Cooperative Protocols for Relay and Interference Channels with Half-Duplex Constraint

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Electrical and Computer Engineering

Waterloo, Ontario, Canada, 2010

AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Hossein Bagheri

Abstract

Enabling cooperation among nodes of a wireless network can significantly reduce the required transmit power as well as the induced intra-network interference. Due to the practical half-duplexity constraint of the cooperating nodes, they are prohibited to simultaneously transmit and receive data at the same time-frequency resource. The purpose of this dissertation is to illustrate the value of cooperation in such an environment. To understand how to cooperate efficiently, information theory is employed as a useful tool, which not only determines the fundamental limits of communication (i.e., capacity) over the considered network, but also provides insights into the design of a proper transmission scheme for that network.

In this thesis, two simple but yet important types of wireless networks, namely *Relay Channel*, and *Interference Channel* are studied. In fact, these models constitute building blocks for larger networks. The first considered channel is a diamond-shaped relay channel consisting of a source, a destination, and two parallel relays. The second analyzed channel is an interference channel composed of two transmitter-receiver pairs with out-of-band transmitter cooperation, also referred to as conferencing encoders. While characterizing the capacity of these channels are difficult, a simpler and a more common approach is to find an achievable scheme for each channel that ensures a small gap from the capacity for all channel parameters.

In chapter 2, the diamond relay channel is investigated in detail. Because of the half-duplex nature of the relays, each relay is either in transmit or receive mode, making four modes possible for the two-relay combination, specifically, 1) broadcast mode (both relays receive) 2,3) routing modes (one relay transmits, another receives) 4) multiple-access mode (both relays transmit). An appropriate scheduling (i.e., timing over the modes) and transmission scheme based on the decode-and-forward strategy are proposed and shown to be able to achieve either the capacity for certain channel conditions or at most 3.6

bits below the capacity for general channel conditions. Particularly, by assuming each transmitter has a constant power constraint over all modes, a parameter Δ is defined, which captures some important features of the channel. It is proven that for $\Delta = 0$ the capacity of the channel can be attained by successive relaying, i.e., using modes 2 and 3 defined above in a successive manner. This strategy may have an infinite gap from the capacity of the channel when $\Delta \neq 0$. To achieve rates as close as 0.71 bits to the capacity, it is shown that the cases of $\Delta > 0$ and $\Delta < 0$ should be treated differently. Using new upper bounds based on the dual problem of the linear program associated with the cut-set bounds, it is proven that the successive relaying strategy needs to be enhanced by an additional broadcast mode (mode 1), or multiple access mode (mode 4), for the cases of $\Delta < 0$ and $\Delta > 0$, respectively. Furthermore, it is established that under average power constraints the aforementioned strategies achieve rates as close as 3.6 bits to the capacity of the channel.

In chapter 3, a two-user Gaussian Interference Channel (GIC) is considered, in which encoders are connected through noiseless links with finite capacities. The setup can be motivated by downlink cellular systems, where base stations are connected via infrastructure backhaul networks. In this setting, prior to each transmission block the encoders communicate with each other over the cooperative links. The capacity region and the sum-capacity of the channel are characterized within some constant number of bits for some special classes of symmetric and Z interference channels. It is also established that properly sharing the total limited cooperation capacity between the cooperative links may enhance the achievable region, even when compared to the case of unidirectional transmitter cooperation with infinite cooperation capacity. To obtain the results, genie-aided upper bounds on the sum-capacity and cut-set bounds on the individual rates are compared with the achievable rate region. The achievable scheme enjoys a simple type of Han-Kobayashi signaling, together with the zero-forcing, and basic relaying techniques.

Acknowledgements

First and foremost, I wish to express my deep gratitude to my supervisor, Professor Amir K. Khandani for providing guidance and inspiration throughout the course of my PhD studies. I have benefited tremendously from his experience and broad knowledge. I also want to thank Amir for the many great opportunities he created for me.

I would like to thank the members of my dissertation committee, Professors Wei Yu, Murat Uysal, Mohamed Oussama Damen, Mary Thompson, and Dana Kulic for taking the time out of their busy schedules to carefully review my thesis, providing me with insightful questions and suggestions.

I was fortunate to collaborate with Seyed Abolfazl Motahari who is an incredible source of knowledge, encouragement, support, and friendship. I have enjoyed sharing ideas and discussing with him.

I offer my regards and blessings to all members of the Coding and Signal Transmission (CST) Lab, particularly Seyed Abolfazl Motahari, Masoud Ebrahimi, Jamshid Abouei, Ghadamali Bagherikaram, Vahid Pourahmadi, Behzad Nourani, and Pooya Mahboobi, who supported me in all respects during the completion of the project.

Further, I am grateful to my friends, Abolfazl Maneshi, Hamid Mohebbi, Ali Ahmadzadeh, and Behzad Behravesh for their advice, time and support.

Finally, and most importantly, I thank my parents, for their endless love and support, without which I would not have succeeded. My special thanks and sincere appreciation are extended to the love of my life, Hajar Sharif, my wife, for all her love and understanding.

Dedications

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To my parents:

Pari Mazaheri and Ahmad Bagheri

and

To my beloved wife:

Hajar Sharif

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List of Abbreviations

IC	Interference Channel
GIC	Gaussian Interference Channel
GZIC	Gaussian Z-Interference Channel
MAC	Multiple Access Channel
BC	Broadcast Channel
AWGN	Additive White Gaussian Noise
HK	Han and Kobayashi
GDOF	Generalized Degrees Of Freedom
MIMO	Multiple-Input Multiple-Output
MISO	Multiple-Input Single-Output
SNR	Signal-to-Noise Ratio
INR	Interference-to-Noise Ratio
MG	Multiplexing Gain

Notation

Boldface Upper-Case Letters	Matrices
Boldface Lower-Case Letters	Vectors
$\mathbf{A}^{\dagger}$	Hermitian transpose of A
$\lvert\mathbf{A}\rvert$	Determinant of the matrix $\mathbf{A}$
$\mathbb{E}[X]$	The expectation of the random variable X
$\bar{x}$	$1-x$
x^{*}	The optimal solution to an optimization problem with an objective function $F(x)$
$a \Rightarrow b$	The link from node a to node b
$x \leftrightarrow y$	The roles of x and y are exchanged in a given function $F(x,y)$
$\log(.)$	Logarithm function to base 2
$\mathcal{C}(x)$	$\frac{1}{2}\log(1+x)$
$h(x)$	entropy of the random variable x If x is continuous, h is the differential entropy
$\mathcal{N}(m,\sigma^2)$	Gaussian distribution with mean m and variance σ^2
$A_{\epsilon}^{(n)}(x^n,y^n)$	The set of ϵ jointly typical sequences of random variables (x^n,y^n), with length n(for the definition, see [28]). Whenever it is clear from the context, the random variables may be omitted from the notation.

Chapter 1

Introduction

Wireless networks play an increasingly indispensable role in modern people's lives. Addressing the high data-rate demand of the network subscribers requires an understanding of the ultimate limits of achievable rates, described by the information-theoretic capacity, in that network. In addition, knowing the important features of a close-to-optimum achievable scheme is essential to fulfill high data-rate requirements. In a network, substantial capacity improvements can be attained by forming a distributed Multi-Input Multi-Output (MIMO) system through enabling cooperation among the users.

Cooperative communication can be implemented either by utilizing additional signal paths that share the medium with the direct transmitter-receiver links or by exploiting the (possibly) available orthogonal resources. In the former case, the broadcast nature of the wireless communication, wherein some intermediate nodes can overhear the signal intended for a receiver, is the key ingredient that makes cooperation possible. However, due to the dynamic range of incoming and outgoing signals and the bulk of ferroelectric components like circulators, nodes are assumed to operate in the *half-duplex* mode, i.e., transmit and receive in different time-frequency resources [12]. Depending on which nodes are in the transmit and which are in the receive modes, different network operating modes are present which need to be properly scheduled for an efficient communication. The *diamond relay channel*, containing a source, a destination, and two parallel relays, is a simple example of such a network that is considered in this thesis. For the latter scenario, some dedicated resources are present for cooperation. In downlink cellular systems for example, base stations are connected via infrastructure backhaul networks [49]. The configuration can be modeled as an interference channel with encoders connected to each other with noise-free finite-capacity links. This type of cooperation is referred to as *conference* [34]. To obtain some insights into the proper achievable scheme for such a network, a two-user interference channel with conferencing encoders is considered in this thesis.

Characterizing the capacity of an information theoretic channel may be difficult. In fact, the capacity is known for only a few number of channels, including the multiple-access and the broadcast channels. A simpler, yet important approach is to find an achievable scheme that ensures a small gap from the capacity of the channel. Recently, Etkin *et al.* characterized the capacity region of the interference channel to within one bit [26]. This new capacity analysis perspective has been used afterwards to reveal the essential ingredients of a proper achievable scheme for different networks (cf. [17, 46–50]).

Motivated by the above considerations, this thesis aims to characterize the capacity of the considered channels, that can serve as the building blocks of larger networks, to within a few number of bits for all ranges of parameters.

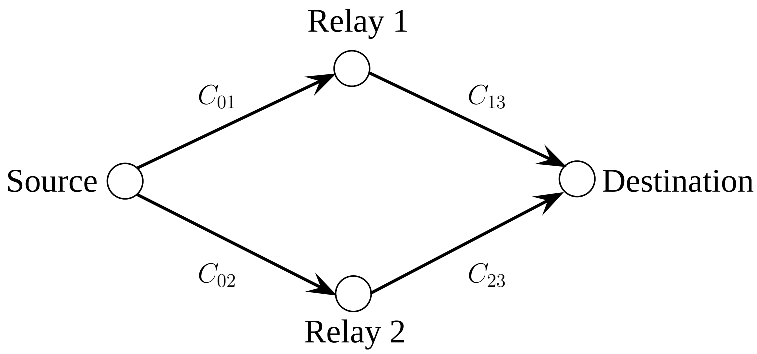
1.1 Diamond Relay Channel

1.1.1 Motivation

Relay-aided wireless systems, also called multi-hop systems, are implemented to increase the coverage and the throughput of communication systems [1]. These systems are becoming important parts of developing wireless communication standards, such as IEEE 802.16j (also known as WiMAX) [2].

From information theoretical point of view, the capacity becomes larger when more relays are added to the system. However, designing optimum strategies, especially in half-duplex systems, is challenging because subtle scheduling, i.e. timing among transmission modes is required to achieve rates near the capacity of such systems.[1] During the last decade, the main stream of research carried out by several researchers dealt with single relay communication systems (cf. [3] and references therein). A simple model for investigating the potential benefits of a system with multiple relays is a dual-hop configuration with two parallel half-duplex relays (see Fig. 1.1). This configuration does not cover all two-relay systems because there are no source-destination and relay-relay links. However, it captures the basic difficulty in finding the best strategy in the system. As will be shown in this

[1] In this thesis, the terms 'transmission modes' and 'network operating modes' are used interchangeably.



$$\Delta \triangleq C_{01}C_{02} - C_{13}C_{23}$$

Figure 1.1: The diamond channel with its fundamental parameter Δ.

thesis, a single strategy falls short of achieving rates near the capacity of the system for all channel realizations.

1.1.2 Prior Works

The single relay channel in which the relay facilitates a point-to-point communication was first studied in [4]. Two important coding techniques, *decode-and-forward* and *compress-and-forward*, were proposed in [5]. In the decode-and-forward scheme, the relay decodes the received message. In the compress-and-forward scheme, the relay sends the compressed (quantized) version of the received data to the destination. Following [5], generalizations to multi-relay networks were investigated by several researchers. A comprehensive survey of the progress in this area can be found in [3].

A simple model for understanding some aspects of the multi-relay networks is a network with two parallel relays, as introduced in [6, 7], and Fig. 1.1. It is assumed that there are no direct links from the source to the destination and also between the relays. This channel is studied in [8–17], and referred to as the *diamond relay channel* in [12].

For full-duplex relays, Schein and Gallager, in [6] and [7], provided upper and lower bounds on the capacity of the diamond channel. In particular, they considered the *amplify-and-forward*, and the decode-and-forward schemes, as well as a hybrid of them based on the time-sharing principle. Kochman, *et al.* proposed a *rematch-and-forward* scheme when different fractions of bandwidth can be allotted to the first and second hops [8]. Rezaei, *et al.* suggested a *combined amplify-and-decode-forward* strategy and proved that their scheme always performs better than the rematch-and-forward scheme [9]. In addition, they showed that the time-sharing between the combined amplify-and-decode-forward and decode-and-forward schemes provides a better achievable rate when compared to the time-sharing between the amplify-and-forward and decode-and-forward, and also between the rematch-and-forward and decode-and-forward, considered in [7], and [8], respectively. Kang and Ulukus employed a combination of the decode-and-forward and compress-and-forward schemes to obtain the capacity of a special class of the diamond channel with a noiseless relay [10]. Ghabeli and Aref in [11] proposed a new achievable rate based on the generalized block Markov encoding [23]. They also showed that their scheme achieves the capacity of a class of deterministic relay networks.

Half-duplex relays are studied in [12–20]. Xue and Sandhu in [12] proposed several schemes including the multi-hop with spatial reuse, scale-forward, broadcast-multiaccess with common message, compress-and-forward, and hybrid methods. These authors demonstrated that the *multi-hop with spatial reuse* protocol can achieve the channel capacity if the parallel links have the same capacity. Unlike [6–10, 12, 17], which assumed no direct link exists between the relays, [13–15] considered such a link. More specifically, Chang, *et al.* proposed a combined dirty paper coding and block Markov encoding scheme [14]. Using numerical examples, they showed that the gap between their proposed strategy and the upper bound is relatively small in most cases. Rezaei, *et al.* considered two scheduling algorithms, namely successive and simultaneous relaying [15]. They derived asymptotic capacity results for the successive relaying and also proposed an achievable rate for the simultaneous relaying using a combination of the amplify-and-forward and decode-and-forward schemes. Avestimehr *et al.* proposed a deterministic model to better analyze the general single-source single-destination and the single-source multi-destination Gaussian networks [16, 17]. Their *quantize-and-map* achievablity scheme is guaranteed to provide a rate that is within a constant number of bits (determined by the graph topology of the network) from the cut-set upper bound. Other related works can be found in [18–21].

1.1.3 Summary of Contributions and Relation to Previous Works

In this thesis, for the diamond relay channel, the setup and assumptions used in [12], with no link between the relays, are followed. In [12], the multi-hop with spatial reuse scheme proved to achieve the capacity of the diamond channel if the capacities of the parallel links in Fig. 1.1 are equal. The scheme is called the *Multi-hopping Decode-and-Forward* (MDF) scheme in this thesis. In the MDF scheme, relays successively forward their decoded messages to the destination. By introducing a fundamental parameter of the channel Δ (see Fig. 1.1), we generalize the optimality condition of the MDF scheme. In particular, we show that whenever $\Delta = 0$, the cut-set upper bound can be achieved. We also show that the MDF scheme cannot have a small gap from the cut-set upper bound for all channel realizations because the optimum strategy is highly related to the value of Δ.

In [17], the aim has been to establish the constant gap argument for the general relay networks with a single source and not to obtain a small gap optimized for a specific channel, such as the diamond channel. For the half-duplex diamond channel, the expression for the gap derived in [17] results in a 30-bit gap. In this thesis, however, we focus on the diamond channel and obtain a smaller gap using our proposed achievablity scheme. In addition, we provide closed-form expressions for the time intervals associated with the transmission modes in the proposed scheduling. Specifically, we show that the expressions are different from those of the cut-set upper bound. This is in contrast to [17], where the constant gap between the cut-set bound and the quantize-and-map scheme was assured for every fixed scheduling, including the optimum scheduling associated with the cut-set upper bound.

In [17], using a different approach (named partial decode-and-forward scheme) than the quantize-and-map scheme, Avestimehr *et al.* showed that the capacity of the *full-duplex* diamond channel can be characterized within 1 bit, regardless of the values of the channel gains. However, applying this scheme to the *half-duplex* diamond channel does not guarantee a constant gap from the channel capacity. We take one further step by providing an achievable scheme that ensures a small gap from the upper bounds for the half-duplex diamond channel. In particular, we show that the gap is smaller than .71 bits, assuming all transmitters have constant power constraints.[2] We also prove that when transmitters have average power constraints instead, the gap is less than 3.6 bits.

1.2 Interference Channel with Conferencing Encoders

Interference limits the throughput of a network consisting of multiple non-cooperative transmitters intending to convey independent messages to their corresponding receivers through a common bandwidth. The way interference is usually dealt with is by either treating it as noise or preventing it by associating different orthogonal dimensions, e.g. time or frequency division, to different users. Since interference has structure, it is possible for a

[2]The *constant* power constraint will be precisely defined in Chapter 2.

receiver to decode some part of the interference and remove it from the received signal. This is indeed the coding scheme proposed by Han-Kobayashi (HK) for the two-user Gaussian Interference Channel (GIC) [30]. The two-user GIC provides a simple example showing that a single strategy against interference is not optimal. In fact, one needs to adjust the strategy according to the channel parameters [31–33]. However, a single suboptimal strategy can be proposed to achieve up to 1 bit per user of the capacity region of the two-user GIC [26].

If the senders can cooperate, interference management can be done more effectively through cooperation. Cooperative links can be either orthogonal or non-orthogonal to the shared medium. Orthogonal cooperation is considered for the interference channel studied in this thesis. In addition, in order to understand some fundamental aspects of the optimal coding scheme (in the sense of having a constant gap to the capacity region), two special cases of the two-user interference channel with conferencing encoders, namely the symmetric GIC and the GZIC in which one transmitter-receiver pair is interference-free are investigated.

1.2.1 Prior Works

Transmitter coordination over orthogonal links is studied for different scenarios with two transmitters (cf. [34–40, 46, 47]). The capacity region of the Multiple Access Channel (MAC) with cooperating encoders is derived in [34], where cooperation is referred to as *conference*. Several achievable rate regions are proposed for the GIC with bidirectional transmitter and receiver cooperation [35]. In the transmit cooperation, the entire message of each transmitter is decoded by the other cooperating transmitter, which apparently limits the performance of the scheme to the capacity of the involved cooperative link. The capacity regions of the compound MAC with conferencing encoders and the GIC with degraded message set, under certain strong interference conditions, are obtained in [36]. The GIC with degraded message set is also termed *Gaussian Cognitive Radio* (GCR) in the literature [37]. The GCR can be considered as an GIC with unidirectional orthogonal cooperation, in which the capacity of the cooperative link is infinity.[3] The capacity region of the GCR is established for the weak interference regime in [38,39]. Recently, the capacity region of the GCR is characterized within 1.87 bits for all ranges of the channel parameters

[3]Technically, the capacity of the cooperative link needs to be equal to the message rate of the user sending data via the cooperative link [34].

[40]. Very recently, in a parallel and independent work, the capacity region of the GIC with bidirectional cooperation is characterized within 6.5 bits [49].[4] For non-orthogonal cooperative links, an achievable rate region is proposed in [42], and the sum-capacity of the GIC is determined up to 18 bits [43].

1.2.2 Summary of Contributions and Relation to Previous Works

Simple achievable schemes based on HK, relaying and zero-forcing techniques are shown to achieve the sum-capacity of the symmetric IC, and the capacity region of the GZIC within 2.13, and 1.71 bits per user, respectively for *all* channel parameters. In this thesis, some important features of the problem are explored one-by-one, and appropriate achievable schemes are proposed accordingly. First, the symmetric IC, with possibly different cooperation link capacities, is studied. In the proposed achievable scheme, each transmitter sends a private and a common message, and zero-forces the part of the other transmitter's private message, communicated over the conference between the encoders. The Generalized Degrees of Freedom (GDOF) of the channel which shows the high SNR behavior of the sum-rate is also characterized. In the second step, the GZIC with unidirectional cooperation is considered. It is demonstrated that the HK scheme together with zero-forcing or relaying can achieve the capacity region up to 1.5 bits per user. Then, based on the observations made in the unidirectional case, the capacity region of the GZIC with bidirectional cooperation is determined up to 1.71 bits per user. Our step-by-step approach to solve the problem is in contrast to the universal strategy of [49], in which the same signaling is used for all channel parameters. Applying the scheme of [49] to the GZIC, five signals should be jointly decoded at each receiver, whereas three signals are required to be jointly decoded in our thesis, which simplifies the transmission scheme. Appropriate compression and sequential decoding techniques are utilized to facilitate such a low complexity decoding. In the achievable schemes, proper power allocation over the employed codewords plays an essential role to achieve the result. Linear Deterministic Model (LDM) proposed in [17] is incorporated to attain such a power allocation. It is illustrated that for some channel parameters, no cooperation or unidirectional cooperation is sufficient to obtain the results. It is also argued that a suitable distribution of total cooperation capacity between the cooperative links can enhance the rate region. In particular, it is demonstrated that

[4] An earlier version of our work containing most of the results is reported in Library and Archives Canada Technical Report UW-ECE 2010-04, Feb. 2010 [45].

the achievable region of the GZIC with limited bidirectional cooperation may outperform the capacity region of the GZIC with infinite unidirectional cooperation, known as the cognitive Z channel. When the noise is the performance-limiting factor instead of the interference, it is shown that treating interference as noise, and not using the cooperative links achieve within 1 bit of the capacity region of the channel for all channel parameters.

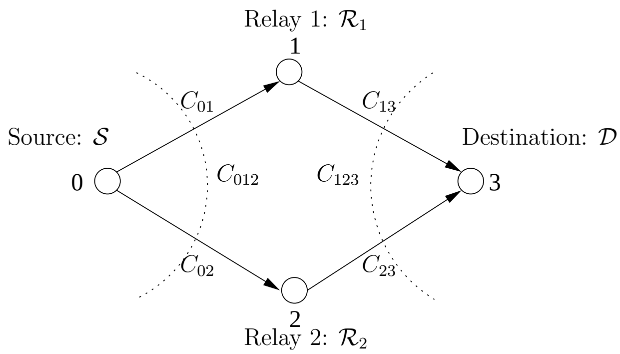
The rest of the thesis is structured as follows. Chapter 2 analyzes the capacity of the diamond channel. Chapter 3 studies the interference channel with conferencing encoders. Finally, chapter 4 outlines a summary of the thesis contributions and discusses possible future research directions.

It is remarked that the results of this dissertation have been published/submitted in [29, 46–48].

Chapter 2

Diamond Relay Channel

In this chapter, a dual-hop communication system composed of a source $\mathcal{S}$ and a destination $\mathcal{D}$ connected through two non-interfering half-duplex relays, $\mathcal{R}_1$ and $\mathcal{R}_2$, is considered. In the literature of Information Theory, this configuration is known as the *diamond channel*. In this setup, four *transmission modes* are present, namely: 1) $\mathcal{S}$ transmits, and $\mathcal{R}_1$ and $\mathcal{R}_2$ listen (broadcast mode), 2) $\mathcal{S}$ transmits, $\mathcal{R}_1$ listens, and simultaneously, $\mathcal{R}_2$ transmits and $\mathcal{D}$ listens. 3) $\mathcal{S}$ transmits, $\mathcal{R}_2$ listens, and simultaneously, $\mathcal{R}_1$ transmits and $\mathcal{D}$ listens. 4) $\mathcal{R}_1$, $\mathcal{R}_2$ transmit, and $\mathcal{D}$ listens (multiple-access mode). Assuming a constant power constraint for all transmitters, a parameter Δ is defined, which captures some important features of the channel. It is proven that for $\Delta = 0$ the capacity of the channel can be attained by successive relaying, i.e., using modes 2 and 3 defined above in a successive manner. This strategy may have an infinite gap from the capacity of the channel when $\Delta \neq 0$. To achieve rates as close as 0.71 bits to the capacity, it is shown that the cases of $\Delta > 0$ and $\Delta < 0$ should be treated differently. Using new upper bounds based on the dual problem of the linear program associated with the cut-set bounds, it is proven that the successive relaying strategy needs to be enhanced by an additional broadcast mode (mode 1), or multiple access mode (mode 4), for the cases of $\Delta < 0$ and $\Delta > 0$, respectively. Furthermore, it is established that under average power constraints the aforementioned strategies achieve rates as close as 3.6 bits to the capacity of the channel.



$$\Delta = C_{01}C_{02} - C_{13}C_{23}$$

Figure 2.1: The diamond channel with its fundamental parameter Δ.

2.1 Problem Statement and Main Results

In this work, a dual-hop communication system, depicted in Fig. 2.1, is considered. The model consists of a source ($\mathcal{S}$), two parallel half-duplex relays ($\mathcal{R}_1$, $\mathcal{R}_2$), and a destination ($\mathcal{D}$), respectively, indexed by 0, 1, 2, and 3 as shown in Fig. 2.1. No link is assumed between Source and Destination, as well as between the relays. The channel gain between node a and b is assumed to be constant, known to all nodes, and is represented by h_{ab} with magnitude $\sqrt{g_{ab}}$.

Due to the half-duplex constraint, four transmission modes exist in the diamond channel where, in every mode, each relay either transmits data to Destination or receives data from Source (see Fig. 2.2). In the figure, $X_a^{(i)}$ and $Y_a^{(i)}$ represent the transmitting and receiving signals at node a corresponding to mode i, respectively. The total transmission time is normalized to one and partitioned into four time intervals (t_1, t_2, t_3, t_4) corresponding to modes $1, 2, 3,$ and 4, with the constraint $\sum_{i=1}^{4} t_i = 1$. The discrete-time baseband representation of the received signals at Relay 1, Relay 2, and Destination are respectively given by:

$$\begin{aligned} Y_1 &= h_{01}X_0 + N_1, \\ Y_2 &= h_{02}X_0 + N_2, \\ Y_3 &= h_{13}X_1 + h_{23}X_2 + N_3, \end{aligned}$$

where N_a is the Gaussian noise at node a with unit variance.

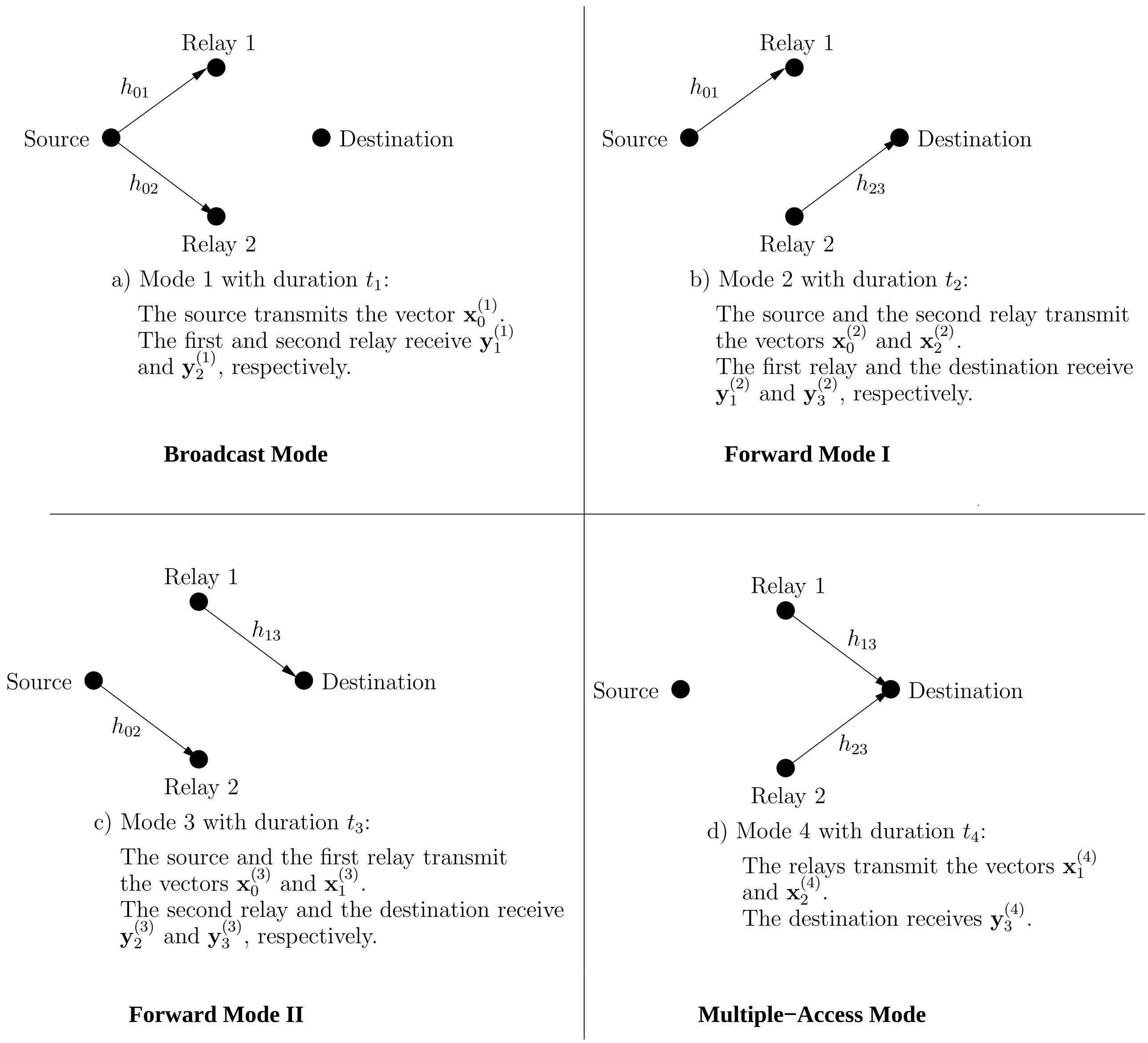


Figure 2.2: Transmission modes for the diamond channel.

Let us assume Source, Relay 1, and Relay 2 consume, respectively, $P_{\mathcal{S}}^{(i)}$, $P_{\mathcal{R}_1}^{(i)}$, and $P_{\mathcal{R}_2}^{(i)}$ amount of power in mode i, i.e.,

$$\begin{aligned}
\frac{1}{t_i} \sum_{t_i \text{ period}} |X_0|^2 &\leq P_{\mathcal{S}}^{(i)}, \\
\frac{1}{t_i} \sum_{t_i \text{ period}} |X_1|^2 &\leq P_{\mathcal{R}_1}^{(i)}, \\
\frac{1}{t_i} \sum_{t_i \text{ period}} |X_2|^2 &\leq P_{\mathcal{R}_2}^{(i)}.
\end{aligned}$$

The total power constraints for Source, Relay 1, and Relay 2 are $P_{\mathcal{S}}$, $P_{\mathcal{R}_1}$, and $P_{\mathcal{R}_2}$, respectively, and are related to the amount of power spent in each mode as follows:

$$\begin{aligned}
\sum_{i=1}^{4} t_i P_{\mathcal{S}}^{(i)} &\leq P_{\mathcal{S}}, \\
\sum_{i=1}^{4} t_i P_{\mathcal{R}_1}^{(i)} &\leq P_{\mathcal{R}_1}, \\
\sum_{i=1}^{4} t_i P_{\mathcal{R}_2}^{(i)} &\leq P_{\mathcal{R}_2}.
\end{aligned}$$

Due to some practical considerations on the power constraints [12], we mainly consider *constant* power constraints for transmitters, i.e., for $i \in \{1, \cdots, 4\}$,

$$\begin{aligned}
P_{\mathcal{S}}^{(i)} &= P_{\mathcal{S}}, \\
P_{\mathcal{R}_1}^{(i)} &= P_{\mathcal{R}_1}, \\
P_{\mathcal{R}_2}^{(i)} &= P_{\mathcal{R}_2}.
\end{aligned} \tag{2.1}$$

Without loss of generality, a unit power constraint is considered for all nodes, i.e., $P_{\mathcal{S}} = P_{\mathcal{R}_1} = P_{\mathcal{R}_2} = 1$. We define the parameters $C_{01}, C_{02}, C_{13}, C_{23}$ as $\mathcal{C}(g_{01}), \mathcal{C}(g_{02}), \mathcal{C}(g_{13}), \mathcal{C}(g_{23})$, respectively. Moreover, C_{012} and C_{123} are defined as:

$$\begin{aligned}
C_{012} &\triangleq \mathcal{C}(g_{01} + g_{02}), \\
C_{123} &\triangleq \mathcal{C}\big((\sqrt{g_{13}} + \sqrt{g_{23}})^2\big).
\end{aligned} \tag{2.2}$$

The case in which transmitters have average power constraints instead of constant power constraints is addressed in section 2.6.

In this work, we are interested in finding communication protocols that operate close to the channel capacity. We introduce an important parameter of the channel Δ as:

$$\Delta \triangleq C_{01}C_{02} - C_{13}C_{23}. \tag{2.3}$$

We categorize all realizations of the diamond channel into three groups based on the sign of Δ (i.e., $\Delta < 0$, $\Delta = 0$, and $\Delta > 0$). As will be shown in the sequel, the sign of Δ plays an important role in designing the optimum scheduling for the channel.

In this setup, the cut-set bounds can be stated in the form of a Linear Program (LP) due to the assumption of constant power constraints for all transmitters. By analyzing the dual program we provide fairly tight upper bounds expressed as single equations corresponding to different channel conditions. Using the dual problem, we prove that when $\Delta = 0$, the MDF scheme achieves the capacity of the diamond channel. Note that $\Delta = 0$ (i.e., $C_{01}C_{02} = C_{13}C_{23}$) includes the previous optimality condition presented in [12] (i.e., $C_{01} = C_{23}$ and $C_{02} = C_{13}$) as a special case. To realize how close the MDF scheme performs to the capacity of the channel when $\Delta \neq 0$, we calculate the gap from the upper bounds. We show that the MDF scheme provides the gap of less than 1.21 bits when applied in the symmetric or some classes of asymmetric diamond channels. More importantly, we explain that the gap can be arbitrarily large for certain ranges of parameters.

By employing new scheduling algorithms we shrink the gap to .71 bits for *all* channel conditions. In particular, for $\Delta < 0$ we add Broadcast (BC) Mode (shown in Fig. 2.2) to the MDF scheme to provide the relays with more reception time. In this three-mode scheme, referred to as Multi-hopping Decode-and-Forward with Broadcast (*MDF-BC*) scheme, the relays decode what they have received from Source and forward the re-encoded information to Destination in Forward Modes I and II. When $\Delta > 0$, Multiple-Access (MAC) mode (shown in Fig. 2.2) in which the relays transmit independent information to Destination is added to the MDF scheme. We call this protocol Multi-hopping Decode-and-Forward with Multiple-Access (*MDF-MAC*) scheme.

The mentioned contributions are associated with the case wherein the transmitters are operating under constant power constraints (2.1). However, for a more general setting in which the transmitters are subject to average power constraints (2.41), it is shown in section 2.6 that the cut-set upper bounds are increased by at most 2.89 bits. Therefore, the proposed achievable schemes guarantee the maximum gap of 3.6 bits from the cut-set upper bounds in the general average power constraint setting.

2.1.1 Coding Scheme

The proposed achievable scheme may employ all four transmission modes as follows:

1. ***Broadcast Mode***: In t_1 fraction of the transmission time, Source broadcasts independent information to Relays 1 and 2 using the superposition coding technique.

2. ***Forward Mode I***: In t_2 fraction of the transmission time, Source transmits new information to Relay 1. At the same time, Relay 2 sends the re-encoded version of part of the data received during Broadcast Mode and/or Forward Mode II of the previous block to Destination.

3. ***Forward Mode II***: In t_3 fraction of the transmission time, Source transmits new information to Relay 2. At the same time, Relay 1 sends the re-encoded version of part of what it has received during Broadcast Mode and/or Forward Mode I of the previous block to Destination.

4. ***Multiple-Access Mode***: In the remaining t_4 fraction of the transmission time, Relays 1 and 2 simultaneously transmit the residual information (corresponding to the previous block) to Destination where, *joint decoding* is performed to decode the received data.

In Broadcast Mode, superposition coding, which is known to be the optimal transmission scheme for the degraded broadcast channel [28], is used to transmit independent data to the relays. The resulting data-rates u and v, respectively associated with Relay 1 and Relay 2 are:

$$u(\eta) = \begin{cases} \mathcal{C}(\eta g_{01}) & \text{if} \quad g_{02} \leq g_{01} \\ C_{01} - \mathcal{C}(\eta g_{01}) & \text{if} \quad g_{01} < g_{02}, \end{cases} \tag{2.4}$$

$$v(\eta) = \begin{cases} C_{02} - \mathcal{C}(\eta g_{02}) & \text{if} \quad g_{02} \leq g_{01} \\ \mathcal{C}(\eta g_{02}) & \text{if} \quad g_{01} < g_{02}. \end{cases} \tag{2.5}$$

The power allocation parameter η determines the amount of Source power used to transmit information to the relay with better channel quality in Broadcast Mode.

In Multiple-Access Mode, a multiple-access channel exists in which the users (relays) have independent messages for Destination. For this channel, joint decoding is optimum, which provides the following rate region [28]:

$$\begin{aligned} R_1 &\le t_4 C_{13}, \\ R_2 &\le t_4 C_{23}, \\ R_1 + R_2 &\le t_4 C_{\mathrm{MAC}}, \end{aligned} \tag{2.6}$$

where R_1, R_2 are the rates that Relay 1 and Relay 2 provide to Destination in Multiple-Access Mode, respectively, and C_{MAC} is defined as:

$$C_{\mathrm{MAC}} \triangleq \mathcal{C}(g_{13} + g_{23}). \tag{2.7}$$

According to the protocol, Relay 1 can receive up to $t_1 u + t_2 C_{01}$ bits per channel use during Broadcast Mode and Forward Mode I. Then the relay has the opportunity to send its received information to Destination in Forward Mode II and Multiple-Access Mode, with the rate $t_3 C_{13} + R_1$. Similarly, Relay 2 can receive and forward messages with the rates $t_1 v + t_3 C_{02}$, and $t_2 C_{23} + R_2$, respectively. Therefore, the maximum achievable rate of the scheme, R, is:

$$R = \max_{\sum_{i=1}^{4} t_i = 1, t_i \ge 0} \left\{ \min\{t_1 u + t_2 C_{01}, t_3 C_{13} + R_1\} + \min\{t_1 v + t_3 C_{02}, t_2 C_{23} + R_2\} \right\}. \tag{2.8}$$

Sections 2.2-2.4 show that employing Forward Modes I and II for $\Delta = 0$, the first three transmission modes for $\Delta < 0$, and the last three transmission modes for $\Delta > 0$ are sufficient to achieve a small gap from the derived upper bounds.

2.1.2 Cut-set Upper Bound and the Dual Program

For general half-duplex networks with K relays, Khojastepour *et al.* proposed a cut-set type upper bound by doing the following steps:

1. Fix the input distribution and scheduling, i.e., $p(X_0, X_1, X_2)$, and t_1, t_2, t_3, t_4 such that $\sum_{i=1}^{4} t_i = 1$.

2. Find the rate $R_{i,j}$ associated with the cut j for each transmission mode i where $i, j \in \{1, \cdots, 2^K\}$.

3. Multiply $R_{i,j}$ by the corresponding time interval t_i.

4. Compute $\sum_{i=1}^{2^K} t_i R_{i,j}$ and minimize it over all cuts.

5. Take the supremum over all input distributions and schedulings.

The preceding procedure can be directly applied to the diamond channel, whose transmission modes are shown in Fig. 2.2. The best input distribution and scheduling lead to:

$$
\begin{aligned}
&C_{\mathrm{DC}} \leq t_1 I(X_0^{(1)}; Y_1^{(1)}, Y_2^{(1)}) + t_2 I(X_0^{(2)}; Y_1^{(2)} | X_2^{(2)}) + t_3 I(X_0^{(3)}; Y_2^{(3)} | X_1^{(3)}) + t_4.0, \\
&C_{\mathrm{DC}} \leq t_1 I(X_0^{(1)}; Y_1^{(1)}) + t_2 \Big(I(X_0^{(2)}; Y_1^{(2)}) + I(X_2^{(2)}; Y_3^{(2)}) \Big) + t_3.0 + t_4 I(X_2^{(4)}; Y_3^{(4)} | X_1^{(4)}), \\
&C_{\mathrm{DC}} \leq t_1 I(X_0^{(1)}; Y_2^{(1)}) + t_2.0 + t_3 \Big(I(X_0^{(3)}; Y_2^{(3)}) + I(X_1^{(3)}; Y_3^{(3)}) \Big) + t_4 I(X_1^{(4)}; Y_3^{(4)} | X_2^{(4)}), \\
&C_{\mathrm{DC}} \leq t_1.0 + t_2 I(X_2^{(2)}; Y_3^{(2)}) + t_3 I(X_1^{(3)}; Y_3^{(3)}) + t_4 I(X_1^{(4)}, X_2^{(4)}; Y_3^{(4)}),
\end{aligned}
$$

where C_{DC} denotes the capacity of the diamond channel. The above bounds do not decrease if each mutual information term is replaced by its maximum value. This substitution simplifies the computation of the upper bound, called R_{up}, by providing the following LP [12]:

$$
\begin{array}{ll}
\text{maximize} & R_{\mathrm{up}} \\
\text{subject to:} & R_{\mathrm{up}} \leq t_1 C_{012} + t_2 C_{01} + t_3 C_{02} + t_4.0 \\
& R_{\mathrm{up}} \leq t_1 C_{01} + t_2 (C_{01} + C_{23}) + t_3.0 + t_4 C_{23} \\
& R_{\mathrm{up}} \leq t_1 C_{02} + t_2.0 + t_3 (C_{02} + C_{13}) + t_4 C_{13} \\
& R_{\mathrm{up}} \leq t_1.0 + t_2 C_{23} + t_3 C_{13} + t_4 C_{123} \\
& \sum_{i=1}^{4} t_i = 1, \;\; t_i \geq 0.
\end{array}
\tag{2.9}
$$

To obtain appropriate single-equation upper bounds on the capacity, we rely on the fact that every feasible point in the dual program provides an upper bound on the primal. Hence, we develop the desired upper bounds by looking at the dual program. In the sequel, we derive the dual program for the LP (2.9).

We start with writing the LP in the standard form as:

$$
\begin{array}{ll}
\text{maximize} & \mathbf{c}^T \mathbf{x} \\
\text{subject to:} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\
& \mathbf{x} \geq 0,
\end{array}
$$

where the unknown vector $\mathbf{x} = [t_1, t_2, t_3, t_4, R_{\text{up}}]^T$, the vectors of coefficients $\mathbf{b} = \mathbf{c} = [0, 0, 0, 0, 1]^T$, and the matrix of coefficients $\mathbf{A}$ is:

$$\mathbf{A} = \begin{pmatrix} -C_{012} & -C_{01} & -C_{02} & 0 & 1 \\ -C_{01} & -(C_{01}+C_{23}) & 0 & -C_{23} & 1 \\ -C_{02} & 0 & -(C_{02}+C_{13}) & -C_{13} & 1 \\ 0 & -C_{23} & -C_{13} & -C_{123} & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Since $\mathbf{A} = \mathbf{A}^T$, it is easy to verify that the primal and dual programs share the same form, i.e.,

$$\begin{array}{ll} \text{minimize} & R_{\text{up}} \\ \text{subject to:} & R_{\text{up}} \geq \tau_1 C_{012} + \tau_2 C_{01} + \tau_3 C_{02} + \tau_4.0 \\ & R_{\text{up}} \geq \tau_1 C_{01} + \tau_2 (C_{01} + C_{23}) + \tau_3.0 + \tau_4 C_{23} \\ & R_{\text{up}} \geq \tau_1 C_{02} + \tau_2.0 + \tau_3 (C_{02} + C_{13}) + \tau_4 C_{13} \\ & R_{\text{up}} \geq \tau_1.0 + \tau_2 C_{23} + \tau_3 C_{13} + \tau_4 C_{123} \\ & \sum_{i=1}^{4} \tau_i = 1,\ \ \tau_i \geq 0. \end{array} \tag{2.10}$$

In the dual program (2.10), τ_i, for $i \in \{1, \cdots, 4\}$ corresponds to the ith rate constraint in the primal LP (2.9). Clearly, the LP (2.9) is *feasible*. Hence, the duality of linear programming ensures that there is no gap between the primal and the dual solutions [27]. However, the benefit of using the dual problem here is that any feasible choice of the vector $\boldsymbol{\tau}$ provides an upper bound to the rate obtained by solving the original LP. This property is known as the weak duality property of LP [27]. Appropriate vectors (i.e., $\boldsymbol{\tau}$'s) in the dual program (2.10) are selected to obtain fairly tight upper bounds. In fact, employing such vectors instead of solving the primal LP (2.9) simplifies the gap analysis. In sections 2.3 and 2.4, these vectors are provided for $\Delta < 0$ and $\Delta > 0$ cases, respectively. In the following sections, we employ the proposed achievable schemes together with the derived upper bounds to characterize the capacity of the diamond channel up to 0.71 bits.

2.2 MDF Scheme and Achieving the Capacity for $\Delta = 0$

In this section, the MDF scheme is described and then proved to be capacity-achieving when $\Delta = 0$.

2.2.1 MDF Scheme

The MDF scheduling algorithm uses two transmission modes: Forward Modes I and II shown in Fig. 2.2 along with the decode-and-forward strategy and can be described as follows:

1. In λ fraction of the transmission time, Source and Relay 2 transmit to Relay 1 and Destination, respectively.

2. In the remaining $\bar{\lambda}$ fraction of the transmission time, Source and Relay 1 transmit to Relay 2 and Destination, respectively.

The achievable rate of the MDF scheme is the summation of the rates of the first and second parallel paths (branches) from Source to Destination, which can be expressed as [12]:

$$R_{\text{MDF}} = \max_{0 \le \lambda \le 1} \Big\{ \min\{\lambda C_{01}, \bar{\lambda} C_{13}\} + \min\{\bar{\lambda} C_{02}, \lambda C_{23}\} \Big\}.$$

The above LP can be re-written as:

$$\begin{aligned} \text{maximize} \quad & R_1 + R_2 \\ \text{subject to:} \quad & R_1 \le \lambda C_{01} \\ & R_1 \le \bar{\lambda} C_{13} \\ & R_2 \le \bar{\lambda} C_{02} \\ & R_2 \le \lambda C_{23} \\ & 0 \le \lambda \le 1, \end{aligned}$$

where R_1 and R_2 denote the rate of the upper and the lower branches, respectively. This LP has three unknowns (R_1, R_2, λ) and six inequalities. The solution turns three out of six inequalities to equality. The optimum time interval λ^* can not be equal to 0 or 1, as both solutions give a zero rate. Hence, three out of the first four inequalities should become equality, which leads to the following achievable rates for different channel conditions:

$$R_{\text{MDF}} = \begin{cases} R^1_{\text{MDF}} = \dfrac{C_{01}(C_{02}+C_{13})}{C_{01}+C_{13}} & \text{if } \Delta \leq 0,\ C_{02} \leq C_{01} \\ R^2_{\text{MDF}} = \dfrac{C_{02}(C_{01}+C_{23})}{C_{02}+C_{23}} & \text{if } \Delta \leq 0,\ C_{02} > C_{01} \\ R^3_{\text{MDF}} = \dfrac{C_{13}(C_{01}+C_{23})}{C_{01}+C_{13}} & \text{if } \Delta > 0,\ C_{23} \leq C_{13} \\ R^4_{\text{MDF}} = \dfrac{C_{23}(C_{02}+C_{13})}{C_{02}+C_{23}} & \text{if } \Delta > 0,\ C_{23} > C_{13}. \end{cases} \tag{2.11}$$

In particular, the achievable rate for the *symmetric* diamond channel, in which $C_{01} = C_{02}$ and $C_{13} = C_{23}$, is:

$$R^{\text{sym}}_{\text{MDF}} = \min\{C_{01}, C_{13}\}.$$

The optimum time interval λ^* is either equal to λ^*_1 or λ^*_2 defined below:

$$\lambda^* = \begin{cases} \frac{C_{13}}{C_{01}+C_{13}} \triangleq \lambda^*_1, \\ \text{or} \\ \frac{C_{02}}{C_{02}+C_{23}} \triangleq \lambda^*_2. \end{cases}$$

Note that if $\lambda^* = \lambda^*_1$, then $\lambda^*_1 C_{01} = \bar{\lambda}^*_1 C_{13}$. Similarly, $\lambda^* = \lambda^*_2$ leads to $\bar{\lambda}^*_2 C_{02} = \lambda^*_2 C_{23}$. In other words, λ^*_i for $i \in \{1, 2\}$ makes the maximum amount of data that can be received by Relay i equal to the maximum amount of data that can be forwarded by Relay i. In this case, branch i (composed of $0 \Rightarrow i \Rightarrow 3$ links) is said to be *fully utilized*.

It is interesting to consider that the case fully utilizing branch 1 or branch 2 leads to the same data-rate. This case occurs when one of the following happens:

$$\begin{cases} \Delta &= 0, \\ C_{01} &= C_{02} \quad \text{if } \Delta < 0, \\ C_{13} &= C_{23} \quad \text{if } \Delta > 0. \end{cases} \tag{2.12}$$

In these situations, one can use either λ^*_1 or λ^*_2 fraction of the transmission time for Forward Mode I and the remaining fraction for Forward Mode II and achieve the same data-rate. It will be shown later that the MDF scheme achieves the capacity of the diamond channel if $\Delta = 0$ and is at most 1.21 bits less than the capacity for the other two cases. It is remarked that $\Delta = 0$ makes both branches fully utilized and all four rates in Eq. (2.11) equal.

2.2.2 MDF is Optimal for $\Delta = 0$

Here, it is explained that R^*_{up}, found by solving the dual-program (2.10), is the same as the MDF rate given in Eq. (2.11) for $\Delta = 0$. It is easy to observe that

$$\boldsymbol{\tau}^* = \left[0, \frac{C_{13}}{C_{01}+C_{13}}, \frac{C_{23}}{C_{02}+C_{23}}, 0\right] \tag{2.13}$$

makes all four rate constraints in the dual-program (2.10) equal to the rate obtained in Eq. (2.11) and satisfies $\sum_{i=1}^{4} \tau_i = 1$. Therefore, the upper bound provided by vector $\boldsymbol{\tau}$ is indeed the capacity of the channel and equals to:

$$C_{\text{DC}} = \frac{C_{01}C_{13}}{C_{01}+C_{13}} + \frac{C_{02}C_{23}}{C_{02}+C_{23}}. \tag{2.14}$$

The result is valid for the Gaussian multiple antenna as well as discrete memoryless channels, and therefore $\Delta = 0$ ensures the optimality of the MDF scheme for those channels too.

2.2.3 MDF Gap Analysis

To investigate how close the MDF scheme performs to the capacity of the diamond channel when $\Delta \neq 0$, the appropriate upper bounds are required, which will be derived in sections 2.3 and 2.4. Therefore, the detailed gap analysis for the MDF scheme is deferred to section 2.7, where it is shown that although a small gap is achievable for some channel conditions, the gap can be large in general. In the following sections, Broadcast and Multiple-Access Modes are added to the MDF algorithm to achieve 0.71 bits of the capacity for $\Delta > 0$ and $\Delta < 0$ cases, respectively.

2.3 MDF-BC Scheme and Achieving within 0.71 Bits of the Capacity for $\Delta < 0$

In the MDF scheme, since both branches cannot be fully utilized when $\Delta < 0$ simultaneously, there exists some unused capacity in the second hop. To efficiently make use of the available resources, Broadcast Mode is added to the MDF scheme. This mode provides the relays with an additional reception time.

2.3.1 Achievable Scheme

The modified protocol uses Broadcast Mode together with Forward Modes I and II. Therefore, by setting $t_4 = 0$ in Eq. (2.8) the maximum achievable rate of the scheme as a function of the power allocation parameter η used in superposition coding is:

$$R_{\mathrm{BC}}(\eta) = \max_{\sum_{i=1}^{3} t_i=1, t_i\geq 0} \left\{ \min\{t_1 u(\eta) + t_2 C_{01}, t_3 C_{13}\} + \min\{t_1 v(\eta) + t_3 C_{02}, t_2 C_{23}\} \right\}.$$

Recall that u, and v, defined respectively in Eqs. (2.4) and (2.5), are the rates associated with Relays 1 and 2 in Broadcast Mode. First, the optimal schedule is obtained, assuming a fixed η, and later an appropriate value for η will be selected. The achievable rate can be written as the following LP:

$$\begin{aligned}
\text{maximize} \quad & R_{\mathrm{BC}} & & \\
\text{subject to:} \quad & R_{\mathrm{BC}} & &\leq t_1(u+v) + t_2 C_{01} + t_3 C_{02} && (2.15)\\
& R_{\mathrm{BC}} & &\leq t_1 u + t_2(C_{01} + C_{23}) + t_3.0 && (2.16)\\
& R_{\mathrm{BC}} & &\leq t_1 v + t_2.0 + t_3(C_{02} + C_{13}) && (2.17)\\
& R_{\mathrm{BC}} & &\leq t_2 C_{23} + t_3 C_{13} && (2.18)\\
& \textstyle\sum_{i=1}^{3} t_i & &= 1 && (2.19)\\
& t_i & &\geq 0. && (2.20)
\end{aligned}$$

For a feasible LP, the solution is at one of the extreme points of the constraint set. One of the extreme points can be obtained by solving a set of linear equations containing Eq. (2.19) and inequalities (2.15)-(2.17) considered as equalities. The solution becomes:

$$\begin{aligned}
t_1 &= \frac{-\Delta}{(C_{01}+C_{13})v + (C_{02}+C_{23})u - \Delta},\\
t_2 &= \frac{C_{13}v + C_{02}u}{(C_{01}+C_{13})v + (C_{02}+C_{23})u - \Delta},\\
t_3 &= \frac{C_{01}v + C_{23}u}{(C_{01}+C_{13})v + (C_{02}+C_{23})u - \Delta},\\
R_{\mathrm{BC}}(\eta) &= \frac{C_{13}(C_{01}+C_{23})v(\eta) + C_{23}(C_{02}+C_{13})u(\eta)}{(C_{01}+C_{13})v(\eta) + (C_{02}+C_{23})u(\eta) - \Delta}. \qquad (2.21)
\end{aligned}$$

It is easy to verify that $\Delta = 0$ makes $t_1 = 0$, and hence leads to the MDF algorithm. Note that in addition to inequalities (2.15)-(2.17), the above extreme point also turns inequality (2.18) into equality. Now, this extreme point is proven to be the solution to the above LP. If one of the elements of vector $\mathbf{t}$ is increased, at least one of the conditions (2.15)-(2.17) provides a smaller rate, compared to the rate obtained by the extreme point. For instance, if t_1 in Eq. (2.21) is increased, then, because of Eq. (2.19), at least one of t_2 and t_3 should be decreased, which in turn reduces the rate associated with the inequality (2.18). This confirms that the extreme point is the optimal solution to the LP with constraints (2.15)-(2.20).

In the following, instead of searching for η^*, which maximizes $R_{BC}(\eta)$, an appropriate value for η is found that not only provides a small gap from the upper bounds, but also simplifies the gap analysis of section 2.3.3. The power allocation parameter η is selected to be either $\eta_1 \triangleq \frac{1}{g_{01}+1}$, or $\eta_2 \triangleq \frac{1}{g_{02}+1}$ for $C_{02} \geq C_{01}$ and $C_{01} \geq C_{02}$ conditions, respectively. As it will be shown in section 2.5, the chosen η produces the same GDOF as the corresponding upper bound, which is a necessary condition in obtaining a small gap. The corresponding u and v for η_1 are:

$$\begin{aligned} u(\eta_1) &= C_{01} - \zeta_1, \\ v(\eta_1) &= C_{012} - C_{01}, \end{aligned} \tag{2.22}$$

and for η_2 are:

$$\begin{aligned} u(\eta_2) &= C_{012} - C_{02}, \\ v(\eta_2) &= C_{02} - \zeta_2. \end{aligned} \tag{2.23}$$

In the above,

$$\zeta_1 \triangleq \mathcal{C}\left(\frac{g_{01}}{g_{01}+1}\right) \leq \frac{1}{2}, \tag{2.24}$$

$$\zeta_2 \triangleq \mathcal{C}\left(\frac{g_{02}}{g_{02}+1}\right) \leq \frac{1}{2}. \tag{2.25}$$

The selected η divides the source power between u and v (considered as the rates of two virtual users in the broadcast channel consisting of $\mathcal{S} \Rightarrow \mathcal{R}_1$ and $\mathcal{S} \Rightarrow \mathcal{R}_2$ links) in such a way that:

1. the sum data-rate (i.e., $u + v$) in the broadcast channel is close to the sum-capacity

of the broadcast channel (i.e., $\max\{C_{01}, C_{02}\}$),

2. the weaker user's rate is close to its capacity. For instance, if $C_{01} \leq C_{02}$, then $u \approx C_{01}$.

Substituting u and v from Eqs. (2.22) and (2.23) into (2.21) leads to the following achievable rates $R^1_{\text{MDF-BC}}$ and $R^2_{\text{MDF-BC}}$ corresponding to η_1 and η_2:

$$\begin{aligned} R^1_{\text{MDF-BC}} &= \frac{C_{13}(C_{01}+C_{23})C_{012} - C_{01}^2 C_{13} + C_{01}C_{02}C_{23} - \zeta_1 C_{23}(C_{02}+C_{13})}{(C_{01}+C_{13})(C_{012}-C_{01}+C_{23}) - \zeta_1(C_{02}+C_{23})}, \\ R^2_{\text{MDF-BC}} &= \frac{C_{23}(C_{02}+C_{13})C_{012} - C_{02}^2 C_{23} + C_{01}C_{02}C_{13} - \zeta_2 C_{13}(C_{01}+C_{23})}{(C_{02}+C_{23})(C_{012}-C_{02}+C_{13}) - \zeta_2(C_{01}+C_{13})}. \end{aligned} \tag{2.26}$$

2.3.2 Upper Bound

Following the discussion in section 2.1.2, we select one of the extreme points of the constraint set (2.10) to obtain a fairly tight upper bound. Below, some insights on how to find an appropriate extreme point are given.

First, Forward Modes I and II play an important role in data transfer from Source to Destination. These two modes let both Source and Destination be simultaneously active, which is important for efficient communication. This implies that *generally* t_2^* and t_3^* are not zero in the original LP (2.9). In addition, $\Delta < 0$ roughly means that the second hop is better than the first hop. In this case, Broadcast Mode helps the relays to collect more data which will be sent to Destination using Forward Modes I and II later. Therefore, Multiple-Access Mode is less important when $\Delta < 0$ and consequently t_4 can be set to zero. Using the complementary slackness theorem of linear programming (cf. [27]), having non-zero t_1, t_2, and t_3 in the original LP translates into having the first three inequalities in the dual program satisfied with equality. Now looking at the dual problem (2.10) with the same structure as the original LP, in order to achieve a smaller objective function, we set τ_2 or τ_3 to zero. This is in contrast to the claim for having both of t_2 and t_3 non-zero in the original LP with the maximization objective. Therefore, the vector $\boldsymbol{\tau}$ with the following properties is selected:

1. Either τ_2 or τ_3 is zero.

2. The first three inequalities are satisfied with equality.

To have a valid $\boldsymbol{\tau}$, we need to make sure that all the elements of vector $\boldsymbol{\tau}$ are non-negative and that $\boldsymbol{\tau}$ satisfies the last condition.

As mentioned earlier, either τ_2 or τ_3 can be set to zero in the dual program (2.10). For instance, setting $\tau_2 = 0$ in the dual-program gives the following LP:

$$\begin{array}{ll} \text{minimize} & \widetilde{R} \\ \text{subject to:} & \widetilde{R} \geq \tau_1 C_{012} + \tau_3 C_{02} + \tau_4 .0 \\ & \widetilde{R} \geq \tau_1 C_{01} + \tau_3 .0 + \tau_4 C_{23} \\ & \widetilde{R} \geq \tau_1 C_{02} + \tau_3 (C_{02} + C_{13}) + \tau_4 C_{13} \\ & \widetilde{R} \geq \tau_1 .0 + \tau_3 C_{13} + \tau_4 C_{123} \\ & \sum_{i=1, i\neq 2}^{4} \tau_i = 1, \;\; \tau_i \geq 0. \end{array} \tag{2.27}$$

Setting the first three inequalities to equalities gives:

$$\begin{aligned} \tau_1^* &= \frac{C_{13}}{C_{012} - C_{02} + C_{13}}, \\ \tau_3^* &= \frac{C_{23}(C_{012} - C_{02}) - C_{13}(C_{012} - C_{01})}{(C_{02} + C_{23})(C_{012} - C_{02} + C_{13})}, \\ \tau_4^* &= \frac{C_{13}(C_{012} - C_{01}) + C_{02}(C_{012} - C_{02})}{(C_{02} + C_{23})(C_{012} - C_{02} + C_{13})}, \\ \widetilde{R}^* &= \frac{(C_{02} + C_{13})C_{23}}{C_{02} + C_{23}} + \frac{C_{13}(C_{01}C_{02} - C_{13}C_{23})}{(C_{02} + C_{23})(C_{012} - C_{02} + C_{13})}. \end{aligned} \tag{2.28}$$

For obtaining a valid result, the following conditions have to be ensured:

1. $\tau_3^* \geq 0$.
   Since $C_{02} \leq C_{012}$, the denominator of τ_3^* is non-negative, therefore, the non-negativity of the nominator has to be guaranteed. This imposes the constraint $\Gamma \geq 0$ on the values of channel parameters, where Γ is defined as:

$$\Gamma \triangleq C_{23}[C_{012} - C_{02}] - C_{13}[C_{012} - C_{01}]. \tag{2.29}$$

2. $\widetilde{R}^* \geq \tau_3^* C_{13} + \tau_4^* C_{123}$.
   To satisfy the following condition:

$$\widetilde{R}^* = \tau_1^* C_{02} + \tau_3^* (C_{02} + C_{13}) + \tau_4^* C_{13} \;\; \geq \tau_3^* C_{13} + \tau_4^* C_{123},$$

it is sufficient to show:

$$(\tau_1^* + \tau_3^*)C_{02} \quad \geq \quad \tau_4^*(C_{123} - C_{13}),$$

which can be equivalently represented as:

$$\frac{C_{02}}{C_{123} - C_{13} + C_{02}} \quad \geq \quad \tau_4^*.$$

The following lemma proves the preceding inequality.

Lemma 2.1. $\tau_4^* \leq \frac{C_{02}}{C_{123}-C_{13}+C_{02}}$ *for* $C_{123} \leq C_{13} + C_{23}$.

Proof. See section 2.9.1 . □

Lemma 2.1 requires $C_{123} \leq C_{13} + C_{23}$, which is not true for $g_{13}g_{23} \leq 4$. To be able to use Lemma 2.1 for the case of $g_{13}g_{23} \leq 4$, we replace either C_{13} by $\hat{C}_{13} \triangleq C_{13} + \delta$ or C_{23} by $\hat{C}_{23} = C_{23} + \delta$ with δ defined as:

$$\delta \triangleq \max\{C_{123} - (C_{13} + C_{23}), 0\}. \tag{2.30}$$

This change provides the desired inequality (i.e., $C_{123} \leq \hat{C}_{13} + C_{23}$ or $C_{123} \leq C_{13} + \hat{C}_{23}$) at the expense of increasing the upper bound. However, we will show in Lemma 2.2 that this increase is always less than δ. We will prove that δ itself is bounded in Lemma 2.3.

Continuing the derivation of the upper bound from the LP (2.27), if $C_{123} \geq C_{13} + C_{23}$, then C_{23} is replaced by $\hat{C}_{23}$. In this case, the dual program (2.27) remains unchanged except for C_{23}. Hence, the set of solutions (2.28) can be used by replacing C_{23} with $\hat{C}_{23}$ and thus the upper bound becomes:

$$\hat{\tilde{R}}^* = \frac{(C_{23} + \delta)(C_{02} + C_{13})}{C_{02} + C_{23} + \delta} + \frac{C_{13}\big(C_{01}C_{02} - C_{13}(C_{23} + \delta)\big)}{(C_{02} + C_{23} + \delta)(C_{012} - C_{02} + C_{13})}. \tag{2.31}$$

Note that the inequality $\hat{\tau}_3^* \geq 0$ holds because $\hat{\Gamma} \geq 0$ simply follows from $\Gamma \geq 0$[1]. According to Lemma 2.1, since $C_{123} = C_{13} + \hat{C}_{23}$, the condition $\hat{\tilde{R}}^* \geq \hat{\tau}_3^* C_{13} + \hat{\tau}_4^* C_{123}$ is satisfied. Lemma 2.2 shows that the enlarged upper bound $\hat{\tilde{R}}^*$ (Eq. (2.31)) is at most δ bits greater than the upper bound of (2.28).

[1]The superscript ˆ is used to indicate parameters associated with $\hat{C}_{23}$. For instance, $\hat{\Gamma}$ has the same formula as Γ in Eq. (2.29), with C_{23} replaced by $\hat{C}_{23}$.

Lemma 2.2. *If $C_{123} \geq C_{13} + C_{23}$, then $\hat{\widetilde{R}}^* - \widetilde{R}^* \leq \delta$.*

Proof. See section 2.9.2. ☐

Therefore, the proposed upper bound for $\Delta \leq 0$ and $\Gamma > 0$ is:

$$R_{\text{up}}^2 = \frac{C_{23}(C_{02} + C_{13})}{C_{02} + C_{23}} + \frac{C_{13}\Delta}{(C_{012} - C_{02} + C_{13})(C_{02} + C_{23})} + \delta.$$

Similarly, when $\Delta \leq 0$ and $\Gamma \leq 0$, τ_3 is set to zero and again the first three inequalities are assumed to be satisfied with equality in the dual-program (2.10). Following the same procedure, the subsequent results are achieved:

$$\begin{aligned}
\tau_1^* &= \frac{C_{23}}{C_{012} - C_{01} + C_{23}}, \\
\tau_2^* &= \frac{C_{13}(C_{012} - C_{01}) - C_{23}(C_{012} - C_{02})}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23})}, \\
\tau_4^* &= \frac{C_{23}(C_{012} - C_{02}) + C_{01}(C_{012} - C_{01})}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23})}, \\
\widetilde{R}^* &= \frac{(C_{01} + C_{23})C_{13}}{C_{01} + C_{13}} + \frac{C_{23}(C_{01}C_{02} - C_{13}C_{23})}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23})}, \\
R_{\text{up}}^1 &= \frac{C_{13}(C_{01} + C_{23})}{C_{01} + C_{13}} + \frac{C_{23}\Delta}{(C_{012} - C_{01} + C_{23})(C_{01} + C_{13})} + \delta. \qquad (2.32)
\end{aligned}$$

In this case, when $C_{123} \geq C_{13} + C_{23}$, $\hat{C}_{13}$ is replaced by $C_{13} + \delta$, it is easy to see that the preceding results can be obtained by exchanging the roles of $C_{01} \leftrightarrow C_{02}$, $C_{13} \leftrightarrow C_{23}$, and $\tau_2 \leftrightarrow \tau_3$ in the results derived for the case of $\Delta \leq 0$ and $\Gamma > 0$.

In order to be able to achieve a small gap from the upper bounds, δ should be bounded. Lemma 2.3 proves that δ is smaller than 0.21 bits.

Lemma 2.3. $\delta \leq \frac{1}{2}\log(\frac{4}{3})$.

Proof. See section 2.9.3. ☐

2.3.3 Gap Analysis

The MDF-BC scheme is proposed for the following regions:

1. $\Delta < 0$, $\Gamma \leq 0$, $C_{02} \geq C_{01}$, and $C_{01} \geq 1$

2. $\Delta < 0$, $\Gamma \geq 0$, $C_{01} \geq C_{02}$, and $C_{02} \geq 1$

For $\Delta < 0$, section 2.7 shows that the MDF scheme provides a small gap from the upper bounds for the remaining regions. Here, the first case is considered. The gap $\kappa^1_{\text{MDF-BC}}$ between the achievable rate $R^1_{\text{MDF-BC}}$ and the upper bound R^1_{up} is:

$$\kappa^1_{\text{MDF-BC}} = \frac{-\zeta_1\Big((C_{012} - C_{01} + C_{23})(C_{13} - C_{23}) + C_{23}(C_{02} + C_{23})\Big)\Delta}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23})\Big((C_{01} + C_{13})(C_{012} - C_{01} + C_{23}) - \zeta_1(C_{02} + C_{23})\Big)} + \delta.$$

In the following lemma, the gap $\kappa^1_{\text{MDF-BC}}$ is proved to be smaller than $\frac{1}{2} + \delta$ bits.

Lemma 2.4. $\kappa^1_{\textit{MDF-BC}} \leq \frac{1}{2} + \delta$.

Proof. See section 2.9.4. □

By exchanging the roles of $g_{01} \leftrightarrow g_{02}$ and $g_{13} \leftrightarrow g_{23}$, the gap for the second case can be easily derived and shown to be less than $\frac{1}{2} + \delta$ bits.

2.4 MDF-MAC Scheme and Achieving within 0.71 Bits of the Capacity for $\Delta > 0$

Similar to section 2.3, a third mode is added to the MDF scheme when $\Delta > 0$ to effectively utilize the unused capacity of the first hop.

2.4.1 Achievable Scheme

Here, Multiple-Access Mode is added to the MDF scheme with independent messages sent from the relays to Destination. This mode provides the relays with an increased transmission time. The modified protocol uses three transmission modes, i.e., Multiple-Access Mode and Forward Modes I and II. Therefore, by setting $t_1 = 0$ in Eq. (2.8) the maximum achievable rate of the scheme, R_{MAC} is:

$$R_{\text{MAC}} = \max_{\sum_{i=2}^4 t_i = 1, t_i \geq 0} \Big\{ \min\{t_2 C_{01}, t_3 C_{13} + R_1\} + \min\{t_3 C_{02}, t_2 C_{23} + R_2\} \Big\}, \qquad (2.33)$$

where R_1 and R_2 are the rates that Relays 1 and 2 provide to Destination in Multiple-Access Mode, respectively. These rates satisfy the multiple-access constraints in (2.6). Lemma 2.5 presents achievable rates, which will be shown to be smaller than the capacity, by at most .71 bits, in section 2.4.3.

Lemma 2.5. *The achievable rates for $\Delta > 0$ together with their corresponding scheduling are as follows:*

$$\begin{aligned} R^1_{MDF\text{-}MAC} &= \frac{C_{01}(C_{02}+C_{13})}{C_{01}+C_{13}} - \frac{C_{02}\Delta}{(C_{01}+C_{13})(C_{MAC}-C_{13}+C_{02})} \quad \textit{for} \quad \Delta > 0,\ \Gamma' \leq 0, \\ R^2_{MDF\text{-}MAC} &= \frac{C_{02}(C_{01}+C_{23})}{C_{02}+C_{23}} - \frac{C_{01}\Delta}{(C_{02}+C_{23})(C_{MAC}-C_{23}+C_{01})} \quad \textit{for} \quad \Delta > 0,\ \Gamma' > 0, \end{aligned} \tag{2.34}$$

$\Gamma' \leq 0$	$\Gamma' > 0$
$t_2 = \frac{C_{13}}{C_{01}+C_{13}}$,	$t_2 = \frac{C_{02}(C_{MAC}-C_{23})+C_{13}C_{23}}{(C_{02}+C_{23})(C_{MAC}-C_{23}+C_{01})}$,
$t_3 = \frac{C_{01}(C_{MAC}-C_{13})+C_{13}C_{23}}{(C_{01}+C_{13})(C_{MAC}-C_{13}+C_{02})}$,	$t_3 = \frac{C_{23}}{C_{02}+C_{23}}$,
$t_4 = \frac{\Delta}{(C_{01}+C_{13})(C_{MAC}-C_{13}+C_{02})}$,	$t_4 = \frac{\Delta}{(C_{02}+C_{23})(C_{MAC}-C_{23}+C_{01})}$,

where

$$\Gamma' \triangleq C_{02}[C_{123} - C_{23}] - C_{01}[C_{123} - C_{13}]. \tag{2.35}$$

Proof. See section 2.9.5. □

It is noted that if $\Delta = 0$, t_4 becomes zero and the scheme is converted to the MDF scheme.

2.4.2 Upper Bound

Following the same procedure as section 2.3.2, the upper bound for the case of $\Delta \geq 0$, $\Gamma' \geq 0$ is attained from (2.28) by exchanging the roles of $C_{01} \leftrightarrow C_{13}$, $C_{02} \leftrightarrow C_{23}$, $\tau_2 \leftrightarrow \tau_3$, and $\tau_1 \leftrightarrow \tau_4$. Similarly, when $\Delta \geq 0$ and $\Gamma' \leq 0$, swapping the positions of $C_{01} \leftrightarrow C_{23}$,

$C_{02} \leftrightarrow C_{13}$, and $\tau_1 \leftrightarrow \tau_4$ in (2.28) provides the upper bound. Therefore:

$$
\begin{aligned}
R_{\text{up}}^3 &= \frac{C_{01}(C_{02}+C_{13})}{C_{01}+C_{13}} + \frac{-C_{02}\Delta}{(C_{123}-C_{13}+C_{02})(C_{01}+C_{13})} + \delta \text{ for } \Gamma' \leq 0, \\
R_{\text{up}}^4 &= \frac{C_{02}(C_{01}+C_{23})}{C_{02}+C_{23}} + \frac{-C_{01}\Delta}{(C_{123}-C_{23}+C_{01})(C_{02}+C_{23})} + \delta \text{ for } \Gamma' > 0. \quad (2.36)
\end{aligned}
$$

2.4.3 Gap Analysis

By comparing the achievable rates (2.34) and the upper bounds (2.36), the gaps κ_{MAC}^1 and κ_{MAC}^2 are respectively calculated for $\Gamma' \leq 0$ and $\Gamma' > 0$ cases as:

$$
\begin{aligned}
\kappa_{\text{MAC}}^1 \triangleq R_{\text{up}}^3 - R_{\text{MDF-MAC}}^1 &= \frac{C_{02}(C_{123}-C_{\text{MAC}})\Delta}{(C_{01}+C_{13})(C_{\text{MAC}}-C_{13}+C_{02})(C_{123}-C_{13}+C_{02})} + \delta, \\
\kappa_{\text{MAC}}^2 \triangleq R_{\text{up}}^4 - R_{\text{MDF-MAC}}^2 &= \frac{C_{01}(C_{123}-C_{\text{MAC}})\Delta}{(C_{02}+C_{23})(C_{\text{MAC}}-C_{23}+C_{01})(C_{123}-C_{23}+C_{01})} + \delta.
\end{aligned}
$$

To show that the above gaps are small, Lemma 2.6 is employed.

Lemma 2.6. $C_{123} - C_{MAC} \leq \frac{1}{2}$.

Proof. See section 2.9.6. □

Considering Lemma 2.6, it is straightforward to show that the gap is at most $\frac{1}{2}+\delta$ bits. Therefore, adding Multiple-Access Mode, with independent messages sent from the relays to Destination, to the MDF scheme ensures the gap of less than .71 bits from the upper bounds for $\Delta > 0$.

2.5 Generalized Degrees of Freedom Characterization

It is interesting to consider the asymptotic capacity of the diamond channel in the high SNR regime. A useful parameter in studying this capacity is the GDOF (cf. [17,26]) defined as:

$$
\text{GDOF}(\boldsymbol{\alpha}) \triangleq \lim_{P \to \infty} \frac{R}{\log P},
$$

where R is the data-rate, P is a channel parameter (can be considered as SNR), and $\boldsymbol{\alpha} = \{\alpha_{01}, \alpha_{02}, \alpha_{13}, \alpha_{23}\}$ with

$$\alpha_{ij} \triangleq \lim_{P\to\infty} \frac{\log(g_{ij})}{\log P} \quad \text{for } i \in \{0,1,2\}, \text{and } j \in \{1,2,3\}.$$

The vector $\boldsymbol{\alpha}$ shows how channel gains scale with P. Based on the above definition, the following approximations are valid:

$$\begin{aligned}
C_{ij} &= \frac{1}{2}\log(1+g_{ij}) \approx \frac{1}{2}\alpha_{ij}\log P, \\
C_{012} &= \frac{1}{2}\log(1+g_{01}+g_{02}) \approx \frac{1}{2}\max\{\alpha_{01},\alpha_{02}\}\log P, \\
C_{123} &= \frac{1}{2}\log\left(1+(\sqrt{g_{13}}+\sqrt{g_{23}})^2\right) \approx \frac{1}{2}\max\{\alpha_{13},\alpha_{23}\}\log P, \\
C_{\mathrm{MAC}} &= \frac{1}{2}\log(1+g_{13}+g_{23}) \approx \frac{1}{2}\max\{\alpha_{13},\alpha_{23}\}\log P, \\
\Gamma &\approx \Big\{\alpha_{23}(\max\{\alpha_{01},\alpha_{02}\}-\alpha_{02}) - \alpha_{13}(\max\{\alpha_{01},\alpha_{02}\}-\alpha_{01})\Big\}(\log P)^2 + \sigma\log(P), \\
\Gamma' &\approx \Big\{\alpha_{02}(\max\{\alpha_{13},\alpha_{23}\}-\alpha_{23}) - \alpha_{01}(\max\{\alpha_{13},\alpha_{23}\}-\alpha_{13})\Big\}(\log P)^2 + \sigma'\log(P),
\end{aligned}$$

where σ and σ' are positive constants. In the following analysis, it is assumed that $(\log P)^2$ terms are dominant, i.e., the coefficients of $(\log P)^2$ for Γ and Γ' are not zero. If this assumption is not valid, MDF scheme achieves the optimum GDOF of the channel. According to the above approximations, it is easy to infer:

$$\begin{cases}
\Gamma \le 0, & \text{if } \alpha_{01} \le \alpha_{02}; \\
\Gamma > 0, & \text{if } \alpha_{01} > \alpha_{02}; \\
\Gamma' \le 0, & \text{if } \alpha_{13} \le \alpha_{23}; \\
\Gamma' > 0, & \text{if } \alpha_{13} > \alpha_{23}.
\end{cases}$$

Therefore, the GDOF associated with the upper bounds is:

$$\begin{aligned}
\mathrm{GDOF}_{\mathrm{up}}^{1} &= \frac{\alpha_{13}(\alpha_{01}+\alpha_{23})}{\alpha_{01}+\alpha_{13}} + \frac{\alpha_{23}(\alpha_{01}\alpha_{02}-\alpha_{13}\alpha_{23})}{(\alpha_{01}+\alpha_{13})(\alpha_{02}-\alpha_{01}+\alpha_{23})}, \\
\mathrm{GDOF}_{\mathrm{up}}^{2} &= \frac{\alpha_{23}(\alpha_{02}+\alpha_{13})}{\alpha_{02}+\alpha_{23}} + \frac{\alpha_{13}(\alpha_{01}\alpha_{02}-\alpha_{13}\alpha_{23})}{(\alpha_{02}+\alpha_{23})(\alpha_{01}-\alpha_{02}+\alpha_{13})}, \\
\mathrm{GDOF}_{\mathrm{up}}^{3} &= \frac{\alpha_{01}(\alpha_{02}+\alpha_{13})}{\alpha_{01}+\alpha_{13}} + \frac{-\alpha_{02}(\alpha_{01}\alpha_{02}-\alpha_{13}\alpha_{23})}{(\alpha_{01}+\alpha_{13})(\alpha_{23}-\alpha_{13}+\alpha_{02})}, \\
\mathrm{GDOF}_{\mathrm{up}}^{4} &= \frac{\alpha_{02}(\alpha_{01}+\alpha_{23})}{\alpha_{02}+\alpha_{23}} + \frac{-\alpha_{01}(\alpha_{01}\alpha_{02}-\alpha_{13}\alpha_{23})}{(\alpha_{02}+\alpha_{23})(\alpha_{13}-\alpha_{23}+\alpha_{01})}.
\end{aligned} \tag{2.37}$$

The GDOF for different achievablity schemes is as follows:
MDF:

$$\begin{aligned}
\mathrm{GDOF}_{\mathrm{MDF}}^{1} &= \frac{\alpha_{01}(\alpha_{02}+\alpha_{13})}{\alpha_{01}+\alpha_{13}}, \\
\mathrm{GDOF}_{\mathrm{MDF}}^{2} &= \frac{\alpha_{02}(\alpha_{01}+\alpha_{23})}{\alpha_{02}+\alpha_{23}}, \\
\mathrm{GDOF}_{\mathrm{MDF}}^{3} &= \frac{\alpha_{13}(\alpha_{01}+\alpha_{23})}{\alpha_{01}+\alpha_{13}}, \\
\mathrm{GDOF}_{\mathrm{MDF}}^{4} &= \frac{\alpha_{23}(\alpha_{02}+\alpha_{13})}{\alpha_{02}+\alpha_{23}}.
\end{aligned} \tag{2.38}$$

MDF-BC:

$$\begin{aligned}
\mathrm{GDOF}_{\text{MDF-BC}}^{1} &= \frac{\alpha_{02}\alpha_{13}(\alpha_{01}+\alpha_{23})-\alpha_{01}^{2}\alpha_{13}+\alpha_{01}\alpha_{02}\alpha_{23}}{(\alpha_{01}+\alpha_{13})(\alpha_{02}-\alpha_{01}+\alpha_{23})}, \\
\mathrm{GDOF}_{\text{MDF-BC}}^{2} &= \frac{\alpha_{01}\alpha_{23}(\alpha_{02}+\alpha_{13})-\alpha_{02}^{2}\alpha_{23}+\alpha_{01}\alpha_{02}\alpha_{13}}{(\alpha_{02}+\alpha_{23})(\alpha_{01}-\alpha_{02}+\alpha_{13})}.
\end{aligned} \tag{2.39}$$

MDF-MAC:

$$\begin{aligned}
\mathrm{GDOF}_{\text{MDF-MAC}}^{1} &= \frac{\alpha_{01}(\alpha_{02}+\alpha_{13})}{\alpha_{01}+\alpha_{13}} - \frac{\alpha_{02}(\alpha_{01}\alpha_{02}-\alpha_{13}\alpha_{23})}{(\alpha_{01}+\alpha_{13})(\alpha_{23}-\alpha_{13}+\alpha_{02})}, \\
\mathrm{GDOF}_{\text{MDF-MAC}}^{2} &= \frac{\alpha_{02}(\alpha_{01}+\alpha_{23})}{\alpha_{02}+\alpha_{23}} - \frac{\alpha_{01}(\alpha_{01}\alpha_{02}-\alpha_{13}\alpha_{23})}{(\alpha_{02}+\alpha_{23})(\alpha_{13}-\alpha_{23}+\alpha_{01})}.
\end{aligned} \tag{2.40}$$

By comparing the upper bounds on the GDOF and the achievable GDOFs, it is easy to see that MDF-BC and MDF-MAC achieve the optimum GDOF of the channel, while the MDF cannot achieve it for all channel parameters.

2.6 Diamond Channel with Average Power Constraints

In this section, it is shown that if the transmitting nodes are subject to average power constraints, each of the cut-set bounds in Eq. (2.9) is increased at most by $\frac{2}{\ln 2}$ bits.[2] This analysis confirms that the achievable schemes proposed in this chapter with constant power constraints are still valid. In other words, they provide a gap of at most $.71 + \frac{2}{\ln 2} \leq 3.6$ bits from the cut-set bounds.

Let $P_{\mathcal{S}}^{(i)*}$, $P_{\mathcal{R}_1}^{(i)*}$, and $P_{\mathcal{R}_2}^{(i)*}$, for $i \in \{1, \cdots, 4\}$ be the optimum power allocated to Source, Relay 1, and Relay 2 in transmission mode i with the corresponding time interval t_i^* leading to the cut-set bound R_0. The following constraints are in effect[3]:

$$\begin{aligned}
\sum_{i=1}^{4} t_i^* P_{\mathcal{S}}^{(i)*} &\leq P_{\mathcal{S}}, \\
\sum_{i=1}^{4} t_i^* P_{\mathcal{R}_1}^{(i)*} &\leq P_{\mathcal{R}_1}, \\
\sum_{i=1}^{4} t_i^* P_{\mathcal{R}_2}^{(i)*} &\leq P_{\mathcal{R}_2}.
\end{aligned} \tag{2.41}$$

Therefore, the cut-set upper bound R_0 satisfies the following constraints:

$$\begin{aligned}
&R_0 \leq t_1^* \mathcal{C}\big((g_{01} + g_{02}) P_{\mathcal{S}}^{(1)*}\big) + t_2^* \mathcal{C}(g_{01} P_{\mathcal{S}}^{(2)*}) + t_3^* \mathcal{C}(g_{02} P_{\mathcal{S}}^{(3)*}), \\
&R_0 \leq t_1^* \mathcal{C}(g_{01} P_{\mathcal{S}}^{(1)*}) + t_2^* \Big(\mathcal{C}(g_{01} P_{\mathcal{S}}^{(2)*}) + \mathcal{C}(g_{23} P_{\mathcal{R}_2}^{(2)*})\Big) + t_4^* \mathcal{C}(g_{23} P_{\mathcal{R}_2}^{(4)*}), \\
&R_0 \leq t_1^* \mathcal{C}(g_{02} P_{\mathcal{S}}^{(1)*}) + t_3^* \Big(\mathcal{C}(g_{02} P_{\mathcal{S}}^{(3)*}) + \mathcal{C}(g_{13} P_{\mathcal{R}_1}^{(3)*})\Big) + t_4^* \mathcal{C}(g_{13} P_{\mathcal{R}_1}^{(4)*}), \\
&R_0 \leq t_2^* \mathcal{C}(g_{23} P_{\mathcal{R}_2}^{(2)*}) + t_3^* \mathcal{C}(g_{13} P_{\mathcal{R}_1}^{(3)*}) + t_4^* \mathcal{C}\Big(\big(\sqrt{g_{13} P_{\mathcal{R}_1}^{(4)*}} + \sqrt{g_{23} P_{\mathcal{R}_2}^{(4)*}}\big)^2\Big).
\end{aligned} \tag{2.42}$$

Suppose that the vector $\mathbf{t}'$ is the solution to the LP (2.9) leading to the rate R_1. If the vector $\mathbf{t}^*$ is used instead of $\mathbf{t}'$ in the LP (2.9), the resulting rate that satisfies the conditions of the LP, called R_2, becomes smaller than R_1. It is clear that the increase in the cut-set bound due to the *average* instead of the *constant* power constraints (compare Eq. (2.1) to Eq. (2.41)), i.e., $R_0 - R_1$ is smaller than $R_0 - R_2$. Here, it is proved that $R_0 - R_2 \leq \frac{2}{\ln 2}$.

Consider each component term in the form of $t_i^* \mathcal{C}(.)$ present in the inequality set (2.42).

[2]Helpful discussions with Mr. Oveis Gharan on the proof of this section are acknowledged.

[3]For the purpose of clarity, here the average powers are not set to unity.

For instance, consider $R_{c,0} \triangleq t_1^* \mathcal{C}(g_{02} P_{\mathcal{S}}^{(1)*})$. The corresponding term in constructing R_2 is $R_{c,2} \triangleq t_1^* \mathcal{C}(g_{02} P_{\mathcal{S}})$. Because of the power constraints (2.41), $R_{c,0} \leq t_1^* \mathcal{C}(g_{02} \frac{P_{\mathcal{S}}}{t_1^*})$. Therefore, it is easy to show:

$$\begin{aligned} R_{c,0} - R_{c,2} &\leq t_1^* \mathcal{C}\Big(\frac{g_{02} P_{\mathcal{S}}(1-t_1^*)}{(1+g_{02}P_{\mathcal{S}})t_1^*}\Big) \\ &\overset{(a)}{\leq} \frac{g_{02} P_{\mathcal{S}}(1-t_1^*)}{2(1+g_{02}P_{\mathcal{S}})\ln 2} \\ &\leq \frac{1}{2\ln 2}, \end{aligned}$$

where (a) is due to the fact that $\mathcal{C}(x) \leq \frac{x}{2\ln 2}$ for any $x \geq 0$. Similar analysis applies to each component term. It is observed that the first and fourth cut-set bounds in inequality set (2.42) have three component terms and the second and third cut-set bounds have four component terms. Therefore, $R_0 - R_2 \leq \frac{2}{\ln 2}$.

2.7 MDF Gap Analysis

We investigate how close the MDF scheme performs to the upper bounds when $\Delta \neq 0$. First, the gap between the MDF scheme and the upper bound is calculated for regions specified in Table I. Then, two special cases are considered.

General Case. We calculate the difference, named κ, between the upper bounds and the rate offered by the MDF scheme from Eq. (2.11) for the cases shown in Table I (see section 2.10):

$$
\begin{aligned}
\kappa_1 &= \frac{-(C_{012}-C_{01})\Delta}{(C_{01}+C_{13})(C_{012}-C_{01}+C_{23})}+\delta,\\
\kappa_2 &= \frac{-(C_{012}-C_{02})\Delta}{(C_{02}+C_{23})(C_{012}-C_{02}+C_{13})}+\delta,\\
\kappa_3 &= \frac{(C_{123}-C_{13})\Delta}{(C_{01}+C_{13})(C_{123}-C_{13}+C_{02})}+\delta,\\
\kappa_4 &= \frac{(C_{123}-C_{23})\Delta}{(C_{02}+C_{23})(C_{123}-C_{23}+C_{01})}+\delta,\\
\kappa_5 &= \frac{-\Delta}{C_{01}+C_{13}}\left(\frac{C_{01}+C_{23}}{C_{02}+C_{23}}-\frac{C_{23}}{C_{012}-C_{01}+C_{23}}\right)+\delta,\\
\kappa_6 &= \frac{-\Delta}{C_{02}+C_{23}}\left(\frac{C_{02}+C_{13}}{C_{01}+C_{13}}-\frac{C_{13}}{C_{012}-C_{02}+C_{13}}\right)+\delta,\\
\kappa_7 &= \frac{\Delta}{C_{01}+C_{13}}\left(\frac{C_{02}+C_{13}}{C_{02}+C_{23}}-\frac{C_{02}}{C_{123}-C_{13}+C_{02}}\right),\\
\kappa_8 &= \frac{\Delta}{C_{02}+C_{23}}\left(\frac{C_{01}+C_{23}}{C_{01}+C_{13}}-\frac{C_{01}}{C_{123}-C_{23}+C_{01}}\right).
\end{aligned}
$$

Note that for the regions associated with κ_7 and κ_8 specified in Table I, $C_{123} \leq C_{13} + C_{23}$ and hence, $\delta = 0$.

To prove that κ_i for $i \in \{1, \cdots, 4\}$ are small, the following lemma is needed:

Lemma 2.7.

$$
\begin{aligned}
C_{012} - \max\{C_{01}, C_{02}\} &\leq \frac{1}{2},\\
C_{123} - \max\{C_{13}, C_{23}\} &\leq 1.
\end{aligned}
$$

Proof. See section 2.9.7. □

For instance, following $\kappa_1 \leq \frac{1}{2} + \delta$ is proved:

$$\begin{aligned}
\kappa_1 &= \frac{(C_{13}C_{23} - C_{01}C_{02})(C_{012} - C_{01})}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23})} + \delta \\
&\overset{(a)}{\leq} \frac{C_{13}C_{23}(C_{012} - C_{01})}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23})} + \delta \\
&\overset{(b)}{\leq} \frac{1}{2}\frac{C_{13}C_{23}}{(C_{01} + C_{13})(C_{012} - C_{01} + C_{23})} + \delta \\
&= \frac{1}{2}\frac{C_{13}}{C_{01} + C_{13}} \times \frac{C_{23}}{C_{012} - C_{01} + C_{23}} + \delta \\
&\leq \frac{1}{2} + \delta,
\end{aligned}$$

where (a) comes from the fact that $\Delta > 0$ for this case. According to the corresponding region shown in Table I, $C_{02} \leq C_{01}$ and therefore (b) is true based on Lemma 2.7.

Lemmas 2.8 and 2.9 prove that $\kappa_5 \leq \frac{1}{2} + \delta$ and $\kappa_7 \leq 1$, respectively. The proof techniques can be easily adopted to correspondingly show that $\kappa_6 \leq \frac{1}{2} + \delta$, and $\kappa_8 \leq 1$.

Lemma 2.8. $\kappa_5 \leq \frac{1}{2} + \delta$.

Proof. See section 2.9.8. □

Lemma 2.9. $\kappa_7 \leq 1$.

Proof. See section 2.9.9. □

Two special cases are also considered:

Symmetric Case. When $C_{01} = C_{02}$ and $C_{13} = C_{23}$, $\Gamma = \Gamma' = 0$ and it can be seen from Table I that the MDF scheme offers a data-rate that is, at most, $1 + \delta$ bits less than the corresponding upper bound.

Partially Symmetric Case. When either $C_{01} = C_{02}$ with $\Delta < 0$, or $C_{13} = C_{23}$ with $\Delta > 0$ occurs, it was seen in section 2.2.1 that fully utilizing branch 1 or branch 2 gives the same achievable rate. Table I shows that in such cases, the gap is less than $1 + \delta$ bits.

Discussion. Multiplexing Gain (MG) of a scheme is defined in [24, 25] as:

$$\mathrm{MG} \triangleq \lim_{\mathrm{SNR} \to \infty} \frac{R}{0.5 \log(\mathrm{SNR})},$$

where R is the achievable rate of the scheme. Using Eq. (2.11), it can be shown that the MDF scheme achieves the multiplexing gain of 1. Avestimehr, *et.al* proposed a broadcast multiple-access scheme for the full-duplex diamond channel and proved that the scheme is within one bit from the cut-set bound [17]. In the half-duplex case, the multiplexing gain of 1 is lost if this approach is followed, leading to an infinite gap between the achievable rate and the upper bound.

It is easy to show that, for the remaining cases (shown in Table I), the gap can be large. For instance, suppose $C_{02} = x$, $C_{13} = C_{23} = \alpha x$ and $C_{01} = \beta x$, with $\alpha > \beta > 1$. In this case $\Delta < 0$, and $\Gamma > 0$ and therefore, the gap κ is:

$$\begin{aligned}
\kappa &= \frac{-\Delta}{C_{02}+C_{23}}\Big(\frac{C_{02}+C_{13}}{C_{01}+C_{13}} - \frac{C_{13}}{C_{012}-C_{02}+C_{13}}\Big) + \delta \\
&= \frac{-\Delta}{C_{02}+C_{23}}\Big(\frac{C_{02}(C_{012}-C_{02}) + C_{13}(C_{012}-C_{01})}{(C_{01}+C_{13})(C_{012}-C_{02}+C_{13})}\Big) + \delta \\
&\overset{(a)}{\geq} \frac{-\Delta}{C_{02}+C_{23}}\Big(\frac{C_{02}(C_{012}-C_{02})}{(C_{01}+C_{13})(C_{012}-C_{02}+C_{13})}\Big) + \delta \\
&\overset{(b)}{\geq} \frac{-\Delta}{C_{02}+C_{23}}\Big(\frac{C_{02}(C_{01}-C_{02})}{(C_{01}+C_{13})(C_{012}-C_{02}+C_{13})}\Big) + \delta \\
&\overset{(c)}{\geq} \frac{-\Delta}{C_{02}+C_{23}}\Big(\frac{C_{02}(C_{01}-C_{02})}{(C_{01}+C_{13})^2}\Big) + \delta \\
&\overset{(d)}{=} \frac{(\alpha^2-\beta)(\beta-1)}{(\alpha+\beta)^2(\alpha+1)}\, x + \delta,
\end{aligned}$$

where in (a) the nominator is decreased by $C_{13}(C_{012} - C_{01})$. To obtain (b), C_{012} in the nominator is replaced by the smaller quantity C_{01}. For (c), C_{012} is substituted by the larger term $C_{01} + C_{02}$ in the denominator. In (d), the assumed values of the capacities in terms of x are substituted. It is clear that the gap increases as x becomes large. GDOF analysis of section 2.5 also confirms that the MDF scheme can have a large gap from the upper bound.

2.8 Summary

In this work, we considered a dual-hop network with two parallel relays in which each transmitting node has a constant power constraint. We categorized the network into three classes based on the fundamental parameter of the network Δ, defined in this chapter. We derived explicit upper bounds for the different classes using the cut-set bound. Based on the upper bounds, we proved that the MDF scheme, which employs two transmission modes (Forward Modes I and II), achieves the capacity of the channel when $\Delta = 0$. Furthermore, we analyzed the gap between the achievable rate of the MDF scheme and the upper bounds, showing that the gap can be large in some ranges of parameters when $\Delta \neq 0$. To guarantee the gap of at most 0.71 bits from the bounds, we added an extra broadcast or multiple-access mode to the baseline MDF scheme for the cases of $\Delta < 0$ and $\Delta > 0$, respectively. In addition, we provided the asymptotic capacity analysis in the high SNR regime. Finally, we argued that when the transmitting nodes operate under average power constraints, the gap between the achievable scheme and the cut-set upper bound is at most 3.6 bits.

2.9 Proofs for Chapter 2

In this section, the proofs of the lemmas used in this chapter are provided.

2.9.1 Proof of Lemma 2.1

We start with the fact that $C_{01} + C_{02} \geq C_{012}$. Rearranging the terms, and multiplying both sides of the inequality by C_{13} give:

$$C_{13}C_{02} \geq C_{13}(C_{012} - C_{01}).$$

By adding $C_{02}(C_{012} - C_{02})$ to both sides and then dividing both sides by $C_{012} - C_{02} + C_{13}$, we obtain:

$$C_{02} \geq \frac{C_{13}(C_{012} - C_{01}) + C_{02}(C_{012} - C_{02})}{C_{012} - C_{02} + C_{13}}.$$

Assuming $C_{123} \leq C_{13} + C_{23}$, we divide the Right Hand Side (RHS) by $C_{02} + C_{23}$ and the Left Hand Side (LHS) by the smaller quantity $C_{123} - C_{13} + C_{02}$ to achieve:

$$\frac{C_{02}}{C_{123} - C_{13} + C_{02}} \quad \geq \quad \frac{C_{13}(C_{012} - C_{01}) + C_{02}(C_{012} - C_{02})}{(C_{012} - C_{02} + C_{13})(C_{02} + C_{23})} = \tau_4^*.$$

This completes the proof.

2.9.2 Proof of Lemma 2.2

$$
\begin{aligned}
\widetilde{\hat{\widetilde{R}}}^* - \widetilde{R}^* &= \frac{\delta C_{02}\Big((C_{02}+C_{13})(C_{012}-C_{02}+C_{13}) - C_{13}(C_{01}+C_{13})\Big)}{(C_{02}+C_{23})(C_{02}+C_{23}+\delta)(C_{012}-C_{02}+C_{13})} \\
&\overset{(a)}{\leq} \frac{\delta C_{02}^2}{(C_{02}+C_{23})^2} \\
&\leq \delta,
\end{aligned}
$$

where in (a), the nominator is increased by replacing $C_{012} - C_{02}$ with C_{01}, using the fact that $C_{012} - C_{02} \leq C_{01}$ (see Eq. (2.2)). In addition, the denominator is decreased by removing δ.

2.9.3 Proof of Lemma 2.3

$$
\begin{aligned}
\delta &= C_{123} - (C_{13} + C_{23}) \\
&= \frac{1}{2}\log\left(\frac{1 + g_{13} + g_{23} + 2\sqrt{g_{13}g_{23}}}{1 + g_{13} + g_{23} + g_{13}g_{23}}\right) \\
&\overset{(a)}{\leq} \frac{1}{2}\log\left(1 + \frac{2\sqrt{g_{13}g_{23}} - g_{13}g_{23}}{1 + 2\sqrt{g_{13}g_{23}} + g_{13}g_{23}}\right) \\
&\overset{(b)}{\leq} \frac{1}{2}\log(\frac{4}{3}),
\end{aligned}
$$

where in (a) the denominator is decreased by replacing $g_{13} + g_{23}$ with the smaller term $2\sqrt{g_{13}g_{23}}$. Defining $x \triangleq \sqrt{g_{13}g_{23}}$, it is easy to show that the maximum of $\log(1 + \frac{2x - x^2}{1+2x+x^2})$, for $0 \leq x \leq 2$, is $x^* = \frac{1}{2}$, i.e., $g_{13}^* g_{23}^* = \frac{1}{4}$, which proves (b).

2.9.4 Proof of Lemma 2.4

It is known that $C_{01}, C_{02} \leq C_{012}$, which proves $0 \leq C_{23}(C_{012} - C_{02})$ and $0 \leq C_{01}(C_{012} - C_{01})$. Since both terms are positive, the sum of them is also positive, i.e., $0 \leq C_{23}(C_{012} - C_{02}) + C_{01}(C_{012} - C_{01})$. By adding and subtracting $(C_{012} - C_{01} + C_{23})C_{13} + C_{01}C_{13}$, the inequality can be rearranged to:

$$0 \leq (C_{012} - C_{01} + C_{23})(C_{01} + C_{13}) + (C_{012} - C_{01})(C_{23} - C_{13}) - C_{23}(C_{02} + C_{13}).$$

As mentioned earlier, Broadcast Mode is used for $\Delta \leq 0$, i.e., $C_{01}C_{02} \leq C_{13}C_{23}$. Therefore, both sides are multiplied by the positive term $-\Delta$ to acquire:

$$0 \leq (C_{13}C_{23} - C_{01}C_{02})\big((C_{012} - C_{01} + C_{23})(C_{01} + C_{13}) + (C_{012} - C_{01})(C_{23} - C_{13}) - C_{23}(C_{02} + C_{13})\big).$$

Now, the positive term $(C_{012} - C_{01} + C_{23})(C_{012} - C_{01})(C_{01} + C_{13})^2$ can be added to the RHS of the inequality to achieve:

$$\begin{aligned} 0 \leq\ & (C_{13}C_{23} - C_{01}C_{02})\big((C_{012} - C_{01} + C_{23})(C_{01} + C_{13}) + (C_{012} - C_{01})(C_{23} - C_{13}) \\ & - C_{23}(C_{02} + C_{13})\big) + (C_{012} - C_{01} + C_{23})(C_{012} - C_{01})(C_{01} + C_{13})^2. \end{aligned}$$

The above inequality can be equivalently stated as:

$$\begin{aligned} & (C_{13}C_{23} - C_{01}C_{02})\Big((C_{012} - C_{01} + C_{23})(C_{13} - C_{23}) + C_{23}(C_{02} + C_{23})\Big) + \\ & \quad C_{01}(C_{02} + C_{23})(C_{01} + C_{13})(C_{012} - C_{01} + C_{23}) \leq (C_{012} - C_{01} + C_{23})^2(C_{01} + C_{13})^2. \end{aligned}$$

Since $1 \leq C_{01}$, the LHS becomes smaller if $C_{01}(C_{02} + C_{23})$ is replaced by $(C_{02} + C_{23})$, leading to:

$$\begin{aligned} & (C_{13}C_{23} - C_{01}C_{02})\Big((C_{012} - C_{01} + C_{23})(C_{13} - C_{23}) + C_{23}(C_{02} + C_{23})\Big) + \\ & \quad (C_{02} + C_{23})(C_{01} + C_{13})(C_{012} - C_{01} + C_{23}) \leq (C_{012} - C_{01} + C_{23})^2(C_{01} + C_{13})^2. \end{aligned}$$

Now as $\zeta_1 \leq \frac{1}{2}$ (see Eq. (2.25)), the following inequality is also true:

$$\begin{aligned} & \zeta_1 \Bigg\{ 2 \times (C_{13}C_{23} - C_{01}C_{02})\Big((C_{012} - C_{01} + C_{23})(C_{13} - C_{23}) + C_{23}(C_{02} + C_{23})\Big) + \\ & (C_{02} + C_{23})(C_{01} + C_{13})(C_{012} - C_{01} + C_{23}) \Bigg\} \leq (C_{012} - C_{01} + C_{23})^2(C_{01} + C_{13})^2. \end{aligned}$$

By rearranging the preceding inequality

$$\frac{\zeta_1(C_{13}C_{23}-C_{01}C_{02})\Big((C_{012}-C_{01}+C_{23})(C_{13}-C_{23})+C_{23}(C_{02}+C_{23})\Big)}{(C_{01}+C_{13})(C_{012}-C_{01}+C_{23})\Big((C_{01}+C_{13})(C_{012}-C_{01}+C_{23})-\zeta_1(C_{02}+C_{23})\Big)} \;\leq\; \frac{1}{2},$$

which completes the proof.

2.9.5 Proof of Lemma 2.5

The optimization (2.33) is an LP and together with the multiple-access constraints (2.6) can be written as follows:

$$\begin{aligned}
\text{maximize} \qquad & R_{\text{MAC}} \\
\text{subject to:} \qquad & R_{\text{MAC}} \leq t_2 C_{01} + t_3 C_{02} \\
& R_{\text{MAC}} - R_1 \leq t_3(C_{02}+C_{13}) \\
& R_{\text{MAC}} - R_2 \leq t_2(C_{01}+C_{23}) \\
& R_{\text{MAC}} - (R_1+R_2) \leq t_2 C_{23} + t_3 C_{13} \\
& R_1 \leq t_4 C_{13} \\
& R_2 \leq t_4 C_{23} \\
& R_1 + R_2 \leq t_4 C_{\text{MAC}} \\
& \textstyle\sum_{i=2}^{4} t_i = 1,\; t_i \geq 0.
\end{aligned}$$

Using Fourier-Motzkin elimination [27], the LP can be equivalently stated as:

$$
\begin{aligned}
\text{maximize} \quad & R_{\text{MAC}} \\
\text{subject to:} \quad & R_{\text{MAC}} \leq t_2 C_{01} + t_3 C_{02} && (2.43) \\
& R_{\text{MAC}} \leq t_3 (C_{02} + C_{13}) + t_4 C_{13} && (2.44) \\
& R_{\text{MAC}} \leq t_2 (C_{01} + C_{23}) + t_4 C_{23} && (2.45) \\
& R_{\text{MAC}} \leq t_2 C_{23} + t_3 C_{13} + t_4 C_{\text{MAC}} && (2.46) \\
& R_{\text{MAC}} \leq t_2 C_{23} + t_3 C_{13} + t_4 (C_{13} + C_{23}) && (2.47) \\
& 2R_{\text{MAC}} \leq t_2 (C_{01} + C_{23}) + t_3 (C_{02} + C_{13}) + t_4 C_{\text{MAC}} && (2.48) \\
& 2R_{\text{MAC}} \leq t_2 C_{23} + t_3 (C_{02} + 2C_{13}) + t_4 (C_{13} + C_{\text{MAC}}) && (2.49) \\
& \sum_{i=2}^{4} t_i = 1, \ t_i \geq 0. && (2.50)
\end{aligned}
$$

Now, it is shown that inequalities (2.47)-(2.49) are redundant. First, since $C_{\text{MAC}} \leq (C_{13} + C_{23})$, the RHS of inequality (2.47) is greater than the RHS of inequality (2.46). Therefore, inequality (2.47) is redundant. Second, inequalities (2.48) and (2.49) are simply obtained by adding inequalities (2.43, 2.46) and (2.44, 2.46), respectively. Therefore, the following LP gives the maximum achievable rate of this scheme:

$$
\begin{aligned}
\text{maximize} \quad & R_{\text{MAC}} \\
\text{subject to:} \quad & R_{\text{MAC}} \leq t_2 C_{01} + t_3 C_{02} && (2.51) \\
& R_{\text{MAC}} \leq t_3 (C_{02} + C_{13}) + t_4 C_{13} && (2.52) \\
& R_{\text{MAC}} \leq t_2 (C_{01} + C_{23}) + t_4 C_{23} && (2.53) \\
& R_{\text{MAC}} \leq t_2 C_{23} + t_3 C_{13} + t_4 C_{\text{MAC}} && (2.54) \\
& \sum_{i=2}^{4} t_i = 1, \ t_i \geq 0. && (2.55)
\end{aligned}
$$

Instead of solving the above LP, a feasible solution that satisfies all the constraints is found. This solution is not necessarily optimum, however it provides us with an achievable rate. For $\Gamma' \leq 0$ inequalities (2.51), (2.52), and (2.54) are set to equalities, leading to:

$$\begin{aligned}
t_2 &= \frac{C_{13}}{C_{01}+C_{13}}, \\
t_3 &= \frac{C_{01}(C_{\text{MAC}}-C_{13})+C_{13}C_{23}}{(C_{01}+C_{13})(C_{\text{MAC}}-C_{13}+C_{02})}, \\
t_4 &= \frac{\Delta}{(C_{01}+C_{13})(C_{\text{MAC}}-C_{13}+C_{02})}, \\
R^1_{\text{MDF-MAC}} &= \frac{C_{01}(C_{02}+C_{13})}{C_{01}+C_{13}} - \frac{C_{02}\Delta}{(C_{01}+C_{13})(C_{\text{MAC}}-C_{13}+C_{02})}.
\end{aligned} \tag{2.56}$$

To ensure that the above results are valid, the inequality (2.53) has to be satisfied. Considering inequalities (2.51) and (2.53), it is sufficient to show that $t_3C_{02} \leq \bar{t}_3C_{23}$. Using the values obtained in Eq. (2.56), this is equivalent to prove:

$$C_{02}\big(C_{01}(C_{\text{MAC}}-C_{13})+C_{13}C_{23}\big) \leq C_{23}\big(\Delta + C_{13}(C_{\text{MAC}}-C_{13}+C_{02})\big).$$

By re-ordering the terms and using the definition of Δ, the above inequality can be alternatively written as:

$$C_{\text{MAC}}\Delta \leq (C_{13}+C_{23})\Delta,$$

which is true since $\Delta > 0$, and $C_{\text{MAC}} = \mathcal{C}(g_{13}+g_{23})$.

For $\Gamma' > 0$, inequalities (2.51), (2.53), and (2.54) are set to equality. In this case, the time intervals and the achievable rate become:

$$\begin{aligned}
t_2 &= \frac{C_{02}(C_{\text{MAC}}-C_{23})+C_{13}C_{23}}{(C_{02}+C_{23})(C_{\text{MAC}}-C_{23}+C_{01})}, \\
t_3 &= \frac{C_{23}}{C_{02}+C_{23}}, \\
t_4 &= \frac{\Delta}{(C_{02}+C_{23})(C_{\text{MAC}}-C_{23}+C_{01})}, \\
R^2_{\text{MDF-MAC}} &= \frac{C_{02}(C_{01}+C_{23})}{C_{02}+C_{23}} - \frac{C_{01}\Delta}{(C_{02}+C_{23})(C_{\text{MAC}}-C_{23}+C_{01})}.
\end{aligned} \tag{2.57}$$

2.9.6 Proof of Lemma 2.6

$$
\begin{aligned}
C_{123} - C_{\mathrm{MAC}} &= \frac{1}{2}\log\left(\frac{1+(\sqrt{g_{13}}+\sqrt{g_{23}})^2}{1+g_{13}+g_{23}}\right) \\
&= \frac{1}{2}\log\left(1+\frac{2\sqrt{g_{13}\ g_{23}}}{1+g_{13}+g_{23}}\right) \\
&\leq \frac{1}{2}\log\left(1+\frac{g_{13}+g_{23}}{1+g_{13}+g_{23}}\right) \\
&\leq \frac{1}{2}.
\end{aligned}
$$

2.9.7 Proof of Lemma 2.7

$$
\begin{aligned}
C_{012} - \max\{C_{01}, C_{02}\} &= \frac{1}{2}\log\left(\frac{1+g_{01}+g_{02}}{1+\max\{g_{01}, g_{02}\}}\right) \\
&= \frac{1}{2}\log\left(1+\frac{\min\{g_{01}, g_{02}\}}{1+\max\{g_{01}, g_{02}\}}\right) \\
&\leq \frac{1}{2}\log\left(1+\frac{\max\{g_{01}, g_{02}\}}{1+\max\{g_{01}, g_{02}\}}\right) \\
&\leq \frac{1}{2},
\end{aligned}
$$

$$
\begin{aligned}
C_{123} - \max\{C_{13}, C_{23}\} &= \frac{1}{2}\log\left(\frac{1+(\sqrt{g_{13}}+\sqrt{g_{23}})^2}{1+\max\{g_{13}, g_{23}\}}\right) \\
&= \frac{1}{2}\log\left(1+\frac{\min\{g_{13}, g_{23}\}+2\sqrt{g_{13}g_{23}}}{1+\max\{g_{13}, g_{23}\}}\right) \\
&\leq \frac{1}{2}\log\left(1+\frac{3\sqrt{g_{13}g_{23}}}{1+\max\{g_{13}, g_{23}\}}\right) \\
&\leq \frac{1}{2}\log\left(1+\frac{3\sqrt{g_{13}g_{23}}}{1+\sqrt{g_{13}g_{23}}}\right) \\
&\leq 1.
\end{aligned}
$$

2.9.8 Proof of Lemma 2.8

In this region, $C_{01} \leq 1$ and $C_{01} \leq C_{02}$, therefore, $0 \leq C_{13}C_{23}(C_{02} - C_{01})(1 - C_{01})$. It is easy to verify that the following inequality is valid:

$$2C_{13}C_{23}\big(C_{01}(C_{02}-C_{01})+0.5(C_{01}+C_{23})\big) \leq (C_{01}+C_{13})(C_{02}+C_{23})\big(C_{23}+.5+C_{01}(C_{02}-C_{01})\big). \tag{2.58}$$

Replacing $C_{13}C_{23}$ by the smaller quantity $(C_{13}C_{23} - C_{01}C_{02})$ in the LHS of the above inequality results in:

$$\begin{aligned}&2(C_{13}C_{23} - C_{01}C_{02})\big(C_{01}(C_{02} - C_{01}) + 0.5(C_{01} + C_{23})\big) \leq \\ &\qquad (C_{01} + C_{13})(C_{02} + C_{23})\big(C_{23} + .5 + C_{01}(C_{02} - C_{01})\big).\end{aligned} \tag{2.59}$$

Since $C_{01} \leq 1$ in the RHS, $C_{01}(C_{02}-C_{01})$ can be substituted by the larger term $(C_{02}-C_{01})$. Hence, the following inequality is true:

$$-2\Delta\big(C_{01}(C_{02}-C_{01})+0.5(C_{01}+C_{23})\big) \leq (C_{01}+C_{13})(C_{02}+C_{23})\big(C_{23}+.5+(C_{02}-C_{01})\big). \tag{2.60}$$

Rearranging the terms leads to:

$$\frac{-\Delta}{C_{01} + C_{13}}\Big(\frac{C_{01} + C_{23}}{C_{02} + C_{23}} - \frac{C_{23}}{C_{02} + 0.5 - C_{01} + C_{23}}\Big) \leq \frac{1}{2}. \tag{2.61}$$

The gap can be further increased by replacing $C_{02}+0.5$ with the smaller term C_{012} according to Lemma 2.7. Therefore:

$$\frac{-\Delta}{C_{01} + C_{13}}\Big(\frac{C_{01} + C_{23}}{C_{02} + C_{23}} - \frac{C_{23}}{C_{012} - C_{01} + C_{23}}\Big) \leq \frac{1}{2}, \tag{2.62}$$

which completes the proof.

2.9.9 Proof of Lemma 2.9

$$\begin{aligned}
\kappa_7 &= \frac{\Delta}{C_{01}+C_{13}}\Big(\frac{C_{02}+C_{13}}{C_{02}+C_{23}} - \frac{C_{02}}{C_{123}-C_{13}+C_{02}}\Big) \\
&\overset{(a)}{\leq} \frac{\Delta}{C_{01}+C_{13}} \times \frac{C_{13}}{C_{02}+C_{23}} + \delta \\
&\overset{(b)}{\leq} \frac{\Delta}{(C_{01}+C_{13})(C_{02}+C_{23})} + \delta \\
&\overset{(c)}{\leq} \frac{C_{01}}{C_{01}+C_{13}} \times \frac{C_{02}}{C_{02}+C_{23}} + \delta \\
&\leq 1+\delta.
\end{aligned}$$

As $C_{123} \leq C_{13} + C_{23}$ in this region, $C_{123} - C_{13}$ is replaced by the larger quantity C_{23} to obtain (a). (b) is valid since $C_{13} \leq 1$ for this scenario. In (c), Δ is substituted by the larger term $C_{01}C_{02}$.

2.10 Gap Analysis Summary

The results related to gap analysis are compactly shown in Table I. For each region specified by some conditions on the link capacities, the corresponding symbols for the upper bound, the achievable rate, and the gap, (i.e., the difference between the upper bound and the achievable rate) are shown [4]. In addition, an upper bound on the value of the gap is given. For instance, for the region specified by $\Delta \leq 0, \Gamma \leq 0$, and $C_{02} \leq C_{01}$ conditions, the upper bound, the achievable rate, and the gap are respectively represented by R^1_{up}, R^1_{MDF}, and κ_1. Using the achievable scheme that leads to R^1_{MDF}, the gap from the upper bound R^1_{up} is less than $\frac{1}{2} + \delta$. Our results, summarized in Table I, indicate that sending *independent* information during each mode together with the decode-and-forward scheme are sufficient to operate close to the capacity of the channel.

[4]The characterizing equation for each symbol used in the table is given in the body of the chapter.

Table 2.1: Summary of the Results: Gap Analysis for Different Regions

<table>
<tr><th colspan="4">Region</th><th>Achievable Rate</th><th>Gap</th><th>Upper Bound on the Gap</th><th>Upper Bound on the Capacity</th></tr>
<tr><td rowspan="3">$\Delta \leq 0$</td><td rowspan="3">$\Gamma \leq 0$</td><td colspan="2">$C_{02} \leq C_{01}$</td><td>R^1_{MDF}</td><td>κ_1</td><td rowspan="3">$\frac{1}{2}+\delta$</td><td rowspan="3">R^1_{up}</td></tr>
<tr><td rowspan="2">$C_{02} \geq C_{01}$</td><td>$C_{01} \leq 1$</td><td>R^2_{MDF}</td><td>κ_5</td></tr>
<tr><td>$C_{01} \geq 1$</td><td>$R^1_{\text{MDF-BC}}$</td><td>$\kappa^1_{\text{MDF-BC}}$</td></tr>
<tr><td rowspan="3">$\Delta \leq 0$</td><td rowspan="3">$\Gamma > 0$</td><td colspan="2">$C_{01} \leq C_{02}$</td><td>R^2_{MDF}</td><td>κ_2</td><td rowspan="3">$\frac{1}{2}+\delta$</td><td rowspan="3">R^2_{up}</td></tr>
<tr><td rowspan="2">$C_{01} \geq C_{02}$</td><td>$C_{02} \leq 1$</td><td>R^1_{MDF}</td><td>κ_6</td></tr>
<tr><td>$C_{02} \geq 1$</td><td>$R^2_{\text{MDF-BC}}$</td><td>$\kappa^2_{\text{MDF-BC}}$</td></tr>
<tr><td rowspan="6">$\Delta > 0$</td><td rowspan="6">$\Gamma' \leq 0$</td><td colspan="2" rowspan="2">$C_{23} \leq C_{13}$</td><td>R^3_{MDF}</td><td>κ_3</td><td>$1+\delta$</td><td rowspan="6">R^3_{up}</td></tr>
<tr><td>$R^1_{\text{MDF-MAC}}$</td><td>$\kappa^1_{\text{MDF-MAC}}$</td><td>$\frac{1}{2}+\delta$</td></tr>
<tr><td rowspan="4">$C_{23} \geq C_{13}$</td><td rowspan="2">$C_{13} \leq 1$, $C_{123} \leq C_{13}+C_{23}$</td><td>$R^4_{\text{MDF}}$</td><td>$\kappa_7$</td><td>1</td></tr>
<tr><td rowspan="3">$R^1_{\text{MDF-MAC}}$</td><td rowspan="3">$\kappa^1_{\text{MDF-MAC}}$</td><td>$\frac{1}{2}$</td></tr>
<tr><td>$C_{13} \leq 1$, $C_{123} \geq C_{13}+C_{23}$</td><td rowspan="2">$\frac{1}{2}+\delta$</td></tr>
<tr><td>$C_{13} \geq 1$</td></tr>
<tr><td rowspan="6">$\Delta > 0$</td><td rowspan="6">$\Gamma' > 0$</td><td colspan="2" rowspan="2">$C_{13} \leq C_{23}$</td><td>R^4_{MDF}</td><td>κ_4</td><td>$1+\delta$</td><td rowspan="6">R^4_{up}</td></tr>
<tr><td>$R^2_{\text{MDF-MAC}}$</td><td>$\kappa^2_{\text{MDF-MAC}}$</td><td>$\frac{1}{2}+\delta$</td></tr>
<tr><td rowspan="4">$C_{13} \geq C_{23}$</td><td rowspan="2">$C_{23} \leq 1$, $C_{123} \leq C_{13}+C_{23}$</td><td>$R^3_{\text{MDF}}$</td><td>$\kappa_8$</td><td>1</td></tr>
<tr><td rowspan="3">$R^2_{\text{MDF-MAC}}$</td><td rowspan="3">$\kappa^2_{\text{MDF-MAC}}$</td><td>$\frac{1}{2}$</td></tr>
<tr><td>$C_{23} \leq 1$, $C_{123} \geq C_{13}+C_{23}$</td><td rowspan="2">$\frac{1}{2}+\delta$</td></tr>
<tr><td>$C_{23} \geq 1$</td></tr>
</table>

Chapter 3

Interference Channel with Conferencing Encoders

In this chapter, a two-user single-antenna Gaussian Interference Channel (GIC) is considered in which encoders are connected through noiseless links with finite capacities. In this setting, prior to each transmission block, the encoders communicate with each other over the cooperative links. New genie-aided upper bounds on the sum-capacity are presented which incorporate the capacities of the cooperative links. Achievable schemes using a simple type of Han-Kobayashi signaling, together with the zero-forcing, and basic relaying techniques are developed for two special classes of the interference channels, namely the symmetric and Z interference channels. Depending on the channel conditions, the communication system is operating under either *interference-limited* or *noise-limited* regime. In this chapter, the former regime is mainly investigated. For the symmetric interference channel with possibly different cooperation capacities, sum-capacity is achieved within 2.13 bits for all channel parameters . In the case of unidirectional cooperation, it is shown that by decoding the known part of the interference instead of canceling it, the sum-capacity can be attained within two bits. For the Gaussian Z-IC, the capacity region and the sum-capacity of the channel are characterized within 1.71 bits per user, and 2 bits in total, respectively. It is also established that properly sharing the total limited cooperation capacity between the cooperative links may enhance the achievable region, even compared to the case of unidirectional transmitter cooperation with infinite cooperation capacity. For the noise-limited regime, it is shown that treating interference as noise achieves the capacity region up to one bit per user for the general interference channel with conferencing encoders.

3.1 System Model and Preliminaries

In this work, a two-user GIC with partial transmit cooperation, as depicted in Fig. 3.1, is considered. The model consists of two transmitter-receiver pairs, in which each transmitter wishes to convey its own data to its corresponding receiver. There exist two noiseless cooperative links with capacities C_{12} and C_{21}, respectively from Encoder 1 to Encoder 2 and vice versa. It is assumed that all nodes are equipped with a single antenna. The input-output relationship for this channel in standard form is expressed as [51]:

$$\begin{aligned} Y_1 &= X_1 + bX_2 + N_1, \\ Y_2 &= aX_1 + X_2 + N_2, \end{aligned} \tag{3.1}$$

where $a, b \geq 0$, and for $i \in \{1, 2\}$, $N_i \sim \mathcal{N}(0, 1)$, i.e., is the Gaussian noise with zero mean and unit variance. The average power constraint of the transmitters are respectively P_1 and P_2. The full channel state information is assumed to be available at both the transmitters and the receivers. For a given block length n, Encoder $i \in \{1, 2\}$ sends its own (random) message index m_i from the index set $\mathcal{M}_i = \{1, 2, ..., M_i = 2^{nR_i}\}$ with rate R_i [bits/channel use]. Each pair (m_1, m_2) occurs with the same probability $\frac{1}{M_1 M_2}$. The result of the conference between the two encoders is two codewords q_{12}^n, q_{21}^n, where for $j \in \{1, 2\}$, $i \neq j$, and each time index $t \in \{1, \cdots, n\}$, $q_{ij}[t]$ is only a function of $(m_i, q_{ji}[1], \cdots q_{ji}[t-1])$. The encoding function f_i maps the message index m_i and q_{ji}^n into a codeword X_i^n chosen from codebook C_i. Therefore:

$$\begin{aligned} X_1^n &= f_1(m_1, q_{21}^n), \\ X_2^n &= f_2(m_2, q_{12}^n). \end{aligned} \tag{3.2}$$

The codewords in each codebook must satisfy the average power constraint $\frac{1}{n} \sum_{t=1}^{n} |X_i[t]|^2 \leq P_i$. Each decoder uses a decoding function $g_i(Y_i^n)$ to decode its desired message index m_i based on its received sequence. Let $\hat{m}_i$ be the output of the decoder. The average probability of error for each decoder is $P_{e_i} = \Pr(\hat{m}_i \neq m_i)$. A rate pair (R_1, R_2) is said to be achievable when there exists an $(M_1, M_2, n, P_{e_1}, P_{e_2})$-code for the GIC consisting of two encoding functions $\{f_1, f_2\}$ and two decoding functions $\{g_1, g_2\}$ such that for sufficiently

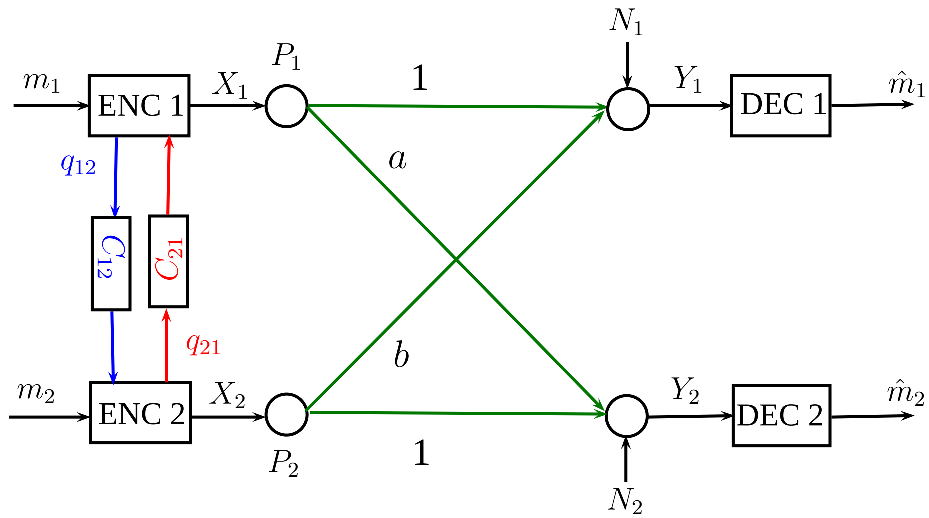


Figure 3.1: The Interference Channel with Conferencing Encoders. The cooperation links are orthogonal to each other and to the communication medium as shown with different colors in the figure. In the proposed achievable schemes, the cooperative links with capacities C_{12}, and C_{21} are used for zero-forcing, and relaying purposes, respectively.

large n:

$$\begin{aligned} R_1 &\leq \tfrac{1}{n}\log(M_1), \\ R_2 &\leq \tfrac{1}{n}\log(M_2), \\ P_e &\leq \epsilon. \end{aligned}$$

In the above, $P_e = \max(P_{e_1}, P_{e_2})$ and $\epsilon > 0$ is a constant that can be chosen arbitrarily small. The capacity region of the GIC with conferencing encoders is the closure of the set of achievable rate pairs. The boundary of the achievable region $\mathcal{R}$ is said to be within (Δ_1, Δ_2) of the boundary of the upper bound region $\mathcal{R}^{\text{up}}$ if for any pair $(R_1^{\text{up}}, R_2^{\text{up}})$ on the boundary of the outer bound, there exists a pair (R_1, R_2) on the achievable region such that $R_1^{\text{up}} - R_1 \leq \Delta_1$, and $R_2^{\text{up}} - R_2 \leq \Delta_2$. In this case, the achievable scheme leading to the region $\mathcal{R}$ is referred to as $\mathcal{R}(\Delta_1, \Delta_2)$ achievable.

If the regions associated with $\mathcal{R}^{\text{up}}$ and $\mathcal{R}(\Delta_1, \Delta_2)$ are polytopes, each facet of the achievable region is compared to its corresponding facet in the upper bound region. Defining $\delta_{R_1} \triangleq R_1^{\text{up}} - R_1$, we have $\delta_{R_1} \leq \Delta_1$. Similarly, for the facet related to $iR_1 + jR_2$, for any $i, j \in \{1, 2, \cdots\}$, we have $\delta_{iR_1+jR_2} \leq i\Delta_1 + j\Delta_2$.

In this work, the interference-limited regime is mainly investigated, i.e., $a^2P_1, b^2P_2 \geq 1$, since otherwise the system is noise limited and is not of much interest. The noise-limited regime will be briefly considered in section 3.5.3.

3.2 Upper Bounds

Theorem 3.1. *The following region is an upper bound on the capacity region of the GIC with conferencing encoders:*

$$R_1^{up} \leq \mathcal{C}(P_1) + \min\left\{C_{12}, \mathcal{C}\left(\frac{b^2P_2 + 2b\sqrt{P_1P_2}}{1+P_1}\right)\right\} \tag{3.3}$$

$$R_2^{up} \leq \mathcal{C}(P_2) + \min\left\{C_{21}, \mathcal{C}\left(\frac{a^2P_1 + 2a\sqrt{P_1P_2}}{1+P_2}\right)\right\} \tag{3.4}$$

$$R_1^{up} + R_2^{up} \leq \mathcal{C}\left(b^2P_2 + \frac{P_1 + 2b\sqrt{P_1P_2}}{1+a^2P_1}\right) + \mathcal{C}\left(a^2P_1 + \frac{P_2 + 2a\sqrt{P_1P_2}}{1+b^2P_2}\right) + C_{12} + C_{21} \tag{3.5}$$

$$R_1^{up} + R_2^{up} \leq \mathcal{C}(\frac{\max\{1-a^2, 0\}P_1}{a^2P_1+1}) + \mathcal{C}\Big((\sqrt{a^2P_1} + \sqrt{P_2})^2\Big) + C_{12} \tag{3.6}$$

$$R_1^{up} + R_2^{up} \leq \mathcal{C}(\frac{\max\{1-b^2, 0\}P_2}{b^2P_2+1}) + \mathcal{C}\Big((\sqrt{b^2P_2} + \sqrt{P_1})^2\Big) + C_{21} \tag{3.7}$$

$$R_1^{up} + R_2^{up} \leq \mathcal{C}\left(P_1P_2(1-ab)^2 + (\sqrt{a^2P_1} + \sqrt{P_2})^2 + (\sqrt{b^2P_2} + \sqrt{P_1})^2\right) \tag{3.8}$$

$$2R_1^{up} + R_2^{up} \leq \mathcal{C}(\frac{\max\{1-a^2, 0\}P_1}{a^2P_1+1}) + \mathcal{C}(a^2P_1 + \frac{P_2 + 2a\sqrt{P_1P_2}}{b^2P_2+1}) + \mathcal{C}\Big((\sqrt{b^2P_2} + \sqrt{P_1})^2\Big) + C_{12} + C_{21} \tag{3.9}$$

$$R_1^{up} + 2R_2^{up} \leq \mathcal{C}(\frac{\max\{1-b^2, 0\}P_2}{b^2P_2+1}) + \mathcal{C}(b^2P_2 + \frac{P_1 + 2b\sqrt{P_1P_2}}{a^2P_1+1}) + \mathcal{C}\Big((\sqrt{a^2P_1} + \sqrt{P_2})^2\Big) + C_{12} + C_{21} \tag{3.10}$$

Proof. The first two bounds are simple applications of cut-set bounds at Transmitter/Receiver

1, and 2, respectively. The rest of the upper bounds are based on the techniques used in [34, 41]. The bounds are similar to the bounds proposed for the GIC with conferencing decoders in [50] with additional terms accounting for the correlation between the transmit signals. Section 3.7.1 provides the detailed proof. □

3.3 Achievable Schemes

The ingredients of a general achievable scheme for the GIC of Fig. 3.1 are composed of the tools used in the IC without conference, i.e., Han-Kobayashi signaling, and the ones associated with the cooperative communications. Cooperation can be helpful in two directions:

1. Canceling the known interference caused by the other party
2. Relaying information for the other party

To operate close to the fundamental limits of the communication, the above ingredients should be properly combined. In the following, some achievable schemes based on a simple type of Han-Kobayashi signaling, zero-forcing, and basic relaying techniques are developed for two special classes of the interference channels, i.e., symmetric and Z interference channels. [1]

3.4 Symmetric Interference Channel

We call an interference channel, symmetric, when

$$a = b$$
$$P_1 = P_2.$$

In this section, the sum-capacity of the symmetric IC is analyzed. The signaling at each transmitter constitutes a private and a common signal corresponding to the HK scheme and private signals that simultaneously provide interference cancellation and relaying benefits.

[1] It is remarked that the binning technique [44] can be used at each encoder to precode its own message against the known interference. This, in general, could enlarge the achievable rate region. However, it is shown that one can achieve close to the capacity without using the binning technique.

In particular, Encoder $i \in \{1, 2\}$ has four Gaussian codebooks, namely private-n codebook C_i^u, private-z codebooks $C_i^{v_i}$ and $C_i^{v_j}$, and common codebook C_i^w, with corresponding codewords u_i, v_i, v_j, and w_i, where $j \in \{1, 2\}, j \neq i$.[2] We use the same power allocation policy for both transmitters. The input to the channel can be written as:

$$
\begin{aligned}
X_1 &= \sqrt{P_u} u_1 + \sqrt{P_w} w_1 + \sqrt{P_{v_p}} v_1 - \sqrt{P_{v_z}} v_2 \\
X_2 &= \sqrt{P_u} u_2 + \sqrt{P_w} w_2 + \sqrt{P_{v_p}} v_2 - \sqrt{P_{v_z}} v_1,
\end{aligned}
$$

where the codewords are assumed to have unit power, and

$$
\begin{aligned}
P_u &= \begin{cases} \frac{1}{a^2} & \text{if} \quad a \leq 1 \\ 0 & \text{if} \quad 1 < a, \end{cases} \\
P_u &+ P_w + P_{v_p} + P_{v_z} \leq P
\end{aligned}
$$

We zero-force v_2 at Receiver 1 and v_1 at Receiver 2, which require $P_{v_z} = a^2 P_{v_p}$. The power is distributed amongst the codewords such that the GDOF of the channel, given in Eq. (3.17), can be achieved by the proposed scheme (cf. section 3.4.1). The received signals after zero-forcing are:

$$
\begin{aligned}
Y_1 &= \sqrt{P_w}(w_1 + a w_2) + \sqrt{\eta P_{v_p}} v_1 + \sqrt{P_u}(u_1 + a u_2) + z_1 \\
Y_2 &= \sqrt{P_w}(w_2 + a w_1) + \sqrt{\eta P_{v_p}} v_2 + \sqrt{P_u}(u_2 + a u_1) + z_2,
\end{aligned}
$$

where

$$
\eta \triangleq (1 - a^2)^2. \tag{3.11}
$$

We provide the same power to the common codeword and codewords involved in zero-forcing, i.e., $P_w = (1 + a^2) P_{v_p}$.

At Receiver i, first w_i, w_j, v_i are jointly decoded and then u_i is decoded. Therefore,

[2] Through out the chapter, it is assumed that all of the employed codebooks are Gaussian and independent of each other.

each receiver forms a MAC.[3] Defining

$$\gamma \triangleq (1+a^2), \tag{3.12}$$

$$P' \triangleq \frac{P-P_u}{2\big(P_u(1+a^2)+1\big)(1+a^2)}, \tag{3.13}$$

$$\zeta \triangleq \min\{a^2, 1\}, \tag{3.14}$$

the MAC constraints after removing the redundant inequalities are:

$$\begin{aligned}
R_{w_1} &\leq \mathcal{C}(\zeta\gamma P') \\
R_{w_2} &\leq \mathcal{C}(\zeta\gamma P') \\
R_{v_1} &\leq \min\{C_{12}, \mathcal{C}(\eta P')\} \\
R_{v_2} &\leq \min\{C_{21}, \mathcal{C}(\eta P')\} \\
R_{w_1}+R_{w_2} &\leq \mathcal{C}\left(\gamma^2 P'\right) \\
R_{w_1}+R_{v_1} &\leq \mathcal{C}\left((\eta+\gamma)P'\right) \\
R_{w_2}+R_{v_2} &\leq \mathcal{C}\left((\eta+\gamma)P'\right) \\
R_{w_1}+R_{v_2} &\leq \mathcal{C}\left((\eta+a^2\gamma)P'\right) \\
R_{w_2}+R_{v_1} &\leq \mathcal{C}\left((\eta+a^2\gamma)P'\right) \\
R_{w_1}+R_{w_2}+R_{v_1} &\leq \mathcal{C}\left((\eta+\gamma^2)P'\right) \\
R_{w_1}+R_{w_2}+R_{v_2} &\leq \mathcal{C}\left((\eta+\gamma^2)P'\right)
\end{aligned}$$

We define the sum-rate as $R_{\text{Sum}} = R' + R_{u_1} + R_{u_2}$ with $R' \triangleq R_{w_1} + R_{w_2} + R_{v_1} + R_{v_2}$. Applying Fourier-Motzkin Elimination (FME) to obtain rate constraints on the sum-rate

[3] It is remarked that in for $1 < a$, decoding v_i instead of zero-forcing it at receiver j , for $i, j \in \{1, 2\}, i \neq j$, may enhance the sum-rate at the expense of increasing decoding complexity. See [46] for more details.

leads to $R' \le \min_{l\in\{1,\cdots,5\}} \{R'_l\}$, where

$$\begin{aligned}
R'_1 &= \mathcal{C}(\zeta\gamma P') + \mathcal{C}\left((\eta+\zeta\gamma)P'\right) + \min\{C_{12}, C_{21}, \mathcal{C}\left(\eta P'\right)\}\\
R'_2 &= \mathcal{C}\left((\eta+\gamma^2)P'\right) + \min\{C_{12}, C_{21}, \mathcal{C}\left(\eta P'\right)\}\\
R'_3 &= 2\mathcal{C}\left((\eta+\zeta\gamma)P'\right)\\
R'_4 &= 2\mathcal{C}\left(\zeta\gamma P'\right) + \min\{C_{12}, \mathcal{C}\left(\eta P'\right)\} + \min\{C_{21}, \mathcal{C}\left(\eta P'\right)\}\\
R'_5 &= \mathcal{C}\left(\gamma^2 P'\right) + \min\{C_{12}, \mathcal{C}\left(\eta P'\right)\} + \min\{C_{21}, \mathcal{C}\left(\eta P'\right)\}
\end{aligned}$$

Theorem 3.2. *The achievable sum-rate is within* 2.13*, and* 1.6 *bits of the sum-capacity for $a \le 1$, and $1 < a$ cases, respectively.*

Proof. See section 3.7.2. □

3.4.1 GDOF Analysis

In this section, the sum-capacity behavior is considered in the high SNR regime by characterizing the GDOF of the channel. It is known that the interference can reduce the available degrees of freedom for data communication [26]. To understand this effect, we define the GDOF by slightly modifying the definition of [46] to account for C_{21}:

$$d(\alpha, \beta_1, \beta_2) = \lim_{\mathrm{SNR}\to\infty} \frac{R_{\mathrm{sum}}(\mathrm{SNR}, \alpha, \beta_1, \beta_2)}{C(\mathrm{SNR})}, \tag{3.15}$$

where $\mathrm{SNR} \triangleq P$, $\beta_1, \beta_2 \ge 0$ are the multiplexing gains of the cooperative links, i.e., $C_{12} = \beta_1 C(P)$ and $C_{21} = \beta_2 C(P)$, and

$$\alpha \triangleq \frac{\log(\mathrm{INR})}{\log(\mathrm{SNR})}, \tag{3.16}$$

with $\mathrm{INR} \triangleq a^2 P$. Defining $\beta^+ \triangleq \beta_1 + \beta_2$ and $\beta^- \triangleq \min(\beta_1, \beta_2)$, we have the following lemma:

Lemma 3.3. *The GDOF of the channel is:*

$$\begin{aligned}
d_{\alpha<1} &= \min\{2, 2\max(\alpha, 1-\alpha) + \beta^+, 2 - \alpha + \beta^-\},\\
d_{\alpha>1} &= \min\{2\alpha, \alpha + \beta^-, 2 + \min(\beta_1, \alpha - 1) + \min(\beta_2, \alpha - 1)\}.
\end{aligned} \tag{3.17}$$

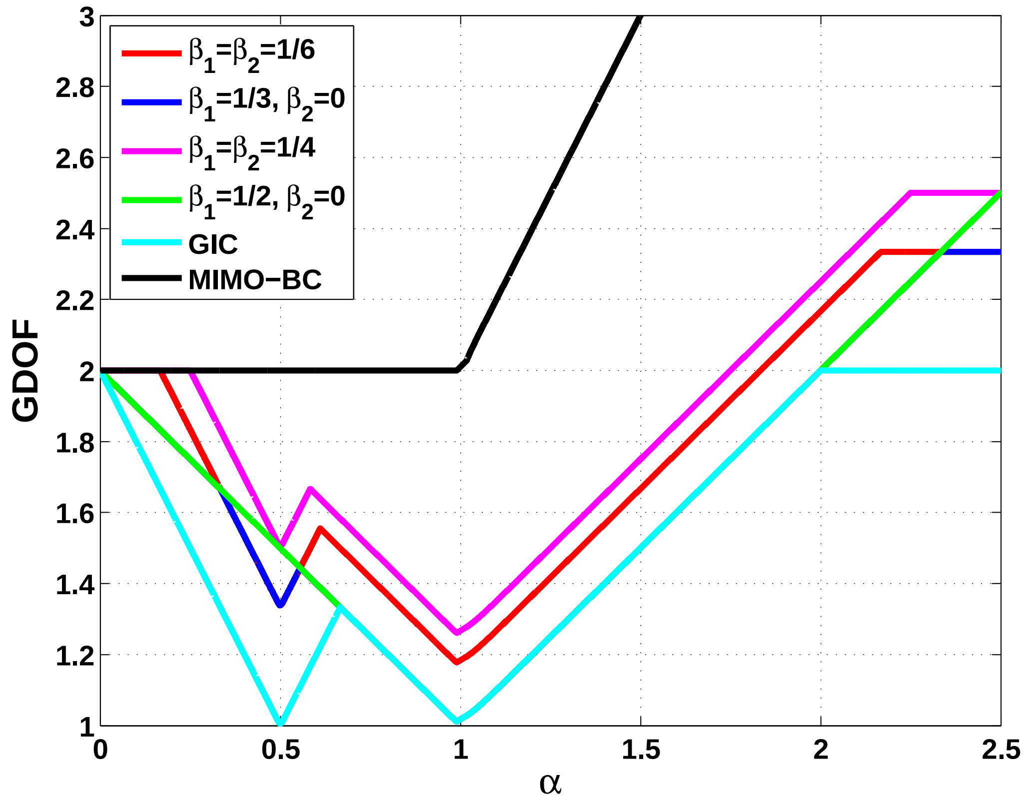


Figure 3.2: The effect of partial cooperation on the GDOF.

Proof. It is straightforward to show that both the lower and the upper bounds provide the GDOF given in Eq. (3.17). □

Fig. 2 shows the GDOF as a function of α, β_1, and β_2.[4] The GDOF associated with GIC and MIMO Broadcast Channel (MIMO-BC) respectively serve as lower and upper bounds on the GDOF for any β_1 and β_2. An interesting observation is unidirectional cooperation never achieves any part of the MIMO-BC curve for $\alpha > 0$. One can easily infer that symmetric bidirectional cooperation outperforms unidirectional cooperation with twice the cooperation capacity.

[4]The GDOF for $\alpha = 1$ is 1 since the MIMO matrix is ill-conditioned, and therefore, it is omitted from the figure.

3.5 Z-Interference Channel

Now, another special class of the GIC shown in Fig. 3.1 is considered in which one transmitter-receiver pair is interference-free, i.e., $b = 0$ in Fig. 3.1.

3.5.1 Upper Bounds

It is convenient to evaluate the upper bounds for the case of GZIC.

Lemma 3.4. *The following region is an upper bound on the capacity region of the GZIC shown in Fig. 3.1:*

$$R_1^{upz} \leq \mathcal{C}(P_1) \tag{3.18}$$

$$R_2^{upz} \leq \mathcal{C}(P_2) + C_{21} \tag{3.19}$$

$$R_2^{upz} \leq \mathcal{C}\left(2a^2P_1 + 2P_2\right) \tag{3.20}$$

$$R_1^{upz}+R_2^{upz} \leq \mathcal{C}\left(\frac{\max\{1-a^2,0\}P_1}{1+a^2P_1}\right)+\mathcal{C}(2a^2P_1+2P_2)+C_{12} \tag{3.21}$$

$$R_1^{upz} + R_2^{upz} \leq \mathcal{C}\left(P_1P_2 + P_1(1+2a^2) + 2P_2\right). \tag{3.22}$$

Proof. By setting $b = 0$ in the bounds introduced in section 3.2, removing redundant inequalities, and replacing the term $(\sqrt{a^2P_1} + \sqrt{P_2})^2$ by the larger term $2a^2P_1 + 2P_2$ in (3.20), (3.21), and (3.22), we can get the above bounds.[5] $\square$

3.5.2 Unidirectional Cooperation

To gain some insight into the essential ingredients of an appropriate achievable scheme, first, it is assumed that one of the cooperative links has zero capacity. Depending on which capacity is zero, two scenarios can occur (see also Fig. 3.3):

1. $C_{21} = 0$ termed as zero-forcing scenario.

2. $C_{12} = 0$ termed as relaying scenario.

[5]The replacement is done to simplify the gap analysis.

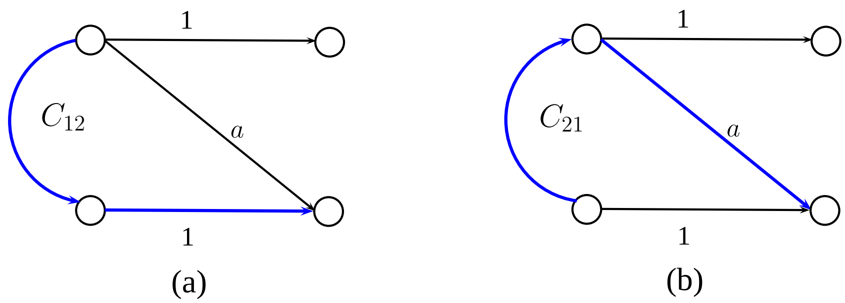


Figure 3.3: Unidirectional Cooperation: (a) Zero-Forcing Scenario, (b) Relaying Scenario. The blue color represents the route of zero-forcing/ relaying.

3.5.2.1 Zero-Forcing Scenario ($C_{21} = 0$)

In this scenario, Transmitter 2 utilizes C_{12} to cancel the known part of the interference (Fig. 3.3 (a)). The rest of the signaling is similar to the one proposed for the conventional GIC [26]. In other words, Encoder 1 makes use of three independent codebooks, namely private codebooks C_1^p and C_1^z, and common codebook C_1^c, with corresponding codewords X_{1p}, X_{1z}, and X_{1c}. Encoder 2 uses two private codebooks C_2^p and C_1^z. X_{1z} is available at both transmitters via the cooperative link and zero-forced at the second receiver. Since Transmitter 2 does not cause any interference on Receiver 1, there is no need to include a common codebook for User 2. The transmit signals are represented as follows:

$$\begin{aligned} X_1 &= X_{1p} + X_{1c} + X_{1z}, \\ X_2 &= X_{2p} - aX_{1z}. \end{aligned} \tag{3.23}$$

Decoders decide on the codeword indices i, j, k, and l according to:

- Decoder 1: $(X_{1p}(i), X_{1z}(j), X_{1c}(k), Y_1) \in A_\epsilon^{(n)}$,
- Decoder 2: $(X_{1c}(k), X_{2p}(l), Y_2) \in A_\epsilon^{(n)}$.

Section 3.7.3 shows the rate region described by (3.24)-(3.26) is achievable:

$$R_1 \leq \min\{\mathcal{C}(P_1), \mathcal{C}(P_{1p}+P_{1c})+C_{12}\} \tag{3.24}$$

$$R_2 \leq \mathcal{C}(\frac{P_{2p}}{d}) \tag{3.25}$$

$$R_1 + R_2 \leq \min\{\mathcal{C}(P_{1p}+P_{1z}), \mathcal{C}(P_{1p})+C_{12}\}+\mathcal{C}(\frac{a^2 P_{1c}+P_{2p}}{d}), \tag{3.26}$$

where

$$d \triangleq 1 + a^2 P_{1p}. \tag{3.27}$$

The following observations are utilized to attain a suitable power allocation for different codewords:

- Since X_{1p} is treated as noise at Receiver 2, we set $P_{1p} = \frac{1}{a^2}$ in order to receive X_{1p} at the level of the Gaussian noise at Receiver 2 [26].
- To make R_2 close to R_2^{upz}, i.e., $\mathcal{C}(P_2)$, we impose the constraint $\frac{P_2}{2} \leq P_{2p}$.
- To make R_1 in (3.24) close to R_1^{upz}, i.e., $\mathcal{C}(P_1)$, we enforce $\frac{P_1 - P_{1p}}{2} \leq P_{1c}$. This requirement is more pronounced when $C_{12} = 0$.

For the last two items, the factor 2 in the denominators ensures a maximum loss of 0.5 bit compared to the case of $P_{2p} = P_2$, and $P_{1c} = P_1 - P_{1p}$. Therefore, to satisfy the above constraints, we select P_{1z} according to:

$$P_{1z} = \min\left(\frac{P_1 - P_{1p}}{2}, \frac{P_2}{2a^2}\right). \tag{3.28}$$

Lemma 3.5. *The preceding achievable scheme is* $\mathcal{R}(0.5, 1)$ *achievable.*

Proof. See section 3.7.3. □

3.5.2.2 Relaying Scenario ($C_{12} = 0$)

Here, it is assumed that $C_{12} = 0$. In this scenario, C_{21} is employed to help Encoder 1 relay some information for User 2 (Fig. 3.3 (b)). Based on the relaying capability (the relay power $a^2 P_1$ and the relay to destination channel gain a), three cases are recognized in this setup and for each case, a different achievable scheme is proposed:

- Non Cooperative Case: $a^2 P_1 \leq P_2 + 1$,
- Common Cooperative Case: $P_2 + 1 < a^2 P_1$, $a^2 \leq P_2 + 1$,
- Private-Common Cooperative Case: $P_2 + 1 < a^2 P_1$, $P_2 + 1 < a^2$.

Before we continue to describe the achievable schemes, we stress that, throughout the chapter, we aim to keep the achievable schemes simple, at the expense of a slight increase in the gap from the upper bound.

3.5.2.2.1 Non Cooperative Case: $a^2P_1 \leq P_2 + 1$: In this case, the cooperative link is not used because it can at most enhance R_2 by one bit. The signaling is similar to the HK signaling developed for the GIC:

$$\begin{aligned} X_1 &= X_{1p} + X_{1c}, \\ X_2 &= X_{2p}. \end{aligned} \tag{3.29}$$

We set $P_{1c} = P_1 - P_{1p}$ with $P_{1p} = \frac{1}{a^2}$ for $a \leq 1$, and $P_{1p} = 0$ for $1 < a$. The decoding rules are:

- Decoder 1: $(X_{1p}(i), X_{1c}(j), Y_1) \in A_\epsilon^{(n)}$,
- Decoder 2: $(X_{1c}(j), X_{2p}(k), Y_2) \in A_\epsilon^{(n)}$,

which lead to the $\mathcal{R}(0, 1.5)$ achievable region below:

$$R_1 \leq \mathcal{C}(P_1) \tag{3.30}$$

$$R_2 \leq \mathcal{C}(\frac{P_2}{d}) \tag{3.31}$$

$$R_1 + R_2 \leq \mathcal{C}(P_{1p}) + \mathcal{C}(\frac{a^2(P_1 - P_{1p}) + P_2}{d}). \tag{3.32}$$

See section 3.7.4.1 for details.

3.5.2.3 Common Cooperative Case: $P_2 + 1 < a^2P_1, a^2 \leq P_2 + 1$

For this case, in addition to the signals transmitted in the non cooperative case, User 1 relays some data communicated over the cooperative link for User 2. The signaling is as follows:

$$\begin{aligned} X_1 &= X_{1p} + X_{1c} + X_{2r}, \\ X_2 &= X_{2p}, \end{aligned} \tag{3.33}$$

where all signals are independent of each other. To find out how to treat X_{2r} at Receiver 1 and how to allocate the Transmitter 1's power between X_{1c} and X_{2r}, the deterministic model shown in Fig. 3.4, is used. The model demonstrates the power level interaction of the interfering signals according to the channel parameters. Since, X_{1p} is received below the noise level of Decoder 2, it is considered as noise at that receiver. When two signals

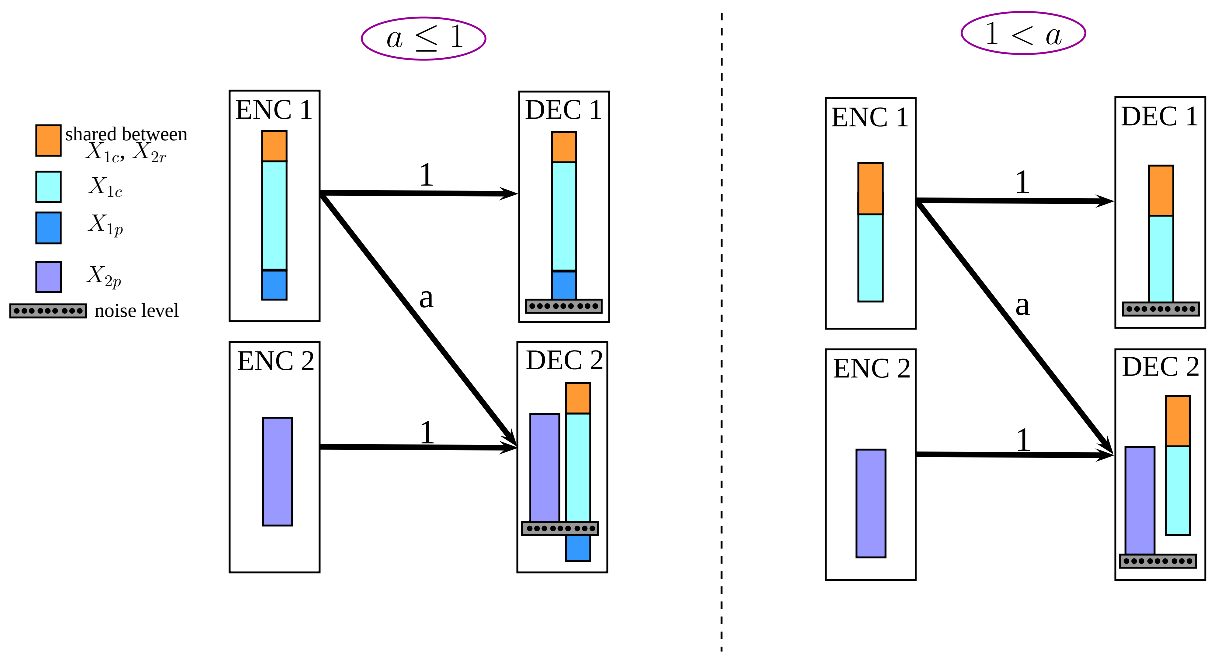


Figure 3.4: LDM for GZIC with $C_{12} = 0$: Common Cooperative Case. The received power level at each receiver is shown for $a \leq 1$, and $1 < a$ situations. Different colors represent different signals. The signals with the same power level should be decoded jointly.

are at the same power level at a decoder, they need to be decoded jointly at that receiver. It is clear that there is not much benefit in having two signals with the same power level intended for the same receiver.[6] Having said that, it is noticed that the relayed signal has to share its power level with the common signal of User 1 to not limit the User 1's rate by P_2 (see power levels at Dec 2 in Fig. 3.4). Therefore, the relayed signal is considered as a common signal, and consequently decoded at both receivers. The decoding rules are:

- Decoder 1: $(X_{1p}(i), X_{1c}(j), X_{2r}(k), Y_1) \in A_\epsilon^{(n)}$,
- Decoder 2: $(X_{1c}(j), X_{2r}(k), X_{2p}(l), Y_2) \in A_\epsilon^{(n)}$.

For allocating power among the codebooks, the LDM suggests X_{2r} and X_{1c} to have the same power level. Hence, we set $P_{2p}=P_2$, $P_{2r}=P_{1c}=\frac{P_1-P_{1p}}{2}$, where $P_{1p}=\frac{1}{a^2}$ for $a\leq 1$, and $P_{1p}=0$ otherwise.

Lemma 3.6. *The following region is $\mathcal{R}(0.5, 1.5)$, and $\mathcal{R}(0.5, 0.5)$ achievable for $a\leq 1$, and $1<a$ cases, respectively:*[7]

$$R_1 \leq \mathcal{C}(\frac{P_1 + P_{1p}}{2}) \tag{3.34}$$

$$R_2 \leq \mathcal{C}(\frac{P_{2p}}{d}) + C_{21} \tag{3.35}$$

$$R_2 \leq \mathcal{C}(\frac{a^2(P_1 - P_{1p}) + 2P_{2p}}{2d}) \tag{3.36}$$

$$R_1 + R_2 \leq \mathcal{C}(P_{1p}) + \mathcal{C}(\frac{a^2(P_1 - P_{1p}) + P_{2p}}{d}) - \frac{1}{2}, \tag{3.37}$$

where d is defined in (3.27).

Proof. See section 3.7.4.2. □

3.5.2.4 Private-Common Cooperative Case: $P_2+1<a^2P_1, P_2+1<a^2$

For this case, the corresponding LDM shown in Fig. 3.5 illustrates that since a is quite large, Transmitter 1 can spend a small amount of power (i.e., less than 1 unit) to relay the

[6]One can infer that in the non cooperative case, the relayed signal is not required (in the constant gap sense). This is because in the corresponding LDM (not shown in the chapter), X_{2p} and X_{2r} would share the same power level.

[7]For $1 < a$ the second condition for R_2 is redundant.

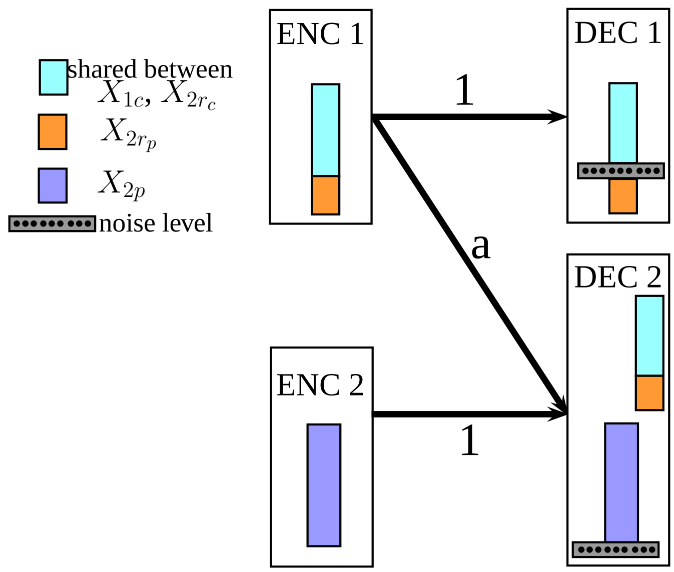


Figure 3.5: LDM for GZIC with $C_{12} = 0$: Private-Common Cooperative Case.

signal X_{2r_p} for User 2 without a decoding requirement at Receiver 1. This technique can be considered as the counterpart of the transmission at the noise level originally proposed for the conventional GIC, and used in this chapter for $a \leq 1$ configurations. This model also suggests that a common signal, named X_{2r_c}, needs to be relayed for User 2, similar to the $a^2 \leq P_2 + 1$ case. Therefore, the following signaling is used:

$$\begin{aligned} X_1 &= X_{1c} + X_{2r_p} + X_{2r_c}, \\ X_2 &= X_{2p}, \end{aligned} \tag{3.38}$$

where X_{1c}, X_{2r_c} are common signals and X_{2p}, X_{2r_p} are private signals for User 2. The decoding rules are:

- Decoder 1: $(X_{1c}(i), X_{2r}(j), Y_1) \in A_\epsilon^{(n)}$,
- Decoder 2: $\big(X_{1c}(i), X_{2r}(j), X_{2r_p}(k), X_{2p}(l), Y_2\big) \in A_\epsilon^{(n)}$.

To avoid the complexity of jointly decoding four signals at Receiver 2, first $X_{1c}, X_{2r_c}, X_{2r_p}$ are jointly decoded assuming X_{2p} as noise and then X_{2p} is decoded. To appropriately allocate the power of Transmitter 1 to different codebooks, the LDM suggests to provide $\min\{1, P_1\}$ amount of power for X_{2r_p} transmission. Since this signal is received below the noise level of Receiver 1, there is no much cost associated with its transmission, and therefore, it can be considered as the first signal to get its share of power. The rest of the power is equally distributed between X_{1c} and X_{2r_c} as the case for $a^2 \leq P_2 + 1$. One question to answer is how to divide the capacity C_{21} between X_{2r_p} and X_{2r_c}? Again, we give the priority to X_{2r_p}, and name its share of the cooperative capacity, C'. Below, $P_1 \leq 1$, and $1 < P_1$ cases are considered, respectively.

If $P_1 \leq 1$, we set $P_{2r_p} = P_1$, $P_{1c} = P_{2r_c} = 0$, and $C' = C_{21}$. It is easy to show $(0, \min\{\mathcal{C}(P_2)+C_{21}, \mathcal{C}(a^2P_1+P_2)\})$ is achievable. Comparing this rate pair with the upper bounds (3.18)-(3.20) proves that $\mathcal{R}(0.5, 0.5)$ is achievable.

Now, it is assumed that $1 < P_1$. We set $P_{2r_p} = 1$, $P_{1c} = P_{2r_c} = \frac{P_1-1}{2}$, and $P_{2p} = P_2$. The decoding error analysis of section 3.7.4.3 states that $R_{2r_p} \leq \mathcal{C}(\frac{a^2}{P_2+1})$, and therefore, we set $C' = \min\{\mathcal{C}(\frac{a^2}{P_2+1}), C_{21}\}$.

Lemma 3.7. *The following region is* $\mathcal{R}(1, 0.5)$ *achievable:*

$$\begin{aligned} R_1 &\leq \mathcal{C}(\frac{P_1 - 1}{4}) \\ R_2 &\leq \mathcal{C}(P_2) + C_{21} \\ R_1 + R_2 &\leq \mathcal{C}(a^2P_1 + P_2) - \frac{1}{2}. \end{aligned} \tag{3.39}$$

Proof. See section 3.7.4.3. □

3.5.3 Bidirectional Cooperation

Now, it is assumed that both of C_{12} and C_{21} can be non-zero. C_{12} and C_{21} are respectively used for the zero-forcing and relaying purposes as suggested by investigation of the unidirectional case. Based on the observations obtained in the unidirectional cooperation case, achievable schemes are proposed to characterize the capacity region of the GZIC with bidirectional cooperation up to 1.71 bits per user. Five scenarios are possible:

A. $a^2P_1 \leq 1$: It is explained that treating interference as noise makes $\mathcal{R}(0, 1)$ achievable. See section 3.7.5.1 for details.

B. $1 \leq a^2P_1 \leq P_2+1$: In this case, since relaying can increase User 2's rate by one bit, C_{21} is not used in this regime (similar to section 3.5.2.2.1). Section 3.7.5.2 shows the zero-forcing technique utilized in section 3.5.2.1 is sufficient to achieve $\mathcal{R}(0.5, 1.5)$.

C. $P_2 + 1 \leq a^2 P_1, a^2 \leq 1$: Both relaying and zero-forcing techniques are used to show $\mathcal{R}(0.5, 1.71)$ is achievable. To avoid the complexity of decoding four signals at Receiver 1, a compressed version of User 1's private signal is zero-forced at Receiver 2.

D. $P_2 + 1 \leq a^2 P_1, 1 \leq a^2 \leq P_2 + 1$: In this regime, a simple combination of zero-forcing and relaying techniques proposed for the unidirectional case is used to prove $\mathcal{R}(1, 1.21)$ is achievable.

E. $P_2+1 \leq a^2 P_1, P_2+1 \leq a^2$: In this case, because Receiver 2 gets a very strong signal from Transmitter 1, there is not much benefit in using C_{12}. In fact, the same scheme employed for $C_{12}=0$ makes $\mathcal{R}(1, 0.5)$ achievable. The gap analysis of section 3.7.4.3 is also valid for the case of $C_{12} > 0$.

It can be seen that for scenarios A, B, and E, the sum-capacity is approximated within 2 bits. In the following, scenarios C and D are further elucidated. Appendices 3.7.5.3, and 3.7.5.4 argue the same gap holds for scenarios C and D, which assures the maximum gap of 2 bits on the sum-capacity for all regimes.

3.5.3.1 $P_2 + 1 \leq a^2 P_1, a \leq 1$

A natural generalization of the schemes proposed for the unidirectional cooperation case, is to consider the following signaling:

$$\begin{aligned} X_1 &= X_{1p} + X_{1c} + X_{1z} + X_{2r}, \\ X_2 &= X_{2p} - aX_{1z}, \end{aligned}$$

where X_{2r} is decoded at both receivers. To avoid the complexity of jointly decoding of four signals at Receiver 1, User 1's private signal is compressed and sent to the other transmitter via the cooperative link with capacity C_{12}.[8] Then, Transmitter 2 zero-forces the compressed version of the signal. In particular, the following signaling is used:

$$\begin{aligned} X_1 &= X_{1p} + X_{1c} + X_{2r}, \\ X_2 &= X_{2p} - \hat{X}_{1p}, \end{aligned} \tag{3.40}$$

[8] Instead of the compression, one might sequentially decode (X_{1c}, X_{1z}, X_{2r}) and X_{1p}, similar to the approach of section 3.5.2.4.

where aX_{1p} is compressed with distortion 1, i.e., $aX_{1p} = \hat{X}_{1p} + Z$, with $Z \sim \mathcal{N}(0,1)$. The compression, imposes the constraint $\mathcal{C}(a^2 P_{1p} - 1) \leq C_{12}$. For the power allocation, it is recalled from the unidirectional cooperation case that at most half of Transmitter 2's power is allocated for zero-forcing to not harm its own maximum rate by more than half a bit. Therefore, the following power allocation is considered:

$$\begin{aligned} P_{1p} &= \frac{\min\{2^{2C_{12}}, \frac{P_2}{2} + 1\}}{a^2}, \\ P_{1c} = P_{2r} &= \frac{P_1 - P_{1p}}{2}, \\ P_{2p} &= P_2 - (a^2 P_{1p} - 1). \end{aligned} \tag{3.41}$$

The term containing C_{12} is due to the compression rate constraint.

Lemma 3.8. *The region given in (3.34)-(3.37) is* $\mathcal{R}(0.5, 1.71)$ *achievable.*

Proof. See section 3.7.5.3. □

3.5.3.2 $P_2 + 1 \leq a^2 P_1, 1 \leq a^2 \leq P_2 + 1$

In this regime, the zero-forcing technique used for $C_{21} = 0$, and the relaying technique used for $C_{12} = 0$ are simply combined, i.e.,

$$\begin{aligned} X_1 &= X_{1z} + X_{1c} + X_{2r}, \\ X_2 &= X_{2p} - aX_{1z}. \end{aligned} \tag{3.42}$$

Similar to the unidirectional case, the following power allocation is used:

$$\begin{aligned} P_{1z} &= \frac{P_2}{2a^2}, \\ P_{1c} = P_{2r} &= \frac{P_1 - P_{1z}}{2}, \\ P_{2p} &= \frac{P_2}{2}. \end{aligned}$$

Lemma 3.9. *The following region is* $\mathcal{R}(1, 1.21)$ *achievable:*

$$R_1 \le \mathcal{C}\left(\frac{P_1 + \frac{P_2}{2a^2}}{2}\right) \tag{3.43}$$

$$R_1 \le \mathcal{C}\left(\frac{P_1 - \frac{P_2}{2a^2}}{2}\right) + C_{12} \tag{3.44}$$

$$R_2 \le \mathcal{C}\left(\frac{P_2}{2}\right) + C_{21} \tag{3.45}$$

$$R_2 \le \mathcal{C}\left(\frac{2a^2 P_1 + P_2}{4}\right) \tag{3.46}$$

$$R_1 + R_2 \le \mathcal{C}(a^2 P_1) + C_{12} - \frac{1}{2} \tag{3.47}$$

$$R_1 + R_2 \le \mathcal{C}(a^2 P_1) + \mathcal{C}\left(\frac{P_2}{2a^2}\right) - \frac{1}{2} \tag{3.48}$$

$$R_1 + R_2 \le \mathcal{C}(P_1) + \mathcal{C}\left(\frac{P_2}{2}\right) - \frac{1}{2}. \tag{3.49}$$

Proof. See section 3.7.5.4. □

3.5.4 Bidirectional v.s. Unidirectional Cooperation: A Numerical Analysis View

In the previous section, it has been observed that unidirectional cooperation is optimum in the constant gap sense for the following two cases: $1 \le a^2 P_1 \le P_2 + 1$, and $P_2 + 1 \le \min\{a^2 P_1, a^2\}$. In this section, however, various numerical results are presented to demonstrate that bidirectional cooperation between the transmitters may provide better rate pairs compared to the unidirectional cooperation with a relatively larger total cooperation capacity. To achieve this goal, it is assumed that the total cooperation capacity $C \triangleq C_{12} + C_{21}$ is fixed and can be arbitrarily distributed between the cooperative links. The achievable region optimized over such distribution is compared to the capacity region or the outer bound region corresponding to the extreme case of unidirectional cooperation , i.e., the Cognitive Radio Z Channel (CRZC).[9]

[9] In the cognitive setup, the cognitive transmitter knows the whole message of the primary user [36]. This property can be modeled as having a GIC with unidirectional transmitter cooperation. In such a configuration, the capacity of the cooperative link suffices to be equal to the message rate of the primary user. This fact is described for the MAC with conferencing encoders in [34]. In this section, it is shown that sharing that required capacity, or even smaller than that amount, between the bidirectional cooperative links can provide better rate pairs depending on the channel parameters.

The CRZC can be either in the form of Fig. 3.3 (a) or Fig. 3.3 (b). The former and the latter forms, respectively called type I and II, serve as the baseline for comparison in Figs. 3.6 and 3.7, respectively. The capacity regions of the type I CRZC (for all a values), and type II CRZC (for $a \leq 1$) are expressed by:

$$R_1^{\mathrm{up}} \leq \mathcal{C}(P_1)$$
$$R_2^{\mathrm{up}} \leq \mathcal{C}(P_2),$$

and the union of the regions over $0 \leq \rho \leq 1$

$$R_1^{\mathrm{up}} \leq \mathcal{C}(\rho P_1)$$
$$R_2^{\mathrm{up}} \leq \mathcal{C}\left(\frac{(\sqrt{a^2(1-\rho)P_1} + \sqrt{P_2})^2}{1 + a^2\rho P_1}\right),$$

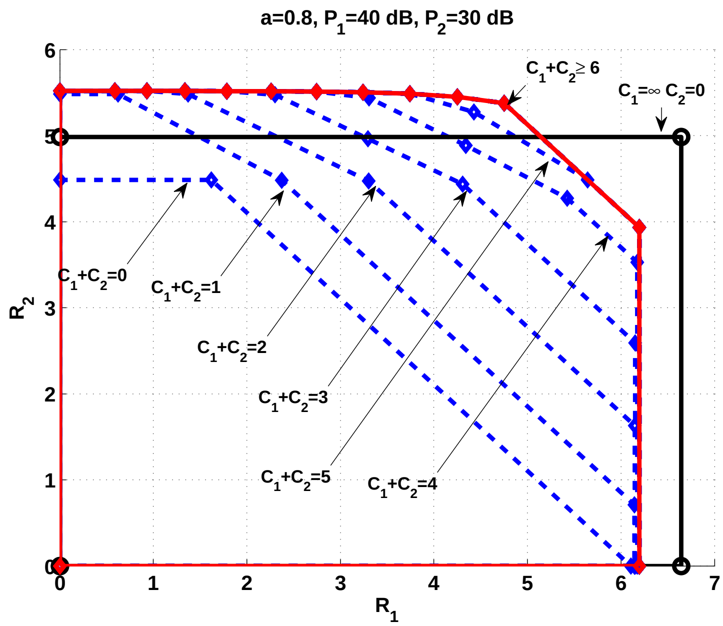
respectively [38]. For type II CRZC with $1 < a$, the upper bound region described by (3.18)-(3.22) with $C_{12} = 0$, and $C_{21} = \infty$ is used.

Figs. 3.6 and 3.7 evaluate the achievable rates for different values of the channel parameters.[10] Looking at Fig. 3.6, it is seen that sharing the cooperative capacity can significantly increase the maximum of R_2, with respect to the type I CRZC, as a gets larger. In contrast, Fig. 3.7 shows bidirectional cooperation with relatively smaller total cooperation capacity can substantially enhance the sum-rate compared to the type II CRZC.

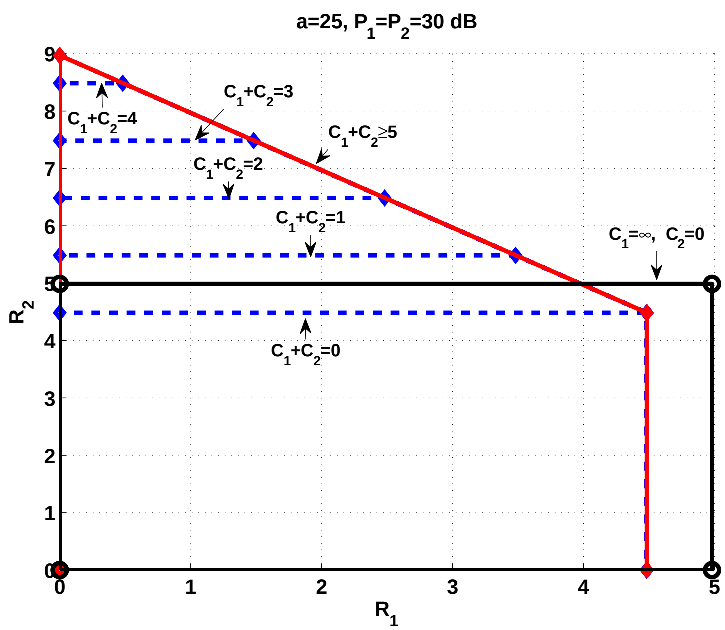
3.6 Summary

The sum-capacity of the symmetric IC has been characterized within 2.13 bits. It has been shown that for GZIC with bidirectional cooperation, $\mathcal{R}(1, 1.71)$ is achievable. As another outcome of the gap analysis, the sum-capacity of the channel has been determined up to 2 bits. To obtain the results, basic communication techniques including Han-Kobayashi, zero-forcing, simple relaying, and transmission at the noise level schemes are employed. In addition, with the aid of signal compression or sequential decoding methods, the decoding complexity of the utilized schemes is limited to jointly decoding of at most three inde-

[10] To make figures more readable, C_1, and C_2 are used instead of C_{12}, and C_{21}, respectively.

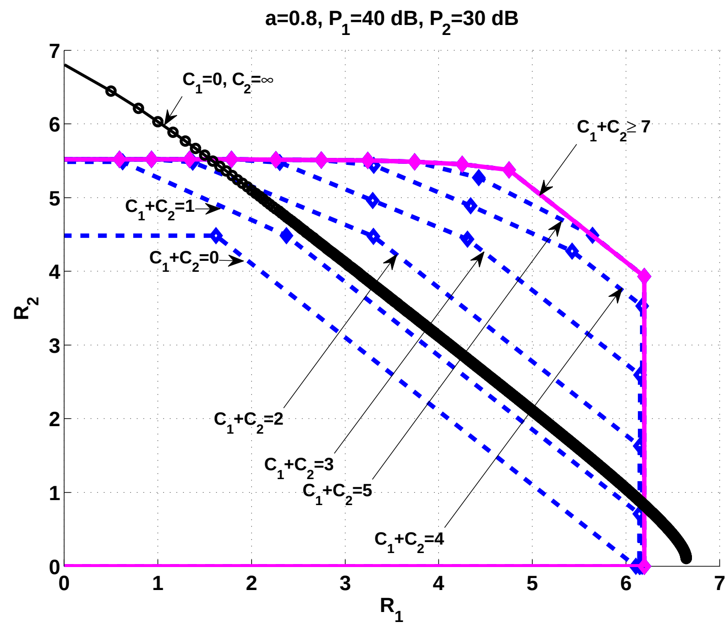


(a)

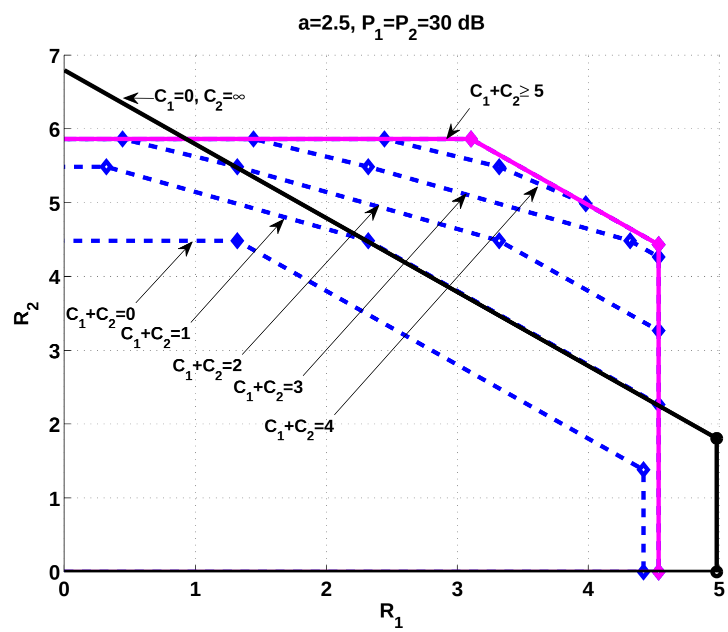


(b)

Figure 3.6: Sharing the cooperative capacity between bidirectional cooperative links can enhance the maximum of R_2 compared to the type I CRZC. For $P_2 + 1 \leq a^2$ (not shown in this figure), the sum-rate can also be increased.



(a)



(b)

Figure 3.7: Sharing the cooperative capacity between bidirectional cooperative links can improve the sum-rate compared to the type II CRZC.

pendent signals at each receiver. It has been observed that for GZIC, unidirectional and bidirectional cooperation almost similarly perform (in the promised constant gap sense) in two scenarios:

1) C_{21} is not required when the relaying power is smaller than the direct transmission power, i.e., $a^2P_1 \leq P_2 + 1$.

2) C_{12} is not necessary when the relaying power and the relay gain are sufficiently large, i.e., $P_2 + 1 \leq a^2P_1$, and $P_2 + 1 \leq a^2$.

Furthermore, it has been shown that properly sharing the total cooperative capacity between the bidirectional links can enhance the achievable rate pairs in some scenarios.

3.7 Proofs for Chapter 3

3.7.1 Proof of Theorem 3.1

In the proofs of different upper bounds, the following statements, respectively denoted by (a)-(i), will be repeatedly used:

- (a): $q^n \triangleq (q_{12}^n, q_{21}^n)$ is a function of the messages m_1 and m_2.
- (b): Removing condition does not decrease the entropy
- (c): Fano's inequality
- (d): X_1^n, X_2^n are functions of $(m_1, q^n), (m_2, q^n)$ pairs, respectively.
- (e): $m_1 \to (q^n, X_1^n) \to Y_1^n$ and $m_2 \to (q^n, X_2^n) \to Y_2^n$ are Markov chains.
- (f): $S_1 \triangleq aX_1 + N_2$, $S_2 \triangleq bX_2 + N_1$, and $S_1' \triangleq aX_1 + N'$.
- (g): $X_1 \leftrightarrow q \leftrightarrow X_2$ is Markov chain, and the fact that X_1, N_1 are independent and also X_2, N_2 are independent. In addition, the channel is additive.
- (h): Conditional version of the maximum entropy theorem. G stands for Gaussian.
- (i): Refer to [41].
- (j): m_1 and m_2 are independent.

- (k): Chain rule of mutual information.

We also define $R_{\text{sum}}^{\text{up}} \triangleq n(R_1 + R_2 - \epsilon_n)$.

3.7.1.1 Proof for the Bound (3.5) on $R_1 + R_2$

$$
\begin{aligned}
R_{\text{sum}}^{\text{up}} &= h(m_1, m_2) - n\epsilon_n \\
&\overset{(a)}{=} h\left(m_1, m_2, q^n(m_1, m_2)\right) - n\epsilon_n \\
&= h(m_1, m_2|q^n) + h(q^n) - n\epsilon_n \\
&= h(m_1|q^n) + h(m_2|q^n, m_1) + h(q^n) - n\epsilon_n \\
&\overset{(b)}{\leq} h(m_1|q^n) + h(m_2|q^n) + h(q^n) - n\epsilon_n \\
&\leq h(m_1|q^n) + h(m_2|q^n) + n(C_{12} + C_{21}) - n\epsilon_n \\
&\overset{(c)}{\leq} I(m_1; Y_1^n|q^n) + I(m_2; Y_2^n|q^n) + n(C_{12} + C_{21}) \\
&\overset{(d)}{=} h(Y_1^n|q^n) - h(Y_1^n|q^n, m_1, X_1^n) + h(Y_2^n|q^n) - h(Y_2^n|q^n, m_2, X_2^n) + n(C_{12} + C_{21}) \\
&\overset{(e)}{=} h(Y_1^n|q^n) - h\left(Y_1^n|q^n, X_1^n\right) + h(Y_2^n|q^n) - h\left(Y_2^n|q^n, X_2^n\right) + n(C_{12} + C_{21}) \\
&\overset{(f)}{\leq} h(Y_1^n, S_1^n|q^n) - h\left(Y_1^n, S_1^n|q^n, X_1^n\right) + nC_{12} + h(Y_2^n, S_2^n|q^n) - h\left(Y_2^n, S_2^n|q^n, X_2^n\right) + nC_{21} \\
&= h(Y_1^n, S_1^n|q^n) - h\left(S_2^n, N_2^n|q^n, X_1^n\right) + nC_{12} + h(Y_2^n, S_2^n|q^n) - h\left(s_1^n, N_1^n|q^n, X_2^n\right) + nC_{21} \\
&\overset{(g)}{=} h(Y_1^n, S_1^n|q^n) - h\left(S_2^n, N_2^n|q^n\right) + h(Y_2^n, S_2^n|q^n) - h\left(s_1^n, N_1^n|q^n\right) + n(C_{12} + C_{21}) \\
&= h(Y_1^n|S_1^n, q^n) - h\left(N_2^n\right) + h(Y_2^n|S_2^n, q^n) - h\left(N_1^n\right) + n(C_{12} + C_{21}) \\
&\overset{(h,b)}{\leq} h(Y_1^n|S_{1G}^n) - h\left(N_2^n\right) + h(Y_2^n|S_{2G}^n) - h\left(N_1^n\right) + n(C_{12} + C_{21}) \\
&\overset{(i)}{\leq} n\left\{\mathcal{C}\left(b^2P_2 + \frac{P_1 + 2b\sqrt{P_1P_2}}{1 + a^2P_1}\right) + \mathcal{C}\left(a^2P_1 + \frac{P_2 + 2a\sqrt{P_1P_2}}{1 + b^2P_2}\right) + C_{12} + C_{21}\right\},
\end{aligned}
$$

3.7.1.2 Proof for the Bounds (3.6) and (3.7) on $R_1 + R_2$

Here, we provide the proof for (3.6). We can simply adapt the proof for the bound (3.7).

$$
\begin{aligned}
R_{\text{sum}}^{\text{up}} &= h(m_1) + h(m_2) - n\epsilon_n \\
&= h(m_1|m_2) + h(m_2) - n\epsilon_n \\
&\overset{(c)}{\leq} I(m_1; Y_1^n|m_2) + I(m_2; Y_2^n) \\
&\leq I(m_1; Y_1^n, q_{12}^n|m_2) + I(m_2; Y_2^n) \\
&\overset{(k)}{=} I(m_1; Y_1^n|m_2, q_{12}^n) + I(m_1; q_{12}^n|m_2) + I(m_2; Y_2^n) \\
&\overset{(d)}{=} I(m_1; Y_1^n|m_2, q_{12}^n, X_2^n) + I(m_1; q_{12}^n|m_2) + I(m_2; Y_2^n) \\
&\overset{(f)}{\leq} I(m_1; Y_1^n, S_1^{'n}|m_2, q_{12}^n, X_2^n) + I(m_1; q_{12}^n|m_2) + I(m_2; Y_2^n) \\
&\leq I(m_1; Y_1^n, S_1^{'n}|m_2, q_{12}^n, X_2^n) + h(q_{12}^n) + I(m_2; Y_2^n) \\
&\leq I(m_1; Y_1^n, S_1^{'n}|m_2, q_{12}^n, X_2^n) + nC_{12} + I(m_2; Y_2^n) \\
&= h(X_1^n + N_1^n, S_1^{'n}|m_2, q_{12}^n, X_2^n) - h(N_1^n, N_2^n) + nC_{12} + I(m_2; Y_2^n) \\
&\overset{(b)}{\leq} h(X_1^n + N_1^n|m_2, q_{12}^n, X_2^n, S_1^{'n}) + h(S_1^{'n}|m_2, q_{12}^n) - h(N_1^n, N_2^n) + nC_{12} + h(Y_2^n) - h(Y_2^n|m_2) \\
&\overset{(b)}{\leq} h(X_1^n + N_1^n|m_2, q_{12}^n, X_2^n, S_1^{'n}) + h(S_1^{'n}|m_2, q_{12}^n) - h(N_1^n, N_2^n) + nC_{12} + h(Y_2^n) - h(Y_2^n|m_2, q_{12}^n) \\
&\overset{(b,f)}{\leq} h(X_1^n + N_1^n|S_1^{'n}) - h(N_1^n, N_2^n) + nC_{12} + h(Y_2^n) \\
&\leq n\left\{\mathcal{C}\left(\frac{\max\{1-a^2, 0\}P_1}{a^2P_1 + 1}\right) + \mathcal{C}\Big((\sqrt{a^2P_1} + \sqrt{P_2})^2\Big) + C_{12}\right\},
\end{aligned}
$$

where, N', N_2 are correlated. The bound is optimized over the correlation coefficient to obtain (3.6) (cf. [41]).

3.7.1.3 Proof for the Bound (3.8) on $R_1 + R_2$

This bound is obtained by allowing full cooperation at both transmitter and receiver sides.

$$\begin{aligned}
R_{\text{sum}}^{\text{up}} &= h(m_1, m_2) - \epsilon_n \\
&\leq I(m_1, m_2; Y_1^n, Y_2^n) \\
&\leq h(Y_1^n, Y_2^n) - h(N_1^n, N_2^n) \\
&\leq n \log |\mathbf{A}| \\
&\overset{(l)}{\leq} n \left\{ \mathcal{C} \left(P_1 P_2 (1-ab)^2 + (\sqrt{a^2 P_1} + \sqrt{P_2})^2 + (\sqrt{b^2 P_2} + \sqrt{P_1})^2 \right) \right\},
\end{aligned}$$

where $\mathbf{A} = \begin{pmatrix} P_1 + b^2 P_2 + 2b\rho' + 1 & aP_1 + (1+ab)\rho' + bP_2 \\ aP_1 + (1+ab)\rho' + bP_2 & a^2 P_1 + P_2 + 2a\rho' + 1 \end{pmatrix}$, with $\rho' \triangleq \mathbb{E}[x_1 x_2]$, which accounts for the correlation between the channel inputs. In (l) each term containing ρ' is maximized separately.

3.7.1.4 Proof for the Bound (3.9) on $2R_1 + R_2$

$$
\begin{aligned}
n(2R_1 + R_2 - \epsilon_n) &= h(m_1, m_2) + h(m_1) - n\epsilon_n \\
&\overset{(a)}{=} h\left(m_1, m_2, q^n(m_1, m_2)\right) + h(m_1) - n\epsilon_n \\
&\leq h(m_1|m_2, q^n) + h(m_2|q^n) + h(m_1) + n(C_{12} + C_{21}) - n\epsilon_n \\
&\overset{(c)}{\leq} I(m_1; Y_1^n|m_2, q^n) + I(m_2; Y_2^n|q^n) + I(m_1; Y_1^n) + n(C_{12} + C_{21}) \\
&\overset{(d)}{=} I(m_1; Y_1^n|m_2, q^n, X_2^n) + I(m_2; Y_2^n|q^n) + I(m_1; Y_1^n) + n(C_{12} + C_{21}) \\
&\leq I(m_1; Y_1^n, S_1^n|m_2, q^n, X_2^n) + I(m_2; Y_2^n, S_2^n|q^n) + I(m_1; Y_1^n) + n(C_{12} + C_{21}) \\
&\overset{(e)}{=} I(m_1; Y_1^n, S_1^n|q^n, X_2^n) + I(m_2; Y_2^n, S_2^n|q^n) + I(m_1; Y_1^n) + n(C_{12} + C_{21}) \\
&\overset{(d)}{=} h(X_1^n + N_1^n, S_1^n|q^n, X_2^n) - h(N_1^n, N_2^n) + I(m_2; Y_2^n, S_2^n|q^n) + I(m_1; Y_1^n) + \\
&\quad + n(C_{12} + C_{21}) \\
&= h(X_1^n + N_1^n, S_1^n|q^n, X_2^n) - h(N_1^n, N_2^n) + h(Y_2^n, S_2^n|q^n) - h(Y_2^n, S_2^n|m_2, q^n) + \\
&\quad + I(m_1; Y_1^n) + n(C_{12} + C_{21}) \\
&\overset{(d)}{=} h(X_1^n + N_1^n, S_1^n|q^n, X_2^n) - h(N_1^n, N_2^n) + h(Y_2^n|S_2^n, q^n) + h(S_2^n|q^n) - \\
&\quad - h(S_1^n, N_1^n|m_2, q^n, X_2^n) + I(m_1; Y_1^n) + n(C_{12} + C_{21}) \\
&\overset{(e)}{=} h(X_1^n + N_1^n|S_1^n, q^n, X_2^n) - h(N_1^n, N_2^n) + h(Y_2^n|S_2^n, q^n) + h(S_2^n|q^n) - \\
&\quad - h(N_1^n|X_2^n, q^n, m_2) + h(Y_1^n) - h(Y_1^n|m_1, q^n) + n(C_{12} + C_{21}) \\
&\overset{(d,e)}{=} h(X_1^n + N_1^n|S_1^n, q^n, X_2^n) - h(N_1^n, N_2^n) + h(Y_2^n|S_2^n, q^n) + h(S_2^n|q^n) - \\
&\quad - h(N_1^n) + h(Y_1^n) - h(S_2^n|X_1, q^n) + n(C_{12} + C_{21}) \\
&= h(X_1^n + N_1^n|S_1^n, q^n, X_2^n) - h(N_1^n, N_2^n) + h(Y_2^n|S_2^n, q^n) - h(N_1^n) + h(Y_1^n) + \\
&\quad + n(C_{12} + C_{21}) \\
&\leq h(X_1^n + N_1^n|S_1^n) - h(N_1^n, N_2^n) + h(Y_2^n|S_2^n) - h(N_1^n) + h(Y_1^n) + n(C_{12} + C_{21}) \\
&\leq n\Big\{\mathcal{C}\big(\frac{\max\{1 - a^2, 0\}P_1}{a^2P_1 + 1}\big) + \mathcal{C}\big(a^2P_1 + \frac{P_2 + 2a\sqrt{P_1P_2}}{b^2P_2 + 1}\big) + \mathcal{C}\Big((\sqrt{b^2P_2} + \sqrt{P_1})^2\Big) \\
&\quad + C_{12} + C_{21}\Big\}
\end{aligned}
$$

Similarly (3.10) can be proved.

3.7.2 Proof of Theorem 3.2

To prove a constant gap, it is sufficient to compare each rate constraint in the achievablity scheme to the upper bound with the same GDOF (cf. section 3.4.1). The key steps of the proof are as follows:

3.7.2.1 $a \leq 1$

1) $R_2 - \min\{R_1, R_5\} \leq \frac{1}{2}$

2) $R_{\text{up}}^5 - R_3 \leq \mathcal{C}(18.15)$

3) For $\mathcal{C}(\eta P') \leq \min\{C_{12}, C_{21}\}$,

$$R_{\text{up}}^5 - \min_{l \in \{1,2,4,5\}} \{R_l\} \leq \mathcal{C}(18.15)$$

4) For $C_{12} \leq \mathcal{C}(\eta P') \leq C_{21}$,

$$R_{\text{up}}^3 - \min_{l \in \{1,2,4,5\}} \{R_l\} \leq \mathcal{C}(14)$$

5) For $C_{21} \leq \mathcal{C}(\eta P') \leq C_{12}$,

$$R_{\text{up}}^4 - \min_{l \in \{1,2,4,5\}} \{R_l\} \leq \mathcal{C}(14)$$

6) For $C_{12}, C_{21} \leq \mathcal{C}(\eta P')$,

$$\begin{aligned} R_{\text{up}}^2 - R_4 &\leq 2 \\ \min\{R_{\text{up}}^3, R_{\text{up}}^4\} - \min_{l \in \{1,2,5\}} \{R_l\} &\leq \mathcal{C}(14) \end{aligned}$$

where R_{up}^k for $k \in \{2, \cdots, 5\}$ correspond to the sum-rate upper bounds given in (3.5)-(3.8), and for $l \in \{1, \cdots, 5\}$, $R_l \triangleq R_l' + 2R_u$.

3.7.2.2 $1 < a$

1) $R_2 - \min\{R_1, R_5\} \leq \frac{1}{2}$

2) $R_{\text{up}}^5 - R_3 \le \mathcal{C}(8.02)$

3) For $\mathcal{C}(\eta P') \le \min\{C_{12}, C_{21}\}$,

$$R_{\text{up}}^5 - \min_{l\in\{1,2,4,5\}}\{R_l\} \le \mathcal{C}(8.02)$$

4) For $C_{12} \le \mathcal{C}(\eta P') \le C_{21}$,

$$R_{\text{up}}^3 - \min_{l\in\{1,2,4,5\}}\{R_l\} \le \mathcal{C}(4.05)$$

5) For $C_{21} \le \mathcal{C}(\eta P') \le C_{12}$,

$$R_{\text{up}}^4 - \min_{l\in\{1,2,4,5\}}\{R_l\} \le \mathcal{C}(4.05)$$

6) For $C_{12}, C_{21} \le \mathcal{C}(\eta P')$,

$$\begin{aligned} R_{\text{up}}^1 - R_4 &\le 1 \\ \min\{R_{\text{up}}^3, R_{\text{up}}^4\} - \min_{l\in\{1,2,5\}}\{R_l\} &\le \mathcal{C}(4.05) \end{aligned}$$

where $R_{\text{up}}^1 \triangleq 2\mathcal{C}(P) + C_{12} + C_{21}$.

The gaps are calculated using the following inequality:

$$\frac{d_1 + d_2 P + d_3 P^2}{d_4 + d_5 P + d_6 P^2} \le \max\{\frac{d_1}{d_4}, \frac{d_2}{d_5}, \frac{d_3}{d_6}\} \text{ for } 0 < d_{i\in\{1,\cdots,6\}}.$$

Note that for both cases, the worst-case gap is obtained by comparing the achievable rate with R_{up}^5, which can be tightened by considering correlation between N_1, N_2 in Fig. 3.1.

3.7.3 Proof of Achievable Rate and Gap Analysis for Zero-Forcing Scenario

The resulting rate constraints from error analysis at Decoder 1 are:

$$
\begin{aligned}
R_{1p} &\leq \mathcal{C}(P_{1p}) \\
R_{1z} &\leq \min\{\mathcal{C}(P_{1z}), C_{12}\} \\
R_{1c} &\leq \mathcal{C}(P_{1c}) \\
R_{1p} + R_{1z} &\leq \mathcal{C}\left(P_{1p} + P_{1z}\right) \\
R_{1p} + R_{1c} &\leq \mathcal{C}(P_{1p} + P_{1c}) \\
R_{1z} + R_{1c} &\leq \mathcal{C}(P_{1z} + P_{1c}) \\
R_{1p} + R_{1z} + R_{1c} &\leq \mathcal{C}(P_1),
\end{aligned}
$$

and the constraints for Decoder 2 are:

$$
\begin{aligned}
R_{2p} &\leq \mathcal{C}(\frac{P_{2p}}{d}) \\
R_{2p} + R_{1c} &\leq \mathcal{C}(\frac{a^2 P_{1c} + P_{2p}}{d}),
\end{aligned}
$$

where d is defined in (3.27). Note that there is no individual rate constraint on R_{1c} at Decoder 2 because the joint typical decoder does not declare an error in the case that only X_{1c} is wrongly decoded [52]. FME is applied to obtain the constraints in terms of $R_1 \triangleq R_{1p} + R_{1z} + R_{1c}$, and $R_2 \triangleq R_{2p}$. Removing the redundant inequalities due to the polymatroid structure of the rate constraints at each decoder, leads to the region (3.24)-(3.26). To simplify the gap analysis, it is noted that the subsequent region is also achievable:

$$
\begin{aligned}
R_1 &\leq \mathcal{C}(\frac{P_1}{2}) && (3.50) \\
R_2 &\leq \mathcal{C}(\frac{P_2}{4}) && (3.51) \\
R_1 + R_2 &\leq \mathcal{C}(\frac{1}{a^2}) + \mathcal{C}(\frac{a^2 P_1 - 1 + P_2}{4}) + C_{12} && (3.52) \\
R_1 + R_2 &\leq \mathcal{C}(\frac{\min\{a^2 P_1 - 1, P_2\} + 2}{2a^2}) + \mathcal{C}(\frac{\max\{a^2 P_1 - 1, P_2\}}{4}). && (3.53)
\end{aligned}
$$

It is clear that the preceding region is smaller than that of (3.24)-(3.26). To obtain this region, we use (3.28) and set $P_{1c} = \frac{P_1 - \frac{1}{a^2}}{2}$ and $P_{2p} = \frac{P_2}{2}$ in the achievable region (3.24)-(3.26). In addition, in getting (3.50) and (3.53) we notice $\frac{P_1}{2} \leq P_{1p} + P_{1c}$ and $\max\{a^2 P_{1c}, P_{2p}\} \leq a^2 P_{1c} + P_{2p}$. In the following, the region is compared to the upper bound to show that $\mathcal{R}(0.5, 1)$ is achievable:

$$
\begin{aligned}
(3.18) - (3.50) &\leq 0.5 \\
(3.19) - (3.51) &\leq 1 \\
(3.21) - (3.52) &\overset{(a)}{\leq} 1.5 \\
(3.18) + (3.19) - (3.53) &\leq 1.5.
\end{aligned}
$$

By $(i) \pm (j)$, we mean the sum/difference of the right hand side of Eqs. (i) and (j). To attain (a), the fact that $\mathcal{C}(\frac{\max\{1-a^2,0\}P_1}{1+a^2 P_1}) \leq \mathcal{C}(\frac{1}{a^2})$ is used. It is remarked that $R_{1p} < 1$ for the case of $1 < a$. Therefore, we can set $P_{1p} = 0$, i.e., we do not use X_{1p} in order to simplify the scheme. It can be shown that $\mathcal{R}(0.5, 0.5)$ is achievable in this case.

3.7.4 Proof of Achievable Rate and Gap Analysis for Relaying Scenario

In this section, the achievable rate and gap analysis corresponding to the three scenarios identified in section 3.5.2.2 are derived in detail.

3.7.4.1 Non-Cooperative Case

The decoding rules lead to the following rate constraints at Decoder 1 and Decoder 2:

Decoder 1:

$$
\begin{aligned}
R_{1p} &\leq \mathcal{C}(P_{1p}) \\
R_{1c} &\leq \mathcal{C}(P_{1c}) \\
R_{1p} + R_{1c} &\leq \mathcal{C}(P_1),
\end{aligned}
$$

Decoder 2:

$$
\begin{aligned}
R_{2p} &\leq \mathcal{C}(\frac{P_2}{2}) \\
R_{2p} + R_{1c} &\leq \mathcal{C}(\frac{P_2 + a^2 P_{1c}}{2}).
\end{aligned}
$$

Noting that $a^2 P_1 - 1 \leq P_2$, it is straightforward to prove the region (3.30)-(3.32), which is obtained by applying FME, is $\mathcal{R}(0, 1.5)$ achievable.

To find the worst case gap, (3.30), (3.31), and (3.32) are respectively compared to the upper bounds (3.18), (3.20), and (3.21). As in the conventional GIC for $1 < a$, the private signal is not needed, and therefore, $\mathcal{R}(0, 1)$ is achievable.

3.7.4.2 Common Cooperative Case (Proof of Lemma 3.6)

The decoding rules impose the following constraints:

Decoder 1:

$$\begin{aligned}
R_{1p} &\leq \mathcal{C}(P_{1p}) && (3.54)\\
R_{1c} &\leq \mathcal{C}(P_{1c}) && (3.55)\\
R_{1p} + R_{1c} &\leq \mathcal{C}(P_{1p} + P_{1c}) && (3.56)\\
R_{1p} + R_{2r} &\leq \mathcal{C}(P_{1p} + P_{2r}) && (3.57)\\
R_{1c} + R_{2r} &\leq \mathcal{C}(P_{1c} + P_{2r}) && (3.58)\\
R_{1p} + R_{1c} + R_{2r} &\leq \mathcal{C}(P_1), && (3.59)
\end{aligned}$$

Decoder 2:

$$\begin{aligned}
R_{2p} &\leq \mathcal{C}(\frac{P_{2p}}{d}) && (3.60)\\
R_{2r} &\leq \min\{C_{21}, \mathcal{C}(\frac{a^2 P_{2r}}{d})\} && (3.61)\\
R_{2p} + R_{2r} &\leq \mathcal{C}(\frac{a^2 P_{2r} + P_{2p}}{d}) && (3.62)\\
R_{2p} + R_{1c} &\leq \mathcal{C}(\frac{P_{2p} + a^2 P_{1c}}{d}) && (3.63)\\
R_{2r} + R_{1c} &\leq \mathcal{C}(\frac{a^2 (P_{2r} + P_{1c})}{d}) && (3.64)\\
R_{2p} + R_{2r} + R_{1c} &\leq \mathcal{C}(\frac{P_{2p} + a^2 (P_{2r} + P_{1c})}{d}), && (3.65)
\end{aligned}$$

where d is defined in (3.27).

Defining $R_1 \triangleq R_{1p} + R_{1c}$ and $R_2 \triangleq R_{2p} + R_{2r}$, applying FME, and removing redundant inequalities, lead to the following rate constraints:

$$R_1 \le \mathcal{C}(P_{1p} + P_{1c}) \tag{3.66}$$
$$R_1 \le \mathcal{C}(P_{1p}) + \mathcal{C}(\frac{a^2(P_{2r} + P_{1c})}{d}) \tag{3.67}$$
$$R_2 \le \mathcal{C}(\frac{P_{2p}}{d}) + C_{21} \tag{3.68}$$
$$R_2 \le \mathcal{C}(\frac{a^2 P_{2r} + P_{2p}}{d}) \tag{3.69}$$
$$R_1 + R_2 \le \mathcal{C}(P_1) + \mathcal{C}(\frac{P_{2p}}{d}) \tag{3.70}$$
$$R_1 + R_2 \le \mathcal{C}(\frac{P_{2p} + a^2 P_{1c}}{d}) + \mathcal{C}(P_{1p} + P_{2r}) \tag{3.71}$$
$$R_1 + R_2 \le \mathcal{C}(\frac{P_{2p} + a^2(P_{2r} + P_{1c})}{d}) + \mathcal{C}(P_{1p}) \tag{3.72}$$
$$R_1 + R_2 \le \mathcal{C}(\frac{P_{2p} + a^2 P_{1c}}{d}) + \mathcal{C}(P_{1p}) + C_{21} \tag{3.73}$$
$$2R_1 + R_2 \le \mathcal{C}(\frac{P_{2p} + a^2 P_{1c}}{d}) + \mathcal{C}(\frac{a^2(P_{2r} + P_{1c})}{d}) + 2\mathcal{C}(P_{1p}) \tag{3.74}$$
$$2R_1 + R_2 \le \mathcal{C}(\frac{P_{2p} + a^2 P_{1c}}{d}) + \mathcal{C}(P_{1p}) + \mathcal{C}(P_1) \tag{3.75}$$
$$R_1 + 2R_2 \le \mathcal{C}(\frac{P_{2p}}{d}) + \mathcal{C}(\frac{P_{2p} + a^2(P_{2r} + P_{1c})}{d}) + \mathcal{C}(P_{1p} + P_{2r}) \tag{3.76}$$
$$R_1 + 2R_2 \le 2\mathcal{C}(\frac{P_{2p}}{d}) + \mathcal{C}(P_{2r} + P_{1c}) + \mathcal{C}(P_{1p} + P_{2r}). \tag{3.77}$$

It is noted that (3.77) is redundant since (3.76) $\le$ (3.77) as verified below for two cases of $a \le 1$, and $1 < a$:

$$\begin{aligned} (3.77) - (3.76) &= (3.60) + (3.58) - (3.65) \\ &\ge (3.60) + (3.64) - (3.65) \ge 0 \qquad \text{for } a \le 1 \\ &= \mathcal{C}(P_2) + \mathcal{C}(P_1) - \mathcal{C}(P_2 + a^2 P_1) \overset{(\star)}{\ge} 0 \quad \text{for } 1 < a, \end{aligned}$$

where to prove $(\star)$, a^2 is replaced by its maximum value, i.e., $P_2 + 1$.

We set $P_{2r} = P_{1c} = \frac{P_1 - P_{1p}}{d}$. This power allocation policy not only does not decrease the maximum rates of both users by more than 0.5 bit, but also can simplify the achievable region by making some inequalities redundant. In particular, if we decrease (3.72) by 0.5 bit, then the region described by (3.34)-(3.37) is achievable. This is due to:

$$\begin{aligned}
(3.66) &\leq (3.67)\\
(3.72) - 0.5 &\overset{(a)}{\leq} (3.70)\\
(3.72) - 0.5 &\leq \min\{(3.71), (3.73)\}\\
(3.67) + (3.72) - 0.5 &\leq (3.74)\\
(3.66) + (3.72) - 0.5 &\leq (3.75)\\
(3.69) + (3.72) - 0.5 &\overset{(b)}{\leq} (3.76).
\end{aligned}$$

To establish (a) and (b), we use the fact that $P_2 + 1 \leq a^2 P_1$. The following steps are proceeded to prove the region is $\mathcal{R}(0.5, 1.5)$ achievable for $a \leq 1$:

$$\begin{aligned}
(3.18) - (3.34) &\leq 0.5\\
(3.19) - (3.35) &\leq 0.5\\
(3.20) - (3.36) &\leq 1.5\\
(3.21) - (3.37) &\leq 1.5,
\end{aligned}$$

and $\mathcal{R}(0.5, 0.5)$ achievable for $1 < a$:

$$\begin{aligned}
(3.18) - (3.34) &\leq 0.5\\
(3.19) - (3.35) &= 0\\
(3.21) - (3.37) &\leq 1.
\end{aligned}$$

It is seen that the worst gap for this case, which is due to (3.36) or (3.37), can be further reduced by sending X_{2r} from both transmitters. To achieve the smaller gap goal, the term $2a^2P_1 + 2P_2$ in the upper bounds (3.20) and (3.21) can also be tightened to $(a\sqrt{P_1} + \sqrt{P_2})^2$.

3.7.4.3 Private-Common Cooperative Case (Proof of Lemma 3.7)

The decoding rules enforce the following rate constraints:

Decoder 1:

$$
\begin{aligned}
R_{1c} &\leq \mathcal{C}(\frac{P_{1c}}{2}) \\
R_{1c} + R_{2r_c} &\leq \mathcal{C}(\frac{P_1 - 1}{2}).
\end{aligned}
$$

The factor 2 in the denominators is because of treating X_{2r_p} as noise.

Decoder 2:

$$
\begin{aligned}
R_{2r_p} &\leq C' \\
R_{2r_c} &\leq \min\{C_{21} - C', \mathcal{C}(\frac{a^2 P_{2r_c}}{P_2 + 1})\} \\
R_{2r_p} + R_{2r_c} &\leq \mathcal{C}(\frac{a^2(P_{2r_c} + 1)}{P_2 + 1}) \\
R_{2r_p} + R_{1c} &\leq \mathcal{C}(\frac{a^2(P_{1c} + 1)}{P_2 + 1}) \\
R_{2r_c} + R_{1c} &\leq \mathcal{C}(\frac{a^2(P_1 - 1)}{P_2 + 1}) \\
R_{2r_p} + R_{2r_c} + R_{1c} &\leq \mathcal{C}(\frac{a^2 P_1}{P_2 + 1}) \\
R_{2p} &\leq \mathcal{C}(P_2),
\end{aligned}
$$

where $C' = \min\{\mathcal{C}(\frac{a^2}{P_2+1}), C_{21}\}$. The expression $P_2 + 1$ in denominators comes from the sequential decoding of $(X_{2r_p}, X_{2r_c}, X_{1c})$, and X_{2p}. The constraint on R_{2p} is obtained due to the assumption that X_{2p} is decoded after $(X_{2r_p}, X_{2r_c}, X_{1c})$ are decoded and their effect is subtracted from the received signal. FME is used to rewrite the constraints in terms of $R_1 \triangleq R_{1c}$, and $R_2 \triangleq R_{2p} + R_{2r_p} + R_{2r_c}$. After removing inactive inequalities according to the operating regime, i.e., $P_2 + 1 \leq a^2$, we attain:

$$\begin{aligned}
R_1 &\leq \mathcal{C}(\frac{P_{1c}}{2}) \\
R_2 &\leq \mathcal{C}(P_2) + C_{21} \\
R_2 &\leq \mathcal{C}\left(a^2(P_{2r_c} + 1) + P_2\right) \\
R_1 + R_2 &\leq \mathcal{C}(\frac{P_1 - 1}{2}) + C' + \mathcal{C}(P_2) \\
R_1 + R_2 &\leq \mathcal{C}(a^2 P_1 + P_2) \\
R_1 + R_2 &\leq \mathcal{C}\left(a^2(P_{1c} + 1) + P_2\right) + C_{21} - C' \\
R_1 + R_2 &\leq \mathcal{C}\left(a^2(P_{1c} + 1) + P_2\right) + \mathcal{C}(\frac{a^2 P_{2r_c}}{P_2 + 1}) \\
2R_1 + R_2 &\leq \mathcal{C}\left(a^2(P_{1c} + 1) + P_2\right) + \mathcal{C}(\frac{P_1 - 1}{2}).
\end{aligned}$$

Substituting the allocated powers provides:

$$R_1 \leq \mathcal{C}(\frac{P_1 - 1}{4}) \tag{3.78}$$

$$R_2 \leq \mathcal{C}(P_2) + C_{21} \tag{3.79}$$

$$R_2 \leq \mathcal{C}(\frac{a^2(P_1 + 1)}{2} + P_2) \tag{3.80}$$

$$R_1 + R_2 \leq \mathcal{C}(\frac{P_1 - 1}{2}) + C' + \mathcal{C}(P_2) \tag{3.81}$$

$$R_1 + R_2 \leq \mathcal{C}(a^2 P_1 + P_2) \tag{3.82}$$

$$R_1 + R_2 \leq \mathcal{C}(\frac{a^2(P_1 + 1)}{2} + P_2) + C_{21} - C' \tag{3.83}$$

$$R_1 + R_2 \leq \mathcal{C}(\frac{a^2(P_1 + 1)}{2} + P_2) + \mathcal{C}(\frac{a^2 P_1 - 1}{2(P_2 + 1)}) \tag{3.84}$$

$$2R_1 + R_2 \leq \mathcal{C}(\frac{a^2(P_1 + 1)}{2} + P_2) + \mathcal{C}(\frac{P_1 - 1}{2}). \tag{3.85}$$

If we deduct 0.5 bit from (3.82), some of the inequalities become redundant, since:

$$\begin{aligned}(3.82) - 0.5 &\leq (3.80), (3.83), (3.84),\\ \min\{(3.78) + (3.79), (3.82) - 0.5\} &\leq (3.81),\\ (3.78) + (3.82) - 0.5 &\leq (3.85),\end{aligned}$$

leading to the achievable region (3.39). Now, the simplified region (3.39) is compared to the upper bounds to show it is $\mathcal{R}(1, 0.5)$ achievable:

$$\begin{aligned}\delta_{R_1} &\leq 1\\ \delta_{R_2} &= 0\\ \delta_{R_1+R_2} &\leq \mathcal{C}(5).\end{aligned}$$

Because $P_2+1 \leq a^2$ in this regime, the upper bound (3.22) is enlarged to $\mathcal{C}(3a^2P_1+P_1+2P_2)$ to be used in proving the gap on the sum-rate.

3.7.5 Gap Analysis for Bidirectional Cooperation

Most of the detailed gap analysis for the bidirectional cooperation case is provided below.

3.7.5.1 $a^2P_1 \leq 1$

Treating interference as noise and not using the cooperative links lead to the following $\mathcal{R}(0, 1)$ achievable region:

$$R_1 \leq \mathcal{C}(P_1) \tag{3.86}$$

$$R_2 \leq \mathcal{C}(\frac{P_2}{a^2P_1 + 1}). \tag{3.87}$$

To prove the gap, we note that

$$\begin{aligned}&(3.18) - (3.86) = 0\\ &(3.20) - (3.87) \leq \mathcal{C}(2a^2P_1 + 2P_2) - \mathcal{C}(\frac{a^2P_1 + P_2 - 1}{2}) \leq 1.\end{aligned}$$

It is easy to show that for the general case of GCI shown in Fig. 3.1, when $a^2P_1, b^2P_2 \leq 1$, the capacity region is achievable within one bit per user. Upper bounds (3.3) and (3.4) are used for comparison purposes.

3.7.5.2 $1 \leq a^2P_1 \leq P_2 + 1$

In this regime, the achievable scheme based on the zero-forcing technique, used for unidirectional case, is shown to be $\mathcal{R}(0.5, 1.5)$ achievable for the bidirectional case. In section 3.7.3, the region (3.50)-(3.53) is proposed to simplify the gap analysis, and also to prove $\mathcal{R}(0.5, 1)$ is achievable for the case of $C_{21} = 0$. When $0 < C_{21}$, we modify the achievable region as well as the gap analysis to show $\mathcal{R}(0.5, 1.5)$ is achievable. It can be readily shown that the region is still achievable if we replace $R_2 \leq \mathcal{C}(\frac{P_2}{2})$ in (3.51) by $R_2 \leq \mathcal{C}(P_2 - \frac{a^2P_1-1}{2})$. We remark that both regions considered in this section and section 3.7.3 are inside the achievable region given in (3.24)-(3.26). We compare the upper bound to the modified region to show $\mathcal{R}(0.5, 1.5)$ is achievable:

$$
\begin{aligned}
(3.18) - (3.50) &\leq 0.5 \\
(3.20) - \mathcal{C}(P_2 - \frac{a^2P_1 - 1}{2}) &\overset{(a)}{\leq} 1.5 \\
(3.21) - (3.52) &\leq 1.5 \\
(3.22) - (3.53) &\leq \mathcal{C}(P_1P_2 + P_1 + 4P_2 + 2) - (3.53) \leq 2.
\end{aligned}
$$

Here, we provide the proof for (a). First we define some parameters:

$$
\begin{aligned}
x &\triangleq a^2P_1 - 1 \\
v &\triangleq \frac{2x + 2P_2 + 3}{-\frac{x}{2} + P_2 + 1} \\
u &\triangleq \mathcal{C}(2a^2P_1 + 2P_2) - \mathcal{C}(P_2 - \frac{a^2P_1 - 1}{2}) \\
&= \frac{1}{2}\log(v).
\end{aligned}
$$

It is easy to see that the derivative of v with respect to x is always positive. Therefore, the maximum value of x provides the maximum value of v, and consequently u. Hence, it is straightforward to show

$$\max_{0\leq x\leq P_2}\{v\} = \frac{4P_2+3}{\frac{P_2}{2}+1} \leq 8,$$

which proves (a).

3.7.5.3 $P_2+1 \leq a^2P_1, a \leq 1$

The received signals for this signaling are:

$$\begin{aligned} Y_1 &= X_{1p} + X_{1c} + X_{2r} + N_1, \\ Y_2 &= X_{2p} + aX_{1c} + aX_{2r} + Z + N_2, \end{aligned}$$

where $N_1, N_2,$ and Z (the compression noise) $\sim \mathcal{N}(0,1)$. The decoding rules impose the same constraints as (3.54)-(3.65) with $P_{2p} = P_2 - (a^2P_{1p} - 1)$ instead of $P_{2p} = P_2$ (due to zero-forcing). Therefore, FME provides the rate region given in (3.66)-(3.76). Here, it is shown that the region (3.34)-(3.37) is achievable. First, it is noted that similar to section 3.7.4.2, (3.71), and (3.73)-(3.75) are redundant. In addition, the power allocation policy (3.41) makes

$$\begin{aligned} (3.66) &\overset{(a)}{\leq} (3.67) \\ (3.72) - 0.5 &\overset{(b)}{\leq} (3.70) \\ (3.69) + (3.72) - 0.5 &\overset{(c)}{\leq} (3.76), \end{aligned}$$

since

(a).

$$\begin{aligned} \mathcal{C}(P_{1p}) + \mathcal{C}(\frac{a^2(P_1-P_{1p})}{2}) &\overset{(\circ)}{\geq} \mathcal{C}(P_{1p} + \frac{a^2(P_1-P_{1p})}{2} + \frac{P_1-P_{1p}}{2}) \\ &\geq \mathcal{C}(\frac{P_1+P_{1p}}{2}), \end{aligned}$$

(b).

$$(3.69) + (3.72) - 0.5 \overset{(\star)}{\leq} \mathcal{C}(\frac{P_{2p} + a^2(P_1 - P_{1p})}{2} + P_{1p} + \frac{P_{1p}P_{2p}}{2} + \frac{(\frac{P_2}{2}+1)(P_1 - P_{1p})}{2}) - 0.5$$
$$\leq \mathcal{C}(P_1 + \frac{P_{2p}}{2} + \frac{P_1 P_2}{4})$$
$$\overset{(\diamond)}{\leq} (3.70),$$

(c).

$$\mathcal{C}(P_{1p}) + \mathcal{C}(\frac{a^2 P_{2r} + P_{2p}}{2}) \overset{(\star)}{\leq} \mathcal{C}(P_{1p} + \frac{a^2 P_{2r} + P_{2p}}{2} + \frac{\frac{P_2}{2}+1}{2} P_{2r} + \frac{P_{1p}P_{2p}}{2})$$
$$\overset{(\diamond)}{\leq} \mathcal{C}(P_{1p} + \frac{P_{2r} + P_{2p}}{2} + \frac{P_{2p}+1}{2} P_{2r} + \frac{P_{1p}P_{2p}}{2})$$
$$\leq 0.5 + \mathcal{C}(\frac{P_{2p}}{2}) + \mathcal{C}(P_{1p} + P_{2r}),$$

where $(\circ)$, $(\star)$, and $(\diamond)$ are correct due to $1 \leq a^2 P_{1p}$, $a^2 P_{1p} \leq \frac{P_2}{2} + 1$, and $\frac{P_2}{2} \leq P_{2p}$, respectively. It is remarked that the above proofs are also valid for the case of $a \leq 1$ in section 3.7.4.2.

To analyze the gap, we note that $\frac{P_2}{2} \leq P_{2p}$ assures

$$(3.18) - (3.34) \leq 0.5$$
$$(3.19) - (3.35) \leq 1$$
$$(3.20) - (3.36) \overset{(\star)}{\leq} \mathcal{C}(\frac{29}{3}),$$

where $(\star)$ is true since $P_2 \leq a^2 P_1 - 1$.

Now, if $P_{1p} = \frac{2^{C12}}{a^2}$, then adding both sides of the next three inequalities verifies that (3.21) − (3.37) ≤ 1.5.

$$\mathcal{C}(\frac{(1-a^2)P_1}{a^2 P_1 + 1}) + C_{12} - \mathcal{C}(\frac{2^{C12}}{a^2}) \leq 0$$
$$\mathcal{C}(a^2 P_1) - \mathcal{C}(\frac{P_{2p} + a^2(P_1 - P_{1p})}{2}) \leq 0.5$$
$$\mathcal{C}(2a^2 P_1 + 2P_2) - \mathcal{C}(a^2 P_1) \leq 1.$$

If $P_{1p} = \frac{P_2+2}{2a^2}$, we have $P_{2p} = \frac{P_2}{2}$, and consequently,

$$\frac{P_{2p} + a^2(P_1 - P_{1p})}{2} = \frac{a^2 P_1 - 1}{2}, \text{ and}$$

$$(3.22) - (3.37) = \frac{1}{2} \log \left(\frac{1 + P_1 P_2 + P_1(1 + 2a^2) + 2P_2}{(1 + \frac{P_2+2}{2a^2})(\frac{a^2 P_1 + 1}{2})} \right)$$
$$\leq 1.5.$$

Therefore, $\mathcal{R}(0.5, 1.71)$ is achievable. It is also observed that (3.37) $\leq$ (3.34) + (3.36), which guarantees that the sum-rate is within 2 bits of the sum-capacity in this regime.

3.7.5.4 $P_2 + 1 \leq a^2 P_1, 1 \leq a^2 \leq P_2 + 1$

The decoding rules impose similar constraints as (3.54)-(3.65) with $R_{1p} \leq \mathcal{C}(P_{1p})$ replaced by $R_{1z} \leq \min\{C_{12}, \mathcal{C}(P_{1z})\}$. FME is applied to write the constraints in the format of $R_1 \triangleq R_{1z} + R_{1c}$, and $R_2 \triangleq R_{2p} + R_{2r}$:

$$R_1 \leq \mathcal{C}(P_{1z} + P_{1c}) \tag{3.88}$$
$$R_1 \leq \mathcal{C}(P_{1c}) + C_{12} \tag{3.89}$$
$$R_2 \leq \mathcal{C}(P_{2p}) + C_{21} \tag{3.90}$$
$$R_2 \leq \mathcal{C}(P_{2p} + a^2 P_{2r}) \tag{3.91}$$
$$R_2 \leq \mathcal{C}(P_{1c} + P_{2r}) + \mathcal{C}(P_{2p}) \tag{3.92}$$
$$R_1 + R_2 \leq \mathcal{C}(P_1) + \mathcal{C}(P_{2p}) \tag{3.93}$$
$$R_1 + R_2 \leq \mathcal{C}(P_{1z} + P_{2r}) + \mathcal{C}(P_{2p} + a^2 P_{1c}) \tag{3.94}$$
$$R_1 + R_2 \leq C_{12} + \mathcal{C}(P_{1c} + P_{2r}) + \mathcal{C}(P_{2p}) \tag{3.95}$$
$$R_1 + R_2 \leq \min\{C_{12}, C(P_{1z})\} + \mathcal{C}(P_{2p} + a^2(P_{1c} + P_{2r})) \tag{3.96}$$
$$R_1 + R_2 \leq \min\{C_{12}, C(P_{1z})\} + \mathcal{C}(P_{2p} + a^2 P_{1c}) + C_{21} \tag{3.97}$$
$$2R_1 + R_2 \leq \min\{C_{12}, C(P_{1z})\} + \mathcal{C}(P_{2p} + a^2 P_{1c}) + C_{12} + \mathcal{C}(P_{1c} + P_{2r}) \tag{3.98}$$
$$2R_1 + R_2 \leq \min\{C_{12}, C(P_{1z})\} + \mathcal{C}(P_{2p} + a^2 P_{1c}) + \mathcal{C}(P_1) \tag{3.99}$$
$$R_1 + 2R_2 \leq \mathcal{C}(P_{1z} + P_{2r}) + 2\mathcal{C}(P_{2p}) + \mathcal{C}(P_{1c} + P_{2r}) \tag{3.100}$$
$$R_1 + 2R_2 \leq \mathcal{C}(P_{1z} + P_{2r}) + \mathcal{C}(P_{2p}) + \mathcal{C}(P_{2p} + a^2(P_{1c} + P_{2r})). \tag{3.101}$$

Again, the employed power allocation can help us to simplify the achievable region to the region described by (3.43)-(3.49). This is because

$$\begin{aligned}
(3.92) &\geq (3.93) - 0.5 \\
(3.94) &= (3.88) + (3.91) \\
(3.95) &\geq (3.93) - 0.5 \\
(3.97) &\geq (3.96) - 0.5 \\
(3.98) &\geq (3.89) + (3.96) - 0.5 \\
(3.99) &\geq (3.88) + (3.96) - 0.5 \\
(3.100) &\geq (3.92) + (3.93) - 0.5 \\
(3.101) &\geq (3.91) + (3.93) - 0.5.
\end{aligned}$$

To obtain the preceding inequalities, we use the fact that $P_2 + 1 \leq a^2 P_1$.

The achievable rate is compared below with the upper bound to prove $\mathcal{R}(1, \mathcal{C}(\frac{13}{3}))$ is achievable:

$$\begin{aligned}
(3.18) - (3.43) &\leq 0.5 \\
(3.18) - (3.44) &\leq 1 \\
(3.19) - (3.45) &\leq 0.5 \\
(3.20) - (3.46) &\leq \mathcal{C}(\frac{13}{3}) \\
(3.21) - (3.47) &\leq 1.5 \\
(3.22) - (3.48) &\overset{(\star)}{\leq} \mathcal{C}(9) \\
(3.22) - (3.49) &\overset{(\diamond)}{\leq} \mathcal{C}(11).
\end{aligned}$$

To achieve $(\star)$, and $(\diamond)$, the upper bound (3.22) is enlarged to $\mathcal{C}(2P_1P_2 + 2P_1 + 2P_2 + a^2P_1)$, and $\mathcal{C}(3P_1P_2 + 3P_1 + 2P_2)$, respectively, as a consequence of $a^2 \leq P_2 + 1$. It is seen that the sum-capacity is determined up to 2 bits in this scenario since $(3.47) \leq (3.44) + (3.46)$.

Chapter 4

Conclusion

In this final chapter, we summarize our contributions and point out several interesting problems for further research.

4.1 Contributions

In this dissertation, we studied cooperative communications for two simple but important models of diamond relay channel and interference channel with conferencing encoders.

4.1.1 Diamond Relay Channel

Diamond relay channel was studied in detail in chapter 2. In particular, the capacity of the channel was characterized for a special class of $\Delta = 0$, and for general channel parameters, it was approximated by at most 3.6 bits. To achieve the results, the decode-and-forward technique together with a proper schedule over the network operating modes were used. The proposed schedule reveals the minimum number of required operating modes in order to guarantee a finite gap from the capacity for all channel parameters. Specifically, the fundamental parameter Δ determines which modes should be included in the schedule. A key step towards the results was the proposed upper bound obtained using the weak duality theorem of linear programming.

4.1.2 Interference Channel with Conferencing Encoders

The two-user GIC was considered in chapter 3, in which encoders are connected through noiseless links with finite capacities. In this setting, prior to each transmission block the encoders communicate with each other over the cooperative links. New genie-aided upper bounds on the sum-capacity were developed which incorporate the capacities of the cooperative links. Two special classes of the GIC was analyzed. First, for the symmetric GIC, with possibly different cooperation link capacities, the sum-capacity was characterized within 2.13 bits gap for all values of the channel parameters. Second, for the GZIC, the capacity region and the sum-capacity of the channel were approximated up to 1.71 bits per user and 2 bits in total, respectively. It was also established that properly sharing the total limited cooperation capacity between the cooperative links may enhance the achievable region, even when compared to the case of unidirectional transmitter cooperation with infinite cooperation capacity. The achievable schemes are based on a simple type of Han-Kobayashi signaling, together with the zero-forcing, and basic relaying techniques.

4.2 Future Directions

Cooperative communications has become a very hot research topic in the communication theory community. In the following, some relevant problems to the setups considered in this thesis are listed.

4.2.1 Signaling over Scheduling

During this thesis, it was assumed that the state sequence and the time-sharing parameters are known to all nodes before transmission. However, the listen-transmit schedule of the relays can carry information by making it dependent on the message to be sent. The references [53, 54] are related to this topic.[1]

4.2.2 Parallel Relay Channel with More than Two Relays

For a parallel relay network, we considered the simplest case of two relays. When more than two relays exist in the system, an important question is: what is a good achievable

[1] Thanks to Professor Wei Yu for directing to reference [54].

scheme? To answer this question, the following observations may be useful. Amplify-and-forward is known to be asymptotically optimum when the number of relays is large [21]. In this thesis, as well as in [17], schemes based on the decode-and-forward strategy have been demonstrated to be close-to-optimum for the two-relay case. Reference [17] also shows a variant of the compress-and-forward scheme can achieve the capacity within a constant gap for a network with any number of relays. Unfortunately, the gap is quite large.

4.2.3 Diamond Channel with Conferencing Relays

The concept of conferencing, used in chapter 3, can also be studied for the relays of the diamond channel as well. To achieve close to the capacity of this channel, combining the decode-and-forward scheme with the schemes used in the MAC with conferencing encoders [34] or in the BC with conferencing decoders [55] may be a good starting point.

4.2.4 Interference Channel with In-band Transmitter Cooperation

The cooperation used in the two-user GIC, considered in chapter 3, was enabled via orthogonal links. If such resources are not available in the system, the broadcast nature of the wireless communication can be used to provide cooperation benefits. However, this requires appropriate scheduling and transmission schemes, which are nice problems to be explored. Reference [43] which analyzed the setup when nodes are full-duplex is a valuable resource.

4.2.5 MIMO-BC without Dirty Paper Coding

The capacity region of the MIMO-BC channel with independent messages, is known to be achieved by Dirty-Paper Coding (DPC) [56]. However, implementing DPC is not an easy task. In [57], the ideas introduced in chapter 3 has been adapted to approximate the capacity region of the two-user broadcast channel, with two transmit and one receive antennae, within one bit, which is attractive since DPC is not required. One interesting research direction is to see if the capacity region of the MIMO-BC can be approximated within some constants without using DPC. The reference [57] also shows that the result is also valid for the case of the two-user broadcast channel with an additional common message for both users. The capacity region of the MIMO-BC with common message is unknown to date. The employed achievable schemes of this thesis may shed some light on this problem.

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