Infinite graphs, graph-like spaces and B-matroids

by

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AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The central theme of this thesis is to prove results about infinite mathematical objects by studying the behaviour of their finite substructures. In particular, we study B-matroids, which are an infinite generalization of matroids introduced by Higgs [13], and graph-like spaces, which are topological spaces resembling graphs, introduced by Thomassen and Vella [27].

Recall that the circuit matroid of a finite graph is a matroid defined on the edges of the graph, with a set of edges being independent if it contains no circuit. It turns out that graph-like continua and infinite graphs both have circuit B-matroids. The first main result of this thesis is a generalization of Whitney's Theorem that a graph has an abstract dual if and only if it is planar. We show that an infinite graph has an abstract dual (which is a graph-like continuum) if and only if it is planar, and also that a graph-like continuum has an abstract dual (which is an infinite graph) if and only if it is planar. This generalizes theorems of Thomassen ([25]) and Bruhn and Diestel ([3]). The difficult part of the proof is extending Tutte's characterization of graphic matroids ([28]) to finitary or co-finitary B-matroids. In order to prove this characterization, we introduce a technique for obtaining these B-matroids as the limit of a sequence of finite minors.

In [29], Tutte proved important theorems about the peripheral (induced and non-separating) circuits of a 3-connected graph. He showed that for any two edges of a 3-connected graph there is a peripheral circuit containing one but not the other, and that the peripheral circuits of a 3-connected graph generate its cycle space. These theorems were generalized to 3-connected binary matroids by Bixby and Cunningham ([1]). We generalize both of these theorems to 3-connected binary co-finitary B-matroids.

Richter, Rooney and Thomassen [22] showed that a locally connected, compact metric space has an embedding in the sphere unless it contains a subspace homeomorphic to K_5 or $K_{3,3}$, or one of a small number of other obstructions. We are able to extend this result to an arbitrary surface Σ; a locally connected, compact metric space embeds in Σ unless it contains a subspace homeomorphic

to a finite graph which does not embed in Σ, or one of a small number of other obstructions.

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Chapter 1

Introduction

1.1 Overview

The first three chapters of this thesis are introductory. The remainder of this chapter introduces some relevant background material. In Chapter 2 we briefly survey the study of compactifications of infinite graphs, and introduce some classes of topological spaces that include but are more general than infinite graphs or their compactifications. Chapter 3 is an introduction to Higgs' infinite generalization of matroids, called B-matroids, and their properties.

The final three chapters contain the bulk of the original work in the thesis, and all of the main results. In Chapter 4, we introduce techniques for viewing a finitary or co-finitary B-matroid as the limit of a sequence of finite matroids, and a graph or graph-like space as the limit of a sequence of finite graphs. Using these techniques, we are able to show that finitary (resp. co-finitary) B-matroids whose every finite minor is graphic are the circuit matroids of graphs (resp. graph-like continua), and thereby obtain a version of Whitney's planarity criterion for infinite graphs and graph-like continua.

Chapter 5 is concerned with proving theorems about the peripheral circuits of 3-connected binary co-finitary B-matroids. Tutte [29] proved that the peripheral circuits of a 3-connected graph generate its cycle space, and that for any edge in a 3-connected graph there are two peripheral circuits whose intersection is exactly that edge. We show analogues of both of these theorems for 3-connected binary co-finitary B-matroids. To prove our versions, we need to understand the behaviour of bridges of circuits in 3-connected binary co-finitary B-matroids, and our sequential technique is again useful in passing from results about bridges in 3-connected binary matroids to the results we require.

Finally, Chapter 6 describes which locally connected, compact metric spaces have embeddings in a given surface. Richter, Rooney and Thomassen [22] showed that a locally connected, compact metric space has an embedding in the sphere unless it contains a subspace homeomorphic to K_5 or $K_{3,3}$, or one of a small number of other obstructions. We extend this result to arbitrary surfaces. Our approach is to attempt to embed the space M by first embedding subspaces of M homeomorphic to finite graphs. These have finitely many (combinatorial) embeddings. We find a finite graph G contained in M such that if a specific embedding Π of G in the desired surface does not extend to an embedding of all of M, then there is another finite graph H_Π contained in M, such that H_Π contains G and Π does not even extend to an embedding of H_Π. Then we show that, if every embedding, Π, of G fails to extend to an embedding of M, we can combine the finite graphs H_Π to obtain a finite graph contained in M that does not embed in the desired surface.

1.2 Preliminaries

1.2.1 Finite graphs and matroids

We will assume that the reader is familiar with the definitions and basic results of graph theory and matroid theory. Generally our terminology and notation will follow Diestel's graph theory textbook [10] and Oxley's matroid theory textbook [20].

The *circuit matroid* of a graph G, denoted $\mathcal{M}(G)$, is a matroid whose ground set is the edge set, $E(G)$, of G and whose independent sets are the subsets of $E(G)$ that do not contain the edge set of a circuit of the graph. The bases of $\mathcal{M}(G)$ are the edge-maximal spanning forests of G, and the co-circuits of $\mathcal{M}(G)$ are the bonds (minimal edge cuts) of G.

Finite graphs G and H are said to be *abstract duals* if there is a bijection between their edge sets such that a set of edges is a circuit in G if and only if it is a bond in H. This definition is symmetric in G and H, since it is equivalent to the statement $\mathcal{M}(G) = \mathcal{M}^*(H)$. Abstract duality is connected with planarity by the following fundamental result of Whitney.

Theorem 1.1
(Whitney [31]) A finite graph G has an abstract dual if and only if it is planar.

We will adopt a common notation for graph (and matroid) minors, which is that $G \prec H$ denotes that G is a minor of H.

Tutte gave a characterization by excluded minors of the *graphic* matroids - the matroids that can be obtained as the circuit matroid of some graph. Let $\tau = \{\mathcal{M}^*(K_5), \mathcal{M}^*(K_{3,3}), U_{2,4}, F_7, F_7^*\}$.

Theorem 1.2
(Tutte [28]) A finite matroid M is graphic if and only if M has no minor in τ.

We will say that a matroid (or, later, a B-matroid) with no minor in τ has no *Tutte minor*. Similarly, we may say that a graph (resp. matroid), finite or otherwise, has no *Kuratowski minor* if it has no K_5 or $K_{3,3}$ (resp. $\mathcal{M}(K_5)$ or $\mathcal{M}(K_{3,3})$) minor. In general, we refer to the set of all matroids (or B-matroids) without any minor in some set χ of matroids as ex(χ).

A *binary matroid* is a matroid with a representation over the binary field. Several well-known characterizations of binary matroids are combined in the following theorem. Additional equivalent statements can be found in [20].

Theorem 1.3
Let M be a matroid. The following are equivalent:

1. M is binary;
2. **(Tutte)** M has no $U_{2,4}$ minor;
3. **(Whitney)** every symmetric difference of circuits of M can be expressed as a disjoint union of circuits of M;
4. every intersection between a circuit and a co-circuit of M contains an even number of elements; and
5. **(Seymour)** there is no pair C, C^* such that C is a circuit and C^* a co-circuit of M and $|C \cap C^*| = 3$.

The *cycle space* $\mathcal{C}(G)$ of a graph G is the subspace of the vector space $\mathbb{Z}_2^{E(G)}$ generated by the characteristic vectors of circuits of G. Similarly, if M is a matroid with ground set S, the cycle space $\mathcal{C}(M)$ of M is the subspace of the vector space $\mathbb{Z}_2^S$ generated by the characteristic vectors of the circuits of M.

Another fundamental result on planarity, *MacLane's planarity criterion*, characterizes planarity of a graph in terms of the algebraic properties of its cycle space.

Theorem 1.4
(MacLane [15]) A finite graph G is planar if and only if $\mathcal{C}(G)$ has a basis such that every $e \in E(G)$ is contained in at most two basis elements.

If G is a graph and H is a subgraph of G, then a *non-degenerate bridge B* of H is a component $N(B)$ of $G - H$, called the nucleus of the bridge, along with the edges incident with H and $N(B)$, and their endpoints in H. A *degenerate bridge* of H is an edge that is not in H but has both endpoints in H. A *peripheral circuit* of G is a circuit that is induced and non-separating or, equivalently, has no degenerate bridges and at most one non-degenerate bridge.

1.2.2 Tools for working with infinite objects

We will make frequent use of *Zorn's Lemma*, which is equivalent to the Axiom of Choice.

Lemma 1.5
(Zorn, [33]) Let $(X, \leq)$ be a partially ordered set such that every chain in X has an upper bound in X. Then X contains at least one maximal element.

Another standard tool for proofs in infinite graph theory is the following lemma, known as *König's infinity lemma*. A proof in English can be found in [10].

Lemma 1.6
(König, [14]) Let $V_1, V_2, \ldots$ be an infinite sequence of disjoint, finite sets. Let G be a graph on their union such that, for $i \geq 1$, every vertex in V_{i+1} has a neighbour in V_i. Either some V_i is empty, or G contains an infinite path $v_1 v_2 \ldots$ such that, for each $i \geq 1$, $v_i \in V_i$.

We will assume the reader is familiar with the definition of the *ordinals*, which we will denote $\mathcal{O}$. The set of countable ordinals will be denoted $\mathcal{O}_0$. We will need the following standard facts.

Theorem 1.7
If X is a countable subset of $\mathcal{O}_0$, then X has a least upper bound in $\mathcal{O}_0$.

Theorem 1.8
(Principle of transfinite induction) For each $\alpha \in \mathcal{O}_0$, let $S(\alpha)$ be a statement which is true or false. If the following statements hold:

1. $S(1)$ is true; and

2. if $\beta \in \mathcal{O}_0$ and $S(\alpha)$ is true for all $\alpha < \beta$, then $S(\beta)$ is also true,

then $S(\alpha)$ is true for every $\alpha \in \mathcal{O}_0$.

1.2.3 Topology

We will assume that the reader is familiar with the basic concepts of point set topology; any terminology not defined here can be found in Munkres' textbook [18].

A topological space X is said to be *locally connected* if for every point $x \in X$, and every open neighbourhood U of x, there is a connected open neighbourhood V of x such that $V \subseteq U$.

A topological space X is *weakly Hausdorff* if, for any two points $x, y \in X$, there are open neighbourhoods U_x of x and U_y of y such that $U_x \cap U_y$ is finite.

A topological space X is *zero-dimensional* if for every pair $\{u, v\}$ of distinct points of X, there is a separation (U, V) of X such that $u \in U$ and $v \in V$.

An *edge* of a topological space is an open subset homeomorphic to $(0, 1)$, and whose closure is homeomorphic to $[0, 1]$.

A *continuum* is a topological space that is compact, connected and Hausdorff.

A *surface* is a connected, compact, Hausdorff topological space, such that each point has an open neighbourhood homeomorphic to the open unit disc in $\mathbb{R}^2$. The famous classification theorem states that every surface is homeomorphic to one of the following:

1. the sphere;
2. the connected sum of finitely many tori; or
3. the connected sum of finitely many real projective planes.

The first two types are the *orientable* surfaces. We say that the sphere has genus 0 and the connected sum of k tori has orientable genus k. The third type are the *non-orientable* surfaces. We say that the connected sum of k real projective planes has non-orientable genus k. The *Euler genus* of a surface Σ is $2k$ if Σ has orientable genus k, and k if Σ has non-orientable genus k.

A graph G (finite or infinite) may be viewed as a topological space $T(G)$ as follows. The point set of $T(G)$ is $V(G) \cup \mathcal{E}(G)$, where $\mathcal{E}(G)$ consists of pairwise disjoint open arcs I_e, one for for each edge $e \in E(G)$. For $e = uv$, the closed arc $\overline{I_e}$ has u and v as its endpoints. The basic open neighbourhoods of a vertex v in $T(G)$ consist of v along with, for each e incident with v, an open subset of $\overline{I_e}$ that contains v. The topological space $T(G)$ is called the *simplicial* topology on G.

Embeddings of finite graphs in surfaces are discussed in detail in [17]. For any topological space T, a function $f : T \to \Sigma$ is an *embedding* of T in Σ if it is a homeomorphism between T and $f(T)$. If G is a graph, then an embedding of $T(G)$ in Σ is also called an embedding of G in Σ. A *face* of an embedding $f : T \to \Sigma$ is a component of $\Sigma \setminus f(T)$. An embedding is *cellular* if every face is homeomorphic to an open disc. Youngs [32] made the observation that every minimum genus embedding of a connected graph is cellular.

An *embedding scheme* is a combinatorial description of an embedding of a graph in a surface. If G is a graph and Σ is a surface, there are only finitely many possible embedding schemes for G in Σ. A theorem of Ringel [23] implies that every cellular embedding of G in Σ is determined up to homeomorphism by its embedding scheme. A detailed description of embedding schemes can be found in Section 3.3 of [17].

If Σ is a fixed surface, then there is a finite list *Forb*(Σ) of graphs, such that any graph G with no embedding in Σ has a subdivision of some graph in Forb(Σ) as a subgraph. This was first proven as part of the Robertson-Seymour graph minors project, but several more elementary direct proofs exist. In particular Mohar [16] gives a constructive proof that takes a similar approach to our argument in Chapter 6.

Chapter 2

Graph-like spaces

2.1 Infinite graphs and compactifications

If G is an infinite graph, the simplicial topology $T(G)$ on G need not be compact. For example, if $R = v_0 v_1 \dots$ is a *ray* (a subgraph isomorphic to a one-way infinite path) in G, then the sequence $v_0, v_1, \dots$ of points in $T(G)$ fails to have a limit point in $T(G)$.

Many theorems that are essentially about the topology of finite graphs fail to generalize to the simplicial topology on infinite graphs for this reason. For example, the planarity criteria of MacLane and Whitney, and Tutte's theorem that the peripheral circuits of a 3-connected graph generate its cycle space all fail in this context. An expository paper of Diestel [9] contains several examples and lists other facts about the cycle space of a finite graph that do not hold for the simplicial topology on infinite graphs. For this reason, we may prefer to study the cycles of an infinite graph G by looking at some compactification of $T(G)$ rather than $T(G)$ itself.

Let G be an infinite graph, and let $\mathcal{R}$ be the set of all rays in G. A *tail* of a ray R is any ray that is a subgraph of R. We define an equivalence relation $\sim$ on $\mathcal{R}$ by saying that $R_1 \sim R_2$ if, for any finite subset U of $V(G)$, R_1 and R_2 have a tail in the same component of $G - U$. An *end* of G is an equivalence class of rays with respect to $\sim$. We denote the set of ends of G by $\Omega(G)$. We say that a vertex v *dominates* an end ω if, for any finite subset U of $V(G)$ not containing v, v is in the same component of $G - U$ as a ray $R \in \omega$.

For example, the double ladder is an infinite graph D defined as follows. Let $R_1 = \dots u_{-1} u_0 u_1 \dots$ and $R_2 = \dots v_{-1} v_0 v_1 \dots$ be two-way infinite paths (also called *double rays*). The graph D has vertices $V(R_1) \cup V(R_2)$ and edges $E(R_1) \cup E(R_2) \cup \{u_j v_j \mid j \in \mathbb{Z}\}$. There are two ends of D, one consisting of all rays going to infinity in the positive direction, and one consisting of all rays going to infinity in the negative direction.

For any infinite graph G, let $|G|$ be the space obtained by adding a point $x(\omega)$ to $T(G)$ for each $\omega \in \Omega(G)$. The basic open neighbourhoods in $|G|$ are given by $U \cup x(\Omega(U))$, where U is any open set in $T(G)$, and $\Omega(U)$ is the set of ends ω of G such that U contains a tail of every ray in ω. If G is locally finite, then $|G|$ is Hausdorff and compact, and is known as the *Freudenthal compactification* of G. The *Alexandroff compactification*, $\mathcal{A}(G)$, is obtained by identifying all of the ends in $|G|$.

We say that a graph G is *finitely separable* if no two vertices of G are joined by infinitely many edge-disjoint paths in G. In this case $|G|$ is compact, and we can form a compact Hausdorff space $\widetilde{G}$ from $|G|$ by identifying each vertex with every end that it dominates. The space $\widetilde{G}$ was introduced by Diestel and Kühn ([11]).

Given a compactification $C(G)$ of G, we can re-define the cycle space of G, saying that a set of edges of G is a circuit if it is exactly the set of edges contained in some homeomorphic image of the unit circle in $C(G)$. We can now attempt to generalize results about the cycle space of finite graphs to cycle spaces obtained in this way from compactifications. An initial result of this type was provided by Bonnington and Richter, who proved a version of MacLane's planarity criterion for $\mathcal{A}(G)$. More recently, Bruhn, Diestel, Kühn, Stein and others have generalized the bulk of the cycle space theory of finite graphs to $|G|$ for locally finite graphs, or to $\widetilde{G}$ for finitely separable graphs (see, for example, [2], [3], [5], [9], [11]).

2.2 Edge spaces and graph-like spaces

An *edge space* (X, E) is a topological space X and a subset $E \subseteq X$ consisting of points e such that e is open but not closed, and the closure of e contains at most two additional points. Notice that all of the topological spaces in the previous section can be converted to edge spaces by taking the edges to be open singletons instead of open intervals. Vella and Richter [30] introduced edge spaces, in part to unify the separate approaches to cycle space theory of Bonnington and Richter (considering cycles in $\mathcal{A}(G)$) and Diestel and Kühn (considering cycles in $\widetilde{G}$).

We will use the following theorem from [30].

Theorem 2.1
(Vella, Richter [30]) Every edge cut in a compact weakly Hausdorff edge space is finite.

A *graph-like space* is a metric space $\mathcal{G}$, whose ground set is $E \cup V$, where E is a collection of pairwise disjoint edges and V is zero-dimensional. Graph-like spaces were introduced by Thomassen and Vella [27]. We will in particular be interested in graph-like continua, that is, compact, connected graph-like spaces.

The simplicial topology of an infinite graph is graph-like, and so is the Freudenthal compactification of a locally finite graph. We can obtain a graph-like continuum from any infinite graph by applying the $\widetilde{G}$ construction from the previous section. Not all graph-like continua arise in this way. For example, additional graph-like continua can be obtained from the Freudenthal compactification of the double ladder by identifying the two ends (this is actually just the Alexandroff compactification of the double ladder), or adding an edge between them.

Graph-like spaces appeal as an object of study because they are more general than any class of compactifications of infinite graphs, yet they have more structure than edge spaces. The next two theorems are examples of fundamental facts about finite graphs that also hold for graph-like continua.

Theorem 2.2
(Thomassen, Vella [27]) Menger's Theorem holds for graph-like continua.

Theorem 2.3
(Rooney [24]) MacLane's Theorem holds for 2-connected graph-like continua.

Let $\mathcal{G}$ be a graph-like continuum with edge-set $E(\mathcal{G})$ and vertex-set $V(\mathcal{G})$. Let X, Y be disjoint subsets of $E(\mathcal{G})$. The *minor* of $\mathcal{G}$ obtained by contracting X and deleting Y, denoted $\mathcal{G}/X\backslash Y$, is defined as follows. Let $\overline{X}$ be the closure in $\mathcal{G}$ of the union of all the edges in X, and let Y' be the union of all the (open) edges in Y. Let $\sim_X$ be the equivalence relation on the points of $\mathcal{G}$ defined by $x \sim_X y$ if x and y are both in the same component of $\overline{X}$. We obtain $\mathcal{G}/X\backslash Y$ by deleting the open set Y' from $\mathcal{G}$ to obtain $\mathcal{G}\backslash Y$, and then taking the quotient of the resulting space by the equivalence relation $\sim_X$ (or vice versa; deletion and contraction commute). It is easy to check that the resulting space is a graph-like continuum with edge set $E(\mathcal{G})\backslash(X \cup Y)$ and vertex set $(V(\mathcal{G})\backslash V(\overline{X})) \cup \mathcal{C}(\overline{X})$ where $\mathcal{C}(\overline{X})$ is the set of components of $\overline{X}$.

Let $\mathcal{G}$ be a graph-like continuum, and let $\mathcal{H}$ be a closed subspace of $\mathcal{G}$. A *bridge* B of $\mathcal{H}$ in $\mathcal{G}$ is the closure of a component $N(B)$ of $\mathcal{G} \backslash \mathcal{H}$.

The following lemma summarizes some useful properties of graph-like continua.

Lemma 2.4

(Thomassen, Vella [27]) A closed subspace of a graph-like continuum is a graph-like continuum. Graph-like continua are locally connected and Hausdorff.

Suppose we take a graph-like continuum $\mathcal{G}$ and replace each open edge with a single point. The resulting space is a compact, weakly Hausdorff edge space, and so we may apply Lemma 2.1. This shows that graph-like continua also have only finite edge cuts.

If $\mathcal{G}$ is a graph-like continuum, a *circuit* of $\mathcal{G}$ is the edge set of a homeomorphic image of the unit circle in $\mathcal{G}$.

Chapter 3

B-matroids

3.1 Definition and background

Perhaps the most natural way to define a matroid with an infinite ground set is simply to use the independence axioms for a finite matroid. A *pre-independence space* is a set S together with a set $\mathcal{I}$ of subsets of S (*independent* sets) such that:

(i1) $\mathcal{I} \neq \emptyset$;
(i2) a subset of an independent set is independent; and
(i3) if I_1 and I_2 are finite independent sets with $|I_2| > |I_1|$, then there exists $x \in I_2 \backslash I_1$ such that $I_1 \cup \{x\} \in \mathcal{I}$.

A subset of S is *dependent* if it is not independent, a *circuit* is a minimal dependent set, and, for $X \subseteq S$, a *basis* of X is a maximal independent set contained in X. A *spanning* set is a set containing a basis of S and a *hyperplane* is a maximal non-spanning set.

If $M = (S, \mathcal{I})$ and $X \subseteq S$, the *deletion* of X from M is given by $M \backslash X = (S \backslash X, \{I \subseteq S \backslash X \mid I \in \mathcal{I}\})$. If X is an independent set then the *contraction* of X from M is given by $M/X = (S \backslash X, \{I \subseteq S \backslash X \mid I \cup X \in \mathcal{I}\})$. If bases of S exist, the *dual* of M is given by $M^* = (S, \{I \subseteq S \mid S \backslash I \text{ spans } S \text{ in } M\})$.

Circuits, bases, spanning sets and hyperplanes may fail to exist in pre-independence spaces, so for many purposes they are an unsatisfactory generalization of finite matroids. An *independence space* (also known as a *finitary B-matroid*) is a pre-independence space that also satisfies:

(i4) if $X \subseteq S$ and every finite subset A of X is independent, then X is independent.

Every dependent set in an independence space contains a circuit and every circuit is finite. Bases, spanning sets and hyperplanes exist in independence spaces. Rado [21] showed that all of the bases of an independence space share the same cardinality.

The class of independence spaces is closed under deletion and contraction, but not under duality. The dual of an independence space is a pre-independence space, and such pre-independences spaces are known as *cofinitary B-matroids*.

Higgs [13] introduced a class of pre-independence spaces that includes independence spaces and is closed under duality. A *B-matroid* is a set S together with a set $\mathcal{I}$ of subsets of S that satisfies (i1), (i2), and also:

(I3) if $T \subset X \subset S$ and T is independent, there is a maximal independent subset (basis) of X containing T; and
(I4) if $X \subset S$, B_1 and B_2 are bases of X, and $x \in B_1 \backslash B_2$, there is some $y \in B_2 \backslash B_1$ such that $(B_1 \backslash \{x\}) \cup \{y\}$ is a basis of X.

It is easy to check that (I3) implies (i3), so every B-matroid is a pre-independence space. It is also easy to check that every independence space is a B-matroid.

Recent work of Bruhn, Diestel, Kriesell, Pendavingh and Wollan [4] shows that the class of B-matroids can be defined by four different axiom systems, one each for the bases, independent sets, circuits and the closure operator. These are similar to the finite matroid axioms, in each case including an appropriate maximality axiom along the lines of (I3).

Higgs showed that B-matroids satisfy some appealing properties of matroids, and Oxley showed that any class of pre-independence spaces that is closed under duality, deletion and contraction must consist of B-matroids.

Theorem 3.1
(Higgs, [13]) Duality is an involution on the class of B-matroids.

Theorem 3.2
(Higgs, [13]) If the Generalized Continuum Hypothesis holds, then all bases of a B-matroid are equicardinal.

Theorem 3.3
(Oxley, [19]) Let S be an infinite set, and let $\mathcal{P}$ be a class of pre-independence spaces such that each member of $\mathcal{P}$ is a pre-independence space defined on a subset of S. If $\mathcal{P}$ is closed under deletion and contraction, and duality is an involution on $\mathcal{P}$, then every pre-independence space in $\mathcal{P}$ is a B-matroid.

3.2 Basic results

In this section we give proofs for B-matroids of some basic results from finite matroid theory. This section is based on joint work with Brendan Rooney.

We begin by proving that the minors of a B-matroid are well-defined B-matroids. Let M be a B-matroid with ground set S, and let $X \subseteq S$. Let B_X be a basis of X. We define $M/X = M/B_X \backslash (X \backslash B_X)$ (recall that the independent sets of M/B_X are the sets I such that $I \cup B_X$ is independent in M)

Lemma 3.4
If M is a B-matroid with ground set S and $X \subseteq S$, then M/X is well-defined (does not depend on the choice of B_X) and is a B-matroid.

Proof We proceed by verifying the B-matroid axioms in M/X. Let B_X be a basis of X. We have that (i1) and (i2) carry over directly from M to M/X.

For (I3), suppose that T is independent in M/X, and $T \subseteq Z \subseteq S \backslash X$. We need to find a basis of Z containing T in M/X. By definition of contraction, $T \cup B_X$ is independent in M. Let B be a basis of $Z \cup B_X$ in M such that $T \cup B_X \subseteq B$. Let $B_Z = B \backslash B_X$. Since B is a basis of $Z \cup B_X$ in M, B_Z is a basis of Z in M/X.

For (I4), suppose that B_1, B_2 are each bases of Z in M/X. We need to show that if $x \in B_1 \backslash B_2$, there is some $y \in B_2 \backslash B_1$ such that $(B_1 \backslash \{x\}) \cup \{y\}$ is also a basis of Z. This follows immediately from applying the same axiom to the bases $B_1 \cup B_X$ and $B_2 \cup B_X$ of $Z \cup B_X$ in M.

Finally, we need to show that if B_X^1, B_X^2 are each bases of X in M, then $M/B_X^1 = M/B_X^2$. Suppose that $J \subseteq S \backslash X$ is an independent set in M/B_X^1, but not in M/B_X^2. In other words, $J \cup B_X^1$ is independent in M, but $J \cup B_X^2$ is dependent. In M, we may choose a basis of $J \cup B_X^2$ containing B_X^2. Let J' be a subset of J such that

$J' \cup B_X^2$ is a basis for $J \cup B_X^2$. Since B_X^1 and B_X^2 are both bases of X in M, $B_1 = J \cup B_X^1$ and $B_2 = J' \cup B_X^2$ are each bases of $J \cup X$ in M. Choose $x \in J \backslash J'$. By (I4), since $x \in B_1 \backslash B_2$, there is some $y \in B_2 \backslash B_1$ such that $(B_1 \backslash \{x\}) \cup y$ is a basis of $J \cup X$ in M. This is a contradiction, because $B_2 \backslash B_1 \subseteq B_X^2$. Therefore a set is independent in M/B_X^1 if and only if it is independent in M/B_X^2, as required. ∎

Next, we show that minors interact with duality in the same way as for finite matroids.

Lemma 3.5
(Higgs, [13]) If M is a B-matroid with ground set S, and X, Y are disjoint subsets of S, then $(M/X \backslash Y)^* = M^*/Y \backslash X$.

Proof It suffices to show that $(M/X)^* = M^* \backslash X$, for then by taking duals on both sides we have $(M \backslash X)^* = M^*/X$ and it follows that $(M/X \backslash Y)^* = ((M/X) \backslash Y)^* = (M/X)^*/Y = (M^* \backslash X)/Y = M^*/Y \backslash X$.

Suppose that J is an independent set in $(M/X)^*$. Then by the definition of duality, $(S \backslash J) \cap (S \backslash X)$ contains a basis B of M/X. If B_X is a basis of X in M, $B \cup B_X$ is a basis of M contained in $S \backslash J$. Therefore, again by the definition of duality, J is an independent set in M^*, and therefore an independent set in $M^* \backslash X$.

Conversely, suppose that J is an independent set in $M^* \backslash X$. Then since J is independent in M^*, $S \backslash J$ contains a basis B of M. Let B_X be a basis of X in M that contains $B \cap X$. Since $(B \backslash X) \cup B_X$ contains B, $B \backslash X$ contains some basis B' of M/X. Since B' is disjoint from J, this shows that J is independent in $(M/X)^*$. ∎

The next several results show that the set of circuits of a B-matroid shares many properties of the set of circuits of a finite matroid.

Lemma 3.6
Every dependent set X in a B-matroid $M = (S, \mathcal{I})$ contains a circuit C_X. The set $\mathcal{C}(M)$ of circuits of M satisfies the finite circuit axioms:

(C1) $\emptyset \notin \mathcal{C}$;

(C2) if $C_1 \in \mathcal{C}$ and $C_2 \subset C_1$, then $C_2 \notin \mathcal{C}$; and

(C3) if $C_1, C_2 \in \mathcal{C}$, $x \in C_1 \cap C_2$, then there is some $C \subseteq (C_1 \cup C_2) - \{x\}$ with $C \in \mathcal{C}$.

Proof Firstly, suppose that X is a dependent set. Consider a basis B_X of X. Since X is dependent there is some $x \in X \backslash B_X$. Since B_X is a basis, $B_X \cup \{x\}$ is a dependent subset of X. Let $C = \{y \in B_X \cup \{x\} \mid (B_X \cup \{x\}) - \{y\} \in \mathcal{I}\}$. We claim that C is a circuit.

Suppose that C is independent. We have that $x \in C \subseteq B_X \cup \{x\}$. Since C is independent we can extend C to a basis of $B_X \cup \{x\}$, B'_X. We know that B_X is a basis of $B_X \cup \{e\}$, $B'_X \backslash B_X = \{x\}$ and, since $B_X \cup \{x\}$ is dependent, there is some $y \in B_X \backslash B'_X$. Therefore by (I4), for $y \in B_X \backslash B'_X$, $(B_X - \{y\}) \cup \{x\}$ is a basis of $B_X \cup \{e\}$. However we have by definition that if $(B_X - \{y\}) \cup \{x\} \in \mathcal{I}$, then $y \in C$ so $y \in B'_X$ and therefore $y \notin B_X \backslash B'_X$, a contradiction. Therefore C is a dependent set. Suppose y is an element of C. Then by definition $B_X \cup \{x\} - \{y\}$ is independent so, since $C \subseteq B_X \cup \{x\}$, $C - \{y\}$ is independent. Therefore C is a circuit.

We have that $\emptyset \in \mathcal{I}$, so $\emptyset \notin \mathcal{C}$ and therefore (C1) holds. If $C_1 \in \mathcal{C}$ and $C_2 \subset C_1$, then, by definition, C_2 must be independent, so (C2) holds.

For (C3), suppose for a contradiction that $(C_1 \cup C_2) - \{x\}$ is independent. Extend $C_1 \cap C_2$ to a basis B_1 of $C_1 \cup C_2$, and notice that $B_2 = (C_1 \cup C_2) - \{x\}$ is also a basis of $C_1 \cup C_2$. Applying (I4), since $x \in B_1 \backslash B_2$ there must be some $y \in B_2 \backslash B_1$ so that $B_1 \backslash \{x\} \cup \{y\}$ is a basis of $C_1 \cup C_2$. Since $y \notin B_1$, suppose without loss of generality that $y \in C_1 \backslash C_2$. Since C_2 is a circuit it cannot be contained in B_1, so there must be some $z \in C_2 \backslash B_1$. We have $(B_1 \backslash \{x\}) \cup \{y, z\} \subseteq B_2$, contradicting the fact that $(B_1 \backslash \{x\}) \cup \{y\}$ is a basis. ■

Lemma 3.7
Suppose that S is any set and $\mathcal{C}$ is a set of finite subsets of $\mathcal{C}$ satisfying (C1), (C2) and (C3). Then $\mathcal{C}$ is the set of circuits of a B-matroid with ground set S.

Proof Let $\mathcal{I}(C)$ be the set of subsets of S that do not contain an element of $\mathcal{C}$. We want to show that these are the independent sets of an independence space (finitary B-matroid) on S. The axioms (i1), (i2) and (i4) are immediate. For (i3), let I_1, I_2 be finite sets in $\mathcal{I}(C)$ with $|I_2| > |I_1|$. Let $\mathcal{C}'$ be the set of elements of $\mathcal{C}'$ that are contained in $I_1 \cup I_2$, and notice that $\mathcal{C}'$ also satisfies (C1), (C2) and (C3). Therefore it is the set of circuits of a finite matroid with ground set $I_1 \cup I_2$, and we may apply (i3) to this finite matroid to obtain $x \in I_2 \backslash I_1$ such that $I_1 \cup x \in \mathcal{I}$, as required. ■

Lemma 3.8
Let $M = (S, \mathcal{I})$ be a B-matroid. If C is a circuit of M and C^* is a circuit of M^*, then $|C \cap C^*| \neq 1$.

Proof Assume that we have $C \in \mathcal{C}$, $C^* \in \mathcal{C}^*$, so that $|C \cap C^*| = 1$. Let $C \cap C^* = \{x\}$. Then $C - \{x\}$ is independent and $C^* - \{x\}$ is independent in M^*, which implies that $S \backslash (C^* - \{x\})$ is a spanning set in M. Therefore we may extend $C - \{x\}$ to a basis B of M that is contained in $S \backslash (C^* - \{x\})$. Since C is a circuit, $x \notin B$, so $B \subseteq (S \backslash C^*)$. This implies that $S \backslash B$ is a basis of M^* and contains C^*, a contradication. Thus $|C \cap C'| \neq 1$. ■

Lemma 3.9
Let M be a B-matroid. If C is a circuit of M and $x, y \in C$, then there is some co-circuit C^* of M so that $C \cap C^* = \{x, y\}$.

Proof Let B be a basis of M containing $C - \{x\}$. Consider the basis $S \backslash B$ of M^*. Since it is a basis, $(S \backslash B) \cup \{y\}$ contains a circuit C^* of M^* that contains y. Consider $C \cap C^*$. We know that $|C \cap C^*| \neq 1$, so $C \cap C' \neq \{y\}$. We also know that $C^* - \{y\} \subseteq S \backslash B$, and $(S \backslash B) \cap C = \{x\}$. Therefore $C \cap C^* = \{x, y\}$, as required. ■

Lemma 3.10
Let C be a dependent set in a B-matroid M. Then C is a circuit of M if and only if, for every distinct $x, y \in C$, there is a co-circuit $C^*_{x,y}$ of M such that $C \cap C^*_{x,y} = \{x, y\}$.

Proof The forward implication is immediate by Lemma 3.9. Conversely, suppose that C is not a circuit. Then some proper subset K of C is a circuit. Choose $x \in K$, $y \in C \backslash K$. If there were a co-circuit $C^*_{x,y}$ with $C \cap C^*_{x,y} = \{x, y\}$, then it would follow that $K \cap C^*_{x,y} = \{x\}$, a contradiction. ■

Lemma 3.11
Let M be a B-matroid, and let $N = M/X\backslash Y$ be a minor of M. If C is a circuit of N, there is a subset C_X of X so that $C \cup C_X$ is a circuit of M. If K is a co-circuit of N, there is a subset K_X of X so that $K \cup K_X$ is a co-circuit of M.

Proof Suppose that C is a circuit of N. Then by definition of deletion, C is a circuit of M/X. By definition of contraction, $C \cup B_X$ is dependent in M for any basis B_X of X. Therefore, there is a circuit $C' \subseteq C \cup B_X$. Now, $C' \cap C$ is dependent in M/X so, since C is a circuit, we must have $C \subseteq C' \subseteq C \cup B_X$. Therefore, if $C_X = C' \cap B_X$, $C \cup C_X$ is a circuit of M, as required. The claim about co-circuits follows immediately by duality (i.e. applying Lemma 3.5). ■

3.3 Strong circuit exchange for B-matroids

In this section we give a proof of the strong circuit exchange axiom for B-matroids. That is, we prove that if C_1, C_2 are circuits of a B-matroid, $x \in C_1 \cap C_2$ and $y \in C_2 \setminus C_1$, then there is a circuit C_3 such that $y \in C_3 \subseteq (C_1 \cup C_2) \setminus \{x\}$. This is a standard result for finite matroids. Work in this section is again joint with Brendan Rooney.

For a matroid M with ground set S, we define the *closure operator* $\mathrm{cl} : 2^S \to 2^S$ by $\mathrm{cl}(X) = X \cup \{x \mid \text{there is a circuit } C \text{ of } M \text{ such that } x \in C \subseteq \{x\} \cup X\}$. We call $X \subseteq S$ *closed* in the matroid M if $\mathrm{cl}(X) = X$.

Lemma 3.12
Let M be a B-matroid. The hyperplanes of M are closed.

Proof Let H be a hyperplane of M. If there is some $x \in \mathrm{cl}\,(H) - H$, then there is a circuit C with $x \in C \subseteq H \cup \{x\}$. However, $S \setminus H$ is a co-circuit and $C \cap (S \setminus H) = \{x\}$, which is a contradiction by Lemma 3.8. ■

Lemma 3.13
If $X_1, X_2 \subseteq S$ are closed sets in a B-matroid M with ground set S, then $X_1 \cap X_2$ is closed.

Proof Suppose that $X_1 \cap X_2$ is not closed. Then $\mathrm{cl}(X_1 \cap X_2) - (X_1 \cap X_2) \neq \emptyset$. So there is some $x \in S \setminus (X_1 \cap X_2)$ such that $(X_1 \cap X_2) \cup \{x\}$ contains a circuit through x. This circuit is contained in $X_1 \cup \{x\}$ and $X_2 \cup \{x\}$. But X_1 and X_2 are both closed, so $x \in X_1$ and $x \in X_2$. Thus $x \in X_1 \cap X_2$, contradicting our choice of x. Thus $\mathrm{cl}(X_1 \cap X_2) = X_1 \cap X_2$. ■

Lemma 3.14
If C is a closed set in a B-matroid M and $x, y \notin C$, then $x \in \mathrm{cl}(C \cup \{y\})$ if and only if $\{x, y\}$ is a circuit of M/C.

Proof Since $x, y \notin C$, $C \cup \{x\}$ does not contain a circuit through x and $C \cup \{y\}$ does not contain a circuit through y, which is to say that $\{x\}$ and $\{y\}$ are both independent in M/C. Now we have that $x \in \mathrm{cl}(C \cup \{y\})$ if and only if $C \cup \{x, y\}$ contains a circuit through both x and y or, in other words, if and only if $\{x, y\}$ is a circuit of M/C. ■

Lemma 3.15
If C is a closed set in a B-matoid M, $x, y \notin C$ and $y \in \mathrm{cl}(C \cup \{x\})$, then $x \in \mathrm{cl}(C \cup \{y\})$.

Proof This follows immediately from Lemma 3.14. ■

Suppose that C is a closed subset of S. Then we define the relation $\sim_c$ on the elements of $S \setminus C$ as $x \sim_c y$ if and only if $y \in \mathrm{cl}(C \cup \{x\})$. It is clear that $x \in \mathrm{cl}(C \cup \{x\})$ and that if $y \in \mathrm{cl}(C \cup \{x\})$ and $z \in \mathrm{cl}(C \cup \{y\})$, then $z \in \mathrm{cl}(C \cup \{y\}) \subseteq \mathrm{cl}(C \cup \{x\})$. Thus $\sim_c$ is both reflexive and transitive. We also have by Lemma 3.15 that $\sim_c$ is symmetric, thus $\sim_c$ is an equivalence relation.

Indeed, by Lemma 3.14, the $\sim_c$ equivalence classes are just the parallel classes of M/C.

Lemma 3.16
Let M be a B-matroid with ground set S. For any set C and any $X \subseteq S \setminus C$, $\mathrm{cl}_M(C \cup X) = C \cup \mathrm{cl}_{M/C}(X)$. In particular, if H is a hyperplane of M/C, then $C \cup H$ is a hyperplane of M.

Proof Suppose $x \in \mathrm{cl}_M(C \cup X)$ with $x \notin C \cup X$. Let B_C be a basis of C and let B be a basis of $C \cup X$ with $B_C \subseteq B$. Since $x \in \mathrm{cl}_M(C \cup X)$, B is also a basis of $C \cup X \cup \{x\}$. Therefore, in M/C, $B \backslash B_C$ is a basis of X and also of $X \cup \{x\}$, and thus $x \in \mathrm{cl}_{M/C}(X)$. Conversely, if $x \in \mathrm{cl}_{M/C}(X)$, let B_X be a basis of X in M/C. Then, in M/C, B_X is also a basis of $X \cup \{x\}$. So $B_C \cup B_X$ is a basis in M of $C \cup X$ and also of $C \cup X \cup \{x\}$, and therefore $x \in \mathrm{cl}_M(C \cup X)$.

If H is a hyperplane of M/C, we have $\mathrm{cl}_M(C \cup H) = C \cup \mathrm{cl}_{M/C}(H) = C \cup H$, so $C \cup H$ is closed. In addition, for $e \notin C \cup H$, $\mathrm{cl}_M(C \cup H \cup \{e\}) = C \cup \mathrm{cl}_{M/C}(H \cup \{e\}) = C \cup (S - C) = S$. So $C \cup H$ is a hyperplane of M. ■

Lemma 3.17
If C is a closed set in a B-matroid M, and $y \notin C$, then there is a hyperplane H of M with $y \notin H$ and $C \subseteq H$.

Proof By Lemma 3.16, it suffices to find a hyperplane H' of M/C with $y \notin H'$. Since C is closed, M/C has no loops. So we may choose a basis B of M/C that contains y. Clearly $H' = \mathrm{cl}_{M/C}(B - \{y\})$ is a hyperplane of M/C not containing y, as required. ■

Lemma 3.18
Let M be a B-matroid with ground set S. If C_1 and C_2 are circuits of M, $x \in C_1 \cap C_2$ and $y \in C_2 \setminus C_1$, then $y \notin \mathrm{cl}^*((S \setminus C_1) \cap (S \setminus C_2)) \cup \{x\})$.

Proof Suppose that $y \in \mathrm{cl}^*((S \setminus C_1) \cap (S \setminus C_2)) \cup \{x\})$. Then $x \in \mathrm{cl}^*((S \setminus C_1) \cap (S \setminus C_2)) \cup \{y\}) \subseteq \mathrm{cl}^*(S \setminus C_1)$. Since $S \setminus C_1$ is a hyperplane of M^* it is closed, so this is a contradiction. ■

Theorem 3.19
Let M be a B-matroid with ground set S. Let C_1, C_2 be circuits of M, with $x \in C_1 \cap C_2$ and $y \in C_2 \setminus C_1$. Then there is some circuit $C_3 \subseteq (C_1 \cup C_2) - \{x\}$ with $y \in C_3$.

Proof We have that $S \setminus C_1$ and $S \setminus C_2$ are hyperplanes of M^* and, by Lemma 3.18, that $y \notin \mathrm{cl}^*(((S \setminus C_1) \cap (S \setminus C_2)) \cup \{x\})$. Since $\mathrm{cl}^*(((S \setminus C_1) \cap (S \setminus C_2)) \cup \{x\})$ is closed, Lemma 3.17 implies that there exists a hyperplane H of M^* with $y \notin H$ and $((S \setminus C_1) \cap (S \setminus C_2)) \cup \{x\} \subseteq H$. Then $C_3 = S \setminus H$ is the required circuit. ■

3.4 Connectivity

Bruhn and Wollan [6] introduced a concept of B-matroid connectivity that extends finite matroid connectivity without using rank functions. Except where otherwise stated, the proofs in this section are those of Bruhn and Wollan (with some small modifications) but are included for completeness.

Let M be a B-matroid with ground set S, and let (X, Y) be a partition of S. If B_X is a basis of X and B_Y is a basis of Y, define $del_{(X,Y)}(B_X, B_Y) = min\{|F| : F \subseteq X, (B_X \cup B_Y) - F \in \mathcal{I}\}$. The following basic facts are from [6].

Lemma 3.20
(Bruhn, Wollan [6]) If B, B' are bases of M with $|B - B'| < \infty$, then $|B' - B| = |B - B'|$.

Proof We proceed by induction on $|B - B'|$. In the base case $|B - B'| = 1$, let $B - B' = \{x\}$. Then, by (I4), there is an element of $B' - B$ so that $B - \{x\} \cup \{y\}$ is a basis. However, $B - \{x\} \cup \{y\} \subseteq B'$ so we must have $B' = B - \{x\} \cup \{y\}$ and therefore $|B' - B| = 1$.

Now suppose the result holds whenever $|B - B'| \leq k$. If $|B - B'| = k + 1$, choose any $x \in B - B'$. By (I4), there is some $y \in B' - B$ so that $B'' = B - \{x\} \cup \{y\}$ is a basis. Now $|B'' - B'| = k$, so $|B' - B''| = k$ and therefore $|B' - B| = k + 1$. ■

Lemma 3.21
(Bruhn, Wollan [6]) Let M be a B-matroid with ground set S. Let (X,Y) be a partition of S, B_X a basis of X and B_Y a basis of Y. Then:

1. $del(B_X, B_Y) = |F|$, for all $F \subseteq B_X$ so that $(B_X \cup B_Y) - F$ is a basis of M;
2. $del_{(X,Y)}(B_X, B_Y) = del_{(Y,X)}(B_Y, B_X)$; and
3. $del(B_X, B_Y) = del(B'_X, B'_Y)$ for every basis B'_X of X and every basis B'_Y of Y.

Proof First, notice that, by Lemma 3.20, if $(B_X \cup B_Y) - F_1$ and $(B_X \cup B_Y) - F_2$ are both bases of M, then $|F_1| = |F_2|$.

From this the first two claims are immediate. For the third, choose $F \subseteq X$ so that $(B_X \cup B_Y) - F$ is a basis. Then $B_X - F$ is a basis of M/Y, and therefore $(B_X \cup B'_Y) - F$ is also a basis of M. By the first claim, there is an $F' \subseteq Y$ so that $(B_X \cup B'_Y) - F'$ is a basis, and $|F| = |F'|$. Now $B'_Y - F$ is a basis of M/X and therefore $(B'_X \cup B'_y) - F'$ is a basis of M and therefore $del(B_X, B_Y) = del(B'_X, B'_Y)$. ■

Therefore we may set $\lambda_M(X) = \lambda_M(X, Y) = del_{(X,Y)}(B_X, B_Y)$. For an integer k we say that (X, Y) is a *k-separation* of M if $\lambda_M(X, Y) \leq k - 1$ and $|X|, |Y| \geq k$. A B-matroid M is *k-connected* if it has no ℓ-separation for any $\ell < k$.

For disjoint subsets X, Y of the ground set of M, we define the connectivity between X and Y, $\lambda_M(X, Y)$, to be the minimum of $\lambda_M(X')$ over all X' such that $X \subseteq X'$ and $X' \cap Y = \phi$.

The connectivity function for B-matroids retains some key properties of the connectivity function for finite matroids. In particular, it is closed under duality, and the univariate (partition) connectivity function is submodular.

Lemma 3.22
(Bruhn, Wollan [6]) If $\lambda_M(X, Y)$ is finite, then

$$\lambda_M(X, Y) = \lambda_{M^*}(X, Y).$$

Lemma 3.23
(Bruhn, Wollan [6]) For all $X, Y \subseteq S$, $\lambda(X) + \lambda(Y) \geq \lambda(X \cap Y) + \lambda(X \cup Y)$.

Let M be a B-matroid with ground set S. We may define a relation $\sim$ on S by saying that $x \sim y$ if there is a circuit of M that contains x and y.

Lemma 3.24
(Bruhn, Wollan [6]) The relation $\sim$ is an equivalence relation. There is a single $\sim$ equivalence class if and only if M is 2-connected.

The main result in [6] is that Tutte's linking theorem holds for finitary or co-finitary B-matroids.

Theorem 3.25
(Bruhn, Wollan [6]) Let M be a finitary or co-finitary B-matroid with ground set S, and let X, Y be disjoint subsets of S. There is a partition (C, D) of $S \setminus (X \cup Y)$ such that $\lambda_{M/C\setminus D}(X, Y) = \lambda_M(X, Y)$

It is unknown whether Tutte's linking theorem holds for general B-matroids. The rest of this section consists of a proof (joint with Brendan Rooney) that the Bixby-Coullard inequality holds for B-matroids. The Bixby-Coullard inequality states that for a matroid M with ground set S, for any $e \in S$, and any disjoint $C, D \subseteq S$ not containing e, $\lambda_{M/e}(C)+\lambda_{M\setminus e}(D) \geq \lambda_M(C\cap D)+\lambda_M(C\cup D\cup\{e\})-1$. For finite matroids, it is easy to prove Tutte's linking theorem from the Bixby-Coullard inequality, but this is not true for B-matroids.

Suppose that M is a B-matroid with ground set S, $e \in S$, $C, D \subseteq S - \{e\}$, and $\lambda(C)$ and $\lambda(D)$ are finite. Define $\lambda_{M\setminus e}(D) = \lambda_M(D) - 1 + k_1$, $\lambda_{M/e}(C) = \lambda_M(C) - 1 + k_2$ and $\lambda_M(C \cup D \cup \{e\}) = \lambda_M(C \cup D) + 1 - k_3$. Then $\lambda_{M/e}(C) + \lambda_{M\setminus e}(D) = \lambda_M(C) + \lambda_M(D) + k_1 + k_2 - 2$. By submodularity, this is at least $\lambda_M(C \cap D) + \lambda_M(C \cup D) + k_1 + k_2 - 2 = \lambda_M(C \cap D) + \lambda_M(C \cup D \cup \{e\}) + k_1 + k_2 + k_3 - 3$. So provided $k_1 + k_2 + k_3 \geq 2$ we will have the Bixby-Coullard inequality: $\lambda_{M/e}(C) + \lambda_{M\setminus e}(D) \geq \lambda_M(C \cap D) + \lambda_M(C \cup D \cup \{e\}) - 1$.

Also, define $\lambda_{M^*\setminus e}(C) = \lambda_{M^*}(C) - 1 + k_1^*$, $\lambda_{M^*/e}(C) = \lambda_{M^*}(C) - 1 + k_2^*$ and $\lambda_{M^*}(C \cup D \cup \{e\}) = \lambda_{M^*}(C \cup D) + 1 - k_3^*$. Notice that the definition of k_i^* is the same as the definition of k_i except that M is replaced with M^* and C is swapped with D. By taking duals, we have $k_1^* = k_2$, $k_2^* = k_1$ and $k_3^* = k_3$. In particular, $k_1^* + k_2^* + k_3^* = k_1 + k_2 + k_3$.

Claim $k_1 = 0$ if and only if e is a co-loop of $M \setminus D$ but not a co-loop of M. Otherwise, $k_1 = 1$. ■

Proof Suppose first that e is not a co-loop of $M \setminus D$. Then choose B_{S-D} to be a basis of $S - D$ avoiding e, and choose B_D to be a basis of D. Choose $T \subseteq B_D$ so that $B = (B_D \cup B_{S-D}) - T$ is a basis of M. Then in $M \setminus e$, B_D is a basis of D, B_{S-D} is a basis of $(S - D) - \{e\}$ and B is a basis of $M \setminus e$. So $\lambda_{M\setminus e}(D) = \lambda_M(D)$ and therefore $k_1 = 1$.

Now suppose that e is a co-loop of $M \setminus D$. Then every basis B_{S-D} of $S - D$ contains e. Choose one such basis, and also choose B_D to be a basis of D. Choose

$T \subseteq B_D$ so that $B = (B_D \cup B_{S-D}) - T$ is a basis of M. In $M \setminus e$, B_D is a basis of D, $B_{S-D} - \{e\}$ is a basis of $(S-D) - \{e\}$ and B is a basis of $M \setminus e$ if and only if e is not a co-loop of M. If e is a co-loop of M, then $B_D \cup (B_{S-D} - \{e\}) - T = B - \{e\}$, so $\lambda_{M\setminus e}(D) = \lambda_M(D)$ and therefore $k_1 = 1$. Otherwise, since $B - e$ is independent and $B_D \cup (B_{S-D} - \{e\})$ is spanning, there is a basis B' of $M \setminus e$ with $B - e \subseteq B' \subseteq (B_{S-D} - \{e\})$. Then $B - B' = \{e\}$ and therefore $B' - B = \{f\}$ for some $f \in T$. Since $B_D \cup (B_{S-D} - \{e\}) - (T - \{f\}) = B'$, we have $\lambda_{M\setminus e}(D) = \lambda_M(D) - 1$ and therefore $k_1 = 0$. ■

Claim $k_2 = 0$ if and only if e is not a loop of M and not a co-loop of $C \cup \{e\}$. Otherwise, $k_2 = 1$. ■

Proof Since $k_2 = k_1^*$, by the previous claim we have that $k_2 = 0$ if and only if e is a co-loop of $M^* \setminus C$ but not a co-loop of M^*. Now, e is a co-loop of $M^* \setminus C$ if and only if it is a loop of M/C if and only if e is contained in some circuit in $C \cup \{e\}$, or in other words is not a co-loop of $C \cup \{e\}$. ■

Claim $k_3 \geq 0$. If $k_1 = 0$, then $k_3 \geq 1$. If $k_1 = k_2 = 0$, then $k_3 = 2$. ■

Proof By submodularity, $\lambda_M(C \cup D \cup \{e\}) \leq \lambda_M(C \cup D) + 1$, so $k_3 \geq 0$. If $k_1 = 0$, then e is a co-loop of $S - D$, and therefore also a co-loop of $S - (C \cup D)$, but not a co-loop of M. Let $B_{C\cup D}$ be a basis of $C \cup D$. Extend $B_{C\cup D}$ to a basis B of $M \setminus e$ which since e is not a co-loop of M is also a basis of M. Now extend $B \cap S - (C \cup D)$ to a basis $B_{S-(C\cup D)}$ of $S - (C \cup D)$, which must contain e. Then $B = B_{C\cup D} \cup B_{S-(C\cup D)} - T$, where $e \in T$ and $\lambda(C \cup D) = |T|$.

Now, $B_{S-(C\cup D)} - \{e\}$ is a basis of $(S - (C \cup D)) - \{e\}$. Either $B_{C\cup D}$ or $B_{C\cup D} \cup \{e\}$ is a basis of $C \cup D \cup \{e\}$. If $B_{C\cup D} \cup \{e\}$ is a basis of $C \cup D \cup \{e\}$, then $(B_{S-(C\cup D)} - \{e\}) \cup (B_{C\cup D} \cup \{e\}) - T = B$ which implies that $\lambda(C \cup D \cup \{e\}) = |T|$ and therefore $k_3 = 1$. If $B_{C\cup D}$ is a basis of $C \cup D \cup \{e\}$, then $(B_{S-(C\cup D)} - \{e\}) \cup (B_{C\cup D}) - (T - \{e\}) = B$ which implies that $\lambda(C \cup D \cup \{e\}) = |T| - 1$ and therefore $k_3 = 2$.

If $k_2 = 0$, then e is not a co-loop of $C \cup \{e\}$ and hence not a co-loop of $C \cup D \cup \{e\}$. Therefore $B_{C\cup D}$ is a basis of $C \cup D \cup \{e\}$ and $k_3 = 2$. ■

Lemma 3.29
The Bixby-Coullard inequality, $\lambda_{M/e}(C) + \lambda_{M\setminus e}(D) \geq \lambda_M(C \cap D) + \lambda_M(C \cup D \cup \{e\}) - 1$, holds for a B-matroid M with ground set S, $C, D \subseteq S$, $e \notin (C \cup D)$.

Proof By the above discussion, we need to show that $k_1 + k_2 + k_3 \geq 2$. We know that each $k_i \geq 0$. If $k_1 = k_2 = 1$, we are done. If not, we may assume that $k_1 = 0$

(if $k_1 = 1$, then $k_1^* = k_2 = 0$ and we prove that $k_1^* + k_2^* + k_3^* \geq 2$ in exactly the same way). By Claim 3.28, $k_3 \geq 1$. Therefore if $k_2 = 1$ we are done. On the other hand, if $k_1 = k_2 = 0$, then, again by Claim 3.28, we have $k_3 = 2$ and we are done. ■

3.5 Binary B-matroids

This section incorporates joint work with Paul Wollan. The following lemma describes a way of extending results about classes of finite matroids defined by excluded minors to classes of B-matroids.

Lemma 3.30
Let $\mathcal{P}$ be a B-matroid property, and $\mathcal{X}$ a set of finite matroids. If:

1. for finite matroids M, $M \in \mathrm{ex}(\mathcal{X})$ if and only if $M \in \mathcal{P}$;
2. $\mathcal{P}$ is closed under minors; and
3. for a B-matroid $M \notin \mathcal{P}$, there is a finite minor N of M such that $N \notin \mathcal{P}$,

then $\mathcal{P} = \mathrm{ex}(\mathcal{X})$.

Proof Suppose that $M \in \mathcal{P}$. Then, by (2), every finite minor N of M is also in $\mathcal{P}$ and therefore, by (1), in $\mathrm{ex}(\mathcal{X})$. Since $\mathcal{X}$ is a set of finite matroids, this implies that $M \in \mathrm{ex}(\mathcal{X})$.

Conversely, suppose that $M \notin \mathcal{P}$. Then, by (3), there is a finite minor N of M such that $N \notin \mathcal{P}$. By (1), $N \notin \mathrm{ex}(\mathcal{X})$, and therefore $M \notin \mathrm{ex}(\mathcal{X})$. ■

Generalizing the standard definition of a binary matroid will yield only finitary matroids. We will instead call a B-matroid *binary* if it has no $U_{2,4}$ minor.

The following result makes it easier to apply Lemma 3.30 to prove that certain B-matroid properties are equivalent to being binary.

Lemma 3.31
Suppose that M is a B-matroid with finite rank. Then either the simplification of M is finite or M has a $U_{2,4}$ minor.

Proof Let B be a basis of the simplification of M. For each $e \notin B$ in the simplification of M, let C_e denote the fundamental circuit for e with respect to B. We claim that for $e \neq f$, $C_e \cap B \neq C_f \cap B$, from which the result follows because B has only finitely many subsets.

Suppose otherwise that $C_e \cap B = C_f \cap B = B_{ef}$. Let $x \in B_{ef}$, and apply circuit elimination to C_e, C_f and $x \in C_e \cap C_f$ to obtain a circuit $C_1 \subseteq C_e \cup C_f$ such that $x \notin C_1$. Since $C_1 \subseteq B \cup \{e, f\}$ and C_1 is distinct from C_e and C_f, we must have $\{e, f\} \subseteq C_1$. Since we are considering a simple matroid, $\{e, f\}$ is not a circuit, so we may choose $y \in C_1 \backslash \{e, f\}$. Now apply circuit elimination to C_e, C_f and $y \in C_e \cap C_f$ to obtain another circuit $C_2 \subseteq C_e \cup C_f$ such that $y \notin C_2$. Since C_1 and C_2 are both circuits and $C_1 \backslash C_2$ is non-empty, $C_2 \backslash C_1$ is also non-empty. Let $z \in C_2 \backslash C_1$.

Note that among the elements $\{e, f, y, z\}$: C_e contains exactly $\{e, y, z\}$; C_f contains exactly $\{f, y, z\}$; C_1 contains exactly $\{e, f, y\}$; and C_2 contains exactly $\{e, f, z\}$. We obtain a $U_{2,4}$ minor of M by taking the restriction of M to $C_e \cup C_f$ then contracting all of the elements except for $\{e, f, y, z\}$. We have already shown that the three element subsets are dependent, and all of the two element subsets except for $\{e, f\}$ are obviously independent. To see that $\{e, f\}$ is independent, note that if it is dependent there is a circuit C_3 contained in $(C_e \cup C_f) \backslash \{y, z\}$, but then applying circuit elimination to C_1, C_3 and e yields a circuit contained in $C_f \backslash \{z\}$, a contradiction. ■

Lemma 3.32
The following are equivalent for a B-matroid M:

1. M is binary ($M \in \mathrm{ex}(\{U_{2,4}\})$);
2. Property $\mathcal{P}_1$: For every circuit C and co-circuit C^* of M, $|C \cap C^*|$ is either infinite or even;
3. Property $\mathcal{P}_2$: For every circuit C and co-circuit C^* of M, $|C \cap C^*| \neq 3$; and
4. Property $\mathcal{P}_3$: For every pair of distinct circuits C_1, C_2 of M, if $C_1 \cap C_2$ is finite, then $C_1 \Delta C_2$ is dependent.

Proof By Theorem 1.3, these statements are all equivalent for finite matroids. We will first show that, for $i \in \{1, 2, 3\}$, $\mathcal{P}_i$ is closed under taking minors, and then that, for $i \in \{1, 2, 3\}$, for any $M \notin \mathcal{P}_i$, there is a finite rank (or corank) minor N of

M such that $N \notin \mathcal{P}_i$, which implies by Lemmas 3.31 and 3.30 that $\mathcal{P}_i$ is equivalent to being binary.

Let $N = M/C\backslash D$ be a minor of M such that C is independent in M. Suppose that $N \notin \mathcal{P}_1$. Then there exist a circuit K and a co-circuit K^* of N such that $|K \cap K^*|$ is odd. Then there exist $C_K \subseteq C$ such that $K \cup C_K$ is a circuit of M, and $D_{K^*} \subseteq D$ such that $K^* \cup D_{K^*}$ is a co-circuit of M. Then $|(K \cup C_K) \cap (K^* \cup D_{K^*})| = |K \cap K^*|$ is odd and therefore $M \notin \mathcal{P}_1$. So $\mathcal{P}_1$ is closed under taking minors, and evidently so is $\mathcal{P}_2$ by the same argument.

Suppose that $N \notin \mathcal{P}_3$. Let K_1 and K_2 be circuits of N such that $K_1 \Delta K_2$ is independent. Then there exist $C_{K_1}, C_{K_2} \subseteq C$ such that $K_1 \cup C_{K_1}$ and $K_2 \cup C_{K_2}$ are circuits of M. Since $K_1 \Delta K_2$ is independent in M/C, and C is independent in M, we must have $(K_1 \cup C_{K_1}) \Delta (K_2 \cup C_{K_2}) \subseteq (K_1 \Delta K_2) \cup C$ independent in M, showing that $M \notin \mathcal{P}_3$. Therefore, $\mathcal{P}_3$ is also closed under taking minors.

Suppose that $M \notin \mathcal{P}_1$. Then there is a circuit C and a co-circuit C^* of M such that $|C \cap C^*|$ is odd. Choose $x \in C \cap C^*$. Then $C\backslash\{x\}$ is independent, so we may extend it to a basis B of M avoiding the co-independent set $C^*\backslash C$. Consider $N = M/(B\backslash(C \cap C^*))$. Observe that $(C \cap C^*)\backslash\{x\}$ is a basis for N, so N has finite rank. Clearly C^* is still a co-circuit of N (since it is disjoint from the elements of M that were contracted), and $C \cap C^*$ is a circuit of N because $C \cap C^* \subseteq B \cup x$, and C is the fundamental circuit for x with respect to B. Therefore N is a finite rank minor of M with $N \notin \mathcal{P}_1$. The argument is identical for $\mathcal{P}_2$.

Finally, suppose that $M \notin \mathcal{P}_3$. Then there are circuits C_1, C_2 of M with $C_1 \cap C_2$ finite and $C_1 \Delta C_2$ independent. Since $C_1 \Delta C_2$ is independent, we may find a cobasis B disjoint from $C_1 \Delta C_2$. Since $C_1 \cap C_2$ is finite, we may delete $B\backslash(C_1 \cap C_2)$ to obtain a finite corank minor of M, in which C_1 and C_2 are still circuits and $C_1 \Delta C_2$ is still independent. ■

3.6 Graphic B-matroids

The lemmas in this section describe B-matroid analogues of the circuit matroid of a graph. Both infinite graphs and graph-like continua have circuit matroids. Since infinite graphs have only finite circuits, their circuit matroids will be finitary, and since graph-like continua have only finite bonds, their circuit matroids will be co-finitary.

Let G be a finite graph. A *k-Tutte-separation* is a partition (E_1, E_2) of the edges of G, such that $|E_1|, |E_2| \geq 3$ and $|V(G[E_1]) \cap V(G[E_2])| = k$. A finite graph is *$k$-Tutte-connected* if it has no ℓ-Tutte-separation for any $\ell < k$. Note that a finite graph with at least four edges is *k-Tutte-connected* if and only if it is k-connected and has girth at least k. We will use these definitions of Tutte-separations and Tutte-connectivity for both infinite graphs and graph-like continua.

Lemma 3.33
Let G be an infinite graph. There is a finitary B-matroid $\mathcal{M}(G)$ whose circuits, co-circuits and bases are the circuits, bonds and edge-maximal spanning forests of G, respectively. If, for any $k \geq 2$, G is k-Tutte-connected, then $\mathcal{M}(G)$ is also k-connected.

Proof By Lemma 3.7, the circuits of G are the circuits of a B-matroid, $\mathcal{M}(G)$. The bases of $\mathcal{M}(G)$ are the maximal circuit avoiding sets of edges of G. A set K is a co-circuit of $\mathcal{M}(G)$ if it is a minimal set that meets every basis. We claim that these sets are exactly the bonds of G. Suppose that X is a set of elements of $\mathcal{M}(G)$ that does not contain any bond of G. Then each component of G is still connected in $G - X$, so any edge-maximal spanning forest of $G - X$ is an edge-maximal spanning forest of G. So any such set X is disjoint from some basis and is therefore co-independent. On the other hand, an edge-maximal spanning forest of G cannot be disjoint from a bond of G. Therefore the co-circuits of $\mathcal{M}(G)$ are exactly the bonds of G.

It remains to show that $\mathcal{M}(G)$ is k-connected. For some $\ell < k$, let (X, Y) be a partition of $E(G)$ with $|X|, |Y| \geq \ell$. We need to show that $\lambda_{\mathcal{M}(G)}(X, Y) \geq \ell$. We define two equivalence relations $\sim_X$ and $\sim_Y$ on the vertices of G, by saying that, for $u, v \in V(G)$, $u \sim_X v$ if there is a uv-path in $G[X]$, and $u \sim_Y v$ if there is a uv-path in $G[Y]$. We now consider two cases.

First, suppose that for every pair u, v of vertices of G, at least one of $u \sim_X v$ or $u \sim_Y v$ holds. We claim that in this case either there is only one $\sim_X$ equivalence class, or there is only one $\sim_Y$ equivalence class. Suppose that there is more than one $\sim_X$ equivalence class, and let u and v be any pair of vertices of G. If they are in different $\sim_X$ equivalence classes, then we are supposing that $u \sim_Y v$, and if they are in the same $\sim_X$ equivalence class there is some w in a different $\sim_X$ equivalence class from either, whence $u \sim_Y w$ and $v \sim_Y w$ imply that $u \sim_Y v$. Therefore the claim holds, and we may suppose without loss of generality that there is only one $\sim_X$ equivalence class. It follows that any basis B_X for X in $\mathcal{M}(G)$ (a spanning tree for $G[X]$) is in fact a spanning tree for G and therefore a basis for $\mathcal{M}(G)$. Let B_Y be any basis for Y. Suppose that $|B_Y| < \ell$. Since $|Y| \geq \ell$, $Y \setminus B_Y$ is non-empty. Let $y \in Y \setminus B_Y$, and consider the fundamental circuit through y with respect to B_Y. It has at most $|B_Y| + 1 \leq \ell < k$ elements, contradicting the fact that G has girth k. Therefore, for any basis B_Y of Y, $\lambda_{\mathcal{M}(G)}(X, Y) \geq |B_Y| \geq \ell$, as required.

Now suppose that there is a pair u, v of vertices of G such that neither $u \sim_X v$

nor $u \sim_Y v$ holds. By Menger's Theorem, there are k internally disjoint uv-paths $P_1, P_2, \ldots, P_k$ in G. Let $F = \bigcup_{i=1}^{k} E(P_i)$. No path consists entirely of edges of X or entirely of edges of Y, so $F \cap X$ and $F \cap Y$ are both independent. Let B_X be a basis of X containing $F \cap X$ and B_Y be a basis of Y containing $F \cap Y$. Since $F \subseteq B_X \cup B_Y$, and no independent subset of F can be obtained by deleting fewer than $k-1$ edges, $\lambda_{\mathcal{M}(G)}(X, Y) \geq k - 1 \geq \ell$, as required. ■

Lemma 3.34

Let $\mathcal{G}$ be a graph-like continuum. There is a co-finitary B-matroid $\mathcal{M}(\mathcal{G})$ whose circuits, co-circuits and bases are the circuits, bonds and spanning trees of $\mathcal{G}$, respectively. If, for any $k \geq 2$, $\mathcal{G}$ is k-connected, then $\mathcal{M}(\mathcal{G})$ is also k-connected.

Proof The bonds of $\mathcal{G}$ are finite, by Lemma 2.1. They also satisfy the finite circuit axioms. For (C3), suppose that X_1, X_2 are closed subsets of $\mathcal{G}$ such that $X_1 \cup X_2 = \mathcal{G}$ and $X_1 \cap X_2$ is contained in the edges of a bond K_1, and Y_1, Y_2 are closed subsets of $\mathcal{G}$ such that $Y_1 \cup Y_2 = \mathcal{G}$ and $Y_1 \cap Y_2$ is contained in the edges of another bond K_2, and $e \in K_1 \cap K_2$. Without loss of generality, suppose that e has one endpoint in $X_1 \cap Y_1$ and the other endpoint in $X_2 \cap Y_2$. Then $X_1 \cap Y_2 \setminus \{e\}$ and $X_2 \cup Y_1 \cup \{e\}$ are both closed subsets of $\mathcal{G}$, their union is $\mathcal{G}$ and their intersection is contained in $(K_1 \cup K_2) \setminus \{e\}$. Therefore there is a bond of $\mathcal{G}$ contained in $(K_1 \cup K_2) \setminus \{e\}$, as required.

Since the bonds of $\mathcal{G}$ are all finite, and satisfy the finite circuit axioms, they are the circuits of a B-matroid by Lemma 3.7. The dual of this B-matroid, $\mathcal{M}(\mathcal{G})$, has the bonds of $\mathcal{G}$ for its co-circuits. Let B be a basis of $\mathcal{M}(\mathcal{G})$. Then B is a minimal subset of $E(\mathcal{G})$ that meets every co-circuit. Every circuit of $\mathcal{G}$ clearly meets every bond at least twice, so B cannot contain a circuit. On the other hand, suppose that there is some $e = xy \in E(\mathcal{G})$ such that $B \cup \{e\}$ does not contain a circuit. Let C_x and C_y be the components of the closure of B containing x and y respectively. Since there is no arc between x and y in the closure of B, C_x and C_y cannot be the same component. Let C'_x be a closed subset of $\mathcal{G}$ containing C_x but disjoint from C_y. Then the set of edges with one endpoint in C'_x and one endpoint not in C'_x is an edge cut of $\mathcal{G}$ and it therefore contains a bond disjoint from B, a contradiction. Therefore B is a maximal circuit avoiding subset of $E(\mathcal{G})$. This also shows that the circuits of $\mathcal{M}(\mathcal{G})$ are the circuits of $\mathcal{G}$.

The proof that $\mathcal{M}(\mathcal{G})$ is k-connected is as above, but considering arcs instead of paths, and using Theorem 2.2. ■

Lemma 3.35
Let G be a graph or a graph-like continuum. Then $\mathcal{M}(G/C\backslash D) = \mathcal{M}(G)/C\backslash D$.

Proof First, suppose that G is a graph. Let B_C be an edge-maximal spanning forest of $G[C]$. Suppose that J is independent in $\mathcal{M}(G/C\backslash D)$. Then J is disjoint from D and does not contain any circuits of G/C. We need to show that $J \cup B_C$ does not contain any circuits of G, so that J is independent in $\mathcal{M}(G)/C\backslash D$. This is clear, because if $J \cup B_C$ contains a circuit K of G, then $K \setminus B_C \subseteq J$ is still the edge set of a closed walk in G/C, and therefore contains a circuit.

Conversely, if J is dependent in $\mathcal{M}(G/C\backslash D)$, then J does contain a circuit K of G/C. Each of the finitely many vertices v of K is either a vertex of G, or can be replaced by a finite path in B_C to yield a circuit of G contained in $J \cup B_C$. This shows that J is dependent in $\mathcal{M}(G)/C\backslash D$.

Now, suppose that G is a graph-like continuum. We will show that the co-circuits of $\mathcal{M}(G/C\backslash D)$ are the same as the co-circuits of $\mathcal{M}(G)/C\backslash D$. Suppose that K is a co-circuit of $\mathcal{M}(G)/C\backslash D$. Then, by Lemma 3.11, there is some subset D' of D such that $K \cup D'$ is a co-circuit of $\mathcal{M}(G)$. Therefore $K \cup D'$ is a bond of G, so G is the union of two closed sets X_1 and X_2 intersecting only in $J \cup D'$. Since C is disjoint from $J \cup D'$, each of the components of the closure of C is entirely contained in X_1 or X_2 and therefore, in G/C, X_1 and X_2 are still closed sets. Every edge in K still has one endpoint in each of X_1 and X_2, so K is a bond of $G/C\backslash D$, as required.

Conversely, suppose that K is a co-circuit of $\mathcal{M}(G/C\backslash D)$. Then K is a bond of $G/C\backslash D$, so there are closed sets X_1 and X_2 in $G/C\backslash D$ whose union is $G/C\backslash D$ and whose intersection is contained in the edges of K. Let X_1' and X_2' be the closed sets in $G\backslash D$ formed by including a whole component of the closure of C in X_i' if its contracted image is contained in X_i. Then X_1' and X_2' are closed sets in G also. The set of edges with one endpoint in each of X_1' and one endpoint in X_2' is a bond in G, contains K, and is otherwise contained in D, so there is some finite subset D' of D such that $K \cup D'$ is a co-circuit of $\mathcal{M}(G)$ and therefore K is a co-circuit of $\mathcal{M}(G)/C\backslash D$, as required. ■

3.7 Matroid intersection for B-matroids

In this section, we will prove a version of the matroid intersection theorem for B-matroids.

Note that Lemma 3.20 implies that, if any subset of a B-matroid has a finite basis, then all of its bases are the same size. Therefore, if M is a B-matroid with ground set S, we can define the *rank function* of M, $r_M : 2^S \to \mathbb{Z} \cup \{\infty\}$, by saying that $r_M(X) = k$, if every basis of X has size k, and $r_M(X) = \infty$, if X has no finite basis.

We will begin by proving the following lemma, which is a verbatim generalization of Edmonds' matroid intersection theorem ([12]) to B-matroids.

Lemma 3.36
Let M_1 and M_2 be two B-matroids with the same countable ground set S. For any integer k, M_1 and M_2 have a common independent set of size k if and only if, for each partition (X, Y) of S, $r_{M_1}(X) + r_{M_2}(Y) \geq k$.

Proof Suppose first that M_1 and M_2 have a common independent set J of size k. Then for any partition (X, Y) of S, $J \cap X$ is an independent subset of X in M_1 and $J \cap Y$ is an independent subset of Y in M_2. Therefore, $r_{M_1}(X) + r_{M_2}(Y) \geq |J \cap X| + |J \cap Y| = |J| = k$.

Now suppose that M_1 and M_2 have no common independent set of size k. Let $S = \{e_1, e_2, \dots\}$. For each $i \in \{1, 2\}$, and each integer j, let $M_i^j = M_i|_{\{e_1,\dots e_j\}}$. Independent sets in M_i^j are also independent in M_i so, for every integer j, M_1^j and M_2^j have no common independent set of size k. For each j, applying the matroid intersection theorem to M_1^j and M_2^j, there is at least one partition (X_j, Y_j) of $\{e_1, \dots e_j\}$ such that $r_{M_1^j}(X_j) + r_{M_2^j}(Y_j) < k$. For each j, let V_j be the set of all such partitions. Form a graph G on the union of the V_j by saying that (X_j, Y_j) is adjacent to (X_{j-1}, Y_{j-1}) if $X_{j-1} \subseteq X_j$ and $Y_{j-1} \subseteq Y_j$. Applying König's infinity lemma to G yields sequences $X_1 \subseteq X_2 \subseteq \dots$ and $Y_1 \subseteq Y_2 \subseteq \dots$ such that $(X_j, Y_j) \in V_j$. Let X be the union of the X_j and let Y be the union of the Y_j. Then (X, Y) is a partition of S and $r_{M_1}(X) + r_{M_2}(Y) < k$, because any finite independent set in X is eventually an independent set in some X_j and any finite independent set in Y is a eventually an independent set in some Y_j. ■

Theorem 3.37
Let M_1 and M_2 be two B-matroids with the same countable ground set S. The following are equivalent:

1. M_1 and M_2 have a countable common independent set;

2. for every positive integer k, M_1 and M_2 have a common independent set of size k; and

3. there is no partition (X, Y) of S such that X has a finite basis in M_1 and Y has a finite basis in M_2.

Proof It is obvious that the first statement implies the second, and Lemma 3.36 shows that the second implies the third. It remains to show the third implication: that if there is no partition (X, Y) of S such that X has a finite basis in M_1 and Y has a finite basis in M_2, then M_1 and M_2 have a countable common independent set.

Suppose that there is no partition (X, Y) of S such that X has a finite basis in M_1 and Y has a finite basis in M_2. We first claim that if J is any finite common independent set of M_1 and M_2, there is some $t \in S$ such that $J \cup \{t\}$ is a common independent set. Suppose otherwise, and let $X' = J \cup \{t \in S \setminus J \mid J \cup \{t\}$ is dependent in $M_1\}$ and $Y' = J \cup \{t \in S \setminus J \mid J \cup \{t\}$ is dependent in $M_2\}$. Then $X' \cup Y' = S$ and J is a finite basis of both X' and Y', so any partition (X, Y) of S such that $X \subseteq X'$ and $Y \subseteq Y'$ provides a contradiction.

Therefore we may start with the empty set and add one element at a time to obtain a sequence $\{J_0, J_1, \ldots\}$ of common independent sets such that, for each $k > 0$, $|J_k| = k$. Let J be the union over $k > 0$ of the J_k. Every finite subset of J is a common independent set. Suppose that J is dependent in M_1. Then there is at least one circuit of M_1 contained in J. Let C be any such circuit. If C were finite it would be independent, so C is countable. Choose any element x of C. Then $C \setminus \{x\}$ is countable, is independent in M_1, and every finite subset of $C \setminus \{x\}$ is a common independent set. If $C \setminus \{x\}$ is not already a common independent set, then it contains a circuit C' of M_2, which, as for C, must be countable. In this case, for any $x' \in C'$, $C' \setminus \{x'\}$ is a countable common independent set. ■

Chapter 4

Matroid and graph limits

4.1 Introduction

In the first section of this chapter, we introduce a technique for viewing a finitary or co-finitary B-matroid as the limit of a sequence of finite matroids. We show that any finitary or co-finitary B-matroid can be obtained in this way. In the following two sections we show that, given a sequence of finite graphs, we can obtain a limit graph that has the same circuits as the finitary limit matroid obtained from the sequence of circuit matroids of the graphs, and a limit graph-like continuum that has the same circuits as the co-finitary limit matroid obtained from the sequence of circuit matroids. It follows that, given a finitary or co-finitary B-matroid, M, whose every finite minor is graphic, we can first construct M as the limit of a sequence of finite matroids then, viewing these matroids as graphs, we can obtain a graph or graph-like continuum with the same circuits as M. This allows us to extend Tutte's characterization of graphic matroids by excluded minors to finitary and co-finitary B-matroids. In the final section, we use this result to obtain a version of Whitney's planarity criterion for infinite graphs and graph-like continua.

4.2 Matroid limits

A sequence $\{M_i\}$, $i \geq 0$ of matroids such that, for each $i \geq 0$, $M_i \prec M_{i+1}$, will be called a *minor sequence* of matroids.

Let $\{M_i\}$ be a minor sequence of matroids and, for each i, let $M_i = (S_i, \mathcal{I}_i)$. Let $S = \cup_{i \in \mathbb{N}} S_i$ and let A be any finite subset of S. There is some $N(A) \in \mathbb{N}$ such that for all $j > N(A)$, $A \subseteq S_j$. If A is independent in M_k for some $k > N(A)$, then it follows immediately from the definition of contraction that A is independent in

M_l for every $l > k$. Therefore, either:

1. for all $j > N(A)$, A is dependent in M_j; or
2. there is some $N'(A) \in \mathbb{N}$ such that for all $j > N'(A)$, A is independent in M_j.

In the first case A is *eventually dependent* and in the second case A is *eventually independent*.

The *finitary limit* of $\{M_i\}$ is given by $[\{M_i\}] = (S, \{I \in S \mid \text{every finite subset } A \text{ of } I \text{ is eventually independent}\})$. The following lemma describes the basic properties of $[\{M_i\}]$.

Lemma 4.1
Let $\{M_i\}$ be a minor sequence of matroids. Then:

1. $[\{M_i\}]$ is a finitary B-matroid;
2. a subset C of S is a circuit of $[\{M_i\}]$ if and only if there is some $N(C) \in \mathbb{N}$ such that, for all $j > N(C)$, $C \subset S_j$ and C is a circuit of M_j; and
3. if A is a finite subset of S and B is a basis for A in $[\{M_i\}]$, then there is some $N(A) \in \mathbb{N}$ such that, for all $j > N(A)$, $A \subseteq S_j$ and B is a basis for A in M_j.

Proof We need to prove the axioms (i1), (i2), (i3) and (i4). Since any independent set of M_1 is eventually independent, (i1) holds, while (i2) follows immediately from (i2) for the matroids M_i. Since every eventually independent set is in $\mathcal{I}$, (i4) follows from the definition of $[\{M_i\}]$. Finally, suppose that I_1 and I_2 are finite independent sets with $|I_2| > |I_1|$. Since I_1 and I_2 are both eventually independent, there is some $j \in \mathbb{N}$ such that both I_1 and I_2 are independent sets in M_j. Since (i3) holds for M_j, there is $x \in I_2 \backslash I_1$ such that $I_1 \cup x$ is independent in M_j, and hence independent in $[\{M_i\}]$ as required for (i3).

If there is some $N(C) \in \mathbb{N}$ such that, for all $j > N$, $C \subset S_j$ and C is a circuit of M_j, then it is clear that C is finite, C is eventually dependent, and every proper subset of C is eventually independent, so C is a circuit of $[\{M_i\}]$. Conversely, suppose C is a circuit of $[\{M_i\}]$. Then by (i4), C is finite, hence eventually dependent. There is a finite number of proper subsets of C, each of which is eventually independent. Therefore eventually C is dependent and every proper subset of C is independent, so C is eventually a circuit, as required.

Let A be a finite subset of S and let B be a basis for A. Since for every $x \in A \backslash B$, $B \cup \{x\}$ is eventually dependent, as soon as $A \subseteq S_j$, $B \cup \{x\}$ is dependent in M_j for every $x \in A \backslash B$. On the other hand, B itself is eventually independent, and therefore eventually a maximal independent subset of A, as required. ∎

Example 4.1
Let $\{M_i\}$ be a minor sequence such that, for each positive integer i, M_i is a circuit with i elements. Let S be the union of the ground sets of the M_i. Every finite subset of S is eventually independent in some M_i, so $[\{M_i\}]$ is a free matroid (every element is a co-loop).

Example 4.2
Let $\{M_i\}$ be a minor sequence such that, for each positive integer i, M_i is isomorphic to $U_{2,i}$. Let S be the union of the ground sets of the M_i. Every subset of S consisting of two elements is eventually independent, and every subset of S consisting of three elements is eventually a circuit, so $[\{M_i\}]$ is $U_{2,\infty}$.

Lemma 4.2
Suppose that E and F are finite subsets of S. Then $[\{M_i\}] \backslash E/F = [\{M_i \backslash E/F\}]$.

Proof Let B be any basis of F in $[\{M_i\}]$. Consider any $A \subseteq S \backslash E/F$. Then by the definitions of deletion and contraction, A is independent in $[\{M_i\}] \backslash E/F$ if and only if $A \cup B$ is independent in $[\{M_i\}]$. If A is independent in $[\{M_i\}] \backslash E/F$, then $A \cup B$ is independent in $[\{M_i\}]$, so for every finite subset A' of A, B is eventually a basis of F and $A' \cup B$ is eventually independent. So every such A' is eventually independent in $M_j \backslash E/F$, and hence A is independent in $[\{M_i \backslash E/F\}]$. On the other hand, if A is dependent in $[\{M_i\}] \backslash E/F$, then $A \cup B$ is dependent in $[\{M_i\}]$, so there is some finite subset A' of A such that $A' \cup B$ is dependent in $[\{M_i\}]$. Eventually $A' \cup B \subset S_j$ and B is a basis of F in M_j, so A' is dependent in $M_j \backslash E/F$, and therefore A' and A are dependent in $[\{M_i \backslash E/F\}]$. ∎

Lemma 4.3
Every countable finitary B-matroid M is the finitary limit of some sequence of matroids $\{M_i\}$.

Proof Let $S = \{e_1, \ldots\}$. For each $i \geq 1$, let $M_i = M|_{\{e_1,\ldots,e_i\}}$. Since any circuit C of M is finite, it is also a circuit of $[\{M_i\}]$. On the other hand, any finite independent set in M is independent in every M_i for which it is contained in S_i, so by (i4) every independent set in M is independent in $[\{M_i\}]$. So $M = [\{M_i\}]$, as required. ■

Lemma 4.4
Let M be a countable finitary binary B-matroid with ground set S, and let $S = C \cup D$ where C is a basis of M and D is its complementary co-basis. Let $\{C_i\}$ and $\{D_i\}$ be arbitrary increasing sequences of finite subsets of C and D respectively, whose unions are C and D. Then $M = [\{M_i\}]$ where $M_i = M/(C \backslash C_i) \backslash (D \backslash D_i)$.

Proof It suffices to show that a finite set X is independent in M if and only if it is independent in $[\{M_i\}]$. Firstly, suppose that X is a finite independent subset of M. Either X is independent in $[\{M_i\}]$, or there is some subset X' of X that is a circuit of $[\{M_i\}]$. Then there is some $N(X')$ such that X' is a circuit in every M_i, $i > N(X')$. Let $j > N(X')$. By definition of contraction, there is a circuit K of M such that $X' \subseteq K \subseteq X' \cup (C \backslash C_j)$. Since M is finitary, K is finite, and hence contained in the ground set of M_k for some $k > j$. Since $k > j > N(X')$, X' is a circuit in M_k, so applying the definition of contraction again we obtain a circuit K' of M such that $X' \subseteq K' \subseteq X' \cup (C \backslash C_k)$. Recall that K is contained in the ground set of M_k, so $K \cap K' \subseteq X'$. By Lemma 3.32, $K \Delta K'$ is dependent, but by choice of K and K', and because $K \cap K' \subseteq X'$, $K \Delta K' \subseteq C$, a contradiction. Therefore X must indeed be independent in $[\{M_i\}]$.

Now suppose that X is a finite dependent subset of M. Then X contains a circuit X' of M. For a sufficiently large j, X' is contained in the ground set of M_j, and since M_j is a minor of M, X' is dependent in M_j. Therefore X is dependent in $[\{M_i\}]$. ■

The *cofinitary limit* of $\{M_i\}$ is given by $[\{M_i^*\}]^*$. This is a cofinitary B-matroid.

Example 4.3
As in Example 4.1, let $\{M_i\}$ be a minor sequence such that, for each positive integer i, M_i is a circuit with i elements. Let S be the union of the ground sets of the M_i. Since M_i is a circuit, M_i^* consists of i parallel elements (each element is independent and each pair of elements is a circuit). Therefore $[\{M_i^*\}]$ also consists of parallel elements, and it follows that $[\{M_i^*\}]^*$ is a circuit.

Example 4.4
As in Example 4.2, let $\{M_i\}$ be a minor sequence such that, for each positive integer i, M_i is isomorphic to $U_{2,i}$. Let S be the union of the ground sets of the M_i. Since M_i is isomorphic to $U_{2,i}$, M_i^* is isomorphic to $U_{i-2,i}$. Every finite subset of S is eventually independent in M_i^*, so $[\{M_i^*\}]$ consists entirely of co-loops. Therefore $[\{M_i^*\}]^*$ is simply a collection of loops.

The remaining lemmas of this section describe the relationship between the circuits of $[\{M_i^*\}]^*$ and the circuits of M_i, for each i.

If $\{S_i\}$ is a sequence of subsets of some set S, we will say that $x \in S$ is *eventually in* $\{S_i\}$ if there is some $N(x)$ such that, for any $j > N(x)$, $x \in S_j$. We will say that $x \in S$ is *infinitely often in* $\{S_i\}$ if there is no $N(x)$ such that, for any $j > N(x)$, $x \notin S_j$. We will denote the set of elements eventually in $\{S_i\}$ by $\{S_i\}_{ev}$ and the set of elements infinitely often in S_i by $\{S_i\}_{inf}$.

Lemma 4.5
Let $\{M_i\}$ be a minor sequence of matroids. Let $j_1 < j_2 < \ldots$ be an infinite increasing sequence of integers, and let C_i be a circuit in M_{j_i}. If the set $C = \{C_i\}_{inf}$ is non-empty, then it is dependent in $[\{M_i^*\}]^*$. Furthermore, C is a circuit of $[\{M_i^*\}]^*$ if and only if for every distinct pair $e, f \in C$ there is a co-circuit K of $[\{M_i^*\}]^*$ such that $C \cap K = \{e, f\}$.

Proof Suppose for contradiction that C is independent. Then C is contained in some basis B of $[\{M_i^*\}]^*$, and disjoint from $B' = S - B$, which is a basis of $[\{M_i^*\}]$. Since C is non-empty, we may choose some $e \in C$. Since B' is a basis, there is a circuit K of $[\{M_i^*\}]$ so that $e \in K \subset B' \cup e$. Since K is a circuit of $[\{M_i^*\}]$ it is finite and, since $K \subset B' \cup e$, $C \cap K = \{e\}$. Each of the finite number of elements of $K \backslash \{e\}$ is not in C, so may be in C_{j_i} for only finitely many i. Therefore there are infinitely many i such that $C_{j_i} \cap K = \{e\}$. Since K is a circuit of M_j^* for sufficiently large j, there is some i so that K is a co-circuit of M_{j_i} and $C_{j_i} \cap K = \{e\}$, a contradiction. So C is dependent.

If C is a circuit and $e, f \in C$ are distinct, there is a co-circuit K with $C \cap K = \{e, f\}$, by Lemma 3.9. Conversely, suppose the property holds. If C is not a circuit, then since we have already shown C to be dependent, there is a circuit C' strictly contained in C. Choose $e \in C'$, $f \in C \backslash C'$. There is a co-circuit K with $C \cap K = \{e, f\}$. However, this implies that $C' \cap K = \{e\}$, which is not possible. ■

Lemma 4.6
Let C be a circuit of $[\{M_i^*\}]^*$. There is an infinite increasing sequence of integers $j_1 < j_2 < \ldots$ and circuits C_i, such that C_i is a circuit in M_{j_i}, and $C = \{C_i\}_{inf} = \{C_i\}_{ev}$.

Proof Choose an arbitrary $e \in C$. Since e is contained in a circuit of $[\{M_i^*\}]^*$, it is not a loop of $[\{M_i^*\}]$. Therefore for some sufficiently large j_1, e is in S_{j_1}, but is not a co-loop of M_{j_1}. Hence e is contained in some circuit C_{j_1} of M_{j_1}.

Now assume we have chosen $j_1, \ldots, j_k$ and $C_1, \ldots, C_k$ for some $k \geq 1$. Let $E_k = (\cup_i C_i) \backslash C$, the set of elements not in C that have been used in some C_i, and let $F_k = (C \backslash e) \cap (\cup_i C_i)$, the set of elements of $C \backslash e$ that have been used in some C_i.

Consider the matroid $N_k = [\{M_i^*\}]^* \backslash E_k / F_k$. Clearly $C \backslash F_k$ is a circuit of N_k containing e. Furthermore, we have $[\{M_i^*\}]^* \backslash E_k / F_k = ([\{M_i^*\}] / E_k \backslash F_k)^* = [\{M_i^* / E_k \backslash F_k\}]^* = [\{(M_i \backslash E_k / F_k)^*\}]^*$, where the outer equalities are properties of duality and the inner equality holds by an earlier lemma because E_k and F_k are finite. Since e is contained in a circuit of N_k, it is not a loop of $N_k^* = [\{(M_i \backslash E_k / F_k)^*\}]$. Therefore for some sufficiently large j_{k+1}, $j_{k+1} > j_k$ and e is not a co-loop of $M_{j_{k+1}} \backslash E_k / F_k$. Hence e is contained in some circuit C'_{k+1} of $M_{j_{k+1}} \backslash E_k / F_k$, which is contained in some circuit C_{k+1} of $M_{j_{k+1}} \backslash E_k$.

Clearly we have integers $j_1 < j_2 < \ldots$ and circuits C_i, such that C_i is a circuit in M_{j_i}, and $\{e \mid e \in C_i$ for infinitely many $i\} \subset C$. By the above lemma, $\{C_i\}_{inf}$ is dependent, so since C is a circuit we must have $C = \{C_i\}_{inf}$.

Finally, suppose some $f \in C$ (distinct from e which we used in the construction and which is in every C_i) is not in $\{C_i\}_{ev}$. Then applying the above lemma to the infinitely many C_i avoiding f we find that $C \backslash f$ is dependent, contradicting the fact that C is a circuit. So every element in C is in all but finitely many of the C_i. ∎

4.3 Graph limits

Suppose that $\{G_i\}$ is a countable sequence of graphs such that $G_i \prec G_{i+1}$. Then we will say that $\{G_i\}$ is a *minor sequence* of graphs.

For simplicity, we will assume that G_1 is a single isolated vertex, and that G_i is obtained by deleting or contracting a single edge e of G_{i+1}. Then we may identify the edges and vertices of G_i with those of G_{i+1}, so that $E(G_{i+1}) = E(G_i) \cup \{e\}$ and there is a surjection $f_i : V(G_{i+1}) \to V(G_i)$, which is the constant function if e is deleted, and, if $e = xy$ is contracted, takes x and y to v_e and fixes every other vertex. Let f be the union of all of the functions f_i.

The limit graph, $[\{G_i\}]$ is obtained as follows. Its vertices are the sequences $\{v_i\}$ such that $v_i \in V(G_i)$ and $f_i(v_{i+1}) = v_i$. Its edges are $\cup_i E(G_i)$. An edge e is incident with a vertex $\{v_i\}$ if for every i such that $e \in E(G_i)$, v_i is incident with e.

Suppose that $e = x_j y_j$ is an edge of G_j. Then either $f^{-1}(x_j) = x_j$, or $f^{-1}(x_j) = \{x_j^1, x_j^2\}$ and, say, $e = x_j^1 y_j$ in G_{j+1}. In the first case, let $g_e(x_j) = x_j$, and in the second case let $g_e(x_j) = x_j^1$. The following is an immediate consequence of the relevant definitions.

Lemma 4.7
If $e = x_j y_j$ with $x_j \neq y_j$ in some G_j, then e is incident with exactly two vertices of $[\{G_i\}]$, namely $\{\ldots,\ f^2(x_j),\ f(x_j),\ x_j,\ g_e(x_j),\ g_e^2(x_j),\ldots\}$ and $\{\ldots, f^2(y_j),\ f(y_j),\ y_j,\ g_e(y_j),\ g_e^2(y_j),\ldots\}$. If e is a loop at x_j in G_j and is still a loop in every G_i with $i > j$, then it is a loop at $\{\ldots, f^2(x_j),\ f(x_j),\ x_j,\ g_e(x_j),\ g_e^2(x_j),\ldots\}$ in $[\{G_i\}]$.

Therefore $G = [\{G_i\}]$ is a well-defined infinite graph.

Example 4.5
For each non-negative integer i, we define H_i as follows. The vertices of H_i are $x_{-i}, \ldots, x_0, \ldots x_i$, $y_{-i}, \ldots, y_0, \ldots y_i$, and z. The edges of H_i consist of a path $x_{-i} \ldots x_0 \ldots x_i$; a path $y_{-i} \ldots y_0 \ldots y_i$; for each $-i \leq j \leq i$, an edge $x_j y_j$; and the edges zx_{-i}, zx_i, zy_{-i} and zy_i. For each positive integer i, H_{i-1} is obtained as a minor of H_i by deleting the edges $x_{-i}y_{-i}$ and $x_i y_i$, and contracting the four edges incident with z. The limit graph $[\{H_i\}]$ consists of two double rays $\ldots x_{-1}x_0x_1 \ldots$ and $\ldots y_{-1}y_0y_1 \ldots$, and rungs $x_j y_j$ for each integer j, forming a double ladder, along with an isolated vertex z.

Lemma 4.8
$\mathcal{M}([\{G_i\}]) = [\{\mathcal{M}(G_i)\}]$.

Proof Suppose that C is a circuit of $[\{\mathcal{M}(G_i)\}]$. Then by the definition of the finitary limit of a sequence of matroids, there is some $N(C)$ such that C is a circuit of $\mathcal{M}(G_i)$ for each $i > N(C)$, which is to say it is the edge set of a circuit of G_i for each $i > N(C)$. Let $j > N(C)$, and let v_j be a vertex of the circuit of G_j whose edge set is C. If C is not a loop, there are two edges $e, f \in C$ incident with v_j in G_j. Since C is also the edge set of a circuit of G_{j+1}, we must have $g_e(v_j) = g_f(v_j)$. Applying the same argument repeatedly shows that e and f are incident with a common vertex $\{\ldots, f^2(x_j), f(x_j), x_j, g_e(x_j), g_e^2(x_j), \ldots\}$ in $[\{G_i\}]$. Therefore, applying this argument to all possible choices of v_j, we see that C is the edge set of a circuit in $[\{G_i\}]$, and therefore C is a circuit of $\mathcal{M}([\{G_i\}])$.

Conversely, suppose that C is a circuit of $\mathcal{M}([\{G_i\}])$ and therefore the edge set of a circuit in $[\{G_i\}]$. Suppose that C is not a loop. Let $e = \{u_i\}\{v_i\}$ be an edge of C. Then $\{u_i\} \neq \{v_i\}$, so we may choose N_e so that, for $j > N_e$, e is an edge of G_j and $u_j \neq v_j$. For such j, e is an edge of G_j with endpoints u_j and v_j. Note that C will be the edge set of a circuit of G_j (and hence will be a circuit of $\mathcal{M}(G_j)$) for every j large enough that $j > N_e$ for every edge e of C. Therefore C is a circuit of $[\{\mathcal{M}(G_i)\}]$. ■

If v_j is a vertex of G_j and v_i a vertex of G_i such that $f^{j-i}(v_j) = v_i$, then call v_j a *forward image* of v_i in G_j, and v_i the *projection* of v_j in G_i. Similarly, for a vertex $\{v_i\}$ of G, $\{v_i\}$ is a forward image of v_i in the limit, and v_i is the projection of $\{v_i\}$ in G_i.

Let B be a finite set of edges of G. Then, for sufficiently large j, $B \subset E(G_j)$. Suppose that C is some component of $G_j - B$. If $G_j = G_{j+1} - e$, then obviously C is contained in some component $\alpha_j(C)$ of $G_{j+1} - B$. If $G_j = G_{j+1}/e$, then if the vertex in G_j that e contracts to, v_e, is not in C, then C is identical to a component $\alpha_j(C)$ of $G_{j+1} - B$, while if v_e is in C, then $C = \alpha_j(C)/e$ for the component $\alpha_j(C)$ of $G_{j+1} - B$ containing both ends of e.

Clearly α_j is a surjection, and $G_j - B$ has finitely many components, so for sufficiently large $k = k(B)$, α_k is a bijection. Furthermore, it is easy to check that if C is a component of $G_k - B$, then $C \prec \alpha_k(C)$. For each component C of G_k, the limit graph $[\{C, \alpha_k(C), \alpha_{k+1}(\alpha_k(C)), \ldots\}]$ is a *limit B-component*, $\alpha(C)$ of G. The limit B-components are not the components of $G - B$: for example, if every G_i is connected, then there is only one limit $\emptyset - component$, but G itself is not guaranteed to be connected.

Lemma 4.9
Let $\{G_i\}$ be a minor sequence of graphs, and let $G = [\{G_i\}]$. For any finite set B of edges of G, every vertex of G and every edge in $E(G) - B$ is contained in exactly one limit B-component.

Proof Let $k = k(B)$ be defined, as above, such that α_j is a bijection for every $j \geq k$. Let e be any edge in $E(G) - B$. Let C be the component of $G_k - B$ that contains e. Then e is contained in $\alpha(C)$, and in no other limit B-component. Let $\{v_i\}$ be a vertex of G. Let C_k be the component of $G_k - B$ that contains v_k. It is easy to see that, for every $j \geq k$, if v_j is in the component C_j of $G_j - B$, then v_{j+1} is in $\alpha_j(C_j)$. So $\{v_i\}$ is also contained in exactly one limit B-component, $\alpha(C_k)$. ■

Lemma 4.10
Let $\{G_i\}$ be a minor sequence of graphs, and let $G = [\{G_i\}]$. Let B_1 and B_2 be finite subsets of edges of G. If C_1 is a limit B_1-component and C_2 a limit B_2-component, then $C_1 \cap C_2$ is a union of limit $(B_1 \cup B_2)$-components.

Proof It suffices to show that every limit $(B_1 \cup B_2)$-component is contained in $C_1 \cap C_2$ or disjoint from it. Choose k to exceed $k(B_1), k(B_2)$ and $k(B_1 \cup B_2)$. Then, for any $j \geq k$, the B_1, B_2 and $(B_1 \cup B_2)$-components of G_j correspond bijectively with the limit components. Let C be a limit $(B_1 \cup B_2)$-component. Then $C = \alpha(C_k)$ for some $(B_1 \cup B_2)$-component C_k of G_k. Some B_1-component C_k^1 of G_k contains C_k, and therefore $C = \alpha(C_k) \subseteq \alpha(C_k^1)$. If $\alpha(C_k^1)$ is C_1, then $C \subseteq C_1$, while if $\alpha(C_k^1)$ is some other limit B_1-component, then $C \cap C_1 = \emptyset$. Similarly, $C \subseteq C_2$ or $C \cap C_2 = \emptyset$. Therefore, C is either contained in $C_1 \cap C_2$ or disjoint from it, as required. ■

4.4 Graph-like limits

We now move to the more difficult task of constructing a graph-like continuum from a minor sequence of graphs, in a way that is consistent with the matroid limit.

Let $\{G_i\}$ be a minor sequence of graphs. The finite edge cut topology (FECT) on $\{G_i\}$ will be denoted $|\{G_i\}|$ and is defined as follows. The ground set is $V(G) \cup E(G)$ where $G = [\{G_i\}]$. A basis for the topology is given by sets of the form $\{e\}$, for any $e \in E(G)$, and $C \cup B$, for any finite set $B \subseteq E(G)$ and limit B-component C.

Example 4.6
For each non-negative integer i, let H_i be as defined in Example 4.5. Let B be any finite set of edges of $[\{H_i\}]$. Let j be sufficiently large that $B \subseteq E(H_j)$. The limit B-component C_z that contains z also contains every forward image of the projection of z in H_j. It follows that if we choose any infinite set of vertices in $|\{H_i\}|$, it meets every basic open neighbourhood of

z. Therefore every ray in $|\{H_i\}|$ converges to z, and $|\{H_i\}|$ is the Alexandroff compactification of the double ladder.

Lemma 4.11
$|\{G_i\}|$ is compact.

Proof By Alexander's Lemma ([7]), it suffices to show that any basic cover has a finite subcover. Equivalently, every collection of complements of basic open sets that has the finite intersection property has non-empty intersection.

Let $\mathcal{C}$ be a collection of complements of basic open sets that has the finite intersection property. There are countably many finite sets of edges each with finitely many limit components, so there are countably many basic open sets, and therefore we may enumerate $\mathcal{C} = \{C_1, C_2, \ldots\}$. Let B_i be the finite set of edges determining the basic open set that is the complement of C_i. By definition, C_i is a union of limit B_i-components.

We will show that, for each $i \geq 1$, we can choose $\{C'_1, \ldots C'_i\}$ such that:

1. for each $1 \leq j \leq i$, $C'_j \subset C_j$;
2. for each $1 \leq j \leq i-1$, $C'_{j+1} \subset C'_j$; and
3. $\{C'_1, \ldots C'_i, C_{i+1}, C_{i+2}, \ldots\}$ has the finite intersection property.

Let $C_1 = C_1^1 \cup \ldots \cup C_1^k$ such that, for each $1 \leq j \leq k$, C_1^j is a limit B_1-component. For each finite subset $\mathcal{C}'$ of $\{C_2, C_3, \ldots\}$, let $J(\mathcal{C}') = \{j \mid 1 \leq j \leq k$ and the intersection of C_1^j and all of the elements of $\mathcal{C}'$ is non-empty$\}$. Suppose that no j, $1 \leq j \leq k$, is in $J(\mathcal{C}')$ for every $\mathcal{C}'$. Then, for each j, $1 \leq j \leq k$, there is some $\mathcal{C}^j$ such that $j \notin J(\mathcal{C}^j)$. However, in that case, the intersection of C_1 with all of the elements in any of the $\mathcal{C}^j$, for all j, $1 \leq j \leq k$, must be empty, which is impossible because $\mathcal{C}$ has the finite intersection property. Therefore there is some j such that $j \in J(\mathcal{C}')$ for every choice of $\mathcal{C}'$, which implies that $\{C_1^j, C_2, C_3, \ldots\}$ has the finite intersection property. Let $C'_1 = C_1^j$.

Now suppose we have chosen $\{C'_1, \ldots C'_{i-1}\}$ as required. By Lemma 4.10, $C'_1 \cap \ldots \cap C'_{i-1} \cap C_i$ is a union of limit $B_1 \cup \ldots \cup B_{i-1} \cup B_i$-components. Let $C'_1 \cap \ldots \cap C'_{i-1} \cap C_i = C_i^1 \cup \ldots \cup C_i^k$ such that, for each $1 \leq j \leq k$, C_1^j is a limit $B_1 \cup \ldots \cup B_{i-1} \cup B_i$-component. Since, by the inductive assumption, $\{C'_1, \ldots C'_{i-1}, C_i, C_{i+1}, \ldots\}$ has the finite intersection property, we may proceed exactly as in the base case to find a j such that $\{C'_1, \ldots C'_{i-1}, C_i^j, C_{i+1}, \ldots\}$ still has the finite intersection property. Let $C'_i = C_i^j$.

Therefore, by induction, we can obtain $\{C'_1, C'_2 \ldots\}$ so that, for $i \geq 1$, $C'_{i+1} \subset C'_i$, $C'_i \subset C_i$, and each C'_i is a limit component of some finite set of edges. Since $C'_i \subset C_i$, it suffices to show that this collection has non-empty intersection.

For $v \in V(G_i)$, let $\mathcal{P}$ be the property "has a forward image in C'_j for every (or, equivalently, infinitely many) $j > i$". Clearly v_1, the single vertex of G_1, has this property. Given $v_k \in G_k$ with property $\mathcal{P}$, since v_k has only one or two forward images in G_{k+1}, we may choose one of them, v_{k+1}, to also have property $\mathcal{P}$. Therefore we obtain a vertex $\{v_i\}$ of G such that each of its projections has property $\mathcal{P}$.

Let B'_j be such that C'_j is a limit B'_j-component. Choose i sufficiently large that $B'_j \subset E(G_i)$ and the B'_j-components of G_i are in bijection with the limit B'_j-components. Every vertex in G_i has every forward image in the same limit B'_j-component, and therefore either has every forward image in C'_j or no forward image in C'_j. Since v_i has property $\mathcal{P}$, all of its forward images are in C'_j, including $\{v_i\}$. So $\{v_i\}$ is in every C'_j, and therefore $\{C'_1, C'_2 \ldots\}$ has non-empty intersection, as required. ∎

Lemma 4.12
Let $\{G_i\}$ be a minor sequence of graphs, and let $G = [\{G_i\}]$. Two vertices $u, v \in V(G)$ are in the same limit B-component for each finite set of edges B if and only if they are in the same component of $V(G)$ (considered as a subspace of $|\{G_i\}|$ with the finite edge cut topology).

Proof If u and v are in the same limit B-component for each finite set of edges B, then every basic open set either contains both of u and v or neither. Let C_u be the component of $V(G)$ containing u. Then C_u is closed in the subspace topology on $V(G)$, so there is an open set V such that $V \cap V(G) = V(G) \backslash C_u$. Since V is an open set not containing u, it does not contain v, so $v \in C_u$, as required.

Conversely, if u and v are in different limit B-components, let C_u be the limit B-component containing u and C' be the union of all the other (finitely many) limit B-components. Then $C_u \cup B$ (basic) and $C' \cup B$ (finite union of basic open sets) are each open, so their intersections with $V(G)$ are a separation with u in one part and v in the other. ∎

Our next goal is to define a graph-like continuum $||\{G_i\}||$ based on $|\{G_i\}|$. We will start by replacing the singleton edges with open intervals to obtain $|\{G_i\}|'$. For each edge e of G, let I_e be an open unit interval, and let $\mathcal{E}$ be the union of all the I_e. The point set of $|\{G_i\}|'$ is $V(G) \cup \mathcal{E}$. If $e = uv$, then the closure $\bar{I}_e$ of I_e will be $I_e \cup \{u, v\}$, where u and v are arbitrarily assigned to the endpoints of the unit interval. If e is a loop at u, then the closure $\bar{I}_e$ of I_e will be the circle $I_e \cup \{u\}$.

A basis for the topology on $G = |\{G_i\}|$ is obtained as follows:

1. for each $e \in E(G)$, every open subinterval of I_e is in the basis; and

2. for each finite subset B of $E(G)$ and each limit B-component C, for each $e \in B$, let u_e be the end of e in C. The set consisting of C together with, for each $e \in B$, a half-open subinterval of $\bar{I}_e$ containing u_e is in the basis.

We do not yet have a graph-like continuum because, whenever $u, v \in V(G)$ are in the same limit B-component for each finite set of edges B, there is no open set of $|\{G_i\}|$ that contains u but not v. The space $||\{G_i\}||$ is obtained from $|\{G_i\}|'$ by identifying all such pairs of points.

Lemma 4.13
Provided it is connected, $||\{G_i\}||$ is a graph-like continuum.

Proof We need to show that $||\{G_i\}||$ is compact (it is obviously Hausdorff).

Let $\mathcal{O}$ be a basic open cover of $||\{G_i\}||$. For any open set O of $||\{G_i\}||$, let $V'(O)$ be the set of vertices of G whose image in $||\{G_i\}||$ is in O, and let $E'(O)$ be the set of edges of G that have non-empty intersection with O. Then we can form an open cover of $|\{G_i\}|$ by replacing every $O \in \mathcal{O}$ with $f(O) = V'(O) \cup E'(O)$. Since $|\{G_i\}|$ is compact, this cover has a finite subcover $\mathcal{F}$. Let $\mathcal{O}'$ consist of all $O \in \mathcal{O}$ such that $f(O) \in \mathcal{F}$. Note that $\mathcal{O}'$ must cover all of the vertices of $||\{G_i\}||$. Note that each basic open set of $||\{G_i\}||$ only contains finitely many vertices without containing the whole of all of their incident edges. Since $\mathcal{O}'$ consists of finitely many basic open sets and covers all of the vertices, it must fail to cover only finitely many edges. However, the part of each edge not covered by $\mathcal{O}'$ is closed, and hence the set of points not covered by $\mathcal{O}'$ is compact. Therefore, finitely many more elements of $\mathcal{O}$ suffice to form a finite subcover. ■

The final theorem of this section shows that the cofinitary matroid limit is consistent with the graph-like limit. This result is used in the following section to show that co-finitary B-matroids whose finite minors are graphic are the circuit matroids of graph-like continua.

Theorem 4.14
Let $\{G_i\}$, $i \in \mathbb{N}$ be a sequence of connected finite graphs such that, for each $i \geq 0$, $G_i \prec G_{i+1}$. The two B-matroids $\mathcal{M}(||\{G_i\}||)$ and $[\{M(G_i)^*\}]^*$ are the same.

Proof It suffices to show that the co-circuits of the two B-matroids are the same. We claim that both are equal to the sets B for which there exists an integer $N(B)$ such that, for every $i > N(B)$, B is a bond of G_i. That these are the co-circuits of $[\{M(G_i)^*\}]^*$ follows directly from Lemma 4.1.

Suppose that B is eventually a bond in G_i. Then there are two limit B-components, C_1 and C_2, and $C_1 \cup B$, $C_2 \cup B$ are each open sets covering $|\{G_i\}|$. They extend in the obvious way to open sets covering $||\{G_i\}||$ and intersecting only in the interior of the edges of B. Therefore, B is an edge cut in $||\{G_i\}||$. It remains to show that every edge cut in $||\{G_i\}||$ contains some eventual bond of G_i.

Now suppose that U, V are open subsets of $||\{G_i\}||$ that intersect only in some set of edges F. Since $||\{G_i\}||$ is a graph-like continuum, F is finite. Suppose that F does not contain any set of edges that is eventually a bond of G_i. Then there is only one limit F-component. Let $\{u_i\} \in U$, $\{v_i\} \in V$. Then there is some N_{uv} such that, for any $i > N_{uv}$, F is contained in $E(G_i)$ and $G_i - F$ contains a path P_i between u_i and v_i. We can choose these paths so that, for each $i > N_{uv}$, $P_i \prec P_{i+1}$. Let $P = |\{P_i\}|$, and let P' be the subspace of $||\{G_i\}||$ consisting of the vertices and edges of P. Note that P' is closed, because it is the union of the closures of its edges - no vertex of P' has a basic open neighbourhood avoiding all of the edges of P'. Therefore $A = P \cap U$, $B = P \cap V$ is a partition of P into two open sets.

Let e_A be any edge in A and e_B be any edge in B. Choose i so that P_i contains e_A and e_B. Then P_i contains a vertex v_i incident with an edge e_A^i of A and an edge e_B^i of B. For any $j > i$, there is a forward image of v_i in P_j incident with e_A^i and a forward image of v_i in P_j incident with e_B^i. Therefore (because the forward images of v_i in P_j form a path) there is a forward image of v_i in P_j, such that v_j is incident with an edge e_A^j of A and an edge e_B^j of B. Therefore there is a vertex of P, $\{v_i\}$, such that infinitely often v_i is incident with edges of both A and B. Therefore every basic open set containing $\{v_i\}$ contains edges in both A and B, a contradiction because $\{v_i\}$ must belong to one of the disjoint open sets A and B. ■

4.5 Whitney's Theorem for B-matroids

Since infinite graphs have finite circuits but infinite bonds, they are not naturally closed under duality. Thomassen [25] solved this problem by ignoring the infinite bonds and saying that H is a dual of G if there is a bijection between their edge sets such that a finite set of edges is a circuit in G if and only if it is a bond in H. If this condition obtains, then we will say that H is a *Thomassen dual* of G. This definition is not symmetric in G and H (G is said to be a pre-dual of H). Thomassen showed that an infinite graph has a Thomassen dual if and only if it is planar and finitely separable.

Bruhn and Diestel [3] restored symmetry, by saying that G is a dual of H if there is a bijection between their edge sets such that a set of edges is a circuit in $\widetilde{G}$ if and only if it is a bond in H (the symmetry is not obvious from the definition). We will say that such G and H are *Bruhn-Diestel* duals. Since a Bruhn-Diestel dual is also a Thomassen dual, their theory has the same restriction to finitely separable graphs. They showed that a finitely separable graph has a dual (also finitely separable) if and only if it is planar.

Notice that infinite graphs have finite circuits but infinite bonds, while graph-like continua have infinite circuits but finite bonds. If we allow general graph-like continua as our duals, there is no need for the restriction to finitely separable graphs. Let G be a graph and H be a graph-like continuum. We will say that G and H are dual if there is a bijection between their edge sets such that a set of edges is a circuit in G if and only if it is a bond in H. In other words, $M(G) = M^*(H)$.

For finite graphs, Whitney's Theorem follows immediately from Kuratowski's Theorem and Tutte's characterization of graphic matroids, which together imply that a graphic matroid is also co-graphic if and only if it has no Kuratowski minor. We will prove a version of Whitney's theorem that refers to Kuratowski minors, rather than directly to the planarity of the graph (graph-like continuum) G, but is equivalent to the usual form of Whitney's Theorem by a result of Richter, Rooney and Thomassen [22]. We first need to generalize Tutte's characterization to B-matroids.

Theorem 4.15
Let M be a countable B-matroid. Then M is the circuit matroid of a graph G if and only if M is finitary and every finite minor of M is graphic.

Proof Suppose that M is finitary and every finite minor of M is graphic. By Lemma 4.3, we can write $M = [\{M_i\}]$ where M_i is a finite minor of M with i elements and $M_i = M_{i+1} \backslash e_{i+1}$. Suppose that the ground set of each M_i is $\{1, \ldots, i\}$. Let $\mathcal{G}_i$ be the set of graphs G_i with unlabelled vertices, no isolated vertices, and edges labelled $\{1, \ldots, i\}$ such that the (labelled) circuits of G_i are exactly the circuits of M_i. Then each $\mathcal{G}_i$ is finite and non-empty and, for any $G_{i+1} \in \mathcal{G}_{i+1}$, we have $G_{i+1} \backslash e_{i+1} \in \mathcal{G}_i$. Therefore, König's Infinity Lemma implies that we may choose a sequence $\{G_i\}$ so that, for every i, $G_i = G_{i+1} \backslash e_{i+1}$. Then $G = [\{G_i\}]$ is an infinite graph and, by Lemma 4.8, $\mathcal{M}(G) = [\{\mathcal{M}(G_i)\}] = M$ so M is the circuit matroid of a graph, as required.

Conversely, suppose that M is the circuit matroid of a graph. Then M is finitary and, by Lemma 3.35, every finite minor of M is graphic. ■

Theorem 4.16
Let M be a countable B-matroid. Then M is the circuit matroid of a graph-like continuum G if and only if M is co-finitary and every finite minor of M is graphic.

Proof Suppose that M is co-finitary and every finite minor of M is graphic. Then M^* is finitary, so by Lemma 4.3, we can write $M^* = [\{M_i^*\}]$ where M_i^* is a finite minor of M^* with i elements and $M_i^* = M_{i+1}^* \backslash e_{i+1}$. Then each $M_i = (M_i^*)^*$ is a finite minor of M. Suppose that the ground set of each M_i is $\{1, \ldots, i\}$. Let $\mathcal{G}_i$ be the set of graphs G_i with unlabelled vertices, no isolated vertices, and edges labelled $\{1, \ldots, i\}$ such that the (labelled) circuits of G_i are exactly the circuits of M_i. Then each $\mathcal{G}_i$ is finite and non-empty, and for any $G_{i+1} \in \mathcal{G}_{i+1}$ we have $G_{i+1}/e_{i+1} \in \mathcal{G}_i$. Therefore, König's Infinity Lemma implies that we may choose a sequence $\{G_i\}$ so that, for every i, $G_i = G_{i+1}/e_{i+1}$. Then $G = ||\{G_i\}||$ is a graph-like continuum and, by Theorem 4.14, $\mathcal{M}(G) = [\{\mathcal{M}(G_i)^*\}]^* = [\{M_i^*\}]^* = M$, so M is the circuit matroid of a graph-like continuum, as required.

Conversely, suppose that M is the circuit matroid of a graph-like continuum. Then M is co-finitary and, by Lemma 3.35, every finite minor of M is graphic. ■

Theorem 4.17
A graph G with countably many edges has no K_5 or $K_{3,3}$ minor if and only if G is dual to a graph-like continuum H. A graph-like continuum G with

countably many edges has no K_5 or $K_{3,3}$ minor if and only if G is dual to a graph H.

Proof Let G be a graph with countably many edges. Then Theorem 4.15 implies that $\mathcal{M}(G)$ is a finitary B-matroid with no Tutte minor. It follows $\mathcal{M}^*(G)$ is a co-finitary B-matroid with no Tutte minor if and only if G (and hence $\mathcal{M}(G)$) has no Kuratowski minor. By Theorem 4.16, there is a graph-like continuum H with $\mathcal{M}(H) = \mathcal{M}^*(G)$ exactly when $\mathcal{M}^*(G)$ is a co-finitary B-matroid with no Tutte minor, completing the proof. The proof of the second statement is identical. ■

We now discuss how this relates to Thomassen and Bruhn-Diestel duality.

Let M be a B-matroid. The set of finite circuits of M determines a new B-matroid, *fin*(M). Similarly, the set of finite co-circuits of M determines a new B-matroid, *cofin*(M).

Lemma 4.18
If N is a finite minor of fin(M) or cofin(M) it is also a minor of M.

Proof Let $N = \text{fin}(M)/C \backslash D$, and consider a dependent set J in N. Let B_C be a basis for C in fin(M). Then by the definition of contraction, $J \cup B_C$ is dependent in fin(M) and therefore there is a finite circuit $C(J)$ of M, $C(J) \subseteq J \cup B_C$. Let C' be the union of $C(J) \cap B_C$ as J ranges over all of the dependent sets of N.

We claim that $N = M/C' \backslash (D \cup (C \backslash C'))$. Firstly, let J be a dependent set in N. Then $C(J) \subseteq C'$, so J is also dependent in $M/C' \backslash (D \cup (C \backslash C'))$. Now let J be a dependent set in $M/C' \backslash (D \cup (C \backslash C'))$, and let $B_{C'}$ be a basis for C' in M. Then $J \cup B_{C'}$ contains a circuit of M and therefore, since both J and C' are finite, $J \cup B_{C'}$ contains a circuit of fin(M). It follows, because $B_{C'} \subseteq C' \subseteq B_C$, that $J \cup B_C$ contains a circuit of fin(M). Therefore, J is dependent in N.

Therefore N is also a minor of M as required. The other part of the lemma follows immediately by duality, since cofin$(M) = (\text{fin}(M^*))^*$. ■

Suppose that $\mathcal{G}$ is a graph-like continuum. The *underlying graph* G of $\mathcal{G}$ has the same vertices and edges as $\mathcal{G}$, and an edge e is incident with a vertex v in G if, in $\mathcal{G}$, the closure of e contains v. In other words, G has the same incidence structure as $\mathcal{G}$, but no information about where rays of $\mathcal{G}$ converge.

Lemma 4.19
Let G be a graph. The following are equivalent:

1. G is finitely separable;
2. G is the underlying graph of a graph-like continuum; and
3. $\mathcal{M}(\widetilde{G}) = \mathrm{cofin}(\mathcal{M}(G))$ and $\mathcal{M}(G) = \mathrm{fin}(\mathcal{M}(\widetilde{G}))$.

Proof We will first show that the first two statements are equivalent, and then that the first statement is equivalent to the third. If G is finitely separable, then $\widetilde{G}$ is a graph-like continuum and, by construction, has G as its underlying graph. Conversely, let x, y be two vertices in the underlying graph G of a graph-like continuum $\mathcal{G}$. Then x and y are separated by a bond K in $\mathcal{G}$ (this follows from the fact that $\mathcal{G}$ is Hausdorff), and therefore they are also separated by K in G. Since graph-like continua have only finite bonds, the underlying graph of a graph-like continuum is always finitely separable.

Now suppose that G is finitely separable. Then, since G is the underlying graph of $\widetilde{G}$, the finite circuits of $\widetilde{G}$ are exactly the circuits of G. A finite set of edges separates $\widetilde{G}$ if and only if it separates G, so $\mathcal{M}(\widetilde{G}) = \mathrm{cofin}(\mathcal{M}(G))$.

Conversely, suppose that both $\mathcal{M}(\widetilde{G}) = \mathrm{cofin}(\mathcal{M}(G))$ and $\mathcal{M}(G) = \mathrm{fin}(\mathcal{M}(\widetilde{G}))$. Let u and v be any two vertices in the same component of G, and let P be any path in G between u and v. Since $\mathcal{M}(G) = \mathrm{fin}(\mathcal{M}(\widetilde{G}))$, and P is independent in $\mathcal{M}(G)$, it must also be independent in $\mathrm{fin}(\mathcal{M}(\widetilde{G}))$. Since P is a finite independent set in $\mathrm{fin}(\mathcal{M}(\widetilde{G}))$, it is also an independent set in $\mathcal{M}(\widetilde{G})$. Therefore, for any edge e of P, there is a co-circuit K of $\mathcal{M}(\widetilde{G})$ that meets P exactly in e. Since $\mathcal{M}(\widetilde{G}) = \mathrm{cofin}(\mathcal{M}(G))$, K is also a finite co-circuit of $\mathcal{M}(G)$. Therefore K is a bond of G that meets P exactly in e, from which it follows that K is a finite set of edges separating u from v. ■

Lemma 4.19 does not imply that all graph-like continua are of the form $\widetilde{G}$ for some finitely separable G. For example, the Alexandroff compactification of the double ladder is a graph-like continuum, $\mathcal{G}$. Its underlying graph G is the double ladder, which is finitely separable, and both of the statements $\mathcal{M}(\widetilde{G}) = \mathrm{cofin}(\mathcal{M}(G))$ and $\mathcal{M}(G) = \mathrm{fin}(\mathcal{M}(\widetilde{G}))$ hold. However, $\mathcal{G} \neq \widetilde{G}$ ($\widetilde{G}$ is the Freudenthal compactification), they just have the same finite circuits.

Recall that H is a Thomassen dual of G if there is a bijection between their edge sets so that the circuits of G are the finite bonds of H. In other words, H is a Thomassen dual of G if $\mathcal{M}(G) = \mathrm{fin}(\mathcal{M}^*(H))$. We can use our version of

Whitney's Theorem to prove Thomassen's version, although we will assume the following result from [25].

Lemma 4.20
(Thomassen [25]) A graph that is not finitely separable has no Thomassen dual.

Theorem 4.21
(Thomassen [25]) G has a Thomassen dual if and only if it is planar and finitely separable.

Proof Firstly, suppose G is planar and finitely separable. Since $\mathcal{M}(G)$ has no Kuratowski minor, Lemma 4.18 implies that $\mathrm{cofin}(\mathcal{M}(G))$ has no Kuratowski minor. Therefore, by Lemma 4.19, $\mathcal{M}(\widetilde{G})$ has no Kuratowski minor. Therefore, by Theorem 4.17, there is a graph H with $\mathcal{M}(\widetilde{G}) = \mathcal{M}^*(H)$. By Lemma 4.19 again, $\mathcal{M}(G) = \mathrm{fin}(\mathcal{M}(\widetilde{G})) = \mathrm{fin}(\mathcal{M}^*(H))$, so H is a Thomassen dual of G.

Conversely, suppose that $\mathcal{M}(G) = \mathrm{fin}(\mathcal{M}^*(H))$ for some graph H. Equivalently, by taking duals on both sides, $\mathcal{M}^*(G) = \mathrm{cofin}(\mathcal{M}(H))$. Since $\mathcal{M}(H)$ has no Tutte minor, Lemma 4.18 implies that $\mathrm{cofin}(\mathcal{M}(H))$ does not have a Tutte minor. Therefore, by Lemma 4.16, $\mathrm{cofin}(\mathcal{M}(H))$ is the circuit matroid of a graph-like continuum, $\mathcal{H}$. Thus we have $\mathcal{M}^*(G) = \mathcal{M}(\mathcal{H})$ which shows, by Theorem 4.17, that G is planar. Lemma 4.21 implies that G is also finitely separable. ■

Let G be a finitely separable graph. Recall that H is Bruhn-Diestel dual to G if there is a bijection between their edge sets so that the circuits of $\widetilde{G}$ are exactly the bonds of H. In other words, H is a Bruhn-Diestel dual of G if $\mathcal{M}(\widetilde{G}) = \mathcal{M}^*(H)$.

Lemma 4.22
Let G be a finitely separable graph. Then H is a Bruhn-Diestel dual of G if and only if G is Thomassen dual to H.

Proof By definition, G is Thomassen dual to H if $\mathcal{M}(H) = \mathrm{fin}(\mathcal{M}^*(G))$ or, equivalently, $\mathcal{M}^*(H) = \mathrm{cofin}(\mathcal{M}(G))$. On the other hand, H is Bruhn-Diestel dual to

G if $\mathcal{M}^*(H) = \mathcal{M}(\widetilde{G})$. By Lemma 4.19, since G is finitely separable, $\mathcal{M}(\widetilde{G}) = \text{cofin}(\mathcal{M}(G))$, so the result is immediate. ■

Theorem 4.23
(Bruhn, Diestel [3]) Suppose G is a countable finitely separable graph. Then G has a Bruhn-Diestel dual if and only if it is planar. Furthermore, the dual graph H is also finitely separable, and G is a Bruhn-Diestel dual of H.

Proof First suppose that G does have a Bruhn-Diestel dual H. Then Lemmas 4.22 and 4.21 imply that H is planar and finitely separable. Applying Lemma 4.19 to H, we have $\mathcal{M}(\widetilde{H}) = \text{cofin}(\mathcal{M}(H)) = (\text{fin}(\mathcal{M}^*(H)))^*$. Since $\mathcal{M}(\widetilde{G}) = \mathcal{M}^*(H)$, it follows by applying Lemma 4.19 to G that $\mathcal{M}(\widetilde{H}) = (\text{fin}(\mathcal{M}(\widetilde{G}))^* = \mathcal{M}^*(G)$, and therefore that G is also a Bruhn-Diestel dual of H. Now Lemmas 4.22 and 4.21 imply that G is also planar.

Conversely, suppose that G is planar. Since $\mathcal{M}(G)$ has no Kuratowski minor, Lemma 4.18 implies that $\text{cofin}(\mathcal{M}(G))$ has no Kuratowski minor. Therefore, by Lemma 4.19, $\mathcal{M}(\widetilde{G})$ has no Kuratowski minor. Now Theorem 4.17 implies that there is a graph H with $\mathcal{M}(\widetilde{G}) = \mathcal{M}^*(H)$. Evidently H is a Bruhn-Diestel dual of G, as required. ■

We proved Theorem 4.17 using the fact that a B-matroid with no Tutte minor also has no Kuratowski minor if and only if its dual has no Tutte minor. This is true for any B-matroids with no Tutte minor, not just finitary or co-finitary ones. It might be hoped that there is a characterization of the B-matroids with no Tutte minor as being the circuit matroids of some class of objects including both graphs and graph-like continua. In that case we ought to be able to show that objects in this class are planar if and only if they have abstract duals from the same class. The final result of the chapter shows that, at least, B-matroids with no Tutte minor are related to graphs and graph-like continua.

Lemma 4.24
If M is a countable B-matroid such that $M \in \text{ex}(\mathcal{T})$, there is a minor sequence $\{G_i\}$ of finite graphs, such that $\text{fin}(M) = \mathcal{M}([\{G_i\}])$ and $\text{cofin}(M) = \mathcal{M}(||\{G_i\}||)$.

Proof Let C be a basis and D be the complementary co-basis of M. Let $\{e_1, e_2, \ldots\}$ be an enumeration of the elements of M, and for each $i \geq 1$, let $C_i = C \cap \{e_1, \ldots e_i\}$ and $D_i = D \cap \{e_1, \ldots e_i\}$. For each $i \geq 1$, let $M_i = M/(C \backslash C_i) \backslash (D \backslash D_i)$. Since M has no Tutte minor, there exists, for each i, a finite graph G_i such that $M_i = \mathcal{M}(G_i)$. Note that Lemma 4.4 does not really require M to be finitary – it simply shows that $\mathrm{fin}(M) = [\{M_i\}]$. Therefore we may apply it to the M_i to show that $\mathrm{fin}(M) = [\{M_i\}]$, and apply it to the $M_i^* = M^*/(D \backslash D_i) \backslash (C \backslash C_i)$ to show that $\mathrm{fin}(M^*) = [\{M_i^*\}]$ and therefore that $\mathrm{cofin}(M) = [\{M_i^*\}]^*$. Finally, Lemma 4.8 and Theorem 4.14 imply that $\mathrm{fin}(M) = \mathcal{M}([\{G_i\}])$ and $\mathrm{cofin}(M) = \mathcal{M}(||\{G_i\}||)$, as required. ■

Chapter 5

Peripheral circuits

5.1 Two theorems of Tutte and their generalizations

Tutte proved the following well-known theorems about the peripheral circuits of a finite graph (recall that a peripheral circuit in a connected finite graph G is an induced circuit C so that $G - C$ remains connected).

Theorem 5.1
(Tutte [29]) If e is an edge of a 3-connected graph G, there are two peripheral circuits of G whose intersection is exactly e and the endpoints of e.

Theorem 5.2
(Tutte [29]) The peripheral circuits of a 3-connected graph generate its cycle space.

If M is a matroid with ground set S, and $T \subseteq S$, then a *bridge* B of T is a component of M/T. A *peripheral circuit* of M is a circuit C such that M/C is connected.

Bixby and Cunningham generalized Tutte's theorems to 3-connected binary matroids. Note that a verbatim generalization of the Theorem 5.1 to matroids is not possible - for example every pair of circuits in F_7^* meet in two elements.

Theorem 5.3
(Bixby, Cunningham [1]) Let e and f be elements of a finite 3-connected binary matroid M. There is a peripheral circuit of M containing e but not f.

Theorem 5.4
(Bixby, Cunningham [1]) The peripheral circuits of a 3-connected binary matroid generate its cycle space.

Let $M = (S, \mathcal{I})$ be a B-matroid, and let $T \subseteq S$. A *T-bridge* is a component of M/T. A *peripheral circuit* is a circuit with exactly one bridge. The *cycle space* $\mathcal{C}(M)$ of M is the space generated by *thin sums* of circuits of M. A thin sum is allowed to have infinitely many summands, but each element of S can appear only finitely often in these summands.

In this chapter we show that the results of Bixby and Cunningham are also true for 3-connected binary B-matroids, provided that they are countable and co-finitary. The co-finitary assumption is necessary, since the theorems fail for 3-connected infinite graphs. Our proofs require countability (in particular it is crucial for our proof of Theorem 5.13), but we have no examples to show that the theorems fail in the uncountable case.

Let $\mathcal{G}$ be a graph-like continuum. The cycle space $\mathcal{C}(\mathcal{G})$ is the space generated by *thin sums* of circuits of $\mathcal{G}$. A consequence of our theorems about 3-connected co-finitary binary B-matroids is that Tutte's theorems generalize to 3-connected graph-like continua. Previously, Bruhn and Stein generalized Theorem 5.1 to compactifications of (some) 3-connected infinite graphs in [5], and Bruhn showed that Theorem 5.2 holds for Freudenthal compactifications of 3-connected locally finite graphs in [2].

5.2 Circuits and bridges in co-finitary binary B-matroids

This section is concerned with the basic properties of circuits and bridges in binary co-finitary B-matroids. The proof of the following lemma uses the same argument used by Vella and Richter in [30] to prove the analogous result for edge spaces.

Lemma 5.5
If $M = (S, \mathcal{I})$ is a co-finitary binary B-matroid, and $X \subseteq S$ has even intersection with every co-circuit of M, then X is a disjoint union of circuits of M.

Proof Let $\mathcal{S}$ be the set of all sets of disjoint circuits contained in X, ordered by set inclusion. We will use Zorn's Lemma to show that $\mathcal{S}$ has a maximal element, and then show that any maximal element must be a set of disjoint circuits whose union is all of X. Let $S_1 \subseteq S_2 \subseteq \ldots$ be an increasing sequence of elements of $\mathcal{S}$, and let $S = \cup_i S_i$. Since any two circuits in S are contained in some S_i together, S is also a set of disjoint circuits contained in X, so $S \in \mathcal{S}$. By Zorn's Lemma, $\mathcal{S}$ has a maximal element, $\mathcal{C}$.

Let X' be the set of elements of X not contained in any member of $\mathcal{C}$. By maximality of $\mathcal{C}$, X' is independent. Let K be any co-circuit of M. By Lemma 3.32, $K \cap X$ is even, as is $K \cap C$ for every $C \in \mathcal{C}$, and therefore $K \cap X'$ is also even. Every non-empty independent set has intersection of size 1 with some co-circuit, so X' must be empty, as required. ■

Lemma 5.6
If M is a binary co-finitary B-matroid, C is a subset of the ground set of $\mathcal{M}$, and K is a circuit, then $K \backslash C$ is a disjoint union of circuits of $\mathcal{M}/C$.

Proof By Lemma 3.32, K has even intersection with each co-circuit of M. Since the co-circuits of M/C are those co-circuits of M that are disjoint from C, $K \backslash C$ has even intersection with each co-circuit of M/C. By Lemma 5.5, $K \backslash C$ is a disjoint union of circuits of M/C. ■

Lemma 5.7
If M is a binary co-finitary B-matroid, and C_1, C_2 are disjoint unions of circuits of M, then $C_1 \Delta C_2$ is a disjoint union of circuits of M.

Proof Let K be a co-circuit of M. Then $K \cap C_1$ and $K \cap C_2$ are even, so $(C_1 \Delta C_2) \cap K$ is even. The result follows by Lemma 5.5. ■

Lemma 5.8
Let M be a co-finitary binary B-matroid with ground set S, $T \subseteq S$, and B a T-bridge. If $T' \subseteq S$ is disjoint from B, then B is contained in a T'-bridge.

Proof It suffices to show that any two distinct elements x and y of B are in a common circuit of M/T'. Certainly there is a circuit C of M/T with $x, y \in C$. It follows that there is some $U \subseteq T$ such that $C \cup U$ is a circuit of M. By Lemma 5.6, $(C \cup U)\backslash T'$ is a disjoint union of circuits of M/T', so we may choose $C_x \subseteq C \cup U$ to be a circuit of M/T' containing x. Now it follows that there is some $U' \subseteq T'$ such that $C_x \cup U'$ is a circuit of M. By Lemma 5.6 again, $(C_x \cup U')\backslash T$ is a disjoint union of circuits of M/T, so we may choose $C'_x \subseteq (C_x \cup U')$ to be a circuit of M/T containing x. Note that C'_x is contained in $C \cup T \cup T'$, so $C'_x \cap B \subseteq C$. However, since B is a T-bridge, any circuit of M/T that meets B must be contained in B, so $C'_x \subseteq C$, and therefore $C'_x = C$. This implies that $C \subseteq (C_x \cup U')$ and thus, since $C \cap U' = \emptyset$, $C \subseteq C_x$, which implies that in particular $y \in C_x$ and C_x is the required circuit of M/T'. ■

5.3 Overlapping bridges and an extension lemma

In this section, we prove an extension lemma that will be crucial in the proofs of our main theorems.

Suppose C is a circuit of a co-finitary binary B-matroid M. If B is a C-bridge, a *B-segment* is a series class consisting of elements of C in $M|_{C \cup B}$. The B-segments form a partition of C, which we will denote $\pi(C, B)$. Two C-bridges C_1 and C_2 *avoid* one another if $C = S_1 \cup S_2$, for some B_1-segment S_1 and some B_2-segment S_2. Bridges that do not avoid one another *overlap*.

If C is a circuit in a B-matroid, we may define its *overlap graph* $\mathcal{O}(C)$ as follows. The vertices of $\mathcal{O}(C)$ are the bridges of C, and two vertices are adjacent if they overlap.

Lemma 5.9
If M is a 3-connected binary co-finitary B-matroid, and C is a circuit of M, then $\mathcal{O}(C)$ is a connected graph.

Proof Suppose otherwise, and let $\mathcal{B}_1'$ be some component of $\mathcal{O}(C)$. Fix a bridge B_2 not in $\mathcal{B}_1'$. Consider two overlapping bridges $B_1, B_1' \in \mathcal{B}_1'$. They both avoid B_2. Suppose there exist B_1- and B_2-segments S_1, S_2 respectively so that $S_1 \cup S_2 = C$, and B_1'- and B_2-segments S_1', S_2' respectively so that $S_1' \cup S_2' = C$. If S_2' is not the same B_2-segment as S_2, then $S_2' \subseteq S_1$, so $S_1 \cup S_1' = C$, which is not possible because B_1 and B_1' overlap. Therefore there is a single B_2-segment S_2 so that every bridge $B_1 \in \mathcal{B}_1'$ has a segment $f(B_1)$ such that $f(B_1) \cup S_2 = C$.

Let A_2 be the intersection over every $B_1 \in \mathcal{B}_1'$ of $f(B_1)$, and let $A_1 = C \backslash A_2$. We claim that every C-bridge either has a segment that contains A_1, or has a segment that contains A_2. Let B be any C-bridge with no segment containing A_2. Clearly B cannot be in $\mathcal{B}_1'$ so, as with B_2 above, there is a fixed B-segment S so that, for each $B_1 \in \mathcal{B}_1'$, there is a B_1-segment S_1 such that $S_1 \cup S = C$. However, for any $B_1 \in \mathcal{B}_1'$, the only B_1-segment meeting A_2 is $f(B_1)$, so we must have $f(B_1) \cup S = C$ for every $B_1 \in \mathcal{B}_1'$. For every $x \in A_1$ there is some $f(B_1)$ avoiding x, so S contains all of A_1.

Let $\mathcal{B}_1$ be the set of C-bridges with a segment containing A_2, and let $\mathcal{B}_2$ be the remaining C-bridges (which all have a segment containing A_1). Let X be the union of A_1 and all of the bridges in $\mathcal{B}_1$ and let Y be the union of A_2 and all of the bridges in $\mathcal{B}_2$. Extend A_1 to a basis S_X of X and extend A_2 to a basis S_Y of Y. We will now show that $S_X \cup S_Y$ contains no circuit except for C ($C = A_1 \cup A_2$, $A_1 \subseteq S_X$ and $A_2 \subseteq S_Y$, so $C \subseteq S_X \cup S_Y$), and hence that (X, Y) is a 2-separation.

Suppose on the contrary that C' is a circuit other than C in $S_X \cup S_Y$. By Lemma 5.6, $C' \backslash C$ is a disjoint union of circuits of M/C. Let C'' be one of these circuits, and let B be the C-bridge that contains C''. There exists some $S \subseteq C$ so that $C'' \cup S$ is a circuit of M. Suppose that $B \in \mathcal{B}_1$ (the argument when $B \in \mathcal{B}_2$ is identical). Since $C'' \cup S$ is a circuit in $C \cup B$, by definition S is a union of B-segments. The only B-segment not contained in S_X is $f(B)$; since S_X is independent, we must have $A_2 \subseteq f(B) \subseteq S$. Therefore $(C'' \cup S) \Delta C$ is contained in $B \cup A_1$ and hence in S_X, but by Lemma 5.7 it is a disjoint union of circuits. Since $(C'' \cup S) \Delta C$ is non-empty, this is a contradiction. ■

If B is a C-bridge, a *primary arc* in B is a circuit of M/C that is contained in B and is not a circuit of M. If A is a primary arc, then a *primary segment* for A is a subset S of C such that $A \cup S$ is a circuit of M. A circuit composed of a primary arc in B and a primary segment is called a *primary circuit* through B.

Lemma 5.10
Let M be a 3-connected, binary, co-finitary B-matroid. Let C be a circuit and B a C-bridge in M. If S is a primary segment for a primary arc A in B, then:

1. $C \backslash S$ is a primary segment for A;
2. no other subset of C is a primary segment for A; and
3. exactly three subsets of $C \cup A$ are circuits: C, $A \cup S$ and $A \cup (C \backslash S)$.

Proof By Lemma 5.7, $(A \cup S)\Delta C = A \cup (C \backslash S)$ is a disjoint union of circuits. No circuit contained in $A \cup C$ can contain a proper, non-empty, subset of A, because A is a circuit of M/C. Therefore $A \cup (C \backslash S)$ is a circuit, as required.

Suppose that some subset T of C not equal to S or $C \backslash S$ is a primary segment for A. Then $(A \cup S)\Delta(A \cup T) = S \Delta T$ is a disjoint union of circuits by Lemma 5.7, which is a contradiction because $S \Delta T$ is a proper non-empty subset of C.

Finally, since A is a circuit of M/C, every circuit of $C \cup A$ contains all of A or none of it, and therefore there are no other circuits. ■

Lemma 5.11
Let M be a 3-connected, binary, co-finitary B-matroid. Let C be a circuit and B be a C-bridge in M. For two distinct B-segments S_1, S_2, there is a primary segment for some primary arc of B that contains S_1 but not S_2.

Proof Let $x \in S_1$ and $y \in S_2$. Let K be a co-circuit that meets C in exactly x and y. Let C_1 be a circuit of M in $C \cup B$ that contains S_1 but not S_2. Since K is finite and meets $C_1 \cap C$ exactly once, K must meet $C_1 \cap B$ an odd number of times. As $C_1 \backslash C = C_1 \cap B$ is a disjoint union of circuits of M/C, there is a primary arc A in B that meets K an odd number of times. Thus one of the primary segments for A must contain x and the other must contain y. ■

It follows that every B-segment is an intersection of primary segments for primary arcs of B.

Let M be a B-matroid, C be a circuit of M, and let B_1, B_2 be C-bridges. Then B_1 and B_2 are *skew* if, for a primary B_1-segment S and a primary B_2-segment T, all of $S \cap T$, $S \backslash T$, $T \backslash S$ and $C \backslash (S \cup T)$ are non-empty. Also, B_1 and B_2 are *k-equipartite* if they both partition C into the same k segments.

Our next goal is to show that overlapping bridges are either skew or 3-equipartite. This is a standard and easy result for finite graphs, and it was generalized to binary matroids by Tutte [28]. We do not know how to prove

Theorem 5.13 directly for B-matroids, so we use the techniques developed in the previous chapter to deduce it from Tutte's theorem on finite binary matroids.

Theorem 5.12
(Tutte [28]) Let M be a finite binary matroid, C a circuit of M and B_1, B_2 overlapping bridges of C. Then B_1 and B_2 are either skew or 3-equipartite.

Theorem 5.13
Let M be a countable cofinitary binary B-matroid, C a circuit of M and B_1, B_2 overlapping bridges of C. Then B_1 and B_2 are either skew or 3-equipartite.

Proof Let F be a basis of M so that $|C \backslash F| = 1$. Enumerate the elements of F as $F = \{e_1, e_2, \ldots\}$ and let F^i be the first i elements of F. Let M_i be the simplification of $M/(F \backslash F^i)$. Note that M_i has finite rank (F^i is a basis) so, by Lemma 3.31, M_i is finite. Since M is cofinitary, M^* is finitary, and we may apply Lemma 4.4 to M^* and $[\{M_i^*\}]$ to verify that $M = [\{M_i^*\}]^*$. Let C^i be the set of elements of C present in the ground set of M_i. Then C^i is a circuit of M_i, because C is the unique circuit in $C^i \cup (F \backslash F^i)$. Choose arbitrarily an element e^1 of B_1 and an element e^2 of B_2, and let B_1^i and B_2^i be the bridges of C^i containing e^1 and e^2 respectively. Notice that, because M_i/C^i is a minor of M/C, $B_1^i \neq B_2^i$.

If B_1^i and B_2^i avoid one another for some fixed i, then so will B_1^j and B_2^j for any $j < i$, because the bridges of C^j in M_j are minors of the bridges of C^i in M_i. Therefore, either B_1^i and B_2^i avoid one another for every i, or there is some N so that B_1^i and B_2^i overlap whenever $i > N$. Combining this observation with Theorem 5.12, it is apparent that at least one of the following occurs: there is some N so that, for $i > N$ both of B_1^i and B_2^i are 3-equipartite; B_1^i and B_2^i avoid one another for every i; there is some i for which B_1^i and B_2^i are skew. To complete the proof, we show that: B_1^i and B_2^i must overlap for some i (since B_1 and B_2 do); that if they are 3-equipartite for every sufficiently large i, then B_1 and B_2 are 3-equipartite; and finally, that if they are skew for some value of i, then B_1 and B_2 are skew.

Claim 1 Two elements x and y of C are in the same B_1-segment of C in M if and only if they are in the same B_1^i-segment of C^i in M_i, for sufficiently large i.

Proof of Claim 1. Since $C^i \cup B_1^i$ is a minor of $C \cup B_1$, it is clear that if x and y are in the same B_1-segment of C, then they are in the same B_1^i-segment of C^i, for every i large enough that they are elements of M_i. For the converse, suppose that x and y are in different B_1-segments of C. By Lemma 5.11, there is a primary segment for some primary arc A in B_1 that contains x but not y. Let K be a co-circuit of M that meets C exactly in $\{x, y\}$.

Note that, since K has even intersection with both primary circuits formed by A, it has odd intersection with A. Now choose a sufficiently large i that K is a co-circuit of M_i, and suppose for a contradiction that x and y are in the same B_1^i-segment of C^i. In that case, $\{x, y\}$ is a co-circuit of $B_1^i \cup C^i$. Thus, there is a co-circuit K' of M_i containing x, y and otherwise disjoint from $B_1^i \cup C^i$; clearly $K' \neq K$. Since M_i is a finite binary matroid, $K \Delta K'$ is a disjoint union of co-circuits. Note that $(K \Delta K') \cap C^i = \emptyset$. Since $K \cap A$ is non-empty and disjoint from K', one co-circuit of the disjoint union of co-circuits contains some $w \in K \cap A$, and, since $K \cap A$ is a proper subset of K, it must also contain some $z \notin B_1^i \cup C^i$. This implies that w and z are in the same component of M_i / C^i, contradicting the fact that w and z are in different C^i-bridges and completing the proof of Claim 1.

Claim 2 If B_1^i and B_2^i are k-equipartite for every sufficiently large i, then B_1 and B_2 are k-equipartite.

Proof of Claim 2. Suppose that B_1^i and B_2^i are k-equipartite for every $i > N$. Let $\{e_1, \ldots, e_k\}$ be elements of C^{N+1}, one from each of the k segments determined by both bridges. For $i > N$ and $j \in \{1, \ldots, k\}$, let S_j^i be the B_1^i-segment (and B_2^i-segment) of C^i containing e_j. For each $j \in \{1, \ldots, k\}$, let S_j be the union over $i > N$ of S_j^i. We claim that $\{S_1, \ldots, S_k\}$ are exactly the B_1-segments of C. It is apparent that $\{S_1, \ldots, S_k\}$ are pairwise disjoint and that every element of C is contained in some S_j. By Claim 1, to check that each S_j is a B_1-segment we need only show that, for a fixed j and increasing $i > N$, $\{S_j^i\}$ is increasing by containment. This follows from the fact that $C^i \cup B_1^i$ is a minor of $C^{i+1} \cup B_1^{i+1}$, but B_1^i has the same number of segments as B_1^{i+1}. Therefore $\{S_1, \ldots, S_k\}$ are indeed the B_1-segments of C, and by the same argument they are also the B_2-segments of C, establishing the claim.

Claim 3 There is some i so that B_1^i and B_2^i overlap.

Proof of Claim 3. Suppose for contradiction that B_1^i and B_2^i avoid one another for every i. Since B_1 and B_2 overlap, they are not 2-equipartite, so Claim 2 shows that there is some i for which B_1^i and B_2^i are not 2-equipartite; choose some such i. Let S_1^i and S_2^i be B_1^i- and B_2^i-segments respectively whose union is C^i. Now choose $j > i$ so that B_1^j and B_2^j are again not 2-equipartite, and choose S_1^j and S_2^j to be B_1^j- and B_2^j-segments respectively whose union is C^j. Since the restrictions of S_1^j and S_2^j to C^i are still B_1^i- and B_2^i-segments respectively, they are candidates to be S_1^i and S_2^i, and, since B_1^i and B_2^i are not 2-equipartite, they must actually be S_1^i and S_2^i. Repeating this process, we obtain a sequence $\{S_1^i, S_2^i\}$ where S_1^i and S_2^i are B_1^i- and B_2^i-segments respectively whose union is C^i, and, for $j > i$, S_1^i and S_2^i are the restrictions of S_1^j and S_2^j to C^i. By Claim 1, the union over all i of S_1^i is contained in a B_1-segment S_1, and the union over all i of S_2^i is contained in a B_2-segment S_2, so B_1 and B_2 avoid one another, the desired contradiction.

Claim 4 If B_1^i and B_2^i are skew for some value of i, then B_1 and B_2 are skew.

Proof of Claim 4. Suppose that B_1^i and B_2^i are skew, and let A_1^i, A_2^i be primary arcs through B_1^i and B_2^i so that the four intersections of a primary segment for A_1^i with a primary segment for A_2^i are all non-empty. Let S_1^i be a primary segment for A_1^i. Then $A_1^i \cup S_1^i$ is a circuit of M_i, and therefore there is a set X of elements of F so that $A_1^i \cup S_1^i \cup X$ is a circuit of M. By Lemma 5.6, $(A_1^i \cup S_1^i \cup X) \backslash C$ is a disjoint union of circuits of M/C. At least one of these, A_1, intersects A_1^i and is therefore contained in B_1 because B_1 is a component of M/C. By definition, A_1 is a primary arc in B_1.

Let S_1 be a primary segment with respect to A_1. By Lemma 5.6 again, the elements of $A_1 \cup S_1$ contained in the ground set of M_i are a disjoint union of circuits. Since these elements are contained in $C^i \cup B_1^i$ and contain an element of A_1^i, they must be exactly one of the primary circuits through A_1^i. The same argument applies with $C \backslash S_1$ in place of S_1, so each of the primary segments with respect to A_1 contains one of S_1^i and $C^i \backslash S_1^i$. Obtaining A_2 in the same way as A_1, we see that every intersection of primary segments for A_1 and A_2 contains an intersection of primary segments for A_1^i and A_2^i, and is thus non-empty, which proves the claim. ■

Now we can prove the required extension lemma.

Lemma 5.14
Let M be a countable 3-connected binary co-finitary B-matroid. Let C be a circuit of M that is not peripheral and B be a C-bridge. Either :

1. there exist circuits C_1, C_2 with $C_1 \Delta C_2 = C$, such that, for $i \in \{1, 2\}$, there is a C_i-bridge B_i that properly contains B; or

2. there exist circuits C_1, C_2, C_3 with $C_1 \Delta C_2 \Delta C_3 = C$, such that, for $i \in \{1,2,3\}$, there is a C_i-bridge B_i that properly contains B.

Proof Let C be a circuit and B be a C-bridge. By Lemma 5.9, if C is not already peripheral, then there is another C-bridge B' that overlaps B.

First suppose that we may choose B' skew to B. There exist primary arcs A and A', through B and B' respectively, so that none of the four intersections between a primary segment for A and a primary segment for A' is empty. Let S_1, S_2 be the primary segments for A'. We claim that $C_1 = A' \cup S_1$ and $C_2 = A' \cup S_2$ have the required properties. Certainly $C_1 \Delta C_2 = C$. So it remains to check that there is a C_1-bridge B_1 with $B \subset B_1$ (and, by the same argument, that there is a C_2-bridge B_2 with $B \subset B_2$).

First note that, by Lemma 5.8, B is contained in a single bridge, B_1, of C_1. We will first show that S_2 is a circuit of M/C_1, and then that there is a circuit in M/C_1 that meets both B and S_2, so that B_1 strictly contains B.

Let $x \in S_2$. By Lemma 5.6, S_2 is a disjoint union of circuits of M/C_1. Let C_x be a circuit of M/C_1 containing x, so $x \in C_x \subseteq S_2$. Then there is some $T \subseteq C_1$ such that $C_x \cup T$ is a circuit of M. By Lemma 5.10, $C_x \cup T$ is either C or C_2, and therefore $C_x = C_2$.

Finally, let the primary segment for A containing x be S. By Lemma 5.6, $(A \cup S) \backslash C_1$ is a disjoint union of circuits of M/C_1. Let C'_x be a circuit of M/C_1 containing x, so $x \in C'_x \subseteq (A \cup S) \backslash C_1$. Then there is some $T \subseteq C_1$ such that $C'_x \cup T$ is a circuit of M. By choice of A and A', $C'_x \cup T$ contains x but not all of C_1 and therefore, by Lemma 5.10, $C'_x \cup T$ cannot be contained in $C \cup A'$. Thus C'_x meets A, as required.

In the other case, no other C-bridge B' is skew to B. Since at least one other C-bridge overlaps B, this implies that C partitions into three B-segments, say S_1, S_2 and S_3. In fact, each of S_1, S_2 and S_3 must consist of a single element. Suppose otherwise that $x, y \in S_1$. Let $\mathcal{B}_1$ be the set of all C-bridges that have a segment containing $C \backslash S_1$, and let $\mathcal{B}_2$ be the set of all C-bridges that have a segment containing S_1. Since every C-bridge is either 3-equipartite with B or avoids it, $\mathcal{B}_1 \cup \mathcal{B}_2$ includes every C-bridge. Let X be the union of S_1 and all of the bridges in $\mathcal{B}_1$, and Y be the union of $S_2 \cup S_3$ and all of the bridges in $\mathcal{B}_2$. As in the proof of Lemma 5.9, we can show that (X, Y) is a 2-separation by extending S_1 to a basis B_X of X and S_2 to a basis B_Y of Y, and checking that no circuit exists in $B_X \cup B_Y$ except for C. This is a contradiction, so each segment has exactly one element.

Any primary arc for B will have S_i and $C \backslash S_i$ as its primary segments, for some

$i \in \{1,2,3\}$. By Lemma 5.11, at least two of S_1, S_2, S_3 must be primary segments for B. Let B' be a C-bridge 3-equipartite with B. Similarly, at least two of S_1, S_2, S_3 must be primary segments for B', so we can assume without loss of generality that S_1 is a primary segment for both B and B'. Let A' be a primary arc in B' with S_1 as one of its primary segments. Let $C' = A' \cup S_2 \cup S_3$, and $C_1 = A' \cup S_1$. As above, there is a C_1-bridge B_1 strictly containing B. By Lemma 5.8, there is a C'-bridge B' with $B \subseteq B'$. Since $A' \cup S_1$ is a circuit and $|S_1| = 1$, we have $|A'| > 1$, and therefore, $|C'| \geq 4$. This implies that either C' is already peripheral, in which case we have the first outcome of the lemma with $C_2 = C'$, or there is a C'-bridge B'' skew to B' and we may apply the first part of the argument to C' and B' to obtain C_2 and C_3 as required. ■

5.4 Main theorems for 3-connected binary co-finitary B-matroids

Our goal in this section is to prove the main theorems of the chapter. Our main tools will be Lemma 5.14 and the following result.

Lemma 5.15
Let $M = (S, \mathcal{I})$ be a countable binary co-finitary B-matroid. Let $\{C_i, B_i\}, i \in \mathbb{N}$, be a sequence in which each C_i is a disjoint union of circuits of M and each B_i is a C_i-bridge, such that $B_i \subset B_{i+1}$. Let $C_{ev} = \{C_i\}_{ev}$, and let $C_{inf} = \{C_{inf}\}$. For any finite set $X \subseteq C_{ev}$, there exists a disjoint union C_X of circuits of M, such that $X \subseteq C_X \subseteq C_{inf}$. Furthermore, C_X has a bridge B_X that properly contains every B_i.

Proof Suppose that Y is a maximal subset of X such that there exists C_Y with $Y \subseteq C_Y \subseteq C_{inf}$, and C_Y is a disjoint union of circuits of M. We will show that in fact $Y = X$. Suppose otherwise that $x \in X \backslash Y$. If there is a circuit C_x containing x and contained in $C_{inf} \backslash Y$, then $C_x \Delta C_Y$ is a disjoint union of circuits by Lemma 5.7, which contradicts the maximality of Y. If no such circuit C_x exists, then x is a co-loop in the restriction of M to $C_{inf} \backslash Y$. Therefore there is co-circuit K of M contained in $\{x\} \cup Y \cup (S \backslash C_{inf})$. Since $\{x\} \cup Y \subseteq C_{ev}$, we may choose N so that for every $i > N$, $\{x\} \cup Y \subseteq C_i$. Furthermore, since K is finite and every element of $S \backslash C_{inf}$ is eventually disjoint from C_i, we may choose some $i > N$ so that $C_i \cap K = \{x\} \cup Y$. However, $C_Y \cap K = Y$, and it follows that $(C_i \Delta C_Y) \cap K = \{x\}$, a contradiction to Lemma 5.7.

It remains to check that C_X has a bridge strictly containing every B_i. It suffices by Lemma 5.8 to check that $C_X \cap B_i = \emptyset$ for every i, which is clearly true because any element of B_i is in every B_j for $j > i$ and thus cannot be in C_{inf}. ■

Theorem 5.16
Let e and f be distinct elements of a countable 3-connected binary co-finitary B-matroid M. There is a peripheral circuit of M containing e but not f.

The proof of Theorem 5.16 will proceed by transfinite induction. We begin with a circuit C_1 containing e but not f and a C_1-bridge B_1 that contains f. For successor ordinals $\alpha + 1$, assume we have already defined (C_α, B_α) such that C_α is a circuit, $e \in C_\alpha$ and B_α is a C_α-bridge. Then if C_α is not already peripheral we will define $(C_{\alpha+1}, B_{\alpha+1})$ so that $C_{\alpha+1}$ is a circuit, $e \in C_{\alpha+1}$, and $B_{\alpha+1}$ is a $C_{\alpha+1}$-bridge such that $B_\alpha \subset B_{\alpha+1}$. For limit ordinals β, assume we have already defined (C_α, B_α) as above, for every $\alpha < \beta$. We will define (C_β, B_β) so that C_β is a circuit, $e \in C_\beta$, and B_β is a C_β-bridge such that $B_\alpha \subset B_\beta$, for every $\alpha < \beta$. Eventually the successor step must be impossible, and therefore we will have a peripheral circuit containing e but not f.

Proof Let S denote the ground set of M. We proceed by transfinite induction. For each ordinal β, if we have not yet found the desired peripheral circuit, we will define (C_β, B_β) such that C_β is a circuit, $e \in C_\beta$, B_β is a C_β-bridge, and $B_\beta \supset B_\alpha$ for every ordinal $\alpha < \beta$. It follows from Lemma 1.7 that it is impossible to have a strictly increasing (by inclusion) sequence $\{B_\alpha\}$ of elements of 2^S where α ranges over all of the countable ordinals, so the desired peripheral circuit must exist. Let C_1 be any circuit of M containing e but not f, and let B_1 be the C_1-bridge containing f.

Successor step. Let $\alpha = \beta - 1$. Assume we have already defined (C_α, B_α) such that C_α is a circuit, $e \in C_\alpha$ and B_α is a C_α-bridge. If C_α is not already peripheral, then we may apply Lemma 5.14 to C_α and B_α. Regardless of which case in the lemma holds, there is an i so that C_i contains e, and we may set (C_β, B_β) to be (C_i, B_i).

Limit step. Assume we have already defined (C_α, B_α) as above for every $\alpha < \beta$. Let $\{\alpha_i\}$ be a countable sequence of ordinals so that $\alpha_i < \beta$, for each i, and every $\alpha < \beta$ has $\alpha < \alpha_i$, for some i. Applying Lemma 5.15 to the sequence $\{C_{\alpha_i}, B_{\alpha_i}\}$ and noting that e is in every C_{α_i}, we obtain a circuit C_β with $e \in C_\beta$ and a C_β-bridge B_β that strictly contains every B_{α_i} and therefore strictly contains every B_α for $\alpha < \beta$. ■

The proof that the peripheral circuits generate the cycle space of a 3-connected binary co-finitary B-matroid is somewhat more complicated than the proof of Theorem 5.16. We will start with an arbitrary cycle space element $\mathcal{Z}$, and show that $\mathcal{Z}$ can be expressed as a thin sum of peripheral circuits. In order to do this, we will work with a fixed enumeration $\{e_1, e_2, \ldots\}$ of the elements of M. Starting with $\mathcal{Z}_0 = \mathcal{Z}$ and an arbitrary $\mathcal{Z}_0$-bridge B_0, we will define $(\mathcal{Z}_i, B_i)$ for each i so that B_i is a $\mathcal{Z}_i$-bridge, $\{e_1, \ldots, e_i\} \subseteq B_i$, and $\mathcal{Z}_i = \mathcal{Z}_{i-1} \Delta P_i^1 \Delta \ldots \Delta P_i^k$ where each P_i^j, $1 \leq j \leq k$, is a peripheral circuit disjoint from B_{i-1}. Since every e_i is contained in all B_l for $l \geq i$, it will be disjoint from all of the P_l^j with $l > i$. Therefore the set of all P_i^j is thin, and its sum is $\mathcal{Z}$.

To ensure that e_i is absorbed into B_i, we will choose a co-circuit K so that $e_i \in K$ and $K \cap B_{i-1} \neq \emptyset$. We prove a lemma showing that we can choose $P_i^1, \ldots, P_i^k$, disjoint from B_{i-1}, so that $(P_i^1 \Delta \ldots \Delta P_i^k) \cap K = \mathcal{Z}_{i-1} \cap K$, whence it is easy to check that we can set $\mathcal{Z}_i = \mathcal{Z}_{i-1} \Delta P_i^1 \Delta \ldots \Delta P_i^k$ and find a $\mathcal{Z}_i$-bridge containing $B_{i-1} \cup K$ for B_i.

The proof of the lemma uses a similar idea to the proof of Theorem 5.16, in the sense that we obtain peripheral circuits by starting from an original circuit (actually, a disjoint union of circuits) and modifying it so that a chosen bridge grows. However, since we need to obtain peripheral circuits with a fixed symmetric difference on the elements in K, a more complex approach is required. Instead of a linear chain of extensions, one for each ordinal, as in the proof of Theorem 5.16, we build at each ordinal an extension tree whose leaves are labelled with circuits (or disjoint unions of circuits) so that the symmetric difference of the circuits on all of the leaves is correct on K. We define a new extension tree at each ordinal, and show that there must be an ordinal where all of the leaves are peripheral circuits.

Our proof borrows heavily from Bruhn's proof of the same result for 3-connected locally finite graphs in [2]. In particular, where we *locally generate* our current cycle space element on a co-circuit K, Bruhn did the same thing with the finite set of edges incident with a chosen vertex. The idea of using an extension tree to obtain a locally generating set of peripheral circuits is also from Bruhn's argument.

Let M be a countable 3-connected co-finitary binary B-matroid with ground set S. Let K be a co-circuit of M. An *extension tree* with respect to K is a finite rooted tree, T, whose vertices are *lists* of subsets of K (that is, finite sequences of subsets of K), along with a label (C_L^T, B_L^T) for each vertex of T, satisfying the following conditions.

1. The root of T is $L_r = (K_0)$, a list of length one, where $K_0 \subseteq K$.

2. Let $L = (K_l, K_{l-1}, \ldots K_0)$ be a node of T. If L is not a leaf, then, for

some integer $k \geq 2$, L has k children, and they are of the form $L^i = (K^i_{l+1}, K_l, K_{l-1}, \ldots K_0)$, $1 \leq i \leq k$. The K^i_{l+1} are distinct and non-empty, their symmetric difference is K_l, and no proper subset has the symmetric difference K_l.

3. Let $L = (K_l, K_{l-1}, \ldots K_0)$ be a node of T. Then C^T_L is a disjoint union of circuits of M, $C^T_L \cap K = K_l$, and B^T_L is a C^T_L-bridge.

4. If L' lies on the path from L to L_r, then $B^T_{L'} \subset B^T_L$.

A *good extension tree* is an extension tree T with the additional property that, for every $L \in V(T)$, L contains no repeated elements. Therefore a good extension tree is finite. Let T be a good extension tree with root (K_0). Let $L_1, \ldots, L_r$ be the leaves of T, with the corresponding first terms of their lists being $K_1, K_2, \ldots, K_r$. Then it follows from the second property of extension trees that K_0 is the symmetric difference of $K_1, K_2, \ldots, K_r$. If C^T_L were a peripheral circuit for every leaf L of T, we would have a set of peripheral circuits whose symmetric difference was K_0. Notice also that every vertex L in T is a list of length $l+1$, where l is the length of the path from L to L_r, and that the path from L to L_r consists of all of the non-empty tails of L.

We define a partial order $\prec$ on the set of all good extension trees with respect to K. If T, T' are good extension trees with respect to K, let $T \prec T'$ if the following conditions hold:

1. for every $L \in V(T)$, either $L \in V(T')$ and $B^T_L \subseteq B^{T'}_L$ or there is some tail L' of L so that $L' \in V(T')$ and $B^T_{L'} \subset B^{T'}_{L'}$; and

2. there is some $L \in V(T)$ such that $L \in V(T')$ and $B^T_L \subset B^{T'}_L$, or $V(T') \backslash V(T) \neq \emptyset$.

Lemma 5.17
Every transfinite sequence $\{T_\alpha\}$, such that every T_α is a good extension tree with respect to K for every countable ordinal α, and $T_\alpha \preceq T_\beta$ for every pair of countable ordinals $\alpha < \beta$, is eventually constant.

Proof For convenience, for any list L and ordinal α, let $B^\alpha_L = B^{T_\alpha}_L$. Let l be the maximum length of a list of subsets of K. For each $i \leq l+1$, we will show that there exists an ordinal α_i such that, for every $\beta > \alpha_i$, L is a vertex of T_β whose path to the root is of length less than i if and only if it is a vertex of T_{α_i} whose path to the root is of length less than i, in which case $B_L^\beta = B_L^{\alpha_i}$.

Choose any ordinal α_0, and let (K_0) be the root of T_{α_0}. Since a list of length one has no non-empty tails, it must also be the root of every T_β for $\beta > \alpha_0$, by definition of $\prec$. Consider the set $\mathcal{O}'_0$ of ordinals $\beta > \alpha_0$ such that $B^\beta_{(K_0)}$ strictly contains $B^\gamma_{(K_0)}$ for every $\gamma < \beta$. Then $\{B^\beta_{(K_0)}\}$, $\beta \in \mathcal{O}'_0$, is a strictly increasing sequence of subsets of a countable set, which implies by Lemma 1.7 that $\mathcal{O}'_0$ is bounded above by a countable ordinal. Let α_1 be the successor of some upper bound for $\mathcal{O}'_0$. Notice that, by the definition of $\prec$ and the second property of extension trees, α_1 has the required properties.

Suppose that, for some $i \geq 1$, we have defined α_i. Let $L = (K_i, K_{i-1}, \ldots K_0)$ be any vertex in T_{α_i} whose path to the root is of length i. Since every non-empty tail of L satisfies $B_L^{\alpha_i} = B_L^\beta$ for every $\beta > \alpha_i$, L must also be in $V(T_\beta)$ for every $\beta > \alpha_i$, by definition of $\prec$. Now consider the set $\mathcal{O}'_i$ of countable ordinals $\beta > \alpha_i$ such that, for any vertex L in T_{α_i} whose path to the root is of length i, B_L^β strictly contains B_L^γ for every $\gamma < \beta$. There are only finitely many such vertices L, so again $\mathcal{O}'_i$ is bounded, and we may define α_{i+1}. Again, α_{i+1} has the required properties, by the definition of $\prec$ and the second property of extension trees.

By definition of α_{l+1}, $T_\beta = T_{\alpha_{l+1}}$ for every $\beta > \alpha_{l+1}$, so the sequence is constant, as required. ■

We are now ready to show that only a finite number of peripheral circuits are needed to “clear" a co-circuit.

Lemma 5.18
Let $\mathcal{Z}$ be a cycle space element, B a $\mathcal{Z}$-bridge and K a co-circuit. There is a set $P^1, \ldots, P^k$ of peripheral circuits disjoint from B so that $(P^1 \Delta \ldots \Delta P^k) \cap K = K \cap \mathcal{Z}$.

Proof We proceed by transfinite induction. For each countable ordinal β, if we have not yet found the desired set of peripheral circuits, we will define T_β to be a good extension tree with respect to K, with root $L_r^\beta = (\mathcal{Z} \cap K)$, such that $B \subseteq B_L^{T_\beta}$ for every $L \in V(T_\beta)$. We will define these trees so that for every $\alpha < \beta$, $T_\alpha \prec T_\beta$. Since, by Lemma 5.17, it is impossible to have a strictly increasing sequence $\{T_\alpha\}$ where α ranges over all of the countable ordinals, there is some β for which we cannot define T_β as described, and therefore the desired peripheral circuits must exist.

For convenience, for any list L and ordinal α, let $B^\alpha_L = B^{T_\alpha}_L$ and $C^\alpha_L = C^{T_\alpha}_L$. Let T_1 be a tree with one node, $L^1_r = (\mathcal{Z} \cap K)$, let $C^1_{L^1_r} = \mathcal{Z}$, and let $B^1_{L^1_r}$ be the bridge of $\mathcal{Z}$ containing B. This is a good extension tree with respect to K.

Successor step. If β is a successor ordinal, let $\alpha = \beta - 1$. Assume we have already defined T_α. If C^α_L is a peripheral circuit for every leaf L of T_α, we may set $P^1, \dots, P^k$ to be this set of peripheral circuits and we are done. Otherwise we may choose some leaf L, such that C^α_L is not a peripheral circuit.

We know that C^α_L is either a circuit that is not peripheral, or a disjoint union of circuits. If C^α_L is a non-peripheral circuit, apply Lemma 5.14 to C^α_L and B^α_L. Of the two or three circuits in the outcome of the lemma, one or more contains elements of K. If C^α_L is a disjoint union of circuits, partition it into circuits, and note that finitely many of these circuits contain elements of K. Either way, we obtain a finite set $C_1, \dots, C_\ell$ of circuits such that, for $1 \le i \le \ell$, C_i has a bridge B_i containing B^α_L.

If $\ell = 1$, then $C_1 \cap K = C^\alpha_L \cap K$ and we may set T_β to be the same tree with the same labels as T_α except for $C^\beta_L = C_1$ and $B^\beta_L = B_1$. Since its vertex set is the same, T_β is still a good extension tree, and since $B^\alpha_L \subset B^\beta_L$, $T_\alpha \prec T_\beta$. If $\ell > 1$, then, for each such i, let L_i be formed by adding $C_i \cap K$ as the first element of L. We may obtain a new extension tree T_β by starting with T_α and adding each L_i as a child of L, with labels $C^\beta_{L_i} = C_i$ and $B^\beta_{L_i} = B_i$. If no L_i has a repeated element, this is a good extension tree with $T_\alpha \prec T_\beta$. Otherwise, choose any $L_i = (C_i \cap K = K_t, K_{t-1}, \dots K_0)$, and $j \neq t$, such that $K_j = C_i \cap K$. Then $L' = (C_i \cap K = K_j, K_{j-1}, \dots K_0)$ is a vertex of T_β on the path between L_i and L^β_r. In this case, we obtain a good extension tree T_β by deleting every vertex apart from L' whose path to the root contains L', and setting $C^\beta_{L'} = C_i$, $B^\beta_{L'} = B_i$. Since T_α was an extension tree, $B^\alpha_{L'} \subset B^\alpha_L \subset B_i$, and we therefore have $T_\alpha \prec T_\beta$ as required.

Limit step. Assume we have already defined T_α for every $\alpha < \beta$. Let $\{\alpha_i\}$, $i \in \mathbb{N}$, be a sequence of ordinals so that $\alpha_i < \beta$ for each i, and every $\alpha < \beta$ has $\alpha < \alpha_i$ for some i. Let l be the maximum length of a list of subsets of K. For each $0 \le i \le l + 1$, we will either obtain a T_β such that $T_\alpha \prec T_\beta$ for every $\alpha < \beta$, as required, or we will define j_i such that, for every $k > j_i$, L is a vertex of T_{α_k} whose path to the root is of length less than i if and only if it is a vertex of $T_{\alpha_{j_i}}$ whose path to the root is of length less than i, in which case $B^{\alpha_k}_L = B^{\alpha_{j_i}}_L$.

Recall that $(\mathcal{Z} \cap K)$ is the root of every T_α. Consider the set N_1 of integers i such that $B^{\alpha_i}_{(\mathcal{Z}\cap K)}$ strictly contains $B^{\alpha_{i-1}}_{(\mathcal{Z}\cap K)}$. If N_1 is infinite, let T_β have root $(\mathcal{Z} \cap K)$ and no other vertices, and apply Lemma 5.15 to the sequence $(C^{\alpha_i}_{(\mathcal{Z}\cap K)}, B^{\alpha_i}_{(\mathcal{Z}\cap K)})$, and the set $\mathcal{Z} \cap K$, to get $(C^{\beta}_{(\mathcal{Z}\cap K)}, B^{\beta}_{(\mathcal{Z}\cap K)})$. Then because $B^{\beta}_{(\mathcal{Z}\cap K)}$ strictly contains every $B^{\alpha_i}_{(\mathcal{Z}\cap K)}$ and hence every $B^{\alpha}_{(\mathcal{Z}\cap K)}$, and $(\mathcal{Z} \cap K)$ is a tail of every vertex of every T_α, we have $T_\alpha \prec T_\beta$ for every $\alpha < \beta$, as required. If instead N_1 is finite, let j_1 be larger than any element of N_1.

Suppose that, for some $i \geq 1$, we have defined j_i as required. Let $L = (K_i, K_{i-1}, \ldots K_0)$ be any vertex in $T_{\alpha_{j_i}}$ whose path to the root is of length i. Since every non-empty tail of L satisfies $B^{\alpha_{j_i}}_L = B^{\alpha_k}_L$ for every $k > j_i$, L must also be in $V(T_{\alpha_k})$ for every $k > j_i$. Now consider the set N_{i+1} of integers $k > j_i$ such that, for any vertex L in $T_{\alpha_{j_i}}$ whose path to the root is of length i, $B^{\alpha_k}_L$ strictly contains $B^{\alpha_{k-1}}_L$.

If N_{i+1} is infinite, let the vertices of T_β be the vertices of $T_{\alpha_{j_i}}$ whose path to the root is of length at most i. For vertices L whose path to the root is of length less than i, let $B^{\beta}_L = B^{\alpha_{j_i}}_L$ and $C^{\beta}_L = C^{\alpha_{j_i}}_L$. For vertices L whose path to the root is of length i, one possibility is that $B^{\alpha_k}_L$ is eventually constant for all $k \geq N(L)$. In that case, let $C^{\beta}_L = C^{\alpha_{N(L)}}_L$ and $B^{\beta}_L = B^{\alpha_{N(L)}}_L$. The other possibility (which occurs for at least one L, because N_{k+1} is infinite) is that $B^{\alpha_k}_L$ is not eventually constant, and then we may apply Lemma 5.15 to the sequence $(C^{\alpha_k}_L, B^{\alpha_k}_L)$ and the set K_L (the first term of L) to get $(C^{\beta}_L, B^{\beta}_L)$. Then because, for every such L, B^{β}_L strictly contains every $B^{\alpha_k}_L$ and hence every B^{α}_L, we have $T_\alpha \prec T_\beta$ for every $\alpha < \beta$, as required. If instead N_{i+1} is finite, let j_{i+1} be larger than any element of N_{i+1}.

Suppose that j_{l+1} exists, then $T_{\alpha_k} = T_{\alpha_{j_{l+1}}}$ for every $k > j_{l+1}$, contradicting the fact that our sequence is strictly increasing. Therefore j_{l+1} does not exist, and therefore there is some $i \leq l+1$ for which we fail to define j_i and instead find T_β as required. ■

Theorem 5.19
The peripheral circuits of a countable 3-connected binary co-finitary B-matroid generate its cycle space.

Proof Let $\mathcal{Z}$ be an arbitrary element of the cycle space of M. We will show that $\mathcal{Z}$ can be expressed as a thin sum of peripheral circuits. Let $\{e_1, e_2, \ldots\}$ be a fixed enumeration of the elements of M. Starting with $\mathcal{Z}_0 = \mathcal{Z}$ and an arbitrary $\mathcal{Z}_0$-bridge B_0, we will define $(\mathcal{Z}_i, B_i)$ for each i so that B_i is a $\mathcal{Z}_i$-bridge, $\{e_1, \ldots, e_i\} \subseteq B_i$, and $\mathcal{Z}_i = \mathcal{Z}_{i-1} \Delta P^1_i \Delta \ldots \Delta P^k_i$ where each P^j_i, $1 \leq j \leq k$, is a peripheral circuit disjoint from B_{i-1}. Since e_i is contained in all B_l, for $l \geq i$, it will be disjoint from all of the P^j_ℓ with $\ell > i$. Therefore the set of all P^j_i is thin, and its sum is $\mathcal{Z}$.

Let K be a co-circuit so that $e_i \in K$ and $K \cap B_{i-1} \neq \emptyset$. Lemma 5.18 implies that there exist peripheral circuits $P_i^1, \ldots, P_i^k$, disjoint from B_{i-1}, so that $(P_i^1 \Delta \ldots \Delta P_i^k) \cap K = \mathcal{Z}_{i-1} \cap K$. Let $\mathcal{Z}_i = \mathcal{Z}_{i-1} \Delta P_i^1 \Delta \ldots \Delta P_i^k$. Since $\mathcal{Z}_i$ is disjoint from B_{i-1}, Lemma 5.8 implies that there exists a $\mathcal{Z}_i$-bridge B_i containing B_{i-1}. Finally, observe that K is a co-circuit of $M/\mathcal{Z}_i$ and therefore, since $(K \cap B_i) \neq \emptyset$, we have $K \subseteq B_i$ and therefore $e_i \in B_i$ as required. ■

5.5 Peripheral circuits in 3-connected graph-like continua

In this section we will show that our results for 3-connected co-finitary binary B-matroids imply generalizations of Tutte's theorems to graph-like continua.

Theorem 5.20
If e is an edge of a 3-connected graph-like continuum $\mathcal{G}$, there are two peripheral circuits of G whose intersection is exactly e.

Proof Let $\mathcal{G}$ be a 3-connected graph-like continuum. Then, by Lemma 3.34 and an easy argument, there is a 3-connected co-finitary binary B-matroid $\mathcal{M}(\mathcal{G})$ with the same circuits and, therefore, the same cycle space as $\mathcal{G}$. Therefore, it is only necessary to show that peripheral circuits in $\mathcal{M}(\mathcal{G})$ are peripheral circuits of $\mathcal{G}$. Suppose that C is a peripheral circuit of $\mathcal{M}(\mathcal{G})$. By definition, $\mathcal{M}(\mathcal{G})/C$ is connected, and by Lemma 3.35 $\mathcal{M}(\mathcal{G})/C = \mathcal{M}(\mathcal{G}/C)$, so the latter is also connected. Therefore each pair of edges of $\mathcal{G}/C$ is in a common cycle, which implies that $\mathcal{G}/C$ has no cutpoints and therefore that, if c is the point to which C is contracted, there is a path between any two points $x, y \in (\mathcal{G}/C) - \{c\}$ that avoids c. Such a path is also a path between x and y in $\mathcal{G} \backslash C$, and thus C is a peripheral circuit of $\mathcal{G}$.

Notice that our proof of Theorem 5.16 implies that if $e = uv$ is an element of a countable 3-connected co-finitary binary B-matroid M, C is a circuit of M with $e \in C$ and B is a C-bridge, there is a peripheral circuit of M containing e and disjoint from B. Let $\mathcal{G}$ be a 3-connected graph-like continuum. Let e be an edge of $\mathcal{G}$, and apply Theorem 5.16 to obtain one peripheral circuit C_1 through e. Let N_u be a neighbourhood of u avoiding v, and N_v be a neighbourhood of v avoiding u. Since $\mathcal{G}$ is 3-connected, N_u contains either part of an edge P_u between u and w_u, where $w_u \notin V(C_1)$, or a vertex $w_u \notin V(C_1)$ and an arc P_u between u and w_u that avoids C_1. Define w_v and P_v in a similar way. Since C_1 is peripheral, w_u and w_v are in the interior of the same C_1-bridge, and there is therefore an arc P_{uv} between them that avoids C. The union of P_u, P_v and P_{uv} contains an arc between u and v that avoids C_1 which, along with e, forms a circuit C_2. Now, since $C_1 - e$ is connected in $\mathcal{G} - C_2$, it is all contained in one bridge B of C_2. By the form of Theorem 5.16 mentioned above, we may now start with C_2 to obtain a second peripheral circuit that meets C_1 in exactly e and its endpoints. ■

Theorem 5.21
The peripheral circuits of a 3-connected graph-like continuum generate its cycle space.

Proof Let $\mathcal{G}$ be a 3-connected graph-like continuum. By Theorem 5.19, the peripheral circuits of $\mathcal{M}(\mathcal{G})$ generate its cycle space. As observed at the beginning of the proof of Theorem 5.20, the peripheral circuits of $\mathcal{M}(\mathcal{G})$ are peripheral circuits of $\mathcal{G}$. By Lemma 3.34, the circuits of $\mathcal{G}$ and the circuits of $\mathcal{M}(\mathcal{G})$ are the same and therefore by definition $\mathcal{G}$ and $\mathcal{M}(\mathcal{G})$ have the same cycle space. It follows that the peripheral circuits of $\mathcal{G}$ generate its cycle space, as required. ■

Chapter 6

Embedding metric spaces in surfaces

6.1 Introduction

In this chapter we give a characterization, for each surface Σ, of the locally connected, compact metric spaces that have an embedding in Σ. The characterization when Σ is the sphere is a result of Richter, Rooney and Thomassen [22]. We show that the general case follows from their result.

We will say that a topological space M *contains* another topological space N if there is an embedding $f : N \to M$. The following is a result of Claytor [8] from 1934, independently rediscovered in 2004 by Thomassen [26].

Theorem 6.1
(Claytor [8], Thomassen [26]) A 2-connected, locally connected, compact topological space M has an embedding in the sphere if and only if M is metrizable and does not contain K_5 or $K_{3,3}$.

The reason that 2-connection is required in the theorem is as follows. Consider the space T formed by the closed unit disc D along with an arc A, such that A is disjoint from D except at one of its endpoints, which is the center of D. The *thumbtack space* T is locally connected, compact and metric, but does not embed in the sphere. Indeed, it does not embed in any surface, because the point at the center of D has no planar neighbourhood.

Let M be a compact, locally connected metric space, and let N be any subspace of M. An N*-bridge* of M is the closure B of a component of $M \setminus N$. The *attachments* of B are the elements of $B \cap N$, and the *residual arcs* of B are the closures in N of the components of $N \setminus B$. If C is a circuit, then two distinct C-bridges, B_1 and B_2, overlap if they are skew or 3-equivalent.

In [22], Richter, Rooney and Thomassen define a family of spaces that resemble the thumbtack space sufficiently closely that they also fail to embed in any surface. These spaces are the *generalized thumbtacks*.

A generalized thumbtack consists of a *web with centre w* and a line segment, one end of which is w, and which is otherwise disjoint from the web. A web W with centre w is a closed connected space in which there is a sequence of disjoint circles $C_0, C_1, \ldots$ satisfying the following two properties:

1. for each $i > 0$, there are two overlapping C_i-bridges in W, one containing $C_0, \ldots, C_{i-1}$ and the other containing $C_{i+1}, C_{i+2}, \ldots$;

2. every neighbourhood of w contains all but finitely many of the C_i.

After explicitly excluding generalized thumbtacks, Claytor's and Thomassen's result can be extended to spaces that are not 2-connected and, after further excluding the disjoint union of the sphere and a point, it extends to all compact, locally connected metric spaces, whether they are connected or not.

Theorem 6.2
(Richter, Rooney, Thomassen [22]) Let M be a compact, locally connected metric space that does not contain a generalized thumbtack or the disjoint union of a sphere and a single point. Then either M embeds in the sphere, or M contains K_5 or $K_{3,3}$.

They conjecture that a similar result holds for higher surfaces, and indeed it does. The main result of this chapter is that, for a fixed surface Σ, if M is a compact, locally connected metric space that does not contain: a generalized thumbtack; the disjoint union of Σ and a point; any surface of lower Euler genus than Σ or any member of Forb(Σ), then M embeds in Σ.

An outline of the proof is as follows. Let Σ be a surface. Let M satisfy the hypotheses of the theorem, and let $\mathcal{G}(M)$ be the set of all finite graphs contained in M. If some $G \in \mathcal{G}(M)$ does not embed in Σ, M contains some $G \in \text{Forb}(\Sigma)$ and we are done. Otherwise, if Σ is orientable, let Σ' be the lowest genus orientable surface in which every $G \in \mathcal{G}(M)$ embeds. If Σ is non-orientable, let Σ' be the lowest Euler genus surface, orientable or not, in which every $G \in \mathcal{G}(M)$ embeds. We will show that in fact M embeds in Σ', and that this implies that M also embeds in Σ (we use here the fact that M does not contain Σ'), completing the proof.

Let $H \in \mathcal{G}(M)$ be a graph that embeds in Σ' but not in any surface of lower Euler genus that was a candidate to be Σ'. Then every embedding of H in Σ' is cellular ([32], see Section 1.2.3). Furthermore, there are only finitely many embedding schemes that may describe the embedding of H in Σ'. Recall from Section 1.2.3 that embeddings with the same embedding scheme are equivalent up to homeomorphism by a result of Ringel [23], so, for a fixed embedding scheme, either every embedding of H in Σ' realizing that embedding scheme extends to an embedding of M or no embedding of H in Σ' realizing that embedding scheme extends to an embedding of M.

We proceed by proving two key lemmas. The first of these says that if H is contained in M, then either a fixed embedding of H in a surface can be extended to an embedding of M in the same surface, or there is a finite graph G containing H and contained in M such that the embedding of H cannot even be extended to an embedding of G. The second says that if we can find embeddings of several different finite graphs $G_1, \ldots G_k$ in M, all of which (in some sense) use the same copy of H in M, then we can find an embedding of a finite graph G that contains all of $G_1, \ldots G_k$ as subgraphs. The result follows from these two lemmas by trying to extend every possible embedding of H.

6.2 Combining finite graphs

Lemma 6.3
Let M be a locally connected, compact metric space, let H be a finite graph, and let $f : H \to M$ be an embedding. Let $G_1, G_2, \ldots G_k$ be finite graphs. For each i, $1 \leq i \leq k$, let H_i be a subgraph of G_i that is a subdivision of H and let $f_i : G_i \to M$ be an embedding such that $f_i|_{H_i} = f$. Then:

1. there is a finite graph G such that, for each i, G contains a subgraph G_i' isomorphic to a subdivision of G_i;
2. the intersection in G of all of the subgraphs G_i' contains a subdivision H' of H;
3. there is an embedding $f' : G \to M$ such that $f'|_{H'} = f$; and

4. for an H'-bridge B in G, $f'(B) \subseteq B_M$ for some $f(H)$-bridge B_M in M. For each $f(H)$-bridge B_M in M there is at most one H'-bridge B in G such that $f'(B) \subseteq B_M$.

Proof We will prove the existence of G and f' satisfying the first three claims of the lemma by induction on k. These claims are:

1. There is a finite graph G such that, for each i, G contains a subgraph G_i' isomorphic to a subdivision of G_i;
2. The intersection in G of all of the subgraphs G_i' contains a subdivision H' of H; and
3. There is an embedding $f' : G \to M$, such that $f'|_{H'} = f$.

First, suppose that $k = 1$. Then $G = G_1$ and $f' = f_1$ satisfy the claims.

Now suppose that the first three claims of the lemma hold for any smaller value of k. Apply this induction hypothesis to $G_1, \ldots G_{k-1}$. This yields a finite graph $G^{(k)}$ such that, for each i, $1 \leq i \leq k-1$, $G^{(k)}$ contains a subgraph $G_i^{(k)}$ isomorphic to a subdivision of G_i. The intersection in $G^{(k)}$ of all of the $G_i^{(k)}$ contains a subdivision $H^{(k)}$ of H, and there is an embedding $f^{(k)} : G^{(k)} \to M$ such that $f^{(k)}|_{H^{(k)}} = f$.

We will define a graph G'. The vertices of G' are all of the vertices of $G^{(k)}$, all of the vertices of G_k, and an additional vertex for each point in the intersection of $f_k(E(G_k) \setminus E(H_k))$ with $f^{(k)}(V(G^{(k)}))$. The edges of G' are all of the edges of $G^{(k)}$, along with a path P_e for each edge $e \in E(G_k) \setminus E(H_k)$. For each $e = uv$, the path P_e is a uv-path, and its interior vertices are the points of $f_k(e) \cap f^{(k)}(V(G^{(k)}))$, in the order in which they are encountered on the arc $f_k(e)$ between u and v. All of these paths form a subdivision G_k'' of G_k. Notice that f_k also describes an embedding f_k' of G_k'' in M.

It is easy to check that G' satisfies the first two conditions in the induction hypothesis, but the embeddings of G_k'' and $G^{(k)}$ cannot immediately be combined, because they may intersect in interior points of edges. Since f_k' and $f^{(k)}$ are embeddings, for each pair of edges $e \in E(G_k'')$ and $f \in E(G^{(k)})$, there are open sets U_e and U_f of M such that U_e does not meet $f_k'(G_k'')$ except in $f_k'(e)$ and U_f does not meet $f^{(k)}(G^{(k)})$ except in $f^{(k)}(f)$. Therefore $U_e \cap U_f$ is an open set of M that only meets the embeddings in $f_k'(e) \cup f^{(k)}(f)$.

Since M is compact and locally connected, $U_e \cap U_f$ has only finitely many components. Let K be one of these components. If K is not disjoint from $f_k'(e) \cap$

$f^{(k)}(f)$, we consider the intersection of K with the closed arc $f_k'(\bar{e})$. Arbitrarily labelling the (images under f_k' of the) endpoints of e to be the beginning and the end of the closed arc $f_k'(\bar{e})$, there is a first and last point of the arc that intersects K. Label these points x and y respectively. We can subdivide the edge e, and replace the arc in $f_k'(\bar{e})$ between x and y with an arc in K between x and y. Doing this for every pair e, f, we obtain the desired graph G and embedding f'.

Finally, we need to modify G so that the fourth condition also holds. Let B_1 and B_2 be H-bridges of G such that $f(B_1)$ and $f(B_2)$ are contained in the same $f(H)$-bridge B_M of M. We can find an arc starting in $f(B_1)$ and ending in $f(B_2)$, and otherwise contained in the interior of B_M. Adding a vertex to G for each endpoint of the arc (subdividing an edge of B_1 or B_2 if necessary), and an edge between the two endpoints, we have combined B_1 and B_2 into a single H-bridge of G. Repeating this as many times as necessary, we can ensure that G and f also satisfy the fourth condition, as required. ■

6.3 Embedding extensions and the main result

We will need the following lemma about distinct bridges of a fixed subspace.

Lemma 6.4
Let M be a compact, locally connected metric space that does not contain $K_{3,\infty}$, and let N be any closed subspace of M. Suppose that $\{B_1, B_2, \ldots\}$ is any set of countably many distinct N-bridges of M. There is a unique point $x \in N$ such that if, for each $i \geq 1$, $x_i \in B_i \setminus N$, then $\{x_i\}$ converges to x.

Proof Let $\{x_i\}$ be a fixed sequence of points such that, for each $i \geq 1$, $x_i \in B_i \setminus N$. Since M is compact, there is at least one point x such that $\{x_i\}$ converges to x. Since M is locally connected, x has a connected neighbourhood. If $x \notin N$, this contradicts the fact that each x_i is in a distinct N-bridge, so $x \in N$.

Suppose that there are distinct points $x, y \in N$, and sequences $\{x_i\}$, $\{y_i\}$ converging to x and y respectively, such that, for each $i \geq 1$, $x_i, y_i \in B_i \setminus N$. Let P_i be an arc in B_i between x_i and y_i. Let ρ be the metric of M, and let z_i be a point on P_i such that $\rho(x_i, z_i) \geq (1/2)\rho(x_i, y_i)$ and $\rho(y_i, z_i) \geq (1/2)\rho(x_i, y_i)$. Since $x \neq y$, the distances between x_i and y_i are not approaching zero, and therefore no subsequence of $\{z_i\}$ can converge to either x or y, and instead some subsequence of $\{z_i\}$ must converge to a third distinct point $z \in N$.

Let $N(x), N(y), N(z)$ be disjoint connected neighbourhoods of x, y and z respectively. Let I be the set of indices i for which $x_i \in N(x)$, $y_i \in N(y)$ and

$z_i \in N(z)$. For each $i \in I$, there is an arc between x and x_i in $N(x)$, an arc between y and y_i in $N(y)$, and an arc between z and z_i in $N(z)$. The union of these three arcs and P_i yields a $K_{1,3}$, $Z_i \subseteq B_i$, with vertices z_i, x, y, z and edges between z_i and each other vertex. Since I is infinite, $\bigcup_{i \in I} Y_i$ is a $K_{3,\infty}$ in M, the desired contradiction. ■

Suppose that N is a subspace of M, and we have an embedding $\Pi : N \to \Sigma$. In order to extend Π to an embedding of M, we need to describe how each of the bridges of N are embedded. In particular, each bridge will be embedded in some closed face of Π. We can break up the problem of extending Π to an embedding of all of M into separate problems of embedding a set of bridges in each closed face of Π. The following key lemma shows that for each of these subproblems, either we can embed the chosen bridges in the given face of Π, or M contains a homeomorph of a finite graph proving otherwise.

Lemma 6.5
Let Σ' be a surface. Let M be a connected, compact, locally connected metric space. Let H be a finite graph and let $f : H \to M$ be an embedding of H in M. Let Π be a cellular embedding of H in Σ'. Let F be a face of Π and let H_F be the subgraph of H that bounds F. Let $\mathcal{B}$ be a subset of the bridges of $f(H_F)$ in M, such that each bridge in $\mathcal{B}$ has at least two attachments. One of the following holds:

1. Π can be extended to an embedding Π' of $f(H)$ along with all of the bridges in $\mathcal{B}$ such that, for every $B \in \mathcal{B}$, $\Pi'(B) \subseteq F$;
2. M contains a generalized thumbtack; or
3. There is a finite graph G and an embedding $f' : G \to M$ such that:
   (a) H is a subgraph of G;
   (b) $f'|_H = f$;
   (c) for every H-bridge B_G of G, there is some $B \in \mathcal{B}$ so that $f'(B_G) \subseteq B$; and
   (d) Π does not extend to an embedding Π' of G in Σ' so that, for every H-bridge B_G of G, $\Pi'(B_G) \subseteq F$.

Proof Let W_F be the facial walk for the face F. We will proceed by induction on the difference between the number of vertices and edges in W_F and the number of vertices and edges in H_F. In the base case, every edge and vertex in H_F occurs exactly once in W_F and so the difference is zero. Let M' be the connected, compact, locally connected metric space formed by $f(H_F)$, the union of every $B \in \mathcal{B}$, and a disjoint open disc D with $f(H_F)$ as its boundary.

Suppose that M' has an embedding $\Pi_{M'}$ in the sphere. Then $\Pi_{M'}(W_F)$ is a circle, and $\Pi_{M'}(D)$ is an open disc with $\Pi_{M'}(W_F)$ as its boundary. Therefore, for each $B \in \mathcal{B}$, $\Pi_{M'}(B)$ is a subset of the face F' of $\Pi_{M'}(W_F)$ in which D is not embedded. Since $\Pi_{M'}(W_F) \cup F'$ is a closed disc, it is homeomorphic to $\Pi(W_F) \cup F$, and therefore we may extend Π to Π' by using $\Pi_{M'}(B)$ for each $B \in \mathcal{B}$.

By Theorem 6.2, if M' does not have an embedding in the sphere, then it contains K_5, $K_{3,3}$, the disjoint union of a sphere and a point, or a generalized thumbtack. If M' contains K_5 or $K_{3,3}$, it is easy to see that we have a finite graph G, as required.

Suppose that M' contains the disjoint union of a sphere S and a point x. Since M' is connected, there is an arc in M' between x and S, so M' also contains a thumbtack. Now suppose that M' contains a generalized thumbtack. The center of the thumbtack cannot be in D, and, if it is in some $B \in \mathcal{B}$, then M also contains a generalized thumbtack. Suppose that $x \in f(H_F)$ is the center of a generalized thumbtack T in M'. The arc that forms the pin of the thumbtack must be contained in some $B \in \mathcal{B}$, and by assumption B has a second attachment y on $f(H_F)$. Using an arc in B between x and y, we can find a finite graph G, as required. This completes the argument in the base case.

Now suppose that the lemma holds whenever there are fewer than k repeated edges or vertices in W_F. Let x be an edge or vertex that is repeated in W_F, and let $\{x_1, \ldots, x_\ell\}$ be the list of occurences of x in W_F. Any sufficiently small open neighbourhood $U(x)$ of $\Pi(x)$ in Σ' avoiding $\Pi'(H) \backslash \Pi'(x)$ partitions into $U_1, \ldots U_\ell$ where, for each $1 \leq i \leq \ell$, the distance in F between U_i and x_i is zero, but for $i \neq j$, the distance between U_i and x_j is not zero. We will say that a subset of Σ' avoids x_i if it is disjoint from some such U_i. Fix some such $U(x)$ and let $U_M(f(x))$ be the open neighbourhood $f \circ \Pi^{-1}(U(x))$ of $f(x)$ in M (recall that f is an embedding of H in M).

We claim that only finitely many bridges of $f(H_F)$ have attachments in $f(x)$ but are not contained in $U_M(f(x))$. Suppose otherwise, and let $y_1, y_2, \ldots$ be a countable set of points of M, outside $U_M(f(x))$, and in different bridges of $f(H_F)$. Since M is compact, there is a limit point y of $y_1, y_2, \ldots$ in M, and, since M is locally connected, it has a connected neighbourhood. If $y \notin f(H_F)$, this is a contradiction to each of the y_i being in different $f(H_F)$-bridges. If $y \in f(H_F)$, we have a contradiction to Lemma 6.4. Let $\chi(x, F)$ be the number of bridges with attachments in $f(x)$ that are not contained in $U_M(f(x))$.

For any subset $\mathcal{B}'$ of $\mathcal{B}$, we will say that a function $g : \mathcal{B}' \to \{1, 2, \ldots \ell\}$ is valid if Π extends to an embedding Π' of $f(H)$ along with all of the bridges in $\mathcal{B}'$, such that for every $B \in \mathcal{B}'$, $\Pi'(B) \subseteq F$ and $\Pi'(B)$ avoids every x_i for $i \neq g(B)$. Obviously if there is any valid function with domain $\mathcal{B}$, then there is an embedding Π' with the properties required by the lemma.

Let $\mathcal{B} = \{B_1, B_2, \ldots\}$. Let V_i be the set of valid functions $g_i : \{B_1, \ldots, B_i\} \to \{1, 2, \ldots \ell\}$. Note that every $g_{i+1} \in V_{i+1}$ extends some $g_i \in V_i$. König's infinity lemma implies that either one of the V_i is empty, or there is a function $g : \mathcal{B} \to \{1, 2, \ldots \ell\}$ such that every restriction of g to a finite subset of its domain is valid. We claim that such a function must in fact be valid. This follows by applying the induction hypothesis to a space where x is replaced by distinct points $x_1, \ldots x_\ell$, and each neighbourhood of x_i is the intersection of a neighbourhood of x with the set of bridges $B \in \mathcal{B}$ such that $g(B) = i$. Therefore there is an embedding as required by the lemma, or some V_i is empty.

Choose i so that V_i is empty. Let $\{h_1, h_2, \ldots h_k\}$ be a list of all functions from $\{B_1, \ldots, B_i\}$ to $\{1, 2, \ldots \ell\}$. For every h_j, since it is not valid, we apply the induction hypothesis to a space where x is replaced by distinct points $x_1, \ldots x_\ell$, and, for each $1 \leq k \leq \ell$, each neighbourhood of x_k is the intersection of a neighbourhood of x with the set of bridges $B \in \{B_1, \ldots, B_i\}$ such that $h_j(B) = k$. This yields a graph G_j for which h_j is not valid. By Lemma 6.3, we combine these to get a finite graph G for which no h_j is valid and an embedding f' of G in M. If $x = x_1x_2$ is an edge, then we modify G by adding, for each bridge $B \in \{B_1, \ldots B_i\}$ such that $f'(G) \cap B \neq \emptyset$, edges e_1, e_2 to G in such a way that $f'(e_1)$, $f'(e_2)$ are each arcs with one endpoint in B and the other endpoints as close on the arc $f(x)$ to $f(x_1)$ and $f(x_2)$ respectively as possible.

Now extend Π to an embedding Π_G of G in every possible way. We will show that, for each possible Π_G, either Π_G extends to the embedding required by the lemma, or there is a finite graph showing that it does not. If no Π_G can be extended, one final application of Lemma 6.3 suffices to prove the lemma.

Let Π_G be some fixed embedding of G. Consider the set $\mathcal{F}'$ of faces of Π_G that are subsets of F. Either each of these faces has fewer repetitions in its boundary than F, or at least one has the same number. If every face has fewer repetitions we proceed, as in the proof of Lemma 6.6, by trying allocations of the bridges of $f'(G)$ that are subsets of the elements of $\mathcal{B}$ to the faces in $\mathcal{F}'$, applying the induction hypothesis to each face individually, and therefore showing that either Π_G extends to the embedding required by the lemma or that there is a finite graph showing otherwise.

Otherwise, suppose that $F_1 \in \mathcal{F}'$ has the same number of repetitions in its boundary as F. This must be because x is an edge and W_{F_1} contains repetitions of an edge x_1 such that x_1 is part of the path in G that represents the edge x in H. Note that $\chi(x_1, F_1) < \chi(x, F)$, because our choice of G guarantees that the H_{F_1}-bridges that attach on $f(x_1)$ are actually H_F-bridges. Repeat the process, with F_1 and x_1 in place of F and x, obtaining G_1 in place of G. Again, if there is an embedding Π_{G_1} of G_1 with a face F_2 that is a subset of F_1 and has the same number of repetitions in its boundary, W_{F_2} contains repetitions of an edge x_2 such that x_2 is part of the path in G_1 that represents the edge x_1 in G. Iterating this process, we either are eventually able to succeed by applying the induction hypothesis as described in the previous paragraph, or we obtain an infinite decreasing sequence of faces $F, F_1, F_2, \ldots$ of embeddings of $H, G, G_1, \ldots$ respectively. This is impossible because then $\chi(x_i, F_i)$ is an infinite decreasing sequence of non-negative integers. ∎

Lemma 6.6

Let Σ' be a surface. Let M' be a connected, compact, locally connected metric space that does not contain a generalized thumbtack or the disjoint union of Σ' and a point. Let H be a finite graph and let $f : H \to M'$ be an embedding. Let Π be a cellular embedding of H in Σ'. Either $\Pi \circ f^{-1} : f(H) \to \Sigma'$ extends to an embedding of M' in Σ', or there exists a finite graph G such that:

1. a subdivision H' of H is a subgraph of G and Π does not extend to an embedding of G in Σ'; and
2. there is an embedding $f' : G \to M'$ such that $f'|_{H'} = f$.

Proof Let $\mathcal{F}(\Pi)$ be the set of faces of Π, and let $\mathcal{B}(H)$ be the set of all bridges of $f(H)$ in M' with at least two attachments. For any $\mathcal{B} \subseteq \mathcal{B}(H)$, we will call a function $g : \mathcal{B} \to \mathcal{F}(\Pi)$ valid if, for all faces F of Π, Π extends to an embedding of $H \cup g^{-1}(F)$ so that every $B \in g^{-1}(F)$ is embedded in F. Notice that if there is a valid function with domain $\mathcal{B}(H)$, then combining the extensions of Π into each face F gives an extension of Π to an embedding of M'.

Let $\mathcal{B}(H) = \{B_1, B_2, \ldots\}$. Let V_i be the set of valid functions $g_i : \{B_1, \ldots, B_i\} \to \mathcal{F}(\Pi)$. Note that every $g_{i+1} \in V_{i+1}$ extends some $g_i \in V_i$. König's infinity lemma implies that either one of the V_i is empty, or there is a function $g : \mathcal{B}(H) \to \mathcal{F}(\Pi)$ such that every restriction of g to a finite subset of its domain is valid. We claim that such a g is itself valid. For each $F \in \mathcal{F}(\Pi)$, we apply Lemma 6.5 to F and the bridges in $g^{-1}(F)$. This gives us an embedding Π' of all of M' except for the bridges of $f(H)$ with only one attachment. Suppose that $x \in f(H)$ is the single attachment of some bridge B_x. Since M does not contain a generalized thumbtack, we can extend Π' by embedding B_x in some face of Π' incident with $\Pi' \circ f^{-1}(x)$. Therefore either some V_i is empty, or there is a valid $g : \mathcal{B}(H) \to \mathcal{F}(\Pi)$.

Suppose that V_i is empty. That is, there is no valid function $g_i : \{B_1, \ldots, B_i\} \to \mathcal{F}(\Pi)$. Let $\{h_1, h_2, \ldots h_k\}$ be a list of all functions from $\{B_1, \ldots, B_i\}$ to $\mathcal{F}(\Pi)$. For each j, $1 \leq j \leq k$, for each $F \in \mathcal{F}(\Pi)$, we may apply Lemma 6.5 to F and the set of bridges $h_j^{-1}(F)$. For a fixed j, since M' does not contain a generalized thumbtack, and h_j is not valid, there is at least one face F_j such that, when applying Lemma 6.5 to F_j and $h_j^{-1}(F_j)$, the third outcome of the lemma holds (if there is more than one such face, choose F_j arbitrarily among them).

Therefore, for each j, there is a finite graph G_j, such that H is a subgraph of G_j and Π does not extend to an embedding Π' of G_j in Σ' such that, for every H-bridge B_{G_j} of G_j, $\Pi'(B_{G_j}) \subseteq F_j$. There is an embedding $f_j : G_j \to M$ such that $f_j|_H = f$ and, for every H-bridge B_{G_j} of G_j, there is some $B_\ell \in h_j^{-1}(F_j)$ such that $f'(B_{G_j}) \subseteq B_\ell$. We may apply Lemma 6.3 to $G_1, G_2, \ldots G_k$.

From Lemma 6.3, we obtain a finite graph G such that, for each i, G contains a subgraph G_i' isomorphic to G_i and the intersection of all of the G_i' contains a subdivision H' of H; and an embedding $f' : G \to M$ such that $f'|_H = f$. For each ℓ, there is at most one H'-bridge B in G such that $f'(B) \subseteq B_\ell$. Recall that G was formed by combining the G_j, which were obtained by applying Lemma 6.5 to the set of bridges $\{B_1, \ldots B_i\}$. Therefore, for any H'-bridge B in G, there is some ℓ, $1 \leq \ell$, such that $f'(B) \subseteq B_\ell$.

We claim that Π does not extend to an embedding of G. Suppose otherwise that Π' is an embedding of G that extends Π. Let $h : \{B_1, \ldots, B_i\} \to \mathcal{F}(\Pi)$ be partially defined by setting $h(B_\ell) = F$ if there is an H-bridge B of G such that $f'(B) \subseteq B_\ell$ and $\Pi'(B) \subseteq F$.

Since $\{h_1, \ldots h_k\}$ is a complete list of functions from $\{B_1, \ldots, B_i\}$ to $\mathcal{F}(\Pi)$, there must be at least one j, $1 \leq j \leq k$ such that h can be extended to h_j. Since G_j' is a subgraph of G, and by the definition of G_j, there is an H'-bridge $B_{G_j'}$ of G_j' such that $\Pi'(B_{G_j'}) \subseteq F'$ for some $F' \neq F_j$. Furthermore (still from the third outcome of Lemma 6.5), there is some $B_\ell \in h_j^{-1}(F_j)$ such that $f'(B_{G_j'}) \subseteq B_\ell$. Consider the H'-bridge B of G that contains $B_{G_j'}$. Since $f'(B)$ is a subset of some $f(H)$-bridge of M', it must be a subset of B_ℓ, and (by the definition of G from Lemma 6.3) B must be the unique H'-bridge with this property. Since $\Pi'(B) \cap F' \neq \emptyset$, we must have $\Pi'(B) \subseteq F'$ and therefore, by defintion of h, $h(B_\ell) = F'$. This implies that $h(B_\ell) = F' \neq h_j(B_\ell)$, so h cannot be extended to h_j, a contradiction. Therefore Π does not extend to an embedding of G. ∎

We are now ready to prove the main result.

Theorem 6.7
Let Σ be a fixed surface and let M be a compact, locally connected metric space that does not contain:

1. a generalized thumbtack;
2. the disjoint union of Σ and a point; or
3. any surface of lower Euler genus than Σ.

Then either M embeds in Σ, or M contains some $G \in \mathrm{Forb}(\Sigma)$.

Proof We will first prove the result for connected M, and then deduce that it is true in general.

Let Σ be a surface. Let M be a connected, compact locally connected metric space that satisfies the hypotheses of the theorem, and let $\mathcal{G}(M)$ be the set of all finite graphs contained in M. If some $G \in \mathcal{G}(M)$ does not embed in Σ, M contains some $G \in \mathrm{Forb}(\Sigma)$ and we are done. Otherwise, if Σ is orientable, let Σ' be the lowest genus orientable surface in which every $G \in \mathcal{G}(M)$ embeds. If Σ is non-orientable, let Σ' be the lowest Euler genus surface, orientable or not, in which every $G \in \mathcal{G}(M)$ embeds. We will show that in fact M embeds in Σ', and that this implies that M also embeds in Σ (we use here the fact that M does not contain Σ'), completing the proof.

Let $H \in \mathcal{G}(M)$ be a graph that embeds in Σ' but not in any surface of lower Euler genus that was a candidate to be Σ'. Since M is connected, we may suppose that every embedding of H in Σ' is cellular ([32], see Section 1.2.3). Furthermore, there are only finitely many embedding schemes that may describe the embedding of H in Σ'. Recall from Section 1.2.3 that embeddings with the same embedding scheme are equivalent up to homeomorphism by a result of Ringel [23], so for a fixed embedding scheme either every embedding of H in Σ' realizing that embedding scheme extends to an embedding of M or no embedding of H in Σ' realizing that embedding scheme extends to an embedding of M.

Let $\{\Pi_1, \Pi_2, \dots \Pi_k\}$ be a set of embeddings of H in Σ', one realizing each possible embedding scheme. For each i, applying Lemma 6.6, either Π_i extends to an embedding of M in Σ', or there is a finite graph G_i and an embedding $f_i : G_i \to M$ as described in the lemma.

First, suppose that there is no i such that Π_i extends to an embedding of M in Σ'. Then by Lemma 6.6 there are finite graphs $G_1, G_2, \dots G_k$ such that, for each i, Π_i does not extend to an embedding of G_i in Σ'. The graphs $G_1, G_2, \dots G_k$ and their embeddings $f_1, f_2, \dots f_k$ satisfy the conditions of Lemma 6.3. Let G and f' be the graph and embedding obtained from that lemma. We claim that G does not embed in Σ'. Any embedding of G must extend some embedding Π_i of H', but, for each i, Π_i does not even extend to an embedding of G'_i in Σ'. So no embedding of G exists and since $G \in \mathcal{G}(M)$, we obtain a contradiction to the choice of Σ'.

Therefore, there is some i such that Π_i extends to an embedding Π' of M in Σ'. If $\Sigma' = \Sigma$, then we are done. Otherwise, since by assumption M does not contain Σ', there is an open neighbourhood U of Σ' homeomorphic to the plane and avoiding the embedded image of M. Removing an open disc contained in U from Σ' yields a space homeomorphic to a subset of Σ, so we are done.

Now we consider the general case. Let M be a compact locally connected metric space that satisfies the hypotheses of the theorem. Since M is compact and locally connected, it has only finitely many components, $M_1, \dots, M_\ell$. For each i, $1 \leq i \leq \ell$, let Σ^i be the lowest genus orientable surface in which M_i embeds, and let Σ^i_N be the lowest genus non-orientable surface in which M_i embeds. The special case already proved implies that if Σ is orientable, then Σ^i has Euler genus equal to or lower than that of Σ, or M_i contains an element of Forb(Σ). Similarly, it implies that if Σ is non-orientable, then Σ^i_N has Euler genus equal to or lower than that of Σ, or M_i contains an element of Forb(Σ).

Assuming no M_i contains an element of Forb(Σ), let Π^i be an embedding of M_i in Σ^i and Π^i_N be an embedding of M_i in Σ^i_N. If $\Pi^i(M_i) = \Sigma^i$ (or, similarly, if $\Pi^i_N(M_i) = \Sigma^i_N$), then M contains the disjoint union of Σ^i and a point, which contradicts one of the assumptions on M. So we may suppose that there is some $x \in \Sigma^i \backslash \Pi^i(M_i)$ (and, similarly, some $x \in \Sigma^i_N \backslash \Pi^i_N(M_i)$). There must be an open neighbourhood of x that avoids $\Pi^i(M_i)$, because M_i is compact. Since Σ^i is a surface, we may choose U_x to be an open neighbourhood of x, homeomorphic to the plane, and avoiding $\Pi^i(M_i)$.

For each i, $1 \leq i \leq \ell$, choose Π_i and Σ_i to be either Π^i and Σ^i or Π^i_N and Σ^i_N. Using the open neighbourhoods we found in the previous paragraph, we may combine these embeddings of $M_1, \dots, M_\ell$ to get an embedding of M in the connected sum of $\Sigma_1, \Sigma_2, \dots, \Sigma_\ell$. Either there is some choice of the Σ_i so that this connected sum is Σ (or can be made in to Σ by taking one more connected sum with a surface), in which case M embeds in Σ, or there is not. If not, then, for each i, $1 \leq i \leq \ell$, let H^i be a graph contained in M_i that does not embed in a lower genus orientable surface than Σ^i and let H^i_N be a graph contained in M_i that does not embed in a lower genus non-orientable surface than Σ^i_N. Applying Lemma 6.3 to each pair H^i, H^i_N, to obtain H_i, and then taking the union of the graphs H_i yields a finite graph that does not embed in Σ, completing the proof. ■

Chapter 7

Future work

Several goals for future research suggest themselves. In particular, any of the following would be interesting.

- A proof of Tutte's Linking Theorem for B-matroids. For finite matroids, a form of the Matroid Intersection Theorem implies the Linking Theorem. Perhaps this is also true for B-matroids?
- An algebraic characterization of B-matroids with no $U_{2,4}$-minor.
- A topological characterization of B-matroids with no Tutte-minor.
- A more substantial theory of planarity and duality for graphs and graph-like spaces including, for example, uniqueness of duality for 3-connected objects, uniqueness of duality up to Whitney flips for 2-connected objects, simultaneous drawings for dual pairs and straight-line embeddings of planar objects.

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