is obtained from G by *reversing* petal B_i. We say that petals B_i, B_j are *consecutive* in G if $V_G(B_i) \cap V_G(B_j) \neq \emptyset$.

Lemma 5.9. *Let $\mathbb{F}$ be a flower of a graph G and let B_1, B_2, B_3, B_4 be petals of $\mathbb{F}$. We can find a non-crossing w-sequence $\mathbb{S}$ of $\mathbb{F}$ such that, for $H := \mathrm{W}_{\mathit{flip}}[G, \mathbb{S}]$, both B_1, B_2 and B_3, B_4 are consecutive petals of $\mathbb{F}$ in H.*

Proof. There exists a flower $\mathbb{F}' = \{B'_1, B'_2, B'_3, B'_4\}$ such that:

- $\mathbb{F}$ is a refinement of $\mathbb{F}'$;
- $B_i \subseteq B'_i$ for $i = 1, 2, 3, 4$;
- $\mathscr{B}_G(B_i) \cap \mathscr{B}_G(B'_i) \neq \emptyset$.

Since $\mathbb{F}'$ has only 4 petals, there is a non-crossing w-sequence $\mathbb{S}'$ of $\mathbb{F}'$ such that, for $H' = \mathrm{W}_{\text{flip}}[G, \mathbb{S}']$, B'_1, B'_2, B'_3, B'_4 appear consecutively in H'. As H can be obtained from H' by possibly reversing some of the petals of $\mathbb{F}'$, the result follows. □

5.4 Proof of Propositions 5.2 and 5.3

Lemma 5.10. *Let $\mathbb{F}$ be a flower of G, let $\mathbb{L}$ be a w-sequence for flower $\mathbb{F}$, and let $H = \mathrm{W}_{\mathit{flip}}[G, \mathbb{L}]$. Consider $Z \subseteq V(G)$, where $|Z| \leq 2$. Then there exists a w-sequence $\mathbb{L}' \odot \mathbb{L}''$ of G such that:*

(1) $H = \mathrm{W}_{\mathit{flip}}[G, \mathbb{L}' \odot \mathbb{L}'']$;

(2) $Z \cap \mathscr{B}_G(X) = \emptyset$ for all $X \in \mathbb{L}'$;

(3) $\mathbb{L}''$ is non-crossing.

Proof. We only consider the case where $Z = \{z_1, z_2\}$ and where both z_1, z_2 are attachments of $\mathbb{F}$ in G, as the other cases are similar. For $i = 1, 2$, there exist consecutive petals B_i, B'_i in G such that $z_i \in \mathscr{B}_G(B_i) \cap \mathscr{B}_G(B'_i)$. Note that H is obtained from G by first permuting the petals of $\mathbb{F}$ and then by reversing a subset of the petals. Since the petals are 2-separations that do not cross any 2-separation of $\mathbb{F}$, we may assume that H is obtained from G by only permuting the petals of $\mathbb{F}$. It follows from Lemma 5.9 that there is a non-crossing w-sequence $\mathbb{L}''$ of H such that, in $H' := \mathrm{W}_{\text{flip}}[H, \mathbb{L}'']$, B_1, B_1' and B_2, B_2' are consecutive. Moreover, we can assume (by possibly reversing petals) that $z_i \in \mathscr{B}_{H'}(B_i) \cap \mathscr{B}_{H'}(B_i')$ for $i = 1, 2$. Let $\mathbb{F}'$ be the flower obtained from $\mathbb{F}$ by replacing, for $i = 1, 2$, petals B_i, B_i' by a unique petal $B_i \cup B_i'$. Then let $\mathbb{L}'$ be a w-sequence for flower $\mathbb{F}'$ such that $\mathrm{W}_{\text{flip}}[G, \mathbb{L}'] = H'$. □

We are now ready for the proof of the first main result.

Proof of Proposition 5.2. We say that a set of sequences $\mathbb{S}_1, \mathbb{S}_2, \mathbb{L}$ satisfies property (P) if there exist graphs H, H', where $\mathbb{S}_1, \mathbb{S}_2, \mathbb{L}$ are w-sequences of G, H', H respectively, and

(1') $H = \mathrm{W}_{\text{flip}}[G, \mathbb{S}_1]$, where $Z \cap \mathscr{B}_G(X) = \emptyset$ for all $X \in \mathbb{S}_1$;

(2') $H' = \mathrm{W}_{\text{flip}}[H, \mathbb{L}]$;

(3') $G' = \mathrm{W}_{\text{flip}}[H', \mathbb{S}_2]$ and $\mathbb{S}_2$ is non-crossing.

As we can choose $\mathbb{S}_1 = \mathbb{S}_2 = \emptyset$ and since G, G' are equivalent, a set of sequences $\mathbb{S}_1, \mathbb{S}_2, \mathbb{L}$ satisfying (P) exists. Lemma 5.8 implies that there exist maximal independent flowers $\mathbb{F}_1, \ldots, \mathbb{F}_k$ and there exists, for all $i \in [k]$, a w-sequence $\mathbb{L}_i$ for $\mathbb{F}_i$ such that $H' = \mathrm{W}_{\text{flip}}[H, \mathbb{L}_1 \odot \cdots \odot \mathbb{L}_k]$.

Among all choices of $\mathbb{S}_1, \mathbb{S}_2, \mathbb{L}_1, \ldots, \mathbb{L}_k$ where $\mathbb{S}_1, \mathbb{S}_2, \mathbb{L}_1 \odot \cdots \odot \mathbb{L}_k$ satisfy property (P), choose one that minimizes k. Suppose $k > 0$. Apply Lemma 5.10 to the sequence $\mathbb{L}_1$ and let $\mathbb{L}_1'$ and $\mathbb{L}_1''$ correspond to $\mathbb{L}'$ and $\mathbb{L}''$ in the statement of the Lemma. Define

$$\hat{\mathbb{S}}_1 := \mathbb{S}_1 \odot \mathbb{L}_1', \qquad \hat{\mathbb{S}}_2 := \mathbb{L}_1'' \odot \mathbb{S}_2.$$

Since flowers $\mathbb{F}_1, \ldots, \mathbb{F}_k$ of H are independent, $\mathbb{L}_1''$ is independent from $\mathbb{L}_2 \odot \cdots \odot \mathbb{L}_k$ (see Proposition 5.6). Therefore, by Remark 5.7,

$$G' = \mathrm{W}_{\text{flip}}[G, \hat{\mathbb{S}}_1 \odot \mathbb{L}_2 \odot \cdots \odot \mathbb{L}_k \odot \hat{\mathbb{S}}_2].$$

Then $\hat{\mathbb{S}}_1, \hat{\mathbb{S}}_2, \mathbb{L}_2 \odot \cdots \odot \mathbb{L}_k$ contradict our choice of $\mathbb{S}_1, \mathbb{S}_2, \mathbb{L}_1, \mathbb{L}_2$. Thus $k = 0$. Note that, if $Z \cap \mathscr{B}_G(X) = \emptyset$ for some $X \in \mathbb{S}_2$, then we can redefine $\mathbb{S}_1$ to be $\mathbb{S}_1 \odot (X)$ and $\mathbb{S}_2$ to be $\mathbb{S}_2 - \{X\}$ ($\mathbb{S}_2$ can be viewed as a set). Hence we may assume that $Z \cap \mathscr{B}_G(X) \neq \emptyset$ for all $X \in \mathbb{S}_2$ and the result follows. □

Remark 5.11. *Let G be a graph, let $\mathbb{S}$ be a non-crossing w-sequence of G, and let $G' = \mathrm{W}_{\mathit{flip}}[G,\mathbb{S}]$. Suppose that there exist $X_1,X_2,X_3 \in \mathbb{S}$, where $X_1 \subset X_2 \subset X_3$ and X_1,X_2,X_3 have distinct boundaries. Suppose that for some vertex z we have $z \in \mathscr{B}_G(X_i)$ for $i = 1,2,3$. Then $\mathscr{B}_{G'}(X_1) \cap \mathscr{B}_{G'}(X_2) \cap \mathscr{B}_{G'}(X_3) = \emptyset$.*

Proof. Because of Remark 5.1, we may assume that X_1,X_2,X_3 appear first in $\mathbb{S}$. Let $H = \mathrm{W}_{\mathrm{flip}}[G,\ (X_1,X_2,X_3)]$. Then $\mathscr{B}_H(X_1) \cap \mathscr{B}_H(X_2) \cap \mathscr{B}_H(X_3) = \emptyset$. The result now follows as $\mathbb{S}$ is non-crossing. $\square$

Lemma 5.12. *Let H,H' be equivalent graphs with $H' = \mathrm{W}_{\mathit{flip}}[H,\mathbb{S}]$ for some non-crossing w-sequence $\mathbb{S}$. Suppose that there exist vertices z in $V(H)$ and z' in $V(H')$ such that $z \in \mathscr{B}_H(X)$ and $z' \in \mathscr{B}_{H'}(X)$ for every $X \in \mathbb{S}$. Then $H' = \mathrm{W}_{\mathit{flip}}[H,\mathbb{S}']$ for some $\mathbb{S}'$ which is a w-star of H with center z and a w-star of H' with center z'.*

Proof. Note that we may swap any X in $\mathbb{S}$ with its complement and maintain the properties of $\mathbb{S}$. Since $\mathbb{S}$ is non-crossing we may assume (after possibly replacing some sets $\mathbb{S}$ by their complement) that $\mathbb{S}$ is laminar, i.e. every two sets in $\mathbb{S}$ are either disjoint or one contains the other. First suppose there exist $X_1,X_2 \in \mathbb{S}$ with $\mathscr{B}_H(X_1) = \mathscr{B}_H(X_2)$. Then we may remove X_1,X_2 from $\mathbb{S}$ and add $X_1 \triangle X_2$. This keeps the w-sequence non-crossing and gives rise to the same graph H'. Hence we may assume that, for every $X_1,X_2 \in \mathbb{S}$, $\mathscr{B}_H(X_1) \cap \mathscr{B}_H(X_2) = \{z\}$ and $\mathscr{B}_{H'}(X_1) \cap \mathscr{B}_{H'}(X_2) = \{z'\}$, and condition (b) in the definition of w-star holds. Suppose that for some $X_1,X_2 \in \mathbb{S}$ we have $X_1 \subset X_2$. By Remark 5.11, there is no set $X_3 \in \mathbb{S}$ where $X_3 \supseteq X_2$ or $\bar{X}_3 \supseteq X_2$. After replacing X_2 by $\bar{X}_2$ the sets in $\mathbb{S}$ satisfy condition (a) of the definition of w-stars. Finally, if any $X \in \mathbb{S}$ contains an edge e where the ends of e are $\mathscr{B}_H(X)$, we may replace X by $X - \{e\}$. Then property (c) of w-stars holds. $\square$

We are now ready for the proof of the second main result.

Proof of Proposition 5.3. Proposition 5.2 implies that there exist a w-sequence $\mathbb{L}$ of G and a graph H with a non-crossing sequence $\mathbb{S}_0$ such that (1) holds in the statement of the proposition, $G' = \mathrm{W}_{\mathrm{flip}}[H,\mathbb{S}_0]$ and $z \in \mathscr{B}_H(X)$ for all $X \in \mathbb{S}_0$. Because of Remark 5.1, we can view $\mathbb{S}_0$ as a set. Hence, $H = \mathrm{W}_{\mathrm{flip}}[G',\mathbb{S}_0]$. Let $\mathbb{L}' = \{X \in \mathbb{S}_0 : z' \notin \mathscr{B}_{G'}(X)\}$ and let $\mathbb{S}_1 := \mathbb{S}_0 - \mathbb{L}'$. Let $H' := \mathrm{W}_{\mathrm{flip}}[H,\mathbb{S}_1]$. Then condition (2) in the statement of the proposition holds. Finally, by Lemma 5.12, there exists a w-sequence $\mathbb{S}$ for H that is a w-star of H with center z and a w-star of H' with center z' and such that $H' = \mathrm{W}_{\mathrm{flip}}[H,\mathbb{S}]$. $\square$

5.5 Proof of Proposition 5.4

Let $\mathbb{F} = \{B_1, \ldots, B_t\}$ be a flower in a graph G. If the petals of $\mathbb{F}$ appear in the order $B_1, \ldots, B_t$ in G, we denote by $\mathbb{F}(G)$ the sequence $(B_1, \ldots, B_t)$. Note that this sequence is not uniquely defined, but we will fix one such sequence for each one of the graphs we are interested in.

To prove Proposition 5.4 we require the following result.

Lemma 5.13. *Let G and G' be equivalent graphs and let P be a path in G. Then there exists a graph H such that:*

(1) $H = \mathrm{W}_{\mathit{flip}}[G, \mathbb{S}_1]$, for some w-sequence $\mathbb{S}_1$ which preserves P, and

(2) $G' = \mathrm{W}_{\mathit{flip}}[H, \mathbb{S}_2]$, for some non-crossing w-sequence $\mathbb{S}_2$.

Proof. By Lemma 5.8, there exists a set of maximal independent flowers $\mathbb{F}_1, \ldots, \mathbb{F}_k$ and there exists, for all $i \in [k]$, a w-sequence $\mathbb{L}_i$ of $\mathbb{F}_i$ such that

$$G' = \mathrm{W}_{\mathrm{flip}}[G, \mathbb{L}_1 \odot \ldots \odot \mathbb{L}_k].$$

By Remark 5.7, it suffices to prove the statement for the case when the graphs G and G' are related by Whitney-flips on a sequence for a flower $\mathbb{F} = \{B_1, \ldots, B_t\}$ of G. By possibly relabeling the petals of $\mathbb{F}$, we may assume that $\mathbb{F}(G) = (B_1, \ldots, B_t)$ and $\{i \in [t] : P \cap B_i \neq \emptyset\} = \{1, \ldots, q\}$, for some $q \in [t]$.

The idea for the proof is the following: first we rearrange the order of the petals $B_2, \ldots,$ B_{q-1} and the petals $B_{q+1}, \ldots, B_t$ independently (using Whitney-flips which preserve P) and we obtain an appropriate graph H. Then we show that we can obtain G' from H by a sequence of pairwise non-crossing Whitney-flips.

Let $\mathbb{F}(G')$ be a sequence corresponding to $\mathbb{F}$ in G', where the first petal in $\mathbb{F}(G')$ is B_1. We define the following index sets:

(a) $I_1 := \{i \in [q-1] - \{1\} : B_q$ precedes B_i in $\mathbb{F}(G')\}$;

(b) $J_1 := \{i \in [q-1] - \{1\} : B_i$ precedes B_q in $\mathbb{F}(G')\}$;

(c) $I_2 := \{i \in [t] - [q] : B_i$ precedes B_q in $\mathbb{F}(G')\}$;

(d) $J_2 := \{i \in [t] - [q] : B_q$ precedes B_i in $\mathbb{F}(G')\}$.

By definition I_1, J_1, I_2, J_2 partition $[t] - \{1, q\}$. We now define the graph H by defining the order in which the petals of $\mathbb{F}$ appear in H. That is, we define the sequence $\mathbb{F}(H)$ corresponding to $\mathbb{F}$ in H, where the first petal in $\mathbb{F}(H)$ is B_1. We are not concerned with reversing some of the petals, as all the Whitney-flips we consider are sequences for $\mathbb{F}$. We define $\mathbb{F}(H)$ such that, for $k = 1, 2$:

(a) B_i precedes B_q in $\mathbb{F}(H)$, for every $i \in I_1 \cup J_1$;

(b) B_q precedes B_i in $\mathbb{F}(H)$, for every $i \in I_2 \cup J_2$;

(c) B_i precedes B_j in $\mathbb{F}(H)$, for every $i \in I_k$ and $j \in J_k$;

(d) the order in $\mathbb{F}(H)$ of petals B_i, for $i \in I_k$, is the reverse of their order in $\mathbb{F}(G)$;

(e) the order in $\mathbb{F}(H)$ of petals B_j, for $j \in J_k$, is the same as their order in $\mathbb{F}(G)$.

First note that H may be obtained from G by a sequence $\mathbb{S}_1$ which preserves P. In fact, we may rearrange petals $B_2, \ldots, B_{q-1}$, so that they satisfy the conditions above, by applying Whitney-flips on a w-sequence $\mathbb{L}_1$ such that, for every $X \in \mathbb{L}_1$, X is the union of petals from $\{B_i : i \in I_1 \cup J_1\}$. Similarly , we may rearrange petals $B_{q+1}, \ldots, B_t$, so that they satisfy the conditions above, by applying Whitney-flips on a w-sequence $\mathbb{L}_2$ such that, for every $X \in \mathbb{L}_2$, X is the union of petals from $\{B_i : i \in I_2 \cup J_2\}$. Then, for $k = 1, 2$, for every $X \in \mathbb{L}_k$ the ends of P are in $V_G(\bar{X})$, hence X preserves P. Thus $\mathbb{S}_1 := \mathbb{L}_1 \odot \mathbb{L}_2$ is the required w-sequence.

It remains to show that G' may be obtained from H by a sequence of non-crossing Whitney-flips. For every $i \in I_1$, let X_i be the union of B_1 and all petals succeeding B_i in $\mathbb{F}(G')$ and let $Y_i := X_i \cup B_i$. For every $i \in I_2$, let X_i be the union of B_q and all petals succeeding B_i and preceding B_q in $\mathbb{F}(G')$; let $Y_i := X_i \cup B_i$. Note that, for every $i \in I_1 \cup I_2$, both X_i and Y_i are formed by petals of $\mathbb{F}$ that are consecutive in H. Consider distinct $i, j \in I_1$ or $i, j \in I_2$ such that B_i precedes B_j in $\mathbb{F}(H)$; by definition of I_1 and I_2, B_j precedes B_i in $\mathbb{F}(G')$. Hence $B_i \in X_j, Y_j$ and $X_i, Y_i \subset X_j, Y_j$. Moreover, for all $i \in I_1$ and $j \in I_2$, B_j precedes B_q and B_q precedes B_i in $\mathbb{F}(G')$ (by definition of the sets I_1 and I_2). Thus $X_i, Y_i \subset \bar{X}_j, \bar{Y}_j$. It follows that the sequence $\mathbb{S}_2$ formed by the concatenation of (X_i, Y_i), for $i \in I_1 \cup I_2$, is a non-crossing w-sequence for H.

Now we show by induction on $|I_1 \cup I_2|$ that $G' = \mathrm{W}_{\text{flip}}[H, \mathbb{S}_2]$ (plus possibly reversing some of the petals of $\mathbb{F}$, but these Whitney-flips are not relevant for the proof). If both I_1 and I_2 are empty, then we are trivially done. Now suppose I_1 is non-empty; we may assume this is the case, by symmetry between I_1 and I_2. Let B_i be the first petal in $\mathbb{F}(H)$ such that $i \in I_1$. Let $H' := \mathrm{W}_{\text{flip}}[H, (X_i, Y_i)]$, $I_1' := I_1 - \{i\}$ and $J_2' := J_2 \cup \{i\}$. Then H' and I_1', J_1, I_2, J_2' satisfy properties (a)-(e) above. Thus, given the sequence $\mathbb{S}_2' := \mathbb{S}_2 - (X_i, Y_i)$, $G' = \mathrm{W}_{\text{flip}}[H', \mathbb{S}_2']$. □

Proof of Proposition 5.4. Among all graphs H as in Lemma 5.13, pick one such that $|\mathbb{S}_2|$ is minimized. As $\mathbb{S}_2$ is non-crossing, every $X \in \mathbb{S}_2$ is a 2-separation in H. If there exists $X \in \mathbb{S}_2$ that preserves P, then $\mathbb{S}_1' = \mathbb{S}_1 \cup X$ and $\mathbb{S}_2' = \mathbb{S}_2 - X$ satisfy $\mathrm{W}_{\text{flip}}[G, \mathbb{S}_1' \odot \mathbb{S}_2'] = \mathrm{W}_{\text{flip}}[G, \mathbb{S}_1 \odot \mathbb{S}_2] = G'$ and violate our choice of H. Thus every $X \in \mathbb{S}_2$ does not preserve P. It follows that, if u, v are the ends of P in H, $|\mathscr{I}_H(X) \cap \{u, v\}| = 1$ and $|\mathscr{I}_H(\bar{X}) \cap \{u, v\}| = 1$ for every $X \in \mathbb{S}_2$. Let $\mathbb{S}_u := \{X \in \mathbb{S}_2 : u \in \mathscr{I}_H(X)\}$ and $\mathbb{S}_v := \{X \in \mathbb{S}_2 : v \in \mathscr{I}_H(X)\}$. Note that $\mathbb{S}_u, \mathbb{S}_v$ partition $\mathbb{S}_2$. Define $\mathbb{S}_2' := \mathbb{S}_u \cup \{\bar{X} : X \in \mathbb{S}_v\}$. Then the sets in $\mathbb{S}_2'$ may be ordered to form a nested w-sequence and, for every $X \in \mathbb{S}_2'$, X does not preserve P, as required. □

5.6 Whitney-flips on grafts

The results in this section are exclusively used in Section 9.6.

5.6.1 Flowers in grafts

Lemma 5.14. *Let (H, T) be a graft and $\mathbb{F} = \{B_1, \ldots, B_t\}$ be a flower of H with attachments $u_1, \ldots, u_t$. Suppose $T = T_a \cup T_b$, where $T_a \subseteq \{u_1, \ldots, u_t\}$, $|T_b| \leq 4$ and, for every $v, w \in T_b$, we have $v \in \mathscr{I}_H(B_i)$ and $w \in \mathscr{I}_H(B_j)$, for distinct $i, j \in [t]$. Then there exists a graft (H', T') equivalent to (H, T) with $|T'| \leq 4$.*

Proof. Note that every graft obtained from (H, T) by Whitney-flips on a sequence for $\mathbb{F}$ satisfies the same hypothesis as (H, T). Let $|T| = 2k$ for some integer k. We may choose a T-join $J = P_1 \triangle P_2 \triangle \cdots P_k$ where $P_1, \ldots, P_k$ are pairwise vertex-disjoint paths of H. Let $\mathbb{B} := \{B \in \mathbb{F} : B \cap P_i \neq \emptyset, \text{ for some } i \in [k]\}$. Let H' be obtained from H by rearranging the petals of $\mathbb{F}$ so that the petals in $\mathbb{B}$ are consecutive in H'. By possibly reversing some of the petals in H' we may obtain a graph H'' where J is the union of at most two paths. Let $T'' := V_{odd}(H''[J])$. Then $|T''| \leq 4$ and (H'', T'') is equivalent to (H, T). □

5.6.2 Caterpillars

A *caterpillar* is a tree obtained by taking a path and adding edges which have exactly one end in common with the path. Let G be a graph and let $\mathbb{S} = (X_1, \ldots, X_k)$ be a nested w-sequence for G. We denote by $\mathrm{Cat}(G, \mathbb{S})$ the graph defined on the vertex set $\cup_{i=1}^{k} \mathscr{B}_G(X_i)$ with edge set $\{e_1, \ldots, e_k\}$, where the ends of e_i are the vertices in $\mathscr{B}_G(X_i)$. Note that $\mathrm{Cat}(G, \mathbb{S})$ is a vertex-disjoint union of caterpillars. Given a graft (G, T) and a w-sequence $\mathbb{S}$ for G, we denote by $\mathrm{W}_{\text{flip}}[(G, T), \mathbb{S}]$ the graft (G', T'), where $G' = \mathrm{W}_{\text{flip}}[G, \mathbb{S}]$ and (G, T) and (G', T') are equivalent.

Lemma 5.15. *Let G be a graph and let $\mathbb{S} = (X_1, \ldots, X_k)$ be a nested w-sequence for G. Let $s, t \in V(G)$ with $s \in \mathscr{I}_G(X_1)$ and $t \in \mathscr{I}_G(\bar{X}_k)$. Let $(G', T) := \mathrm{W}_{\text{flip}}[(G, \{s, t\}), \mathbb{S}]$. Then $T = \{s, t\} \cup V_{odd}(\mathrm{Cat}(G', \mathbb{S}))$.*

Proof. Let us proceed by induction on k. The result is trivially true for $k = 0$. Thus let us assume that $k \geq 1$ and that the result holds for $k - 1$.

Let $\mathbb{S}' = (X_1, \ldots, X_{k-1})$ and define $(H, T') := \mathrm{W}_{\text{flip}}[(G, \{s, t\}), \mathbb{S}']$. By induction

$$T' = V_{odd}(\mathrm{Cat}(H, \mathbb{S}')) \cup \{s, t\}. \tag{5.1}$$

We have $(G', T) = \mathrm{W}_{\text{flip}}[(H, T'), (X_k)]$. Let u, v denote the vertices in $\mathscr{B}_H(X_k) = \mathscr{B}_{G'}(X_k)$. Since $\mathrm{Cat}(G', \mathbb{S})$ is obtained from $\mathrm{Cat}(H, \mathbb{S}')$ by adding vertices u, v (if not already in it) and edge uv,

$$V_{odd}(\mathrm{Cat}(G', \mathbb{S})) = V_{odd}(\mathrm{Cat}(H, \mathbb{S}')) \triangle \{u, v\}. \tag{5.2}$$

We claim that it suffices to prove that $T \triangle T' = \{u, v\}$ as this implies that

$$\begin{aligned}
T =& T' \triangle \{u, v\} & \\
=& (V_{odd}(\mathrm{Cat}(H, \mathbb{S}')) \cup \{s, t\}) \triangle \{u, v\} & \text{by (5.1)} \\
=& (V_{odd}(\mathrm{Cat}(H, \mathbb{S}')) \triangle \{u, v\}) \cup \{s, t\} & \\
=& (V_{odd}(\mathrm{Cat}(G, \mathbb{S})) \cup \{s, t\}, & \text{by (5.2)}
\end{aligned}$$

as required. Let J be a T'-join of H. Then T is defined as $V_{odd}(G'[J])$, hence J is a T-join of G'. Therefore

$$T' = V_{odd}(H[J \cap X_k]) \triangle V_{odd}(H[J \setminus X_k]), \text{ and} \tag{5.3}$$

$$T = V_{odd}(G'[J \cap X_k]) \triangle V_{odd}(G'[J \setminus X_k]). \tag{5.4}$$

As $H[X_k] = G'[X_k]$, (5.3), (5.4) imply that

$$T \triangle T' = V_{odd}(H[J \setminus X_k]) \triangle V_{odd}(G'[J \setminus X_k]). \tag{5.5}$$

Since $T' \setminus \{t\} \subset V_H(X_k)$ we may assume (after possibly interchanging the role of u and v) that

$$V_{odd}(H[J \setminus X_k]) = \{u, t\}. \tag{5.6}$$

As $G' = \mathrm{W}_{\text{flip}}[H, (X_k)]$, it follows that

$$V_{odd}(G'[J \setminus X_k]) = \{v, t\}. \tag{5.7}$$

Then (5.5), (5.6) and (5.7) imply that $T \triangle T' = \{u, v\}$, as required. $\square$

Chapter 6

Proofs of the even cycle isomorphism results

In this chapter we prove the results about split siblings and quad siblings stated in Chapter 4 using results from Chapter 5.

6.1 Proof of Theorem 4.3 - split siblings

We say that split-templates

$$\mathbb{T} = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2, \mathbb{S}) \quad \text{and} \quad \mathbb{T}' = (H_1', v_1', \alpha_1', H_2', v_2', \alpha_2', \mathbb{S}') \tag{6.1}$$

are *compatible* if:

(a) H_i, H_i' are equivalent, for $i = 1, 2$, and

(b) $\alpha_i \triangle \alpha_i'$ forms a cut of H_1, for $i = 1, 2$.

Note that, by Theorem 1.1, $\text{cut}(H_1) = \text{cut}(H_2) = \text{cut}(H_1') = \text{cut}(H_2')$.

Lemma 6.1. *Let $\mathbb{T}$ and $\mathbb{T}'$ be compatible split-templates. Let (G_1, Σ_1), (G_2, Σ_2) be the siblings arising from $\mathbb{T}$ and (G_1', Σ_1'), (G_2', Σ_2') be the siblings arising from $\mathbb{T}'$. Then, for $i = 1, 2$, (G_i, Σ_i) and (G_i', Σ_i') are equivalent.*

Proof. Let us assume that $\mathbb{T}, \mathbb{T}'$ are as described in (6.1). Then, by construction,

$$\mathrm{cut}(G_1) = \mathrm{span}\big(\mathrm{cut}(H_1) \cup \{\alpha_1\}\big) \quad \text{and} \quad \mathrm{cut}(G_1') = \mathrm{span}\big(\mathrm{cut}(H_1) \cup \{\alpha_1'\}\big).$$

By hypothesis, $\alpha_1 \triangle \alpha_1' \in \mathrm{cut}(H_1)$. Hence, $\mathrm{cut}(G_1) = \mathrm{cut}(G_1')$. It follows from Theorem 1.1 that G_1 and G_1' are equivalent. Similarly, G_2 and G_2' are equivalent. It follows that $(G_1', \Sigma_1), (G_2', \Sigma_2)$ are siblings. As the matching signature pair for G_1', G_2' is unique up to signature exchange, (G_i, Σ_i) and (G_i', Σ_i') are equivalent, for $i = 1, 2$. □

Lemma 6.2. *Every split-template has a compatible split-template which is simple or nova.*

Proof. Suppose that $\mathbb{T} := (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2, \mathbb{S})$ is a split-template.

Claim 1. *There is a template $(H_1', v_1, \alpha_1, H_2', v_2, \alpha_2, \mathbb{S}')$ which is compatible with $\mathbb{T}$ and has the property that $\mathbb{S}'$ is a w-star of H_1', H_2'.*

Proof. The proof follows easily from Proposition 5.3, since H_1 and H_2 are equivalent. ◇

Choose a split-template $\mathbb{T}' = (H_1', v_1', \alpha_1', H_2', v_2', \alpha_2', \mathbb{S}')$ with the following properties:

(M1) $\mathbb{T}'$ is compatible with $\mathbb{T}$;

(M2) for $i = 1, 2$, $\mathbb{S}'$ is a w-star of H_i' with center v_i';

(M3) $|\cup\{X : X \in \mathbb{S}'\}|$ is minimized among all split-templates satisfying (M1) and (M2).

Such a split-template exists because of Claim 1. We may assume that $\mathbb{S}' \neq \emptyset$ for otherwise $\mathbb{T}'$ is simple and we are done. We will show that $\mathbb{T}'$ is nova. As (N1) (from the definition of nova) holds, it suffices to prove (N2). Let $X' \subseteq X \in \mathbb{S}'$, where $\mathscr{B}_{H_1'}(X') = \mathscr{B}_{H_1'}(X) = \{v_1', w\}$, for some vertex w. Let us assume that we chose X' to be an inclusion-wise minimal subset with that property. It suffices to show for (N2) (as we can interchange the role of H_1' and H_2') that there exists $\{v_1', w\}$-handcuffs included in X' in $(H_1', \alpha_1' \triangle \alpha_2')$.

Claim 2. *None of the following holds:*

(1) $\delta_{H_1'}(v_1') \cap X' \cap \alpha_1'$ *is empty;*

(2) $(\delta_{H_1'}(v_1') \cap X') - \alpha_1'$ *is empty;*

(3) *we can partition X' into Z,Z' such that $\mathscr{B}_{H_1'}(X') = \mathscr{B}_{H_1'}(Z) = \mathscr{B}_{H_1'}(Z')$ and $\alpha_1' \cap X' = \delta_{H_1'}(v_1') \cap Z$.*

Proof. Define

$$D := \begin{cases} \emptyset & \text{if (1) holds} \\ \delta_{H_1'}(v_1') & \text{if (2) holds} \\ \delta_{H_1'}(\mathscr{I}_{H_1'}(Z)) & \text{if (3) holds.} \end{cases}$$

Let $\tilde{\alpha} = \alpha_1' \triangle D$, let $\tilde{H} := \mathrm{W}_{\text{flip}}[H_1', (X')]$ and let $\tilde{\mathbb{S}} = \mathbb{S}' - \{X\} \cup \{X - X'\}$. There is a vertex $\tilde{v}$ of $\tilde{H}$ where $\delta_{\tilde{H}}(\tilde{v}) \supseteq \tilde{\alpha}$. Since $\mathbb{S}$ is non-crossing, $H_2' = \mathrm{W}_{\text{flip}}[\tilde{H}, \tilde{\mathbb{S}}]$. Hence, (M2) holds for $\tilde{\mathbb{T}} := (\tilde{H}, \tilde{v}, \tilde{\alpha}, H_2', v_2', \alpha_2', \tilde{\mathbb{S}})$. Since D is a cut of H_1', (M3) holds for $\tilde{\mathbb{T}}$. As $|\cup\{X : X \in \tilde{S}\}| < |\cup\{X : X \in \mathbb{S}'\}|$, this contradicts our choice (M3). $\diamond$

Claim 3. *There exists a circuit $C \subseteq X'$ of H_1' avoiding w with $|C \cap \alpha_1'|$ odd.*

Proof. We claim that otherwise (1),(2), or (3) of Claim 2 must hold, giving a contradiction. Let G be the graph obtained from $H_1'[X']$ by splitting v_1' into $v_1'^+, v_1'^-$ according to α_1'. Every $(v_1'^-, v_1'^+)$-path P of $H[X']$ avoiding w is a required circuit. Hence, we may assume that no such path exists. It follows that w is a cut-vertex separating $v_1'^-$ and $v_1'^+$ in $G[X']$. Let Z,Z' be the partition of X' such that $V_{G[X']}(Z) \cap V_{G[X']}(Z') = \{w\}$ and $v_1'^- \in G[Z]$, $v_1'^+ \in G[Z']$. Then (3) holds. $\diamond$

By Claim 3 and by reversing the role of H_1' and H_2', we deduce that there exists an odd circuit C_1 (respectively C_2) included in X' using v_1' (respectively w) and avoiding w (respectively v_1'). Consider first the case where C_1 and C_2 have at least one common vertex in H_1'. As $\alpha_1' \subseteq \delta_{H_1'}(v_1')$ and $\alpha_2' \subseteq \delta_{H_1'}(w)$, we may assume, after possibly redefining C_1, that C_1 and C_2 intersect in exactly one vertex or intersect in a path. Hence, in that case $(C_1, C_2, \emptyset)$ form $\{v_1', w\}$-handcuffs included in X' in $(H_1', \alpha_1' \triangle \alpha_2')$, as required. Consider now the case where C_1 and C_2 have no common vertex in H_1'. As X' was selected to be inclusion-wise minimal, there exists a path $P \subset X'$ joining C_1 and C_2 which avoids v_1' and w. For an inclusion-wise minimal such P, (C_1, C_2, P) form $\{v_1', w\}$-handcuffs in $(H_1', \alpha_1' \triangle \alpha_2')$ as required. $\square$

Proof of Theorem 4.3. By definition, (G_1, Σ_1) and (G_2, Σ_2) arise from a split-template $(H_1, v_1, \alpha_1, H_2, v_2, \alpha_2)$. Let T_1, T_2 be the matching terminal pair for G_1, G_2. Lemma 2.4

implies that G_1 and G_2 are 2-connected, except for possible loops. Consider first the case where $H_1 \setminus \mathrm{loop}(H_1)$ is not 2-connected. Then for some $X \subseteq E(G_1)$, $\mathscr{B}_{G_1}(X) = T_1$. It follows, from the argument in Section 4.5.3, that $(G_1, \Sigma_1), (G_2, \Sigma_2)$ can be reduced. Hence, H_1 is 2-connected, except for possible loops, and so is H_2. It follows that $\mathbb{T} = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2, \mathbb{S})$ is a split-template for some w-sequence $\mathbb{S}$ of H_1, where $H_2 = \mathrm{W}_{\text{flip}}[H_1, \mathbb{S}]$. Lemma 6.2 implies that there exists a split-template $\mathbb{T}'$ which is simple or nova and compatible with $\mathbb{T}$. Let $(G_1', \Sigma_1'), (G_2', \Sigma_2')$ arise from $\mathbb{T}'$. By definition $(G_1', \Sigma_1'), (G_2', \Sigma_2')$ are simple twins or nova twins. By Lemma 6.1, for $i = 1, 2$, (G_i', Σ_i') is equivalent to (G_i, Σ_i). It follows that $(G_1, \Sigma_1), (G_2, \Sigma_2)$ are simple or nova siblings. □

6.2 Proof of Theorem 4.6 - quad siblings

To prove Theorem 4.6 we require some preliminary results. Similarly to the proof for split siblings, we define compatible quad-templates. The different types of quad siblings arise from different types of templates.

6.2.1 Templates

Remark 6.3. *Suppose that* $\mathbb{T} = (H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2)$ *is a quad-template and* (G_1, Σ_1), (G_2, Σ_2) *are the quad siblings arising from* $\mathbb{T}$. *Then* α_{3-i} *and* β_{3-i} *are signatures of* (G_i, Σ_i), *for* $i = 1, 2$.

Proof. For $i = 1, 2$, vertex v_i of H_i gets split into vertices v_i^-, v_i^+ of G_i. By construction, $\alpha_i = \delta_{G_i}(v_i^-)$, for $i = 1, 2$. As $v_1^- \in T_1$, Theorem 3.1 implies that α_1 is a signature of (G_2, Σ_2). Similarly β_1 is a signature of (G_2, Σ_2). By symmetry, α_2, β_2 are signatures of (G_1, Σ_1). □

We say that two quad-templates

$$\begin{gathered} \mathbb{T} = (H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2) \\ \text{and} \\ \mathbb{T}' = (H_1', v_1', w_1', \alpha_1', \beta_1', H_2', v_2', w_2', \alpha_2', \beta_2') \end{gathered} \tag{6.2}$$

are *compatible* if, for $i = 1, 2$:

(a) H_i is equivalent to H_i';

(b) $\alpha_i \Delta \alpha_i'$ is a cut of H_1;

(c) $\beta_i \Delta \beta_i'$ is a cut of H_1.

Note that, by Theorem 1.1, $\text{cut}(H_1) = \text{cut}(H_2) = \text{cut}(H_1') = \text{cut}(H_2')$.

Lemma 6.4. *Let $\mathbb{T}$ and $\mathbb{T}'$ be compatible quad-templates. Let (G_1, Σ_1), (G_2, Σ_2) and (G_1', Σ_1'), (G_2', Σ_2') be quad siblings arising from $\mathbb{T}$ and $\mathbb{T}'$ respectively. Then (G_i, Σ_i) and (G_i', Σ_i') are equivalent, for $i = 1, 2$.*

Proof. Let $\mathbb{T}$, $\mathbb{T}'$ be compatible quad-templates defined as in (6.2). Fix $i \in [2]$. Let $v_i^-, v_i^+ \in V(G_i)$ be obtained by splitting v_i according to α_i in H_i. We first show that G_i is equivalent to G_i' by showing that $\text{cut}(G_i) = \text{cut}(G_i')$. Let $C = \alpha_i \Delta \alpha_i'$. By definition of compatible templates, C is a cut of H_i', hence it is a cut of G_i'. Moreover, by construction, $\delta_{G_i}(v_i^-) = \alpha_i$. Thus $\delta_{G_i}(v_i^-) = C \Delta \alpha_i'$ is a cut of G_i'. Similarly, we can show that $\delta_{G_i}(v_i^+)$ is a cut of G_i'. By symmetry between v_i and w_i, we have that $\delta_{G_i}(w_i^-)$, $\delta_{G_i}(w_i^+)$ are cuts of G_i'. Moreover, for every $u \in V(G_i)$, if $u \neq v_i^-, v_i^+, w_i^-, w_i^+$, then $\delta_{G_i}(u)$ is a cut of H_i, hence a cut of H_i' and a cut of G_i'. Thus $\delta_{G_i}(u)$ is a cut of G_i' for every $u \in V(G_i)$. As the cuts of G_i are generated by its fundamental cuts (i.e. the cuts of the form $\delta_{G_i}(u)$, for $u \in V(G_i)$), this shows that $\text{cut}(G_i) \subseteq \text{cut}(G_i')$. By symmetry between $\mathbb{T}$ and $\mathbb{T}'$, we have that $\text{cut}(G_i') = \text{cut}(G_i)$, thus G_i and G_i' are equivalent. As G_i is equivalent to G_i', Σ_1, Σ_2 is a matching signature pair for G_1', G_2'. By Proposition 3.7, the matching signature pair is unique up to resigning, thus (G_i, Σ_i) and (G_i', Σ_i') are equivalent. □

Let $(H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2)$ be a quad-template. If H_1, H_2 are 2-connected, except for possible loops, we have that $H_2 = \text{W}_{\text{flip}}[H_1, \mathbb{S}]$ for some w-sequence $\mathbb{S}$. We abuse terminology slightly and say that $(H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2, \mathbb{S})$ is a *quad-template*. (This is only well defined for the case where H_1, H_2 are 2-connected up to loops).

Consider a template $\mathbb{T} = (H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2, \mathbb{S})$, where $\mathbb{S} = (X_1, \ldots, X_k)$ for some $k \geq 0$ and $X_i \neq \emptyset$ for every $i \in [k]$. We say that $\mathbb{T}$ is of *type I* if:

(TIa) $X_i \cap X_j = \emptyset$, for every $i, j \in [k]$, $i \neq j$;

(TIb) $H_i[X_j] \setminus \mathscr{B}_{H_i}(X_j)$ is non-empty and connected, for every $i = 1, 2$ and $j \in [k]$;

(TIc) $\mathscr{B}_{H_i}(X_j) = \{v_i, w_i\}$, for $i = 1,2$ and $j \in [k]$.

We say that $\mathbb{T}$ is of *type II* if:

(TIIa) $k = 1$ or $k = 2$;

(TIIb) if $k = 2$, X_1 is disjoint from X_2;

(TIIc) $v_i \in \mathscr{B}_{H_i}(X_j)$, for $i = 1,2$ and $j \in [k]$;

(TIId) $w_1 \in \mathscr{I}_{H_1}(X_1)$;

(TIIe) if $k = 1$, $w_2 \in \mathscr{I}_{H_2}(\bar{X}_1 - \text{loop}(H_2))$;

(TIIf) if $k = 2$, $w_2 \in \mathscr{I}_{H_2}(X_2)$.

6.2.2 The proof

A signed graph (G,Σ) is *ec-standard* if ecycle(G,Σ) is 3-connected and, for every (G',Σ') equivalent to (G,Σ), (G',Σ') does not contain a blocking vertex. To prove Theorem 4.6 we require the following four results, which will be proved at the end of the chapter.

Lemma 6.5. *Suppose that $(G_1,\Sigma_1),(G_2,\Sigma_2)$ are quad siblings arising from a quad-template $\mathbb{T}$ of type I. Suppose that* ecycle(G_1,Σ_1) *is* 3*-connected. Then (G_1,Σ_1), (G_2,Σ_2) are either shuffle, tilt or twist twins.*

Lemma 6.6. *Suppose that $(G_1,\Sigma_1),(G_2,\Sigma_2)$ are Δ-irreducible ec-standard quad siblings arising from a quad-template $\mathbb{T}$ of type II. Then $(G_1,\Sigma_1),(G_2,\Sigma_2)$ are either widget or gadget twins.*

Lemma 6.7. *Suppose that $(G_1,\Sigma_1),(G_2,\Sigma_2)$ are Δ-irreducible ec-standard quad siblings arising from a quad-template $\mathbb{T} = (H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2, \mathbb{S})$. Then there exists a template $\mathbb{T}'$ which is compatible with $\mathbb{T}$ and is of type I or type II.*

Lemma 6.8. *Let $\mathbb{T} = (H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2)$ be a quad-template. Let (G_1,Σ_1), (G_2,Σ_2) be the quad siblings arising from $\mathbb{T}$. If (G_1,Σ_1) and (G_2,Σ_2) are ec-standard and Δ-irreducible, then either $(G_1,\Sigma_1),(G_2,\Sigma_2)$ are shuffle, tilt, twist, gadget or widget siblings or H_1, H_2 are* 2*-connected, except for the possible presence of loops.*

Proof of Theorem 4.6. Let M be a 3-connected non-graphic matroid and $(G_1,\Sigma_1),(G_2,\Sigma_2)$ be quad siblings representating M. By Remark 2.9, $(G_1,\Sigma_1),(G_2,\Sigma_2)$ are ec-standard. If they are Δ-reducible we are done. Thus in the remainder of the proof we will assume that (G_1,Σ_1), (G_2,Σ_2) are Δ-irreducible quad siblings. Suppose that they arise from a quad-template $(H_1,v_1,w_1,\alpha_1,\beta_1,H_2,v_2,w_2,\alpha_2,\beta_2)$. By Lemma 6.8, either (G_1,Σ_1), (G_2,Σ_2) fall into one of the cases $(1)-(5)$ in the statement of the theorem, or H_1,H_2 are 2-connected, except for the presence of loops. Therefore we may assume that $H_2 = \mathrm{W}_{\text{flip}}[H_1,\mathbb{S}]$ for some w-sequence $\mathbb{S}$ of H_1 and $(G_1,\Sigma_1),(G_2,\Sigma_2)$ arise from a quad-template $\mathbb{T}$ with w-sequence $\mathbb{S}$. By Lemma 6.7, there exists a quad-template $\mathbb{T}'$ compatible with $\mathbb{T}$ which is of type I or of type II. Let (G_1',Σ_1') and (G_2',Σ_2') be the quad siblings arising from $\mathbb{T}'$. If $\mathbb{T}'$ is of type I then, by Lemma 6.5, (G_1',Σ_1') and (G_2',Σ_2') are shuffle, tilt or twist siblings. If $\mathbb{T}'$ is of type II then, by Lemma 6.6, (G_1',Σ_1') and (G_2',Σ_2') are widget or gadget twins. Finally, by Lemma 6.4, (G_i,Σ_i) and (G_i',Σ_i') are equivalent for $i=1,2$. Therefore the result follows. □

The proofs of Lemma 6.5, Lemma 6.6, Lemma 6.7 and Lemma 6.8 are given in Section 6.2.4. First we require some technical results.

6.2.3 Technical lemmas

Recall that a set X is a 3-$(0,1)$-separation of a signed graph (G,Σ) if X is a 3-separation of G such that $(G[X],\Sigma\cap X)$ is bipartite and $(G[\bar{X}],\Sigma - X)$ is non-bipartite.

Lemma 6.9. *Let (G_1,Σ_1), (G_2,Σ_2) be ec-standard siblings. Let X be a* 3-$(0,1)$-*separation in both (G_1,Σ_1) and (G_2,Σ_2). Then (G_1,Σ_1), (G_2,Σ_2) are Δ-reducible.*

Proof. Let $\mathscr{B}_{G_1}(X)=\{u_1,u_2,u_3\}$ and $\mathscr{B}_{G_2}(X)=\{u_1',u_2',u_3'\}$. We claim that we can relabel the vertices in $\mathscr{B}_{G_1}(X)$ so that every (u_i,u_j)-path in $G_1[X]$ is a $(u_i,' u_j')$-path in $G_2[X]$, for every choice of $i,j\in[3], i\neq j$. Consider $i,j\in[3], i\neq j$. Let P be a (u_i,u_j)-path in $G_1[X]$. Let Q be a (u_i,u_j)-path in $G_1[\bar{X}]$ of the same parity as P. Note that some such Q exists as ecycle(G_1,Σ_1) is 3-connected and $(G_1[\bar{X}],\Sigma_1\cap\bar{X})$ is non-bipartite. By the choice of Q, $C := P\cup Q$ is an even circuit of ecycle(G_1,Σ_1), hence a circuit of ecycle(G_2,Σ_2). As $(G_2[X],\Sigma_2\cap X)$ is bipartite, every cycle in $G_2[X]$ is even, hence $G_2[P]$ does not contain any cycle. Therefore P is a (u_s',u_t')-path in $G_2[X]$ for some $s,t\in[3], s\neq t$. The same argument holds for every choice of $i, j \in [3], i \neq j$. Now, if P_1 is a (u_i, u_j)-path and P_2 is a (u_j, u_h)-path in $G_1[X]$, for distinct $i, j, h \in [3]$, then P_1 is a (u'_s, u'_t)-path in $G_2[X]$ and P_2 is a (u'_q, u'_r)-path in $G_2[X]$. We cannot have $\{s,t\} = \{q,r\}$, as otherwise $P_1 \cup P_2$ would be an even cycle in (G_2, Σ_2) and a path in G_1. Therefore P_1, P_2 share exactly one end in $G_2[X]$, say u'_t. Thus, we can reindex u_j as u_t. Similarly we can reindex all the vertices in $\mathscr{B}_{G_1}[X]$ as desired. Note that, in particular, $G_1[X] = G_2[X]$. For $i = 1,2$, let Σ'_i be a resigning of (G_i, Σ_i) such that $\Sigma'_i \cap X = \emptyset$. Define $Y := X \cup \{e \in E(G_1) : e \notin \Sigma'_i, e = (u_i, u_j) \text{ for some } i, j \in [3]\}$. Now we can apply a Δ-reduction to Y. □

Lemma 6.10. *Let H be a graph and let s_1, s_2 be distinct vertices of H. Let $\varphi_i \subseteq \delta_H(s_i)$, for $i = 1,2$. Suppose that $\varphi_1 \Delta \varphi_2$ is a non-empty cut of H such that $\varphi_1 \Delta \varphi_2 \neq \delta_H(s_2)$. Then there exists $Y \subseteq E(H)$ such that the following hold:*

(1) $\mathscr{B}_H(Y) \subseteq \{s_1, s_2\}$;

(2) $\mathscr{I}_H(Y) \neq \emptyset$;

(3) $\delta_H(s_1) \cap Y = \varphi_1 - \varphi_2$;

(4) for $\hat{\varphi}_2 := \varphi_2$ or $\hat{\varphi}_2 := \varphi_2 \Delta \delta_H(s_2)$, $\delta_H(s_2) \cap Y = \hat{\varphi}_2 - \varphi_1$.

Proof. As $\varphi_1 \Delta \varphi_2$ is a non-empty cut of H, $\varphi_1 \Delta \varphi_2 = \delta_H(U)$ for some $U \subset V(H)$, where $U \neq \emptyset, V(H)$. If $s_1 \in U$, we can pick $V(H) - U$ instead of U. Thus we may assume that $s_1 \notin U$. If $s_2 \notin U$, let $\hat{\varphi}_2 := \varphi_2$ and $W := U$, otherwise let $\hat{\varphi}_2 := \varphi_2 \Delta \delta_H(s_2)$ and $W := U - \{s_2\}$. Thus $s_1, s_2 \notin W$ and $\delta_H(W) = \varphi_1 \Delta \hat{\varphi}_2$. Define $Y := \{(u,v) \in E(H) : \{u,v\} \cap W \neq \emptyset\}$. Conditions (3) and (4) in the statement are satisfied by construction. Note that $U \neq \{s_2\}$, as $\varphi_1 \Delta \varphi_2 \neq \delta_H(s_2)$. Hence W is non-empty and $\mathscr{I}_H(Y)$ is non-empty. For every $v \in W$, $\delta_H(v) \subseteq Y$, hence $v \notin \mathscr{B}_H(Y)$. Moreover, for every $v \notin W \cup \{s_1, s_2\}$, $\delta_H(v) \cap Y = \emptyset$, hence $v \notin \mathscr{B}_H(Y)$. Hence $\mathscr{B}_H(Y) \subseteq \{s_1, s_2\}$. □

Lemma 6.11. *Let H be a graph and s_1, s_2, s_3 be distinct vertices of H. Let $\varphi_i \subseteq \delta_H(s_i)$, for $i = 1,2,3$. Suppose that $\varphi_1 \Delta \varphi_2 \Delta \varphi_3$ is a non-empty cut of H. Suppose moreover that $\varphi_1 \Delta \varphi_2 \Delta \varphi_3$ is not equal to any of the sets $\delta_H(s_2), \delta_H(s_3), \delta_H(\{s_2, s_3\})$. Then there exists $Y \subseteq E(H)$ such that the following hold:*

(1) $\mathscr{B}_H(Y) \subseteq \{s_1, s_2, s_3\}$;

(2) $\mathscr{I}_H(Y) \neq \emptyset$*;*

(3) $\delta_H(s_1) \cap Y = \varphi_1 - (\varphi_2 \cup \varphi_3)$*;*

(4) for $\hat{\varphi}_2 := \varphi_2$ *or* $\hat{\varphi}_2 := \varphi_2 \Delta \delta_H(s_2)$*,* $\delta_H(s_2) \cap Y = \hat{\varphi}_2 - (\varphi_1 \cup \varphi_3)$*;*

(5) for $\hat{\varphi}_3 := \varphi_3$ *or* $\hat{\varphi}_3 := \varphi_3 \Delta \delta_H(s_3)$*,* $\delta_H(s_3) \cap Y = \hat{\varphi}_3 - (\varphi_1 \cup \varphi_2)$.

Proof. As $\varphi_1 \Delta \varphi_2 \Delta \varphi_3$ is a non-empty cut of H, $\varphi_1 \Delta \varphi_2 \Delta \varphi_3 = \delta_H(U)$ for some $U \subset V(H)$, where $U \neq \emptyset, V(H)$. If $s_1 \in U$, we can pick $V(H) - U$ instead of U. Thus we may assume that $s_1 \notin U$. For $i = 2,3$, define $\hat{\varphi}_i := \varphi_i$ if $s_i \notin U$ and $\hat{\varphi}_i = \delta_H(s_i) \Delta \varphi_i$ otherwise. Let $W := U - \{s_2, s_3\}$. Thus $s_1, s_2, s_3 \notin W$ and $\delta_H(W) = \varphi_1 \Delta \hat{\varphi}_2 \Delta \hat{\varphi}_3$. Define $Y := \{(u,v) \in E(H) : \{u,v\} \cap W \neq \emptyset\}$. By construction, $\delta_H(s_1) \cap Y = \varphi_1 - (\varphi_2 \Delta \varphi_3)$. If $e \in \varphi_2 \cap \varphi_3$, then $e = (s_2, s_3)$ and $e \notin \varphi_1$. Thus $\varphi_1 - (\varphi_2 \Delta \varphi_3) = \varphi_1 - (\varphi_2 \cup \varphi_3)$ and condition (3) holds. Conditions (4) and (5) follow similarly. It follows from the hypothesis of the lemma that U is not contained in $\{s_2, s_3\}$. Hence W is non-empty and $\mathscr{I}_H(Y)$ is non-empty. For every $v \in W$, $\delta_H(v) \subseteq Y$, hence $v \notin \mathscr{B}_H(Y)$. Moreover, for every $v \notin W \cup \{s_1, s_2, s_3\}$, $\delta_H(v) \cap Y = \emptyset$, hence $v \notin \mathscr{B}_H(Y)$. It follows that $\mathscr{B}_H(Y) \subseteq \{s_1, s_2, s_3\}$. □

Remark 6.12. *Let* $\mathbb{T} = (H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2, \mathbb{S})$ *be a quad-template. Suppose that* $\mathbb{S} = (X_1, \ldots, X_k)$ *and* $\mathscr{B}_{H_1}(X_1) \cap \{v_1, w_1\} = \emptyset$*. Let* $\mathbb{T}' = (\mathrm{W}_{flip}[H_1, \mathbb{S}], v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2, \mathbb{S}')$*, where* $\mathbb{S}' = (X_2, \ldots, X_k)$*. Then* $\mathbb{T}'$ *is a quad-template and* $\mathbb{T}$ *and* $\mathbb{T}'$ *are compatible.*

Suppose that $\mathbb{T} = (H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2, \mathbb{S})$ is a quad-template. If we substitute α_i (respectively β_i) with $\delta_{H_i}(v_i) \Delta \alpha_i$ (respectively $\delta_{H_i}(w_i) \Delta \beta_i$) for $i = 1$ or $i = 2$ we obtain a quad-template $\mathbb{T}'$ giving rise to the same quad siblings as $\mathbb{T}$. We say that $\mathbb{T}'$ is obtained from $\mathbb{T}$ by a *swap* on v_i (respectively w_i). We will make repeated use of swaps in the next section.

Lemma 6.13. *Let* (G_1, Σ_1)*,* (G_2, Σ_2) *be ec-standard and* Δ*-irreducible quad siblings arising from a quad-template* $\mathbb{T} = (H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2)$*. Let* X *be a* k*-separation of* H_1 *and* H_2*, for* $k \leq 2$*. Let* $Y := E(H_1) - (X \cup \mathrm{loop}(H_1))$*. Suppose that* $\mathscr{I}_{H_i}(X), \mathscr{I}_{H_i}(Y) \neq \emptyset$ *and* $v_i, w_i \in V(H_i[X])$*, for* $i = 1,2$*. Suppose moreover that, for* $h = 1$ *or* $h = 2$*,* $\mathscr{I}_{H_h}(X) \cap \{v_h, w_h\} \neq \emptyset$*. Let* $j = 3 - h$*. Then* $\mathscr{B}_{H_j}(X) = \{v_j, w_j\}$ *and all the sets* $\alpha_j \cap Y$*,* $(\delta_{H_j}(v_j) - \alpha_j) \cap Y$*,* $\beta_j \cap Y$ *and* $(\delta_{H_j}(w_j) - \beta_j) \cap Y$ *are non-empty. In particular,* X *is a* 2*-separation in* H_1 *and* H_2*.*

Proof. To simplify the notation we prove the result for the case $h = 1$. Thus we may assume that $v_1 \in V(H_1[X])$, $w_1 \in \mathscr{I}_{H_1}(X)$ and $v_2, w_2 \in V(H_2[X])$. Suppose for contradiction that $w_2 \in \mathscr{I}_{H_2}(X)$ or that $w_2 \in \mathscr{B}_{H_2}(X)$ but one of the sets $\beta_2 \cap Y$, $(\delta_{H_2}(w_2) - \beta_2) \cap Y$ is empty. If $w_2 \in \mathscr{I}_{H_2}(X)$, then $\beta_2 \cap Y = \emptyset$. Thus either $\beta_2 \cap Y = \emptyset$ or $w_2 \in \mathscr{B}_{H_2}(X)$ and $\delta_{H_2}(w_2) \cap Y \subseteq \beta_2$. In the second case, we may substitute β_2 with $\delta_{H_2}(w_2)\Delta\beta_2$ (this is just a swap), reducing to the case $\beta_2 \cap Y = \emptyset$. As $w_1 \in \mathscr{I}_{H_1}(X)$, we have $\beta_1 \cap Y = \emptyset$. For $i = 1,2$, let v_i be split into vertices v_i^- and v_i^+ of G_i. Define w_i^-, w_i^+ similarly. Recall that β_i is a signature of (G_{3-i}, Σ_{3-i}) for $i = 1,2$. Every edge in $\beta_i \cap \text{loop}(H_i)$ is also in $\alpha_{3-i}\Delta\beta_{3-1}$ (by definition of unfolding). Thus every edge in $\beta_i \cap \text{loop}(H_i)$ is either a (v_{3-i}^-, v_{3-i}^+) edge or a (w_{3-i}^-, w_{3-i}^+) edge in G_{3-i}. This implies that, for $i = 1,2$, $(G_i[Y], \Sigma_i \cap Y)$ is bipartite and Y is a k_i-separation of G_i for $k_i \leq 3$. As $\text{ecycle}(G_1, \Sigma_1)$ is 3-connected, Y is not a 1- or a 2-separation in G_1 or G_2, by Lemma 2.4. Thus $k_1 = k_2 = 3$. Moreover, Y is not a 3-$(0,0)$-separation in (G_i, Σ_i), for $i = 1,2$, for otherwise (G_i, Σ_i) would contain a blocking vertex. Thus Y is a 3-$(0,1)$-separation in (G_1, Σ_1) and (G_2, Σ_2). By Lemma 6.9, (G_1, Σ_1) and (G_2, Σ_2) are Δ-reducible, a contradiction. This implies that $w_2 \in \mathscr{B}_{H_2}(X)$ and the sets $\beta_2 \cap Y$ and $(\delta_{H_2}(w_2) - \beta_2) \cap Y$ are non-empty. By symmetry between v_2 and w_2, $v_2 \in \mathscr{B}_{H_2}(X)$ and the sets $\alpha_2 \cap Y$ and $(\delta_{H_2}(v_2) - \alpha_2) \cap Y$ are non-empty. □

6.2.4 Proofs of Lemmas 6.5, 6.6, 6.7 and 6.8

Proof of Lemma 6.5. Let $\mathbb{T} = (H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2, \mathbb{S})$ be a quad-template of type I, where $\mathbb{S} = (X_1, \ldots, X_k)$ for some $k \geq 0$. For $i = 1,2$, let $\Gamma_i := \alpha_i \Delta \beta_i$. By definition of quad siblings, (H_1, Γ_1) and (H_2, Γ_2) are equivalent. Thus $\Gamma_1 \Delta \Gamma_2 = \alpha_1 \Delta \beta_1 \Delta \alpha_2 \Delta \beta_2$ is a cut of H_1. Let $\Gamma_1 \Delta \Gamma_2 = \delta_{H_1}(U)$ for some $U \subseteq V(H_1)$. By possibly swapping on v_1 or w_1, we may assume that $v_1, w_1 \notin U$.

Case 1: Suppose $k \geq 1$. Let $X_{k+1}, \ldots, X_t$ be a partition of $E(H_1) - (X_1 \cup \ldots \cup X_k \cup \text{loop}(H_1))$ into minimal 2-separations having as boundary $\{v_1, w_1\}$ plus possibly edges with ends v_1, w_1. Let $U_j = U \cap V_{H_1}(X_j)$, for every $j \in [t]$. As $v_1, w_1 \notin U$ and X_j, X_h are disjoint for every distinct $j, h \in [t]$, the sets $U_1, \ldots, U_t$ are all disjoint. Suppose that $U_j \neq \emptyset$ for some $j \in [t]$. Thus $(\Gamma_1 \Delta \Gamma_2) \cap X_j$ is a non-empty cut of $H_1[X_j]$. By Lemma 6.10, there exists a set $Y \subseteq X_j$ such that $\mathscr{B}_{H_1}(Y) \subseteq \{v_1, w_1\}$; $\mathscr{I}_{H_1}(Y) \neq \emptyset$; $\delta_{H_1}(v_1) \cap Y = (\Gamma_1 \Delta \Gamma_2) \cap \delta_{H_1}(v_1) \cap X_j$; $\delta_{H_1}(w_1) \cap Y = (\Gamma_1 \Delta \Gamma_2) \cap \delta_{H_1}(w_1) \cap X_j$. As $H_1[X_j] \setminus \{v_1, w_1\}$ is connected, $Y = X_j$ and $U_j = \mathscr{I}_{H_1}(X_j)$. Thus for every $j \in [t]$, either $U_j = \emptyset$ or $U_j = \mathscr{I}_{H_1}(X_j)$. Therefore $U =$

$\cup_{i\in I}\mathscr{I}_{H_1}(X_i)$, for some $I \subseteq [t]$. Define the following index sets: $I_1 := ([t]-[k])\cap I$; $I_2 := [k]-I$; $I_3 := [k]\cap I$; $I_4 := [t]-([k]\cup I)$. Note that I_1, I_2, I_3, I_4 partition $[t]$. The idea is that for each 2-separation X_j with $\mathscr{B}_{H_i}(X_j) = \{v_i, w_i\}$, there are four possible choices, depending whether, when going from H_1 to H_2, we resign, flip, resign and flip or do not perform any operation in $H_1[X_j]$. Now partition the edges in $\text{loop}(H_1)\cap\Gamma_1$ as $L_1\cup L_2$, where $e\in L_1$ if $e\in\alpha_1\cap\alpha_2$ or $e\in\beta_1\cap\beta_2$ and $e\in L_2$ otherwise. Finally define $Y_1 := \cup_{j\in I_1}(X_j)\cup L_1$; $Y_2 := \cup_{j\in I_2}(X_j)$; $Y_3 := \cup_{j\in I_3}(X_j)\cup L_2$; $Y_4 := \cup_{j\in I_4}(X_j)$. Then (G_1,Σ_1) and (G_2,Σ_2) form a shuffle with partition Y_1, Y_2, Y_3, Y_4.

Case 2: Suppose $k=0$. This implies that $H_1 = H_2$. In this case we may also assume that $v_2, w_2 \notin U$ (by possibly swapping on v_2, w_2). We now have different cases depending on the cardinality of $\{v_1,w_1\}\cap\{v_2,w_2\}$.

Case 2.1: Suppose $\{v_1,w_1\} = \{v_2,w_2\}$. Then, similarly to case 1, we obtain a shuffle (where the sets Y_2 and Y_3 are empty).

Case 2.2: Suppose $\{v_1,w_1\}\cap\{v_2,w_2\} = \{v_1\} = \{v_2\}$. This implies that $\delta(U) \subseteq \delta(v_1)\cup\delta(w_1)\cup\delta(w_2)$. Moreover, $\delta(w_1)\cap\delta(U) = \delta(w_1)\cap\Gamma_1$ and $\delta(w_2)\cap\delta(U) = \delta(w_2)\cap\Gamma_2$. Define $Y_1 := E(H_1[U])\cup\delta(U)$ and $Y_2 := E(H_1)-(Y_1\cup\text{loop}(H_1))$. If $e\in\text{loop}(H_i)-(\alpha_i\cup\beta_i)$, then e is an even loop of (G_i,Σ_i), contradicting the fact that $\text{ecycle}(G_i,\Sigma_i)$ is 3-connected. Thus every loop of H_i is either in α_i or in β_i (but not both, by definition of unfolding). Moreover (G_i,Σ_i) do not have parallel edges of the same parity. It follows that $|\text{loop}(H_i)|\le 4$ and every edge in $\text{loop}(H_1)$ is in exactly one of α_1,β_1 and in exactly one of α_2,β_2. If $\text{loop}(H_1)\cap\beta_1\cap\alpha_2$ is non-empty, let $e\in\text{loop}(H_1)\cap\beta_1\cap\alpha_2$. Similarly, if they exist, define edges $f,g,h\in\text{loop}(H_1)$ as follows: $f\in\beta_1\cap\beta_2$; $g\in\alpha_1\cap\alpha_2$; $h\in\alpha_1\cap\beta_2$. Then (G_1,Σ_1) and (G_2,Σ_2) are related by a twist with partition $Y_1, Y_2, \{e,f,g,h\}$.

Case 2.3: Suppose $\{v_1,w_1\}\cap\{v_2,w_2\} = \emptyset$. This implies that $\delta(U)\subseteq\delta(v_1)\cup\delta(w_1)\cup\delta(v_2)\cup\delta(w_2)$. Moreover, $\delta(v_i)\cap\delta(U) = \delta(v_i)\cap\Gamma_i$ and $\delta(w_i)\cap\delta(U) = \delta(w_i)\cap\Gamma_i$ for $i=1,2$. Define $Y_1 := E(H_1[U])\cup\delta(U), Y_2 := E(H_1)-(Y_1\cup\text{loop}(H_1))$ and, if they exist, edges $e,f,g,h\in\text{loop}(H_1)$ as follows: $e\in\alpha_1\cap\alpha_2$; $f\in\alpha_1\cap\beta_2$; $g\in\beta_1\cap\alpha_2$; $h\in\beta_1\cap\beta_2$. Then (G_1,Σ_1) and (G_2,Σ_2) are related by a tilt with partition $Y_1, Y_2, \{e,f,g,h\}$. □

Proof of Lemma 6.6. Let $\mathbb{T} = (H_1,v_1,w_1,\alpha_1,\beta_1,H_2,v_2,w_2,\alpha_2,\beta_2,\mathbb{S})$ be a quad template of type II. Fix $i=1$ or $i=2$. If $e\in\text{loop}(H_i)-(\alpha_i\cup\beta_i)$, then e is an even loop of (G_i,Σ_i), contradicting the fact that $\text{ecycle}(G_i,\Sigma_i)$ is 3-connected. Thus every loop of H_i is either in α_i or in β_i (but not both, by definition of unfolding). Moreover, for $i=1,2$, (G_i,Σ_i) do not have parallel edges of the same parity. It follows that $|\operatorname{loop}(H_i)| \leq 4$ and every edge in $\operatorname{loop}(H_1)$ is in exactly one of α_1, β_1 and in exactly one of α_2, β_2. Thus we will not consider the behavior of the loops of H_1 any further in this proof. Now we consider two cases, depending on whether $|\mathbb{S}| = 1$ or $|\mathbb{S}| = 2$.

Case 1: Suppose that $|\mathbb{S}| = 1$. We will show that (G_1, Σ_1) and (G_2, Σ_2) are widget twins. In this case, $H_2 = \mathrm{W}_{\text{flip}}[H_1, X]$ for some 2-separation X of H_1, and $v_i \in \mathscr{B}_{H_i}(X)$, for $i = 1, 2$. Moreover, $w_1 \in \mathscr{I}_{H_1}(X)$ and, for $Y := \bar{X} - \operatorname{loop}(H_1)$, $w_2 \in \mathscr{I}_{H_2}(Y)$. For $i = 1, 2$, let z_i be the vertex in $\mathscr{B}_{H_i}(X)$ distinct from v_i. By swapping the role of X and Y and of H_1 and H_2, we may assume that $\delta_{H_1}(v_1) \cap X = \delta_{H_2}(v_2) \cap X$. Define $\varphi_1 := (\alpha_1 \Delta \alpha_2) \cap X$ and $\varphi_2 := \beta_1 \cap X$. Let $H := H_1[X]$. We have $\varphi_1 \subseteq \delta_H(v_1)$ and $\varphi_2 \subseteq \delta_H(w_1)$. Moreover, $\varphi_1 \Delta \varphi_2 = (\alpha_1 \Delta \alpha_2 \Delta \beta_1) \cap X$. By definition of quad siblings, $\alpha_1 \Delta \alpha_2 \Delta \beta_1 \Delta \beta_2$ is a cut of H_1. As $\beta_2 \cap X$ is empty, $C_1 := (\alpha_1 \Delta \alpha_2 \Delta \beta_1) \cap X$ is a cut of H. First suppose that C_1 is empty. Then all the edges in $\beta_1 - \operatorname{loop}(H_1)$ are either in α_1 or in α_2 (but not both). As (G_1, Σ_1) does not contain parallel edges of the same parity, there cannot be two edges in $\beta_1 \cap \alpha_1$ or in $\beta_1 \cap \alpha_2$. If H_1 contains a (v_1, w_1) edge in $\beta_1 \cap \alpha_1$ (respectively in $\beta_1 \cap \alpha_2$) call such an edge e (respectively f). Let $\gamma = (X \cap \alpha_1) - \{e\}$. As C_1 is empty, $\alpha_2 \cap X = \gamma \cup \{f\}$.

Now suppose that C_1 is non-empty. If $\delta_H(w_1) = C_1$, we may swap on w_1 and reduce to the case where $C_1 = \emptyset$ (as $\delta_H(w_1) = \delta_{H_1}(w_1)$). Thus we may assume that $C_1 \neq \delta_{H_1}(w_1)$. By Lemma 6.10, there exists $Z \subseteq X$ such that $\mathscr{B}_H(Z) \subseteq \{v_1, w_1\}$, $\mathscr{I}_H(Z) \neq \emptyset$, $\delta_H(v_1) \cap Z = \varphi_1 - \varphi_2$ and for $\hat{\varphi}_2 = \varphi_2$ or $\hat{\varphi}_2 = \varphi_2 \Delta \delta_H(w_1)$, we have $\delta_H(w_1) \cap Z = \hat{\varphi}_2 - \varphi_1$. Note that Z is a 2-separation in H_1, because $\mathscr{B}_H(Z) \subseteq \{v_1, w_1\}$ and H_1 is 2-connected except for loops. Let $\hat{Z} := E(H_1) - (\operatorname{loop}(H_1) \cup \{(v_1, w_1) \in E(H_1)\})$. The condition $\delta_H(w_1) \cap Z = \hat{\varphi}_2 - \varphi_1$ implies that either $\delta_H(w_1) \cap Z \subseteq \beta_2$ or $\delta_H(w_1) \cap Z \subseteq \delta_H(w_1) - \beta_2$. Hence $\hat{Z}$ violates Lemma 6.13.

We conclude that, by possibly swapping on w_1, $\beta_1 - \operatorname{loop}(H_1) = \{e, f\}$, $\alpha_1 \cap X = \gamma \cup \{e\}$ and $\alpha_2 \cap X = \gamma \cup \{f\}$. Now we proceed to consider the structure of $H_1[Y]$. We assume that every edge with endpoints v_1, z_1 in H_1 is in X. Define sets $\varphi_1 = \alpha_1 \cap Y$, $\varphi_2 = \alpha_2 \cap Y$ and $\varphi_3 = \beta_2 \cap Y$. As β_1 does not intersect Y, $C_2 := (\alpha_1 \Delta \alpha_2 \Delta \beta_1 \Delta \beta_2) \cap Y = \varphi_1 \Delta \varphi_2 \Delta \varphi_3$. As $\alpha_1 \Delta \alpha_2 \Delta \beta_1 \Delta \beta_2$ is a cut of H_1, we have that C_2 is a cut of $H_1[Y]$. If $C_2 = \emptyset$, then every edge in β_2 is either contained in α_1 or in α_2. Similarly for the edges in $\alpha_1 \cap Y$ and in $\alpha_2 \cap Y$. As there are no (v_1, z_1) edges in Y, we have $\beta_2 - \operatorname{loop}(H_1) = \{a, c\}$ for two edges $a = (v_1, w_2)$ and $c = (z_1, w_2)$ in H_1 (if they exist). Moreover, $\alpha_1 \cap Y = \{a\}$ and $\alpha_2 \cap Y = \{c\}$. Let $Z = Y - \{a, c\} - \{(v_1, w_2), (z_1, w_2) \in E(H_1)\}$. Then all the sets $\alpha_1 \cap Z$, $\alpha_2 \cap Z$,

$\beta_1 \cap Z$, $\beta_2 \cap Z$, are empty. Therefore, if $\mathscr{I}_{H_1}(Z)$ is non-empty, Z is a 3-$(0,1)$-separation of (G_1,Σ_1) and (G_2,Σ_2) and (G_1,Σ_1), (G_2,Σ_2) are Δ-reducible by Lemma 6.9, contradiction. Hence Z is empty and $Y = \{a,b,c,d\}$, where $b = (v_1,w_2)$, $d = (z_1,w_2)$ (if they exist) and $b,d \notin \alpha_1 \cup \alpha_2 \cup \beta_2$. We conclude that, in the case $C_2 = \emptyset$, (G_1,Σ_1) and (G_2,Σ_2) are widget twins.

Now suppose that $C_2 \neq \emptyset$. Let $H := H_1[Y]$. If C_2 is equal to one of the sets $\delta_H(w_2), \delta_H(z_1)$, $\delta_H(\{z_1,w_2\})$, we may swap on w_2 or v_2 and reduce to the case where $C_2 = \emptyset$ (as $\delta_H(z_1) \subset \delta_{H_2}(v_2)$). Therefore we may assume that $\varphi_1, \varphi_2, \varphi_3$ satisfy the hypotheses of Lemma 6.11. Hence there exists a set $W \subseteq Y$ such that $\mathscr{B}_H(W) \subseteq \{v_1,z_1,w_2\}$, $\mathscr{I}_H(W) \neq \emptyset$, and

(a) $\delta_H(v_1) \cap W = \alpha_1 - (\alpha_2 \cup \beta_2)$;

(b) either $\delta_H(z_1) \cap W = \alpha_2 - (\alpha_1 \cup \beta_2)$, or $\delta_H(z_1) \cap W = \delta_H(z_1) - (\alpha_1 \cup \alpha_2 \cup \beta_2)$;

(c) either $\delta_H(w_2) \cap W = \beta_2 - (\alpha_1 \cup \alpha_2)$, or $\delta_H(w_2) \cap W = \delta_H(w_2) - (\alpha_1 \cup \alpha_2 \cup \beta_2)$.

Therefore W is a 3-$(0,1)$-separation of (G_1,Σ_1) and (G_2,Σ_2). By Lemma 6.9, (G_1,Σ_1) and (G_2,Σ_2) are Δ-reducible, a contradiction.

Case 2: $|\mathbb{S}| = 2$. We will show that (G_1,Σ_1) and (G_2,Σ_2) are gadget twins. In this case $H_2 = \mathrm{W}_{\text{flip}}[H_1,(Y,Z)]$ for some disjoint 2-separations Y,Z of H_1, where $v_i \in \mathscr{B}_{H_i}(Y) \cap \mathscr{B}_{H_i}(Z)$, for $i = 1,2$, $w_1 \in \mathscr{I}_{H_1}(Y)$ and $w_2 \in \mathscr{I}_{H_2}(Z)$. For $i = 1,2$, let z_i be the vertex in $\mathscr{B}_{H_i}(Y)$ distinct from v_i and u_i the vertex in $\mathscr{B}_{H_i}(Z)$ distinct from v_i. For $X := E(H_1) - (Y \cup Z \cup \mathrm{loop}(H_1))$, $\mathscr{B}_{H_i}(X) = \{v_i,u_i,z_i\}$, for $i = 1,2$. Moreover, we can choose Y and Z so that all the edges in H_1 with both ends in $\{v_1,z_1,u_1\}$ are contained in X. By construction, $\delta_{H_1}(v_1) \cap X = \delta_{H_2}(v_2) \cap X$. Moreover $\delta_{H_1}(z_1) \cap Y = \delta_{H_2}(v_2) \cap Y$ and $\delta_{H_1}(u_1) \cap Z = \delta_{H_2}(v_2) \cap Z$. Define $\varphi_1 = \alpha_2 \cap Y$, $\varphi_2 = \alpha_1 \cap Y$ and $\varphi_3 = \beta_1 \cap Y$. Let $H := H_1[Y]$. So $\varphi_1 \subseteq \delta_H(z_1)$, $\varphi_2 \subseteq \delta_H(v_1)$ and $\varphi_3 \subseteq \delta_H(w_1)$. Note that $C := \varphi_1 \Delta \varphi_2 \Delta \varphi_3 = (\alpha_1 \Delta \alpha_2 \Delta \beta_1 \Delta \beta_2) \cap Y$. As $\alpha_1 \Delta \alpha_2 \Delta \beta_1 \Delta \beta_2$ is a cut of H_1, we have that C is a cut of H.

If $C = \emptyset$, then every edge in β_1 is either contained in α_1 or in α_2. Similarly for the edges in $\alpha_1 \cap Y$ and in $\alpha_2 \cap Y$. As there are no (v_1,z_1) edges in Y, we have $\beta_1 - \mathrm{loop}(H_1) = \{a_1,c_1\}$ for two edges $a_1 = (v_1,w_1)$ and $c_1 = (z_1,w_1)$ in H_1 (if they exist). Moreover, $\alpha_1 \cap Y = \{a_1\}$ and $\alpha_2 \cap Y = \{c_1\}$. Let $W = Y - \{a_1,c_1\} - \{(v_1,w_1),(z_1,w_1) \in E(H_1)\}$. Then all the sets $\alpha_1 \cap W$, $\alpha_2 \cap W$, $\beta_1 \cap W$, $\beta_2 \cap W$, are empty. Therefore, if $\mathscr{I}_{H_1}(W)$ is non-empty, W is a 3-$(0,1)$-separation of (G_1,Σ_1) and (G_2,Σ_2) and (G_1,Σ_1), (G_2,Σ_2) are

Δ-reducible by Lemma 6.9, a contradiction. Hence W is empty and $Y = \{a_1, b_1, c_1, d_1\}$, where $b_1 = (v_1, w_1)$, $d_1 = (z_1, w_1)$ (if they exist) and $b_1, d_1 \notin \alpha_1 \cup \alpha_2 \cup \beta_1$.

Now suppose that $C \neq \emptyset$. If C is equal to one of the sets $\delta_H(v_1), \delta_H(w_1), \delta_H(\{v_1, w_1\})$, we may swap on v_1 or w_1 and reduce to the case $C = \emptyset$. Therefore we may assume that $\varphi_1, \varphi_2, \varphi_3$ satisfy the hypothesis of Lemma 6.11. Hence there exists a set $W' \subseteq Y$ such that $\mathscr{B}_H(W') \subseteq \{v_1, z_1, w_1\}$, $\mathscr{I}_H(W') \neq \emptyset$, and

(a) $\delta_H(z_1) \cap W' = (\alpha_2 \cap Y) - (\alpha_1 \cup \beta_1)$;

(b) either $\delta_H(v_1) \cap W' = (\alpha_1 \cap Y) - (\alpha_2 \cup \beta_1)$, or $\delta_H(v_1) \cap W' = \delta_H(v_1) - (\alpha_1 \cup \alpha_2 \cup \beta_1)$;

(c) either $\delta_H(w_1) \cap W' = (\beta_1 \cap Y) - (\alpha_1 \cup \alpha_2)$, or $\delta_H(w_1) \cap W' = \delta_H(w_1) - (\alpha_1 \cup \alpha_2 \cup \beta_1)$.

Therefore W' is a 3-$(0,1)$-separation of (G_1, Σ_1) and (G_2, Σ_2). By Lemma 6.9, (G_1, Σ_1) and (G_2, Σ_2) are Δ-reducible, a contradiction. We deduce that, up to swaps on v_1, w_1, $Y = \{a_1, b_1, c_1, d_1\}$, with the conditions on $\alpha_1, \beta_1, \alpha_2, \beta_2$ established before. Now consider the structure of $H_1[Z]$. Define $\varphi_1 = \alpha_1 \cap Z$, $\varphi_2 = \alpha_2 \cap Z$ and $\varphi_3 = \beta_1 \cap Z$. Then with an argument similar to the one above, we conclude that, up to possible swaps on v_2, w_2, $Z = \{a_2, b_2, c_2, d_2\}$, where the ends of a_2, b_2 are v_1, w_2 and the ends of c_2, d_2 are u_1, w_2. Moreover, $\beta_2 - \text{loop}(H_1) = \{a_2, c_2\}$, $\alpha_1 \cap Z = \{a_2\}$ and $\alpha_2 \cap Z = \{c_2\}$.

Let $\gamma := \alpha_1 \cap X$. As $(\alpha_1 \Delta \alpha_2) \cap X$ is a cut of $H_1[X]$, either $\alpha_2 \cap X = \gamma$ or $\alpha_2 \cap X = (\delta_{H_2}(v_2) \cap X) - \gamma$. In the second case, $\alpha_1 \Delta \beta_1 \Delta \alpha_2 \Delta \beta_2 = \delta_{H_2}(v_2) \cap X$, which is not a cut of H_2, contradiction. It follows that $\alpha_2 \cap X = \gamma$ and (G_1, Σ_1) and (G_2, Σ_2) are gadget twins. □

Proof of Lemma 6.7. Let $\mathbb{T} = (H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2, \mathbb{S})$ and $\mathbb{S} = (X_1, \ldots, X_k)$. By Proposition 5.2 applied to H_1 and $Z = \{v_1, w_1\}$, there exists a graph H such that:

- $H = \mathrm{W}_{\text{flip}}[H_1, \mathbb{S}_1]$ for some w-sequence $\mathbb{S}_1$ of H_1, where $\{v_1, w_1\} \cap \mathscr{B}_{H_1}(X) = \emptyset$ for all $X \in \mathbb{S}_1$, and
- $H_2 = \mathrm{W}_{\text{flip}}[H, \mathbb{S}_2]$ for some non-crossing w-sequence $\mathbb{S}_2$ such that, for all $X \in \mathbb{S}_2$, $\{v_1, w_1\} \cap \mathscr{B}_{H_1}(X) \neq \emptyset$.

Let $\mathbb{T}' = (H, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2, \mathbb{S}_2)$. By Remark 6.12, $\mathbb{T}'$ is a quad-template and $\mathbb{T}$, $\mathbb{T}'$ are compatible. Thus we may assume that (G_1, Σ_1),(G_2, Σ_2) arise from a template $\mathbb{T} = (H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2, \mathbb{S})$, where $\mathbb{S} = (X_1, \dots, X_k)$ is non-crossing, and for all $X \in \mathbb{S}$, $\{v_1, w_1\} \cap \mathscr{B}_{H_1}(X) \neq \emptyset$. Similarly we may assume that, for all $X \in \mathbb{S}$, $\{v_2, w_2\} \cap \mathscr{B}_{H_2}(X) \neq \emptyset$. We will also assume that every Whitney-flip in $\mathbb{S}$ is non-trivial, that is, $\mathscr{I}_{H_1}(X) \neq \emptyset$ for every $X \in \mathbb{S}$.

First suppose that, for every $X \in \mathbb{S}$, $\mathscr{B}_{H_i}(X) = \{v_i, w_i\}$, for $i = 1, 2$. We show that in this case we can find a w-sequence $\mathbb{S}'$ for H_1 such that $\mathbb{T}' := (H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_2, w_2, \alpha_2, \beta_2, \mathbb{S}')$ is a quad-template of type I. As $\mathbb{T}'$ is trivially compatible with $\mathbb{T}$, this would prove the statement for this case. Suppose that there exists $X \in \mathbb{S}$ such that $H_i[X] \setminus \mathscr{B}_{H_i}(X)$ is not connected. Since $\mathbb{S}$ is non-crossing, we may rearrange the sets in $\mathbb{S}$ in any order. Hence we may assume that $X = X_1$. As H_1 is 2-connected except for loops, there exists a partition $Y_1, \dots, Y_s$ of X such that $\mathscr{B}_{H_i}(Y_j) = \{v_i, w_i\}$ and $H_i[Y_j] \setminus \mathscr{B}_{H_i}(Y_j)$ is connected for every $i = 1, 2$ and $j \in [s]$. Therefore, we can replace $\mathbb{S}$ with $(Y_1, \dots, Y_s, X_2, \dots, X_k)$. Hence we may assume that $H_i[X_j] \setminus \mathscr{B}_{H_i}(X_j)$ is connected for every $i = 1, 2$ and $j \in [k]$. If there exist $i, j \in [k]$, $i \neq j$ such that $X_i \cap X_j \neq \emptyset$, then $X_i = X_j$. Thus we may just remove X_i and X_j from $\mathbb{S}$. This will lead to a w-sequence $\mathbb{S}'$ with the required properties.

Now suppose that there exists $X \in \mathbb{S}$ with $\mathscr{B}_{H_i}(X) \neq \{v_i, w_i\}$, for $i = 1$ or $i = 2$. We will show that in this case we can find a compatible quad-template of type II.

Claim 1. *Let $X \in \mathbb{S}$ such that $|\mathscr{B}_{H_i}(X) \cap \{v_i, w_i\}| = 1$ and $|\mathscr{I}_{H_i}(X) \cap \{v_i, w_i\}| = 1$ for $i = 1$ or $i = 2$. Then for $j = 3 - i$ and $Y := \bar{X} - \text{loop}(H_i)$, $|\mathscr{B}_{H_j}(X) \cap \{v_j, w_j\}| = 1$ and $|\mathscr{I}_{H_j}(Y) \cap \{v_j, w_j\}| = 1$.*

Proof. To simplify the notation we prove the claim for the case $i = 1$. Thus we may assume that $v_1 \in \mathscr{B}_{H_1}(X)$ and $w_1 \in \mathscr{I}_{H_1}(X)$. As $\mathscr{B}_{H_2}(Z) \cap \{v_2, w_2\} \neq \emptyset$ for every $Z \in \mathbb{S}$, we have $\mathscr{B}_{H_2}(X) \cap \{v_2, w_2\} \neq \emptyset$. Thus we may assume that $v_2 \in \mathscr{B}_{H_2}(X)$. Suppose for contradiction that X violates the statement, that is, $w_2 \in V(H_2[X])$. Note that we may choose X such that for no other $X' \in \mathbb{S}$ do we have $X \subseteq X'$ or $X \cap \bar{X}' = \emptyset$. By this choice, $H_1[Y] = H_2[Y]$. If there exists an edge e with ends $\mathscr{B}_{H_1}(X)$, we will assume that such an edge is in X. By Lemma 6.13, $w_2 \in \mathscr{B}_{H_2}(X)$ and the sets $\beta_2 \cap Y$ and $(\delta_{H_2}(w_2) - \beta_2) \cap Y$ are non-empty. Thus $\mathscr{B}_{H_2}(X) = \{v_2, w_2\}$. By symmetry between v_2 and w_2, we may assume that $\delta_{H_1}(v_1) \cap Y = \delta_{H_2}(v_2) \cap Y$. Define $\varphi_1 = (\alpha_1 \Delta \alpha_2) \cap Y$ and $\varphi_2 = \beta_2 \cap Y$. Then $\varphi_1 \subseteq \delta_{H_2}(v_2)$ and $\varphi_2 \subseteq \delta_{H_2}(w_2)$. Moreover, $C := \varphi_1 \Delta \varphi_2$ is a cut of $H_2[Y]$. As there is no (v_2, w_2) edge in

Y, the sets φ_1, φ_2 are disjoint. Moreover, the sets $\beta_2 \cap Y$ and $(\delta_{H_2}(w_2) - \beta_2) \cap Y$ are non-empty, thus C is non-empty and $C \neq \delta_{H_2}(w_2)$. Let $H := H_2[Y]$. By Lemma 6.10, there exists a set $Z \subset Y$ such that $\mathscr{B}_H(Z) \subseteq \{v_2, w_2\}$; $\mathscr{I}_H(Z) \neq \emptyset$; $\delta_H(v_2) \cap Y = \varphi_1$; and for $\hat{\varphi}_2 = \varphi_2$ or $\hat{\varphi}_2 = \varphi_2 \Delta \delta_H(w_2)$, $\delta_H(w_2) \cap Y = \hat{\varphi}_2$. Define $W := E(H_1) - (Z \cup \text{loop}(H_1))$. Then W contradicts Lemma 6.13. $\diamond$

Now we can conclude the proof. We have already considered the case in which, for every $X \in \mathbb{S}$, $\mathscr{B}_{H_i}(X) = \{v_i, w_i\}$, for $i = 1,2$. Thus we have that for some $X \in \mathbb{S}$ and $i = 1$ or $i = 2$, $|\mathscr{B}_{H_i}(X) \cap \{v_i, w_i\}| = 1$ and $|\mathscr{I}_{H_i}(X) \cap \{v_i, w_i\}| = 1$. Let $Y := \bar{X} - \text{loop}(H_j)$, for $j = 3-i$. By Claim 1, $|\mathscr{B}_{H_j}(X) \cap \{v_j, w_j\}| = 1$ and $|\mathscr{I}_{H_j}(Y) \cap \{v_j, w_j\}| = 1$. Thus we may assume that $v_1 \in \mathscr{B}_{H_1}(X)$, $w_1 \in \mathscr{I}_{H_1}(X)$, $v_2 \in \mathscr{B}_{H_2}(X)$ and $w_2 \in \mathscr{I}_{H_2}(Y)$. Now suppose that there exists $X' \in \mathbb{S}$ such that $w_1 \in \mathscr{B}_{H_1}(X')$. Let $Y' := \bar{X}' - \text{loop}(H_1)$. As $w_1 \in \mathscr{I}_{H_1}(X)$, X is not contained in X' and X' is not disjoint from X. As $\mathbb{S}$ is non-crossing, by possibly swapping X' with Y', we may assume that $X' \subset X$. Thus $v_1 \notin \mathscr{I}_{H_1}(X')$. Moreover, as $w_2 \in \mathscr{I}_{H_2}(Y)$ and $Y \subset Y'$, we have $w_2 \in \mathscr{I}_{H_2}(Y')$. Therefore, by the choice of $\mathbb{S}$, $v_2 \in \mathscr{B}_{H_2}(X')$. Hence X' violates Claim 1. This shows that for every $X \in \mathbb{S}$, $w_1 \notin \mathscr{B}_{H_1}(X)$. By symmetry between H_1 and H_2, for every $X \in \mathbb{S}$, $w_2 \notin \mathscr{B}_{H_2}(X)$. Moreover, as $\mathscr{B}_{H_i}(X) \cap \{v_i, w_i\} \neq \emptyset$, for $i = 1,2$, we have $v_i \in \mathscr{B}_{H_i}(X)$ for every $X \in \mathbb{S}$ and $i = 1,2$. Lemma 5.12 implies that there exists a w-sequence $\mathbb{S}'$ of H_1 with $H_2 = \text{W}_{\text{flip}}[H_1, \mathbb{S}']$ and that $\mathbb{S}'$ is a star of H_i with center v_i, for $i = 1,2$. Let $\mathbb{S}' = (Y_1, \ldots, Y_h)$. For distinct $Y, Y' \in \mathbb{S}'$, Y and Y' are disjoint. It follows that if $h \geq 3$, then for some $Y \in \mathbb{S}$, $w_i \notin \mathscr{I}_{H_i}(Y)$, for $i = 1,2$. Hence $\bar{Y} - \text{loop}(H_1)$ contradicts Lemma 6.13. Therefore $h = 1$ or $h = 2$ and $(H_1, v_1, w_1, \alpha_1, \beta_1, H_2, v_1, w_2, \alpha_2, \beta_2, \mathbb{S}')$ is a quad-template of type II, as required. $\square$

Proof of Lemma 6.8. Suppose that $H_1 \setminus \text{loop}(H_1)$ is not 2-connected. This is equivalent to $H_2 \setminus \text{loop}(H_2)$ not being 2-connectred, as H_1 and H_2 are equivalent. For $i = 1,2$, let τ_i be the tree of blocks of $H_i \setminus \text{loop}(H_i)$. So the vertices of τ_i are partitioned into sets A_i and $\mathbf{B}_i$, where A_i is the set of the cut-vertices and $\mathbf{B}_i$ is the set of blocks of $H_i \setminus \text{loop}(H_i)$. Note that, as H_1, H_2 are equivalent, there is a bijection between the vertices in $\mathbf{B}_1$ and the vertices in $\mathbf{B}_2$. By Lemma 2.4(2), for $i = 1,2$, $G_i \setminus \text{loop}(G_i)$ does not contain 1-separations. Thus $A_i \subseteq \{v_i, w_i\}$, for $i = 1,2$. In particular this implies that at most one vertex in $\mathbf{B}_i$ is not a leaf of τ_i, for $i = 1,2$. Hence there exists $X \in \mathbf{B}_1$ which is a leaf of both τ_1 and τ_2. By symmetry between v_1 and w_1, we may assume that $\mathscr{B}_{H_1}(X) = \{v_1\}$. Similarly we may assume that $\mathscr{B}_{H_2}(X) = \{v_2\}$. Note that $|X| \geq 2$, as otherwise X would be a bridge of G_1.

If for $i=1$ or $i=2$, $w_i \in V_{H_i}(Y)$, for $Y=X$ or $Y=E(H_1)-(X\cup \text{loop}(H_1))$, we derive a contradiction by Lemma 6.13. Therefore, by symmetry between H_1 and H_2, we may assume that $w_1 \in \mathscr{I}_{H_1}(X)$ and $w_2 \notin V_{H_2}(X)$.

Claim 2. $H_1[X]=H_2[X]$.

Proof. As H_1 and H_2 are equivalent and $H_1[X],H_2[X]$ are 2-connected, by Lemma 5.2 there exists a graph H such that:

- $H=\mathrm{W}_{\text{flip}}[H_1[X],\mathbb{S}_1]$ for some w-sequence $\mathbb{S}_1$, where $v_1,w_1 \notin \mathscr{B}_{H_1}(Y)$ for all $Y\in\mathbb{S}_1$, and
- $H_2[X]=\mathrm{W}_{\text{flip}}[H,\mathbb{S}_2]$ for some non-crossing w-sequence $\mathbb{S}_2$ such that, for all $Y\in\mathbb{S}_2$, $\mathscr{B}_{H_1}(Y)\cap\{v_1,w_1\}\neq\emptyset$.

Suppose that $\mathbb{S}_1=(Y_1,\dots,Y_k)$. Then either Y_1 or $X-Y_1$ is a 2-separation in H_1 and $(\mathrm{W}_{\text{flip}}[H_1,Y_1],\, v_1,w_1,\alpha_1,\beta_1,H_2,v_2,w_2,\alpha_2,\beta_2)$ is a quad-template which is compatible with $\mathbb{T}$. By Lemma 6.4, proving the statement for a compatible quad-template leads to a proof for the original template. Thus, by repeating this reasoning on $Y_2,\dots,Y_k$, we may assume that $\mathbb{S}_1=\emptyset$. Therefore $H_2[X]=\mathrm{W}_{\text{flip}}[H_1[X],\mathbb{S}]$ for a non-crossing w-sequence $\mathbb{S}$, where for every $Y\in\mathbb{S}$, $\mathscr{B}_{H_1}(Y)\cap\{v_1,w_1\}\neq\emptyset$. Consider $Y\in\mathbb{S}$. If $v_2\notin\mathscr{B}_{H_2}(Y)$, then either Y or $X-Y$ is a 2-separation of H_2 and $(H_1,v_1,w_1,\alpha_1,\beta_1,\mathrm{W}_{\text{flip}}[H_2,Y],\, v_2,\, w_2,\alpha_2,\beta_2)$ is a quad-template which is compatible with $\mathbb{T}$. Thus we may assume that $v_2\in\mathscr{B}_{H_2}(Y)$, for every $Y\in\mathbb{S}$. In particular, this implies that for every $Y\in\mathbb{S}$, both Y and $X-Y$ are 2-separations in H_2. As $w_2\notin H_2[X]$, we have $w_2\notin V_{H_2}(Y),V_{H_2}(X-Y)$, for every $Y\in\mathbb{S}$. Note that we may assume that, for every $Y\in\mathbb{S}$, $\mathscr{I}_{H_i}(Y),\mathscr{I}_{H_i}(X-Y)\neq\emptyset$, for $i=1,2$, otherwise the Whitney-flip on Y is trivial and may be omitted. As $\mathscr{B}_{H_1}(Y)\cap\{v_1,w_1\}\neq\emptyset$ and $v_1,w_1\in V_{H_1}(X)$, either $v_1,w_1\in V_{H_1}(Y)$ or $v_1,w_1\in V_{H_1}(X-Y)$. By Lemma 6.13, $v_1,w_1\in\mathscr{B}_{H_1}(Y)$ and all the sets $\alpha_1\cap Y,(\delta_{H_1}(v_1)-\alpha_1)\cap Y,\beta_1\cap Y,(\delta_{H_1}(w_1)-\beta_1)\cap Y$ are non-empty. Fix a minimal $Y\in\mathbb{S}$. We may assume that no edge (v_1,w_1) is in Y. Either $\alpha_2\cap Y\subseteq\delta_{H_1}(v_1)$ or $\alpha_2\cap Y\subseteq\delta_{H_1}(w_1)$. In the first case, define $\varphi_1=(\alpha_1\Delta\alpha_2)\cap Y$ and $\varphi_2=\beta_1\cap Y$. In the second case, define $\varphi_1=\alpha_1\cap Y$ and $\varphi_2=(\beta_1\Delta\alpha_2)\cap Y$. In both cases, $\varphi_1\subseteq\delta_{H_1}(v_1)$ and $\varphi_2\subseteq\delta_{H_1}(w_1)$. By definition of quad siblings, $\alpha_1\Delta\beta_1\Delta\alpha_2\Delta\beta_2$ is a cut of H_1. As $\beta_2\cap Y=\emptyset$, this implies that $C:=(\alpha_1\Delta\beta_1\Delta\alpha_2)\cap Y$ is a cut of $H_1[Y]$. As all the sets $\alpha_1\cap Y,(\delta_{H_1}(v_1)-\alpha_1)\cap Y,\beta_1\cap Y,(\delta_{H_1}(w_1)-\beta_1)\cap Y$ are non-empty and there is no edge with ends v_1, w_1 in Y, C is a non-empty cut. Moreover, $C \neq \delta_{H_1}(w_1)$ and $\varphi_1 \cap \varphi_2 = \emptyset$. By Lemma 6.10, there exists $Z \subseteq Y$ such that the following hold:

- $\mathscr{B}_{H_1}(Z) \subseteq \{v_1, w_1\}$;
- $\mathscr{I}_{H_1}(Z) \neq \emptyset$;
- $\delta_{H_1}(v_1) \cap Z = \varphi_1$;
- for $\hat{\varphi}_2 = \varphi_2$ or $\hat{\varphi}_2 = \varphi_2 \Delta \delta_{H_1}(w_1)$, $\delta_{H_1}(w_1) \cap Z = \hat{\varphi}_2$.

Therefore, for $i = 1,2$, $(G_i[Z], \Sigma_i \cap Z)$ is bipartite and Z is a k_i-separation of G_i, where $k_i \leq 3$. By Lemma 2.4, $k_1 = k_2 = 3$ and Z is a 3-$(0,1)$-separation in both (G_1, Σ_1) and (G_2, Σ_2). By Lemma 6.9, (G_1, Σ_1), (G_2, Σ_2) are Δ-reducible, a contradiction. We conclude that $\mathbb{S} = \emptyset$ and $H_1[X] = H_2[X]$. $\diamond$

As X is a leaf of τ_1 and $w_1 \in \mathscr{I}_{H_1}(X)$, no block of $H_1 \setminus \text{loop}(H_1)$ has as boundary $\{w_1\}$. Thus for every $Y \in \mathbf{B}_1$, $\mathscr{B}_{H_1}(Y) = \{v_1\}$. Suppose that, for some $Y \in \mathbf{B}_2$, $\mathscr{B}_{H_2}(Y) = \{w_2\}$. Thus $v_2 \notin V_{H_2}(Y)$, $\mathscr{B}_{H_1}(Y) = \{v_1\}$ and $w_1 \notin V_{H_1}(Y)$, contradicting Lemma 6.13. It follows that, for every $Y \in \mathbf{B}_i$, $\mathscr{B}_{H_i}(Y) = \{v_i\}$, for $i = 1,2$. If $|\mathbf{B}_1| \geq 3$, then for some $Y \in \mathbf{B}_1$, $w_i \notin \mathscr{I}_{H_i}(Y)$, for $i = 1,2$, contradicting Lemma 6.13. Thus $\mathbf{B}_1 = \{X, Y\}$ for some set Y, $w_1 \in \mathscr{I}_{H_1}(X)$, $w_2 \in \mathscr{I}_{H_2}(Y)$ and $\mathscr{B}_{H_i}(X) = \{v_i\}$, for $i = 1,2$. By Claim 2, $H_1[X] = H_2[X]$. By symmetry between H_1 and H_2, we also have $H_1[Y] = H_2[Y]$. In particular this implies that w_2 is a vertex of H_1 and $H_2 \setminus \text{loop}(H_2)$ is obtained by identifying a vertex $x \in V(H_1[X])$ with a vertex $y \in V(H_1[Y])$. Define paths P_x and P_y as follows. If $x = v_1$, let P_x be a (w_1, v_1)-path in $H_1[X]$, otherwise let P_x be an (x, v_1)-path in $H_1[X]$. If $y = v_1$, let P_y be a (w_2, v_1)-path in $H_1[Y]$, otherwise let P_y be a (y, v_1)-path in $H_1[Y]$. It follows that P_x, P_y are non-empty and $P := P_x \cup P_y$ is a path of H_1. As x is an end of P_x and y is an end of P_y, P is also a path of H_2. For $i = 1,2$, construct a graph H_i' by adding to H_i an edge Ω with ends the ends of P in H_i. Note that H_1' is now 2-connected, except for the possible presence of loops. We show that H_1' and H_2' are equivalent by showing that they have the same cycles. By construction, $P \cup \Omega$ is a cycle in both H_1', H_2'. Let C be a cycle of H_1'. If $\Omega \notin C$, C is a cycle of H_1 and H_2 and we are done. If $\Omega \in C$, then $C' := C \Delta (P \cup \Omega)$ is a cycle of H_1' not using Ω, hence it is a cycle of H_2'. It follows that $C = C' \Delta (P \cup \Omega)$ is a cycle of H_2'. We conclude that H_1', H_2' are equivalent. Define a w-sequence for H_1' as follows:

$$\mathbb{S} := \begin{cases} \emptyset & \text{if } x = v_1 \text{ and } y = v_1 \\ (X) & \text{if } x \neq v_1 \text{ and } y = v_1 \\ (Y) & \text{if } x = v_1 \text{ and } y \neq v_1 \\ (X,Y) & \text{if } x \neq v_1 \text{ and } y \neq v_1. \end{cases}$$

Then $H_2' = \mathrm{W}_{\text{flip}}[H_1', \mathbb{S}]$. For $i = 1,2$, if P is $(\alpha_i \Delta \beta_i)$-even, define $\alpha_i' := \alpha_i$, otherwise set $\alpha_i' := \alpha_i \Delta \delta_{H_i}(\mathscr{I}_{H_i}(Y))$. With this choice, $P \cup \Omega$ is an $(\alpha_i' \Delta \beta_i)$-even cycle in H_i', for $i = 1,2$. Therefore $(H_1', \alpha_1' \Delta \beta_1)$, $(H_2', \alpha_2' \Delta \beta_2)$ have the same even cycles. Moreover, $\alpha_i \subseteq \delta_{H_i'}(v_i)$. It follows that $\mathbb{T}' := (H_1', v_1, w_1, \alpha_1', \beta_1, H_2', v_2, w_2, \alpha_2', \beta_2, \mathbb{S})$ is a quad-template. Moreover $\mathbb{T}'$ is of type I if $\mathbb{S} = \emptyset$ and of type II in the other three cases. Let $\mathbb{T}'' := (H_1, v_1, w_1, \alpha_1', \beta_1, H_2, v_2, w_2, \alpha_2', \beta_2)$. Then $\mathbb{T}''$ and $\mathbb{T}$ are compatible quad-templates. Let (G_1', Σ_1'), (G_2', Σ_2') (respectively (G_1'', Σ_1''), (G_2'', Σ_2'')) be the quad siblings arising from $\mathbb{T}'$ (respectively $\mathbb{T}''$). By Lemma 6.5 and Lemma 6.6, (G_1', Σ_1'), (G_2', Σ_2') are either shuffle, tilt, twist, widget or gadget siblings. For $i = 1,2$, $(G_i'', \Sigma_i'') = (G_i', \Sigma_i') \setminus \Omega$, therefore (G_1'', Σ_1''), (G_2'', Σ_2'') are either shuffle, tilt, twist, widget or gadget siblings. As $\mathbb{T}$ and $\mathbb{T}''$ are compatible, the statement follows by Lemma 6.4. $\square$

Chapter 7

Finding excluded minors

7.1 Excluded minors with low connectivity

Recall that we only consider binary matroids in this work. It is easy to find the disconnected excluded minors for the classes of even cycle and even cut matroids. We say that a matroid M is the 1-*sum* of two matroids M_1 and M_2 if:

(a) $E(M_1)$ and $E(M_2)$ are disjoint;

(b) $E(M) = E(M_1) \cup E(M_2)$;

(c) C is a circuit of M if and only if C is a circuit of M_1 or a circuit of M_2.

We denote the 1-sum of M_1 and M_2 by $M_1 \oplus_1 M_2$. Note that, if X is a 1-separation of a matroid M, then $M = M|_X \oplus_1 M|_{\bar{X}}$ (where $M|_X$ denotes the restriction of M to X, i.e. the matroid $M \setminus \bar{X}$).

Lemma 7.1. *A disconnected matroid M is an excluded minor for the class of even cycle matroids if and only if $M = M_1 \oplus_1 M_2$ for two minimally non-graphic matroids M_1 and M_2.*

Proof.

Claim 1. *If $M = M_1 \oplus_1 M_2$, where M_1 is an even cycle matroid and M_2 is graphic, then M is an even cycle matroid.*

Proof. Suppose that $M = M_1 \oplus_1 M_2$, where M_1 is an even cycle matroid and M_2 is graphic. Then M_1 has a signed graph representation (G_1, Σ_1) and M_2 has a graph representation G_2. Construct a graph G by identifying one vertex of G_1 with one vertex of G_2. Then for every circuit C of G either $C \subseteq E(G_1)$ or $C \subseteq E(G_2)$. Every circuit of ecycle(G, Σ_1) is either a Σ_1-even circuit of G or the union of two Σ_1-odd circuits of G sharing at most one vertex. As $E(G_2) \cap \Sigma_1$ is empty, every Σ_1-odd circuit of G is contained in G_1. It follows that C is a circuit of ecycle(G, Σ_1) if and only if C is a circuit of M_1 or a circuit of M_2. Hence (G, Σ_1) is a signed graph representation of M and M is an even cycle matroid. $\diamond$

Claim 2. *If $M = M_1 \oplus_1 M_2$ for two minimally non-graphic matroids M_1 and M_2, then M is not an even cycle matroid.*

Proof. Suppose that $M = M_1 \oplus_1 M_2$ for two minimally non-graphic matroids M_1 and M_2. Suppose for contradiction that M is an even cycle matroid, with a signed graph representation (G, Σ). For $i = 1, 2$, let $G_i := G[E(M_i)]$ and $\Sigma_i := \Sigma \cap E(M_i)$. Then (G_i, Σ_i) is a signed graph representation of M_i, for $i = 1, 2$. If there exist a Σ_1-odd circuit C_1 in G_1 and a Σ_2-odd circuit C_2 in G_2, then $C_1 \cup C_2$ is a circuit of M, contradicting the fact that $M = M_1 \oplus_1 M_2$. Hence for some $i \in [2]$, (G_i, Σ_i) is bipartite. It follows that $M_i = \text{cycle}(G_i)$, contradicting the fact that M_i is non-graphic. $\diamond$

Let M be a disconnected matroid which is an excluded minor for the class of even cycle matroids. Then $M = M_1 \oplus_1 M_2$ for some matroids M_1, M_2. Moreover, by minimality of M, M_1 and M_2 are even cycle matroids. By Claim 1 and by symmetry between M_1 and M_2, M_1 and M_2 are not graphic, hence they each contain one of the excluded minors for graphic matroids. By Claim 2 and by minimality of M, M_1 and M_2 are minimally non-graphic matroids. The other direction of the statements follows immediately from Claim 2. $\square$

Lemma 7.2. *A disconnected matroid M is an excluded minor for the class of even cut matroids if and only if $M = M_1 \oplus_1 M_2$ for two minimally non-cographic matroids M_1, M_2.*

We omit the proof of Lemma 7.2, as it is similar to the proof of Lemma 7.1.

We now briefly discuss excluded minors for the class of even cycle matroids which are connected but not 3-connected. We do not have a complete list of excluded minors that are not 3-connected, we just give an example in Lemma 7.3.

We say that a matroid M is the 2-*sum* of two matroids M_1 and M_2 on an element e, where e is not a loop of M_1 and M_2, if:

(a) $E(M_1) \cap E(M_2) = \{e\}$;

(b) $E(M) = E(M_1) \triangle E(M_2)$;

(c) C is a circuit of M if and only if one of the following holds: C is a circuit of $M_1 \setminus e$; C is a circuit of $M_2 \setminus e$; $(C - E(M_i)) \cup \{e\}$ is a circuit of M_{3-i}, for $i = 1, 2$.

We denote the 2-sum of M_1 and M_2 by $M_1 \oplus_2 M_2$. Note that $M_1 \oplus_2 M_2$ contains a minor isomorphic to M_1 and a minor isomorphic to M_2. If X is a 2-separation of a matroid M, then M is the 2-sum of two matroids with ground sets $X \cup \{e\}$ and $\bar{X} \cup \{e\}$ respectively, for some element $e \notin E(M)$. The following two constructions provide a signed graph representation of a matroid which is the two sum of two even cycle matroids, provided that the two even cycle matroids have some special properties.

Construction 1: Suppose that M_1 is an even cycle matroid and M_2 is a graphic matroid such that $E(M_1) \cap E(M_2) = \{e\}$. Suppose that there exist representations (G_1, Σ_1) and G_2 of M_1 and M_2 respectively, such that e is neither a loop of G_1 nor of G_2. Let G be the graph obtained from G_1 and G_2 by identifying the endpoints of e in G_1 with the endpoints of e in G_2 and then deleting both copies of e. Let Σ be any signature of (G_1, Σ_1) such that $e \notin \Sigma$. Then $\text{ecycle}(G, \Sigma) = M_1 \oplus_2 M_2$.

Construction 2: Suppose that M_1 and M_2 are even cycle matroids such that $E(M_1) \cap E(M_2) = \{e\}$. Suppose that there exist representations (G_1, Σ_1) and (G_2, Σ_2) of M_1 and M_2 respectively, such that e is an odd loop in both G_1 and G_2. Let G be the graph obtained by identifying a vertex of G_1 with a vertex of G_2 and then deleting both copies of e. Let $\Sigma := \Sigma_1 \triangle \Sigma_2$. Then $\text{ecycle}(G, \Sigma) = M_1 \oplus_2 M_2$.

Lemma 7.3. *The* 2*-sum of* R_{10} *and a minimally non-graphic matroid is an excluded minor for the class of even cycle matroids.*

Proof. Let $M := M_1 \oplus_2 R_{10}$, where M_1 is a minimally non-graphic matroid. First we show that every minor of M is an even cycle matroid. Let f be any element of M. If $f \in E(M_1)$, then both $M_1 \setminus f$ and M_1/f are graphic matroids; moreover $M \setminus f = (M_1 \setminus f) \oplus_2 R_{10}$ and $M/f = (M_1/f) \oplus_2 R_{10}$. As M_1 is 3-connected (every minimally non-graphic matroid is), no graph representing $M_1 \setminus f$ and no graph representing M_1/f contains any loops. Moreover, no signed graph representation of R_{10} contains a loop. Hence, by Construction 1, $M \setminus f$ and M/f are even cycle matroids. The same argument holds for $M \setminus f$ if $f \in E(R_{10})$, as $R_{10} \setminus f$ is isomorphic to cycle$(K_{3,3})$, for every $f \in E(R_{10})$. For every $f \in E(R_{10})$, R_{10}/f is isomorphic to cut$(K_{3,3})$. As discussed in Appendix B, for every element e of $K_{3,3}$, cut$(K_{3,3})$ has a signed graph representation where e is an odd loop. Moreover, again by the results in Appendix B, so does M_1. It follows, by Construction 2, that M/f is an even cycle matroid.

Now we show that M is not an even cycle matroid. Suppose for contradiction that M has a signed graph representation (G,Σ). Let $X := E(M_1) - E(R_{10})$. Define graphs $G_1 := G[X]$ and $G_2 := G[\bar{X}]$. For $i = 1,2$, let $\Sigma_i := \Sigma \cap E(G_i)$. Then (G_1,Σ_1) is a representation of $M_1 \setminus e$, for some element $e \in E(M_1)$ and (G_2,Σ_2) is a representation of $R_{10} \setminus e$, for some element $e \in E(R_{10})$. Note that both $M_1 \setminus e$ and $R_{10} \setminus e$ are connected. By Lemma 2.7, X is a k-separation of G, for some $k \leq 3$, and one of the following occurs:

(1) $k = 3$ and both (G_1,Σ_1), (G_2,Σ_2) are bipartite;

(2) $k = 2$ and exactly one of (G_1,Σ_1), (G_2,Σ_2) is bipartite;

(3) $k = 1$ and both (G_1,Σ_1), (G_2,Σ_2) are non-bipartite.

If case (1) occurs, then one of the vertices in $\mathscr{B}_G(X)$ is a blocking vertex of (G,Σ). By Remark 2.9, M is a graphic matroid, contradicting the fact that M_1 is a minor of M. If case (2) occurs, then (G,Σ) is obtained from (G_1,Σ_1) and (G_2,Σ_2) by Construction 1, but this is not possible as neither M_1 nor R_{10} is graphic. If case (3) occurs, then (G,Σ) is obtained from (G_1,Σ_1) and (G_2,Σ_2) by Construction 2, but again this is not possible, as no signed graph representation of R_{10} contains an odd loop. □

7.2 Disjoint odd circuits do not fix the representation

We already discussed the fact that degenerate even cycle matroids may have a large number of inequivalent representations. Degenerate even cycle matroids have representations with blocking pairs, which have at most two disjoint odd circuits. One might hope that having many disjoint odd circuits implies uniqueness of representation. This is, for example, the case for signed-graphic matroids (which were defined in Chapter 1). Slilaty [32] proved that if any representation (G,Σ) of a signed-graphic matroid M has three vertex-disjoint odd circuits, then (G,Σ) is the *unique* representation of M. This is not the case for even cycle matroids.

Remark 7.4. *For every integer k, there exists a signed graph (G,Σ) with the property that:*

(1) every signed graph equivalent to (G,Σ) has k vertex-disjoint odd circuits, and

(2) $\mathrm{ecycle}(G,\Sigma)$ *has at least two inequivalent representations.*

Proof. Let $\mathbb{T} = (H_1, v_1, \alpha_1, H_2, v_2, \alpha_2, \mathbb{S})$ be a split-template which is nova. Let (G_1,Σ_1), (G_2,Σ_2) be the siblings arising from $\mathbb{T}$. Because of Remark 4.4, we may assume that $\Sigma_1 = \Sigma_2 = \alpha_1 \triangle \alpha_2$. Suppose $\mathbb{S} = \{X_1,\ldots,X_k\}$ for some integer k. Because of (N2) (in the definition of nova), for every $j \in [k]$, there exists an odd circuit $C_j \subseteq X_j$ of (H_1,Σ_1) avoiding v_1. In particular, C_j remains an odd circuit of (G_1,Σ_1). Thus odd circuits $C_1,\ldots,C_k$ of (G_1,Σ_1) are pairwise vertex disjoint. Moreover, it is easy to select H_1 so that the only 2-separations of H_1 are given by $\mathbb{S}$. Then G_1 is 3-connected. Hence, (1) holds with $(G,\Sigma) = (G_1,\Sigma_1)$. Moreover, $\mathrm{ecycle}(G_1,\Sigma_1) = \mathrm{ecycle}(G_2,\Sigma_2)$, thus (2) holds as required. □

7.3 Stabilizers

We now discuss stabilizers, a concept introduced by Whittle in [39]. Stabilizers were introduced in the setting of matroids representable over some field $\mathbf{F}$, to deal with inequivalent matrix representations over $\mathbf{F}$. Let $\mathscr{M}$ be a class of matroids representable over some field $\mathbf{F}$. A matroid $N \in \mathscr{M}$ *stabilizes* $\mathscr{M}$ if, for every 3-connected matroid $M \in \mathscr{M}$ containing N as a minor, a matrix representation (over $\mathbf{F}$) of M is determined uniquely by a matrix representation of N. For example, for every field $\mathbf{F}$, the matroid $U_{2,4}$ stabilizes the class of $\mathbf{F}$-representable matroids with no $U_{2,5}$ or $U_{3,5}$ minor. In our context, representations are not matrices, but signed graphs and grafts. We define a notion of stabilizers, similar to the one introduced by Whittle, for even cycle and even cut matroids.

7.3.1 Stabilizers for even cycle matroids

Consider a matroid M and let $N := M \setminus I/J$ be a minor of M. Then M is a *major* of N.

Let M be an even cycle matroid with a representation (G,Σ). Then

$$\text{ecycle}(G,\Sigma)\setminus I/J = \text{ecycle}(H,\Gamma)$$

where $(H,\Gamma) = (G,\Sigma)\setminus I/J$. We say that (G,Σ) is an *extension* to M of the representation (H,Γ) of N, or alternatively that (H,Γ) *extends* to M.

Let N be a k-connected even cycle matroid. Suppose that, for all k-connected majors M of N and for every equivalence class $\mathscr{F}$ of representations of N, the set $\mathscr{F}'$ of extensions of $\mathscr{F}$ to M is the union of at most ℓ equivalence classes. Then we say that N is a *stabilizer* of *order* ℓ for k-connected matroids.

In Chapter 8 we prove that every 3-connected non-degenerate even cycle matroid is a stabilizer of order 2 for 3-connected matroids (Theorem 8.1). This implies, in particular, the following.

Corollary 7.5. *Let M be a* 3*-connected even cycle matroid which contains as a minor a non-degenerate* 3*-connected matroid N. Then the number of equivalence classes of the representations of M is at most twice the number of equivalence classes of the representations of N.*

Note that order 2 is the best we can hope for. In fact, consider split siblings (G_1,T_1) and (G_2,T_2) where, for $i=1,2$, Ω is an edge of G_i with ends T_i. Let Σ_1,Σ_2 be a corresponding signature pair. Let $M = \text{ecycle}(G_1,\Sigma_1)$ and let $N = M/\Omega$. Let $\mathscr{F}$ be the set of representations equivalent to $(G_1,\Sigma_1)/\Omega$. Then, if $\mathscr{F}'$ is the set of extensions of $\mathscr{F}$ to M, $\mathscr{F}'$ contains the inequivalent signed graphs (G_1,Σ_1) and (G_2,Σ_2).

There is, however, a condition that ensures that an even cycle matroid is a stabilizer of order 1 for 2-connected matroids. Consider a signed graph (G,Σ) and suppose there exists a partition $\mathscr{C}_1,\mathscr{C}_2$ of the odd circuits of (G,Σ) and graphs G_1,G_2 equivalent to G such that, for $i=1,2$, $v_i \in V(G_i)$ intersects all circuits in $\mathscr{C}_i$. Then we call the pair (G_1,v_1) and (G_2,v_2) an *intercepting pair* for (G,Σ). If (G,Σ) has a blocking pair v_1,v_2, then (G,v_1), (G,v_2) is an intercepting pair for (G,Σ). Hence having no intercepting pair is a stronger property than being non-degenerate. In Chapter 8 we prove that even cycle matroids that have no representations with an intercepting pair are stabilizers of order 1 for 2-connected matroids (Theorem 8.2). In particular, this implies the following.

Corollary 7.6. *Let M be a* 2*-connected even cycle matroid which contains as a minor a* 2*-connected matroid N for which none of the representations have an intercepting pair.*

Then the number of equivalence classes of the representations of M is at most the number of equivalence classes of the representations of N.

Note that, if a representation (G,Σ) has no blocking pair and G is 3-connected, then (G,Σ) has no intercepting pair. As an application of Corollary 7.6, consider the class of even cycle matroids which contain R_{10} as a minor. All 6 representations of R_{10} are of the form $\big(K_5,E(K_5)\big)$, thus none of them contain an intercepting pair. Hence, 2-connected even cycle matroids which contain R_{10} as a minor have at most 6 inequivalent representations.

7.3.2 Stabilizers for even cut matroids

Let M be an even cut matroid with a representation (G,T). Then

$$\text{ecycle}(G,T)\setminus D/C = \text{ecycle}(H,R)$$

where $(H,R) = (G,T)/D\setminus C$. We say that (G,T) is an *extension* to M of the representation (H,R) of N, or alternatively that (H,R) *extends* to M.

Let N be a k-connected even cut matroid. Suppose that, for all k-connected majors M of N and for every equivalence class $\mathscr{F}$ of representations of N, the set $\mathscr{F}'$ of extensions of $\mathscr{F}$ to M is the union of at most ℓ equivalence classes. Then we say that N is a *stabilizer* of *order* ℓ for k-connected matroids.

In Chapter 9 we prove that every 3-connected non-degenerate even cut matroid is a stabilizer of order 2 for 3-connected matroids (Theorem 9.1). This implies, in particular, the following.

Corollary 7.7. *Let M be a* 3*-connected even cut matroid which contains as a minor a* 3*-connected matroid N which is non-degenerate. Then the number of equivalence classes of the representations of M is at most twice the number of equivalence classes of the representations of N.*

In Chapter 9 we will introduce an operation on grafts that shows that order 2 is the best we can hope for. As for even cycle matroids, excluding a particular configuration assures that an even cut matroid is a stabilizer of order 1 for 2-connected matroids. Consider a graft (G,T) and suppose there exist graphs G_1,G_2 equivalent to G and paths P_1,P_2 in

G_1, G_2 respectively, such that $T = V_{odd}(G[P_1 \triangle P_2])$. We call the pair (G_1, P_1) and (G_2, P_2) a *reaching pair* for (G,T). When $G_1 = G_2 = G$, $|T| \leq 4$ and $\text{ecut}(G,T)$ is degenerate. Hence having no reaching pair is a stronger property than being non-degenerate. In Chapter 9 we show that even cut matroids that have no representations with a reaching pair are stabilizers of order 1 for 2-connected matroids (Theorem 9.2). In particular, we have the following.

Corollary 7.8. *Let M be a 2-connected even cut matroid which contains as a minor a 2-connected matroid N for which none of the representations have a reaching pair. Then the number of equivalence classes of the representations of M is at most the number of equivalence classes of the representations of N.*

As an application of Corollary 7.8, consider the class of even cut matroids which contain R_{10} as a minor. Recall that every representation of R_{10} is isomorphic to the graft in Figure 1.11 and the representations of R_{10} partition into 10 equivalence classes. The graft obtained by contracting the pin in the graft in Figure 1.11 is 3-connected and has six terminals, hence has no reaching pair. We will show that the property of having a reaching pair is closed under minors; it follows that no representation of R_{10} has a reaching pair. Hence every 2-connected even cut matroid containing R_{10} as a minor has at most 10 inequivalent representations.

7.3.3 Use of stabilizers

Why are we interested in stabilizer theorems? Suppose M is a 2-connected minimally non-even cycle matroid containing, for example, R_{10} as a minor. Then no representation of R_{10} extends to M. Suppose we can show that, for any representation (G,Σ) of R_{10}, there exists a 2-connected matroid N such that:

(P1) R_{10} is a minor of N;

(P2) N is a minor of M;

(P3) (G,Σ) does not extend to N;

(P4) $|E(N)|$ is small (compared to R_{10}).

The stabilizer theorem implies that N has one fewer representation than R_{10}. Thus we may repeat this process until we eliminate all the representations and conclude that M is small, compared with R_{10}. If the stabilizer theorem didn't hold, N might have had more representations than R_{10}; thus we wouldn't be gaining anything by eliminating (G,Σ).

Chapter 8

Stabilizer theorem for even cycle matroids

8.1 Main results

In this chapter we prove the following two results.

Theorem 8.1. *Let N be a 3-connected non-degenerate even cycle matroid. Let M be a 3-connected major of N. For every equivalence class $\mathscr{F}$ of representations of N, the set of extensions of $\mathscr{F}$ to M is the union of at most two equivalence classes.*

Theorem 8.2. *Let N be a 2-connected even cycle matroid with the property that no representation of N has an intercepting pair. Let M be a 2-connected major of N. For every equivalence class $\mathscr{F}$ of representations of N, the set of extensions of $\mathscr{F}$ to M is contained in one equivalence class.*

8.2 The proof

Consider a matroid M and let $N := M \setminus I/J$ be a minor of M. If $J = \emptyset$ and $|I| = 1$ then M is a *column major* of N. If $I = \emptyset$ and $|J| = 1$ then M is a *row major* of N.

A set $\mathscr{F}$ of representations of an even cycle matroid is *closed under equivalence* if, for every $(H,\Gamma) \in \mathscr{F}$ and (H',Γ') equivalent to (H,Γ), we have that $(H',\Gamma') \in \mathscr{F}$. Note that, if (G,Σ) and (G',Σ') are equivalent, then so are $(G,\Sigma) \setminus I/J$ and $(G',\Sigma') \setminus I/J$.

Remark 8.3. *Let $\mathscr{F}$ be a set of representations of an even cycle matroid N and let M be a major of N. If $\mathscr{F}$ is closed under equivalence, then so is the set $\mathscr{F}'$ of extensions of $\mathscr{F}$ to M.*

Proof. Let $(G,\Sigma) \in \mathscr{F}'$ and let (G',Σ') be equivalent to (G,Σ). We have $N = M \setminus I/J$, for some $I,J \subseteq E(M)$. Note that $(H,\Gamma) := (G,\Sigma) \setminus I/J$ and $(H',\Gamma') := (G',\Sigma') \setminus I/J$ are equivalent. Since $(G,\Sigma) \in \mathscr{F}'$, $(H,\Gamma) \in \mathscr{F}$. As $\mathscr{F}$ is closed under equivalence, $(H',\Gamma') \in \mathscr{F}$. Hence, by definition, $(G',\Sigma') \in \mathscr{F}'$. □

Let $\mathscr{F}$ be an equivalence class of signed graphs and let N be the corresponding even cycle matroid. We say that $\mathscr{F}$ is *stable* if for all row and column majors M of N which satisfy the following properties:

(a) M is non-graphic;

(b) M has no loop or co-loop,

the set of extensions of $\mathscr{F}$ to M is an equivalence class. If in the previous definition we consider only row (respectively column) majors M of N, then we say that $\mathscr{F}$ is *row stable* (respectively *column stable*). Hence, an equivalence class is stable if and only if it is both row and column stable.

Lemma 8.4. *Equivalence classes of signed graphs are column stable.*

We postpone the proof until Section 8.3.

Consider split siblings $(G_1,T_1),(G_2,T_2)$ where, for $i = 1,2$, Ω is an edge of G_i with ends T_i. Let Σ_1,Σ_2 be a corresponding signature pair. Let $M = \text{ecycle}(G_1,\Sigma_1)$ and let $N = M/\Omega$. Let $\mathscr{F}$ be the set of representations equivalent to $(G_1,\Sigma_1)/\Omega$. Then (G_1,Σ_1) and (G_2,Σ_2) are two inequivalent representations of M which extend representations of $\mathscr{F}$. In particular, $\mathscr{F}$ is not row stable. Thus, equivalence classes are not row stable in general. Moreover, Remark 7.4 shows that equivalence classes need not be row stable, even if there are an arbitrary number of vertex disjoint odd circuits in every signed graph in the equivalence class. However, in the previous example, $(G_1,\Sigma_1)/\Omega$ has an intercepting pair. To have an inductive argument on signed graphs with no intercepting pair, we need to know that, if a signed graph (H,Γ) has no intercepting pair, so does every major of (H,Γ).

Remark 8.5. *If (G,Σ) has an intercepting pair, then so does every minor (H,Γ) of (G,Σ).*

Proof. Since (G,Σ) has an intercepting pair, there exists a partition of the odd circuits of (G,Σ) into $\mathscr{C}_1,\mathscr{C}_2$ and there exists, for $i=1,2$, a graph G_i equivalent to G with a vertex $v_i \in V(G_i)$ that intersects all circuits in $\mathscr{C}_i$. We have $(H,\Gamma) = (G,\Sigma)\setminus I/J$ for some $I,J \subseteq E(G)$. For $i=1,2$: let $H_i = G_i \setminus I/J$, let $\mathscr{D}_i := \{C-J | C \in \mathscr{C}_i \text{ and } C \cap I = \emptyset\}$, and let w_i be the vertex of H_i which corresponds to the component of $G[J]$ containing v_i. Since G_1,G_2 are equivalent to G, H_1,H_2 are equivalent to H. The odd circuits of (H,Γ) are contained in $\mathscr{D}_1 \cup \mathscr{D}_2$ and, for $i=1,2$, vertex w_i of H_i intersects all circuits in $\mathscr{D}_i$. Hence, (H,Γ) has an intercepting pair. □

By definition, if a signed graph has intercepting pair, then so does every equivalent signed graph. Hence, we may talk about equivalence classes having an intercepting pair.

Lemma 8.6. *Equivalence classes without intercepting pairs are row stable.*

We postpone the proof until section 8.3. The last two results we require are the following.

Lemma 8.7. *Let N be an even cycle matroid and let $\mathscr{F}$ be an equivalence class of representations of N. Let M be a row major of N with no loops or co-loops. Suppose that the set $\mathscr{F}'$ of extensions of $\mathscr{F}$ to M is non-empty. Then $\mathscr{F}'$ is either an equivalence class or the union of two equivalence classes $\mathscr{F}_1$, $\mathscr{F}_2$ and any $(G_1,\Sigma_1) \in \mathscr{F}_1$, $(G_2,\Sigma_2) \in \mathscr{F}_2$ are split siblings which arise from a split-template $(H_1,v_1,\alpha_1,H_2,v_2,\alpha_2)$, where $(H_i,\alpha_1 \triangle \alpha_2) \in \mathscr{F}$, for $i=1,2$.*

Lemma 8.8. *Let $(G_1,\Sigma_1),(G_2,\Sigma_2)$ arise from a nova-template $\mathbb{T} = (H_1,v_1,\alpha_1,H_2,v_2,\alpha_2,\mathbb{S})$. Suppose that, for $i=1,2$, $\text{ecycle}(H_i,\alpha_1\triangle\alpha_2)$ and $\text{ecycle}(G_i,\Sigma_i)$ are 3-connected. Suppose also that, for $i=1,2$, there exists $\Omega \in E(G_i)$ such that $(G_i,\Sigma_i)/\Omega = (H_i,\alpha_1\triangle\alpha_2)$. Suppose finally that no signed graph equivalent to $(H_1,\alpha_1\triangle\alpha_2)$ has a blocking pair. Then, for $i=1,2$, (G_i,Σ_i) has no intercepting pairs.*

We postpone the proofs until section 8.4.

Assuming correctness of Lemmas 8.4, 8.6, 8.7 and 8.8, we can now prove Theorem 8.1 and Theorem 8.2.

Proof of Theorem 8.2. Let N be a 2-connected even cycle matroid, where none of the representations of N has an intercepting pair. Let M be a 2-connected major of N. Then there exists a sequence of 2-connected matroids $N_1, \dots, N_k$, where $N = N_1$, $M = N_k$ and, for $i \in [k-1]$, N_{i+1} is a row or column major of N_i (see [25], page 290; see also [3]). In particular, N_i has no loops or co-loops, for every $i \in [k]$. Let $\mathscr{F}$ be an equivalence class of the representations of N that extends to M and, for every $j \in [k]$, define $\mathscr{F}_j$ to be the set of extensions of $\mathscr{F}$ to N_j. It suffices to show that, for all $j \in [k]$, $\mathscr{F}_j$ is an equivalence class. Let us proceed by induction. As $N_1 = N$, the result holds for $j = 1$. Suppose that the result holds for $j \in [k-1]$. By Remark 8.5, $\mathscr{F}_j$ does not have an intercepting pair. Therefore, by Lemma 8.4 and Lemma 8.6, $\mathscr{F}_j$ is stable. It follows that $\mathscr{F}_{j+1}$ is an equivalence class. □

Proof of Theorem 8.1. Let N be a 3-connected non-degenerate even cycle matroid. Let M be a 3-connected major of N. It follows (see [28]) that there is a sequence of 3-connected matroids $N_1, \dots, N_k$, where $N = N_1$, $M = N_k$ and, for every $i \in [k-1]$, N_{i+1} is a row or column major of N_i. In particular, N_i has no loops or co-loops for any $i \in [k]$. Let $\mathscr{F}$ be an equivalence class of representations of N that extends to M. For every $j \in [k]$, define $\mathscr{F}_j$ to be the set of extensions of $\mathscr{F}$ to N_j. It suffices to show that, for all $j \in [k]$, $\mathscr{F}_j$ is either

(a) an equivalence class, or

(b) the union of two equivalence classes without intercepting pairs.

Let us proceed by induction. As $N_1 = N$, the result holds for $j = 1$. Suppose that the result holds for $j \in [k-1]$.

Consider the case where N_{j+1} is a column major of N_j. If (a) holds for $\mathscr{F}_j$, then Lemma 8.4 implies that (a) holds for $\mathscr{F}_{j+1}$. If (b) holds for $\mathscr{F}_j$, then Lemma 8.4 and Remark 8.5 imply that either (a) or (b) holds for $\mathscr{F}_{j+1}$.

Consider the case where N_{j+1} is a row major of N_j. Suppose first that (a) holds for $\mathscr{F}_j$. Then Lemma 8.7 implies that either (a) holds for $\mathscr{F}_{j+1}$ or $\mathscr{F}_{j+1} = \mathscr{F}' \cup \mathscr{F}''$, where $\mathscr{F}', \mathscr{F}''$ are equivalence classes which satisfy the following: any $(G_1, \Sigma_1) \in \mathscr{F}'$, $(G_2, \Sigma_2) \in \mathscr{F}''$ are split siblings which arise from a template $(H_1, v_1, \alpha_1, H_2, v_2, \alpha_2)$, where $(H_i, \alpha_1 \triangle \alpha_2) \in \mathscr{F}_j$ for $i = 1, 2$. Remark 4.4 implies that $N_j = \text{ecycle}(H_i, \alpha_1 \triangle \alpha_2)$, for $i = 1, 2$. Lemma 2.4 implies that H_1, H_2 are 2-connected, except for possible loops. Theorem 4.3 implies that (G_1, Σ_1) and (G_2, Σ_2) are simple siblings or nova siblings. Because of Remark 8.3, we may assume that (G_1,Σ_1) and (G_2,Σ_2) are either simple twins or nova twins. However, the former case does not occur, for otherwise Remark 4.5 implies that (G_1,Σ_1) has a blocking pair. Lemma 8.8 implies that $\mathscr{F}$ and $\mathscr{F}'$ have no intercepting pair. Hence, (b) holds for $\mathscr{F}_{j+1}$. Suppose now that (b) holds for $\mathscr{F}_j$. Then Lemma 8.6 implies that either of (a) or (b) holds for $\mathscr{F}_{j+1}$. □

8.3 Proof of Lemmas 8.4 and 8.6

As a consequence of Remark 3.8 we obtain the following.

Remark 8.9.

(1) Suppose that $\mathrm{ecycle}(G_1,\Sigma_1) = \mathrm{ecycle}(G_2,\Sigma_2)$. *If an odd cycle of* (G_1,Σ_1) *is a cycle of* G_2*, then* G_1 *and* G_2 *are equivalent.*

(2) Suppose that $\mathrm{ecut}(G_1,T_1) = \mathrm{ecut}(G_2,T_2)$. *If any odd cut of* (G_1,T_1) *is a cut of* G_2*, then* G_1 *and* G_2 *are equivalent.*

Lemma 8.10. *Let* $(G_1,\Sigma_1),(G_2,\Sigma_2)$ *be signed graph siblings and let* $\Omega \in E(G_1)$*. For* $i = 1,2$*, let* $(H_i,\Gamma_i) := (G_i,\Sigma_i)\setminus\Omega$*. Suppose that* (H_1,Γ_1) *and* (H_2,Γ_2) *are equivalent. Then, for* $i = 1,2$*,* Ω *is either a bridge of* G_i *or a signature of* (G_i,Σ_i)*. In particular,* Ω *is a co-loop of* $\mathrm{ecycle}(G_1,\Sigma_1)$.

Proof. We prove the statement for $i = 1$. Remark 8.9(1) implies that no odd cycle of (G_1,Σ_1) is a cycle of G_2. Since H_1 and H_2 are equivalent, $\mathrm{cycle}(H_1) = \mathrm{cycle}(H_2)$. It follows that all odd cycles of (G_1,Σ_1) use Ω. Hence, after possibly a signature exchange, $\Sigma_1 \subseteq \{\Omega\}$. Similarly, we may assume that $\Sigma_2 \subseteq \{\Omega\}$. If Ω is a bridge of G_1, we are done. Suppose otherwise. If $\Sigma_1 = \emptyset$, then there exists an even cycle C of (G_1,Σ_1) using Ω; hence Ω is not a bridge of G_2 and $\Sigma_2 \neq \{\Omega\}$. But then $\Sigma_1 = \Sigma_2 = \emptyset$ and $\mathrm{cycle}(G_1) = \mathrm{cycle}(G_2)$, a contradiction. □

Lemma 8.10 has a counterpart for even cuts. We shall omit the proof of the following observation as the proof is analogous to that of Lemma 8.10.

Lemma 8.11. *Let $(G_1,T_1),(G_2,T_2)$ be graft siblings and let $\Omega \in E(G_1)$. For $i=1,2$, let $(H_i,R_i) := (G_i,T_i)/\Omega$. Suppose that (H_1,R_1) and (H_2,R_2) are equivalent. Then, for $i=1,2$, either Ω is a loop of G_i or $|T_i| = 2$ and T_i are the ends of Ω in G_i. In particular, Ω is a co-loop of* $\operatorname{ecut}(G_1,T_1)$.

The last two lemmas imply the following result,

Lemma 8.12. *Let N be an even cycle matroid and $\mathscr{F}$ an equivalence class of representations of N. Let M be a row or column major of N which is not graphic. Suppose that the unique element Ω in $E(M) - E(N)$ is not a loop or a co-loop of M. Let $\mathscr{F}'$ be the set of extensions of $\mathscr{F}$ to M and consider $(G_1,\Sigma_1),(G_2,\Sigma_2) \in \mathscr{F}'$.*

(1) If M is a column major of N, then $(G_1,\Sigma_1),(G_2,\Sigma_2)$ are equivalent.

(2) If M is a row major of N, then $(G_1,\Sigma_1),(G_2,\Sigma_2)$ are either equivalent or split siblings. Moreover, in the latter case, let T_1 (respectively T_2) denote the ends of Ω in G_1 (respectively G_2). Then T_1,T_2 is the matching terminal pair for G_1,G_2.

Proof. **(1)**. Follows from Lemma 8.10 as M has no co-loop. **(2)**. We may assume that G_1 and G_2 are not equivalent. Then there exists a unique matching terminal pair T_1,T_2 for G_1,G_2. For $i=1,2$, let $(H_i,R_i) = (G_i,T_i)/\Omega$. Then $\operatorname{ecut}(H_1,R_1) = \operatorname{ecut}(H_2,R_2)$. Moreover, (G_1,Σ_1) and (G_2,Σ_2) are both in $\mathscr{F}'$, hence $H_1 = G_1/\Omega$ and $H_2 = G_2/\Omega$ are equivalent. It follows that (H_1,R_1) and (H_2,R_2) are equivalent. Lemma 8.11 implies that, for $i=1,2$, either Ω is a loop of G_i or T_i are the ends of Ω in G_i. If the latter case occurs for both $i=1,2$, then $(G_1,T_1),(G_2,T_2)$ are split siblings and we are done. Now suppose that Ω is a loop of G_i, for $i=1$ or $i=2$. Then every cut of G_i is a cut of H_i, hence a cut of H_{3-i} (as H_1 and H_2 are equivalent). It follows that every cut of G_i is a cut of G_{3-i}. By Remark 8.9(2), every cut of (G_i,T_i) is even. Therefore T_i is empty. By Theorem 3.1, Σ_{3-i} is empty and M is graphic, a contradiction. □

Proof of Lemma 8.4. It follows immediately from Lemma 8.12(1). □

Proof of Lemma 8.6. Let N be an even cycle matroid and let M be a row extension of N, i.e. $N = M/\Omega$ for some $\Omega \in E(M)$. Let $\mathscr{F}$ be an equivalence class of representations of N and let $\mathscr{F}'$ be the extension of $\mathscr{F}$ to M. Suppose for a contradiction that there exist inequivalent signed graphs $(G_1,\Sigma_1),(G_2,\Sigma_2) \in \mathscr{F}'$. Lemma 8.12(2) implies that $(G_1,\Sigma_1),(G_2,\Sigma_2)$

are split siblings which arise from a split-template $(H_1, v_1, \alpha_1, H_2, v_2, \alpha_2)$ where, for $i = 1,2$, $H_i = G_i/\Omega$. Remark 4.4 states that $\alpha_1 \triangle \alpha_2$ is a signature of (G_i, Σ_i) for $i = 1,2$. Hence, $(H_i, \alpha_1 \triangle \alpha_2) \in \mathscr{F}$, for $i = 1,2$. It follows that (H_1, v_1) and (H_2, v_2) form an intercepting pair of $(H_1, \alpha_1 \triangle \alpha_2)$, a contradiction. □

8.4 Proof of Lemmas 8.7 and 8.8

Proof of Lemma 8.7. For some $\Omega \in E(M)$, we have $N = M/\Omega$. Suppose for a contradiction that there exist, for $i = 1,2,3$, $(G_i, \Sigma_i) \in \mathscr{F}'$, where G_1, G_2, G_3 are inequivalent. For any distinct $i, j \in [3]$, let T_i, T_j be the matching terminal pair for G_i and G_j. Lemma 8.12(2) implies that the ends of Ω in G_i are T_i. It follows that $(G_1, T_1), (G_2, T_2), (G_3, T_3)$ are pairwise siblings.

For $i = 1,2$, let $v_i \in T_i$ and let $B_i = \delta_{G_i}(v_i)$. Theorem 3.1 implies that B_1 and B_2 are signatures of (G_3, Σ_3). Hence, $B_1 \triangle B_2$ is a cut of G_3. As $\Omega \notin B_1 \triangle B_2$, the cut $B_1 \triangle B_2$ is even in (G_3, T_3). It follows that $B_1 \triangle B_2$ is an even cut of (G_1, T_1). Hence, $B_1 \triangle (B_1 \triangle B_2) = B_2$ is a cut of G_1. But now Remark 8.9(2) implies that G_1 and G_2 are equivalent, a contradiction. □

Before we proceed to prove Lemma 8.8 we shall need a preliminary definition and an observation. An edge of a graph G that is a petal of a flower of G with at least four petals is said to be a *petal edge*.

Remark 8.13. *Let (G, Σ) be a signed graph and let (H, Γ) be obtained from (G, Σ) by contracting a petal edge.*

(1) If no signed graph equivalent to (G, Σ) has a blocking pair, then no signed graph equivalent to (H, Γ) has a blocking pair.

(2) If ecycle(G, Σ) *is 3-connected, then so is* ecycle(H, Γ).

(3) If (G, Σ) has a handcuff-separation and ecycle(G, Σ) *is 3-connected, then (H, Γ) has a handcuff-separation.*

Proof. **(1)** Suppose some signed graph (H', Γ') equivalent to (H, Γ) has a blocking pair. As e is a petal edge, there exists a signed graph (G', Σ') equivalent to (G, Σ) such that

$(H',\Gamma')=(G',\Sigma')/e$. Let $\mathbb{F}$ be the maximal flower of G' containing e. Let u,v be the ends of e in G'. As (H',Γ') has a blocking pair, $\Sigma'-\text{loop}(G')\subseteq\delta_{G'}(u)\cup\delta_{G'}(v)\cup\delta_{G'}(w)$, for some $w\in V(G')$. Let x be a connector of $\mathbb{F}$ distinct from u,v,w (x exists because, by definition of petal edge, $\mathbb{F}$ has at least four petals). Let G'' be obtained from G' by inserting e between the two petals of $\mathbb{F}$ incident with x and leaving the order of the other petals unchanged. Then $\Sigma'-\text{loop}(G'')$ is incident to two vertices in G'', so (G'',Σ') has a blocking pair and is equivalent to (G,Σ). **(2)** Follows from Lemma 2.4. **(3)** Let X be a handcuff-separation of (G,Σ). In particular, $|X|\geq 3$. As e is a petal edge, the ends of e in G are not $\mathscr{B}_G(X)$. Thus, if $e\in X$, then $X-\{e\}$ is a handcuff-separation of (H,Γ) and, if $e\notin X$ and $|E(H)-X|\geq 2$, then X is a handcuff-separation of (H,Γ). If $e\notin X$ and $E(H)-X=\{f\}$ for some edge f, then e,f are series edges in G, hence ecycle(G,Σ) is not 3-connected. □

Proof of Lemma 8.8. We will prove that (G_1,Σ_1) has no intercepting pair. Suppose for a contradiction that this is not the case and that (G_1,Σ_1) has an intercepting pair (G,v) and (G',v'), i.e. G and G' are equivalent to G_1 and every odd circuit of (G_1,Σ_1) either uses the vertex v in G or uses the vertex v' in G'. It follows that $(G_1,\Sigma_1)\setminus\left[\delta_G(v)\cup\delta_{G'}(v')\right]$ is bipartite. Hence, we can find $\alpha\subseteq\delta_G(v)$ and $\alpha'\subseteq\delta_{G'}(v')$ such that $\alpha\triangle\alpha'$ is a signature of (G_1,Σ_1). Lemma 2.4 implies that G and G' are 2-connected, up to loops. By Proposition 5.3 we may assume that, for some w-star $\mathbb{S}'$ of G, $G'=\mathrm{W}_{\text{flip}}[G,\mathbb{S}']$ and $\mathbb{T}:=(G,v,\alpha,G',v',\alpha',\mathbb{S}')$ is a split-template. Lemma 6.2 implies that there exists a split-template $\hat{\mathbb{T}}:=(\hat{G},\hat{v},\hat{\alpha},\hat{G}',\hat{v}',\hat{\alpha}')$ compatible with $\mathbb{T}$ which is simple or nova. Since $\hat{\mathbb{T}}$ is compatible with $\mathbb{T}$, both $\alpha\triangle\hat{\alpha}$ and $\alpha'\triangle\hat{\alpha}'$ are cuts of G and $\hat{G}$. It follows that $\hat{\alpha}\triangle\hat{\alpha}'$ is a signature of $(\hat{G},\Sigma_1)$. Observe that $\hat{\mathbb{T}}$ is not simple, for otherwise $\hat{v}$ is a blocking vertex of $(\hat{G},\Sigma_1)$, contradicting our hypothesis. Hence, $\hat{\mathbb{T}}$ is nova and, in particular, $(\hat{G},\Sigma_1)$ must have a handcuff-separation.

Recall that, by hypothesis, (G_1,Σ_1) and (G_2,Σ_2) arise from a nova-template $(H_1,v_1,\alpha_1,H_2,v_2,\alpha_2,\mathbb{S})$. Lemma 2.4 implies that H_1 is 2-connected, up to loops. The remainder of the proof is organized as follows: we first describe the set of all possible 2-separations of H_1, then we deduce the set of all possible 2-separations of G_1, and we conclude that no signed graph equivalent to (G_1,Σ_1) has a handcuff-separation, which provides us with the desired contradiction. Because of Remark 8.13, we can assume that (G_1,Σ_1) has no petal edge.

Let $X_1,\ldots,X_k$ denote the sets in $\mathbb{S}$ and let $X_0:=E(H_1)-(X_1\cup\ldots\cup X_k)$. For every

$i \in [k]$, $\mathscr{B}_{H_1}(X_i) = \{v_1, w_i\}$, for some vertex w_i. For $i \in [2]$ and $v \in V_{H_1}$, we define

$$\mu_i(v) := \delta_{H_1}(v) \cap \alpha_i \qquad \text{and} \qquad \overline{\mu}_i(v) := \delta_{H_1}(v) - \alpha_i.$$

Recall that $\Sigma_1 = \alpha_1 \triangle \alpha_2$ and that, for $i = 1, 2$, $\alpha_i \subseteq \delta_{H_i}(v_i) \cup \mathrm{loop}(H_i)$. This implies the following.

Claim 1. $\Sigma_1 \subseteq \left[\bigcup_{i \in [k]} \mu_2(w_i) \cap X_i\right] \cup \mu_1(v_1) \cup \mu_2(v_1) \cup \mathrm{loop}(H_1)$.

In particular, Claim 1 implies that $k \geq 2$, for otherwise v_1, w_1 is a blocking pair of (H_1, Σ_1), contradicting our hypothesis. Let Z be a 2-separation of H_1, where $Z \notin \mathbb{S}$. Denote by z_1, z_2 the vertices in $\mathscr{B}_{H_1}(Z)$. As $X_1, \ldots, X_k$ are pairwise disjoint sets, after possibly replacing Z by $\bar{Z}$, Z has to be a separation of one of the following types:

(T1) for all $i \in [k]$, either $Z \supseteq X_i$ or $\bar{Z} \supseteq X_i$;

(T2) for some $i_1 \in [k]$ and every $i_2 \in [k]$ such that $i_1 \neq i_2$, we have

$$Z \cap X_{i_1} \neq \emptyset \quad \bar{Z} \cap X_{i_1} \neq \emptyset \quad \bar{Z} \supseteq X_{i_2};$$

(T3) for some $i_1, i_2, i_3 \in [k]$ we have

$$Z \cap X_{i_1} \neq \emptyset \quad \bar{Z} \cap X_{i_1} \neq \emptyset \quad Z \supseteq X_{i_2} \quad \bar{Z} \supseteq X_{i_3};$$

(T4) for some distinct $i_1, i_2 \in [k]$, we have

$$Z \cap X_{i_1} \neq \emptyset \quad \bar{Z} \cap X_{i_1} \neq \emptyset \quad Z \cap X_{i_2} \neq \emptyset \quad \bar{Z} \cap X_{i_2} \neq \emptyset.$$

Claim 2. *There is no* 2*-separation* Z *of type (T3) or (T4).*

Proof. Suppose for a contradiction that Z is of type (T4). Without loss of generality, we may assume that $z_1 \in \mathscr{I}_{H_1}(X_{i_1})$ and $z_2 \in \mathscr{I}_{H_1}(X_{i_2})$. It follows that (after possibly replacing Z with $\bar{Z}$) there is a flower with petals $Z \cap X_{i_2}, \bar{Z} \cap X_{i_2}, X_0, \bar{Z} \cap X_{i_1}, Z \cap X_{i_1}$ in that order and with attachments v_1, z_2, w_2, w_1, z_1, in that order as well. Claim 1 implies that $\Sigma_1 \subseteq \mu_1(v_1) \cup \mu_2(w_1) \cup \mu_2(w_2)$. Then after rearranging the petals we obtain a signed graph with a blocking pair, a contradiction. Suppose for a contradiction that Z is of type (T3). Since H_1 is 2-connected, $H_1[X_0]$ is connected. In particular, there exists a circuit C of H_1 such that $C \cap X_{i_1} = \emptyset$ and $C \cap X_{i_2}, C \cap X_{i_3} \neq \emptyset$. We may assume that $z_1 \in \mathscr{I}_{H_1}(X_{i_1})$. Because of C, X_{i_2}, X_{i_3} are either both contained in Z or both contained in $\bar{Z}$, a contradiction. $\diamond$

Claim 3. *Let Z be a 2-separation of H_1. Then (after possibly replacing Z by $\bar{Z}$) one of the following holds:*

(1) $Z \subseteq X_i$, for some $i \in [k]$;

(2) for all $i \in [k]$, either $Z \supseteq X_i$ or $\bar{Z} \supseteq X_i$.

Proof. By Claim 2, Z is of type (T1) or (T2). In the former case we have outcome (2), hence we may assume that Z is of type (T2). Let $i := i_1$. Suppose that outcome (1) does not hold. Then $Z \cap \bar{X}_i \neq \emptyset$. It follows that H_1 has a flower with petals $X_i \cap Z, X_i - Z, Z - X_i, E(H_1) - (X_i \cup Z)$. Moreover, $\mathscr{B}_{H_1}(Z - X_i) = \{w_i, z_1\}$, where $z_1 \neq v_1$. Note that $\mu_1(z_1) \cap Z = \mu_2(z_1) \cap Z = \emptyset$ and $\mu_1(w_i) = \emptyset, \mu_2(w_i) \subseteq X_i$. Hence $Z - X_i$ contains no odd cycle of (H_1, Σ_1). It follows, from Lemma 2.4, that $Z - X_i$ consists of a single edge e. But then e is a petal edge of (H_1, Σ_1), hence also of (G_1, Σ_1), contradicting our assumption that (G_1, Σ_1) has no petal edge. $\diamond$

Recall that Ω is the edge in $E(G_1) - E(H_1)$. Denote by v_1^-, v_1^+ the ends of edge Ω in G_1.

Claim 4. *Let Z' be a 2-separation of G_1. Denote by z'_1, z'_2 the vertices in $\mathscr{B}_{G_1}(Z')$. Then (after possibly replacing Z' by $\bar{Z}'$ and interchanging the role of z'_1 and z'_2) one of the following holds:*

(1) $Z' = \{\Omega, e\}$, where e, Ω are parallel edges of G_1;

(2) $Z' \subset X_i$, for some $i \in [k]$, and $z'_1 = w_1, z_2 \notin \{v_1^-, v_1^+\}$;

(3) for all $i \in [k]$, $\bar{Z}' \supseteq X_i$ and $z'_1 \in \{v_1^-, v_1^+\}$.

Proof. Let $Z := Z' - \{\Omega\}$. Suppose $|Z| = 1$; then $Z = \{\Omega, e\}$ for some $e \in E(H_1)$. As G_1 has no series edges, e and Ω are in parallel in G_1 and (1) holds. Otherwise Z is a 2-separation of H_1 (recall that H_1 is 2-connected, except for possible loops). Consider first the case where Z satisfies outcome (1) of Claim 3, i.e. $Z \subseteq X_i$ for some $i \in [k]$. Let z_1, z_2 be the vertices in $\mathscr{B}_{H_1}(Z)$, where, for $j = 1, 2$, vertex z_j of H_1 corresponds to vertex z'_j of G_1. It follows from Claim 1 and Lemma 2.4 that $\{z_1, z_2\} \cap \{w_i, v_1\} \neq \emptyset$. Suppose that $w_i \notin \{z_1, z_2\}$. Lemma 2.4 implies that $\mu_1(v_1) \cap X_i, \bar{\mu}_1(v_1) \cap X_i$ are both non-empty. This implies that Z' is not a 2-separation of G_1, a contradiction. Thus we may assume that $z_1 = w_1$. Suppose for a contradiction that $z'_2 \in \{v_1^-, v_1^+\}$. Then $z_2 = v_1$. By the property (N2) of novae, Z

must be a handcuff-separation of (H_1, Σ_1). It follows that $\mu_1(v_1) \cap X_i, \bar{\mu}_1(v_1) \cap X_i$ are both non-empty. But this implies that Z' is not a 2-separation of G_1, a contradiction. Hence, Z satisfies outcome (2) of Claim 3. Since Z' is a 2-separation of G_1, there do not exist i_1, i_2 such that $X_{i_1} \subseteq Z$ and $X_{i_2} \subseteq \bar{Z}$. Finally it follows, from Claim 1 and Lemma 2.4, that $v_1 \in \{z_1, z_2\}$, and we obtain outcome (3). $\diamond$

It can now be readily checked from Claim 4 that no signed graph equivalent to (G_1, Σ_1) has a handcuff-separation, completing the proof. $\square$

Chapter 9

Stabilizer theorem for even cut matroids

9.1 Main results

In this chapter we prove the following two results.

Theorem 9.1. *Let N be a* 3*-connected non-degenerate even cut matroid. Let M be a* 3*-connected major of N. For every equivalence class $\mathscr{F}$ of representations of N, the set of extensions of $\mathscr{F}$ to M is the union of at most two equivalence classes.*

Theorem 9.2. *Let N be a* 2*-connected even cut matroid with the property that every representation of N has no reaching pair. Let M be a* 2*-connected major of N. For every equivalence class $\mathscr{F}$ of representations of N, the set of extensions of $\mathscr{F}$ to M is contained in one equivalence class.*

The proof of Theorem 9.2 is given in the next section. To prove Theorem 9.1 we need to introduce and characterize an operation on grafts. This is done in Section 9.3. The proof of the Theorem 9.1 follows (in Section 9.4). The last two sections of the chapter are dedicated to proving Lemmas that are needed to prove the two main theorems.

9.2 Proof of Theorem 9.2

A set $\mathscr{D}$ of representations of an even cut matroid is *closed under equivalence* if, for every $(H,R) \in \mathscr{D}$ and (H',R') equivalent to (H,R), we have that $(H',R') \in \mathscr{D}$. Note that, if (G,T)

and (G',T') are equivalent then so are $(G,T)/D\setminus C$ and $(G',T')/D\setminus C$.

Remark 9.3. *Let $\mathscr{D}$ be a set of representations of an even cut matroid N and let M be a major of N. If $\mathscr{D}$ is closed under equivalence, then so is the set $\mathscr{D}'$ of extensions of $\mathscr{D}$ to M.*

Proof. Let $(G,T)\in\mathscr{D}'$ and let (G',T') be equivalent to (G,T). We have $N=M\setminus D/C$ for some $D,C\subseteq E(M)$. Moreover, $(H,R):=(G,T)/D\setminus C$ and $(H',R'):=(G',T')/D\setminus C$ are equivalent. Since $(G,T)\in\mathscr{D}'$, $(H,R)\in\mathscr{D}$. As $\mathscr{D}$ is closed under equivalence, $(H',R')\in\mathscr{D}$. Hence, by definition, $(G',T')\in\mathscr{D}'$. □

Let $\mathscr{D}$ be an equivalence class of grafts and let N be the corresponding even cut matroid. We say that $\mathscr{D}$ is *stable* if, for all row and column majors M of N which satisfy the following properties:

i. M is not cographic;

ii. M has no loop or co-loop,

the set of extensions of $\mathscr{D}$ to M is an equivalence class. If in the previous definition we consider only row (respectively column) majors M of N, then we say that $\mathscr{D}$ is *row stable* (respectively *column stable*). Hence, an equivalence class is stable if and only if it is both row and column stable.

Lemma 9.4. *Equivalence classes of grafts are column stable.*

We postpone the proof until Section 9.5.

In general, equivalence classes are not row stable. We will show how this follows from the operation we introduce in the next section. Recall the definition of reaching pair given in Section 7.3.2. By definition, if a graft has a reaching pair then so does any equivalent graft. Hence, we may talk about an equivalence class having a reaching pair.

Remark 9.5. *If (G,T) has a reaching pair, so does every minor (H,R) of (G,T).*

Proof. Since (G,T) has a reaching pair, there exists, for $i=1,2$, a graph G_i equivalent to G and a path P_i in G_i such that $T=V_{odd}(G[P_1\triangle P_2])$. By induction, it suffices to prove the statement for the cases $(H,R)=(G,T)\setminus e$ and $(H,R)=(G,T)/e$, for some $e\in E(G)$.

First, suppose that $(H,R) = (G,T) \setminus e$. If e is an odd bridge of G, then R is empty and the statement is trivially true (taking as reaching pair $(H,\emptyset),(H,\emptyset)$). If e is not an odd bridge of G, then $R = T$. If e is an even bridge of G, then $e \notin P_1 \triangle P_2$ and e is a bridge of G_1 and G_2. Thus in this case we may assume that $e \notin P_1, P_2$ (if $e \in P_1 \cap P_2$, we may replace G_1, G_2 with some equivalent graphs and P_i with $P_i - e$, for $i = 1,2$). For $i = 1,2$, let v_i, w_i be the ends of P_i in G_i and $H_i := G_i \setminus e$. Let Q_i be a (v_i,w_i)-path in H_i (Q_i exists, as either e is not a bridge of G or $e \notin P_1, P_2$). Then $V_{odd}(G[P_i]) = V_{odd}(G[Q_i])$, for $i = 1,2$. Therefore $T = V_{odd}(H[Q_1 \triangle Q_2])$ and (H_1,Q_1), (H_2,Q_2) is a reaching pair for (H,T).

Now suppose that $(H,R) = (G,T)/e$. Note that, if J is a T-join of G, then $J - \{e\}$ is an R-join of H. For $i = 1,2$, define $H_i := G_i/e$ and $Q_i := P_i - e$. Then Q_i is a $\{v_i,w_i\}$-join of H_i, for some $v_i, w_i \in V(H_i)$. Let Q'_i be a (v_i,w_i)-path in H_i. As H_1, H_2 are equivalent to H, $V_{odd}(H[Q'_1 \triangle Q'_2]) = V_{odd}(H[Q_1 \triangle Q_2])$. As $Q_1 \triangle Q_2 = (P_1 \triangle P_2) - \{e\}$, the statement follows. □

We introduced reaching pairs because of the following result.

Lemma 9.6. *Equivalence classes without reaching pairs are row stable.*

We postpone the proof until section 9.5.

Proof of Theorem 9.2. Let N be a 2-connected non-degenerate even cut matroid. Let M be a 2-connected major of N. Then there exists a sequence of 2-connected matroids $N_1, \dots, N_k$, where $N = N_1$, $M = N_k$ and, for all $i \in [k-1]$, N_{i+1} is a row or column major of N_i (see [25], page 290; see also [3]). In particular, N_i has no loops or co-loops, for any $i \in [k]$. Let $\mathscr{D}$ be an equivalence class of representations of N which extends to M and, for every $j \in [k]$, define $\mathscr{D}_j$ to be the set of extensions of $\mathscr{D}$ to N_j. It suffices to show that, for all $j \in [k]$, $\mathscr{D}_j$ is an equivalence class. Let us proceed by induction. As $N_1 = N$, the result holds for $j = 1$. Suppose that the result holds for $j \in [k-1]$. By Remark 9.5, $\mathscr{D}_j$ does not have a reaching pair. Therefore, by Lemma 9.4 and Lemma 9.6, $\mathscr{D}_j$ is stable. It follows that $\mathscr{D}_{j+1}$ is an equivalence class. □

9.3 Clip siblings

We now introduce an operation on grafts which preserves even cuts. Consider a pair of equivalent graphs H_1 and H_2. Suppose that $P_i \subset E(H_i)$ is a path in H_i, for $i = 1,2$. For

$i = 1,2$, let G_i be obtained from H_i by adding an edge Ω with endpoints the ends of P_i. Since H_1 and H_2 are equivalent, $\text{cycle}(H_1) = \text{cycle}(H_2)$. Moreover,

$$\text{ecycle}(G_1, \{\Omega\}) = \text{cycle}(H_1) = \text{cycle}(H_2) = \text{ecycle}(G_2, \{\Omega\}).$$

In particular, $(G_1, \{\Omega\}), (G_2, \{\Omega\})$ are either equivalent or siblings. Let T_1, T_2 be a matching terminal pair for G_1, G_2. If $(G_1, T_1), (G_2, T_2)$ are inequivalent we say that the tuple $\mathbb{T} = (H_1, P_1, H_2, P_2)$ is a *clip-template* and that (G_1, T_1),(G_2, T_2) (respectively $(G_1, \{\Omega\})$, $(G_2, \{\Omega\})$) are *clip siblings* which *arise* from $\mathbb{T}$. An explicit characterization of clip siblings is given in Section 9.3.1.

Remark 9.7. *Let $\mathbb{T} = (H_1, P_1, H_2, P_2)$ be a clip-template and let (G_1, T_1), (G_2, T_2) be clip siblings that arise from $\mathbb{T}$. Then, for $i = 1,2$, we have $T_i = V_{odd}(G_i[P_1 \Delta P_2])$.*

Proof. As $P_i \cup \Omega$ is an odd cycle of $(G_i, \{\Omega\})$ for $i = 1,2$, by Theorem 3.1, we have $T_i = V_{odd}(G_i[P_{3-i} \cup \Omega]) = V_{odd}(G_i[P_{3-i}]) \Delta V_{odd}(G_i[\Omega])$. As Ω and P_i have the same ends in G_i, we have $V_{odd}(G_i[\Omega]) = V_{odd}(G_i[P_i])$. It follows that $T_i = V_{odd}(G_i[P_{3-i}]) \Delta V_{odd}(G_i[P_i]) = V_{odd}(G_i[P_1 \Delta P_2])$. □

Consider clip siblings $(G_1, T_1), (G_2, T_2)$ arising from a clip-template (H_1, P_1, H_2, P_2). Let Ω be the edge in $E(G_1) - E(H_1)$. Let $M = \text{ecut}(G_1, T_1)$ and let $N = M/\Omega$. Then (H_1, T_1) is a representation of N. Let $\mathscr{D}$ be the set of representations equivalent to (H_1, T_1). Then (G_1, T_1) and (G_2, T_2) are two inequivalent representations of M which extend representations of $\mathscr{D}$. In particular, $\mathscr{D}$ is not row stable. Thus, equivalence classes of grafts are not row stable in general.

9.3.1 A characterization of clip siblings

The main result of the section is the following.

Theorem 9.8. *Let M be a 3-connected even cut matroid with representations (G_i, T_i) for $i = 1,2$. Suppose that $(G_1, T_1), (G_2, T_2)$ are clip siblings arising from a clip-template $\mathbb{T} = (H_1, P_1, H_2, P_2)$, where $\text{ecut}(H_1, T_1)$ is 3-connected and is not cographic. Then (G_1, T_1), (G_2, T_2) are either basic siblings or strip siblings.*

It remains to define the terms "basic siblings" and "strip siblings". Consider a clip-template (H_1,P_1,H_2,P_2). If $H_2 = \mathrm{W}_{\text{flip}}(H_1,\mathbb{S})$ for some w-sequence $\mathbb{S}$, we slightly abuse notation and say that $(H_1,P_1,H_2,P_2,\mathbb{S})$ is a clip-template. We will always assume that $\mathbb{S}$ is a w-sequence in this case.

Consider a clip-template $\mathbb{T} = (H_1,P_1,H_2,P_2,\mathbb{S})$. If $\mathbb{S} = \emptyset$ (that is $H_1 = H_2$) then $\mathbb{T}$ is a *basic* template and $(G_1,T_1),(G_2,T_2)$ arising from $\mathbb{T}$ are *basic twins*. By Remark 9.7, $T_i = V_{odd}(H_i[P_1 \triangle P_2])$, for $i = 1,2$. As P_1,P_2 are both paths in H_1,H_2, this implies that $|T_1|,|T_2| \leq 4$. Therefore:

Remark 9.9. *Basic twins are degenerate.*

We say that a clip-template $\mathbb{T} = (H_1,P_1,H_2,P_2,\mathbb{S})$ is a *strip-template* if the following hold:

(a) $\mathbb{S} = (X_1,\ldots,X_k)$ is a nested w-sequence for H_1;

(b) P_i has one end in $\mathscr{I}_{H_i}(X_1)$ and the other end in $\mathscr{I}_{H_i}(\bar{X}_k)$, for $i = 1,2$.

In this case we say that the grafts $(G_1,T_1),(G_2,T_2)$ arising from $\mathbb{T}$ are *strip twins*.

We say that $(G_1,T_1),(G_2,T_2)$ are basic (respectively strip) *siblings* if, for $i = 1,2$, there exists (G_i',T_i') equivalent to (G_i,T_i) such that $(G_1',T_1'),(G_2',T_2')$ are basic (respectively strip) twins.

9.3.2 Proof of Theorem 9.8

We say that clip-templates:

$$\mathbb{T} = (H_1,P_1,H_2,P_2,\mathbb{S}) \quad \text{and} \quad \mathbb{T}' = (H_1',P_1',H_2',P_2',\mathbb{S}') \tag{9.1}$$

are *compatible* if:

(a) H_i,H_i' are equivalent, for $i = 1,2$, and

(b) $P_i \triangle P_i'$ is a cycle of H_1, for $i = 1,2$.

Note that, by Theorem 1.1, $\mathrm{cycle}(H_1) = \mathrm{cycle}(H_2) = \mathrm{cycle}(H_1') = \mathrm{cycle}(H_2')$.

Lemma 9.10. *Let $\mathbb{T}$ and $\mathbb{T}'$ be compatible templates. Let (G_1,T_1), (G_2,T_2) arise from $\mathbb{T}$ and let (G_1',T_1'), (G_2',T_2') arise from $\mathbb{T}'$. Then, for $i=1,2$, (G_i,T_i) and (G_i',T_i') are equivalent.*

Proof. Let us assume that $\mathbb{T},\mathbb{T}'$ are as described in (9.1). Then, by construction, $\text{cycle}(G_1) = \text{span}\big(\text{cycle}(H_1)\cup\{P_1\cup\Omega\}\big)$ and $\text{cycle}(G_1') = \text{span}\big(\text{cycle}(H_1)\cup\{P_1'\cup\Omega\}\big)$. By hypothesis, $(P_1\cup\Omega)\triangle(P_1'\cup\Omega) = P_1\triangle P_1' \in \text{cycle}(H_1)$. Hence, $\text{cycle}(G_1) = \text{cycle}(G_1')$. It follows from Theorem 1.1 that G_1 and G_1' are equivalent. Similarly, G_2 and G_2' are equivalent. It follows that $(G_1', V_{odd}(G_1'[J_1]))$ and $(G_2', V_{odd}(G_2'[J_2]))$ (where J_i is a T_i-join of G_i, for $i=1,2$) are siblings. As the matching terminal pair for G_1',G_2' is unique (by Proposition 3.7), (G_i,T_i) and (G_i',T_i') are equivalent, for $i=1,2$. $\square$

Lemma 9.11. *Let $\mathbb{T} = (H_1,P_1,H_2,P_2,\mathbb{S})$ be a clip-template. Then $\mathbb{T}$ has a compatible clip-template which is basic or strip.*

Proof. Suppose that $\mathbb{T} = (H_1,P_1,H_2,P_2,\mathbb{S})$ is a clip-template. By Proposition 5.4, there exists a graph H such that

(1) $H = \mathrm{W}_{\text{flip}}[H_1,\mathbb{S}_1]$, for some w-sequence $\mathbb{S}_1$ which preserves P_1, and

(2) $H_2 = \mathrm{W}_{\text{flip}}[H,\mathbb{S}_2]$, for some nested w-sequence $\mathbb{S}_2$, where every $X\in\mathbb{S}_2$ does not preserve P_1.

Now let $\mathbb{S}_3 := (X\in\mathbb{S}_2 : X \text{ preserves } P_2)$ and $\mathbb{S}_4 := \mathbb{S}_2 - \mathbb{S}_3$. Then $\mathbb{T}' = (H,P_1,\mathrm{W}_{\text{flip}}[H_2,\mathbb{S}_3],P_2,\mathbb{S}_4)$ and $\mathbb{T}$ are compatible clip-templates. Moreover, $\mathbb{S}_4 = (X_1,\dots,X_k)$ is nested (as it is a subsequence of $\mathbb{S}_2$) and every $X\in\mathbb{S}_4$ does not preserve P_1 and P_2. This implies that, for $i=1,2$ and for every $j\in[k]$, P_i has one end in $\mathscr{I}_{H_i}(X_j)$ and one end in $\mathscr{I}_{H_i}(\bar{X}_j)$. As $X_1\subset X_2\subset\cdots\subset X_k$, this implies that $\mathbb{T}'$ is a clip-template, if $\mathbb{S}_4$ is non-empty. If $\mathbb{S}_4$ is empty, then $\mathbb{T}'$ is basic and we are done. $\square$

Proof of Theorem 9.8. Proposition 2.5 implies that H_1 and H_2 are 2-connected, except for the possible presence of a single pin. Suppose that e is a pin of H_1 (and H_2). Let v_i be the head of e in H_i and Ω the edge in $E(G_1)-E(H_1)$. If e is a pin neither in G_1 nor in G_2, then, for $i=1,2$, Ω is incident to v_i in G_i and $\delta_{G_i}(v_i) = \{e,\Omega\}$. Thus $e\in P_1\cap P_2$. By Remark 9.7, $T_i = V_{odd}(G_i[P_1\triangle P_2])$, hence $v_i\notin T_i$. Therefore $\delta_{G_i}(v_i)$ is an even cut of (G_i,T_i) and e,Ω are parallel elements of $\text{ecut}(G_1,T_1)$, a contradiction. It follows that e is not a pin in one of G_1,

G_2. As a pin can be moved anywhere by a Whitney-flip, we will ignore the position of e. Hence we may assume that there exists a w-sequence $\mathbb{S}$ for H_1 such that $H_2 = \mathrm{W}_{\text{flip}}[H_1,\mathbb{S}]$. It follows that $\mathbb{T} = (H_1,P_1,H_2,P_2,\mathbb{S})$ is a clip-template. Lemma 9.11 implies that there exists a clip-template $\mathbb{T}'$ which is basic or strip and compatible with $\mathbb{T}$. Let $(G_1',T_1'),(G_2',T_2')$ arise from $\mathbb{T}'$. By definition $(G_1',T_1'),(G_2',T_2')$ are basic twins or strip twins. By Lemma 9.10, for $i = 1,2$, (G_i',T_i') is equivalent to (G_i,T_i). It follows that $(G_1,T_1),(G_2,T_2)$ are basic or strip siblings. □

9.4 Proof of Theorem 9.1

The last two results we require to prove Theorem 9.1 are the following.

Lemma 9.12. *Let N be an even cut matroid and let $\mathscr{D}$ be an equivalence class of representations of N. Let M be a row major of N with no loops or co-loops. Suppose that the set $\mathscr{D}'$ of extensions of $\mathscr{D}$ to M is non-empty. Then $\mathscr{D}'$ is either an equivalence class or the union of two equivalence classes $\mathscr{D}_1$, $\mathscr{D}_2$ and any $(G_1,T_1) \in \mathscr{D}_1$, $(G_2,T_2) \in \mathscr{D}_2$ are clip siblings which arise from a clip-template (H_1,P_1,H_2,P_2), where $(H_i,V_{odd}(H_i[P_1 \triangle P_2])) \in \mathscr{D}$, for $i = 1,2$.*

Lemma 9.13. *Let $(G_1,T_1),(G_2,T_2)$ arise from a strip-template $\mathbb{T} = (H_1,P_1,H_2,P_2,\mathbb{S})$. Suppose that* $\mathrm{ecut}(H_1,T_1)$ *and* $\mathrm{ecut}(G_1,T_1)$ *are 3-connected and (H_1,T_1) is non-degenerate. Then (G_1,T_1), (G_2,T_2) have no reaching pair.*

We postpone the proofs of these lemmas until Section 9.6.

Proof of Theorem 9.1. Let N be a 3-connected non-degenerate even cut matroid. Let M be a 3-connected major of N. It follows [28] that there is a sequence of 3-connected matroids $N_1,\dots,N_k$, where $N = N_1$, $M = N_k$ and, for all $i \in [k-1]$, N_{i+1} is a row or column major of N_i. In particular, N_i has no loop or co-loop, for any $i \in [k]$. Let $\mathscr{D}$ be an equivalence class of representations of N which extends to M and, for every $j \in [k]$, define $\mathscr{D}_j$ to be the set of extensions of $\mathscr{D}$ to N_j. It suffices to show that, for all $j \in [k]$, $\mathscr{D}_j$ is either

(a) an equivalence class, or

(b) the union of two equivalence classes without reaching pairs.

Let us proceed by induction. As $N_1 = N$, the result holds for $j = 1$. Suppose that the result holds for $j \in [k-1]$.

Consider the case where N_{j+1} is a column major of N_j. If (a) holds for $\mathscr{D}_j$, then Lemma 9.4 implies that (a) holds for $\mathscr{D}_{j+1}$. If (b) holds for $\mathscr{D}_j$, then Lemma 9.4 and Remark 9.5 imply that either (a) or (b) holds for $\mathscr{D}_{j+1}$.

Consider the case where N_{j+1} is a row major of N_j. Suppose first that (a) holds for $\mathscr{D}_j$. Lemma 9.12 implies that either (a) holds for $\mathscr{D}_{j+1}$ or $\mathscr{D}_{j+1} = \mathscr{D}^1 \cup \mathscr{D}^2$, where $\mathscr{D}^1, \mathscr{D}^2$ are equivalence classes. Moreover, any $(G_1, T_1) \in \mathscr{D}^1$, $(G_2, T_2) \in \mathscr{D}^2$ are clip siblings which arise from a clip-template (H_1, P_1, H_2, P_2), where $(H_i, V_{odd}(H_i[P_1 \triangle P_2])) \in \mathscr{D}_j$, for $i = 1, 2$. Remark 9.7 implies that $N_j = \text{ecut}(H_i, V_{odd}(H_i[P_1 \triangle P_2]))$, for $i = 1, 2$. Theorem 9.8 implies that (G_1, T_1) and (G_2, T_2) are basic siblings or strip siblings. Because of Remark 9.3, we may assume that (G_1, T_1) and (G_2, T_2) are either basic twins or strip twins. The former case does not occur, for otherwise Remark 9.9 implies that $|T_1| \leq 4$ and $\text{ecut}(H_1, T_1)$ is degenerate. Lemma 9.13 implies that $\mathscr{D}^1, \mathscr{D}^2$ have no reaching pair. Hence, (b) holds for $\mathscr{D}_{j+1}$. Suppose now that (b) holds for $\mathscr{D}_j$. Then Lemma 9.6 and Remark 9.5 imply that either (a) or (b) holds for $\mathscr{D}_{j+1}$. □

9.5 Proof of Lemmas 9.4 and 9.6

Lemma 9.14. *Let N be an even cut matroid and $\mathscr{D}$ be an equivalence class of representations of N. Let M be a row or column major of N that is not cographic. Suppose that the unique element Ω in $E(M) - E(N)$ is not a loop or a co-loop of M. Let $\mathscr{D}'$ be the set of extensions of $\mathscr{D}$ to M and $(G_1, T_1), (G_2, T_2) \in \mathscr{D}'$.*

(1) If M is a column major of N, then $(G_1, T_1), (G_2, T_2)$ are equivalent.

(2) If M is a row major of N, then $(G_1, T_1), (G_2, T_2)$ are either equivalent or clip siblings. Moreover, in the latter case, $\Sigma_1 = \Sigma_2 = \{\Omega\}$ is the matching signature pair for G_1, G_2.

Proof. **(1)**. Follows from Lemma 8.11, as M has no co-loop. **(2)**. We may assume that G_1, G_2 are not equivalent. Then there is a unique (up to signature exchange) matching signature pair Σ_1, Σ_2 for G_1, G_2. For $i = 1, 2$, let $(H_i, \Gamma_i) = (G_i, \Sigma_i) \setminus \Omega$. As $\text{ecycle}(H_1, \Gamma_1) = \text{ecycle}(H_2, \Gamma_2)$ and H_1, H_2 are equivalent, it follows that $(H_1, \Gamma_1), (H_2, \Gamma_2)$ are equivalent.

Lemma 8.10 implies that, for $i = 1,2$, either Ω is a bridge of G_i or a signature of (G_i, Σ_i). If the latter case occurs for both $i = 1$ and $i = 2$, then $(G_1, T_1), (G_2, T_2)$ are clip siblings and we are done. Now suppose that Ω is a bridge of G_i, for $i = 1$ or $i = 2$. Then every cycle of G_i is a cycle of H_i, hence a cycle of H_{3-i} (as H_1 and H_2 are equivalent). It follows that every cycle of G_i is a cycle of G_{3-i}. By Remark 8.9(1), every cycle of (G_i, Σ_i) is even. Therefore $\Sigma_i' = \emptyset$ is a signature of (G_i, Σ_i). By Proposition 3.7 and Theorem 3.1, T_{3-i} is empty and M is cographic, a contradiction. □

Proof of Lemma 9.4. It follows from part (1) of Lemma 9.14. □

Proof of Lemma 9.6. Let N be an even cut matroid and let M be a row extension of N, i.e. $N = M/\Omega$ for some $\Omega \in E(M)$. Suppose that M is not cographic and Ω is not a loop or co-loop of M. Let $\mathscr{D}$ be an equivalence class of representations of N with no reaching pair and let $\mathscr{D}'$ be the extension of $\mathscr{D}$ to M. Suppose for a contradiction that there exist inequivalent grafts $(G_1, T_1), (G_2, T_2) \in \mathscr{D}'$. Lemma 9.14(2) implies that $(G_1, T_1), (G_2, T_2)$ are clip siblings which arise from a clip-template (H_1, P_1, H_2, P_2), where, for $i = 1,2$, $H_i = G_i \setminus \Omega$. Remark 9.7 states that $T_i = V_{odd}(G_i[P_1 \triangle P_2])$, for $i = 1,2$. Hence, $(H_i, T_i) \in \mathscr{D}$, for $i = 1,2$. It follows that (H_1, P_1) and (H_2, P_2) form a reaching pair of (H_1, T_1), a contradiction. □

9.6 Proof of Lemmas 9.12 and 9.13

Proof of Lemma 9.12. For some $\Omega \in E(M)$, we have $N = M/\Omega$. Suppose for a contradiction that there exist, for $i = 1,2,3$, $(G_i, T_i) \in \mathscr{D}'$, where G_1, G_2, G_3 are inequivalent. For any distinct $i, j \in [3]$, let Σ_{ij}, Σ_{ji} be the matching signature pair for G_i and G_j. Lemma 9.14(2) implies that Ω is a signature of (G_1, Σ_{ij}), for every i, j. It follows that $(G_1, \{\Omega\}), (G_2, \{\Omega\}), (G_3, \{\Omega\})$ are pairwise siblings. For $i = 1,2$, let P_i be a path in G_i forming a cycle with Ω. Let $C_i := P_i \cup \Omega$. Theorem 3.1 implies that C_1 and C_2 are T_3-joins of (G_3, T_3). Hence, $C_1 \triangle C_2$ is a cycle of G_3. As $\Omega \notin C_1 \triangle C_2$, the cycle $C_1 \triangle C_2$ is even in $(G_3, \{\Omega\})$. It follows that $C_1 \triangle C_2$ is an even cycle of $(G_1, \{\Omega\})$. Hence, $C_1 \triangle (C_1 \triangle C_2) = C_2$ is a cycle of G_1. But now Remark 8.9(1) implies that G_1 and G_2 are equivalent, a contradiction. □

The proof of Lemma 9.13 is quite complicated and requires some results and definitions. Given a graft (G, T), we say that a 2-separation X of G is *simple* if $\mathscr{I}_G(X) = \{u\}$, for some $u \in T$ such that u has degree two in G. We say that a graft (G,T) is *well behaved* if (G,T) is non-degenerate, G is 2-connected and, for every 2-separation X of G, either X or $\bar{X}$ is simple.

Lemma 9.15. *If (G,T) is well behaved, then (G,T) does not have a reaching pair.*

Proof. Suppose for contradiction that (G,T) has a reaching pair (G_1,P_1),(G_2,P_2). Thus $T = V_{odd}(G[P_1 \triangle P_2])$. For $i = 1,2$, let $\mathbb{S}_i$ be a w-sequence for G such that $G_i = \mathrm{W}_{\text{flip}}[G,\mathbb{S}_i]$. As (G,T) is well behaved, we may assume that X is simple for every $X \in \mathbb{S}_i$. It follows that, for $i = 1,2$, every $X \in \mathbb{S}_i$ is a 2-separation in G_{3-i}. We may assume that every $X \in \mathbb{S}_i$ does not preserve P_i. Consider $X \in \mathbb{S}_1$; let $\{u\} = \mathscr{I}_G(X)$. As X does not preserve P_1, u is an end of P_1. As $T = V_{odd}(G[P_1 \triangle P_2])$ and $u \in T$, u is not an end of P_2, hence X preserves P_2. It follows that both P_1 and P_2 are paths in G_1, hence $|V_{odd}(G_1[P_1 \triangle P_2])| \leq 4$. As $P_1 \triangle P_2$ is a T-join of G, the graft $(G_1, V_{odd}(G_1[P_1 \triangle P_2])$ is equivalent to (G,T), so (G,T) is degenerate, a contradiction. □

Recall that X is a 2-$(0,0)$-separation (respectively a 2-$(0,1)$-separation) of a graft (G,T) if X is a 2-separation of G, $\mathscr{I}_G(X) \cap T$ is empty and $\mathscr{I}_G(\bar{X}) \cap T$ is empty (respectively, $\mathscr{I}_G(\bar{X}) \cap T$ is non-empty). We say that a graft (G,T) is *nice* if the following hold:

(a) G is 2-connected;

(b) every graft (G',T') equivalent to (G,T) contains no 2-$(0,0)$ or 2-$(0,1)$-separation.

Note that, in particular, nice grafts do not contain even cuts of size two.

A graft (G,T) is a *clean strip* if the following hold:

(a) there exists an edge Ω of G such that $(H,T) := (G,T) \setminus \{\Omega\}$ is nice and non-degenerate;

(b) there exists a nested sequence $\mathbb{S} = (X_1,\ldots,X_k)$ in H;

(c) $T = T' \cup T_c$, where $T_c \subseteq \cup_{i=1}^{k} \mathscr{B}_H(X_i)$ and $v_2, w_2 \in T' \subseteq \{v_1,v_2,w_1,w_2\}$, for distinct vertices $v_1, v_2 \in \mathscr{I}_H(X_1)$ and $w_1, w_2 \in \mathscr{I}_H(\bar{X}_k)$;

(d) the ends of Ω are v_1, w_1.

Lemma 9.16. *Let $\mathbb{T} = (H_1, P_1, H_2, P_2, \mathbb{S})$ be a strip-template. Let (G_1, T_1), (G_2, T_2) be the strip siblings arising from $\mathbb{T}$. If (H_1, T_1) is nice and non-degenerate, then (G_1, T_1) and (G_2, T_2) are clean strips.*

Proof. To simplify the notation we prove the statement for $i = 1$. Let $\mathbb{S} = (X_1, \dots, X_k)$ and, for $i = 1, 2$, let v_i, w_i be the ends of P_i in H_i. By definition of strip-template, we have $v_1 \in \mathscr{I}_{H_1}(X_1)$ and $w_1 \in \mathscr{I}_{H_1}(\bar{X}_k)$. Let Ω be the edge in $E(G_1) - E(H_1)$. Then the ends of Ω are v_1, w_1. Note that $\mathscr{I}_{H_1}(X_1) = \mathscr{I}_{H_2}(X_1)$ and $\mathscr{I}_{H_1}(\bar{X}_k) = \mathscr{I}_{H_2}(\bar{X}_k)$, thus v_2, w_2 are vertices of H_1. By Lemma 5.15, $V_{odd}(H_1[P_2]) = \{v_2, w_2\} \cup V_{odd}(\mathrm{Cat}(H_1, \mathbb{S}))$. Therefore $T_1 = \{v_1, w_1\} \triangle (\{v_2, w_2\} \cup V_{odd}(\mathrm{Cat}(H_1, \mathbb{S})))$. Hence $T_1 \cap \mathscr{I}_{H_1}(X_1) = \{v_1\} \triangle \{v_2\}$. As (H_1, T_1) is nice, $T_1 \cap \mathscr{I}_{H_1}(X_1)$ is non-empty. It follows that v_1 and v_2 are distinct vertices of H_1 and $v_1, v_2 \in T_1$. Similarly, w_1 and w_2 are distinct vertices of H_1 and $w_1, w_2 \in T_1$. $\square$

Lemma 9.17. *Let (G, T) be a clean strip. Then (G, T) contains a well behaved graft as a minor.*

We postpone the proof of Lemma 9.17 until the end of the section. We are now ready to prove Lemma 9.13.

Proof of Lemma 9.13. Let $(G_1, T_1), (G_2, T_2)$ arise from a strip-template $\mathbb{T} = (H_1, P_1, H_2, P_2, \mathbb{S})$, where $\mathrm{ecut}(H_1, T_1)$ and $\mathrm{ecut}(G_1, T_1)$ are 3-connected and $\mathrm{ecut}(H_1, T_1)$ is non-degenerate. By symmetry between (G_1, T_1) and (G_2, T_2), it suffices to show that (G_1, T_1) has no reaching pair. If H_1 is 2-connected, then, by Proposition 2.5, (H_1, T_1) is nice. By Lemma 9.16, (G_1, T_1) is a clean strip. By Lemma 9.17, (G_1, T_1) contains a well behaved graft as a minor. By Lemma 9.15, such a minor does not have a reaching pair, hence the result follows by Remark 9.5.

Now suppose that H_1 is not 2-connected. By Proposition 2.5, (H_1, T_1) contains a pin e and H_1/e is 2-connected. If $(H_1, T_1)/e$ is nice and non-degenerate, we may apply Lemmas 9.16, 9.17 and 9.15 to $(G_1, T_1)/e$ and deduce that $(G_1, T_1)/e$ has no reaching pair. By Remark 9.5 it follows that (G_1, T_1) has no reaching pair. Thus it suffices to show that $(H_1, T_1)/e$ is non-degenerate and nice.

Suppose for contradiction that $(H_1, T_1)/e$ is degenerate. Then there exists a graft $(\hat{H}, \hat{T})$ equivalent to $(H_1, T_1)/e$ with $|\hat{T}| \leq 4$. As e is a pin of (H_1, T_1), it follows that there exists a graft (H, T) equivalent to (H_1, T_1) such that $(\hat{H}, \hat{T}) = (H, T)/e$. As (H_1, T_1) is non-degenerate, $|T| \geq 6$. Moreover, $|\hat{T}| = |T|$ or $|\hat{T}| = |T| - 2$, thus $|T| = 6$, $|\hat{T}| = 4$ and both ends of e in H are in T. Let t be a vertex of T which is not an end of e. Consider the graft (H',T') obtained from T by a Whitney-flip moving e to be incident to t. Then $|T'| = 4$ and (H',T') is equivalent to (H_1,T_1), a contradiction.

It remains to show that $(H_1,T_1)/e$ is nice. We already showed that H_1/e is 2-connected. Now suppose that X is a 2-$(0,0)$ or a 2-$(0,1)$-separation in $(H_1,T_1)/e$. By Proposition 2.5, X is not a 2-$(0,0)$ or a 2-$(0,1)$-separation in (H_1,T_1). It follows that $X \cup \{e\}$ is a 2-$(1,1)$-separation in (H_1,T_1) and the vertices in $T_1 \cap \mathscr{I}_{H_1}(X)$ are the ends of e. Thus X is a 2-$(0,1)$-separation in the graft obtained from (H_1,T_1) by moving e to a vertex in $\mathscr{B}_{H_1}(X)$, a contradiction. □

To conclude the chapter it remains to prove Lemma 9.17. To obtained the desired minor for Lemma 9.17, we require the following reduction. Consider a nice graft (G,T) and a 2-separation X of G with the following properties:

(i) either $G[X]$ is 2-connected or $G[X-\{e\}]$ is 2-connected for some $e \in X$; in the second case, the end of e in $\mathscr{I}_G(X)$ is not in T;

(ii) $|\mathscr{I}_G(X) \cap T| = 1$.

The graft (H,R) is obtained from (G,T) by *cleaning* X if H is obtained from G by replacing X with a triangle $\{f,g,h\}$, where the ends of f are $\mathscr{B}_G(X)$ and the vertex that g,h share has degree two. If there are any edges parallel to f in H, we delete all such edges. We let $R := T - \mathscr{I}_G(X) \cup \mathscr{I}_H(\{g,h\})$. Note that, by this definition, (H,R) is also nice and (H,R) is a minor of (G,T).

Lemma 9.18. *Suppose that (H,R) is obtained from (G,T) by cleaning X. If (G,T) is non-degenerate, then so is (H,R).*

Proof. Suppose for contradiction that (H,R) is degenerate. Let f,g,h be the edges in H substituting X. Let (H',R') be equivalent to (H,R) with $|R'| \leq 4$. Then $\{f,g,h\}$ is also a triangle in H'. Consider the graft (G',T') obtained by substituting such a triangle with X, so that (H',R') is obtained from (G',T') by cleaning X. Then $|T'| = |R'| \leq 4$ and (G',T') is equivalent to (G,T), a contradiction. □

The following is easy.

Remark 9.19. *Let (G,T) be a graft and X be a 2-separation of G. Let $(G',T') = \mathrm{W}_{flip}[(G,T), X]$. Then $\mathscr{I}_G(X) \cap T = \mathscr{I}_{G'}(X) \cap T'$ and $\mathscr{I}_G(\bar{X}) \cap T = \mathscr{I}_{G'}(\bar{X}) \cap T'$. Moreover, if $|\mathscr{I}_G(X) \cap T|$ is odd, then the following hold:*

(1) *if $\mathscr{B}_G(X) \subset T$, then $\mathscr{B}_{G'}(X) \cap T'$ is empty;*

(2) *if $\mathscr{B}_G(X) \cap T = \{v\}$, then $\mathscr{B}_{G'}(X) \cap T' = \{v'\}$, where v' is the vertex in $\mathscr{B}_{G'}(X)$ incident to $\delta_G(v) \cap X$.*

Proof of Lemma 9.17. Let (G,T) be a clean strip. Let (H,T), Ω, T_c, v_1, v_2, w_1, w_2 and $\mathbb{S} = (X_1, \ldots, X_k)$ be defined as in the definition of clean strip. We prove the statement by induction on the number of non-simple 2-separations in (G,T). If every 2-separation in (G,T) is simple, then we are trivially done.

Claim 1. *We may assume that every 2-separation Y of G, with $Y \subset X_1$, is simple.*

Proof. Suppose that Y is a non-simple 2-separation of G with $Y \subset X_1$. Pick Y to be minimal with this property. Then $v_1 \notin \mathscr{I}_H(Y)$, as the edge Ω is incident to v_1. As $T \cap \mathscr{I}_H(Y)$ is non-empty, it follows that $T \cap \mathscr{I}_H(Y) = \{v_2\}$. If v_2 is a cut-vertex of $G[Y]$, then Y partitions into sets Y_1, Y_2 with $\{v_2\} = V_H(Y_1) \cap V_H(Y_2)$. As $\mathscr{I}_H(Y_i) \cap T$ is empty, for $i = 1,2$, it follows that Y_1, Y_2 are each formed by a single edge and Y is simple. As Y is not simple, v_2 is not a cut-vertex of $G[Y]$. Now suppose that $G[Y]$ has a cut-vertex $x \neq v_2$. Then Y partitions into sets Y_1, Y_2, with $\{x\} = V_H(Y_1) \cap V_H(Y_2)$ and $v_2 \in \mathscr{I}_H(Y_1)$. It follows that $\mathscr{I}_G(Y_2) \cap T$ is empty, hence Y_2 is a single edge. Moreover, $G[Y_1]$ is 2-connected, for otherwise we contradict the choice of Y (Y_1 cannot be formed by two series edges, for otherwise these edges would be in series with Y_2 and (H,T) would contain an even cut of size two). Hence we may clean Y. The resulting graft is a clean strip, hence the result follows by induction. $\diamond$

By symmetry between X_1 and $\bar{X}_k$, we may also assume that every 2-separation Y of G with $Y \subset \bar{X}_k$ is simple.

Claim 2. *If Y is a 2-separation of G with $Y \subset X_k - X_1$, then $Y = \{e, f\}$, for two series edges e, f of G.*

Proof. Suppose that Y is a 2-separation of G with $Y \subset X_k - X_1$. Thus $x \in \mathscr{I}_H(Y)$ for some $x \in T_c$. By definition of clean strip, $x \in \mathscr{B}_H(X_p)$ for some $p \in [k]$. Note that $p \neq 1, k$, as $x \in \mathscr{I}_H(Y)$ and $Y \subset X_k - X_1$. As $x \in \mathscr{I}_H(Y)$, the sets $X_p \cap Y$ and $Y - X_p$ are non-empty. Moreover $X_1 \subset X_p$ and $Y \cap X_1$ is empty, so $X_p - Y$ is also non-empty. Finally $\bar{X}_k$ is contained in both $\bar{X}_p$ and in $\bar{Y}$, hence $\bar{X}_p \cap \bar{Y}$ is non-empty. It follows that Y and X_p cross. By Remark 5.5, there exists a partition Z_1, Z_2, Z_3, Z_4 of $E(H)$ such that $X_p = Z_1 \cup Z_2$, $Y = Z_2 \cup Z_3$ and one of the following occurs:

(1) $\mathscr{B}_H(Z_i) = \mathscr{B}_H(X_p)$, for every $i \in [4]$;

(2) $\{Z_1, Z_2, Z_3, Z_4\}$ is a flower of H.

The first case cannot occur, as $x \in \mathscr{B}_H(X_p) - \mathscr{B}_H(Y)$. Hence we have $X_1 \subseteq X_p - Y = Z_1$ and $\bar{X}_k \subseteq \bar{X}_p \cap \bar{Y} = Z_4$. If both Z_2 and Z_3 are single edges, then they are in series in G and we are done. Now suppose that one of them, say Z_2, has non-empty interior. It follows that $y \in \mathscr{I}_H(Z_2)$, for some $y \in T_c$. Let $q \in [k]$ such that $y \in \mathscr{B}_H(X_q)$. Then y is a cut-vertex of $H[Z_2]$ and not a cut-vertex of H, as $\mathscr{B}_H(X_q)$ separates v_1 from v_2 in H. Hence Z_2 partitions into sets W_1, W_2 where $(Z_1, W_1, W_2, Z_3, Z_4)$ is a flower of H and $V_H(W_1) \cap V_H(W_2) = \{y\}$. A similar argument holds for every $z \in T_c \cap \mathscr{I}_H(Z_3)$. It follows that $Z_2 \cup Z_3$ partitions into sets $B_1, \ldots, B_\ell$, where $(Z_1, B_1, \ldots, B_\ell, Z_4)$ is a flower of H and, for every $i \in [\ell]$, $\mathscr{I}_H(B_i) \cap T$ is empty. It follows that $B_1, \ldots, B_\ell$ are series edges in H. Hence, if $\ell \geq 3$, two of these edges form an even cut, a contradiction. The result follows. $\diamond$

Note that, in particular, Claim 2 implies that every 2-separation in $X_k - X_1$ is simple. By Claims 1 and 2 and by symmetry between X_1 and $\bar{X}_k$, we conclude that every 2-separation Y of G with either $Y \subset X_1$, or $Y \subset \bar{X}_k$, or $Y \subset X_k - X_1$, is simple. Hence every non-simple 2-separation Y of G crosses either X_1 or $\bar{X}_k$. By symmetry between X_1 and $\bar{X}_k$ we may assume that there exists a non-simple 2-separation Y in G_1 which crosses X_1. Choose Y to be inclusion-wise minimal with such properties. We conclude the proof by showing that we can clean Y and obtain a clean strip. We will assume that $H[X_1]$ does not partition into sets U_1, U_2, where $v_1, v_2 \in \mathscr{I}_H(U_1)$ and $\{U_1, U_2, \bar{X}_1\}$ is a flower of H, as otherwise we may redefine X_1 to be $X_1 - U_2$. The same holds for $\bar{X}_k$. Moreover, we may assume that X_1 and Y_1 cross in a non-trivial way, i.e. every set $X_1 \cap Y$, $X_1 - Y$, $Y - X_1$, $\bar{X}_1 \cap \bar{Y}$ is not an edge with endpoints $\mathscr{B}_H(Y)$ (otherwise we may clean Y as in Claim 1). Therefore there exists a partition Z_1, Z_2, Z_3, Z_4 of $E(H)$ such that $X_1 = Z_1 \cup Z_2$, $Y = Z_2 \cup Z_3$ and one of the following occurs:

(1) $\mathscr{B}_H(Z_i) = \mathscr{B}_H(X_1)$ and $\mathscr{I}_H(Z_i)$ is non-empty, for every $i \in [4]$;

(2) $\{Z_1, Z_2, Z_3, Z_4\}$ is a flower of H.

As $X_1 = Z_1 \cup Z_2$, we have $v_1 \in \mathscr{I}_H(Z_1 \cup Z_2)$ and $w_1 \in \mathscr{I}_H(Z_3 \cup Z_4)$. As Y is a 2-separation in G_1, either $v_1 \in V_H(Z_1)$ and $w_1 \in V_H(Z_4)$ or $v_1 \in V_H(Z_2)$ and $w_1 \in V_H(Z_3)$. By possibly swapping Y with its complement, we may assume that $v_1 \in V_H(Z_1)$ and $w_1 \in V_H(Z_4)$.

We first show that case (1) does not occur. As each set Z_i has a non-empty interior, it is a 2-separation in H. Hence $\mathscr{I}_H(Z_i) \cap T$ is non-empty for every $i \in [4]$. It follows that $v_2 \in \mathscr{I}_H(Z_2)$. Suppose that $k \geq 2$, i.e. there is a 2-separation X_2 in $\mathbb{S}$ with $X_1 \subset X_2$. We may assume that X_1 and X_2 have distinct boundaries. Let $\{a_1, a_2\}$ be the boundary of X_2. As $X_1 \subset X_2$, $a_1, a_2 \in V_H(Z_3 \cup Z_4)$. If $a_1, a_2 \in V_H(Z_4)$, then $Z_3 \subset X_2 - X_1$ and $T \cap \mathscr{I}_H(Z_3)$ is empty, a contradiction. Hence $a_1 \in \mathscr{I}_H(Z_3)$; then there exists a (v_1, v_2)-path in $H \setminus \{a_1, a_2\}$, a contradiction. It follows that $\mathbb{S} = (X_1)$, hence $T_c \subseteq \mathscr{B}_H(X_1)$. As $|T| \geq 6$ (because $\text{ecut}(H,T)$ is non-degenerate), we have $T = \{v_1, v_2, w_1, w_2\} \cup \mathscr{B}_H(X_1)$. By Remark 9.19, it follows that $\mathrm{W}_{\text{flip}}[(H,T), X_1]$ has four terminals, contradicting the fact that $\text{ecut}(H,T)$ is non-degenerate.

We conclude that, for every 2-separation Y which crosses X_1, case (2) occurs. We claim that Y does not cross X_k. Suppose it does. Then, by a similar argument to the one above (applied to $\bar{X}_k$), there exists a flower (W_1, W_2, W_3, W_4) of H with $\bar{X}_k = W_1 \cup W_2$ and $Y = W_2 \cup W_3$. Let $\mathbb{F}$ be a maximal flower that is a refinement of (Z_1, Z_2, Z_3, Z_4) and $\mathbb{F}'$ be a maximal flower that is a refinement of (W_1, W_2, W_3, W_4). As Y crosses both X_1 and $\bar{X}_k$, we have $\mathbb{F} = \mathbb{F}'$. Hence $\mathbb{F}$ is a flower of H and X_1 and $\bar{X}_k$ are each the union of at least two petals of $\mathbb{F}$. As (by the assumption above) $H[X_1]$ does not partition into sets U_1, U_2 with $v_1, v_2 \in \mathscr{I}_H(U_1)$ and $\{U_1, U_2, \bar{X}\}$ a flower of H, we have that v_1, v_2 are not in the interior of the same petal of $\mathbb{F}$. Similarly, w_1, w_2 are not in the interior of the same petal of $\mathbb{F}$. Moreover, as $\mathbb{F}$ is a refinement of (Z_1, Z_2, Z_3, Z_4), v_1 and w_1 are in distinct petals of $\mathbb{F}$. For every $X \in \mathbb{S}$, the vertices in $\mathscr{B}_H(X)$ are attachments of $\mathbb{F}$, as there is no (v_1, w_1)-path in $H - \mathscr{B}_H(X)$. Hence T_c is contained in the set of attachments of $\mathbb{F}$. By Lemma 5.14, (H,T) is degenerate, a contradiction. We conclude that Y does not cross $\bar{X}_k$, hence $\bar{X}_k \subset Z_4$.

Now suppose that $\mathscr{I}_H(Z_3)$ is non-empty. It follows that Z_3 is a 2-separation of G with $Z_3 \subset X_k - X_1$. By Claim 2, Z_3 is formed by two series edges $\{e, f\}$. In this case let e be the edge with one end in $V_H(Z_2)$. If $\mathscr{I}_H(Z_3)$ is empty, then Z_3 is composed of a single edge; call this edge e. It follows that $\mathscr{I}_H(Z_2)$ is not a singe edge, as otherwise either $Z_2 \cup Z_3$ are three series edges (and (H,T) contains an even cut of size two), or Y is simple. Hence $T \cap \mathscr{I}_H(Z_2) = \{v_2\}$. By minimality of Y, $Z_3 = \{e\}$. Let x be the end of e in $V_H(Z_2)$. If $x \notin T$, then we may clean Y in (G,T) and, by induction, obtain a clean strip. If $x \in T$, let $(G',T') = \mathrm{W}_{\text{flip}}[(G,T),(Z_2,Y)]$. Then, by Remark 9.19, we may clean Y in (G',T') and obtain a clean strip. $\square$

Chapter 10

Future work and open problems

10.1 Isomorphism Problem

For the Isomorphism Problem, we started by relating even cut siblings with even cycle siblings. We defined two classes of even cycle siblings (Shih siblings and quad siblings) and solved the Isomorphism Problem for these classes. The next step would be to prove the Isomorphism Conjecture 4.2. The results in Chapter 3 imply that a proof of the Isomorphism Conjecture would solve the Isomorphism Problem for both even cycle and even cut matroids. However, we would like to have a solution to the Isomorphism Problem for even cut matroids where all the operations involved preserve the even cuts. This is not the case for sequences of Lovász-flips, as discussed in Section 3.1.

Consider the following basic operation on graft with four terminals: let (G,T) be a graft with $|T| = 4$; let (H,Γ) be obtained from (G,T) by folding with some pairing. Let (H',Γ') be obtained from (H,Γ) by either one Whitney-flip or a signature exchange, where $\Gamma' \subseteq \delta_{H'}(u) \cup \delta_{H'}(v) \cup \text{loop}(H')$, for some vertices $u,v \in V(H')$. Let (G',T') be obtained from (H',Γ') by unfolding on u,v. Then (G,T) and (G',T') are quad siblings and $\text{ecut}(G,T) = \text{ecut}(G',T')$; we say that (G,T) and (G',T') are related by a *basic operation*. For example, tilt and twist twins are related by a simple operation and shuffle twins are related by a sequence of basic operations. In Section 4.4 we conjecture that, up to Whitney-flips, Lovász-flips, signature exchanges and reductions, signed graphs siblings are related by one of a set of possible operations. The following asks which operations we need to describe the relation between graft siblings.

Open Problem 1. *Up to Whitney-flips, basic operations and reductions, what are the operations needed to define the relation between two grafts representing the same even cut matroid?*

Note that sequences of Lovász-flips on signed graphs give rise to examples like the one described in Section 2.4.3 (and represented in Figure 2.1); sequences of basic operations on grafts gives rise to examples like the one described in Section 2.4.3 and represented in Figure 2.2. To answer Open Problem 1 we will have to take into account pairs of siblings like the ones in Figure 2.1, which, for even cut matroids, arise in pairs.

10.2 Excluded Minor Problem

The work in Chapters 8 and 9 provides tools toward solving the following problems.

Open Problem 2. *What are the excluded minors for the class of even cycle matroids?*

Open Problem 3. *What are the excluded minors for the class of even cut matroids?*

However, an answer to these two problems will certainly be quite hard to attain. We may instead focus on a more specific problem.

Let $\mathscr{E}$ be the class of even cycle matroids that contain R_{10} as a minor. Theorem 8.2 implies that, for every matroid M in $\mathscr{E}$ and any fixed R_{10}-minor in M, every equivalence class of representations of M arises uniquely from an equivalence class of representations of the minor. This makes the following problem more approachable than Problem 2.

Open Problem 4. *What are the excluded minors for the class $\mathscr{E}$?*

In other words, we are asking which are the matroids M such that every proper minor of M is either an even cycle matroid or does not contain R_{10} as a minor. Note that this does not imply that M itself is an excluded minor for the class of even cycle matroids.

When looking for excluded minors for a class of matroids, it is often useful to first consider only excluded minors which contain a specific matroid (in our case, R_{10}) as a minor. We may then focus on finding excluded minors for the class of even cycle matroids which do not contain R_{10} as a minor. The tools needed to solve Problem 4 would likely also be useful in applying Theorem 8.2 or Theorem 8.1 to other classes of non-degenerate even cycle matroids, to solve the analogue of Problem 4 for them.

As R_{10} is also an even cut matroid and every graft representation of R_{10} has no reaching pair, we may ask the analogous question for even cut matroids. Let $\mathscr{E}'$ be the class of even cut matroids which contain R_{10} as a minor.

Open Problem 5. *What are the excluded minors for the class $\mathscr{E}'$?*

Our initial motivation to study even cycle and even cut matroids was to prove Seymour's Conjecture 1.5. Guenin showed that Seymour's conjecture holds for even cycle and even cut matroids (see [14]). Seymour (see [29]) showed that the property of being 1-flowing is closed under duality. Hence Seymour's conjecture also holds for duals of even cycle matroids and duals of even cut matroids. Therefore, to prove Seymour's conjecture we need to know something about the matroids that are not even cycle, duals of even cycle, even cut or duals of even cut matroids. This is in general a very hard problem; even knowing the excluded minors for the basic classes, finding the excluded minors for their union will not be easy.

We may focus on a more specific problem, like solving Seymour's conjecture for matroids containing R_{10} as a minor. Let $\mathscr{E}^*$ be the class of matroids that are duals of matroids in $\mathscr{E}$ and $(\mathscr{E}')^*$ be the class of matroids that are duals of matroids in $\mathscr{E}'$. Let $\mathscr{E}_U$ be the union of $\mathscr{E}$, $\mathscr{E}'$, $\mathscr{E}^*$ and $(\mathscr{E}')^*$. The matroid R_{10} is self-dual; thus any matroid in $\mathscr{E}_U$ contains R_{10} as a minor. To prove Seymour's conjecture for matroids containing R_{10} as a minor, we would want to know which are the binary matroids outside the class $\mathscr{E}_U$.

Open Problem 6. *What are the excluded minors for the class $\mathscr{E}_U$?*

10.3 More Open Problems

In Section 2.5 we proved that degenerate even cycle matroids are projections of graphic matroids and degenerate even cut matroids are projections of cographic matroids. We do not know whether the converse is true. We do not have any evidence for either a positive or negative answer to this question.

Open Problem 7. *Let M be an even cycle matroid which is the projection of a graphic matroid. Is M degenerate?*

Open Problem 8. *Let M be an even cut matroid which is the projection of a cographic matroid. Is M degenerate?*

Theorem 8.1 implies that, if M is a 3-connected even cycle matroid which contains as a minor a 3-connected non-degenerate even cycle matroid N, then the number of inequivalent representations of M is at most twice the number of inequivalent representations of N. By the work of Geelen, Gerards and Whittle [11], we know that every minor closed class of binary matroids has a finite number of excluded minors. It follows that there exists a constant c such that every non-degenerate even cycle matroid contains a non-degenerate minor of size at most c. However, we would like a more precise result, with a small constant and possibly a characterization of the minimally non-degenerate even cycle matroids.

Open Problem 9. *Which are the excluded minors for the class of degenerate even cycle matroids?*

Note that, if (G,Σ) is a signed graph with no blocking pair, it is very likely true that (G,Σ) contains a small minor (H,Γ) with no blocking pair. However, this does not necessarily imply that every other representation of $\text{ecycle}(H,\Gamma)$ has no blocking pair.

We conclude this section with the analogue of Problem 9 for even cut matroids.

Open Problem 10. *Which are the excluded minors for the class of degenerate even cut matroids?*

APPENDICES

Appendix A

Recognition

In this appendix we present an algorithm to find signed graph representations of a given binary matroid. Given a matrix representation over $GF(2)$ of a binary matroid M, the algorithm returns the list of all representations of M as an even cycle matroid. If M is not an even cycle matroid, the algorithm returns an empty list. The running time of the algorithm is exponential in the rank of the matroid. We also present an analogous algorithm for even cut matroids.

A.1 Even cycle matroids

Let A be a binary matrix with r rows and x be a non-zero column of A. Let M be the binary matroid with matrix representation A. Let e be the element of M corresponding to column x of A. Then a matrix representation of M/e is the matrix A/x, where A/x is obtained from A by:

(a) row reducing A so that column e has exactly one non-zero element in row r;

(b) deleting row r and column e.

Let y be row r at the end of step (a) (i.e. the row that is deleted); then we denote $S_x(A) := \{f \in E(M) - \{e\} : y_f = 1\}$. We have the following algorithm for recognizing even cycle matroids, based on the fact that even cycle matroids are lifts of graphic matroids.

- **Input:** Binary matrix representation A of a binary matroid M of rank r.
- **Output:** All representations of M as an even cycle matroid, up to equivalence.
- **Algorithm:**
  - (i) Set $\mathbb{L} := \emptyset$.
  - (ii) For all non-zero binary vectors x of size r do:
    - (1) add x to A to obtain a matrix A';
    - (2) check if $M(A'/x)$ is graphic: if so, $\mathbb{L} := \mathbb{L} \cup (G, S_x(A'))$, where G is a graph representation of $M(A'/x)$.
  - (iii) Return $\mathbb{L}$.

We claim that the above algorithm returns an empty set if M is not an even cycle matroid, and returns all representations of M, up to equivalence, if M is an even cycle matroid.

Suppose that M is a binary matroid with matrix representation A. Let M' be obtained from M by adding a binary non-zero element e. Let A' be the matrix representing M', where column e of A' has exactly one non-zero element, in row r. Suppose M'/e is graphic with representation G. Then M' is an even cycle matroid represented by $(G', S_e(A') \cup \{e\})$, where G' is obtained from G by adding a loop e. It follows that M is an even cycle matroid with representation (G, S_e). Hence if the algorithm returns a non-empty list, then M is an even cycle matroid and each signed graph in the list is a representation of M.

Now suppose M is an even cycle matroid with representations $(G_1, \Sigma_1), \ldots, (G_k, \Sigma_k)$. For every $i \in [k]$, we may obtain a signed graph (G_i', Σ_i') from (G_i, Σ_i) by adding an odd loop e_i; let $M_i := \text{ecycle}(G_i, \Sigma_i)$. Then, for every $i \in [k]$, $M_i \setminus e_i = M$ and M_i/e_i is a graphic matroid represented by G_i. By Whitney's Theorem, all the representations of M_i are equivalent to G_i. Hence the algorithm returns, up to equivalence, all the representations of M.

In Appendix B we present some even cycle matroids with their representations. The representations were obtained with the above algorithm (implemented in maple).

The algorithm above is exponential in the rank, as there are $2^r - 1$ binary vectors to check. Step (2) in the algorithm is polynomial, as proved by Tutte in [35].

A.2 Even cut matroids

The algorithm for recognizing even cut matroids is analogous to the algorithm for even cycle matroids and relies on the fact that, if (G,T) is a graft and e is an odd bridge of (G,T), then $\text{ecut}(G,T)/e$ is cographic with representation G/e.

- **Input:** Binary matrix representation A of a binary matroid M of rank r.
- **Output:** All representations of M as an even cut matroid, up to equivalence.
- **Algorithm:**
  - (i) Set $\mathbb{L} := \emptyset$.
  - (ii) For all non-zero binary vectors x of size r do:
    - (1) add x to A to obtain a matrix A';
    - (2) check if $M(A'/x)$ is cographic: if so, $\mathbb{L} := \mathbb{L} \cup (G,T)$, where G is a graph representation of $M(A'/x)$ and $T = V_{odd}(G[S_x(A')])$.
  - (iii) Return $\mathbb{L}$.

This algorithm returns an empty set if M is not an even cut matroid, and returns all representations of M, up to equivalence, if M is an even cut matroid.

Suppose that M is a binary matroid with matrix representation A. Let M' be obtained from M by adding a binary non-zero element e. Let A' be the matrix representing M', where column e of A' has exactly one non-zero element, in row r. Suppose M'/e is cographic with representation G. Then M' is an even cut matroid represented by (G',T'), where G' is obtained from G by adding a bridge e and $T' = V_{odd}(G'[S_e(A) \cup \{e\}])$. It follows that M is an even cut matroid with representation (G,T), where $(G,T) = (G',T')/e$. We conclude that, if the algorithm returns a non-empty list, then M is an even cut matroid and each graft in the list is a representation of M.

Let M be an even cut matroid with representations $(G_1,T_1),\dots,(G_k,T_k)$. For every $i \in [k]$, we may obtain a graft (G_i',T_i') from (G_i,T_i) by uncontracting an odd bridge e_i; let $M_i := \text{ecut}(G_i,T_i)$. Then, for every $i \in [k]$, $M_i \setminus e_i = M$ and M_i/e_i is a cographic matroid represented by G_i. By Whitney's Theorem, all the representations of M_i are equivalent to G_i. Hence the algorithm returns, up to equivalence, all the representations of M.

In Appendix B we present some even cut matroids with their representations. The representations were obtained with the above algorithm (implemented in maple).

Appendix B

Some interesting matroids

In this appendix we define some interesting matroids, namely the minimally non-graphic and minimally non-cographic matroids, the matroids in Conjecture 1.5 and R_{10}, which is repeatedly used as an example in this work.

$\mathbf{F_7}$. The Fano plane. It has the following partial matrix representation.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

F_7 is minimally non-graphic and minimally non-cographic. It is both an even cycle and an even cut matroid. It has 7 inequivalent representations as an even cycle matroid, all isomorphic to the signed graph in Figure B.1. Each one of these representation arises from a different choice for the element to be an odd loop. It has 7 inequivalent representations as an even cut matroid, all isomorphic to the signed graph in Figure B.2. Each one of these representation arises from a different choice for the element to be a pin. Note that every representation of F_7 arises from a planar graph, as every graph with 7 edges is planar. It follows that we may obtain every graft representation of F_7 from a signed graph representation of F_7 by the construction in Section 2.4.1.

$\mathbf{F_7^*}$. Dual of F_7. It is minimally non-graphic and minimally non-cographic. It is both an even cycle and an even cut matroid. It has 14 inequivalent representations as an even cycle matroid, represented in Figure B.3; 7 of the representations are isomorphic to the signed graph (a) and the other 7 to (b). It has 14 inequivalent representations as an even cut matroid, represented in Figure B.4; 7 of the representations are isomorphic to the graft (a) and the other 7 to (b).

$M(K_5)$. Cycle matroid of K_5. It has the following partial matrix representation.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

It is a minimally non-cographic matroid. It is an even cut matroid with 10 inequivalent graft representations, all isomorphic to the graft in Figure B.5. Each one of these representation arises from a different choice for the element to be a pin.

$M(K_5)^*$. Dual of $M(K_5)$. It is a minimally non-graphic matroid. $M(K_5)^*$ is an even cycle matroid which has 52 inequivalent representations as an even cycle matroid, represented in Figure B.6; 15 representations are isomorphic to the signed graph (a), 15 to , 10 to (c) and the remaining 12 to (d).

$M(K_{3,3})$. Cycle matroid of $K_{3,3}$. It has the following partial matrix representation.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

It is a minimally non-cographic matroid. It is an even cut matroid with 22 inequivalent graft representations, represented in Figure B.7; 9 of the representations are isomorphic to the graft (a), 6 to (b), 6 to (c) and the remaining one to (d).

$M(K_{3,3})^*$. The dual of $M(K_{3,3})$. It is a minimally non-graphic matroid. $M(K_{3,3})^*$ is an even cycle matroid with 15 inequivalent representations, represented in Figure B.8; 9 representations are isomorphic to the signed graph (a) and the other 6 to the signed graph (b).

R_{10}. Both an even cycle and an even cut matroid. R_{10} is self-dual, hence it is also the dual of an even cycle matroid and the dual of an even cut matroid. It has 6 representations as an even cycle matroid, all isomorphic to the signed graph $(K_5, E(K_5))$. It has 10 inequivalent representations as an even cut matroid, all isomorphic to the graft in Figure B.9.

$\mathbf{AG(3,2)}$. It has the following partial matrix representation.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

It is minimally non-1-flowing. $AG(3,2)$ is both an even cycle and an even cut matroid. There are, up to equivalence, 7 signed graphs representing $AG(3,2)$, all isomorphic to the signed graph in Figure B.10(a). By the construction in Section 2.4.1 and the fact that every graph with 8 edges is planar, $AG(3,2)$ also has 7 representations as an even cut matroid, all isomorphic to the graft in Figure B.10(b).

$\mathbf{T_{11}}$. It has the following partial matrix representation.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

It is minimally non-1-flowing. It is not an even cycle matroid. T_{11} is an even cut matroid with, up to equivalence, 10 representations, all isomorphic to the graft in Figure B.11.

$\mathbf{T_{11}^*}$. Dual of T_{11}. It is minimally non-1-flowing. It is an even cycle matroid with, up to equivalence, one representation as in Figure B.12. It is not an even cut matroid.

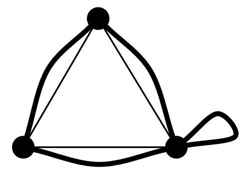


Figure B.1: Even cycle representation of F_7. Bold edges are odd.

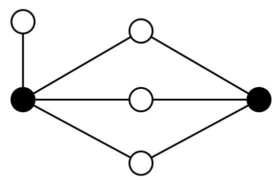


Figure B.2: Even cut representation of F_7. White vertices are terminals.

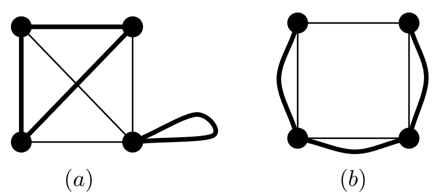


Figure B.3: Even cycle representations of F_7^*. Bold edges are odd.

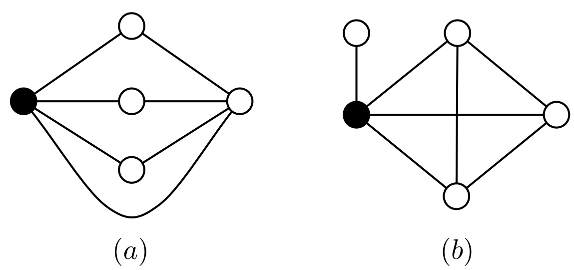


Figure B.4: Even cut representations of F_7^*. White vertices are terminals.

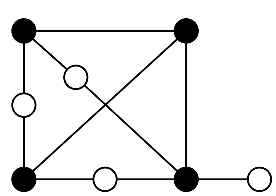


Figure B.5: Even cut representation of $M(K_5)$. White vertices are terminals.

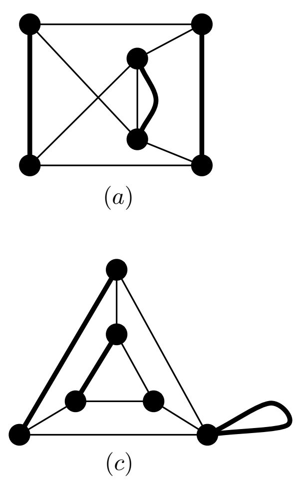


Figure B.6: Even cycle representations of $M(K_5)^*$. Bold edges are odd.

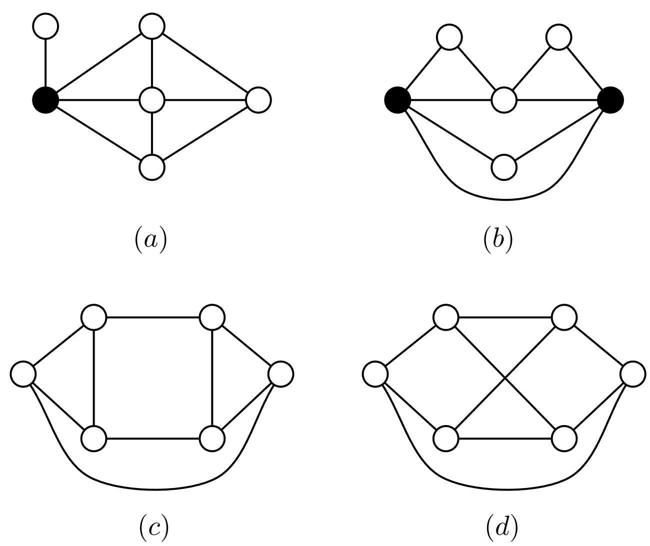
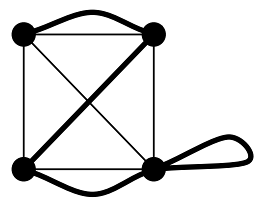


Figure B.7: Even cut representations of $M(K_{3,3})$. White vertices are terminals.



(a) (b)

Figure B.8: Even cycle representations of $M(K_{3,3})^*$. Bold edges are odd.

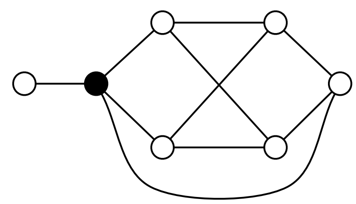
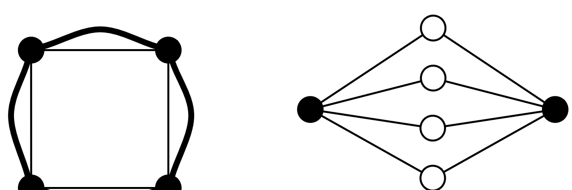


Figure B.9: Even cut representation of R_{10}. White vertices are terminals.



(a) (b)

Figure B.10: Even cycle and even cut representations of $AG(3,2)$. Bold edges are odd, white vertices are terminals.

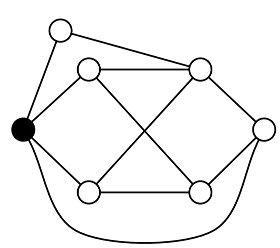


Figure B.11: Even cut representation of T_{11}. White vertices are terminals.

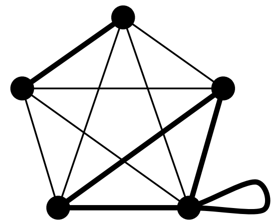


Figure B.12: Even cycle representation of T_{11}^{*}. Bold edges are odd.

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