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Stability of Impulsive Switched Systems in Two Measures

by

Benjamin Turnbull

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Abstract

This thesis introduces the notion of using stability analysis in terms of two measures for impulsive switched systems. Impulsive switched systems are defined in the context of hybrid system theory and the motivation for the study of these systems is presented. The motivation for studying stability in two measures is also given, along with the definitions of stability, uniform stability, and uniform asymptotic stability in one and two measures.

The results presented are a sets of sufficient stability criteria for linear and nonlinear systems. For autonomous linear systems, there are criteria for stability and asymptotic stability using a particular family of choices for the two measures. There is an additional stronger set of criteria for asymptotic stability using one measure, for comparison. There is also a proposed method for finding the asymptotic stability of a non-autonomous system in one measure. The method for extending these criteria to linearized systems is also presented, along with stability criteria for such systems. The criteria for nonlinear systems cover stability, uniform stability, and uniform asymptotic stability, considering state-based and time-based switching rules in different ways.

The sufficient stability criteria that were found were used to solve four instructive examples. These examples show how the criteria are applied, how they compare, and what the shortcomings are in certain situations. It was found that the method of using two measures produced stricter stability requirements than a similar method for one measure. It was still found to be a useful result that could be applied to the stability analysis of an actual impulsive switched system.

Acknowledgements

I would like to thank my supervisor, Xinzhi Liu, for his unwavering support and Zhigang Zhang for his help with many things in the later stages of completing this work.

Dedication

This is dedicated to my parents and to Fanny Yuen.

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Chapter 1

Introduction

Stability theory is the foundation of control methodologies that we use to make a mechanical or electrical system accomplish a desired task with a desired precision. The fundamentals of stability theory have been around for many years, and the field has grown to the point where we can model almost any process as a dynamical system and develop an algorithm to monitor and control that system's behaviour in real time. This thesis adds to our understanding of the stability of impulsive switched systems using two measures.

At the most basic level, stability theory deals with the question of whether the states of an evolving dynamical system will stay within a given set of boundaries. Although this seems limited, for many physical problems with complicated dynamics, the important questions we need to answer are based on this requirement. Although it may seem desirable to find explicit solutions to the evolution of a dynamical system, this is difficult in practice and probably unnecessary. Consider a very simple problem, the design of the temperature control of an oven. Depending on its intended use, the oven will need to maintain an internal temperature within a given tolerance and may be required to increase or decrease the temperature at a specific rate. A mathematical model of the oven, its heating elements, its contents, and the influence of the outside world would be too complex to sove explicitly, so solving for the exact time-varying internal temperature function would be effectively impossible. It is also unnecessary. Instead of an exact solution to the control problem, we make the desired solution a stable point of the system, such that once the temperature gets close enough to the desired temperature, it will stay within a tolerated bound. A more complex problem is the design of a robot arm for use in a manufacturing process. The arm itself, along with its sensors and motors, compose a physical system that is modeled as a dynamical system of equations. This model will never be a perfect representation of the sum of the physical components, but it must be precise enough for the robot to perform a desired function. This is essentially to be in a particular place and orientation at a particular time. Solving the equations of motion for a robot arm would be difficult

and ultimately futile, since they are never a perfect representation of the physical system. It is better to arrange that the destination of the robot arm is an asymptotically stable point of the system. In this case, we use the related concept of asymptotic stability so that the state of the system will approach the desired state in a predictable way.

When formulating a dynamical system to describe a physical system, one of the many considerations is whether to use continuous or discrete dynamics [38, 40]. The former is a system is governed by a rule prescribing the instantaneous rate of change of the states, while the latter is a system governed by a rule prescribing the states at the next time step. In many cases, it is obvious which is the better choice; we pick the best choice and modify the necessary parts of our model to fit. We may decide that discrete dynamics are inappropriate because they require instantaneous events that are not possible in the physical world (requiring, for example, infinite acceleration), or we may decide that continuous processes can be effectively modeled by a discrete process with an arbitrarily small time step (as in all computer simulations). We could, for example, approximate the occasional step function with a steep ramp function or choose a small enough time step so that a discrete variable appears to evolve continuously. In some situations, however, trying to make the system fit into one of these two categories is not only mathematically awkward, but it fundamentally changes the character of the system.

Hybrid systems are used when neither continuous nor discrete dynamics are appropriate, and we wish to combine the two approaches. There are many different examples of hybrid systems, which can be found in [40]. One good example is temperature control in an oven. For a typical household oven, the temperature will be set by a simple thermostat. The temperature of the oven's interior is naturally modeled using a continuous system; we can simplify it to the exponential decay of Newton's law of cooling or we can use the heat equation to keep track of the temperature in space as well as time. The thermostat, however, is a simple switch that either turns the heating elements on or off, depending on the temperature. This is fundamentally not a continuous function. Although, in this simple case, we could just decide that the temperature is the most important state to consider and remodel the state of the oven's element to be a continuous variable taking values between "on" and "off," we can also use a hybrid system. In some situations, using a hybrid system will be unavoidable. This is often the case with a supervisory control model, as shown in [40] and used in [11]. If the control input for the system changes the system abruptly in fundamental ways, we must accept this as a discrete change, even thought we may want to use continuous variables for other states of the system.

It is often desirable to turn a system that could be modeled accurately as a continuous system into a hybrid system. Usually we are motivated to do this by a control problem where we want to give occasional impulsive control inputs, but not apply continuous control. When controlling a switched system, it is particularly useful practical to make impulsive control inputs, as detailed in [12].

Switched systems and impulsive switched systems are subcategories of hybrid systems, which are further explained in Chapter 2. The purpose of studying these systems is to extend the results of stability theory for continuous systems to systems that have switching and impulses. This will enable us to include models of systems where the equations governing the system can change abruptly, and the values of the states can change as well.

Stability analyses using two measures were developed to unify several different stability criteria. The mathematical definitions of stability involving two measures are provided in Chapter 2. The essential idea is that by using one set of stability criteria, we can evaluate the stability of a system in a traditional manner. A system is stable if all of the states begin close enough to a desired state, and they remain close to that state; or a system is partially stable if all the states begin close enough to a desired state, some of the states will remain close.

In addition to the information on hybrid system theory, Chapter 2 also contains some background on stability in terms of two measures. This area of stability theory, best presented in [18], concerns unifying different characterizations of stability. Rather than considering the value of the states themselves to determine stability, we create two measures based on the states and consider stability with respect to these. This way, by defining the measures the right way, we can consider the stability of other aspects of the system besides just the states. This makes it easier to determine the stability of anything other than a point, such as a periodic cycle or an invariant set. We can also consider partial stability, where we only look at the stability behaviour of a subset of the states. By formulating stability results for impulsive switched systems in two measures, it will be easier to make these types of dynamical system models adhere to more exotic stability requirements.

The results in Chapter 3 are sufficient conditions for stability and asymptotic stability of linear impulsive switched systems. Since linear systems are common simple models to use, and their stability is easy to verify, the results are worked out based on specific properties of the equations governing the systems, from which it is possible to make quadratic Lyapunov functions. In Chapter 4, we cover nonlinear systems, which have more general requirements based on finding Lyapunov functions, rather than the governing equations themselves. In both cases, we consider multiple Lyapunov functions, where each subsystem of the switched system has its own independent Lyapunov function. This method makes it easier to use existing methods for finding Lyapunov functions and applying them to impulsive switched systems.

Chapter 5 contains examples pertaining to the stability criteria in Chapters 3 and 4. With these examples, we illustrate the potential utility of the results, based on how easy it is to demonstrate the stability of a particular system, and whether the stability criteria are overly restrictive and fail to be satisfied by many systems which are, in fact, stable. Although this work does not constitute anything close to a complete characterization of stability of impulsive switched systems in two measures, it does present useful and applicable

results that can be extended in many ways.

Chapter 2

Background

2.1 Dynamical Systems

2.1.1 Definitions and Terminology

Dynamical systems are the broad category of mathematical objects we are studying. In general, a dynamical system consists of a number of states with a rule that defines how those states will change a short time in the future (see [16, 32] and references therein for details). As such, many dynamical systems consist of ordinary differential equations (ODEs) such as

$$\dot{x} = f(t, x),\ t \in [0, \infty) \tag{2.1}$$

or a difference equation such as

$$x(k+1) = f(k, x(k)),\ k \in \{0, 1, \ldots\}. \tag{2.2}$$

In these two cases, the new state of the system depends only on the current state, unlike systems with time delay, which we will not study here. A trajectory of the system is a (multivalued) function returning the value of each of the states at a given time. In most cases, these trajectories are distinguished by initial values, where the state vector $x(t)$ or x_k is defined for a specific t or k (often 0). In some simple cases it is possible, using only the dynamical system along with this information, to explicitly write an expression for the system trajectory. Take, for example, the following ODE,

$$\dot{x} = Ax, \tag{2.3}$$

where A is a constant real valued matrix. It is classified as linear because $f(t, x) = Ax$ is linear in x and autonomous because $f(t, x) = Ax$ does not depend directly on t. Starting

with any initial condition $x(t_0) = x_0$, we can explicitly write the solution as

$$x(t) = e^{A(t-t_0)}x_0 \tag{2.4}$$

using the matrix exponential [39].

Linear Systems

In the simplest form there is no dependence on time, so the system is autonomous. We can examine the stability of the system by explicitly finding the solution to

$$x(t) = e^{A(t-t_0)}x_0, \tag{2.5}$$

where $x(t_0) = x_0$. Since the eigenvalues of the matrix exponential are exactly the same as the exponentials of the eigenvalues of the matrix A, we can determine stability entirely by examining the eigenvalues of A.

A slightly more complicated example is the non-autonomous linear system given by

$$\dot{x} = A(t)x \tag{2.6}$$

In this case, it is not alway possible to write the solution in an explicit closed form. It is also not immediately clear whether the eigenvalues of $A(t)$ will provide a complete picture. In fact, it can be shown by a simple counterexample that, even with eigenvalues that always have negative real part, the solution will not necessarily be stable. The key is to construct an example with constant eigenvalues but time varying eigenvectors. Certain trajectories will be able to grow without limit.

Example 1. *[10] Consider the system given by*

$$\begin{cases} \dot{x_1} = (-1 + 1.5\cos^2 t)x_1 + (1 - 1.5\sin t\cos t)x_2 \\ \dot{x_2} = (-1 - 1.5\sin t\cos t)x_1 + (-1 + 1.5\sin^2 t)x_2 \end{cases} \tag{2.7}$$

We can calculate the eigenvalues and see that they are actually constant valued at

$$\lambda_1 = -\frac{1}{4} + \frac{\sqrt{7}}{4}i, \quad \lambda_2 = -\frac{1}{4} - \frac{\sqrt{7}}{4}i \tag{2.8}$$

However, we can see that there is a solution

$$x(t) = \begin{pmatrix} e^{t/2}\cos t \\ -e^{t/2}\sin t \end{pmatrix} \tag{2.9}$$

that is, nevertheless, unstable.

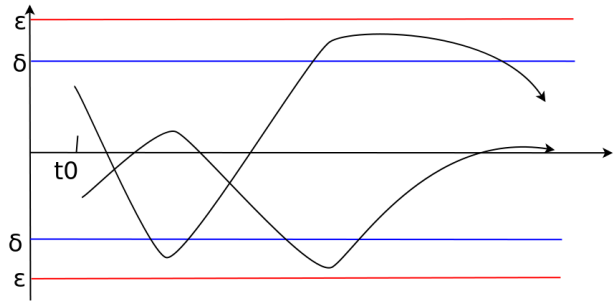

Figure 2.1: Possible trajectories of a stable system

Existence/Uniqueness of the Solution of Dynamical Systems

If all dynamical systems based on ODEs could be explicitly solved as easily as (2.3), they would not comprise an interesting field of study. Before we can study the trajectories of the general system (2.1), we must consider the existence and uniqueness of these solutions using the following theorem.

Theorem 1. *[Existence and Uniqueness][39] Let D be an open set in $\mathbb{R}^{n+1}$. If $f : D \mapsto \mathbb{R}^n$ is locally Lipschitz on D, then, given any $(t_0, x_0) \in D$, there exists a $\delta > 0$ such that (2.1) along with the initial value $x(t_0) = x_0$ has a unique solution $x(t, t_0, x_0)$ defined on $[t_0 - \delta, t_0 + \delta]$.*

Stability of Dynamical Systems

We are primarily concerned with the stability of dynamical systems. There are several different ways to characterize stability. Before we look at that, however, we must note that a system is only stable with respect to a certain trajectory. Usually, the trajectory of interest is the solution $x(t) = 0$. If we wish to consider a particular solution $x^*(t)$, we can simply reformulate the problem using a change of variables so that $y(t) = x(t) - x^*(t)$; hence, 0 is a solution. With this in mind, the simplest form of stability is that the trajectories will remain within a certain bound, provided they begin close enough to the zero solution. In addition to the mathematical definitions, Figure 2.1 illustrates how trajectories stay within an arbitrary bound ϵ, provided they start within a stricter bound δ.

Definition 2. *[Stability,S1] $x = 0$ is stable if for any $\epsilon > 0$ and $t_0 \in \mathbb{R}^+$, there exists a $\delta = \delta(t_0, \epsilon) > 0$ such that for any solution $x(t) = x(t, t_0, x_0)$, of (2.1), where $x(t_0) = x_0$, $\|x_0\| < \delta$ implies $\|x(t)\| < \epsilon$ for all $t \geq t_0$.*

Definition 3. *[Asymptotic Stability,S2] $x = 0$ is asymptotically stable if it is stable and there exists a $\sigma > 0$ such that $\|x_0\| < \sigma$ implies $x(t) \to 0$ as $t \to \infty$.*

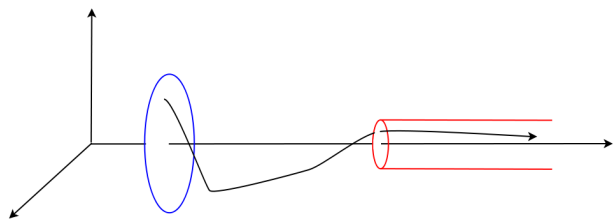

Figure 2.2: A trajectory of an asymptotically stable system.

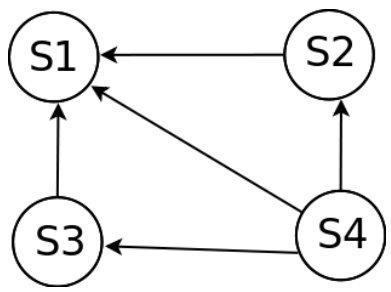

Figure 2.3: Relations of stability definitions S1–S4

See Figure 2.1.1.

Definition 4. *[Uniform Stability,S3] $x = 0$ is uniformly stable if it is stable for a δ independent of t_0.*

Definition 5. *[Uniform Asymptotic Stability,S4] $x = 0$ is uniformly asymptotically stable if it is uniformly stable and there exists $\sigma > 0$ such that $\|x_0\| < \sigma$ implies for any $\eta > 0$, there is a $T = T(\eta) > 0$ such that $\|x(t)\| < \eta$ if $t \geq t_0 + T$ for any initial time t_0.*

See Figure 2.1.1, which shows how these definitions relate to one another.

Definition 6. *$x = 0$ is unstable if it does not satisfy the definition of stability.*

There are many other forms of stability that we will be unable to cover. For instance, one may specify robustness for any stability condition, which requires that the system be stable even if there is some parameter uncertainty or disturbance present.

Lyapunov Functions

The primary tool we will use in stability analysis is the Lyapunov function. This general concept applies to most dynamical systems, and it is a familiar approach to take. We begin with a few definitions.

Suppose that the function $W(x) \in C[D, R^+]$, $W(0) = 0$; $V(t,x) \in C[R^+ \times D, R^+]$ and $V(t,0) \equiv 0$ where $D \subseteq R^n$.

Definition 7. *The function $W(x)$ is said to be positive definite if*

$$W(x) \begin{cases} > 0 & \text{for } x \in D, x \neq 0, \\ = 0 & \text{for } x = 0. \end{cases}$$

$W(x)$ is said to be positive semi-definite if $W(x) \geq 0$ for $x \in D$.

$W(x)$ is said to be negative semi-definite, if $W(x) \leq 0$ for $x \in D$.

Definition 8. *The function $V(t,x)$ is said to be positive definite, if there exists a positive definite function $W(x)$ such that $V(t,x) \geq W(x)$ and $V(t,0) \equiv 0$. The functioin $V(t,x)$ is said to be negative definite, if $-V(t,x)$ is positive definite.*

Theorem 9 ([31])**.** *The necessary and sufficient condition for the zero solution of system being stable is that there exists a positive definite function $V(t,x) \in C[R^+ \times G_H, R^+]$ such that along the solution of (2.1)*

$$\left.\frac{dV}{dt}\right|_{(2.1)} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t,x) \leq 0$$

holds, where $G_H = \{(t,x), t \geq t_0, \|x\| < H = constant\}$.

For more about the Lyapunov functions, one can see [20] and references therein.

2.2 Hybrid Systems

The terms "hybrid system" and "switched system" are used to describe dynamical systems with continuous and discrete components. An example would be a system of the following form:

$$\begin{cases} \dot{x} = f(t,x,\alpha) \\ \Delta\alpha = g(t,x,\alpha) \end{cases} . \tag{2.10}$$

Here, x represents a vector of continuous states, and α represents a vector of discrete states. The functions f and g can take on many different forms, but it is important to note that the evolution of the continuous and discrete states are not independent of each other. These systems are highly generalized, so it is common to only study particular classes.

2.2.1 Switched Systems

Switched systems are a subset of hybrid dynamical systems consisting of a finite number of continuous subsystems, see [21] and references therein. Each subsystem is itself a dynamical system based on an ordinary differential equation as follows:

$$\dot{x} = f(t, x). \tag{2.11}$$

The one discrete state variable is a switching rule that is used to change the governing dynamics. Hence, if we have a set of equations of the form $\dot{x} = f_\alpha(t, x)$, for $\alpha \in \{1, 2, \ldots, m\}$, the switched system becomes

$$\dot{x} = f_(\sigma(t, x(t)))(t, x) \tag{2.12}$$

Here, the switching rule, $\sigma : \mathbb{R}^+ \times \mathbb{R}^n \to \{1, 2, \ldots, m\}$ depends on both the state and time. There are other possibilities: the switching signal may depend on time or state alone, the switching signal may additionally depend on its own previous values, and the switching signal may have a delay ($\sigma(t - r, x(t - r))$. In all the cases where the switching does not depend solely on time, the signal may be either synchronous or asynchronous [38]. An asynchronous signal may change in value at any time, according to the arguments. A synchronous switching signal may only change value at certain predetermined times, t_k, with $t_k - t_{k-1} > \Delta t_{\min} > 0$ and $\lim_{k\to\infty} t_k = \infty$. It is not, however, required that the subsystem switches every t_k. Hence, the switching rule may depend on the value of its arguments for an entire interval or only a certain time within that interval. Synchronous switching rules tend to be better behaved because having pre-determined switching times enforces a minimum activation time, $\Delta t_{\min}$, which prevents chatter or infinitely fast switching. Chatter is a problem because it can lead to instability, and this behavior cannot be implemented in a real system. Here, we will be concerned with time-based switching rules as well as synchronous and asynchronous switching rules based on the state and time, although some results will be generally applicable. It is also possible to consider switching rules that involve memory of past states. When there is a memory of past time, this is usually considered time delay, such as studied in [30] and [41]. However, we can also require that the system remembers past state values [23].

2.2.2 Switched Systems with Impulses

An impulsive switched system is a more general type of hybrid system, the major difference being that hybrid systems have impulses that occur at the switching times. A general example is as follows:

$$\begin{aligned} \dot{x} &= f_{\sigma(t,x(t))}(t, x), & t \neq t_k \\ \Delta x &= I_{\sigma(t_k,x(t_k))}(t_k, x(t_k)), & t = t_k \end{aligned} \tag{2.13}$$

Note that here, although we mention t_k, σ may be an asynchronous signal meaning that the times, t_k, are simply not known *a priori*. The equation, as written is also ambiguous in terms of whether the impulse for subsystem α is applied at the end of an interval where the continuous portion of subsystem α was activated or at the beginning. Arguments may be made for both cases, and it is even possible to make the impulse depend on both the previous and next active subsystem. For our purposes, we will define

$$\Delta x = x(t_k^+) - x(t_k) \tag{2.14}$$

In this case, if there are m continuous subsystems, one can simply create a hybrid system with m^2 subsystems of which there are only m different continuous portions, but each has a different impulse. Rather than deal with these different situations, we will consider the case where a subsystem α, with the continuous portion active on the interval $(t_{k-1}, t_k]$ has the impulse applied at t_k such that $x(t_k^+) - x(t_k) = I_\alpha(t_k, x(t_k))$. Here, $x(t_k^+) = \lim_{t \to t_k^+} x(t)$.

Hybrid systems are useful because they provide great flexibility for modeling and control problems. The impulses may represent actual disturbances as a result of switching the dynamics of the system, or they may simply be control impulses meant to stabilize the system. A system with the same continuous dynamics and periodically applied impulses is still a hybrid system. For more about the systems with impulses one can see [17, 19, 15, 42, 44, 1] and references therein.

2.2.3 Stability of Switched Systems

Even without involving impulses, stability results for switched systems require more care than systems based on ODEs. It is possible to take several subsystems that would be stable at the origin on their own and devise a switching rule that renders them unstable as a switched system. A good example found in [26] has been modified and included in Chapter 5. By running this same situation in the reverse time direction, it is possible to stabilize several otherwise unstable subsystems through careful switching. In general, to get around this problem, we introduce dwell time conditions. These conditions are worth some detailed study [34], but their main purpose is to allow subsystems to be activated for a long enough period of time that they can stabilize the system. In fact, rather than looking at which subsystems are stable and unstable, it is just as informative to look at how long the stable subsystems are active and how quickly we switch away from an unstable system.

Because of the basic problem of switching causing instability, the Lyapunov function approach has some problems. In the example where instability occurs, the subsystems are both stable, but not using the same Lyapunov function. This is crucial because, finding a global Lyapunov function for a switched system – a function that works for all the different

subsystems – can lead to stability conditions for arbitrary switching rules, such as the one listed in [23]. Even if we directly try to solve a linear switched system, we end up with a trajectory given by the product of an arbitrary number of matrix exponentials (for each switching mode that occurred). As shown by [5], however, this is not a trivial problem. The only time we have a hope of doing anything with the expression is if we have a periodic switching rule, in which case we can collapse each matrix multiplier in a single period to a single matrix, and the system essentially becomes a matrix to an arbitrary power. We can then, and only then, glean some information on stability from the eigenvalues.

The problem with global Lyapunov functions is that they are hard to find. There are already some existing techniques for finding Lyapunov functions for standard systems, so it would be helpful if we could simply use a different one for each subsystem. Of course, the major problem is that the level sets of the Lyapunov functions cross each other, so, under certain switching rules, the system state can switch to higher and higher level sets. Usually it is not possible to consider arbitrary switching signals, but we need to impose rules such as dwell time conditions. There is also a question of how to keep track of the magnitude of the state variable when there are several different functions being used. This task is addressed different ways, for example, by [29] and [13].

2.3 Stability with multiple measures

There are different types of stability in addition to the ones covered already. In some situations, we do not need to know if all the states of the system are stable. There are a number of other options including partial stability, stability of invariant and conditionally invariant sets, and orbital stability. The way these differ from standard stability results is simply in the measure that is used in the stability definition. Instead of using $\|x\|$, we can use $d(x, A)$, the distance of x to an invariant set A, for example. We can then generalize the concept so that all these types of stability can be described by first defining two measures, $h(t, x)$, and $h_0(t, x)$, see [18, 28].

2.3.1 Definitions

In most stability results, we need to make some form of comparison with functions satisfying particular properties. The most useful classes are as follows:

$$K = \{a \in C[\mathbb{R}^+, \mathbb{R}^+] : a(u) \text{ is strictly increasing in } u \text{ and } a(0) = 0\}, \tag{2.15}$$

$$K_0 = \{d \in C[\mathbb{R}^+, \mathbb{R}^+] : d(u) > 0 \text{ for } u > 0 \text{ and } d(0) = 0\}, \tag{2.16}$$

$$CK = \{a \in C[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+] : a(t,\cdot) \in K \text{ and } a(\cdot,u) \text{ is continuous.}, \tag{2.17}$$

$$PCK = \{a \in C[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+] : a(t,\cdot) \in K \text{ and } a(\cdot,u) \text{ is piecewise continuous.}, \tag{2.18}$$

and

$$\begin{aligned}\Gamma = \ & \{h : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+,\ h(\cdot,x) \text{ is piecewise continuous, } h(t,\cdot) \text{ is continuous, and} \\ & \inf_x h(t,x) = 0 \text{ for each } t \in \mathbb{R}^+\}.\end{aligned} \tag{2.19}$$

Finally, for h_0 and h, there is always an order of precedence in the stability definitions. Therefore, to assure we are using the correct measures in the correct position, we require that h_0 is finer that h according to the following definitions.

Definition 10. *h_0 is finer than h if there is a constant $\sigma > 0$ and a function $\varphi \in PCK$ such that $h_0(t,x) < \sigma$ implies $h(t,x) \leq \varphi(t, h_0(t,x))$*

Definition 11. *h_0 is uniformly finer than h if there is a constant $\sigma > 0$ and a function $\varphi \in K$ such that $h_0(t,x) < \sigma$ implies $h(t,x) \leq \varphi(h_0(t,x))$*

An example of this relationship can be found in the case of partial stability.

The following definitions can be found in [18, 28]. For nonlinear systems, the basic definitions for stability are as follows:

Definition 12. *[(h_0,h)-stable] A system is (h_0,h)-stable if, for any $\epsilon > 0$, and trajectory $x(t) = x(t,t_0,x_0)$, such that $x(t_0) = x_0$, we can find $\delta = \delta(t_0,\epsilon) > 0$ such that $h_0(t_0,x_0) < \delta$ implies $h(t,x(t)) < \epsilon$ for $t \geq t_0$.*

Definition 13. *A system is (h_0,h)-uniformly stable if it is (h_0,h)-stable and δ is independent of t_0.*

Definition 14. *A system is (h_0,h)-uniformly asymptotically stable if it is (h_0,h)-uniformly stable and for each $\epsilon > 0$, there are constants δ_0 and T such that $h_0(t_0,x_0) < \delta_0$ implies $h(t,x(t)) < \epsilon$ for $t \geq t_0 + T$. These constants are also independent of t_0.*

There are equivalent definitions corresponding to other types of stability that are not required here.

Definition 15. *A system is partially stable if for any $\epsilon > 0$, there is a $\delta = \delta(t_0,\epsilon) > 0$ such that $\|x_0\| < \epsilon$ implies $\|x(t)\|_s < \delta$ for all future time $t \geq t_0$. The truncated norm is given by $\|x\|_s = \sqrt{x_1^2 + x_2^2 + \cdots + x_s^2}$, where $s < n$.*

In this case, $\|x\|_s$ takes on the role of h and $\|x\|$ takes on the role of h_0.

In order to use these measures, we also need to define some common mathematical properties of functions given in terms of functions measures $h_0, h \in \Gamma$.

Definition 16. *V is h-positive definite if there is a $\rho > 0$ and a $b \in K$ such that $h(t,x) < \rho$ implies $b(h(t,x)) \leq V(t,x)$.*

Definition 17. *V is weakly h-decrescent if there is a $\delta > 0$ and an $a \in PCK$ such that $h(t,x) < \delta$ implies $a(t, h(t,x)) \leq V(t,x)$.*

Definition 18. *V is h-decrescent if there is a $\delta > 0$ and an $a \in K$ such that $h(t,x) < \delta$ implies $a(h(t,x)) \leq V(t,x)$.*

Definition 19. *We say $(t,x) \in S(h,\rho)$ when (t,x) satisfy $h(t,x) < \rho$.*

2.3.2 (h_0, h)-Stability Results for Non-Hybrid Systems

Basic Results

Consider (2.1) as a basic nonlinear system. The basic stability results using two measures can be found as follows.

Theorem 20 ([28])**.** *Assume $V \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+]$ and $h, h_0 \in \Gamma$ satisfy the following conditions:*

1. *h_0 is finer than h;*
2. *$V(t,x)$ is locally Lipschitzian in x and h-positive definite;*
3. *$D^+V(t,x) \leq 0,\ (t,x) \in S(h,\rho)$;*
4. *$V(t,x)$ is h_0-weakly decrescent.*

Then the system is (h_0, h)-stable.

First, we find a function $a \in CK$ (continuous in t and strictly increasing in x), so that

$$V(t_0, x_0) \leq a(t_0, h_0(t_0, x_0)), \text{ if } h_0(t_0, x_0) < \delta_0, \tag{2.20}$$

which is guaranteed by V being weakly h_0-decrescent. Since V is also h-positive definite,

$$b(h(t,x)) \leq V(t,x), \text{ if } h(t,x) \leq \rho_0, \tag{2.21}$$

where $b \in K$. Since h_0 is finer than h, we can find δ_1 sufficiently small so that $h_0(t_0, x_0) < \delta_1$ implies

$$h(t_0, x_0) \leq \varphi(t_0, h_0(t_0, x_0)) \tag{2.22}$$

and $\varphi(t_0, \delta_1) < \rho_0$. Therefore we have bounds placed on $h_0(t_0, x_0)$ such that the solution is initially contained in the region where the required assumptions are true. Now, let $\epsilon > 0$ and $t_0 \in \mathbb{R}^+$, assuming $\epsilon < \rho_0$ without loss of generality. We find δ_2 such that $a(t_0, \delta_2) < b(\epsilon)$. If we let $\delta = \min\{\delta_0, \delta_1, \delta_2\}$, then when $h_0(t_0, x_0) < \delta$, we have $b(h(t_0, x_0)) < V(t_0, x_0)$ from δ_1, $V(t_0, x_0) < a(t_0, h_0(t_0, x_0))$ from δ_0, and $a(t_0, h_0(t_0, x_0)) < b(\epsilon)$ from δ_2, so $h(t_0, x_0) < \epsilon$ at the start. Assume that there exists a solution $x(t) = x(t, t_0, x_0)$ and $t_1 > t_0$ such that $h(t_1, x(t_1)) = \epsilon$, but $h(t, x(t)) < \epsilon$ for $t \in [t_0, t)$. The assumptions on V imply that $V(t, x(t)) \leq 0$ for $[t_0, t_1]$. This implies V is nonincreasing on the interval, leading to the contradiction that

$$b(\epsilon) = b(h(t_1, x(t_1))) \leq V(t_1, x(t_1)) \leq V(t_0, x(t_0)) < b(\epsilon) \tag{2.23}$$

So no such t_1 can exist and the system is stable.

Theorem 21 ([18]). *Assume $V \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+]$ and $h, h_0 \in \Gamma$ satisfy the following conditions:*

1. *h_0 is uniformly finer than h;*
2. *$V(t, x)$ is locally Lipschitzian in x, h-positive definite, and h_0-decrescent;*
3. *$D^+V(t, x) \leq 0,\ (t, x) \in S(h, \rho),\ C \in \mathcal{K}$;*

Then the system is (h_0, h)-uniformly stable.

The criteria for uniform asymptotic stability are simpler than the criteria for asymptotic stability alone.

Theorem 22 ([18]). *Assume $V \in C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+]$ and $h, h_0 \in \Gamma$ satisfy the following conditions:*

1. *h_0 is uniformly finer than h;*
2. *$V(t, x)$ is locally Lipschitzian in x, h-positive definite, and h_0-decrescent;*
3. *$D^+V(t, x) \leq C(h_0(t, x)),\ (t, x) \in S(h, \rho),\ C \in K$;*

Then the system is (h_0, h)-uniformly asymptotically stable.

As is the case in standard stability analysis, for asymptotic stability alone, there are some additional requirements.

Theorem 23 ([28]). *If*

1. $h_0, h \in \Gamma$ *and* h_0 *is finer than* h*;*
2. $V(t,x)$ *is locally Lipschitzian in* x*,* h*-positive definite,* h_0*-weakly decrescent and*
$$D^+V(t,x) \leq -C(h(t,x)) \tag{2.24}$$
in a local region, with $C \in \mathcal{K}$*;*
3. $h \in C^1[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^+]$ *and the magnitude of the derivative of* h *along a trajectory,* $|h'(t,x)| = |h_t(t,x) + h_x(t,x) \cdot f(t,x)$*, is bounded in the region*

Then the system is (h_0, h)*-asymptotically stable.*

2.3.3 Stability Analysis of Switched Systems

When dealing with hybrid systems, traditional stability theorems do not directly apply. This is similar to the situation where a linear, non-autonomous system's stability does not depend on its eigenvalues alone. Although $A(t)$ may be a stable matrix for all t, it can rotate in such a way so that a trajectory can constantly be in a position where it is increasing.

In addition, there are several more categories of stability analysis that are not present in the standard theory. The most obvious stability problem is determining the stability properties of a hybrid system, given all the subsystems and a specific switching rule. This is essentially the same problem as a continuous dynamical system, and we must determine the stability for different initial data. Quite often, however, a hybrid model is designed precisely because the switching rule is either something unpredictable, but falling within certain parameters, or something that is designed to achieve the desired stability properties. We may even want stability under all possible switching rules.

Stability Under all Switching Rules

Stability under all possible switching rules necessitates the most strict stability requirements, but many can be deduced immediately: the continuous portion of all subsystems must form a stable system, and the impulsive portions; taken as discrete dynamical systems, must also be stable. These are not the only requirements, but they already limit the possibilities. The best method to deal with this problem is to use a global Lyapunov function approach, as described later.

Stability Under Constrained Switching Rules

Stability under constrained switching allows for more flexibility in the types of subsystems used. The goal with this approach is to allow some unstable elements in the continuous and discrete dynamics, that will be compensated for by a selection of requirements on the switching rule. Two important criteria commonly used for this analysis are the average dwell time and the ergodicity. The average dwell time, τ, is defined in terms of the number of discontinuities (switch operations), $N_\sigma(t_0, t)$, for the switching signal σ between the times t_0 and t:

$$N_0^- + \frac{(t - t_0)}{\tau} \leq N_\sigma(t_0, t) \leq N_0^+ \frac{(t - t_0)}{\tau}. \tag{2.25}$$

The ergodicity criteria are that for all subsystems, α and for all t_0, the set

$$\{t|\sigma(t) = \alpha\} \bigcap [t_0, t_0 + T] \tag{2.26}$$

is non-empty. This ensures that a minimum number of switches occur and also that no subsystem is over used or under used. A finite average dwell time ensures that infinitely fast switching cannot occur. This is an important condition to eliminate chattering, where the subsystems start to switch infinitely quickly, which would not occur in any real system. These conditions can also be made more complex, so that certain subsystems must be activated for a given portion of the total elapsed time. These types of rules may be useful in situations where the switching rule is unknown, but it satisfies these sorts of properties due to the properties of the real system being modeled.

Stability Using Supervisory Control

When we are attempting to design a switching rule that will stabilize a hybrid system, there is often a supervisory control problem, where we have a control "supervisor" that chooses the correct subsystem at any particular time. This is a similar problem to the previous one, only now we are looking to actually design a specific switching rule, rather than come up with some general constraints on the switching that might be satisfied even if we do not have total control over what the switching rule will be. Generally, the switching rules obtained will be state-based, since the system presumably does not have dynamics that can be calculated trivially, and the switching signal must rely on information in the emergent behaviour.

Difficulties with Autonomous Linear Systems

For autonomous systems, the only types of stability to consider are Lyapunov stability and asymptotic stability, which is equivalent to exponential stability for these systems. We can

also consider all of these in the context of h_0, h-stability, although only certain interpretations apply. Since particular Lyapunov functions are often used for linear systems, rather than using a general $V(x)$, we will use standard quadratic Lyapunov functions, $x^T Px$, but also consider the difference when it satisfies different properties.

The simplest type of hybrid system uses linear equations for both the continuous and impulsive portions as follows:

$$\begin{cases} x' = A_{i(t,x)}x, & t \neq t_k \\ \Delta x = B_{i(t,x)}x, & t = t_k. \end{cases} \tag{2.27}$$

The switching signal may be time and state dependent, so $i : \mathbb{R}_+ \times \mathbb{R}^n \to I = \{1, 2, \ldots, m\}$. Although the system may be asynchronous, t_k represents the right boundary of the kth switching interval, $(t_{k-1}, t_k]$. For ease of notation, we also use i_k to represent the active subsystem in this interval.

Since the equations are linear, we are in a position to explicitly solve the equations. Each subsystem, α has a solution $x(t) = e^{A_\alpha x_0(t-t_0)}x_0$. Therefore, we can write the full solution to an initial value problem as follows:

$$x(t) = e^{A_{i_1}(t-t_0)}x_0, \quad t \in (t_0, t_1] \tag{2.28}$$

$$x(t) = e^{A_{i_2}(t-t_1)}(B_{i_1} + I)e^{A_{i_1}(t-t_0)}x_0, \quad t \in (t_1, t_2] \tag{2.29}$$

$$x(t) = e^{A_{i_k}(t-t_{k-1})}(B_{i_{k-1}} + I)e^{A_{i_{k-1}}(t_{k-2}-t_{k-1})} \cdots (B_{i_1} + I)e^{A_{i_1}(t-t_0)}x_0, \quad t \in (t_{k-1}, t_k] \tag{2.30}$$

The stability problem essentially becomes one of an infinite matrix product. There are a few conclusions we can come to immediately. Firstly, if we want stability under arbitrary switching it is necessary for each linear subsystem to be stable since an unstable subsystem may be run for an arbitrary amount of time. Secondly, the matrices for the impulses must satisfy $\lambda_{\min}\{B_\alpha + I\} \leq 1$. Otherwise, consider a subsystem α such that x^* is an eigenvector of B_α with eigenvalue λ, $|\lambda| > 1$. Starting in any subsystem α, let $x(0) = x^*$. If we switch at time t_1 a short time later, $x(t_1) = x^* + x_\epsilon$, where $|x_\epsilon|$ may be arbitrarily small. Therefore, $x(t_1^+) = (I + B_\alpha)x(t_1) = \lambda x^* + (I + B_\alpha)x_\epsilon$. By switching immediately to another subsystem and then back to subsystem α fast enough, the system will be unstable. Note also the situation when $I + B_\alpha$ is singular: if the impulse is applied at exactly the right time, the state may be moved immediately to the equilibrium point. Other than these conclusions, there are no other obvious criteria for stability under arbitrary switching rules revealed by looking at (2.30).

Directly using (2.30) can be helpful in very specific special cases. Ideally, the switching signal would be periodic. If this is the case, the times $\Delta t_k = t_k - t_{k-1}$ are all known ahead of time. In addition, $x(t)$ becomes equivalent to the infinite product of a single matrix,

which itself is the product of all matrices $M_k = (B_{i_k} + I)e^{A_{i_k}(t_{k-1}-t_k)}$ that make up a single period. The eigenvalues of this matrix must lie strictly inside the unit circle for asymptotic stability and on, or inside, the unit circle for stability.

If we generalize this case only a little to the case where the dwell time for each subsystem is constant, but the order of the subsystems in arbitrary, this problem is much harder. Interestingly, Blondel and Tsitsiklis showed that the boundedness of an infinite arbitrary product of only two different matrices was undecidable [5]. For aperiodic switching signals, this approach is not practical.

2.3.4 Lyapunov Functions

Lyapunov functions are also a common method of finding stability results for hybrid systems, but they need to be used in different ways.

Global Lyapunov Functions

A global Lyapunov function behaves in a similar way to a Lyapunov function for standard dynamical systems, since is must be positive definite, and the derivative along trajectories of all subsystems must be less than or equal to 0. It is particularly useful because the switching signal becomes irrelevant under a global Lyapunov function. Even with a hybrid system with impulses, we only need to make sure that the impulses are bounded and that they do not occur too frequently, and we can conclude stability or asymptotic stability in a similar way.

An example of where we can get sufficient conditions is in the case of the following switched linear system with an asynchronous state-based switching rule:

$$\dot{x} = Ax(t) + Bu(t) \tag{2.31}$$

$$\begin{aligned} u_1(t) &= L_1 x(t) \\ u_2(t) &= L_2 x(t) \\ &\vdots \\ u_m(t) &= L_m x(t) \end{aligned} \tag{2.32}$$

$$\tag{2.33}$$

In this case, we attempt to find a switching rule to stabilize the system. The necessary and sufficient conditions for such a rule to exist can be found using a Lyapunov function

that works on each subsystem for some values of x. The problem of stabilizing the system is shown to be equivalent to finding a matrix P such that the matrices

$$\begin{aligned} Z_1 &= (A+BL_1)^T P + P(A+BL_1) \\ Z_2 &= (A+BL_2)^T P + P(A+BL_2) \\ &\vdots \\ Z_n &= (A+BL_m)^T P + P(A+BL_m) \end{aligned} \tag{2.34}$$

$$\tag{2.35}$$

form a set such that for any $x \neq 0$, $x^T Z_\alpha x < 0$ for some Z_α. Hence, it is possible to use a single Lyapunov function that does not work for each subsystem for all values of x, but still stabilizes the system for a particular switching rule.

This form of global Lyapunov function is less strict, but it is difficult to find and verify. Usually, another condition will have to be satisfied that is sufficient to show that the set of matrices satisfies the desired properties. Another issue is with applying impulses to the system. Certainly impulses can be accommodated if they adhere to requirements on maximum absolute size and frequency, but it is difficult to integrate them seamlessly into this method so that impulses may act as a desired property of the controller and stabilize the system.

Multiple Lyapunov Functions

It is sometimes difficult to find a global Lyapunov function for a particular system, so it would be useful to use standard methods to find multiple Lyapunov functions for each of the subsystems involved. With this method, we must be much more careful about what properties the switching signal satisfies. We can use Lyapunov theory to determine whether each subsystem is stable or unstable, but, as we know, this alone is not sufficient to determine stability. We can see how multiple Lyapunov functions work in Chapter 3 and 4.

2.3.5 Special Cases

Periodic switching signals are common for many different models, and they have enough useful properties to be considered separately. For an autonomous system, in particular, we can consider the trajectory over each period. In the case of linear systems, we can evaluate this and actually apply the time period as a mapping, effectively converting the hybrid dynamical system into a discrete dynamical system.

2.3.6 Necessary and Sufficient Conditions

There are necessary and sufficient conditions that have been developed for switched autonomous linear systems with no impulses [23]. This is based on the problem of robust asymptotic stability for polytopic uncertain linear time-variant systems. These systems are of the form:

$$x_{k+1} = A(k)x_k, \tag{2.36}$$

where $A(k) \in \text{Conv}\{A_1, A_2, \ldots, A_m\}$. $\text{Conv}\{\cdot\}$ is the convex combination. Hence, $A(k)$ can be made from any convex combination of subsystem state matrices. The result is that system (2.36) is robustly asymptotically stable if and only if there exists an integer l such that

$$\|A_{i_1} A_{i_2} \cdots A_{i_n}\| < 1 \tag{2.37}$$

For all l-tuples of $A_{i_j} \in \{A_1, A_2, \ldots, A_N\}$. The ∞ norm for matrices is used.

This problem is equivalent to the stability of a discrete switched linear system:

$$x_{k+1} = A_{\sigma(k)} x_k \tag{2.38}$$

This can also be extended to the continuous case we are interested in, where the following are equivalent:

1. The switched linear system
$$\dot{x}(t) = A_{\sigma(t)} x(t) \tag{2.39}$$
with $A_{\sigma(t)} \in \{A_1, \ldots, A_m\}$ is asymptotically stable under arbitrary switching;

2. the linear time-variant system
$$\dot{x}(t) = A(t)x(t) \tag{2.40}$$
with $A(t) \in \text{Conv}\{A_1, \ldots, A_m\}$, is robustly asymptotically stable; and

3. there exist a full column rank matrix $L \in \mathbb{R}^{m \times n}$, $m \geq n$, and a family of matrices $\{\tilde{A}_i \in \mathbb{R}^{m \times n} : i \in \mathcal{I}\}$ (the set of subsystems) with strictly negative row dominating diagonal, i.e., for each $\tilde{A}_i$, $i \in \mathcal{I}$ its elements satisfying
$$\hat{a}_{kk} + \sum_{k \neq l} |\hat{a}_{kl}| < 0, k = 1, \ldots, m \tag{2.41}$$
such that $LA_i = \tilde{A}_i L$ for all i.

These conditions are directly from [23].

Chapter 3

Stability of Linear Impulsive Switched Systems

3.1 Problem Formulation

In this chapter we will study the stability of autonomous linear systems in the following form:

$$\begin{aligned} \dot{x} &= A_{\sigma(t,x)}x, \quad && t \neq t_k \\ \Delta x &= B_{\sigma(t,x)}x, \quad && t = t_k, \end{aligned} \tag{3.1}$$

where the switching signal σ takes on values $\alpha \in \{1, 2, \ldots, m\} = \mathbb{L}$, and A_α and B_α are constant real matrices for all $\alpha \in \mathbb{L}$.

The method used is an adaptation of the method in [28], which formulates sufficient stability conditions for nonlinear impulsive systems (with a single continuous mode, not hybrid systems) in two measures. We will begin by presenting the stability conditions using the standard single measure definitions of stability, however. The reasons for the choice are, firstly, to better introduce the material, and, secondly, because, for a linear system, we expect to be able to characterize the stability based on the properties of the matrices that make up the subsystem, and not based on a general Lyapunov function. For two measure stability, the Lyapunov functions are closely linked to the measures h_0 and h, so we need to choose these together. In the section on stability in terms of two measures, we will discuss this issue further.

Since we are using ordinary one measure stability, we will use quadratic Lyapunov functions to determine stability. We use multiple Lyapunov functions, which means we can use standard methods to find the functions, but we require extra conditions to ensure stability. Recall that, to choose a quadratic Lyapunov function for a continuous linear system, we solve an equation of the form $A^T P + PA = Q$ for an unknown positive definite

symmetric matrix P. We choose P such that $A^T P + PA = \pm Q$, where Q is positive definite. The sign of Q depends on whether A is a stable matrix, meaning the eigenvalues are entirely in the left half complex plane. If A is stable, we use the negative sign, and, if A is unstable (with eigenvalues entirely in the right half complex plane), we use the positive sign. If A has eigenvalues with real part zero or of mixed sign, we choose to add cI, a multiple of the identity, such that $A + cI$ has eigenvalues of positive real part. Hence, $(A^T + cI)P + P(A + cI) = Q$, so $A^T P + PA = Q - 2cP$, and we use the P found in this equation. These are all linear equations, and solving them provides a clear and unambiguous method for finding Lyapunov functions. From now on, when we refer to P_α and Q_α associated with subsystem α, we mean the solution to the appropriate equation involving A_α and any $Q_\alpha > 0$:

$$P_\alpha > 0 \text{ such that } \begin{cases} A_\alpha^T P_\alpha + P_\alpha A_\alpha = Q_\alpha, & A_\alpha \text{ is unstable;} \\ A_\alpha^T P_\alpha + P_\alpha A_\alpha = -Q_\alpha, & A_\alpha \text{ is stable;} \\ A_\alpha^T P_\alpha + P_\alpha A_\alpha = Q_\alpha - 2cP_\alpha, & A_\alpha \text{ is neither stable nor unstable,} \\ & \text{but } A_\alpha + cI \text{ is unstable.} \end{cases} \tag{3.2}$$

In the following sections, for a matrix A, we will use $\lambda(A)$ to denote all the eigenvalues; $\lambda_{\max}(A)$, the maximum eigenvalue; and $\lambda_{\min}(A)$, the minimum eigenvalue.

3.2 Stability Criteria in One Measure

For the special case where $h = h_0 = \|x\|$, we will start with the criteria for stability. Because we are dealing with linear non-autonomous systems, the stability will automatically be global uniform stability. We do not make any explicit assumptions about the switching rule, but the criteria implicitly require it to satisfy certain properties. We can achieve these using either a time- or space-based switching rule. The difference between using different switching rules is also covered in Chapter 4, with nonlinear systems.

Before we present the stability results, we need the following lemma.

Lemma: For a symmetric Q and positive definite P, $x^{\mathrm{T}} Q x \leq \lambda_{\max}(P^{-1}Q) x^{\mathrm{T}} P x$.

This follows by taking $x^{\mathrm{T}} Q x = x^{\mathrm{T}} P^{-1} Q P x \leq x^{\mathrm{T}} \lambda_{\max}(P^{-1}Q) P x$. We get equality if and only if x or Px is an eigenvector of eigenvalue $\lambda_{\max}(P^{-1}Q)$ for $P^{-1}Q$. We can show this by decomposing either x or Px into a linear combination of eigenvectors of $P^{-1}Q$.

Theorem 1. *Assume that*

1. *there exist positive definite matrices P_α for each $\alpha \in \mathbb{L}$;*

2. *there exist* $\mu_\alpha \geq \Delta t_\alpha \lambda_{\max}(P_\alpha^{-1}(A_\alpha^T P_\alpha + P_\alpha A_\alpha))$ *and* $\nu_{\alpha,\beta} \geq \lambda_{\max}(P_\alpha^{-1}(B_\alpha^T P_\beta + B_\alpha^T P_\beta B_\alpha + P_\beta B_\alpha + P_\beta - P_\alpha))$ *with* Δt_α *a conservative estimate of the activation time of the subsystem for every* $\alpha, \beta \in \mathbb{L}$ *such that*

$$\mu_\alpha + \nu_{\alpha,\beta} \leq 0; \textit{and}$$

3. *there exists* $\gamma_\alpha \geq 0$ *for every* $\alpha, \beta \in \mathbb{L}$ *such that*

$$\mu_\alpha + \ln(1 + \nu_{\alpha,\beta}) \leq -\gamma_\alpha. \tag{3.3}$$

Then the zero solution of system 3.1 is uniformly stable.

Before proceeding with the proof, note that there are two aspects of condition 2 that relate to the switching rule. The conservative estimate of the activation time of the subsystem should be taken to mean that Δt_α is the maximum activation time of a subsystem if $\mu_\alpha \geq 0$ and that Δt_α is the minimum activation time of a subsystem if $\mu_\alpha < 0$. If the switching rule is a known function of time, these can be changed to specific values for specific intervals. See Chapter 4 for details. The conditions required for all ordered pairs α, β only need to be verified if it is possible, based on the switching rule, for a switch from subsystem α to subsystem β to occur.

Proof. Let $V_\alpha(x(t)) = x(t)^T P_\alpha x(t)$, $Q_\alpha = A_\alpha^T P_\alpha + P_\alpha A_\alpha$, $\alpha \in \mathbb{L}$ and $V(t,x) = V_{\sigma(t,x(t))}(x(t))$. If the subsystem α is activated at time t, we have

$$\begin{aligned}
\dot{V}(x(t)) &= \dot{V}_\alpha(x) \\
&= x^T(A_\alpha^T P_\alpha + P_\alpha A_\alpha)x && (3.4)\\
&\leq x^T Q_\alpha x \leq \lambda_{\max}(P_\alpha^{-1} Q) V_\alpha(x) \\
&\leq \frac{\mu_\alpha}{\Delta t_\alpha} V_\alpha(x). && (3.5)
\end{aligned}$$

If the system is switching from subsystem α to β, we have

$$\begin{aligned}
V(x(t^+)) - V(x(t)) &= \Delta V_{\alpha,\beta}(x) \\
&= V_\beta(x^+) - V_\alpha(x) && (3.6)\\
&= x^{+T} P_\beta x^+ - x^T P_\alpha x && (3.7)\\
&= x^T((I + B_\alpha)^T P_\beta (I + B_{[23]\alpha}))x - x^T P_\alpha x \\
&\leq \nu_{\alpha,\beta} V_\alpha(x). && (3.8)
\end{aligned}$$

Let $\epsilon > 0$ be given. Choose σ_1 such that $\sigma_1 = e^{-1} b(\epsilon)$ with $b(\epsilon) = \lambda_{PM}^2 \epsilon^2$ where $\lambda_{PM} = \max_{\alpha \in \mathbb{L}}\{\sqrt{\lambda(P_\alpha)}\}$. Using the condition on the sum of μ_α and $\nu_{\alpha,\beta}$, we know that

$\mu_\alpha \leq 1$, since otherwise, $\nu_{\alpha,\beta}$ must be less than -1, which is contradicted by condition 2 in the theorem.

Note that the choice of σ_1 implies that when $V(x) < \sigma_1$, $\|x(t)\| < \epsilon$.

There exists a $\sigma_2 = \sigma_2(\epsilon) > 0$ such that

$$\sigma_2 + \nu\sigma_2 < \sigma_1,$$

where $\nu = \max_{\alpha,\beta\in\mathbb{L}}\{\nu_{\alpha,\beta}\}$.

Choose $\sigma_0 = \min\{\sigma_1 e^{-1}, \sigma_2\}$. There exists a $\delta = \delta(\epsilon) > 0$ such that

$$V_q(x_0) < \sigma_0 \text{ if } |x_0| < \delta$$

where q is the first active subsystem after the initial condition. This notation avoids any potential problems with having to choose the time intervals based on the initial condition. We assume the initial condition is $x(\tau_0^+) = x_0$, where τ_0 may or may not be a switching time (using τ_0^+ eliminates any ambiguity).

Let $x(t) = x(t, \tau_0, x_0)$ be a solution of (3.1) with $\|x_0\| < \delta$. Then $\|x_0\| < \epsilon$, assume that, for all t, $\|x(t)\| < \epsilon$. If not, there will be some $\tilde{t} > \tau_0$ such that $\|x(\tilde{t}^+)\| \geq \epsilon$. Set $j = \max\{k|t_k \leq \tilde{t}\}$. If $j > q$, for any $k = q+1, \ldots, j$, suppose subsystem α is active in $(t_{k-1}, t_k]$, we have

$$\int_{V_\alpha(x(t_{k-1}^+))}^{V_\alpha(x(t_k))} \frac{ds}{s} \leq \mu_\alpha$$

Therefore,

$$\ln\left(\frac{V_\alpha(x(t_k))}{V_\alpha(x(t_{k-1}^+))}\right) \leq \mu_\alpha$$

A similar integration from t_k to t_k^+ with system switching from system α to β leads to

$$\ln\left(\frac{V_\beta(x(t_k^+))}{V_\alpha(x(t_k))}\right) \leq \ln\left(\frac{V_\alpha(x(t_k))(1+\nu_{\alpha,\beta})}{V_\alpha(x(t_k))}\right)$$

By (3.3) we know

$$\ln\left(\frac{V_\beta(x(t_k^+))}{V_\alpha(x(t_{k-1}^+))}\right) = \ln\left(\frac{V_\alpha(x(t_k))}{V_\alpha(x(t_{k-1}^+))}\right) + \ln\left(\frac{V_\alpha(x(t_k))(1+\nu_{\alpha,\beta})}{V_\alpha(x(t_k))}\right) \leq -\gamma_\alpha < 0$$

Hence, $V_\beta(x(t_k^+)) \leq V_\alpha(x(t_{k-1}^+)) \leq V_q(t_q^+, x)$. We now claim that

$$V(t_q^+, x) < \sigma_1$$

In this first time interval, we assume that the qth subsystem is activated. There are two possibilities. The first is that $\mu_q \leq 0$. In this case, the condition on the derivatives along trajectories implies $V(x(t_q)) \leq V(x_0) < \sigma_0$. Because of our choice of σ_0, we have that

$$V(x(t_q^+)) \leq \sigma_0 + \nu\sigma_0 < \sigma_1 \tag{3.9}$$

The other case is that $\mu_q > 0$, we must have $\nu_{q,\sigma(x(t_q^+))} < 0$ by condition 2, which implies $V(x(t_q^+)) \leq V(x(t_q))$, and

$$\int_{\sigma_0}^{V_q(x(t_q))} \frac{ds}{s} \leq \int_{V_q(x_0)}^{V_q(x(t_q))} \frac{ds}{s} \leq \mu_q \frac{t_q - \tau_0}{\Delta t_q} < \mu_q.$$

However, because $\mu_q \leq 1$, we know that

$$\int_{\sigma_0}^{\sigma_1} \frac{ds}{s} = \ln\left(\frac{\sigma_1}{\sigma_0}\right) \geq 1 \geq \mu_q$$

Therefore, $V(x(t_q)) < \sigma_1$, which implies $V(t_q^+, x) < \sigma_1$, since the impulse is stabilizing in this case.

We have now established that the Lyapunov functions decrease on the endpoints of the intervals. We now assume that there is a time $\tilde{t}$ where $\|x\| \geq \epsilon$. This occurs in some interval j, and $\tilde{t} \in (\hat{t}, t_{j+1})$, where $\hat{t} = \max\{\tau_0, t_j\}$. Suppose the αth subsystem is activated in this interval. If $\tilde{t} = \hat{t}$ or if $\tilde{t} > \hat{t}$ and $\mu_\alpha \leq 0$, then we have

$$V(x(\tilde{t}^+)) \leq V(x(\hat{t}^+)) < \sigma_1,$$

which leads to the following contradiction

$$b(\epsilon) \leq b(\|x(\tilde{t}^+)\|) \leq V(x(\tilde{t}^+)) < \sigma_1 \leq b(\epsilon).$$

The other contradiction is when $\mu_\alpha > 0$, in which case we do the same integration as above

$$\int_{V_\alpha(x(\hat{t}^+))}^{V_\alpha(x(\tilde{t}))} \frac{ds}{s} \leq \mu_\alpha.$$

The contradiction arrives because this integral should be larger than integrating from σ_1 to $e\sigma_1$. By the original choice of σ_1, if $\|x\| \geq \epsilon$, the Lyapunov function will have to be greater than $e\sigma_1$. But, once again, since there is a factor of e between the two values, this integral will be greater than or equal to μ_{j+1}

$$\mu_\alpha \geq \int_{V_\alpha(x(\hat{t}^+))}^{V_\alpha(x(\tilde{t}))} \frac{ds}{s} > \int_{\sigma_1}^{e\sigma_1} \frac{ds}{s} = 1 \geq \mu_\alpha.$$

Thus we must have $\|x\| < \epsilon$ and therefore system (3.1) is uniformly stable. □

Although the notation was very cumbersome in the above proof, it will be useful for the results in two measures because we need to generalize what $V_\alpha(t, x)$ means. This proof, in fact, is very similar to the corresponding one that will follow in the section on two measure stability. For this reason, rather than including the extension to these criteria that will satisfy asymptotic stability, we will look at a different set of criteria, using a slightly different approach, that will be sufficient to show asymptotic stability. This result is included later in this chapter.

3.3 Stability Criteria in Two Measures

3.3.1 Quadratic Lyapunov Functions in Two Measures

Partial stability and the stability of invariant sets are two relevant considerations for linear systems, so it is necessary to consider stability using two measures. In order to continue using the quadratic Lyapunov functions, we need to take care how these measures are defined. We have a quadratic version with some symmetric P, which is of the form:

$$V(x) = x^T P x$$

Positive definite means $V(x) > a(h(x))$. Decrescent means $V(x) < b(h_0(x))$. The functions a and b can be any suitable functions in K. The upper bound is relatively simple, and it can be achieved for any quadratic Lyapunov function based on a symmetric matrix. The other major condition on V is that

$$\dot{V}(x) = x^T(A^T P + PA)x \leq \lambda V(x).$$

For a linear system, we will generally be concerned with stability results related to measures involving different subspaces of $\mathbb{R}^n$. For partial stability, we will use $h_0(x) = \|x\|$ and $h(x) = \|x\|_s = \sqrt{x_1^2 + x_2^2 + \cdots + x_s^2}$. The second measure is the norm on a subspace of $\mathbb{R}^n$. Similarly with invariant sets, we will be interested in the null space of the matrices A_α. Because we are dealing with switched systems, there are other possibilities, including stable orbits or different types of invariant sets, but, in order to have some structure on the measures and Lyapunov functions, we must make some assumptions. If we are only dealing with subspaces, we can essentially use the same quadratic Lyapunov functions. We simply take the subspace we wish to use for $h(x)$ and find a basis for $\mathbb{R}^n$ that separates the subspace from its complement. It is then only necessary to express the Lyapunov function in terms of the lower dimensional system. Since the Lyapunov functions will be essentially the same, we do not need to adjust the stability conditions very much.

3.3.2 Stability Criteria

For h_0, h-stability of a linear hybrid system, we can make many simplifications. Stability is automatically global, since we are dealing with a linear system and uniform, since it is autonomous. We present the theorem in a more general form that will help with

Theorem 2. *If the following criteria are satisfied, the hybrid autonomous linear system will be (h_0, h)-stable.*

1. *$h_0, h \in \Gamma$ and h_0 is finer than h;*
2. *there exist functions $a \in PCK$, $b \in K$ and constants $\delta_0, \alpha_0 > 0$ such that for all $\alpha \in \mathbb{L}$*
$$\lambda_{PM}\|x\| \leq a(h_0(t,x)) \text{ if } (t,x) \in S(h_0, \delta_0),$$
and
$$\lambda_{Pm}\|x\| \geq b(h(t,x)) \text{ if } h(t,x) < \alpha_0,$$
where $\lambda_{PM} = \max_{k\in\mathbb{L}}\{\lambda(P_k)\}$, $\lambda_{Pm} = \min_{k\in\mathbb{L}}\{\lambda(P_k)\}$;
3. *there exists $\mu_\alpha \geq \Delta t_\alpha \lambda_{\max}(P_\alpha^{-1}(A_\alpha^T P_\alpha + P_\alpha A_\alpha))$ and $\nu_{\alpha,\beta} \geq \lambda_{\max}(P_\alpha^{-1}(B_\alpha^T P_\beta + B_\alpha^T P_\beta B_\alpha + P_\beta B_\alpha + P_\beta - P_\alpha))$ with Δt_α a conservative estimate of the activation time of the subsystem for every $\alpha, \beta \in \mathbb{L}$ such that*
$$\mu_\alpha + \nu_{\alpha,\beta} \leq 0.$$
4. *there exists $\gamma_\alpha \geq 0$ for each $\alpha \in \mathbb{L}$*
$$\mu_\alpha + \ln(1 + \nu_{\alpha,\beta}) \leq -\gamma_\alpha.$$

Proof. Let $V_\alpha(x) = x^T P_\alpha x$, $Q_\alpha = A_\alpha^T P_\alpha + P_\alpha A_\alpha$, $\alpha \in \mathbb{L}$ and $V(x(t)) = V_{\sigma(t,x)}(x)$. If the αth subsystem is activated at time t, we have

$$\begin{aligned}
\dot{V}(x(t)) &= \dot{V}_\alpha(x) \\
&= x^T(A_\alpha^T P_\alpha + P_\alpha A_\alpha)x \\
&\leq x^T Q_\alpha x \\
&\leq \lambda_{\max}(P_\alpha^{-1} Q_\alpha) V_\alpha(x) \\
&\leq \frac{\mu_\alpha}{\Delta t_\alpha} V_\alpha(x)
\end{aligned}$$

If the system is switching from the subsystem α to β, we have

By condition 1, there exists $\varphi \in PCK$ and $\delta_1 > 0$ such that

$$h(t,x) \leq \varphi(t, h_0(t,x)) < \alpha_0 \text{ whenever } h_0(t,x) < \delta_1.$$

Let $\rho_0 = \frac{\rho}{\lambda_B}$ with $\lambda_B = \max_{k\in\mathbb{L}}\{\sqrt{\lambda(I+B_k)}\}$. Then $(t_k, x) \in s(h, \rho_0)$ implies $(t_k, x + B(t_k, x)) \in s(h, \rho)$, $k = 1, 2, \ldots$;

Let $\epsilon > 0$ with $0 < \epsilon < \rho^* = \min\{\rho_0, \alpha_0\}$. Choose $\sigma_1 = e^{-1}b(\epsilon)$. Using the condition on the sum of μ_α and $\nu_{\alpha,\beta}$, we know that $\mu_\alpha \leq 1$, since otherwise, $\nu_{\alpha,\beta}$ must be less than -1, which is contradicted by the condition on ΔV.

Note that the choice of σ_1 implies that when $V(x) < \sigma_1$, $h(t,x) < \epsilon$.

There exists a $\sigma_2 = \sigma_2(\epsilon) > 0$ such that

$$\sigma_2 + \nu\sigma_2 < \sigma_1,$$

where $\nu = \max_{\alpha,\beta\in\mathbb{L}}\{\nu_{\alpha,\beta}\}$.

Choose $\sigma_0 = \min\{\sigma_1 e^{-1}, \sigma_2\}$. There exists a $\delta = \delta(\epsilon) > 0$ such that

$$V(x_0) < \sigma_0 \text{ if } \|h_0(t_0, x_0)\| < \delta.$$

Let $x(t) = x(t, \tau_0, x_0)$ be a solution of (3.1) with $h_0(t_0, x_0) < \delta$. Then $h(t_0, x_0) < \epsilon$. Assume that, for all t, $h(t, x(t)) < \epsilon$. If not, there will be some $\tilde{t} > \tau_0$ such that $h(t, x(\tilde{t}^+) \geq \epsilon$. Set $j = \max\{k | t_k \leq \tilde{t}\}$. If $j > q$, then for $k = q+1, \ldots, j$, suppose subsystem α is active in $(t_{k-1}, t_k]$, we have

$$\int_{V_\alpha(x(t_{k-1}^+))}^{V_\alpha(x(t_k))} \frac{ds}{s} \leq \int_{V_\alpha(x(t_{k-1}^+))}^{V_\alpha(x(t_k))} \frac{ds}{s} \leq \mu_\alpha.$$

Therefore,

$$\ln\left(\frac{V_\alpha(x(t_k))}{V_\alpha(x(t_{k-1}^+))}\right) \leq \mu_\alpha.$$

A similar integration from t_k to t_k^+ leads to

$$\ln\left(\frac{V_\beta(x(t_k^+))}{V_\alpha(x(t_k))}\right) \leq \ln\left(\frac{V_\alpha(x(t_k^+))(1+\nu_{\alpha,\beta})}{V_\alpha(x(t_k))}\right).$$

This means we know

$$\ln\left(\frac{V_\beta(x(t_k^+))}{V_\alpha(x(t_{k-1}^+))}\right) \leq 0.$$

Hence, $V_\beta(x(t_k^+)) \leq V_\alpha(x(t_{k-1}^+)) \leq V_q(x(t_q^+))$. We now claim that

$$V(x(t_q^+)) < \sigma_1.$$

In this first time interval, we assume that the qth subsystem is activated. There are two possibilities. The first is that $\mu_q \leq 0$. In this case, the condition on the derivatives along trajectories implies $V(x(t_q)) \leq V(x_0) < \sigma_0$. Because of our choice of σ_0, we have

$$V(x(t_q^+)) \leq \sigma_0 + \nu\sigma_0 < \sigma_1 \tag{3.10}$$

The other case is that $\mu_q > 0$, in which case we quantify the difference between $V_q(x_0)$ and $V_q(x(t_q))$ by integrating:

$$\int_{\sigma_0}^{V_q(x(t_q))} \frac{ds}{s} \leq \int_{V_q(x_0)}^{V_q(x(t_q))} \frac{ds}{s} \leq \mu_q \frac{t_q - \tau_0}{\Delta t_q} < \mu_q$$

However, because $\mu_q \leq 1$, we know that

$$\int_{\sigma_0}^{\sigma_1} \frac{ds}{s} = \ln\left(\frac{\sigma_1}{\sigma_0}\right) \geq 1 \geq \mu_q$$

Therefore, $V_q(x(t_q)) < \sigma_1$, which implies $V_q(x(t_q^+)) < \sigma_1$, since the impulse is stabilizing in this case. Thus our claim is true.

We have now established that the Lyapunov functions decrease on the endpoints of the intervals. We now assume that there is a time $\tilde{t}$ where $\|x\| \geq \epsilon$. This occurs in some interval j, and $\tilde{t} \in (\hat{t}, t_{j+1})$, where $\hat{t} = \max\{\tau_0, t_j\}$. Suppose the αth subsystem is activated in this interval. If $\tilde{t} = \hat{t}$ or if $\tilde{t} > \hat{t}$ and $\mu_{j+1} \leq 0$, then we have $V(x(\tilde{t})) < \sigma_1$, but this is only possible if $\|x\| < \epsilon$. The other contradiction is when $\mu_{j+1} > 0$, in which case we do the same integration as above:

$$\int_{V_\alpha(x(\hat{t}^+))}^{V_\alpha(x(\tilde{t}))} \frac{ds}{s} \leq \mu_\alpha$$

The contradiction arrives because this integral should be larger than integrating from σ_1 to $e\sigma_1$. By the original choice of σ_1, if $\|x\| \geq \epsilon$, the Lyapunov function will have to be greater than $e\sigma_1$. But, once again, since there is a factor of e between the two values, this integral will be greater than or equal to μ_{j+1}

$$\mu_\alpha \geq \int_{V_\alpha(x(\hat{t}^+))}^{V_\alpha(x(\tilde{t}))} \frac{ds}{s} > \int_{\sigma_1}^{e\sigma_1} \frac{ds}{s} \geq \mu_\alpha \tag{3.11}$$

which is a contradiction. Thus we must have $\|x\| < \epsilon$ and therefore system (3.1) is (h_0, h) stable. □

Remark 1. Since δ_1 is independent of t, so we can also get the uniform stability of the system.

To show how this theorem would be used to show a stability result, we present a corollary showing the conditions for partial stability.

Corollary 1. *If the following criteria are satisfied, the hybrid autonomous linear system will be partially stable with respect to* $\|x\|_s$.

1. *There exists* $\mu_\alpha \geq \Delta t_\alpha \lambda_{\max}(P_\alpha^{-1}(A_\alpha^T P_\alpha + P_\alpha A_\alpha))$ *and* $\nu_{\alpha,\beta} \geq \lambda_{\max}(P_\alpha^{-1}(B_\alpha^T P_\beta + B_\alpha^T P_\beta B_\alpha + P_\beta B_\alpha + P_\beta - P_\alpha))$ *with* Δt_α *a conservative estimate of the activation time of the subsystem for every* $\alpha, \beta \in \mathbb{L}$ *such that*

$$\mu_\alpha + \nu_{\alpha,\beta} \leq 0, and$$

2. *there exists* $\gamma_\alpha \geq 0$ *for each* $\alpha \in \mathbb{L}$

$$\mu_\alpha + \ln(1 + \nu_{\alpha,\beta}) \leq -\gamma_\alpha.$$

Proof. Denote $h_0 = \|x\|$, $h = \|x\|_s$. It can be seen that $h_0, h \in \Gamma$ and h_0 is finer than h, V is h-positive definite and weakly h_0-decrescent. Then all the conditions of Theorem 2 are satisfied. This completes the proof. □

The next result is on (h_0, h)-asymptotic stability.

Theorem 3. *Assume all the requirements for stability are satisfied and*

1. Δt_k *is bounded and there exist* $d \in K$ *and* $\alpha_1 > 0$ *such that*

$$\lambda_{PM}\|x\| \leq d(h(t,x)) \text{ if } h(t,x) < \alpha_1.$$

2. *for every* $\beta, M > 0$, *there is a positive integer* N *such that*

$$\sum_{k=q+1}^{q+N} \beta \min_{k \in \mathbb{L}}\{\gamma_k\} > M, \ \forall q \geq 0.$$

Then system (3.1) is (h_0, h)*-asymptotically stable.*

Proof. By Theorem 2 the system is (h_0, h)-uniformly stable. Thus for $\tilde{\rho} = \min\{\rho^*, \sigma_1\}$ there exists a $\tilde{\delta} = \tilde{\delta}(\tilde{\rho}) > 0$ such that $h_0(\tau_0, x_0) < \tilde{\delta}$ implies $h(t, x(t)) < \tilde{\rho}$, $t \geq \tau_0$, where $x(t) = x(t, \tau_0, x_0)$ is any solution of (3.1) with $h_0(\tau_0, x_0) < \tilde{\delta}$. It is easy to see, in view of condition 1, that system (3.1) is also (h, h)-uniform stable. Let $\epsilon \in (0, \tilde{\rho})$ be given and

define $\delta = \delta(\epsilon)$ as in the definition of (h, h)-uniform stability. From conditons (ii) there exists $N > 0$ such that

$$\sum_{k=q+1}^{q+N} \gamma_{\sigma(t_k)} b_{\sigma(t_k)}(\delta) > d(\tilde{\rho}) \tag{3.12}$$

Note that this is not immediately guaranteed by our assumption, which uses a constant β and M, but, given a finite number of subsystems, we can choose β to be the smallest value of $b_{\sigma(t_k)}(\delta)$. Let $\Delta t_k < \Delta t$ for all $k = 1, 2, \ldots$, and choose $T = (N+1)\Delta t$. Then for any $t_0 \in R_+$ let $x(t) = x(t, \tau_0, x_0)$ be a solution of (3.1) with $h_0(\tau_0, x_0) < \tilde{\delta}$. It is sufficient to show that there exists $t^* \in [\tau_0, \tau_0 + T]$ such that $h(t^*, x(t^*)) < \delta$. For the sake of contradiction, assume $h(x) \geq \delta$ for the entire interval $[\tau_0, \tau_0 + T]$. This means that integrating between any endpoints of a switching interval

$$\int_{V_{\sigma(t_k)}(x(t_{k-1}^+))}^{V_{\sigma(t_{k+1})}(x(t_k^+))} \frac{ds}{s} \leq -\gamma_{\sigma(t_k)}$$

yields

$$V_{\sigma(t_k)}(x(t_{k-1}^+)) \leq V_{\sigma(t_{k+1})}(x(t_k^+)) - \gamma_{\sigma(t_k)} b_{\sigma(t_k)}(\delta).$$

Let $q = \min\{k : t_k \geq \tau_0\}$. Then by the choice of T, $t_{q+N} \in [\tau, \tau_0 + T]$. Then by (3.12) we obtain

$$V_{\sigma(t_{q+N+1})}(x(t_{q+N}^+)) \leq d(\tilde{\rho}) - \sum_{k=q+1}^{q+N} \gamma_{\sigma(t_k)} b_{\sigma(t_k)}(\delta)$$

which leads to the contradiction

$$V_{\sigma(t_{q+N+1})}(x(t_{q+N}^+)) < 0.$$

Thus we must have $h(t^*, x(t^*)) < \delta$ for some $t^* \in [\tau_0, \tau_0 + T]$ and hence $h(t, x(t)) < \epsilon$ for $t \geq \tau_0 + T$. So the system (3.1) is (h_0, h)-asymptotically stable. □

3.4 Other Results

Although these results are sufficient to cover basic stability properties of linear systems, there are a couple of other results we should at least consider.

3.4.1 Asymptotic Stability in One Measure

These criteria are based on [13], but we will also consider the stabilizing effect impulses have on the overall system. Because we are not concerned with using consistent notation with a similar result in two measures, the notation will be different here.

We define the average dwell time, by stating that the number of discontinuities, $N_\sigma(t_0, t)$ must satisfy

$$N_0^- + \frac{(t - t_0)}{\tau_a} \leq N_\sigma(t_0, t) \leq N_0^+ + \frac{(t - t_0)}{\tau_a}, \tag{3.13}$$

where τ_a is the average dwell time.

We categorize the subsystems α into two classes:

1. $S^- = \{\alpha | \lambda_\alpha \tau_a + \ln(\rho(1 + \nu_\alpha)) < 0\}$ and
2. $S^+ = \{\alpha | \lambda_\alpha \tau_a + \ln(\rho(1 + \nu_\alpha)) \geq 0\}$.

Here, $\rho = \max\left(\frac{\lambda_{\max}(P_\alpha)}{\lambda_{\min}(P_\alpha)}\right)$. The total time a subsystem α is used between time t_0 and time t is given by $T_\alpha(t_0, t)$. We define $T^-(t_0, t)$ and $T^+(t_0, t)$, respectively as the total times of activation of the subsystems in the two classes. We also define

1. for $\alpha \in S^-$, we let $lLambda^-$ denote the $\max_\alpha(\lambda_\alpha \tau_a + \ln(\rho(1 + \nu_\alpha))$ and
2. for $\alpha \in S^+$, we let Λ^+ denote the $\max_\alpha(\lambda_\alpha \tau_a + \ln(\rho(1 + \nu_\alpha))$.

Theorem 4. *Assume we have a system of the form (2) and we can find positive definite matrices P_α and the functions φ_α and $\psi_{j,\alpha}$, $j = 1, 2, 3$ associated with them in a local region. Assume also that all of the definitions above. If*

$$\Lambda^- + q\Lambda^+ < 0 \tag{3.14}$$

for some $q \geq 0$ and if

$$T^+(t_0, t) \leq qT^-(t_0, t) \tag{3.15}$$

then the trivial solution will be exponentially stable.

Proof. Find a switched Lyapunov function of the form

$$V_{i_k}(x(t)) = x^{\mathrm{T}} P_{i_k} x, \quad i_k \in \{1, 2, \ldots, m\}, \tag{3.16}$$

where each $P_{i_k} > 0$.

We take the derivative along trajectories:

$$\begin{aligned} \dot{V}_{i_k}(x(t)) &= x^T(A_{i_k}^T P_{i_k} + P_{i_k} A_{i_k})x \\ &\leq x^T Q_{i_k} x \\ &\leq \lambda_{\max}(P_{i_k}^{-1} Q_{i_k}) V_{i_k}(x) \\ &\leq \lambda_{i_k} V_{i_k}(x(t)) \end{aligned} \tag{3.17}$$

Therefore,

$$V_{i_k}(x(t)) \leq V_{i_k}(x(t_{k-1}^+))e^{\lambda_{i_k}(t-t_{k-1})}, \quad t \in (t_{k-1}, t_k]. \tag{3.18}$$

Considering the impulses, we see that

$$\begin{aligned}
\Delta V_{i_k}(x(t_k)) &= \Delta V_{i_k}(x) && (3.19)\\
&= V_{i_k}(x^+) - V_{i_k}(x) && (3.20)\\
&= x^{+T} P_{i_k} x^+ - x^T P_{i_k} x && (3.21)\\
&= x^T((I + B_{i_k})^T P_{i_k} + (I + B_{i_k}))x - x^T P_{i_k} x && (3.22)\\
&\leq \nu_{i_k} V_{i_k}(x(t_k)) && (3.23)
\end{aligned}$$

Therefore,

$$V_{i_k}(t_k^+) \leq (1 + \nu_{i_k}) V_{i_k}(t_{k-1}^+) e^{\lambda_{i_k}(t-t_{k-1})}. \tag{3.24}$$

We can further simplify the equation using $w(t) = x^{\mathrm{T}}(t)x(t)$ and ρ, obtaining

$$w(t_k^+) \leq \rho(t + \nu_{i_k}) w(t_{k-1}^+) e^{\lambda_{i_k}(t-t_{k-1})}. \tag{3.25}$$

As a result, we find the solution on the first interval satisfies

$$w(t) \leq w(t_0^+)\rho e^{\lambda_{i_1}(t-t_0)}, \quad t \in (t_0, t_1]. \tag{3.26}$$

The kth interval satisfies

$$w(t) \leq w(t_0^+)\rho^k(k-1+\nu_{i_1}+\nu_{i_2}+\cdots+\nu_{i_k}) \exp[\lambda_{i_1}(t_1-t_0)+\lambda_{i_1}(t_2-t_1)+\cdots+\lambda_{i_1}(t-t_{k-1})], \tag{3.27}$$

for $t \in (t_{k-1}, t_k]$. Note that we multiply by a factor of ρ each time we switch to a different subsystem.

Now we collect the contributions of each subsystem:

$$\begin{aligned}
w(t) \leq\ & w(t_0^+)\rho \exp\left[T_1(t_0,t)\left(\lambda_1 + \ln(\rho(1+\nu_1))\left(\frac{1}{\tau_a} + \frac{N_1}{T_1(t_0,t)}\right)\right)\right. && (3.28)\\
& \left. + \cdots + T_m(t_0,t)\left(\lambda_m + \ln(\rho(1+\nu_m))\left(\frac{1}{\tau_a} + \frac{N_m}{T_m(t_0,t)}\right)\right)\right], && (3.29)
\end{aligned}$$

where $N_\alpha = N_0^+$ or N_0^- as appropriate to maintain the inequality. Since we are only interested in the asymptotics, and the terms with N_α do not depend on t, we can take them out of the exponential as the constant C_1. We also collect the subsystems of the two different types together, obtaining:

$$\begin{aligned}
w(t) &\leq w(t_0^+)\rho C_1 \exp\left[T^-(t_0,t)\Lambda^- + T^+(t_0,t)\Lambda^+\right]\\
&\leq w(t_0^+)\rho C_1 \exp\left[T^-(t_0,t)(\Lambda^- + q\Lambda^+)\right]\\
&\leq w(t_0^+)\rho C_1 \exp\left[\frac{1}{1+q}(\Lambda^- + q\Lambda^+)(t-t_0)\right]. && (3.30)
\end{aligned}$$

With this inequality, we see that exponential stability is satisfied. □

For this result, we used a different approach, where we did not consider each possible switch from one subsystem to another. This meant we needed the parameter ρ as a "worst case." It would also be possible to do this in a similar way to the other criteria. The major aspect that cannot easily be reconciled, however, is the fact that this proof requires very explicit upper bounds on the growth of the system during each interval. This is difficult to estimate when we are using two arbitrary measures, which is why it is easier to make sure the Lyapunov functions do not increase on each interval. For standard stability results, however, this method allows more interesting cases.

3.4.2 Non-Autonomous Systems

Since we have only covered autonomous systems, it is natural to ask about non-autonomous systems. As mentioned in Chapter 2, these systems present certain difficulties, even when they are not also impulsive switched systems.

The systems of interest are described as follows:

$$\begin{array}{ll} \dot{x} = A_{\sigma(t,x(t))}(t)x(t), & t \neq t_k \\ \Delta x = B_{\sigma(t,x(t))}(t_k)x(t_k), & t = t_k \end{array} \tag{3.31}$$

with the same assumptions on the switching signal σ.

In this case, the Lyapunov functions, $V(t,x)$, depend on time. If we also need to incorporate measures $h(t,x)$ and $h_0(t,x)$ in a meaningful way. We will again want to restrict the possibilities for h and h_0, in order to be able to specify Lyapunov functions more exactly, but, in this case, we must remember that sets invariant under $A(t)$ are dependent on time. This will require a different approach from simply making a change of variables to isolate two subspaces in which we are interested. It might be possible to incorporate a time varying change of variable matrix in order to have the Lyapunov function constantly expressed in a basis that separates the invariant set. In order to deal with partial stability, we could attempt the same approach as before.

First, we recall the conditions for exponential stability of a standard non-autonomous system

$$\dot{x} = A(t)x(t).$$

We require a continuously differentiable positive definite matrix $P(t)$ such that

1. $c_1\|x\|^2 \leq x^T P(t)x \leq c_2\|x\|^2$ and
2. $-P'(t) = A^T(t)P(t) + P(t)A(t) + Q(t), \quad Q(t) > 0.$

Let $P(t)$ be a C^1 positive definite matrix such that

$$-\dot{P}(t) = A^T(t)P(t) + P(t)A(t) + Q(t)$$

$Q(t)$ is normally required to be positive definite, and, in addition, constants are required so that $c_1\|x\|^2 \leq V(t,x) \leq c_2\|x\|^2$.

3.4.3 Stability Criteria

We propose the following stability criteria without proof, but as template for what is likely to be found possible.

1. $h_0, h \in \Gamma$ and h_0 is finer than h;
2. there are $\lambda_B \in R$ such that $\lambda_B \leq \lambda_{\max}\{B_\alpha(t)\}$ for all $\alpha \in \{1, 2, \ldots m\}$ and $t \in R_+$;
3. there exists constant $0 < \rho_0 < \rho$ such that $(t_k, x) \in s(h, \rho_0)$ implies $(t_k, x + I(t_k, x)) \in s(h, \rho)$, $k = 1, 2, \ldots$;
4. there exist functions $a \in PCK$, $b \in K$ and constants $\delta_0, \alpha_0 > 0$ such that for all $i \in \{1, 2, \ldots, m\}$
$$\lambda_{PM}(t)\|x\|^2 \leq a(t, h_0(t,x)) \text{ if } h_0 < \delta_0,$$
and
$$\lambda_{Pm}(t)\|x\|^2 \geq b(h(t,x)) \text{ if } h(t,x) < \alpha_0$$
where $\lambda_{PM}(t) = \max_{k\in\mathbb{L}} \lambda(P_k(t))$ and $\lambda_{Pm}(t) = \min_{k\in\mathbb{L}} \lambda(P_k(t))$
5. for each $\alpha \in \mathbb{L}$, there exists $P_\alpha(t)$ such that
$$-\dot{P}_\alpha(t) = A_\alpha^T(t)P_\alpha(t) + P_\alpha(t)A_\alpha(t) + Q_\alpha(t)$$
for a symmetric matrix Q_α;
6. Q_α above satisfies properties such that the eigenvalues of $P_\alpha^{-1}(t)Q_\alpha(t)$ are bounded for all time and
$$-x^T Q_\alpha(t)x \leq \frac{\mu_\alpha}{\Delta t_\alpha} x^T P_\alpha(t)x;$$
7. there exists $\nu_{\alpha,\beta} \geq \lambda_{\max}(P_\alpha^{-1}(t)(B_\alpha^T P_\beta(t) + B_\alpha^T P_\beta(t)B_\alpha + P_\beta(t)B_\alpha + P_\beta(t) - P_\alpha(t))$ for $\alpha, \beta \in \mathbb{L}$; and
8. there exists $\gamma_\alpha \geq 0$, such that
$$\mu_\alpha + \ln(1 + \nu_{\alpha,\beta}) \leq -\gamma_\alpha.$$

Then the zero solution of system (3.31) is (h_0, h)-stable.

Remark 2. Note that for time-dependent switching, we can find ν_k and μ_k more easily because we do not require the global convergence of the eigenvalues, but only need to look at them for one interval.

Although we do not have a complete proof, we can still look through the steps of the previous results and see that things do not change too much.

$$\begin{aligned} V_\beta(t_k^+, x(t_k^+)) - V_\alpha(t_k, x(t_k)) &= x^{+T} P_\beta(t_k) x^+ - x^T P_\alpha(t_k) x \\ &= x^T((I + B_\alpha(t_k))^T P_\beta(t_k) + (I + B_\alpha(t_k)))x - x^T P_\alpha(t_k) x \\ &\leq \nu_{\alpha,\beta} V_\alpha(t_k, x) \end{aligned} \tag{3.32}$$

We chose $V_\alpha(t, x)$ to be h-positive definite and h_0-decrescent. Therefore,

$$V_\alpha(t, x) \geq b_\alpha(h(x)) \tag{3.33}$$

We use this bound to find an estimate for δ based on ϵ

For one measure, we simply used an explicit formula for σ_1 that satisfied our requirements. In this slightly more complicated case, we must use the inverse of b such that $\sigma_1 = e^{-1} b^{-1}(\epsilon)$. Here, we choose from the finite list of b_α to minimize σ_1. Using the condition on the sum of μ and ν, we know that $\mu_\alpha \leq 1$, since otherwise, $\nu_{\alpha,\beta}$ must be less than -1, which is contradicted by the condition on ΔV.

The only other major difference when using two measures is that we have to choose δ with respect to h_0, which requires the decrescentness of V_α

Note that the choice of σ_1 implies that when $V(x) < \sigma_1$, $h(x) < \epsilon$. We have to make a further refinements with $\sigma_1 < \sigma_0(1 + |\nu|)$ and $\sigma_1 > e\sigma_0$, where ν is chosen from $\nu_{\alpha,\beta}$ such that this is true of all subsystems. We choose δ such that $V(x) < \sigma_0$ when $h_0(x) < \delta$. We now let $h_0(x_0) < \delta$.

Since $h(x_0) < \epsilon$, assume that, for all t, $h(x(t)) < \epsilon$. If not, there will be some $\tilde{t} > \tau_0$ such that $h(x(\tilde{t}^+) \geq \epsilon$. Set $j = \max\{k | t_k \leq \tilde{t}\}$. If $j > q$,

$$\int_{V_\alpha(x(t_{k-1}^+)}^{V_\alpha(x(t_k)} \frac{ds}{s} \leq \mu_\alpha. \tag{3.34}$$

Therefore,

$$\ln\left(\frac{V_\alpha(x(t_k))}{V_\alpha(x(t_{k-1}^+))}\right) \leq \mu_\alpha. \tag{3.35}$$

A similar integration from t_k to t_k^+ leads to

$$\ln\left(\frac{V_\alpha(x(t_k^+))}{V_\alpha(x(t_k))}\right) \leq \ln\left(\frac{V_\alpha(x(t_k^+))(1 + \nu_{\alpha,\beta})}{V_\alpha(x(t_k))}\right). \tag{3.36}$$

This means we know

$$\ln\left(\frac{V_\alpha(x(t_k^+))}{V_\alpha(x(t_{k-1}^+))}\right) \leq 0. \tag{3.37}$$

Hence, $V(x(t_k^+)) \leq V(x(t_{k-1}^+)) \leq V(x(t_q^+))$. We now claim that

$$V_q(x(t_q^+)) < \sigma_1. \tag{3.38}$$

In this first time interval, there are two possibilities. The first is that $\mu_q \leq 0$. In this case, the condition on the derivatives along trajectories implies $V(x(t_q)) \leq V(x_0) < \sigma_0$. Because of our choice of σ_0, we have that

$$V(x(t_q^+)) \leq \sigma_0 + |\nu_q|\sigma_0 < \sigma_1 \tag{3.39}$$

The other case is that $\mu_q > 0$, in which case we quantify the difference between $V_q(x_0)$ and $V_q(x(t_1))$ by integrating:

$$\int_{\sigma_0}^{V_q(x(t_q))} \frac{ds}{s} \leq \int_{V_q(x_0)}^{V_q(x(t_q))} \frac{ds}{s} \leq \mu_q \frac{t_q - \tau_0}{\Delta t_q} < \mu_q \tag{3.40}$$

However, because $\mu_q \leq 1$, we know that

$$\int_{\sigma_0}^{\sigma_1} \frac{ds}{s} = \ln\left(\frac{\sigma_1}{\sigma_0}\right) \geq 1 \geq \mu_q \tag{3.41}$$

Therefore, $V_q(x(t_q)) < \sigma_1$, which implies $V_q(x(t_q^+)) < \sigma_1$.

We have now established that the Lyapunov functions decrease on the endpoints of the intervals. We now assume that there is a time $\tilde{t}$ where $\|x\| \geq \epsilon$. This occurs in some interval j, and $\tilde{t} \in (\hat{t}, t_{j+1})$, where $\hat{t} = \max\{\tau_0, t_j\}$. If $\tilde{t} = \hat{t}$ or if $\tilde{t} > \hat{t}$ and $\mu_{j+1} \leq 0$, then we have $V(x(\tilde{t}) < \sigma_1$, but this is only possible if $\|x\| < \epsilon$. The other contradiction is when $\mu_{j+1} > 0$, in which case we do the same integration as above:

$$\int_{V_j(x(\hat{t}^+))}^{V_j(x(\tilde{t}))} \frac{ds}{s} \leq \mu_{j+1} \tag{3.42}$$

The contradiction arrives because this integral should be larger than integrating from σ_1 to $e\sigma_1$. By the original choice of σ_1, if $\|x\| \geq \epsilon$, the Lyapunov function will have to be greater than $e\sigma_1$. But, once again, since there is a factor of e between the two values, this integral will be greater than or equal to μ_{j+1}

$$\mu_{j+1} \geq \int_{V_j(x(\hat{t}^+))}^{V_j(x(\tilde{t}))} \frac{ds}{s} > \int_{\sigma_1}^{e\sigma_1} \frac{ds}{s} \geq \mu_{j+1} \tag{3.43}$$

Remark 3. Note that for time-dependent switching, we can find ν_k and μ_k more easily because we do not require the global convergence of the eigenvalues, but only need to look at them for one interval.

3.4.4 Two Measures

Dealing with non-autonomous linear systems in two measures is a little more difficult because we must accept the possibilities that the measures will be time dependent and take on many forms. For example, we could consider stability with respect to the time varying subspace given by the kernel of $A_\alpha(t)$ in one of the subsystems. Rather than get into the detail of choosing the correct versions of h_0 and h, we will move on to more general nonlinear non-autonomous systems in the next chapter.

3.5 Linearization

Before looking at stability results for general nonlinear systems, we will see how the linear approach can also apply. In many cases, it is sufficient to use quadratic Lyapunov functions for each of the subsystems. The method of linearization used here is similar to the one used for switched systems in [13]

Assume we can express the nonlinear system as a system with autonomous linear parts, so that subsystem α takes the form

$$\begin{cases} x' = A_\alpha x + g_\alpha(t,x), & t \neq t_k \\ \Delta x = B_\alpha x + J_\alpha(t,x), & t = t_k \end{cases}, \tag{3.44}$$

where A_α and B_α are real matrices. We have an autonomous linear impulsive switched system with nonlinear disturbances.

Let $V_\alpha(t,x) = x^{\mathrm{T}} P_\alpha x$ for positive definite matrices P_α for each subsystem. We compute the derivative along trajectories:

$$\dot{V}_\alpha(t,x) = x^{\mathrm{T}}(P_\alpha A_\alpha + A_\alpha^{\mathrm{T}} P_\alpha)x + 2x^{\mathrm{T}} P_\alpha g_\alpha(t,x) \tag{3.45}$$

We will need the non-linear part to be insignificant. It is, therefore, useful if we can find conditions that create a bound based on the quadratic Lyapunov function we are using. We use the assumptions in the following lemma.

Lemma 24. *If the nonlinear part $g_\alpha(t,x)$ satisfies*

$$\lim_{\|x\|\to 0} \frac{\|g_\alpha(t,x)\|}{\|x\|} = 0 \tag{3.46}$$

then there exists a function $\varphi_\alpha(t)$ such that $x^{\mathrm{T}} P_\alpha g_\alpha(t,x) \leq \varphi_\alpha(t) x^{\mathrm{T}} P_\alpha x$ in a local region about the origin.

If we make the assumption (3.46), we let $\epsilon > 0$ and find δ such that $\|x\| \leq \delta$ implies $\|g_\alpha(t,x)\| \leq \epsilon\|x\|$. Therefore, $x^{\mathrm{T}} P_\alpha g_\alpha(t,x) \leq \epsilon \frac{\lambda_{\max}(P_\alpha)}{\lambda_{\min}(P_\alpha)} x^{\mathrm{T}} P_\alpha x$. Another way to reach the same conclusion is to assume that g_α is Lipschitz, so $\|g_\alpha(t,x)\| \leq L(t)\|x\|$.

During impulses, we can make a similar calculation

$$\begin{aligned}\Delta V_{\alpha,\beta}(t_k,x) = \; & x^{\mathrm{T}}(P_\beta B_\alpha + B_\alpha^{\mathrm{T}} P_\beta B_\alpha + B_\alpha^{\mathrm{T}} P_\beta)x + x^{\mathrm{T}}(P_\beta - P_\alpha)x + 2x^{\mathrm{T}} P_\beta J_\alpha(t_k^+,x) \\ & + 2x^{\mathrm{T}} B_\alpha^{\mathrm{T}} P_\beta J_\alpha(t_k^+,x) + J_\alpha(t_k^+,x)^{\mathrm{T}} P_\beta J_\alpha(t_k^+,x).\end{aligned} \tag{3.47}$$

For the nonlinear terms, we can assume there are functions $\psi_i(t_k)$, for $i = 1,2,3$ such that the following are true locally:

$$x^{\mathrm{T}} P_\alpha J_\alpha(t_k^+,x) \leq \psi_1(t_k) x^{\mathrm{T}} P_\alpha x; \tag{3.48}$$

$$x^{\mathrm{T}} B_\alpha^{\mathrm{T}} P_\alpha J_\alpha(t_k^+,x) \leq \psi_2(t_k) x^{\mathrm{T}} P_\alpha x; \tag{3.49}$$

$$J_\alpha(t_k^+,x)^{\mathrm{T}} P_\alpha J_\alpha(t_k^+,x) \leq \psi_3(t_k) x^{\mathrm{T}} P_\alpha x. \tag{3.50}$$

Similar constraints on J_α to those found for g_α will ensure that these functions can be found.

3.5.1 Stability Criteria

We will use some definitions to help clarify the stability criteria. For the continuous portions of the trajectories, we let $Q_{1,\alpha} = (P_\alpha A_\alpha + A_\alpha^{\mathrm{T}} P_\alpha)$

$$\dot{V}_\alpha(t,x) = x^{\mathrm{T}}(P_\alpha A_\alpha + A_\alpha^{\mathrm{T}} P_\alpha)x + 2x^{\mathrm{T}} P_\alpha g_\alpha(t,x) \tag{3.51}$$

Assume $x^{\mathrm{T}} P_\alpha g_\alpha(t,x) \leq \varphi(t) x^{\mathrm{T}} P_\alpha x$, and then find $\lambda_\alpha(t)$ such that

$$\lambda_{\max}(P_\alpha^{-1} Q_{1,\alpha}) + 2\varphi_\alpha(t) \leq \lambda_\alpha(t). \tag{3.52}$$

Since t only affects the nonlinear parts, assume also that there exists λ_α such that $\lambda_\alpha(t) \leq \lambda_\alpha$.

For the discrete portion, let $P_\beta B_\alpha + B_\alpha^{\mathrm{T}} P_\beta B_\alpha + B_\alpha^{\mathrm{T}} P_\beta + P_\beta - P_\alpha = Q_{2,\alpha\beta}$. We find $\nu_\alpha\beta$ such that

$$\lambda_{\max}(P_\alpha^{-1} Q_{2,\alpha\beta}) + 2\psi_{1,\alpha}(t_k) + 2\psi_{2,\alpha}(t_k) + \psi_{3,\alpha}(t_k) \leq \nu_\alpha(t_k). \tag{3.53}$$

Since t only affects the nonlinear parts, assume we can find $\nu_\alpha \geq \nu_\alpha(t_k)$ for all t_k.

We are now ready to present a theorem. In order to achieve uniform asymptotic stability, we simply use the criteria for the linear system, but replace μ_α and ν_α from those theorems with the appropriate values.

Theorem 5. *If the following criteria are satisfied, the impulsive switched nonlinear system will be (h_0, h)-stable.*

1. $h_0, h \in \Gamma$ *and* h_0 *is finer than* h*;*
2. *there exist functions* $a \in PCK$*,* $b \in K$ *and constants* $\delta_0, \alpha_0 > 0$ *such that for all* $\alpha \in \mathbb{L}$
$$\lambda_{PM}\|x\| \leq a(h_0(t,x)) \text{ if } h_0(t,x) < \delta_0,$$
and
$$\lambda_{Pm}\|x\| \geq b(h(t,x)) \text{ if } h(t,x) < \alpha_0,$$
where $\lambda_{PM} = \max_{k\in\mathbb{L}}\{\lambda(P_k)\}$*,* $\lambda_{Pm} = \min_{k\in\mathbb{L}}\{\lambda(P_k)\}$*;*
3. *the system is linearizable in such a way that there exists* λ_α *as an upper bound to* $\lambda_\alpha(t)$ *defined in (3.52)and* ν_α *as an upper bound to* $\nu_\alpha(t)$ *defined in (3.53);*
4. *let* $\mu_\alpha = \lambda_\alpha \Delta t_\alpha$*, where* Δt_α *a conservative estimate of the activation time of the subsystem for every* $\alpha, \beta \in \mathbb{L}$ *such that*
$$\mu_\alpha + \nu_{\alpha,\beta} \leq 0; \text{ and}$$
5. *there exists* $\gamma_\alpha \geq 0$ *for each* $\alpha \in \mathbb{L}$
$$\mu_\alpha + \ln(1 + \nu_{\alpha,\beta}) \leq -\gamma_\alpha.$$

Proof. As we have already included the effects of the nonlinear parts within the parameters required for stability, the proof proceeds in the same manner as the other linear system results. □

3.6 Conclusion

In this chapter, stability criteria for both autonomous and non-autonomous systems with impulse have been obtained. Criteria of partial stability for these systems have also been established.

Chapter 4

Stability of Impulsive Non-Linear Switched Systems in Two Measures

4.1 Problem Formulation

Consider a system given by

$$\begin{cases} x' = f_{i(k)}(t,x), & t \in (t_{k-1}, t_k] \\ \Delta x = I_{i(k)}(t,x), & t = t_k \end{cases}. \tag{4.1}$$

The switching signal $i : \mathbb{N} \to \mathbb{L} = \{1, 2, \ldots, m\}$ indicates which subsystem is active in the time interval $(t_k, t_{k+1}]$. To begin with, we use a predictable switching rule: one that is a function of time only. For stability, we say that (4.1) is (h_0, h)-stable if given $\epsilon > 0$ and $\tau_0 \in \mathbb{R}^+$, there exists a $\delta = \delta(\tau_0, \epsilon) > 0$ such that $h_0(\tau_0, x_0) < \delta$ implies $h(t, x(t)) < \epsilon, t \geq \tau_0$, where $x(t) = x(t, \tau_0, x_0)$ is a solution with $x(\tau_0) = x_0$.

Since we are dealing with multiple Lyapunov functions, we are not interested in the change of a particular V_α after an impulse, but instead interested in the change from V_α to V_β, where α is the initial switching mode and β is the new mode. Therefore, we define $\Delta V_{\alpha,\beta}$ as

$$\Delta V_{\alpha,\beta}(t_k, x) = V_\beta(t_k^+, x + I_\alpha(t_k^+, x)) - V_\alpha(t_k, x).$$

With this one change, the stability criteria for system (4.1) are almost identical to those for a common Lyapunov function.

4.2 Stability Criteria for Non-Linear Systems Based on Time-Dependent Switching

Theorem 6. *If the following conditions hold, system (4.1) will be (h_0, h)-stable. If there is a set of Lyapunov functions $V_\alpha(t,x)$ that are piecewise continuous in t and locally Lipschitzian in x such that*

1. *$h, h_0 \in \Gamma$ and h_0 is finer than h;*
2. *there exist constants $0 < \rho_0 < \rho$ such that $(t,x) \in s(h,\rho_0)$ implies $(t_k, x+I(t_k, x(t_k))) \in S(h,\rho)$, $k \in \mathbb{L}$;*
3. *$V_\alpha \in \nu_0$ is h-positive definite and weakly h_0-decrescent for all subsystems α;*
4. *the switching signal i is either time-based, or the active subsystem in each time interval can be predicted in advance.*
5. *there exists a constant M and, for every switching time, k, there exists a constant $\mu_k \leq M$ and a function $C_k \in K_0$ such that*

$$D^+ V_{i(k)}(t,x) \leq \frac{\mu_k}{\Delta t_k} C_k(V_{i(k)}(t,x)), \quad (t,x) \in s(h,\rho), \tag{4.2}$$

   where $\Delta t_k = t_k - t_{k-1}$ and $i(k) \in \mathbb{L}$ is the subsystem activated during the interval $(t_{k-1}, t_k]$;

6. *for every k, there is a constant ν_k and a function $d_k \in K_0$ such that*

$$\Delta V_{i(k),i(k+1)}(t_k, x) \leq \nu_k d_k(V_{i(k)}(t_k, x), \quad (t_k, x)) \in s(h,\rho); \tag{4.3}$$

7. *the constants μ_k and ν_k both satisfy*

$$\mu_k + \nu_k \leq 0 \tag{4.4}$$

   for all $k \in \mathbb{L}$;

8. *there exists a constant l such that, for every k, there exists a constant $l_k \geq l > 0$ such that*

$$l_k > \mu_k \sup_{\sigma \in (0, l_k]} \{C_k(\sigma)\}; \text{ and} \tag{4.5}$$

9. *there exist constants* $\gamma_k \geq 0$ *such that*

$$\mu_k + \int_{\sigma}^{\sigma+\nu_k d_k(\sigma)} \frac{ds}{C_k(s)} \leq -\gamma_k, \quad \forall \sigma \in (0, l_k) \tag{4.6}$$

and

$$\nu_k C_k(\sigma) + \mu_k d_k(\sigma) \leq 0, \quad \forall \sigma \in (0, l_k). \tag{4.7}$$

Proof. Since $V(t,x) = V_{i(k)}(t,x)$ with $t \in (t_{k-1}, t_k]$ is h-positive definite and weakly h_0-decrescent for all subsystems, thus for any $\alpha \in \mathbb{L}$, there exist functions $a_\alpha \in PCK$, $b_\alpha \in K$, and constants $\delta_{0,\alpha}$ and ξ_α such that

$$V_\alpha(t,x) \leq a_\alpha(t, h_0(t,x)) \text{ if } h_0(t,x) < \delta_{0,\alpha}$$

and

$$V_\alpha(t,x) \geq b_\alpha(h(t,x)) \text{ if } h(t,x) < \xi_\alpha \tag{4.8}$$

With a finite number of subsystems, we can then define $\xi = \min\{\xi_\alpha\}$ and $\delta_0 = \min\{\delta_{\alpha,0}\}$. Using condition 1, we define δ_1 according to the fact that

$$h(t,x) \leq \varphi(t, h_0(t,x)) < \xi \text{ whenever } h_0(t,x) < \delta_1$$

Let $\epsilon > 0$ be given and assume, without loss of generality, that $0 < \epsilon < \rho^* = \min\{\rho_0, \xi, b^{-1}(l)\}$, where $b_\alpha^{-1}(l_\alpha) = \epsilon_0$ such that $b_\alpha(\epsilon_0) < l_\alpha$ for all $\alpha \in \mathbb{L}$. It is for this reason that l_α must have a lower bound. Define $\sigma_{1,\alpha} = b_\alpha(\epsilon_0)e^{-p_\alpha}$. With (4.5), we choose p_α such that $\mu_\alpha \sup_{\sigma \in (0,l_k]} \{C_\alpha(\sigma)\} + \sigma_{1,\alpha} < l_\alpha$ (again, this is possible provided a lower bound on l_α). Let the initial time τ_0 also be given. From (4.3) and (4.4), for any $k \in \mathbb{N}$, we derive

$$\begin{aligned} -V_{i(k)}(t_k, x) &\leq \Delta V_{i(k),i(k+1)} \\ &\leq \nu_k d_k(V_{i(k)}(t_k,x)) \\ &\leq -\mu_k d_k(V_{i(k)}(t_k,x)) \text{ if } h(t_k,x) < \rho. \end{aligned}$$

Divide through the inequality by $-V_{i(k)}(t_k,x)d_k(V_{i(k)}(t_k,x))$ (a negative quantity) and integrate with respect to $V_{i(k)}(t_k,x)$ represented by the variable s:

$$\int_{\sigma_{1,i(k)}}^{b_{i(k)}(\epsilon)} \frac{ds}{d_k(s)} \geq \mu_k \int_{\sigma_{1,i(k)}}^{b_{i(k)}(\epsilon)} \frac{ds}{s} = \mu_k \ln\left(\frac{b_{i(k)}(\epsilon)}{\sigma_{1,i(k)}}\right) \geq \mu_k, \quad k = 1, 2, \ldots. \tag{4.9}$$

Using the same integration with $\sigma_{2,i(k)} = \sigma_{1,i(k)}e^{-1}$, we conclude

$$\int_{\sigma_{2,i(k)}}^{\sigma_{1,i(k)}} \frac{ds}{d_k(s)} \geq \mu_k, \quad k = 1, 2, \ldots. \tag{4.10}$$

Define $\sigma_1 = \min_{\alpha\in\mathbb{L}}\{\sigma_{1,\alpha}\}$ and $\sigma_2 = \min_{\alpha\in\mathbb{L}}\{\sigma_{2,\alpha}\}$. Let q be the first switching time after the initial time, τ_0, so $q = \min\{k : t_k \geq \tau_0\}$. For ease of notation, we will also use q to index other constants associated with the interval, such as C_q and μ_q. Since $d_k \in K_0$ (i.e., continuous, $d_k(0) = 0$, and $d_k(s) \neq 0$ if $s \neq 0$), there exists a $\sigma_{3,q} = \sigma_{3,q}(\tau_0, \epsilon) > 0$ such that

$$\sigma_{3,q} + |\nu_q| d_q(\sigma_{3,q}) < \sigma_1. \tag{4.11}$$

Define $\sigma_0 = \min\{\sigma_2, \sigma_{3,q}\}$, noting $\sigma_2 < \sigma_1$ by definition. Using the decrescence of the multiple Lyapunov function, there exists a $\delta_2 = \delta_2(\tau_0, \epsilon) > 0$ such that

$$V_{i(q)}(\tau_0^+, x_0) < \sigma_0 \text{ if } h_0(\tau_0, x_0) < \delta_2. \tag{4.12}$$

Let $\delta = \min\{\delta_{0,q}, \delta_{1,q}, \delta_2\}$ and consider a solution to (4.1), $x(t) = x(t, \tau_0, x_0)$ with switching signal i. Then the choice of δ implies $h(\tau_0, x_0) < \epsilon$ since $V_q(\tau_0^+, x_0) < \sigma_0 \leq \sigma_{1,q} = b_q(\epsilon)e^{-p_q} < b_q(\epsilon)$. $V_{i(q)}$ is h-positive definite, and $b_q \in K$, so $h(\tau_0, x_0) < \epsilon$. Now suppose that $h(t, x(t))$ eventually exceeds ϵ. Hence, there is a $\tilde{t}$ such that $h(\tilde{t}^+, x(\tilde{t}^+)) \geq \epsilon$ and $h(t, x(t)) < \epsilon$ for $t \in [\tau_0, \tilde{t})$. This covers the cases when $h(t, x(t))$ exceeds ϵ on a continuous trajectory and when it does so immediately following an impulse. Now, set $j = \max\{k : t_k \leq \tilde{t}\}$. First, consider the time immediately after the first t_q, the first switch on or after the initial time. We claim that

$$V_{i(q+1)}(t_q^+, x(t_q^+)) < \sigma_1.$$

If $t_q = \tau_0$, this is true by the definition of σ_0, which is smaller than σ_1. Therefore, we assume, $t_{q-1} < \tau_0 < t_q$. If, on the one hand, $\mu_q \leq 0$, then (4.2) implies $V_{i(q)}$ is decreasing, so $V_{i(q)}(t_q, x(t_q)) \leq V_{i(q)}(\tau_0^+, x(\tau_0^+)) < \sigma_0 < \sigma_1$. Using (4.11) and (4.3) leads to the conclusion that

$$\begin{aligned} V_{i(q+1)}(t_q^+, x(t_q^+)) &\leq V_{i(q)}(t_q, x(t_q)) + \nu_q d_q(V_{i(q)}(t_q, x(t_q))) \\ &\leq \sigma_0 + |\nu_q| d_q(\sigma_0) < \sigma_1. \end{aligned}$$

On the other hand, if $\mu_q > 0$, then (4.4) implies $\nu_q < 0$, so $V_{i(q+1)}(t_q^+, x(t_q^+)) \leq V_{i(q)}(t_q, x(t_q))$. Using (4.2) and integrating as before, we have

$$\int_{\sigma_0}^{V_{i(q)}(t_q, x(t_q))} \frac{ds}{C_q(s)} \leq \int_{V_{i(q)}(\tau_0^+, x(\tau_0^+))}^{V_{i(q)}(t_q, x(t_q))} \frac{ds}{C_q(s)} \leq \mu_q \frac{t_q - \tau_0}{\Delta t_q} < \mu_q. \tag{4.13}$$

With (4.7) and (4.4), we can rearrange and integrate to obtain

$$\int_{\sigma_0}^{\sigma_1} \frac{ds}{C_q(s)} \geq \frac{-\nu_q}{\mu_q} \int_{\sigma_0}^{\sigma_1} \frac{ds}{d_q(s)} \geq \int_{\sigma_0}^{\sigma_1} \frac{ds}{d_q(s)}$$

since both σ_0 and σ_1 are less than l_q. This, in combination with (4.10), indicates that

$$\int_{\sigma_0}^{\sigma_1} \frac{ds}{C_q(s)} \geq \mu_q.$$

Therefore, (4.13) tells us that $V_{i(q)}(t_q, x(t_q)) < \sigma_1$; thus, $V_{i(q+1)}(t_q^+, x(t_q^+)) < \sigma_1$.

Assume for $k = q+1, \ldots, j$,

$$V_{i(k)}(t_{k-1}^+, x(t_{k-1}^+)) < \sigma_1. \tag{4.14}$$

Since we have not reached $\tilde{t}$, $h(t, x(t)) < \epsilon < \rho$, so we can use (4.2). Hence,

$$\int_{V_{i(k)}(t_{k-1}^+, x(t_{k-1}^+)}^{V_{i(k)}(t_k, x(t_k))} \frac{ds}{C_k(s)} \leq \mu_k. \tag{4.15}$$

Using (4.3) and the fact that $C_k \in K_0$,

$$\int_{V_{i(k)}(t_k, x(t_k)}^{V_{i(k+1)}(t_k^+, x(t_k^+))} \frac{ds}{C_k(s)} \leq \int_{V_{i(k)}(t_k, x(t_k)}^{V_{i(k)}(t_k, x(t_k)) + \nu_k d_k(V_{i(k)}(t_k, x(t_k))} \frac{ds}{C_k(s)}.$$

We claim that $V_{i(k)}(t_k, x(t_k)) < l_k$. If not, there exists a t^* such that $t_{k-1} < t^* \leq t_k$, $V_{i(k)}(t^*, x(t^*)) = l_k \leq V_{i(k)}(t_k, x(t_k))$ and $V_{i(k)}(t, x(t)) < l_k$ for all $t_{k-1} < t < t^*$. Using our assumption (4.14), (4.5), and (4.15), note that

$$\begin{aligned}
& (V_{i(k)}(t^*, x(t^*)) - \sigma_1) \frac{1}{\sup_{s \in (0,l]}\{C_k(s)\}} \\
< \; & (V_{i(k)}(t^*, x(t^*)) - V_{i(k)}(t_{k-1}^+, x(t_{k-1}^+))) \frac{1}{\sup_{s \in (0,l]}\{C_k(s)\}} \\
\leq \; & \int_{V_{i(k)}(t_{k-1}^+, x(t_{k-1}^+)}^{V_{i(k)}(t^*, x(t^*))} \frac{ds}{C_k(s)} \\
\leq \; & \mu_k
\end{aligned}$$

So

$$V_{i(k)}(t^*, x(t^*)) < \mu_k \sup_{s \in (0,l]} \{C_k(s)\} + \sigma_1 < l_k,$$

which is a contradiction.

Hence, $V_{i(k)}(t_k, x(t_k)) < l_k$, and we can apply (4.6) to show that

$$\int_{V_{i(k)}(t_{k-1}^+, x(t_{k-1}^+))}^{V_{i(k+1)}(t_k^+, x(t_k^+))} \frac{ds}{C_k(s)} \leq 0.$$

Hence,

$$V_{i(k+1)}(t_k^+, x(t_k^+)) \leq V_{i(k)}(t_{k-1}^+, x(t_{k-1}^+)).$$

Since $V_{i(q+1)}(t_q^+, x(t_q^+)) < \sigma_1$, this justifies (4.14). Hence, we have shown that

$$V_{i(j+1)}(t_j^+, x(t_j^+)) \leq V_{i(q+1)}(t_q^+, x(t_q^+)).$$

Now set $\hat{t} = \max\{\tau_0, t_j\}$, and we have determined that

$$V_{i(j+1)}(\hat{t}^+, x(\hat{t}^+)) < \sigma_1, \tag{4.16}$$

where $i(j+1)$ is the subsystem that applies, since $\tilde{t} \in [\hat{t}, t_{j+1})$. If $\tilde{t} = \hat{t}$ or if $\tilde{t} > \hat{t}$ and $\mu_{j+1} \leq 0$, then

$$V_{i(j+1)}(\tilde{t}^+, x(\tilde{t}^+)) \leq V_{i(j+1)}(\hat{t}^+, x(\hat{t}^+)) < \sigma_1,$$

From (4.2) and (4.16). Therefore,

$$b_{i(j+1)}(\epsilon) \leq b_{i(j+1)}(h(\tilde{t}^+, x(\tilde{t}^+))) \leq V_{i(j+1)}(\tilde{t}^+, x(\tilde{t}^+)) < \sigma_1 \leq b_{i(j+1)}(\epsilon), \quad \forall \alpha,$$

which is a contradiction. Focusing instead on $\tilde{t} > \hat{t}$ and $\mu_{j+1} > 0$, then by (4.2), (4.7), and (4.9),

$$\mu_{j+1} \geq \int_{V_{i(j+1)}(\hat{t}^+, x(\hat{t}^+))}^{V_{i(j+1)}(\tilde{t}, x(\tilde{t}))} \frac{ds}{C_{j+1}(s)} > \int_{\sigma_1}^{b_{i(j+1)}(\epsilon)} \frac{ds}{C_{j+1}(s)} \geq \int_{\sigma_1}^{b_{i(j+1)}(\epsilon)} \frac{ds}{d_{j+1}(s)} \geq \mu_{j+1},$$

which is a contradiction. Therefore, time $\tilde{t}$ cannot exist, so the trajectory is stable. □

By the similar methods in Section 3, we get the following criteria for partial stability of the trivial solution of (4.1).

Corollary 2. *If the following conditions hold, system (4.1) will be partially stable with respect to x_s.*

1. *The switching signal i is either time-based, or the active subsystem in each time interval can be predicted in advance.*

2. *There exists a constant M and, for every switching time, k, there exists a constant $\mu_k \leq M$ and a function $C_k \in K_0$ such that*

$$D^+ V_{i(k)}(t, x) \leq \frac{\mu_k}{\Delta t_k} C_k(V_{i(k)}(t, x)), \quad (t, x) \in s(h, \rho),$$

where $\Delta t_k = t_k - t_{k-1}$ and $i(k) \in I$ is the subsystem activated during the interval $(t_{k-1}, t_k]$;

3. *For every k, there is a constant ν_k and a function $d_k \in K_0$ such that*

$$\Delta V_{i(k),i(k+1)}(t_k, x) \leq \nu_k d_k(V_{i(k)}(t_k, x), \quad (t_k, x)) \in s(h, \rho);$$

4. *The constants μ_k and ν_k both satisfy*

$$\mu_k + \nu_k \leq 0$$

for all $k \in \mathbb{L}$;

5. *There exists a constant l such that, for every k, there exists a constant $l_k \geq l > 0$ such that*

$$l_k > \mu_k \sup_{\sigma \in (0, l_k]} \{C_k(\sigma)\};$$

6. *There exist constants γ_k such that*

$$\mu_k + \int_{\sigma}^{\sigma + \nu_k d_k(\sigma)} \frac{ds}{C_k(s)} \leq -\gamma_k, \quad \forall \sigma \in (0, l_k)$$

and

$$\nu_k C_k(\sigma) + \mu_k d_k(\sigma) \leq 0, \quad \forall \sigma \in (0, l_k).$$

4.3 Stability Criteria for Non-Linear Systems Based on Space-Dependent Switching

The previous result only works for a time-based switching rule, or one in which a lot is known about the switching signal in advance. However, we can modify the list of conditions to include different switching schemes. These modifications will also lead to a stricter set of requirements for the system to satisfy. In such a case, the switching times will not be predetermined. To avoid the stability problems that will result from chatter, we also need to have some basic conditions on the activation times of subsystems. Instead of looking at the interval k as the basic unit of analysis, we look at the active subsystem.

Let an admissible pair of subsystems be defined as follows: $(\alpha, \beta) \in \mathcal{A}$, the set of admissible subsystems pairs if, according to the switching rule, it is possible for subsystem α to be active immediately before subsystem β. In general, we have to look at the transition between all m^2 possible pairs of subsystems, but it may be possible to ignore certain possibilities in a case where the order of the subsystems is predetermined, but the activation times are not.

Theorem 7. *If the following conditions hold, system (4.1) will be (h_0, h)-stable.*

1. *$h, h_0 \in \Gamma$ and h_0 is finer than h*

2. *There exist constants $0 < \rho_0 < \rho$ such that $(t, x) \in s(h, \rho_0)$ implies $(t, x + I(x)) \in s(h, \rho)$;*

3. *$V_\alpha \in \nu_0$ is h-positive definite and h_0-decrescent for all $\alpha \in \mathbb{L}$;*

4. *For every $\alpha \in \mathbb{L}$, there exists a constant μ_α and a function $C_\alpha \in K_0$ such that*

$$D^+V_\alpha(t, x) \leq \frac{\mu_\alpha}{\Delta t_\alpha} C_\alpha(V_\alpha(t, x)), \quad (t, x) \in s(h, \rho), \tag{4.17}$$

   where Δt_α is a conservative estimate of the dwell time for the subsystem: if $\mu_\alpha < 0$, Δt_α is the minimum activation time of subsystem α; if $\mu_\alpha > 0$, Δt_α is the maximum activation time of the subsystem.

5. *For every $\alpha, \beta \in \mathbb{L}$, there is a constant $\nu_{\alpha\beta}$ and a function $d_{\alpha\beta} \in K_0$ such that*

$$\Delta V_\beta(t, x) \leq \nu_{\alpha,\beta} d_{\alpha,\beta}(V_\alpha(t, x), \quad (t, x)) \in s(h, \rho). \tag{4.18}$$

   Note that t is used since the precise switching times cannot be defined in advance.

6. *For all pairs α, β as defined above, the constants μ_α and $\nu_{\alpha\beta}$ both satisfy*

$$\mu_\alpha + \nu_{\alpha,\beta} \leq 0; \tag{4.19}$$

7. *For every $\alpha \in \mathbb{L}$, there exists a constant $l_\alpha > 0$ such that*

$$l_\alpha > \sup_{\sigma \in (0, l_\alpha]} \{C_\alpha(\sigma)\mu_\alpha\}; \tag{4.20}$$

8. *There exist constants γ_α and $l > 0$ such that*

$$\mu_\alpha + \int_\sigma^{\sigma + \nu_{\alpha,\beta} d_{\alpha,\beta}(\sigma)} \frac{ds}{C_\alpha(s)} \leq -\gamma_\alpha, \quad \forall \sigma \in (0, l_\alpha) \tag{4.21}$$

   and

$$\nu_\alpha C_\alpha(\sigma) + \mu_{\alpha,\beta} d_{\alpha,\beta}(\sigma) \leq 0, \quad \forall \sigma \in (0, l_\alpha) \tag{4.22}$$

   for all $\alpha, \beta \in \mathbb{L}$.

Proof. Since $V_\alpha(t, x)$ is h-positive definite and weakly h_0-decrescent for all subsystems, α, there exist functions $a_\alpha \in PCK$, $b_\alpha \in K$, and constants $\delta_{0,\alpha}$ and ξ_α such that

$$V_\alpha(t, x) \leq a_\alpha(t, h_0(t, x)) \text{ if } h_0(t, x) < \delta_{0,\alpha}$$

and

$$V_\alpha(t,x) \geq b_\alpha(h(t,x)) \text{ if } h(t,x) < \xi_\alpha \tag{4.23}$$

With a finite number of subsystems, we can then define $\xi = \min\{\xi_\alpha\}$ and $\delta_0 = \min\{\delta_{\alpha,0}\}$. Using condition 1, we define δ_1 according to the fact that

$$h(t,x) \leq \varphi(t, h_0(t,x)) < \xi \text{ whenever } h_0(t,x) < \delta_1.$$

Let $\epsilon > 0$ be given and assume, without loss of generality, that $0 < \epsilon < \rho^* = \min\{\rho_0, \xi, b^{-1}(l)\}$, where $b^{-1}(l) = \epsilon_0$ such that $b_\alpha(\epsilon_0) < l_\alpha$ for all α. Define $\sigma_{1,\alpha} = b_\alpha e^{-p_\alpha}$. With (4.20), we choose p_α such that $\sup\limits_{\sigma\in(0,l_\alpha]} \{C_\alpha(\sigma)\mu_\alpha\} + \sigma_{1,\alpha} < l_\alpha$ for all α. Let the initial time τ_0 also be given. From (4.18) and (4.19), we derive

$$\begin{aligned} -V_\alpha(t,x) &\leq \Delta V_{\alpha,\beta} \\ &\leq \nu_{\alpha,\beta} d_{\alpha,\beta}(V_\alpha(t,x)) \\ &\leq -\mu_\alpha d_{\alpha,\beta}(V_\alpha(t,x)) \text{ if } h(t,x) < \rho. \end{aligned}$$

Divide through the inequality by $-V_\alpha(t,x)d_{\alpha,\beta}(V_\alpha(t,x))$ (a negative quantity) and integrate with respect to $V_\alpha(t,x)$ represented by the variable s:

$$\int_{\sigma_{1,\alpha}}^{b_\alpha(\epsilon)} \frac{ds}{d_{\alpha,\beta}(s)} \geq \mu_\alpha \int_{\sigma_{1,\alpha}}^{b_\alpha(\epsilon)} \frac{ds}{s} = \mu_\alpha \ln\left(\frac{b_\alpha(\epsilon)}{\sigma_{1,\alpha}}\right) = \mu_\alpha. \tag{4.24}$$

Using the same integration with $\sigma_{2,\alpha} = \sigma_{1,\alpha}e^{-1}$, we conclude

$$\int_{\sigma_{2,\alpha}}^{\sigma_{1,\alpha}} \frac{ds}{d_{\alpha,\beta}(s)} \geq \mu_\alpha. \tag{4.25}$$

Define $\sigma_1 = \min_\alpha\{\sigma_{1,\alpha}\}$ and $\sigma_2 = \min_\alpha\{\sigma_{2,\alpha}\}$. Let t_q be the first switching time after the initial time, τ_0. Although the switching times are not known in advance, they are still well defined from the dynamics. Unlike in the time-based case where most indexes were the time intervals, we use q as an index to represent the active subsystem active in the time interval $(t_{q-1}, t_q]$ and q' to represent the subsystem active in the interval $(t_q, t_{q+1}]$. In addition, define all associated quantities $C_q, \mu_q, \nu_{q,q'}, d_{q,q'}, \ldots$ for the first subsystems. Since $d_{\alpha,\beta} \in K_0$ (i.e., continuous, $d_{\alpha,\beta}(0) = 0$, and $d_{\alpha,\beta}(s) \neq 0$ if $s \neq 0$), there exists a $\sigma_{3,q} = \sigma_{3,q}(\tau_0, \epsilon) > 0$ such that

$$\sigma_{3,q} + |\nu_{q,q'}| d_{q,q'}(\sigma_{3,q}) < \sigma_{1,q}. \tag{4.26}$$

Define $\sigma_0 = \min\{\sigma_2, \sigma_{3,q}\}$, noting $\sigma_2 < \sigma_1$ by definition. Using the decrescence of the multiple Lyapunov function, there exists a $\delta_2 = \delta_2(\tau_0, \epsilon) > 0$ such that

$$V_q(\tau_0^+, x_0) < \sigma_0 \text{ if } h_0(\tau_0, x_0) < \delta_2.$$

Let $\delta = \min\{\delta_{0,i(q)}, \delta_{1,i(q)}, \delta_2\}$ and consider a solution to (4.1), $x(t) = x(t, \tau_0, x_0)$ with switching signal i. Then the choice of δ implies $h(\tau_0, x_0) < \epsilon$ since $V_q(\tau_0^+, x_0) < \sigma_0 \leq \sigma_{1,q} = b_q(\epsilon)e^{-1} < b_q(\epsilon)$. V_q is h-positive definite, and $b_q \in K$, so $h(\tau_0, x_0) < \epsilon$. Now suppose that $h(t, x(t))$ eventually exceeds ϵ. Hence, there is a $\tilde{t}$ such that $h(\tilde{t}^+, x(\tilde{t}^+)) \geq \epsilon$ and $h(t, x(t)) < \epsilon$ for $t \in [\tau_0, \tilde{t})$. This covers the cases when $h(t, x(t))$ exceeds ϵ on a continuous trajectory and when it does so immediately following an impulse. Now, set $j = \max\{k : t_k \leq \tilde{t}\}$, which we may do if the switching times are well defined as a result of the switching rule. First, consider the time immediately after the first t_q, the first switch on or after the initial time. We claim that

$$V_{i(q+1)}(t_q^+, x(t_q^+)) < \sigma_1.$$

If $t_q = \tau_0$, this is true by the definition of σ_0, which is smaller than σ_1. Therefore, we assume $t_{q-1} < \tau_0 < t_q$. If, on the one hand, $\mu_q \leq 0$, then (4.17) implies V_q is decreasing, so $V_q(t_q, x(t_q)) \leq V_q(\tau_0^+, x(\tau_0^+)) < \sigma_0 < \sigma_1$. Using (4.26) and (4.18) leads to the conclusion that

$$\begin{aligned} V_{q+1}(t_q^+, x(t_q^+)) &\leq V_q(t_q, x(t_q)) + \nu_{q,q'} d_{q,q'}(V_{i(q)}(t_q, x(t_q))) \\ &\leq \sigma_0 + |\nu_{q,q'}| d_{q,q'}(\sigma_0) < \sigma_1. \end{aligned}$$

On the other hand, if $\mu_q > 0$, then (4.19) implies $\nu_{q,q'} < 0$, so $V_{q'}(t_q^+, x(t_q^+)) \leq V_q(t_q, x(t_q))$. Using (4.17) and integrating as before, we have

$$\int_{\sigma_0}^{V_q(t_q, x(t_q))} \frac{ds}{C_q(s)} \leq \int_{V_q(\tau_0^+, x(\tau_0^+))}^{V_q(t_q, x(t_q))} \frac{ds}{C_q(s)} \leq \mu_q \frac{t_q - \tau_0}{\Delta t_q} < \mu_q. \tag{4.27}$$

With (4.22) and (4.19), we can rearrange and integrate to obtain

$$\int_{\sigma_0}^{\sigma_1} \frac{ds}{C_q(s)} \geq \frac{-\nu_{q,q'}}{\mu_q} \int_{\sigma_0}^{\sigma_1} \frac{ds}{d_{q,q'}(s)} \geq \int_{\sigma_0}^{\sigma_1} \frac{ds}{d_{q,q'}(s)}$$

since both σ_0 and σ_1 are less than l_q. This, in combination with (4.25), indicates that

$$\int_{\sigma_0}^{\sigma_1} \frac{ds}{C_q(s)} \geq \mu_q.$$

Therefore, (4.27) tells us that $V_q(t_q, x(t_q)) < \sigma_1$ and thus $V_{q'}(t_q^+, x(t_q^+)) < \sigma_1$. Assume for all $k = q+1, \ldots, j$,

$$V_{i(k)}(t_{k-1}^+, x(t_{k-1}^+)) < \sigma_1. \tag{4.28}$$

Since we have not reached $\tilde{t}$, $h(t, x(t)) < \epsilon < \rho$, we can use (4.17). Hence,

$$\int_{V_{i(k)}(t_{k-1}^+, x(t_{k-1}^+))}^{V_{i(k)}(t_k, x(t_k))} \frac{ds}{C_k(s)} \leq \frac{t_k - t_{k-1}}{\Delta t_{i(k)}} \mu_{i(k)} \leq \mu_{i(k)}. \tag{4.29}$$

This slight difference is the result of defining conservative activation time estimates. Using (4.18) and the fact that $C_{i(k)} \in K_0$,

$$\int_{V_{i(k)}(t_k,x(t_k)}^{V_{i(k+1)}(t_k^+,x(t_k^+))} \frac{ds}{C_{i(k)}(s)} \leq \int_{V_{i(k)}(t_k,x(t_k)}^{V_{i(k)}(t_k,x(t_k))+\nu_{i(k),i(k+1)}d_{i(k),i(k+1)}(V_{i(k)}(t_k,x(t_k)))} \frac{ds}{C_{i(k)}(s)}.$$

We claim that $V_{i(k)}(t_k, x(t_k)) < l_{i(k)}$. If not, there exists a t^* such that $t_{k-1} < t^* \leq t_k$, $V_{i(k)}(t^*, x(t^*)) = l_{i(k)} \leq V_{i(k)}(t_k, x(t_k))$ and $V_{i(k)}(t, x(t)) < l_{i(k)}$ for all $t_{k-1} < t < t^*$. Using our assumption (4.28), (4.20), and (4.29), note that

$$\begin{aligned}
& (V_{i(k)}(t^*, x(t^*)) - \sigma_1)\frac{1}{\sup_{s\in(0,l]}\{C_{i(k)}(s)\}} \\
< \; & (V_{i(k)}(t^*, x(t^*)) - V_{i(k)}(t_{k-1}^+, x(t_{k-1}^+)))\frac{1}{\sup_{s\in(0,l]}\{C_{i(k)}(s)\}} \\
\leq \; & \int_{V_{i(k)}(t_{k-1}^+,x(t_{k-1}^+)}^{V_{i(k)}(t^*,x(t^*))} \frac{ds}{C_{i(k)}(s)} \\
\leq \; & \mu_{i(k)}
\end{aligned}$$

So

$$V_{i(k)}(t^*, x(t^*)) < \mu_{i(k)} \sup_{s\in(0,l]} \{C_{i(k)}(s)\} + \sigma_1 < l_{i(k)},$$

which is a contradiction. Hence, $V_{i(k)}(t_k, x(t_k)) < l_{i(k)}$, and we can apply (4.21) to show that

$$\int_{V_{i(k)}(t_{k-1}^+,x(t_{k-1}^+))}^{V_{i(k+1)}(t_k^+,x(t_k^+))} \frac{ds}{C_{i(k)}(s)} \leq 0.$$

Hence,

$$V_{i(k+1)}(t_k^+, x(t_k^+)) \leq V_{i(k)}(t_{k-1}^+, x(t_{k-1}^+)).$$

Since $V_{i(q+1)}(t_q^+, x(t_q^+)) < \sigma_1$, this justifies (4.28). Hence, we have shown that

$$V_{i(j+1)}(t_j^+, x(t_j^+)) \leq V_{i(q+1)}(t_q^+, x(t_q^+)).$$

Now set $\hat{t} = \max\{\tau_0, t_j\}$, and we have determined that

$$V_{i(j+1)}(\hat{t}^+, x(\hat{t}^+)) < \sigma_1, \tag{4.30}$$

where $i(j+1)$ is the subsystem that applies, since $\tilde{t} \in [\hat{t}, t_{j+1})$. If $\tilde{t} = \hat{t}$ or if $\tilde{t} > \hat{t}$ and $\mu_{i(j+1)} \leq 0$, then

$$V_{i(j+1)}(\tilde{t}^+, x(\tilde{t}^+)) \leq V_{i(j+1)}(\hat{t}^+, x(\hat{t}^+)) < \sigma_1,$$

From (4.17) and (4.30). Therefore,

$$b_{i(j+1)}(\epsilon) \leq b_{i(j+1)}(h(\tilde{t}^+, x(\tilde{t}^+))) \leq V_{i(j+1)}(\tilde{t}^+, x(\tilde{t}^+)) < \sigma_1 \leq b_\alpha(\epsilon), \quad \forall \alpha,$$

which is a contradiction. Focusing instead on $\tilde{t} > \hat{t}$ and $\mu_{i(j+1)} > 0$, then by (4.17), (4.22), and (4.24),

$$\mu_{i(k)} \geq \int_{V_{i(j+1)}(\hat{t}^+, x(\hat{t}^+))}^{V_{i(j+1)}(\tilde{t}, x(\tilde{t}))} \frac{ds}{C_{i(j+1)}(s)} > \int_{\sigma_1}^{b_{i(j+1)}(\epsilon)} \frac{ds}{C_{i(j+1)}(s)} \geq \int_{\sigma_1}^{b_{i(j+1)}(\epsilon)} \frac{ds}{d_{i(j+1),i(j+2)}(s)} \geq \mu_{i(j+1)},$$

which is a contradiction. Therefore, time $\tilde{t}$ cannot exist, so the trajectory is stable. □

By the similar methods in Section 3, we get the following criteria for partial stability of the trivial solution of (4.1) with state-dependent switching.

Corollary 3. *If the following conditions hold, system (4.1) will be partially stable with respect to x_s.*

1. *For every $\alpha \in \mathbb{L}$, there exists a constant μ_α and a function $C_\alpha \in K_0$ such that*

$$D^+ V_\alpha(t,x) \leq \frac{\mu_\alpha}{\Delta t_\alpha} C_\alpha(V_\alpha(t,x)), \quad (t,x) \in s(h,\rho),$$

   where Δt_α is a conservative estimate of the dwell time for the subsystem: if $\mu_\alpha < 0$, Δt_α is the minimum activation time of subsystem α; if $\mu_\alpha > 0$, Δt_α is the maximum activation time of the subsystem.

2. *For every $\alpha\beta \in \mathbb{L}$, there is a constant $\nu_{\alpha\beta}$ and a function $d_{\alpha\beta} \in K_0$ such that*

$$\Delta V_\beta(t,x) \leq \nu_{\alpha,\beta} d_{\alpha,\beta}(V_\alpha(t,x), \quad (t,x)) \in s(h,\rho).$$

   Note that t is used since the precise switching times cannot be defined in advance.

3. *For every $\alpha, \beta \in \mathbb{L}$, the constants μ_α and $\nu_{\alpha\beta}$ both satisfy*

$$\mu_\alpha + \nu_{\alpha,\beta} \leq 0;$$

4. *For every $\alpha \in \mathbb{L}$, there exists a constant $l_\alpha > 0$ such that*

$$l_\alpha > \sup_{\sigma \in (0, l_\alpha]} \{C_\alpha(\sigma)\mu_\alpha\};$$

5. *There exist constants γ_α and $l > 0$ such that*

$$\mu_\alpha + \int_\sigma^{\sigma + \nu_{\alpha,\beta} d_{\alpha,\beta}(\sigma)} \frac{ds}{C_\alpha(s)} \leq -\gamma_\alpha, \quad \forall \sigma \in (0, l_\alpha)$$

   and

$$\nu_\alpha C_\alpha(\sigma) + \mu_{\alpha,\beta} d_{\alpha,\beta}(\sigma) \leq 0, \quad \forall \sigma \in (0, l_\alpha)$$

   for every $\alpha, \beta \in \mathbb{L}$.

4.3.1 Uniform Stability Criteria for Non-Linear Systems

Theorem 8. *If we are looking for (h, h_0)-uniform stability, we need to adjust the conditions to avoid time dependence. We have the following conditions as before:*

1. *$h, h_0 \in \Gamma$ and h_0 is finer than h;*
2. *There exist constants $0 < \rho_0 < \rho$ such that $(t, x) \in s(h, \rho_0)$ implies $(t_k, x+I(t_k, x(t_k))) \in s(h, \rho)$, k=1,2,. . . ;*
3. *$V_\alpha \in \nu_0$ is h-positive definite and h_0-decrescent for all subsystems α;*
4. *The switching signal i is either time-based, or the active subsystem in each time interval can be predicted in advance;*
5. *There exists a constant M and, for every switching time, k, there exists a constant $\mu_k \leq M$ and a function $C_k \in K_0$ such that*
$$D^+V_{i(k)}(t, x) \leq \frac{\mu_k}{\Delta t_k} C_k(V_{i(k)}(t, x)), \quad (t, x) \in s(h, \rho),$$
where $\Delta t_k = t_k - t_{k-1}$ and $i(k) \in I$ is the subsystem activated during the interval $(t_{k-1}, t_k]$;
6. *For every $k \in \mathbb{L}$, there is a constant ν_k and a function $d_k \in K_0$ such that*
$$\Delta V_{i(k),i(k+1)}(t_k, x) \leq \nu_k d_k(V_{i(k)}(t_k, x), \quad (t_k, x)) \in s(h, \rho),$$
where for any η there exists $\alpha > 0$ such that $d_k(\sigma) < \eta$ for all $\sigma \in [0, \alpha]$.
7. *The constants μ_k and ν_k both satisfy*
$$\mu_k + \nu_k \leq 0$$
for all $k \in \mathbb{L}$;
8. *There exists a constant l such that, for every $k \in \mathbb{L}$, there exists a constant $l_k \geq l > 0$ such that*
$$l_k > \sup_{\sigma \in (0, l_k]} \{C_k(\sigma) u_k\};$$
9. *There exist constants γ_k with $k \in \mathbb{L}$ such that*
$$\mu_k + \int_\sigma^{\sigma + \nu_k d_k(\sigma)} \frac{ds}{C_k(s)} \leq -\gamma_k, \quad \forall \sigma \in (0, l_k)$$
and
$$\nu_k C_k(\sigma) + \mu_k d_k(\sigma) \leq 0, \quad \forall \sigma \in (0, l_k).$$

Proof. We begin the proof the same way as for the non-uniform stability result. This time, we use the fact that

$$V_\alpha(t,x) \leq a_\alpha(h_0(t,x)) \; if \; h_0(t,x) < \delta_{0,\alpha}$$

δ_1 can be determined the same way as before, since the calculations to find σ_1 and σ_2 do not have any time dependence. When choosing σ_0, however, we need to take into account that the initial subsystem could be any possibility:

$$\sigma_3 + |\nu_\alpha| d_\alpha(\sigma_3) < \sigma_1$$

Given that there are a limited number of possibilities for ν_α and d_α, we can satisfy this condition easily. Hence, we can choose σ_0 and δ in such a way that it is not dependent on τ_0, which is the requirement for uniform stability. □

For uniform stability of the system using a state-based switching rule, we have the same general approach.

4.4 Conclusion

In this chapter, Stability criteria nonlinear switched systems with impulse in two measure have been obtained. Criteria of partial stability for these systems have also been carried out.

Chapter 5

Evaluation, Examples, and Discussion

In the previous two chapters, we proved theorems concerning sufficient conditions for stability of impulsive switched systems. In order to show that these results have more than a purely mathematical significance, we will use some examples to illustrate their potential utility. Having established that the results are mathematically valid, it remains to show that they are easily verifiable and not overly exclusive, in the sense that they are only satisfied in special circumstances unlikely to actually arise. It is difficult to quantify such vague statements, however, so we hope the examples will serve the goal of demonstrating the utility of the results.

5.1 Autonomous Linear Systems

The simplest examples we can use are two-dimensional linear systems with periodic switching rules. Although such examples do not require elaborate theory to find stability criteria, they serve to demonstrate the need for the criteria that have been developed. The first instructive example has two different stable subsystems, one with a stabilizing impulse, and one with a destabilizing impulse.

Example 2. *Consider the switched impulsive linear system*

$$\begin{cases} \dot{x} = A_i x & t \neq t_k \\ \Delta x = B_i x & t = t_k \end{cases} \tag{5.1}$$

with subsystems given as follows:

$$A_1 = \begin{pmatrix} -1 & -4 \\ 5 & -1 \end{pmatrix} \quad A_2 = \begin{pmatrix} -1 & -5 \\ 4 & -1 \end{pmatrix}, \text{ and} \tag{5.2}$$

$$B_1 = \begin{pmatrix} -\frac{3}{4} & 0 \\ 0 & -\frac{3}{4} \end{pmatrix} \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.3}$$

Will the system be stable under a periodic switching rule?

There are two issues that can arise related to the stability of this system. The first is that the continuous subsystems, although stable, may create an unstable trajectory given the appropriate switching rule. The second is that the unstable impulse will be used too frequently.

Proceeding with our analysis, we first obtain the standard quadratic Lyapunov functions associated with each subsystem by solving the linear equations $A_i^T P_i + P_i A_i = -I$. For the small value of $n = 2$, this can be done by hand, but it is possible to make an algorithm to solve for higher dimensions. We obtain $x^T P_1 x$ and $x^T P_2 x$, where

$$P_1 = \begin{pmatrix} \frac{47}{84} & \frac{1}{84} \\ \frac{1}{84} & \frac{19}{42} \end{pmatrix}; \quad P_2 = \begin{pmatrix} \frac{19}{42} & -\frac{1}{84} \\ -\frac{1}{84} & \frac{47}{84} \end{pmatrix}. \tag{5.4}$$

By choosing Lyapunov functions this way, when we calculate λ_1, we are calculating the maximum eigenvalue of $P_1^{-1}(P_1 A_1 - A_1^{\mathrm{T}} P_1) = -P_1^{-1}$. We get eigenvalues of approximately -1.783 and -2.553, and we choose $\lambda_1 = -1.783$.

Now we calculate the νs. If we allow the possibility to switch from one subsystem to the same subsystem (causing only an impulse to be applied), we will find four values. We calculate $P_\alpha^{-1}(P_\beta B_\alpha + B_\alpha^{\mathrm{T}} P_\beta B_\alpha + B_\alpha^{\mathrm{T}} P_\beta + P_\beta - P_\alpha)$, where α is the initial subsystem, and β is the new subsystem. After some more calculation and choosing the largest eigenvalue each time, we obtain

$$(\nu_{\alpha\beta}) = \begin{pmatrix} -\frac{15}{16} & -0.9222 \\ 3.973 & 3 \end{pmatrix}. \tag{5.5}$$

The first observation is that switching from the subsystem with the destabilizing impulse leads to higher values (the values in the second row). More subtle is the observation that, when switching to the same system again, the value is lower than when switching to a different system (the diagonal elements are switches to the same system; the off diagonals are switches to the other system). This is because the value of ν has to take into account the worst case scenario of switching from one system to another, and the effect this has on the Lyapunov function.

In order to satisfy the stability requirements, using the rules that we established in Chapter 3, we should choose a switching rule such that $\mu_\alpha + \nu_{\alpha\beta} \leq 0$. This turns out to be a very strict requirement and is somewhat over prescriptive for the simple example that we are considering here. Using this requirement we would find that $\Delta t_2 = 2.23$ as derived from $\Delta t_2 \lambda_2 + \nu_{21} = -0.003 < 0$. Note that this is the solution to that equation that is the worst case corresponding to the largest possible value for $\nu_{\alpha\beta}$. The value of

Δt_1 can be anything, since it corresponds to the stable subsystem and impulse. The other requirement, $\mu_\alpha + \ln(1+\nu_{\alpha\beta}) < 0$ is less strict than the first requirement above, and, hence, is automatically satisfied for our chosen value of Δt_2.

The above requirements lead to a limiting value for $\Delta t_2 = 2.23$. However, if we perform a simulation with this value of Δt, the graphical result becomes effectively trivial, and it is difficult to see the stabilizing behaviour of the system. To better illustrate the stabilizing behaviour, we have chosen a value of $\Delta t_1 = \Delta t_2 = 0.05$, which is a much less strict dwell time condition. Figure 5.1 shows the results of running the system through a number of impulses, starting at $x_0^T = (0, 1)$. The system is clearly stable, with alternating stabilizing and destabilizing impulses, and, moreover, it is clear that a dwell time of 2.23 is much longer than is required.

For the example in Figure 5.1, we have set the ratio of stabilizing to destabilizing impulses at $q = 1$. It is instructive to increase this ratio to see the unstable behaviour. Figure 5.2 shows the same system with $q = 2$, which takes more time to spiral to the origin. Figure 5.3 plots the behaviour of the same system with $q = 3$. In this case, the system is not stable and spirals away from the origin, which is why the scales have been multiplied by a factor of approximately 1000. This method of showing how stability is related to q is also illustrated in the next example.

In fact, we can still get a stable solution for $q = 2$, but $q = 3$ fails.

Example 3. *This is a classic example of a switched system with stable subsystems, but switching modes that lead to instability. In this example, however, we will add an impulse to make the system stable for all average dwell times and, hence, for arbitrary switching.*

$$\begin{cases} \dot{x} = A_i x & t \neq t_k \\ \Delta x = B_i x & t = t_k \end{cases} ; \tag{5.6}$$

$$A_1 = \begin{pmatrix} -1 & -100 \\ 10 & -1 \end{pmatrix} \quad A_2 = \begin{pmatrix} -1 & -10 \\ 100 & -1 \end{pmatrix} \tag{5.7}$$

$$B_1 = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \quad B_2 = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \tag{5.8}$$

We find P_1 and P_2 by solving the linear equations, obtaining:

$$P_1 = \begin{pmatrix} \frac{551}{2002} & -\frac{45}{2002} \\ -\frac{45}{2002} & \frac{5501}{2002} \end{pmatrix} \quad P_2 = \begin{pmatrix} \frac{5501}{2002} & \frac{45}{2002} \\ \frac{45}{2002} & \frac{551}{2002} \end{pmatrix} \tag{5.9}$$

By choosing Lyapunov functions this way, when we calculate λ_1, we are calculating the maximum eigenvalue of $P^{-1}(PA - A^{\mathrm{T}}P) = -P^{-1}$. So, we get $\lambda_1 = \lambda_2 \approx -\frac{1}{2.748}$. In both cases, the eigenvalues of P_α are approximately 0.275 and 2.748. Hence, $\rho \approx \frac{2.748}{0.275} \approx 9.99$.

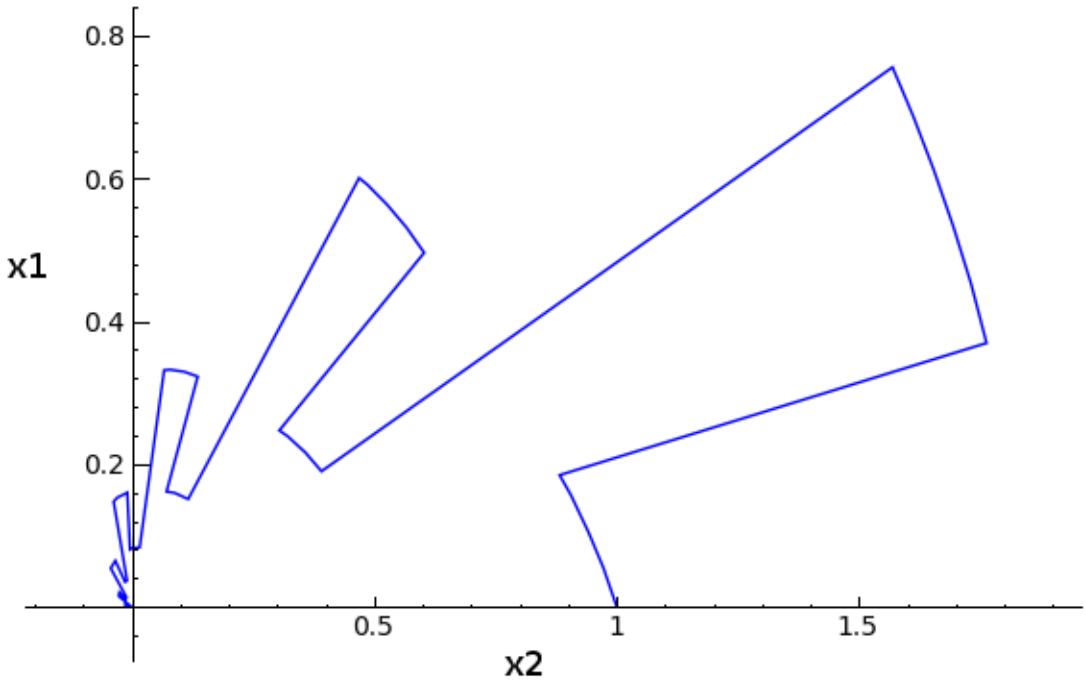

Figure 5.1: The system in Example 2 with $q = 1$.

Now we calculate the νs. We start by letting $b = -\frac{2}{3}$. For system 1, we calculate $P_1^{-1}(P_1B_1 + B_1^{\mathrm{T}}P_1B_1 + B_1^{\mathrm{T}}P_1)$ and obtain

$$\begin{pmatrix} -\frac{8}{9} & 0 \\ 0 & -\frac{8}{9} \end{pmatrix} \tag{5.10}$$

The answer is the same for system 2.

Therefore, $\nu_1 = \nu_2 = -\frac{8}{9}$. Hence, $\rho(1 + \nu_1) \approx 1.11 > 1$, but we are close. The impulse size is not enough to satisfy the condition for stability under arbitrary switching. Using $b = -0.68$, we get $\nu_1 = \nu_2 \approx 1.02$, and using $b = -0.7$, we get $\nu_1 = \nu_2 \approx 0.90$, which does satisfy the condition.

Figure 5.4 shows the system evolution for $b = -2/3$, which is unstable as expected, although the system oscillates rapidly. For the plots of this system, we plotted $\|x\|$. Although the large number of switches makes it impossible to distinguish the continuous trajectory from the impulses, we can get an idea of the rate of divergence. For $b = -0.7$ we get stability, as expected, which is confirmed in Figure 5.5. We also checked $b = -0.68$, which is predicted to be very close to the stability requirement. As shown in Figure 5.6, this system is also stable, indicating that the stability requirement for this system is actually a little stricter than necessary. The reason for this behaviour is that we assume the

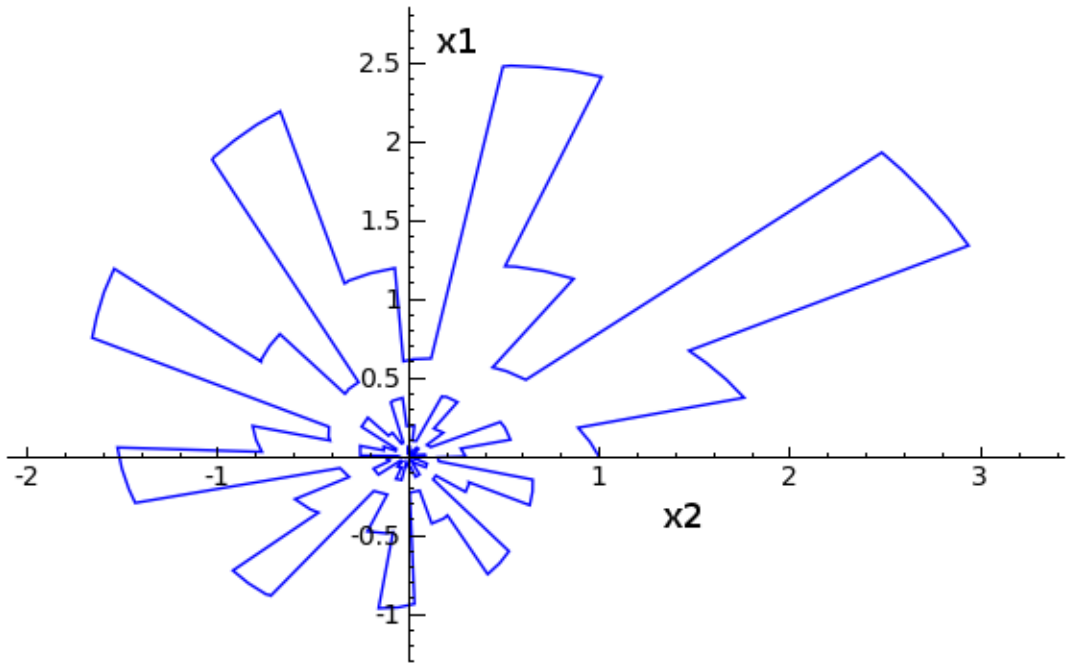

Figure 5.2: The system in Example 2 with $q = 2$.

systems stabilize very slowly with respect to their Lyapunov functions, so that the right switching will cause $x^{\mathrm{T}}x$ to increase by a factor of ρ in each time interval. In reality, the stable subsystems do not allow an increase this high.

Example 4. *Consider one more linear example*

$$\begin{aligned} \dot{x} &= A_{\sigma(t,x)}x, \qquad t \neq t_k \\ \Delta x &= B_{\sigma(t,x)}x, \quad t = t_k, \end{aligned} \tag{5.11}$$

$$A_1 = \begin{bmatrix} \frac{1}{4} & -1 \\ 1 & \frac{1}{4} \end{bmatrix}, \; A_2 = \begin{bmatrix} -\frac{1}{3} & 1 \\ -1 & -\frac{1}{3} \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} -\frac{3}{4} & 0 \\ 0 & -\frac{3}{4} \end{bmatrix}.$$

It is easy to see that the underlying continuous system is an unstable focus. Take

$$P_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \; P_2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}.$$

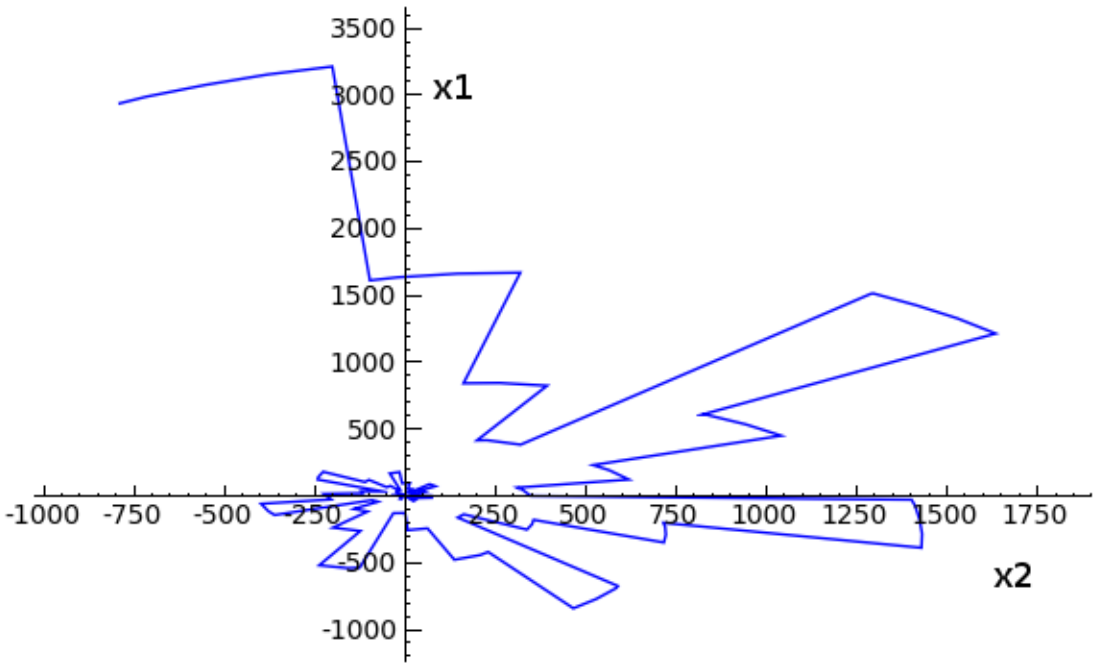

Figure 5.3: The system in Example 2 with $q = 3$.

Then we have

$$P_1^{-1}(A_1'P_1 + P_1A_1) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$P_2^{-1}(A_2'P_2 + P_2A_2) = \begin{bmatrix} -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} \end{bmatrix}$$

$$P_1^{-1}(B_1'P_1 + B_1'P_1B_1 + P_1B_1 + P_1 - P_1) = \begin{bmatrix} -0.9375 & 0 \\ 0 & -0.9375 \end{bmatrix}$$

$$P_1^{-1}(B_1'P_2 + B_1'P_2B_1 + P_2B_1 + P_2 - P_1) = \begin{bmatrix} -0.9688 & 0 \\ 0 & -0.9688 \end{bmatrix}$$

$$P_2^{-1}(B_2'P_1 + B_2'P_1B_2 + P_1B_2 + P_1 - P_2) = \begin{bmatrix} -0.8750 & 0 \\ 0 & -0.8750 \end{bmatrix}$$

$$P_2^{-1}(B_2'P_2 + B_2'P_2B_2 + P_2B_2 + P_2 - P_2) = \begin{bmatrix} -0.9375 & 0 \\ 0 & -0.9375 \end{bmatrix}$$

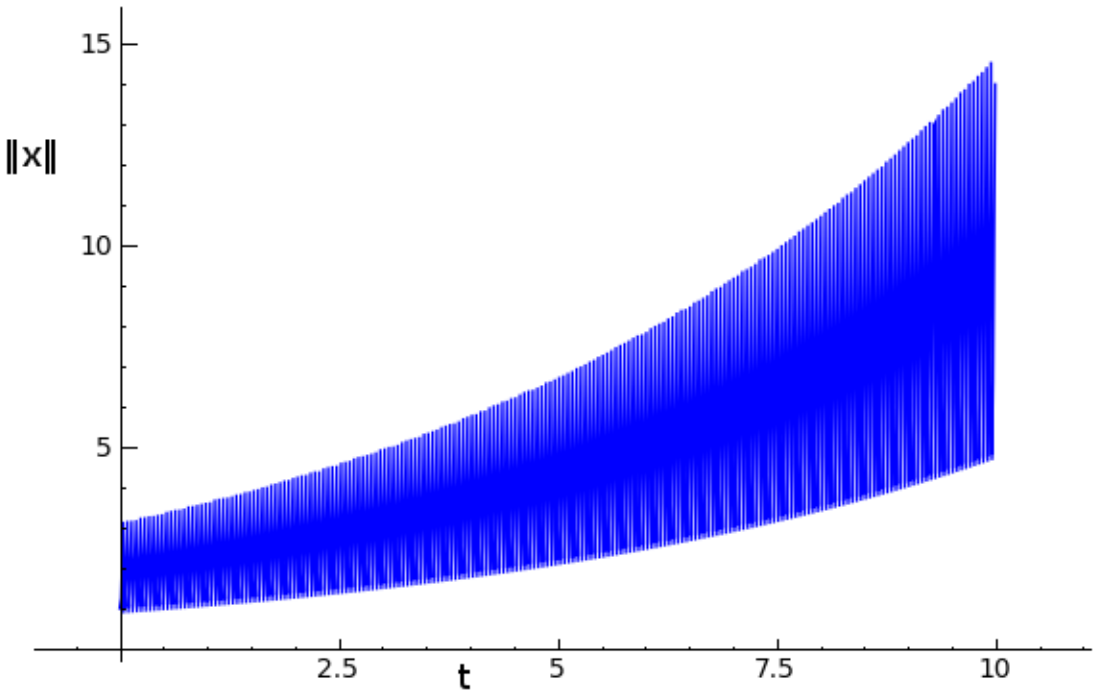

Figure 5.4: Example 3 with $b = -2/3$

If take $\Delta t_\alpha = 1$, then we can choose $\mu_1 = 0.5$, $\mu_2 = -0.6$, $\nu_{1,1} = -0.93$, $\nu_{1,2} = -0.96$, $\nu_{2,1} = -0.87$, $\nu_{2,2} = -0.93$, $\gamma_1 = \gamma_2 = 2$, then all the conditions of Theorem 1 in Section 3 are satisfied. Hence the zero solution of the system is stable. For this system, we have plotted the subsystems separately. Figure 5.7 shows the unstable subsystem with two curves for x_1 and x_2. Likewise, Figure 5.8 shows the stable subsystem. In this case, the switched system without impulses is stable, as shown in Figure 5.9, but the impulsive switches system converges more rapidly in Figure 5.10.

5.2 Nonlinear Systems

Example 5. *Consider the following example*

$$\begin{aligned} \dot{x} &= A_{\sigma(t,x)}x + f_{\sigma(t,x)}(t,x), \quad && t \neq t_k \\ \Delta x &= B_{\sigma(t,x)}x, && t = t_k, \end{aligned} \tag{5.12}$$

$$A_1 = \begin{bmatrix} \frac{1}{4} & -1 \\ 1 & \frac{1}{4} \end{bmatrix}, \; A_2 = \begin{bmatrix} -\frac{1}{3} & 1 \\ -1 & -\frac{1}{3} \end{bmatrix},$$

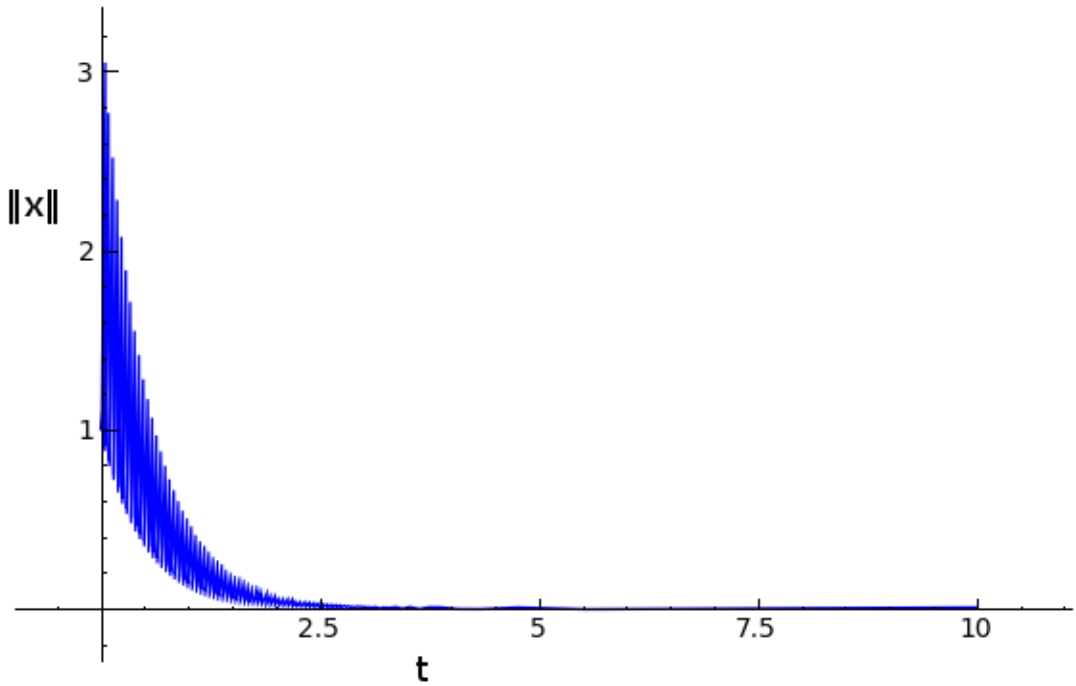

Figure 5.5: Example 3 with $b = -0.7$

$$f_1 = \begin{bmatrix} \frac{1}{8}x_1 \sin x_2 \\ \frac{1}{8}x_2 \cos x_2 \end{bmatrix}, \ f_2 = \begin{bmatrix} -\frac{1}{6}x_1 \cos x_2 \\ -\frac{1}{6}x_2 \sin x_1 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} -\frac{3}{4} & 0 \\ 0 & -\frac{3}{4} \end{bmatrix}.$$

It is easy to see that the underlying continuous system is an unstable focus. Let $V_1(x) = \frac{1}{2}(x_1^2 + x_2^2)$, $V_2(x) = \frac{1}{4}(x_1^2 + x_2^2)$. Then

$$V_1'(x) = x_1(\frac{1}{4}x_1 - x_2 + \frac{1}{8}x_1 \sin x_2) + x_2(x_1 + \frac{1}{4}x_2 + \frac{1}{8}x_2 \cos x_2) \leq \frac{3}{4}V(x), \ t \neq t_k;$$

$$V_2'(x) = \frac{1}{2}x_1(-\frac{1}{3}x_1 + x_2 - \frac{1}{6}x_1 \cos x_2) + \frac{1}{2}x_2(-x_1 - \frac{1}{3}x_2 - \frac{1}{6}x_2 \sin x_1) \leq -\frac{1}{3}V(x), \ t \neq t_k;$$

$$V_1(x^+) - V_1(x) = \frac{1}{2}[(x_1 - \frac{3}{4}x_1)^2 + (x_2 - \frac{3}{4}x_2)^2] - \frac{1}{2}(x_1^2 + x_2^2) = -\frac{15}{16}V_1(x)$$

$$V_2(x^+) - V_1(x) = \frac{1}{4}[(x_1 - \frac{3}{4}x_1)^2 + (x_2 - \frac{3}{4}x_2)^2] - \frac{1}{2}(x_1^2 + x_2^2) = -\frac{31}{32}V_1(x)$$

$$V_1(x^+) - V_2(x) = \frac{1}{2}[(x_1 - \frac{3}{4}x_1)^2 + (x_2 - \frac{3}{4}x_2)^2] - \frac{1}{4}(x_1^2 + x_2^2) = -\frac{7}{8}V_2(x)$$

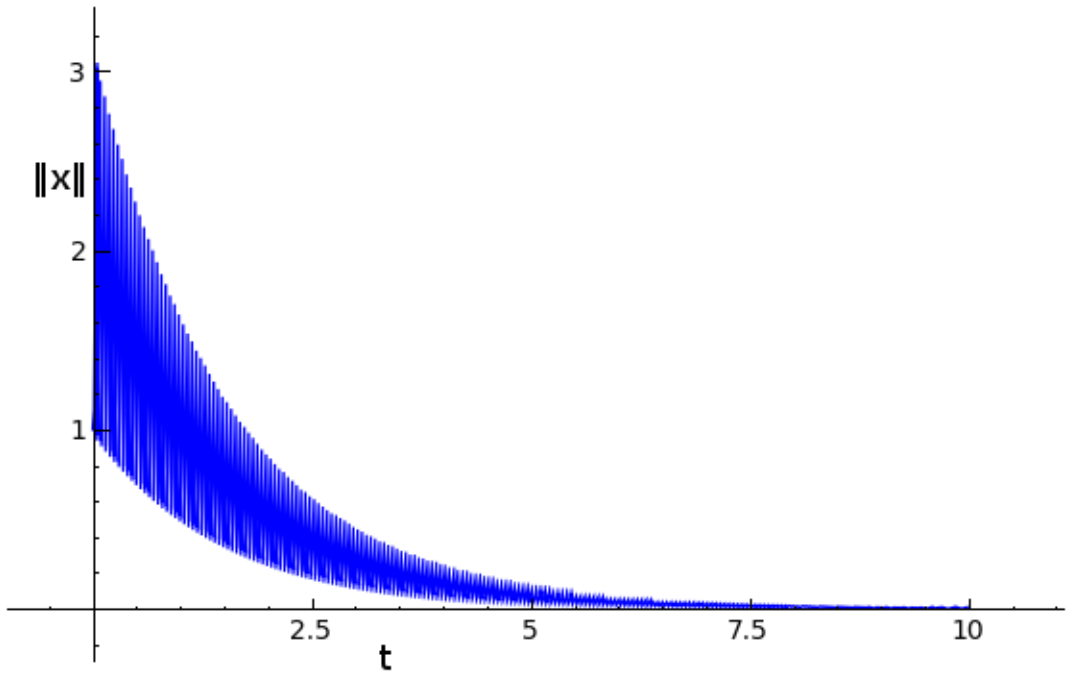

Figure 5.6: Example 3 with $b = -0.68$

$$V_2(x^+) - V_2(x) = \frac{1}{4}[(x_1 - \frac{3}{4}x_1)^2 + (x_2 - \frac{3}{4}x_2)^2] - \frac{1}{4}(x_1^2 + x_2^2) = -\frac{15}{16}V_2(x)$$

If take $\Delta t_\alpha = 1$, then we can choose $\mu_1 = 0.75$, $\mu_2 = -0.3$, $\nu_{1,1} = -0.93$, $\nu_{1,2} = -0.96$, $\nu_{2,1} = -0.87$, $\nu_{2,2} = -0.93$, $C_1(s) = C_2(s) = d_1(s) = d_2(s) = s$, $\forall s \in R$, l_1 and l_2 be any positive real numbers, $\gamma_1 = 2$, $\gamma_2 = 2.2$ then all the conditions of Theorem 6 in Chapter 5 are satisfied. Hence the zero solution of the system is uniformly asymptotically stable. The simulation result can be seen in Figure 5.11. The nonlinear system.

5.3 Evaluation of results

The above examples serve to illustrate some of the shortcomings of the sufficient conditions that have been developed. Example 2 shows that the requirements are far too strict for certain systems. The system we chose only had a destabilizing impulse. Although it would be possible to destabilize the system through a choice of switching times, a dwell time of 2.23 is far too strict. There is no need for a dwell time in such a system that would require the trajectory to orbit the origin more than once. This occurred partly because the

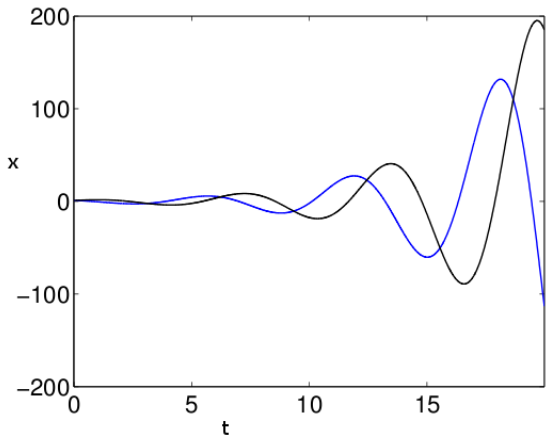

Figure 5.7: The trajectory of the first subsystem of Example 4

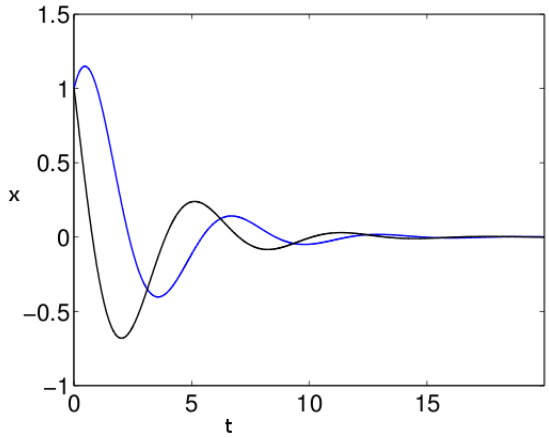

Figure 5.8: The trajectory of the second subsystem of Example 4

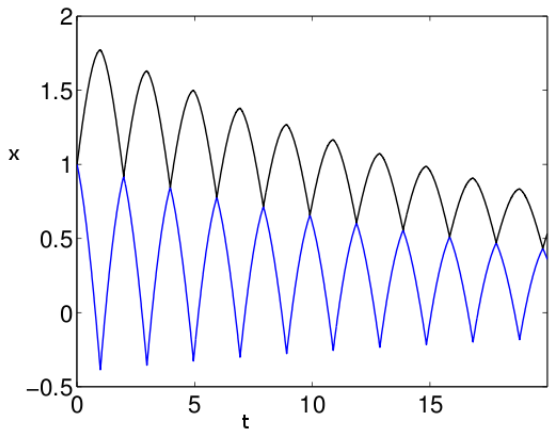

Figure 5.9: The trajectory of the switched system in Example 4, with no impulses

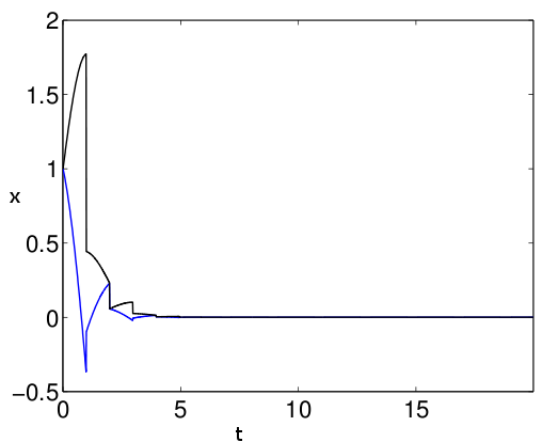

Figure 5.10: Evolution of the states in Example 4 with impulses.

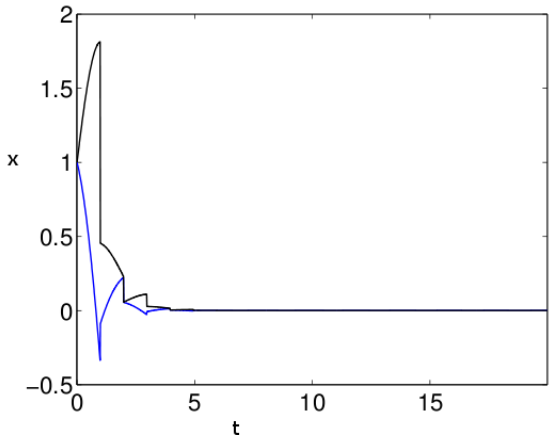

Figure 5.11: Evolution of the states for Example 5.

example is not suited to stability criteria that require the Lyapunov function to decrease for each interval, and partly because the criteria are simply stricter than they need to be.

Example 3 uses an alternative method that works much better for the linear system. This method works because we can keep track of increases and decreases to $\|x\|$ across different intervals. The size of the impulse that was found was very nearly the smallest that would lead to stability under the switching rule that destabilized the switched system without impulses.

The last two examples show how the criteria can work a little better for a system with a nonlinear subsystem as the destabilizing element. It is also a slightly nonlinear example, although this does not change the graphical result much, as can be seem by comparing Figures 5.10 and 5.11.

Chapter 6

Conclusions and Future Work

6.1 Conclusions

Having completed this project, we have found sufficient criteria for stability and asymptotic stability of linear impulsive switched systems which work in the context of stability using two measures. We have also found similar criteria that apply to nonlinear systems, covering stability, uniform stability, and uniform asymptotic stability. Since these results both use a multiple Lyapunov function approach, traditional methods for obtaining Lyapunov functions still apply. Using illustrative examples, we showed that these results can be easily verified, although a particular set of criteria may fail to be satisfied for certain systems with stable dynamics.

6.2 Further Study

There are many different directions to take with this research, due to the wide variety of results and techniques available for the study of either hybrid systems or (h_0, h)-stability. We could study hybrid systems with time delay, random variables, or different classes of switching behaviour. Rather than name all the possibilities, we will just mention some of the more immediate possible extensions to this work.

Different Stability Results

We only covered the most basic stability results for linear and nonlinear systems. In particular, we have not covered instability and non-uniform asymptotic stability for nonlinear systems. We could also cover instability for linear systems. Since we have found sufficient

conditions, it is important to compare what is sufficient for instability to the present results, to see how much leeway there is for systems whose stability properties cannot be determined by these methods.

Robustness is also an important aspect of stability. There are several different approaches to consider, including linear quadratic inequalities, used in [25] as an extension to the simple quadratic Lyapunov functions applied to a situation where uncertainty is involved. We could also consider the work in [43], which looks at robust stability under the constraint of a cost function as well.

LaSalle's Invariance Principle

There has been some study of LaSalle's invariance principle and how it applies to switched systems in [7], [37], and [3]. These methods provide an extra tool in determining stability because a Lyapunov function that does not strictly decrease can still be helpful in proving asymptotic stability. It is also possible to use this tool when considering two measure stability, as demonstrated in [27].

Bibliography

[1] M.S. Alwan and X. Liu. Stability of singularly perturbed switched systems with time delay and impulsive effects. *Nonlinear Analysis*, 71(9):4297–4308, 2009. 11

[2] A. Bacciotti and L. Mazzi. From artstein–sontag theroem to the min-projection strategy. *Transactions of the Institute of Measurement and Control OnlineFirst*, doi:10.1177/0142331208095427, 2009.

[3] A. Bacciotti and L. Mazzi. An invariance principle for nonlinear switched systems. *Systems and Control Letters*, 2005. 69

[4] A. Bemporad, G. Ferrari-Trecate, and M. Morari. Observability and controllability of piecewise affine and hybrid systems. *IEEE Transactions on Automatic Control*, 45(10):1864–1876, 2000.

[5] Vincent D. Blondel and Jown N. Tsitsiklis. The boundedness of all products of a pair of matrices is undecidable. *Systems and Control Letters*, 2000. 12, 19

[6] M.S. Branicky. Stability of switched and hybrid systems. In *IEEE conference on decision and control*, volume 4, pages 3498–3498. INSTITUTE OF ELECTRICAL ENGINEERS INC (IEE), 1994.

[7] Daizhan Cheng, Jinhuan Wang, and Xiaoming Hu. An estension of lasalle's invariance principle and its application to multi-agent consensus. *IEEE Transactions on Automatic Control*, 2008. 69

[8] Ming-Li Chiang and Li-Chen Fu. Variable Structure Based Switching Adaptive Control for a Class of Unknown Switched Linear Systems. In *2009 AMERICAN CONTROL CONFERENCE, VOLS 1-9*, PROCEEDINGS OF THE AMERICAN CONTROL CONFERENCE, pages 3019–3024, 345 E 47TH ST, NEW YORK, NY 10017 USA, 2009. IEEE. American Control Conference 2009, St Louis, MO, JUN 10-12, 2009.

[9] Gustav S. Christensen. A simple and fast algorithm for determining asymptotic stability of linear autonomous systems. *Canadian Journal of Electrical and Computer Engineering*, 2003.

[10] Gustav S. Christensen. Uniform asymptotic stability of linear non-autonomous systems. *Canadian Journal of Electrical and Computer Engineering*, 2003. 6

[11] J. E. R. Cury, B. H. Krogh, and T. Niinomi. Synthesis of supervisory controllers for hybrid systems based on approximating automata. *IEEE Transactions on Automatic Control*, 1998. 2

[12] S. S. Ge and Zhendong Sun. Switched controllability via bumpless transfer input and constrained switching. *IEEE Transactions on Automatic Control*, 2008. 2

[13] Zhi-Hong Guan, David J. Hill, and Xuemin Shen. On hybrid impulsive and switching systems and application to nonlinear control. *IEEE Transactions on Automatic Control*, 2005. 12, 32, 39

[14] Zhi-Hong Guan, David J. Hill, and Jing Yao. A hybrid impulsive and switching control strategy for synchronization of nonlinear systems and application to chua's chaotic circuit. *International Journal of Bifurcation and Chaos*, 2006.

[15] Z. Guang-Deng and WU Yu-Qiang. Robust Exponential Stability of Discrete Time Perturbed Impulsive Switched Systems. *ACTA AUTOMATICA SINICA*, 2006. 11

[16] H.K. Khalil and JW Grizzle. *Nonlinear systems*. Prentice hall Upper Saddle River, NJ, 1996. 5

[17] V. Lakshmikantham, D. Bainov, and P.S. Simeonov. *Theory of impulsive differential equations*. World Scientific, 1989. 11

[18] V. Lakshmikantham and X. Z. Liu. *Stability Analysis in terms of Two Measures*. World Scientific, River Edge, NJ, 1993. 3, 12, 13, 15

[19] V. Lakshmikantham and X.Z. Liu. Impulsive hybrid systems and stability theory. *Dynamic Systems and Applications*, 7(1):1–10, 1998. 11

[20] Xiaoxin Liao, Liqiu Wang, and Pei Yu. *Stability of dynamical systems*. Elsevier, 2007. 9

[21] D. Liberzon. *Switching in systems and control*. Springer, 2003. 10

[22] H. Lin and P. J. Antsaklis. Stability and persistent disturbance attenuation properties for a class of networked control systems: switched system approach. *International Journal of Control*, 78(18):1447–1458, 2005.

[23] Hai Lin and Panos J. Antsaklis. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Transactions on Automatic Control*, 2009. 10, 12, 21, 24

[24] Bin Liu and Horacio J. Marquez. Razumikhin-type stability theorems for discrete delay systems. *Automatica*, 2007.

[25] Duyu Liu, Xinzhi Liu, and Shouming Zhong. Delay-dependent robust stability and control synthesis for uncertain switched neutral systems with mixed delays. *Applied Mathematics and Computation*, 2008. 69

[26] Jun Liu, Xinzhi Liu, and Wei-Chau Xie. Delay-dependent robust control for uncertain switched systems with time-delay. *Nonlinear Analysis: Hybrid Systems*, 2008. 11

[27] Xinzhi Liu. On (h_0, h)-stability of autonomous systems. *Journal of Applied Mathematics and Stochastic Analysis*, 1992. 69

[28] Xinzhi Liu. Further extensions of the direct method and stability of impulsive systems. *Nonlinear World*, 1994. 12, 13, 14, 16, 22

[29] Xinzhi Liu and Hongtao Zhang. Robust stability of impulsive switched systems with disturbance. *ANZIAM J.*, 2006. 12

[30] Yubin Liu and Weizhen Feng. Razumikhin-Lyapunov functional method for the stability of impulsive switched systems with time delay. *Math. Comput. Modelling*, 49(1-2):249–264, 2009. 10

[31] A.M. Lyapunov. *The General Problem of Stability of Motion.* Fizmatgiz, Moscow, 1950. 9

[32] Richard K. Miller and Anthony N. Michel. *Ordinary differential equations.* Academic press, 1982. 5

[33] Gilles Millerioux and Jamal Daafouz. An observer-based approach for input-independent global chaos synchronization of discrete-time switched systems. *IEEE Transactions on Circuits and Systems—I: Fundamental Theory and Applications*, 2003.

[34] Sayan Mitra and Daniel Liberzon. Stability of hybrid automata with average dwell time: An invariant approach. In *43rd IEEE Conference on Decision and Control.* 11

[35] A. Nerode and W. Kohn. Models for hybrid systems: Automata, topologies, controllability, observability. *Hybrid systems*, pages 317–356, 1993.

[36] X. Nicollin, A. Olivero, J. Sifakis, and S. Yovine. An approach to the description and analysis of hybrid systems. *Hybrid Systems*, pages 149–178, 1993.

[37] Ricardo G. Sanfelice, Rafal Goebel, and Andrew R. Teel. Invariance principles for hybrid systems with connections to detectability and asyptotic stability. *IEEE Transactions on Automatic Control*, 2007. 69

[38] Andrey V. Savkin and Robin J. Evans. *Hybrid Dynamical Systems: Controller and Sensor Switching Problems*. Birkhäuser, Boston, 2002. 2, 10

[39] Willian J. Terrell. *Stability and Stabilization*. Princeton University Press, Princeton, NJ, 2009. 6, 7

[40] Arjan van der Schaft and Hans Schumacher. *An Introduction to Hybrid Dynamical Systems*. Springer-Verlag, 2000. 2

[41] Zhengrong Xiang and Ronghao Wang. Robust l_∞ reliable control for uncertain nonlinear switched systems with time delay. *Applied Mathematics and Computation*, 2009. 10

[42] H. Xu, X. Liu, and K.L. Teo. A LMI approach to stability analysis and synthesis of impulsive switched systems with time delays. *Nonlinear Analysis: Hybrid Systems*, 2(1):38–50, 2008. 11

[43] Honglei Xu, Kok Lay Teo, and Xinzhi Liu. Robust stability analysis of guaranteed cost control for impulsive switched systems. *IEEE Transactions on Systems, Man, and Cybernetics—Part B: Cybernetics*, 2008. 69

[44] C. Yang and W. Zhu. Stability analysis of impulsive switched systems with time delays. *Mathematical and Computer Modelling*, 50(7-8):1188–1194, 2009. 11

[45] H. Ye, A.N. Michel, and L. Hou. Stability theory for hybrid dynamical systems. *IEEE Transactions on Automatic Control*, 43(4):461–474, 1998.

[46] A. A. Zevin and Mark A. Pinsky. General solution of stability problem for plane linear switched systems and differential inclusions. *IEEE Transaction Automatic Control*, 2008.

[47] Liguo Zhang, Ying Lu, Yangzhou Chen, and Nikos E. Mastorakis. Robust uniformly ultimate boundedness control for uncertain switched systems. *Computers and Mathematics with Applications*, 2008.