Enumeration of Factorizations in the Symmetric Group: From Centrality to Non-centrality

by

Craig A. Sloss

A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Combinatorics & Optimization

Waterloo, Ontario, Canada, 2011

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The character theory of the symmetric group is a powerful method of studying enumerative questions about factorizations of permutations, which arise in areas including topology, geometry, and mathematical physics. This method relies on having an encoding of the enumerative problem in the centre $Z(n)$ of the algebra $\mathbb{C}[\mathfrak{S}_n]$ spanned by the symmetric group $\mathfrak{S}_n$. This thesis develops methods to deal with permutation factorization problems which cannot be encoded in $Z(n)$. The (p, q, n)-dipole problem, which arises in the study of connections between string theory and Yang-Mills theory, is the chief problem motivating this research.

This thesis introduces a refinement of the (p, q, n)-dipole problem, namely, the (a, b, c, d)-dipole problem. A Join-Cut analysis of the (a, b, c, d)-dipole problem leads to two partial differential equations which determine the generating series for the problem. The first equation determines the series for $(a, b, 0, 0)$-dipoles, which is the initial condition for the second equation, which gives the series for (a, b, c, d)-dipoles. An analysis of these equations leads to a process, recursive in genus, for solving the (a, b, c, d)-dipole problem for a surface of genus g. These solutions are expressed in terms of a natural family of functions which are well-understood as sums indexed by compositions of a binary string.

The combinatorial analysis of the $(a, b, 0, 0)$-dipole problem reveals an unexpected fact about a special case of the (p, q, n)-dipole problem. When $q = n - 1$, the problem may be encoded in the centralizer $Z_1(n)$ of $\mathbb{C}[\mathfrak{S}_n]$ with respect to the subgroup $\mathfrak{S}_{n-1}$. The algebra $Z_1(n)$ has many combinatorially important similarities to $Z(n)$ which may be used to find an explicit expression for the genus polynomials for the $(p, n-1, n)$-dipole problem for all values of p and n, giving a solution to this case for all orientable surfaces.

Moreover, the algebraic techniques developed to solve this problem provide an algebraic approach to solving a class of non-central problems which includes problems such as the non-transitive star factorization problem and the problem of enumerating Z_1-decompositions of a full cycle, and raise intriguing questions about the combinatorial significance of centralizers with respect to subgroups other than $\mathfrak{S}_{n-1}$.

Acknowledgements

This thesis would not have been possible without the guidance of my advisor, Dr. David M. Jackson. He not only provided mentorship during my graduate studies, but also sparked my interest in enumerative combinatorics through his undergraduate teaching. Were it not for his inspiring lectures, I would likely be working in an entirely different area of mathematics.

The faculty members and graduate students of the Department of Combinatorics and Optimization are to be commended for their role in providing an enriching environment in which to conduct research.

Finally, the Natural Sciences and Engineering Research Council of Canada has provided valuable funding which permitted me to conduct the research which is the subject of this thesis.

Dedication

This thesis is dedicated to my parents, Keith and Lianne Sloss. They have instilled in me a lifelong love of knowledge and learning, the culmination of which is the document you see before you. Mom and Dad, the boy you once read *Goodnight Moon* to is now reading books on algebraic combinatorics, and has even written one of his own.

Table of Contents

List of Tables

List of Figures

List of Symbols

$A_\lambda(i,j)$	"same cycle" basis element for $Z_2(n)$, page 59
$\langle\cdot,\cdot\rangle_n$	standard inner product on $\mathbb{C}[\mathfrak{S}_n]$, page 20
$B_\lambda(i,j)$	"different cycle" basis element of $Z_2(n)$, page 60
C	cut operator for $(p,n-1,n)$-dipoles, page 71
C'	cut operator for $(a,b,0,0)$-dipoles, page 77
C''	cut operator for (a,b,c,d)-dipoles, page 81
χ^λ_μ	irreducible character indexed by λ, evaluated at μ, page 20
$\mathcal{C}_i(j)$	compositions of the integer j into i positive parts, page 140
$\mathcal{C}_i(R)$	string compositions of the binary string R into i non-empty parts, page 140
$\mathbf{c}_\lambda$	vector of contents of any tableau of shape λ, page 93
$\mathcal{C}_\lambda$	conjugacy class of $\mathfrak{S}_n$ consisting of permutations of cycle type λ, page 8
$c^{\nu,k}_{\lambda,i,\mu,j}$	connection coefficients for $Z_1(n)$, page 92
$c^\nu_{\lambda,\mu}$	connection coefficients of $Z(n)$, page 10
$c_{\mu,j}$	content of the box containing n of any tableau in $\mathrm{SYT}(\mu,j)$, page 93
$\mathbf{c}_T$	content vector $(c_T(1),c_T(2),\ldots,c_T(n))$, page 23
$c_T(i)$	content of i in the tableau T; column of i - row of i, page 23
$\mathbb{C}[\mathfrak{S}_n]$	complex group algebra generated by $\mathfrak{S}_n$, page 8
$\mathcal{D}$	set of all rooted dipoles, page 14

$\delta_{i,j}$	Kronecker delta; equal to 1 if $i = j$ and 0 otherwise, page 20
$\hat{\mathcal{D}}$	set of all labelled dipoles, page 14
d_λ	degree of the irreducible representation of $\mathfrak{S}_n$ indexed by λ, page 20
$d_{n,g}$	number of rooted dipoles with n edges in a surface of genus g, page 14
$\hat{d}_{n,g}$	number of labelled dipoles with n edges in a surface of genus g, page 14
$d_{n,g}^{p,q}$	number of (p,q,n)-dipoles in a surface of genus g, page 37
$e(T)$	semi-normal unit corresponding to the tableau T, page 22
e_k	elementary symmetric function of degree k, page 25
ϵ	empty string in $\{\bullet, \circ\}^*$, page 140
$E(t)$	generating series for elementary symmetric functions, $\prod_{i\geq 1}(1+tx_i)$, page 25
$\mathcal{F}_\lambda$	Ferrers diagram of shape λ, page 21
$\Gamma^{\lambda,i}$	orthogonal idempotent in $Z_1(n)$, $\sum_{T\in \mathrm{SYT}(\lambda,i)} e(T)$, page 86
$\gamma_{\mu,j}^{\lambda,i}$	generalized character of the symmetric group, page 88
$g(D)$	genus of the surface in which the dipole D is embedded, page 14
$G_{\mathcal{S}}$	group algebra element $\sum_{\pi\in\mathcal{S}}\pi$ for $\mathcal{S}\subseteq\mathfrak{S}_n$, page 8
h_k	complete symmetric function of degree k, page 25
$H_\lambda(x)$	generating series for irreducible characters of $\mathfrak{S}_n$ indexed by hook partitions, evaluated at the conjugacy class indexed by λ, page 29
$H(t)$	generating series for complete symmetric functions, $\prod_{i\geq 1}(1-tx_i)^{-1}$, page 25
$i \in \lambda$	i is a part of λ, page 7
J	join operator for $(p, n-1, n)$-dipoles, page 71
J'	join operator for $(a,b,0,0)$-dipoles, page 77
J''	join operator for (a,b,c,d)-dipoles, page 81
J_n	Jucys-Murphy element, $\sum_{1\leq i\leq n-1}(i,n)$, page 23

K_λ	standard basis for $Z(n)$, $\sum_{\pi \in \mathcal{C}_\lambda} \pi$, page 9
$[K_\lambda]G$	coefficient of K_λ in $G \in Z(n)$, page 9
$K_{\lambda,i}$	standard basis element for $Z_1(n)$, page 58
$[K_{\lambda,i}]G$	coefficient of $K_{\lambda,i}$ in $G \in Z_1(n)$, page 58
L	linear operator $x^n y^k \mapsto n^{-1}\binom{n-1}{k}^{-1} x^n$, page 31
$\lambda(\pi)$	cycle type of the permutation π, page 7
$\Lambda(D)$	multiset of cyclic binary strings corresponding to the non-root faces of the (a, b, c, d)-dipole D, page 76
$\lambda(D)$	half-face-degree sequence of the dipole D, page 14
$\Lambda[[x_1, x_2, \ldots]]$	ring of symmetric functions in a countable number of indeterminates, page 25
$\Lambda[[x_1, \ldots, x_n]]$	ring of symmetric functions in n indeterminates, page 25
$\Lambda[x_1, \ldots, x_n]$	ring of symmetric polynomials in n indeterminates, page 25
$\Lambda^{(1)}[x_2, \ldots, x_n]$	ring of almost symmetric polynomials, page 92
$m(\pi)$	number of cycles of the permutation π, page 7
$m_i(\lambda)$	multiplicity of i as a part of the integer partition λ, page 7
$m(\lambda)$	number of parts of the partition λ, page 7
$n(D)$	number of edges of the dipole D, page 14
$n(\lambda)$	sum of parts of the integer partition λ, page 7
$p(D)$	p-value of the dipole D, page 37
$\phi(v, x_1, \ldots, x_i)$	functions arising in the definition of τ-functions, page 141
$\phi_{i,j}$	ϕ_S when S is a string containing i white dots and j black dots, page 142
ϕ_S	specialization of ϕ corresponding to a string $S \in \{\bullet, \circ\}^*$, page 142
$[\pi]$	coefficient extraction operator, $[\pi] : \sum_{\sigma \in \mathfrak{S}_n} g_\sigma \sigma \mapsto g_\pi$, page 8
p_k	power sum symmetric function of degree k, page 26

Ψ	generating series for $(p, n-1, n)$-dipoles, page 70
Ψ'	generating series for $(a, b, 0, 0)$-dipoles, page 77
Ψ''	generating series for (a, b, c, d)-dipoles, page 76
$\Psi_{\mathcal{D}}$	generating series for unlabelled rooted dipoles, page 31
$q(D)$	q-value of the dipole D, page 37
$R(D)$	binary string corresponding to the root face of the (a, b, c, d)-dipole D, page 76
(S)	multiset of cyclic shifts of the binary string S, page 76
$\mathfrak{S}_n$	symmetric group acting on $\{1, \ldots, n\}$, page 6
$\mathcal{S}(\bullet, \circ)$	set of all cyclic binary strings on the symbols $\bullet, \circ$, page 76
$\mathrm{sgn}(\pi)$	signum of the permutation π, page 22
$\sigma^{(2)}(\lambda)$	sum of squares of contents of a tableau of shape λ, page 27
$\sigma(\lambda)$	sum of contents of a tableau of shape λ, page 27
$\mathrm{SSYT}(\lambda)$	set of semi-standard Young tableaux of shape λ, page 26
$\mathfrak{S}_{\mathcal{T}}$	group of permutations of the set $\mathcal{T}$, page 6
$\mathrm{SYT}(\lambda)$	set of standard Young tableaux of shape λ, page 21
$\mathrm{SYT}(\lambda, i)$	set of standard Young tableaux of shape λ in which n appears in a row of length i, page 86
T^*	tableau obtained by deleting n from the tableau T, page 22
$\tau_{R,k}$	solution to $(\partial/\partial y - C')\tau_{R,k} = g_{\bullet R}\phi_k$, page 143
$\tau_{R,S}$	solution to $(\partial/\partial v - C'')\tau_{R,S} = g_{\bullet R}\phi_S$, page 146
$\tau_{R,i,j}$	$\tau_{R,S}$ when S has i occurrences of $\circ$ and j occurrences of $\bullet$, page 149
$\vdash$	"is an integer partition of", page 7
$\models$	"is a composition of", page 34
X^λ	group algebra element corresponding to the irreducible character indexed by λ, page 20

$\overline{z}$	complex conjugate of z, page 20
$Z_H(\mathcal{T})$	centralizer of $\mathbb{C}[\mathfrak{S}_{\mathcal{T}}]$ with respect to the subgroup H of $\mathfrak{S}_{\mathcal{T}}$, page 55
$Z_k(n)$	centralizer of $\mathbb{C}[\mathfrak{S}_n]$ with respect to $\mathfrak{S}_{n-k}$, page 55
$Z(n)$	centre of $\mathbb{C}[\mathfrak{S}_n]$, page 9

Chapter 1

Introduction

An enumerative question about a permutation π for which the answer is a non-negative integer $f(\pi)$ is said to be **central** if the formal sum

$$\sum_{\pi \in \mathfrak{S}_n} f(\pi)\pi$$

lies in the centre of the group algebra: that is, if it commutes with every element of the symmetric group $\mathfrak{S}_n$. This thesis examines the extent to which two techniques used by algebraic combinatorialists to solve central problems, namely, the character theory of the symmetric group and Join-Cut analysis, may be generalized to approach **non-central** problems. Central methods have been used to solve a number of important problems in which the answer is the number of ways of decomposing a permutation π as a product of factors with specified properties. Examples of this include enumerating embeddings of graphs in orientable surfaces (in which products of a permutation and a fixed point free involution are considered), and ramified covers of the sphere by a surface of genus g (in which products of transpositions are considered). The attempt to generalize these techniques is motivated primarily by the (p, q, n)-dipole problem, which was introduced by Constable *et al.* [2] in 2002, and arises in the study of duality between gauge theory and string theory. These dipoles appear as the summation indices in two-point functions for Berenstein-Maldacena-Nastase operators which arise in Yang-Mills theory. The (p, q, n)-dipole problem is demonstrably non-central, and will be employed throughout the thesis to illustrate the techniques used to approach non-central problems.

The main contributions of this thesis to the development of approaches to non-central problems are as follows. Chapter 5 describes how a combinatorial analysis of the (p, q, n)-dipole problem leads to a pair of partial differential equations which determine the generating series for a more general problem (Theorems 5.3.1 and 5.3.2). Chapter 8 describes a recursive method for solving these equations, which expresses the genus g solution as a linear combination of functions for which combinatorial formulas are known (Theorems 8.2.9 and 8.2.11). Chapter 6 formulates an algebraic approach for a special class of non-central problems, namely, those which have encodings in $\mathbb{C}[\mathfrak{S}_n]$ which commute with $\mathfrak{S}_{n-1}$. Chapter 7 applies this approach to the $(p, n-1, n)$-dipole problem (Theorem 7.1.9), the star factorization problem (Theorem 7.2.1), and a non-central generalization of the cycle decomposition problem (Theorem 7.3.2). The detailed content of each chapter is as follows.

Chapter 2 gives a description of the centre $Z(n)$ of the symmetric group algebra, and an account of the known techniques which have been used to solve several enumerative problems which may be expressed in terms of products of the standard basis elements of $Z(n)$. The relevant aspects of character theory of the symmetric group are introduced, and the connection coefficients of $Z(n)$ are expressed as sums of irreducible characters. Thus, central factorization problems are reduced to the matter of evaluating irreducible characters. In many important cases, the characters which arise can be expressed in closed form. Several known solutions to combinatorial problems are presented as an illustration of these techniques, with a particular emphasis on the problem of enumerating 2-cell embeddings of a dipole in an orientable surface. This problem generalizes to the (p, q, n)-dipole problem.

Chapter 3 describes the non-central permutation factorization problems used throughout the thesis both to motivate and demonstrate the techniques being developed. It is this chapter which contains a complete definition of each problem and an account of the history of attempts at solving it. **(a)** The first such problem is the (p, q, n)-dipole problem mentioned above. **(b)** The second non-central problem is the problem of enumerating factorizations of a permutation into "star-transpositions," the set of all transpositions of the form (i, n), which can be considered either with or without imposing the condition that the factorization is transitive. The requirement that the symbol n be present in every transposition disrupts the centrality of the problem. Though both the transitive and non-transitive versions of the problem have been solved (the former by Goulden and Jackson [11] and the latter by Lascoux and Thibon [30]), this problem is included because it raises interesting questions about centrality; in particular, the transitive version of the problem turns out to be unexpectedly central. This chapter also includes a discussion of Join-Cut analysis, a non-character based approach to studying permutation factorization problems used by Goulden and Jackson to solve the transitive star factorization problem (among numerous other problems). In this approach, considering the effect of multiplication by a transposition allows one to write down a partial differential equation for the generating series. **(c)** This chapter also introduces a problem called the Z_1-factorization problem. This problem is included because it is a natural generalization of a central problem, and it is natural to expect that a successful extension of central techniques should provide an approach to the Z_1-factorization problem.

Chapter 4 contains definitions and elementary results for the centralizers of the symmetric group algebra. The centralizers are subalgebras of the symmetric group algebra which are natural generalizations of the centre, and perform two functions in this thesis. First, they serve as a metric of non-centrality; *i.e.* they provide a language to describe "how non-central" a given problem is. Second, they serve as an algebraic framework for approaching non-central problems. Of particular interest is the centralizer $Z_1(n)$ of $\mathbb{C}[\mathfrak{S}_n]$ with respect to $\mathfrak{S}_{n-1}$. This algebra is the non-central algebra which is "closest" to being central, while having enough structure to permit significant results to be obtained.

Chapter 5 contains a description of Join and Cut operators which determine the generating series for the (p, q, n)-dipole problem. (In fact, they determine the series for a slightly more general problem, called the (a, b, c, d)-dipole problem.) The analysis is combinatorial in nature, and consists of considering how the face structure and the values of p and q change when a new edge is added to a dipole. Two formal partial differential equations arise from this analysis: the first determines the generating series Ψ'' for the (a, b, c, d)-dipole problem, and the second determines the initial condition Ψ' for this equation. The series Ψ' corresponds to the special case of $(p, 1, n)$-dipoles, and is remarkable because the form of the equation suggests that the $(p, 1, n)$-dipole problem, while still being non-central, is in fact "less non-central" than it initially appears. This second observation is exploited in Chapter 7, and the general (a, b, c, d) operators are analyzed in more detail in Chapter 8

Chapter 6 introduces a set of orthogonal idempotents for $Z_1(n)$ which are defined as sums of Young's semi-normal units over a restricted class of standard Young tableaux. The coefficients of these idempotents in the standard basis for $Z_1(n)$ are the generalized characters introduced by Strahov [42]. This allows the connection coefficients of $Z_1(n)$ to be expressed in terms of generalized characters. Two methods of evaluating generalized characters are then discussed. The first generalizes a technique used by Diaconis and Greene [4] to evaluate ordinary characters. This technique reduces the computation of generalized characters to a problem of evaluating a function, symmetric in all but one of the indeterminates, at the contents of a tableau. The second is an application of Strahov's analogue of the Murnaghan-Nakayama rule for generalized characters.

Chapter 7 explains how the results of Chapter 6 can be applied to some of the problems described in Chapter 3. The $(p, n-1, n)$-dipole problem, the non-transitive star factorization problem, and the general Z_1-factorization problem can all be expressed as sums over generalized characters. In the case of the $(p, n-1, n)$-dipole problem, the generalized characters appearing in the sum can all be evaluated explicitly, yielding a solution for the $(p, n-1, n)$-dipole problem for all values of p, n, and for all orientable surfaces. In addition, this chapter discusses the applicability of the material from Chapter 6 to other non-central problems such as non-transitive star factorizations and Z_1-factorizations of a full cycle. It is evident that there is a $Z_1(n)$ combinatorial context that is analogous to the $Z(n)$ case, and whose structure and significant elements are known and understood.

Chapter 8 contains an analysis of the partial differential equations which give the generating series for the (a, b, c, d)-dipole problem. It gives a procedure, recursive in genus, for computing a series valid for all values of a, b, c and d, in which a substantial amount of information about the face structure of the dipoles is recorded. The series obtained are combinatorial in nature, being sums over objects called string compositions. This procedure is then applied to find series solutions for the (a, b, c, d)-dipole problem on the torus and double torus.

Finally, Chapter 9 discusses related unresolved questions, as well as some new questions which follow from the work done in this thesis, and which form the basis for future work.

Chapter 2

Combinatorial Applications of the Centre of $\mathbb{C}[\mathfrak{S}_n]$

This chapter gives an overview of combinatorial applications of the algebra $\mathbb{C}[\mathfrak{S}_n]$ spanned by the symmetric group $\mathfrak{S}_n$, and the centre of this algebra. It is not intended to be a comprehensive account of the representation theory of $\mathfrak{S}_n$; the emphasis is on presenting results whose non-central analogues are described in Chapter 6, and doing so in such a way that the generalizations of these results emerge naturally.

Elementary definitions and notation regarding the centre of $\mathfrak{S}_n$ are given in Section 2.1. Section 2.2 contains descriptions of some of the combinatorial problems that have been studied using the centre of $\mathbb{C}[\mathfrak{S}_n]$. Two classes of problems are presented here: the enumeration of maps embedded in an orientable surface, and the enumeration of ramified covers of the sphere by a surface of genus g. This section explains how the solutions to these problems may be expressed as a product of basis elements in the centre of $\mathbb{C}[\mathfrak{S}_n]$. One example of a map enumeration problem, the problem of enumerating rooted dipoles, is singled out for special attention, because the generalization of this problem described in Chapter 3 is a major object of study in this thesis. Throughout this chapter, the rooted dipole problem will be used as an example to illustrate the techniques being described.

Section 2.3 describes the techniques used to perform the central computations arising in Section 2.2. In particular, it is an account of some standard results in the representation theory of $\mathfrak{S}_n$. It describes one method of constructing the irreducible representations (and hence, the irreducible characters) of $\mathfrak{S}_n$, namely, Young's semi-normal representation. While there exist many different approaches to the construction of irreducible representations, this approach is presented here because Young's semi-normal units have a key role to play in Chapter 6. Young's semi-normal units are used to derive expressions for the connection coefficients of the centre as sums involving irreducible characters of the symmetric group; thus, the solutions to the problems described in Section 2.2 are reduced to the problem of computing irreducible characters of $\mathfrak{S}_n$.

Section 2.4 addresses the question of how the irreducible characters of $\mathfrak{S}_n$ may be determined. Though explicit formulas for the characters of $\mathfrak{S}_n$ are not known in general, in many combinatorial applications the characters which arise can be determined explicitly. This section gives some examples of cases in which explicit formulas for characters are known, and gives a brief description of some of the techniques used to arrive at these formulas. The section closes by demonstrating how these formulas have been used to give an explicit solution to the rooted dipole problem, and giving an expression for the generating series for this problem which is useful in Chapters 5 and 8.

2.1 The Group Algebra $\mathbb{C}[\mathfrak{S}_n]$ and its centre

The definitions and results in this section are classical, and appear in numerous sources, such as James and Kerber [25], Sagan [39], and Macdonald [32]. Given a set $\mathcal{T}$ of positive integers, let $\mathfrak{S}_{\mathcal{T}}$ denote the group of permutations of the set $\mathcal{T}$ in which group multiplication is composition of the permutations as functions; this is called the **symmetric group** on the ground set $\mathcal{T}$. When $\mathcal{T} = \{1, 2, \ldots, n\}$, the notation $\mathfrak{S}_n$ is used in place of $\mathfrak{S}_{\mathcal{T}}$. Throughout this thesis, a right-to-left convention is used, *i.e.* if π_1 and π_2 are permutations, then $\pi_1\pi_2$ is the permutation obtained by applying π_2 followed by π_1.

There are three commonly used notations to specify a permutation. The first is **two-line notation**, in which the numbers $1, 2, \ldots, n$ are written along the first line, and $\pi(i)$ is written below i; *i.e.*

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}.$$

The **one-line notation** for π is obtained by deleting the top row from the two-line notation and regarding the permutation as the sequence $\pi = \pi(1)\,\pi(2)\,\cdots\,\pi(n)$.

The third notation for π is obtained by considering the **functional digraph** of π. This is the directed graph whose vertices are $\{1, \ldots, n\}$, in which there is an edge from i to j if and only if $\pi(i) = j$. Since π is a permutation, every vertex has in-degree and out-degree equal to one; the only such directed graphs are those in which every component is a directed cycle. As an example, the functional digraph of $\pi = 231465$ is given in Figure 2.1. A permutation may thus also be specified by listing the cycles of its functional digraph. The cycle containing i may be written as $(i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i))$, where k is the **length** of the cycle. (A cycle of length k can thus be represented in k different ways, depending on which element is chosen to be the first element of the cycle.) Writing one representative for each cycle of π gives a **disjoint cycle representation** for π; for example, if $\pi = 231465$,

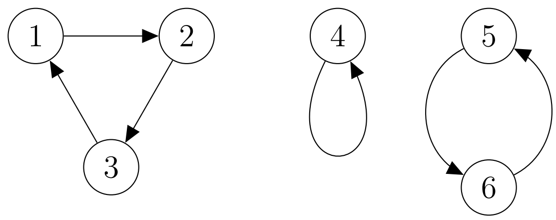


Figure 2.1: Functional digraph of the permutation whose one-line notation is 231465.

then one disjoint cycle representation of π would be $(1,2,3)(4)(5,6)$. Fixed points, which give a cycle of length 1, are usually omitted from a disjoint cycle representation.

Permutations may be classified according to the lengths of cycles appearing in their disjoint cycle representations. To make this notation more precise, the following notation is used.

1. Let $\mathbb{P}$ denote the set of positive integers. A **partition** λ of n is an element $(\lambda_1, \ldots, \lambda_m) \in \mathbb{P}^m$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$. The notation $\lambda \vdash n$ is used to indicate that λ is a partition of n.

2. Each λ_i is called a **part** of λ.

3. The **multiplicity** $m_i(\lambda)$ of i as a part in λ is the number of times i appears in λ.

4. The notation $m(\lambda)$ denotes the total number of parts of λ, *i.e.* $m(\lambda) = \sum_{i \geq 1} m_i(\lambda)$. In situations where sets consisting of partitions of any integer are being considered, let $n(\lambda) = \sum_{1 \leq i \leq m} \lambda_i$.

5. Partitions will often be specified by listing their parts and multiplicities, using expressions of the form $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots)$.

6. The assertion $i \in \lambda$ is equivalent to $m_i(\lambda) > 0$.

7. If $i \in \lambda$, then $\lambda \setminus i$ is the partition obtained by reducing $m_i(\lambda)$ by one. For any i, $\lambda \cup i$ is the partition obtained by increasing $m_i(\lambda)$ by one.

The **cycle type** $\lambda(\pi)$ of a permutation π is the partition given by the lengths of the cycles of π in its disjoint cycle representation. The notation $m(\pi)$ denotes the number of cycles in the disjoint cycle representation of π, which is the number of parts of the partition $\lambda(\pi)$.

Given two permutations π and σ, the **conjugate** of π with respect to σ is the permutation $\sigma\pi\sigma^{-1}$. Conjugation has a natural combinatorial interpretation: the vertices of the functional digraph of π are relabelled according to the permutation σ; *i.e.* the vertex

i is replaced with $\sigma(i)$. Consequently, the cycle type of π is invariant under conjugation. Conversely, if two permutations π_1 and π_2 have the same cycle type, it is always possible to find a permutation σ such that $\pi_1 = \sigma\pi_2\sigma^{-1}$. Thus, the sets

$$\mathcal{C}_\mu := \{\pi \in \mathfrak{S}_n : \lambda(\pi) = \mu\}$$

are the **conjugacy classes** of $\mathfrak{S}_n$. By considering the size of the stabilizer of a conjugacy class,

$$|\mathcal{C}_\lambda| = \frac{n!}{|\mathrm{Aut}(\lambda)|},$$

where $|\mathrm{Aut}(\lambda)| = \prod_{i\geq 1} i^{m_i(\lambda)} m_i(\lambda)!$.

2.1.1 A basis for the centre of $\mathbb{C}[\mathfrak{S}_n]$

Let $\mathbb{C}[\mathfrak{S}_n]$ be the **group algebra** of $\mathfrak{S}_n$ over $\mathbb{C}$. The group algebra is the complex span of $\mathfrak{S}_n$, in which two basis elements are multiplied according to the multiplication in $\mathfrak{S}_n$. This can be extended distributively to define multiplication of any two elements of $\mathbb{C}[\mathfrak{S}_n]$, thereby giving a ring structure to $\mathbb{C}[\mathfrak{S}_n]$.

The group algebra of $\mathfrak{S}_n$ is a natural algebraic context in which to study combinatorial questions about factorizations of permutations. Given a set $\mathcal{S} \subseteq \mathfrak{S}_n$, let

$$G_\mathcal{S} := \sum_{\pi\in\mathcal{S}} \pi \in \mathbb{C}[\mathfrak{S}_n].$$

If $\mathcal{T} \subseteq \mathfrak{S}_n$ is a second set of permutations, then

$$G_\mathcal{S} G_\mathcal{T} = \left(\sum_{\sigma\in\mathcal{S}} \sigma\right)\left(\sum_{\tau\in\mathcal{S}} \tau\right) = \sum_{\substack{\sigma\in\mathcal{S},\\ \tau\in\mathcal{T}}} \sigma\tau.$$

Thus, the coefficient of a permutation σ in $G_\mathcal{S} G_\mathcal{T}$ is the number of ways in which σ may be obtained as a product of a permutation in $\mathcal{S}$ with a permutation $\mathcal{T}$. Given an element $G \in \mathbb{C}[\mathfrak{S}_n]$, let $[\pi]G$ denote the coefficient of π in G; in other words, if $G = \sum_{\sigma\in\mathfrak{S}_n} G_\sigma \sigma$, then $[\pi]G = G_\pi$. (The operation $[\pi]$ is a linear functional on $\mathbb{C}[\mathfrak{S}_n]$.) With this notation, the above observation reads as follows.

Lemma 2.1.1 (Encoding lemma for the Permutation Factorization Problem)**.** *Let $\mathcal{S}, \mathcal{T} \subseteq \mathfrak{S}_n$. The number of factorizations of a permutation π as $\pi = \sigma\tau$ such that $\sigma \in \mathcal{S}$ and $\tau \in \mathcal{T}$ is*

$$[\pi] G_\mathcal{S} G_\mathcal{T}.$$

(This lemma has an obvious generalization which allows for an arbitrary number of factors.) This observation, while simple, is a fundamental bridge between the combinatorial problem of enumerating factorizations, and the algebraic techniques which are used to study them; it is the chief motivation for a combinatorialist to study the algebraic properties of $\mathbb{C}[\mathfrak{S}_n]$. This chapter describes some techniques that have been used when $\mathcal{S}$ and $\mathcal{T}$ are conjugacy classes; the remainder of the thesis develops techniques to deal with situations in which they are not.

To understand why the problem of computing $G_{\mathcal{S}}G_{\mathcal{T}}$ is substantially more tractable when $\mathcal{S}$ and $\mathcal{T}$ are conjugacy classes than when $\mathcal{S}$ and $\mathcal{T}$ are arbitrary, let $K_\lambda = G_{\mathcal{C}_\lambda}$ denote the sum of elements in $\mathcal{C}_\lambda$. For any permutation $\pi \in \mathfrak{S}_n$, $\pi K_\lambda \pi^{-1} = K_\lambda$, since $\mathcal{C}_\lambda$ is a conjugacy class. Expressing this as $\pi K_\lambda = K_\lambda \pi$, and observing that we may sum this expression over an arbitrary linear combination of permutations in $\mathfrak{S}_n$, it is clear that K_λ commutes with every element of $\mathbb{C}[\mathfrak{S}_n]$. Let

$$Z(n) = \{G \in \mathbb{C}[\mathfrak{S}_n] : \pi G = G\pi \text{ for all } \pi \in \mathfrak{S}_n\}$$

denote the set of elements of $\mathbb{C}[\mathfrak{S}_n]$ which commute with everything in $\mathbb{C}[\mathfrak{S}_n]$ — this is the **centre** of $\mathbb{C}[\mathfrak{S}_n]$. (Equivalently, the centre is the set of elements which are invariant under conjugation by $\mathfrak{S}_n$.) It is clear from the definition that $Z(n)$ is a commutative subalgebra of $\mathbb{C}[\mathfrak{S}_n]$. The elements $\{K_\lambda\}_{\lambda \vdash n}$ form a basis for $Z(n)$, called the **standard basis** for $Z(n)$. It is clear that the set $\{K_\lambda\}_{\lambda \vdash n}$ is linearly independent; to see that it spans $Z(n)$, it suffices to show that for any $G \in Z(n)$, $[\pi_1]G = [\pi_2]G$ whenever π_1 is conjugate to π_2. Let σ be such that $\pi_1 = \sigma^{-1}\pi_2\sigma$. Then

$$[\pi_1]G = [\pi_1]\sigma G \sigma^{-1} = [\pi_1] \sum_{\tau \in \mathfrak{S}_n} ([\tau]G)\sigma\tau\sigma^{-1} = [\sigma^{-1}\pi_1\sigma]G = [\pi_2]G;$$

hence, $\{K_\lambda\}_{\lambda \vdash n}$ is a basis for $Z(n)$. If $G \in Z(n)$, let $[K_\lambda]G$ denote the coefficient of K_λ in G.

Recall that, as defined in the introduction, an enumerative problem about the permutation π whose answer is the non-negative integer $f(\pi)$ is said to be **central** if $\sum_{\pi \in \mathfrak{S}_n} f(\pi)\pi \in Z(n)$. The earlier discussion, in which conjugation of a permutation was interpreted combinatorially as a relabelling of the functional digraph of a permutation, provides some combinatorial intuition about when a problem is central: for $\sum_{\pi \in \mathfrak{S}_n} f(\pi)\pi$ to be invariant under conjugation, the problem must not change if labels on the functional digraphs of the pemutations involved are arbitrarily rearranged. Equivalently, none of the elements of the ground set $\{1, 2, \ldots, n\}$ may be distinguished in any way. Thus, for example, a problem which refers to the length of the cycle containing n is not central, since such a reference distinguishes the label n.

It is helpful to introduce a stronger notion. A problem is said to be **naturally central** if there exists an explicit expression for $\sum_{\pi \in \mathfrak{S}_n} f(\pi)\pi$ as a monomial in the standard basis elements $\{K_\lambda\}_{\lambda\vdash n}$. When such an expression exists, the methods described in this chapter may be used to approach the problem. The distinction between centrality and natural centrality will be made clearer in Chapter 3, which contains an example of a problem (the transitive star factorization problem) which is central, but not naturally central.

It is now possible to state the general form of a central factorization problem. All the combinatorial problems described in the next section are special cases of this problem.

Problem 2.1.2 (The Central Factorization Problem)**.** *Given a permutation $\pi \in \mathfrak{S}_n$ and partitions $\lambda, \mu \vdash n$, determine the number of pairs of permutations (σ_1, σ_2) such that*

1. $\sigma_1\sigma_2 = \pi$,
2. σ_1 *has cycle type* λ, *and*
3. σ_2 *has cycle type* μ.

Since $Z(n)$ is an algebra, the product $K_\lambda K_\mu$ lies in $Z(n)$, and since $\{K_\lambda\}_{\lambda\vdash n}$ is a basis for $Z(n)$, there exist constants $c^\nu_{\lambda,\mu}$ such that

$$K_\lambda K_\mu = \sum_{\nu\vdash n} c^\nu_{\lambda,\mu} K_\nu.$$

The constants $c^\nu_{\lambda,\mu}$ are called the **connection coefficients** of $Z(n)$. Computing these coefficients gives, in principle, a solution to any naturally central enumerative question; in particular, these coefficients give a solution to any combinatorial problem which can be encoded using Lemma 2.1.1 such that $\mathcal{S}$ and $\mathcal{T}$ are conjugacy classes. It is worth recording this observation in the following.

Corollary 2.1.3 (Encoding lemma for the Central Factorization Problem)**.** *Let $\mu, \nu \vdash n$. The number of factorizations of a permutation $\pi \in \mathfrak{S}_n$ as $\pi = \sigma\tau$ such that $\sigma \in \mathcal{C}_\mu$ and $\tau \in \mathcal{C}_\nu$ is*

$$[\pi]K_\mu K_\nu = [K_{\lambda(\pi)}]K_\mu K_\nu = c^{\lambda(\pi)}_{\mu,\nu},$$

where $c^\lambda_{\mu,\nu}$ are the connection coefficients of $Z(n)$.

The next section discusses some combinatorial problems which may be encoded in such a manner, and Sections 2.3 and 2.4 discuss the techniques used to determine the connection coefficients.

2.2 Central Methods in Enumerative Combinatorics

This section introduces two classes of combinatorial problems to which central techniques have been applied: enumeration of maps in orientable surfaces, and enumeration of ramified coverings of the sphere. Each problem is defined, and an explanation of how it can be encoded in $Z(n)$ is provided. From a combinatorial perspective, there are broadly two reasons for doing so:

1. In cases in which the central encoding can be explicitly evaluated, a closed form solution to the problem can be obtained.

2. Algebraic relations may reveal previously unseen connections between two problems, proving the existence of a bijection between sets of combinatorial objects. In such cases, the problem of finding a natural bijection becomes an important problem in its own right, the study of which often reveals further combinatorial structure.

Even in cases where the central encoding of a problem cannot be evaluated, it is still useful for the purpose of providing a computational method of solving small cases of the problem it encodes, which could be used to conjecture a solution to be proved using other techniques.

2.2.1 Enumeration of Rooted Maps in Orientable Surfaces

The first major class of enumerative questions which have been solved using central techniques is the question of enumerating maps in an orientable surface. A **surface** is a compact, connected 2-manifold. Such surfaces are locally orientable; however, in this thesis, unqualified use of the term "surface" should be taken to refer to orientable surfaces. It should be noted that maps in non-orientable surfaces are encoded not using the centre of $\mathbb{C}[\mathfrak{S}_n]$ but rather the double coset algebra of the hyperoctahedral group, and thus do not fit within the present framework of centrality and non-centrality. A brief discussion regarding the possibility of extending the results of this thesis to the non-orientable case appears in Chapter 9.

The Classification Theorem for Surfaces states that every orientable surface is homeomorphic to the connected sum of a sphere with g tori, for some $g \geq 0$. The integer g is the **genus** of the surface. A **map** is an embedding of a connected graph G into a surface Σ such that if G is deleted, Σ decomposes into a union of regions, each of which is homeomorphic to an open disc. Each of these regions is called a **face** of the embedding. Two maps $\mathcal{M}$ and $\mathcal{N}$ are equivalent if there is a homeomorphism of Σ which takes $\mathcal{M}$ to $\mathcal{N}$. Maps satisfy the **Euler-Poincaré** formula; *i.e.*, if the map has v vertices, e edges, f faces, and is in a surface of genus g, then

$$v - e + f = 2 - 2g.$$

Since v and e are determined by the underlying graph of the map, one of the main consequences of this formula is that the surface in which the graph is embedded is completely determined by the number of faces of the embedding, and *vice versa*. A **rooted map** is a map in which an edge and one of the vertices to which it is incident are distinguished. In diagrams, the root edge is indicated by an arrow pointing away from the root vertex.

The generating series for rooted maps has been given by Jackson and Visentin (see [24] and [23]). They encode rooted maps in terms of **rotation systems**; this encoding is summarized below, and may be found in detail in Chapter 10 of Tutte's *Graph Theory* [43]. Given a rooted map, assign the label 1 to the root edge, and the labels $2, \ldots, n$ to the non-root edges in an arbitrary manner. Assign an arbitrary direction to each of the non-root edges. (Observe that there are $(n-1)!2^{n-1}$ ways of doing this, so the number of rooted maps may easily be obtained by enumerating the number of maps decorated in this manner.) For an oriented edge labelled i, mark the "head" of the edge with the symbol i^+, and the "tail" of the edge with i^-. For each vertex, a counterclockwise tour of its neighbourhood gives a cyclic ordering of the labels assigned to the ends of the edges incident to it. Regard this as a cyclic permutation of these labels. This gives a permutation $\nu \in \mathfrak{S}_{2n}$, called the rotation system of the decorated map. (This process is illustrated in Figure 2.2.) Let ϵ_n denote the canonical fixed-point-free involution

$$\epsilon_n := (1^+1^-)(2^+2^-)\cdots(n^+n^-)$$

in $\mathfrak{S}_{2n}$. Let $\phi = \nu\epsilon_n$. Then each cycle of ϕ represents the edge labels encountered during a clockwise tour of one of the faces of the map. The cycle types $\lambda(\nu)$ and $\lambda(\phi)$ are the vertex degree sequence and face degree sequence of the map, respectively. For the map to be connected, it is necessary and sufficient for the group generated by ν and ϵ_n to act transitively on $\{1^+, 1^-, \ldots, n^+, n^-\}$. This gives the following encoding.

Lemma 2.2.1 (Encoding for decorated rooted maps)**.** *Let $\lambda_v, \lambda_f \vdash n$. Then the number of decorated rooted maps with vertex degree sequence λ_v and face degree sequence λ_f is equal to the number of pairs (ν, ϕ) such that*

1. $\nu\epsilon_n = \phi$,
2. $\lambda(\nu) = \lambda_v$ *and* $\lambda(\phi) = \lambda_f$, *and*
3. *the group generated by ϕ and ϵ_n acts transitively on* $\{1^+, 1^-, \ldots, n^+, n^-\}$.

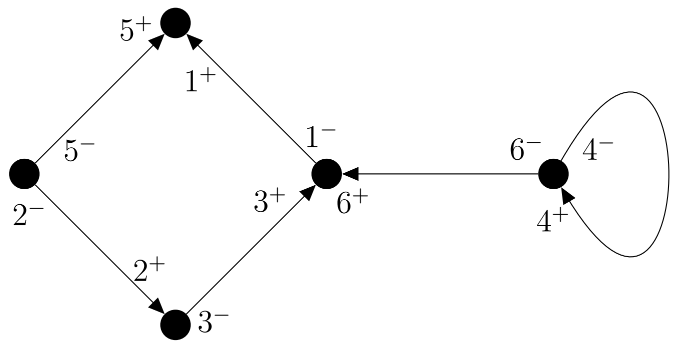


Figure 2.2: Rotation system $(1^-, 3^+, 6^+)(1^+, 5^+)(2^-, 5^-)(2^+, 3^-)(4^+, 4^-, 6^-)$ of a decorated map. Multiplication by the canonical fixed-point-free involution ϵ_6 gives the face permutation $\varphi = (1^-, 5^+, 2^-, 3^-, 6^+, 4^+, 6^-)(1^+, 3^+, 2^+, 5^-)(4^-)$.

When condition 3 is not enforced, the objects being enumerated are referred to as **pre-maps**. Since the generating series for maps may be obtained by taking the logarithm of the generating series for pre-maps, it suffices to solve the problem without condition 3. Furthermore, since $\nu \in \mathcal{C}_{\lambda_v}$ if and only if $\nu^{-1} \in \mathcal{C}_{\lambda_v}$, then the condition $\nu\epsilon_n = \phi$ may be replaced with $\nu\phi = \epsilon_n$. In light of Corollary 2.1.3, the problem reduces to the following.

Lemma 2.2.2. *The number of decorated, rooted pre-maps with vertex degree sequence λ_v and face degree sequence λ_f is*

$$[K_{(2^n)}]K_{\lambda_v}K_{\lambda_f} = c^{(2^n)}_{\lambda_v,\lambda_f}.$$

This Lemma allows the map enumeration problem to be approached using central methods. In particular, it is the starting point for Jackson and Visentin's [23] derivation of the general form for the generating series for maps in orientable surfaces. Even when the connection coefficients arising in Lemma 2.2.2 cannot be evaluated explicitly, algebraic relationships between them can reveal combinatorial structure. A notable example of this is Jackson and Visentin's proof of the existence of a bijection between 4-face-regular maps and the set of all maps in surfaces of lower genus, with some additional decoration.

Rooted Dipoles

An important special class of maps in orientable surfaces is the class of **rooted dipoles**, which are rooted maps with two vertices and no loops. It is of course possible to consider two-vertex maps with loops, but for the purpose of this thesis the term "dipole" shall refer to one without loops, unless otherwise specified. A **labelled dipole** is a dipole in which the edges are labelled from the set $\{1, 2, \ldots, n\}$, with the convention that the edge having the label 1 is regarded as the root. Let $\mathcal{D}$ and $\hat{\mathcal{D}}$ denote the sets of all rooted dipoles, and all labelled dipoles, respectively. Given a dipole D of either type, let $n(D)$ denote its number of edges, let $g(D)$ denote the genus of the surface in which it is embedded. Let $d_{n,g}$ denote the number of rooted dipoles with n edges in a surface of genus g, and let $\hat{d}_{n,g}$ denote the number of labelled dipoles with n edges in a surface of genus g. The number of labelled dipoles in a given surface is equal to $(n-1)!$ times the number of unlabelled rooted dipoles, since this is the number of ways of labelling the non-root edges. An example of a dipole with labelled edges is illustrated in Figure 2.3.

Dipoles, as a special case of the map enumeration problem, are of relevance to this thesis for two reasons. First, the problem of enumerating dipoles is one in which central methods give an explicit result; this problem is used as an example in the present chapter to demonstrate central techniques. Second, a physically significant non-central generalization of the dipole problem is defined in Chapter 3, and serves as motivation for the non-central techniques developed in this thesis.

Kwak and Lee [28] describe a method of encoding a labelled dipole as a pair of permutations (σ_1, σ_2) where each $\sigma_i \in \mathcal{C}_{(n)}$; the encoding is a modification of the encoding for all maps in terms of rotation systems, and relies on the fact that a dipole has exactly two vertices and no loops. The permutation σ_2 is the full-cycle permutation obtained by reading the edge labels encountered in a counterclockwise circulation of the root vertex; the permutation σ_1 is the full-cycle permutation obtained by reading the edge labels encountered while moving counterclockwise around the non-root vertex. The pair (σ_1, σ_2) shall be referred to as the **vertex permutation pair** of the dipole.

The edge labels induce a labelling of the half of the corners of the faces in the following manner: if an edge has the label k, then the corner which is counterclockwise from that edge around the root vertex also receives the label k. The **face permutation** of a dipole is the permutation in which each cycle is the cyclic permutation obtained by reading the root corner labels encountered during a clockwise boundary tour of a face. For example, the face permutation of the dipole in Figure 2.3 is $(1245)(36)$. The face permutation of a dipole may be obtained algebraically as the product of the two vertex permutations. In a loopless dipole, every face has even degree, so it is convenient to use the notation $\lambda(D)$ to denote the partition corresponding to *half* the degrees of the faces of a dipole D. If D is encoded by the vertex permutation pair (σ_1, σ_2), then $\lambda(D) = \lambda(\sigma_1\sigma_2)$.

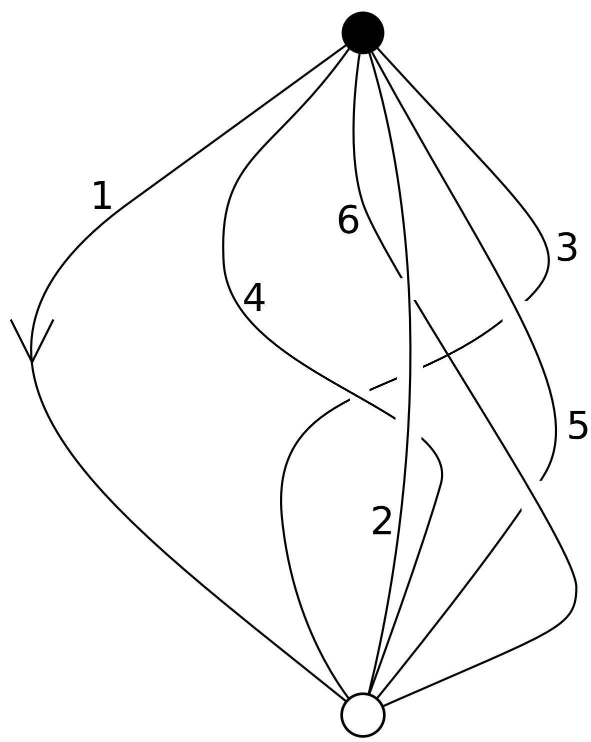


Figure 2.3: An example of a genus 2 labelled dipole with vertex permutations $((165423), (146253))$ and face permutation $(1245)(36)$.

If m is the number of faces of a dipole on n edges, then the genus of the surface in which it is embedded can be obtained using the Euler-Poincaré formula: $g = \frac{1}{2}(n-m)$. Consequently, it suffices to determine the number of dipoles with m faces. Therefore, the dipole problem is encoded in $Z(n)$ as follows:

Problem 2.2.3 (The labelled dipole problem)**.** *Determine the number of pairs* $(\sigma_1, \sigma_2) \in \mathfrak{S}_n$ *such that*

1. $\sigma_1, \sigma_2 \in \mathcal{C}$, *and*
2. $\sigma_1\sigma_2$ *has* $n-2g$ *cycles.*

Corollary 2.1.3 immediately expresses this as a problem in $Z(n)$, namely,

$$d_{n,g} = \sum_{\substack{\lambda \vdash n \\ m(\lambda)=n-2g}} |\mathcal{C}_\lambda| [K_\lambda] K_{(n)}^2 = \sum_{\substack{\lambda \vdash n \\ m(\lambda)=n-2g}} |\mathcal{C}_\lambda| c_{(n),(n)}^{\lambda}.$$

Section 2.4 gives an account of how a result of Jackson [21] on enumerating permutation factorizations may be applied to determine the numbers $d_{n,g}$ explicitly.

2.2.2 Ramified Covers of the Sphere by a Surface of Genus g

A second major class of central problems is the enumeration of ramified covers of the sphere by a connected surface of genus g — this is known as the **Hurwitz problem**. This section discusses the **Double Hurwitz problem**, the combinatorics of which was studied extensively by Goulden, Jackson and Vakil in [13], because a special case of this problem can be solved explicitly using central methods. It also has connections to the transitive star factorization problem discussed in Chapter 3.

The following terminology pertaining to ramified covers may be found in any of the papers cited throughout this section, or in an introductory algebraic geometry text such as [44]. Let f be a meromorphic function from a surface of genus g to the sphere, which is regarded for these purposes as the compactification of the complex plane $\mathbb{C}$ *via* the addition of the point ∞. All but a finite number of points on the sphere will have d preimages under f, for some positive integer d — this is the **degree** of f. The finite set of points with a preimage smaller than d are called the **branch points** of f. If z is a branch point and $f^{-1}(z) = \{a_1, \ldots, a_k\}$, there is a partition $\alpha = (\alpha_1, \ldots, \alpha_k) \vdash d$ such that α_i is the multiplicity of a_i as a solution to $f(a_i) = z$; this is called the **branching type** of z. If $\alpha_i > 1$, then a_i is a **ramification point** with ramification index α_i. The various parameters of a cover of this type are related by the **Riemann-Hurwitz** formula: if the ramification points of f are $a_1, \ldots, a_k$ with a_i having ramification index α_i, then

$$2g - 2 = -2d + \sum_{1 \leq i \leq k} (\alpha_i - 1).$$

The double Hurwitz problem deals with the enumeration of ramified covers in which at most 2 of the branch points, typically taken to be the points 0 and ∞, have preimages containing fewer than $d-1$ points — let $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be the branching types of 0 and ∞, respectively. There are r additional branch points whose preimage contains exactly $d-1$ points — these are called **simple** branch points, and they necessarily have branching type $(2, 1^{d-2})$. In this case, the Riemann-Hurwitz formula reduces to

$$2g - 2 = -m - n + r.$$

The number of covers satisfying these conditions is denoted by $H^g_{\alpha,\beta}$ This problem was combinatorialized by Hurwitz in 1891 [18]. (An English translation of his combinatorialization may be found in [13].) The encoding is as follows:

Problem 2.2.4 (Double Hurwitz Problem)**.** *Let $\alpha, \beta \vdash d$ with $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_n)$. Then $H^g_{\alpha,\beta}$ is equal to $\frac{1}{d!}\alpha_1! \cdots \alpha_m! \beta_1! \cdots \beta_n!$ times the number of factorizations $(\sigma, \tau_1, \ldots, \tau_r, \pi)$ such that*

1. $\sigma \in \mathcal{C}_\beta$, $\pi \in \mathcal{C}_\alpha$, *and* $\tau_i \in \mathcal{C}_{(2,1^{d-2})}$,
2. $\tau_1 \cdots \tau_r \sigma = \pi$,
3. $r = m + n + 2g - 2$, *and*
4. *the group generated by* $\{\sigma_1, \tau_1, \ldots, \tau_r\}$ *acts transitively on* $\{1, \ldots, d\}$.

When $\beta = (1^d)$ (and hence σ is always the identity), this is known as the **Single Hurwitz Problem**. Explicit expressions for the generating series for single Hurwitz numbers have been given for the sphere [8], torus [10] and [12], and double torus [9]. Recursions which determine the generating series for double Hurwitz numbers are also known [13]. Some of these results are obtained by the use of character theory, and others use a technique called Join-Cut analysis. Consideration of the effect of multiplication by a transposition on a permutation of cycle type λ leads to a partial differential equation for the generating series for the problem, which is then either solved or, in cases where a closed form solution is not known, analyzed to deduce information about the problem. An illustration of the use of this technique appears in Chapter 3 as part of a discussion of the transitive star factorization problem. Although Join-Cut analysis does not rely on the character theory of the symmetric group, central methods still play a role in providing computational data which can lead to a conjectured form for the solution.

The special case of the double Hurwitz problem in which $\alpha = (d)$ merits additional commentary, because in this case, an explicit formula can be derived using entirely central methods, as is done in Section 3 of [13]. When $\pi \in \mathcal{C}_{(d)}$, the factors involved in any factorization of π will necessarily act transitively on $\{1, \ldots, d\}$, since the group generated by π alone acts transitively on this set. Consequently, there is no need to enforce condition 4 in Problem 2.2.4, so it suffices to compute

$$[K_\beta](K_{(2,1^{d-2})})^r K_{(d)}.$$

Various results appearing later in this chapter may be applied to this problem, yielding the following.

Theorem 2.2.5 (Goulden, Jackson and Vakil [13])**.** *Let $r = n + 2g - 1$, and let $\beta \vdash d$. Then*

$$H^g_{(d),\beta} = r! d^{r-1} [t^{2g}] \frac{t/2}{\sinh(t/2)} \prod_{k \geq 1} \left(\frac{\sinh(kt/2)}{kt/2} \right)^{m_k(\beta)}.$$

Aside from the applications to topology and algebraic geometry described in detail above, it should be noted that enumerative questions about permutation factorizations arise in other areas as well. An example of an application to theoretical physics is discussed in Chapter 3. Permutation factorizations arise in computer science in the study of sorting networks, which can be modelled as a factorization of a permutation into transpositions (see, for example, [1]). Permutation factorizations form the basis for a problem in communication complexity which was studied by Harvey [17] and then used to find a non-trivial lower bound on the number of queries made by an algorithm for the matroid intersection problem.

2.3 Character Theory of $\mathfrak{S}_n$ and the Connection Coefficients of $Z(n)$

This section addresses the question of how to compute the connection coefficients $c^\lambda_{\mu,\nu}$. This is done by constructing a basis for $Z(n)$ consisting of elements which are orthogonal and idempotent with respect to ring multiplication in $\mathbb{C}[\mathfrak{S}_n]$. Such a basis is known to exist because $Z(n)$ is a commutative semi-simple algebra. Once this basis is constructed, the familiar strategy for computing $K_\mu K_\nu$, where K_μ is the sum of permutations of cycle type μ, is as follows.

1. Express K_μ and K_ν in terms of the basis of orthogonal idempotents.
2. Perform the (now trivial) multiplication in this basis.
3. Invert the basis change, expressing $K_\mu K_\nu$ in the standard basis for $Z(n)$.

An orthogonal idempotent basis for $Z(n)$ may be constructed using the representation theory of $\mathfrak{S}_n$. Section 2.3.1 contains definitions and elementary results related to the theory of representations and irreducible characters of $\mathfrak{S}_n$. Section 2.3.2 gives an account of the construction of an example of an irreducible representation, namely, Young's semi-normal representation. Section 2.3.3 demonstrates how these results are used to find expressions for $c_{\mu,\nu}^{\lambda}$ in terms of irreducible characters. The combinatorial problems of Section 2.2 are thereby reduced to the problem of computing irreducible characters of $\mathfrak{S}_n$.

2.3.1 Representations and Characters

Let $\mathrm{GL}_d(\mathbb{C})$ denote the group of invertible $d \times d$ complex matrices. A **representation** of $\mathfrak{S}_n$ is a homomorphism $\rho : \mathfrak{S}_n \to \mathrm{GL}_d(\mathbb{C})$. The integer d is called the **degree** of the representation, denoted by $\deg(\rho)$. Two representations ρ and ρ' of degree d are said to be **equivalent** if there exists a $d \times d$ matrix T such that $\rho(\pi) = T\rho'(\pi)T^{-1}$ for all $\sigma \in \mathfrak{S}_n$.

A representation induces an $\mathfrak{S}_n$-action on the vector space $\mathbb{C}^d$, namely, $g\mathbf{v} := \rho(g)\mathbf{v}$ for $g \in \mathfrak{S}_n$ and $\mathbf{v} \in \mathbb{C}^d$. In other words, every matrix representation of $\mathfrak{S}_n$ gives rise to an $\mathfrak{S}_n$-module. Conversely, given a d-dimensional vector space which is also an $\mathfrak{S}_n$-module, a matrix representation ρ can be constructed by defining $\rho(g)$ to be the matrix representation of the linear transformation $\mathbf{v} \mapsto g\mathbf{v}$. Consequently, a representation may be regarded as either a matrix representation or a module, depending on which perspective is most convenient at the time.

Given a matrix representation ρ, define the **character** of ρ to be the function $\chi^\rho : \mathfrak{S}_n \to \mathbb{C}$ given by

$$\chi^\rho(\pi) := \mathrm{trace}(\rho(\pi))$$

for $\pi \in \mathfrak{S}_n$. It will be convenient to regard the characters as elements of $\mathbb{C}[\mathfrak{S}_n]$, namely,

$$X^\rho := \frac{\deg(\rho)}{n!} \sum_{\pi \in \mathfrak{S}_n} \chi^\rho(\pi)\pi.$$

For any $\sigma, \pi \in \mathfrak{S}_n$,

$$\chi^\rho(\sigma\pi\sigma^{-1}) = \mathrm{trace}(\rho(\sigma)\rho(\pi)\rho(\sigma)^{-1}) = \mathrm{trace}(\rho(\pi)) = \chi^\rho(\pi),$$

so $\chi^\rho(\pi)$ depends only on the conjugacy class of π. Consequently, $\chi^\rho(\pi)$ is written as χ^ρ_μ whenever $\lambda(\pi) = \mu$, and X^ρ is an element of $Z(n)$, *i.e.*

$$X^\rho = \frac{\deg(\rho)}{n!} \sum_{\lambda \vdash n} \chi^\rho_\lambda K_\lambda.$$

A representation is said to be **irreducible** if, when regarded as a $\mathfrak{S}_n$-module, it has no proper nontrivial submodules. A character is said to be irreducible if its corresponding representation is irreducible. There are two facts about irreducible characters that are fundamental to the present approach to computing $c^\lambda_{\mu,\nu}$. The first is that the number of irreducible characters of $\mathfrak{S}_n$ is equal to the number of conjugacy classes of $\mathfrak{S}_n$. Thus, irreducible characters may be indexed by partitions of n; the notation $\chi^\lambda_\mu = \chi^\rho_\mu$ and $X^\lambda = X^\rho$ will be used when ρ is the irreducible representation indexed by λ. Writing $\deg(\rho) = d_\lambda$ when ρ is the irreducible representation indexed by λ,

$$X^\lambda = \frac{d_\lambda}{n!} \sum_{\mu \vdash n} \chi^\lambda_\mu K_\mu.$$

The second fact pertains to the standard inner product $\langle \cdot, \cdot \rangle_n$ on $\mathbb{C}[\mathfrak{S}_n]$ given by

$$\langle F, G \rangle_n = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} ([\pi]F)\overline{([\pi]G)},$$

where $\overline{z}$ denotes the complex conjugate of z. (In combinatorial applications, the coefficients are often real numbers, and the conjugate is usually dropped from this definition when this is the case.) With respect to this inner product, the irreducible characters of $\mathfrak{S}_n$ are orthonormal, *i.e.*

$$\langle X^\lambda, X^\mu \rangle = \delta_{\lambda,\mu},$$

where $\delta_{i,j}$ is the Kronecker delta. The fact that $X^\lambda \in Z(n)$, the number of X^λ's is equal to the dimension of $Z(n)$, and the X^λ's are orthogonal (and hence linearly independent) implies that $\{X^\lambda\}_{\lambda \vdash n}$ forms a basis for $Z(n)$, which will be called the **character basis** of $Z(n)$. Since this basis is orthonormal, it is routine to invert the change of basis transformation and express the standard basis in terms of the character basis, namely,

$$K_\mu = \sum_{\lambda \vdash n} \frac{|\mathcal{C}_\mu| \chi^\lambda_\mu}{d_\lambda} X^\lambda. \tag{2.1}$$

1	3	4	6
2	7		
5	9		
8			

Figure 2.4: A standard Young tableaux of shape $(4,2,2,1)$. Without the labels on the boxes, this is the Ferrers diagram $\mathcal{F}_{(4,2,2,1)}$.

2.3.2 Young's Semi-normal Units

This section contains a description of Young's semi-normal units, elements of $\mathbb{C}[\mathfrak{S}_n]$ from which the irreducible representations and characters of $\mathfrak{S}_n$ may be obtained. The method presented here is singled out for two reasons. First, an obvious consequence of the construction given here is that the basis $\{X^\lambda\}_{\lambda \vdash n}$ is not only orthogonal with respect to the inner product; it is also orthogonal with respect to ring multiplication. Furthermore, X^λ is idempotent, so the character basis is the one which will allow the connection coefficients of $Z(n)$ to be computed according to the strategy laid out in the preamble of Section 2.3. Second, the generalization of the character-theoretic approach appearing in Chapter 6 will appear quite natural if the characters of $\mathfrak{S}_n$ are constructed in the manner of this section.

Young's semi-normal units are defined in terms of combinatorial objects called tableaux. Given a partition $\lambda \vdash n$, the **Ferrers diagram** $\mathcal{F}_\lambda$ is a diagram consisting of n boxes arranged in $m(\lambda)$ rows such that, if the parts of λ are ordered such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m(\lambda)}$, then there are λ_i boxes in row i, justified to the left margin. An illustration of a Ferrers diagram is given in Figure . A **standard Young tableau** is a bijective assignment of the integers $\{1, \ldots, n\}$ to the boxes of $\mathcal{F}_\lambda$ such that the labels on the boxes increase to the right along rows, and down columns. The set of all standard Young tableaux of shape λ is denoted by $\mathrm{SYT}(\lambda)$.

Given a tableaux T, let R_i denote the set of integers appearing in row i of T for $1 \leq i \leq m(\lambda)$, and let C_j denote the set of integers appearing in column j of T for $1 \leq j \leq \lambda_1$. Then the **row group** of T is

$$\mathcal{R}(T) := \mathfrak{S}_{R_1} \times \mathfrak{S}_{R_2} \times \cdots \times \mathfrak{S}_{R_{m(\lambda)}},$$

and the **column group** is

$$\mathcal{C}(T) := \mathfrak{S}_{C_1} \times \mathfrak{S}_{C_2} \times \cdots \times \mathfrak{S}_{C_{\lambda_1}}.$$

Define the group algebra elements

$$R_T := \sum_{\pi \in \mathcal{R}(T)} \pi$$

and

$$C_T := \sum_{\pi \in \mathcal{C}(T)} \operatorname{sgn}(\pi)\pi,$$

where $\operatorname{sgn}(\pi)$ is the signum of π, equal to 1 if π is a product of an even number of transpositions, and -1 otherwise. Let T^* be the tableau obtained by deleting the box with label n from T. The approach to the representations of $\mathfrak{S}_n$ presented here relies on the following set of group algebra elements indexed by tableaux.

Definition 2.3.1 (Young's semi-normal units)**.** Let T be a standard Young tableau of shape $\lambda \vdash n$. The **semi-normal unit**, $e(T)$, is given by

$$e(T) = \frac{d_\lambda}{n!} e(T^*) R_T C_T e(T^*)$$

when T has at least one box, and $e(T) = 1$ if T is the tableau consisting of a single box.

The semi-normal units may be used to define the **semi-normal basis** for $\mathbb{C}[\mathfrak{S}_n]$. The basis elements corresponding to tableaux of shape λ can be used to describe a set of irreducible $\mathfrak{S}_n$-modules, indexed naturally by λ, called the **semi-normal representation**. However, the additional terminology and notation needed to describe this basis is not needed in this thesis, and the reader is referred to James and Kerber [25] for details. The facts about this representation and the semi-normal units which are used in this thesis are as follows.

Lemma 2.3.2. *Let $\lambda \vdash n$. Then the following results hold.*

1. *The degree of the irreducible representation indexed by λ is*

$$d_\lambda = |\mathrm{SYT}(\lambda)|.$$

2. *If T is a standard Young tableau of shape λ, then the coefficient of the identity permutation in $e(T)$ is $d_\lambda/n!$.*

3. *Let T, S be standard Young tableaux of shape λ. Then $e(T)e(S) = \delta_{T,S}e(T)$.*

4. *Let $\lambda \vdash n$. Then*

$$X^\lambda = \sum_{T \in \mathrm{SYT}(\lambda)} e(T).$$

0	1	2	3
-1	0		
-2	-1		
-3			

Figure 2.5: Contents of the boxes of a Young tableau of shape $(4, 2, 2, 1)$.

An obvious corollary of this expression for X^λ given in this Lemma, and the orthogonal idempotency of the semi-normal units is that $\{X^\lambda\}_{\lambda \vdash n}$ is in fact a basis for $Z(n)$ consisting of orthogonal idempotents. There are two additional facts about the semi-normal units which will be of use later.

The first has to do with the relationship between the Young idempotents and another set of group algebra elements called the **Jucys-Murphy elements** J_n, defined as follows:

$$J_n := \sum_{1 \leq i \leq n-1} (i, n).$$

The Jucys-Murphy elements, like any group algebra element, may be regarded as the linear operator on $\mathbb{C}[\mathfrak{S}_n]$ given by $g \mapsto J_n g$. The semi-normal units are eigenvectors of these operators, and their eigenvalues may be defined combinatorially. Given a standard Young tableau T in which the element i appears in the box in row j and column k, let $c_T(i) = k - j$ be the **content** of i in T. Figure 2.5 provides an illustration of this definition.Then the following fact holds.

Lemma 2.3.3 (Murphy [34])**.** *Let T be a standard Young tableau of shape $\lambda \vdash n$, and let $k \leq n$. Then*

$$J_k e(T) = c_T(k) e(T).$$

It is often convenient to refer to the vector $\mathbf{c}_T = (c_T(1), c_T(2), \ldots, c_T(n))$ as the **content vector** of T.

The second additional fact is that the semi-normal units in $\mathbb{C}[\mathfrak{S}_{n-1}]$ have a useful description in terms of the semi-normal units in $\mathbb{C}[\mathfrak{S}_n]$, and this expression can be stated in terms of the combinatorial operation of deleting a box from a tableau:

Lemma 2.3.4. *If T is a tableau with n boxes, then $e(T^*) = \sum_S e(S)$, where the sum is over all tableaux S such that T^* is obtained by deleting the box of S containing n.*

Now that the facts from representation theory which play a role in this thesis have been identified, the next section will demonstrate how they may be used to solve combinatorial problems which are naturally central.

2.3.3 Expressing Connection Coefficients in terms of Characters

The results of the two preceding sections can be combined to express the connection coefficients of $Z(n)$ in terms of characters. By Equation (2.1),

$$K_\mu K_\nu = \left(\sum_{\rho \vdash n} \frac{|\mathcal{C}_\mu|\chi^\rho_\mu}{d_\rho} X^\rho\right)\left(\sum_{\kappa \vdash n} \frac{|\mathcal{C}_\nu|\chi^\kappa_\nu}{d_\kappa} X^\kappa\right).$$

Since $X^\rho X^\kappa = \delta_{\rho,\kappa}$, then

$$K_\mu K_\nu = |\mathcal{C}_\mu||\mathcal{C}_\nu| \sum_{\rho \vdash n} \frac{\chi^\rho_\mu \chi^\rho_\nu}{d^2_\rho} X^\rho.$$

By the definition of X^ρ,

$$K_\mu K_\nu = \frac{|\mathcal{C}_\mu||\mathcal{C}_\nu|}{n!} \sum_{\lambda \vdash n}\sum_{\rho \vdash n} \frac{\chi^\rho_\mu \chi^\rho_\nu \chi^\rho_\lambda}{d_{\mu'}} K_\lambda.$$

This gives the following expression for the connection coefficients of the centre of $\mathbb{C}[\mathfrak{S}_n]$.

Lemma 2.3.5 (Connection coefficients of $Z(n)$)**.** *Let $c^\lambda_{\mu,\nu}$ be such that $K_\mu K_\nu = \sum_{\lambda \vdash n} c^\lambda_{\mu,\nu} K_\lambda$. Then*

$$c^\lambda_{\nu,\mu} = \frac{|\mathcal{C}_\mu||\mathcal{C}_\nu|}{n!} \sum_{\rho \vdash n} \frac{\chi^\rho_\mu \chi^\rho_\nu \chi^\rho_\lambda}{d_\rho}.$$

This lemma provides the solution to any naturally central problem for which the characters arising in the expression for $c^\lambda_{\nu,\mu}$ can be evaluated. Explicit expressions for characters are only known for special choices of λ, μ and ν. Some instances of these are given in the next section, leading to explicit expressions for the solution to a number of combinatorial problems.

2.4 Methods for Determining Characters

This section contains an account of two methods which may be used to compute the values of irreducible characters in various special, combinatorially-relevant cases. One method, described in Section 2.4.2, uses properties of the Jucys-Murphy elements to express characters as evaluations of symmetric polynomials at the contents of a Ferrers diagram. Another method, described in Section 2.4.3, uses the Frobenius formula to translate the problem of evaluating characters into a question about symmetric functions, and then uses the Jacobi-Trudy formula to give an expression for the Schur symmetric functions introduced by this method. Both methods use the terminology of symmetric functions, which is given in Section 2.4.1. This chapter closes, in Section 2.4.4, with an illustration of how these results have been used to find an explicit solution to the loopless dipole problem.

2.4.1 The Ring of Symmetric Functions

The definitions and results in this section are standard, and may be found in many sources such as Macdonald [32]. Results are stated here without proof, and the reader is referred to Macdonald for details. Let $\mathbb{C}[[x_1, x_2, \ldots]]$ denote the ring of formal power series in a countable number of indeterminates. For any bijection σ of the positive integers, define an action of σ on $\mathbb{C}[[x_1, x_2, \ldots]]$ by

$$\sigma f(x_1, x_2, \ldots) := f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots).$$

If $\sigma f = f$ for all such σ, then f is a **symmetric function** . Let $\Lambda[[x_1, x_2, \ldots]]$ denote the set of symmetric functions; this is a subalgebra of $\mathbb{C}[[x_1, x_2, \ldots]]$. It is also useful to discuss symmetric functions on a finite set of indeterminates; these may be obtained by evaluating a symmetric function at $x_i = 0$ for $i > n$, where n is the desired number of indeterminates. In this case, $f(x_1, \ldots, x_n)$ is written instead of $f(x_1, x_2, \ldots)$. The subalgebra of symmetric functions in n indeterminates is denoted by $\Lambda[[x_1, \ldots, x_n]]$, and the notation $\Lambda[x_1, \ldots, x_n]$ denotes the algebra of symmetric polynomials.

There are many sets of symmetric functions which generate Λ and are algebraically independent. Those which are most relevant to the present topic are:

1. **Elementary symmetric functions**: Let $E(t) := \prod_{i \geq 1}(1 + tx_i)$, and define $e_k := [t^k]E(t)$. Alternatively,

$$e_k := \sum_{1 \leq i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

2. **Complete symmetric functions**: Let $H(t) := \prod_{i \geq 1}(1 - tx_i)^{-1}$, and define $h_k := [t^k]H(t)$. Alternatively,

$$h_k := \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

3. **Power sum symmetric functions**: Let $p_k := \sum_{i\geq 1} x_i^k$. The generating series $P(t) = \sum_{k\geq 1} p_k t^{k-1}$ for power sum symmetric functions may be expressed in terms of the generating series for complete symmetric functions, *i.e.*

$$P(t) = \frac{d}{dt}\log(H(t)) = \frac{H'(t)}{H(t)}. \tag{2.2}$$

Taking f to be one of e, h or p, and $\lambda \vdash n$, let $f_\lambda = \prod_{i\geq 1} f_{\lambda_i}$. In this notation, $\{f_\lambda\}$, where λ ranges over all partitions of non-negative integers, forms a linear basis for Λ in each of these three cases.

A fourth important class of symmetric functions, which forms a linear basis for Λ, is the class of Schur symmetric functions. Given a partition λ, a **semi-standard Young tableau** of shape λ is a (not necessarily bijective) assignment of positive integers to the cells of the Ferrers diagram $\mathcal{F}_\lambda$ which is strictly increasing from left to right, and weakly increasing down columns. Let $\mathrm{SSYT}(\lambda)$ denote the set of semi-standard Young tableaux of shape λ. Given $T \in \mathrm{SSYT}(\lambda)$, let x_T be the product of x_i where i ranges over the labels in the cells of T. The final type of symmetric function used in this thesis is the following.

4. **Schur symmetric function**: Let

$$s_\lambda := \sum_{T\in \mathrm{SSYT}(\lambda)} x_T.$$

2.4.2 Using Jucys-Murphy elements to evaluate characters

The method for computing characters presented in this subsection is due to Diaconis and Greene [4]. It relies on the following fact which reduces the problem of evaluating the character χ^μ to the problem of evaluating a symmetric function at the contents of a tableau of shape μ.

Lemma 2.4.1. *Let $\lambda \vdash n$ and let $f \in \Lambda[x_1, \ldots, x_n]$ be such that $K_\lambda = f(J_1, J_2, \ldots, J_n)$. Then*

$$\chi^\mu_\lambda = \frac{d_\mu}{|\mathcal{C}_\lambda|} f(\mathbf{c}_\mu),$$

where $\mathbf{c}_\mu$ denotes the content vector of any tableau of shape μ.

This may be proven in a manner similar to the proof of Lemma 6.2.2, which appears later in this thesis. While this lemma does not appear explicitly in [4], it is a convenient way to summarize the Diaconis-Greene approach. The method relies on having an explicit expression for the symmetric function f; the existence of f is guaranteed by the following result.

Theorem 2.4.2 (Jucys [26])**.**

$$Z(n) = \Lambda[J_1, J_2, \ldots, J_n].$$

One containment is obvious: when evaluated at the Jucys-Murphy elements, the elementary symmetric functions become

$$\begin{aligned} e_k(J_1, \ldots, J_k) &= \sum_{2 \le i_1 < i_2 < \cdots < i_k} J_{i_1} J_{i_2} \cdots J_{i_k} \\ &= \sum_{\substack{\pi \in \mathfrak{S}_n, \\ m(\pi) = k}} \pi, \end{aligned}$$

which is an element of $Z(n)$. Since the elementary symmetric functions generate $\Lambda[x_1, \ldots, x_n]$, then $\Lambda[x_1, \ldots, x_n] \subseteq Z(n)$. The other containment has a more substantial proof, which may be found in [26].

Diaconis and Greene are able to obtain explicit formulas for special cases of evaluations of characters in the following cases. These results are most easily stated by introducing the following notation. For $\lambda \vdash n$, fix a tableau T of shape λ and define

$$\sigma(\lambda) := \sum_{1 \le j \le n} c_T(j) = \frac{1}{2} \sum_{1 \le i \le m(\lambda)} \left(\lambda_i^2 - (2i-1)\lambda_i \right),$$

the sum of contents of a tableau of shape λ. Let

$$\sigma^{(2)}(\lambda) := \sum_{1 \le j \le n} c_T(j)^2 = \sum_{1 \le i \le m(\lambda)} \left(\lambda_i(i^2 - i + 1) + 3\binom{\lambda_i}{2} + 2\binom{\lambda_i}{3} - i\lambda_i^2 \right),$$

the sum of squares of contents of a tableau of shape λ. (Observe that both quantities depend only on the shape of T.)

Lemma 2.4.3. *Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash n$. Then:*

1. $\chi^{\mu}_{(n)} = \begin{cases} (-1)^k & \text{if } \mu = (n-k, 1^k), \\ 0 & \text{otherwise.} \end{cases}$
2. $\chi^{\mu}_{(2,1^{n-2})} = \binom{n}{2}^{-1} d_\mu \sigma(\mu)$.
3. $\chi^{\mu}_{(3,1^{n-3})} = \frac{1}{2}\binom{n}{3}^{-1} d_\mu \left(\sigma^{(2)}(\mu) - \binom{n}{2}\right)$
4. $\chi^{\mu}_{(2,2,1^{n-4})} = \frac{1}{6}\binom{n}{4}^{-1} d_\mu \left(\sigma(\lambda)^2 - 3\sigma^{(2)}(\lambda) + 2\binom{n}{2}\right)$.

These results are also obtainable using the non-central refinement of the Diaconis-Greene method described in Chapter 7. All these character evaluations may be obtained as a corollary of later results. The first of these character evaluations is notable because it implies that in sums involving characters indexed by a full cycle, many terms vanish. The non-vanishing terms correspond to partitions of the form $(n-k, 1^k)$, which are referred to as **hook partitions**.

2.4.3 Using the Jacobi-Trudy identity to evaluate characters

A second method for evaluating characters relies on the following result, which expresses the irreducible characters of $\mathfrak{S}_n$ as change-of-basis coefficients in the ring of symmetric functions:

Theorem 2.4.4 (Frobenius). *Let $\lambda \vdash n$. Then*

$$s_\lambda = \sum_{\mu \vdash n} \frac{\chi_\mu^\lambda}{|\mathrm{Aut}(\mu)|} p_\mu.$$

This method involves resolving the Schur functions in the power sum basis, and extracting the coefficients in this basis to obtain the irreducible characters. It is a classical method which may be found in many sources; a convenient summary of the technique, together with examples of characters which may be computed using this method, appears in a paper by Jackson [21]. One of the primary methods of resolving Schur functions in terms of power sum symmetric functions is the following:

Theorem 2.4.5 (Jacobi-Trudy identity). *Let $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$. Then*

$$s_\lambda = \det[h_{\lambda_i - i + j}]_{m \times m},$$

where $[a_{i,j}]_{m\times m}$ denotes the $m \times m$ matrix whose entry in the i^{th} row and j^{th} column is $a_{i,j}$.

Integrating and exponentiating Equation 2.2 allows complete symmetric functions to be expressed in terms of the power sum basis, namely,

$$h_k = \sum_{\lambda \vdash k} \frac{p_\lambda}{|\mathrm{Aut}(\lambda)|}.$$

Consequently, for any partitions for which the determinant in Theorem 2.4.5 can be evaluated explicitly, Theorem 2.4.4 allows the irreducible characters indexed by that partition to be evaluated. One important special case, evaluation of characters corresponding to hook partitions, plays an important role in this thesis and appears in [21]:

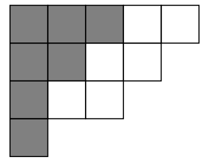


Figure 2.6: Illustration of a skew diagram λ/μ where $\lambda = (5, 4, 3, 1)$ and $\mu = (3, 2, 1, 1)$. This is a rim hook of height 2.

Lemma 2.4.6. *Let $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$, and let $0 \le k \le n-1$. Then*

$$\chi_\lambda^{(n-k,1^k)} = [x^k](1+x)^{-1} \prod_{1 \le j \le m} (1 - (-x)^{\lambda_j}).$$

Since the series appearing in this Lemma is used often in this thesis, it is convenient to denote it by

$$H_\lambda(x) := (1+x)^{-1} \prod_{1 \le j \le m} (1 - (-x)^{\lambda_j}).$$

Lemma 2.4.6 may also be proven inductively using the Murnaghan-Nakayama rule. In order to state this rule, some additional terminology is needed. Given two partitions $\lambda, \mu \vdash n$ such that $\mathcal{F}_\mu \subseteq \mathcal{F}_\lambda$, the **skew diagram** λ/μ is the diagram obtained from $\mathcal{F}_\lambda$ by removing the boxes corresponding to $\mathcal{F}_\mu$. A skew diagram is a **rim hook** if it is connected and contains no 2×2 box as a subdiagram. The **height** of a rim hook λ/μ is equal to one less than number of rows occupied by λ/μ, and is denoted by $\langle \lambda/\mu \rangle$. These concepts are illustrated in Figure 2.6. Then the following is true.

Theorem 2.4.7 (Murnaghan-Nakayama Rule)**.** *Let $\lambda, \mu \vdash n$, and let i be a part of μ. Then*

$$\chi_\mu^\lambda = \sum_\rho (-1)^{\langle \lambda/\rho \rangle} \chi_{\mu \setminus i}^\rho,$$

where the sum is over $\rho \vdash n - i$ such that λ/ρ is a rim hook.

(This rule appears in numerous sources. See, for example, Chapter 4 of Sagan [39] for a particularly clear account.)

2.4.4 The number of loopless dipoles in a surface of genus g

As a demonstration of how the algebraic techniques described in this chapter may be used to solve a concrete combinatorial problem, this section gives an explicit expression for the generating series for the number of loopless dipoles with n edges in a surface of genus g, as well as a more refined generating series which keeps track of half-face-degree sequence. The formula for this number may be found explicitly in the literature in [28], or as an obvious corollary of the following result.

Theorem 2.4.8 (Jackson, [21])**.** *For a partition $\lambda \vdash n$, let e_k^λ denote the number of permutations with k cycles which are the product of a permutation of cycle type λ and the cycle $(1, 2, \ldots, n)$. Let $x_\lambda = \prod_{1 \le i \le m(\lambda)} x_i$, and let ϕ_z denote the linear operator defined by $\binom{z}{k} \mapsto z^k$. Then*

$$z + \sum_{k,n \ge 1} \sum_{\lambda \vdash n} e_k^\lambda x_\lambda \frac{y^n}{n!} \phi_z(z^k) = z \exp\left(\sum_{i \ge 1} \frac{y^i x_i}{i} ((1+z)^i - z^i) \right).$$

Jackson also gave a generalization of this formula to the case in which there is an arbitrary number of factors — specifically, for the number of factorizations of the form $\pi = \sigma_1 \ldots \sigma_r$ where $\pi \in \mathcal{C}_\lambda$, $\sigma_1 \in \mathcal{C}_{(n)}$ and σ_i has t_i cycles, for a given sequence of positive integers $(t_2, \ldots, t_r)$. This formula appears in [22].

The analysis presented here for the special case of rooted dipoles gives a more refined result than the one obtainable as a corollary of this theorem since it gives a generating series which keeps track of the full face-degree sequence instead of just the number of faces; however, the proof techniques used are essentially the same as the ones used in the proof of Theorem 2.4.8.

By the encoding used in Section 2.2 and Lemma 2.3.5, the number of labelled dipoles with half-face-degree sequence λ is given by

$$|\mathcal{C}_\lambda| c_{(n),(n)}^\lambda = \frac{|\mathcal{C}_\lambda|(n-1)!}{n} \sum_{\nu \vdash n} \frac{\chi_{(n)}^\nu \chi_{(n)}^\nu \chi_\lambda^\nu}{d_\nu}.$$

Applying part 1 of Lemma 2.4.3,

$$|\mathcal{C}_\lambda| c_{(n),(n)}^\lambda = \frac{|\mathcal{C}_\lambda|(n-1)!}{n} \sum_{0 \le k \le n-1} \frac{1}{d_{(n-k,1^k)}} \chi_\lambda^{(n-k,1^k)}.$$

Since $d_{(n-k,1^k)}$ is the number of standard Young tableaux of shape $(n-k, 1^k)$, then

$$|\mathcal{C}_\lambda| c_{(n),(n)}^\lambda = \frac{|\mathcal{C}_\lambda|(n-1)!}{n} \sum_{0 \le k \le n-1} \frac{1}{\binom{n-1}{k}} \chi_\lambda^{(n-k,1^k)}.$$

Applying Lemma 2.4.6,

$$|\mathcal{C}_\lambda| c^{\lambda}_{(n),(n)} = \frac{|\mathcal{C}_\lambda|(n-1)!}{n} \sum_{0 \le k \le n-1} \frac{1}{\binom{n-1}{k}} [y^k] H_\lambda(y).$$

At this point, it is convenient to introduce the linear functional L_n defined by

$$L_n : y^k \mapsto \frac{1}{(n+1)\binom{n}{k}}$$

so that the expression for the number of dipoles of face type λ becomes

$$|\mathcal{C}_\lambda| c^{\lambda}_{(n),(n)} = |\mathcal{C}_\lambda|(n-1)! L_{n-1} H_\lambda(y).$$

Given a partition λ, let f_λ denote the product of indeterminates f_i corresponding to the parts of λ, *i.e.*

$$f_\lambda := \prod_{1 \le i \le m(\lambda)} f_i.$$

Consider the generating series for unlabelled rooted dipoles in which a face of degree $2i$ is marked by the indeterminate f_i, namely,

$$\begin{aligned}
\Psi_{\mathcal{D}} &:= \sum_{D \in \mathcal{D}} \frac{x^{n(D)}}{n(D)!} u^{2g(D)} f_{\lambda(D)} \\
&= \sum_{n \ge 1} \sum_{\lambda \vdash n} \frac{x^n}{n!(n-1)!} u^{n-m(\lambda)} f_\lambda |\mathcal{C}_\lambda| c^{\lambda}_{(n),(n)} \\
&= \sum_{n \ge 1} \frac{x^n}{n!} L_{n-1} \sum_{\lambda \vdash n} u^{n-m(\lambda)} f_\lambda |\mathcal{C}_\lambda| H_\lambda(y).
\end{aligned}$$

(Recall that the number of unlabelled rooted dipoles differs from the number of labelled dipoles by a factor of $(n-1)!$.) Define the linear operator L by

$$L = \sum_{n \ge 1} x^n L_{n-1} [x^n];$$

in other words, $L : x^n y^k \mapsto n^{-1} \binom{n-1}{k}^{-1} x^n$. Then the generating series becomes

$$\Psi_{\mathcal{D}} = L \left(\sum_{n \ge 1} \frac{x^n}{n!} \sum_{\lambda \vdash n} u^{n-m(\lambda)} f_\lambda |\mathcal{C}_\lambda| H_\lambda(y) \right).$$

Since

$$u^{n-m(\lambda)} f_\lambda H_\lambda(y) = u^n (1+y)^{-1} \prod_{1 \le j \le m(\lambda)} \frac{f_{\lambda_j}}{u} (1 - (-y)^{\lambda_i}),$$

the series to which L is applied is just, up to a factor of $(1+y)^{-1}$, the generating series for permutations of $\{1,\ldots,n\}$ in which n is recorded exponentially in xu, and a cycle of length i is recorded ordinarily with $u^{-1}f_i(1-(-y)^i)$. Thus, the following holds true.

Lemma 2.4.9. *The generating series for unlabelled rooted dipoles is given by*

$$\Psi_{\mathcal{D}} = L\left((1+y)^{-1}\left(\exp\left(\sum_{i\geq 1}\frac{x^i u^{i-1}}{i}f_i(1-(-y)^i)\right)-1\right)\right).$$

In addition to serving as a demonstration of central techniques, this series, in the general form given here, makes an appearance in Chapters 5 and 8 as the initial condition for differential equations which determine the solution to the (p,q,n)-dipole problem. To address the problem of determining $d_{n,g}$, set $f_i = z$ for all i to "forget" information about degree sequence and record only the number of faces. Since genus may be determined from the number of edges and number of faces, to determine $d_{n,g}$, calculations may be simplified by setting $u=1$ and extracting the coefficient of z^{n-2g} instead of u^{2g}. In other words,

$$\begin{aligned}
d_{n,g} &= n![x^n z^{n-2g}]L\left((1+y)^{-1}\left(\exp\left(\sum_{i\geq 1}\frac{x^i}{i}z(1-(-y)^i)\right)-1\right)\right)\\
&= n![x^n z^{n-2g}]L\left((1+y)^{-1}\left(\exp\left(z\log(1-x)^{-1}-z\log(1+xy)^{-1}\right)-1\right)\right)\\
&= n![x^n z^{n-2g}]L\left((1+y)^{-1}\left(\frac{1+xy}{1-x}\right)^z-(1+y)^{-1}\right).
\end{aligned}$$

This expression may be simplified using an integral representation of the linear operator L. Such a representation may be found by considering the integral

$$\int_0^1 s^a(1-s)^b ds = \frac{a!b!}{(a+b+1)!} = \frac{1}{\binom{a+b}{b}}\frac{1}{a+b+1}.$$

The operation

$$f(x,y)\longmapsto \int_0^1 s^{-1}f\left(xs,\frac{1-s}{s}\right)ds$$

satisfies

$$\begin{aligned}
x^n y^k &\longmapsto \int_0^1 s^{-1}(xs)^n\left(\frac{1-s}{s}\right)^k ds\\
&= x^n\int_0^1 s^{n-k-1}(1-s)^k ds\\
&= \frac{x^n}{n\binom{n-1}{k}}\\
&= L(x^n y^k).
\end{aligned}$$

Extending linearly gives the following.

Lemma 2.4.10. *Let $f(x,y)$ be a formal power series in the indeterminates x, y. Then*

$$L(f(x,y)) = \int_0^1 s^{-1} f\left(xs, \frac{1-s}{s}\right) ds.$$

Applying the integral form of L, the expression for the number of dipoles in a surface of genus g becomes

$$\begin{aligned}
d_{n,g} &= n![x^n z^{n-2g}] \int_0^1 \left(1 + \frac{x}{1-xs}\right)^z - 1 ds \\
&= n![x^n z^{n-2g}] \sum_{i \geq 1} \binom{z}{i} x^i \int_0^1 \left(\frac{1}{1-xs}\right)^i ds \\
&= n![x^n z^{n-2g}] \left(z \log((1-x)^{-1}) + \sum_{i \geq 2} \binom{z}{i} \frac{x^{i-1}}{i-1} \left(\frac{1}{(1-x)^{i-1}} - 1 \right) \right) \\
&= [z^{n-2g}] \left((n-1)!z + \sum_{2 \leq i \leq n} \binom{z}{i} \frac{n!}{i-1} \binom{n-1}{i-2} \right) \\
&= [z^{n-2g}](n-1)! \binom{z}{1} + [z^{n-2g}] \sum_{2 \leq i \leq n} \binom{z}{i} (n-1)! \binom{n}{i-1} \\
&= (n-1)![z^{n-2g}] \sum_{1 \leq i \leq n} \binom{z}{i} \binom{n}{i-1}.
\end{aligned}$$

Thus, the loopless dipole problem is an example of a problem which can be solved explicitly using character theory. The proof of Theorem 2.4.8 makes use of the fact that evaluating the sum of all characters indexed by partitions with a given number of cycles is easier than evaluating one specific character, and this is what allows for such an explicit form of the solution when it is only the number of cycles of the factors that is of concern. This is a natural specialization in applications such as map enumeration, since summing over all permutations with a fixed number of cycles corresponds to summing over all maps in a given surface.

2.4.5 Central decompositions of a full cycle

Central methods can also be applied to deal with the more refined case in which the cycle type of the factors is specified. The cases in which the most explicit results are possible are those in which at least one of the factors is a full cycle, since in this case Lemma 2.4.3 allows the character sums involved to be simplified considerably.

A complete solution to Problem 2.1.2 in the case when π is a full cycle, *i.e.*, determining the connection coefficient $c^{(n)}_{\lambda,\mu} = [K_{(n)}]K_\lambda K_\mu$, was determined by Goupil and Schaeffer [15] using central methods. In the following, the notation $(i_1, \ldots, i_\ell) \models g_1$ indicates that $(i_1, \ldots, i_\ell)$ is a **composition** of g_1, *i.e.* $(i_1, \ldots, i_\ell)$ is a sequence of non-negative integers which sum to g_1.

Theorem 2.4.11 (Goupil and Schaeffer [15]). *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and let $\mu = (\mu_1, \ldots, \mu_m)$ be partitions of n, and let $2g = n + 1 - m(\lambda) - m(\mu)$. Then*

$$c^{(n)}_{\lambda,\mu} = n \sum_{g_1+g_2=g} \frac{(\ell + 2g_1 - 1)!(m + 2g_2 - 1)!}{|\mathrm{Aut}(\lambda)||\mathrm{Aut}(\mu)|2^{2g}} \sum_{\substack{(i_1,\ldots,i_\ell)\models g_1 \\ (j_1,\ldots,j_m)\models g_2}} \prod_{1\le k\le \ell} \binom{\lambda_k}{2i_k + 1} \prod_{1\le k\le m} \binom{\mu_k}{2j_k + 1}.$$

Alternatively,

$$c^{(n)}_{\lambda,\mu} = \frac{n}{\prod_i m_i(\lambda)!m_i(\mu)!2^{2g}} \sum_{g_1+g_2=g} S_{g_1}(\lambda)S_{g_2}(\mu).$$

where

$$S_g(x_1, \ldots, x_k) = (k + 2g - 1)! \sum_{(i_1,\ldots,i_k)\models g} \prod_{1\le j\le k} \frac{1}{2i_j + 1} \binom{x_j - 1}{2i_j}$$

is a symmetric polynomial.

A combinatorial proof of this result, in the special case $g = 0$, was given by Goulden and Jackson [7] in 1992 by establishing a bijection between the factorizations under consideration and a class of two-coloured rooted trees. Extensions of the Goupil-Schaeffer result to deal with an arbitrary number of factors were given by Poulalhon and Schaeffer [38] in 2002 and by Irving [19] in 2006. In 2008, Schaeffer and Vassilieva [40] gave a combinatorial proof of Jackson's [22] formula in the case of decomposing a full cycle into two factors with a specified number of cycles. This is an example of a situation in which an enumerative formula, derived algebraically, pointed the way towards a bijective proof of formula and revealed the combinatorial structure of the objects being enumerated.

The interplay between algebraic and combinatorial enumeration is further illustrated by the history of the Harer-Zagier [16] formula for the number of representations of a genus g surface as a polygon with $2n$ sides, which arose in their study of the Euler characteristic of the moduli space of curves. In the terminology of maps, this is the problem of enumerating maps with n edges and a single face in a surface of genus g. Harer and Zagier's original proof used an integral over the space of hermitian matrices, and a second proof using character theory was given by Jackson [21]. Subsequently, a purely combinatorial proof of the same result was given by Goulden and Nica [14]. Given the history of the relationship between algebraic and combinatorial approaches to permutation factorization problems, it is reasonable to expect that the algebraic study of non-central problems will lead to similar combinatorial insights in the future.

2.4.6 Concluding comments

There are three essential facts about the centre of $\mathbb{C}[\mathfrak{S}_n]$ which make it possible to find such an explicit form for the solution to the rooted dipole problem (and, more generally, to prove Theorem 2.4.8):

1. there is an expression for the connection coefficients in terms of the irreducible characters of $\mathfrak{S}_n$, namely, Lemma 2.3.5;
2. irreducible characters, when evaluated at the conjugacy class of full-cycle permutations, vanish unless they are the characters indexed by hook partitions — see Lemma 2.4.3;
3. the evaluation of an irreducible character indexed by a hook at an arbitrary class is known, and can be expressed as the coefficient of a polynomial generating series — see Lemma 2.4.6.

In the context of developing techniques to deal with non-central problems, one of the main results of this thesis is that there is a class of non-central problems for which statements analogous to all three of these facts hold true. These statements will be given and proven in Chapter 6, and these results will be applied to solve some combinatorial problems in Chapter 7. First, however, the next chapter discusses some examples of non-central problems which serve to motivate the development of non-central techniques.

Chapter 3

Non-central Permutation Factorization Problems

This chapter describes some permutation factorization problems which can be encoded as elements of $\mathbb{C}[\mathfrak{S}_n]$, but to which the techniques described in Chapter 2 cannot be applied. Section 3.1 describes a non-central problem, the (p, q, n)-dipole problem, for which solutions are only known for surfaces of genus at most two. This problem serves as the chief motivation for developing methods to deal with non-central problems. Section 3.2 contains a description of the transitive star factorization problem and a natural generalization, the G-factorization problem, which is parametrized by a graph G. (The star factorization problem is the case in which G is the complete bipartite graph $K_{1,n-1}$.) The problem of enumerating transitive star factorizations was fully solved by Goulden and Jackson [11]. While it is not obvious from the statement of this problem that it is central, the formula given by Goulden and Jackson depends only on the cycle type of the permutation being factorized. In other words, this is a central problem which is not "naturally central." Though this problem has been solved, it is of relevance to this thesis since the non-central approach used to solve it has inspired the approach to the (p, q, n)-dipole problem described in Chapter 5. Furthermore, since the non-transitive version of the star factorization problem is non-central, the solution to this problem raises questions about the relationship between centrality and transitivity. Finally, Section 3.3 introduces the Z_1-factorization problem, which is a natural generalization of Problem 2.1.2. Like the (p, q, n)-dipole problem, the role of the Z_1-factorization problem in this thesis is to motivate and demonstrate the non-central techniques which are developed.

3.1 The (p,q,n)-dipole problem

The (p,q,n)-dipole problem is a combinatorial problem arising in mathematical physics, and it is one of the primary motivators for the development of non-central techniques in this thesis. A combinatorial definition of the (p,q,n)-dipole problem is given in Section 3.1.1. Section 3.1.2 gives an encoding of the problem in $\mathbb{C}[\mathfrak{S}_n]$ and demonstrates that the problem is non-central. Section 3.1.3 gives an account of the history of the problem, including the motivation from mathematical physics behind its introduction together with descriptions of the methods that have been previously applied to the problem.

3.1.1 Definition of the Problem

The (p,q,n)-dipole problem is about rooted dipoles with a second distinguished edge. (In diagrams, the root edge will be denoted by an *arrow* on the edge, pointing away from the root vertex. The distinguished edge will be denoted by a *dashed line*.) Given such a dipole, the neighbourhoods of each vertex can be partitioned into four regions as follows.

- **Region 1** is the part of the neighbourhood of the root vertex encountered while travelling counterclockwise from the root edge to the distinguished edge.
- **Region 2** is the remainder of the neighbourhood of the root vertex.
- **Region 3** is the part of the neighbourhood of the non-root vertex encountered while travelling counterclockwise from the root edge to the distinguished edge.
- **Region 4** is the remainder of the neighbourhood of the non-root vertex.

These regions are illustrated in Figure 3.1.

The partition of the neighbourhoods of the root vertices into these four regions permits the following definition to be made.

Problem 3.1.1 ((p,q,n)-dipole problem)**.** *Let D be a rooted dipole with an additional distinguished edge. The p-**value** of D, denoted by $p(D)$, is one plus the number of edges intersecting the interior of Region 1. The q-**value** of D, denoted by $q(D)$, is one plus the number of edges intersecting the interior of Region 3. A dipole with specified values for p, q, and number of edges n is referred to as a (p,q,n)**-dipole**. Determine the number $d_{n,g}^{p,q}$ of of (p,q,n)-dipoles in a surface of genus g.*

As an example, the dipole shown in Figure 3.1.1 is a $(3,4,6)$-dipole. Some additional terminology which will be useful is as follows. An edge which is neither the root edge nor

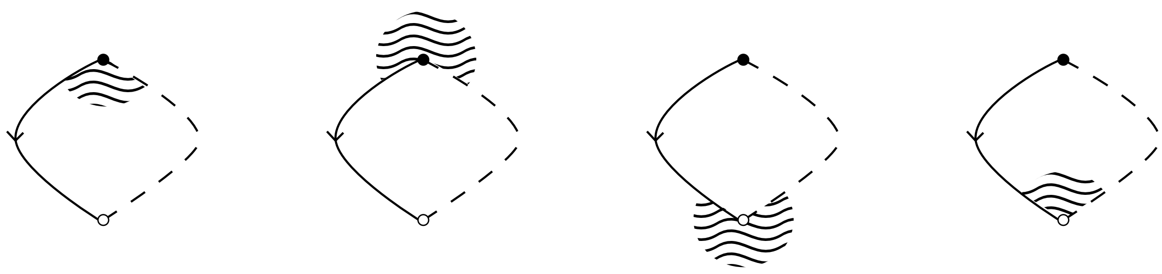


Figure 3.1: Illustration of the definitions of the regions into which the neighbourhoods of the vertices of a rooted dipole with a second distinguished edge are divided.

the second distinguished edge shall be referred to as an **ordinary edge**. The root edge / vertex pair uniquely identifies a root face, namely, the face encountered when moving counterclockwise from the root edge around the root vertex. (The corner of the root face which is incident with the root edge and vertex is called the **root corner**.)

There are two natural bijections from the set of rooted dipoles to itself which lead to simple relations among the numbers $d_{n,g}^{p,q}$. The first bijection, interchanging the vertices of the dipole, exchanges the role of p and q; in other words,

$$d_{n,g}^{p,q} = d_{n,g}^{q,p}. \tag{3.1}$$

The second bijection interchanges the root edge and the second distinguished edge. If a dipole has $p-1$ edges in Region 1, then its image under this bijection has $(n-2)-(p-1) = n-p-1$ edges in Region 1. A similar argument about the q-value of the dipole yields the following:

$$d_{n,g}^{p,q} = d_{n,g}^{n-p,n-q}. \tag{3.2}$$

Given these relations, it suffices to solve the (p, q, n)-dipole problem when $p \leq q$ and $p + q \leq n$.

In some cases it will be more convenient to enumerate (p, q, n)-dipoles in which each of the ordinary edges has been assigned a unique label from the set $\{1, 2, \ldots, n-2\}$. A dipole in which the ordinary edges are labelled shall be referred to as a **labelled** (p, q, n)**-dipole**. Let $\hat{d}_{n,g}^{p,q}$ denote the number of labelled (p, q, n)-dipoles in a surface of genus g. Clearly,

$$d_{n,g}^{p,q} = \frac{1}{(n-2)!}\hat{d}_{n,g}^{p,q}.$$

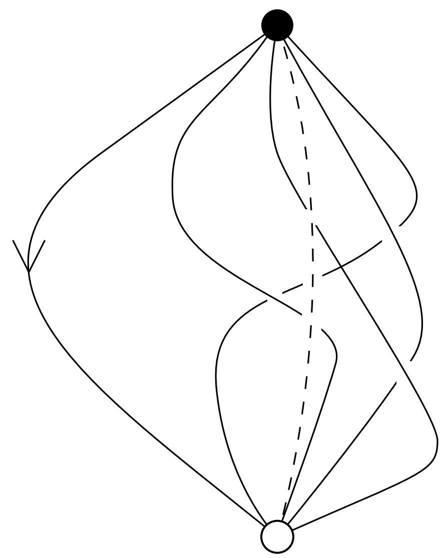


Figure 3.2: An example of a $(3, 4, 6)$-dipole.

3.1.2 Non-Centrality of the (p, q, n)-dipole Problem

The labelled (p, q, n)-dipole problem may be encoded as an element of $\mathbb{C}[\mathfrak{S}_n]$ as a straightforward refinement of Kwak and Lee's encoding (Section 2.2.1). Assign the label n to the root edge, and $n-1$ to the second distinguished edge. Consequently, any pair of vertex permutations (σ_1, σ_2) which corresponds to a (p, q, n)-dipole must also satisfy the requirement that $\sigma_1^q(n) = n - 1$ and $\sigma_2^p(n) = n - 1$. Applying Lemma 2.1.1 proves the following.

Lemma 3.1.2 (Encoding of the labelled (p, q, n)-dipole problem in $\mathbb{C}[\mathfrak{S}_n]$)**.** *Let $\pi \in \mathfrak{S}_n$, and let $1 \leq p, q \leq n-1$. Then the number of labelled (p, q, n)-dipoles with face permutation π is given by*

$$[\pi] \left(\sum_{\substack{\sigma_1 \in \mathcal{C}_{(n)} \\ \sigma_1^q(n) = n-1}} \sigma_1 \right) \left(\sum_{\substack{\sigma_2 \in \mathcal{C}_{(n)} \\ \sigma_2^p(n) = n-1}} \sigma_2 \right).$$

Since neither the left nor right factor in this expression commutes with $\mathfrak{S}_n$, this encoding does not lie in the centre of the group algebra. To answer the question of whether or not an alternate central encoding of this problem exists, consider the specific case of computing $(1, q, 4)$-dipoles for various values of q. The face permutations for $(1, q, 4)$-dipoles are given in Table 3.1.

	σ_1	$\sigma_2 = (1243)$	$\sigma_2 = (1432)$
$q = 1$	(1243)	(14)(23)	(134)(2)
	(1432)	(1)(234)	(13)(24)
$q = 2$	(1423)	(134)(2)	(124)(3)
	(1324)	(142)(3)	(1)(234)
$q = 3$	(1342)	(1)(2)(3)(4)	(123)(4)
	(1234)	(132)(4)	(1)(2)(3)(4)

Table 3.1: Face permutations of $(1, q, 4)$-dipoles for various values of q

In the case of $(1, 2, 4)$-dipoles, for example, the group algebra element encoding the face permutations is

$$(134) + (124) + (142) + (234).$$

In this element, four members of $\mathcal{C}_{(3,1)}$ appear with coefficient one, and four appear with coefficient 0; consequently, it does not lie in $Z(4)$. Thus, the (p, q, n)-dipole problem is fundamentally non-central, in contrast to problems such as the transitive star factorization problem discussed in Section 3.2 which is central without having an explicit central encoding.

3.1.3 Previous Approaches to the Problem

The (p, q, n)-dipole problem was first introduced in a physical context by Constable, Freedman, Headrick, Minwalla, Motl, Postnikov and Skiba [2]. Maps arise in this context as embeddings of Feynman diagrams in an orientable surface. Particle interactions are modelled as sums of terms indexed by these diagrams. From a combinatorial point of view, this is an evaluation of the generating series of these diagrams.

The particular case of (p, q, n)-dipoles corresponds physically to the free two-point functions of the Berenstein-Maldacena-Nastase operators

$$O_k^n := \sum_{0 \le \ell \le n} x^\ell \mathrm{Tr}(\phi Z^\ell \psi Z^{n-\ell})$$

and

$$\overline{O}_{k'}^n := \sum_{0 \le \ell \le n} y^\ell \mathrm{Tr}(\overline{Z}^\ell \overline{\psi} \overline{Z}^{n-\ell} \overline{\phi}),$$

where n is a positive integer, k, k' are integers, $x = \exp(2\pi i k/n)$, $y = \exp(-2\pi i k'/n)$, and Z, ψ and ϕ are $N \times N$ hermitian matrices. The two-point functions are given by

$$(xy)^{-1} \sum_{\substack{\sigma \in \mathfrak{S}_n, \\ \sigma(1)=1}} N^{m(\sigma^{-1} C^{-1} \sigma C)} \sum_{2 \le i \le n} x^{i-1} y^{\sigma(i)-1}, \tag{3.3}$$

where $C = (12 \ldots n)$ is the canonical full cycle in $\mathfrak{S}_n$. The equivalence of this problem with the description of the (p, q, n)-dipole problem given in Definition 3.1.1 and Lemma 3.1.2 can be seen by observing that

$$\{\sigma^{-1} C^{-1} \sigma : \sigma \in \mathfrak{S}_n, \sigma(1) = 1\} = \mathcal{C}_{(n)}.$$

Thus, the pair $(\sigma^{-1} C^{-1} \sigma, C)$ is just the vertex permutation pair encoding a rooted dipole in which the edges have been given a canonical labelling. (The root is given the label 1, and the other edges are labelled $2, 3, \ldots, n$ in counterclockwise order around the root vertex.) The exponent of z in Expression (3.3) records the number of faces of such a dipole. The exponent of x is $i - 1 = C^i(1)$; in other words, regarding i as the second distinguished edge of the dipole, the exponent of x records the p-value of the dipole. The q-value may be determined by identifying the location of the distinguished edge label i on the cycle

$$(1, \sigma^{-1}(n), \sigma^{-1}(n-1), \ldots, \sigma^{-1}(2))$$

which encodes the non-root vertex. If $j = \sigma(i)$, then $\sigma^{-1}(j) = i$ is the $(n-j+1)^{\text{th}}$ element following 1 on this cycle. Thus, $n - q = \sigma(i) - 1$, so the exponent of y records $n - q$. Thus, up to a change in notational convention, determining the coefficients of Expression 3.3 is equivalent to the statement of the (p, q, n)-dipole problem given in Problem 3.1.1.

Constable *et al.* give an explicit expression for Expression (3.3) as a sum over **block-reduced permutations**. A permutation $\pi \in \mathfrak{S}_n$ is said to be block-reduced if it satisfies the following conditions:

1. $\pi(1) = 1$,
2. $\pi^{-1}(i+1) \neq \pi^{-1}(i) + 1$ for $1 \leq i \leq n-1$, and
3. $\pi(n) \neq n$.

Block-reduced permutations have a natural combinatorial interpretation in terms of dipoles. Consider the dipole with vertex permutation pair $(\pi C^{-1} \pi^{-1}, C)$. If $\pi^{-1}(i+1) = \pi^{-1}(i) + 1$ and $1 \leq i \leq n-1$, then

$$\pi C^{-1} \pi^{-1} C(i) = \pi C^{-1} \pi^{-1}(i+1) = \pi C^{-1}(\pi^{-1}(i) + 1) = i;$$

thus, i is a fixed point of the face permutation of the dipole, when $1 \leq i \leq k-1$. (Similarly, the condition $\pi(n) \neq n$ ensures that n is not a fixed point of the face permutation.) Thus, a block-reduced permutation corresponds to a dipole having no faces of degree 2. (A face of degree two is referred to as a **digon**.) The set of block-reduced permutations in $\mathfrak{S}_k$ corresponding to a dipole of genus g is denoted by BR_k^g.

The method used by Constable *et al.* is to construct the set of all dipoles from the set of block-reduced permutations by replacing each edge in a dipole having no digons with some number b_i of digons, while keeping track of how the insertion of edges changes the p- and q-values of the dipole. They give the following expression for the generating series for (p, q, n)-dipoles.

Theorem 3.1.3 (Constable *et al*[2])**.** *The coefficient in (3.3) corresponding to dipoles of genus g is given by*

$$\sum_{k\geq 1}\sum_{\pi\in \mathrm{BR}_k^g}\sum_{1\leq i\leq k+1}\left(\sum x^{b_1+\cdots+b_{i-1}}y^{b'_1+\cdots b'_{\pi(i)-1}}\frac{1-(xy)^{b_i+1}}{1-xy}\right),$$

where the inner summation is over all sequences $b_1, \ldots, b_{k+1}$ satisfying $b_1, b_{k+1} \geq 0$, $b_i > 0$ for $2 \leq i \leq k$, $b_1 + \cdots b_{k+1} = n+1$, $b'_i = b_{\pi^{-1}(i)}$ for $1 \leq i \leq k$ and $b'_{k+1} = b_{k+1}$.

Results about the cardinality of BR_k^g give an idea of how many terms arise in the expression given in this theorem.

Lemma 3.1.4 (Constable *et al*[2])**.** *If $\pi \in \mathrm{BR}_k^g$, then $k \leq 4g$. Furthermore,*

$$|\mathrm{BR}_{4g}^g| = \frac{(4g-1)!!}{2g+1}$$

and

$$|\mathrm{BR}_{4g-1}^g| = \frac{4g-1}{3}|\mathrm{BR}_{4g}^g|.$$

Constable *et al.* use Theorem 3.1.3 to solve the (p, q, n)-dipole problem asymptotically for genus 1 and 2. (In the asymptotic case, the only terms arising in the formula of Theorem 3.1.3 are those for which $k = 4g$.) Subsequently, Visentin and Wieler [45] gave exact expressions for the coefficients of Expression (3.3) on the torus and double torus. Their method is combinatorial: they perform a case analysis, one case for each dipole with no digons, and determine the coefficients of the expression in Theorem 3.1.3 by enumerating the integer compositions indexing the innermost summation of this expression which correspond to particular choices of p and q. This results in sums of products of binomial coefficients, which, when simplified using Maple, yield the following.

Theorem 3.1.5 (Visentin and Wieler [45])**.** *Suppose $p \leq q$ and $p + q \leq n$. Then the number of (p, q, n)-dipoles on the torus is*

$$\begin{aligned} pq(n-p-q) + \frac{p(p-1)(3q-p-1)}{6} &\qquad \text{if } p+q<n,\\ \binom{p+1}{4} + \binom{q+1}{4} + \frac{p(p-1)(3q-p-1)}{6} &\qquad \text{if } p+q=n. \end{aligned}$$

Theorem 3.1.6 (Visentin and Wieler [45]). *Suppose $p \le q$ and $p+q < n$. Then the number of (p,q,n)-dipoles on the double torus is*

$$\begin{aligned}
&\frac{1}{12}pq(n-p-q)\left(\binom{n-p-1}{2}\left(2\binom{n-p-q}{2}+(q-2)(q+1)\right)\right. \\
&\quad\left. + \binom{n-q-1}{2}\left(2\binom{n-p-q}{2}+(p-2)(p+1)\right)\right) + \sum_{1\le i\le p-1}(p-i)(q-i)\binom{n-i}{4} \\
&\quad + \frac{1}{4}pq\binom{n-p-q+1}{3}(2(p-1)(q-1)-(n-p-q)(n-p-q-2)) \\
&\quad - \binom{p}{3}\left(3n\binom{q}{3}+\frac{1}{280}p^4-\frac{11}{280}p^3+\frac{1}{120}p^3q+\frac{1}{40}p^2-\frac{1}{20}p^2q^2+\frac{1}{10}p^2q\right. \\
&\quad\left. + \frac{17}{20}pq^2-\frac{91}{120}pq-\frac{1}{4}pq^3+\frac{43}{280}p+\frac{7}{30}q^2+\frac{3}{4}q^3-\frac{17}{20}q-\frac{1}{3}q^4+\frac{3}{35}\right).
\end{aligned}$$

Theorem 3.1.7 (Visentin and Wieler [45]). *Suppose $p \le q$ and $p+q = n$. Then the number of (p,q,n)-dipoles on the double torus is*

$$\begin{aligned}
&\frac{1}{336}\binom{p+1}{5}(21p^3-155p^2+338p+456)+\frac{1}{8}\binom{q+1}{6}(3q^2-q-6) \\
&\quad + \frac{1}{2880}pq(p-1)(84p^3q-132p^3-281p^2q+5p^2q^3+10p^2q^2+458p^2-102p \\
&\quad - 210pq^2+209pq+55pq^3+214q+440q^2-692-310q^3+60q^4).
\end{aligned}$$

These formulas will be used in this thesis as a means of verifying the results appearing in Chapters 5, 7 and 8.

3.2 Powers of Jucys-Murphy Elements and G-Factorizations

This section discusses some results pertaining to a seemingly non-central problem, the transitive star factorization problem.

Problem 3.2.1 (Transitive Star Factorization Problem). *Let $\pi \in \mathfrak{S}_n$ and let $r \ge 1$. Determine the number of sequences $(\tau_1, \dots, \tau_r) \in \mathfrak{S}_n^r$ which satisfy*

1. *$\tau_1\tau_2\cdots\tau_r = \pi$,*
2. *each τ_i is a transposition of the form (n, j) for some $1 \le j \le n-1$, and*

3. the group generated by $\{\tau_1, \dots, \tau_r\}$ acts transitively on $\{1, \dots, n\}$.

This problem is typically regarded as encoding a ramified cover of the sphere by a surface of genus g in the manner described in Section 2.2. By the Riemann-Hurwitz formula, $r = n + m - 2 + 2g$, and thus the problem may be regarded as parametrized by either the number of factors or the genus of the covering surface. Based solely on the statement of this problem, it does not appear that the problem is central. Indeed, given the distinguished role played by the symbol n, one would expect that an element of $\mathbb{C}[\mathfrak{S}_n]$ encoding this problem would not commute with permutations for which n is not a fixed point. However, Goulden and Jackson [11] have shown that this problem is in fact central. The history of results pertaining to this problem, and a discussion of the questions raised by Goulden and Jackson's result is given in Section 3.2.1

A problem closely related to Problem 3.2.1 is the following.

Problem 3.2.2 (Non-transitive Star Factorization Problem). *Let $\pi \in \mathfrak{S}_n$ and let $r \geq 1$. Determine the number of sequences $(\tau_1, \dots, \tau_r) \in \mathfrak{S}_n^r$ which satisfy*

1. *$\tau_1\tau_2 \cdots \tau_r = \pi$, and*
2. *each τ_i is a transposition of the form (n, j) for some $1 \leq j \leq n-1$.*

To solve the non-transitive version of the problem, it suffices to compute the r^{th} power of the Jucys-Murphy element J_n. (Consequently, Problem 3.2.1 is referred to as the problem of computing **transitive powers of Jucys-Murphy elements**.) Problem 3.2.2 is relevant to this thesis as an example of a problem which is clearly non-central; indeed, J_n^1 does not commute with permutations which do not fix n. Thus, the transitivity condition is essential to the centrality of Problem 3.2.1. The problem of determining J_n^r has, in principle, been solved. Results of Lascoux and Thibon [30] related to computing power sums of Jucys-Murphy elements may be used to do this. These results are discussed in Section 3.2.2.

Finally, Section 3.2.3 introduces the G-factorization problem, which generalizes the (transitive) star factorization problem.

3.2.1 Transitive Star Factorizations

The problem of enumerating transitive star factorizations was first introduced by Pak [37]. He solved the problem for the special case in which $n = km + 1$ for some k and m, $\pi(n) = n$, π is of cycle type $(k^m, 1)$, and in which the number of factors is minimal. (Minimal factorization correspond to the genus 0 case.) Pak did not explicitly require the factorizations to be transitive, though for permutations of this sort, any factorization into star transpositions is necessarily transitive. Indeed, as long as π has no fixed points other than n, any star factorization is transitive. His result is as follows.

Theorem 3.2.3 (Pak [37]). *Let $n = km + 1$, and $k \geq 2$. Let $\pi \in \mathcal{C}_{(k^m,1)}$ be such that $\pi(n) = n$. The minimal number of factors in a star factorization of π is $(k+1)m$, and the number of minimal star factorizations (and minimal transitive star factorizations) of π is*

$$\frac{k^m(km+m)!}{(km+1)!}.$$

This result was generalized by Irving and Rattan [20] to deal with minimal transitive factorizations of an arbitrary permutation:

Theorem 3.2.4 (Irving and Rattan [20]). *Let $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$, and let $\pi \in \mathcal{C}_\lambda$. The minimal number of factors in a transitive star factorization of π is $n + m - 2$, and the number of minimal transitive star factorizations of π is*

$$\frac{(n+m-2)!}{n!} \prod_{1 \leq i \leq m} \lambda_i.$$

Irving and Rattan also make the observation that the non-transitive version of the problem can be solved by disregarding the fixed points of π and regarding the problem as taking place in the symmetric group on a smaller group of symbols, obtaining the following.

Theorem 3.2.5 (Irving and Rattan [20]). *Let $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$, and let $\pi \in \mathcal{C}_\lambda$ be such that π has exactly k fixed points among the symbols $\{1, \ldots, n-1\}$. The number of minimal star factorizations of π is*

$$\frac{(n+m-2(k+1)))!}{(n-k)!} \prod_{1 \leq i \leq m} \lambda_i.$$

The approaches of both Pak, and of Irving and Rattan are combinatorial. The proofs involve establishing a bijection between the sets of factorizations and certain sets of two-coloured plane-planted trees, and then enumerating these trees. What is particularly interesting about the result of Irving and Rattan is that the solution to the minimal transitive star factorization problem does not treat the length of the cycle containing the distinguished element n differently from the lengths of the other cycles. That is, despite the non-central appearance of the statement of the problem, the solution is in fact central.

The transitive star factorization problem was solved for all genera by Goulden and Jackson [11] in 2007. Let $\mathcal{C}_{\lambda,i}$ denote the set of permutations in $\mathcal{C}_\lambda$ such that the element n appears on a cycle of length i. Let $c_{g,\lambda,i}$ denote the number of transitive factorizations of $\pi \in \mathcal{C}_{\lambda,i}$ into $n + m(\lambda) - 2 + 2g$ star transpositions. The method of Goulden and Jackson is to derive a differential equation for the generating series

$$\Psi := \sum_{\substack{n \geq i \geq 1,\\ m,g \geq 0}} nt^n \frac{u^{n+m-2+2g}}{(n+m-2+2g)!} x^{2g} z_i \sum_{\substack{\lambda \vdash n,\\ m(\lambda)=m}} |\mathcal{C}_{\lambda,i}| c_{g,\lambda,i} y_{\lambda \setminus i},$$

and then to obtain a solution of the equation. In this series, the indeterminates have the following meaning:

- t marks the size of the set on which the symmetric group acts,
- u marks the number of factors,
- x marks twice the genus of the covering surface,
- z_i marks a cycle of length i containing n, and
- y_i marks a cycle of length i not containing n.

The method used to obtain the differential equation for this series is called a **Join-Cut analysis** because it is based on the observation that multiplication of a permutation by a transposition (i, j) has the effect of either joining the cycles containing i and j into one cycle, or splitting a cycle containing both i and j into two smaller cycles. Specifically, multiplying the permutation π by (i, j), the effect on the cycle(s) containing i and j is either

$$(i,j)(i,\pi(i),\ldots,\pi^{-1}(i))(j,\pi(j),\ldots,\pi^{-i}(j)) = (i,\pi(i),\ldots,\pi^{-1}(i),j,\pi(j),\ldots,\pi^{-1}(j)) \tag{3.4}$$

or

$$(i,j)(i,\pi(i),\ldots,\pi^{-1}(j),j,\pi(j),\ldots,\pi^{-1}(i)) = (i,\pi(i),\ldots\pi^{-i}(j))(j,\pi(j),\ldots\pi^{-1}(i)). \tag{3.5}$$

For the particular case of transitive star factorizations, Goulden and Jackson obtain the following.

Theorem 3.2.6 (Goulden and Jackson [11])**.** *Let*

$$\Delta = t\frac{\partial}{\partial t}t\sum_{i\geq 1} z_{i+1}\frac{\partial}{\partial z_i} + \sum_{i,j\geq 1} z_i y_j \frac{\partial}{\partial z_{i+j}} + x^2 \sum_{i,j\geq 1} j z_{i+j}\frac{\partial^2}{\partial z_i \partial y_j}.$$

Then

$$\Delta\Psi = \frac{\partial \Psi}{\partial u}.$$

This equation is obtained by considering how the addition of the final factor in a transitive star factorization affects the various parameters of the factorization. Suppose (k, n) is the final factor in a transitive star factorization. One of the following three cases occurs.

1. The element k does not appear in any of the other factors in the factorization. In this case, the removal of (k,n) results in a non-transitive factorization; however, it may be regarded as a transitive star factorization in the symmetric group on the symbols $\{1,\ldots,n\}\setminus\{k\}$. Multiplying by (k,n) increases the length of the cycle containing n by one, so this case corresponds to the term $t\frac{\partial}{\partial t}t\sum_{i\geq 1} z_{i+1}\frac{\partial}{\partial z_i}$.

2. The element k does appear in one of the other factors, and when it is removed the result is a transitive star factorization in which k and n appear on the same cycle. In this case, multiplication by (k,n) cuts the cycle into two smaller cycles, one containing k and the other containing n. This case corresponds to the term $\sum_{i,j\geq 1} z_i y_j \frac{\partial}{\partial z_{i+j}}$.

3. The element k does appear in one of the other factors, and when it is removed the result is a transitive star factorization in which k and n appear on different cycles. In this case, multiplication by (k,n) joins the cycle into one cycle containing both n and k. This case corresponds to the term $x^2\sum_{i,j\geq 1} j z_{i+j}\frac{\partial^2}{\partial z_i \partial y_j}$.

Goulden and Jackson solve the problem by exhibiting a solution to the equation in Theorem 3.2.6 in terms of the series $\xi(x) = 2x^{-1}\sinh(x/2)$.

Theorem 3.2.7 (Goulden and Jackson [11])**.** *Let*

$$W(\boldsymbol{z}) = \sum_{\ell\geq 1} z_\ell \xi(\ell u x)\xi(ux)^{\ell-2}u^{\ell-1}t^\ell,$$

and let $Z = t\frac{\partial}{\partial t}W(\boldsymbol{z})$, *and* $Y = \xi(ux)^2u^2W(\boldsymbol{y})$. *Then*

$$\Psi = Ze^Y.$$

The coefficients of this series may be expressed in terms of the functions ϕ_λ defined by

$$\phi_\lambda(x) := \frac{x^{n(\lambda)+m(\lambda)-2}\xi(x)^{n(\lambda)-2}}{n(\lambda)!}\prod_{i\geq 1}\xi(\lambda_i x)$$

as follows:

Theorem 3.2.8 (Goulden and Jackson [11])**.** *Let* $\lambda \vdash n$ *be a partition with* m *parts, and let* i *be a part of* λ. *Let* $r = n+m-2+2g$. *Then*

$$c_{g,\lambda,i} = r!\prod_{i\geq 1}\lambda_i [x^r]\phi_\lambda(x) \tag{3.6}$$

$$= \frac{r!}{n!}\prod_{i\geq 1}\lambda_i \sum_{\beta\vdash g}\frac{\xi_{2\beta}q_{2\beta}(\lambda)}{1^{m_1(\beta)}m_1(\beta)!2^{m_2(\beta)}m_2(\beta)!\cdots}, \tag{3.7}$$

where $\xi_j = [x^j]\log(\xi(x))$, q_j *is the symmetric function defined by* $q_j = p_j + p_1 - 2$, *and* 2β *is the partition obtained from* β *by multiplying every part by 2.*

(The function ϕ_λ often appears in the literature scaled by a factor of $|\mathrm{Aut}(\lambda)|^{-1}$; however, for the purpose of the present discussion it is more convenient to omit this factor from the definition.) The formula obtained by Goulden and Jackson is particularly notable because it does not depend on i: in other words, the transitive star factorization problem is in fact a central problem. Goulden and Jackson posed the problem of finding a simple, *a priori* proof of centrality of this problem; that is, a proof that does not rely on explicit evaluation of the coefficients of the generating series. Such a proof is given in Chapter 4. (A second proof was given independently by Féray [5].)

3.2.2 Non-transitive Powers of Jucys-Murphy Elements

A differential operator for the non-transitive star factorization problem may be obtained by a straightforward adjustment of Goulden and Jackson's argument for the transitive case. Let $\hat{c}_{r,\lambda,i}$ denote the number of factorizations of $\pi \in \mathcal{C}_{\lambda,i}$ into r factors, and let

$$\hat{\Psi} := \sum_{\substack{r\geq 0,\\ n\geq 1}} \frac{x^r}{r!} \sum_{\substack{\lambda\vdash n,\\ i\in\lambda}} |\mathcal{C}_{\lambda,i}| \hat{c}_{r,\lambda,i} g_i f_{\lambda\setminus i}$$

be the generating series in which x records the number of factors, g_i records the length of the cycle containing n, and the f-type indeterminates record the lengths of the cycles not containing n. Since the first case in the proof of Theorem 3.2.6 only arises when transitivity is required, this series satisifes the differential equation

$$\frac{\partial \hat{\Psi}}{\partial x} = \hat{\Delta}\hat{\Psi},$$

where

$$\hat{\Delta} := \sum_{i,j\geq 1} g_i f_j \frac{\partial}{\partial g_{i+j}} + \sum_{i,j\geq 1} j g_{i+j} \frac{\partial^2}{\partial g_i \partial f_j}. \tag{3.8}$$

The initial condition for this equation is

$$\hat{\Psi}|_{x=0} = \sum_{n\geq 1} g_1 f_1^{n-1} = \frac{g_1}{1-f_1},$$

since the only factorizations having zero factors are factorizations of the identity. The non-transitive analogue of the star factorization problem is important not only because it is an example of a non-central problem, but also because in Chapter 5 it is shown that the Join-Cut operator corresponding to a special case of the (p,q,n)-dipole problem bears a striking similarity to the operator for non-transitive star factorizations.

The non-transitive star factorization problem may be solved by using a result of Lascoux and Thibon [30] which expresses power sum symmetric functions evaluated at Jucys-Murphy elements in the standard basis for $Z(n)$. The group algebra element encoding the non-transitive star factorization problem is

$$J_n^r = p_r(J_2, \ldots, J_n) - p_r(J_2, \ldots, J_{n-1}).$$

The expansion for power sums of Jucys-Murphy elements in terms of the standard basis for $Z(n)$ is as follows.

Theorem 3.2.9 (Lascoux and Thibon [30])**.** *Let $n \geq 2$ and $r \geq 1$. Then*

$$p_r(J_2, \ldots, J_n) = \sum_{1 \leq k \leq r+1} \sum_{\substack{\lambda \vdash k, \\ m(\lambda) \leq r-k+2}} r! \prod_{i \geq 1} \lambda_i \binom{n-k+m_1(\lambda)}{m_1(\lambda)} [x^r] \phi_\lambda(x) K_{\lambda \cup 1^{n-k}}. \tag{3.9}$$

Lascoux and Thibon prove this result by analyzing a differential operator which corresponds to multiplication by a power sum of Jucys-Murphy elements. (This method differs from analyzing the operator given in (3.8) because the problem of computing $p_r(J_2, \ldots, J_n)$, being central, only requires one set of indeterminates, in contrast to the problem of computing J_n^r *via* differential methods which requires both "f" and "g" indeterminates.) Chapter 7 contains an approach to computing J_n^r that is an alternative to taking the difference of two sums of the form given in Equation (3.9).

3.2.3 The G-factorization problem

A natural generalization of the (transitive) star factorization problem is the following.

Problem 3.2.10 ((Transitive) G-factorization problem)**.** *Let G be a graph on vertex set $\{1, \ldots, n\}$, let $\pi \in \mathfrak{S}_n$, and let $r \geq 1$. Determine the number of sequences $(\tau_1, \ldots, \tau_r) \in \mathfrak{S}_n^r$ which satisfy*

1. *$\tau_1 \tau_2 \cdots \tau_r = \pi$,*
2. *(the group generated by $\{\tau_1, \ldots, \tau_r\}$ acts transitively on $\{1, \ldots, n\}$), and*
3. *each τ_k is a transposition of the form (i, j) where $\{i, j\}$ is an edge of G.*

Taking G to be the graph in which vertex n has degree $n-1$ and every other vertex has degree 1, this gives the (transitive) star factorization problem. There are two other notable special cases of this problem. Setting G to be the complete graph on n vertices

	Complete graph	Star graph	Path
Transitive	Central	Central	Non-central
Non-transitive	Central	Non-central	Non-central

Table 3.2: Centrality / non-centrality of various versions of the G-factorization problem.

gives the problem of factorizations into transpositions which has been well-studied both in its transitive form (see [8], [10], [9] and [12], for example) and its non-transitive form (see [6], [22], [3] and [33], for example). Setting G to be a path gives the problem of enumerating factorizations into adjacent transpositions, which was studied in its non-transitive form by Stanley [41].

It is clear that, in the non-transitive case, the G-factorization problem is non-central unless G is the complete graph. In the transitive case, given the centrality of the transitive star factorization problem, the answer to whether or not the problem is central is less clear. Table 3.2 summarizes the answer to the question of centrality for various special graphs, and for both the transitive and non-transitive cases.

An elementary argument may be used to show that, for many graphs G, the transitive G-factorization problem is non-central. A complete characterization of the graphs G for which the transitive G-factorization is central is not yet known. At present, the only graphs for which this problem is known to be central are the complete graph and the star graph.

Theorem 3.2.11 (S.)**.** *Let G be a connected graph. If the complement of G contains a Hamilton path, then the transitive G-factorization is non-central.*

Proof. It suffices to show that there exists a full-cycle permutation which has a transitive G-factorization into $n-1$ transpositions, and another full-cycle permutation which does not. In other words, the solution to this problem (regarded as an element of $\mathbb{C}[\mathfrak{S}_n]$) is not constant on the class $\mathcal{C}_{(n)}$. Since G is connected, it has a spanning tree T containing $n-1$ edges. Let $\tau_1, \ldots, \tau_{n-1}$ be the transpositions corresponding to the edges of T. Since each transposition in this list is a "join," then the product of this list of transpositions is a full cycle.

Suppose, without loss of generality, that the Hamilton path in the complement visits the vertices in the order 1, 2, 3, ..., n. It suffices to show that the full-cycle permutation $(1, 2, \ldots, n)$ admits no G-factorizations with $n-1$ factors. To prove this, it suffices to show that if $\tau_1, \ldots, \tau_{n-1}$ is a factorization of $(1, 2, \ldots, n)$, then at least one of the τ_k must be of the form $(i, i+1)$ for some $1 \leq i \leq n-1$ — and this edge does not lie in the graph G. This is proven in the following.

Lemma 3.2.12. *Let $n \geq 3$. Suppose $\pi \in \mathcal{C}_{(n)} = \tau_1 \cdots \tau_{n-1}$ is a factorization of a full cycle into transpositions. Then there are at least two transpositions $\tau_a = (i_a, j_a)$ and $\tau_b = (i_b, j_b)$ such that $\pi(i_a) = j_a$ and $\pi(i_b) = j_b$.*

Proof. This result may be proven using induction on n. When $n = 3$, suppose that $\pi = (a, b, c)$. Then the only three factorizations of π into two transpositions are

$$(a, b, c) = (a, b)(b, c),$$
$$(a, b, c) = (c, a)(a, b),$$

and

$$(a, b, c) = (b, c)(c, a),$$

each of which satisfy the conclusion of the lemma.

Next, suppose that $n \geq 4$, and that $\pi = \tau_1 \cdots \tau_{n-1}$ is a factorization into $n-1$ transpositions. Let $\tau_1 = (i, j)$. Every transposition in this factorization must be a "join." In order for τ_1 to be a join, the disjoint cycle representation of $\tau_2 \cdots \tau_r$ must consist of exactly two cycles, one of which contains i, and the other of which contains j. These cycles are denoted by $C_i = (i, c_2, \ldots, c_k)$ and $C_j = (j, d_2, \ldots d_{n-k})$ respectively, with the notational convention that $c_1 = i$ and $d_1 = j$. Then

$$\pi = (i, j)C_iC_j = (i, c_2, \ldots, c_k, j, d_2, \ldots, d_{n-k}). \tag{3.10}$$

None of the transpositions τ_ℓ for $2 \leq \ell \leq n-1$ can be of the form (c_a, d_b), since the existence of a transposition of this form would require some other factor to be a cut because c_a and d_b end up on different cycles of $\tau_2 \cdots \tau_{n-1}$. Thus, without loss of generality, we may assume that

$$C_i = \tau_2\tau_3 \cdots \tau_k$$

and

$$C_j = \tau_{k+1} \cdots \tau_{n-1}.$$

Furthermore, the number of factors in each of these two expressions is one less than the length of the corresponding cycle.

If both C_i and C_j have length greater than 3, then by induction, there is at least one transposition of the form (c_a, c_{a+1}) among the transpositions $\{\tau_2, \ldots, \tau_k\}$, and at least one transposition of the form (d_b, d_{b+1}) among $\{\tau_{k+1}, \ldots, \tau_{n-1}\}$. In light of the expression for π given in Equation (3.10), the result is proven in this case.

To finish the proof, the three remaining cases are examined as follows:

- If one of the cycles, say C_j, has length 1, then C_i must have length at least 3. Applying the inductive hypothesis to C_i yields one of the desired transpositions, and the second is $\tau_1 = (j,i)$ since $\pi = (i,j)(i,c_2,\dots c_{n-1}) = (j,i,c_2,\dots c_{n-1})$.

- If both C_i and C_j are both of length 2, then the factorization of π is of the form

$$\pi = (i,j)(i,c_2)(j,d_2) = (i,c_2,j,d_2),$$

so (i,c_2) and (j,d_2) are the desired transpositions.

- If one of the cycles, say C_j, is of length 2 and C_i has length strictly greater than 2, then

$$\pi = (i,j)(i,c_2,\dots c_{n-2})(j,d_2) = (i,c_2,\dots c_{n-2},j,d_2),$$

so (j,d_2) is one of the desired transpositions; the other is obtained by applying induction to C_i.

□

Applying this Lemma, at least one of the factors in a factorization of $(1,2,\dots,n)$ must be of the form $(i,i+1)$, and since the edge $\{i,i+1\}$ lies in the complement of G, there are no G-factorizations of $(1,2,\dots,n)$. □

This result demonstrates that for many graphs G, the transitive G-factorization problem is a source of properly non-central problems.

3.3 Z_1-factorizations

This section introduces a natural non-central generalization of the Central Factorization Problem (Problem 2.1.2), called the Z_1-factorization problem. The reason for selecting this name will be made apparent in Chapter 4.

Problem 3.3.1 (The Z_1-factorization problem)**.** *Given a permutation $\pi \in \mathfrak{S}_n$, partitions $\lambda,\mu \vdash n$ and parts i,j of λ and μ respectively, determine the number of pairs of permutations (σ_1,σ_2) such that*

1. $\sigma_1\sigma_2 = \pi$,
2. *σ_1 has cycle type λ, with n appearing on a cycle of length i, and*
3. *σ_2 has cycle type μ, with n appearing on a cycle of length j.*

There are two main reasons for interest in this problem. First, as a natural generalization of Problem 2.1.2, it is important as a test of the effectiveness of the non-central techniques developed in this thesis. Specifying the length of the cycle containing n is the simplest non-central generalization of the central factorization problem; therefore, one would expect that the problem would retain enough structure to be interesting. In this respect, the Z_1-factorization problem is the "next step up" from the central factorization problem, and if the methods generalizing central techniques presented in this thesis are to be considered successful, they should at the very least yield an approach to solving this problem that is comparable to central methods. Given that the central version of Problem 3.3.1 is considerably simplified when factorizations of a full cycle are being considered, this is also a natural specialization to consider in the non-central case. Thus, it is natural to ask whether there is a non-central analogue of Theorem 2.4.11. This question is addressed in Chapter 7.

The second reason for interest in an algebraic solution to the Z_1-factorization problem is that it is reasonable to expect that the solution to such a problem will hint at future combinatorial results on permutation factorization. Much like Shaeffer and Vassilieva's combinatorial proof [40] of a special case of Theorem 2.4.8, and Goulden and Jackson's combinatorial proof [7] of a special case of Theorem 2.4.11, it is reasonable to expect that non-central analogues of these combinatorial constructions also exist. Giving an algebraic solution to Problem 3.3.1 (particularly in the case when π is a full cycle) would provide an important hint of what these analogues might be.

Chapter 4

Centralizers of $\mathbb{C}[\mathfrak{S}_n]$

The first step in developing an algebraic approach to the non-central problems described in the preceding chapter is to identify algebras which can play a role analogous to that played by $Z(n)$ in the case of central problems. The algebras which play this role are the centralizer algebras of $\mathbb{C}[\mathfrak{S}_n]$. Section 4.1 uses the problems described in Chapter 3 to motivate this choice, and presents definitions and elementary results pertaining to centralizer algebras. Section 4.2 describes some important properties of these algebras. In particular, it contains a description of how to construct the basis for a centralizer algebra in general, and gives explicit constructions of the bases of two specific centralizers which play a prominent role in this thesis. The problems of Chapter 3 are then encoded as products of these basis elements. Finally, Section 4.3 shows how the language of centralizers can be used to give an *a priori* proof of centrality of the transitive star factorization problem.

4.1 Motivation and Definition

To motivate the choice of centralizers as the algebraic context for approaching non-central problems, recall that Problem 3.3.1 can be solved by evaluating products of elements of the form

$$\sum_{\substack{\sigma \in \mathcal{C}_\lambda, \\ n \text{ is on a cycle of } \sigma \text{ of length } i}} \sigma. \tag{4.1}$$

From a combinatorial point of view, given any permutation $\sigma \in \mathcal{C}_\lambda$ such that n appears on a cycle of length i, any relabelling of its functional digraph that does not relabel the element n will preserve both the cycle type and the length of the cycle containing n. In algebraic terms, if $\pi \in \mathfrak{S}_n$ is such that $\pi(n) = n$, then the expression (4.1) is invariant under conjugation by π. Of course, the set of permutations fixing n form a subgroup of

$\mathfrak{S}_n$, namely, $\mathfrak{S}_{n-1}$. Consequently, it is natural to consider the subset of $\mathbb{C}[\mathfrak{S}_n]$ consisting of elements invariant under conjugation by a specified subgroup:

Definition 4.1.1 (Centralizer)**.** Let $\mathcal{T}$ be a finite set, and let H be a subgroup of $\mathfrak{S}_{\mathcal{T}}$. The **centralizer of $\mathbb{C}[\mathfrak{S}_{\mathcal{T}}]$ with respect to** H is the set

$$Z_H(\mathcal{T}) := \{g \in \mathbb{C}[\mathfrak{S}_{\mathcal{T}}] : \sigma g \sigma^{-1} = g \text{ for all } \sigma \in H\}.$$

As a notational shorthand, if $\mathcal{T} = \{1, \ldots, n\}$, then $Z_H(\mathcal{T})$ is denoted by $Z_H(n)$. If, in addition, $H = \mathfrak{S}_{\{1,\ldots,k\}}$ for some $k \leq n$, then $Z_H(\mathcal{T})$ is denoted by $Z_{n-k}(n)$. The following observations follow immediately from the definition of $Z_H(\mathcal{T})$ and are thus stated without proof.

Lemma 4.1.2. *1. $Z_H(\mathcal{T})$ is a subalgebra of $\mathbb{C}[\mathfrak{S}_{\mathcal{T}}]$.*

2. $Z_0(n)$ is the centre of $\mathbb{C}[\mathfrak{S}_n]$.

3. $Z_0(n) \subseteq Z_1(n) \subseteq \cdots \subseteq Z_n(n) = \mathbb{C}[\mathfrak{S}_n]$.

The study of this subalgebra is motivated by the observation that the the problems of Chapter 3 may be encoded by elements of $Z_H(\mathcal{T})$ for appropriate choices of $\mathcal{T}$ and H.

Example 4.1.3 ((p, q, n)-dipoles)**.** Recall that in order to solve the (p, q, n)-dipole problem, it suffices to find an expression for the element

$$g := \sum_{\substack{\pi_1 \in \mathcal{C}_{(n)}, \\ \pi_1^q(n) = n-1}} \sum_{\substack{\pi_2 \in \mathcal{C}_{(n)}, \\ \pi_1^p(n) = n-1}} \pi_1 \pi_2.$$

For any $\sigma \in \mathfrak{S}_{\{1,\ldots,n-2\}}$, note that

$$\sigma g \sigma^{-1} = \sum_{\substack{\pi_1 \in \mathcal{C}_{(n)}, \\ \pi_1^q(n) = n-1}} \sum_{\substack{\pi_2 \in \mathcal{C}_{(n)}, \\ \pi_1^p(n) = n-1}} \sigma \pi_1 \sigma^{-1} \sigma \pi_2 \sigma^{-1}.$$

Since $(\sigma \pi_2 \sigma^{-1})^p = \sigma \pi_2^p \sigma^{-1}$ and both n and $n-1$ are fixed points of σ, then $(\sigma \pi_2 \sigma^{-1})^p(n) = n-1$ if and only if $\pi_2^p(n) = n-1$. Similarly, $(\sigma \pi_1 \sigma^{-1})^q(n) = n-1$ if and only if $\pi_1^q(n) = n-1$, so $\sigma g \sigma^{-1} = g$. Consequently, $g \in Z_2(n)$.

Example 4.1.4 (G-factorizations)**.** Let G be a graph with vertex set $\{1, \ldots, n\}$. To solve the G-factorization problem, it suffices to find an expression for

$$g := \sum_{\substack{\tau_i=(u,v),\\ \{u,v\}\in E(G)}} \tau_1 \cdots \tau_r.$$

Let $\sigma \in \mathrm{Aut}(G)$. Since $\sigma(u,v)\sigma^{-1} = (\sigma(u), \sigma(v))$, and $\{u,v\} \in E(G)$ if and only if $\{\sigma(u), \sigma(v)\} \in E(G)$, then $\sigma g \sigma^{-1} = g$. Consequently, $g \in Z_{\mathrm{Aut}(G)}(n)$.

Example 4.1.5 (Powers of Jucys-Murphy elements)**.** As an important special case of Example 4.1.4, take G to be the graph in which vertex n has degree $n-1$, and every other vertex has degree 1. Then the element g is the r^{th} power of the Jucys-Murphy element J_n. Since $\mathrm{Aut}(G) = \mathfrak{S}_{n-1}$, then $J_n^r \in Z_1(n)$.

In addition to the combinatorial applications listed above, centralizers of $\mathbb{C}[\mathfrak{S}_n]$ are of interest because of the role they play in an alternative derivation of the irreducible representations of the symmetric group due to Okounkov and Vershik [35]. In their approach, centralizers are used to prove the Branching Rule and the Murnaghan-Nakayama rule.

4.2 Properties of Centralizers

Having identified an appropriate algebraic context for studying the problems of Chapter 3, the next step is to study the elementary properties of this algebra. The first task is to identify a natural linear basis for $Z_H(n)$. This basis is natural in the sense that the problems of Chapter 3 have a simple encoding as products of basis elements. Following the construction of this basis is a description of some combinatorially natural transformations relating various centralizers of the form $Z_{\mathfrak{S}_\alpha}(n)$.

4.2.1 A Basis for $Z_H(n)$

Recall that sums over $\mathfrak{S}_n$-conjugacy classes form a basis for the centre of the group algebra. This idea may be generalized as follows. Let $\star$ denote the action of conjugation by H on $\mathfrak{S}_n$, *i.e.* $\star$ is defined by $h \star \pi = h\pi h^{-1}$ for $h \in H, \pi \in \mathfrak{S}_n$. Let $\mathcal{O}_1, \ldots, \mathcal{O}_m$ denote the orbits of $\mathfrak{S}_n$ with respect to this action. Define the elements Ω_i by

$$\Omega_i := \sum_{\pi \in \mathcal{O}_i} \pi$$

for $1 \leq i \leq m$. A useful alternative expression for Ω_i is

$$\Omega_i = \frac{|\mathcal{O}_i|}{|H|} \sum_{h \in H} h \star \pi$$

where π is a fixed element of $\mathcal{O}_i$. (The notation $\mathcal{O}_\pi$ and Ω_π will be used to denote the orbit and group algebra sum, respectively, corresponding to the orbit containing π.) First, observe that for any $h \in H$,

$$h\Omega_i h^{-1} = \sum_{\pi \in \mathcal{O}_i} h\pi h^{-1} = \sum_{\pi \in \mathcal{O}_i} h \star \pi = \sum_{\pi \in \mathcal{O}_i} \pi = \Omega_i,$$

so that $\Omega_i \in Z_H(n)$. Furthermore, since the orbits partition $\mathfrak{S}_n$, the set $\{\Omega_i\}_{1 \leq i \leq m}$ is linearly independent.

To show that $\{\Omega_i\}_{1 \leq i \leq m}$ spans $Z_H(n)$, it suffices to show that for any $g \in Z_H(n)$, if π_1 and π_2 are in the same orbit, then $[\pi_1]g = [\pi_2]g$. Let $h \in H$ be such that $\pi_2 = h \star \pi_1$. Write g as

$$g = \sum_{\pi \in \mathfrak{S}_n} g_\pi \pi.$$

Since $hgh^{-1} = g$, then

$$[\pi_2]g = [\pi_2]hgh^{-1} = [\pi_2] \sum_{\pi \in \mathfrak{S}_n} g_\pi h \star \pi = g_{\pi_1} = [\pi_1]g.$$

This result is summarized in the following.

Lemma 4.2.1. *Let H be a subgroup of $\mathfrak{S}_n$. Let $\mathcal{O}_1, \ldots, \mathcal{O}_m$ be the orbits of $\mathfrak{S}_n$ with respect to the action of H given by $h \star \pi = h\pi h^{-1}$. For $1 \leq i \leq m$, let*

$$\Omega_i = \sum_{\pi \in \mathcal{O}_i} \pi.$$

Then the set $\{\Omega_i\}_{1 \leq i \leq m}$ is a linear basis for $Z_H(n)$.

The set $\{\Omega_i\}_{1 \leq i \leq m}$ will be referred to as the **standard basis** for $Z_H(n)$. By taking $H = \mathfrak{S}_n$, this result specializes to the known basis for the centre of $\mathbb{C}[\mathfrak{S}_n]$, since in this case, the orbits of $\mathfrak{S}_n$ with respect to the action $\star$ are conjugacy classes. Since many of the problems of interest lie in $Z_1(n)$ and $Z_2(n)$, the bases for these algebras will now be described more explicitly.

Standard basis for $Z_1(n)$

For $Z_1(n)$, the orbits of $\mathfrak{S}_n$ with respect to $\mathfrak{S}_{n-1}$ are of the following form.

Definition 4.2.2. Let $\lambda \vdash n$, and let i be a part of λ. Define $\mathcal{C}_{\lambda,i}$ by

$$\mathcal{C}_{\lambda,i} := \{\sigma \in \mathcal{C}_\lambda : n \text{ is on a cycle of length } i\}.$$

Consequently, the basis elements of $Z_1(n)$ are of the following form.

Definition 4.2.3. Let $\lambda \vdash n$ and let i be a part of λ. Let

$$K_{\lambda,i} := \sum_{\pi \in \mathcal{C}_{\lambda,i}} \pi.$$

By Lemma 4.2.1,

Lemma 4.2.4. *The set $\{K_{\lambda,i}\}_{\lambda \vdash n, i \in \lambda}$ is a basis for $Z_1(n)$.*

Notably, this basis for $Z_1(n)$ contains the Jucys-Murphy element J_n; specifically, $K_{(2,1^{n-2}),2} = J_n$; hence, the non-transitive star factorization problem may be encoded in $Z_1(n)$ as $K^r_{(2,1^{n-2}),2}$. Furthermore, Problem 3.3.1 may be naturally encoded in $Z_1(n)$ as well. Let $[K_{\lambda,i}]G$ denote the coefficient of $K_{\lambda,i}$ in the element $G \in Z_1(n)$. Then the following statement is true.

Corollary 4.2.5. *Let $\lambda, \mu, \nu \vdash n$, and let i, j and k be parts of λ, μ and ν respectively. If $\pi \in \mathcal{C}_{\nu,k}$, then the solution to Problem 3.3.1 is given by*

$$[\pi] K_{\lambda,i} K_{\mu,j} = [K_{\nu,k}] K_{\lambda,i} K_{\mu,j}.$$

A further consequence of Lemma 4.2.4 is that it leads to an elementary proof that $Z_1(n)$ is commutative. Define the operator $\mathrm{Inv} : \mathbb{C}[\mathfrak{S}_n] \to \mathbb{C}[\mathfrak{S}_n]$ by $\mathrm{Inv}(\pi) = \pi^{-1}$ for $\pi \in \mathfrak{S}_n$, extending linearly to $\mathbb{C}[\mathfrak{S}_n]$. If $G = \sum_{\pi \in \mathfrak{S}_n} G_\pi \pi$ and $H = \sum_{\sigma \in \mathfrak{S}_n} H_\sigma \sigma$ are arbitrary elements of $\mathbb{C}[\mathfrak{S}_n]$, then

$$\begin{aligned}
\mathrm{Inv}(GH) &= \sum_{\pi \in \mathfrak{S}_n} \sum_{\sigma \in \mathfrak{S}_n} G_\pi H_\sigma \mathrm{Inv}(\pi\sigma) \\
&= \sum_{\pi \in \mathfrak{S}_n} \sum_{\sigma \in \mathfrak{S}_n} G_\pi H_\sigma \sigma^{-1} \pi^{-1} \\
&= \sum_{\sigma \in \mathfrak{S}_n} H_\sigma \sigma^{-1} \sum_{\pi \in \mathfrak{S}_n} G_\pi \pi^{-1} \\
&= \mathrm{Inv}(H)\mathrm{Inv}(G).
\end{aligned}$$

Considering the action of Inv on the standard basis for $Z_1(n)$ leads to the following.

Lemma 4.2.6. *For any $G \in Z_1(n)$, $\mathrm{Inv}(G) = G$. Consequently, $Z_1(n)$ is commutative.*

Proof. Taking the inverse of a permutation does not change its cycle type, nor does it change the length of the cycle containing n, so $\mathrm{Inv}(K_{\lambda,i}) = K_{\lambda,i}$. Extending linearly to $Z_1(n)$, we find that $\mathrm{Inv}(G) = G$ for all $G \in Z_1(n)$. Thus, if $G, H \in Z_1(n)$,

$$GH = \mathrm{Inv}(GH) = \mathrm{Inv}(H)\mathrm{Inv}(G) = HG.$$

$\square$

Commutativity of $Z_1(n)$ is important, because it permits the construction of a basis of orthogonal idempotents from which the connection coefficients for $Z_1(n)$ may be determined. This is done in Chapter 6.

It will, of course, prove useful to know the size of each $\mathfrak{S}_{n-1}$-orbit as well.

Lemma 4.2.7.

$$|\mathcal{C}_{\lambda,i}| = \frac{(n-1)! i m_i(\lambda)}{|\mathrm{Aut}(\lambda)|}.$$

Proof. For each element in $\mathcal{C}_{\lambda,i}$, there are $\binom{n-1}{i-1}$ choices for the elements on the cycle containing n other than n, and $(i-1)!$ ways to order them on the cycle. There are $|\mathcal{C}_{\lambda\backslash i}|$ ways to choose the remaining cycles, so the total number of elements in $\mathcal{C}_{\lambda,i}$ is

$$\binom{n-1}{i-1}(i-1)!|\mathcal{C}_{\lambda\backslash i}| = \frac{(n-1)! i m_i(\lambda)}{|\mathrm{Aut}(\lambda)|}.$$

$\square$

Standard basis for $Z_2(n)$

The orbits of $\mathfrak{S}_n$ with respect to conjugation by $\mathfrak{S}_{n-2}$ may be categorized into one of the two types, depending on whether or not n and $n-1$ lie on the same cycle. The two types of basis elements are defined as follows.

Definition 4.2.8. Let $\lambda \vdash n$, let i be a part of λ, and let $1 \leq j < i$. Define the set

$$\mathcal{A}_\lambda(i,j) := \{\sigma \in \mathcal{C}_\lambda : n \text{ and } n-1 \text{ are on the same cycle of length } i, \sigma^j(n) = n-1\},$$

and let

$$A_\lambda(i,j) := \sum_{\sigma \in \mathcal{A}_\lambda(i,j)} \sigma,$$

with the notational convention that $A_\lambda(i,j) = 0$ if i is not a part of λ, or if $j \geq i$.

Definition 4.2.9. Let $\lambda \vdash n$, and let i and j be parts of λ such that either $i \neq j$ or $m_i(\lambda) \geq 2$. Define the set

$\mathcal{B}_\lambda(i,j) := \{\sigma \in \mathcal{C}_\lambda : n$ is on a cycle of length i and $n-1$ is on a different cycle of length $j\}$

and let

$$B_\lambda(i,j) = \sum_{\sigma \in B_\lambda(i,j)} \sigma,$$

with the convention that $B_\lambda(i,j) = 0$ if i and j do not satisfy the condition stated above.

With these definitions, the following Lemma is true.

Lemma 4.2.10. *The set* $\{A_\lambda(i,j)\}_{\lambda \vdash n, i \in \lambda, 1 \leq j < i} \cup \{B_\lambda(i,j)\}_{\lambda \vdash n; i,j \in \ \lambda}$ *is a basis for* $Z_2(n)$.

Bases for $Z_k(n)$ for $k \geq 3$ may be defined in a similar manner (see [35]), though centralizers corresponding to larger values of k are not used in this thesis.

With this notation in hand, it is now possible to express the solution to Problem 3.1.1 as a product of standard basis elements for $Z_2(n)$.

Corollary 4.2.11. *Let* $\pi \in \mathfrak{S}_n$*, and let* $1 \leq p, q \leq n-1$*. Then the number of labelled* (p,q,n)*-dipoles with face permutation* π *is given by*

$$[\pi] A_{(n)}(n,q) A_{(n)}(n,p).$$

It is worth noting that $Z_2(n)$ is not commutative. For example,

$$\begin{aligned} A_{(5)}(5,2)A_{(2,1^3)}(2,1) &= [(51423)+(51432)+(52413)+(52431)+(53412)+(53421)](45) \\ &= (14)(235)+(14)(253)+(24)(135)+(24)(153)+(34)(125)+(34)(152) \\ &= B_{(3,2)}(3,2), \end{aligned}$$

but

$$\begin{aligned} A_{(2,1^3)}(2,1)A_{(5)}(5,2) &= (45)[((51423)+(51432)+(52413)+(52431)+(53412)+(53421)] \\ &= (234)(15)+(243)(15)+(134)(25)+(143)(25)+(124)(35)+(142)(35) \\ &= B_{(3,2)}(2,3). \end{aligned}$$

The fact that $Z_2(n)$ is not commutative means that it does not have a basis of orthogonal idempotents, so one of the main techniques used to study the connection coefficients of the centre does not have an analogue in $Z_2(n)$. The effect of interchanging the order of multiplication in $Z_2(n)$ can be determined by considering he action of Inv on the standard basis, as follows.

Lemma 4.2.12. $\mathrm{Inv}(A_\lambda(i,j)) = A_\lambda(i, i-j)$ *and* $\mathrm{Inv}(B_\lambda(i,j)) = B_\lambda(i,j)$.

Proof. To invert a permutation in disjoint cycle notation, reverse the cycles. Consequently, if $\pi \in \mathcal{B}_\lambda(i,j)$ then $\pi^{-1} \in \mathcal{B}_\lambda(i,j)$, and if $\pi \in \mathcal{A}_\lambda(i,j)$ then $\pi^{-1} \in \mathcal{A}_\lambda(i,i-j)$, from which the result follows. $\square$

Lemma 4.2.13. *The following relations among the basis elements of $Z_2(n)$ hold.*

1. $A_\lambda(i,j)A_\mu(i',j') = \mathrm{Inv}(A_\mu(i',i'-j')A_\lambda(i,i-j))$,
2. $A_\lambda(i,j)B_\mu(i',j') = \mathrm{Inv}(B_\mu(i',j')A_\lambda(i,i-j))$, *and*
3. $B_\lambda(i,j)B_\mu(i',j') = \mathrm{Inv}(B_\mu(i',j')B_\lambda(i,j))$.

Proof. For the first relation, we have

$$\begin{aligned} A_\lambda(i,j)A_\mu(i',j') &= \mathrm{Inv}\mathrm{Inv}(A_\lambda(i,j)A_\mu(i',j')) \\ &= \mathrm{Inv}(\mathrm{Inv}(A_\mu(i',j'))\mathrm{Inv}(A_\lambda(i,j))) \\ &= \mathrm{Inv}(A_\mu(i',i'-j')A_\lambda(i,i-j)). \end{aligned}$$

The other relations are proven similarly. $\square$

The size of the orbits of $\mathfrak{S}_n$ with respect to conjugation by $\mathfrak{S}_{\{1,\ldots,n-2\}}$ may be obtained in a manner similar to the proof of Lemma 4.2.7, so a proof has been omitted.

Lemma 4.2.14.

$$|\mathcal{A}_\lambda(i,j)| = \frac{(n-2)!}{\mathrm{Aut}(\lambda \setminus i)}$$

and

$$|\mathcal{B}_\lambda(i,j)| = \frac{(n-2)!}{\mathrm{Aut}(\lambda \setminus i \setminus j)}.$$

4.2.2 Relationships between different centralizers

This section discusses some of the natural relationships between centralizers of various types. Although many results in this section may seem intuitively obvious, detailed proofs are given because having precise definitions of the operators involved can be used to give an *a priori* proof of centrality for the transitive star factorization problem. For any $\alpha \subseteq \{1,\ldots,n\}$ such that $|\alpha| = k$, there is a natural isomorphism between $Z_{\mathfrak{S}_\alpha}(n)$ and $Z_{n-k}(n)$ obtained by "relabelling" the elements of the set α to $\{1,\ldots,k\}$. Consequently, in order to study centralizers with respect to subgroups of the form $\mathfrak{S}_\alpha$, it suffices to study $Z_{n-k}(n)$.

To make this notion more precise, suppose the elements of α are $\alpha_1 < \alpha_2 < \cdots < \alpha_k$, and that the elements of $\{1, \ldots, n\} \setminus \alpha$ are $\beta_1 < \beta_2 < \cdots < \beta_{n-k}$. Given a permutation $\pi \in \mathfrak{S}_{n-k}$, define the permutation $\sigma_{\alpha,\pi}$ in one-line notation by

$$\sigma_{\alpha,\pi} = \alpha_1 \alpha_2 \ldots \alpha_k \beta_{\pi(1)} \cdots \beta_{\pi(n-k)},$$

and let $T_{\alpha,\pi} : \mathbb{C}[\mathfrak{S}_n] \to \mathbb{C}[\mathfrak{S}_n]$ be the linear transformation defined by

$$T_{\alpha,\pi}(\rho) = \sigma_{\alpha,\pi} \rho \sigma_{\alpha,\pi}^{-1}$$

for $\rho \in \mathfrak{S}_n$. This operator preserves group multiplication, and it is clear that this is invertible (indeed, $T_{\alpha,\pi}^{-1}(\rho) = \sigma_{\alpha,\pi}^{-1} \rho \sigma_{\alpha,\pi}$), so $T_{\alpha,\pi}$ is an algebra automorphism of $\mathbb{C}[\mathfrak{S}_n]$.

To show that $Z_{\mathfrak{S}_\alpha}(n)$ is isomorphic to $Z_{n-k}(n)$, it suffices to show that the image of $Z_{n-k}(n)$ under any $T_{\alpha,\pi}$ is $Z_{\mathfrak{S}_\alpha}(n)$, and that the image of $Z_{\mathfrak{S}_\alpha}(n)$ under $T_{\alpha,\pi}^{-1}$ is $Z_{n-k}(n)$. Each basis element of $Z_{n-k}(n)$ is of the form

$$\Omega = \sum_{\rho \in \mathfrak{S}_k} \rho \rho_0 \rho^{-1}$$

for some permutation ρ_0. Then

$$\begin{aligned} T_{\alpha,\pi}\Omega &= \sum_{\rho \in \mathfrak{S}_k} \sigma_{\alpha,\pi} \rho \rho_0 \rho^{-1} \sigma_{\alpha,\pi}^{-1} \\ &= \sum_{\rho \in \mathfrak{S}_k} \left(\sigma_{\alpha,\pi} \rho \sigma_{\alpha,\pi}^{-1} \right) \left(\sigma_{\alpha,\pi} \rho_0 \sigma_{\alpha,\pi}^{-1} \right) \left(\sigma_{\alpha,\pi} \rho \sigma_{\alpha,\pi}^{-1} \right)^{-1} \end{aligned}$$

Let $\rho' = \sigma_{\alpha,\pi} \rho \sigma_{\alpha,\pi}^{-1}$. Since ρ fixes $\{k+1, \ldots, n\}$ if and only if ρ' fixes $\{1, \ldots, n\} \setminus \alpha$, then

$$T_{\alpha,\pi}\Omega = \sum_{\rho' \in \mathfrak{S}_\alpha} \rho' \left(\sigma_{\alpha,\pi} \rho_0 \sigma_{\alpha,\pi}^{-1} \right) \rho'^{-1},$$

which is a basis element of $Z_{\mathfrak{S}_\alpha}(n)$. Showing the reverse inclusion is similar, thus the following result holds.

Lemma 4.2.15. *Let $1 \le k \le n$ and let $\alpha \subseteq \{1, \ldots, n\}$, with $k = |\alpha|$. Then*

$$T_{\alpha,\pi} : Z_{n-k}(n) \to Z_{\mathfrak{S}_\alpha}(n)$$

is an algebra isomorphism for any $\pi \in \mathfrak{S}_{n-|\alpha|}$.

Since $T_{\alpha,\pi}$ is an isomorphism for any choice of π, it is useful to also consider the transformation obtained by "averaging" over all choices of π, namely, the linear transformation

$$T_\alpha := \frac{1}{(n - |\alpha|)!} \sum_{\pi \in \mathfrak{S}_{n-|\alpha|}} T_{\alpha,\pi}.$$

This leads to the following result.

Corollary 4.2.16. *Let $\alpha \subseteq \{1, \ldots, n\}$, with $|\alpha| = k$. Then*

$$T_\alpha : Z_{n-k}(n) \to Z_{\mathfrak{S}_\alpha}(n)$$

is a vector space isomorphism.

It should be noted that when $|\alpha| > k$, the operators $T_{\alpha,\pi}$ and T_α are still well-defined, although they are not isomorphisms.

Recall that centralizers with respect to a given subgroup H are naturally included in the centralizers with respect to any subgroup of H. In particular,

$$Z(n) \subset Z_1(n) \subset Z_2(n) \subset \cdots Z_0(n) = \mathbb{C}[\mathfrak{S}_n].$$

Any $\mathfrak{S}_n$-orbit can be partitioned into $\mathfrak{S}_k$-orbits for any $k \leq n$ — thus, an element of $Z(n)$ can be regarded as an element of $Z_{n-k}(n)$ which is the sum of the $\mathfrak{S}_k$-orbits forming said partition. Conversely, there is a natural linear transformation from $Z_{n-k}(n)$ to $Z(n)$ obtained by sending the basis element of $Z_{n-k}(n)$ corresponding to the $\mathfrak{S}_k$-orbit $\mathcal{O}$ to the basis element of $Z(n)$ corresponding to the $\mathfrak{S}_n$-orbit containing $\mathcal{O}$. This transformation can be described in terms of the transformations T_α as follows.

Suppose that

$$\Omega_\pi = \frac{|\mathcal{O}_\pi|}{k!} \sum_{\sigma \in \mathfrak{S}_k} \sigma \pi \sigma^{-1}$$

is a basis element of $Z_{n-k}(n)$. Under the transformation described in the preceding paragraph, if π has cycle type λ, this element maps to

$$K_\lambda = \frac{|\mathcal{C}_\lambda|}{n!} \sum_{\rho \in \mathfrak{S}_n} \rho \pi \rho^{-1}$$

in $Z(n)$. A standard result from bijective combinatorics associates a permutation $\rho \in \mathfrak{S}_n$ with a triple (α, ρ_1, ρ_2) where α is the k-element subset of $\{1, \ldots, n\}$ obtained by applying ρ to $\{1, \ldots, k\}$, ρ_1 is the permutation of $\mathfrak{S}_k$ which gives the ordering of α in the first k positions of the one-line notation for ρ, and ρ_2 gives the ordering of $\{1, \ldots, n\} \setminus \alpha$ among the last $n-k$ positions of the one-line notation for ρ. Under this bijection, $\rho = \sigma_{\alpha,\rho_2}\rho_1$, so that

$$\begin{aligned}
K_\lambda &= \frac{|\mathcal{C}_\lambda|}{n!} \sum_{\rho \in \mathfrak{S}_n} \rho\pi\rho^{-1} \\
&= \frac{|\mathcal{C}_\lambda|}{n!} \sum_{\substack{\alpha \subset \{1,\dots,n\}, \\ |\alpha| = k}} \sum_{\rho_1 \in \mathfrak{S}_k} \sum_{\rho_2 \in \mathfrak{S}_{n-k}} \sigma_{\alpha,\rho_2} \rho_1 \pi \rho_1^{-1} \sigma_{\alpha,\rho_2}^{-1} \\
&= \frac{|\mathcal{C}_\lambda|}{n!} \sum_{\substack{\alpha \subset \{1,\dots,n\}, \\ |\alpha| = k}} \sum_{\rho_2 \in \mathfrak{S}_{n-k}} T_{\alpha,\rho_2} \left(\sum_{\rho_1 \in \mathfrak{S}_k} \rho_1 \pi \rho_1^{-1} \right) \\
&= \frac{|\mathcal{C}_\lambda|(n-k)!k!}{n!|\mathcal{O}_\pi|} \sum_{\substack{\alpha \subset \{1,\dots,n\}, \\ |\alpha| = k}} T_\alpha(\Omega_\pi).
\end{aligned}$$

Consequently, defining the operator T_k by

$$T_k = \sum_{\substack{\alpha \subset \{1,\dots,n\} \\ |\alpha| = k}} T_\alpha,$$

permits the following statement relating the centralizers with respect to $\mathfrak{S}_{n-k}$ to the centre of $\mathbb{C}[\mathfrak{S}_n]$.

Lemma 4.2.17. *Let $k \leq n$, and let $g \in Z_{n-k}(n)$. Then $T_k(g) \in Z(n)$. In particular, if $\mathcal{O}_\pi$ is the $\mathfrak{S}_{n-k}$-orbit containing π and $\Omega_\pi = \sum_{\sigma \in \mathcal{O}_\pi} \sigma$, then*

$$T_k(\Omega_\pi) = \binom{n}{k} \frac{|\mathcal{O}_\pi|}{|\mathcal{C}_\lambda|} K_\lambda,$$

where λ is the cycle type of π.

One special case of this Lemma is worth additional attention. If $\lambda \vdash k$ for some $k \leq n$, then the element K_λ of the centre $Z(k)$ may also be regarded as an element of the centralizer $Z_{n-k}(n)$. Applying the preceding lemma to K_λ yields the following.

Corollary 4.2.18. *Let $\lambda \vdash k \leq n$. Then*

$$T_k(K_\lambda) = \frac{(m_1(\lambda) + n - k)!}{(n-k)!m_1(\lambda)!} K_{\lambda 1^{n-k}}.$$

4.3 Centrality of the Transitive Star Factorization Problem

The elementary results regarding centralizer algebras which were developed in the preceding two sections may be used to provide an *a priori* proof of the centrality of the transitive star factorization problem. This proof uses a more general definition of the Jucys-Murphy elements than that given in Section 2.3. Let $\alpha \subseteq \{1, \ldots, n-1\}$. Define

$$J_n(\alpha) = \sum_{i \in \alpha} (i, n).$$

In this notation, $J_n = J_n(\{1, \ldots, n-1\})$. The proof begins with the observation, appearing in Goulden and Jackson's paper [11], that the principle of Inclusion-Exclusion may be used to obtain the following expression for transitive powers of Jucys-Murphy elements:

$$\sum_{\alpha \in \{1 \ldots, n-1\}} (-1)^{|\alpha|} J_n(\{1, \ldots, n-1\} \setminus \alpha)^r. \tag{4.2}$$

Recall that a factorization into star transpositions is transitive if and only if it involves every star transposition at least once. The element $J_n(\{1, \ldots, n-1\} \setminus \alpha)^r$ gives all factorizations which fail to include every (in) for $i \in \alpha$; thus, Expression (4.2) corresponds to factorizations which include every star transposition. These are precisely the transitive factorizations. By a change of index, Expression (4.2) can also be written as

$$\sum_{\alpha \in \{1 \ldots, n-1\}} (-1)^{n-1-|\alpha|} J_n(\alpha)^r. \tag{4.3}$$

The element $J_n(\alpha)$ lies in $Z_{\mathfrak{S}_\alpha}(n)$; thus, so does $J_n(\alpha)^r$. By Corollary 4.2.16, if $|\alpha| = k$, $J_n(\alpha)^r$ can be regarded as the image under T_α of an element of $Z_{n-k}(n)$. Indeed, for any $\pi \in \mathfrak{S}_k$, $J_n(\alpha) = T_{\alpha \cup \{n\}, \pi}(J_{k+1})$, so (4.3) becomes

$$\sum_{1 \le k \le n-1} \sum_{\substack{\alpha \in \{1 \ldots, n-1\}, \\ |\alpha| = k}} (-1)^{n-1-k} T_{\alpha \cup \{n\}} J_{k+1}^r.$$

However, J_{k+1}^r may be expressed as the difference of two power sum symmetric functions in the Jucys-Murphy elements. Hence, this expression becomes

$$\sum_{1 \le k \le n-1} \sum_{\substack{\alpha \in \{1 \ldots, n-1\} \\ |\alpha| = k}} (-1)^{n-1-k} T_{\alpha \cup \{n\}} \left(p_r(J_2, \ldots, J_{k+1}) - p_r(J_2, \ldots, J_k)\right),$$

adopting the notational convention that when $k=1$, $p_r(J_2,\dots,J_k)=0$. Rearranging this sum gives

$$T_{\{1,\dots,n\}}(p_r(J_2,\dots,J_n)) \\ + \sum_{1\le k\le n-2}(-1)^{n-1-k}\left(\sum_{\substack{\alpha\subseteq\{1,\dots,n-1\}\\|\alpha|=k}} T_{\alpha\cup\{n\}} + \sum_{\substack{\beta\subseteq\{1,\dots,n-1\}\\|\beta|=k+1}} T_{\beta\cup\{n\}}\right) p_r(J_2,\dots,J_{k+1}).$$

Since $p_r(J_2,\dots,J_{k+1})\in Z(k+1)$, it is an element of $Z_{n-k-1}(n)$. Thus, if $|\beta|=k+1$, then $T_{\beta\cup\{n\}}p_r(J_2,\dots,J_{k+1}) = T_\beta p_r(J_2,\dots,J_{k+1})$. Indeed, if $\beta=\{\beta_1<\beta_2<\cdots<\beta_{k+1}\}$, then for any $i\le k+1$,

$$T_\beta J_i = J_{\beta_i}(\{\beta_1,\dots,\beta_{i-1}\}) = T_{\beta\cup\{n\}}J_i.$$

Consequently, the expression for transitive star factorizations becomes

$$\sum_{1\le k\le n-1}(-1)^{n-1-k}T_{k+1}(p_r(J_2,\dots,J_{k+1})).$$

By Lemma 4.2.17, this expression lies in the centre of $\mathbb{C}[\mathfrak{S}_n]$.

A second *a priori* proof of centrality of the transitive star factorization problem was given independently by Féray [5] using the semigroup algebra of partial permutations introduced by Ivanov and Kerov [27]. This algebra is spanned by pairs of the form (π, A) where A is a set of positive integers and π is a permutation of A. Multiplication in this algebra is done according to the rule

$$(\pi_1, A_1)(\pi_2, A_2) = (\pi_1\pi_2, A_1\cup A_2).$$

Feray defines analogues of the Jucys-Murphy elements in this algebra, called partial Jucys-Murphy elements, and shows that symmetric functions in these elements are invariant under the group action

$$\tau\star(\pi, A) = (\tau\pi\tau^{-1}, \tau(A)).$$

Centrality is then deduced by applying the transformation $(\pi, A)\mapsto\pi$. An advantage of Féray's approach over the one presented in this chapter is that it allows the formula for the number of transitive star factorizations to be deduced from Lascoux and Thibon's [30] expression (see Theorem 3.2.9) for power sums of Jucys-Murphy elements.

Chapter 5

Differential Operators for the (p, q, n)-dipole problem

This chapter begins the development of techniques to solve the non-central problems described in Chapter 3. It is an account of how a Join-Cut argument may be used to determine a partial differential equation whose solution, in turn, can be used to determine the generating series for the (p, q, n)-dipole problem. The approach developed here is a result of the synthesis of two ideas. The first, due to Kwak and Shim [29], is a Join-Cut approach to the ordinary dipole problem; this is described in Section 5.1. The second idea, originating in Goulden and Jackson's approach to enumerating transitive star factorizations as described in Section 3.2, is to record non-central information *via* the introduction of an additional set of indeterminates.

Section 5.2 gives the differential equation corresponding to a specific case of the (p, q, n)-dipole problem when q is fixed to be $n - 1$. The form of this differential equation suggests that, although the general (p, q, n)-dipole problem is encoded as an element of $Z_2(n)$, the special case of $(p, n-1, n)$-dipoles may be encoded in $Z_1(n)$. This observation is then given an algebraic explanation in addition to the combinatorial one by providing an explicit encoding of the problem in $Z_1(n)$.

Section 5.3 is an account of a generalization of the approach used in Section 5.2 to deal with the (p, q, n)-dipole problem in general. This section introduces a refinement of the problem, namely, the (a, b, c, d)-dipole problem, and describes a pair of differential equations. The first equation determines the generating series for $(a, b, 0, 0)$-dipoles, which is used as the initial condition for the second equation, which then determines the generating series for the (a, b, c, d)-dipole problem.

5.1 The differential approach to ordinary dipoles

The technique for analyzing (p, q, n)-dipoles used in this chapter is based on a technique used by Kwak and Shim [29] to analyze ordinary dipoles in locally orientable surfaces. This section contains an overview of their method when restricted to to the orientable case. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$, recall that $f_\lambda = \prod_{1 \le i \le m} f_{\lambda_i}$. Let $\hat{\mathcal{D}}_n$ denote the set of labelled dipoles on n edges, and define the polynomial D_n by

$$D_n = \sum_{D \in \hat{\mathcal{D}}_n} u^{2g(D)} f_{\lambda(D)}.$$

(Recall that $\lambda(D)$ is the half-face-degree sequence of D.) Kwak and Shim relate the polynomial D_n to the polynomial D_{n-1} by observing that every dipole with n edges may be obtained by adding an edge to a dipole with $n-1$ edges, possibly with the addition of a "handle" to the underlying surface.

Suppose D is a dipole with $n-1$ edges, with half-face-degree sequence λ. If an edge is to be added to D, its position is determined by selecting one of the $n-1$ corners incident with the root vertex, and one of the $n-1$ corners incident with the non-root vertex. There are two cases to consider. If these two corners are part of the same face (having degree j) then the new edge lies entirely within that face and cuts it into two smaller faces, say of degree i and $j - i + 1$ for some $1 \le i \le j$. There are $m_j(\lambda)$ choices for the face of half-degree j, and j choices for the corner to which the one end of the edge is added. (The location of the other end is then determined by the choice of i.) The two faces may be distinguished, for example, by identifying the face of degree i as the face encountered when moving counterclockwise from the new edge around the root vertex. Thus, edges which cut a face in two correspond to the linear operator

$$C_{\text{dipoles}} : f_\lambda \mapsto \sum_{j \ge 1} m_j(\lambda) j \frac{f_\lambda}{f_j} \sum_{1 \le i \le j} f_i f_{j-i+1}.$$

The second case occurs if the two corners are on different faces, say of degrees i and j. Adding an edge whose ends lie in two different faces necessitates the addition of an extra handle to the surface in which the dipole is embedded, increasing the genus by two. The two faces are then joined to form a larger face of half-degree $i + j + 1$. If i and j are different, there are $m_i(\lambda) m_j(\lambda)$ choices for the two faces, and ij choices for the position of the ends of the new edge. If $i = j$, the condition that the faces be distinct means there are $m_i(\lambda)(m_i(\lambda) - 1)$ choices for the two faces, and i^2 choices for the position of the ends of the new edge. Thus, edges which join two faces correspond to the linear operator

$$J_{\text{dipoles}} : f_\lambda \mapsto \sum_{\substack{i,j \ge 1, \\ i \ne j}} m_i(\lambda) m_j(\lambda) ij \frac{f_\lambda f_{i+j+1}}{f_i f_j} + \sum_{i \ge 1} m_i(\lambda)(m_i(\lambda) - 1) i^2 \frac{f_\lambda f_{2i+1}}{f_i^2}.$$

In this notation,

$$D_n = (C_{\text{dipoles}} + u^2 J_{\text{dipoles}})D_{n-1}$$

when $n \geq 2$. Starting with the initial condition $D_1 = f_1$, Kwak and Shim give explicit expressions for D_n for small values of n by repeatedly applying the transformation $C_{\text{dipoles}} + J_{\text{dipoles}}$.

While Kwak and Shim regarded C and J as algebraic substitutions, these operations are more conveniently expressed as differential operators. The cut operator can be expressed as

$$C_{\text{dipoles}} = \sum_{j \geq 1} \left(\sum_{1 \leq i \leq j} f_i f_{j-i+1} \right) j \frac{\partial}{\partial f_j},$$

and the join operator can be expressed as

$$J_{\text{dipoles}} = \sum_{i,j \geq 1} ij f_{i+j+1} \frac{\partial^2}{\partial f_i \partial f_j}.$$

The formal power series $D = \sum_{n \geq 0} D_{n+1} \frac{x^n}{n!}$ then satisfies the partial differential equation

$$\frac{\partial D}{\partial x} = (C_{\text{dipoles}} + u^2 J_{\text{dipoles}})D,$$

which is obtained by multiplying the equation $(C_{\text{dipoles}} + u^2 J_{\text{dipoles}})D_{n-1} = D_n$ by $\frac{x^{n-2}}{(n-2)!}$ and summing over $n \geq 2$. The initial condition is $D|_{x=0} = D_1$.

5.2 Differential Operators for the $(p, n-1, n)$-dipole problem

The Join-Cut analysis of ordinary dipoles may be extended to deal with (p, q, n)-dipoles by introducing indeterminates to keep track of extra "non-central" information. It is necessary to consider how these indeterminates are affected by the addition of an edge to a dipole. This section concentrates on a special case, in which q is fixed at $n-1$. By the symmetry of the problem, this is equivalent to enumerating $(n-p, 1, n)$-dipoles (which is the version of the problem which satisfies Visentin and Wieler's [45] condition that $p + q \leq n$), though it is more convenient for the present purpose to adopt the convention that $q = n - 1$. An alternative perspective on $(p, n-1, n)$-dipoles is that they are, in the language introduced in Chapter 3, the set of all dipoles having no edges with an end in Region 4. (The diagram illustrating the definition of the various regions has been copied to this chapter as Figure 5.1 for the reader's convenience.)

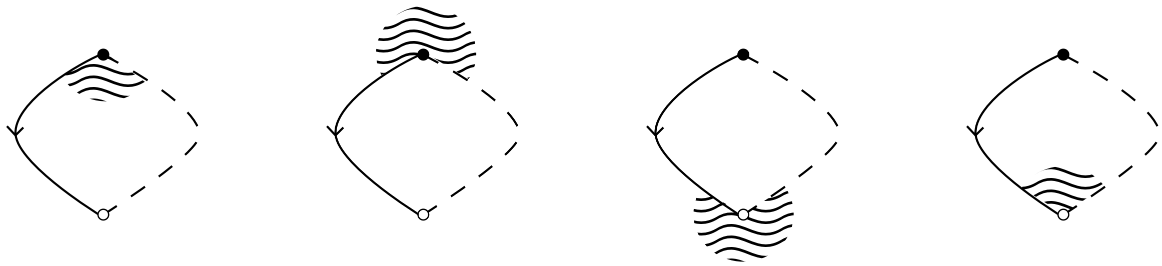


Figure 5.1: Notation for the regions into which the neighbourhoods of the vertices of a rooted dipole with a second distinguished edge are divided.

The **root face** of a rooted dipole is the face encountered when moving counterclockwise from the root edge around the root vertex, and the **root corner** is the corner of this face which is incident with both the root edge and root vertex. Let $r(D)$ denote half the degree of the root face of the rooted dipole D. The case of $(p, n-1, n)$-dipoles is important for two reasons. First, it provides useful clues about how to proceed with the more general analysis. Second, the form of the differential equation for $(p, n-1, n)$-dipoles suggests that the problem is "less non-central" than it initially appears.

Let

$$\Psi = \sum_{\substack{D \in \mathcal{D}, \\ q(D)=n(D)-1}} \frac{y^{p(D)-1}}{(p(D)-1)!} \frac{x^{n(D)-p(D)}}{(n(D)-p(D))!} u^{2g(D)} g_{r(D)} f_{\lambda(D)\setminus r(D)}$$

be the generating series for all $(p, n-1, n)$-dipoles in which g_i marks a root face of degree $2i$, and f_i marks a non-root face of degree $2i$. Let

$$\Psi_p = (p-1)![y^{p-1}]\Psi$$

be the generating series corresponding to a specified value of p. Deleting the edge of a $(p, n-1, n)$-dipole which is nearest to the root edge in a counterclockwise circulation of the root vertex results in a $(p-1, n-2, n-1)$-dipole, possibly in a surface of lower genus. Thus, all $(p, n-1, n)$-dipoles may be obtained by adding a single edge to a $(p-1, n-2, n-1)$-dipole such that one end of the new edge lies in the root corner. The other end of the new edge must lie in Region 3, for otherwise the result would be a $(p, n-2, n)$-dipole. Thus, in order to determine Ψ_p from Ψ_{p-1}, it suffices to classify the dipoles resulting from this edge addition according to the degree of their root face and the degree sequence of the non-root faces.

As in the case with ordinary dipoles, two cases must be considered. Suppose that a $(p-1, n-2, n-1)$-dipole is marked with the monomial $g_i f_\lambda$. The end of the new edge which is affixed to the root corner is necessarily on the root face, marked by g_i. Suppose that the other end of the new edge is also affixed to a corner of the root face. In this case, the root face is cut into two smaller faces — a replacement root face of half-degree $j+1$, and a new non-root face of half-degree $i-j$ for some $1 \le j \le i-1$. (The case in which the replacement root face has half-face-degree 1 is absent because the root face could only be a digon if the new edge had one end in Region 4, which is forbidden by the restriction that $q = n-1$.) Hence, this case corresponds to the cut operator

$$C := \sum_{i \ge 2} \sum_{1 \le j \le i-1} g_{j+1} f_{i-j} \frac{\partial}{\partial g_i} = \sum_{i \ge 1} \sum_{j \ge 1} g_{i+1} f_j \frac{\partial}{\partial g_{i+j}}.$$

The second case to consider is the one in which the ends of the new edge are affixed to a corner on the root face (say, of half-degree i) and a non-root face (say, of half-degree j). The location of the end affixed to the root face is determined by the restriction that the new edge be added at the root corner. On the other hand, there are j choices of a corner to which to affix the other end. The addition of the new edge necessitates the addition of a handle to the underlying surface, and joins these two faces into one larger root face, of half-degree $i+j+1$. Thus, this case corresponds to the join operator

$$J := \sum_{i \ge 1} \sum_{j \ge 1} j g_{i+j+1} \frac{\partial^2}{\partial g_i \partial f_j}.$$

Let $\Delta = C + u^2 J$. Then

$$\Psi_p = \Delta \Psi_{p-1} \tag{5.1}$$

when $p \ge 2$. The $p = 1$ case forces the root face to have half-degree 1, so that

$$\Psi_1 = \Psi|_{y=0} = \sum_{\substack{D \in \mathcal{D}, \\ q(D)=n(D)-1, \\ p(D)=1}} \frac{x^{n(D)-1}}{(n(D)-1)!} u^{2g(D)} g_1 f_{\lambda(D) \setminus 1}.$$

Since the root face may be contracted to a single edge, which may then be designated as a root edge, this series becomes

$$\Psi_1 = \Psi|_{y=0} = g_1 \sum_{D \in \mathcal{D}} \frac{x^{n(D)}}{n(D)!} u^{2g(D)} f_{\lambda(D)},$$

the series for ordinary dipoles, which was determined in Chapter 2 using central methods (see Lemma 2.4.9). Multiplying Equation (5.1) by $y^{p-2}/(p-2)!$, summing over $p \ge 2$, and using the fact that $\partial \Psi_1 / \partial y = 0$ yields the following.

Theorem 5.2.1 (S.)**.** *The generating series for $(p, n-1, n)$-dipoles is the unique solution to the partial differential equation*

$$\Delta\Psi = \frac{\partial\Psi}{\partial y},$$

with initial condition

$$\Psi|_{y=0} = g_1 L\left((1+y)^{-1}\left(\exp\left(\sum_{i\geq 1}\frac{x^i u^{i-1}}{i} f_i(1-(-y)^i)\right)-1\right)\right).$$

(Here, and in proofs of other similar results, uniqueness follows from the fact that the coefficients of Ψ are completely determined from the initial condition and the recurrence of Equation (5.1).)

An unexpected consequence of this result it that the recurrence given in Equation (5.1) relies on only one piece of non-central information — the degree of the face containing the corner labelled with n. In other words, despite initially appearing to lie in $Z_2(n)$, Theorem 5.2.1 suggests that the $(p, n-1, n)$-dipole problem is closer to being central than it seems, and that there should be a method to solve it using $Z_1(n)$. This observation is significant, because the commutative algebra $Z_1(n)$ is substantially easier to work with than the non-commutative $Z_2(n)$. Of course, although Theorem 5.2.1 suggests that there exists a $Z_1(n)$ encoding for the $(p, n-1, n)$-dipole problem, it does not provide such an encoding explicitly.

A $Z_1(n)$-encoding of the $(p, n-1, n)$-dipole problem can be found by revisiting the proof of Lemma 3.1.2. The strategy is to encode a dipole not by its pair of vertex permutations, but rather by the vertex permutation ν corresponding to the non-root vertex and the face permutation ρ, isolating those pairs for which $\nu\rho^{-1}$ gives a root vertex permutation corresponding to a specified value of p. In this approach, it is notationally more convenient to consider dipoles in which the ordinary edges are not labelled, and which can therefore be given a canonical labelling. Specify a canonical cycle C_p for the vertex permutation at the root vertex with the property that $C_p^p(n) = n-1$, say

$$C_p = (n, 1, \ldots, p-1, n-1, p, \ldots, n-2).$$

Using the encoding of a dipole as a pair of vertex permutations, the number of $(p, n-1, n)$-dipoles with face permutation π is given, in the notation of Definition 4.2.8, by

$$[\pi]A_{(n)}(n, n-1)C_p.$$

What makes this expression "close" to being in $Z_1(n)$ is the fact that

$$A_{(n)}(n, n-1) = \sum_{\substack{\sigma\in\mathcal{C}_{(n)},\\ \sigma^{-1}(n)=n-1}} \sigma \;=\; \sum_{\substack{\sigma\in\mathcal{C}_{(n-1,n)},\\ \sigma(n)=n}} \sigma(n, n-1) = K_{(n-1,1),1}(n, n-1).$$

The presence of the permutation C_p is still a barrier to doing this computation in $Z_1(n)$, but this can be rectified *via* the observation that

$$[\pi]A_{(n)}(n, n-1)C_p = [C_p^{-1}]\pi^{-1}A_{(n)}(n, n-1).$$

(This is the algebraic statement corresponding to the combinatorial observation that encoding dipoles as a face/vertex permutation pair is equivalent to encoding them as a pair of vertex permutations.) Thus, if $S \subset \mathfrak{S}_n$, then the number of $(p, n-1, n)$-dipoles whose face permutation is an element of S is given by

$$\sum_{\pi \in S}[\pi]A_{(n)}(n, n-1)C_p = \sum_{\pi \in S}[C_p^{-1}]\pi^{-1}A_{(n)}(n, n-1) = [C_p^{-1}]\left(\sum_{\pi \in S}\pi^{-1}\right)K_{(n-1,1),1}(n, n-1).$$

Thus, whenever the set S is invariant under conjugation by $\mathfrak{S}_{n-1}$, the product

$$\left(\sum_{\pi \in S}\pi^{-1}\right)K_{(n-1,1),1}$$

may be computed within $Z_1(n)$, instead of $Z_2(n)$.

It remains to determine the effect on $Z_1(n)$ basis elements of multiplication by the transposition $(n, n-1)$ and extraction of the coefficient of C_p^{-1}. Since every basis element of $Z_1(n)$ can be expressed as a sum of basis elements in $Z_2(n)$,

$$[C_p^{-1}]K_{\lambda,i}(n, n-1) = [C_p^{-1}]\sum_{1 \le j \le i-1} A_\lambda(i, j)(n, n-1) + [C_p^{-1}]\sum_{j \in \lambda \setminus i} B_\lambda(i, j)(n, n-1).$$

If n and $n-1$ are on the same cycle, multiplication by $(n, n-1)$ cuts this cycle in two. In particular, no full cycles can appear in the product $A_\lambda(i, j)(n, n-1)$, so the first sum in this expression is zero. If n and $n-1$ are on different cycles, multiplication by $(n, n-1)$ will join them into one cycle, *i.e.*

$$B_\lambda(i, j)(n, n-1) = A_{\lambda \setminus \{i,j\} \cup (i+j)}(i+j, j).$$

Thus,

$$\begin{aligned}
[C_p^{-1}]K_{\lambda,i}(n, n-1) &= [C_p^{-1}]A_{\lambda \setminus \{i,j\} \cup (i+j)}(i+j, j) \\
&= \begin{cases} 1 & \text{if } \lambda = (p, n-p) \text{ and } i = p, \\ 0 & \text{otherwise} \end{cases} \\
&= [K_{(p,n-p),p}]K_{\lambda,i}.
\end{aligned}$$

Combining these facts gives the following encoding for the $(p, n-1, n)$-dipole problem in $Z_1(n)$.

Lemma 5.2.2 (S.). *Let $\lambda \vdash n$, and let i be a part of λ. Then the number of $(p, n-1, n)$-dipoles (with unlabelled ordinary edges) having face degree sequence 2λ in which the root face has degree $2i$ is given by*

$$[K_{(p,n-p),p}]K_{\lambda,i}K_{(n-1,1),1}.$$

The existence of this encoding for the $(p, n-1, n)$-dipole problem indicates that a more thorough study of $Z_1(n)$ is warranted, in order to develop the tools necessary to analyze the product $K_{(n-1,1),1}K_{\lambda,i}$. These tools are developed in Chapter 6, and in Chapter 7 they are used to solve the $(p, n-1, n)$-dipole problem for all orientable surfaces.

Although the $Z_1(n)$ encoding of the $(p, n-1, n)$-dipole problem given in Lemma 5.2.2 is the most useful one for solving the problem algebraically, there are some questions which it does not answer. Typically, Join-Cut behaviour is observed when iterated multiplication by transpositions is involved, and Lemma 5.2.2 does not involve transpositions. The resemblance between the operator $C+u^2J$ for the $(p, n-1, n)$-dipole problem and the operator given in Equation (3.8) for the non-transitive star factorization problem suggests that there should be an encoding for the $(p, n-1, n)$-dipole problem involving Jucys-Murphy elements. The relationship between Jucys-Murphy elements and the $(p, n-1, n)$-dipole problem may be identified by determining an algebraic recursion for the $(p, n-1, n)$-dipole problem, which can be used to give a second proof of Theorem 5.2.1, as sketched below. The verification of each of the following facts is routine.

- The element $A_{(n)}(n, n-1)$ may be obtained from $A_{(n-1)}(n-1, n-2)$ by relabelling $n-2$ to $n-1$, relabelling $n-1$ to n, and then multiplying by the Jucys-Murphy element J_{n-2}. Thus,

  $$A_{(n)}(n, n-1) = J_{n-2}R_n(A_{(n-1)}(n-1, n-2))$$

  where R_n is the linear relabelling operator defined by

  $$R_n(\pi) = (n-2, n-1, n)\pi(n, n-1, n-2).$$

- The element $A_{(n)}(n, p)$ may be obtained from $A_{(n-1)}(n-1, p-1)$ by relabelling according to R_n, multiplying on the right by $(n-2, n)$ to insert the symbol $n-2$ immediately after n on the cycle. This results in an element of $Z_3(n)$. Thus, summing over relabellings of $n-2$ to j for $1 \le j \le n-2$ gives an element of $Z_2(n)$:

  $$A_{(n)}(n, p) = T\left(R_n(A_{(n-1)}(n-1, p-1))(n-2, n)\right),$$

  where T is the linear operator defined by

  $$T(\pi) = \sum_{1 \le j \le n-2} (j, n-2)\pi(j, n-2).$$

- Since $A_{(n)}(n, n-1) \in Z_2(n)$, then

$$\begin{aligned} &A_{(n)}(n, n-1)A_{(n)}(n, p) \\ &\quad = T(A_{(n)}(n, n-1)R_n(A_{(n-1)}(n-1, p-1))(n-2, n)) \\ &\quad = T(J_{n-2}R_n(A_{(n-1)}(n-1, n-2)A_{(n-1)}(n-1, p-1))(n-2, n)), \end{aligned}$$

  which expresses the algebraic encoding for $(p, n-1, n)$-dipoles in terms of the algebraic encoding for $(p-1, n-2, n-1)$-dipoles.

- Equation (5.1) can then be obtained by applying the transformation

$$\Phi_{1,n} : \pi \mapsto g_{r(\pi)} f_{\ell(\pi) \setminus r(\pi)},$$

  where $r(\pi)$ is the length of the cycle of π containing n, to this recursion. A second proof of Theorem 5.2.1 can now be obtained by using the fact that $\Phi_{1,n}T = \Phi_{1,n}$ since the conjugations done by T do not change the length of the cycle containing n, and making an argument similar to the one for non-transitive star factorizations.

5.3 Differential Operators for the (p, q, n)-dipole problem

5.3.1 A refinement of the problem: (a, b, c, d)-dipoles

In order to enumerate (p, q, n)-dipoles, it will be helpful to consider a refinement of the problem which classifies the non-root, non-distinguished edges of a dipole D into one of four possible types, defined as follows.

- An a-**edge** is an edge whose ends lie in Regions 2 and 3.
- A b-**edge** is an edge whose ends lie in Regions 1 and 3.
- A c-**edge** is an edge whose ends lie in Regions 2 and 4.
- A d-**edge** is an edge whose ends lie in Regions 1 and 4.

These definitions are illustrated in Figure 5.3.1. Let $a(D), b(D), c(D)$ and $d(D)$ denote the number of a, b, c and d-edges of D, respectively. The p and q values of a dipole can be recovered from this information *via* the observation that

$$p(D) = b(D) + d(D) + 1$$

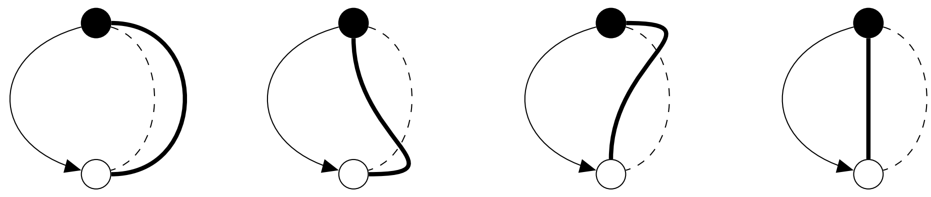


Figure 5.2: Classification of the ordinary edges, indicated by thickened lines, of a rooted dipole with a second distinguished edge.

and

$$q(D) = a(D) + b(D) + 1.$$

The total number of edges in a dipole is

$$n(D) = a(D) + b(D) + c(D) + d(D) + 2.$$

To enumerate dipoles with respect to the number of a, b, c and d edges, label the corners of the dipole in Region 1 with a black dot, $\bullet$, and the corners in Region 2 with a white dot, $\circ$. This associates a unique binary string $R(D)$ to the root face, namely, the string encountered during a counterclockwise boundary walk of the root face, starting at the root corner. Given an element $S \in \{\bullet, \circ\}^*$, let (S) denote the multiset of cyclic shifts of S. For example,

$$(\circ\bullet\circ\bullet) = (\bullet\circ\bullet\circ) = \{2\circ\bullet\circ\bullet, 2\bullet\circ\bullet\circ\}.$$

Such a set shall be referred to as a **cyclic binary string**. Let $\mathcal{S}(\bullet, \circ)$ denote the set of all cyclic binary strings on the symbols $\bullet$ and $\circ$. Each non-root face of a dipole may be associated with the cyclic binary string encountered during a counterclockwise boundary walk of the face. Let $\Lambda(D)$ denote the multiset of cyclic binary strings corresponding to the non-root faces of D.

Consider two infinite sets of indeterminates: $\{g_S\}_{S \in \{\bullet,\circ\}^*}$, indexed by binary strings, and $\{f_{(S)}\}_{(S) \in \mathcal{S}(\bullet,\circ)}$, indexed by cyclic binary strings. Let $f_{\Lambda(D)} = \prod_{(S) \in \Lambda(D)} f_{(S)}$. The generating series for (a, b, c, d)-dipoles is

$$\Psi'' := \sum_{D \in \mathcal{D}} \frac{x^{a(D)+1}}{(a(D)+1)!} \frac{y^{b(D)}}{b(D)!} \frac{v^{c(D)+d(D)}}{(c(D)+d(D))!} w^{d(D)} u^{2g(D)} g_{R(D)} f_{\Lambda(D)}.$$

The number of (p, q, n)-dipoles may be recovered from this series as a sum of coefficients. Since $c + d = n - q - 1$, $p = b + d + 1$, and $a = q - b - 1$, the sum over all values of a, b, c and d corresponding to fixed p and q is

$$\sum_{0 \le b \le p-1} (n-q-1)!b!(q-b)![v^{n-q-1} y^b w^{p-1-b} x^{q-b}]\Psi''.$$

5.3.2 Differential equations for the (a, b, c, d)-dipole problem

The strategy for analyzing the generating series Ψ'' is as follows.

1. By considering the deletion of edges having an end in Region 4, obtain a differential equation for Ψ'' whose initial condition is the series Ψ' for $(a, b, 0, 0)$-dipoles, namely,

$$\Psi' = \sum_{\substack{D\in\mathcal{D},\\ c(D)=d(D)=0}} \frac{x^{a(D)+1}}{(a(D)+1)!}\frac{y^{b(D)}}{b(D)!}u^{2g(D)}g_{R(D)}f_{\Lambda(D)}.$$

2. By considering the deletion of b-edges, obtain a differential equation for the $(a, b, 0, 0)$-series Ψ' whose initial condition is the series for $(a, 0, 0, 0)$-dipoles.

3. Since $(a, 0, 0, 0)$-dipoles can be regarded as a regular rooted dipoles with a "doubled" edge, this series may be computed using central methods.

The partial differential equation which determines Ψ' may be obtained by an argument similar to that of the proof of Theorem 5.2.1. The resulting equation is as follows.

Theorem 5.3.1 (S.)**.** *The generating series Ψ' for $(a, b, 0, 0)$-dipoles is the unique solution to the partial differential equation*

$$(C' + u^2 J')\Psi' = \frac{\partial \Psi'}{\partial y},$$

where

$$C' = \sum_{R\in\{\circ,\bullet\}^*}\left(\sum_{2\le i\le \ell(R)} g_{R_1\cdots R_i} f_{(R_1 R_{i+1}\cdots R_{\ell(R)})}\right)\frac{\partial}{\partial g_R}$$

and

$$J' = \sum_{R\in\{\circ,\bullet\}^*}\sum_{(S)\in\mathcal{S}(\circ,\bullet)}\left(\sum_{S\in(S)} g_{RR_1S}\right)\frac{\partial^2}{\partial g_R \partial f_{(S)}},$$

with initial condition

$$\Psi'|_{y=0} = g_\bullet \sum_{D\in\mathcal{D}} \frac{x^{n(D)}}{n(D)!}u^{2g(D)} f_{\lambda'(D)}$$

where $\lambda'(D) = (\circ^{\lambda_1}, \circ^{\lambda_2}, \ldots, \circ^{\lambda_{m(\lambda)}})$ *when D has face degree sequence* $(2\lambda_1, 2\lambda_2, \ldots, 2\lambda_{m(\lambda)})$.

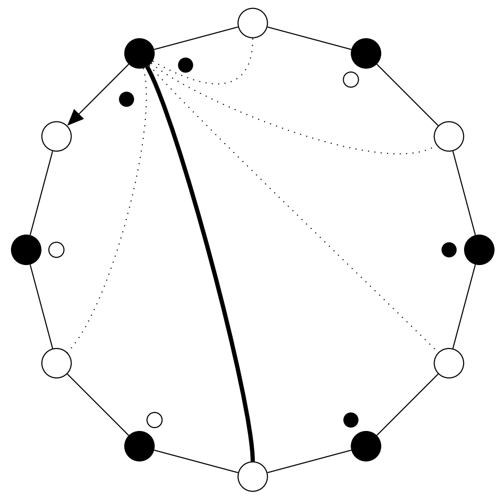


Figure 5.3: A new b-edge, indicated by the thickened edge, is added to an $(a, b, 0, 0)$-dipole in a way that cuts the root face, marked by the indeterminate $g_{\bullet\circ\circ\bullet\bullet\circ}$ into two faces, one of which is a new root face marked by $g_{\bullet\circ\circ}$, and the other of which is a non-root face marked by $f_{(\bullet\bullet\circ\bullet)}$. The dotted lines indicate other possible valid choices for the new b-edge.

Proof. Let $\Psi'_b = b![y^b]\Psi'$. As with the proof of Theorem 5.2.1, it suffices to show that

$$\Psi'_b = (C' + u^2 J')\Psi'_{b-1}$$

for $b \geq 1$. The set of all $(a, b, 0, 0)$-dipoles is generated uniquely from the set of $(a, b-1, 0, 0)$-dipoles by adding a b-edge e such that one end of e is affixed to the root vertex at the root corner of the dipole. Suppose that D is a $(a, b-1, 0, 0)$-dipole encoded by the monomial

$$u^{2g} g_R \prod_i f_{(S^{(i)})},$$

where $R = R_1 \ldots R_{\ell(R)}$. The operators C' and J' may be obtained by considering how this monomial changes when e is added to D. The analysis splits into two cases depending on whether the non-root end of e is added to a corner of the root face, or to a corner of a non-root face.

In the first case, the non-root end of e is attached to a corner of the root face. Thus, the root face is cut into a smaller root face and a non-root face, and the genus is unchanged. (This case is illustrated in Figure 5.3.) The edge e also divides the root corner into two corners each labelled with the symbol R_1, one of which is the new root corner, and the other of which is a new non-root corner. To determine the monomial which encodes the resulting $(a, b, 0, 0)$ dipole, consider the symbols encountered on a counterclockwise boundary tour of the root face starting at the root corner. The non-root end of e was added to a corner immediately following R_i for some $2 \leq i \leq \ell(R)$. (It cannot be added to the corner following R_1, since otherwise it would be a d-edge.) Thus, on a boundary tour of the root face, once the symbol R_i is encountered, the edge e completes the boundary tour, so the root face of the resulting dipole is encoded by $g_{R_1 \cdots R_i}$. To determine the encoding of the new non-root face, consider a counterclockwise boundary tour starting at the newly-created non-root corner, marked by R_1. The first edge encountered is e, and the next symbol encountered will be R_{i+1}, after which the boundary tour visits the remaining corners of the former root face, ending when the newly-created non-root corner is reached. Hence, this face is encoded by $f_{(R_1 R_i \cdots R_{\ell(R)})}$. The differential operator corresponding to replacing g_R with $g_{R_1 \cdots R_i} f_{(R_1 R_i \cdots R_{\ell(R)})}$ for some $2 \leq i \leq \ell(R)$ is

$$C' = \sum_{R \in \{\circ, \bullet\}^*} \left(\sum_{2 \leq i \leq \ell(R)} g_{R_1 \cdots R_i} f_{(R_1 R_{i+1} \cdots R_{\ell(R)})} \right) \frac{\partial}{\partial g_R}.$$

In the second case, illustrated in Figure 5.4, the non-root end of e is attached to a corner of a non-root face, marked by $f_{(S)}$, where $S = S_1 \cdots S_{\ell(S)}$ is some fixed representative element of (S). (For any given (S), the number of choices for a face with (S) as the sequence of symbols encountered on a counterclockwise boundary walk is equal to the degree of $f_{(S)}$ in $\prod_i f_{(S^{(i)})}$.) To join e to corners of two different faces, it is necessary to add a handle to the surface, increasing genus by 1. Adding the edge e joins the two faces marked by g_R and $f_{(S)}$ into a larger root face. Consider a counterclockwise boundary tour of this new face, starting at the root corner. Since the non-root end of e was added to a non-root face, this tour will first visit corners marked by $R_1, R_2, \ldots, R_{\ell(R)}$ before visiting the newly-created non-root corner, marked by R_1, followed by the edge e. The boundary walk then continues around the face marked by $f_{(S)}$, with the sequence of symbols encountered given by some element of the set (S). In terms of the fixed representative S, the sequence is $S_i \cdots S_{\ell(S)} S_1 \cdots S_{i-1}$ for some $1 \leq i \leq \ell(S)$, followed by the edge e which returns to the root corner, ending the boundary tour. Thus, the new root face is encoded by

$$g_{R R_1 S_i \cdots S_{\ell(S)} S_1 \cdots S_{i-1}}.$$

Summing over all cyclic shifts of S gives the following operator corresponding to this case.

$$J' = \sum_{R \in \{\circ, \bullet\}^*} \sum_{(S) \in \mathcal{S}(\circ, \bullet)} \left(\sum_{S \in (S)} g_{R R_1 S} \right) \frac{\partial^2}{\partial g_R \partial f_{(S)}},$$

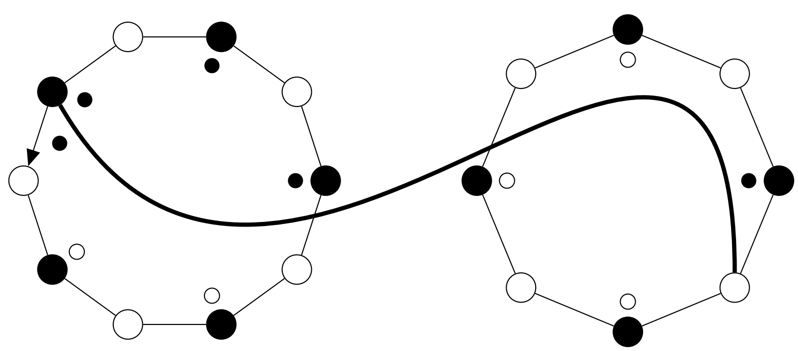


Figure 5.4: A new b-edge, indicated by the thickened line, is added to an $(a, b, 0, 0)$-dipole such that the root face, marked by $g_{\bullet\circ\circ\bullet\bullet}$, is joined to a non-root face, marked by $f_{(\circ\circ\circ\bullet)}$. The resulting face is marked by $g_{\bullet\circ\circ\bullet\bullet\bullet\bullet\circ\circ\circ}$, and this operation requires the addition of a handle to the surface in which the dipole is embedded.

As for the initial condition,

$$\begin{aligned}\Psi'_{y=0} &= \sum_{\substack{D\in\mathcal{D},\\ b(D)=c(D)=d(D)=0}} \frac{x^{a(D)+1}}{(a(D)+1)!} u^{2g(D)} g_\bullet f_{\Lambda(D)} \\ &= g_\bullet \sum_{D\in\mathcal{D}} \frac{x^{n(D)}}{n(D)!} u^{2g(D)} f_{\lambda'(D)},\end{aligned}$$

since contracting the root face of a $(a,0,0,0)$ dipole results in an ordinary rooted dipole with $a(D)+1$ edges. $\square$

While the initial condition may be written more explicitly by replacing f_i with $f_{(\circ^i)}$ in Lemma 2.4.9, it is often convenient for later analysis of the series Ψ' to leave the initial condition in the form given in the statement of this Theorem. The series Ψ' is the initial condition for a partial differential equation for the series Ψ'' corresponding to the full (a,b,c,d)-dipole problem. A similar analysis gives the following equation for Ψ''.

Theorem 5.3.2 (S.)**.** *The generating series Ψ'' for (a,b,c,d)-dipoles is the unique solution to the partial differential equation*

$$(C'' + u^2 J'')\Psi'' = \frac{\partial \Psi''}{\partial v}$$

where

$$C'' = \sum_{R\in\{\circ,\bullet\}^*} \left(\sum_{2\le i\le \ell(R)} w^{\delta_{R_i,\bullet}} g_{R_1R_i\cdots R_{\ell(R)}} f_{(R_2\cdots R_i)} + w g_{R_1} f_{(R)} \right) \frac{\partial}{\partial g_R}$$

and

$$J'' = \sum_{R\in\{\circ,\bullet\}^*} \sum_{(S)\in\mathcal{S}(\circ,\bullet)} \left(\sum_{S_1S_2\cdots S_{\ell(S)}\in(S)} w^{\delta_{S_1,\bullet}} g_{R_1S_1\cdots S_{\ell(S)}S_1R_2\cdots R_{\ell(R)}} \right) \frac{\partial^2}{\partial g_R \partial f_{(S)}},$$

with initial condition $\Psi''|_{v=0} = \Psi'$.

Proof. Let $\Psi''_m = m![v^m]\Psi''$ be the generating series for (a,b,c,d)-dipoles in which $c+d=m$. In other words, this is the series for dipoles in which there are m edges having an end in Region 4. The set of dipoles having m edges in Region 4 is uniquely generated by adding an edge e, with one end in Region 4, to a dipole D having $m-1$ edges in Region 4 in a canonical way. This edge will be added so that its non-root end is added to the corner which is the first one encountered as one travels clockwise around the non-root vertex starting from the root edge. Placing the new edge in this manner ensures that it

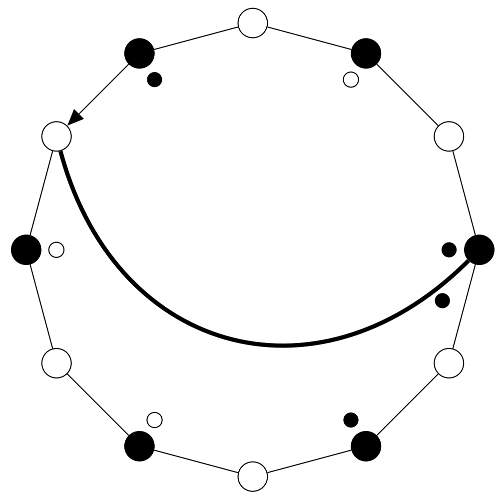


Figure 5.5: A d-edge is added to an (a, b, c, d)-dipole in a manner which cuts the root face, originally marked by $g_{\bullet\circ\circ\bullet\bullet\circ}$, into a new root face, marked by $g_{\bullet\bullet\circ}$, and a non-root face, marked by $f_{(\circ\circ\bullet\bullet)}$. This edge is known to be a d-edge, as opposed to a c-edge, since its root end is added to a corner marked by $\bullet$. Thus, this case also adds a factor of w to the monomial encoding the dipole.

will be part of the root face, being the next edge encountered after the root edge on a counterclockwise boundary tour of the root face. The root end of e will be added to a corner which is marked either with a $\bullet$ or with a $\circ$. If it is added to a corner marked with a $\bullet$, then it is a d-edge. If it is added to a corner marked with a $\circ$, then it is a c-edge. Suppose that D is encoded by the monomial

$$u^{2g} g_R \prod_i f_{(S^{(i)})},$$

where $R = R_1 \cdots R_{\ell(R)}$. As before, the analysis splits into two cases depending on whether the root end of e is added to a corner of the root face, or of a non-root face.

First, consider the case when the root end of e is added to a corner of the root face, say, the corner marked by R_i where $1 \leq i \leq \ell(R)$. (This case is illustrated in Figure 5.5.) In this case, the root face is cut into two faces, one of which is the new root face, and one of which is a non-root face. When $i = 1$, the addition of e is a d edge, and creates a new root face which is a digon marked by R_1. In this case, the corners of the non-root face are marked by the same sequence of symbols as the old root face, so the contribution in this case is

$$w g_{R_1} f_{(R)} \prod_i f_{(S^{(i)})}.$$

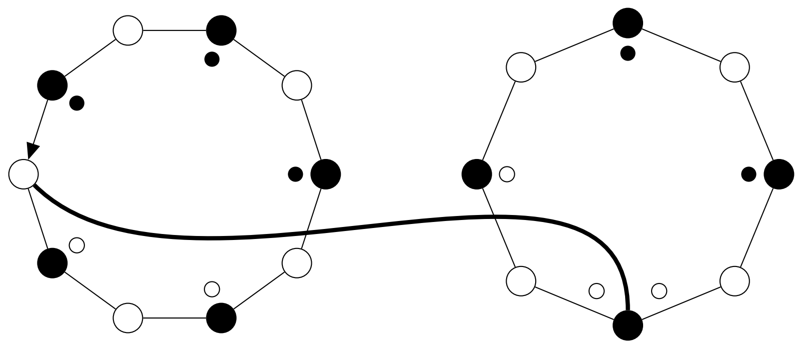


Figure 5.6: A c-edge is added to an (a, b, c, d)-dipole such that the root face is joined to a non-root face. Originally, the two faces were marked by $g_{\bullet\circ\circ\bullet\bullet}$ and $f_{(\circ\bullet\bullet\circ)}$, and the resulting face is marked by $g_{\bullet\circ\bullet\bullet\circ\circ\circ\circ\bullet\bullet}$. This edge is known to be a c-edge, as opposed to a d-edge, since its root end was added to a corner marked by $\circ$, so no additional factor of w is contributed.

The general case occurs if $2 \le i \le \ell(R)$. Consider a counterclockwise boundary tour of the new root face, starting at the root corner (marked by R_1). After travelling along the root edge, the edge e is encountered, which goes to the corner marked by R_i. Continuing the boundary walk, the symbols $R_{i+1}, \ldots R_\ell$ are encountered, after which the boundary walk is complete. As for the newly-created non-root face, since non-root faces are encoded up to cyclic equivalence any starting point for the boundary walk may be chosen. It is convenient to start at the corner counterclockwise from the non-root end of e. Starting from this corner, the corners encountered are marked by $R_2, R_3, \ldots, R_i$, upon which the edge e is encountered, returning to the starting point. Thus, the contribution from this case is

$$w^{\delta_{R_i,\bullet}} g_{R_1 R_i \cdots R_{\ell(R)}} f_{(R_2 \cdots R_i)} \prod_i f_{(S^{(i)})}.$$

Summing over all cases, the differential operator corresponding to a cut edge is

$$C'' = \sum_{R \in \{\circ,\bullet\}^*} \left(\sum_{2 \le i \le \ell(R)} w^{\delta_{R_i,\bullet}} g_{R_1 R_i \cdots R_{\ell(R)}} f_{(R_2 \cdots R_i)} + w g_{R_1} f_{(R)} \right) \frac{\partial}{\partial g_R}.$$

Next, consider the case when the root end of e is added to a corner of a non-root face, as illustrated in Figure 5.6. Suppose this non-root face is marked by $f_{(S)}$. (The number of choices for a given (S) is equal to the degree of $f_{(S)}$ in $\prod_i f_{(S^{(i)})}$.) The choice of corner of the non-root face picks out one specific string from the set of cyclic binary strings (S), say, the string $S_1 \cdots S_{\ell(S)}$ such that S_1 is the corner to which e is added. Then, on a counterclockwise boundary tour of the new root face, the corner labels are encountered in the order

$$R_1, S_1, S_2, \ldots, S_{\ell(S)}, S_1, R_2, R_3, \ldots, R_{\ell(R)}.$$

If $S_1 = \bullet$, e is a d-edge and an additional power of w is needed. If $S_1 = \circ$, e is a c-edge and the power of w remains the same. Summing over all choices of a cyclic binary string in S, the differential operator corresponding to this case is.

$$J'' = \sum_{R \in \{\circ,\bullet\}^*} \sum_{(S) \in \mathcal{S}(\circ,\bullet)} \left(\sum_{S_1 S_2 \cdots S_{\ell(S)} \in (S)} w^{\delta_{S_1,\bullet}} g_{R_1 S_1 \cdots S_{\ell(S)} S_1 R_2 \cdots R_{\ell(R)}} \right) \frac{\partial^2}{\partial g_R \partial f_{(S)}}.$$

□

The combinatorial analysis of the (p, q, n)-dipole problem presented in this Chapter leads to two natural questions, which are the subject of the remainder of the thesis.

1. How can the revelation that the $(p, n-1, n)$-dipole problem lies in the algebra $Z_1(n)$ as opposed to $Z_2(n)$ (Lemma 5.2.2) be used to approach this special case of the (p, q, n)-dipole problem? The algebra needed to do this is developed in Chapter 6, leading to a full solution of the $(p, n-1, n)$-dipole problem for all orientable surfaces in Chapter 7.

2. What information about the general (p, q, n)-dipole problem can be gleaned from analysis of the partial differential equations in Theorems 5.3.1 and 5.3.2? This analysis is conducted in Chapter 8, and while it does not lead to a closed-form expression for Ψ'', it provides a process, recursive in g, for determining the generating series for (a, b, c, d)-dipoles in a surface of genus g, from which the number of (p, q, n)-dipoles in the surface can be determined.

Chapter 6

Connection Coefficients for $Z_1(n)$

This chapter contains a detailed examination of the connection coefficients of $Z_1(n)$. The additional attention given to this algebra is warranted because of the observation made in the preceding chapter that a special case of the (p, q, n)-dipole problem, the case in which $p = n - 1$, may be solved using the algebra $Z_1(n)$ rather than $Z_2(n)$. Given that $Z_1(n)$ is commutative, one would expect to be able to make substantially more progress using algebraic techniques in the $p = n - 1$ case than in the general case.

The results of this chapter may be summarized as follows: the algebra $Z_1(n)$ possesses properties analogous to the three "essential" properties of the centre which are summarized at the end of Chapter 2. First, Section 6.1 contains a combinatorially natural definition of a basis of orthogonal idempotents for $Z_1(n)$. This basis is shown to be related to the generalized characters defined by Strahov [42]; hence, the connection coefficients of $Z_1(n)$ may be computed in terms of generalized characters. Second, Section 6.2 shows how the Diaconis-Greene [4] method for evaluating characters may be extended to evaluate generalized characters. In particular, much like ordinary characters, generalized characters indexed by non-hook partitions vanish when evaluated at full cycles. Finally, Section 6.3 shows how Strahov's generalization of the Murnaghan-Nakayama Rule may be used to evaluate generalized characters indexed by hook partitions.

6.1 Orthogonal Idempotents for $Z_1(n)$

6.1.1 Defining the Z_1-idempotents

Recall that a basis of orthogonal idempotents for $Z(n)$ is given by

$$X^\lambda = \sum_{T \in \mathrm{SYT}(\lambda)} e(T),$$

where $e(T)$ is the semi-normal unit corresponding to the tableau T. Since the standard basis for $Z_1(n)$ is indexed by pairs (λ, i) where $\lambda \vdash n$ and i is a part of λ, in order to find elements of $Z_1(n)$ which play a role analogous to the role played by X^λ, it is natural to look for elements which are indexed by the same set. Given a tableau of shape λ, consider the placement of the symbol n. Since it is the greatest symbol, it must appear at the end of a row and the bottom of a column. Thus, the placement of n identifies a part of λ, and conversely, given a part i of λ, there is a unique row of length i in which n may be placed. Let $\mathrm{SYT}(\lambda, i)$ denote the set of standard Young tableaux of shape λ in which n appears at the end of a row of length i. This permits the following definition to be made.

Definition 6.1.1 (Z_1-idempotents)**.** Let $\lambda \vdash n$ and let i be a part of λ. The **Z_1-idempotents** are the elements in $\mathbb{C}[\mathfrak{S}_n]$ given by

$$\Gamma^{\lambda,i} := \sum_{T \in \mathrm{SYT}(\lambda,i)} e(T).$$

The following facts about $\Gamma^{\lambda,i}$ are immediate from the definition, and they justify the choice of $\Gamma^{\lambda,i}$ as a natural candidate in the search for a useful basis for $Z_1(n)$:

1. They are indeed idempotent, and are orthogonal with respect to ring multiplication. This follows from the fact that the elements $e(T)$ are orthogonal and idempotent.

2. The set $\{\Gamma^{\lambda,i}\}_{\lambda \vdash n, i \in \lambda}$ has the same cardinality as the standard basis $\{K_{\lambda,i}\}_{\lambda \vdash n, i \in \lambda}$ for $Z_1(n)$. Furthermore, this set is linearly independent, due to the linear independence of the set $\{e(T)\}$ as T ranges over all tableaux with n boxes.

3. The Z_1-idempotents can be viewed as a "partition" of the central idempotents in the following sense:
$$X^\lambda = \sum_{i \in \lambda} \Gamma^{\lambda,i}.$$
This is analogous to the obvious fact that the standard bases for $Z(n)$ and $Z_1(n)$ are related by
$$K_\lambda = \sum_{i \in \lambda} K_{\lambda,i}.$$

It is not immediately obvious from the definition of the Z_1-idempotents that they in fact lie in $Z_1(n)$. Once this is shown, combined with the second observation above, it will prove that $\{\Gamma^{\lambda,i}\}_{\lambda \vdash n, i \in \lambda}$ is a basis for $Z_1(n)$. In fact, it is possible to find an explicit expression for the Z_1-idempotents as a product of an idempotent in $Z(n)$ and an idempotent in $Z(n-1)$. In the following, the notation $i_-(\lambda)$ denotes the partition $(\lambda \setminus i \cup (i-1))$ if i is a part of λ.

Lemma 6.1.2. *Let $\lambda \vdash n$ and i be a part of λ. Then*

$$\Gamma^{\lambda,i} = X^{\lambda} X^{i_-(\lambda)},$$

where the product is taken in $\mathbb{C}[\mathfrak{S}_n]$ and every element in the support of $X^{i_-(\lambda)}$ is regarded as having the element n as a fixed point. Consequently, $\Gamma^{\lambda,i} \in Z_1(n)$, and $\{\Gamma^{\lambda,i}\}_{\lambda \vdash n, i \in \lambda}$ is a basis for $Z_1(n)$.

Proof. By Lemma 2.3.2,

$$X^{\lambda} X^{i_-(\lambda)} = \left(\sum_{T \in \mathrm{SYT}(\lambda)} e(T) \right) \left(\sum_{S \in \mathrm{SYT}(i_-(\lambda))} e(S) \right).$$

For a fixed $S \in \mathrm{SYT}(i_-(\lambda))$, apply Lemma 2.3.4 and then use the fact that the $e(T)$'s are orthogonal idempotents to obtain

$$\begin{aligned} \sum_{T \in \mathrm{SYT}(\lambda)} e(T)e(S) &= \sum_{T \in \mathrm{SYT}(\lambda)} \sum_{\substack{S_0 \in \mathrm{SYT}(\lambda), \\ S_0^* = S}} e(T)e(S_0) \\ &= \sum_{\substack{T \in \mathrm{SYT}(\lambda), \\ T^* = S}} e(T). \end{aligned}$$

(Recall that, given a tableau T, the tableau T^* is obtained by deleting the box containing n.) Summing over all $S \in \mathrm{SYT}(i_-(\lambda))$ gives

$$X^{\lambda} X^{i_-(\lambda)} = \sum_{S \in \mathrm{SYT}(i_-(\lambda))} \sum_{\substack{T \in \mathrm{SYT}(\lambda), \\ T^* = S}} e(T).$$

Since the tableaux $T \in \mathrm{SYT}(\lambda)$ with the property that $T^* \in \mathrm{SYT}(i_-(\lambda))$ are those in which the symbol n appears at the end of a row of length i, then

$$X^{\lambda} X^{i_-(\lambda)} = \sum_{T \in \mathrm{SYT}(\lambda,i)} e(T) = \Gamma^{\lambda,i}.$$

□

Since $\Gamma^{\lambda,i} \in Z_1(n)$, there exist coefficients $\gamma^{\lambda,i}_{\mu,j}$ such that

$$\Gamma^{\lambda,i} = \frac{d_{\lambda}}{n!} \sum_{\substack{\mu \vdash n, \\ j \in \mu}} \gamma^{\lambda,i}_{\mu,j} K_{\mu,j},$$

This is enshrined in the following.

	$\mathcal{C}_{(3),3}$	$\mathcal{C}_{(2,1),1}$	$\mathcal{C}_{(2,1),2}$	$\mathcal{C}_{(1^3),1}$
Size of class	2	1	2	1
$6\Gamma^{(3),3}$	1	1	1	1
$3\Gamma^{(2,1),1}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	1
$3\Gamma^{(2,1),2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	1
$6\Gamma^{(1^3),1}$	1	-1	-1	1

Table 6.1: Coefficents of scaled Z_1-idempotents for $\mathfrak{S}_3$, $\frac{n!}{d_\lambda}\Gamma^{\lambda,i}$.

	$\mathcal{C}_{(4),4}$	$\mathcal{C}_{(3,1),3}$	$\mathcal{C}_{(3,1),1}$	$\mathcal{C}_{(2,2),2}$	$\mathcal{C}_{(2,1,1),2}$	$\mathcal{C}_{(2,1,1),1}$	$\mathcal{C}_{(1,1,1,1),1}$
Size of class	6	6	2	3	3	3	1
$24\Gamma^{(4),4}$	1	1	1	1	1	1	1
$8\Gamma^{(3,1),3}$	$-\frac{2}{3}$	$\frac{1}{3}$	-1	$-\frac{2}{3}$	$\frac{4}{3}$	0	2
$8\Gamma^{(3,1),1}$	$-\frac{1}{3}$	$-\frac{1}{3}$	1	$-\frac{1}{3}$	$-\frac{1}{3}$	1	1
$12\Gamma^{(2,2),2}$	0	-1	-1	2	0	0	2
$8\Gamma^{(2,1,1),2}$	$\frac{1}{3}$	$-\frac{1}{3}$	1	$-\frac{1}{3}$	$\frac{1}{3}$	-1	1
$8\Gamma^{(2,1,1),1}$	$\frac{2}{3}$	$\frac{1}{3}$	-1	$-\frac{2}{3}$	$-\frac{4}{3}$	0	2
$24\Gamma^{(1,1,1,1),1}$	-1	1	1	1	-1	-1	1

Table 6.2: Coefficients of scaled Z_1-idempotents for $\mathfrak{S}_4$, $\frac{n!}{d_\lambda}\Gamma^{\lambda,i}$.

Definition 6.1.3. Let $\lambda, \mu \vdash n$ and let i and j be parts of λ and μ, respectively. Define

$$\gamma_{\mu,j}^{\lambda,i} := \frac{n!}{d_\lambda}[K_{\mu,j}]\Gamma^{\lambda,i}.$$

(The scaling factor $n!/d_\lambda$ is included for later convenience.)

As an example, the various values of $\gamma_{\mu,j}^{\lambda,i}$ for $Z_1(3)$ and $Z_1(4)$ are given in Tables 6.1 and 6.2, respectively. The values given here were computed in two ways: using Definition 6.1.1 and using the expression from Lemma 6.1.2.

For the Z_1-idempotents to be useful from a combinatorial point of view, it is necessary to also know how to express $K_{\mu,j}$ as a linear combination of the Z_1-idempotents. An examination of Tables 6.1 and 6.2 suggests that the idempotent basis is orthogonal not only with respect to ring multiplication, but also with respect to the standard inner product on $\mathbb{C}[\mathfrak{S}_n]$. If true in general, this fact would make it routine to express $K_{\mu,j}$ in terms of the idempotent basis. Thus, the next task is to prove orthogonality.

6.1.2 Generalized Characters of the Symmetric Group

Orthogonality of the Z_1-idempotents with respect to the inner product can be proven *via* an application of a result of Strahov regarding generalized characters of the symmetric group [42]. Strahov uses the term **generalized character** to refer to the zonal spherical functions of the Gelfand pair $(\mathfrak{S}_n \times \mathfrak{S}_{n-1}, \mathrm{diag}(\mathfrak{S}_{n-1}))$, where the diagonal $\mathrm{diag}(G)$ of a group G is defined by

$$\mathrm{diag}(G) = \{(g, g) : g \in G\}.$$

Strahov gives the following expression for generalized characters in terms of ordinary characters.

Definition 6.1.4 (Generalized character)**.** Let $\lambda \vdash n$, and let i be a part of λ. Let $\pi \in \mathfrak{S}_n$. The **generalized character** indexed by (λ, i) evaluated at π is given by

$$\gamma^{\lambda,i}(\pi) = \frac{d_{i_-(\lambda)}}{(n-1)!} \sum_{\sigma_2 \in \mathfrak{S}_{n-1}} \chi^\lambda(\pi\sigma_2^{-1})\chi^{i_-(\lambda)}(\sigma_2).$$

(Strahov's usage of the term "generalized character" is different from other uses of this term appearing in the literature, such as those in which the term refers to any integral combination of characters.) Generalized characters are in fact the scaled coefficients of $\Gamma^{\lambda,i}$; specifically,

Lemma 6.1.5. *Let $\lambda \vdash n$ and let i be a part of λ. Let $\pi \in \mathcal{C}_{\mu,j}$. Then*

$$\gamma^{\lambda,i}(\pi) = \gamma^{\lambda,i}_{\mu,j}.$$

Proof. This may be proved routinely using Lemma 6.1.2.

$$\begin{aligned}
\gamma^{\lambda,i}_{\mu,j} &= \frac{n!}{d_\lambda}[K_{\mu,j}]X^\lambda X^{i_-(\lambda)} \\
&= \frac{d_{i_-(\lambda)}}{(n-1)!}[\pi] \sum_{\sigma_1 \in \mathfrak{S}_n} \sum_{\sigma_2 \in \mathfrak{S}_{n-1}} \chi^\lambda(\sigma_1)\chi^{i_-(\lambda)}(\sigma_2)\sigma_1\sigma_2 \\
&= \frac{d_{i_-(\lambda)}}{(n-1)!} \sum_{\substack{(\sigma_1,\sigma_2) \in \mathfrak{S}_n \times \mathfrak{S}_{n-1}, \\ \sigma_1\sigma_2 = \pi}} \chi^\lambda(\sigma_1)\chi^{i_-(\lambda)}(\sigma_2) \\
&= \frac{d_{i_-(\lambda)}}{(n-1)!} \sum_{\sigma_2 \in \mathfrak{S}_{n-1}} \chi^\lambda(\pi\sigma_2^{-1})\chi^{i_-(\lambda)}(\sigma_2) \\
&= \gamma^{\lambda,i}(\pi).
\end{aligned}$$

□

The equivalence of Definition 6.1.3 and Definition 6.1.4 means that the coefficients of $\Gamma^{\lambda,i}$ satsify the properties of zonal spherical functions. (A list of these properties may be found in Chapter VII of Macdonald [32].) Of particular relevance to the present task is the fact that these coefficients are orthogonal with respect to the standard inner product on $\mathbb{C}[\mathfrak{S}_n]$. This is the content of the following result.

Corollary 6.1.6. *Let $\lambda, \mu \vdash n$ and let i, j be parts of λ and μ, respectively. Then*

$$\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\gamma^{\lambda,i}(\sigma)\gamma^{\mu,j}(\sigma) = \frac{1}{n!}\sum_{\nu\vdash n,k\in\nu}|\mathcal{C}_{\nu,k}|\gamma^{\lambda,i}_{\nu,k}\gamma^{\mu,j}_{\nu,k} = \frac{d_{i_-(\lambda)}}{d_\lambda}\delta_{\lambda,\mu}\delta_{i,j}$$

These relations permit $K_{\lambda,i}$ to be written in the Z_1-idempotent basis. Let T be the operator

$$T = \sum_{\lambda\vdash n,i\in\lambda}\sum_{\mu\vdash n,j\in\mu}\frac{d_\lambda}{n!}\gamma^{\lambda,i}_{\mu,j}K_{\mu,j}[K_{\lambda,i}] = \sum_{\lambda\vdash n,i\in\lambda}\Gamma^{\lambda,i}[K_{\lambda,i}]$$

so that

$$\Gamma^{\lambda,i} = TK_{\lambda,i}.$$

Define T^* by

$$T^* = \sum_{\nu\vdash n,k\in\nu}\sum_{\rho\vdash n,\ell\in\rho}\frac{|\mathcal{C}_{\rho,\ell}|}{d_{k_-(\nu)}}\gamma^{\nu,k}_{\rho,\ell}K_{\nu,k}[K_{\rho,\ell}].$$

Then

$$\begin{aligned}
T^*T &= \sum_{\nu\vdash n,k\in\nu}\sum_{\rho\vdash n,\ell\in\rho}\frac{|\mathcal{C}_{\rho,\ell}|}{d_{k_-(\nu)}}\gamma^{\nu,k}_{\rho,\ell}K_{\nu,k}\sum_{\lambda\vdash n,i\in\lambda}\frac{d_\lambda}{n!}\gamma^{\lambda,i}_{\rho,\ell}[K_{\lambda,i}]\\
&= \sum_{\nu\vdash n,k\in\nu}\sum_{\lambda\vdash n,i\in\lambda}\frac{d_\lambda}{d_{k_-(\nu)}}\left(\frac{1}{n!}\sum_{\rho\vdash n,\ell\in\rho}|\mathcal{C}_{\rho,\ell}|\gamma^{\nu,k}_{\rho,\ell}\gamma^{\lambda,i}_{\rho,\ell}\right)K_{\nu,k}[K_{\lambda,i}]\\
&= \sum_{\nu\vdash n,k\in\nu}\sum_{\lambda\vdash n,i\in\lambda}\frac{d_\lambda}{d_{k_-(\nu)}}\langle\Gamma^{\nu,k},\Gamma^{\lambda,i}\rangle K_{\nu,k}[K_{\lambda,i}]\\
&= \sum_{\lambda\vdash n,i\in\lambda}K_{\lambda,i}[K_{\lambda,i}]\\
&= I,
\end{aligned}$$

the identity operator on $Z_1(n)$. Since $Z_1(n)$ is finite dimensional, then $TT^* = I$ and

$T^* = T^{-1}$. The $K_{\lambda,i}$ basis can be expressed in terms of the $\Gamma^{\mu,j}$ basis as follows:

$$\begin{aligned} K_{\lambda,i} &= TT^{-1}K_{\lambda,i} \\ &= T \sum_{\mu \vdash n, j \in \mu} \frac{|\mathcal{C}_{\lambda,i}|}{d_{j_-(\mu)}} \gamma_{\lambda,i}^{\mu,j} K_{\mu,j} \\ &= \sum_{\mu \vdash n, j \in \mu} \frac{|\mathcal{C}_{\lambda,i}|}{d_{j_-(\mu)}} \gamma_{\lambda,i}^{\mu,j} \Gamma^{\mu,j}. \end{aligned}$$

Thus, the following has been proven.

Lemma 6.1.7. *Let $\lambda \vdash n$ and let i be a part of λ. Then*

$$K_{\lambda,i} = \sum_{\mu \vdash n, j \in \mu} \frac{|\mathcal{C}_{\lambda,i}|}{d_{j_-(\mu)}} \gamma_{\lambda,i}^{\mu,j} \Gamma^{\mu,j}.$$

Of particular interest in later applications is the specialization of this result which expresses the identity permutation 1 in terms of the generalized character basis. Since the coefficient of the identity permutation in $e(T)$ is $d_\lambda/n!$ when T is of shape λ, then

$$\gamma_{(1^n),1}^{\mu,j} = \frac{n!}{d_\mu}[K_{(1^n),1}]\Gamma^{\mu,j} = |\mathrm{SYT}(\mu, j)|.$$

However, the number of tableaux of shape μ in which n is at the end of a row of length j is the same as the number of tableaux of shape $j_-(\mu)$, so

$$\gamma_{(1^n),1}^{\mu,j} = d_{j_-(\mu)}.$$

Consequently,

Corollary 6.1.8.

$$1 = K_{(1^n),1} = \sum_{\mu \vdash n, j \in \mu} \Gamma^{\mu,j}.$$

The preceding results allow the connection coefficients for $Z_1(n)$ to be expressed in terms of generalized characters. Let $\lambda, \mu \vdash n$ and let i and j be parts of λ and μ respectively. The product $K_{\lambda,i}K_{\mu,j}$ may be written in the Z_1-idempotent basis as

$$K_{\lambda,i}K_{\mu,j} = |\mathcal{C}_{\lambda,i}||\mathcal{C}_{\mu,j}| \sum_{\nu \vdash n, k \in \nu} \sum_{\rho \vdash n, \ell \in \rho} \frac{\gamma_{\lambda,i}^{\nu,k}}{d_{k_-(\nu)}} \frac{\gamma_{\mu,j}^{\rho,\ell}}{d_{\ell_-(\rho)}} \Gamma^{\nu,k}\Gamma^{\rho,\ell}.$$

By orthogonal idempotency of the Z_1-idempotents, this may be written as

$$K_{\lambda,i}K_{\mu,j} = |\mathcal{C}_{\lambda,i}||\mathcal{C}_{\mu,j}| \sum_{\rho \vdash n, \ell \in \rho} \frac{\gamma_{\lambda,i}^{\rho,\ell}\gamma_{\mu,j}^{\rho,\ell}}{d_{\ell_-(\rho)}^2} \Gamma^{\rho,\ell}.$$

Extracting the coefficient of $K_{\nu,k}$ yields the following.

Theorem 6.1.9 (S.). *Let $\lambda, \mu, \nu \vdash n$ and let i, j and k be parts of λ, μ and ν, respectively. Let the constants $c_{\lambda,i,\mu,j}^{\nu,k}$ be defined by*

$$K_{\lambda,i} K_{\mu,j} = \sum_{\nu \vdash n, k \in \nu} c_{\lambda,i,\mu,j}^{\nu,k} K_{\nu,k}.$$

Then

$$c_{\lambda,i,\mu,j}^{\nu,k} = \frac{|\mathcal{C}_{\lambda,i}||\mathcal{C}_{\mu,j}|}{n!} \sum_{\rho \vdash n, \ell \in \rho} \frac{\gamma_{\lambda,i}^{\rho,\ell} \gamma_{\mu,j}^{\rho,\ell} \gamma_{\nu,k}^{\rho,\ell}}{d_{\ell-(\rho)}} \frac{d_\rho}{d_{\ell-(\rho)}}.$$

6.2 Evaluating Generalized Characters at Special Orbits

6.2.1 Generalizing the Diaconis-Greene technique

From a combinatorial point of view, the usefulness of the expression for the connection coefficients of $Z_1(n)$ given in Theorem 6.1.9 depends on having explicit expressions for the generalized characters appearing in the sum. This section demonstrates that the technique used by Diaconis and Greene [4] to evaluate ordinary characters by evaluating symmetric polynomials at the contents of a tableau (*i.e.* applying Lemma 2.4.3) may be extended to evaluate generalized characters at particular types of orbits, such as those corresponding to transpositions, cycles of length 3, cycles of length n and cycles of length $n-1$. This technique relies on Lemma 2.3.3, which states that the semi-normal unit $e(T)$ is an eigenvector of the Jucys-Murphy element J_k, and its eigenvalue is the content of the box containing k in the tableau T.

Let $\Lambda^{(1)}[x_2, \ldots, x_n]$ denote the ring polynomials that are invariant under permutations of $x_2, \ldots, x_{n-1}$. Such a polynomial will be called an **almost symmetric** polynomial, and may be regarded as a polynomial in x_n whose coefficients are symmetric polynomials in the variables $x_2, \ldots, x_{n-1}$. Suppose that $f \in \Lambda^{(1)}[x_2, \ldots, x_n]$ is an almost symmetric polynomial such that

$$f(J_2, \ldots, J_n) = \sum_{\substack{\lambda \vdash n \\ i \in \lambda}} a_{\lambda,i} K_{\lambda,i}.$$

Expressing the identity element $K_{(1^n),1}$ of $Z_1(n)$ in terms of generalized characters,

$$\begin{aligned}\sum_{\substack{\lambda\vdash n,\\ i\in\lambda}} a_{\lambda,i}K_{\lambda,i} &= f(J_2,\ldots,J_n)K_{(1^n),1}\\ &= \sum_{\substack{\mu\vdash n\\ j\in\mu}} f(J_2,\ldots,J_n)\Gamma^{\mu,j}.\end{aligned}$$

By the definition of $\Gamma^{\mu,j}$,

$$\begin{aligned}f(J_2,\ldots,J_n)\Gamma^{\mu,j} &= \sum_{T\in\mathrm{SYT}(\mu,j)} f(J_2,\ldots,J_n)e(T)\\ &= \sum_{T\in\mathrm{SYT}(\mu,j)} f(c_T(2),c_T(3),\ldots,c_T(n))e(T)\\ &= \sum_{T\in\mathrm{SYT}(\mu,j)} f(\mathbf{c}_{j_-(\mu)},c_{\mu,j})e(T),\end{aligned}$$

where $\mathbf{c}_\lambda$ denotes the content vector of any tableau of shape λ, and $c_{\mu,j}$ is the content of the box containing n in any tableau in $\mathrm{SYT}(\mu,j)$. The quantity $c_{\mu,j}$ depends only on μ and j, and is given by

$$c_{\mu,j} = j - \sum_{k\geq j} m_k(\mu).$$

The quantity $f(\mathbf{c}_{j_-(\mu)},c_{\mu,j})$ is well-defined since f is symmetric in $x_2,\ldots,x_{n-1}$. Thus,

$$f(J_2,\ldots,J_n)\Gamma^{\mu,j} = f(\mathbf{c}_{j_-(\mu)},c_{\mu,j})\Gamma^{\mu,j},$$

and

$$\sum_{\substack{\lambda\vdash n,\\ i\in\lambda}} a_{\lambda,i}K_{\lambda,i} = \sum_{\substack{\mu\vdash n\\ j\in\mu}} f(\mathbf{c}_{j_-(\mu)},c_{\mu,j})\Gamma^{\mu,j}.$$

On the other hand, by Lemma 6.1.7, the standard basis for $Z_1(n)$ may also be expressed in the generalized character basis as follows:

$$\sum_{\substack{\lambda\vdash n,\\ i\in\lambda}} a_{\lambda,i}K_{\lambda,i} = \sum_{\substack{\lambda\vdash n\\ i\in\lambda}} a_{\lambda,i}\sum_{\substack{\mu\vdash n\\ j\in\mu}} \frac{|\mathcal{C}_{\lambda,i}|}{d_{j_-(\mu)}}\gamma^{\mu,j}_{\lambda,i}\Gamma^{\mu,j}.$$

Comparing coefficients gives the following:

Lemma 6.2.1 (S.)**.** *Let $\mu\vdash n$ and let j be a part of μ. Let $f\in\Lambda^{(1)}[x_2,\ldots,x_n]$ be such that $f(J_2,\ldots,J_n) = \sum_{\substack{\lambda\vdash n\\ i\in\lambda}} a_{\lambda,i}K_{\lambda,i}$. Then $\Gamma^{\mu,j}$ is an eigenvector of $f(J_2,\ldots,J_n)$ with eigenvalue $f(\mathbf{c}_{j_-(\mu)},c_{\mu,j})$, and*

$$\sum_{\substack{\lambda\vdash n,\\ i\in\lambda}} a_{\lambda,i}\frac{|\mathcal{C}_{\lambda,i}|}{d_{j_-(\mu)}}\gamma^{\mu,j}_{\lambda,i} = f(\mathbf{c}_{j_-(\mu)},c_{\mu,j}).$$

An important special case of this lemma is as follows.

Corollary 6.2.2 (S.). *Let $\lambda, \mu \vdash n$ and let i and j be parts of λ and μ, respectively. Let $f \in \Lambda^{(1)}[x_2, \ldots, x_n]$ be such that $f(J_2, \ldots, J_n) = K_{\lambda,i}$. Then the generalized characters are given by the formula*

$$\gamma_{\lambda,i}^{\mu,j} = \frac{d_{j_-(\mu)}}{|\mathcal{C}_{\lambda,i}|} f(\mathbf{c}_{j_-(\mu)}, c_{\mu,j})$$

This provides a more precise notion of the "special" orbits referred to in the title of this section. It is possible to evaluate generalized characters at any orbit for which there is an explicit expression for that orbit's standard basis element as an almost-symmetric polynomial in $J_1, \ldots, J_n$. The existence of an almost symmetric polynomial satisfying the condition $f(J_2, \ldots, J_n) = K_{\lambda,i}$ is guaranteed by the following result.

Theorem 6.2.3 (Olshanski [36]). *Let J_k be the Jucys-Murphy element $\sum_{1 \le i < k}(i, k)$. Then*

$$\Lambda^{(1)}[J_2, \ldots, J_n] = Z_1(n).$$

A further consequence of Corollary 6.2.2 is that it reveals a useful symmetry property of generalized characters. Consider the following.

Definition 6.2.4. Let $\lambda \vdash n$. The **conjugate** of λ is the partition λ^* whose Ferrers diagram $\mathcal{F}_{\lambda^*}$ is the reflection of $\mathcal{F}_\lambda$ in the line $y = -x$. Let $i \in \lambda$. The conjugate of the marked partition (λ, i) is the pair (λ^*, i^*), where i^* is the number of parts of λ greater than or equal to i.

Then the following relation holds.

Corollary 6.2.5 (S.). *Let $\lambda \vdash n$ have m parts, and let i be a part of λ. Let $\mu \vdash n$, and $j \in \mu$. Then*

$$\gamma_{\lambda,i}^{\mu^*,j^*} = (-1)^{n-m}\gamma_{\lambda,i}^{\mu,j}.$$

Proof. Let $f(x_2, \ldots, x_n)$ be an almost symmetric function such that $f(J_2, \ldots, J_n) = K_{\lambda,i}$. Since the degree of any monomial in f is equal to the number of transpositions in some permutation of cycle type λ, then every monomial must either have even degree (if $n - m$ is even), or odd degree (if $n - m$ is odd). Since the contents of any tableau of shape (μ, j) are the negative of the contents of a tableau of shape (μ^*, j^*), then

$$\begin{aligned}
\gamma_{\lambda,i}^{\mu^*,j^*} &= \frac{d_{j^*_-(\mu^*)}}{|\mathcal{C}_{\lambda,i}|} f(\mathbf{c}_{j^*_-(\mu^*)}, c_{\mu^*,j^*}) \\
&= \frac{d_{j_-(\mu)}}{|\mathcal{C}_{\lambda,i}|} f(-\mathbf{c}_{j_-(\mu)}, -c_{\mu,j}) \\
&= (-1)^{n-m}\gamma_{\lambda,i}^{\mu,j}.
\end{aligned}$$

□

6.2.2 Some explicit evaluations of generalized characters

The following are some examples of standard basis elements of $Z_1(n)$ for which there are explicit expressions as almost-symmetric polynomials in $J_2, \ldots, J_n$.

Lemma 6.2.6. *Let* ≥ 3. *Then*

1. $K_{(2,1^{n-2}),2} = J_n$.
2. $K_{(2,1^{n-2}),1} = p_1(J_2, \ldots, J_{n-1}) = J_2 + J_3 + \cdots + J_{n-1}$.
3. $K_{(3,1^{n-3}),3} = J_n^2 - (n-1)K_{(1^n),1}$.
4. $K_{(2,2,1^{n-4}),2} = p_1(J_2, \ldots, J_{n-1})J_n - J_n^2 + (n-1)K_{(1^n),1}$.
5. $K_{(3,1^{n-3}),1} = p_2(J_2, \ldots, J_{n-1}) - \binom{n-1}{2}K_{(1^n),1}$.
6. $K_{(2,2,1^{n-4}),1} = \frac{1}{2}(p_1(J_2, \ldots, J_{n-1})^2 - 3p_2(J_2, \ldots, J_{n-1})) + \binom{n-1}{2}K_{(1^n),1}$.
7. $K_{(n),n} = e_{n-1}(J_2, \ldots, J_n) = J_2 J_3 \cdots J_n$.
8. $K_{(n-1,1),1} = e_{n-2}(J_2, \ldots, J_{n-1}) = J_2 J_3 \cdots J_{n-1}$.

Proof. Expressions 1, 2, 7 and 8 are obvious. For Expression 3, observe that

$$\begin{aligned} J_n^2 &= \sum_{1 \leq i < n} \sum_{1 \leq j < n} (i,n)(j,n) \\ &= \sum_{\substack{1 \leq i,j < n \\ i \neq j}} (j,i,n) + (n-1)K_{(1^n),1} \\ &= K_{(3,1^{n-3}),3} + (n-1)K_{(1^n),1}, \end{aligned}$$

from which the result follows. To prove expression 4, first observe that

$$\begin{aligned} p_1(J_2, \ldots, J_{n-1})J_n &= \sum_{\{i,j\} \subset \{1,\ldots,n-1\}} \sum_{1 \leq k < n} (i,j)(k,n) \\ &= \sum_{1 \leq k < n-1} \left(\sum_{\{i,j\} \subset \{1,\ldots,n-1\} \setminus k} (i,j)(k,n) + \sum_{i \in \{1,\ldots,n-1\} \setminus k} (k,i,n) \right) \\ &= K_{(2,2,1^{n-4}),2} + K_{(3,1^{n-3}),3}. \end{aligned}$$

The result follows after applying part 3. For expression 5,

$$\begin{aligned}
p_2(J_2,\ldots,J_{n-1}) &= \sum_{1\le k\le n-1}\ \sum_{1\le i,j<k}(i,k)(j,k)\\
&= \sum_{1\le k\le n-1}\ \sum_{\substack{1\le i,j<k\\ i\neq j}}(j,i,k) + \sum_{1\le k\le n-1}\ \sum_{1\le i<k} K_{(1^n),1}\\
&= K_{(3,1^{n-3}),1} + \binom{n-1}{2}K_{(1^n),1},
\end{aligned}$$

from which the result follows. For expression 6,

$$\begin{aligned}
(J_2+\cdots+J_{n-1})^2 &= \sum_{\{i,j\}\subset\{1,\ldots,n-1\}}(i,j)\Bigg((i,j) + \sum_{k\in\{1,\ldots,n-1\}\setminus\{i,j\}}(i,k) + \sum_{k\in\{1,\ldots,n-1\}\setminus\{i,j\}}(j,k)\\
&\qquad + \sum_{\{k,\ell\}\subset\{1,\ldots,n-1\}\setminus\{i,j\}}(k,\ell)\Bigg)\\
&= \binom{n-1}{2}K_{(1^n),1} + 3\sum_{\{i,j,k\}\subset\{1,\ldots,n-1\}}((i,k,j)+(i,j,k)) + 2K_{(2,2),1}\\
&= \binom{n-1}{2}K_{(1^n),1} + 3K_{(3,1^{n-3}),1} + 2K_{(2,2),1}.
\end{aligned}$$

The result then follows by applying expression 5. □

Using these results, together with with Lemma 6.2.1, gives expressions for some evaluations of the generalized characters. Recall that $\sigma(\lambda)$ denotes the sum of contents of a tableau of shape λ, and that $\sigma^{(2)}(\lambda)$ denotes the sum of squares of contents of a tableau of shape λ. The expressions given in Lemma 6.2.6 lead to the following.

Theorem 6.2.7. *Let $n\ge 3, \mu\vdash n$, and let j be a part of μ. Then*

1. $$\gamma^{\mu,j}_{(2,1^{n-2}),2} = \frac{1}{n-1}c_{\mu,j}d_{j_-(\mu)}.$$

2. $$\gamma^{\mu,j}_{(2,1^{n-2}),1} = \binom{n-1}{2}^{-1}\sigma(j_-(\mu))d_{j_-(\mu)}.$$

3. $$\gamma^{\mu,j}_{(3,1^{n-3}),3} = \frac{1}{2}\binom{n-1}{2}^{-1}(c^2_{\mu,j}-n+1)d_{j_-(\mu)}.$$

4.

$$\gamma^{\mu,j}_{(2,2,1^{n-4}),2} = \frac{1}{n-1}\binom{n-2}{2}^{-1}(\sigma(j_-(\mu))c_{\mu,j} - c^2_{\mu,j} + n - 1)d_{j_-(\mu)}.$$

5.

$$\gamma^{\mu,j}_{(3,1^{n-3}),1} = \frac{1}{2}\binom{n-1}{3}^{-1}\left(\sigma^{(2)}(j_-(\mu)) - \binom{n-1}{2}\right)d_{j_-(\mu)}.$$

6.

$$\gamma^{\mu,j}_{(2,2,1^{n-4}),1} = \frac{1}{6}\binom{n-1}{4}^{-1}(\sigma(j_-(\mu))^2 - 3\sigma^{(2)}(j_-(\mu)) + (n-1)(n-2))d_{j_-(\mu)}$$

7.

$$\gamma^{\mu,j}_{(n),n} = \begin{cases} (-1)^k\frac{n-k-1}{n-1} & \textit{if } \mu = (n-k,1^k), j = n-k; \\ (-1)^k\frac{k}{n-1} & \textit{if } \mu = (n-k,1^k), j = 1; \\ 0 & \textit{otherwise.} \end{cases}$$

8.

$$\gamma^{\mu,j}_{(n-1,1),1} = \begin{cases} (-1)^k & \textit{if } \mu = (n-k-1,2,1^{k-1}) \textit{ and } j = 2; \\ (-1)^k & \textit{if } \mu = (n-k,1^k) \textit{ and } j = n-k; \\ (-1)^{k-1} & \textit{if } \mu = (n-k,1^k) \textit{ and } j = 1; \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Formulas 1 through 6 are immediate from Lemma 6.2.1 and Lemma 6.2.6. The value $\gamma^{\mu,j}_{(n),n}$ is proportional to the product of contents of a tableau of shape μ. Since a box in row 2 and column 2 will have a content of zero, then $\gamma^{\mu,j}_{(n),n} = 0$ unless μ is of the form $(n-k,1^k)$ for some $0 \le k \le n-1$. If $0 \le k \le n-2$ and $j = n-k$, then there are $\binom{n-2}{k}$ standard Young tableaux of shape $j_-(\mu)$. The product of contents of a tableau of shape $(n-k,1^k)$ is $k!(-1)^k(n-k-1)!$. Thus,

$$\gamma^{(n-k,1^k),n-k}_{(n),n} = \frac{\binom{n-2}{k}k!(-1)^k(n-k-1)!}{(n-1)!} = (-1)^k\frac{n-k-1}{n-1}.$$

If $1 \le k \le n-1$ and $j = 1$, then there are $\binom{n-2}{k-1}$ standard Young tableaux of shape $j_-(\mu)$. Thus,

$$\gamma^{(k,1^{n-k}),1}_{(n),n} = \frac{1}{(n-1)!}\binom{n-2}{k-1}k!(-1)^k(n-k-1)! = (-1)^k\frac{k}{n-1}.$$

The value of $\gamma^{\mu,j}_{(n-1,1),1}$ is proportional to the product of contents the boxes containing labels 2 through $n-1$ in a tableau of type (μ,j). This result will be zero unless any label

Cycle type (μ, j)	$d_{j-(\mu)}$	$\prod_{2\leq i\leq n-1} c_T(i)$
$(n-k-1, 2, 1^{k-1}), 2$	$\binom{n-2}{k}$	$(n-k-2)!(-1)^k k!$
$(n-k, 1^k), n-k$	$\binom{n-2}{k}$	$(n-k-2)!(-1)^k k!$
$(n-k, 1^k), 1$	$\binom{n-2}{k-1}$	$(n-k-1)!(-1)^{k-1}(k-1)!$

Table 6.3: Calculations needed to evaluate $\gamma^{\mu,j}_{(n-1,1),1}$.

other than n occupies a box in the second row and second column. Thus, the only tableaux giving a nonzero result are either hook tableaux or tableaux of shape $(n-k-1, 2, 1^k)$ in which n appears at the end of the row of length 2. Determining the number of tableaux in each case, as well as the product of their contents, can be done as in the case for $\gamma^{\mu,j}_{(n),n}$; the resulting values are given in Table 6.3, and they yield the results in the statement of the theorem. □

6.2.3 Relationships between generalized characters and ordinary characters

In addition to evaluating generalized characters, the technique used in this section may also be used to prove two identities relating the generalized characters and ordinary characters of the symmetric group. The first such identity results from the observation that for any $\lambda \vdash n$,

$$\sum_{i\in\lambda} K_{\lambda,i} = K_\lambda.$$

Let $f_{\lambda,i} \in \Lambda^{(1)}[x_2, \ldots, x_n]$ be such that $f_{\lambda,i}(J_2, \ldots, J_n) = K_{\lambda,i}$, and let $f_\lambda \in \Lambda[x_2, \ldots, x_n]$ be a symmetric polynomial in $x_2, \ldots, x_n$ such that $f_\lambda(J_2, \ldots, J_n) = K_\lambda$. Thus,

$$\sum_{i\in\lambda} f_{\lambda,i}(J_2, \ldots, J_n) = f_\lambda(J_2, \ldots, J_n).$$

For any $\mu \vdash n$ and $j \in \mu$, let $T \in \mathrm{SYT}(\mu, j)$. Then

$$\begin{aligned} f_\lambda(\mathbf{c}_\mu)e(T) &= f_\lambda(J_2, \ldots, J_n)e(T) \\ &= \sum_{i\in\lambda} f_{\lambda,i}(J_2, \ldots, J_n)e(T) \\ &= \sum_{i\in\lambda} f_{\lambda,i}(\mathbf{c}_{j-(\mu)}, c_{\mu,j})e(T), \end{aligned}$$

so

$$f_\lambda(\mathbf{c}_\mu) = \sum_{i\in\lambda} f_{\lambda,i}(\mathbf{c}_{j_-(\mu)}, c_{\mu,j}).$$

Applying Lemma 2.4.1 and Corollary 6.2.2 yields the following.

Lemma 6.2.8 (S.). *Let $\lambda, \mu \vdash n$ and let j be a part of μ. Then*

$$\chi^\mu_\lambda = \frac{d_\mu}{|\mathcal{C}_\lambda| d_{j_-(\mu)}} \sum_{i\in\lambda} |\mathcal{C}_{\lambda,i}| \gamma^{\mu,j}_{\lambda,i}.$$

This result, along with Theorem 6.2.7, may be used to obtain the character evaluations given in Lemma 2.4.3. This is not the ideal method of obtaining the results of Lemma 2.4.3, since these results are more easily obtained directly using Lemma 2.4.1; rather, this observation is useful as a verification of the evaluations of generalized characters appearing in Theorem 6.2.7. This observation is a further justification that the theory developed in this chapter is a natural non-central refinement of character theory, since it is possible to recover results about characters from results about generalized characters.

A second relationship between generalized characters and ordinary characters may be obtained from the observation that

$$X^\lambda = \sum_{i\in\lambda} \Gamma^{\lambda,i},$$

which follows from Lemma 2.3.2 and Definition 6.1.1. Since X^λ is central, $[K_{\mu,j}]X^\lambda = [K_\mu]X^\lambda$ for all j, and comparing coefficients on both sides of this equation yields the following.

Lemma 6.2.9. *Let $\lambda, \mu \vdash n$. For any $j \in \mu$,*

$$\chi^\lambda_\mu = \sum_{i\in\lambda} \gamma^{\lambda,i}_{\mu,j}.$$

(Although these coefficients are scaled, the scaling factor of $\frac{d_\lambda}{n!}$ is the same on both the left and right side of this equation.) This relationship is of particular interest because different choices of j yield different, but equal, expressions for χ^λ_μ. For example, if $\mu = (2, 1^{n-2})$, taking $j = 1$ yields

$$\chi^\lambda_{(2,1^{n-2})} = \binom{n-1}{2}^{-1} \sum_{i\in\lambda} \sigma_{i_-(\lambda)} d_{i_-(\lambda)}.$$

However, taking $j = 2$ yields

$$\chi^{\lambda}_{(2,1^{n-2})} = \frac{1}{n-1} \sum_{i \in \lambda} c_{\lambda,i} d_{i_-(\lambda)}.$$

These expressions are both equal to the quantity given in Lemma 2.4.3, resulting in the following unexpected combinatorial identity.

Corollary 6.2.10. *For any $\lambda \vdash n$,*

$$\binom{n}{2}^{-1} d_\lambda \sigma_\lambda = \frac{1}{n-1} \sum_{i \in \lambda} c_{\lambda,i} d_{i_-(\lambda)} = \binom{n-1}{2}^{-1} \sum_{i \in \lambda} \sigma_{i_-(\lambda)} d_{i_-(\lambda)}.$$

It is not immediately clear whether there is a direct combinatorial explanation for this identity. Similar, but more complicated, identities may be obtained from Theorem 6.2.7 by making other choices of μ and j. Lemma 6.2.9, along with part 8 of Theorem 6.2.7, may be used to give a new derivation of an old result: the formula for evaluating ordinary characters at the conjugacy class indexed by $(n-1,1)$ is as follows.

Corollary 6.2.11. *Let $\lambda \vdash n$. Then*

$$\chi^{\lambda}_{(n-1,1)} = \begin{cases} 1 & \textit{if } \lambda = (n), \\ (-1)^n & \textit{if } \lambda = (1^n), \\ (-1)^k & \textit{if } \lambda = (n-k-1, 2, 1^{k-1}), \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The only choices for λ for which $\gamma^{\lambda,i}_{(n-1,1),1}$ is non-zero are $\lambda = (n-k-1, 2, 1^{k-1})$ or $\lambda = (n-k, 1^k)$. If $\lambda = (n-k-1, 2, 1^{k-1})$, then

$$\chi^{\lambda}_{(n-1,1)} = \gamma^{(n-k-1,2,1^{n-k}),n-k-1} + \gamma^{(n-k-1,2,1^{n-k}),2} + \gamma^{(n-k-1,2,1^{n-k}),1} = 0 + (-1)^k + 0.$$

If $\lambda = (n)$, then the only term arising in the sum is $\gamma^{(n),n}_{(n-1,1),1} = 1$. If $\lambda = (1^n)$, then the only term arising in the sum is $\gamma^{(1^n),1}_{(n-1,1),1} = (-1)^n$. If $\lambda = (n-k, 1^k)$ and $1 \le k \le n-2$, then

$$\chi^{\lambda}_{(n-1,1)} = \gamma^{(n-k,1^k),n-k}_{(n-1,1),1} + \gamma^{(n-k,1^k),1}_{(n-1,1),1} = (-1)^k + (-1)^{k-1} = 0,$$

proving the result. ☐

This result is of interest because it is not clear how to obtain $\chi^{\lambda}_{(n-1,1)}$ *via* an application of Lemma 2.4.1, since there is not a sufficiently simple expression for $K_{(n-1,1)}$ in terms of Jucys-Murphy elements. (Diaconis and Greene [4] compute it instead by using the branching rule for ordinary characters.) In other words, this is a case in which a non-trivial evaluation of an ordinary character may be obtained in a fairly simple manner by considering the theory of generalized characters.

Having determined $\chi^{\mu}_{(n-1,1)}$ and $\gamma^{\mu,j}_{(n-1,1),1}$, it is now possible to determine the generalized character $\gamma^{\mu,j}_{(n-1,1),n-1}$ by applying Lemma 6.2.8. The result is as follows.

Corollary 6.2.12. *Let $\mu \vdash n$ and let j be a part of μ. Then*

$$\gamma^{\mu,j}_{(n-1,1),n-1} = \begin{cases} 1 & \text{if } \mu = (n), j = n, \\ (-1)^n & \text{if } \mu = (1^n), j = 1, \\ \frac{(-1)^{k+1}}{n-1} & \text{if } \mu = (n-k, 1^k), 1 \leq k \leq n-2, j = n-k, \\ \frac{(-1)^k}{n-1} & \text{if } \mu = (n-k, 1^k), 1 \leq k \leq n-2, j = 1, \\ \frac{(-1)^k}{k(n-k-2)} & \text{if } \mu = (n-k-1, 2, 1^{k-1}), j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Rearranging the equation in Lemma 6.2.8 gives

$$\gamma^{\mu,j}_{(n-1,1),n-1} = \frac{1}{n-1}\left(\frac{nd_{j-(\mu)}}{d_\mu}\chi^{\mu}_{(n-1,1)} - \gamma^{\mu,j}_{(n-1,1),1}\right).$$

Substituting the known values for $\chi^{\mu}_{(n-1,1)}$ and $\gamma^{\mu,j}_{(n-1,1),1}$ gives the result. □

6.3 Evaluation of Generalized Characters Corresponding to Hook Partitions

6.3.1 The Murnaghan-Nakayama Rule for Generalized Characters

One of the combinatorially important results in the theory of ordinary characters of the symmetric group is Lemma 2.4.6, which gives the generating series for $\chi^{(n-k,1^k)}_\lambda$. The next step in developing the theory of generalized characters is to find a result which is analogous to this lemma. In light of parts 7 and 8 of Theorem 6.2.7, it is natural to look for generating series for three types of generalized characters: those of the form $\gamma^{(n-k,1^k),n-k}_{\mu,j}$, $\gamma^{(n-k,1^k),1}_{\mu,j}$ and $\gamma^{(n-k-1,2,1^{k-1}),2}_{\mu,j}$. The generalized characters in $\mathfrak{S}_5$ of these types are given in Table 6.4. The method used in this section to evaluate these characters relies on a theorem, due to Strahov [42], which expresses the generalized characters of $\mathfrak{S}_n$ in terms of the ordinary irreducible characters of symmetric groups of lower order.

Lemma		$\mathcal{C}_{(5),5}$	$\mathcal{C}_{(4,1),4}$	$\mathcal{C}_{(4,1),1}$	$\mathcal{C}_{(3,2),3}$	$\mathcal{C}_{(3,2),2}$	$\mathcal{C}_{(3,1,1),3}$
6.3.2	$120\,\Gamma^{(5),5}$	1	1	1	1	1	1
	$30\,\Gamma^{(4,1),4}$	$-\frac{3}{4}$	$\frac{1}{4}$	-1	$-\frac{3}{4}$	$-\frac{3}{4}$	$\frac{5}{4}$
	$20\,\Gamma^{(3,1,1),3}$	$\frac{1}{2}$	$-\frac{1}{4}$	1	$-\frac{1}{2}$	$\frac{3}{4}$	0
	$30\,\Gamma^{(2,1,1,1),2}$	$-\frac{1}{4}$	$\frac{1}{4}$	-1	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$
6.3.3	$30\,\Gamma^{(4,1),1}$	$-\frac{1}{4}$	$-\frac{1}{4}$	1	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
	$20\,\Gamma^{(3,1,1),1}$	$\frac{1}{2}$	$\frac{1}{4}$	-1	$\frac{1}{2}$	$-\frac{3}{4}$	0
	$30\,\Gamma^{(2,1,1,1),1}$	$-\frac{3}{4}$	$-\frac{1}{4}$	1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{5}{4}$
	$120\,\Gamma^{(1^5),1}$	1	-1	-1	-1	-1	1
6.3.4	$24\,\Gamma^{(3,2),2}$	0	$-\frac{1}{2}$	-1	0	$\frac{3}{2}$	-1
	$24\,\Gamma^{(2,2,1),2}$	0	$\frac{1}{2}$	1	0	$-\frac{3}{2}$	-1

Lemma		$\mathcal{C}_{(3,1,1),1}$	$\mathcal{C}_{(2,2,1),2}$	$\mathcal{C}_{(2,2,1),1}$	$\mathcal{C}_{(2,1,1,1),2}$	$\mathcal{C}_{(2,1,1,1),1}$	$\mathcal{C}_{(1^5),1}$
6.3.2	$120\,\Gamma^{(5),5}$	1	1	1	1	1	1
	$30\,\Gamma^{(4,1),4}$	0	$\frac{1}{4}$	-1	$\frac{9}{4}$	1	3
	$20\,\Gamma^{(3,1,1),3}$	0	-1	-1	$\frac{3}{2}$	-1	3
	$30\,\Gamma^{(2,1,1,1),2}$	1	$-\frac{1}{4}$	1	$\frac{1}{4}$	-1	1
6.3.3	$30\,\Gamma^{(4,1),1}$	1	$-\frac{1}{4}$	1	$-\frac{1}{4}$	1	1
	$20\,\Gamma^{(3,1,1),1}$	0	-1	-1	$-\frac{3}{2}$	1	3
	$30\,\Gamma^{(2,1,1,1),1}$	0	$\frac{1}{4}$	-1	$-\frac{9}{4}$	-1	3
	$120\,\Gamma^{(1^5),1}$	1	1	1	-1	-1	1
6.3.4	$24\,\Gamma^{(3,2),2}$	0	1	-1	0	1	3
	$24\,\Gamma^{(2,2,1),2}$	0	1	-1	0	-1	3

Table 6.4: Evaluations of generalized characters of $\mathfrak{S}_5$ corresponding to results appearing in this section.

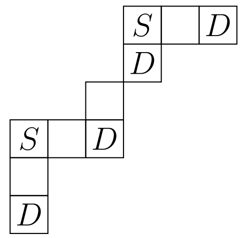


Figure 6.1: Illustration of the terminology used in the Murnaghan-Nakayama Rule for generalized characters. This broken border strip has two components, one having height 3 and the other having height 1. The height of the entire diagram is 4. Sharp corners are indicated with an S, and dull boxes are indicated by a D.

Strahov uses the following terminology in the statement of his theorem. These definitions are illustrated in Figure 6.1. A skew partition λ/ν is called a **broken border strip** if it contains no 2×2 boxes. (Thus, a broken border strip which is also connected is a rim hook. Two boxes in a Ferrers diagram whose corners touch are not considered to be connected.) A **sharp corner** in a skew diagram is a box which has a box both below it and to the right. A **dull box** has boxes neither to the right nor below it. Let $\mathrm{SC}(\lambda/\nu)$ and $\mathrm{DB}(\lambda/\nu)$ denote the set of sharp corners and dull boxes of λ/ν, respectively. Recall that the height of a rim hook λ/ν, denoted by $\langle\lambda/\nu\rangle$, is equal to the greatest row occupied by λ/ν minus the least row occupied by λ/ν. If λ/ν is a broken border strip, $\langle\lambda/\nu\rangle$ is defined to be the sum of heights of its connected components. Given a skew diagram λ/ν and a part i of λ, the number $\varphi_{\lambda/\nu,i}$ is defined by

$$\varphi_{\lambda/\nu,i} = (-1)^{\langle\lambda/\nu\rangle} \prod_{s \in \mathrm{SC}(\lambda/\nu)} [c_{\lambda,i} - c(s)] \prod_{\substack{d \in \mathrm{DB}(\lambda/\nu) \\ d \neq \lambda/i_-(\lambda)}} [c_{\lambda,i} - c(d)]^{-1}$$

when λ/ν is a broken border strip, and zero otherwise.

Strahov's result is as follows.

Theorem 6.3.1 (Murnaghan-Nakayama Rule for Generalized Characters)**.** *Let $\lambda, \rho \vdash n$. Let i be a part of λ, and let j be a part of ρ. Then*

$$\gamma^{\lambda,i}_{\mu,j} = \sum_{\substack{\nu \subseteq i_-(\lambda) \\ \nu \vdash n-j}} \varphi_{\lambda/\nu,i} \chi^{\nu}_{\mu\backslash j}.$$

The reason this theorem is particularly useful for evaluating $\gamma^{(n-k,1^k),n-k}_{\mu,j}$, $\gamma^{(n-k,1^k),1}_{\mu,j}$ and $\gamma^{(n-k-1,2,1^{k-1}),2}_{\mu,j}$ is that in all three cases, $i_-(\lambda)$ is a hook partition. Thus, every $\nu \subseteq i_-(\lambda)$ is also a hook partition, and $\chi^{\nu}_{\mu\backslash j}$ may be evaluated using Lemma 2.4.6.

6.3.2 Evaluation of $\gamma_{\mu,j}^{(n-k,1^k),n-k}$ and $\gamma_{\mu,j}^{(n-k,1^k),1}$

To determine the values of $\gamma_{\mu,j}^{(n-k,1^k),n-k}$ and $\gamma_{\mu,j}^{(n-k,1^k),1}$, it is helpful to deal with the cases when $\lambda = (n)$ or $\lambda = (1^n)$ separately; the reason is that in these cases, there are no sharp corners and the only dull box corresponds to the distinguished part of λ, so the product appearing in the definition of φ is empty. If $\lambda = (n)$, the height of λ/ν is 0, and thus $\varphi_{(n/\nu),n} = 1$. If $\lambda = (1^n)$ and $\nu = (1^{n-j})$, the height of λ/ν is $j-1$, and thus $\varphi_{(1^n/\nu),1} = (-1)^{j-1}$. Thus,

$$\gamma_{\mu,j}^{(n),n} = \chi_{\mu\backslash j}^{(n-j)} = 1$$

and

$$\gamma_{\mu,j}^{(1^n),1} = (-1)^{j-1}\chi_{\mu\backslash j}^{(1^{n-j})} = (-1)^{n-m(\mu)}.$$

The general case occurs if the partition $(n-k, 1^k)$ is a proper hook, *i.e.* if $1 \le k \le n-2$. To compute $\gamma_{\mu,j}^{(n-k,1^k),n-k}$ in this case, consider the partitions $\nu \vdash n-j$ satisfying $\nu \subseteq (n-k-1, 1^k)$. These are all of the form $\nu = (n-j-\ell, 1^\ell)$ for some $0 \le \ell \le n-j-1$. However, in some cases, not all partitions of this form will contribute to the sum. For large values of $n-j$, the values of ℓ for which there is a contribution to the expression for $\gamma_{\mu,j}^{(n-k,1^k),n-k}$ are limited by the restriction that $\nu \subseteq (n-k-1, 1^k)$. Thus, there are three cases to consider: if $n-j$ is small enough that none of the choices for ν can hit either end of the hook $(n-k-1, 1^k)$, if the size of $n-j$ permits exactly one end of the hook to be hit by some ν, and if $n-j$ is large enough that for each end of the hook there is some ν which hits it. These cases are stated in Table 6.5, and the range of validity for ℓ is indicated for each.

As for the value of $\varphi_{\lambda/\nu,i}$, the height of $(n-k, 1^k)/(n-j-\ell, 1^\ell)$ is equal to $k-\ell-1$ when $\ell < k$, and 0 when $\ell = k$. The partition $(n-k, 1^k)/(n-j-\ell, 1^\ell)$ has exactly two dull boxes: one at the end of the first row, and one at the bottom of the first column. Providing $\ell < k$, both boxes remain in $(n-k, 1^k)/(n-j-\ell, 1^\ell)$, and thus

$$\varphi_{(n-k,1^k)/(n-j-\ell,1^\ell),n-k} = \frac{(-1)^{k-\ell-1}}{(n-k-1)-(-k)} = \frac{(-1)^{k-\ell-1}}{n-1}.$$

There is one exceptional value of ℓ, namely, $\ell = k$, for which the dull box at the bottom of the first column is removed with ν. When this occurs, $\varphi_{(n-k,1^k)/(n-j-k,1^k),n-k} = 1$. The cases in which this applies are indicated in Table 6.5. Collecting the cases and applying Theorem 6.3.1 gives the following expressions for $\gamma_{\mu,j}^{(n-k,1^k),n-k}$.

Lemma 6.3.2 (S.). *Suppose $0 \le k \le n-2$ and $1 \le j \le n-1$. Let $\mu \vdash n$ and let j be a*

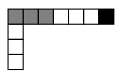
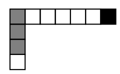
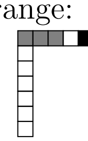
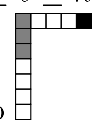
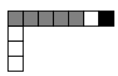
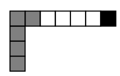
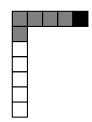
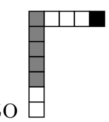
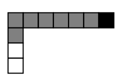
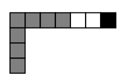
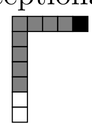
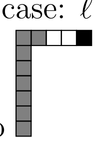
	$k \leq n-k-1$	$k \geq n-k-1$
ν can hit neither end of λ	Occurs when $n-j \leq k$ Valid range: $0 \leq \ell \leq n-j-1$  to 	Occurs when $n-j < n-k$ Valid range: $0 \leq \ell \leq n-j-1$  to 
ν can hit exactly one end of λ	Occurs when $k < n-j < n-k$ Valid range: $0 \leq \ell \leq k$ Exceptional case: $\ell = k$  to 	Occurs when $n-k \leq n-j \leq k$ Valid range: $k-j < \ell \leq n-j-1$  to 
ν can hit both ends of λ	Occurs when $n-j \geq n-k$ Valid range: $k-j < \ell \leq k$ Exceptional case: $\ell = k$  to 	Occurs when $n-j > k$ Valid range: $k-j < \ell \leq k$ Exceptional case: $\ell = k$  to 

Table 6.5: Ranges of validity ℓ, where $\nu = (n-j-\ell, 1^\ell)$, in the computation of $\gamma_{\mu,j}^{(n-k,1^k),n-k}$. The diagrams illustrate the extreme points of the range in each case: shaded boxes indicate ν, and the black box is the distinguished box.

part of μ. If $k \le n-k-1$, then

$$\gamma_{\mu,j}^{(n-k,1^k),n-k} = \begin{cases} \sum_{0\le\ell\le n-j-1} \frac{(-1)^{k-\ell-1}}{n-1}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)} & \text{if } (n-j)\le k, \\ \sum_{0\le\ell<k} \frac{(-1)^{k-\ell-1}}{n-1}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)} + \chi_{\mu\backslash j}^{(n-j-k,1^k)} & \text{if } k<(n-j)<n-k, \\ \sum_{k-j<\ell<k} \frac{(-1)^{k-\ell-1}}{n-1}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)} + \chi_{\mu\backslash j}^{(n-j-k,1^k)} & \text{if } n-k\le(n-j). \end{cases}$$

If $k \ge n-k-1$, then

$$\gamma_{\mu,j}^{(n-k,1^k),n-k} = \begin{cases} \sum_{0\le\ell\le n-j-1} \frac{(-1)^{k-\ell-1}}{n-1}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)} & \text{if } (n-j)<n-k \\ \sum_{k-j<\ell\le n-j-1} \frac{(-1)^{k-\ell-1}}{n-1}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)} & \text{if } n-k\le(n-j)\le k \\ \sum_{k-j<\ell<k} \frac{(-1)^{k-\ell-1}}{n-1}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)} + \chi_{\mu\backslash j}^{n-j-k,1^k} & \text{if } k<(n-j). \end{cases}$$

(Although the argument used to derive these formulas assumed that $k \ge 1$, they also agree with the special case when $k = 0$.) The expressions for $\gamma_{\mu,j}^{(n-k,1^k),1}$ may be derived in two different ways. The first method is to use Theorem 6.3.1, in the same manner as the proof of Lemma 6.3.2. (This is the method used in the proof below.) The second method is to use Lemma 6.3.2 along with the symmetry property of Corollary 6.2.5. Using either method gives the following.

Lemma 6.3.3. *Suppose $1 \le k \le n-1$ and $1 \le j \le n-1$. Let $\mu \vdash n$ and let j be a part of μ. If $k \le n-k-1$, then*

$$\gamma_{\mu,j}^{(n-k,1^k),1} = \begin{cases} \sum_{0\le\ell\le n-j-1} \frac{(-1)^{k-\ell}}{n-1}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)} & \text{if } (n-j)\le k, \\ \sum_{0\le\ell<k} \frac{(-1)^{k-\ell}}{n-1}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)} & \text{if } k<(n-j)<n-k, \\ \sum_{k-j<\ell<k} \frac{(-1)^{k-\ell}}{n-1}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)} + (-1)^{j-1}\chi_{\mu\backslash j}^{(n-k,1^{k-j})} & \text{if } (n-j)\ge n-k. \end{cases}$$

If $k \ge n-k-1$, then

$$\gamma_{\mu,j}^{(n-k,1^k),1} = \begin{cases} \sum_{0\le\ell\le n-j-1} \frac{(-1)^{k-\ell}}{n-1}\chi_{\mu,j}^{(n-j-\ell,1^\ell)} & \text{if } (n-j)<n-k \\ \sum_{k-j<\ell\le n-j-1} \frac{(-1)^{k-\ell}}{n-1}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)} + (-1)^{j-1}\chi_{\mu\backslash j}^{(n-k,1^{k-j})} & \text{if } n-k\le(n-j)\le k \\ \sum_{k-j<\ell<k} \frac{(-1)^{k-\ell}}{n-1}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)} + (-1)^{j-1}\chi_{\mu\backslash j}^{(n-k,1^{k-j})} & \text{if } (n-j)>k. \end{cases}$$

Proof. The case analysis is summarized in Table 6.6. For values of ℓ for which the removal of ν does not remove the box at the end of the first row,

$$\varphi_{(n-k,1^k)/(n-j-\ell,1^\ell),1} = \frac{(-1)^{k-\ell-1}}{(-k)-(n-k-1)} = \frac{(-1)^{k-\ell}}{n-1}.$$

The exceptional case, in which the box at the end of the first row is removed, occurs when $\ell = k-j$; in these cases, $\varphi_{(n-k,1^k)/(n-k,1^{k-j}),1} = (-1)^{j-1}$.

□

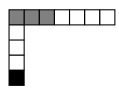
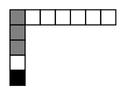
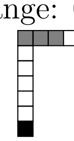
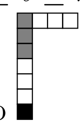
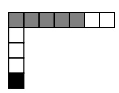
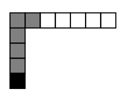
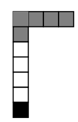
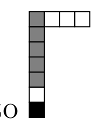
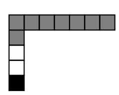
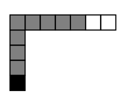
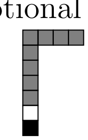
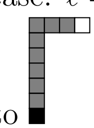
	$k \leq n-k-1$	$k \geq n-k-1$
ν can hit neither end of λ	Occurs when $n-j \leq k$ Valid range: $0 \leq \ell \leq n-j-1$  to 	Occurs when $n-j < n-k$ Valid range: $0 \leq \ell \leq n-j-1$  to 
ν can hit exactly one end of λ	Occurs when $k < n-j < n-k$ Valid range: $0 \leq \ell < k$  to 	Occurs when $n-k \leq n-j \leq k$ Valid range: $k-j \leq \ell \leq n-j-1$ Exceptional case: $\ell = k-j$  to 
ν can hit both ends of λ	Occurs when $n-j \geq n-k$ Valid range: $k-j \leq \ell < k$ Exceptional case: $\ell = k-j$  to 	Occurs when $n-j \geq k$ Valid range: $k-j \leq \ell < k$ Exceptional case: $\ell = k-j$  to 

Table 6.6: Ranges of validity for ℓ, where $\nu = (n-j-\ell, 1^\ell)$, in the computation of $\gamma_{\mu,j}^{(n-k,1^k),1}$.

6.3.3 Evaluation of $\gamma_{\mu,j}^{(n-k-1,2,1^{k-1}),2}$

The generalized characters $\gamma_{\mu,j}^{(n-k-1,2,1^{k-1}),2}$ for $1 \le k \le n-3$ may be evaluated in a manner similar to the generalized character evaluations done in the preceding section. Prior to stating the formula for $\gamma_{\mu,j}^{(n-k-1,2,1^{k-1}),2}$ in the general case, two exceptional cases must be dealt with. Let $\lambda = (n-k-1, 2, 1^{k-1})$. First, suppose $j = 1$. In this case, the only partition $\nu \subseteq (n-k-1, 1^k)$ such that $\nu \vdash n-j$ is $(n-k-1, 1^k)$. Removing this from λ leaves only a single box, so $\varphi_{\lambda/\nu,2} = 1$ in this case. Thus,

$$\gamma_{\mu,j}^{(n-k-1,2,1^{k-1}),2} = \chi_{\mu\backslash j}^{(n-k-1,1^k)}.$$

Second, suppose $j = n$. In this case, the formula for $\gamma_{(n),n}^{\lambda}$ has already been given by Lemma 6.2.7, namely,

$$\gamma_{(n),n}^{(n-k-1,2,1^{k-1}),2} = 0.$$

The formula in the general case is as follows.

Lemma 6.3.4 (S.). *Let $\mu \vdash n$ and let j be a part of μ. Suppose $1 \le k \le n-3$ and $2 \le j \le n-1$. If $k \le n-k-2$, then $\gamma_{\mu,j}^{(n-k-1,2,1^{k-1}),2}$ is equal to*

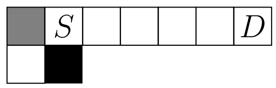
$$\begin{cases} \sum_{0\le \ell \le n-j-1} \frac{(-1)^{k-\ell}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)}}{k(n-k-2)} & \text{if } (n-j) \le k, \\ \sum_{0\le \ell < k} \frac{(-1)^{k-\ell}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)}}{k(n-k-2)} - \frac{\chi_{\mu\backslash j}^{(n-j-k,1^k)}}{n-k-2} & \text{if } k < (n-j) < (n-k-1), \\ \sum_{k-j+1<\ell<k} \frac{(-1)^{k-\ell}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)}}{k(n-k-2)} + \frac{(-1)^j\chi_{\mu\backslash j}^{(n-k-1,1^{k-j+1})}}{k} & \\ \qquad\qquad - \frac{\chi_{\mu\backslash j}^{(n-j-k,1^k)}}{n-k-2} & \text{if } (n-j) \ge (n-k-1). \end{cases}$$

If $k \ge n-k-2$, then $\gamma_{\mu,j}^{(n-k-1,2,1^{k-1}),2}$ is equal to

$$\begin{cases} \sum_{0\le \ell \le n-j-1} \frac{(-1)^{k-\ell}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)}}{k(n-k-2)} & \text{if } (n-j) \le (n-k-2), \\ \sum_{k-j+1<\ell\le n-j-1} \frac{(-1)^{k-\ell}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)}}{k(n-k-2)} + \frac{(-1)^j\chi_{\mu\backslash j}^{(n-k-1,1^{k-j+1})}}{k} & \text{if } (n-k-2) < (n-j) \le k, \\ \sum_{k-j+1<\ell<k} \frac{(-1)^{k-\ell}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)}}{k(n-k-2)} + \frac{(-1)^j\chi_{\mu\backslash j}^{(n-k-1,1^{k-j+1})}}{k} & \\ \qquad\qquad - \frac{1}{n-k-2}\chi_{\mu\backslash j}^{(n-j-k,1^k)} & \text{if } (n-j) \ge (k+1). \end{cases}$$

Proof. The proof is by a case analysis similar to the proofs of Lemmas 6.3.2 and 6.3.3. Throughout the following, $\lambda = (n-k-1, 2, 1^{k-1})$. Each case is illustrated with a diagram in which grey boxes indicate ν, the black box indicates the distinguished box, and white boxes indicate $2_{-}(\lambda)/\nu$. Sharp corners and dull boxes of λ/ν are indicated on diagrams by S and D, respectively, with the exception of the distinguished box, which is always a dull box.

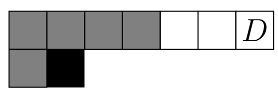
The cases $k = 1$ and $k = n-3$ are treated differently than $2 \leq k \leq n-4$, but give the same result. Suppose $k = 1$. (The $k = n-3$ case is nearly identical.) If $j = n-1$, then $(n-j) \leq k$, and a typical diagram is of the following form.



In this case, the formula in the statement of the Theorem gives

$$\gamma^{(n-2,2),2}_{(n-1,1),n-1} = \frac{(-1)^k}{k(n-k-2)} = \frac{(-1)^k}{n-3},$$

which agrees with the value computed in Corollary 6.2.12. When $3 \leq j \leq n-2$, then $k < (n-j) < (n-k-1)$. In this case, the only partitions $\nu \vdash n-j$ which are contained in $(n-2,1)$ are $(n-j)$ and $(n-j-1,1)$:

 and  .

In both cases, λ/ν has height 0, no sharp corners, and one dull box (of content $n-3$) aside from the dull box corresponding to the distinguished part 2, so

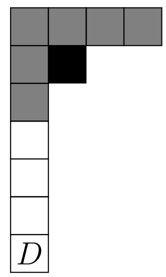
$$\varphi_{\lambda/\nu,2} = \frac{1}{0-(n-k-2)} = -\frac{1}{n-3},$$

which agrees with the stated formula for this case. When $j = 2$, $(n-j) \geq (n-k-1)$, and the diagrams corresponding to the two possibilities for ν are

 and  .

The argument proceeds as in the case when $3 \leq j \leq n-2$, with the exception that when $\nu = (n-j)$, λ/ν has no dull boxes, so $\varphi_{\lambda/\nu,2} = 1$. This agrees with the stated formula.

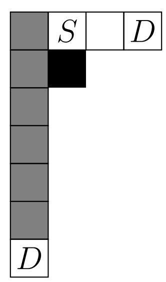
For the general case, $k \leq 2 \leq n-4$, there are six cases to consider. As they are all similar, one case is singled out here for a detailed presentation. This case has been selected because it illustrates all the peculiarities which must be taken into account when computing $\gamma_{\mu,j}^{(n-k-1,2,1^{k-1}),2}$, but did not arise when computing $\gamma_{\mu,j}^{(n-k,1^k),n-k}$ and $\gamma_{\mu,j}^{(n-k,1^k),1}$. Suppose $k \geq n-k-2$ and $(n-k-2) < (n-j) \leq k$. Let $\nu = (n-j-\ell, 1^\ell)$. The range of ℓ for which $\nu \subseteq 2_{-}(\lambda)$ is $k-j+1 \leq \ell \leq n-j-1$. When $\ell = k-j+1$, $\nu = (n-k-1, 1^{k-j+1})$, so λ/ν has height j, no sharp corners, and one dull box (of content $-k$) aside from the distinguished dull box, as illustrated here:



Thus, in this case,

$$\varphi_{\lambda/\nu,2} = \frac{(-1)^j}{k}.$$

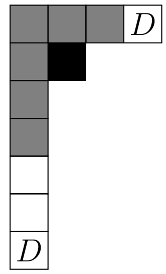
When $\ell = n-j-1$, $\nu = (1^{n-j})$, so λ/ν has height $k-\ell$, one sharp corner of content 1, and two dull boxes aside from the distinguished box, having contents $(n-k-2)$ and $-k$.



Thus,

$$\varphi_{\lambda/\nu,2} = \frac{(-1)^{k-\ell}(0-1)}{0-(-k))(0-(n-k-2)} = \frac{(-1)^{k-\ell}}{k(n-k-2)}.$$

Finally, when $k-j+1 < \ell < n-j-1$, λ/ν has height $k-\ell-1$, no sharp corners, and two dull boxes aside from the distinguished box, having contents $(n-k-2)$ and $-k$.



.

Thus,

$$\varphi_{\lambda/\nu,2} = \frac{(-1)^{k-\ell-1}}{(0-(-k))(0-(n-k-2))} = \frac{(-1)^{k-\ell}}{k(n-k-2)}.$$

Combining these cases gives the formula in the statement of the Theorem. $\square$

Chapter 7

Combinatorial Applications of $Z_1(n)$

This chapter explores some of the combinatorial consequences of the results given in Chapter 6. Three applications of the theory of $Z_1(n)$ and generalized characters are given here. The first, given in Section 7.1, is a solution to the $(p, n-1, n)$-dipole problem for all orientable surfaces, using the $Z_1(n)$-encoding of this problem from Chapter 5. Much like the case for ordinary diploles, the genus polynomials for $(p, n-1, n)$-dipoles can be expressed as linear combinations of binomial coefficients of the form $\binom{t+i}{j}$, where t is an indeterminate. The second application, given in Section 7.2, is a new approach to determining non-transitive powers of Jucys-Murphy elements. The third application, given in Section 7.3, gives an approach to the Z_1-factorization problem in the special case when the permutation being factorized is a full cycle. This is a refinement of previous work decomposing a full cycle into permutations of specified cycle types in that it also allows the length of the cycle containing n in each factor to be specified.

7.1 The number of $(p, n-1, n)$-dipoles in a genus g surface

7.1.1 A general form for the generating series

Recall from Lemma 5.2.2 that the number of $(p, n-1, n)$-dipoles having face degree sequence 2λ and a root face of degree $2i$ is given by

$$d^{p,n-1}_{\lambda,i} = [K_{(p,n-p),p}]K_{\lambda,i}K_{(n-1,1),1}.$$

The material from Chapter 6, specifically, Theorem 6.1.9, allows this quantity to be expressed in terms of generalized characters as follows.

$$d_{\lambda,i}^{p,n-1} = \frac{|\mathcal{C}_{\lambda,i}|(n-2)!}{n!} \sum_{\substack{\rho \vdash n, \\ \ell \in \rho}} \frac{\gamma_{\lambda,i}^{\rho,\ell} \gamma_{(n-1,1),1}^{\rho,\ell} \gamma_{(p,n-p),p}^{\rho,\ell}}{d_{\ell_-(\rho)}} \frac{d_\rho}{d_{\ell_-(\rho)}}.$$

Applying the formula for $\gamma_{(n-1,1),1}^{\rho,\ell}$ given in Lemma 6.2.7, the terms of this summation vanish unless (ρ, ℓ) is one of three types: $((n-k, 1^k), n-k)$, $((n-k, 1^k), 1)$, or $((n-k-1, 2, 1^{k-1}), 2)$. The expression for $d_{\lambda,i}^{p,n-1}$ splits into a sum over these three cases, as given in the following.

Theorem 7.1.1 (S.). *Let $\lambda \vdash n$ and let $i \in \lambda$. Let $1 \leq p \leq n-1$. Then the number of $(p, n-1, n)$-dipoles (with unlabelled ordinary edges) having face degree sequence 2λ and a root face of degree $2i$ is given by*

$$d_{\lambda,i}^{p,n-1} = \frac{|\mathcal{C}_{\lambda,i}|(n-2)!}{n!} \left(A_{n,p}^{\lambda,i} + B_{n,p}^{\lambda,i} + C_{n,p}^{\lambda,i} \right),$$

where

$$A_{n,p}^{\lambda,i} = \sum_{0 \leq k \leq n-2} \frac{(-1)^k \gamma_{\lambda,i}^{(n-k,1^k),n-k} \gamma_{(p,n-p),p}^{(n-k,1^k),n-k}}{d_{(n-k-1,1^k)}} \frac{n-1}{n-k-1},$$

$$B_{n,p}^{\lambda,i} = \sum_{1 \leq k \leq n-1} \frac{(-1)^{k-1} \gamma_{\lambda,i}^{(n-k,1^k),1} \gamma_{(p,n-p),p}^{(n-k,1^k),1}}{d_{(n-k,1^{k-1})}} \frac{n-1}{k},$$

and

$$C_{n,p}^{\lambda,i} = \sum_{1 \leq k \leq n-3} \frac{(-1)^k \gamma_{\lambda,i}^{(n-k-1,2,1^{k-1}),2} \gamma_{(p,n-p),p}^{(n-k-1,2,1^{k-1}),2}}{d_{(n-k-1,1^k)}} \frac{nk(n-k-2)}{(n-k-1)(k+1)}.$$

The ratios $\frac{d_\rho}{d_{\ell_-(\rho)}}$ have been determined by a routine determination of the number of standard Young tableaux in each case. Although the quantity $d_{\ell_-(\rho)}$ appearing in the denominator of each summand could be evaluated similarly, it is convenient for later purposes to leave it as it stands.

Much like the case of ordinary dipoles, the expression given in Theorem 7.1.1 has a more explicit form when summed over all partitions having m parts. By the Euler-Poincaré formula, this gives the number of $(p, n-1, n)$-dipoles in an orientable surface of genus $\frac{n-m}{2}$. Let

$$d_m^{p,n-1} = \sum_{\substack{\lambda \vdash n, \\ m(\lambda)=m, i \in \lambda}} d_{\lambda,i}^{p,n-1}.$$

Define $F_{n,p}$ by

$$F_{n,p} := \frac{1}{n!} \sum_{\substack{\lambda \vdash n, \\ m(\lambda)=m, i \in \lambda}} |\mathcal{C}_{\lambda,i}| F^{\lambda,i}_{n,p}$$

where F is either A, B or C. Then

$$d^{p,n-1}_m = (n-2)!(A_{n,p} + B_{n,p} + C_{n,p}).$$

This expression may be simplified by evaluating expressions of the form

$$\sum_{\substack{\lambda \vdash n, \\ m(\lambda)=m, i \in \lambda}} \frac{|\mathcal{C}_{\lambda,i}| \gamma^{\rho,\ell}_{\lambda,i}}{d_{\ell_-(\rho)}} \tag{7.1}$$

In light of Lemma 6.2.8, the quantity is just, up to to scaling, the sum

$$\sum_{\lambda \vdash n} \frac{|\mathcal{C}_\lambda| \chi^\rho_\lambda}{d_\rho}.$$

Consequently, the expression in Equation (7.1) may be determined from known results regarding sums of ordinary characters. However, it may also be derived from the theory developed in Chapter 6 in the following manner.

Lemma 7.1.2. *Let $\rho \vdash n$ and let ℓ be a part of ρ. Let T be any standard Young tableau of shape ρ. Then*

$$\sum_{\substack{\lambda \vdash n, \\ m(\lambda)=m, i \in \lambda}} \frac{|\mathcal{C}_{\lambda,i}|}{d_{\ell_-(\rho)}} \gamma^{\rho,\ell}_{\lambda,i} = e_{n-m}(\mathbf{c}_\rho) = [t^m] \prod_{1 \le i \le n} (t + c_i(T)).$$

Proof. It is routine to show that the elementary symmetric polynomial $e_{n-m}(x_2, \ldots, x_n)$ evaluated at the Jucys-Murphy elements is given by

$$e_{n-m}(J_2, \ldots, J_n) = \sum_{\substack{\lambda \vdash n \\ m(\lambda)=m}} K_\lambda = \sum_{\substack{\lambda \vdash n \\ m(\lambda)=m, i \in \lambda}} K_{\lambda,i}.$$

Thus, by Lemma 6.2.1,

$$\sum_{\substack{\lambda \vdash n \\ m(\lambda)=m, i \in \lambda}} \frac{|\mathcal{C}_{\lambda,i}|}{d_{\ell_-(\rho)}} \gamma^{\rho,\ell}_{\lambda,i} = e_{n-m}(\mathbf{c}_\rho).$$

The result then follows by using the generating series for elementary symmetric functions. $\square$

The two cases of interest to the $(p, n-1, n)$-dipole problem are as follows.

Corollary 7.1.3. *Let $\rho = (n-k, 1^k)$, and let $\ell \in \{1, n-k\}$. Then*

$$\sum_{\substack{\lambda \vdash n,\\ m(\lambda)=m, i\in\lambda}} \frac{|\mathcal{C}_{\lambda,i}|}{d_{\ell_-(\rho)}} \gamma_{\lambda,i}^{\rho,\ell} = [t^m] n! \binom{t+n-k-1}{n}.$$

Proof. The contents along the first row of a tableau of shape $(n-k, 1^k)$ are $0, 1, 2, \dots, n-k-1$, and the contents in the first column (excluding the box in the first row) are $-1, -2, \dots, -k$. Thus,

$$\prod_{1\le i\le n} (t + c_i(T)) = \prod_{-k\le i\le n-k-1} (t+i),$$

from which the result follows. □

Corollary 7.1.4. *Let $\rho = (n-k-1, 2, 1^{k-1})$ and let $\ell \in \{n-k-1, 2, 1\}$. Then*

$$\sum_{\substack{\lambda \vdash n,\\ m(\lambda)=m, i\in\lambda}} \frac{|\mathcal{C}_{\lambda,i}|}{d_{\ell_-(\rho)}} \gamma_{\lambda,i}^{\rho,\ell} = [t^m](n-1)! t \binom{t+n-k-2}{n-1}.$$

Proof. This is obtained in a similar manner to the preceding Corollary, except there is an additional box of content 0 (the one at the end of the row of length 2), and one box of content $n-k-1$ has been removed. □

Combining these Corollaries with Theorem 7.1.1, along with routine simplification, gives the following expression for the generating series for $(p, n-1, n)$-dipoles as a linear combination of binomial coefficients in the indeterminate t.

Theorem 7.1.5 (S.)**.** *The number of $(p, n-1, n)$-dipoles (with unlabelled ordinary edges) in an orientable surface of genus g is given by*

$$d_{n-2g}^{p,n-1} = (n-2)! [t^{n-2g}] D_{n,p}(t)$$

where

$$D_{n,p}(t) = \binom{t+n-1}{n} + \sum_{1\le k\le n-2} (-1)^k (n-1) \left(\frac{\gamma_{(p,n-p),p}^{(n-k,1^k),n-k}}{n-k-1} - \frac{\gamma_{(p,n-p),p}^{(n-k,1^k),1}}{k} \right) \binom{t+n-k-1}{n}$$
$$+ \binom{t}{n} + \sum_{1\le k\le n-3} (-1)^k \frac{k(n-k-2)\gamma_{(p,n-p),p}^{(n-k-1,2,1^{k-1}),2}}{(n-k-1)(k+1)} t \binom{t+n-k-2}{n-1}.$$

7.1.2 Genus polynomials for $(p, n-1, n)$-dipoles

While the generating series given in Theorem 7.1.5 is most compactly expressed with some of the generalized characters left unevaluated, in order to obtain series for specific values of p it is necessary to evaluate the generalized characters which arise. In this section, formulas for these generalized characters are provided.

The first case to consider is when $p = 1$. Although $(1, n-1, n)$-dipoles are simply rooted dipoles on $n-1$ edges in which the root has been "doubled," and this problem may be solved using central methods, it is useful to record the form of the series in Theorem 7.1.5 when $p = 1$ both for completeness and as a verification of its correctness. When $p = 1$, all the generalized characters appearing in the series may be computed using Theorem 6.2.7. The result is as follows:

$$\begin{aligned} D_{n,1}(t) = &\binom{t+n-1}{n} + \sum_{1\le k\le n-2} \frac{(n-1)^2}{k(n-k-1)}\binom{t+n-k-1}{n} \\ &+ \binom{t}{n} + \sum_{1\le k\le n-3} \frac{k(n-k-2)}{(n-k-1)(k+1)} t\binom{t+n-k-2}{n-1}. \end{aligned}$$

When $2 \le p \le n-1$, the values of $\gamma^{(n-k,1^k),n-k}_{(n-p,p),p}$, $\gamma^{(n-k,1^k),1}_{(n-p,p),p}$ and $\gamma^{(n-k-1,2,1^{k-1}),2}_{(n-p,p),p}$ which arise in the expression for the number of $(p, n-1, n)$-dipoles can be evaluated using Lemmas 6.3.2, 6.3.3, 6.3.4 and 2.4.3. Taking $\mu = (n-p, p)$ and $j = p$, the partition $\mu \setminus j$ arising in these expressions is just $(n-p)$. Thus, $\chi^{(n-j-\ell,1^\ell)}_{\mu\setminus j} = (-1)^\ell$. After routine simplification, the expressions for these generalized characters reduce to the following.

Lemma 7.1.6 (S.). *Let $0 \le k \le n-2$ and $1 \le p \le n-1$. If $k \le n-k-1$, then*

$$\gamma^{(n-k,1^k),n-k}_{(n-p,p),p} = \begin{cases} (-1)^{k-1}\frac{n-p}{n-1} & \text{if } (n-p) \le k, \\ (-1)^k \frac{n-k-1}{n-1} & \text{if } k < (n-p) < n-k, \\ (-1)^k \frac{n-p}{n-1} & \text{if } n-k \le (n-p). \end{cases}$$

If $k \ge n-k-1$, then

$$\gamma^{(n-k,1^k),n-k}_{(n-p,p),p} = \begin{cases} (-1)^{k-1}\frac{n-p}{n-1} & \text{if } (n-p) < n-k, \\ (-1)^{k-1} \frac{n-k-1}{n-1} & \text{if } n-k \le (n-p) \le k, \\ (-1)^k \frac{n-p}{n-1} & \text{if } k < (n-p). \end{cases}$$

Lemma 7.1.7 (S.). *Let $1 \le k \le n-1$, and let $1 \le p \le n-1$. If $k \le n-k-1$, then*

$$\gamma^{(n-k,1^k),1}_{(n-p,p),p} = \begin{cases} (-1)^{k}\frac{n-p}{n-1} & \text{if } (n-p) \le k, \\ (-1)^k \frac{k}{n-1} & \text{if } k < (n-p) < n-k, \\ (-1)^{k-1} \frac{n-p}{n-1} & \text{if } n-k \le (n-p). \end{cases}$$

If $k \geq n-k-1$, then

$$\gamma_{(n-p,p),p}^{(n-k,1^k),1} = \begin{cases} (-1)^k \frac{n-p}{n-1} & \text{if } (n-p) < n-k, \\ (-1)^{k-1} \frac{k}{n-1} & \text{if } n-k \leq (n-p) \leq k, \\ (-1)^{k-1} \frac{n-p}{n-1} & \text{if } k < (n-p). \end{cases}$$

Lemma 7.1.8 (S.)**.** *Let $1 \leq k \leq n-3$, and let $2 \leq p \leq n-1$. If $k \leq n-k-2$, then*

$$\gamma_{(n-p,p),p}^{(n-k-1,2,1^{k-1}),2} = \begin{cases} (-1)^k \frac{n-p}{k(n-k-2)} & \text{if } (n-p) \leq k, \\ 0 & \text{if } k < (n-p) < n-k-1, \\ (-1)^{k+1} \frac{n-p}{k(n-k-2)} & \text{if } (n-p) \geq n-k-1. \end{cases}$$

If $k \geq n-k-2$, then

$$\gamma_{(n-p,p),p}^{(n-k-1,2,1^{k-1}),2} = \begin{cases} (-1)^k \frac{n-p}{k(n-k-2)} & \text{if } (n-p) \leq n-k-2, \\ 0 & \text{if } n-k-2 < (n-p) \leq k, \\ (-1)^{k+1} \frac{n-p}{k(n-k-2)} & \text{if } (n-p) > k. \end{cases}$$

These generalized character evaluations can be used to give explicit expressions for the generating series for $(p, n-1, n)$-dipoles with respect to the number of faces, as follows.

Theorem 7.1.9 (S.)**.** *Let $n \geq 4$. When $2 \leq p \leq \frac{n-1}{2}$, the generating series for $(p, n-1, n)$-dipoles is*

$$D_{n,p}(t) = \binom{t+n-1}{n} + \sum_{p \leq k \leq n-p-1} \frac{(n-1)(n-p)}{k(n-k-1)} \binom{t+n-k-1}{n} + \binom{t}{n} - \sum_{p-1 \leq k \leq n-p-1} \frac{(n-p)}{(n-k-1)(k+1)} t \binom{t+n-k-2}{n-1}.$$

When $\max\{2, \frac{n-1}{2}\} \leq p \leq n-1$, the generating series for $(p, n-1, n)$ dipoles is

$$D_{n,p}(t) = \binom{t+n-1}{n} - \sum_{n-p \leq k \leq p-1} \frac{(n-1)(n-p)}{k(n-k-1)} \binom{t+n-k-1}{n} + \binom{t}{n} + \sum_{n-p \leq k \leq p-2} \frac{(n-p)}{(n-k-1)(k+1)} t \binom{t+n-k-2}{n-1}.$$

Proof. The proof proceeds by evaluating the generalized characters appearing in Theorem 7.1.5 using Lemmas 7.1.6, 7.1.7 and 7.1.8. First, consider the case when $2 \leq p \leq \frac{n-1}{2}$. In this case, when $1 \leq k < p$,

$$\begin{aligned} \frac{\gamma_{(p,n-p),p}^{(n-k,1^k),n-k}}{n-k-1} - \frac{\gamma_{(p,n-p),p}^{(n-k,1^k),1}}{k} &= \frac{(-1)^k}{n-1} - \frac{(-1)^k}{n-1} \\ &= 0. \end{aligned}$$

When $p \leq k \leq n-p-1$,

$$\begin{aligned} \frac{\gamma_{(p,n-p),p}^{(n-k,1^k),n-k}}{n-k-1} - \frac{\gamma_{(p,n-p),p}^{(n-k,1^k),1}}{k} &= \frac{(-1)^k(n-p)}{(n-1)(n-k-1)} - \frac{(-1)^{k-1}(n-p)}{(n-1)k} \\ &= \frac{(-1)^k(n-p)}{k(n-k-1)}. \end{aligned}$$

When $n-p \leq k \leq n-2$,

$$\begin{aligned} \frac{\gamma_{(p,n-p),p}^{(n-k,1^k),n-k}}{n-k-1} - \frac{\gamma_{(p,n-p),p}^{(n-k,1^k),1}}{k} &= \frac{(-1)^{k-1}}{n-1} - \frac{(-1)^{k-1}}{n-1} \\ &= 0. \end{aligned}$$

When $k < p-1$ or $k \geq n-p$, then $\gamma_{(p,n-p),p}^{(n-k-1,2,1^{k-1}),2} = 0$. For $p-1 \leq k \leq n-p-1$,

$$\gamma_{(n-p,p),p}^{(n-k-1,2,1^{k-1}),2} = \frac{(-1)^{k-1}(n-p)}{k(n-k-2)}.$$

Combining these facts gives the result in the statement of the Theorem.

Next, consider the case when $\max\{2, \frac{n-1}{2}\} \leq p \leq n-1$. In this case, when $1 \leq k < n-p$,

$$\begin{aligned} \frac{\gamma_{(p,n-p),p}^{(n-k,1^k),n-k}}{n-k-1} - \frac{\gamma_{(p,n-p),p}^{(n-k,1^k),1}}{k} &= \frac{(-1)^k}{n-1} - \frac{(-1)^k}{n-1} \\ &= 0. \end{aligned}$$

When $n-p \leq k \leq p-1$,

$$\begin{aligned} \frac{\gamma_{(p,n-p),p}^{(n-k,1^k),n-k}}{n-k-1} - \frac{\gamma_{(p,n-p),p}^{(n-k,1^k),1}}{k} &= \frac{(-1)^{k-1}(n-p)}{(n-1)(n-k-1)} - \frac{(-1)^k(n-p)}{(n-1)k} \\ &= \frac{(-1)^{k-1}(n-p)}{k(n-k-1)}. \end{aligned}$$

When $p \le k \le n-2$,

$$\frac{\gamma_{(p,n-p),p}^{(n-k,1^k),n-k}}{n-k-1} - \frac{\gamma_{(p,n-p),p}^{(n-k,1^k),1}}{k} = \frac{(-1)^{k-1}}{n-1} - \frac{(-1)^{k-1}}{n-1}$$
$$= 0.$$

When $k < n-p$ or $k > p-2$, then $\gamma_{(p,n-p),p}^{(n-k-1,2,1^{k-1}),2} = 0$. For $n-p \le k \le p-2$,

$$\gamma_{(n-p,p),p}^{(n-k-1,2,1^{k-1}),2} = \frac{(-1)^k(n-p)}{k(n-k-2)}.$$

Combining these facts gives the result in the statement of the Theorem. $\square$

Expressions for these series at small values of n are given in Appendix A. Examination of these tables suggests an unexpected symmetry in $(p, n-1, n)$-dipoles:

Conjecture 7.1.10. *Let $p, p' \ge 2$ and let $p + p' = n + 1$. Then*

$$D_{n,p}(t) = D_{n,p'}(t).$$

In other words, the number of $(p, n-1, n)$-dipoles in a surface of genus g is equal to the number of $(p', n-1, n)$-dipoles in the same surface.

This observation cannot be explained by either of the "obvious" symmetries of Equations (3.1) and (3.2). Theorem 7.1.9 provides a useful starting point to providing an algebraic proof of this conjecture, and naturally, the more difficult problem of finding a combinatorial proof remains an interesting open problem.

Although Theorem 7.1.9 gives a full solution to the $(p, n-1, n)$-dipole problem for all orientable surfaces, there is an intermediate specialization of Theorem 7.1.1 which has a particularly interesting form. Let

$$d_\lambda^{n,p} = \sum_{i \in \lambda} d_{\lambda,i}^{p,n-1}$$

be the number of $(p, n-1, n)$ dipoles with face degree sequence 2λ. In other words, this is a specialization which "forgets" the degree of the face containing the label n, but retains all other information about the face degree sequence. Let

$$F_{n,p}^{\lambda} = \sum_{i \in \lambda} |\mathcal{C}_{\lambda,i}| F_{n,p}^{\lambda,i}$$

when F is one of A, B or C. By Theorem 7.1.9,

$$d_\lambda^{n,p} = \frac{(n-2)!}{n!}(A_{n,p}^\lambda + B_{n,p}^\lambda + C_{n,p}^\lambda).$$

The terms in this sum may be simplified by using Lemma 6.2.8, which relates sums of generalized characters to ordinary characters according to the formula

$$\sum_{i \in \lambda} |\mathcal{C}_{\lambda,i}| \gamma_{\lambda,i}^{\mu,j} = \frac{|\mathcal{C}_\lambda| d_{j_-(\mu)}}{d_\mu} \chi_\lambda^\mu.$$

Thus,

$$A_{n,p}^\lambda = |\mathcal{C}_\lambda| \sum_{0 \le k \le n-2} (-1)^k \gamma_{(p,n-p),p}^{(n-k,1^k),n-k} \frac{n-1}{(n-k-1)d_{(n-k,1^k)}} \chi_\lambda^{(n-k,1^k)},$$

$$B_{n,p}^\lambda = |\mathcal{C}_\lambda| \sum_{1 \le k \le n-1} (-1)^k \gamma_{(p,n-p),p}^{(n-k,1^k),1} \frac{n-1}{k d_{(n-k,1^k)}} \chi_\lambda^{(n-k,1^k)},$$

and

$$C_{n,p}^\lambda = |\mathcal{C}_\lambda| \sum_{1 \le k \le n-3} (-1)^k \gamma_{(p,n-p),p}^{(n-k-1,2,1^{k-1}),2} \frac{nk(n-k-2)}{(k+1)(n-k-1)d_{(n-k-1,2,1^{k-1})}} \chi^{(n-k-1,2,1^{k-1})}.$$

The generalized characters remaining in these expressions are the same generalized characters which were evaluated as part of the proof of Theorem 7.1.9. These generalized character evaluations lead to the following result.

Theorem 7.1.11 (S.). *Let $n \ge 4$ and $\lambda \vdash n$. When $2 \le p \le \frac{n-1}{2}$, the number of $(p, n-1, n)$-dipoles with face degree sequence 2λ is*

$$d_\lambda^{p,n-1} = \frac{|\mathcal{C}_\lambda|}{n(n-1)} \left(1 + \sum_{p \le k \le n-p-1} \frac{(n-p)(n-1)}{k(n-k-1)} \frac{\chi_\lambda^{n-k,1^k}}{d_{(n-k,1^k)}} \right.$$
$$\left. + (-1)^{n-m(\lambda)+1} - \sum_{p-1 \le k \le n-p-1} \frac{n(n-p)}{(k+1)(n-k-1)} \frac{\chi_\lambda^{(n-k-1,2,1^k)}}{d_{(n-k-1,2,1^{k-1})}} \right).$$

When $\max\{2, \frac{n-1}{2}\} \le p \le n-1$, the number of $(p, n-1, n)$-dipoles with face degree sequence 2λ is

$$d_\lambda^{p,n-1} = \frac{|\mathcal{C}_\lambda|}{n(n-1)} \left(1 - \sum_{n-p \le k \le p-1} \frac{(n-p)(n-1)}{k(n-k-1)} \frac{\chi_\lambda^{n-k,1^k}}{d_{(n-k,1^k)}} \right.$$
$$\left. + (-1)^{n-m(\lambda)+1} + \sum_{p-1 \le k \le n-p-1} \frac{n(n-p)}{(k+1)(n-k-1)} \frac{\chi_\lambda^{(n-k-1,2,1^k)}}{d_{(n-k-1,2,1^{k-1})}} \right).$$

The most notable aspect of this Theorem is that, although it was derived by working in the algebra $Z_1(n)$, the formula it gives involves not generalized characters, but ordinary characters of the symmetric group. Moreover, it is not known how to obtain this expression by working only within the centre of $\mathbb{C}[\mathfrak{S}_n]$. Because irreducible characters are linearly independent, one consequence of Theorem 7.1.11 is to demonstrate that the symmetry suggested by Conjecture 7.1.10 does not exist when information about face degree sequence is retained. (For example, when $p \leq \frac{n-1}{2}$, the coefficients of characters of the form $\chi_\lambda^{(n-k,1^k)}$ are positive, but their coefficients are negative when $p \geq \frac{n-1}{2}$.) On the other hand, the binomial coefficients appearing in Theorem 7.1.9 are not linearly independent as polynomials in z. Thus, the conjectured symmetry of $(p, n-1, n)$-dipoles is only present when summing over all dipoles in a given surface.

7.2 Non-transitive powers of Jucys-Murphy elements

Though the problem of enumerating non-transitive factorizations into star transpositions may be derived from Theorem 3.2.9, the results of Chapter 6 provide an alternative approach to this problem. J_n^r may be computed by regarding J_n^r as a linear operator on $\mathbb{C}[\mathfrak{S}_n]$ applied to the identity element, which can be written in the Z_1-idempotent basis using Corollary 6.1.8 as follows:

$$J_n^r = J_n^r K_{(1^n),1} = J_n^r \sum_{\substack{\mu \vdash n \\ j \in \mu}} \Gamma^{\mu,j}.$$

Since $J_n = K_{(2,1^{n-2}),2}$, then by Lemma 6.2.1, $\Gamma^{\mu,j}$ is an eigenvector of J_n with eigenvalue $c_{\mu,j}$. Thus,

$$J_n^r = \sum_{\substack{\mu \vdash n \\ j \in \mu}} c_{\mu,j}^r \Gamma^{\mu,j}.$$

By the definition of $\gamma_{\mu,j}^{\lambda,i}$, this may be written in the standard basis as

$$J_n^r = \sum_{\substack{\lambda \vdash n \\ i \in \lambda}} \sum_{\substack{\mu \vdash n \\ j \in \mu}} \frac{d_\mu}{n!} \gamma_{\lambda,i}^{\mu,j} c_{\mu,j}^r K_{\lambda,i}.$$

Extracting coefficients yields the following expression for powers of Jucys-Murphy elements in terms of generalized characters.

Theorem 7.2.1 (S.). *Let $\lambda \vdash n$ and let i be a part of λ. For $\pi \in \mathcal{C}_{\lambda,i}$, the number of factorizations of π into r star transpositions is given by*

$$[K_{\lambda,i}] J_n^r = \sum_{\substack{\mu \vdash n \\ j \in \mu}} \frac{d_\mu}{n!} \gamma_{\lambda,i}^{\mu,j} c_{\mu,j}^r.$$

	$[K_{(4)}]$	$[K_{(3,1),3}]$	$[K_{(3,1),1}]$	$[K_{(2,2),2}]$	$[K_{(2,1,1),2}]$	$[K_{(2,1,1),1}]$	$[K_{(1^4),1}]$
$r=0$	0	0	0	0	0	0	1
$r=1$	0	0	0	0	1	0	0
$r=2$	0	1	0	0	0	0	3
$r=3$	1	0	0	0	5	2	0
$r=4$	0	8	3	4	0	0	15
$r=5$	15	0	0	0	31	20	0

Table 7.1: Coefficients of J_4^r for small values of r.

As a verification of this result, the values of generalized characters given in Table 6.2 may be used to obtain explicit expressions when $n=4$, as follows.

Example 7.2.2 (Powers of J_4)**.** When $n=4$ and $r \geq 1$, the expression for J_4^r in the Z_1-idempotent basis becomes the following:

$$J_4^r = 3^r\Gamma^{(4),4} + 2^r\Gamma^{(3,1),3} + (-1)^r\Gamma^{(3,1),1} + \Gamma^{(2,1,1),2} + (-2)^r\Gamma^{(2,1,1),1} + (-3)^r\Gamma^{(1,1,1,1),1}.$$

Extracting coefficients gives

$$\begin{aligned}
[K_{(4),4}]J_4^r &= \frac{1}{24}(3^r - 2^{r+1} + (-1)^{r+1} + 1 - (-2)^{r+1} - (-3)^r),\\
[K_{(3,1),3}]J_4^r &= \frac{1}{24}(3^r + 2^r + (-1)^{r+1} - 1 + (-2)^r + (-3)^r),\\
[K_{(3,1),1}]J_4^r &= \frac{1}{24}(3^r - 3\cdot 2^r + 3(-1)^r + 3 - 3(-2)^r + (-3)^r),\\
[K_{(2,2),2}]J_4^r &= \frac{1}{24}(3^r - 2^{r+1} + (-1)^{r+1} - 1 + (-2)^{r+1} + (-3)^r),\\
[K_{(2,1,1),2}]J_4^r &= \frac{1}{24}(3^r + 2^{r+2} + (-1)^{r+1} + 1 - (-2)^{r+2} - (-3)^r),\\
[K_{(2,1,1),1}]J_4^r &= \frac{1}{24}(3^r + 3(-1)^r - 3 - (-3)^r),\\
[K_{(1^4),1}]J_4^r &= \frac{1}{24}(3^r + 3\cdot 2^{r+1} + 3(-1)^r + 3 - 3(-2)^{r+1} + (-3)^r).
\end{aligned}$$

Table 7.1 gives values of these coefficients for small values of r. These values agree with those obtained by direct computation of J_4^r.

While Theorem 7.2.1 is the most general statement that can be made with regards to non-transitive star factorizations, the presence of the generalized character $\gamma_{\lambda,i}^{\mu,j}$ means that it does not give an explicit formula in general. However, several specializations of this theorem do lead to more explicit results. The remainder of this section will describe the results arising from three specializations: first, restricting to (λ, i) for which $\gamma_{\lambda,i}^{\mu,j}$ is known explicitly; second, summing the result over all permutations in a given conjugacy class, and third, summing the result over all permutations having k cycles.

Special $\mathfrak{S}_{n-1}$ conjugacy classes

Theorem 7.2.1 can be used to give more explicit expressions for the coefficients of J_n^r in cases when the generalized characters arising in the expression for J_n^r can be evaluated. For example, Theorem 6.2.7 gives simple expressions for $\gamma_{(n),n}^{\mu,j}$ and $\gamma_{(n-1,1),1}^{\mu,j}$, while Corollary 6.2.12 gives a simple expression for $\gamma_{(n-1,1),n-1}^{\mu,j}$. Thus, Theorem 7.2.1 gives the following expressions for the number of factorizations of a full cycle into star transpositions in these three special cases.

Corollary 7.2.3. *Let $r \geq 1$. The number of factorizations of a full cycle of length n into r star transpositions is given by*

$$[K_{(n),n}]J_n^r = \frac{2^n(r+1)!}{n!(n-1)}[x^{r+1}]\sinh\left(\frac{(n-1)x}{2}\right)\sinh\left(\frac{x}{2}\right)^{n-1}. \tag{7.2}$$

The number of factorizations of a permutation $\pi \in \mathcal{C}_{(n-1,1),1}$ into r star transpositions is given by

$$[K_{(n-1,1),1}]J_n^r = \frac{2^n r!}{n!}[x^r]\sinh\left(\frac{(n-1)x}{2}\right)\sinh\left(\frac{x}{2}\right)^{n-1}. \tag{7.3}$$

The number of factorizations of a permutation $\pi \in \mathcal{C}_{(n-1,1),n-1}$ into r star transpositions is given by

$$[K_{(n-1,1),n}]J_n^r = \frac{r!}{n!(n-1)}[x^r]\left(n(e^{(n-1)x} + (-1)^n e^{-(n-1)x}) - 2^n \sinh\left(\frac{(n-1)x}{2}\right)\sinh\left(\frac{x}{2}\right)^{n-1}\right). \tag{7.4}$$

Proof. As the proof is similar in all three cases, details are provided only for Equation (7.4), which is the most complicated of the three. By Corollary 6.2.12, the only marked partitions (μ, j) which make a non-zero contribution to the formula of Theorem 7.2.1 are those of the form $((n-k,1^k), n-k)$, $((n-k,1^k),1)$ and $((n-k-1,2,1^{k-1}),2)$. For marked partitions of the form $((n-k-1,2,1^{k-1}),2)$, the quantity $c_{\mu,j}$ is zero, so when $r \geq 1$, these partitions do not contribute to the sum. Thus,

$$\begin{aligned}
[K_{(n-1,1),n-1}]J_n^r &= \frac{1}{n!}(n-1)^r + \frac{1}{n!}(-1)^n(1-n)^r \\
&\quad + \sum_{1\le k\le n-2} \binom{n-1}{k} \frac{(-1)^{k+1}}{n!(n-1)}(n-k-1)^r \\
&\quad + \sum_{1\le k\le n-2} \binom{n-1}{k} \frac{(-1)^{k}}{n!(n-1)}(-k)^r \\
&= \frac{r!}{n!(n-1)}[x^r]\Bigg((n-1)e^{(n-1)x} + (-1)^n(n-1)e^{-(n-1)x} \\
&\qquad\qquad + (1-e^{(n-1)x}) \sum_{1\le k\le n-2} \binom{n-1}{k}(-e^{-x})^k \Bigg).
\end{aligned}$$

By the binomial theorem,

$$\sum_{1\le k\le n-2} \binom{n-1}{k}(-e^{-x})^k = (1-e^{-x})^{n-1} - 1 + (-1)^n e^{-(n-1)x}.$$

Substituting this into the expression for $[K_{(n-1,1),n-1}]J_n^r$ and simplifying gives

$$[K_{(n-1,1),n-1}]J_n^r = \frac{r!}{n!(n-1)}[x^r]\left(ne^{(n-1)x} + (-1)^n ne^{-(n-1)x} + (1-e^{(n-1)x})(1-e^{-x})^{n-1}\right).$$

(Constant terms may be disregarded when extracting the coefficient of x^r, since $r \ge 1$.) Finally, the observation that

$$(1-e^{-x})^{n-1} = e^{-\frac{(n-1)x}{2}}\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^{n-1} = 2^{n-1}e^{-\frac{(n-1)x}{2}} \sinh\left(\frac{x}{2}\right)^{n-1}$$

gives the stated result.

$\square$

Since factorizations of a full cycle are necessarily transitive, the formula given in Equation (7.2) must coincide with the formula given by Goulden and Jackson (Theorem 3.2.6) for the same problem. Indeed, in the case of factorizations of a full cycle, the Goulden-Jackson formula reduces to the following:

$$[K_{(n),n}]J_n^r = \frac{2^{n-1}r!}{n!}[x^r]\sinh\left(\frac{nx}{2}\right)\sinh\left(\frac{x}{2}\right)^{n-2}.$$

The equivalence of this expression with the expression given in Corollary 7.2.3 follows from the fact that $(r+1)[x^{r+1}] = [x^r]\frac{\partial}{\partial x}$. Star factorizations of a permutation in $\mathcal{C}_{(n-1,1),1}$ are also always transitive, and Equation (7.3) also agrees with the formula given by Goulden and Jackson [11]. Equation (7.4) is notable for corresponding to a case in which not all the factorizations which are enumerated by this formula are transitive. Thus, Equation (7.4) is a result which is not obtainable from Theorem 3.2.6.

Sums over all permutations in a $\mathfrak{S}_n$ conjugacy class

The presence of the term $\sinh\left(\frac{(n-1)x}{2}\right)\sinh\left(\frac{x}{2}\right)^{n-1}$ in both Equations (7.3) and (7.4), together with the observation that $\mathcal{C}_{(n-1,1),1} \cup \mathcal{C}_{(n-1,1),n-1} = \mathcal{C}_{(n-1,1)}$, suggests that the generating series for the sum of $[\pi]J_n^r$ over all $\pi \in \mathcal{C}_{(n-1,1)}$ will simplify considerably. Indeed, this quantity is given by

$$\begin{aligned}
\sum_{\pi\in\mathcal{C}_{(n-1,1)}} [\pi]J_n^r &= |\mathcal{C}_{(n-1,1),1}|[K_{(n-1,1),1}]J_n^r + |\mathcal{C}_{(n-1,1),n-1}|[K_{(n-1,1),n-1}]J_n^r \\
&= \frac{r!}{n-1}[x^r](e^{(n-1)x} + (-1)^n e^{-(n-1)x}) \\
&= \begin{cases} \frac{2}{n-1} r![x^r]\sinh((n-1)x) & \text{if } n \text{ is odd,} \\ \frac{2}{n-1} r![x^r]\cosh((n-1)x) & \text{if } n \text{ is even.} \end{cases} \\
&= \begin{cases} 2(n-1)^{r-1} & \text{if } n+r \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The remarkable simplicity of this expression suggests that a direct combinatorial explanation should exist, though one is not currently known. This example suggests that using known results about sums of generalized characters may be used to find the following specialization of Theorem 7.2.1.

Theorem 7.2.4 (S.). *Let $\lambda \vdash n$, and $r \geq 1$. The number of sequences $(\tau_1, \ldots, \tau_r)$ such that each τ_i is a star transposition and $\prod_{1\leq i\leq r} \tau_i \in \mathcal{C}_\lambda$ is given by*

$$\frac{|\mathcal{C}_\lambda|}{n!}\sum_{\mu\vdash n}\chi_\lambda^\mu\left(\sum_{j\in\mu} d_{j-(\mu)}c_{\mu,j}^r\right).$$

Proof. The number of sequences satisfying the stated properties is

$$\sum_{i\in\lambda}|\mathcal{C}_{\lambda,i}|[K_{\lambda,i}]J_n^r.$$

By Theorem 7.2.1, this expression is equal to

$$\sum_{i\in\lambda} |\mathcal{C}_{\lambda,i}| \sum_{\substack{\mu\vdash n,\\ j\in\mu}} \frac{d_\mu}{n!}\gamma^{\mu,j}_{\lambda,i} c^r_{\mu,j}.$$

Interchanging the order of summation, this becomes

$$\sum_{\substack{\mu\vdash n,\\ j\in\mu}} \frac{d_\mu}{n!} c^r_{\mu,j} \sum_{i\in\lambda} |\mathcal{C}_{\lambda,i}|\gamma^{\mu,j}_{\lambda,i}.$$

By Lemma 6.2.8, for any $j \in \mu$,

$$\sum_{i\in\lambda} |\mathcal{C}_{\lambda,i}|\gamma^{\mu,j}_{\lambda,i} = \frac{|\mathcal{C}_\lambda| d_{j_-(\mu)}}{d_\mu}\chi^\mu_\lambda,$$

from which the result follows. □

Like Theorem 7.1.11, this is another example of a result in which an expression involving only ordinary characters has been obtained as a result of working in the algebra $Z_1(n)$. Again, it is not clear how to obtain this result by working only in the centre of $\mathbb{C}[\mathfrak{S}_n]$.

Sums over all permutations with a specified number of cycles

As a further specialization, Lemma 7.1.2 suggests that the formula for products of Jucys-Murphy elements has a particularly elegant form, in terms of evaluation of symmetric functions at the contents of tableaux, when we are only concerned with the number of cycles of permutations appearing in the product, as opposed to their cycle type. (Since the factorizations involved in this case are not necessarily transitive, this is a new result which is not given by the Goulden-Jackson formula.) In other words, Lemma 7.1.2 gives the following natural generalization of Corollary 7.2.3.

Corollary 7.2.5 (S.)**.** *The number of sequences $(\pi, \tau_1, \ldots, \tau_r)$ such that π has k cycles, $\tau_i = (j, n)$ for some $1 \le j \le n-1$, and $\pi = \tau_1 \cdots \tau_r$ is given by*

$$\sum_{\substack{\mu\vdash n,\\ j\in\mu}} \frac{d_\mu d_{j_-(\mu)} c^r_{\mu,j}}{n!} e_{n-k}(\mathbf{c}_\mu) = [t^k] \sum_{\substack{\mu\vdash n,\\ j\in\mu}} \frac{d_\mu d_{j_-(\mu)} c^r_{\mu,j}}{n!} \prod_{\Box\in\mathcal{F}_\mu} (t + c(\Box)).$$

Proof. This number is given by

$$\begin{aligned}\sum_{\substack{\lambda\vdash n\\ m(\lambda)=k}}\sum_{i\in\lambda}|\mathcal{C}_{\lambda,i}|[K_{\lambda,i}]J_n^r &= \sum_{\substack{\lambda\vdash n\\ m(\lambda)=k}}\sum_{i\in\lambda}|\mathcal{C}_{\lambda,i}|\sum_{\substack{\mu\vdash n\\ j\in\mu}}\frac{d_\mu}{n!}\gamma_{\lambda,i}^{\mu,j}c_{\mu,j}^r\\ &= \sum_{\substack{\mu\vdash n\\ j\in\mu}}\frac{d_\mu d_{j_-(\mu)}c_{\mu,j}^r}{n!}\left(\sum_{\substack{\lambda\vdash n\\ m(\lambda)=k}}\sum_{i\in\lambda}\frac{|\mathcal{C}_{\lambda,i}|}{d_{j_-(\mu)}}\gamma_{\lambda,i}^{\mu,j}\right),\end{aligned}$$

from which the result follows by an application of Lemma 7.1.2. □

7.3 Z_1-decompositions of a full cycle

While the general solution to the Z_1-factorization problem (Problem 3.3.1) is given by the formula for the connection coefficients of $Z_1(n)$ in Theorem 6.1.9, the solution may be made much more explicit in the special case of Z_1-factorizations of a full cycle. Much like the central case, this is made possible by the fact that there is a simple, explicit formula for a generalized character evaluated at a full cycle.

To state the generating series for Z_1-factorizations of a cycle concisely, it is first helpful to express the generating series for generalized characters corresponding to hook partitions in a form analogous to the generating series for ordinary hook partition characters, namely,

$$H_\lambda(x) = (1+x)^{-1}\prod_{1\le i\le m(\lambda)}(1-(-x)^{\lambda_i}).$$

These expressions are as follows.

Lemma 7.3.1 (S.). *Let $\mu\vdash n$ and let j be a part of μ. Define $R_{n,j}$ by*

$$R_{n,j}(x) := \frac{(n-1)+nx+(-x)^j}{1+x}.$$

Then, for $0\le k\le n-2$,

$$\gamma_{\mu,j}^{(n-k,1^k),n-k} = \frac{1}{n-1}[x^k]R_{n,j}(x)H_{\mu\setminus j}(x). \tag{7.5}$$

Define $S_{n,j}$ by

$$S_{n,j}(x) := (-1)^{j-1}\frac{(-1)^j x+nx^j+(n-1)x^{j+1}}{1+x}.$$

Then, for $1 \le k \le n-1$,

$$\gamma_{\mu,j}^{(n-k,1^k),1} = \frac{1}{n-1}[x^k]S_{n,j}(x)H_{\mu\backslash j}(x). \tag{7.6}$$

Proof. This result may be proven by using the fact, from Lemma 2.4.6, that

$$\chi_\mu^{(n-k,1^k)} = [x^k]H_\mu(x),$$

together with the formulas for generalized characters given in Lemmas 6.3.2 and 6.3.3. Details are provided for the evaluation of $\gamma_{\mu,j}^{(n-k,1^k),n-k}$ when $k \le n-k-1$.

The series $R_{n,j}(x)$ may be expanded as

$$(n-1) + \sum_{0<\ell<j} (-1)^{\ell+1} x^\ell.$$

Based on the expression for $\gamma_{\mu,j}^{(n-k,1^k),n-k}$ given in Lemma 6.3.2, it is natural to split the proof into three subcases: when $(n-k) \le (n-j)$, when $k < (n-j) < n-k$, and when $(n-j) \le k$. First, when $k \ge j$,

$$\begin{aligned}
\frac{1}{n-1}[x^k]R_{n,j}(x)H_{\mu\backslash j}(x) &= [x^k]H_{\mu\backslash j}(x) + \sum_{0<\ell<j} \frac{(-1)^{\ell+1}}{n-1}[x^{k-\ell}]H_{\mu\backslash j}(x) \\
&= \chi_{\mu\backslash j}^{(n-j-k,1^k)} + \sum_{0<\ell<j} \frac{(-1)^{\ell+1}}{n-1}\chi_{\mu\backslash j}^{(n-j-k+\ell,1^{k-\ell})} \\
&= \chi_{\mu\backslash j}^{(n-j-k,1^k)} + \sum_{k-j<\ell<k} \frac{(-1)^{k-\ell+1}}{n-1}\chi_{\mu\backslash j}^{(n-j-\ell,1^\ell)} \\
&= \gamma_{\mu,j}^{(n-k,1^k),n-k}.
\end{aligned}$$

In the other two cases, the range $0 < \ell < j$ on the index of summation is truncated due to the fact that the summand is zero for certain values of ℓ. When $k < (n-j) < n-k$, since $j > k$,

$$\begin{aligned}\frac{1}{n-1}[x^k]R_{n,j}(x)H_{\mu\setminus j}(x) &= [x^k]H_{\mu\setminus j}(x) + \sum_{0<\ell<j} \frac{(-1)^{\ell+1}}{n-1}[x^k]x^\ell H_{\mu\setminus j}(x) \\ &= [x^k]H_{\mu\setminus j}(x) + \sum_{0<\ell\le k} \frac{(-1)^{\ell+1}}{n-1}[x^{k-\ell}]H_{\mu\setminus j}(x) \\ &= \chi_{\mu\setminus j}^{(n-j-k,1^k)} + \sum_{0<\ell\le k} \frac{(-1)^{\ell+1}}{n-1}\chi_{\mu\setminus j}^{(n-j-k+\ell,1^{k-\ell})} \\ &= \chi_{\mu\setminus j}^{(n-j-k,1^k)} + \sum_{0\le\ell<k} \frac{(-1)^{k-\ell+1}}{n-1}\chi_{\mu\setminus j}^{(n-j-\ell,1^{\ell})} \\ &= \gamma_{\mu,j}^{(n-k,1^k),n-k}.\end{aligned}$$

The final subcase, when $(n-j) \le k$, relies on the fact that $H_{\mu\setminus j}(x)$ is a polynomial of degree $n-j-1$, so $[x^i]H_{\mu\setminus j}(x) = 0$ when $i \ge n-j$. In particular, $[x^k]H_{\mu\setminus j}(x) = 0$. Furthermore, $j \ge n-k > k$. Thus, in this case,

$$\begin{aligned}\frac{1}{n-1}[x^k]R_{n,j}(x)H_{\mu\setminus j}(x) &= [x^k]H_{\mu\setminus j}(x) + \sum_{0<\ell<j} \frac{(-1)^{\ell+1}}{n-1}[x^k]x^\ell H_{\mu\setminus j}(x) \\ &= \sum_{k-n+j+1\le\ell\le k} \frac{(-1)^{\ell+1}}{n-1}[x^{k-\ell}]H_{\mu\setminus j}(x) \\ &= \sum_{k-n+j+1\le\ell\le k} \frac{(-1)^{\ell+1}}{n-1}\chi_{\mu,j}^{(n-j-k+\ell,1^{k-\ell})} \\ &= \sum_{0\le\ell\le n-j-1} \frac{(-1)^{k-\ell+1}}{n-1}\chi_{\mu,j}^{(n-j-\ell,1^{\ell})} \\ &= \gamma_{\mu,j}^{(n-k,1),n-k}.\end{aligned}$$

Thus, when $k \le n-k-1$, for all values of j,

$$\frac{1}{n-1}[x^k]R_{n,j}(x)H_{\mu\setminus j}(x) = \gamma_{\mu,j}^{(n-k,1^k),n-k}.$$

The case when $k \ge n-k-1$, as well as the expression for $\gamma_{\mu,j}^{(n-k,1^k),1}$ may be verified in a similar manner. □

This lemma may be used to simplify the generating series for Z_1-factorizations of a full cycle. By Lemma 6.1.9, the number of pairs (σ_1, σ_2) such that $\sigma_1 \in \mathcal{C}_{\lambda,i}$, $\sigma_2 \in \mathcal{C}_{\mu,j}$ and $\sigma_1\sigma_2$

is a full cycle is given by

$$[K_{(n),n}]K_{\lambda,i}K_{\mu,j} = \frac{|\mathcal{C}_{\lambda,i}||\mathcal{C}_{\mu,j}|}{n!} \sum_{\rho\vdash n, \ell\in\rho} \frac{\gamma_{\lambda,i}^{\rho,\ell}\gamma_{\mu,j}^{\rho,\ell}\gamma_{(n),n}^{\rho,\ell}}{d_{\ell_-(\rho)}} \frac{d_\rho}{d_{\ell_-(\rho)}}.$$

By part 7 of Theorem 6.2.7,

$$[K_{(n),n}]K_{\lambda,i}K_{\mu,j} = \frac{|\mathcal{C}_{\lambda,i}||\mathcal{C}_{\mu,j}|}{n!} \left(\sum_{0\le k\le n-2} \frac{\gamma_{\lambda,i}^{(n-k,1^k),n-k}\gamma_{\mu,j}^{(n-k,1^k),n-k}(-1)^k}{\binom{n-2}{k}} \right.$$
$$\left. + \sum_{1\le k\le n-1} \frac{\gamma_{\lambda,i}^{(n-k,1^k),1}\gamma_{\mu,j}^{(n-k,1^k),1}(-1)^k}{\binom{n-2}{k-1}} \right).$$

Using Lemma 7.3.1,

$$[K_{(n),n}]K_{\lambda,i}K_{\mu,j} = \frac{|\mathcal{C}_{\lambda,i}||\mathcal{C}_{\mu,j}|}{(n-1)^2 n!} \left(\sum_{0\le k\le n-2} \frac{(-1)^k}{\binom{n-2}{k}} [x^k y^k] R_{n,i}(x) H_{\lambda\setminus i}(x) R_{n,j}(y) H_{\mu\setminus j}(y) \right.$$
$$\left. + \sum_{1\le k\le n-1} \frac{(-1)^k}{\binom{n-2}{k-1}} [x^k y^k] S_{n,i}(x) H_{\lambda\setminus i}(x) S_{n,j}(y) H_{\mu\setminus j}(y) \right).$$

This has proven the following.

Theorem 7.3.2 (S.)**.** *Let $\lambda, \mu \vdash n$. Let i be a part of λ and let j be a part of μ. The number of factorizations of a full cycle $C = \sigma_1\sigma_2$ such that $\sigma_1 \in \mathcal{C}_{\lambda,i}$ and $\sigma_2 \in \mathcal{C}_{\mu,j}$ is given by*

$$\frac{|\mathcal{C}_{\lambda,i}||\mathcal{C}_{\mu,j}|}{(n-1)^2 n!} \sum_{1\le k\le n-1} \frac{(-1)^{k-1}}{\binom{n-2}{k-1}} [x^k y^k] T_{n,i,j}(x,y) H_{\lambda\setminus i}(x) H_{\mu\setminus j}(y),$$

where

$$T_{n,i,j} = xyR_{n,i}(x)R_{n,j}(y) - S_{n,i}(x)S_{n,j}(y),$$

and $R_{n,i}$ and $S_{n,j}$ are given in Lemma 7.3.1.

For comparison, the analogous expression for the central version of the problem is

$$[K_{(n)}]K_\lambda K_\mu = \frac{|\mathcal{C}_{\lambda,i}||\mathcal{C}_{\mu,j}|}{n!} \sum_{0\le k\le n-1} \frac{(-1)^k}{\binom{n-1}{k}} [x^k y^k] H_\lambda(x) H_\mu(y). \tag{7.7}$$

The similarity between these two expressions suggests that further analysis of the series in Theorem 7.3.2 could lead to non-central analogues of the results about central decompositions of the full cycle which were proven using Equation (7.7).

Chapter 8

Analysis of the differential operators for the (p, q, n)-dipole problem

Although the algebraic techniques developed in the two preceding chapters can be used to solve the $(p, n-1, n)$-dipole problem, an effective algebraic approach to the (p, q, n)-dipole is not yet known. In this chapter, an analysis of the differential equations developed in Chapter 5 is used to find partial information about the generating series for (a, b, c, d)-dipoles. While a closed form solution to these equations is not yet known, this chapter describes a process, recursive in g, for determining the generating series for (a, b, c, d)-dipoles in a surface of genus g. This process is then used to obtain expressions for these series for small values of g.

This chapter begins with a differential analysis of the $(p, n-1, n)$-dipole problem. Although this problem has been solved in Chapter 7, this analysis is valuable because the relative simplicity of the equation for $(p, n-1, n)$-dipoles makes it possible to determine solutions more easily than in the general case. Since partial differential equations arising in algebraic combinatorics are often solved by conjecturing a solution and demonstrating that it satisfies the equation, the analysis of the $(p, n-1, n)$-dipole equation is useful for developing conjectures about the general case.

Using the analysis of $(p, n-1, n)$-dipoles as motivation, Section 8.2 describes analogues of the functions arising in Section 8.1 which correspond to the cases of $(a, b, 0, 0)$-dipoles and (a, b, c, d)-dipoles. Section 8.3 demonstrates how these functions may be used to give explicit expressions for the generating series for (a, b, c, d)-dipoles in surfaces of small genus.

8.1 Analysis of the partial differential equation for $(p, n-1, n)$-dipoles

8.1.1 Overview of the General Strategy

The solution to the $(p, n-1, n)$-dipole problem is given by Theorem 5.2.1, namely,

$$(C + u^2 J)\Psi = \frac{\partial \Psi}{\partial y},$$

where

$$C = \sum_{i\geq 2} \sum_{1\leq j\leq i-1} g_{j+1} f_{i-j} \frac{\partial}{\partial g_i} = \sum_{i\geq 1} \sum_{j\geq 1} g_{i+1} f_j \frac{\partial}{\partial g_{i+j}}.$$

and

$$J = \sum_{i\geq 1} \sum_{j\geq 1} j g_{i+j+1} \frac{\partial^2}{\partial g_i \partial f_j}.$$

The standard approach to solving any equation of the form

$$\Delta \Psi = \frac{\partial \Psi}{\partial t},$$

where Δ is a linear differential operator, is to determine the eigenvectors of Δ, and then to write the initial condition as a linear combination of these eigenvectors. The analysis of the differential equation for $(p, n-1, n)$-dipoles is made difficult by the fact that $(C+u^2J)$ has no eigenvectors. Indeed, defining $I(g_i f_\lambda)$ to be the sum of indices of $g_i f_\lambda$, applying $(C + u^2 J)$ to any power series F in the indeterminates $\{g_i\}$ and $\{f_j\}$ causes the quantity

$$\min\{I(g_i f_\lambda) : g_i f_\lambda \text{ appears with nonzero coefficient in } F\}$$

to increase strictly. Thus, a different approach is needed.

The approach to determining Ψ used in this section is to find a recurrence which determines the part of the series corresponding to $(p, n-1, n)$-dipoles in a surface of genus g, namely, $\Psi^{(g)} := [u^{2g}]\Psi$. While this process will not determine Ψ completely, it can be used to solve the problem for surfaces of small genus.

Applying $[u^{2g}]$ to the differential equation for Ψ when $g \geq 1$,

$$\left(\frac{\partial}{\partial y} - C\right) \Psi^{(g)} = J \Psi^{(g-1)}. \tag{8.1}$$

When $g = 0$, there is only one dipole on n edges, and it is a $(1, n-1, n)$-dipole in which every face has half-degree 1. Thus,

$$\Psi^{(0)} = \sum_{n \geq 2} \frac{x^{n-1}}{(n-1)!} g_1 f_1^{n-1} = g_1(\exp(f_1 x) - 1).$$

Approaching the problem by using the recurrence in Equation 8.1 simplifies the analysis by reducing the problem of solving a second-order partial differential equation to solving a first-order partial differential equation, at the cost of adding the inhomogeneous term $J\Psi^{(g-1)}$ to the equation.

A second technique to simplify the analysis is to subtract the constant term from the generating series so that Equation 8.1 may be considered with respect to a vanishing initial condition. Define $\hat{\Psi}^{(g)} = \Psi^{(g)} - \Psi^{(g)}|_{y=0}$, so that $\hat{\Psi}^{(g)}|_{y=0} = 0$. Since $\Psi^{(g)}|_{y=0}$ is the series corresponding to dipoles in which $p = 1$ (and hence the root face has half-degree 1), $\left(\frac{\partial}{\partial y} - C\right) \Psi^{(g)}|_{y=0} = 0$. Thus,

$$\left(\frac{\partial}{\partial y} - C\right) \Psi^{(g)} = \left(\frac{\partial}{\partial y} - C\right) \hat{\Psi}^{(g)}$$

The analysis of the operator $\frac{\partial}{\partial y} - C$ is facilitated by two observations. First, both the indeterminate x and all the indeterminates f_i are constant with respect to $\frac{\partial}{\partial y} - C$. Second, $\frac{\partial}{\partial y} - C$ is linear. Thus, if $\Psi^{(g-1)}$ is known, in order to solve Equation 8.1, it suffices to solve equations of the form

$$\left(\frac{\partial}{\partial y} - C\right) \Psi^{(g)} = g_k h(y), \tag{8.2}$$

where $h(y)$ is an element of some suitably chosen family of functions $\mathcal{F}$. In order to apply the recursion in Equation 8.1 repeatedly, the class $\mathcal{F}$ must be chosen such that solutions to Equation 8.2 are also expressable in terms of $\mathcal{F}$.

8.1.2 The Sequence $\{\phi_i\}$

A suitable choice for the class $\mathcal{F}$ is the sequence of functions $\{\phi_i\}_{i \geq 0}$ defined by $\phi_0(y) = 1$, and

$$\phi_j(y) = \exp(f_1 y) \int_0^y \exp(-f_1 \xi) \phi_{j-1}(\xi) d\xi \tag{8.3}$$

for $j \geq 1$. These functions satisfy the following properties.

Lemma 8.1.1. *1. $\phi_j(0) = 0$ for all $j \geq 1$.*

2. $\frac{\partial \phi_j(y)}{\partial y} = \phi_{j-1}(y) + f_1\phi_j(y)$ *for all* $j \geq 1$.

3. $\phi_j(y) = [\zeta^j]\Phi(\zeta)$, *where* $\Phi(\zeta)$ *is the series given by*

$$\Phi(\zeta) = \frac{f_1 + \zeta \exp((f_1 + \zeta)y))}{f_1 + \zeta}.$$

4. $\frac{\partial \phi_j(y)}{\partial f_1} = j\phi_{j+1}(y)$.

5. $[y^k]\phi_j(y)|_{f_1=1} = \frac{1}{k!}\binom{k-1}{j-1}$.

Proof. The first statement is obvious from the definition of ϕ_j. For the second statement, differentiating Equation (8.3) with respect to y gives

$$\begin{aligned}
\frac{\partial \phi_j}{\partial y} &= f_1 \exp(f_1 y) \int_0^y \exp(-f_1\xi)\phi_{j-1}(\xi)d\xi + \exp(f_1 y)\exp(-f_1 y)\phi_{j-1}(y) \\
&= f_1\phi_j(y) + \phi_{j-1}(y).
\end{aligned}$$

To prove the third statement, define $\Phi = \sum_{j\geq 0} \phi_j(y)\zeta^j$. Multiplying the equation from statement 2 by ζ^j and summing over $j \geq 1$,

$$\frac{\partial \Phi}{\partial y} = (\zeta + f_1)\Phi - f_1$$

with $\Phi(0) = 1$. It is routine to check that $\frac{f_1+\zeta\exp((f_1+\zeta)y)}{f_1+\zeta}$ is the unique solution to this differential equation and initial condition.

To prove the fourth statement, first note that

$$\frac{\partial \Phi}{\partial f_1} = \frac{1 + y\zeta\exp((f_1+\zeta)y)}{f_1+\zeta} - \frac{f_1 + \zeta\exp((f_1+\zeta)y)}{(f_1+\zeta)^2}$$

and

$$\frac{\partial \Phi}{\partial \zeta} = \frac{\exp((f_1+\zeta)y) + \zeta y\exp((f_1+\zeta)y)}{f_1+\zeta} - \frac{f_1 + \zeta\exp((f_1+\zeta)y)}{(f_1+\zeta)^2}.$$

Thus,

$$\begin{aligned}
\frac{\partial \phi_j}{\partial f_1} &= [\zeta^j]\frac{\partial \Phi}{\partial f_1} \\
&= [\zeta^j]\frac{\partial \Phi}{\partial \zeta} - [\zeta^j]\frac{\exp((f_1+\zeta)y) - 1}{f_1+\zeta} \\
&= (j+1)[\zeta^{j+1}]\Phi - [\zeta^{j+1}]\frac{\zeta\exp((f_1+\zeta)y) - \zeta}{f_1+\zeta}.
\end{aligned}$$

Since

$$\Phi - \frac{\zeta \exp((f_1+\zeta)y) - \zeta}{f_1+\zeta} = \frac{f_1+\zeta}{f_1+\zeta} = 1,$$

then for $j \geq 0$, $[\zeta^{j+1}]\frac{\zeta \exp((f_1+\zeta)y)-\zeta}{f_1+\zeta} = [\zeta^{j+1}]\Phi$. Thus,

$$\frac{\partial \phi_j}{\partial f_1} = (j+1)[\zeta^{j+1}]\Phi - [\zeta^{j+1}]\Phi = j\phi_{j+1},$$

proving statement 4.

Proving the fifth statement is now a straightforward computation:

$$\begin{aligned}
[y^k]\phi_j|_{f_1=1} &= [y^k\zeta^j]\frac{1+\zeta\exp((1+\zeta)y)}{1+\zeta} \\
&= [\zeta^{j-1}]\frac{(1+\zeta)^k}{k!(1+\zeta)} \\
&= \frac{1}{k!}\binom{k-1}{j-1}.
\end{aligned}$$

□

Part 2 of Lemma 8.1.1 is a key feature of the sequence $\{\phi_i\}$ which allows the operator $\frac{\partial}{\partial y}$ to be dealt with in a fairly simple manner. To make this notion more precise, consider the umbral composition $\circ\phi$ defined as follows: let

$$s^i \circ \phi = \phi_i,$$

and extend linearly to any formal power series in s. Using property 2 from Lemma 8.1.1,

$$\begin{aligned}
\frac{\partial}{\partial y}(s^i \circ \phi) &= \frac{\partial \phi_i}{\partial y} \\
&= \phi_{i-1} + f_1\phi_i \\
&= s^{i-1} \circ \phi + f_1 s^i \circ \phi \\
&= ((s^{-1} + f_1)s^i) \circ \phi.
\end{aligned}$$

Extending linearly to the whole ring yields the following.

Lemma 8.1.2. *For any formal power series $H(s)$,*

$$\frac{\partial}{\partial y}(H(s) \circ \phi) = ((s^{-1} + f_1)H(s)) \circ \phi.$$

In light of this lemma, working in the pre-image of the umbral composition effectively removes one of the differential operators from the equation and replaces it with an algebraic operation, namely, multiplication by $(s^{-1} + f_1)$.

8.1.3 Preimage of $g_{j+1}\phi_k$ under $\frac{\partial}{\partial y} - C$

The umbral composition $\circ\phi$ may be used to define a generating series for the solutions to

$$\left(\frac{\partial}{\partial y} - C\right)\Psi = g_{j+1}\phi_k(y).$$

Define the series $F_j(t)$ and $G_j(t)$ by

$$F_j(t) = \sum_{i\geq 1} f_{i+j}t^i$$

and

$$G_j(t) = \sum_{i\geq 1} g_{i+j}t^i.$$

Let

$$T = \Xi\left(sG_1\exp((r+vF_1)s)\right)\circ\phi,$$

where Ξ is the linear functional defined by $\Xi(v^i) = i!$. Then the following holds.

Theorem 8.1.3 (S.). *The series T satisfies*

$$\left(\frac{\partial}{\partial y} - C\right)T = G_1\exp(rs)\circ\phi.$$

Proof. First, an application of Lemma 8.1.2 gives

$$\frac{\partial T}{\partial y} = \Xi((s^{-1} + f_1)sG_1\exp((r+vF_1)s))\circ\phi.$$

Since C commutes with both Ξ and $\circ\phi$,

$$\begin{aligned}
CT &= \Xi\left(\sum_{i\geq 1}\sum_{j\geq 1} g_{i+1}f_j\frac{\partial}{\partial g_{i+j}}sG_1\exp((r+vF_1)s)\right)\circ\phi \\
&= \Xi\left(\sum_{i\geq 1}\sum_{j\geq 1} g_{i+1}f_jt^{i+j-1}s\exp((r+vF_1)s)\right)\circ\phi \\
&= \Xi\left(s\sum_{i\geq 1} g_{i+1}t^i\sum_{j\geq 1} f_jt^{j-1}\exp((r+vF_1)s)\right)\circ\phi \\
&= \Xi\left(sG_1(F_1+f_1)\exp((r+vF_1)s)\right)\circ\phi.
\end{aligned}$$

Thus,

$$\begin{aligned}\left(\frac{\partial}{\partial y} - C\right) T &= \Xi \left(G_1(1 - sF_1) \exp((r + vF_1)s)\right) \circ \phi \\ &= \Xi \left(1 - \frac{\partial}{\partial v}\right) G_1 \exp((r + vF_1)s) \circ \phi.\end{aligned}$$

To complete the proof, it suffices to determine the action of $\Xi\left(1 - \frac{\partial}{\partial v}\right)$ on any power series $H(v)$. Consider the action of $\Xi(1 - \frac{\partial}{\partial v})$ on monomials v^j. When $j = 0$,

$$\Xi \left(1 - \frac{\partial}{\partial v}\right) v^0 = \Xi(v^0) = 1.$$

When $j \geq 1$,

$$\begin{aligned}\Xi \left(1 - \frac{\partial}{\partial v}\right) v^j &= \Xi(v^j - jv^{j-1}) \\ &= j! - j(j-1)! \\ &= 0.\end{aligned}$$

Extending linearly, $\Xi\left(1 - \frac{\partial}{\partial v}\right) H(v) = H(0)$. Thus, the stated result is obtained by evaluating the series

$$G_1 \exp((r + vF_1)s) \circ \phi$$

at $v = 0$. □

Define $\tau_{j,k} = k![t^j r^k]T$. These coefficients are the functions which are needed in order to apply the recurrence of Equation 8.1. More precisely, the following statement is true.

Corollary 8.1.4. *Let $j \geq 1, k \geq 0$. The series $\tau_{j,k}$ is the unique solution to the partial differential equation*

$$\left(\frac{\partial}{\partial y} - C\right) \tau_{j,k} = g_{j+1}\phi_k$$

with initial condition $\tau_{j,k}|_{y=0} = 0$.

Proof. By Theorem 8.1.3,

$$\begin{aligned}\left(\frac{\partial}{\partial y} - C\right) \tau_{j,k} &= k![t^j r^k] \left(\frac{\partial}{\partial y} - C\right) T \\ &= k![t^j r^k] G_1 \exp(rs) \circ \phi \\ &= g_{j+1}\phi_k.\end{aligned}$$

To check that the initial condition is satisfied, first note that $(s^i \circ \phi)|_{y=0} = 0$ when $i \geq 1$, and $(s^0 \circ \phi)|_{y=0} = 1$. Thus, $f(s) \circ \phi|_{y=0} = f(0)$, so

$$\tau_{j,k}|_{y=0} = k![t^j r^k]\Xi(sG_1 \exp(r + vF_1)s)|_{s=0} = 0.$$

□

8.1.4 The number of $(p, n-1, n)$-dipoles in surfaces of small genus

This section demonstrates how the functions $\tau_{j,k}$ may be used to find solutions to the $(p, n-1, n)$-dipole problem for small genus but arbitrary p and n. To obtain the series for $(p, n-1, n)$-dipoles without keeping track of face degree, set all f and g type indeterminates to 1. For any formal power series F, let $\langle F \rangle$ denote the series obtained by setting all f and g indeterminates to 1. In order to extract coefficients to find formulae for the number of $(p, n-1, n)$-dipoles, the following lemma is useful.

Lemma 8.1.5. *Setting $f_i = 1$ and $g_i = 1$ for all $i \geq 1$, the series $\tau_{j,k}$ becomes*

$$\langle \tau_{j,k} \rangle = s^{k+1}(1+s)^{j-1} \circ \phi|_{f_1=1}.$$

Consequently,

$$(p-1)![y^{p-1}]\langle \tau_{j,k} \rangle = \sum_{0 \leq i \leq j-1} \binom{j-1}{i} \binom{p-2}{k+i}.$$

Proof. This is a straightforward computation using the definition of $\tau_{j,k}$:

$$\begin{aligned}
\langle \tau_{j,k} \rangle &= k![t^j r^k]\Xi\left(\frac{st}{1-t} \exp\left(\left(r + \frac{vt}{1-t}\right) s\right)\right) \circ \phi|_{f_1=1} \\
&= [t^j]\Xi\left(\frac{ts^{k+1}}{1-t} \sum_{i \geq 0} \frac{(vts)^i}{i!(1-t)^i}\right) \circ \phi|_{f_1=1} \\
&= [t^{j-1}]\left(\frac{s^{k+1}}{1-t} \frac{1}{1 - \frac{ts}{1-t}}\right) \circ \phi|_{f_1=1} \\
&= [t^{j-1}]\frac{s^{k+1}}{1-(1+s)t} \circ \phi|_{f_1=1} \\
&= s^{k+1}(1+s)^{j-1} \circ \phi|_{f_1=1}.
\end{aligned}$$

Extracting the coefficient of $[y^{p-1}]$,

$$\begin{aligned}(p-1)![y^{p-1}]\langle \tau_{j,k}\rangle &= [y^{p-1}] \sum_{0\le i\le j-1} \binom{j-1}{i} s^{k+1+i} \circ \phi|_{f_1=1} \\ &= \sum_{0\le i\le j-1} \binom{j-1}{i}\binom{p-2}{k+i},\end{aligned}$$

following an application of property 5 in Lemma 8.1.1. □

$(p, n-1, n)$-dipoles in the torus

The functions $\tau_{j,k}$ form a natural family of functions in which to express the generating series for $(p, n-1, n)$-dipoles, as illustrated in the following example. To determine the genus 1 solution, first compute

$$J\Psi^{(0)} = g_3 x \exp(f_1 x) = x \exp(f_1 x) g_3 \phi_0.$$

In order to solve

$$\left(\frac{\partial}{\partial y} - C\right) \hat{\Psi}^{(1)} = x \exp(f_1 x) g_3 \phi_0,$$

apply Corollary 8.1.4 to obtain

$$\hat{\Psi}^{(1)} = x \exp(f_1 x) \tau_{2,0}.$$

Thus,

$$\Psi^{(1)} = x \exp(f_1 x)\tau_{2,0} + [u^2]\Psi_0.$$

Lemma 8.1.5 may be used to find the number of $(p, n-1, n)$-dipoles on the torus when $p \ge 2$, namely,

$$\begin{aligned}(p-1)!(n-p)![y^{p-1}x^{n-p}]\langle \Psi^{(1)}\rangle &= (p-1)!(n-p)![y^{p-1}x^{n-p}] \exp(x)\langle \tau_{2,0}\rangle \\ &= (n-p)\left(\binom{1}{0}\binom{p-2}{0} + \binom{1}{1}\binom{p-2}{1}\right) \\ &= (n-p)(p-1).\end{aligned}$$

This result agrees with the formula given by Visentin and Wieler (Theorem 3.1.5) for the equivalent problem of $(n-p, 1, n)$-dipoles. The generating series for $(p, n-1, n)$-dipoles in surfaces of higher genera may be computed in a similar manner. Details of the calculations for the equivalent problem of $(a, b, 0, 0)$-dipoles on the double torus are given in Section 8.3

8.2 τ-type functions for the $(a,b,0,0)$- and (a,b,c,d)-dipole problems

The main barrier in generalizing the preceding analysis of the differential operators for the $(p, n-1, n)$-dipole problem to the more refined $(a,b,0,0)$- and (a,b,c,d)-dipole problems is that no function is known which can play a role analogous to the function T from the $(p, n-1, n)$ case. However, a closer examination of the functions $\tau_{j,k}$ provides valuable insight about how the analysis for the $(p, n-1, n)$ case may be generalized. By the definition of $\tau_{j,k}$,

$$\begin{aligned}
\tau_{j,k} &= k![r^k t^j]T \\
&= [t^j]\Xi\left(s^{k+1}G_1\exp(vsF_1)\right)\circ\phi \\
&= [t^j]\left(s^{k+1}G_1\sum_{i\geq 0}s^iF_1^i\right)\circ\phi \\
&= [t^j]\sum_{i\geq 0}\phi_{i+k+1}G_1F_1^i \\
&= [t^j]\sum_{i\geq 1}\phi_{k+i}\sum_{a_1,\ldots,a_i\geq 1}t^{a_1+\cdots+a_i}g_{a_1+1}\prod_{2\leq \ell\leq i}f_{a_\ell+1}.
\end{aligned}$$

Let $\mathcal{C}_i(j)$ be the set of compositions of the integer j into i positive parts. Then the functions $\tau_{j,k}$ may be regarded as the following sum indexed by these combinatorial objects.

Lemma 8.2.1.

$$\tau_{j,k} = \sum_{i\geq 1}\phi_{k+i}\sum_{(a_1,\ldots,a_i)\in\mathcal{C}_i(j)}g_{a_1+1}\prod_{2\leq \ell\leq i}f_{a_\ell+1}.$$

This section will define analogous τ-like functions, indexed not by integers but by binary strings, for the $(a,b,0,0)$- and (a,b,c,d)-dipole problems. This generalization is combinatorial in nature in that it proceeds by replacing the set indexing the summation in Lemma 8.2.1 with a different set of combinatorial objects called **string compositions**, which are defined as follows.

Definition 8.2.2. Let $R \in \{\bullet,\circ\}^*$ be a binary string. An ordered sequence $(\rho_1, \rho_2, \ldots, \rho_i)$ such that each $\rho_i \in \{\bullet,\circ\}^* \setminus \epsilon$ and $\rho_1\rho_2\cdots\rho_i = R$ is called a string composition of R into k parts. Let $\mathcal{C}_i(R)$ denote the set of string compositions of R into i parts.

(The symbol $\epsilon \in \{\bullet,\circ\}^*$ denotes the empty string.) Before the generalized τ-functions can be defined, it is first necessary to define generalizations of the ϕ functions which are indexed by binary strings. This is done in Section 8.2.1. Following that, the τ-like functions for the $(a,b,0,0)$-dipole problem are defined in Section 8.2.2, and for the (a,b,c,d)-dipole problem in Section 8.2.3.

8.2.1 ϕ-functions

The following family of functions provides an appropriate generalization of the sequence $\{\phi_i\}$ arising in the analysis of the $(p, n-1, n)$-dipole problem.

Definition 8.2.3 (General ϕ-function)**.** Let $i \geq 1$, and let $v, x_1, \ldots, x_i$ be indeterminates. Define

$$\phi(v, x_1, \ldots, x_i) = \sum_{n \geq i} h_{n-i}(x_1, \ldots, x_i) \frac{v^n}{n!},$$

where $h_j(x_1, \ldots, x_k)$ is the complete symmetric function of total degree j in k indeterminates. The function $\phi(v)$ shall denote the constant function $\phi(v) = 1$, corresponding to the case $i = 0$.

(If $j < 0$, the convention $h_j(x_1, \ldots x_k) = 0$ is used.) The usefulness of these functions lies in the fact that they satisfy the following property, which generalizes part 2 of Lemma 8.1.1.

Lemma 8.2.4. *For $i \geq 1$,*

$$\frac{\partial}{\partial v}\phi(v, x_1, \ldots, x_i) - x_i \phi(v, x_1, \ldots, x_i) = \phi(v, x_1, \ldots, x_{i-1}).$$

Proof. For $n \geq i$, $x_i h_{n-i}(x_1, \ldots, x_i)$ is the sum over monomials of degree $n-i+1$ in the variables $x_1, \ldots, x_i$ such that x_i appears with degree at least 1. However, $h_{n-i+1}(x_1, \ldots, x_i) - h_{n-i+1}(x_1, \ldots, x_{i-1})$ is the sum over the same set of monomials. When $n = i - 1$, both $h_{n-i+1}(x_1, \ldots, x_i)$ and $h_{n-i+1}(x_1, \ldots, x_{i-1})$ are equal to 1, and $x_i h_{n-i}(x_1, \ldots, x_i) = 0$. Multiplying the equation

$$x_i h_{n-i}(x_1, \ldots, x_i) = h_{n-i+1}(x_1, \ldots, x_i) - h_{n-i+1}(x_1, \ldots, x_{i-1})$$

by $\frac{v^n}{n!}$ and summing over $n \geq i - 1$ yields the desired result. $\square$

An alternative form of this Lemma expresses these functions as an integral recursion, which is parallel to the definition of the sequence $\{\phi_i\}$ from Section 8.1.

Lemma 8.2.5. *For $i \geq 1$,*

$$\phi(v, x_1, \ldots, x_i) = \exp(v x_i) \int_0^v \exp(-\xi x_i) \phi(\xi, x_1, \ldots, x_{i-1}) d\xi,$$

with $\phi(v) = 1$ when $i = 0$.

Proof. When $i \geq 1$,

$$\begin{aligned}
&\exp(vx_i)\int_0^v \exp(-\xi x_i)\phi(\xi, x_1, \ldots, x_{i-1})d\xi \\
&\quad = \exp(vx_i)\int_0^v \exp(-\xi x_i)\left(\frac{\partial}{\partial \xi}\phi(\xi, x_1, \ldots, x_i) - x_i\phi(\xi, x_1, \ldots, x_i)\right)d\xi \\
&\quad = \exp(vx_i)\int_0^v \frac{\partial}{\partial \xi}\exp(-\xi x_i)\phi(v, x_1, \ldots, x_i)d\xi \\
&\quad = \phi(v, x_1, \ldots, x_i).
\end{aligned}$$

□

This Lemma implies that the sequence $\{\phi_i\}$ from Section 8.1 may be recovered from the general ϕ-functions by defining $\phi_i = \phi(y, f_1, \ldots, f_1)$, where there are i copies of the indeterminate f_1.

Two specializations of ϕ are used in the analysis of the differential operators corresponding to (a, b, c, d)-dipoles and $(a, b, 0, 0)$-dipoles. Let $S \in \{\bullet, \circ\}^i$, say, $S = S_1 S_2 \cdots S_i$. Define the function $\phi_S(v)$ to be the evaluation of $\phi(v, x_1, \ldots, x_i)$ at

$$x_j = \begin{cases} f_{(\circ)} & \text{if } S_j = \circ, \\ w f_{(\bullet)} & \text{if } S_j = \bullet. \end{cases}$$

Of course, since ϕ is symmetric in $x_1, \ldots, x_i$, in order to specify $\phi_S(v)$, only two parameters are needed — let $\phi_{i,j}(v) = \phi_S(v)$ whenever S is a string consisting of i white dots and j black dots. (Nevertheless, in many cases it will often be more natural to index ϕ by S instead of i and j; thus, both notations are used.) The specialization of Lemma 8.2.4 to $\phi_S(v)$ is as follows.

Corollary 8.2.6. *Let $i \geq 1$ and let $S \in \{\circ, \bullet\}^{i-1}$. Let $S_i \in \{\circ, \bullet\}$. If $S_i = \circ$, then*

$$\frac{\partial \phi_{SS_i}(v)}{\partial v} - f_{(\circ)}\phi_{SS_i}(v) = \phi_S(v).$$

If $S_i = \bullet$, then

$$\frac{\partial \phi_{SS_i}(v)}{\partial v} - f_{(\bullet)} w \phi_{SS_i}(v) = \phi_S(v).$$

A further specialization is the functions $\phi_i(y)$, defined to be the evaluation of $\phi(v, x_1, \ldots, x_i)$ at $v = y$ and $x_j = f_{(\bullet)}$ for all j. (These functions are just the sequence $\{\phi_i\}$ from Section 8.1 in which f_1 has been replaced by $f_{(\bullet)}$. From this point onward, $\phi_i(y)$ always denotes the specialization obtained by evaluating x_j at $f_{(\bullet)}$, and the functions from Section 8.1 are no longer used.) In this case, Lemma 8.2.4 specializes as follows.

Corollary 8.2.7. *Let $i \geq 0$. Then*

$$\frac{\partial \phi_{i+1}}{\partial y} - f_{(\bullet)}\phi_{i+1} = \phi_i.$$

8.2.2 Preimage of $g_R\phi_i$ under the operator $\frac{\partial}{\partial y} - C'$

Recall that the cut operator C' corresponding to the $(a, b, 0, 0)$-dipole problem is given by

$$C' = \sum_{R \in \{\circ,\bullet\}^*} \left(\sum_{2 \leq i \leq \ell(R)} g_{R_1 \cdots R_i} f_{(R_1 R_{i+1} \cdots R_{\ell(R)})} \right) \frac{\partial}{\partial g_R}.$$

Define the function $\tau_{R,k}(y)$ to be the unique solution to the partial differential equation

$$\left(\frac{\partial}{\partial y} - C' \right) \tau_{R,k} = g_{\bullet R}\phi_k(y), \tag{8.4}$$

with initial condition $\tau_{R,k}(0) = 0$. Theorem 8.2.9 gives an expression for these functions as a sum over string compositions. This theorem is proven by conjecturing an expression for $\tau_{R,k}$, and then demonstrating that the conjectured formula satisfies Equation (8.4).

In order to motivate the expression for $\tau_{R,k}$ given in Theorem 8.2.9, consider that the functions $\tau_{R,k}$ satisfy the following recursion. (In this recursion, no initial condition is needed, since when $\ell(R) = 1$, the following formula completely determines $\tau_{R,k}$.)

Lemma 8.2.8. *Let $R = R_1 R_2 \cdots R_{\ell(R)} \in \{\circ, \bullet\}^* \setminus \epsilon$, and $k \geq 0$. Then*

$$\tau_{R,k} = g_{\bullet R}\phi_{k+1} + \sum_{1 \leq i \leq \ell(R)-1} f_{(\bullet R_{i+1} \cdots R_{\ell(R)})} \tau_{R_1 \cdots R_i, k+1}. \tag{8.5}$$

Proof. Applying $\frac{\partial}{\partial y} - C'$ to $g_{\bullet R}\phi_{k+1}$,

$$\left(\frac{\partial}{\partial y} - C' \right) g_{\bullet R}\phi_{k+1} = g_{\bullet R}\frac{\partial \phi_{k+1}}{\partial y} - \phi_{k+1} \left(\sum_{1 \leq i \leq \ell(R)} g_{\bullet R_1 \cdots R_i} f_{(\bullet R_{i+1} \cdots R_{\ell(R)})} \right).$$

By the definition of $\tau_{R,k}$,

$$\left(\frac{\partial}{\partial y} - C' \right) \sum_{1 \leq i \leq \ell(R)-1} f_{(\bullet R_{i+1} \cdots R_{\ell(R)})} \tau_{R_1 \cdots R_i, k+1} = \sum_{1 \leq i \leq \ell(R)-1} f_{(\bullet R_{i+1} \cdots R_{\ell(R)})} g_{\bullet R_1 \cdots R_i} \phi_{k+1}.$$

Summing these equations,

$$\left(\frac{\partial}{\partial y} - C'\right)\left(g_{\bullet R}\phi_{k+1} + \sum_{1\le i\le \ell(R)-1} f_{(\bullet R_{i+1}\cdots R_{\ell(R)})}\tau_{R_1\cdots R_i,k+1}\right)$$
$$= g_{\bullet R}\frac{\partial \phi_{k+1}}{\partial y} - g_{\bullet R} f_{(\bullet)}\phi_{k+1}$$
$$= g_{\bullet R}\phi_k.$$

Since the right side of Equation (8.5) vanishes when $y = 0$, then it must be the unique solution to $\left(\frac{\partial}{\partial y} - C'\right)\tau_{R,k} = g_{\bullet R}\phi_k$. □

Although, strictly speaking, Lemma 8.2.8 is not needed to prove Theorem 8.2.9, it is helpful in conjecturing the expression for $\tau_{R,k}$ appearing in the statement of the theorem. To motivate the form of the answer, consider the following non-deterministic algorithm for producing a monomial in the g and f indeterminates.

1. Start with a string $R \in \{0,1\}^*$, whose bits are $R_1, R_2, \ldots, R_{\ell(R)}$.
2. Select one of two operations: a "g-cut" or an "f-cut."
3. If an f-cut was selected, select a number i between 1 and the length of R. Add the factor $f_{(\bullet R_{i+1}\cdots R_{\ell(R)})}$ to the monomial, and remove the bits $R_{i+1}, \ldots, R_{\ell(R)}$ from R. Go back to Step 2.
4. If a g-cut was selected, add the factor $g_{\bullet R}$ to the monomial. Multiply by ϕ_{k+j}, where j is the total number of cuts performed, and end the algorithm.

According to the recursive formula given in Lemma 8.2.8, the sum over all possible outcomes of this algorithm should give $\tau_{R,k}$. This motivates the following, more explicit, expression for $\tau_{R,k}$.

Theorem 8.2.9 (S.). *Let $R \in \{\bullet, \circ\}^* \setminus \epsilon$, and $k \ge 0$. Then*

$$\tau_{R,k} = \sum_{i\ge 1}\phi_{k+i}\sum_{(\rho_1,\ldots,\rho_i)\in\mathcal{C}_i(R)} g_{\bullet\rho_1}\prod_{2\le j\le i} f_{(\bullet\rho_j)}. \tag{8.6}$$

Proof. Let $\rho_{1,1}, \rho_{1,2}, \ldots, \rho_{1,\ell(\rho_1)}$ be the bits of ρ_1. Applying C' to the right side of Equation (8.6) gives

$$\begin{aligned}
&\sum_{i\geq 1} \phi_{k+i} \sum_{(\rho_1,\ldots,\rho_i)\in\mathcal{C}_i(R)} \sum_{1\leq n\leq \ell(\rho_1)} g_{\bullet\rho_{1,1}\cdots\rho_{1,n}} f_{(\bullet\rho_{1,n+1}\cdots\rho_{1,\ell(\rho)})} \prod_{2\leq j\leq i} f_{(\bullet\rho_j)} \\
&\quad = f_{(\bullet)} \sum_{i\geq 1} \phi_{k+i} \sum_{(\rho_1,\ldots,\rho_i)\in\mathcal{C}_i(R)} g_{\bullet\rho_1} \prod_{2\leq j\leq i} f_{(\bullet\rho_j)} \\
&\qquad + \sum_{i\geq 1} \phi_{k+i} \sum_{(\rho_1,\ldots,\rho_i)\in\mathcal{C}_i(R)} \sum_{1\leq n\leq \ell(\rho_1)-1} g_{\bullet\rho_{1,1}\cdots\rho_{1,n}} f_{(\bullet\rho_{1,n+1}\cdots\rho_{1,\ell(\rho)})} \prod_{2\leq j\leq i} f_{(\bullet\rho_j)} \\
&\quad = f_{(\bullet)} \sum_{i\geq 1} \phi_{k+i} \sum_{(\rho_1,\ldots,\rho_i)\in\mathcal{C}_i(R)} g_{\bullet\rho_1} \prod_{2\leq j\leq i} f_{(\bullet\rho_j)} \\
&\qquad + \sum_{i\geq 1} \phi_{k+i} \sum_{(\rho_1,\ldots,\rho_{i+1})\in\mathcal{C}_{i+1}(R)} g_{\bullet\rho_1} \prod_{2\leq j\leq i+1} f_{(\bullet\rho_j)},
\end{aligned}$$

by relabelling $\rho_{1,1}\cdots\rho_{1,n}$ to be ρ_1, and $\rho_{1,n+1}\cdots\rho_{1,\ell(\rho)}$ to be ρ_{i+1}. Thus,

$$\begin{aligned}
&\left(\frac{\partial}{\partial y} - C'\right) \sum_{i\geq 1} \phi_{k+i} \sum_{(\rho_1,\ldots,\rho_i)\in\mathcal{C}_i(R)} g_{\bullet\rho_1} \prod_{2\leq j\leq i} f_{(\bullet\rho_j)} \\
&\quad = \sum_{i\geq 1} \left(\frac{\partial\phi_{k+i}}{\partial y} - f_{(\bullet)}\phi_{k+i}\right) \sum_{(\rho_1,\ldots,\rho_i)\in\mathcal{C}_i(R)} g_{\bullet\rho_1} \prod_{2\leq j\leq i} f_{(\bullet\rho_j)} \\
&\qquad - \sum_{i\geq 1} \phi_{k+i} \sum_{(\rho_1,\ldots,\rho_{i+1})\in\mathcal{C}_{i+1}(R)} g_{\bullet\rho_1} \prod_{2\leq j\leq i+1} f_{(\bullet\rho_j)} \\
&\quad = \sum_{i\geq 1} \phi_{k+i-1} \sum_{(\rho_1,\ldots,\rho_i)\in\mathcal{C}_i(R)} g_{\bullet\rho_1} \prod_{2\leq j\leq i} f_{(\bullet\rho_j)} \\
&\qquad - \sum_{i\geq 2} \phi_{k+i-1} \sum_{(\rho_1,\ldots,\rho_i)\in\mathcal{C}_i(R)} g_{\bullet\rho_1} \prod_{2\leq j\leq i} f_{(\bullet\rho_j)} \\
&\quad = \phi_k g_{\bullet R}.
\end{aligned}$$

Since the right side of Equation (8.6) vanishes when $x = 0$, then by uniqueness of the solution to the partial differential equation, it must be equal to $\tau_{R,k}$. □

There is a natural bijection between i-part compositions of the string R, and i-part compositions of the integer $\ell(R)$, namely,

$$(\rho_1, \ldots, \rho_i) \leftrightarrow (\ell(\rho_1), \ldots, \ell(\rho_i)).$$

Consequently, specializing the series $\tau_{R,k}$ by making the substitutions $g_R \mapsto g_{\ell(R)}$ and $f_{(S)} \mapsto f_{\ell(S)}$ gives the function

$$\sum_{i\geq 1} \phi_{k+i}(x) \sum_{(\rho_1,\ldots,\rho_i)\in\mathcal{C}_i(R)} g_{\ell(\rho_1)+1} \prod_{2\leq j\leq i} f_{\ell(\rho_j)+1} = \sum_{i\geq 1} \phi_{k+i}(x) \sum_{(a_1,\ldots,a_i)\in C_{\ell(R),i}} g_{a_1+1} f_{a_2+1} \cdots f_{a_i+1},$$

which is the analogue $\tau_{j,k}$ of $\tau_{R,k}$ which arises in Section 8.1 in the analysis of the "simplifed" Join-Cut operator for $(p, n-1, n)$-dipoles.

8.2.3 Preimage of $g_R \phi_S(v)$ under the operator $\frac{\partial}{\partial v} - C''$

Recall that the cut operator for (a, b, c, d)-dipoles is given by

$$C'' = \sum_{R \in \{\circ, \bullet\}^*} \left(\sum_{2 \le i \le \ell(R)} w^{\delta_{R_i, \bullet}} g_{R_1 R_i \cdots R_{\ell(R)}} f_{(R_2 \cdots R_i)} + w g_{R_1} f_{(R)} \right) \frac{\partial}{\partial g_R}.$$

Define the function $\tau_{R,S}$ to be the unique solution to the partial differential equation

$$\left(\frac{\partial}{\partial v} - C'' \right) \tau_{R,S} = g_{\bullet R} \phi_S(v)$$

which satisfies the initial condition $\tau_{R,S}|_{v=0} = 0$. (Notationally, these are distinguished from the τ-functions in the preceding section by the fact that they are indexed by a pair of binary strings, as opposed to a binary string and an integer.) The functions $\tau_{R,S}$ may be determined using the following recursion.

Lemma 8.2.10. *Let $R, S \in \{\circ, \bullet\}^*$, and let $t \in \{\circ, \bullet\}$. Let $R = R_1 R_2 \cdots R_{\ell(R)}$. Then*

$$\tau_{tR,S}(v) = \phi_{tS}(v) g_{\bullet tR} + \sum_{1 \le i \le \ell(R)} f_{(tR_1 \cdots R_i)} w^{\delta_{R_i, \bullet}} \tau_{R_i \cdots R_{\ell(R)}, tS} + w f_{(\bullet tR)} \tau_{\epsilon, tS}, \tag{8.7}$$

and

$$\tau_{\epsilon,S} = g_\bullet \phi_{\bullet S}.$$

Proof. For the case of $\tau_{\epsilon,S}$,

$$\begin{aligned} \left(\frac{\partial}{\partial v} - C'' \right) g_\bullet \phi_{\bullet S} &= g_\bullet \frac{\partial}{\partial v} \phi_{\bullet S} - w g_\bullet f_{(\bullet)} \phi_{\bullet S}(v) \\ &= g_\bullet \phi_S(v), \end{aligned}$$

applying Corollary 8.2.6. Since $g_\bullet \phi_{\bullet S}(0) = 0$, then $\tau_{\epsilon,S} = g_\bullet \phi_{\bullet S}$.

For the general case, apply $\left(\frac{\partial}{\partial v} - C''\right)$ to the right side of Equation 8.7 to obtain

$$\begin{aligned}
&\left(\frac{\partial}{\partial v} - C''\right)\left(\phi_{tS}(v)g_{\bullet tR} + \sum_{1\le i\le \ell(R)} f_{(tR_1\cdots R_i)}w^{\delta_{R_i,\bullet}}\tau_{R_i\cdots R_{\ell(R)},tS} + wf_{(\bullet tR)}\tau_{\epsilon,tS}\right)\\
&= g_{\bullet tR}\frac{\partial}{\partial v}\phi_{tS}(v) - \phi_{tS}(v)\left(w^{\delta_{t,\bullet}}g_{\bullet tR}f_{(t)} + \sum_{1\le i\le \ell(R)} w^{\delta_{R_i,\bullet}}g_{\bullet R_i\cdots R_{\ell(R)}}f_{(tR_1\cdots R_i)} + wg_{\bullet}f_{(\bullet tR)}\right)\\
&\quad + \sum_{1\le i\le \ell(R)} f_{(tR_1\cdots R_i)}w^{\delta_{R_i,\bullet}}g_{\bullet R_i\cdots R_{\ell(R)}}\phi_{tS}(v) + wf_{(\bullet tR)}g_{\bullet}\phi_{tS}(v)\\
&= g_{\bullet tR}\frac{\partial}{\partial v}\phi_{tS}(v) - \phi_{tS}(v)f_{(t)}w^{\delta_{t,\bullet}}g_{\bullet tR}\\
&= g_{\bullet tR}\phi_S(v),
\end{aligned}$$

by Corollary 8.2.6. Since $\tau_{tR,S}(0) = 0$, then the left side of Equation 8.7 is the unique solution to $\left(\frac{\partial}{\partial v} - C''\right)\tau_{tR,S}(v) = g_{\bullet tR}\phi_S(v)$ which vanishes at $v = 0$, proving the lemma. $\square$

The first six τ functions, computed using the recursive formula of Lemma 8.2.10, are as follows.

$$\begin{aligned}
\tau_{\bullet,S} &= g_{\bullet\bullet}\phi_{\bullet S} + wg_{\bullet}f_{(\bullet\bullet)}\phi_{\bullet\bullet S}\\
\tau_{\circ,S} &= g_{\bullet\circ}\phi_{\circ S} + wg_{\bullet}f_{(\bullet\circ)}\phi_{\bullet\circ S}\\
\tau_{\bullet\bullet,S} &= g_{\bullet\bullet\bullet}\phi_{\bullet S} + (wg_{\bullet\bullet}f_{(\bullet\bullet)} + wg_{\bullet}f_{(\bullet\bullet\bullet)})\phi_{\bullet\bullet S} + w^2g_{\bullet}f^2_{(\bullet\bullet)}\phi_{\bullet\bullet\bullet S}\\
\tau_{\circ\bullet,S} &= g_{\bullet\circ\bullet}\phi_{\circ S} + (wg_{\bullet\bullet}f_{(\bullet\circ)} + wg_{\bullet}f_{(\bullet\circ\bullet)})\phi_{\bullet\circ S} + w^2g_{\bullet}f_{(\bullet\bullet)}f_{(\circ\bullet)}\phi_{\bullet\bullet\circ S}\\
\tau_{\bullet\circ,S} &= g_{\bullet\bullet\circ}\phi_{\bullet S} + wg_{\bullet}f_{(\bullet\bullet\circ)}\phi_{\bullet\bullet S} + g_{\bullet\circ}f_{(\bullet\circ)}\phi_{\circ\bullet S} + wg_{\bullet}f^2_{(\bullet\circ)}\phi_{\bullet\circ\bullet S}\\
\tau_{\circ\circ,S} &= g_{\bullet\circ\circ}\phi_{\circ S} + wg_{\bullet}f_{(\bullet\circ\circ)}\phi_{\bullet\circ S} + g_{\bullet\circ}f_{(\circ\circ)}\phi_{\circ\circ S} + wg_{\bullet}f_{(\circ\circ)}f_{(\bullet\circ)}\phi_{\bullet\circ\circ S}
\end{aligned}$$

Based on the experience with the functions $\tau_{R,k}$, it is natural to attempt to conjecture a similar closed form for the functions $\tau_{R,S}$. Given a string composition $(\rho_1, \ldots, \rho_k)$, define the non-negative integer $c(\rho_1, \ldots, \rho_k)$ by

$$c(\rho_1, \ldots, \rho_k) = |\{i \ge 2 : \text{the first bit of } \rho_i \text{ is } \bullet\}|.$$

Theorem 8.2.11 (S.). *Let $R \in \{\bullet, \circ\}^* \setminus \epsilon$. Let $S \in \{\bullet, \circ\}^*$. Given a string composition $(\rho_1, \ldots, \rho_k)$, let $\rho_{j,1}$ denote the first bit of ρ_j. Then*

$$\tau_{R,S} = \sum_{k\ge 1}\sum_{(\rho_1,\ldots,\rho_k)\in\mathcal{C}_k(R)} w^{c(\rho_1,\ldots,\rho_k)}(g_{\bullet\rho_k}\phi_{\rho_{k,1}\cdots\rho_{1,1}S}(v) + wg_{\bullet}f_{(\bullet\rho_k)}\phi_{\bullet\rho_{k,1}\cdots\rho_{1,1}S}(v))\prod_{1\le j\le k-1} f_{(\rho_j\rho_{j+1,1})}. \tag{8.8}$$

Proof. Let

$$A_{1,k} = \sum_{(\rho_1,\ldots,\rho_k)\in\mathcal{C}_k(R)} w^{c(\rho_1,\ldots,\rho_k)} g_{\bullet\rho_k} \phi_{\rho_{k,1}\cdots\rho_{1,1}S} \prod_{1\le j\le k-1} f_{(\rho_j\rho_{j+1,1})}$$

and

$$A_{2,k} = \sum_{(\rho_1,\ldots,\rho_k)\in\mathcal{C}_k(R)} w^{c(\rho_1,\ldots,\rho_k)+1} g_{\bullet} f_{(\bullet\rho_k)} \phi_{\bullet\rho_{k,1}\cdots\rho_{1,1}S} \prod_{1\le j\le k-1} f_{(\rho_j\rho_{j+1,1})}$$

so that the right side of Equation (8.8) becomes

$$\sum_{k\ge 1}(A_{1,k}+A_{2,k}).$$

Applying C'' to the first of these expressions gives

$$C''A_{1,k} = B_{1,k}+B_{2,k}+B_{3,k},$$

Where

$$B_{1,k} = \sum_{(\rho_1,\ldots,\rho_k)\in\mathcal{C}_k(R)} w^{c(\rho_1,\ldots,\rho_k)+1} g_{\bullet} f_{(\bullet\rho_k)} \phi_{\rho_{k,1}\cdots\rho_{1,1}S} \prod_{1\le j\le k-1} f_{(\rho_j\rho_{j+1,1})},$$

$$B_{2,k} = \sum_{(\rho_1,\ldots,\rho_k)\in\mathcal{C}_k(R)} w^{c(\rho_1,\ldots,\rho_k)+\delta_{\rho_{k,1},\bullet}} g_{\bullet\rho_k} f_{(\rho_{k,1})} \phi_{\rho_{k,1}\cdots\rho_{1,1}S} \prod_{1\le j\le k-1} f_{(\rho_j\rho_{j+1,1})},$$

and

$$B_{3,k} = \sum_{(\rho_1,\ldots,\rho_k)\in\mathcal{C}_k(R)} \sum_{2\le i\le\ell(\rho_k)} w^{c(\rho_1,\ldots,\rho_k)+\delta_{\rho_{k,i},\bullet}} g_{\bullet\rho_{k,i}\cdots\rho_{k,\ell(\rho_k)}} f_{(\rho_{k,1}\cdots\rho_{k,i})} \phi_{\rho_{k,1}\cdots\rho_{1,1}S} \prod_{1\le j\le k-1} f_{(\rho_j\rho_{j+1,1})}.$$

Applying Corollary 8.2.6,

$$\begin{aligned}\frac{\partial}{\partial v}A_{1,k} - B_{2,k} &= \sum_{(\rho_1,\ldots,\rho_k)\in\mathcal{C}_k(R)} w^{c(\rho_1,\ldots,\rho_k)} g_{\bullet\rho_k}\left(\frac{\partial}{\partial v}\phi_{\rho_{k,1}\cdots\rho_{1,1}S}\right.\\ &\qquad\left. - w^{\delta_{\rho_{k,1},\bullet}} f_{(\rho_{k,1})}\phi_{\rho_{k,1}\cdots\rho_{1,1}S}\right) \prod_{1\le j\le k-1} f_{(\rho_j\rho_{j+1,1})}\\ &= \sum_{(\rho_1,\ldots,\rho_k)\in\mathcal{C}_k(R)} w^{c(\rho_1,\ldots,\rho_k)} g_{\bullet\rho_k} \phi_{\rho_{k-1,1}\cdots\rho_{1,1}S} \prod_{1\le j\le k-1} f_{(\rho_j\rho_{j+1,1})}.\end{aligned}$$

To analyze $B_{k,3}$, given any $(\rho_1,\ldots,\rho_k)\in\mathcal{C}_k(R)$ and $2\le i\le\ell(\rho_k)$, define the string composition $(\rho'_1,\ldots,\rho'_k,\rho'_{k+1})\in\mathcal{C}_{k+1}(R)$ by $\rho'_j=\rho_j$ when $1\le j\le k-1$, $\rho'_k=\rho_{k,1}\cdots\rho_{k,i-1}$ and $\rho'_{k+1}=\rho_{k,i}\cdots\rho_{k,\ell(\rho_k)}$. Since $c(\rho'_1,\ldots,\rho'_{k+1}) = c(\rho_1,\ldots,\rho_k)+\delta_{\rho_{k,i},\bullet}$, then

$$B_{3,k} = \sum_{(\rho'_1,\ldots,\rho'_{k+1})\in\mathcal{C}_{k+1}(R)} w^{c(\rho'_1,\ldots,\rho'_{k+1})} g_{\bullet\rho'_{k+1}} \phi_{\rho'_{k,1}\cdots\rho'_{1,1}S} \prod_{1\le j\le k} f_{(\rho'_j\rho'_{j+1,1})}.$$

In other words, for $k \geq 2$,

$$\frac{\partial}{\partial v} A_{1,k} - B_{2,k} - B_{3,k-1} = 0.$$

Next, apply $\left(\frac{\partial}{\partial v} - C''\right)$ to $A_{2,k}$ and use Corollary 8.2.6 to obtain

$$\begin{aligned}
\left(\frac{\partial}{\partial v} - C''\right) A_{2,k} &= \sum_{(\rho_1,\ldots,\rho_k)\in\mathcal{C}_k(R)} w^{c(\rho_1,\ldots,\rho_k)+1} g_\bullet f_{(\bullet\rho_k)} \left(\frac{\partial}{\partial v}\phi_{\bullet\rho_{k,1}\cdots\rho_{1,1}S}\right. \\
&\qquad \left. - w f_{(\bullet)}\phi_{\bullet\rho_{k,1}\cdots\rho_{1,1}S}\right) \prod_{1\leq j\leq k-1} f_{(\rho_j\rho_{j+1,1})} \\
&= \sum_{(\rho_1,\ldots,\rho_k)\in\mathcal{C}_k(R)} w^{c(\rho_1,\ldots,\rho_k)+1} g_\bullet f_{(\bullet\rho_k)} \phi_{\rho_{k,1}\cdots\rho_{1,1}S} \prod_{1\leq j\leq k-1} f_{(\rho_j\rho_{j+1,1})} \\
&= B_{1,k}.
\end{aligned}$$

Consequently, applying $\left(\frac{\partial}{\partial v} - C''\right)$ to the right side of Equation (8.8) yields

$$\begin{aligned}
\left(\frac{\partial}{\partial v} - C''\right) \sum_{k\geq 1} A_{1,k} + A_{2,k} &= \sum_{k\geq 1} \left(\frac{\partial}{\partial v} A_{1,k} - B_{1,k} - B_{2,k} - B_{3,k} + \left(\frac{\partial}{\partial v} - C''\right) A_{2,k}\right) \\
&= \frac{\partial}{\partial v} A_{1,1} - B_{2,1} \\
&= g_{\bullet R} \frac{\partial}{\partial v}\phi_{R_1 S} - w^{\delta_{R_1,\bullet}} g_{\bullet R} f_{(R_1)} \phi_{R_1 S} \\
&= g_{\bullet R}\phi_S,
\end{aligned}$$

again using Corollary 8.2.6. Since the right side of Equation (8.8) vanishes when $v = 0$, it is the unique solution to $\left(\frac{\partial}{\partial v} - C''\right) \tau_{R,S} = g_{\bullet R}\phi_S$.

□

This formula demonstrates that $\tau_{R,S}$ depends on S only through the ϕ functions. Since the ϕ functions depend only on the number of occurrences of $\bullet$ and $\circ$ in S, the notation $\tau_{R,i,j}$ is sometimes used to denote $\tau_{R,S}$ when S is a binary string consisting of i occurrences of $\circ$ and j occurrences of $\bullet$. The contributions to $\tau_{R,S}$ from various compositions of small strings is given in Table 8.1. By way of verification, this table agrees with the earlier expressions for $\tau_{R,S}$, which were computed recursively.

The importance of the functions $\tau_{R,S}$ is that, because of the linearity of $\frac{\partial}{\partial v} - C''$, the solution to any equation of the form

$$\left(\frac{\partial}{\partial v} - C''\right) F = G$$

String Composition	Contribution to $\tau_{R,S}$
$(\bullet)$	$g_{\bullet\bullet}\phi_{\bullet S} + wg_{\bullet}f_{(\bullet\bullet)}\phi_{\bullet\bullet S}$
$(\circ)$	$g_{\bullet\circ}\phi_{\circ S} + wg_{\bullet}f_{(\bullet\circ)}\phi_{\bullet\circ S}$
$(\bullet\bullet)$	$g_{\bullet\bullet\bullet}\phi_{\bullet S} + wg_{\bullet}f_{(\bullet\bullet\bullet)}\phi_{\bullet\bullet S}$
$(\bullet,\bullet)$	$w(g_{\bullet\bullet}\phi_{\bullet\bullet S} + wg_{\bullet}f_{(\bullet\bullet)}\phi_{\bullet\bullet\bullet S})f_{(\bullet\bullet)}$
$(\circ\bullet)$	$g_{\bullet\circ\bullet}\phi_{\circ S} + wg_{\bullet}f_{(\bullet\circ\bullet)}\phi_{\bullet\circ S}$
$(\circ,\bullet)$	$w(g_{\bullet\bullet}\phi_{\bullet\circ S} + wg_{\bullet}f_{(\bullet\bullet)}\phi_{\bullet\bullet\circ S})f_{(\circ\bullet)}$
$(\bullet\circ)$	$g_{\bullet\bullet\circ}\phi_{\bullet S} + wg_{\bullet}f_{(\bullet\bullet\circ)}\phi_{\bullet\bullet S}$
$(\bullet,\circ)$	$(g_{\bullet\circ}\phi_{\circ\bullet S} + wg_{\bullet}f_{(\bullet\circ)}\phi_{\bullet\circ\bullet S})f_{(\bullet\circ)}$
$(\circ\circ)$	$g_{\bullet\circ\circ}\phi_{\circ S} + wg_{\bullet}f_{(\bullet\circ\circ)}\phi_{\bullet\circ S}$
$(\circ,\circ)$	$(g_{\bullet\circ}\phi_{\circ\circ S} + wg_{\bullet}f_{(\bullet\circ)}\phi_{\bullet\circ\circ S})f_{(\circ\circ)}$

Table 8.1: Contributions to $\tau_{R,S}$ from various small string compositions.

can be expressed as a linear combination of $\{\tau_{R,S}\}_{R,S\in\{\bullet,\circ\}^*}$, provided G is expressed as a linear combination of terms of the form $g_R\phi_S(v)$. Moreover, Theorem 8.2.11 demonstrates that the τ-functions can in turn be expressed as a linear combination of terms of the from $g_R\phi_S(v)$, allowing this process to be iterated. Thus, $\{\tau_{R,S}\}_{R,S\in\{\bullet,\circ\}^*}$ is a natural family of functions in which to express the genus g solutions the (a,b,c,d)-dipole problem. These expressions are finite linear combinations of τ-functions in which the coefficients are polynomials with non-negative integer coefficients. The next section is a demonstration of how this may be done for surfaces of small genus.

8.3 Solutions for Small Genus

This section contains a demonstration of how the results of Section 8.2 can be used to find the generating series for $(a,b,0,0)$-dipoles on the torus and double torus, and (a,b,c,d)-dipoles on the torus. The series for (a,b,c,d)-dipoles on the double torus appears in Appendix B.

8.3.1 $(a,b,0,0)$-dipoles on the torus and double torus

Much like the strategy used in Section 8.1, the genus g solution to the $(a,b,0,0)$-dipole problem, $\Psi'^{(g)} = [u^{2g}]\Psi'$, may be determined by solving the equation

$$\left(\frac{\partial}{\partial y} - C'\right)\Psi'^{(g)} = J\Psi'^{(g-1)} \tag{8.9}$$

when $g \geq 1$. When $g = 0$,

$$\Psi'^{(0)} = g_\bullet \sum_{a \geq 0} \frac{x^{a+1}}{(a+1)!} f_{(\circ)}^{a+1} = g_\bullet(\exp(xf_{(\circ)}) - 1).$$

Let $\hat{\Psi}'^{(g)} = \Psi'^{(g)} - \Psi'^{(g)}|_{y=0}$. When $y = 0$, the series Ψ' corresponds to dipoles with no b-edges, for which the root face is a digon. Thus, $C'\Psi'^{(g)}|_{y=0} = 0$, so solving Equation (8.9) is equivalent to solving

$$\left(\frac{\partial}{\partial y} - C'\right)\hat{\Psi}'^{(g)} = J\Psi'^{(g-1)} \tag{8.10}$$

with the initial condition $\hat{\Psi}'^{(g)} = 0$.

When solving for the genus 1 solution, Equation 8.10 becomes

$$\begin{aligned}\left(\frac{\partial}{\partial y} - C'\right)\Psi'^{(1)} &= Jg_\bullet(\exp(xf_{(\circ)}) - 1) \\ &= g_{\bullet\bullet\circ}\, x\exp(xf_{(\circ)}),\end{aligned}$$

with initial condition $\hat{\Psi}'^{(1)}|_{y=0} = 0$. By Theorem 8.2.9, the solution to this equation is

$$\hat{\Psi}'^{(1)} = \tau_{\bullet\circ,0}\, x\exp(xf_{(\circ)}),$$

where

$$\tau_{\bullet\circ,0} = \phi_1(y)g_{\bullet\bullet\circ} + \phi_2(y)g_{\bullet\bullet}f_{(\bullet\circ)}.$$

The series $\Psi'^{(1)}|_{y=0}$ can be obtained from the initial condition given in Theorem 5.3.1. Using the fact that genus 1 dipoles with n edges must have half face degree sequence either $(3, 1^{n-3})$ or $(2, 2, 1^{n-4})$,

$$\begin{aligned}\Psi'^{(1)}|_{y=0} &= [u^2]g_\bullet \sum_{D \in \mathcal{D}} \frac{x^{n(D)}}{n(D)!} u^{2g(D)} f_{\lambda'(D)} \\ &= g_\bullet\left(\sum_{n \geq 3} \frac{x^n}{n!} D_{(3,1^{n-3})} f_{(\circ\circ\circ)} f_{(\circ)}^{n-3} + \sum_{n \geq 4} D_{(2,2,1^{n-4})} f_{(\circ\circ)}^2 f_{(\circ)}^{n-4}\right),\end{aligned}$$

where D_λ is the number of rooted dipoles with face degree sequence 2λ. Although D_λ is a known quantity which can be computed by central methods, it simplifies computations if it is left unevaluated while solving the differential equations for (a, b, c, d)-dipoles. One additional benefit of this is that the results obtained will appear as "linear combinations of central problems," and identify which parts of the solutions arise from the central initial condition and which parts arise from the non-central aspects of the problem. Since $\Psi'^{(1)} = \hat{\Psi}'^{(1)} + \Psi'^{(1)}|_{y=0}$, the following result has now been proven.

Theorem 8.3.1 (S.). *The generating series for $(a, b, 0, 0)$-dipoles on the torus is*

$$\Psi'^{(1)} = \tau_{\bullet\circ,0} x \exp(x f_{(\circ)}) + g_{\bullet} \left(\sum_{n \geq 3} \frac{x^n}{n!} D_{(3,1^{n-3})} f_{(\circ\circ\circ)} f_{(\circ)}^{n-3} + \sum_{n \geq 4} D_{(2,2,1^{n-4})} f_{(\circ\circ)}^2 f_{(\circ)}^{n-4} \right)$$

In order to determine the genus 2 solution for Ψ', the first step is to determine $J'\Psi'^{(1)}$. Since $\Psi'^{(1)}$ is expressed in terms of the functions ϕ_i, it is first necessary to determine the action of $\frac{\partial}{\partial f_{(\bullet)}}$ on ϕ_i. The coefficients of ϕ_i are, from definition, given by

$$\left[\frac{y^n}{n!} \right] \phi_i = [t^{n-i}](1 - t f_{(\bullet)})^{-i}$$

Computing the derivative with respect to $f_{(\bullet)}$ gives

$$\begin{aligned} \frac{\partial}{\partial f_{(\bullet)}} \left[\frac{y^n}{n!} \right] \phi_i &= [t^{n-i-1}] i (1 - t f_{(\bullet)})^{-(i+1)} \\ &= i \left[\frac{y^n}{n!} \right] \phi_{i+1}(x). \end{aligned}$$

Since this holds for all $n \geq i$, then the following result has been proven.

Lemma 8.3.2. *Let $i \geq 1$. Then*

$$\frac{\partial \phi_i}{\partial f_{(\bullet)}} = i \phi_{i+1}.$$

Using this lemma, the following computation may be done:

$$\begin{aligned} J'\Psi'^{(1)} = {} & x \exp(x f_{(\circ)}) (g_{\bullet\bullet\circ\bullet\bullet} \phi_2(x) + x g_{\bullet\bullet\circ\bullet\circ} \phi_1(x) + 2 g_{\bullet\bullet\bullet\bullet} f_{(\bullet\circ)} \phi_3(x) + g_{\bullet\bullet\bullet\bullet\circ} \phi_2(x) \\ & \qquad + g_{\bullet\bullet\bullet\circ\bullet} \phi_2(x) + x g_{\bullet\bullet\bullet\circ} f_{(\bullet\circ)} \phi_2(x)) \\ & + \sum_{n \geq 3} \frac{x^n}{n!} D_{(3,1^{n-3})} (3 g_{\bullet\bullet\circ\circ\circ} f_{(\circ)}^{n-3} + (n-3) g_{\bullet\bullet\circ} f_{(\circ\circ\circ)} f_{(\circ)}^{n-4}) \\ & + \sum_{n \geq 4} \frac{x^n}{n!} D_{(2,2,1^{n-4})} (4 g_{\bullet\bullet\circ\circ} f_{(\circ\circ)} f_{(\circ)}^{n-4} + (n-4) g_{\bullet\bullet\circ} f_{(\circ\circ)}^2 f_{(\circ)}^{n-5}). \end{aligned}$$

Proceeding as in the genus 1 case leads to the following.

Theorem 8.3.3 (S.). *The generating series for $(a,b,0,0)$-dipoles on the double torus is*

$$\begin{aligned}\Psi'^{(2)} &= x\exp(xf_{(\circ)})(\tau_{\bullet\circ\bullet\bullet,2} + x\tau_{\bullet\circ\bullet\bullet,1} + 2f_{(\bullet\circ)}\tau_{\bullet\bullet\bullet,3} + \tau_{\bullet\bullet\bullet\circ,2} + \tau_{\bullet\bullet\circ\bullet,2} + xf_{(\bullet\circ)}\tau_{\bullet\bullet\circ,2})\\ &\quad + 3\tau_{\bullet\circ\circ\circ,0}\sum_{n\geq 3}\frac{x^n}{n!}D_{(3,1^{n-3})}f_{(\circ)}^{n-3} + \tau_{\bullet\circ,0}f_{(\circ\circ\circ)}\sum_{n\geq 4}\frac{(n-3)x^n}{n!}D_{(3,1^{n-3})}f_{(\circ)}^{n-4}\\ &\quad + 4\tau_{\bullet\circ\circ,0}f_{(\circ\circ)}\sum_{n\geq 4}\frac{x^n}{n!}D_{(2,2,1^{n-4})}f_{(\circ)}^{n-4} + \tau_{\bullet\circ,0}f_{(\circ\circ)}^2\sum_{n\geq 5}\frac{(n-4)x^n}{n!}D_{(2,2,1^{n-4})}f_{(\circ)}^{n-5}\\ &\quad + [u^4]g_{\bullet}\sum_{D\in\mathcal{D}}\frac{x^{n(D)}}{n(D)!}u^{2g(D)}f_{\lambda'(D)}.\end{aligned}$$

Extracting Coefficients from the $(a,b,0,0)$-dipole series

In order to extract coefficients from the series appearing in Theorems 8.3.1 and 8.3.3, the first step is to determine the coefficients of $\langle\tau_{S,k}\rangle$. The functions $\langle\tau_{S,k}\rangle$ are expressed in terms of $\langle\phi_i\rangle$, whose coefficients are, from definition, given by

$$\left[\frac{y^n}{n!}\right]\langle\phi_i\rangle = [t^{n-i}](1-t)^{-i} = \binom{n-1}{n-i},$$

with the convention that $\binom{n}{k}=0$ when $k<0$. When setting the f- and g-type indeterminates to 1, $\tau_{R,k}$ has the following more explicit form.

Corollary 8.3.4. *Let $R\in\{\bullet,\circ\}^*\setminus\epsilon$, and $k\geq 0$. Then*

$$\langle\tau_{R,k}\rangle = \sum_{i\geq 1}\binom{\ell(R)-1}{i-1}\langle\phi_{k+i}(x)\rangle$$

and

$$\left[\frac{y^n}{n!}\right]\langle\tau_{R,k}\rangle = \binom{\ell(R)+b-2}{\ell(R)+k-1}.$$

Proof. The first expression follows from Theorem 8.2.9, along with the observation that the number of string compositions of R into i parts is equal to the number of integer compositions of $\ell(R)$ into i parts, which is $\binom{\ell(R)-1}{i-1}$. Extracting the coefficient of $\frac{y^n}{n!}$,

$$\begin{aligned}\left[\frac{y^n}{n!}\right]\langle\tau_{R,k}\rangle &= \sum_{i\geq 1}\binom{\ell(R)-1}{i-1}\binom{n-1}{n-k-i}\\ &= \binom{\ell(R)+n-2}{\ell(R)+k-1},\end{aligned}$$

by Vandermonde's identity. □

This result may be used, along with Theorem 8.3.1, to determine the number of $(a,b,0,0)$-dipoles on the torus when $b \geq 1$:

$$\left[\frac{y^b}{b!}\frac{x^{a+1}}{(a+1)!}\right]\langle \Psi'^{(1)}\rangle = \langle \tau_{\bullet\circ,0}\rangle x\exp(x) = b(a+1).$$

This agrees with the computation of the number of $(p,n-1,n)$-dipoles on the torus done in Section 8.1, since a $(p,n-1,n)$-dipole has $p-1$ b-edges, and $n-p-1$ a-edges. The coefficients for $(a,b,0,0)$ dipoles on the double torus may be obtained in a similar manner:

$$\begin{aligned}\left[\frac{y^b}{b!}\frac{x^{a+1}}{(a+1)!}\right]\langle \Psi'^{(2)}\rangle = 3(a+1)\binom{b+2}{5} &+ a(a+1)\binom{b+2}{4} + 2(a+1)\binom{b+1}{5}\\ &+ a(a+1)\binom{b+1}{4} + \left(3\binom{b+2}{3} + (a-2)b\right)d_{(3,1^{a-2})}\\ &+ \left(4\binom{b+1}{2} + (a-3)b\right)d_{(2,2,1^{a-3})},\end{aligned}$$

adopting the convention that $d_{(3,1^{a-2})} = 0$ when $a < 2$, and $d_{(2,2,1^{a-3})} = 0$ when $a < 3$.

8.3.2 (a,b,c,d)-dipoles on the torus

Again, let $\Psi''^{(g)} = [u^{2g}]\Psi''$ so that

$$\left(\frac{\partial}{\partial v} - C''\right)\Psi''^{(g)} = J''\Psi''^{(g-1)} \tag{8.11}$$

when $g \geq 1$, with

$$\begin{aligned}\Psi''^{(0)} &= \sum_{a\geq 0}\sum_{d\geq 0}\frac{x^{a+1}}{(a+1)!}\frac{(vw)^d}{d!}g_\bullet f_{(\bullet)}^d f_{(\circ)}^{a+1}\\ &= g_\bullet(\exp(xf_{(\circ)})-1)(wf_{(\bullet)}\phi_\bullet(v)+1),\end{aligned}$$

using the fact that, from the definition,

$$\phi_\bullet(v) = \frac{1}{wf_{(\bullet)}}\left(\exp(wvf_{(\bullet)})-1\right).$$

In order to convert Equation (8.11) to a differential equation whose initial condition is zero, let

$$\hat{\Psi}''^{(g)} = \Psi''^{(g)} - \Psi''^{(g)}|_{v=0} = \Psi''^{(g)} - \Psi'^{(g)}.$$

Then $\hat{\Psi}''^{(g)}$ is the solution to the equation

$$\left(\frac{\partial}{\partial v} - C''\right)\hat{\Psi}''^{(g)} = J''\Psi''^{(g-1)} + C''\Psi'^{(g)}. \tag{8.12}$$

In order to compute $J''\Psi''^{(g-1)}$, it is first necessary to determine the action of $\frac{\partial}{\partial f_{(\circ)}}$ and $\frac{\partial}{\partial f_{(\bullet)}}$ on ϕ_S. If S is a string of with i instances of $\circ$ and j instances of $\bullet$,

$$\begin{aligned}\left[\frac{v^n}{n!}\right]\phi_S &= \left[\frac{v^n}{n!}\right]\phi_{i,j} \\ &= [t^{n-i-j}](1-tf_{(\circ)})^{-i}(1-twf_{(\bullet)})^{-j}.\end{aligned}$$

If $i = 0$, then $\frac{\partial\phi_{0,j}}{\partial f_{(\circ)}} = 0$. For $i > 1$, and any $n \geq i$,

$$\begin{aligned}\left[\frac{v^n}{n!}\right]\frac{\partial\phi_{i,j}}{\partial f_{(0)}} &= [t^{n-i}]\frac{\partial}{\partial f_{(\circ)}}(1-tf_{(\circ)})^{-i}(1-tyf_{(\bullet)})^{-j} \\ &= i[t^{n-i-1}](1-tf_{(\circ)})^{-(i+1)}(1-tyf_{(\bullet)})^{-j} \\ &= i\left[\frac{v^n}{n!}\right]\phi_{i+1,j}.\end{aligned}$$

Thus,

$$\frac{\partial\phi_{i,j}}{\partial f_{(\circ)}} = i\phi_{i+1,j}$$

when $i > 0$. A similar argument may be used to determine $\frac{\partial}{\partial f_{(\bullet)}}\phi_{i,j}$, leading to the following.

Lemma 8.3.5. *Let $i, j \geq 0$. Then*

$$\frac{\partial\phi_{i,j}}{\partial f_{(\circ)}} = i\phi_{i+1,j}$$

and

$$\frac{\partial\phi_{i,j}}{\partial f_{(\bullet)}} = jw\phi_{i,j+1}.$$

To determine the generating series for (a,b,c,d)-dipoles on the torus, first, determine $J''\Psi''^{(0)} + C''\Psi'^{(1)}$. Using Lemma 8.3.5,

$$\begin{aligned}J''\Psi''^{(0)} &= g_{\bullet\circ\circ}(wf_{(\bullet)}\phi_\bullet(v)+1)x\exp(xf_{(\circ)}) \\ &\quad + g_{\bullet\bullet\bullet}(w^2\phi_\bullet(v) + w^3 f_{(\bullet)}\phi_{\bullet\bullet}(v))(\exp(xf_{(\circ)})-1).\end{aligned}$$

Using the expression for $\Psi'^{(1)}$ given in Theorem 8.3.1,

$$\begin{aligned}C''\Psi'^{(1)} &= (wg_\bullet f_{(\bullet\bullet\circ)} + wg_{\bullet\bullet\circ}f_{(\bullet)} + g_{\bullet\circ}f_{(\bullet\circ)})\phi_1(y)x\exp(xf_{(\circ)})\\ &+ (wg_\bullet f_{(\bullet\bullet)} + wg_{\bullet\bullet}f_{(\bullet)})f_{(\bullet\circ)}\phi_2(y)x\exp(xf_{(\circ)})\\ &+ wg_\bullet f_{(\bullet)}\left(\sum_{n\geq 3}\frac{x^n}{n!}D_{(3,1^{n-3})}f_{(\circ\circ\circ)}f_{(\circ)}^{n-3} + \sum_{n\geq 4}\frac{x^n}{n!}D_{(2,2,1^{n-4})}f_{(\circ\circ)}^2 f_{(\circ)}^{n-4}\right).\end{aligned}$$

Expressing the solution to Equation (8.12) in terms of the functions $\tau_{R,S}$ (using the fact that $\tau_{\epsilon,S} = g_\bullet\phi_{\bullet S}(v)$ and adding the initial condition gives the following.

Theorem 8.3.6 (S.). *The generating series for (a,b,c,d)-dipoles on the torus is*

$$\begin{aligned}\Psi''^{(1)} &= (wf_{(\bullet)}\tau_{\circ\circ,\bullet} + \tau_{\circ\circ,\epsilon})x\exp(xf_{(\circ)})\\ &+ (w^2\tau_{\bullet\bullet,\bullet} + w^3 f_{(\bullet)}\tau_{\bullet\bullet,\bullet\bullet})(\exp(xf_{(\circ)}) - 1)\\ &+ (wf_{(\bullet\bullet\circ)}g_\bullet\phi_\bullet(v) + wf_{(\bullet)}\tau_{\bullet\circ,\epsilon} + f_{(\bullet\circ)}\tau_{\circ,\epsilon})\phi_1(y)x\exp(xf_{(\circ)})\\ &+ (wf_{(\bullet\bullet)}g_\bullet\phi_\bullet(v) + wf_{(\bullet)}\tau_{\bullet,\epsilon})f_{(\bullet\circ)}\phi_2(y)x\exp(xf_{(\circ)})\\ &+ wf_{(\bullet)}g_\bullet\phi_\bullet(v)\left(\sum_{n\geq 3}\frac{x^n}{n!}D_{(3,1^{n-3})}f_{(\circ\circ\circ)}f_{(\circ)}^{n-3} + \sum_{n\geq 4}\frac{x^n}{n!}D_{(2,2,1^{n-4})}f_{(\circ\circ)}^2 f_{(\circ)}^{n-4}\right) + \Psi'^{(1)},\end{aligned}$$

where the functions $\tau_{R,S}$ are given in Table 8.1.

Extracting Coefficients from the (a,b,c,d)-dipole series

Suppose $c+d>0$. Then the term $\Psi'^{(1)}$ in $\Psi''^{(1)}$ may be disregarded. When information about face structure is forgotten,

$$\begin{aligned}\langle\Psi''^{(1)}\rangle &= xe^x(\langle\phi_{1,0}\rangle + \langle\phi_{2,0}\rangle + 2w\langle\phi_{1,1}\rangle + 2w\langle\phi_{2,1}\rangle + w^2\langle\phi_{1,2}\rangle + w^2\langle\phi_{2,2}\rangle)\\ &+ (e^x - 1)(w^2\langle\phi_{0,2}\rangle + 3w^3\langle\phi_{0,3}\rangle + 3w^4\langle\phi_{0,4}\rangle + w^5\langle\phi_{0,5}\rangle)\\ &+ xe^x\langle\phi_1(y)\rangle(\langle\phi_{1,0}\rangle + 2w\langle\phi_{0,1}\rangle + 2w\langle\phi_{1,1}\rangle + w^2\langle\phi_{0,2}\rangle + w^2\langle\phi_{1,2}\rangle)\\ &+ xe^x\langle\phi_2(y)\rangle(2w\langle\phi_{0,1}\rangle + w^2\langle\phi_{0,2}\rangle)\\ &+ w\langle\phi_{0,1}\rangle\left(\sum_{n\geq 3}\frac{x^n}{n!}D_{(3,1^{n-3})} + \sum_{n\geq 4}\frac{x^n}{n!}D_{(2,2,1^{n-4})}\right).\end{aligned}$$

Extracting the coefficients of x^{a+1} and y^b may be done as in the $(a,b,0,0)$ case:

$$\begin{aligned}\left[\frac{x^{a+1}y^b}{(a+1)!b!}\right]\langle\Psi''^{(1)}\rangle &= (a+1)\delta_{b,0}(\langle\phi_{1,0}\rangle + \langle\phi_{2,0}\rangle + 2w\langle\phi_{1,1}\rangle + 2w\langle\phi_{2,1}\rangle + w^2\langle\phi_{1,2}\rangle + w^2\langle\phi_{2,2}\rangle) \\ &\quad + \delta_{b,0}(w^2\langle\phi_{0,2}\rangle + 3w^3\langle\phi_{0,3}\rangle + 3w^4\langle\phi_{0,4}\rangle + w^5\langle\phi_{0,5}\rangle) \\ &\quad + (a+1)(1-\delta_{b,0})(\langle\phi_{1,0}\rangle + 2w\langle\phi_{0,1}\rangle + 2w\langle\phi_{1,1}\rangle + w^2\langle\phi_{0,2}\rangle + w^2\langle\phi_{1,2}\rangle) \\ &\quad + (a+1)(1-\delta_{b,0})(b-1)(2w\langle\phi_{0,1}\rangle + w^2\langle\phi_{0,2}\rangle) \\ &\quad + w\langle\phi_{0,1}\delta_{b,0}\rangle(D_{(3,1^{a-2})} + D_{(2,2,1^{a-3})}).\end{aligned}$$

To continue, it is necessary to extract coefficients of the form

$$\left[w^d\frac{v^{c+d}}{(c+d)!}\right] w^k\langle\phi_{i,j}\rangle.$$

From the definition, $\left[\frac{v^{c+d}}{(c+d)!}\right]\langle\phi_{i,j}\rangle$ is just the complete symmetric function $h_{c+d-i-j}$ in $i+j$ indeterminates, with i indeterminates set to 1 and j indeterminates set to w. Thus,

$$\begin{aligned}\left[w^d\frac{v^{c+d}}{(c+d)!}\right] w^k\langle\phi_{i,j}\rangle &= [w^{d-k}][t^{c+d-i-j}](1-t)^{-i}(1-wt)^{-j} \\ &= [t^{c+k-i-j}](1-t)^{-i}\binom{d-k+j-1}{d-k} \\ &= \binom{c+k-j-1}{c+k-j-i}\binom{d-k+j-1}{d-k}.\end{aligned}$$

This leads to the following.

Theorem 8.3.7 (S.)**.** *When $c+d>0$, the number of (a,b,c,d)-dipoles on the torus is given by*

$$\begin{aligned}&(a+1)\delta_{b,0}\left(\binom{c-1}{c-1}\binom{d-1}{d} + \binom{c-1}{c-2}\binom{d-1}{d} + 2\binom{c-1}{c-1}\binom{d-1}{d-1} + 2\binom{c-1}{c-2}\binom{d-1}{d-1} + \binom{c-1}{c-1}\binom{d-1}{d-2} + \binom{c-1}{c-2}\binom{d-1}{d-2}\right) \\ &\quad + \delta_{b,0}\left(\binom{c-1}{c}\binom{d-1}{d-2} + 3\binom{c-1}{c}\binom{d-1}{d-3} + 3\binom{c-1}{c}\binom{d-1}{d-4} + \binom{c-1}{c}\binom{d-1}{d-5}\right) \\ &\quad + (a+1)\binom{b-1}{b-1}\left(\binom{c-1}{c-1}\binom{d-1}{d} + 2\binom{c-1}{c}\binom{d-1}{d-1} + 2\binom{c-1}{c-1}\binom{d-1}{d-1} + \binom{c-1}{c}\binom{d-1}{d-2} + \binom{c-1}{c-1}\binom{d-1}{d-2}\right) \\ &\quad + (a+1)\binom{b-1}{b-2}\left(2\binom{c-1}{c}\binom{d-1}{d-1} + \binom{c-1}{c}\binom{d-1}{d-2}\right) \\ &\quad + \delta_{b,0}\binom{c-1}{c}\binom{d-1}{d-1}(D_{(3,1^{a-2})} + D_{(2,2,1^{a-3})}).\end{aligned}$$

8.3.3 Concluding comments on the differential approach to (a,b,c,d)-dipoles

Although closed form solutions to the equations given in Theorems 5.3.1 and 5.3.2 are not known, this chapter has demonstrated that the differential operators $C'+u^2J'$ and $C''+u^2J''$

are sufficiently well-understood that they can be used to develop a process, recursive in genus, for determining the generating series for (a, b, c, d) dipoles in a specified surface. While this process could, in principle, be applied to determine the solution for surfaces other than the surfaces considered by Visentin and Wieler, in practice the calculations needed to do this are quite lengthy, albeit simple. Nevertheless, the Join-Cut approach has several appealing features.

- The differential approach is more likely to lead to a proof technique for the general case than a combinatorial case analysis. If solutions to the partial differential equations can be conjectured, then it suffices to demonstrate that the conjectured solutions satisfy the equations.

- The differential approach records more data about the dipoles than existing methods. Specifically, it allows us to keep track of the numbers of four different types of edges, as opposed to tracking only p, q and n.

- The solutions are expressed as sums over combinatorial objects, namely, string compositions. As a result, the coefficients of the series computed using differential methods are expressed naturally as sums of non-negative integers.

Chapter 9

Concluding Comments and Areas for Future Work

This thesis has examined the extent to which two of the techniques used to solve central problems, character-based methods and Join-Cut analysis, can be generalized to deal with non-central problems, with the (p, q, n)-dipole problem (or more precisely, its refinement to the (a, b, c, d)-dipole problem) as a motivating example. The Join-Cut analysis of the (a, b, c, d)-dipole problem conducted in Chapter 5 resulted in two partial differential equations which determine the generating series for the problem. Non-central information was handled through the introduction of additional classes of indeterminates to keep track of the additional information. Moreover, this analysis identified a special case of the (p, q, n)-dipole problem, the case when $q = n - 1$, as one that is "less non-central" than the general problem, in the sense of having an encoding in $Z_1(n)$ as opposed to $Z_2(n)$. In Chapter 8, an analysis of these partial differential equations led to a method, recursive in genus, for determining the generating series for (a, b, c, d)-dipoles.

With regards to generalizing character-based methods, Chapters 6 and 7 demonstrated that the algebra $Z_1(n)$ can be used to solve a class of non-central problems much in the same manner as $Z(n)$ is used to solve central problems. Orthogonal idempotents for $Z_1(n)$ were constructed, and techniques used to evaluate their coefficients in the standard basis were developed. A key combinatorial insight which made this possible was the interpretation of Strahov's generalized characters of $\mathfrak{S}_n$ as the coefficients of a group algebra sum indexed by the set of standard Young tableaux in which the position of the symbol n is specified. These techniques were then used to give a full solution to an open problem (the $(p, n-1, n)$-dipole problem), to give an alternative solution to a previously-solved problem (the non-transitive star factorization problem), and to provide an avenue to approach a new problem which is a natural non-central generalization of a well-studied central problem (Z_1-decompositions of a full cycle). Because ordinary characters can be recovered from generalized characters by summing over all choices of a distinguished part of the indexing partition, specializations of the $(p, n-1, n)$-dipole problem and the transitive star factorization problem which "forget" non-central information lead to expressions which are linear combinations of ordinary characters. It is not known how to obtain such expressions by working only within the centre of $\mathbb{C}[\mathfrak{S}_n]$.

The research done in this thesis has led to a number of further questions to form the basis of future research. Several of these have already been commented on in the thesis, and are collected here for convenience.

1. Find algebraic and combinatorial proofs of Conjecture 7.1.10.
2. Find a closed form solution to the partial differential equations in Theorems 5.3.1 and 5.3.2.
3. Find a combinatorial explanation for the identities which arise as a consequence of the relationship between generalized characters and ordinary characters, such as Corollary 6.2.10.
4. Solve the problem of Z_1-decompositions of a full cycle. An analysis of the expression given in Theorem 7.3.2 is a likely starting point for approaching this problem.
5. Study the asymptotics of generalized characters, and consequently, the asymptotics of the combinatorial problems whose solutions can be expressed in terms of generalized characters.

In addition to these questions, there are three areas for future research that warrant a more extensive discussion. These are discussed in the following sections.

9.1 Study of other Centralizer Algebras

A natural extension of the work done in this thesis is to perform an analysis of centralizer algebras other than $Z_1(n)$, analogous to what was done in Chapters 6 and 7. The algebra $Z_2(n)$ is an obvious candidate for further study, not only because it is the "next step up" from the study of $Z_1(n)$, but also because an understanding of $Z_2(n)$ would lead to a complete solution of the (p, q, n)-dipole problem. The main difficulty in the study of $Z_2(n)$ is that it is not commutative, and thus cannot have a basis of orthogonal idempotents. Determining the centre of $Z_2(n)$ would be an important first step.

An additional line of inquiry would be to study centralizers of the form $Z_H(n)$ where H is not a subgroup of the form $\mathfrak{S}_k$. One particularly appealing example is $H = \langle \mathfrak{S}_{n-2}, (n-1, n) \rangle$

(or, more generally, taking H to be a Young subgroup of the form $\langle \mathfrak{S}_{\{1,\ldots,k\}}, \mathfrak{S}_{\{k+1,\ldots,n\}}\rangle$). It is clear that the conjugacy classes with respect to H are unions of $\mathfrak{S}_{n-2}$-conjugacy classes. Specifically, there are two types of conjugacy classes with respect to H:

$$\mathcal{A}_\lambda(i,j) \cup \mathcal{A}_\lambda(i,i-j)$$

and

$$\mathcal{B}_\lambda(i,j) \cup \mathcal{B}_\lambda(j,i).$$

Consequently, the elements of the form

$$A_\lambda(i,j) + A_\lambda(i,i-j)$$

and

$$B_\lambda(i,j) + B_\lambda(j,i)$$

form a basis for $Z_H(n)$. Notably, the conjugacy classes with respect to H are closed under taking inverses, and thus the proof of Lemma 4.2.6 may be used to show that $Z_H(n)$ is a commutative subalgebra of $Z_2(n)$. Hence, the avenue of studying $Z_H(n)$ *via* the construction of a basis of orthogonal idempotents is available. The study of $Z_H(n)$ would be an important first step in the study of $Z_2(n)$, since the centre of $Z_2(n)$ lies in $Z_H(n)$. Indeed, since $(n, n-1) = A_{(2,1^{n-2})}(2,1)$, in any central element of $Z_2(n)$, the coefficient of $A_\lambda(i,j)$ must equal the coefficient of $A_\lambda(i,i-j)$, and the coefficient of $B_\lambda(i,j)$ must equal the coefficient of $B_\lambda(j,i)$. This containment is strict: the element $A_{(5)}(5,3) + A_{(5)}(5,2)$ of $Z_H(5)$ does not commute with the element $A_{(5)}(5,3)$ of $Z_2(5)$.

The algebra $Z_H(n)$ is of combinatorial interest for several reasons. First, for any n, the element

$$A_{(2n)}(2n,n)^2$$

lies in $Z_H(2n)$. Thus, a special case of the (p,q,n)-dipole problem, the case of $(n,n,2n)$-dipoles, may be encoded in $Z_H(2n)$. Second, instances of the non-transitive G-factorization problem (Problem 3.2.10) for which H is the automorphism group of G may be encoded in $Z_H(n)$. Examples of graphs which have H as their automorphism group include:

1. The complete graph, K_n, with the edge $\{n-1, n\}$ removed.
2. The complete bipartite graph $K_{2,n-2}$.
3. The complete bipartite graph $K_{2,n-2}$ with an edge added between the two vertices of degree $n-2$.

Taking G to be either the second or third graph defines a problem which could be viewed as a natural generalization of the star factorization problem, since star factorizations correspond to the graph $K_{1,n-1}$. Taking G to be the first graph defines a problem which has a particularly simple combinatorial description:

Problem 9.1.1 (Forbidden Transposition Problem). *Let $\pi \in \mathfrak{S}_n$ and $r \geq 0$. Determine the number of sequences $(\tau_1, \ldots, \tau_r)$ such that:*

1. *Each τ_i is a transposition,*
2. *$\tau_i \neq (n-1, n)$,*
3. *and $\prod_{1 \leq i \leq r} \tau_i = \pi$.*

A combinatorial analysis of the G-factorization problem for these three graphs could provide an important first step in the study of $Z_H(n)$.

9.2 Further study of the relationship between the character-based and differential approaches to non-central problems

The fact that both algebraic and differential approaches may be used to approach the same problem suggests that there is a relationship between the two methods which should be more well-understood. Each of the two methods has its own strengths and weaknesses. The algebraic approach is very effective at fully solving special cases, such as the $(p, n-1, n)$-dipole problem, but does not appear to be easy to extend to the (p, q, n)-case. The differential approach applies to the (p, q, n)-problem for arbitrary p and q, but it relies on the introduction of a more difficult problem, the (a, b, c, d)-dipole problem. Furthermore, it provides only a recursive method for computing solutions for a surface of genus g, as opposed to giving solutions for all orientable surfaces. The strengths of one method could be used to hint at ways of rectifying the weaknesses of the other method. Some specific instances where further inquiry is warranted are discussed in the following.

Although the (p, q, n)-dipole problem can be encoded as a product of basis elements in $Z_2(n)$, currently there is no known partial differential equation which describes the generating series for (p, q, n)-dipoles using only a "Z_2" amount of information. Currently, the Join-Cut analysis of the (p, q, n)-dipole problem requires the introduction of an ancillary problem, the (a, b, c, d)-dipole problem which is significantly more non-central than the (p, q, n)-dipole problem. Indeed, the introduction of (a, b, c, d)-dipoles is a map-theoretic refinement of the problem for which there is no known algebraic refinement, *i.e.* an encoding of the (a, b, c, d)-dipole problem as a product of basis elements in some centralizer algebra. Finding a combinatorial argument leading to a partial differential equation for (p, q, n)-dipoles which relies only on the cycle type of the dipole and the positions of n and $n-1$ would be an important step forward in solving the (p, q, n)-dipole problem. If such operators could be found, they would likely be much easier to analyze than the operators for the (a,b,c,d)-dipole problem, since their indeterminates would be indexed by integers as opposed to combinatorial objects.

Although the $(p,n-1,n)$-dipole problem can be solved by algebraic means, the solution given in Theorem 7.1.1 does not yet lead to a closed-form generating series which is a solution to the partial differential equation given in Theorem 5.2.1. The main barrier is the lack of a k-independent generating series for the generalized characters $\gamma_{\mu,j}^{(n-k-1,2,1^{k-1}),2}$ similar to the series given in Lemma 7.3.1. If such a series were known, then algebraic manipulations similar to the ones used in the proof of Lemma 2.4.9 could be used to write down a closed form for Ψ. Having a closed form expression for Ψ would assist in conjecturing closed forms for the generating series Ψ' and Ψ'' for $(a,b,0,0)$-dipoles and (a,b,c,d)-dipoles, respectively.

Another method of gaining greater understanding of the relationship between the algebraic and differential approaches is through the introduction of a non-central generalization of the Frobenius transformation. One possible definition of a non-central Frobenius transformation is

$$\Phi_1 : \pi \mapsto g_{r(\pi)} f_{\lambda(\pi)\backslash r(\pi)}, \tag{9.1}$$

where $\pi \in \mathfrak{S}_n$, $r(\pi)$ is the length of the cycle of π containing n, and g_r and f_λ are power sum symmetric functions in two different sets of indeterminates. With this definition, if $\hat{\Delta}$ is the differential operator for the non-transitive star factorization problem given by Equation 3.8, then

$$\hat{\Delta}\Phi_1(\Gamma^{\mu,j}) = c_{\mu,j}\Phi_1(\Gamma^{\mu,j}). \tag{9.2}$$

In other words, the Z_1-idempotents give the eigenfunctions of $\hat{\Delta}$. A study of the functions $\Phi_1(\Gamma^{(\mu,j)})$ would provide a non-central analogue of the differential approaches to central problems due to Goulden [6] and Lascoux and Thibon [30]. From a combinatorial point of view, this is of interest because of the similarity between $\hat{\Delta}$ and the operator $(C+J)$ for the $(p,n-1,n)$-dipole problem, given in Theroem 5.2.1. More precisely, if G_+ denotes the operator given by $G_+(g_r) = g_{r+1}$, then when restricted to the space of functions which are linear in the g-indeterminates,

$$(C+J) = G_+\hat{\Delta}.$$

Thus, studying the action of G_+ on the functions $\Phi_1(\Gamma^{\mu,j})$ would provide an alternative approach to the $(p,n-1,n)$-dipole problem, since the action of $\hat{\Delta}$ is well-understood.

It should be noted that Strahov [42] introduces a different notion of a non-central Frobenius transform, namely

$$\pi \mapsto t^{r(\pi)-1} f_{\lambda(\pi)\backslash r(\pi)}.$$

Using this definition, Strahov is able to convert the Murnaghan-Nakayama rule for generalized characters into a Z_1-analogue of the Jacobi-Trudi formula. However, this definition does not satisfy Equation (9.2), so there is still value in studying the transformation given in (9.1).

9.3 The non-orientable version of the (p, q, n)-dipole problem

A natural extension of the (p, q, n)-dipole problem in orientable surfaces is to consider the problem for all surfaces. Some care must be taken in defining the (p, q, n)-dipole problem for a non-orientable surface, since the values of p and q are defined with respect to an orientation on each vertex. This difficulty may be rectified by taking an open set containing the root edge and both vertices, and uniformly choosing an orientation on this set.

Some work on the non-orientable version of the (p, q, n)-dipole problem has been done by Liu and Yang [31] using topological methods. (Although the definition of (p, q, n)-dipoles used in this paper is not made explicit, it appears to be the one described in the preceding paragraph.) They are able to determine the number of (p, q, n)-dipoles on the projective plane and the Klein bottle. Their method is to embed the dipole in a $2n$-sided polygonal representation of the surface such that each side of the $2n$-gon is bisected by an edge of the dipole. This induces an identification of the edges of the $2n$-gon, which can be expressed as a sequence over the alphabet $\{a_i, a_i^{-1}\}_{1 \leq i \leq n}$ in which each symbol appears exactly once. The values of p and q force the symbols corresponding to the second distinguished edge to appear in fixed locations in this sequence. The surface can then be identified by determining whether or not this sequence contains certain excluded subsequences; for example, if the surface is the projective plane, then there are no subsequences of the form $xyx^{-1}y$. The number of dipoles on the surface is then determined by enumerating the number of sequences which correspond to the given surface.

APPENDICES

Appendix A

Generating Series for $(p, n-1, n)$-dipoles

n	p	$D_{n,p}$
4	1	$t^2 + t^4$
4	2	$2t^2$
4	3	$2t^2$
5	1	$5t^3 + t^5$
5	2	$3t + 3t^3$
5	3	$2t^2 + 4t^3$
5	4	$3t + 3t^3$
6	1	$8t^2 + 15t^4 + t^6$
6	2	$20t^2 + 4t^4$
6	3	$18t^2 + 6t^4$
6	4	$18t^2 + 6t^4$
6	5	$20t^2 + 4t^4$
7	1	$84t^3 + 35t^5 + t^7$
7	2	$40t + 75t^3 + 5t^5$
7	3	$32t + 80t^3 + 8t^5$
7	4	$36t + 75t^3 + 9t^5$
7	5	$32t + 80t^3 + 8t^5$
7	6	$40t + 75t^3 + 5t^5$

n	p	$D_{n,p}$
8	1	$180t^2 + 469t^4 + 70t^6 + t^8$
8	2	$504t^2 + 210t^4 + 6t^6$
8	3	$460t^2 + 250t^4 + 10t^6$
8	4	$468t^2 + 240t^4 + 12t^6$
8	5	$468t^2 + 240t^4 + 12t^6$
8	6	$460t^2 + 250t^4 + 10t^6$
8	7	$504t^2 + 210t^4 + 6t^6$
9	1	$3044t^3 + 1869t^5 + 126t^7 + t^9$
9	2	$1260t + 3283t^3 + 490t^5 + 7t^7$
9	3	$1080t + 3318t^3 + 630t^5 + 12t^7$
9	4	$1140t + 3255t^3 + 630t^5 + 15t^7$
9	5	$1104t + 3304t^3 + 616t^5 + 16t^7$
9	6	$1140t + 3255t^3 + 630t^5 + 15t^7$
9	7	$1080t + 3318t^3 + 630t^5 + 12t^7$
9	8	$1260t + 3283t^3 + 490t^5 + 7t^7$
10	1	$8064t^2 + 26060t^4 + 5985t^6 + 210t^8 + t^{10}$
10	2	$24352t^2 + 14952t^4 + 1008t^6 + 8t^8$
10	3	$22568t^2 + 16366t^4 + 1372t^6 + 14t^8$
10	4	$22872t^2 + 16002t^4 + 1428t^6 + 18t^8$
10	5	$22800t^2 + 16100t^4 + 1400t^6 + 20t^8$
10	6	$22800t^2 + 16100t^4 + 1400t^6 + 20t^8$
10	7	$22872t^2 + 16002t^4 + 1428t^6 + 18t^8$
10	8	$22568t^2 + 16366t^4 + 1372t^6 + 14t^8$
10	9	$24352t^2 + 14952t^4 + 1008t^6 + 8t^8$
11	1	$193248t^3 + 152900t^5 + 16401t^7 + 330t^9 + t^{11}$
11	2	$72576t + 234540t^3 + 53865t^5 + 1890t^7 + 9t^9$
11	3	$64512t + 232832t^3 + 62832t^5 + 2688t^7 + 16t^9$
11	4	$66528t + 231252t^3 + 62181t^5 + 2898t^7 + 21t^9$
11	5	$65664t + 232320t^3 + 61992t^5 + 2880t^7 + 24t^9$
11	6	$66240t + 231500t^3 + 62265t^5 + 2860t^7 + 25t^9$
11	7	$65664t + 232320t^3 + 61992t^5 + 2880t^7 + 24t^9$
11	8	$66528t + 231252t^3 + 62181t^5 + 2898t^7 + 21t^9$
11	9	$64512t + 232832t^3 + 62832t^5 + 2688t^7 + 16t^9$
11	10	$72576t + 234540t^3 + 53865t^5 + 1890t^7 + 9t^9$

n	p	$D_{n,p}$
12	1	$604800t^2 + 2286636t^4 + 696905t^6 + 39963t^8 + 495t^{10} + t^{12}$
12	2	$1932480t^2 + 1529000t^4 + 164010t^6 + 3300t^8 + 10t^{10}$
12	3	$1811808t^2 + 1610640t^4 + 201474t^6 + 4860t^8 + 10t^{10}$
12	4	$1829664t^2 + 1590600t^4 + 203112t^6 + 5400t^8 + 24t^{10}$
12	5	$1825488t^2 + 1596140t^4 + 201684t^6 + 5460t^8 + 28t^{10}$
12	6	$1826640t^2 + 1594500t^4 + 202230t^6 + 5400t^8 + 30t^{10}$
12	7	$1826640t^2 + 1594500t^4 + 202230t^6 + 5400t^7 + 30t^{10}$
12	8	$1825488t^2 + 1596140t^4 + 201684t^6 + 5460t^8 + 28t^{10}$
12	9	$1829664t^2 + 1590600t^4 + 203112t^6 + 5400t^8 + 24t^{10}$
12	10	$1811808t^2 + 1610640t^4 + 201474t^6 + 4860t^8 + 18t^{10}$
12	11	$1932480t^2 + 1529000t^4 + 164010t^6 + 3300t^8 + 10t^{10}$
13	1	$68428800t^2 + 292271616t^4 + 109425316t^6 + 8691683t^8 + 183183t^{10} + 1001t^{12} + t^{14}$
13	2	$6652800t + 25152996t^3 + 7665955t^5 + 439593t^7 + 5445t^9 + 11t^{11}$
13	3	$6048000t + 24798840t^3 + 8498050t^5 + 563640t^7 + 8250t^9 + 20t^{11}$
13	4	$6168960t + 24736932t^3 + 8421435t^5 + 580041t^7 + 9405t^9 + 27t^{11}$
13	5	$6128640t + 24779392t^3 + 8422480t^5 + 576576t^7 + 9680t^9 + 32t^{11}$
13	6	$6148800t + 24752420t^3 + 8429575t^5 + 576345t^7 + 9625t^9 + 35t^{11}$
13	7	$6134400t + 24773496t^3 + 8421930t^5 + 577368t^7 + 9750t^9 + 36t^{11}$
13	8	$6148800t + 24752420t^3 + 8429575t^5 + 576345t^7 + 9625t^9 + 35t^{11}$
13	9	$6128640t + 24779392t^3 + 8422480t^5 + 576576t^7 + 9680t^9 + 32t^{11}$
13	10	$6168960t + 24736932t^3 + 8421435t^5 + 580041t^7 + 9405t^9 + 27t^{11}$
13	11	$6048000t + 24798840t^3 + 8498050t^5 + 563640t^7 + 8250t^9 + 20t^{11}$
13	12	$6652800t + 25152996t^3 + 7665955t^5 + 439593t^7 + 5445t^9 + 11t^{11}$

Appendix B

Generating Series for (a, b, c, d)-dipoles on the double torus

In order to determine the generating series for (a, b, c, d)-dipoles on the double torus, it is first necessary to compute $C''\Psi'^{(2)}$ and $J''\Psi''^{(1)}$, where

$$C'' = \sum_{R \in \{\circ,\bullet\}^*} \left(\sum_{2 \le i \le \ell(R)} w^{\delta_{R_i,\bullet}} g_{R_1 R_i \cdots R_{\ell(R)}} f_{(R_2 \cdots R_i)} + w g_{R_1} f_{(R)} \right) \frac{\partial}{\partial g_R},$$

$$\begin{aligned}
\Psi'^{(2)} &= x \exp(x f_{(\circ)}) (\tau_{\bullet\circ\bullet\bullet,2} + x\tau_{\bullet\circ\bullet\bullet,1} + 2 f_{(\bullet\circ)} \tau_{\bullet\bullet\bullet,3} + \tau_{\bullet\bullet\bullet\circ,2} + \tau_{\bullet\bullet\circ\bullet,2} + x f_{(\bullet\circ)} \tau_{\bullet\bullet\circ,2}) \\
&+ 3\tau_{\bullet\circ\circ\circ,0} \sum_{n\ge3} \frac{x^n}{n!} D_{(3,1^{n-3})} f_{(\circ)}^{n-3} + \tau_{\bullet\circ,0} f_{(\circ\circ\circ)} \sum_{n \ge 4} \frac{(n-3)x^n}{n!} D_{(3,1^{n-3})} f_{(\circ)}^{n-4} \\
&+ 4\tau_{\bullet\circ\circ,0} f_{(\circ\circ)} \sum_{n\ge4} \frac{x^n}{n!} D_{(2,2,1^{n-4})} f_{(\circ)}^{n-4} + \tau_{\bullet\circ,0} f_{(\circ\circ)}^2 \sum_{n\ge5} \frac{(n-4)x^n}{n!} D_{(2,2,1^{n-4})} f_{(\circ)}^{n-5} \\
&+ [u^4] g_\bullet \sum_{D \in \mathcal{D}} \frac{x^{n(D)}}{n(D)!} u^{2g(D)} f_{\lambda'(D)},
\end{aligned}$$

$$J'' = \sum_{R \in \{\circ,\bullet\}^*} \sum_{(S) \in \mathcal{S}(\circ,\bullet)} \left(\sum_{S_1 S_2 \cdots S_{\ell(S)} \in (S)} w^{\delta_{S_1,\bullet}} g_{R_1 S_1 \cdots S_{\ell(S)} S_1 R_2 \cdots R_{\ell(R)}} \right) \frac{\partial^2}{\partial g_R \partial f_{(S)}},$$

and

$$
\begin{aligned}
\Psi''^{(1)} = {} & (wf_{(\bullet)}\tau_{\circ\circ,\bullet} + \tau_{\circ\circ,\epsilon})x\exp(xf_{(\circ)}) \\
& + (w^2\tau_{\bullet\bullet,\bullet} + w^3 f_{(\bullet)}\tau_{\bullet\bullet,\bullet\bullet})(\exp(xf_{(\circ)}) - 1) \\
& + (wf_{(\bullet\bullet\circ)}g_\bullet\phi_\bullet(v) + wf_{(\bullet)}\tau_{\bullet\circ,\epsilon} + f_{(\bullet\circ)}\tau_{\circ,\epsilon})\phi_1(y)x\exp(xf_{(\circ)}) \\
& + (wf_{(\bullet\bullet)}g_\bullet\phi_\bullet(v) + wf_{(\bullet)}\tau_{\bullet,\epsilon})f_{(\bullet\circ)}\phi_2(y)x\exp(xf_{(\circ)}) \\
& + wf_{(\bullet)}g_\bullet\phi_\bullet(v)\left(\sum_{n\geq 3}\frac{x^n}{n!}D_{(3,1^{n-3})}f_{(\circ\circ\circ)}f_{(\circ)}^{n-3} + \sum_{n\geq 4}\frac{x^n}{n!}D_{(2,2,1^{n-4})}f_{(\circ\circ)}^2 f_{(\circ)}^{n-4}\right) + \Psi'^{(1)},
\end{aligned}
$$

To compute $C''\Psi'^{(2)}$, the functions $\tau_{R,j}$ arising in $\Psi'^{(2)}$ may be determined, using Theorem 8.2.9, by listing the string compositions of R and determining the contribution each one makes to $\tau_{R,j}$. This is done in the following tables. The action of C'' on each contribution is also determined. Thus, to obtain $\tau_{R,j}$, sum the terms in the second column of each of the following tables. To obtain $C''\tau_{R,j}$, sum the terms in the third column.

$R = \bullet\circ\bullet\bullet$

String composition	Contribution to $\tau_{\bullet\circ\bullet\bullet,j}$	Contribution to $C''\tau_{\bullet\circ\bullet\bullet,j}$
$\bullet\circ\bullet\bullet$	$\phi_{j+1}g_{\bullet\bullet\circ\bullet\bullet}$	$\phi_{j+1}(wg_{\bullet\bullet\circ\bullet\bullet}f_{(\bullet)} + g_{\bullet\circ\bullet\bullet}f_{(\bullet\circ)} + wg_{\bullet\bullet\bullet}f_{(\bullet\circ\bullet)}$ $+wg_{\bullet\bullet}f_{(\bullet\circ\bullet\bullet)} + wg_\bullet f_{(\bullet\bullet\circ\bullet\bullet)})$
$\bullet,\circ\bullet\bullet$	$\phi_{j+2}g_{\bullet\bullet}f_{(\bullet\circ\bullet\bullet)}$	$\phi_{j+2}f_{(\bullet\circ\bullet\bullet)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_\bullet f_{(\bullet\bullet)})$
$\bullet\circ,\bullet\bullet$	$\phi_{j+2}g_{\bullet\bullet\circ}f_{(\bullet\bullet\bullet)}$	$\phi_{j+2}f_{(\bullet\bullet\bullet)}(wg_{\bullet\bullet\circ}f_{(\bullet)} + g_{\bullet\circ}f_{(\bullet\circ)} + wg_\bullet f_{(\bullet\bullet\circ)})$
$\bullet\circ\bullet,\bullet$	$\phi_{j+2}g_{\bullet\bullet\circ\bullet}f_{(\bullet\bullet)}$	$\phi_{j+2}f_{(\bullet\bullet)}(wg_{\bullet\bullet\circ\bullet}f_{(\bullet)} + g_{\bullet\circ\bullet}f_{(\bullet\circ)}$ $+wg_{\bullet\bullet}f_{(\bullet\circ\bullet)} + wg_\bullet f_{(\bullet\bullet\circ\bullet)})$
$\bullet,\circ,\bullet\bullet$	$\phi_{j+3}g_{\bullet\bullet}f_{(\bullet\circ)}f_{(\bullet\bullet\bullet)}$	$\phi_{j+3}f_{(\bullet\circ)}f_{(\bullet\bullet\bullet)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_\bullet f_{(\bullet\bullet)})$
$\bullet,\circ\bullet,\bullet$	$\phi_{j+3}g_{\bullet\bullet}f_{(\bullet\circ\bullet)}f_{(\bullet\bullet)}$	$\phi_{j+3}f_{(\bullet\circ\bullet)}f_{(\bullet\bullet)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_\bullet f_{(\bullet\bullet)})$
$\bullet\circ,\bullet,\bullet$	$\phi_{j+3}g_{\bullet\bullet\circ}f_{(\bullet\bullet)}^2$	$\phi_{j+3}f_{(\bullet\bullet)}^2(wg_{\bullet\bullet\circ}f_{(\bullet)} + g_{\bullet\circ}f_{(\bullet\circ)} + wg_\bullet f_{(\bullet\bullet\circ)})$
$\bullet,\circ,\bullet,\bullet$	$\phi_{j+4}g_{\bullet\bullet}f_{(\bullet\circ)}f_{(\bullet\bullet)}^2$	$\phi_{j+4}f_{(\bullet\circ)}f_{(\bullet\bullet)}^2(wg_{\bullet\bullet}f_{(\bullet)} + wg_\bullet f_{(\bullet\bullet)})$

$R = \bullet\bullet\bullet$

String composition	Contribution to $\tau_{\bullet\bullet\bullet,j}$	Contribution to $C''\tau_{\bullet\bullet\bullet,j}$
$\bullet\bullet\bullet$	$\phi_{j+1}g_{\bullet\bullet\bullet\bullet}$	$\phi_{j+1}(wg_{\bullet\bullet\bullet\bullet}f_{(\bullet)} + wg_{\bullet\bullet\bullet}f_{(\bullet\bullet)} + wg_{\bullet\bullet}f_{(\bullet\bullet\bullet)} + wg_\bullet f_{(\bullet\bullet\bullet\bullet)})$
$\bullet,\bullet\bullet$	$\phi_{j+2}g_{\bullet\bullet}f_{(\bullet\bullet\bullet)}$	$\phi_{j+2}f_{(\bullet\bullet\bullet)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_\bullet f_{(\bullet\bullet)})$
$\bullet\bullet,\bullet$	$\phi_{j+2}g_{\bullet\bullet\bullet}f_{(\bullet\bullet)}$	$\phi_{j+2}f_{(\bullet\bullet)}(wg_{\bullet\bullet\bullet}f_{(\bullet)} + wg_{\bullet\bullet}f_{(\bullet\bullet)} + wg_\bullet f_{(\bullet\bullet\bullet)})$
$\bullet,\bullet,\bullet$	$\phi_{j+3}g_{\bullet\bullet}f_{(\bullet\bullet)}^2$	$\phi_{j+3}f_{(\bullet\bullet)}^2(wg_{\bullet\bullet}f_{(\bullet)} + wg_\bullet f_{(\bullet\bullet)})$

$R = \bullet\bullet\bullet\circ$

String composition	Contribution to $\tau_{\bullet\bullet\bullet\circ,j}$	Contribution to $C''\tau_{\bullet\bullet\bullet\circ,j}$
$\bullet\bullet\bullet\circ$	$\phi_{j+1}g_{\bullet\bullet\bullet\bullet\circ}$	$\phi_{j+1}(wg_{\bullet\bullet\bullet\bullet\circ}f_{(\bullet)} + wg_{\bullet\bullet\bullet\circ}f_{(\bullet\bullet)} + wg_{\bullet\bullet\circ}f_{(\bullet\bullet\bullet)}$ $+g_{\bullet\circ}f_{(\bullet\bullet\bullet\circ)} + wg_{\bullet}f_{(\bullet\bullet\bullet\bullet\circ)})$
$\bullet, \bullet\bullet\circ$	$\phi_{j+2}g_{\bullet\bullet}f_{(\bullet\bullet\bullet\circ)}$	$\phi_{j+2}f_{(\bullet\bullet\bullet\circ)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$
$\bullet\bullet, \bullet\circ$	$\phi_{j+2}g_{\bullet\bullet\bullet}f_{(\bullet\bullet\circ)}$	$\phi_{j+2}f_{(\bullet\bullet\circ)}(wg_{\bullet\bullet\bullet}f_{(\bullet)} + wg_{\bullet\bullet}f_{(\bullet\bullet)} + wg_{\bullet}f_{(\bullet\bullet\bullet)})$
$\bullet\bullet\bullet, \circ$	$\phi_{j+2}g_{\bullet\bullet\bullet\bullet}f_{(\bullet\circ)}$	$\phi_{j+2}f_{(\bullet\circ)}(wg_{\bullet\bullet\bullet\bullet}f_{(\bullet)} + wg_{\bullet\bullet\bullet}f_{(\bullet\bullet)}$ $+wg_{\bullet\bullet}f_{(\bullet\bullet\bullet)} + wg_{\bullet}f_{(\bullet\bullet\bullet\bullet)})$
$\bullet, \bullet, \bullet\circ$	$\phi_{j+3}g_{\bullet\bullet}f_{(\bullet\bullet)}f_{(\bullet\bullet\circ)}$	$\phi_{j+3}f_{(\bullet\bullet)}f_{(\bullet\bullet\circ)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$
$\bullet, \bullet\bullet, \circ$	$\phi_{j+3}g_{\bullet\bullet}f_{(\bullet\bullet\bullet)}f_{(\bullet\circ)}$	$\phi_{j+3}f_{(\bullet\bullet\bullet)}f_{(\bullet\circ)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$
$\bullet\bullet, \bullet, \circ$	$\phi_{j+3}g_{\bullet\bullet\bullet}f_{(\bullet\bullet)}f_{(\bullet\circ)}$	$\phi_{j+3}f_{(\bullet\bullet)}f_{(\bullet\circ)}(wg_{\bullet\bullet\bullet}f_{(\bullet)} + wg_{\bullet\bullet}f_{(\bullet\bullet)} + wg_{\bullet}f_{(\bullet\bullet\bullet)})$
$\bullet, \bullet, \bullet, \circ$	$\phi_{j+4}g_{\bullet\bullet}f^2_{(\bullet\bullet)}f_{(\bullet\circ)}$	$\phi_{j+4}f^2_{(\bullet\bullet)}f_{(\bullet\circ)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$

$R = \bullet\bullet\circ\bullet$

String composition	Contribution to $\tau_{\bullet\bullet\circ\bullet,j}$	Contribution to $C''\tau_{\bullet\bullet\circ\bullet,j}$
$\bullet\bullet\circ\bullet$	$\phi_{j+1}g_{\bullet\bullet\bullet\circ\bullet}$	$\phi_{j+1}(wg_{\bullet\bullet\bullet\circ\bullet}f_{(\bullet)} + wg_{\bullet\bullet\circ\bullet}f_{(\bullet\bullet)} + g_{\bullet\circ\bullet}f_{(\bullet\bullet\circ)}$ $+wg_{\bullet\bullet}f_{(\bullet\bullet\circ\bullet)} + wg_{\bullet}f_{(\bullet\bullet\bullet\circ\bullet)})$
$\bullet, \bullet\circ\bullet$	$\phi_{j+2}g_{\bullet\bullet}f_{(\bullet\bullet\circ\bullet)}$	$\phi_{j+2}f_{(\bullet\bullet\circ\bullet)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$
$\bullet\bullet, \circ\bullet$	$\phi_{j+2}g_{\bullet\bullet\bullet}f_{(\bullet\circ\bullet)}$	$\phi_{j+2}f_{(\bullet\circ\bullet)}(wg_{\bullet\bullet\bullet}f_{(\bullet)} + wg_{\bullet\bullet}f_{(\bullet\bullet)} + wg_{\bullet}f_{(\bullet\bullet\bullet)})$
$\bullet\bullet\circ, \bullet$	$\phi_{j+2}g_{\bullet\bullet\bullet\circ}f_{(\bullet\bullet)}$	$\phi_{j+2}f_{(\bullet\bullet)}(wg_{\bullet\bullet\bullet\circ}f_{(\bullet)} + wg_{\bullet\bullet\circ}f_{(\bullet\bullet)}$ $+g_{\bullet\circ}f_{(\bullet\bullet\circ)} + wg_{\bullet}f_{(\bullet\bullet\bullet\circ)})$
$\bullet, \bullet, \circ\bullet$	$\phi_{j+3}g_{\bullet\bullet}f_{(\bullet\bullet)}f_{(\bullet\circ\bullet)}$	$\phi_{j+3}f_{(\bullet\bullet)}f_{(\bullet\circ\bullet)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$
$\bullet, \bullet\circ, \bullet$	$\phi_{j+3}g_{\bullet\bullet}f_{(\bullet\bullet\circ)}f_{(\bullet\bullet)}$	$\phi_{j+3}f_{(\bullet\bullet\circ)}f_{(\bullet\bullet)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$
$\bullet\bullet, \circ, \bullet$	$\phi_{j+3}g_{\bullet\bullet\bullet}f_{(\bullet\circ)}f_{(\bullet\bullet)}$	$\phi_{j+3}f_{(\bullet\circ)}f_{(\bullet\bullet)}(wg_{\bullet\bullet\bullet}f_{(\bullet)} + wg_{\bullet\bullet}f_{(\bullet\bullet)} + wg_{\bullet}f_{(\bullet\bullet\bullet)})$
$\bullet, \bullet, \circ, \bullet$	$\phi_{j+4}g_{\bullet\bullet}f^2_{(\bullet\bullet)}f_{(\bullet\circ)}$	$\phi_{j+4}f^2_{(\bullet\bullet)}f_{(\bullet\circ)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$

$R = \bullet\bullet\circ$

String composition	Contribution to $\tau_{\bullet\bullet\circ,j}$	Contribution to $C''\tau_{\bullet\bullet\circ,j}$
$\bullet\bullet\circ$	$\phi_{j+1}g_{\bullet\bullet\bullet\circ}$	$\phi_{j+1}(wg_{\bullet\bullet\bullet\circ}f_{(\bullet)} + wg_{\bullet\bullet\circ}f_{(\bullet\bullet)} + g_{\bullet\circ}f_{(\bullet\bullet\circ)} + wg_{\bullet}f_{(\bullet\bullet\bullet\circ)})$
$\bullet, \bullet\circ$	$\phi_{j+2}g_{\bullet\bullet}f_{(\bullet\bullet\circ)}$	$\phi_{j+2}f_{(\bullet\bullet\circ)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$
$\bullet\bullet, \circ$	$\phi_{j+2}g_{\bullet\bullet\bullet}f_{(\bullet\circ)}$	$\phi_{j+2}f_{(\bullet\circ)}(wg_{\bullet\bullet\bullet}f_{(\bullet)} + wg_{\bullet\bullet}f_{(\bullet\bullet)} + wg_{\bullet}f_{(\bullet\bullet\bullet)})$
$\bullet, \bullet, \circ$	$\phi_{j+3}g_{\bullet\bullet}f_{(\bullet\bullet)}f_{(\bullet\circ)}$	$\phi_{j+3}f_{(\bullet\bullet)}f_{(\bullet\circ)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$

$R = \bullet\circ\circ\circ$

String composition	Contribution to $\tau_{\bullet\circ\circ\circ,j}$	Contribution to $C''\tau_{\bullet\circ\circ\circ,j}$
$\bullet\circ\circ\circ$	$\phi_{j+1}g_{\bullet\bullet\circ\circ\circ}$	$\phi_{j+1}(wg_{\bullet\bullet\circ\circ\circ}f_{(\bullet)} + g_{\bullet\circ\circ\circ}f_{(\bullet\circ)} + g_{\bullet\circ\circ}f_{(\bullet\circ\circ)}$ $+g_{\bullet\circ}f_{(\bullet\circ\circ\circ)} + wg_{\bullet}f_{(\bullet\bullet\circ\circ\circ)})$
$\bullet, \circ\circ\circ$	$\phi_{j+2}g_{\bullet\bullet}f_{(\bullet\circ\circ\circ)}$	$\phi_{j+2}f_{(\bullet\circ\circ\circ)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$
$\bullet\circ, \circ\circ$	$\phi_{j+2}g_{\bullet\bullet\circ}f_{(\bullet\circ\circ)}$	$\phi_{j+2}f_{(\bullet\circ\circ)}(wg_{\bullet\bullet\circ}f_{(\bullet)} + g_{\bullet\circ}f_{(\bullet\circ)} + wg_{\bullet}f_{(\bullet\bullet\circ)})$
$\bullet\circ\circ, \circ$	$\phi_{j+2}g_{\bullet\bullet\circ\circ}f_{(\bullet\circ)}$	$\phi_{j+2}f_{(\bullet\circ)}(wg_{\bullet\bullet\circ\circ}f_{(\bullet)} + g_{\bullet\circ\circ}f_{(\bullet\circ)} + g_{\bullet\circ}f_{(\bullet\circ\circ)} + wg_{\bullet}f_{(\bullet\bullet\circ\circ)})$
$\bullet, \circ, \circ\circ$	$\phi_{j+3}g_{\bullet\bullet}f_{(\bullet\circ)}f_{(\bullet\circ\circ)}$	$\phi_{j+3}f_{(\bullet\circ)}f_{(\bullet\circ\circ)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$
$\bullet, \circ\circ, \circ$	$\phi_{j+3}g_{\bullet\bullet}f_{(\bullet\circ\circ)}f_{(\bullet\circ)}$	$\phi_{j+3}f_{(\bullet\circ)}f_{(\bullet\circ\circ)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$
$\bullet\circ, \circ, \circ$	$\phi_{j+3}g_{\bullet\bullet\circ}f^2_{(\bullet\circ)}$	$\phi_{j+3}f^2_{(\bullet\circ)}(wg_{\bullet\bullet\circ}f_{(\bullet)} + g_{\bullet\circ}f_{(\bullet\circ)} + wg_{\bullet}f_{(\bullet\bullet\circ)})$
$\bullet, \circ, \circ, \circ$	$\phi_{j+4}g_{\bullet\bullet}f^3_{(\bullet\circ)}$	$\phi_{j+4}f^3_{(\bullet\circ)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$

$R = \bullet\circ$

String composition	Contribution to $\tau_{\bullet\circ,j}$	Contribution to $C''\tau_{\bullet\circ,j}$
$\bullet\circ$	$\phi_{j+1}g_{\bullet\circ\bullet}$	$\phi_{j+1}(g_{\bullet\circ\bullet}f_{(\circ)} + wg_{\bullet\bullet}f_{(\circ\bullet)} + wg_{\bullet}f_{(\bullet\circ\bullet)})$
$\bullet, \circ$	$\phi_{j+2}g_{\bullet\bullet}f_{(\bullet\circ)}$	$\phi_{j+2}f_{(\bullet\circ)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$

$R = \bullet\circ\circ$

String composition	Contribution to $\tau_{\bullet\circ\circ,j}$	Contribution to $C''\tau_{\bullet\circ\circ,j}$
$\bullet\circ\circ$	$\phi_{j+1}g_{\bullet\bullet\circ\circ}$	$\phi_{j+1}(wg_{\bullet\bullet\circ\circ}f_{(\bullet)} + g_{\bullet\circ\circ}f_{(\bullet\circ)} + g_{\bullet\circ}f_{(\bullet\circ\circ)} + wg_{\bullet}f_{(\bullet\bullet\circ\circ)})$
$\bullet, \circ\circ$	$\phi_{j+2}g_{\bullet\bullet}f_{(\bullet\circ\circ)}$	$\phi_{j+2}f_{(\bullet\circ\circ)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$
$\bullet\circ, \circ$	$\phi_{j+2}g_{\bullet\bullet\circ}f_{(\bullet\circ)}$	$\phi_{j+2}f_{(\bullet\circ)}(wg_{\bullet\bullet\circ}f_{(\bullet)} + g_{\bullet\circ}f_{(\bullet\circ)} + wg_{\bullet}f_{(\bullet\bullet\circ)})$
$\bullet, \circ, \circ$	$\phi_{j+3}g_{\bullet\bullet}f^2_{(\bullet\circ)}$	$\phi_{j+3}f^2_{(\bullet\circ)}(wg_{\bullet\bullet}f_{(\bullet)} + wg_{\bullet}f_{(\bullet\bullet)})$

$J''\Psi''^{(1)}$ may be computed by using the expressions for τ-functions given in Table 8.1. The result of applying J'' to each term of $\Psi''^{(1)}$ is given in the following table; to obtain $J''\Psi''^{(1)}$, sum the third column.

τ	Contributions to $\Psi''^{(1)}$	Action of J''
$\tau_{\circ\circ,\bullet}$	$wxe^{xf_{(\circ)}}g_{\bullet\circ\circ}f_{(\bullet)}\phi_{1,1}$	$w^2xe^{xf_{(\circ)}}g_{\bullet\bullet\bullet\circ\circ}f_{(\bullet)}(\phi_{1,1}+wf_{(\bullet)}\phi_{1,2})$
		$+wxe^{xf_{(\circ)}}g_{\bullet\circ\circ\circ\circ}(x\phi_{1,1}+\phi_{2,1})$
	$w^2xe^{xf_{(\circ)}}g_{\bullet}f_{(\bullet)}f_{(\bullet\circ\circ)}\phi_{1,2}$	$w^3xe^{xf_{(\circ)}}g_{\bullet\bullet\bullet}f_{(\bullet\circ\circ)}(\phi_{1,2}+2wf_{(\bullet)}\phi_{1,3})$
		$+w^2xe^{xf_{(\circ)}}g_{\bullet\circ\circ}f_{(\bullet)}f_{(\bullet\circ\circ)}(x\phi_{1,2}+\phi_{2,2})$
		$+w^2xe^{xf_{(\circ)}}f_{(\bullet)}\phi_{1,2}(wg_{\bullet\bullet\circ\circ\bullet}+g_{\bullet\circ\circ\bullet\circ}+g_{\bullet\circ\bullet\circ\circ})$
	$wxe^{xf_{(\circ)}}g_{\bullet\circ}f_{(\bullet)}f_{(\circ\circ)}\phi_{2,1}$	$w^2xe^{xf_{(\circ)}}g_{\bullet\bullet\bullet\circ}f_{(\circ\circ)}(\phi_{2,1}+wf_{(\bullet)}\phi_{2,2})$
		$+wxe^{xf_{(\circ)}}g_{\bullet\circ\circ\circ}f_{(\bullet)}f_{(\circ\circ)}(x\phi_{2,1}+2\phi_{3,1})$
		$+2wxe^{xf_{(\circ)}}f_{(\bullet)}\phi_{2,1}g_{\bullet\circ\circ\circ\circ}$
	$w^2xe^{xf_{(\circ)}}g_{\bullet}f_{(\bullet)}f_{(\circ\circ)}f_{(\bullet\circ)}\phi_{2,2}$	$w^3xe^{xf_{(\circ)}}g_{\bullet\bullet\bullet}f_{(\circ\circ)}f_{(\bullet\circ)}(\phi_{2,2}+2wf_{(\bullet)}\phi_{2,3})$
		$+w^2xe^{xf_{(\circ)}}g_{\bullet\circ\circ}f_{(\bullet)}f_{(\circ\circ)}f_{(\bullet\circ)}(x\phi_{2,2}+2\phi_{3,2})$
		$+2w^2xe^{xf_{(\circ)}}f_{(\bullet)}f_{(\bullet\circ)}\phi_{2,2}g_{\bullet\circ\circ\circ}$
		$+w^2xe^{xf_{(\circ)}}f_{(\bullet)}f_{(\circ\circ)}\phi_{2,2}(wg_{\bullet\bullet\circ\bullet}+g_{\bullet\circ\bullet\circ})$
$\tau_{\circ\circ,\epsilon}$	$xe^{xf_{(\circ)}}g_{\bullet\circ\circ}\phi_{1,0}$	$xe^{xf_{(\circ)}}g_{\bullet\circ\circ\circ\circ}(x\phi_{1,0}+\phi_{2,0})$
	$wxe^{xf_{(\circ)}}g_{\bullet}f_{(\bullet\circ\circ)}\phi_{1,1}$	$wxe^{xf_{(\circ)}}g_{\bullet\circ\circ}f_{(\bullet\circ\circ)}(x\phi_{1,1}+\phi_{2,1})$
		$+w^3xe^{xf_{(\circ)}}g_{\bullet\bullet\bullet}f_{(\bullet\circ\circ)}\phi_{1,2}$
		$+wxe^{xf_{(\circ)}}\phi_{1,1}(wg_{\bullet\bullet\circ\circ\bullet}+g_{\bullet\circ\circ\bullet\circ}+g_{\bullet\circ\bullet\circ\circ})$
	$xe^{xf_{(\circ)}}g_{\bullet\circ}f_{(\circ\circ)}\phi_{2,0}$	$xe^{xf_{(\circ)}}g_{\bullet\circ\circ\circ}f_{(\circ\circ)}(x\phi_{2,0}+2\phi_{3,0})$
		$+2xe^{xf_{(\circ)}}g_{\bullet\circ\circ\circ\circ}\phi_{2,0}$
	$wxe^{xf_{(\circ)}}g_{\bullet}f_{(\bullet\circ)}f_{(\circ\circ)}\phi_{2,1}$	$wxe^{xf_{(\circ)}}g_{\bullet\circ\circ}f_{(\bullet\circ)}f_{(\circ\circ)}(x\phi_{2,1}+2\phi_{3,1})$
		$+w^3xe^{xf_{(\circ)}}g_{\bullet\bullet\bullet}f_{(\bullet\circ)}f_{(\circ\circ)}\phi_{2,2}$
		$+wxe^{xf_{(\circ)}}f_{(\bullet\circ)}\phi_{2,1}(wg_{\bullet\bullet\circ\bullet}+g_{\bullet\circ\bullet\circ})$
		$+2wxe^{xf_{(\circ)}}f_{(\circ\circ)}\phi_{2,1}g_{\bullet\circ\circ\circ}$
$\tau_{\bullet\bullet,\bullet}$	$w^2(e^{xf_{(\circ)}}-1)g_{\bullet\bullet\bullet}\phi_{0,2}$	$2w^4(e^{xf_{(\circ)}}-1)g_{\bullet\bullet\bullet\bullet\bullet}\phi_{0,3}+xw^2e^{xf_{(\circ)}}g_{\bullet\circ\circ\bullet\bullet}\phi_{0,2}$
	$w^3(e^{xf_{(\circ)}}-1)g_{\bullet}f_{(\bullet\bullet\bullet)}\phi_{0,3}$	$3w^5(e^{xf_{(\circ)}}-1)g_{\bullet\bullet\bullet}f_{(\bullet\bullet\bullet)}\phi_{0,4}+w^3xe^{xf_{(\circ)}}g_{\bullet\circ\circ}f_{(\bullet\bullet\bullet)}\phi_{0,3}$
		$+3w^4(e^{xf_{(\circ)}}-1)g_{\bullet\bullet\bullet\bullet\bullet}\phi_{0,3}$
	$w^3(e^{xf_{(\circ)}}-1)g_{\bullet\bullet}f_{(\bullet\bullet)}\phi_{0,3}$	$3w^5(e^{xf_{(\circ)}}-1)g_{\bullet\bullet\bullet\bullet}f_{(\bullet\bullet)}\phi_{0,4}+w^3xe^{xf_{(\circ)}}g_{\bullet\circ\circ\bullet}f_{(\bullet\bullet)}\phi_{0,3}$
		$+2w^4(e^{xf_{(\circ)}}-1)g_{\bullet\bullet\bullet\bullet\bullet}\phi_{0,3}$
	$w^4(e^{xf_{(\circ)}}-1)g_{\bullet}f^2_{(\bullet\bullet)}\phi_{0,4}$	$4w^6(e^{xf_{(\circ)}}-1)g_{\bullet\bullet\bullet}f^2_{(\bullet\bullet)}\phi_{0,5}+w^4xe^{xf_{(\circ)}}g_{\bullet\circ\circ}f^2_{(\bullet\bullet)}\phi_{0,4}$
		$+4w^5(e^{xf_{(\circ)}}-1)g_{\bullet\bullet\bullet\bullet}f_{(\bullet\bullet)}\phi_{0,4}$

τ	Contributions to $\Psi''^{(1)}$	Action of J''
$\tau_{\bullet\bullet,\bullet\bullet}$	$w^3(e^{xf_{(\circ)}}-1)g_{\bullet\bullet\bullet}f_{(\bullet)}\phi_{0,3}$	$w^4(e^{xf_{(\circ)}}-1)g_{\bullet\bullet\bullet\bullet\bullet}(\phi_{0,3}+3wf_{(\bullet)}\phi_{0,4})$
		$+w^3xe^{xf_{(\circ)}}g_{\bullet\circ\circ\bullet\bullet}f_{(\bullet)}\phi_{0,3}$
	$w^4(e^{xf_{(\circ)}}-1)g_{\bullet}f_{(\bullet\bullet\bullet)}f_{(\bullet)}\phi_{0,4}$	$w^5(e^{xf_{(\circ)}}-1)g_{\bullet\bullet\bullet}f_{(\bullet\bullet\bullet)}(\phi_{0,4}+4wf_{(\bullet)}\phi_{0,5})$
		$w^4xe^{xf_{(\circ)}}g_{\bullet\circ\circ}f_{(\bullet\bullet\bullet)}f_{(\bullet)}\phi_{0,4}$
		$+3w^5(e^{xf_{(\circ)}}-1)g_{\bullet\bullet\bullet\bullet\bullet}f_{(\bullet)}\phi_{0,4}$
	$w^4(e^{xf_{(\circ)}}-1)g_{\bullet\bullet}f_{(\bullet\bullet)}f_{(\bullet)}\phi_{0,4}$	$w^5(e^{xf_{(\circ)}}-1)g_{\bullet\bullet\bullet\bullet}f_{(\bullet\bullet)}(\phi_{0,4}+4wf_{(\bullet)}\phi_{0,5})$
		$+w^4xe^{xf_{(\circ)}}g_{\bullet\circ\circ\bullet}f_{(\bullet\bullet)}f_{(\bullet)}\phi_{0,4}$
		$+2w^5(e^{xf_{(\circ)}}-1)g_{\bullet\bullet\bullet\bullet}f_{(\bullet)}\phi_{0,4}$
	$w^5(e^{xf_{(\circ)}}-1)g_{\bullet}f^2_{(\bullet\bullet)}f_{(\bullet)}\phi_{0,5}$	$w^6(e^{xf_{(\circ)}}-1)g_{\bullet\bullet\bullet}f^2_{(\bullet\bullet)}(\phi_{0,5}+5wf_{(\bullet)}\phi_{0,6})$
		$+w^5xe^{xf_{(\circ)}}g_{\bullet\circ\circ}f^2_{(\bullet\bullet)}f_{(\bullet)}\phi_{0,5}$
		$+4w^6(e^{xf_{(\circ)}}-1)g_{\bullet\bullet\bullet\bullet}f_{(\bullet\bullet)}f_{(\bullet)}\phi_{0,5}$
$\tau_{\bullet\circ,\epsilon}$	$\phi_1(y)xe^{xf_{(\circ)}}wg_{\bullet\bullet\circ}f_{(\bullet)}\phi_{0,1}$	$xe^{xf_{(\circ)}}w^2g_{\bullet\bullet\bullet\bullet\circ}(f_{(\bullet)}\phi_2(y)\phi_{0,1}+\phi_1(y)\phi_{0,1}$
		$+wf_{(\bullet)}\phi_1(y)\phi_{0,2})$
		$+\phi_1(y)x^2e^{xf_{(\circ)}}wg_{\bullet\circ\circ\bullet\circ}f_{(\bullet)}\phi_{0,1}$
	$\phi_1(y)xe^{xf_{(\circ)}}w^2g_{\bullet}f_{(\bullet\bullet\circ)}f_{(\bullet)}\phi_{0,2}$	$xe^{xf_{(\circ)}}w^3g_{\bullet\bullet\bullet}f_{(\bullet\bullet\circ)}(f_{(\bullet)}\phi_2(y)\phi_{0,2}+\phi_1(y)\phi_{0,2}$
		$+2wf_{(\bullet)}\phi_1(y)\phi_{0,3})$
		$+\phi_1(y)x^2e^{xf_{(\circ)}}w^2g_{\bullet\circ\circ}f_{(\bullet\bullet\circ)}f_{(\bullet)}\phi_{0,2}$
		$+\phi_1(y)xe^{xf_{(\circ)}}w^2f_{(\bullet)}\phi_{0,2}(wg_{\bullet\bullet\bullet\circ\bullet}+wg_{\bullet\bullet\circ\bullet\bullet}+g_{\bullet\circ\bullet\bullet\circ})$
	$\phi_1(y)xe^{xf_{(\circ)}}wg_{\bullet\circ}f_{(\bullet\circ)}f_{(\bullet)}\phi_{1,1}$	$xe^{xf_{(\circ)}}w^2g_{\bullet\bullet\bullet\circ}f_{(\bullet\circ)}(f_{(\bullet)}\phi_2(y)\phi_{1,1}+\phi_1(y)\phi_{1,1}$
		$+wf_{(\bullet)}\phi_1(y)\phi_{1,2})$
		$+\phi_1(y)xe^{xf_{(\circ)}}wg_{\bullet\circ\circ\circ}f_{(\bullet\circ)}f_{(\bullet)}(x\phi_{1,1}+\phi_{2,1})$
		$+\phi_1(y)xe^{xf_{(\circ)}}wf_{(\bullet)}\phi_{1,1}(wg_{\bullet\bullet\circ\bullet\circ}+g_{\bullet\circ\bullet\circ\circ})$
	$\phi_1(y)xe^{xf_{(\circ)}}w^2g_{\bullet}f^2_{(\bullet\circ)}f_{(\bullet)}\phi_{1,2}$	$xe^{xf_{(\circ)}}w^3g_{\bullet\bullet\bullet}f^2_{(\bullet\circ)}(\phi_2(y)f_{(\bullet)}\phi_{1,2}+\phi_1(y)\phi_{1,2}$
		$+2w\phi_1(y)f_{(\bullet)}\phi_{1,3})$
		$+\phi_1(y)xe^{xf_{(\circ)}}w^2g_{\bullet\circ\circ}f^2_{(\bullet\circ)}f_{(\bullet)}(x\phi_{1,2}+\phi_{2,2})$
		$+2\phi_1(y)xe^{xf_{(\circ)}}w^2f_{(\bullet\circ)}f_{(\bullet)}\phi_{1,2}(wg_{\bullet\bullet\circ\bullet}+g_{\bullet\circ\bullet\circ})$
$\tau_{\circ,\epsilon}$	$xe^{xf_{(\circ)}}\phi_1(y)g_{\bullet\circ}f_{(\bullet\circ)}\phi_{1,0}$	$wxe^{xf_{(\circ)}}\phi_2(y)g_{\bullet\bullet\bullet\circ}f_{(\bullet\circ)}\phi_{1,0}$
		$+xe^{xf_{(\circ)}}\phi_1(y)g_{\bullet\circ\circ\circ}f_{(\bullet\circ)}(x\phi_{1,0}+\phi_{2,0})$
		$+xe^{xf_{(\circ)}}\phi_1(y)\phi_{1,0}(g_{\bullet\circ\bullet\circ\circ}+wg_{\bullet\bullet\circ\bullet\circ})$
	$wxe^{xf_{(\circ)}}\phi_1(y)g_{\bullet}f^2_{(\bullet\circ)}\phi_{1,1}$	$w^2xe^{xf_{(\circ)}}g_{\bullet\bullet\bullet}f^2_{(\bullet\circ)}(\phi_2(y)\phi_{1,1}+w\phi_1(y)\phi_{1,2})$
		$+wxe^{xf_{(\circ)}}g_{\bullet\circ\circ}f^2_{(\bullet\circ)}\phi_1(y)(x\phi_{1,1}+\phi_{2,1})$
		$+2wxe^{xf_{(\circ)}}f_{(\bullet\circ)}\phi_{1,1}\phi_1(y)(g_{\bullet\circ\bullet\circ}+wg_{\bullet\bullet\circ\bullet})$

τ	Contributions to $\Psi''^{(1)}$	Action of J''
$\tau_{\bullet,\epsilon}$	$wxe^{xf_{(\circ)}}\phi_2(y)g_{\bullet\bullet}f_{(\bullet)}f_{(\bullet\circ)}\phi_{0,1}$	$w^2xe^{xf_{(\circ)}}g_{\bullet\bullet\bullet\bullet}f_{(\bullet\circ)}(2f_{(\bullet)}\phi_3(y)\phi_{0,1}+\phi_2(y)\phi_{0,1}$
		$+wf_{(\bullet)}\phi_2(y)\phi_{0,2})$
		$+wx^2e^{xf_{(\circ)}}\phi_2(y)g_{\bullet\circ\circ\bullet}f_{(\bullet)}f_{(\bullet\circ)}\phi_{0,1}$
		$+wxe^{xf_{(\circ)}}\phi_2(y)\phi_{0,1}f_{(\bullet)}(g_{\bullet\circ\bullet\circ\bullet}+wg_{\bullet\bullet\circ\bullet\bullet})$
	$w^2xe^{xf_{(\circ)}}\phi_2(y)g_{\bullet}f_{(\bullet)}f_{(\bullet\circ)}f_{(\bullet\bullet)}\phi_{0,2}$	$w^3xe^{xf_{(\circ)}}g_{\bullet\bullet\bullet}f_{(\bullet\circ)}f_{(\bullet\bullet)}(2f_{(\bullet)}\phi_3(y)\phi_{0,2}+\phi_2(y)\phi_{0,2}$
		$+2wf_{(\bullet)}\phi_2(y)\phi_{0,3})$
		$+w^2x^2e^{xf_{(\circ)}}\phi_2(y)g_{\bullet\circ\circ}f_{(\bullet)}f_{(\bullet\circ)}f_{(\bullet\bullet)}\phi_{0,2}$
		$+w^2xe^{xf_{(\circ)}}\phi_2(y)\phi_{0,2}f_{(\bullet)}f_{(\bullet\bullet)}(g_{\bullet\circ\bullet\circ}+wg_{\bullet\bullet\circ\bullet})$
		$+2w^3xe^{xf_{(\circ)}}\phi_2(y)\phi_{0,2}g_{\bullet\bullet\bullet\bullet}f_{(\bullet)}f_{(\bullet\circ)}$

The remaining terms contributing to $\Psi''^{(1)}$ are:

Contributions to $\Psi''^{(1)}$	Action of J''
$wxe^{xf_{(\circ)}}\phi_1(y)g_{\bullet}f_{(\bullet\bullet\circ)}\phi_{0,1}$	$w^2xe^{xf_{(\circ)}}g_{\bullet\bullet\bullet}f_{(\bullet\bullet\circ)}(\phi_2(y)\phi_{0,1}+w\phi_1(y)\phi_{0,2})$
	$+wx^2e^{xf_{(\circ)}}\phi_1(y)g_{\bullet\circ\circ}f_{(\bullet\bullet\circ)}\phi_{0,1}$
	$+wxe^{xf_{(\circ)}}\phi_1(y)\phi_{0,1}(wg_{\bullet\bullet\bullet\circ\bullet}+wg_{\bullet\bullet\circ\bullet\bullet}+g_{\bullet\circ\bullet\bullet\circ})$
$wxe^{xf_{(\circ)}}\phi_2(y)g_{\bullet}f_{(\bullet\bullet)}f_{(\bullet\circ)}\phi_{0,1}$	$w^2xe^{xf_{(\circ)}}g_{\bullet\bullet\bullet}f_{(\bullet\bullet)}f_{(\bullet\circ)}(2\phi_3(y)\phi_{0,1}+w\phi_2(y)\phi_{0,2})$
	$+wx^2e^{xf_{(\circ)}}\phi_2(y)g_{\bullet\circ\circ}f_{(\bullet\bullet)}f_{(\bullet\circ)}\phi_{0,1}$
	$+wxe^{xf_{(\circ)}}\phi_2(y)\phi_{0,1}(2wg_{\bullet\bullet\bullet\bullet}+wg_{\bullet\bullet\circ\bullet}+g_{\bullet\circ\bullet\circ})$
$wg_{\bullet}f_{(\bullet)}f_{(\circ\circ\circ)}f_{(\circ)}^{n-3}\phi_{0,1}$	$w^2g_{\bullet\bullet\bullet}f_{(\circ\circ\circ)}f_{(\circ)}^{n-3}(\phi_{0,1}+wf_{(\bullet)}\phi_{0,2})$
	$+wf_{(\bullet)}\phi_{0,1}(3g_{\bullet\circ\circ\circ\circ}f_{(\circ)}^{n-3}+(n-3)g_{\bullet\circ\circ}f_{(\circ\circ\circ)}f_{(\circ)}^{n-4})$
$wg_{\bullet}f_{(\bullet)}f_{(\circ\circ)}^2f_{(\circ)}^{n-4}\phi_{0,1}$	$w^2g_{\bullet\bullet\bullet}f_{(\circ\circ)}^2f_{(\circ)}^{n-4}(\phi_{0,1}+wf_{(\bullet)}\phi_{0,2})$
	$+wf_{(\bullet)}\phi_{0,1}(4g_{\bullet\circ\circ\circ}f_{(\circ\circ)}f_{(\circ)}^{n-4}+(n-4)g_{\bullet\circ\circ}f_{(\circ\circ)}^2f_{(\circ)}^{n-5})$

Finally, the contributions from $\Psi'^{(1)}$ are:

Term of $\Psi'^{(1)}$	Action of J''
$xe^{xf_{(\circ)}}g_{\bullet\bullet\circ}\phi_1(y)$	$xe^{xf_{(\circ)}}(xg_{\bullet\circ\circ\bullet\circ}\phi_1(y)+wg_{\bullet\bullet\bullet\bullet\circ}\phi_2(y))$
$xe^{xf_{(\circ)}}g_{\bullet\bullet}f_{(\bullet\circ)}\phi_2(y)$	$xe^{xf_{(\circ)}}(xg_{\bullet\circ\circ\bullet}f_{(\bullet\circ)}\phi_2(y)+wg_{\bullet\bullet\circ\bullet\bullet}\phi_2(y)+g_{\bullet\circ\bullet\circ\bullet}\phi_2(y)+2wg_{\bullet\bullet\bullet\bullet}f_{(\bullet\circ)}\phi_3(y))$
$g_{\bullet}f_{(\circ\circ\circ)}f_{(\circ)}^{n-3}$	$3g_{\bullet\circ\circ\circ\circ}f_{(\circ)}^{n-3}+(n-3)g_{\bullet\circ\circ}f_{(\circ\circ\circ)}f_{(\circ)}^{n-4}$
$g_{\bullet}f_{(\circ\circ)}^2f_{(\circ)}^{n-4}$	$4g_{\bullet\circ\circ\circ}f_{(\circ\circ)}f_{(\circ)}^{n-4}+(n-4)g_{\bullet\circ\circ}f_{(\circ\circ)}^2f_{(\circ)}^{n-5}$

The generating series $\Psi''^{(2)}$ may be obtained from the above results by replacing $g_{\bullet R}\phi_{i,j}$ with $\tau_{R,i,j}$ whenever it appears. In order to simplify the presentation, $\langle\Psi''^{(2)}\rangle$ is given below, instead of $\Psi''^{(2)}$. In the following tables, the following definitions are used:

$$D_3 := \sum_{n\geq 3} D_{(3,1^{n-3})}\frac{x^n}{n!},$$

$$D_{2,2} := \sum_{n\geq 4} D_{(2,2,1^{n-4})}\frac{x^n}{n!},$$

$$D_3^* := \sum_{n\geq 4}(n-3)D_{(3,1^{n-3})}\frac{x^n}{n!},$$

$$D_{2,2}^* := \sum_{n\geq 5}(n-4)D_{(2,2,n^{n-4})}\frac{x^n}{n!}.$$

The coefficients of each τ function appearing in $\langle \Psi''^{(2)} \rangle$ are as follows.

τ-function	Coefficient in $\langle \Psi''^{(2)} \rangle$
$\langle \tau_{\epsilon,0,0} \rangle$	$wxe^x(5\langle\phi_6\rangle + (13+2x)\langle\phi_5\rangle + (11+5x)\langle\phi_4\rangle + (3+4x)\langle\phi_3\rangle + x\langle\phi_2\rangle)$ $+3w(\langle\phi_1\rangle + 3\langle\phi_2\rangle + 3\langle\phi_3\rangle + \langle\phi_4\rangle)D_3 + w(\langle\phi_1\rangle + \langle\phi_2\rangle)(D_3^* + D_{2,2}^*)$ $+4w(\langle\phi_1\rangle + 2\langle\phi_2\rangle + \langle\phi_3\rangle)D_{2,2} + w[u^4]\sum_{D\in\mathcal{D}} u^{2g(D)} \frac{x^{n(D)}}{n(D)!}$
$\langle \tau_{\circ,0,0} \rangle$	$xe^x(\langle\phi_5\rangle + (2+x)\langle\phi_4\rangle + (1+2x)\langle\phi_3\rangle) + 3(\langle\phi_1\rangle + 2\langle\phi_2\rangle + \langle\phi_3\rangle)D_3$ $+4(\langle\phi_1\rangle + \langle\phi_2\rangle)D_{2,2}$
$\langle \tau_{\bullet,0,0} \rangle$	$wxe^x(5\langle\phi_6\rangle + (12+2x)\langle\phi_5\rangle + (9+4x)\langle\phi_4\rangle + (2+2x)\langle\phi_3\rangle + x\langle\phi_2\rangle)$ $+3w(\langle\phi_2\rangle + 2\langle\phi_3\rangle + \langle\phi_4\rangle)D_3 + w(\langle\phi_1\rangle + \langle\phi_2\rangle)(D_3^* + D_{2,2}^*)$ $+4w(\langle\phi_2\rangle + \langle\phi_3\rangle)D_{2,2}$
$\langle \tau_{\circ\circ,0,0} \rangle$	$3(\langle\phi_1\rangle + \langle\phi_2\rangle)D_3 + 4\langle\phi_1\rangle D_{2,2} + D_3^* + D_{2,2}^*$
$\langle \tau_{\circ\circ,0,1} \rangle$	$wx^2e^x(\langle\phi_1\rangle + \langle\phi_2\rangle) + w(D_3^* + D_{2,2}^*)$
$\langle \tau_{\circ\circ,0,2} \rangle$	$w^2x^2e^x(\langle\phi_1\rangle + \langle\phi_2\rangle)$
$\langle \tau_{\circ\circ,0,3} \rangle$	w^3xe^x
$\langle \tau_{\circ\circ,0,4} \rangle$	$2w^4xe^x$
$\langle \tau_{\circ\circ,0,5} \rangle$	w^5xe^x
$\langle \tau_{\circ\circ,1,1} \rangle$	$wx^2e^x(1 + \langle\phi_1\rangle)$
$\langle \tau_{\circ\circ,1,2} \rangle$	$w^2x^2e^x(1 + \langle\phi_1\rangle)$
$\langle \tau_{\circ\circ,2,1} \rangle$	$wxe^x(1 + \langle\phi_1\rangle) + wx^2e^x$
$\langle \tau_{\circ\circ,2,2} \rangle$	$w^2xe^x(1 + \langle\phi_1\rangle) + w^2x^2e^x$
$\langle \tau_{\circ\circ,3,1} \rangle$	$2wxe^x$
$\langle \tau_{\circ\circ,3,2} \rangle$	$2w^2xe^x$
$\langle \tau_{\circ\bullet,0,0} \rangle$	$xe^x(\langle\phi_4\rangle + (1+x)\langle\phi_3\rangle) + \langle\phi_1\rangle(D_3^* + D_{2,2}^*)$
$\langle \tau_{\bullet\circ,0,0} \rangle$	$wxe^x(\langle\phi_5\rangle + (2+x)\langle\phi_4\rangle + (1+2x)\langle\phi_3\rangle) + 3w(\langle\phi_2\rangle + \langle\phi_3\rangle)D_3$ $+4w\langle\phi_2\rangle D_{2,2}$
$\langle \tau_{\bullet\bullet,0,0} \rangle$	$wxe^x(4\langle\phi_5\rangle + (5+x)\langle\phi_4\rangle + \langle\phi_3\rangle + x\langle\phi_2\rangle)$
$\langle \tau_{\bullet\bullet,0,1} \rangle$	$w^2xe^x(\langle\phi_2\rangle + 2\langle\phi_3\rangle) + w^2(D_3 + D_{2,2})$
$\langle \tau_{\bullet\bullet,0,2} \rangle$	$2w^3xe^x(\langle\phi_1\rangle + 2\langle\phi_2\rangle + 2\langle\phi_3\rangle) + w^3(D_3 + D_{2,2})$
$\langle \tau_{\bullet\bullet,0,3} \rangle$	$2w^4xe^x(\langle\phi_1\rangle + \langle\phi_2\rangle)$
$\langle \tau_{\bullet\bullet,0,4} \rangle$	$4w^5(e^x - 1)$
$\langle \tau_{\bullet\bullet,0,5} \rangle$	$9w^6(e^x - 1)$
$\langle \tau_{\bullet\bullet,0,6} \rangle$	$5w^7(e^x - 1)$
$\langle \tau_{\bullet\bullet,1,1} \rangle$	$w^2xe^x\langle\phi_2\rangle$
$\langle \tau_{\bullet\bullet,1,2} \rangle$	$(2 + 2\langle\phi_1\rangle + \langle\phi_2\rangle)w^3xe^x$
$\langle \tau_{\bullet\bullet,1,3} \rangle$	$2w^4xe^x(1 + \langle\phi_1\rangle)$
$\langle \tau_{\bullet\bullet,2,2} \rangle$	$2w^3xe^x$
$\langle \tau_{\bullet\bullet,2,3} \rangle$	$2w^4xe^x$

τ-function	Coefficient in $\langle\Psi''^{(2)}\rangle$
$\langle\tau_{\circ\circ\circ,0,0}\rangle$	$3w\langle\phi_1\rangle D_3 + 4D_{2,2}$
$\langle\tau_{\circ\circ\circ,0,1}\rangle$	$4wD_{2,2}$
$\langle\tau_{\circ\circ\circ,1,0}\rangle$	$x^2e^x\langle\phi_1\rangle$
$\langle\tau_{\circ\circ\circ,1,1}\rangle$	$wx^2e^x\langle\phi_1\rangle$
$\langle\tau_{\circ\circ\circ,2,0}\rangle$	$x^2e^x + xe^x\langle\phi_1\rangle$
$\langle\tau_{\circ\circ\circ,2,1}\rangle$	$wx^2e^x + (2+\langle\phi_1\rangle)wxe^x$
$\langle\tau_{\circ\circ\circ,2,2}\rangle$	$2w^2xe^x$
$\langle\tau_{\circ\circ\circ,3,0}\rangle$	$2xe^x$
$\langle\tau_{\circ\circ\circ,3,1}\rangle$	$2wxe^x$
$\langle\tau_{\circ\circ\bullet,0,0}\rangle$	$x^2e^x\langle\phi_2\rangle$
$\langle\tau_{\circ\circ\bullet,0,1}\rangle$	$wx^2e^x\langle\phi_2\rangle$
$\langle\tau_{\circ\circ\bullet,0,3}\rangle$	w^3xe^x
$\langle\tau_{\circ\circ\bullet,0,4}\rangle$	w^4xe^x
$\langle\tau_{\circ\bullet\circ,0,1}\rangle$	$wxe^x\langle\phi_2\rangle$
$\langle\tau_{\circ\bullet\circ,0,2}\rangle$	$w^2xe^x\langle\phi_2\rangle$
$\langle\tau_{\circ\bullet\circ,1,1}\rangle$	$2wxe^x\langle\phi_1\rangle$
$\langle\tau_{\circ\bullet\circ,1,2}\rangle$	$w^2xe^x\langle\phi_1\rangle$
$\langle\tau_{\circ\bullet\circ,2,1}\rangle$	wxe^x
$\langle\tau_{\circ\bullet\circ,2,2}\rangle$	w^2xe^x
$\langle\tau_{\bullet\circ\circ,0,0}\rangle$	$3w\langle\phi_2\rangle D_3 + 4w\langle\phi_1\rangle D_{2,2}$
$\langle\tau_{\circ\bullet\bullet,0,0}\rangle$	$xe^x(\langle\phi_3\rangle + x\langle\phi_2\rangle)$
$\langle\tau_{\bullet\circ\bullet,0,0}\rangle$	$wxe^x(\langle\phi_4\rangle + (1+x)\langle\phi_3\rangle)$
$\langle\tau_{\bullet\circ\bullet,0,1}\rangle$	$w^2xe^x\langle\phi_2\rangle$
$\langle\tau_{\bullet\circ\bullet,0,2}\rangle$	$w^3xe^x\langle\phi_2\rangle$
$\langle\tau_{\bullet\circ\bullet,1,1}\rangle$	$2w^2xe^x\langle\phi_1\rangle$
$\langle\tau_{\bullet\circ\bullet,1,2}\rangle$	$2w^3xe^x\langle\phi_1\rangle$
$\langle\tau_{\bullet\circ\bullet,2,1}\rangle$	w^2xe^x
$\langle\tau_{\bullet\circ\bullet,2,2}\rangle$	w^3xe^x
$\langle\tau_{\bullet\bullet\circ,0,0}\rangle$	$wxe^x(\langle\phi_4\rangle + (1+x)\langle\phi_3\rangle)$
$\langle\tau_{\bullet\bullet\circ,1,0}\rangle$	$wxe^x\langle\phi_2\rangle$
$\langle\tau_{\bullet\bullet\circ,1,1}\rangle$	$w^2xe^x(\langle\phi_1\rangle + \langle\phi_2\rangle)$
$\langle\tau_{\bullet\bullet\circ,1,2}\rangle$	$w^3xe^x\langle\phi_1\rangle$
$\langle\tau_{\bullet\bullet\circ,2,1}\rangle$	w^2xe^x
$\langle\tau_{\bullet\bullet\circ,2,2}\rangle$	w^3xe^x
$\langle\tau_{\bullet\bullet\bullet,0,0}\rangle$	$wxe^x(2\langle\phi_3\rangle + 3\langle\phi_4\rangle)$
$\langle\tau_{\bullet\bullet\bullet,0,1}\rangle$	$w^2xe^x(3\langle\phi_2\rangle + 2\langle\phi_3\rangle)$
$\langle\tau_{\bullet\bullet\bullet,0,2}\rangle$	$3w^3xe^x\langle\phi_2\rangle$
$\langle\tau_{\bullet\bullet\bullet,0,4}\rangle$	$10w^5(e^x-1)$
$\langle\tau_{\bullet\bullet\bullet,0,5}\rangle$	$8w^6(e^x-1)$

τ-function	Coefficient in $\langle \Psi''^{(2)} \rangle$
$\langle \tau_{\circ\circ\circ\circ,0,0} \rangle$	$3D_3$
$\langle \tau_{\circ\circ\circ\circ,0,1} \rangle$	$3wD_3$
$\langle \tau_{\circ\circ\circ\circ,1,0} \rangle$	$x^2 e^x$
$\langle \tau_{\circ\circ\circ\circ,2,0} \rangle$	$3xe^x$
$\langle \tau_{\circ\circ\circ\circ,1,1} \rangle$	$wx^2 e^x$
$\langle \tau_{\circ\circ\circ\circ,2,1} \rangle$	$3wxe^x$
$\langle \tau_{\circ\circ\bullet\circ,0,0} \rangle$	$x^2 e^x \langle \phi_1 \rangle$
$\langle \tau_{\circ\circ\bullet\circ,0,1} \rangle$	$wx^2 e^x \langle \phi_1 \rangle$
$\langle \tau_{\circ\circ\bullet\circ,1,1} \rangle$	wxe^x
$\langle \tau_{\circ\circ\bullet\circ,1,2} \rangle$	$w^2 xe^x$
$\langle \tau_{\circ\circ\bullet\bullet,0,2} \rangle$	$xw^2 e^x$
$\langle \tau_{\circ\circ\bullet\bullet,0,3} \rangle$	$w^3 xe^x$
$\langle \tau_{\circ\bullet\circ\circ,1,0} \rangle$	$xe^x \langle \phi_1 \rangle$
$\langle \tau_{\circ\bullet\circ\circ,1,1} \rangle$	$wxe^x (1 + \langle \phi_1 \rangle)$
$\langle \tau_{\circ\bullet\circ\circ,1,2} \rangle$	$w^2 xe^x$
$\langle \tau_{\circ\bullet\circ\bullet,0,0} \rangle$	$xe^x \langle \phi_2 \rangle$
$\langle \tau_{\circ\bullet\circ\bullet,0,1} \rangle$	$wxe^x \langle \phi_2 \rangle$
$\langle \tau_{\circ\bullet\bullet\circ,0,1} \rangle$	$wxe^x \langle \phi_1 \rangle$
$\langle \tau_{\circ\bullet\bullet\circ,0,2} \rangle$	$w^2 xe^x \langle \phi_1 \rangle$
$\langle \tau_{\bullet\circ\circ\circ,0,0} \rangle$	$3\langle \phi_1 \rangle D_3$
$\langle \tau_{\bullet\circ\circ\bullet,1,1} \rangle$	$w^2 xe^x$
$\langle \tau_{\bullet\circ\circ\bullet,1,2} \rangle$	$w^3 xe^x$
$\langle \tau_{\bullet\circ\bullet\circ,1,0} \rangle$	$wxe^x \langle \phi_1 \rangle$
$\langle \tau_{\bullet\circ\bullet\circ,1,1} \rangle$	$w^2 xe^x \langle \phi_1 \rangle$
$\langle \tau_{\bullet\circ\bullet\bullet,0,0} \rangle$	$wxe^x (\langle \phi_3 \rangle + (1+x)\langle \phi_2 \rangle)$
$\langle \tau_{\bullet\circ\bullet\bullet,0,1} \rangle$	$w^2 xe^x (\langle \phi_1 \rangle + \langle \phi_2 \rangle)$
$\langle \tau_{\bullet\circ\bullet\bullet,0,2} \rangle$	$w^3 xe^x \langle \phi_1 \rangle$
$\langle \tau_{\bullet\bullet\circ\circ,1,1} \rangle$	$w^2 xe^x$
$\langle \tau_{\bullet\bullet\circ\circ,1,2} \rangle$	$w^3 xe^x$
$\langle \tau_{\bullet\bullet\circ\bullet,0,0} \rangle$	$wxe^x (\langle \phi_2 \rangle + \langle \phi_3 \rangle)$
$\langle \tau_{\bullet\bullet\circ\bullet,0,1} \rangle$	$w^2 xe^x \langle \phi_1 \rangle$
$\langle \tau_{\bullet\bullet\circ\bullet,0,2} \rangle$	$w^3 xe^x \langle \phi_1 \rangle$
$\langle \tau_{\bullet\bullet\bullet\circ,0,0} \rangle$	$wxe^x \langle \phi_3 \rangle$
$\langle \tau_{\bullet\bullet\bullet\circ,0,1} \rangle$	$w^2 xe^x (\langle \phi_1 \rangle + \langle \phi_2 \rangle)$
$\langle \tau_{\bullet\bullet\bullet\circ,0,2} \rangle$	$w^3 xe^x \langle \phi_1 \rangle$
$\langle \tau_{\bullet\bullet\bullet\bullet,0,3} \rangle$	$8w^4 (e^x - 1)$
$\langle \tau_{\bullet\bullet\bullet\bullet,0,4} \rangle$	$6w^5 (e^x - 1)$

References

[1] O. Angel, A. Holroyd, D. Romik, and B. Virág, *Random sorting networks*, Adv. Math. **215** (2007), 839 –868. 18

[2] N. R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov, and W. Skiba, *PP-wave string interactions from perturbative Yang-Mills theory*, J. High Enegry Phys. **7** (2002), no. 017, 56 pp. 1, 40, 42

[3] J. Dénes, *The representation of a permutation as the product of a minimal number of transpositions, and its connection with the theory of graphs*, Publ. Math. Inst. Hungar. Acad. Sci. **4** (1959), 63 – 70. 50

[4] P. Diaconis and C. Greene, *Applications of Murphy's elements*, Tech. Report 335, Stanford University, Stanford, California, 1989. 3, 26, 85, 92, 100

[5] V. Féray, *Partial Jucys-Murphy elements and star factorizations*, http://arxiv.org/abs/0904.4854 (2009). 48, 66

[6] I. P. Goulden, *A differential operator for symmetric functions and the combinatorics of multiplying transpositions*, Trans. Amer. Math. Soc. **344** (1994), 421–440. 50, 163

[7] I. P. Goulden and D. M. Jackson, *The combinatorial relationship between trees, cacti and certain connection coefficients for the symmetric group*, European J. Combin. **13** (1992), 357 – 365. 34, 53

[8] ———, *Transitive factorisations into transpositions and holomorphic mappings on the sphere*, Proc. Amer. Math. Soc. **125** (1997), no. 1, 51 – 60. 17, 50

[9] ———, *The number of ramified coverings of the sphere by the double torus, and a general form for higher genera*, J. Combin. Theory Ser. A **88** (1999), 259 – 275. 17, 50

[10] ———, *A proof of a conjecture for the number of ramified coverings of the sphere by the torus*, J. Combin. Theory Ser. A **88** (1999), 246 – 258. 17, 50

[11] ———, *Transitive powers of Young-Jucys-Murphy elements are central*, J. Algebra **321** (2009), no. 7, 1826 – 1835. 2, 36, 44, 45, 46, 47, 65, 125

[12] I. P. Goulden, D. M. Jackson, and A. Vainshtein, *The number of ramified coverings of the sphere by the torus and surfaces of higher genera*, Ann. Comb. **4** (2000), 27 – 46. 17, 50

[13] I. P. Goulden, D. M. Jackson, and R. Vakil, *Towards the geometry of double Hurwitz numbers*, Adv. Math. **198** (2005), 43 – 92. 16, 17, 18

[14] I. P. Goulden and A. Nica, *A direct bijection for the Harer-Zaiger formula*, J. Combin. Theory Ser. B **111** (2005), 224 – 238. 34

[15] A. Goupil and G. Schaeffer, *Factoring n-cycles and counting maps of given genus*, European J. Combin. **19** (1998), 819 – 834. 34

[16] J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math. **85** (1986), 457 – 485. 34

[17] N. J. A. Harvey, *Matroid intersection, pointer chasing, and Young's semi-normal representation of* $\mathfrak{S}_n$, Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms (New York), ACM, 2008, pp. 542–549. 18

[18] A. Hurwitz, *Ueber Riemann'sche flächen mit gegebenen verzweigungspunkten*, Math. Ann. (1891), 1 – 60. 17

[19] J. Irving, *On the number of factorizations of a full cycle*, J. Combin. Theory Ser. A **113** (2006), 1549 –1554. 34

[20] J. Irving and A. Rattan, *Minimal factorizations of permutations into star transpositions*, Discrete Math. (2009), no. 6, 1435 – 1442. 45

[21] D. M. Jackson, *Counting cycles in permutations by group characters, with an application to a topological problem*, Trans. Amer. Math. Soc. **299** (1987), no. 2, 785 – 801. 16, 28, 30, 34

[22] ———, *Some combinatorial problems associated with products of conjugacy classes of the symmetric group*, J. Combin. Theory Ser. A **49** (1988), 363 – 369. 30, 34, 50

[23] D. M. Jackson and T. I. Visentin, *A character theoretic approach to embeddings of rooted maps in an orientable surface of given genus*, Trans. Amer. Math. Soc. **322** (1990), no. 1, 343–363. 12, 13

[24] ———, *An Atlas of the Smaller Maps in Orientable and Nonorientable Surfaces*, Chapman and Hall / CRC Press, 2001. 12

[25] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley, 1981. 6, 22

[26] A.-A. A. Jucys, *Symmetric polynomials and the centre of the symmetric group ring*, Rep. Math. Phys. **5** (1974), no. 1, 107 – 112. 27

[27] S. Kerov and V. Ivanov, *The algebra of conjugacy classes in symmetric groups, and partial permutations*, J. Math. Sci. **107** (2001), no. 5, 4212 – 4230. 66

[28] J. H. Kwak and J. Lee, *Genus polynomials of dipoles*, Kyungpook Math. J. **33** (1993), no. 1, 115–125. 14, 30

[29] J. H. Kwak and S. H. Shim, *Total embedding distributions for bouquets of circles*, Discrete Math. **248** (2002), 93 – 108. 67, 68

[30] A. Lascoux and J.-Y. Thibon, *Vertex operators and the class algebras of symmetric groups*, J. Math. Sci. **121** (2004), no. 3, 2380 – 2392. 2, 44, 49, 66, 163

[31] L. Liu and Y. Yang, *Classification of (p, q, n)-dipoles on nonorientable surfaces*, Electron. J. Combin. **17** (2010), no. 12, 6 pp. 164

[32] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second ed., Oxford University Press, 1995. 6, 25, 90

[33] P. Moszkowski, *A solution to a problem of Dénes: a bijection between trees and factorizations of cyclic permutations*, Europ. J. Combin. **10** (1989), 13–16. 50

[34] G. E. Murphy, *A new construction of Young's semi-normal representation of the symmetric groups*, J. Algebra **69** (1980), 287 – 297. 23

[35] A. Okounkov and A. Vershik, *A new approach to representation theory of symmetric groups*, Selecta Math. (1996). 56, 60

[36] G. I. Olshanski, *Extension of the algebra $U(g)$ for infinite-dimensional classical lie algebras g and the Yangians $Y(gl(m))$*, Soviet Math. Dokl. **36** (1988), no. 569 – 573. 94

[37] I. Pak, *Reduced decompositions on permutations in terms of star transpositions, generalized Catalan numbers and k-ary trees*, Discrete Math. **204** (1998), 329 – 335. 44, 45

[38] D. Poulalhon and G. Schaeffer, *Factorizations of large cycles in the symmetric group*, Discrete Math. **254** (2002), 433 – 458. 34

[39] B. E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, second ed., Springer, 2001. 6, 29

[40] G. Schaeffer and E. Vassilieva, *A bijective proof of Jackson's formula for the number of factorizations of a cycle*, J. Combin. Theory Ser. A **115** (2008), 903 – 924. 34, 53

[41] R. P. Stanley, *On the number of reduced decompositions of elements of Coxeter groups*, European J. Combin. **5** (1984), 359–372. 50

[42] E. Strahov, *Generalized characters of the symmetric group*, Adv. Math. **212** (2007), 109–142. 3, 85, 89, 101, 163

[43] W. T. Tutte, *Graph Theory*, Encyclopedia of Mathematics and its Applications, vol. 21, Addison-Wesley, 1984. 12

[44] K. Ueno, *An Introduction to Algebraic Geometry*, Translations of Mathematical Monographs, vol. 166, American Mathematical Society, Providence, RI, 1995. 16

[45] T. I. Visentin and S. W. Wieler, *On the genus distribution of* (p, q, n)*-dipoles*, Electron. J. Combin. **14** (2007). 42, 43, 69

Index