# A New Class of Cycle Inequality for the Time-Dependent Traveling Salesman Problem

by

John White

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners. I understand that my thesis may be made electronically available to the public.

**Abstract**

The Time-Dependent Traveling Salesman Problem is a generalization of the well-known Traveling Salesman Problem, where the cost for travel between two nodes is dependent on the nodes and their position in the tour. Inequalities for the Asymmetric TSP can be easily extended to the TDTSP, but the added time information can be used to strengthen these inequalities. We look at extending the Lifted Cycle Inequalities, a large family of inequalities for the ATSP. We define a new inequality, the Extended Cycle (X-cycle) Inequality, based on cycles in the graph. We extend the results of Balas and Fischetti for Lifted Cycle Inequalities to define Lifted X-cycle Inequalities. We show that the Lifted X-cycle Inequalities include some inequalities which define facets of the submissive of the TDTS Polytope.

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# Chapter 1

# Background

The Traveling Salesman Problem (TSP) is a classic problem in combinatorial optimization, and perhaps the problem best known to the general public. For the traveling salesman, the problem is to find the shortest path to visit each of a set of cities, given the distances between them. For the mathematician, the problem is to find the least cost Hamiltonian path through a graph with edge weights. The Time-Dependent Traveling Salesman Problem (TDTSP) is a generalization of the TSP, where the cost between nodes is dependent on the position in the sequence in which the arc is taken. The first known formalization of the TDTSP was to solve a scheduling problem in the brewing industry with time-dependent set-up costs [19]. With the added time dependency, the TDTSP can not only model the TSP, but also other problems in routing, such as the Traveling Deliveryman Problem, or minimum latency problem, and other variants of one-machine scheduling.

Polyhedral study of the TSP polytope has been quite successful in creating programs to routinely solve TSP instances with hundreds or thousands of nodes. The history of solving the TDTSP is less successful (in terms of solving TDTSP instances), though numerous formulations have been found. Picard and Queyranne introduced several integer programming formulations, and a quadratic programming formulation, and showed a method to solve the TDTSP on instances with up to 20 nodes [19]. One of these formulations, the so-called *flow formulation* or *three index* formulation, will be the basis of this work, and will be called the Picard–Queyranne formulation. Vander Weil and Sahindis introduced another quadratic programming formulation [20]. Gouveis and Voß-linearized the Vander Weil–Sahindis formulation, and showed it was equivalent to the Picard–Queyranne formulation, in a paper comparing formulations for the TDTSP [11].

This work will be presented in five chapters. The first chapter reviews necessary background on polyhedra, cutting plane techniques for solving Mixed Integer Linear Programs, lifted inequalities, monotonizations of polyhedra, and the Picard-Queyranne formulation for the TDTSP. In the second chapter, we summarize some results on lifted cycle inequalities for the asymmetric TSP. New results begin in the third chapter, with the derivation of a new cycle inequality for the TDTSP, which we will call the x-cycle inequality. In chapter four, we present results on lifting the x-cycle inequality. Finally, in chapter five, we show a sub-family of x-cycle inequality that has high-dimension.

## 1.1 Polyhedra

We will assume a basic understanding of graph theory and linear programming. In this section, we will give the basics of integer programming and polyhedral theory. This material can be found in any integer programming textbook (e.g. [21]) or some survey papers on valid inequalities for integer programs (e.g. [7, 8]). A mixed integer linear program has the form:

$$\max \ c^T x + h^T y \tag{1.1a}$$
$$\text{subject to} \ Ax + Gy \le b \tag{1.1b}$$
$$x \ge 0 \text{ and integer} \tag{1.1c}$$
$$y \ge 0 \tag{1.1d}$$

with the row vectors $c \in \mathbb{Q}^n$ and $h \in \mathbb{Q}^p$, column vector $b \in \mathbb{Q}^m$, and matrices $A \in \mathbb{Q}^{m\times n}$ and $G \in \mathbb{Q}^{m\times p}$. When $n = 0$, the problem is a *linear program*; when $p = 0$, the problem is a *(pure) integer program*; otherwise, it is a *mixed integer linear program*. We will only be considering pure integer programs, unless otherwise indicated an integer program is a pure integer program. The feasible set of solutions $S$ described by (1.1b)–(1.1d) is the *mixed integer linear set* when $p \ge 1$ and the *pure integer linear set* when $p = 0$. The feasible set of solutions of a system of linear inequalities, without integrality constraints, is a *polyhedron*.

Linear programming is an important tool in optimization and operations research, and has been extensively studied since Dantzig's initial work in the 1950s. Implementations of Dantzig's simplex method and interior point methods can solve most large problems in reasonable time; however, the decision variables of linear programs are continuous and cannot model the discrete decisions required by combinatorial problems. Integer programs have this expressive power, but they cannot be efficiently solved in general. Polyhedral theory, used in part to study linear programs, and linear programming itself are used to solve integer programs by the cutting plane methods, which we will describe later.

In the remainder of this section, let $P$ be a polyhedron, $P := \{x \in \mathbb{R}^n \mid Ax \le b\} \subset \mathbb{R}^n$, with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. When $A$ and $b$ are rational, $P$ is a *rational polyhedron*. A polyhedron $C := \{x \in \mathbb{R}^n \mid Ax \le 0\}$ is a *polyhedral cone*. For any set $S \subseteq \mathbb{R}^n$, the *convex hull* of $S$, $conv(S)$ is the smallest convex set containing $S$. $conv(S)$ is also the set of all convex combinations of points in $S$, $conv(S) := \{x \in \mathbb{R}^n \mid x = \sum_{i=1}^k \lambda_i x^i, \sum_{i=1}^k \lambda_i = 1, \lambda \ge 0, x^1, \cdots, x^k \in S, k \ge 0\}$. A *polytope* is the convex hull of a finite set of points in $\mathbb{R}^n$. The *conic hull* of a non-empty set $S \subseteq \mathbb{R}^n$ is $cone(S) := \{x \in \mathbb{R}^n \mid x = \sum_{i=1}^k \lambda_i x^i, \lambda \ge 0, x^1, \cdots, x^k \in S, k \ge 0\}$.

If $S$ is a finite set, we say that $cone(S)$ is *finitely generated*. Every polyhedral cone is finitely generated, and every finitely generated cone is polyhedral.

A well-known theorem due to Minkowski and Weyl states that every polyhedron $P$ is the sum of a polytope $Q$ and a finitely generated cone $C$. Here, the sum of polyhedra is the sum of their points, $Q + C := \{x \in \mathbb{R}^n \mid x = q + r \text{ for some } q \in Q \text{ and } r \in R\}$. A consequence of this theorem, due to Meyer, is fundamental to integer programming:

**Theorem 1.** *Given a polyhedron $P := \{(x, y) \mid Ax + Gy \leq b\}$, where $A$, $G$ are rational matrices and $b$ a rational vector, and let $S := \{(x, y) \in P \mid x \text{ integral}\}$, then $conv(S)$ is a rational polyhedron. That is, there exists $A'$, $G'$ are rational matrices and $b'$ a rational vector, such that $conv(S) = \{(x, y) \mid A'x + G'y \leq b'\}$.*

If $S$ is the mixed integer linear set of a mixed integer linear program, and if the polyhedral representation of $conv(S)$ were known, then the MILP could be solved as a linear program. Unfortunately, the polyhedral representation may not be known explicitly, and may be exponentially larger than the representation of $P$. The techniques for solving MILPs will use the polyhedral representation implicitly.

A few such polyhedra we will use are the STS polytope, the convex hull of Hamiltonian cycles on an undirected graph; the ATS polytope, the convex hull of Hamiltonian dicycles on a directed graph; and the TDTS polytope. The TDTS polytope we use will be defined by the PQ formulation, to be specified later.

The *affine hull* of a set $S \subseteq \mathbb{R}^n$, $aff(S)$, is the smallest affine set containing $S$. An *affine combination* of points $x^1, \ldots, x^k \in \mathbb{R}^n$ is $\sum_{i=1}^k \lambda_k x^k$ with $\sum_{i=1}^k \lambda_k = 1$, $k \geq 1$ and $\lambda \in \mathbb{R}^k$. Points $x^1, \ldots, x^k \in \mathbb{R}^n, k \geq 1$ are *affinely independent* if and only if the unique solution to $\sum_{i=1}^k \lambda_k x^k = 0$ is $\lambda_1 = \cdots = \lambda_k = 0$. Define the dimension of $S$ to be the dimension of the affine hull of $S$, $dim(S) := dim(aff(S))$. Showing $d + 1$ affinely independent points of a polyhedron is enough to show that the polyhedron has dimension at least $d$.

An inequality $a^i x \leq b_i$ of $Ax \leq b$ for which $a^i x = b_i$ for all $x \in P$ is an *implicit equality* of the system. Let $A(i)x \leq b(i)$ be the sub-system of $Ax \leq b$ without row $i$ (without $a^i x \leq b_i$). If $\{x \in \mathbb{R}^n \,|\, Ax \leq b\} = \{x \in \mathbb{R}^n \,|\, A(i)x \leq b(i)\}$, then we call $a^i x \leq b_i$ a *redundant inequality* of the system. We can partition the inequalities of $Ax \leq b$ into the implicit equalities, $A^= x \leq b^=$, and the rest, $A^< x \leq b^<$, so that $P = \{x \in \mathbb{R}^n \mid A^= x = b^=, A^< x \leq b^<\}$. As a consequence of this partition, $dim(P) = n - rank(A^=)$.

We call an inequality $\alpha x \leq \alpha_0$ *valid* for a polyhedron $P$ if every point $x \in P$ satisfies $\alpha x \leq \alpha_0$. A *face* $F$ of a polyhedron $P$ is $F := P \cap \{x \in \mathbb{R}^n \mid \alpha x = \alpha_0\}$, where $\alpha x \leq \alpha_0$ is a valid inequality for $P$. The inequality $\alpha x \leq \alpha_0$ is said to *define* the face $F$.

A face is an intersection of two polyhedra, $P$ and the hyperplane $\alpha x = \alpha_0$, so a face is also a polyhedron. We call a face $F$ of $P$ *proper* if $F$ is non-empty and properly contained in $P$. The maximal proper faces of $P$ are called *facets*; if $P$ has dimension $d$, then facets of $P$ have dimension $d - 1$. Often, the non-negativity constraints, such as $x_i \geq 0$ are facet-defining; we call these the *trivial facets*. Facets defined by inequalities which are not the non-negativity constraints are called *non-trivial*.

Let $I^<$ index the rows of $A^< x \leq b^<$. Assuming $A^< x \leq b^<$ does not contain redundant inequalities, then the facets of $P$ are $\mathcal{F} := \{F_i := \{x \in P \mid a^i x = b_i\}\ i \in I^<\}$. A polyhedron with $m$ facets will require at least $m$ inequalities in a linear inequality system that represents it.

## 1.2 Solving MILP by Cutting Planes

We can now describe a general method for solving mixed integer linear programs, based on Theorem 1. Let $S$ be the mixed integer linear set. The theorem implies that $conv(S)$ has a polyhedral representation, which we will use implicitly. First, solve the linear programming relaxation of a mixed integer linear program, that is, we drop the integrality requirement from (1.1c) and solve the relaxed problem. If $(\bar{x}, \bar{y})$ is the optimum of the LP relaxation, and if $\bar{x}$ is integral, then $(\bar{x}, \bar{y}) \in S$ and is optimal for the original problem. Otherwise, we find a valid inequality $\alpha x + \beta y \leq \alpha_0$ of $conv(S)$ which $(\bar{x}, \bar{y})$ violates, that is, $\alpha \bar{x} + \beta \bar{y} > \alpha_0$. We add this inequality to the constraints of the linear relaxation, and repeat. The inequalities $\alpha x + \beta y \leq \alpha_0$ found in each iteration "cuts" the current optimum from the relaxation, thus we refer to them as the *cutting planes*. Facets are, in some sense, the best possible cuts we could hope to make, since all of the facets would define $conv(S)$.

There are some variations of this method, such as finding a feasible point of the stronger relaxation, rather than a new optimum; adding multiple cutting planes; or using cutting planes to get a "good" approximation before using an enumerative method. In general, cutting plane methods build a polyhedral representation of $conv(S)$ iteratively. We've omitted a rather large detail—the identification of cutting planes—to which we will return.

A cutting plane method was used by Dantzig, Fulkerson and Johnson to solve the first large-scale TSP with one city in each of the (then) 48 states and the District of Columbia [9, 2]. They identified their inequalities by "ingenuity" (to borrow the term used by Gomory in [10]) and not algorithmically, and their method was not yet general enough for solving any TSP. The result, though, started decades of successes in solving ever-larger TSPs. (For a complete history of the solving of TSPs, see chapter 2 of [2]).

Gomory noted that the method of Dantzig, et al. lacked a way to systematically find cutting planes [10]. Research in identifying inequalities moved in two directions. Gomory gave a general method for finding cutting planes for general MILPs from the optimal basis of the LP relaxation [10]; this and other techniques for general MILPs are summarized in [7, 8], among others. Others worked to find cutting planes for specific problems, using the structure of the problem to derive new families of valid inequalities. For cutting planes specific to the TSP, see [13, 5]. These inequalities are only useful if there is a way to derive a specific inequality that will be a cutting plane; this second problem is referred to as *the separation problem*. The identification of new cutting planes, advances in computing power, and the use of cutting planes with other solving techniques lead to a rapid rise in the size of solvable TSP instances. To date, the largest TSP ever solved has 85,900 nodes [2].

## 1.3 Lifted Inequalities

In some cases, it may be easy to identify valid inequalities of a face of a polyhedron which define a high-dimensional face. We next describe the technique of *lifting* for strengthening the inequalities to make valid inequalities of the polyhedron which also define a high-dimensional face. This technique was introduced in [17] and generalized in [18]. Assume $P \subseteq [0,1]^n$ is a rational polyhedron, and let $S := conv(P \cap \mathbb{Z}^n)$. For some $j = 1, \dots, n$, let $S^0 := S \cap \{x \mid x_j = 0\}$ and $S^1 := S \cap \{x \mid x_j = 1\}$. Assume that $S^1 \neq \emptyset$. Let

$$\alpha x := \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i x_i \leq \alpha_0$$

be a valid inequality for $S^0$.

We call $\alpha x + \alpha_j x_j \leq \alpha_0$ a *lifted* inequality for $S$ if $\alpha_j$ is chosen so that $\alpha_j = \alpha_0 - \max_{x \in S^1} \alpha x$. By this choice of $\alpha_j$, the lifted inequality is valid for $S$. If the face of $S^0$ defined by $\alpha x \leq \alpha_0$ has dimension $d$, then the face of $S$ defined by $\alpha x + \alpha_j x_j \leq \alpha_0$ has dimension at least $d+1$. In particular, if $\alpha x \leq \alpha_0$ defines a facet of $S^0$, then the lifted inequality $\alpha x + \alpha_j x_j \leq \alpha_0$ defines a facet of $S$.

Let $S[\{j, j+1, \dots, n\}] := S \cap \{x \mid x_j = x_{j+1} = \cdots = x_n = 0\}$, and let $\alpha x := \sum_{i=1}^{j-1} \leq \alpha_0$ be valid for $S[\{j, j+1, \dots, n\}]$. We can find an inequality $\alpha x + \sum_{i=j}^{n} \alpha_i x_i \leq \alpha_0$, valid for $S$, by lifting $\alpha x \leq \alpha_0$ *sequentially*, first finding $\alpha_j$ for $\alpha x + \alpha_j x_j \leq \alpha_0$ valid for $S[\{j+1, \dots, n\}]$, and then recursively finding $\alpha_{j+1}, \dots, \alpha_n$.

This differs from a *simultaneous* lifting of $\alpha x \leq \alpha_0$, where the coefficients $\alpha_j, \dots, \alpha_n$ are chosen to optimize some function $f(\alpha_j, \dots, \alpha_n)$ such that $\alpha x + \sum_{i=j}^{n} \alpha_i x_i \leq \alpha_0$ is valid for $S$. We will not consider simultaneously lifted inequalities (see [15] for an overview of general simultaneous lifting) and unless otherwise noted, lifting means sequential lifting.

## 1.4 Monotonization of Polyhedra

The polyhedra studied in polyhedral approaches to integer programs are often less than full-dimensional, making the analysis of the facial structure difficult. It is often more convenient to study a larger and related full-dimensional polyhedron. Monotonizations of polyhedra are natural candidates, which are (under mild assumption) full-dimensional and preserve some of the properties of the original polyhedron. For our definition of monotonization, we follow the definitions and results of Balas and Fischetti in [3], who give their results in a general context. Let $P \subseteq \mathbb{R}^N$, and $N = \{1, 2, \dots, n\}$, and assume $P$ is in the form $P := \{x \in \mathbb{R}^N \mid Ax \leq b\}$. Let $A^= x = b^=$ be a full row rank equality system for $P$, with $r := n - dim(P)$ rows. For any partition $[N^L, N^U]$ of $N$, let $B_j$ be the lower (upper) bound for $x_j$ for $j \in N^L$ ($j \in N^U$). That is, for any $x \in P$,

$$B_j \in \mathbb{R} \cup \{-\infty\} \text{ and } x_j \geq B_j \text{ for } j \in N^L$$
$$B_j \in \mathbb{R} \cup \{+\infty\} \text{ and } x_j \leq B_j \text{ for } j \in N^U.$$

We define the *general monotonization* of $P$ (or $g\text{-}mon(P)$) as

$$g\text{-}mon(P) := \{y \in \mathbb{R}^N \mid B_j \leq y_j \leq x_j, j \in N^L \text{ and } x_j \leq y_j \leq B_j, j \in N^U \text{ for some } x \in P\}.$$

$g\text{-}mon(P)$ generalizes two standard monotonizations of polytopes in the non-negative orthant: $sub(P)$, the *submissive* or *downward monotonization* of $P$, and $dom(P)$, the *dominant* or *upward monotonization* of P. When $P \subseteq \mathbb{R}^N_+$, $N^U = \emptyset$, $B_j = 0$, $\forall j \in N^L = N$, then

$$g\text{-}mon(P) = \{y \in \mathbb{R}^N \mid 0 \leq y_j \leq x_j, j \in N_L, \text{ for some } x \in P\} = sub(P),$$

and when $P \subset \mathbb{R}^N_+$, $N^L = \emptyset$, $B_j = +\infty$, $\forall j \in N^U = N$, then

$$g\text{-}mon(P) = \{y \in \mathbb{R}^N \mid y_j \geq x_j, j \in N_U, \text{ for some } x \in P\} = dom(P).$$

We want to study full-dimensional polytopes, so we require the dimension of $g\text{-}mon(P)$. Let $Q := \{j \in N \mid |B_j| < \infty, x_j = B_j, \forall x \in g\text{-}mon(P)\}$, then $dim(g\text{-}mon(P)) = n - |Q|$ [3]. For our purposes, we will only consider the submissive of the TDTS polytope, which is full-dimensional. In the remainder of this section, we assume that any monotonized polyhedron is full-dimensional.

The natural (and pertinent) question is under what circumstances are facets equivalent: when is a facet of the polytope also a facet of a monotonization of the polytope, and vice versa. Let $\alpha x \leq \alpha_0$ be a valid inequality that defines a non-trivial facet of $g\text{-}mon(P)$. Let $N^0 := \{j \in N \mid \alpha_j = 0\}$, and let $A^=_0$ be the $r \times |N^0|$ sub-matrix of $A^=$, whose columns are indexed by $N_0$. Grötschel and Pulleyblank defined the inequality to be *support reduced* (with respect to $P$) if $A^=_0$ has full row rank [14]. Balas and Fischetti proved the following property of inequalities which are *not* support reduced [3]:

**Lemma 2.** *Let $\alpha x \leq \alpha_0$ define a non-trivial face $\tilde{F}$ of* g-mon$(P)$*. If $\alpha x \leq \alpha$ is not support reduced, then there exists $\lambda \in \mathbb{R}^r \setminus \{0\}$ such that $\lambda A^= \tilde{x} = \lambda b^=$ for all $\tilde{x} \in \tilde{F}$.*

This fact proves the following, which helps to identify that an inequality is support reduced, when $A^= x = b^=$ is given [3]:

**Theorem 3.** *Let $\alpha x \leq \alpha_0$ define a non-trivial facet of* g-mon$(P)$*. Then either $\alpha x \leq \alpha_0$ is support reduced, or else there exists $\mu \in \mathbb{R}^r \setminus \{0\}$ such that $(\alpha, \alpha_0) = \mu(A^=, b^=)$.*

Inequalities which are support reduced, in other words, are those which are not implied by the equality system of the polyhedron. Being support reduced is itself a necessary and sufficient condition for a facet-defining inequality of $P$ to be facet-defining for *g-mon*$(P)$ [3]:

**Theorem 4.** *Let $\alpha x \leq \alpha_0$ be a valid inequality for* g-mon$(P)$*, that defines a non-trivial facet of $P$. Then $\alpha x \leq \alpha_0$ defines a facet of* g-mon$(P)$ *if and only if it is support reduced.*

Next, we would like to consider the opposite direction: when a non-trivial facet of *g-mon*$(P)$ also defines a facet of $P$. Define $N_0$ and $A_0^=$ as before. Let $\alpha x \leq \alpha_0$ define a non-trivial facet of *g-mon*$(P)$, and let $F$ be the face of $P$ induced by $\alpha x \leq \alpha_0$, $F := \{x \in P \mid \alpha x = \alpha_0\}$. Suppose that $\alpha x \leq \alpha_0$ is support reduced, and w.l.o.g. let $N_0 = \{j_1, \ldots, j_q\}$, with $q = |N_0|$, and let $\{j_1, \ldots, j_r\}$ be indices of the columns of a basis of $A^=$. For $x^1, x^2 \in P$, let $\Delta(x^1, x^2) := \{j \in N \mid x_j^1 \neq x_j^2\}$. Balas and Fischetti call the inequality $\alpha x \leq \alpha_0$ *strongly support reduced* if it is support reduced, and if for every $k \in \{r+1, \ldots, q\}$ there exists a pair $x^1, x^2 \in F$ such that $\{j_k\} \subseteq \Delta(x^1, x^2) \subseteq \{j_1, \ldots, j_k\}$. Balas and Fischetti proved the following of strongly support reduced inequalities [3]:

**Theorem 5.** *Let $\alpha x \leq \alpha_0$, where $\alpha_0 \neq 0$, define a non-trivial facet of* g-mon$(P)$*. If $\alpha x \leq \alpha_0$ is strongly support reduced with respect to $P$, then it also defines a facet of $P$.*

This sufficient condition appears to be the only known conditions under which a facet of *g-mon*$(P)$ is also a facet of $P$, perhaps because it is relatively easily proven for many polyhedra. In the case of STS polytope and ATS polytope, easily checked properties of the graph induced by the arcs with non-zero coefficients in the inequality are enough to show that an inequality is strongly support reduced in many cases [3, 4]. No such properties are known for the TDTS polytope, and may be difficult to show in some cases.

## 1.5 Integer Programming Formulation for the TDTSP

Rather than solving the TDTSP as a shortest Hamiltonian path problem, we will solve the problem as a constrained shortest path problem on the multipartite directed graph introduced by Picard and Queyranne in [19]. This formulation has been a popular formulation for the TDTSP, and has a strong linear relaxation [11]. We will call the multipartite graph the *extended network*. Let $N := \{1, \ldots, n\}$, and let $K(N)$ be the complete graph on vertices $N$.

Let $G := (V, A)$ be the extended network for $K(N)$. Let $V$, the set of nodes, consist of a source node 0, a sink node $T$, and intermediate nodes $\{(i,t)|i, t \in N\}$. The pair $(i, t)$ indicates visiting node $i$ in position $t$ of a tour through $K(N)$. For each $t \in N$, the intermediate nodes $\{(i, t) \mid i \in N\}$ constitute a *time layer* in the extended network. Each arc is a triple consisting of the tail and head nodes,

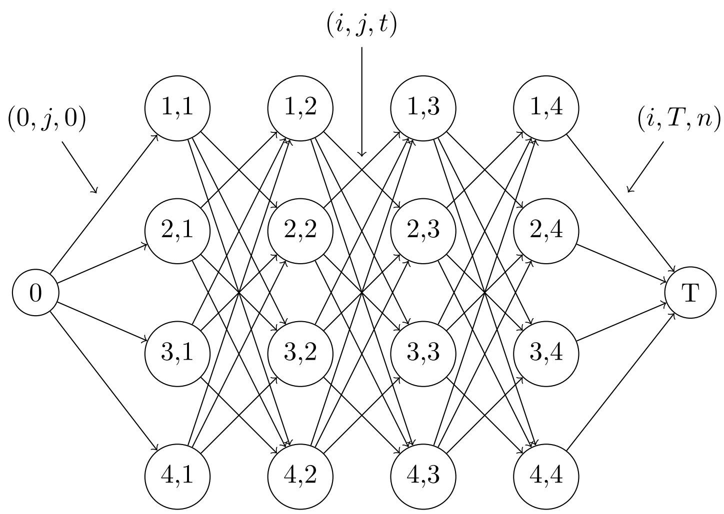


Figure 1.1: Extended network for $n = 4$.

and a time. The arc set $A$ consists of three types of arcs. Arcs $\{(0, j, 0) \,|\, j \in N\}$ go from the source node 0 to time layer 1. Arcs $\{(i, j, t) \,|\, i, j \in N, i \neq j, 1 \leq t \leq n-1\}$ go from time layer $t$ to $t+1$. Arcs $\{(i, T, n) \,|\, i \in N\}$ go from time layer $n$ to the sink node $T$. An example of the extended network for $n = 4$ is given in Figure 1.1.

We denote by $G(n)$ the subgraph of $G$ induced by $V \setminus \{0, T\}$. Any path through $G(n)$ will consist of nodes $((v_1, 1), (v_2, 2), \ldots, (v_n, n))$ and can be abbreviated by the sequence $(v_1, v_2, \ldots, v_n)$. Any path with the node sequence $(v_t \mid t \in N, v_i \neq v_j, \forall i \neq j)$ will be called a *sequence path* or *s-path*, following the terminology from [19]. An s-path is the complete ordering of the elements of $N$, so there is a one-to-one correspondence between s-paths of $G(n)$ and Hamiltonian paths in the complete graph on $n$ vertices.

Let $c^0_{0,j}$ be the cost of beginning a tour in node $j$ for $j \in N$, let $c^t_{i,j}$ be the cost of going from $i \in N$ in position $t$ of the sequence to $j \in N$ (with $i \neq j$), and let $c^n_{i,T}$ be the cost of ending a tour in node $i \in N$. Let $x^t_{i,j}$ denote the flow on arc $(i, j, t)$. Picard and Queyranne introduced the following integer programming formulation of the TDTSP, as a constrained shortest path problem on the extended network $G$ [19]:

$$
\begin{aligned}
\min \quad & \sum_{j\in N} c^0_{0,j}x^0_{0,j} + \sum_{t=1}^{n-1}\sum_{i\in N}\sum_{j\in N(i)} c^t_{i,j}x^t_{i,j} + \sum_{i\in N} c^n_{i,T}x^n_{i,T} && && (1.2a)\\
\text{subject to} \quad & \sum_{j\in N} x^0_{0,j} = 1 && && (1.2b)\\
& x^0_{0,j} = \sum_{k\in N(j)} x^1_{j,k} && \forall j\in N && (1.2c)\\
& \sum_{i\in N(j)} x^t_{i,j} = \sum_{k\in N(j)} x^{t+1}_{j,k} && \forall j\in N,\ t=1\ldots n-2 && (1.2d)\\
& \sum_{i\in N(j)} x^{n-1}_{i,j} = x^n_{j,T} && \forall j\in N && (1.2e)\\
& x^0_{0,j} + \sum_{t=1}^{n-1}\sum_{i\in N(j)} x^t_{i,j} = 1 && \forall j\in N && (1.2f)\\
& x \geq 0 \text{ and integer} && && (1.2g)
\end{aligned}
$$

Abeledo et al. used equations (1.2b) and (1.2e) to eliminate the variables corresponding to the arcs from source to the first time layer and from the last time layer to the sink [1]. By adjusting the costs on the first and last arc layer, this gives an equivalent formulation for flows in $G(n)$:

$$
\begin{aligned}
\min \quad & \sum_{t=1}^{n-1}\sum_{i\in N}\sum_{j\in N(i)} c^t_{i,j}x^t_{i,j} && && (1.3a)\\
\text{subject to} \quad & \sum_{i\in N}\sum_{j\in N(j)} x^1_{i,j} = 1 && && (1.3b)\\
& \sum_{i\in N(j)} x^t_{i.j} = \sum_{k\in N(j)} x^{t+1}_{j,k} && \forall j\in N,\ t=1\ldots n-2 && (1.3c)\\
& \sum_{k\in N(j)} x^1_{j,k} + \sum_{t=1}^{n-1}\sum_{i\in N(j)} x^t_{i,j} = 1 && \forall j\in N && (1.3d)\\
& x \geq 0 \text{ and integer} && && (1.3e)
\end{aligned}
$$

Abeledo et al. showed that the system of equations (1.3c) and (1.3d) has rank $n^2 - n$, or full row rank [1]. We will define the TDTS polytope as the convex hull of s-paths in $G(n)$.

## 1.6 Intent

In this thesis, we want to derive a new class of inequalities for the TDTS polytope based on cycles in the complete graph. Balas and Fischetti, in [3, 4] have shown that cycle inequalities for the ATS polytope are facet-defining under weak assumptions and that the lifted cycle inequalities include many interesting inequalities. Also, they showed necessary and sufficient conditions for the lifted coefficients of the inequalities. Our inequalities will be derived from the lifted cycle inequality for the TDTS polytope, which Abeledo, et al. have used with success in solving TDTSP instances [1]; we hope that this will extend to the inequalities we define.

# Chapter 2

# Lifted Cycle Inequalities for ATS Polytope

In this chapter, we will summarize some results on the lifted cycle inequalities, a large family of inequalities for the ATS polytope. We begin with notation and definitions from [4]. In this section, let $G = (N, A)$ be a digraph on $|N| = n$ nodes. Let $P$ be the ATS polytope, the convex hull of incidence vectors of Hamiltonian dicycles (tours) in $G$. Let $x(H) := \sum_{a \in H} x_a$ for all $H \subseteq A$ and let $x(S,T) := \sum\{x_{i,j} \mid i \in S, j \in T, i \neq j\}$ for $S, T \subset N$. We denote $\delta^+, \delta^-, \gamma$ in the usual way:

$$\begin{aligned}
\delta^+(S) &:= \{(i,j) \in A \mid i \in S, j \in N \setminus S\} \\
\delta^-(S) &:= \{(i,j) \in A \mid i \in N \setminus S, j \in S\} \\
\gamma(S) &:= \{(i,j) \in A \mid i \in S, j \in S\}.
\end{aligned}$$

Then the ATS polytope is the convex hull of $0-1$ points in $\mathbb{R}^A$ satisfying

$$x(\delta^+(\{i\})) = 1 \qquad \forall i \in N \tag{2.1a}$$

$$x(\delta^-(\{i\})) = 1 \qquad \forall i \in N \tag{2.1b}$$

$$x(\gamma(S)) \leq |S| - 1 \; \forall S \subset N, 2 \leq |S| \leq n-2 \tag{2.1c}$$

Equations (2.1a) and (2.1b) are the *degree equalities* and inequality (2.1c) is the *subtour elimination inequality.*

Denote by $\tilde{P}$ the submissive of the ATS polytope: the convex hull of all $0-1$ points in $\mathbb{R}^A$ satisfying the *degree inequalities*

$$\begin{aligned}
x(\delta^+(\{i\})) \leq 1 \; \forall i \in N \\
x(\delta^-(\{i\})) \leq 1 \; \forall i \in N
\end{aligned}$$

and the subtour elimination inequality. Note that $\tilde{P}$, as defined here, is equal to $sub(P)$, as defined by section 1.4, for many cases, including when defined over the complete graph [4]. For $F \subseteq A$, let $\tilde{P}[F] := \{x \in \tilde{P} \mid x_a = 0 \; \forall a \in F\}$.

We will consider all directed cycles to be simple. For $S \subset N$ and $S = \{i_1, i_2, \ldots, i_s\}$, let $C := \{(i_1, i_2), (i_2, i_3), \ldots, (i_{s-1}, i_s), (i_s, i_1)\}$ be a directed cycle visiting all of the nodes of $S$. Use $i_{j+1}$ ($i_{j-1}$) to denote the successor (predecessor) of $i_j$ in the cycle; in particular, $i_{s+1} \equiv i_1$ and $i_0 \equiv i_s$. A *chord* of $C$ is an arc $(i_a, i_b) \in A$ such that $i_b \neq i_{a+1}$. Let $R$ denote the set of all the chords of $C$.

Grötschel introduced and showed that the inequality $x(C) \leq |C| - 1$ defines a facet of $\tilde{P}[R]$ [12]. For $R = \{a_1, a_2, \ldots, a_m\}$, then the *lifted cycle inequality* for the ATS polytope is

$$\alpha x := x(C) + \sum_{i=1}^{m} \alpha_{a_i} x_{a_i} \leq \alpha_0 := |C| - 1.$$

The coefficients $\alpha_{a_i}$ are computed sequentially, that is $\alpha_{a_j}$ is the maximum value for which the inequality

$$x(C) + \sum_{i=1}^{j-1} \alpha_{a_i} x_{a_i} + \alpha_{a_j} x_{a_j} \leq |C| - 1$$

is valid for $\tilde{P}[\{a_{j+1}, \ldots, a_m\}]$. By standard lifting theory [21], the lifted cycle inequality defines a facet of $\tilde{P}$.

Different sequences of the chords may lead to different inequalities. The coefficients take values $0, 1$, or $2$, depending on the position of the chord in the sequence. The coefficient is largest if the chord is lifted first, and is a monotonic non-increasing function of its position in the sequence (with the position of the other chords fixed). By this monotonicity property, there is a canonical ordering for a lifting sequence of a given lifted cycle inequality, with all of the chords (in any order) with coefficient 2 lifted first, followed by those with coefficient 1 (in any order), and finally those with coefficient 0 (in any order). If we consider swapping two adjacent chords in a sequence, either their coefficients stay the same, or the coefficient of the chord lifted earlier increases and the other decreases. The second case cannot occur if a chord with coefficient 2 is lifted earlier or a chord with coefficient 0 is lifted later, so a series of swaps leading to the canonical order will not change the final inequality.

Lifted cycle inequalities were studied in the context of $\tilde{P}$ and not $P$, since the former has full dimension and the latter does not. Whether lifted cycle inequalities defined facets of $P$ remained an open question until Balas and Fischetti's results on general monotonizations of polyhedra [3] (summarized in Section 1.4.)

This result specializes the results on monotonization of polyhedra to the ATS polytope. The following notation and results are from [3]. Associate with the directed graph $G = (V, A)$ the bipartite graph $B[G] = (V^+ \cup V^-, E)$ with two nodes $v^+ \in V^+$ and $v^- \in V^-$ for each node $v$ of $G$ and the edge $[i^+, j^-] \in E$ for each arc $(i, j)$ of $G$. Since the coefficient matrix of the equality system of the ATS polytope (2.1a) and (2.1b) is the node-edge incidence matrix of $B[G]$, a subset $\tilde{A}$ of arcs of $G$ is a basis of the equality system if and only if $\tilde{A}$ is a spanning tree of $B[G]$.

We next define two auxiliary sub-graphs of $G$, based on an inequality $\alpha x \leq \alpha_0$ which defines a facet of *g-mon*$(P)$. Let

$$\begin{array}{ll} A^0 := \{(i,j) \in A \mid \alpha_{i,j} = 0\} & A^+ := A \setminus A^0 \\ G^0 := (V, A^0) & G^+ := (V, A^+) \end{array}$$

$G^+$ is called the *support graph* of $\alpha x \leq \alpha_0$, while $G^0$ is the complement of $G^+$. Applying the definitions of support reduced inequality and $G^0$ to the characterization of bases of the equality system of $\tilde{P}$, $\alpha x \leq \alpha_0$ is support reduced with respect to $P$ if and only if $B[G^0]$ is connected. The following specializes Theorem 4 to the ATS polytope:

**Theorem 6.** *Let $\alpha x \leq \alpha_0$ be a non-trivial facet-defining inequality for $P$, that is valid for* g-mon$(P)$. *Then $\alpha x \leq \alpha_0$ defines a facet of* g-mon$(P)$ *if and only if the bipartite graph $B[G^0]$ is connected.*

By further specializing this condition, we can find a sufficient condition for $\alpha x \leq \alpha_0$ to be strongly support reduced. When $B[G^+]$ has an isolated node $h$ and $G^0$ has an arc $a^* \notin \delta(h)$, then the arcs $\delta(h) \cup a^*$ are a spanning tree of $B[G^0]$.

Let $G_h := G - \{h\}$. Call two arcs of $G_h$ *$\alpha$-adjacent* in $G_h$ if they are contained in a tour $T$ of $G_h$ such that $\alpha(T) = \alpha_0$. Define another associated graph, the $\alpha$-adjacency graph $G_h^* := (V^*, E^*)$, with a vertex for every arc in $A^0 \setminus \delta(h)$, and an edge for every $a, b \in V^*$ such that $a$ and $b$ are $\alpha$-adjacent in $G_h^*$. Then for $G_h^*$:

**Lemma 7.** *Let $\alpha x \leq \alpha_0$ be valid for sub$(P)$, with $G^+$ having an isolated node $h$ and $G^0$ having an arc $a^* \notin \delta(h)$. If the $\alpha$-adjacency graph $G_h^*$ is connected, then $\alpha x \leq \alpha_0$ is strongly supported reduced.*

Notice that this result holds for sub$(P)$, not the more general *g-mon*$(P)$. Finally, A sufficient condition for $G_h^*$ to be connected is

**Theorem 8.** *Let $\alpha x \leq \alpha_0$ define a non-trivial facet of sub$(P)$ and a proper face of $P$. Assume $G^+$ has two isolated nodes, $h$ and $k$. If the bipartite graph $B[G^0 - \{h,k\}]$ is connected, then $\alpha x \leq \alpha_0$ is strongly support reduced.*

One sufficient condition for $B[G^0 - \{h,k\}]$ to be connected is the existence of isolated nodes of $G^+$:

**Corollary 9.** *Let $\alpha x \leq \alpha_0$ define a non-trivial facet of sub$(P)$ and a proper face of $P$. If $G^+$ has three isolated nodes, then $\alpha x \leq \alpha_0$ defines a facet of $P$.*

Finally, for any lifted cycle inequality, the nodes not in the cycle will be isolated in $G^+$. It follows immediately that

**Theorem 10.** *The lifted cycle inequalities define facets of $P$ for all $|C| \leq n-3$.*

Finally, we summarize the results of [4] to determine coefficients for lifted cycle inequalities based on patterns of chords. Denote by $V(F)$ the nodes spanned by an arc set $F \subseteq A$. A chord set $H \subset R$ is *2-liftable* if there exists a chord sequence that produces lifted coefficient 2 for every chord in $H$. We may assume such a sequence is canonical; then $H$ is 2-liftable if and only if $x(C) + 2x(H) \leq |C| - 1$ is a valid inequality for $\tilde{P}[R \setminus H]$. Showing $H$ does not contain certain patterns of chords is necessary and sufficient to show that $H$ is 2-liftable.

Define two arcs as *compatible* if there exists a tour containing both. Arcs $(i, j)$ and $(u, v)$ are compatible if and only if $i \neq u$, $j \neq v$ and they do not form a 2-cycle. Two arcs which are not compatible are *incompatible*. For a chord $(i_a, i_b) \in R$, the *internal* nodes of $C$ with respect to $(i_a, i_b)$ are the nodes $\{i_a, i_{a+1}, \ldots, i_{b-1}, i_b\}$, and the *external* nodes of $C$ with respect to $(i_a, i_b)$ are the nodes $\{i_{b+1}, \ldots, i_{a-1}\}$.

For two chords $(i_a, i_b)$ and $(i_c, i_d)$, we say that $(i_c, i_d)$ *crosses* $(i_a, i_b)$ if $(i_a, i_b)$ and $(i_c, i_d)$ are compatible and if $i_c$ and $i_d$ are not both internal or external with respect to $(i_a, i_b)$. This is a symmetric relationship: $(i_c, i_d)$ crosses $(i_a, i_b)$ if and only if $(i_a, i_b)$ crosses $(i_c, i_d)$. Two chords which do not cross are *non-crossing*; all incompatible chords are non-crossing.

A *noose* in $C \cup R$ is a simple, alternating (in direction) cycle $Q := \{a_1, b_1, a_2, b_2, \ldots, a_q, b_q\}$ of $2q \geq 4$ distinct arcs $a_i \in R$ and $b_i \in V(C)$ for $i = 1, \ldots, q$ with all arcs adjacent in $Q$ incompatible (including $a_1$ and $b_q$) and pairwise non-crossing.

Balas and Fischetti proved the following of 2-liftable chord sets [4]:

**Theorem 11.** *A chord set $H \subset R$ is 2-liftable if and only if $C \cup H$ contains no pair of crossing chords and no noose.*

**Theorem 12.** *Let $(i_a, i_b)$ be a chord of $C$ such that $\alpha_{i_a, i_b} = 2$. The following chords have coefficient 0:*

$$\begin{array}{ll} (i_j, i_{a+1}) & j = b, b+1, \ldots, a-1 \\ (i_{b-1}, i_l) & l = b+1, b+2, \ldots a \end{array}$$

**Corollary 13.** *If chord $(i_a, i_{a+2})$ has coefficient 2, then all chords incident to $i_{a+1}$ will have coefficient 0.*

The lifted cycle inequalities generalize a number of inequalities for the ATS polytope, including the $D_k^+$, $D_k^-$ and odd CAT inequalities (see [5] for a description of these inequalities.) Lifted cycle inequalities do not include the subtour elimination inequalities, since the first chord lifted always has coefficient 2. Balas and Fischetti showed new, large families of facet defining inequalities for the ATS polytope which were used to derive new facet defining inequalities of the STS polytope. Existence of these chords patterns can be checked in polynomial time, suggesting efficient separation routines should be easy to develop; however, separation remains an open question in general.

# Chapter 3

# The X-cycle Inequality

In this chapter, we will briefly review previous work at defining inequalities for the TDTS polytope which adapt the idea of a cycle inequality, and then define a new inequality, which we call the x-cycle inequality, which is derived from the lifted subtour elimination constraint for the TDTSP.

## 3.1 Other cycle-type inequalities for the TDTS polytope

A given cycle inequality for the ATSP excludes from the solutions paths which revisit the nodes in the cycle. There are multiple ways to adapt this idea to the TDTSP. One method is to exclude from the solutions paths which go through node $(i, t)$, then $r-1$ other nodes, and then $(i, t+r)$. These are called *r-cycles*, and have been used in [1, 6, 16]. Abeledo, et al. defined $r$-cycle inequalities for the PQ formulation [1]. Let $X$ be a subset of connected nodes of the extended network, $X \subseteq V$, not containing 0 or $T$. For $e \in \delta^-(X)$, define the set of *compatible arcs* of $e$ with respect to $X$ as the subset $C(X,e) \subseteq \delta^+(X)$, such that for each $f \in C(X,e)$, there exists an s-path entering $X$ for the first time by $e$ and leaving $X$ for the first time by $f$. Let $((i,t),(u_i,t{+}1),\ldots,(u_{r-1},t{+}r{-}1),(i,t{+}r)$ be a minimal $r$-cycle, and let $X := \{(u_1,t{+}1),\ldots,(u_{r-1},t{+}r{-}1)\}$. The *r-cycle elimination inequality* is

$$x^t_{i,u_1} \leq x(C(X),(i,u_1,t)).$$

They show that 2-cycle elimination constraints are facet-defining for $n \geq 6$, and conjecture that $r$-cycle inequalities are facet-defining in general [1].

Bigras, et al. used a reformulation of the PQ formulation [6]. Solving the reformulated MILP was aided by solving a subproblem, also an MILP. The constraints of this subproblem were strengthened by adding *r-cycle inequalities* for an $r$-cycle $((i_1,t),(i_2,t+1),\ldots,(i_r,t+r-1),(i_1,t+r))$ the type

$$x^t_{i_1,i_2} + x^{t+1}_{i_2,i_3} + \cdots + x^{t+r-1}_{i_r,i_1} \leq r-1.$$

Finally, Miranda-Bront, et al. [16] added r-cycle inequalities to the formulation introduced in [20]. This formulation requires the *assignment variables* $y_i^t$, where

$$y_i^t = \begin{cases} 1 & \text{if node } i \text{ is visited in position } t \\ 0 & \text{otherwise} \end{cases}$$

They introduce the *time-dependent cycle inequalities* for an r-cycle $((i_1, t), (i_2, t+1), \ldots, (i_r, t+r-1), (i_1, t+r))$

$$\sum_{j=1}^{r} x_{i_j, i_j+1}^{t+j-1} \leq \sum_{j=1}^{r-1} y_{i_j}^{t+j}.$$

These *time-dependent cycle inequalities* can be lifted: the lifted TD cycle inequalities are facet-defining [16].

## 3.2 A new class of inequality

Our approach will be to find an inequality which excludes from the solutions s-paths which revisit the nodes a subset $S \subset N$ for which we choose an order, like cycle inequalities for the ATSP. We will use the subtour elimination inequality to derive the *x-cycle inequality.* For the ATSP, the subtour elimination inequality (or subtour elimination constraint, SEC) for a subset $S$ of the nodes can be expressed equivalently as:

$$x(S, S) \leq |S| - 1 \text{ , or,} \tag{3.1a}$$
$$x(S, \bar{S}) \geq 1 \tag{3.1b}$$

The *internal* (3.1a) and *external* (3.1b) forms are expressed in terms of the arcs within the set or leaving the set, respectively. The external form may be derived from the internal form (and vice-versa) by substituting the degree constraint equality (2.1a) over the nodes $v \in S$.

We begin by deriving the internal form of the Lifted Subtour Elimination Constraint for the TDTSP (Lifted SEC), defined by Abeledo, et al. [1]. For $S \subset N$, $1 < |S| < n$, the LSEC is

$$\sum_{t=|S|}^{n-1} \sum_{i \in S} \sum_{j \notin S} x_{i,j}^t + \sum_{j \in S} \sum_{i \in N(j)} x_{i,j}^{n-1} \geq 1 \text{ , or,} \tag{3.2a}$$
$$\sum_{t=|S|}^{n-1} \sum_{i \in S} \sum_{j \notin S} x_{i,j}^t \geq 1 - \sum_{j \in S} \sum_{i \in N(j)} x_{i,j}^{n-1}. \tag{3.2b}$$

None of the terms of (3.2a) or (3.2b) are contained within the set $S$, but by repeated application of the degree constraints of the TDTSP, we can get an equivalent form. An equivalent to the degree constraint (1.3d) is

$$\sum_{t=1}^{n-1} \sum_{j \in N(i)} x_{i,j}^t + x_{i,T}^n = 1 \qquad \forall i \in N. \tag{3.3}$$

Split the summation over $t$ into $t = 1 \dots |S| - 1$ and $t = |S| \dots n-1$,

$$\sum_{t=1}^{|S|-1} \sum_{j \in N(i)} x_{i,j}^t + \sum_{t=|S|}^{n-1} \sum_{j \in N(i)} x_{i,j}^t + x_{i,T}^n = 1 \qquad \forall i \in N. \tag{3.4}$$

Now split the second summation into $j \in N(i) \cap S$ and $j \notin S$,

$$\sum_{t=1}^{|S|-1} \sum_{j \in N(i)} x_{i,j}^t + \sum_{t=|S|}^{n-1} \sum_{j \in N(i) \cap S} x_{i,j}^t + \sum_{t=|S|}^{n-1} \sum_{j \notin S} x_{i,j}^t + x_{i,T}^n = 1 \qquad \forall i \in S$$

$$\sum_{t=|S|}^{n-1} \sum_{j \notin S} x_{i,j}^t = 1 - \sum_{t=|S|}^{n-1} \sum_{j \in N(i) \cap S} x_{i,j}^t - x_{i,T}^n - \sum_{t=1}^{|S|-1} \sum_{j \in N(i)} x_{i,j}^t \qquad \forall i \in S.$$

Summing over $i \in S$ yields

$$\sum_{i \in S} \sum_{t=|S|}^{n-1} \sum_{j \notin S} x_{i,j}^t = |S| - \sum_{i \in S} \sum_{t=|S|}^{n-1} \sum_{j \in N(i) \cap S} x_{i,j}^t - \sum_{i \in S} x_{i,T}^n - \sum_{i \in S} \sum_{t=1}^{|S|-1} \sum_{j \in N(i)} x_{i,j}^t. \tag{3.5}$$

Since the left-hand side of (3.2b) and (3.5) are equal,

$$|S| - \sum_{i \in S} \sum_{t=|S|}^{n-1} \sum_{j \in N(i) \cap S} x_{i,j}^t - \underbrace{\sum_{i \in S} x_{i,T}^n}_{(a)} - \sum_{i \in S} \sum_{t=1}^{|S|-1} \sum_{j \in N(i)} x_{i,j}^t \geq 1 - \underbrace{\sum_{i \in S} \sum_{h \in N(i)} x_{h,i}^{n-1}}_{(b)}. \tag{3.6}$$

The inequality can be simplified. Summing the flow constraints (1.2e) over $i \in S$ gives

$$\sum_{i \in S} \sum_{h \in N(i)} x_{h,i}^{n-1} = \sum_{i \in S} x_{i,T}^n, \tag{3.7}$$

eliminating summations (a) and (b) in (3.6). The inequality becomes, after rearranging,

$$\sum_{t=1}^{|S|-1} \sum_{i \in S} \sum_{j \in N(i)} x_{i,j}^t + \sum_{t=|S|}^{n-1} \sum_{i \in S} \sum_{j \in N(i) \cap S} x_{i,j}^t \leq |S| - 1, \text{ or,} \tag{3.8a}$$

$$\sum_{t=1}^{|S|-1} \sum_{i \in S} \sum_{j \notin S} x_{i,j}^t + \sum_{t=1}^{n-1} \sum_{i \in S} \sum_{j \in N(i) \cap S} x_{i,j}^t \leq |S| - 1. \tag{3.8b}$$

Inequalities (3.8a) and (3.8b) are equivalent to (3.2a), but expressed in terms of arcs inside $S$.

We can now find a cycle-type inequality for the subset $S$. Just as an s-path is the representation of a Hamiltonian path in the complete graph when embedded in the extended network, we need also terms relating cycles of the complete graph to their representation in the extended network.

**Definition 1.** For a subset $S \subset N$ with $|S| \geq 3$, let $(i_1, \ldots, i_s)$ be an order of the elements of $S$. Let $i_{j+1}$ $(i_{j-1})$ be the successor (predecessor) of $i_j$ in this order, in particular $i_{s+1} \equiv i_1$ and $i_0 \equiv i_s$. Define the *extended cycle* (or *x-cycle*) $C$ in the extended network $G(n)$ to be the subgraph of $G(n)$ induced by arcs $\{(i_j, i_{j+1}, t) \mid j = 1, \ldots, |S|, t = 1, \ldots, n-1\}$.

We will refer to these arcs as the *x-cycle arcs*. The actual number of nodes and arcs in $C$ as a subgraph of $G(n)$ is less important than the size of the $S \subset N$ from which it is defined. We will refer to the size of the x-cycle as the size of the set $S$, $|C| := |S|$. For convenience, we will denote by $l_{jk}$ the number of x-cycle arcs from $i_j$ to $i_k$, so $i_{j+l_{jk}} \equiv i_k$.

**Definition 2.** For two non-consecutive nodes $i_j$ and $i_{j+l}$ for $l = 2, \ldots, |C|-1$ in an x-cycle $C$, we define the arc $(i_j, i_{j+l}, t)$ for any $t$ with $1 \leq t \leq n-1$ to be an *extended chord* (or *x-chord*) of length $l$.

Let $R$ be the set of all x-chords of an x-cycle $C$ at times $t = 1 \ldots n-1$. We will denote by $K_l^t$ the set of all length $l$ x-chords at time $t$, that is, $K_l^t := \{(i_j, i_{j+l}, t) \mid j = 1, \ldots, s\}$. We also may refer to $K_l^t$ as the $l$-x-chords at $t$.

**Definition 3.** We define *tangent* to an x-cycle $C$ as any arc in the extended network whose tail is in $V(C)$ but whose head is not in $V(C)$, that is, $(i_j, v, t)$ with $i_j \in V(C), v \notin V(C), 1 \leq t \leq n-1$ is a tangent.

Let $T$ be the set of tangents in times 1 to $|C|-1$, $T := \{(i_j, v, t) \mid i_j \in V(C), v \notin V(C), t = 1, \ldots, |C|-1\}$.

Let $P$ be the TDTS polytope, and let $\tilde{P}$ be the submissive of $P$. The following *x-cycle inequality* is derived from the internal form of the Lifted SEC (3.8b) by dropping the arcs $R \cup T$. The inequality

$$x(C) = \sum_{t=1}^{n-1} \sum_{j=1}^{s} x_{i_j, i_{j+1}}^t \leq s - 1 \tag{3.9}$$

is valid for $\tilde{P}[R \cup T]$. We will use (3.9) as the basis for developing lifted cycle inequalities for the TDTS polytope.

Alternately, we could choose the inequality

$$\sum_{t=1}^{n-1} \sum_{j=1}^{s} x_{i_j, i_{j+1}}^t + \sum_{t=1}^{s-1} \sum_{j=1}^{s} \sum_{v \notin C} x_{i_j, v}^t \leq s - 1 \tag{3.10}$$

as the basis for deriving lifted x-cycle inequalities. We've chosen (3.9) for a number of reasons. We will show that (3.10) is a lifting of (3.9), so the latter is more general. Liftings of (3.10) are less “interesting” in so far as most of the lifted coefficients will be 1, regardless of the lifting order. Finally, some of the lifting sequences of (3.10) will yield the Lifted SEC, while any lifting sequence of (3.9) with an x-chord first will *not* lead to the Lifted SEC.

Before presenting any results on lifted x-cycle inequalities, we should establish some conventions for the figures which accompany the inequalities. For simplicity of the figures, and to emphasize the relationship between x-cycles in the extended network and cycles in the complete graph, sub-s-paths in the extended network and subgraphs of the extended network will be shown as sub-paths and subgraphs of the complete graph. For a particular set of arcs $\{(u, v, t) \,|\, t = 1, \ldots, n-1\}$ in the extended network, it will be represented by the arc $(u, v)$ in the figure. To show a subset of the arcs $\{(u, v, t) \,|\, t = 1, \ldots, n-1\}$ for specific times $t$, the arc $(u, v)$ will be annotated with these time indices in the figure. We will often omit isolated vertices from figures. Any number of intermediate nodes between two nodes in a cycle will be indicated by a dotted line.

# Chapter 4

# Lifting X-chords and Tangents

In this chapter, we will establish some rules for determining the lifted coefficients of the x-chords and tangents, similar to the results of Balas and Fischetti on lifted coefficients for lifted cycle inequalities. Where possible, we will extend their results on patterns of arcs to the time-dependent context.

Let $R \cup T := \{x_1, x_2, \ldots, x_m\}$ be a chosen lifting sequence of the x-chords and tangents of an x-cycle $C$. Assume that the inequality $x(C)+\sum_{i=1}^{j} \alpha_j x_j \leq |C|-1$ is valid for $\tilde{P}[\{x_{j+1}, \ldots, x_m\}]$. In order to determine $\alpha_{j+1}$, in general, we would need to solve

$$\max \left\{ x(C) + \sum_{i=1}^{j} \alpha_i x_i \;\middle|\; x \in \tilde{P}[\{x_{j+2}, \ldots, x_m\}],\, x_{j+1} = 1 \right\}. \tag{4.1}$$

We would like to be able to find the maximum coefficient for $x_{j+1}$ without solving this maximization problem.

For the lifted cycle inequalities of the ATS polytope, the flow constraints limit the paths that can be chosen and provide necessary and sufficient conditions for the sets of chords which can be 2-lifted. For the lifted x-cycle inequalities of the TDTS polytope we will show a weaker result: the upper limit of a lifted coefficient, given the coefficients of other arcs (x-chords and tangents) in the inequality. Both results rely on particular patterns of arcs. When we say that a lifted coefficient in a given sequence is *maximal*, we mean that in the given sequence it takes the same coefficient as if it were first in sequence. For the ATSP, all chords could take at most coefficient 2, so maximal-lifting is equivalent to 2-lifting.

We cannot give as strong a result as for the ATSP case, because the patterns of arcs are much less general. Recall that for the ATSP case, a pair of crossing chords cannot both be 2-lifted, and any chord set containing chords which are pairwise crossing cannot be 2-lifted. In the TDTSP case, we will have to strengthen the conditions under which a pair of crossing x-chords cannot be maximally-lifted to take into account the times of the x-chords. Because of this added condition, there are patterns of three or more x-chords for which each pair could be maximally-lifted, but all cannot be maximally-lifted; sufficient conditions for a maximally-liftable set of chords must include such patterns.

We will need to revise the terms introduced in [4] to take into account the time coordinate of arcs in the extended network:

**Definition 4.** Two arcs $(u, v, t_1), (x, y, t_2) \in A$ are *compatible* if they can both appear in some s-path. It is easy to check that $(u, v, t_1)$ and $(x, y, t_2)$ are compatible if $u \neq x$, $v \neq y$, $t_1 \neq t_2$, and $(u, v)$ and $(x, y)$ do not form a 2-cycle in $K(n)$, and if one of these conditions hold: (i) $t_1 \neq t_2 \pm 1$ (ii) $t_2 = t_1 + 1$, $v = x$ and $u \neq y$, or (iii) $t_1 = t_2 + 1$, $y = u$, and $v \neq x$ . Two arcs which are not compatible are called *incompatible*.

Since we are working with the submissive of the TDTS polytope, $\tilde{P}$, we need to show subsets of s-paths, or sub-s-paths, in the polytope. By definition, a sub-s-path is a set of arcs which are all compatible.

**Definition 5.** Let $(i_a, i_{a+l}, t) \in R$, then the *interior* of $C$ with respect to $(i_a, i_{a+l})$ are the nodes $V(C_{i_a,i_{a+l}}) := \{(i_a, t), (i_{a+l}, t), \ldots, (i_{a-1}, t) | t = 1, \ldots, n-1\} \subset V$. We will call $(i_j, t_2) \in V(C_{i_a,i_{a+l}})$ an *internal node* with respect to $(i_a, i_{a+l}, t)$. The *exterior* of $C$ with respect to $(i_a, i_{a+l}, t)$ are the nodes $V(C) \setminus V(C_{i_a,i_{a+l}}) = \{(i_{a+1}, t), \ldots, (i_{a+l-1}, t) \mid t = 1, \ldots, n-1\}$. We will call $(i_j, t) \in V(C) \setminus V(C_{i_a,i_{a+l}})$ an *external node* with respect to $(i_a, i_{a+l}, t)$.

Notice that the arcs $\{(i_a, i_{a+l}, t) \mid t = 1, \ldots, n-1\}$ together with the x-cycle arcs spanning $V(C_{i_a,i_{a+l}})$ form another x-cycle, which we denote $C_{i_a,i_{a+l}}$. As before, an x-chord $(i_c, i_d, t_2)$ crosses the x-chord $(i_a, i_b, t_1)$ if one, but not both, of $i_c$ and $i_d$ are internal with respect to $(i_a, i_b, t_1)$.

As in the ATSP case, the flow constraints and the degree constraints of the TDTSP formulation will exclude arcs from being used in any s-path containing a particular x-chord or tangent. Any s-path which uses the x-chord $(i_a, i_b, \bar{t})$ cannot use the sets of x-cycle arcs $\{(i_a, i_{a+1}, t) \mid t = 1, \ldots, n-1\}$ and $\{(i_{b-1}, i_b, t) \mid t = 1, \ldots, n-1\}$, and in addition, it cannot use more than $l_{ab} - 1$ of the x-cycle arcs internal to the x-chord. In total, any s-path which uses the x-chord $(i_a, i_b, \bar{t})$ can use at most $|C| - 3$ x-cycle arcs.

Any s-path (or sub-s-path) which uses the tangent $(i_a, v, \bar{t})$ cannot use the arcs $\{(i_a, i_{a+1}, t) \mid t = 1, \ldots, n-1\}$. In addition, if $\bar{t} \leq |C| - 1$, any path containing $(i_a, v, \bar{t})$ cannot cover all of the nodes before the tangent, so a path containing $(i_a, v, \bar{t})$ can contain at most $|C| - 2$ x-cycle arcs. Figure 4.1 shows a simplified x-cycle with an x-chord and tangent, and the arcs which could be included in a sub-s-path with the maximum number of x-cycle arcs.

From the definition of compatible arcs, we can show

**Lemma 14.** *Let $R \cup T := (x_1, x_2, \ldots, x_m)$ be a chosen lifting order. If $x_j$ and $x_{j+1}$ are not compatible, then the lifted coefficients of $x_j$ and $x_{j+1}$ will be the same if they are swapped in the order.*

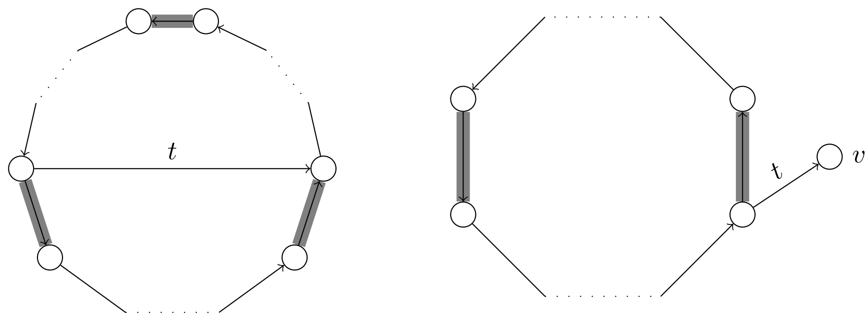


Figure 4.1: An x-cycle with an x-chord, left, and tangent, right. The non-highlighted arcs show what a sub-s-path containing the maximum number of x-cycle arcs will look like.

*Proof.* Let $\tilde{x}^j$ be the path which maximizes (4.1) for determining the coefficient $\alpha_j$ and let $\tilde{x}^{j+1}$ be the path which maximizes (4.1) for determining the coefficient $\alpha_{j+1}$. Since $x_j$ and $x_{j+1}$ are not compatible, $\tilde{x}^j_{j+1} = 0$ and $\tilde{x}^{j+1}_j = 0$, the following equalities hold:

$$\max_{\substack{x \in \tilde{P}[\{x_{j+1},\ldots,x_m\}] \\ x_j = 1}} \left\{ x(C) + \sum_{i=1}^{j-1} \alpha_i x_i \right\} = \max_{\substack{x \in \tilde{P}[\{x_{j+2},\ldots,x_m\}] \\ x_j = 1}} \left\{ x(C) + \sum_{\substack{i=1, \\ i \neq j}}^{j+1} \alpha_i x_i \right\},$$

$$\max_{\substack{x \in \tilde{P}[\{x_{j+2},\ldots,x_m\}] \\ x_{j+1} = 1}} \left\{ x(C) + \sum_{i=1}^{j} \alpha_j x_j \right\} = \max_{\substack{x \in \tilde{P}[\{x_j, x_{j+2},\ldots,x_m\}] \\ x_{j+1} = 1}} \left\{ x(C) + \sum_{i=1}^{j-1} \alpha_i x_i \right\}.$$

Then the lifted coefficients of $x_j$ and $x_{j+1}$ will be the same if they are swapped in the order. $\square$

The maximum number of x-cycle arcs in a given sub-s-path will establish the maximum coefficients that chords and tangents may take, whether lifted first or after other arcs have been lifted. We will first look at x-chords lifted alone, extending the results of [4] where possible, and showing the maximum coefficients for the x-chords in particular patterns. Then we will give results on tangents lifted alone, and finally, some results on x-chords and tangents lifted together.

## 4.1 Maximum lifted coefficients of x-chords

As stated earlier, any sub-s-path containing an x-chord $(i_a, i_b, t)$ can contain at most $|C| - 3$ x-cycle arcs; however, in some cases, the time dependencies do not allow a sub-s-path with all $|C|-3$ x-cycle arcs. The maximum lifted coefficient for an x-chord follow naturally from this fact:

**Lemma 15.** *The lifted coefficient of a chord $x_j = (i_a, i_{a+l}, t)$ (when lifted first) is at least 3 if $n-l<t<l$.*

*Proof.* The sub-s-path in $\tilde{P}[(R\setminus\{x_j\}]\cup T]$ that maximizes $x(C)$ with $x_j = 1$ can use at most $|C|-3$ x-cycle arcs, from the flow and degree constraints. We will show that by the time constraints, such a path cannot exist. Notice that any sub-s-path containing $|C|-3$ x-cycle arcs and an x-chord can be divided into two parts: Int, covering the nodes interior to the x-chord; and Ext, covering the nodes exterior to the x-chord. We may either take Int followed by Ext; or vice-versa. The Int will require $|C|-l$ arcs to cover all of its nodes; the Ext will require $l-2$ arcs to cover all of its nodes.

Suppose we take Int followed by Ext. Since $n-l<t$ and $|C|\le n$, then $|C|-l<t$. If we begin Int at time 1—regardless of the arc we choose to start Int—we cannot include the x-chord at time $t$ in such a sub-s-path. The earliest that we can end Int is at time $t$, that is, ending with the x-chord. Next we take Ext immediately following Int. Ext will end at time $t+l-1$, since it requires 1 arc to go from $i_{a+l}$ to $i_{a+1}$ and $l-2$ arcs to cover the exterior nodes. But since $t>n-l$, $t+l-1>n-1$, that is, there is not enough time to cover all of the external nodes.

Instead, we try Ext followed by Int. If we start Ext at time 1, it ends at time $l-2$; then the arc at time $l-1$ will be an x-chord between the Ext and Int. The first arc in Int will be at time $l$. But since $t<l$, we cannot use the x-chord in Int if we start Ext at time 1 or later.

Since there does not exist a sub-s-path with $|C|-3$ x-cycle arcs and the x-chord, the maximum lifted coefficient is at least 3. □

In the following proofs, we'll use the terms Int to mean any sub-s-path covering the internal nodes with respect to an x-chord, and Ext to mean any sub-s-path covering the external nodes. We do not, however, have unbounded lifted coefficients. In fact,

**Theorem 16.** *The maximum lifted coefficient of any x-chord is 3.*

*Proof.* Let $x_1 := (i_a, i_{a+l}, t) \in R$ be an x-chord which is first in the lifting order. Let $\tilde{x}$ any sub-s-path in $\tilde{P}[\{x_2,\ldots,x_m\}]$ which attains the maximum of (4.1). We would like a lower limit on $\tilde{x}(C)$. We define the following sub-s-paths

- $\textsc{Ext}_1 := \begin{cases} \emptyset & \text{if } t\le 2 \\ (i_{a+1}, i_{a+2}, 1), \ldots, (i_{a+t-2}, i_{a+t-1}, t-2) & \text{if } t \le l-1 \\ (i_{a+1}, i_{a+2}, 1), \ldots, (i_{a+l-2}, i_{a+l-1}, l-2) & \text{if } t\ge l \end{cases}$
- $\textsc{Int} := \{(i_a, i_{a+l}, t), (i_{a+l}, i_{a+l+1}, t+1), \ldots, (i_{a-2}, i_{a-1}, t+|C|-l-1)\}$
- $\textsc{Ext}_2 := \begin{cases} \emptyset & \text{if } t\ge l \\ (i_{a+t}, i_{a+t+1}, t+|C|-l), \ldots, (i_{a+l-2}, i_{a+l-1}, |C|-1) & \text{otherwise} \end{cases}$

INT contains the x-chord and the interior x-cycle arcs. $\textsc{Ext}_1$, if non-empty, contains the first $t-2$ x-cycle arcs of those exterior to the x-chord, or all of the x-cycle arcs of those exterior to the x-chord; $\textsc{Ext}_2$, if non-empty, contains the remaining exterior arcs. In all cases, $\textsc{Ext}_1 \cup \textsc{Int} \cup \textsc{Ext}_2$ contains at least $|C|-4$ x-cycle arcs, and they are all compatible. Let $\bar{x}$ be the characteristic vector of the sub-s-path $\textsc{Ext}_1 \cup \textsc{Int} \cup \textsc{Ext}_2$. Since $\bar{x} \in \tilde{P}[\{x_2, \ldots, x_m\})]$ and $\bar{x}_1 = 1$, $\tilde{x}(C) \geq \bar{x}(C) = |C|-4$. Since $\alpha_1 = |C|-1-\tilde{x}(C)$, then $\alpha_1 \leq 3$ and the result is shown. □

Notice that if either of the sub-s-paths in Lemma 15 existed, then the maximum lifted coefficient would be 2:

**Lemma 17.** *The maximum lifted coefficient of an x-chord $x_j = (i_a, i_{a+l}, t)$ is 2 if $t \leq |C|-l$ or $t \leq n-l$ or $t \geq l$.*

*Proof.* It is enough to show that the inequality

$$x(C) + 3x^t_{i_a, i_{a+l}} \leq |C|-1$$

is not valid for $\tilde{P}[(R \setminus \ \{x_j\}) \cup T]$. We show a sub-s-path containing $|C|-3$ x-cycle arcs and the x-chord.

If $t \leq |C|-l$, the sub-s-path

$$\begin{aligned}\{(i_{a-|C|+l+2}, i_{a-|C|+l+3}, 1), \ldots, (i_{a-1}, i_a, t-1),\\ (i_a, i_{a+l}, t), (i_{a+l}, i_{a+l+1}, t+1), \ldots, (i_{a-|C|+l}, i_{a-|C|+l+1}, |C|-l),\\ (i_{a+1}, i_{a+2}, |C|-l+2), \ldots, (i_{a+l-2}, i_{a+l-1}, |C|-1)\}\end{aligned}$$

has $|C|-3$ x-cycle arcs. If $|C|-l+1 \leq t \leq n-l$, the sub-s-path

$$\begin{aligned}\{(i_{a+l+1}, i_{a+l+2}, t-|C|+l+1), \ldots, (i_{a-1}, i_a, t-1), (i_a, i_{a+l}, t),\\ (i_{a+1}, i_{a+2}, t+2), \ldots, (i_{a+l-2}, i_{a+l-1}, t+l-1)\}\end{aligned}$$

has $|C|-3$ x-cycle arcs. Notice that these are the sub-s-paths with INT containing $|C|-l$ arcs (the x-chord and $|C|-l-1$ x-cycle arcs), followed by EXT containing $l-2$ x-cycle arcs.

If $t \geq l$, then the sub-s-path

$$\begin{aligned}\{(i_{a+1}, i_{a+2}, 1), \ldots, (i_{a+l-2}, i_{a+l-1}, l-2),\\ (i_a, i_{a+l}, t), (i_{a+l}, i_{a+l+1}, t+1), \ldots, (i_{a-2}, i_{a-1}, t+|C|-l-1)\}\end{aligned}$$

has $|C|-3$ x-cycle arcs. (Here, we have EXT before INT.) □

We can show necessary and sufficient conditions for 3-liftable x-chords:

**Theorem 18.** *An x-chord $(i_a, i_{a+l}, t)$ has maximum lifted coefficient 3 if and only if $n-l < t < l$.*

*Proof.* Sufficiency is proved by Lemma 15. Necessity is proved by Lemma 17. □

We can also guarantee that all x-chords have maximum lifted coefficient for appropriately sized cycles:

**Corollary 19.** *If $2|C| \leq n$, the maximum lifted coefficient of any x-chord $x_j = (i_a, i_{a+l}, t)$ (when lifted first) is 2.*

*Proof.* This follows from Theorem 18: if $2|C| \leq n$, then there do not exist any x-chords $(i_a, i_{a+l}, t)$ with $n - l < t < l$. □

Having established the maximum lifted coefficient for a single x-chord, we will now look at patterns of x-chords. The chord patterns we consider will have to take into account both the position of the x-chords in the x-cycle and their times. These results will be expressed in terms of the maximum sum of the coefficients of the x-chords. This is not to suggest that they are being simultaneously lifted; rather, it is a general way to state the maximum lifted coefficients of a set of x-chords, given a set of lifted x-chords.

## 4.2 Patterns of non-crossing x-chords

We will first consider patterns of non-crossing x-chords. We extended the definition of a noose from [4] to sets of $q \geq 2$ x-chords for which there exists a sub-s-path containing all $q$ x-chords and $|C| - 2q$ x-cycle arcs.

**Definition 6.** Let $U = \{(i_{a_1+l_1}, i_{a_1}, t_1), \ldots, (i_{a_q+l_q}, i_{a_q}, t_q)\} \subset R$ of $q \geq 2$ pairwise compatible x-chords, with $i_{a_j} = i_{a_{j+1}+l_{j+1}+1}$ and $i_{a_j}, i_{a_j+l_j}$ external nodes with respect to the other x-chords of $U$ for all $j = 1, \ldots, q$.

Let $j_1, \ldots, j_q$ be the sequence such that $t_{j_1} < t_{j_2} < \cdots < t_{j_q}$. Define $\underline{t_{j_k}}$ and $\underline{t'_{j_k}}$ by:

$$\underline{t_{j_1}} = \left\{ \begin{array}{rl} 1 & \text{if } t_{j_1} \leq l_{j_1}, \\ t_{j_1} - l_{j_1} + 1 & \text{otherwise} \end{array} \right\}$$

$$\underline{t_{j_k}} = \left\{ \begin{array}{rl} \underline{t'_{j_{k-1}}} + 2 & \text{if } t_{j_k} \leq \underline{t'_{j_{k-1}}} + l_{j_k} + 1, \\ \infty & \text{if } t_{j_k} \leq \underline{t'_{j_{k-1}}} + 1 \text{ or } \underline{t'_{j_{k-1}}} = \infty, \\ t_{j_k} - l_{j_k} + 1 & \text{otherwise} \end{array} \right\} \text{for k = 2, \ldots, q}$$

$$\underline{t'_{j_k}} = \left\{ \begin{array}{rl} \underline{t_{j_k}} + l_{j_k} - 1 & \text{if } \underline{t_{j_k}} \neq \infty, \\ \infty & \text{otherwise} \end{array} \right\} \qquad \text{for k = 1, \ldots, q}$$

$U$ is a *tight noose* if $\underline{t_{j_k}}$ and $\underline{t'_{j_k}}$ are defined (not $\infty$) for all $k = 1, \ldots, q$ and $\underline{t'_{j_q}} \leq n - 1$.

In the ATSP case, a noose is defined as both chords and cycle arcs; if we were to consider the x-chords in $U$ as chords in the cycle (rather than in the extended network), these chords and cycle arcs would be a noose by the definition in [4]. Notice that the first order of the indices orders the x-chords around the x-cycle, and the sequence $j_1, j_2, \ldots, j_q$ orders the x-chords in time.

The second part, defining $\underline{t_{j_k}}$ and $\underline{t'_{j_k}}$, guarantees we can find a sub-s-path containing all of the x-chords and a maximum number of x-cycle arcs. A sub-s-path $\mathcal{P}$ containing all of the x-chords in a tight noose $U$ can be decomposed into sub-s-paths $\mathcal{P}_{j_1}, \mathcal{P}_{j_2}, \ldots, \mathcal{P}_{j_q}$, each containing one x-chord of $U$. $\underline{t_{j_k}}$ defines the time of the first arc in $\mathcal{P}_{j_k}$, given that $\mathcal{P}_{j_1}, \ldots, \mathcal{P}_{j_{k-1}}$ contain $\overline{l_{j_1}, \ldots, l_{j_{k-1}}}$ arcs, respectively. $\underline{t'_{j_k}}$ is the time of the last arc in $\mathcal{P}_k$ if the first arc is at time $\underline{t_{j_k}}$ and it contains $\overline{l_{j_k}}$ arcs.

**Theorem 20.** *If $U = \{(i_{a_1+l_1}, i_{a_1}, t_1), \ldots, (i_{a_q+l_q}, i_{a_q}, t_q)\} \subset R$ is a tight noose, then the inequality*

$$x(C) + \sum_{j=1}^{q} \alpha_j x^{t_j}_{i_{a_j+l_j}, i_{a_j}} \leq |C| - 1$$

*is not valid for $\tilde{P}[(R \setminus U) \cup T]$ if $\sum_{j=1}^{q} \alpha_j \geq 2q$.*

*Proof.* We need to show a sub-s-path $\mathcal{P}$ which violates the inequality. We assume that $\alpha$ has been chosen so that each $\alpha_j$ is a valid coefficient of the x-chord $(i_{a_j+l_j}, i_{a_j}, t_j)$; i.e., there does not exist a sub-s-path in $\tilde{P}[(R \setminus U) \cup T]$ with less than $q$ of the arcs in $U$ which violates the inequality. By this assumption, $\mathcal{P}$ must contain all of $U$ and at least $|C| - 2q$ x-cycle arcs.

Let $U = \{(i_{a_j+l_j}, i_{a_j}, t_j) \mid j = 1, \ldots, q\}$ be a tight noose. Define $C_j$ for $j = 1, \ldots, q$ as the x-cycle arcs spanning the nodes of $C$ that are internal with respect to $(i_{a_j+l_j}, i_{a_j}, t_j)$:

$$C_j := \{(i_{a_j}, j_{a_j+1}, t), (i_{a_j+1}, j_{a_j+2}, t), \ldots, (i_{a_j+l_j-1}, j_{a_j+l_j}, t) \mid t = 1, \ldots, n-1\}.$$

By the flow constraints, any sub-s-path containing $(i_{a_j+l_j}, i_{a_j}, t_j)$ cannot contain $(i_{a_j-1}, i_{a_j}, t)$ or $(i_{a_j+l_j}, i_{a_j+l_j+1}, t)$ for any times $t = 1, \ldots, n-1$ nor more than $l_j - 1$ x-cycle arcs of $C_j$; then $\mathcal{P}$ must contain exactly $|C| - 2q$ x-cycle arcs.

Build $\mathcal{P}$ containing $U$ and $|C| - 2q$ x-cycle arcs, from sub-s-paths $\mathcal{P}_j$ where $(i_{a_j+l_j}, i_{a_j}, t_j) \in \mathcal{P}_j$ and $c \in P_j \cap C \Rightarrow c \in C_j$. The definitions of $\underline{t_{j_k}}$ and $\underline{t'_{j_k}}$ give the earliest time for which $\mathcal{P}_{j_k}$ can start and end, respectively. Since $(i_{a_j+l_j}, i_{a_j}, t_j)$ is the only arc in $\mathcal{P}_j$ with a fixed time, in general the arcs of $C_j$ in $\mathcal{P}_j$ can be chosen to have any time before or after $t_j$, as long as $\mathcal{P}_{j-1}$ and $\mathcal{P}_{j+1}$ can still have $l_{j-1} - 1$ and $l_{j+1} - 1$ cycle arcs from $C_{j-1}$ and $C_{j+1}$, respectively.

Assume that sub-s-path $\mathcal{P}_{j_{k-1}}$ ends at time $\underline{t'_{j_{k-1}}}$. The next sub-s-path may start at the earliest at $\underline{t'_{j_{k-1}}}+2$, since non-consecutive x-cycle arcs require at least one arc between them in any s-path containing both. A sub-s-path starting at $\underline{t'_{j_{k-1}}}+2$ and containing $l_{j_k}$ arcs will end at time $\underline{t'_{j_{k-1}}}+1$. If $t_{j_k} \leq \underline{t'_{j_{k-1}}} + l_{j_k} + 1$, then there exists a valid sub-s-path starting at $\underline{t'_{j_{k-1}}}+1$ and taking $l_{j_k}$ consecutive

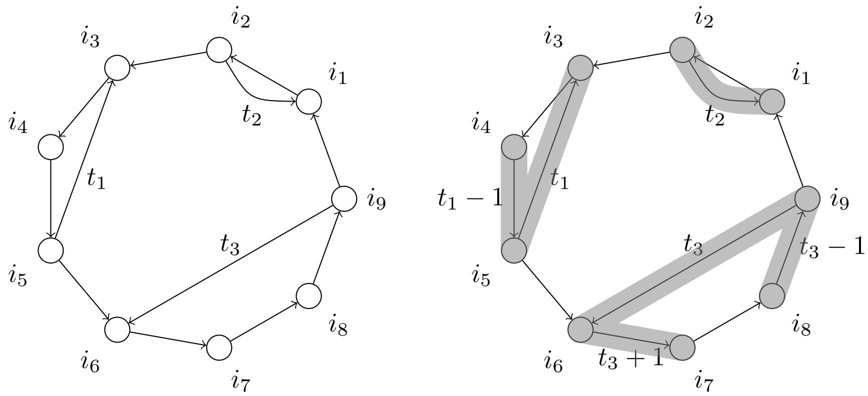


Figure 4.2: A tight noose in an x-cycle with 9 nodes, and with a sub-s-path $\mathcal{P}$ highlighted.

arcs, including the x-chord $(i_{a_{j_k}+l_{j_k}}, i_{a_{j_k}}, t_{j_k})$. If $t_{j_k} \leq \underline{t'_{j_{k-1}}} + 1$, then the x-chord is incompatible with the arcs of $\mathcal{P}_{j_{k-1}}$. In all other cases, the earliest sub-s-path containing the x-chord and $l_{j_k}$ consecutive arcs before it will start at $t_{j_k} - l_{j_k} + 1$, thus ending at $t_{j_k}$.

The conditions on $\underline{t_i}$ and $\underline{t'_i}$ guarantee that when $\mathcal{P}_{j_1}, \ldots, \mathcal{P}_{j_k}$ have their maximum number of x-cycle arcs, $\mathcal{P}_{j_{k+1}}$ can be chosen so that $\mathcal{P}_{j_1}, \ldots, \mathcal{P}_{j_k}, \mathcal{P}_{j_{k+1}}$ is a sub-s-path and $\mathcal{P}_{j_{k+1}}$ has $l_{j_{k+1}} - 1$ x-cycle arcs. For $(i_{a_j+l_j}, i_{a_j}, t_j)$, then $\mathcal{P}_j$ is one of:

$$(i_{a_j+l_j}, i_{a_j}, t_j), (i_{a_j}, i_{a_j+1}, t_j + 1), \ldots, (i_{a_j+l_j-2}, i_{a_j+l_j-1}, t_j + l_j - 1)$$

$$\begin{aligned}&(i_{a_j+l_j-k}, i_{a_j+l_j-k+1}, t_j - k), \ldots, (i_{a_j+l_j-1}, i_{a_j+l_j}, t_j - 1), (i_{a_j+l_j}, i_{a_j}, t_j),\\ &\qquad (i_{a_j}, i_{a_j+1}, t_j + 1), \ldots, (i_{a_j+l_j-k-2}, i_{a_j+l_j-k-1}, t_j + l_{i_j} - k - 1)\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for some } k = 1, \ldots, l_j - 1\end{aligned}$$

$$(i_{a_j+1}, i_{a_j+2}, t_j - l_{i_j}), \ldots, (i_{a_j+l_j-1}, i_{a_j}, t_j - 1), (i_{a_j+l_j}, i_{a_j}, t_j)$$

Since each $\mathcal{P}_j$ includes $l_j - 1$ cycle arcs in $C_j$, $\mathcal{P} := \bigcup_{i=1}^{q} \mathcal{P}_j$ will contain $|C| - 2q$ x-cycle arcs and all of $U$. $\mathcal{P}$ will be a point in $\tilde{P}[(R \setminus U) \cup T]$ which violates the inequality. □

Notice that a tight noose may include an x-chord which is 3-liftable. For example, when $n = 8$ and $|C| = 8$, the x-chord $(1, 6, 4)$ is 3-liftable, and it is in the tight noose $\{(3, 2, 1), (5, 4, 7), (1, 6, 4)\}$ (see Figure 4.3). When all x-chords are at most 2-liftable, i.e., when $2|C| \leq n$, then the tight noose is the only pattern of non-crossing x-chords which cannot be 2-lifted. Another set of non-crossing x-chords that cannot be 2-lifted would contradict the necessary and sufficient conditions for a set of 2-liftable chords for the ATSP lifted cycle inequalities.

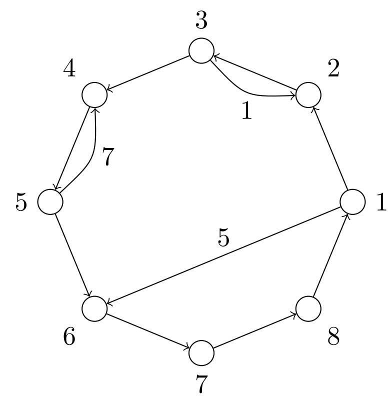


Figure 4.3: Tight noose with 3-liftable x-chord $(1, 6, 5)$

## 4.3 Patterns of crossing x-chords

Next, we will consider lifting two x-chords, say, $(i_a, i_{a+l_1}, t_1)$ and $(i_b, i_{b+l_2}, t_2)$, when one (but not both) of $i_a$ or $i_{a+l_1}$ is external with respect to $(i_b, i_{b+l_2}, t_2)$. In the ATSP case, the chords $(i_a, i_{a+l_1})$ and $(i_b, i_{b+l_2})$ (if compatible) would not be 2-liftable, because there always exists a sub-path containing both chords and at least $|C| - 4$ cycle arcs. Any sub-s-path containing both x-chords cannot contain any of the x-cycle arcs

$$\{(i_a, i_{a+1}, t), (i_{a+l_1-1}, i_{a+l_1}, t), (i_b, i_{b+1}, t), (i_{b+l_2-1}, i_{b+l_2}, t) \mid t = 1, \ldots, n-1\}.$$

We need conditions under which a sub-s-path containing both x-chords and at least $|C| - 4$ x-cycle arcs exists.

Recall that we denote by $l_{jk}$ the number of x-cycle arcs from $i_j$ to $i_k$, that is, $i_{j+l_{jk}} = i_k$.

**Definition 7.** Let $(i_a, i_b, t_1), (i_c, i_d, t_2) \in R$ be compatible. $(i_c, i_d, t_2)$ is a *reachable crossing x-chord* of $(i_a, i_b, t_1)$ if

1. $i_c$ internal and $i_d$ external with respect to $(i_a, i_b, t_1)$, $t_2 = t_1 + 1 + l_{bc}$, and either
   - (a) $t_1 - l_{ca} + 1 \geq 1$ and $t_2 + l_{ad} + l_{db} - 2 \leq n - 1$, or
   - (b) $t_1 - l_{ca} - l_{ad} + 2 \geq 1$ and $t_2 + l_{db} - 1 \leq n - 1$
2. $i_c$ external and $i_d$ internal with respect to $(i_a, i_b, t_1)$, $t_1 = t_2 + 1 + l_{da}$, and either
   - (a) $t_2 - l_{ac} + 1 \geq 1$ and $t_1 + l_{cb} + l_{bd} - 2 \leq n - 1$, or
   - (b) $t_2 - l_{ac} - l_{cb} + 2 \geq 1$ and $t_1 + l_{bd} - 1 \leq n - 1$

The definition requires that the x-chords be crossing, that is, either $i_c$ or $i_d$ internal to $(i_a, i_b, t_1)$, but not both. The relationship between $t_1$ and $t_2$ ensures that there is a sub-path containing both x-chords and only x-cycle arcs between them. The final conditions ensure that there is enough time to complete a sub-s-path containing $|C|-4$ x-cycle arcs and both x-chords. By definition, reachable crossing x-chords are not both maximally-liftable:

**Theorem 21.** *If a pair of x-chords $(i_a, i_b, t_1), (i_c, i_d, t_2) \in R$ are reachable crossing chords, then the sum of their lifted coefficients is at most 3.*

*Proof.* It is enough to show that inequality

$$x(C) + (4-\alpha)x^{t_1}_{i_a,i_b} + \alpha x^{t_2}_{i_c,i_d} \leq |C| - 1$$

for $\alpha = 1, 2, 3$ is not valid for $\tilde{P}[R \setminus \{(i_a, i_b, t_1), (i_c, i_d, t_2)\} \cup T]$. We will show a sub-s-path which violates the inequality.

If either of $(i_a, i_b, t_1)$ or $(i_c, i_d, t_2)$ is not 3-liftable, then the inequality is not valid for $\alpha = 1$ or $\alpha = 3$, respectively. One of the sub-s-paths in Corollary 17 is sufficient to show the result.

In each case in the definition, a sub-s-path containing both chords and at least $|C| - 4$ x-cycle arcs must be shown. Depending on which case holds, we can use one of the following sets to show the inequality is violated. Note that the set is indexed by the case in the definition for which it applies:

$$\begin{aligned}\mathcal{P}_{1a} = \{&(i_{c+1}, i_{c+2}, t_M), \ldots, (i_{a-1}, i_a, t_1 - 1),\\ &(i_a, i_b, t_1), (i_b, i_{b+1}, t_1 + 1), \ldots, (i_{c-1}, i_c, t_2 - 1),\\ &(i_c, i_d, t_2), (i_d, i_{d+1}, t_2 + 1), \ldots, (i_{b-2}, i_{b-1}, t'_M),\\ &(i_{a+1}, i_{a+2}, t_m), \ldots, (i_{d-2}, i_{d-1}, t'_m)\}\end{aligned} \tag{4.2}$$

$$\begin{aligned}\mathcal{P}_{1b} = \{&(i_{a+1}, i_{a+2}, t_m), \ldots, (i_{d-2}, i_{d-1}, t'_m),\\ &(i_{c+1}, i_{c+2}, t_M), \ldots, (i_{a-1}, i_a, t_1 - 1),\\ &(i_a, i_b, t_1), (i_b, i_{b+1}, t_1 + 1), \ldots, (i_{c-1}, i_c, t_2 - 1),\\ &(i_c, i_d, t_2), (i_d, i_{d+1}, t_2 + 1), \ldots, (i_{b-2}, i_{b-1}, t'_M)\}\end{aligned} \tag{4.3}$$

$$\begin{aligned}\mathcal{P}_{2a} = \{&(i_{a+1}, i_{a+2}, t_M), \ldots, (i_{c-1}, i_c, t_2 - 1),\\ &(i_c, i_d, t_2), (i_d, i_{d+1}, t_2 + 1), \ldots, (i_{a-1}, i_a, t_1 - 1),\\ &(i_a, i_b, t_1), (i_b, i_{b+1}, t_1 + 1), \ldots, (i_{d-2}, i_{d-1}, t'_M),\\ &(i_{c+1}, i_{c+2}, t_m), \ldots, (i_{b-2}, i_{b-1}, t'_m)\}\end{aligned} \tag{4.4}$$

$$\begin{aligned}\mathcal{P}_{2b} = \{&(i_{c+1}, i_{c+2}, t_m), \ldots, (i_{b-2}, i_{b-1}, t'_m),\\ &(i_{a+1}, i_{a+2}, t_M), \ldots, (i_{c-1}, i_c, t_2 - 1),\\ &(i_c, i_d, t_2), (i_d, i_{d+1}, t_2 + 1), \ldots, (i_{a-1}, i_a, t_1 - 1),\\ &(i_a, i_b, t_1), (i_b, i_{b+1}, t_1 + 1), \ldots, (i_{d-2}, i_{d-1}, t'_M)\}\end{aligned} \tag{4.5}$$

Each set contains both x-chords and $|C|-4$ x-cycle arcs. Each arc is compatible with the other arcs in its set. If the time coordinates $t_M$, $t'_M$, $t_m$ and $t'_m$ are in $\{1,\ldots,n-1\}$, then these sets are sub-s-paths which violate the inequality. I will refer to the part of the set containing the chords as the *major* subset, and the remainder the *minor* subset (disregarding their actual length.)

- case 1(a) holds: for $\mathcal{P}_{1a}$, it is enough to show that $t_M \geq 1$ and $t'_m \leq n-1$. For this subset, $t_M = t_1 - l_{ac} + 1$ and by taking the minor subset immediately after the major subset, $t'_m = t_2 + l_{bd} + l_{da} - 2$. By case 1(a), $t_M \geq 1$ and $t'_m \leq n-1$.
- case 1(b) holds: for $\mathcal{P}_{1b}$, it is enough to show that $t_m \geq 1$ and $t'_M \leq n-1$. For this subset, $t'_M = t_2 + l_{bd} - 1$ and by taking the major subset immediately after the minor subset, $t_m = t_1 - l_{ca} - l_{ad} + 2$. By case 1(b), $t_m \geq 1$ and $t'_M \leq n-1$.
- case 2(a) holds: for $\mathcal{P}_{2a}$, it is enough to show that $t_M \geq 1$ and $t'_m \leq n-1$. For this subset, $t_M = t_2 - l_{ac} + 1$ and by taking the minor subset immediately after the major subset, $t'_m = t_1 + l_{bd} + l_{cb} - 2$. By case 2(a), $t_M \geq 1$ and $t'_m \leq n-1$.
- case 2(b) holds: for $\mathcal{P}_{2b}$, it is enough to show that $t_m \geq 1$ and $t'_M \leq n-1$. For this subset, $t'_M = t_1 + l_{bd} - 1$ and by taking the major subset immediately after the minor sub-s-path, $t_m = t_2 - l_{ac} - l_{cb} + 2$. By case 2(b), $t_m \geq 1$ and $t'_M \leq n-1$. □

We can strengthen this result. For some crossing x-chords, there exists a sub-s-path containing both x-chords and $|C|-3$ x-cycle arcs, in which case, the sum of the coefficients of the x-chords is at most 2.

**Theorem 22.** *Let $(i_j, i_{j+l}, t)$ be an x-chord of $C$. The sum of the lifted coefficients of $(i_j, i_{j+l}, t)$ and the reachable crossing x-chords*

1. $\{(i_{j+l+k}, i_{j+1}, t+k+1) | \forall k = \max(0, |C|-l-t), \ldots, \min(|C|-l-1, n-l-t)\}$
2. $\{(i_{j+l-1}, i_{j-k}, t-k-1) | \forall k = \max(t+|C|-l-n, 0), \ldots, \min(t-l, |C|-l-1)\}$

*is at most 2.*

*Proof.* Let $(i_j, i_{j+l}, t)$ be an x-chord of the x-cycle $C$.

Assuming that $(i_{j+l+k}, i_{j+1}, t+k+1)$ is in the first set for some $k = \max(0, |C|-l-t) \ldots \min(|C|-l-1, n-l-t)$, we show that for $\alpha = 0, 1, 2$, the inequality
$$x(C) + (3-\alpha)x^{t}_{i_j, i_{j+l}} + \alpha x^{t+k+1}_{i_{j+l+k}, i_{j+1}} \leq |C| - 1$$
is not valid for $\tilde{P}[(R \setminus \{(i_j, i_{j+l}, t), (i_{j+l+k}, i_{j+1}, t+k+1)\}) \cup T]$. Define $\mathcal{P}_1$ as

$$\begin{aligned}\mathcal{P}_1 := \{&(i_{j+l+k+1}, i_{j+l+k}, t-s+l+k+1), \ldots, (i_{j-1}, i_j, t-1), (i_j, i_{j+l}, t),\\ &(i_{j+l}, i_{j+l+1}, t+1), \ldots, (i_{j+l+k-1}, i_{j+l+k}, t+k), (i_{j+l+k}, i_{j+1}, t+k+1),\\ &(i_{j+1}, i_{j+2}, t+k+2), \ldots, (i_{j+l+k-1}, i_{j+l+k}, t+l+k-1)\}.\end{aligned}$$

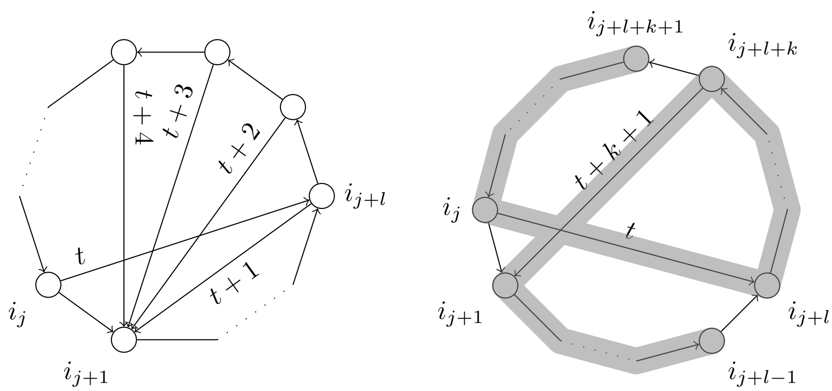


Figure 4.4: Reachable crossing x-chords of the first type of Theorem 22 (left), and the sub-s-path $\mathcal{P}_1$ highlighted (right).

Since $i_{j+1}$ is external with respect to $(i_j, i_{j+l}, t)$, $i_{j+l+k}$ must be internal with respect to $(i_j, i_{j+l}, t)$ and not equal to $i_j$ for the x-chords to cross and be compatible. This is true for $k = 0, \ldots, |C| - l - 1$. Since $t - |C| + l + k + 1 \geq 1 \iff k \geq |C| - l - t$ and $t + l + k - 1 \leq n - 1 \iff k \leq n - l - t$, the first and last arcs in $\mathcal{P}_1$ are between 1 and $n - 1$. $\mathcal{P}_1$ is a sub-s-path with $|C| - 3$ x-cycle arcs, so the inequality is violated by this sub-s-path, and the result holds.

Assuming that $(i_{j-l-k}, i_{j-k}, t - k - 1)$ is in the second set for some $k = \max(t + |C| - l - n, 0) \ldots \min(t - l, |C| - l - 1)$, we show that for $\alpha = 0, 1, 2$, the inequality

$$x(C) + (3 - \alpha)x^t_{i_j, i_{j+l}} + \alpha x^{t-k-1}_{i_{j+l-k}, i_{j-k}} \leq |C| - 1$$

is not valid for $\tilde{P}[(R \setminus \{(i_j, i_{j+l}, t), (i_{j-l-k}, i_{j-k}, t - k - 1)\}) \cup T]$. Define $\mathcal{P}_2$ as

$$\begin{aligned}\mathcal{P}_2 := \{&(i_{j+1}, i_{j+2}, t-k-l+2), \ldots, (i_{j+l-2}, i_{j+l-1}, t-k-2), (i_{j+l-1}, i_{j-k}, t-k-1),\\ &(i_{j-k}, i_{j-k+1}, t-k), \ldots, (i_{j-1}, i_j, t-1), (i_j, i_{j+l}, t),\\ &(i_{j+l}, i_{j+l+1}, t+1), \ldots, (i_{j-k-2}, i_{j-k-1}, t + |C| - l - k - 1)\}.\end{aligned}$$

Since $i_{j+l-1}$ is external with respect to $(i_j, i_{j+l}, t)$, $i_{j-k}$ must be internal with respect to $(i_j, i_{j+l}, t)$ and not equal to $i_{j-l}$ for the x-chords to cross and be compatible. This is true for $k = 0, \ldots, |C| - l - 1$. Since $t - k - l + 2 \geq 1 \iff k \leq t - l - 1$ and $t + |C| - l - k - 1 \leq n - 1 \iff k \geq t + |C| - l - n$, the first and last arcs of $\mathcal{P}_2$ are between 1 and $n - 1$. $\mathcal{P}_2$ is a sub-s-path with $|C| - 3$ x-cycle arcs, so the inequality is violated by this sub-s-path, and the result holds. □

All of our results on crossing x-chords have been on pairs of crossing x-chords. To finish this section, we will show that there are patterns of x-chords which are not pairwise reachable crossing x-chords, but which cannot all be maximally-lifted. Assume $C$ is an x-cycle of size at least 5, and that $2|C| \leq n$. First,

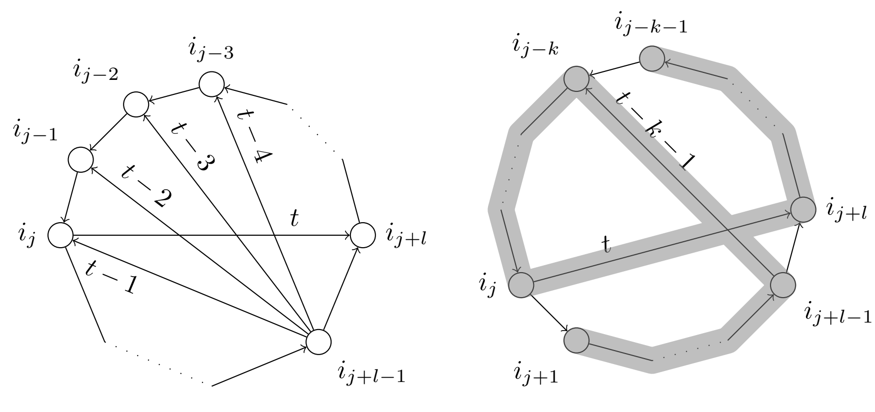


Figure 4.5: Reachable crossing chords of the second type of Theorem 22 (left), and the sub-s-path $\mathcal{P}_2$ highlighted (right).

we choose $(i_a, i_b, 1)$ to be a chord of length at most $|C| - 2$. Next, we choose $(i_c, i_d, t_2)$ so that $i_c$ is internal to $(i_a, i_b, 1)$, $i_c \neq i_{a-1}$ and $i_c \neq i_{a-2}$. Choose $i_d$ so that $i_{a+1} \neq i_d$, $i_{a+2} \neq i_d$ and $i_d$ is external to $(i_a, i_b, 1)$, and let $t_2 := l_{bc} + 2$. For this choice of x-chords, $(i_a, i_b, 1)$ and $(i_c, i_d, l_{bc} + 2)$ will not be reachable crossing x-chords, since the x-cycle arcs between $i_c$ and $i_a$ cannot be covered before taking the x-chord $(i_a, i_b, 1)$. Let the third x-chord be $(i_{a-1}, i_{a+1}, t_3)$, with $t_3 := l_{bc} + l_{db} + l_{ca} = l_{da}$. $(i_{a-1}, i_{a+1}, l_{da})$ does not cross $(i_c, i_d, l_{bc} + 2)$, so they may both be 2-lifted. Although $(i_a, i_b, 1)$ and $(i_{a-1}, i_{a+1}, l_{da})$ cross, their times are such that there is no sub-s-path that contains both x-chords and all of the x-cycle arcs between $i_b$ and $i_{a-1}$, so they are not reachable crossing x-chords, so they could both be 2-lifted.

However, by the choice of x-chords, they cannot all be 2-lifted. The sub-s-path is

$$\begin{aligned}
\{&(i_a, i_b, 1), (i_b, i_{b+1}, 2), \ldots, (i_{c-1}, i_c, l_{bc} + 1), \\
&\quad (i_c, i_d, l_{bc} + 2), (i_d, i_{d+1}, l_{bc} + 3), \ldots, (i_{b-2}, i_{b-1}, l_{bc} + l_{db} + 1), \\
&\qquad (i_{c+1}, i_{c+2}, l_{bc} + l_{db} + 3), \ldots, (i_{a-2}, i_{a-1}, l_{da} - 1), \\
&\qquad (i_{a-1}, i_{a+1}, l_{da}), (i_{a+1}, i_{a+2}, l_{da} + 1), \ldots, (i_{d-2}, i_{d-1}, |C| - 1)\}
\end{aligned}$$

contains all three x-chords and $|C| - 5$ x-cycle arcs (illustrated in Figure 4.6), the inequality $x(C) + 2x^1_{i_a, i_b} + 2x^{l_{bc}+2}_{i_c, i_d} + 2x^{l_{da}}_{i_{a-1}, i_{a+1}} \leq |C| - 1$ is therefore not valid. Note that since the sub-s-path requires arcs in times 1 to $|C| - 1$, such arrangements exist even in x-cycles for which $2|C| > n$. Any sufficient conditions for a maximally-liftable set of x-chord will need to take such arrangements into consideration, and what they may be is an open question.

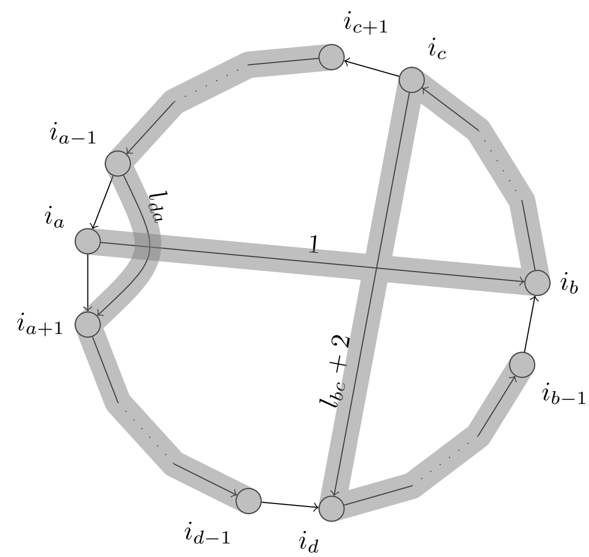


Figure 4.6: Three x-chords of which any two, but not all three, may be 2-lifted.

## 4.4 Maximum lifted coefficients of tangents

Recall that we defined $T$ to be the set of tangents of an x-cycle $C$ at times $t = 1, \ldots, |C| - 1$. At the beginning of the chapter, we noted that tangent $(i_j, v, \bar{t})$ and any of the x-cycle arcs $\{(i_j, i_{j+1}, t) \mid t = 1, \ldots, n-1\}$ could not both be in the same sub-s-path. In order to find the lifted coefficient of a tangent, and assuming the tangent is lifted first, we want to find a sub-s-path containing the tangent and as many x-cycle arcs as possible. Since we are only considering the tangents in times before $|C|-1$, all of the remaining x-cycle arcs cannot be covered before taking the tangent, so some of the remaining x-cycle arcs must be covered later. This implies that the maximum lifted coefficient of any tangent in $T$ is 1:

**Theorem 23.** *For any tangent $(i_j, v, t)$ with $i_j \in V(C)$, $v \notin V(C)$ and $1 \le t \le |C|-1$, the lifted coefficient when $(i_j, v, t)$ is the first arc lifted in a lifted x-cycle inequality is 1.*

*Proof.* As mentioned before, we can use at most $|C| - 2$ x-cycle arcs, so the lifted coefficient of the tangent is at least 1.

Next, we show that a sub-s-path containing the tangent and $|C| - 2$ x-cycle arcs is always possible. Choose the following set of compatible arcs:

$$\{(i_{j-t}, i_{j-t+1}, 1), \ldots, (i_{j-1}, i_j, t-1), (i_j, v, t),$$
$$(i_{j+1}, i_{j+2}, t+2), \ldots, (i_{j-t-2}, i_{j-t-1}, |C|)\}.$$

If $|C| = n$, then there wouldn't be any tangents. Since $|C| \leq n-1$, this set of arcs is therefore a sub-s-path, and shows that the maximum lifted coefficient of the tangent is at most 1. $\square$

The sub-s-path given in the proof can be modified to show that any pair of tangents may both be 1-lifted.

**Corollary 24.** *Let $(i_j, v, t_1)$ and $(i_k, w, t_2)$ be two compatible tangents in $T$; then the maximum lifted coefficient of both tangents is 1.*

*Proof.* We already know that the maximum lifted coefficient of a single tangent is 1, so we need to show that

$$x(C) + x_{i_j,v}^{t_1} + x_{i_k,w}^{t_2} \leq |C| - 1$$

is valid for $\tilde{P}[R \cup (T \setminus \{(i_j, v, t_1), (i_k, w, t_2)\}]$.

Assume that the inequality is not valid. Then there exists a sub-s-path containing both tangents and at least $|C| - 2$ x-cycle arcs.

Assume without loss of generality that $t_1 < t_2$. This sub-s-path must contain the following set of arcs:

$$\{(i_{j-t}, i_{j-t+1}, 1), \ldots, (i_{j-1}, i_j, t-1), (i_j, v, t_1),$$
$$(i_{j+1}, i_{j+2}, t_2 - l_{jk} + 1), \ldots, (i_{k-1}, i_k, t_2 - 1), (i_k, x, t_2)\}.$$

Since this set contains $|C| - 1$ arcs, $t_2 \geq |C|$. This is a contradiction, since $t_2 \leq |C| - 1$ for this tangent to be in $T$. $\square$

Since any pair of tangents in $T$ may be 1-lifted, we will proceed to show how lifted x-chords affect lifting tangents and how lifted tangents affect lifting x-chords. All of the sub-s-paths given in the following proofs will have $|C| - 3$ x-cycle arcs, an x-chord and a tangent. We can assume that the x-chord is lifted to coefficient 2: any such sub-s-path can be easily modified to be a sub-s-path with $|C| - 3$ x-cycle arcs and the x-chord, implying that the maximum lifted coefficient of the x-chord is 2.

We call a tangent external with respect to an x-chord if it is incident to an x-cycle node in the exterior of the x-chord, and internal with respect to an x-chord if it is incident to an x-cycle node in the interior of the x-chord. Both give conditions under which both an x-chord and tangent may not be maximally lifted. For x-chord $(i_a, i_b, t_1)$ and tangent $(i_j, v, t_2)$, this is equivalent to saying that the inequality $x(C) + 2x_{i_a,i_b}^{t_1} + x_{i_j,v}^{t_2} \leq |C| - 1$ is valid for $\tilde{P}[(R \setminus \{(i_a, i_b, t_1)\}) \cup (T \setminus \{(i_j, v, t_2)\})]$.

First, the case of an x-chord and its external tangents:

**Lemma 25.** *Let $(i_j, i_{j+l}, t)$ be an x-chord with coefficient 2. Then any tangent of*

$$\{(i_{j+l-1}, v, t_2) \,|\, l-1 \leq t_2 \leq |C|-1, \exists k : \max(0, s+t-n-l) \leq k \leq s-l-1, v \notin C\}$$

*has maximum lifted coefficient 0.*

*Proof.* For any tangent $(i_{j+l-1}, v, t_2)$ in the set, we show a sub-s-path which violates the inequality

$$x(C) + 2x^t_{i_j,i_{j+l}} + x^{t_2}_{i_{j+l-1},v} \leq |C| - 1.$$

Take the following set of arcs:

$$\begin{aligned}&\{(i_{j+1}, i_{j+2}, t_2 - l + 2), \ldots, (i_{j-l-2}, i_{j-l-1}, t_2 - 1), (i_{j+l-1}, v, t_2),\\ &(v, i_{j-k}, t_2 + 1), (i_{j-k}, i_{j-k+1}, t_2 + 2), \ldots, (i_{j-1}, i_j, t - 1),\\ &(i_j, i_{j+l}, t), (i_{j+l}, i_{j+l+1}, t + 1), \ldots, (i_{j-k-2}, i_{j-k-1}, t + s - k - l - 1)\}.\end{aligned}$$

Since $t_2 \geq l - 2$, the subset spanning the nodes external to the x-chord is a sub-s-path: the first arc in the set is at time $t_2 - l + 2 \geq 1$. By definition of this set of tangents, there exists a $k$ with $\max(0, |C| + t - n - l) \leq k \leq |C| - l - 1$. The last arc in the set is at time $t + |C| - k - l - 1 \leq n - 1$, so the subset spanning the nodes internal to the x-chord begins at some node between $i_{j+l+1}$ and $i_j$. This set of arcs is a sub-s-path. The entire set is a sub-s-path containing $|C| - 3$ x-cycle arcs, so the inequality is violated by this sub-s-path. □

Lemma 25 is illustrated in Figure 4.7. Now a similar result for an x-chord and its internal tangents:

**Lemma 26.** *Let $(i_j, i_{j+l}, t)$ be an x-chord with coefficient 2. Then any tangent of*

$$\{(i_{j+l+k}, v, t+k+1) \,|\, \max(0, |C|-t-l) \leq k \leq \min(|C|-l-2, |C|-t-2), \forall v \notin C\}$$

*has maximum lifted coefficient 0.*

*Proof.* For any tangent in $(i_{j+l+k}, v, t + k + 1)$ in the set, we show a sub-s-path which would violate the inequality

$$x(C) + 2x^t_{i_j,i_{j+l}} + x^{t+k+1}_{i_{j+l+k},v} \leq |C| - 1.$$

Take the following set of arcs:

$$\begin{aligned}&\{(i_{j+l+k+1}, i_{j+l+k+2}, t - s + k + l + 1), \ldots, (i_{j-1}, i_j, t - 1),\\ &(i_j, i_{j+l}, t), (i_{j+l}, i_{j+l+1}, t + 1), \ldots, (i_{j+l+k-1}, i_{j+l+k}, t + k),\\ &(i_{j+l+k}, v, t + k + 1), (v, i_{j+1}, t + k + 2), \ldots, (i_{j+l-2}, i_{j+l-3}, t + k + l)\}.\end{aligned}$$

Since $\max(0, |C| - t - l) \leq k \leq \min(|C| - l - 2, |C| - t - 2)$, $t - |C| + k + l + 1 \geq 1$ and $t + k + l \leq n - 1$. The arcs are all compatible, so this is a sub-s-path. It contains $|C| - 3$ x-cycle arcs, so the inequality is violated by this sub-s-path. □

Lemma 26 is illustrated in Figure 4.8.

We can also give the conditions under which an x-chord cannot be 2-lifted, given a 1-lifted tangent, essentially restating the previous theorems for a given tangent rather than a given x-chord. First, the x-chords for which a given tangent is external:

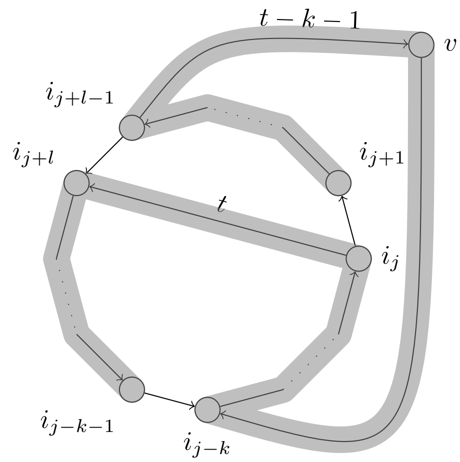


Figure 4.7: X-chord $(i_j, i_{j+l}, t)$ and the external tangent $(i_{j+l+1}, v, t-k-1)$, with the sub-s-path with $|C|-3$ x-cycle arcs highlighted.

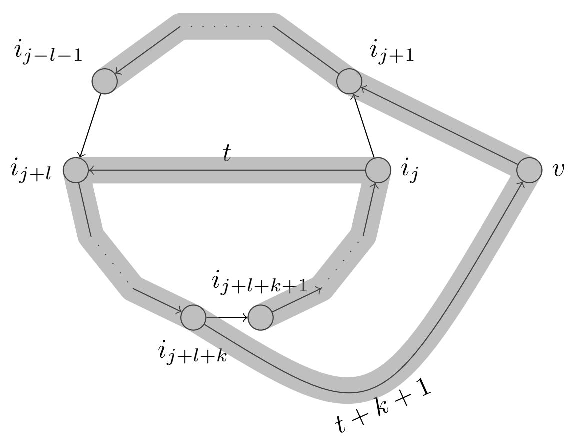


Figure 4.8: X-chord $(i_j, i_{j+l}, t)$ and the internal tangent $(i_{j+l+k}, v, t+k+1)$, with the sub-s-path with $|C|-3$ x-cycle arcs highlighted.

**Theorem 27.** *Let $(i_j, v, t) \in T$. If $(i_j, v, t)$ has coefficient 1, then any x-chord in*

$$\{(i_{j-l+1}, i_{j+1}, t_2) \mid 2 \leq l \leq \min(s-1, t+1), t+2 \leq t_2\}$$

*can take coefficient at most 1.*

*Proof.* Let $(i_j, v, t)$ be a 1-lifted tangent, and $(i_{j-l+1}, i_{j+1}, t_2)$ be an x-chord as defined. Then it is enough show that the inequality

$$x(C) + x^t_{i_j,v} + 2x^{t_2}_{i_{j-l+1},i_{j+1}} \leq |C| - 1$$

is not valid for $\tilde{P}[(R \setminus \{(i_{j-l+1}, i_{j+1}, t_2)\}) \cup (T \setminus \{(i_j, v, t)\})]$.

We again show a sub-s-path that violates the inequality, in two parts: Ext with $l-2$ x-cycle arcs external to the x-chord and the tangent, and Int with the x-chord and $s-l-1$ x-cycle arcs internal to the x-chord. Let $m := l+t_2-t-3$. Define Ext and Int as follows:

$$\textsc{Ext} := \{(i_{j-l+2}, i_{j-l+3}, t-l+2), \ldots, (i_{j-1}, i_j, t-1), (i_j, v, t)\}$$

$$\textsc{Int} := \begin{cases} \{(i_{j-m}, i_{j-m+1}, t+2), \ldots, (i_{j-l}, i_{j-l+1}, t_2-1), (i_{j-l+1}, i_{j+1}, t_2), \\ \quad (i_{j+1}, i_{j+2}, t_2+1), \ldots, (i_{j-m-2}, i_{j-m-1}, t+s-l+1)\} & \text{if } t_2 \leq t+s-l+1 \\ \{(i_{j+2}, i_{j+3}, t_2-s+l-1), \ldots, (i_{j-l}, i_{j-l+1}, t_2-1), (i_{j-l+1}, i_{j+1}, t_2)\} & \text{otherwise} \end{cases}$$

It remains to show that Ext, Int and Ext $\cup$ Int are sub-s-paths. By choice of these sets, if the times of the arcs are between 1 and $n-1$, then they are sub-s-paths. Ext has compatible arcs in times $t-l+2 \geq 1$ to $t$. Int has compatible arcs from either $t-2$ to $t+|C|-l+1 \leq n-1$ or $t+|C|-l+2 \geq t+2$ to $n-1$. Ext $\cup$ Int are a set of compatible arcs, and contain $|C|-3$ x-cycle arcs, so the inequality is not valid. □

Next, the x-chords for which a given tangent is internal:

**Theorem 28.** *Let $(i_j, v, t) \in T$ and $2 \leq t \leq |C|-1$. If $(i_j, v, t)$ has coefficient 1, then any x-chord in*

$$\{(i_{j-k-l}, i_{j-k}, t-k-1) \mid \max(2, |C|-t+1) \leq l \leq |C|-1,\ t+l \leq n,\ 0 \leq k \leq t-2\}$$

*can take lifted coefficient at most 1.*

*Proof.* Let $(i_j, v, t)$ be a tangent with $2 \leq t \leq |C|-1$, and $(i_{j-k-l}, i_{j-k}, t-k-1)$ be an x-chord as defined. Then it is enough show that the inequality

$$x(C) + x^t_{i_j,v} + 2x^{t-k-1}_{i_{j-k-l},i_{j-k}} \leq |C| - 1$$

is not valid for $\tilde{P}[(R \setminus \{(i_{j-k-l}, i_{j-k}, t-k-1)\}) \cup (T \setminus \{(i_j, v, t)\})]$.

We show a sub-s-path that violates the inequality, in two parts: Int with the x-chord, tangent, and $|C|-l-1$ x-cycle arcs internal to the x-chord, and

$\textsc{Ext}$ with $l-2$ x-cycle arcs external to the x-chord. Define the sets $\textsc{Int}$ and $\textsc{Ext}$ as follows:

$$\begin{aligned}\textsc{Int} := \{&(i_{j+1}, i_{j+2}, t-s+l), \ldots, (i_{j-k-l-1}, i_{j-k-l}, t-k-2),\\ &(i_{j-k-l}, i_{j-k}, t-k-1),\\ &(i_{j-k}, i_{j-k+1}, t-k), \ldots, (i_{j-1}, i_j, t-1), (i_j, v, t)\}\\ \textsc{Ext} := \{&(i_{j-k-l+1}, (i_{j-k-l+2}, t+2), \ldots, (i_{j-k-2}, i_{j-k-1}, t+l-1)\}\end{aligned}$$

It remains to show that $\textsc{Int}$, $\textsc{Ext}$ and $\textsc{Int} \cup \textsc{Ext}$ are sub-s-paths. The arcs in these sets are all compatible, so if the times of the arcs are between 1 and $n-1$, then they are sub-s-paths. $\textsc{Ext}$ has x-cycle arcs from time $t+2$ through $t+l-1 \leq n-1$. $\textsc{Int}$ has arcs from time $t-|C|+l \geq 1$ through $t$. $\textsc{Int} \cup \textsc{Ext}$ contains $|C|-3$ x-cycle arcs, so $\textsc{Int} \cup \textsc{Ext}$ violates the inequality. □

Notice that for $2|C| \leq n$, the condition $t+l \leq n$ is always met.

We use these results to describe the types of inequalities which can be constructed when the tangents appear first in the lifting order, that is, when the tangents have lifted coefficient 1. This is also equivalent to starting lifted cycle inequalities from the inequality (3.10).

**Lemma 29.** *For any x-cycle $C$, if the tangents $\{(i_j, v, t) \mid i_j \in V(C), v \notin V(C), 1 \leq t \leq |C|-1\}$ all have coefficient 1, the maximum lifted coefficient of the x-chords $\{(i_j, i_{j+l}, t) \mid 2 \leq l \leq |C|-1,\ 1 \leq t \leq |C|-l,\ l+1 \leq t \leq n-1\}$ is 1.*

*Proof.* For any x-chord $(i_j, i_{j+l}, t)$ in the set, it is enough to show that the inequality

$$x(C) + \sum_{t=1}^{|C|-1} \sum_{j=1}^{|C|} \sum_{v \notin V(C)} x^t_{i_j,v} + 2x^t_{i_j,i_{j+l}} \leq |C|-1$$

is not valid for $\tilde{P}[R \setminus \{(i_j, i_{j+l}, t)\}]$. We show a sub-s-path consisting of three parts: $\textsc{Int}$ with $|C|-l-1$ x-cycle arcs interior to the x-chord and the x-chord, $\textsc{Ext}$ with the $l-2$ x-cycle arcs exterior to the x-chord, and a tangent in time $\leq |C|-1$. We consider the x-chords in three cases:

Case 1: $\{(i_j, i_{j+l}, t) \mid i_j \in V(C),\ 2 \leq l \leq |C|-1,\ 1 \leq t \leq |C|-l\}$ Choose the sub-s-path $\textsc{Int}$ so that it contains the x-chord $(i_j, i_{j+l}, t)$ in this set and covers all other possible x-cycle arcs as early as possible. Since $t \leq |C|-l$, then we can start $\textsc{Int}$ at time 1 and end at time $|C|-l$, and include the x-chord in those arcs:

$$\begin{aligned}\textsc{Int} := \{&(i_{j-t+1}, i_{j-t+2}, 1), \ldots, (i_{j-1}, i_j, t-1), (i_j, i_{j+l}, t),\\ &(i_{j+l}, i_{j+l+1}, t+1), \ldots, (i_{j-t-2}, i_{j-t-1}, |C|-l)\}\end{aligned}$$

The sub-s-path $\textsc{Ext}$ will have to cover the remaining $l-2$ x-cycle arcs compatible with this choice for $\textsc{Int}$. We take this sub-s-path as early as possible, leaving time for a tangent between them:

$$\textsc{Ext} := \{(i_{j+1}, i_{j+2}, |C|-l+3), \ldots, (i_{j+l-2}, i_{j+l-1}, |C|)\}$$

Choose any compatible tangent in time $|C|-l+1$, then the sub-s-path $\textsc{Int} \cup \{(i_{j+l}, v, s-l+1)\} \cup \textsc{Ext}$ will be a sub-s-path covering $|C|-3$ x-cycle arcs and the x-chord. The only question that remains is whether the tangent $(i_{j+l}, v, s-l+1)$ will have coefficient 1 in the inequality. Since $2 \leq l \leq s-1$, then $2 \leq s-l+1 \leq s-1$, and the tangent will have coefficient 1.

Case 2: For x-chords $\{(i_j, i_{j+l}, t) | i_j \in V(C),\ 2 \leq l \leq |C|-1,\ l+1 \leq t \leq |C|\}$, we choose the sub-s-path with $\textsc{Ext}$ as early as possible:

$$\textsc{Ext} := \{(i_{a+1}, i_{a+2}, 1), \ldots, (i_{a+l-2}, i_{a+l-1}, l-2)\}.$$

Now choose $\textsc{Int}$ so it follows $\textsc{Ext}$ immediately, with a tangent between. Since $l+1 \leq t \leq |C|$, an x-chord at time $t$ with $l+1 \leq t \leq |C|$ can be included in this part of the sub-s-path:

$$\begin{aligned}\textsc{Int} :=& \{(i_{a-t+l+1}, i_{a-t+l+2}, l+1), \ldots, (i_{a-1}, i_a, t-1),\\ & (i_a, i_{a+l}, t), (i_{a+l}, i_{a+l+1}, t+1), \ldots, (i_{a-t+l-1}, i_{a-t+l}, |C|)\}\end{aligned}$$

Choose any compatible tangent in time $l-1$, then the sub-s-path $\textsc{Ext} \cup \{(i_j, v, l-1)\} \cup \textsc{Int}$ will be a sub-s-path covering $|C|-3$ x-cycle arcs and the x-chord. Again, we just need that the tangent chosen has coefficient 1 in the inequality. Since $l \leq |C|-1$, then $l-1 \leq |C|-2$, and the tangent has coefficient 1.

Case 3: For x-chords $\{(i_j, i_{j+l}, t) \mid i_j \in V(C),\ 2 \leq l \leq |C|-1,\ |C|+1 \leq t \leq n-1\}$, we choose the sub-s-path with $\textsc{Ext}$ as early as possible, as for case 3:

$$\textsc{Ext} := \{(i_{a+1}, i_{a+2}, 1), \ldots, (i_{a+l-2}, i_{a+l-1}, l-2)\}.$$

Now we choose $\textsc{Int}$ so that it come after $\textsc{Ext}$, but with enough time between them so that the x-chord in time $t$ with $|C|+1 \leq t$ will be the last arc:

$$\textsc{Int} := \{(i_{a+l+1}, i_{a+l+2}, t-|C|+l), \ldots, (i_{a-1}, i_a, t-1), (i_a, i_{a+l}, t)\}$$

Since $t \geq |C|+1$, the time of the first arc of $\textsc{Int}$ is $t-|C|+l \geq l+1$, to this choice of $\textsc{Ext}$ and $\textsc{Int}$ are all compatible arcs. Choose any compatible tangent in time $l-1$, then the sub-s-path $\textsc{Ext} \cup \{(i_j, v, l-1)\} \cup \textsc{Int}$ will be a sub-s-path covering $|C|-3$ x-cycle arcs and the x-chord. As for case 3, the tangent will have coefficient 1. □

In the case of $2|C| \leq n$, we can improve this:

**Lemma 30.** *For any x-cycle $C$ with $2|C| \leq n$, if the tangents $\{(i_j, v, t) \mid i_j \in V(C), v \notin V(C), 1 \leq t \leq |C|-1\}$ all have coefficient 1, the maximum lifted coefficient of the x-chords $\{(i_j, i_{j+l}, t) \mid 2 \leq l \leq |C|-2, 1 \leq t \leq n-1\}$, $\{(i_j, i_{j-1}, t) \mid 1 \leq t \leq |C|-2\}$ and $\{(i_j, i_{j-1}, t) \mid |C| \leq t \leq n-1\}$ is 1.*

*Proof.* The case of x-chords $\{(i_j, i_{j+l}, t) \mid 2 \leq l \leq |C|-1,\ 1 \leq t \leq |C|-l\}$ and $\{(i_j, i_{j+l}, t) \mid 2 \leq l \leq |C|-1,\ l+1 \leq t \leq n-1\}$ is covered by Lemma 29, so we will show the result for the remaining x-chords. We noted that in particular, we have either (i) $2 \le l \le |C|-2$ and $|C|-l+1 \le t \le l$, or (ii) $l = |C|-1$ and $2 \le t \le |C|-2$. We take the sub-s-path Int ending with the x-chord:

$$\textsc{Int} := \{(i_{a+l+1}, i_{a+l+2}, t-|C|+l+1), \ldots, (i_{a-1}, i_a, t-1), (i_a, i_{a+l}, t)\}.$$

Then take the sub-s-path Ext covering the remaining external nodes, leaving time to take a tangent between Ext and Int:

$$\textsc{Ext} := \{(i_{a+1}, i_{a+2}, t+3), \ldots, (i_{a-2}, i_{a-1}, t+l)\}.$$

Since $t + l \le 2l \le 2(|C|-1) \le n-1$, there is enough time to cover all $l-2$ compatible exterior x-cycle arcs. Finally, we can choose any tangent $(i_{j+l}, v, t+1)$, and by assumptions on $t$, $t+1 \le |C|-1$, so the tangent with have coefficient 1 in the inequality, and thus the maximum inequality for the x-chord is 1. □

Next, we can show that if all of the $l$-x-chords in times $1 \le t \le |C|-l$ and $l+1 \le t \le n-1$ have coefficient 1, the remaining x-chords will have maximum lifted coefficient 1:

**Lemma 31.** *If the x-chords*

$$\begin{aligned} Q :=& \{(i_j, i_{j+l}, t) \mid i_j \in V(C),\ 2 \le l \le |C|-1,\ t = 1, \ldots, |C|-l\} \cup \\ & \{(i_j, i_{j+l}, t) \mid i_j \in V(C),\ 2 \le l \le |C|-1,\ t = l+1, \ldots, n-1\} \end{aligned}$$

*have coefficient 1, then the x-chords $R \setminus Q$ have maximum lifted coefficient 1.*

*Proof.* It is enough to show for any x-chord $(i_j, i_{j+l}, t)$ with $|C|-l+1 \le t \le l$ that the inequality

$$x(C) + x(Q) + 2x^t_{i_j, i_{j+l}} \le |C| - 1$$

is not valid for $\tilde{P}[(R \setminus (Q \cup \{(i_j, i_{j+l}, t)\})) \cup T]$. We show a sub-s-path in three parts, which is a generalization of the sub-s-path in the proof of Theorem 16, and two additional x-chords. For $0 \le k \le \min(t-2, |C|-l-2)$, define the following sets of arcs:

$$\begin{aligned} \textsc{Ext}_1 :=& \{(i_{a+1}, i_{a+2}, 1), \ldots, (i_{a+t-k-2}, i_{a+t-k-1}, t-k-2)\} \\ \textsc{Int} :=& \{(i_{a-k}, i_{a-k+1}, t-k), \ldots, (i_{a-1}, i_a, t-1), (i_a, i_{a+l}, t), \\ & (i_{a+l}, i_{a+l+1}, t+1), \ldots, (i_{a-k-2}, i_{a-k-1}, t+|C|-l-k-1)\} \\ \textsc{Ext}_2 :=& \{(i_{a+t-k}, i_{a+t-k+1}, t+|C|-l-k+1), \ldots, (i_{a+l-2}, i_{a-l-1}, |C|-1) \end{aligned}$$

The two x-chords compatible with $\textsc{Ext}_1 \cup \textsc{Int} \cup \textsc{Ext}_2$ are $(i_{a+t-k-1}, i_{a-k}, t-k-1)$ and $(i_{a-k-1}, i_{a+t-k}, t+|C|-l-k)$. If we can show that there exists $k$ so that these x-chords have coefficient 1, that is, if they are in the set $Q$, then the sub-s-path $\textsc{Ext}_1 \cup \{(i_{a+t-k-1}, i_{a-k}, t-k-1)\} \cup \textsc{Int} \cup \{(i_{a-k-1}, i_{a+t-k}, t+|C|-l-k)\} \cup \textsc{Ext}_2$ violates the inequality.

The length of $(i_{a+t-k-1}, i_{a-k}, t-k-1)$ is $|C|-t-1$, so we need to show that the inequality

$$|C| - (|C|-t-1) + 1 = t \le t-k-1 \le |C|-t-1$$

is violated for some value of $k$. Since $k \geq 0$, then $t > t-k-1$, and $(i_{a+t-k-1}, i_{a-k}, t-k-1) \in Q$.

The length of $(i_{a-k-1}, i_{a+t-k}, t + |C| - l - k)$ is $t + 1$, so we need to show that the inequality

$$|C| - (t+1) + 1 = |C| - t \leq t + |C| - l - k \leq t + 1$$

is violated for some value of $k$. Since $k \leq |C| - l - 2$, then $t + |C| - l - k > t + 1$, and $(i_{a-k-1}, i_{a+t-k}, t + |C| - l - k) \in Q$. □

Finally, we can put these results together to show a lifting of all of the tangents and x-chords, which gives the Lifted SEC.

**Theorem 32.** *The lifted subtour elimination constraints on subsets $S \subset N$ of the TDTS polytope are lifted x-cycle inequalities.*

*Proof.* Let $Q$ be defined as in Lemma 31. Lift the x-chords and tangents in the following order:

1. all tangents, in any order
2. $Q$, in any order
3. $R \setminus Q$

From Corollary 24, all of the tangents will have coefficient 1. From Lemma 29, the x-chords in $Q$ have maximum lifted coefficient 1. From Lemma 31, the x-chords in $R \setminus Q$ will have maximum lifted coefficient 1. The Lifted Subtour Elimination Constraint achieves this maximum for all of the arcs in $R \cup T$, so it is a valid lifting. Thus the Lifted SEC is a lifted x-cycle inequality. □

Notice that this differs from a result of Balas and Fischetti, which says that the ATS polytope subtour elimination constraint is *not* a lifted cycle inequality [4]. Also, if the first arc lifted is *any* x-chord, the lifted x-cycle inequality will never be the Lifted SEC, since its coefficient will be at least 2.

# Chapter 5

# Facet-Defining Properties of Lifted X-Cycle Inequalities and a Family of Inequalities

In the previous chapter, we showed that the Lifted SEC was a lifted x-cycle inequality. In this chapter, we will show that other liftings of the x-cycle inequality are facet-defining for the monotonized TDTS polytope; in particular, some liftings in which some coefficients are at least 2. We will define a family of lifted x-cycle inequalities, which we call the nest inequalities. The fact that this family is facet-defining will rely on the fact that for each 2-lifted x-chord, there may be multiple new sub-s-paths in the face defined by the inequality. Finding a set of additional sub-s-paths in the face which are affinely independent, and thus showing the dimension of the face, is non-trivial. We will also show that lifted x-cycle inequalities which define facets of the monotonized polytope are support reduced.

In the following, as before, let $\tilde{P}$ be the submissive of $P$, the TDTS polytope on $G(n)$. $\tilde{P}[X]$, for a set of arcs $X$, is $\tilde{P}[X] := \{y \in \tilde{P} \mid y_a = 0, \forall a \in X\}$. By lifting theory, an inequality which is facet-defining for $\tilde{P}[X]$ and is sequentially lifted for any order of $X$ will be facet-defining for $\tilde{P}$.

Starting with the x-cycle inequality, however, will not necessarily yield a facet-defining inequality of $\tilde{P}$. The x-cycle inequality

$$x(C) = \sum_{t=1}^{n-1} \sum_{j=1}^{|C|} x^t_{i_j,i_{j+1}} \leq |C| - 1 \tag{5.1}$$

is not strong enough to be facet-defining for $\tilde{P}[R \cup T]$.

**Claim 33.** *The inequality*

$$x(C) + \sum_{t=n-|C|+1}^{n-1} \sum_{u \notin C} \sum_{j=1}^{|C|} x^t_{u,i_j} \leq |C| - 1$$

*is valid for* $\tilde{P}[R \cup T]$ *and stronger than the cycle inequality* (5.1).

*Proof.* Let $M = \{(u, i_j, t) \mid u \notin C, i_j \in C, n - |C| + 1 \leq t \leq n - 1\}$. Trivially, this inequality is valid for any sub-s-path in $\tilde{P}[R \cup T]$ not containing any arcs in $M$ by the flow constraints. It remains to show validity when such an arc is included in the sub-s-path.

Take any sub-s-path $\mathcal{P} \in \tilde{P}[R \cup T]$ that includes an arc $(u, i_j, t) \in M$ and covers all of the nodes in the x-cycle. $\mathcal{P}$ cannot cover all of the nodes in the x-cycle with arcs after $(u, i_j, t)$, so some part of the x-cycle must be covered by arcs at an earlier time. Assume $\mathcal{P}$ contains $q$ x-cycle arcs following $(u, i_j, t)$. $\mathcal{P}$ can contain at most $|C| - q - 2$ x-cycle arcs before $(u, i_j, t)$, since it can contain neither $(i_{j-1}, i_j, t)$ nor $(i_{j+q}, i_{j+q+1}, t)$ at any time $t$, so the inequality (33) holds for $\mathcal{P}$.

The claim holds similarly for sub-s-paths $\mathcal{P} \in \tilde{P}[R \cup T]$ that include more than one arc of $M$ and cover all of the nodes of the x-cycle. Let $\mathcal{P}_1, \ldots, \mathcal{P}_k$ be the sub-s-paths of $\mathcal{P}$ starting with an arc $(u_k, i_{j_k}, t_k) \in M$ followed by $q_k$ x-cycle arcs. $\mathcal{P}_1, \ldots, \mathcal{P}_k$ cannot contain any of the x-cycle arcs $(i_{j_k - 1}, i_{j_k}, t)$ or $(i_{j_k + q_k}, i_{j_k + q_k + 1}, t)$ for all times $t$. Further, since $n - |C| + 1 \leq t_k \leq n - 1$, $\mathcal{P}_1, \ldots, \mathcal{P}_k$ cannot cover all of the nodes in the x-cycle, and there must be another sub-path, call it $\mathcal{P}_0$, covering the remaining nodes. $\mathcal{P}$ cannot contain the x-cycle arcs immediately preceding and immediately following the x-cycle arcs of $\mathcal{P}_0$ (irrespective of time), so the inequality (33) holds for $\mathcal{P}$. □

It might seem a bit odd that we would start deriving the lifted x-cycle inequalities from a weak inequality when inequality (33) is stronger; however, the inequalities lifted from (33) are largely the same, regardless of lifting order. The lifted coefficients of x-chords starting from this inequality will mostly be 1, and some inequalities lifted from this inequality will be the Lifted SEC. This is the same reason as that for not using inequality (3.10), and the proofs are similar.

However, the family of lifted x-cycle inequalities contains some inequalities which are facet-defining for $\tilde{P}$. The lifting of some x-chords to coefficient 2 will increase the dimension of the face by more than one; by choosing a particular set of x-chords to lift first, we can get an inequality which defines a facet of its lower-dimension polytope, and then all inequalities lifted from it will define facets of the polytope. We have chosen a set of arcs to lift first which are all pairwise incompatible. In addition, the face of the lifted inequality will have sub-s-paths which we can easily show are affinely independent by a simple row-reduction argument.

**Definition 8.** Let $K_2^r$ be the set of 2-x-chords of $C$ at $r$ and $K_3^r$ be the set of 3-x-chords of $C$ at $r$, with $2 \leq r \leq |C| - 3$. Define the *nest inequality* to be the following inequality

$$x(C) + 2x(K_2^r) + 2x(K_3^r) \leq |C| - 1. \tag{5.2}$$

Any inequality which includes coefficient 2 for the x-chords in sets $K_2^r$ and $K_3^r$ for some $r$ with $2 \leq r \leq |C| - 3$ is said to be in the family of nest inequalities.

Since all of $K_2^r, K_3^r$ are pairwise incompatible, by Lemma 14 $K_2^r \cup K_3^r$ can be lifted in any order and all would take the maximum coefficient. For $2|C| \leq n$, this maximum coefficient is 2 and the nest inequality is a lifted x-cycle inequality. In the remainder of this section, we will look at the properties of the family of nest inequalities. We will show that the family of nest inequalities are facet-defining for $\tilde{P}$. We will also show that the nest inequality is *not* a closed form: different inequalities can be derived from different lifting orders of $R \setminus (K_2^r \cup K_3^r) \cup T$.

## 5.1 Facet-Defining Properties

To show that they are facet-defining for $\tilde{P}$, we need to show that our choice of lifting $K_2^r$ and $K_3^r$ allows for enough affinely independent points to show that the nest inequality defines a facet.

**Lemma 34.** *The family of nest inequalities on an x-cycle $C$, with $|C| \geq 6$ and $2|C| \leq n$, is facet-defining for $\tilde{P}$.*

*Proof.* By lifting theory, it is enough to show that the nest inequality is facet-defining for $\tilde{P}[R \setminus (K_2^r \cup K_3^r) \cup T]$. To prove this claim, it is sufficient to show a set of linearly independent sub-s-paths in the face induced by the nest inequality, one for each arc in $\tilde{P}[R \setminus \{K_2^r \cup K_3^r\} \cup T]$. Partition the arcs into the following sets:

1. $T_2 = \{(i_j, v, t) \mid \forall i_j \in V(C), \forall v \notin V(C), |C| \leq t \leq n-1\}$, tangents in times $|C|$ to $n-1$
2. $E_1 = \{(u, i_j, t) \mid \forall u \notin V(C), \forall i_j \in V(C), 1 \leq t \leq n-|C|\}$, entering arcs in times 1 to $n-|C|$
3. $E_1 = \{(u, i_j, t) \mid \forall u \notin V(C), \forall i_j \in V(C), 1 \leq n-|C|+1 \leq n-1\}$, entering arcs in times $n-|C|+1$ to $n-1$
4. $K_2^r$
5. $Y_1 = \{(u, v, t) \mid \forall u, v \notin V(C), 1 \leq t \leq n-|C|\}$, arcs non-incident to the x-cycle in times 1 to $n-|C|$
6. $Y_2 = \{(u, v, t) \mid \forall u, v \notin V(C), n-|C|+1 \leq t \leq n-1\}$, arcs non-incident to the x-cycle in times $n-|C|+1$ to $n-1$
7. $K_3^r$
8. x-cycle arcs in times 1 to $|C|-2$
9. x-cycle arcs in times $|C|-1$ to $n-2$
10. x-cycle arcs in time $n-1$

For each arc $(i_j, v, t)$ in 1, take the sub-s-path

$$\{(i_{j-|C|+1}, i_{j-|C|+2}, t-|C|+1), \ldots, (i_{j-1}, i_j, t-1), (i_j, v, t)\}. \tag{5.3a}$$

For each arc $(u, i_j, t)$ in 2, take the sub-s-path

$$\{(u, i_j, t), (i_j, i_{j+1}, t+1), \ldots, (i_{j-2}, i_{j-1}, t+|C|-1)\}. \tag{5.3b}$$

For each arc $(u, i_j, t)$ in 3, take the sub-s-path

$$\begin{aligned}&\{(i_{j-r}, i_{j-r+1}, 1), \ldots, (i_{j-2}, i_{j-1}, r-1), (i_{j-1}, i_{j+1}, r),\\ &(i_{j+1}, i_{j+2}, r+1), \ldots, (i_{j-r-2}, i_{j-r-1}, |C|-2), (u, i_j, t)\}.\end{aligned} \tag{5.3c}$$

For each arc $(i_j, i_{j+2}, r)$ in 4, take the sub-s-path

$$\begin{aligned}&\{(i_{j-r-1}, i_{j-r}, 1), \ldots, (i_{j-1}, i_j, r-1), (i_j, i_{j+2}, r),\\ &\{(i_{j+2}, i_{j+3}, r+1), \ldots, (i_{j-r-1}, i_{j-r}, |C|-2)\}.\end{aligned} \tag{5.3d}$$

For each arc $(u, v, t)$ in 5, take the sub-s-path

$$\{(u, v, t), (i_1, i_2, n-|C|+2), \ldots, (i_{|C|-1}, i_{|C|}, n-1)\}. \tag{5.3e}$$

For each arc $(u, v, t)$ in 6, take the sub-s-path

$$\{(i_1, i_2, 1), \ldots, (i_{|C|-1}, i_|C|, |C|-1), (u, v, t)\}. \tag{5.3f}$$

For each arc $(i_j, i_{j+3}, r)$ in 7, take the sub-s-path

$$\begin{aligned}&\{(i_{j-r+1}, i_{j-r+2}, 1), \ldots, (i_{j-1}, i_j, r-1), (i_j, i_{j+3}, r),\\ &(i_{j+4}, i_{j+5}, r+1), \ldots, (i_{j-r-1}, i_{j-r}, |C|-3), (i_{j+1}, i_{j+2}, n-1)\}.\end{aligned} \tag{5.3g}$$

For each arc $(i_j, i_{j+1}, t)$ in 8, take the sub-s-path

$$\{(i_j, i_{j+1}, t), (i_{j+1}, i_{j+2}, t+1), \ldots, (i_{j-2}, i_{j-1}, t+|C|-2)\}. \tag{5.3h}$$

For each arc $(i_j, i_{j+1}, t)$ in 9, take the sub-s-path

$$\begin{aligned}&\{(i_{j-r}, i_{j-r+1}, 1), \ldots, (i_{j-2}, i_{j-1}, r-1), (i_{j-1}, i_{j+2}, r),\\ &(i_{j+2}, i_{j+3}, r+1), \ldots, (i_{j-r-3}, i_{j-r-2}, |C|-3), (i_j, i_{j+1}, t)\}.\end{aligned} \tag{5.3i}$$

For each arc $(i_j, i_{j+1}, n-1)$ in 10, take the sub-s-path

$$\begin{aligned}&\{(i_{j-r}, i_{j-r+1}, 2), \ldots, (i_{j-2}, i_{j-1}, r-1), (i_{j-1}, i_{j+2}, r),\\ &(i_{j+2}, i_{j+3}, r+1), \ldots, (i_{j-r-2}, i_{j-r-1}, |C|-2), (i_j, i_{j+1}, n-1)\}.\end{aligned} \tag{5.3j}$$

To prove that this set of sub-s-paths is linearly independent, it is enough to show that the matrix whose rows are the incidence vectors of the sub-s-paths (5.3a)–(5.3j) is full row rank. Assuming appropriate ordering of the columns within each set, the matrix will appear as Figure 5.1. By subtracting the rows of 7 corresponding to the rows of 9, the matrix reduces to Figure 5.2. Next, we show the row-reduction steps for the rows of 10. In the following, rows of the matrix are specified by the sets of arcs for which they are incidence vectors; a negative sign before an arc indicates that $-1$ appears in that position in the vector.

The row of 10 in Figure 5.2 containing chord $(i_j, i_{j+3}, r)$ is:

$$\begin{aligned}&\{(i_{j-r+2}, i_{j-r+3}, 2),\ldots,(i_{j-1}, i_j, r-1), (i_j, i_{j+3}, r),\\ &\quad(i_{j+3}, i_{j+4}, r+1),\ldots,(i_{j-r-1}, i_{j-r}, |C|-2), (i_{j+1}, i_{j+2}, n-1)\}\end{aligned} \tag{5.4a}$$

Subtract from 5.4a the row of 7 with the chord $(i_j, i_{j+3}, r)$:

$$\begin{aligned}&-\{(i_{j-r+1}, i_{j-r+2}, 1),\ldots,(i_{j-1}, i_j, r-1), (i_j, i_{j+3}, r),\\ &\quad(i_{j+4}, i_{j+5}, r+1),\ldots,(i_{j-r-1}, i_{j-r}, |C|-3), (i_{j+1}, i_{j+2}, n-1)\}\end{aligned} \tag{5.4b}$$

to yield the row:

$$\{-(i_{j-r+1}, i_{j-r+2}, 1), (i_{j-r-1}, i_{j-r}, |C|-2)\} \tag{5.4c}$$

Add to 5.4c the row from 8 leading with $(i_{j-r+1}, i_{j-r+2}, 1)$ and subtract the row from 8 leading with $(i_{j-r+2}, i_{j-r+3}, 2)$:

$$\begin{aligned}&+\{(i_{j-r+1}, i_{j-r+2}, 1),(i_{j-r+2}, i_{j-r+3}, 2),\ldots,(i_{j-r-1}, i_{j-r}, |C|-2)\ \}\\ &\qquad-\{(i_{j-r+2}, i_{j-r+3}, 2),\ldots,(i_{j-r-1}, i_{j-r}, |C|-2),(i_{j-r}, i_{j-r+1}, |C|-1)\}\end{aligned}$$

to yield:

$$\{(i_{j-r-1}, i_{j-r}, |C|-2), -(i_{j-r}, i_{j-r+1}, |C|-1)\} \tag{5.4d}$$

Subtract from 5.4d the row from 8 leading with $(i_{j-r-1}, i_{j-r}, |C|-2)$:

$$-\{(i_{j-r-1}, i_{j-r}, |C|-2), (i_{j-r}, i_{j-r+1}, |C|-1), \ldots, (i_{j-r-3}, i_{j-r-2}, 2|C|-4)\}$$

to yield:

$$\{-(i_{j-r}, i_{j-r+1}, |C|-1), -(i_{j-r+1}, i_{j-r+2}, |C|-1), \ldots, -(i_{j-r-3}, i_{j-r-2}, 2|C|-4)\} \tag{5.4e}$$

By adding appropriate rows of 9 to (5.4e), this becomes:

$$\{-(i_{j-r}, i_{j-r+1}, n-1), -(i_{j-r+1}, i_{j-r+2}, n-1), \ldots, -(i_{j-r-3}, i_{j-r-2}, n-1)\} \tag{5.4f}$$

At the end of these row-reduction steps, the row will appear as (5.4f): $-1$ in the columns of $x^{n-1}(C)$ except for $(i_{j-r-2}, i_{j-3}, n-1)$ and zeros elsewhere. Through repeated application of this row-reduction to all of the rows of 10, and permutation of the columns, the sub-matrix in the $x^{n-1}(C)$ columns of the rows of 10 will be $I - \mathbb{1}$ (where $\mathbb{1}$ is the square matrix of ones). The final array will appear as Figure 5.3, which is full row rank. $\square$

| | $x(T)$ | $x(E_1)$ | $x(E_2)$ | $x^r(K_2)$ | $x(Y_1)$ | $x(Y_2)$ | $x^r(K_3)$ | $x^1(C)$ | $x^2(C)$ | $\dots$ | $x^{r-1}(C)$ | $x^r(C)$ | $x^{r+1}(C)$ | $\dots$ | $x^{s-3}(C)$ | $x^{s-2}(C)$ | $x^{s-1}(C)$ | $\dots$ | $x^{n-2}(C)$ | $x^{n-1}(C)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $I$ | | | $I$ | | | | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ |
| 2 | | $I$ | | | | | | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ |
| 3 | | | $I$ | | | | | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ |
| 4 | | | | $I$ | | | | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ |
| 5 | | | | | $I$ | | | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ |
| 6 | | | | | | $I$ | | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ |
| 7 | | | | | | | $I$ | $*$ | $*$ | $\dots$ | $*$ | | $*$ | $\dots$ | $*$ | | | | | $I$ |
| 8 | | | | | | | | $I$ | $I$ | $\dots$ | $I$ | $I$ | $I$ | $\dots$ | $I$ | $I$ | $*$ | | | |
| | | | | | | | | | $\ddots$ | | | | | | | | $\vdots$ | $\ddots$ | | |
| | | | | | | | | | | | | | | | | $I$ | $*$ | $\dots$ | $*$ | |
| 9 | | | | | | | $I$ | $*$ | $*$ | $\dots$ | $*$ | | $*$ | $\dots$ | $*$ | | $I$ | | | |
| | | | | | | | $I$ | $*$ | $*$ | $\dots$ | $*$ | | $*$ | $\dots$ | $*$ | | | $\ddots$ | | |
| | | | | | | | $I$ | $*$ | $*$ | $\dots$ | $*$ | | $*$ | $\dots$ | $*$ | | | | $I$ | |
| 10 | | | | | | | $I$ | | $*$ | $\dots$ | $*$ | | $*$ | $\dots$ | $*$ | $*$ | | | | $I$ |

Figure 5.1: Array of sub-s-paths in the face defined by the nest inequality

Since we have shown that the family of nest inequalities defines a facet of $\tilde{P}$, we can show that they are support-reduced by Theorem 3. In fact, we will show a more general result, that any lifted x-cycle inequality which defines a facet of $\tilde{P}$ is support reduced.

**Lemma 35.** *Lifted x-cycle inequalities defined for x-cycles of size $\leq n-2$ which define facets of $\tilde{P}$ are support reduced.*

*Proof.* Let $A^=$ be the matrix of flow conservation constraints (1.3c) and $B^=$ be the matrix of degree constraints (1.3d), so $\begin{bmatrix} A^= \\ B^= \end{bmatrix} x = \begin{bmatrix} 0 \\ e \end{bmatrix}$ is the equality system for the TDTS polytope.

Assume the inequality $\alpha x = x(C) \leq |C| - 1 = \alpha_0$ is not support reduced, then by Theorem 3, there exist $\mu \in \mathbb{R}^{n^2-2n}$ and $\gamma \in \mathbb{R}^n$ such that $(\alpha, \alpha_0) = \mu(A^=, 0) + \gamma(B^=, e)$. Let $(j,t)$ for $j = 1 \dots n, t = 1 \dots n-2$ index the rows of the flow constraints $A^=$ (and components of $\mu$) and let $j$ for $j = 1 \dots n$ index the rows of the degree constraints $B^=$ (and components of $\gamma$.) The column of $\begin{bmatrix} A^= \\ B^= \end{bmatrix}$ for arc $(u, v, n-1)$ with $u, v \in N$ and $u \neq v$ will have $-1$ in row $(u, n-2)$ of $A^=$, 1 in row $u$ of $B^=$, and zero elsewhere.

| | $x(T)$ | $x(E_1)$ | $x(E_2)$ | $x^r(K_2)$ | $x(Y_1)$ | $x(Y_2)$ | $x^r(K_3)$ | $x^1(C)$ | $x^2(C)$ | $\dots$ | $x^{r-1}(C)$ | $x^r(C)$ | $x^{r+1}(C)$ | $\dots$ | $x^{s-3}(C)$ | $x^{s-2}(C)$ | $x^{s-1}(C)$ | $\dots$ | $x^{n-2}(C)$ | $x^{n-1}(C)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $I$ | | | $I$ | | | | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ |
| 2 | | $I$ | | | | | | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ |
| 3 | | | $I$ | | | | | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ |
| 4 | | | | $I$ | | | | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ |
| 5 | | | | | $I$ | | | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ |
| 6 | | | | | | $I$ | | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ | $*$ | $\dots$ | $*$ | $*$ |
| 7 | | | | | | | $I$ | $*$ | $*$ | $\dots$ | $*$ | | $*$ | $\dots$ | $*$ | | | | | $I$ |
| 8 | | | | | | | | $I$ | $I$ | $\dots$ | $I$ | $I$ | $I$ | $\dots$ | $I$ | $I$ | $*$ | | | |
| | | | | | | | | | $\ddots$ | | | | | | | | $\vdots$ | $\ddots$ | | |
| | | | | | | | | | | | | | | | | $I$ | $*$ | $\dots$ | $*$ | |
| 9 | | | | | | | | | | | | | | | | | $I$ | | | $-I$ |
| | | | | | | | | | | | | | | | | | | $\ddots$ | | $\vdots$ |
| | | | | | | | | | | | | | | | | | | | $I$ | $-I$ |
| 10 | | | | | | | $I$ | | $*$ | $\dots$ | $*$ | | $*$ | $\dots$ | $*$ | $*$ | | | | $I$ |

Figure 5.2: After first row reduction step

Consider the following types of arcs in time $n-1$:

$$\begin{aligned}
\text{x-cycle arcs } & \{(i_j, i_{j+1}, n-1),\ j = 1 \dots |C|\} \\
\text{tangents } & \{(i_j, v, n-1),\ j = 1 \dots |C|, v \notin V(C)\} \\
\text{entering arcs } & \{(u, i_j, n-1),\ j = 1 \dots |C|, u \notin V(C)\} \\
\text{non-incident arcs } & \{(u, v, n-1),\ u, v \notin V(C), u \neq v\}
\end{aligned}$$

X-cycle arcs have a coefficient 1 in $\alpha$, the rest are 0. For these arcs, then, the following system must hold if $(\alpha, \alpha_0) = \mu(A^=, 0) + \gamma(B^=, e)$:

$$-\mu_{i_j,n-2} + \gamma_{i_{j+1}} = 1,\ j = 1 \dots |C| \tag{5.5}$$

$$-\mu_{i_j,n-2} + \gamma_v = 0,\ j = 1 \dots |C|, \forall v \notin V(C) \tag{5.6}$$

$$-\mu_{u,n-2} + \gamma_{i_j} = 0,\ j = 1 \dots |C|, \forall u \notin V(C) \tag{5.7}$$

$$-\mu_{u,n-2} + \gamma_v = 0,\ \forall u, v \notin V(C), u \neq v \tag{5.8}$$

Since $\mu_{u,n-2} = \gamma_v$ for all pairs of nodes $u$ and $v$ outside the cycle, then $\mu_{u,n-2} = \gamma_u = \kappa$, $\forall u \notin C$ for some $\kappa$. Then by equation (5.6), $\mu_{i_j,n-2} = \kappa, \forall j = 1 \dots |C|$, and by equation (5.7) $\gamma_{i_j} = \kappa, \forall j = 1 \dots |C|$. But this is a contradiction, by equation (5.5).

Thus, $\mu, \gamma$ do not exist, and $x(C) \leq |C| - 1$ is support reduced. $\square$

| | $x(T)$ | $x(E_1)$ | $x(E_2)$ | $x^r(K_2)$ | $x(Y_1)$ | $x(Y_2)$ | $x^r(K_3)$ | $x^1(C)$ | $x^2(C)$ | $\ldots$ | $x^{r-1}(C)$ | $x^r(C)$ | $x^{r+1}(C)$ | $\ldots$ | $x^{s-3}(C)$ | $x^{s-2}(C)$ | $x^{s-1}(C)$ | $\ldots$ | $x^{n-2}(C)$ | $x^{n-1}(C)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $I$ | | | $I$ | | | | $*$ | $*$ | $\ldots$ | $*$ | $*$ | $*$ | $\ldots$ | $*$ | $*$ | $*$ | $\ldots$ | $*$ | $*$ |
| 2 | | $I$ | | | | | | $*$ | $*$ | $\ldots$ | $*$ | $*$ | $*$ | $\ldots$ | $*$ | $*$ | $*$ | $\ldots$ | $*$ | $*$ |
| 3 | | | $I$ | | | | | $*$ | $*$ | $\ldots$ | $*$ | $*$ | $*$ | $\ldots$ | $*$ | $*$ | $*$ | $\ldots$ | $*$ | $*$ |
| 4 | | | | $I$ | | | | $*$ | $*$ | $\ldots$ | $*$ | $*$ | $*$ | $\ldots$ | $*$ | $*$ | $*$ | $\ldots$ | $*$ | $*$ |
| 5 | | | | | $I$ | | | $*$ | $*$ | $\ldots$ | $*$ | $*$ | $*$ | $\ldots$ | $*$ | $*$ | $*$ | $\ldots$ | $*$ | $*$ |
| 6 | | | | | | $I$ | | $*$ | $*$ | $\ldots$ | $*$ | $*$ | $*$ | $\ldots$ | $*$ | $*$ | $*$ | $\ldots$ | $*$ | $*$ |
| 7 | | | | | | | $I$ | $*$ | $*$ | $\ldots$ | $*$ | | $*$ | $\ldots$ | $*$ | | | | | $*$ |
| 8 | | | | | | | | $I$ | $I$ | $\ldots$ | $I$ | $I$ | $I$ | $\ldots$ | $I$ | $I$ | $*$ | | | |
| | | | | | | | | | $\ddots$ | | | | | | | | $\vdots$ | $\ddots$ | | |
| | | | | | | | | | | | | | | | | $I$ | $*$ | $\ldots$ | $*$ | |
| 9 | | | | | | | | | | | | | | | | | $I$ | | | $*$ |
| | | | | | | | | | | | | | | | | | | $\ddots$ | | $\vdots$ |
| | | | | | | | | | | | | | | | | | | | $I$ | $*$ |
| 10 | | | | | | | | | | $\ldots$ | | | | $\ldots$ | | | | | | $I - \mathbb{1}$ |

Figure 5.3: After row-reducing rows 10, and permutation of $x^{n-1}(C)$ columns.

Showing that lifted cycle inequalities are *strongly* support reduced is more difficult. In the case of lifted cycle inequalities for the ATS polytope, the graphic properties of the support graph (or its complement) of the cycle inequality alone are sufficient to show it is strongly support reduced in some cases. This does not depend on the lifting sequence (equivalently, the final inequality.) A graphic interpretation of a basis of the equality system of the TDTS polytope is unknown, and in some cases the complete lifted x-cycle inequality may need to be known explicitly. For 0-lifted x-chords, in particular, the definition of strongly support reduced inequalities requires that either (i) the x-chord be in the initial basis $\{j_1 \ldots j_q\}$ of $\begin{bmatrix} A^= \\ B^= \end{bmatrix}$, or (ii) that there exist two s-paths $x^1$ and $x^2$ in the face of the inequality, which (minimally) do not differ in their x-cycle arcs but do differ in the x-chord.

Any s-path which uses x-cycle arcs immediately before and after the x-chord cannot be one of these s-paths, since the only compatible arc between the two x-cycle arcs is the x-chord. Every 0-lifted x-chord must then either be in the initial basis or be contained in two s-paths in the face of the inequality which do not use an x-cycle arc immediately before *and* after it. Some 0-lifted x-chords with this property can be identified, such as the reachable crossing x-chords of a 2-x-chord, but in general, it would require knowing the complete lifting sequence to determine which 0-lifted x-chord must be in the basis and which are not.

## 5.2 Lifting Properties

In this section, we will describe some of the lifting properties of the nest inequalities. Since nest inequalities have coefficient 2 on $K_2^r \cup K_3^r$, we will assume that $K_2^r$ and $K_3^r$ are lifted first, and show the maximum lifted coefficients of the chords and tangents which remain. In particular, we show that 2-lifting $K_2^r \cup K_3^r$ will leave some x-chords which are 2-liftable and some tangents which are 1-liftable. Different sequences of the remaining arcs will therefore give different final inequalities, in general. Identifying the classes of x-chords and tangents with maximum lifted coefficient 0 or 1 is a necessary first step for lifting the x-chords and tangents. These sets can be easily described, as shown below, and checking if an x-chord or tangent is in one of them is easy, simplifying the lifting of the remaining x-chords and tangents.

First we consider reachable crossing x-chords and nooses of $C \cup K_2^r \cup K_3^r$. Given that the x-chords of $K_2^r \cup K_3^r$ have 1 or 2 external nodes, there are relatively few crossing x-chords and nooses. From Theorem 22, the following x-chords are all reachable crossing x-chords of $K_2^r$, and have maximum coefficient 0:

$$\begin{aligned}
&\{(i_{j+2+k}, i_{j+1}, r+k+1) \\
&k = \max(|C|-r-2, 0) \ldots \min(n-r-2, |C|-3), i_j \in V(C)\} && (5.9a)\\
\text{if } r \geq 2 \ &\{(i_{j+1}, i_{j-k}, r-k-1) \\
&k = \max(r+|C|-n-2, 0) \ldots \min(|C|-3, r-2), i_j \in V(C)\} && (5.9b)
\end{aligned}$$

The following x-chords are reachable crossing chords with respect to $K_3^r$, and have maximum coefficient 0:

$$\begin{aligned}
&\{(i_{j+3+k}, i_{j+1}, r+k+1) \\
&k = \max(|C|-r-3, 0) \ldots \min(n-r-3, |C|-4), i_j \in V(C)\} && (5.10a)\\
\text{if } r \geq 3 \ &\{(i_{j+2}, i_{j-k}, r-k-1) \\
&k = \max(r+|C|-n-3, 0) \ldots \min(|C|-4, r-3), i_j \in V(C)\} && (5.10b)
\end{aligned}$$

The following are reachable crossing x-chords with respect to $K_3^r$, and have maximum coefficient 1:

$$\begin{aligned}
&\{(i_{j+3+k}, i_{j+2}, r+k+1) \\
&k = \max(|C|-r-3, 0) \ldots \min(n-r-3, |C|-4), i_j \in V(C)\} && (5.11a)\\
\text{if } r \geq 3 \ &\{(i_{j+1}, i_{j-1-k}, r-k-1) \\
&k = \max(r+|C|-n-3, 0) \ldots \min(|C|-4, r-3), i_j \in V(C)\} && (5.11b)
\end{aligned}$$

Since $r \leq |C|-3$ and $n \geq 2|C|$, we can state explicitly the range of $k$ in each sets. For example, $|C|-r-2 \geq |C|-|C|+3-2 \geq 1 \Rightarrow \max(|C|-r-2, 0) = |C|-r-2$ and $n-r-2 \geq 2|C|-|C|+3-2 \geq |C|-1 \Rightarrow \min(n-r-2, |C|-3) = |C|-3$ in (5.9a). Applying this to all of the sets (5.9a)–(5.11b), the following sets of x-chords have maximum lifted coefficient 0:

$$\{(i_{j+2+k}, i_{j+1}, r+k+1) \mid |C|-r-2 \le k \le |C|-3,\ i_j \in V(C)\} \tag{5.12a}$$
$$\text{if } r \ge 2\quad \{(i_{j+1}, i_{j-k}, r-k-1) \mid 0 \le k \le r-2,\ i_j \in V(C)\} \tag{5.12b}$$
$$\{(i_{j+3+k}, i_{j+1}, r+k+1) \mid |C|-r-3 \le k \le |C|-4,\ i_j \in V(C)\} \tag{5.12c}$$
$$\text{if } r \ge 3\quad \{(i_{j+2}, i_{j-k}, r-k-1) \mid 0 \le k \le r-3,\ i_j \in V(C)\} \tag{5.12d}$$

and the following sets of x-chords have maximum lifted coefficient 1:

$$\{(i_{j+3+k}, i_{j+2}, r+k+1) \mid |C|-r-3 \le k \le |C|-4, i_j \in V(C)\} \tag{5.13a}$$
$$\text{if } r \ge 3\quad \{(i_{j+1}, i_{j-1-k}, r-k-1) \mid 0 \le k \le r-3,\ i_j \in V(C)\} \tag{5.13b}$$

The x-chords in each set, for a given value of $k$, have the same length. Each is a set of $l(k)$-x-chords, where the length $l(k)$ is a function of $k$, and will help to clarify which x-chords have maximum coefficient 0 or 1. For example, by letting $l = |C| - k - 1$ in the set (5.12a) it simplifies to $K_l^{r+|C|-l}$ for $2 \le l \le r+1$. Simplifying sets (5.12a)–(5.12d), the crossing x-chords with maximum lifted coefficient 0 are:

$$\{K_l^{r+|C|-l} \mid 2 \le l \le r+1\} \quad \text{from (5.12a)} \tag{5.14a}$$
$$\text{if } r \ge 2\quad \{K_l^{r-|C|+l} \mid |C|-r+1 \le l \le |C|-1\} \quad \text{from (5.12b)} \tag{5.14b}$$
$$\{K_l^{r+|C|-l-1} \mid 2 \le l \le r+1\} \quad \text{from (5.12c)} \tag{5.14c}$$
$$\text{if } r \ge 3\quad \{K_l^{r-|C|+l+1} \mid |C|-r+1 \le l \le |C|-2\} \quad \text{from (5.12d)} \tag{5.14d}$$

The crossing x-chords with maximum lifted coefficient 1 are:

$$\{K_l^{r+|C|-l} \mid 3 \le l \le r+2\} \quad \text{from (5.13a)} \tag{5.15a}$$
$$\text{if } r \ge 3\quad \{K_l^{r-|C|+l+1} \mid |C|-r+1 \le l \le |C|-2\} \quad \text{from (5.13b)} \tag{5.15b}$$

We can also apply the definition of a tight noose (definition 6) to find another set of x-chords with maximum lifted coefficient of 1. Since there are no x-chords in the exterior of an x-chord of $K_2^r$, we need to only consider tight nooses containing x-chord(s) from $K_3^r$. All of $K_3^r$ has been lifted in the same time, so no noose containing one x-chord $K_3^r$ can contain more than one x-chord of $K_3^r$. The only non-crossing x-chords external to $(i_j, i_{j+3}, r) \in K_3^r$ are $\{(i_{j+2}, i_{j+1}, t)\ 1 \le t \le n-1\}$. For times $1 \le t \le r-2$ and $|C|-1 \le t \le n-1$, each of these forms a tight noose with $(i_j, i_{j+3}, r)$; the x-chords

$$\text{if } r \ge 3\quad \{(i_{j+2}, i_{j+1}, t) \mid 1 \le t \le r-2\} = \{K_{|C|-1}^t \mid 1 \le t \le r-2\} \tag{5.16a}$$
$$\{(i_{j+2}, i_{j+1}, t) \mid |C|-1 \le t \le n-1,\ j \in V(C)\} = \{K_{|C|-1}^t \mid |C|-1 \le t \le n-1\} \tag{5.16b}$$

have maximum lifted coefficient 1.

In addition, we can apply lemmata 26 and 25 to find the set of tangents which will have maximum lifted coefficient 0. By applying lemma 25 to $K_2^r$, the tangents

$$\begin{aligned}\{(i_{j+1}, v, t) \mid r+2 \leq t \leq |C|-1,\, i_j \in V(C),\, v \notin V(C)\} = \\ \{(i_j, v, t) \mid r+2 \leq t \leq |C|-1,\, i_j \in V(C),\, v \notin V(C)\}\end{aligned} \tag{5.17a}$$

will have maximum coefficient 0. By applying lemma 25 to $K_3^r$, the tangents

$$\begin{aligned}\{(i_{j+2}, v, t) \mid r+3 \leq t \leq |C|-1,\, i_j \in V(C),\, v \notin V(C)\} = \\ \{(i_j, v, t) \mid r+3 \leq t \leq |C|-1,\, i_j \in V(C),\, v \notin V(C)\}\end{aligned} \tag{5.18a}$$

will have maximum coefficient 0. The set in (5.18a) is a subset of (5.17a), so the set in (5.17a) are sufficient to describe these tangents with maximum lifted coefficient of zero.

By applying lemma 26 to $K_2^r$, the following tangents

$$\begin{aligned}\{(i_{j+2+k}, v, r+1+k) \mid k = \max(0, |C|-r-2) \ldots \min(|C|-4, |C|-r-2) \\ i_j \in V(C),\, v \notin V(C)\}\end{aligned} \tag{5.19}$$

have maximum coefficient 0. By applying lemma 26 to $K_3^r$, the tangents

$$\begin{aligned}\{(i_{j+3+k}, v, r+1+k) \mid k = \max(0, |C|-r-3) \ldots \min(|C|-5, |C|-r-2) \\ i_j \in V(C),\, v \notin V(C)\}\end{aligned} \tag{5.20}$$

will have maximum lifted coefficient 0. As before, we can eliminate some of the min and max expressions and simply the set of tangents to

$$\{(i_j, v, |C|-1) \mid i_j \in V(C), v \notin V(C)\} \tag{5.21a}$$

$$\begin{aligned}\{(i_j, v, r+1+k) \mid |C|-r-3 \leq k \leq \min(|C|-5, |C|-r-2) \\ i_j \in V(C), v \notin V(C)\}\end{aligned} \tag{5.21b}$$

The range of times on the tangents in (5.21b) is $|C|-2 \ldots \min(r+|C|-4, |C|-1)$, so both sets (5.21a) and (5.21b) will be subsets of (5.17a).

With these sets determined, we can easily check the number of x-chords which cannot be 2-lifted and the number of tangents which cannot be 1-lifted. This number is dependent on the choice of $r$, and Table 5.1 summarizes the size of the x-chord sets (5.14a)–(5.16b) for the minimum and maximum values of $r$. It can be easily checked that each of the sets is disjoint. The total number of 0-liftable x-chords and 1-liftable x-chords is given. Notice that $n|C|^2-2n|C|-|C|^2$ x-chords remain after lifting $K_2^r \cup K_3^r$, so there remain some 2-liftable x-chords for any choice of $r$.

For $r = 2$, there are $|C|(n-|C|)(|C|-4)$ tangents in set (5.17a); for $r = |C|-3$, there are $|C|(n-|C|)$ tangents in set (5.17a). (The factor $|C|(n-|C|)$ is a constant, for the number of tangents in one time period.) There a total of $|C|(n-|C|)(|C|-1)$ tangents to be lifted, so there remain some 1-liftable tangents for any choice of $r$. This shows that, in general, the lifting order of the arcs remaining after lifting $K_2^r \cup K_3^r$ will determine the final inequality.

| set | size of set for $r = 2$ | size of set for $r = \|C\| - 3$ |
|---|---|---|
| (5.14a) | $2\|C\|$ | $\|C\|^2 - 3\|C\|$ |
| (5.14b) | $\|C\|$ | $\|C\|^2 - 4\|C\|$ |
| (5.14c) | $2\|C\|$ | $\|C\|^2 - 3\|C\|$ |
| (5.14d) | | $\|C\|^2 - 4\|C\|$ |
| total | $5\|C\|$ | $4\|C\|^2 - 14\|C\|$ |
| (5.15a) | $3\|C\|$ | $\|C\|^2 - 3\|C\|$ |
| (5.15b) | | $\|C\|^2 - 5\|C\|$ |
| (5.16a) | | $\|C\|^2 - 5\|C\|$ |
| (5.16b) | $n\|C\| - \|C\|^2 + \|C\|$ | $n\|C\| - \|C\|^2 + \|C\|$ |
| total | $n\|C\| - \|C\|^2 + 4\|C\|$ | $n\|C\| + 2\|C\|^2 - 12\|C\|$ |

Table 5.1: Size of sets (5.14a)–(5.16b) for $r = 2$ and $r = |C| - 3$.

# Chapter 6

# Conclusion

In this work, we have begun to investigate a new class of inequalities for the TDTSP. We have shown a new inequality, the x-cycle inequality, which lifts the ATSP cycle inequality to the TDTSP formulation. We presented rules for helping to determine the lifted coefficients of arcs in the x-cycle, the x-chords, and leaving the x-cycle, the tangents, to strengthen the inequality. From these rules, we showed that lifted x-cycle inequalities include the Lifted Subtour Elimination Constraint for the TDTSP. We also showed a new family of inequalities which are facet-defining for $\tilde{P}$, and showed that all lifted x-cycle inequalities which define facets of $\tilde{P}$ are support reduced, a necessary condition for showing that they define facets of $P$.

Naturally, much work remains in fully describing the lifted x-cycle inequalities. We are lacking the conditions under which these inequalities are facet-defining for the polytope $P$ and not just its submissive. This may require showing that the inequalities are strongly support reduced, or may require a new sufficient condition for inequalities that define facets of the submissive to define facets of the polytope. This may also be possible for a closed-form lifted x-cycle inequality, as it is for the Lifted SEC. We are also lacking necessary and sufficient conditions for the lifted coefficients of the x-chords and tangents of a given x-cycle. Finally, we do not offer anything in the way of separation routines to use lifted x-cycle inequalities in a cutting plane approach to solving the TDTSP. This may be easy for some closed form inequalities, as it stands now with lifted cycle inequalities for the ATSP. Despite what is unknown about lifted x-cycle inequalities, the lifting properties suggest that a wide variety of inequalities can be derived from this family, and they merit further investigation.

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