

On Schnyder's Theorem

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The central topic of this thesis is Schnyder's Theorem. Schnyder's Theorem provides a characterization of planar graphs in terms of their poset dimension, as follows: a graph G is planar if and only if the dimension of the incidence poset of G is at most three. One of the implications of the theorem is proved by giving an explicit mapping of the vertices to \mathbb{R}^2 that defines a straightline embedding of the graph. The other implication is proved by introducing the concept of normal labelling. Normal labellings of plane triangulations can be used to obtain a realizer of the incidence poset. We present an exposition of Schnyder's theorem with his original proof, using normal labellings. An alternate proof of Schnyder's Theorem is also presented. This alternate proof does not use normal labellings, instead we use some structural properties of a realizer of the incidence poset to deduce the result.

Some applications and a generalization of one implication of Schnyder's Theorem are also presented in this work. Normal labellings of plane triangulations can be used to obtain a barycentric embedding of a plane triangulation, and they also induce a partition of the edge set of a plane triangulation into edge disjoint trees. These two applications of Schnyder's Theorem and a third one, relating realizers of the incidence poset and canonical orderings to obtain a compact drawing of a graph, are also presented. A generalization, to abstract simplicial complexes, of one of the implications of Schnyder's Theorem was proved by Ossona de Mendez. This generalization is also presented in this work. The concept of order labelling is also introduced and we show some similarities of the order labelling and the normal labelling. Finally, we conclude this work by showing the source code of some implementations done in Sage.

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Chapter 1

Introduction

The main topic of this thesis is concerned with planarity, a topological property of graphs. We will begin by briefly describing the origins of graph theory and *simplicial complexes*, and mentioning how part of the origin of topology relates to them.

We start by mentioning the problem of the bridges of Königsberg, a problem that consisted in finding a traversal through each of the bridges in Königsberg exactly once. The solution to this problem was due to Euler in 1736 and is considered to be the first problem in graph theory. His solution also contains what is considered to be the first topological argument [19]. In 1847 Listing, a student of Gauss, introduced the term *Topologie* in a publication closely related to the above mentioned result of Euler. A generalization of Euler's formula, which holds in particular for planar graphs, was given by Listing in 1862, in this publication he introduced the term *complexes* [6], which we may think of as a "higher dimensional" analogue of a graph.

We say a graph is planar if it can be embedded in the plane, informally speaking, if there is a drawing of the graph in the plane so that no two edges cross, except at mutual endpoints. The study of planar graphs has a huge range of influence, from recreational puzzles (see Chapter 8 of [6]) to practical problems such as VLSI design. Solving another recreational problem, namely the Four Colour Problem, was of central importance in the development of graph theory. Mohar and Thomassen [28] consider that The Four Colour Conjecture was the main source of inspiration for results on planar graphs.

1.1 Characterizations of planar graphs

Our focus for this thesis is an important theorem of Schnyder from 1989, which gives a characterization of planar graphs. There have been many other results that characterize planar graphs, we will now mention some of these characterizations.

One of the most famous results in graph theory, Kuratowski's theorem [23], is the first known characterization of planar graphs. He proved his result in 1930. It states that a graph is planar if and only if it does not have a subdivision of the complete bipartite graph $K_{3,3}$ or the complete graph K_5 . A similar result by Wagner ([41]) was proved independently in 1937.

In the year 1937, MacLane provided an algebraic characterization of planarity [26].

His characterization is in terms of the cycle space of the graph and states that a graph is planar if and only if the cycle space of the graph admits a simple basis.

A first characterization using an order relation was due to De Fraysseix and Rosenstiehl, published in 1985. They characterized planar graphs in terms of an order relation on the vertex set called the *Trémaux order* [14], which is an order defined in terms of a depth-first-search tree.

It was the year 1989 when Schnyder presented the theorem we are interested in [33]. This result characterizes planar graphs in terms of poset dimension. It is a parametric characterization, in the sense that the dimension d of a poset associated to a graph determines whether the given graph is planar or not. The theorem states that a graph is planar if and only if $d \leq 3$. Schnyder's theorem has been very influential in several different fields, as can be seen from the high number of citations of this paper (currently about 148 in google scholar and 132 in citeseerX). We will state Schnyder's theorem precisely in Section 1.3.

Another parametric characterization of planar graphs was given in 1990 by Colin de Verdière [9] (or [10] for an English version). In this characterization he introduces a parameter μ , called the Colin de Verdière number, which is associated to a graph. This parameter is defined in terms of the maximum multiplicity of the second smallest eigenvalue of a set of matrices associated to the graph. The result states that the graph is planar if and only if $\mu \leq 3$. There are also some other topological properties of graphs captured by this parameter, Colin de Verdière also showed in his publication that $\mu \leq 1$ if and only if the graph is a disjoint union of paths and $\mu \leq 2$ if and only if the graph is outerplanar. In a theorem by Lovasz and Schrijver [25] they show that the parameter μ captures yet another topological property of graphs, namely, that $\mu \leq 4$ if and only if the graph is linklessly embeddable in \mathbb{R}^3 .

There have been many other characterizations of planarity. We refer the reader to [42], [18], [1], [11], [22], [2], and [24] for some other such results.

The existence of convex drawings of planar graphs was proven by Steinitz in 1922 [36]. A result of Tutte [40], also related to convex drawings, from 1963 shows there exists such drawings for 3-connected planar graphs. An algorithmic procedure that obtains convex drawings can be obtained from Tutte's result. In 1989 Schnyder showed in his publication that there exist barycentric embeddings for maximal planar graphs, which also implies the existence of convex drawings for plane triangulations.

1.2 Posets

We now introduce some definitions from order theory to state Schnyder's Theorem.

Given a set A , a *relation* R on A is a subset of $A \times A$. We write aRb if $(a, b) \in R$, and in this case we say that b is related to a . Note that b can be related to more than one element of A .

Example 1.2.1. Let $A = \{1, 2, 3\}$, and let

$$R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}.$$

Note that in this example 3 is related to 1, 2 and to itself.

Let R be a relation defined on a set A . The relation R is *reflexive* if aRa for all $a \in A$. We say that the relation R is *symmetric* if aRb implies bRa for all $a, b \in A$. We will call R *antisymmetric* if aRb and bRa imply $a = b$ for all $a, b \in A$. The relation R will be called *transitive* if aRb and bRc imply aRc for all $a, b, c \in A$.

There are two particular kinds of relations that we will be play a central role in this thesis. The first kind of relation is defined below.

Definition 1.2.2. A *partial order* of a set A is a reflexive, antisymmetric and transitive relation on A . Whenever we define more than one partial order on a set, we will use sub indices to distinguish between partial orders.

Notation. We will usually denote partial order relations by \leq . If \leq is a partial order of a set A , $a, b \in A$ and $a \leq b$, we will sometimes use the notation $b \geq a$ to be equivalent to $a \leq b$. In this same context, $a < b$ will denote $a \leq b$ and $a \neq b$, similarly, $b > a$ will denote $b \geq a$ and $b \neq a$.

Example 1.2.3.

1. Let A and R be defined as in Example 1.2.1. It is not so hard to check that R is a partial order of A . In this example we can observe that $2 \leq 3$.
2. Let $A = \{1, 2, 3, 4\}$, and let

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (4, 4)\}.$$

We can verify that R is a partial order of A .

Definition 1.2.4. Let A be a set and \leq be a partial order on A . For $a, b \in A$, we say that a and b are *comparable* if $a < b$ or $b < a$. If neither $a < b$ and $b < a$ hold then we say that a and b are *incomparable*. A *linear order* of a set A is a partial order of A so that every pair of elements of A are comparable.

Given a finite set $A = \{a_1, a_2, \dots, a_l\}$, we will denote a linear order \leq of A by

$$\leq: a_{i_1} a_{i_2} \cdots a_{i_l},$$

which means that $a_{i_j} \leq a_{i_k}$ if and only if $j \leq k$.

Example 1.2.5. If we let A and R to be defined as in Example 1.2.1 we can verify that every pair of elements of A is comparable, hence R is a linear order of A . In fact we can denote R by $\leq: 123$. In the other hand, if we consider A and R as in Example 1.2.3, 2, we can observe that 1 and 3 are comparable, as $1 \leq 3$. Also observe that in this example $3 \not\leq 4$ and $4 \not\leq 3$, so 3 and 4 are incomparable.

Definition 1.2.6. A *partially ordered set* or a *poset* is a pair $P = (A, \leq)$, where A is a set and \leq is a partial order of A . The set A is called the *ground set* of P . We will only be concerned with posets where the ground set is finite.

Let (A, \leq) be a poset and let \leq be a linear order, say $\leq: a_1 a_2 \cdots a_l$. We will define the function $\text{ord}_{\leq} : A \rightarrow \mathbb{N}$ as $\text{ord}_{\leq}(a_i) = i$. The element a of A for which $\text{ord}_{\leq}(a)$ is maximum will be called the *maximum element with respect to \leq* and will be denoted by $\max_{\leq} A$. Similarly, the *minimum element with respect to \leq* is defined to be the element b of A for which $\text{ord}_{\leq}(b)$ is minimum and will be denoted by $\min_{\leq} A$. If $S \subseteq A$, $\max_{\leq} S$ will refer to $\max_{\leq|_S} S$, where $\leq|_S = \{(a, b) \in \leq \mid a, b \in S\}$, that is, the restriction of \leq to S .

It was the same year when the first characterization of planar graphs appeared, 1930, when Szpilrajn introduced the concept of linear extension of a poset [37], and in 1941 Dushnik and Miller introduced the concept of dimension of a poset [16], according to Trotter [38] this concept has been of much importance in set theory and in combinatorics. These two concepts are now defined.

Definition 1.2.7. Let $P = (A, \leq)$ be a poset. A linear extension of P is a linear order \leq on A so that $\leq \subseteq \leq$. A *realizer* $R = \{\leq_1, \dots, \leq_k\}$ of P is a set of linear orders on A so that

$$\bigcap_{i=1}^k \leq_i = \leq.$$

We may observe that each element of R is a linear extension of \leq . Since A is finite, we can see that the number of linear orders on A is $n!$. So if we let R be the set of all linear extensions of P , we can observe that R is a realizer of P , and in particular $|R| \leq n!$. Furthermore, any realizer R' of P will be a subset of R . The dimension of a poset P , denoted $\dim(P)$, is defined to be the minimum cardinality of its realizers.

We now introduce graphs to relate posets to them. A *graph* $G = (V, E)$ consists of a finite set V of *vertices* and a set E of *edges*. The edges are unordered pairs of distinct vertices.

Definition 1.2.8. Given a graph $G = (V, E)$, we define the *incidence poset* $I = (X, \leq_G)$ of G , where $X = V \cup E$ and $a <_G b$ if and only if $a \in V$, $b \in E$ and a is an endpoint of b . We also define the *dimension* of G , $\dim(G)$, to be the dimension of its incidence poset I .

1.3 Schnyder's Theorem

We now state Schnyder's Theorem.

Theorem 1.3.1 (Schnyder, 1989). *A graph G is planar if and only if $\dim(G) \leq 3$.*

Schnyder's Theorem states that it is necessary and sufficient that the dimension of a graph is at most 3 for the graph to be planar. In fact, one implication of the theorem is a consequence of a theorem of Babai and Duffus from 1981 [5]. In his publication, Schnyder introduces the concept of *normal labellings* of plane triangulations. This concept can be used to obtain a partition of the edge set of a plane triangulation into three edge disjoint acyclic digraphs. Each of these digraphs induce a partial order on the set of vertices so that if we consider any linear extension of each of these partial orders we obtain a realizer for the incidence poset. For more details about these concepts and a proof of Schnyder's

theorem the reader is referred to Chapter 2. An alternate, and more direct, proof of the theorem can be found in Chapter 4.

Applications of Schnyder's Theorem can be found in several areas, such as graph drawings, graph algorithms and even ring theory [3]. For instance, in [12] three linear time algorithms for planar graphs based on Schnyder's theorem are given. One algorithm tests 4-connectedness of plane triangulations, another algorithm enumerates triangles, the third algorithm tests 3-connectedness of planar graphs.

In the field of graph drawing, one of the consequences of Schnyder's Theorem is that any planar graph has a straightline drawing in an $\mathcal{O}(n) \times \mathcal{O}(n)$ grid [33]. In fact, Schnyder gave an algorithm that finds this drawing in $\mathcal{O}(n)$ time [34]. Many subsequent papers on graph drawing have used ideas from Schnyder's theorem, for example a result of Dhandapani [15] proves that Schnyder's algorithm can be generalized to obtain a *greedy drawing* of a plane triangulation. A greedy drawing is a drawing of a plane triangulation in which there is a path $v_1 \dots v_k$ between any pair of vertices v_1 and v_k so that while traversing the path towards v_k the Euclidean distance to v_k decreases. Some other applications are described in more detail in Chapter 3.

There have been several generalizations of Schnyder's Theorem, concerning different aspects of the result. There are a couple of generalizations of Schnyder's Theorem due to Brightwell and Trotter [7, 8]. In their first result they show that the dimension of the incidence poset of a 3-connected planar graph, now taking into account all the faces of the graph, is always equal to 4, and if one of the faces is omitted, then the dimension decreases to 3. In their second result, they prove that the dimension is at most 4 when considering a planar map.

A generalization of another aspect of Schnyder's result is given by Felsner in [17]. In this publication it is shown that 3-connected planar graphs admit a tree decomposition, similar to the one obtained by Schnyder for plane triangulations. A consequence of this result is the existence of a convex drawing of the graph on an $\mathcal{O}(f) \times \mathcal{O}(f)$ grid, where f is the number of faces of the graph.

In [30, 31], Ossona de Mendez generalizes one implication of Schnyder's result to abstract simplicial complexes, that is, given finite a collection R of d linear orders whose intersection is empty he defines the simplicial complex $\Sigma(R)$ induced by these linear orders and shows that $\Sigma(R)$ is realizable in \mathbb{R}^{d-1} . Schnyder's result is the particular case in which we consider 3 linear orders, in this case the simplicial complex induced by the linear orders consists of the vertices, edges and interior faces of a plane triangulation. Some aspects of this generalization are explored in more detail in Chapter 5.

1.4 Outline of this work

In Chapter 2 we give Schnyder's original proof of his theorem. We start by proving that graphs of dimension at most 3 are planar. For this, an embedding in the $\mathcal{O}(2^n) \times \mathcal{O}(2^n)$ grid is given. After this, the concepts of order labelling and normal labelling are introduced and it is shown that the order labelling defines a normal labelling. Some properties of normal labellings are shown and it is proved that every plane triangulation admits a normal labelling. These concepts are used afterwards to prove the converse implication

in Schnyder's theorem.

In Chapter 3 some applications of Schnyder's theorem are presented. We start by showing that every plane triangulation admits a barycentric embedding. In this embedding each vertex is mapped to a point in the first octant lying on the plane $x+y+z=1$, in fact it is a straight line embedding in the above mentioned plane. The concept of canonical order is also discussed in this chapter. We see how it is related to minimal realizers of the incidence poset of a graph. We also show an algorithm that obtains a straight line embedding of a planar graph using a canonical order. We conclude the chapter by mentioning applications of Schnyder's theorem in tree decomposition.

Chapter 4 is devoted to giving an alternate proof of Schnyder's theorem. This proof is done without using the concept of normal labelling. The central idea of the proof is to analyze properties of the realizers of a graph. Another very important idea is how closely related the realizers of a graph are to those of the graph after contracting a very particular edge. The content of this chapter has been accepted for publication [4].

As we mentioned before, Ossona de Mendez generalizes one of the implications of Schnyder's theorem to abstract simplicial complexes. In Chapter 5 we introduce the concept of abstract simplicial complexes and geometric simplicial complex. We also present a proof of Ossona de Mendez's theorem. Finally, we define a natural generalization of order labelling for simplicial complexes and we show some results that resemble the properties of a normal labelling.

Chapter 6 is devoted to showing the source code of some implementations done in Sage. One of the functions implemented obtains 3-representations for plane triangulations, using ideas from Chapter 4. We also show the source code of functions that obtain a barycentric embedding and an embedding using the canonical ordering. Both of these embeddings are described in Chapter 3. We conclude this chapter by showing two functions; the first one obtains the complex induced by a d -representation and the second one obtains an embedding of the complex induced by 4-representations.

Chapter 2

Schnyder's Theorem

Schnyder's Theorem has been stated in the previous chapter. We will now work towards proving the theorem. The following sections will show how the dimension of a graph is closely related to the topological property of the graph being planar.

2.1 Graphs of dimension 3 are planar

In this section we will introduce the concept of d -representations and see how these are related to the dimension of the incidence poset. We conclude this section by proving that graphs of dimension at most 3 are planar.

Definition 2.1.1. Given a sequence of total orders \leq_1, \dots, \leq_k on a finite set V so that

$$\bigcap_{i=1}^k \leq_i = \emptyset, \quad (2.1)$$

we say that the sequence of orders is a k -dimensional representation on V . We will call equation (2.1) the *vertex property*. Let $a_i = \max_{\leq_i} V$. The representation is said to be *standard* if for every $j \neq i$, $\text{ord}_{\leq_j}(a_i) \leq k - 1$. The elements a_i will be called *exterior* and all the other elements of V will be called *interior*.

Let G be a graph and let \leq_1, \dots, \leq_k be a k -dimensional representation on V_G . The k -dimensional representation is said to satisfy the *edge property* if for every edge $xy \in E_G$ and for every vertex $z \in V_G \setminus \{x, y\}$ there is an order \leq_j so that $x <_j z$ and $y <_j z$.

As we will see in the following proposition, the dimension of a graph G and k -dimensional representations of the vertex set of G satisfying the edge property are closely related.

Proposition 2.1.2 (Schnyder, [33]). *Let G be a graph. Then $\dim(G) \leq k$ if and only if there exists a k -dimensional representation of V , \leq_1, \dots, \leq_k , satisfying the edge property for E_G .*

Proof. Let $\{\leq_1, \dots, \leq_k\}$ be a realizer of the incidence poset of G . Consider the restrictions of these orders to the set of vertices, $\leq_1|_V, \dots, \leq_k|_V$. We must have $\bigcap_{i=1}^k \leq_i|_V = \emptyset$,

otherwise we would obtain a pair of vertices x and y of G so that $x \leq_G y$, which is a contradiction. So the restriction to V of the orders in the realizer of the incidence poset of G forms a k -dimensional representation of V . Now, let xy be an edge of G , and let $z \in V \setminus \{x, y\}$. If we assume that the edge property does not hold for xy and that z in $\{\leq_i \mid_V\}$, then $z \leq_i \mid_V x$ or $z \leq_i \mid_V y$ holds for all i . And since $x \leq_i xy$ and $y \leq_i xy$ for all i , by transitivity we get $z \leq_i xy$ for all i . Hence $z \leq_G xy$, a contradiction.

Now assume that there exists a k -dimensional representation of V , say \leq_1, \dots, \leq_k , satisfying the edge property for E_G . We will extend each linear order \leq_i to a total order \leq_i on $V \cup E$ by inserting the edge xy right after the largest of x and y . In the following argument, we denote an edge xy as (xy) only to be able to tell it apart from an edge and a vertex in a linear order. The way to extend the linear order is as follows. If the order before inserting the edge $(x_j x_k)$ is $\leq'_i: x_1 \cdots x_n$ and assuming $j < k$, then the resulting order \leq''_i after inserting the edge $(x_j x_k)$ will be $\leq''_i: x_1 \cdots x_k (x_j x_k) x_{k+1} \cdots x_n$. Observe that x_{k+1} might be an edge or a vertex, in either case, we will have $\text{ord}_{\leq''_i}(x_{k+1}) = \text{ord}_{\leq'_i}(x_{k+1}) + 1$. Now, we will prove that $\{\leq_i\}$ is a realizer of the incidence poset of G .

Let $\leq = \bigcap_i \leq_i$ and assume that $a < b$. We will show that $a \in V$, $b \in E$ and that a and b are incident by analyzing the different possibilities for a and b .

Case I: Both a and b are vertices. Note that this is not possible since $\leq \mid_V = \emptyset$.

Case II: Both a and b are edges. Assume that $a = wx$ and $b = yz$ are two distinct edges. We may assume, without loss of generality, that w is an endpoint of a which is not an endpoint of b . Since the edge property holds, there is an order \leq_i so that $y \leq_i w$ and $z \leq_i w$. So this implies that $y \leq_i w$ and $z \leq_i w$, and by construction of \leq_i we have $yz \leq_i w$. We also know, by construction of \leq_i , that $w \leq_i wx$. So finally by transitivity we get $yz \leq_i wx$, so both a and b cannot be edges.

Case III: a is an edge and b is a vertex. Assume that $a = wx$. Note that b cannot be incident with a , since by construction of \leq_i we have $\text{ord}_{\leq_i}(x) \leq \text{ord}_{\leq_i}(wx)$, that is, $x \leq_i wx$ for all i . So b is not incident with w . Since the vertex property holds, there must be an order \leq_j in which $b \leq_j w$. This implies $b \leq_j w$. Together with $w \leq_j wx$ this implies $b \leq_j wx$, a contradiction.

Case IV: $a \in V$ and $b \in E$. If $a \in V$, $b \in E$ and $a \leq b$ then a must be an endpoint of b . Otherwise, assume that a is not an endpoint of $b = wx$. It follows from the edge property that there is an order \leq_j so that $w \leq_j a$ and $x \leq_j a$. So $w \leq_j a$ and $x \leq_j a$, and by construction $wx \leq_j a$, a contradiction.

So it follows that if $a \leq b$ then $a \in V$, $b \in E$ and a is an endpoint of b . □

We have seen how the dimension of a graph $G = (V, E)$ is related to the k -dimensional representations of V .

Let $G = (V_G, E_G)$ be a graph and $H = (V_H, E_H)$ be a subgraph of G . A k -dimensional representation of V_G that satisfies the edge property for G is also a k -dimensional representation of V_H that satisfies the edge property for H . Therefore, by Proposition 2.1.2, if $H \subseteq G$ then $\dim(H) \leq \dim(G)$.

We also see that the representation of a vertex set can represent more than one graph. We call the *maximal graph admitted by the representation* the graph for which we cannot add more edges without violating the edge property.

Definition 2.1.3. A *straight line embedding* of a graph G in \mathbb{R}^2 is an injective function $f : V \rightarrow \mathbb{R}^2$ so that for every pair of distinct edges wx and yz of G we have that the straight line joining $f(w)$ and $f(x)$ does not intersect the line segment joining $f(y)$ and $f(z)$, except at a mutual endpoint if it exists.

Proposition 2.1.4 (Schnyder, [33]). *Every graph $G = (V, E)$ of dimension at most three is planar.*

Proof. Let G be a graph of dimension 1. It follows from Proposition 2.1.2 that there is a 1-dimensional representation of V . Since the edge property must hold, we can see that there cannot be any edges in G . We can also observe that since the vertex property must hold, then G must have at most one vertex.

Now, let G be a graph of dimension 2. Let \leq_1, \leq_2 be a 2-dimensional representation of G . Say $\leq_1: v_1 v_2 \cdots v_n$. In order for the vertex property to hold we must have $v_n = \min_{\leq_2} V$. A similar argument shows that $v_{n-1} = \min_{\leq_2} V \setminus \{v_n\}$. By the repeated application of this argument we get that $\leq_2: v_n v_{n-1} \cdots v_2 v_1$. Since the edge property must hold, then every edge must be of the form $v_i v_{i+1}$, otherwise if $v_j v_{j+k} \in E$ and $k > 1$ the edge property is not satisfied for v_{j+1} and $v_j v_{j+k}$. So G is a subgraph of a path.

Now assume that G has dimension 3, and let \leq_1, \leq_2, \leq_3 be a 3 dimensional representation of G . To show G is planar we will exhibit a straight line embedding of G in the plane. Let $f : V \rightarrow \mathbb{R}^2$ be defined as $f(v) = (2^{\text{ord}_{\leq_1}(v)}, 2^{\text{ord}_{\leq_2}(v)})$. We will denote $2^{\text{ord}_{\leq_i}(v)}$ with v_i .

Claim. If $x, y \in V$ satisfy $y <_1 x$ and $x <_2 y$, then no vertex $z \neq x, y$ will be mapped by f to the triangle T delimited by (x_1, x_2) , (y_1, y_2) and (x_1, y_2) .

Proof (Claim). To prove that no vertex z will be mapped by f to T , let us assume the opposite and see what happens. As $f(z) \in T$, then

$$\frac{x_1 + y_1}{2} \leq z_1 \leq x_1 \quad \text{or} \quad \frac{x_2 + y_2}{2} \leq z_2 \leq y_2.$$

If the first inequality holds we get $\frac{x_1}{2} < z_1 \leq x_1$. As z_1 is a power of 2 then we must have $z_1 = x_1$, a contradiction. It can be proven in a similar way that the second inequality does not hold, so such z does not exist. □

Now we will prove that it is not possible for two edges to cross by considering two possible cases. First assume that two non adjacent edges cross, say edges xy and uv . We may assume, without loss of generality, that $x = \max_{\leq_1} \{x, y, u, v\}$. By the edge property we have $y >_i u, v$; $u >_j x, y$ and $v >_k x, y$. As $x >_1 u, v$ then $j, k \in \{2, 3\}$. Also note that if $y >_1 u, v$ then we get no intersection, so $i \in \{2, 3\}$. Now we have $j = k$, otherwise we get $i = j$ or $i = k$ hence $y >_j u >_j y$ or $y >_k v >_k y$ a contradiction. If $j = k = 2$ then we get no intersection, so $j = k = 3$ and $i = 2$. Observe that if $x_2 > y_2$, and since $y_2 > u_2, v_2$ then

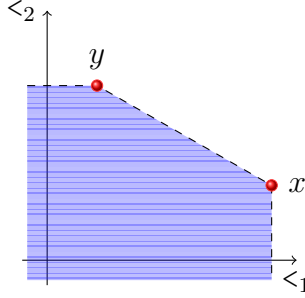


Figure 2.1: Vertices u and v must lie in the shaded region.

there would be no intersection, hence $y_2 > x_2$. That means that $u_1, v_1 < x_1$ and $u_2, v_2 < y_2$. So the only possible region where f could map u and v is $\{(s, t) | s < x_1 \text{ and } t < y_2\}$. It follows from the Claim we proved above that there is no vertex mapped to the triangle defined by $(x_1, x_2), (x_1, y_2), (y_1, y_2)$. Therefore the region where f can map u and v is reduced to the one shown in Figure 2.1, so there is no intersection, a contradiction.

Finally, let us treat the case where the edges, say xy and yz , are adjacent. One of x and z must be contained in the line segment joining the other two vertices. Without loss of generality we may assume that such vertex is z . As to not contradict the Claim we proved, we must have that $x_1 < z_1 < y_1$ and $x_2 < z_2 < y_2$ or $y_1 < z_1 < x_1$ and $y_2 < z_2 < x_2$. By the edge property we must have $x <_3 z$ and $y <_3 z$. First we will assume that $x_1 < z_1 < y_1$ and $x_2 < z_2 < y_2$. But this contradicts the vertex property, since $x <_i z$ for all i . We can proceed in a similar way in the case where $y_1 < z_1 < x_1$ and $y_2 < z_2 < x_2$. \square

Our aim in the following sections will be to prove the converse of Proposition 2.1.4. We will begin by introducing the concept of *dual orders* and a special type of labelling called *order labelling*. After this we define another type of labelling called *normal labelling*. We show that an order labelling is a normal labelling, as well as some properties of normal labelling. Once we have these tools, we show that every planar triangulation admits a normal labelling. Finally, we use all this to show that a normal labelling induces a 3-dimensional representation of the vertex set of a graph, hence proving that the graph has poset dimension at most 3.

2.2 Dual orders and order labelling

We introduce some partial orders that are obtained from a 3-dimensional representation. These partial orders will be helpful as tools to show how a 3-dimensional representation induces a normal labelling.

Let \leq_1, \leq_2, \leq_3 be a 3-dimensional representation of a set V . We define 3 dual relations $\leq_1^*, \leq_2^*, \leq_3^*$ on V as follows. For two elements $x, y \in V$, $x \leq_i^* y$ if and only if $x \leq_i y$ and $x \geq_j y$ for $j \neq i$.

The dual relations of a 3-dimensional representation are partial orders. One can observe that these relations are reflexive and antisymmetric. Now, let us prove that the relations are transitive. Let $x, y, z \in V$ so that $x \leq_i^* y$ and $y \leq_i^* z$. This implies $x \leq_i y$,

$y \leq_i z$ and $x \geq_j y$, $y \geq_j z$ for $j \neq i$. By transitivity in each of \leq_l we conclude that $x \leq_i z$ and $x \geq_j z$ for $j \neq i$. Hence $x \leq_i^* z$.

Let $G = (V, E)$ be the maximal graph admitted by a 3-dimensional representation of V . It follows from the vertex property that each pair of vertices x, y of G is comparable in exactly one of the dual orders. This allows us to define three disjoint sets of arcs $A_i \stackrel{\text{def}}{=} \{(x, y) : x \leq_i^* y, xy \in E\}$. Note that these sets of arcs induce a partition of the edge set of G into 3 sets, namely $E_i \stackrel{\text{def}}{=} \{xy \in E \mid (x, y) \in A_i\}$.

Lemma 2.2.1 (Schnyder, [33]). *Let \leq_1, \leq_2, \leq_3 be a 3-dimensional representation of V and let G be the maximal graph admitted by this 3-dimensional representation. If we let*

$$A'_i \stackrel{\text{def}}{=} \left\{ (x, y) \in V^2 \mid y = \min_{\leq_i} \{w \in V \mid x <_i^* w\} \right\},$$

then $A'_i = A_i$.

Proof. First we shall prove that $A_i \subseteq A'_i$. We will proceed by contradiction, let $(x, y) \in A_i$ and assume that y is not the minimum element of $\{w \in V \mid x <_i^* w\}$ with respect to \leq_i . Let $z = \min_{\leq_i} \{w \in V \mid x <_i^* w\}$. Now, let us observe that $z <_i y$ and since $x <_i^* z$ it follows that $z <_j x$ for $j \neq i$. This contradicts the edge property, since there is no order in which z is greater than both x and y . So we must have that $y = \min_{\leq_i} \{w \in V \mid x <_i^* w\}$.

Now we will prove that $A'_i \subseteq A_i$. Let $(x, y) \in A'_i$. Clearly $x \leq_i^* y$. We will now prove that xy is an edge of G . It suffices to show that the edge property is satisfied for xy and any vertex different from x and y . Let $z \in V \setminus \{x, y\}$. If $z <_i x$ then we use the vertex property, so there is an order $<_j$, $j \neq i$, so that $x <_j z$, and since $x <_i^* y$ it follows that $y <_j x <_j z$ as desired. Now, if $x <_i z <_i y$ it follows from the minimality of y that $x \not<_i^* z$, so there must be another order $<_j$, $j \neq i$, so that $x <_j z$. For this same order we have $y <_j x$ and so $y <_j x <_j z$ as desired. Finally, if $y <_i z$, then we have $x <_i y <_i z$. So the edge property is satisfied, hence $xy \in E_G$. □

Lemma 2.2.2 (Schnyder, [33]). *Let \leq_1, \leq_2, \leq_3 be a 3-dimensional representation of V and let G be the maximal graph admitted by this 3-dimensional representation. Then the digraph $D_i = (V, A_i)$ is acyclic and each vertex has outdegree at most one. If the representation is standard, then the outdegree of each interior vertex in D_i is one and the only non trivial component of D_i is that containing $a_i \stackrel{\text{def}}{=} \max_{<_i} V$.*

Proof. We can see that D_i is acyclic. Otherwise, assume that $x_1 x_2 \dots x_l x_1$ is a cycle in D_i . This implies $x_1 \leq_i^* x_2 \leq_i^* \dots \leq_i^* x_l \leq_i^* x_1$. So in particular $x_1 \leq_i x_2 \leq_i \dots \leq_i x_l \leq_i x_1$, implying that $x_1 = x_2 = \dots = x_l$, a contradiction. It follows from Lemma 2.2.1 that at each vertex there is at most one outgoing edge, since for each x , if $(x, y) \in A_i$ then y is unique.

Now assume that the representation is standard. Let x be an interior vertex. Since the representation is standard $\{w \in V \mid x <_i^* w\}$ is nonempty, as a_i is an element of this set. So by Lemma 2.2.1, the outdegree of every interior vertex is exactly one in this case. So given any interior vertex x we can find a directed path in D_i starting at x and ending at a_i .

Another thing to observe when the representation is standard is that the indegree of a_j for $j \neq i$ is 0. This is because $x \leq_j^* a_j$ for all $x \in V$ and for each partial order \leq_j^* . The other components, if any, would consist of isolated vertices, namely a_j , $j \neq i$. □

We next prove a proposition that will be useful in showing that the order labelling defines a normal labelling.

Proposition 2.2.3 (Schnyder, [33]). *Let \leq_1, \leq_2, \leq_3 be a 3-dimensional representation of V and let G be the maximal graph admitted by this 3-dimensional representation. The graph G is a plane triangulation if and only if the representation is standard.*

Proof. Assume that G is a plane triangulation and let a_i denote the maximum element of V with respect to \leq_i . We will show that the representation is standard. We proceed by contradiction. Assume that there is an order \leq_j and $a_i, i \neq j$, so that a_i is not among the smallest 2 elements in \leq_j . Let x and y denote the two smallest elements in \leq_j and let k be the index not i or j . We should first note that $a_k \neq a_i$, otherwise the vertex property would not hold for a_i and one of x or y . We may assume, without loss of generality, that $x \neq a_k$. Note that it is not possible that $a_i a_k \in E_G$ since if we choose $a_i a_k$ and x the edge property is not satisfied.

We have seen that $a_i a_k \notin E_G$. Now we will show that G is a proper subgraph of a planar graph, thus having a contradiction, since plane triangulations are maximal planar graphs. Let us define three total orders of V denoted by $\leq_l, l = 1, 2, 3$, as follows,

$$\leq_l = \begin{cases} \leq_l & \text{if } l \neq j \\ a_k a_i (\leq_j|_{V \setminus \{a_k, a_i\}}) & \text{if } l = j \end{cases},$$

where $\leq_j|_{V \setminus \{a_k, a_i\}}$ denotes the restriction of \leq_j to $V \setminus \{a_k, a_i\}$. Observe that in \leq_j the relative orders of elements in $V \setminus \{a_i, a_k\}$ are preserved. So if $x, y \in V \setminus \{a_k, a_i\}$ then the vertex property holds. Now, if one of $x, y \in \{a_k, a_i\}$ the vertex property holds, because a_k and a_i are the maximum elements of \leq_k and \leq_i respectively and they are the two minimal elements in \leq_j . So $\{\leq_l\}_{l=1,2,3}$ defines a 3-dimensional representation of V .

Let H be the maximal graph induced by $\{\leq_l\}_{l=1,2,3}$. Let $uv \in E_G$; we will show that $uv \in E_H$. Let $z \in V \setminus \{u, v\}$. If $z \notin \{a_i, a_k\}$, then the edge property holds, since the positions of a_i and a_k are decreased in \leq_j , and the relative order of the other vertices is preserved. If $z = a_i$ or $z = a_k$, then the edge property for uv and z would still hold in the order \leq_i or \leq_k respectively. So $G \subseteq H$. Now, observe that $a_i a_k \in E_H$, since the edge property holds in \leq_j for any $z \in V \setminus \{a_i, a_k\}$. So $G \not\subseteq H$, a contradiction.

Conversely, assume that the 3-dimensional representation is standard. In this case we know that G , the maximal induced graph, is planar by Proposition 2.1.4. We can see, from Lemma 2.2.2, that in each of the digraphs $D_i = (V, A_k)$ there is an outgoing edge from each of the $n - 3$ interior vertices. Thus there are at least $3n - 9$ edges. Also note that if the representation is standard, the three exterior vertices are pairwise adjacent. This totals $3n - 6$ edges, and hence G is a plane triangulation. \square

Let \leq_1, \leq_2, \leq_3 be a standard 3-dimensional representation of V and let G be the plane triangulation induced by these orders. If $x_0, x_1, x_2 \in V$ are pairwise adjacent vertices, then for each x_i there exists an order \leq_l so that $x_i = \max_{\leq_l} \{x_0, x_1, x_2\}$. The assertion holds because of the edge property, that is, using the edge property for the edge $x_{i+1} x_{i+2}$ and the vertex x_i , where the indices are taken modulo 3. We may also observe that such order is unique, otherwise, at least one of the two remaining vertices would not be above the other two in any order, thus contradicting the edge property.

We will assume that the positions of the vertices of G in the plane are given by the embedding used in the proof of Proposition 2.1.4. An edge $xy \in E_G$ will be called an *interior edge* if at least one of x or y is an interior vertex. Similarly, a triangle x, y, z in G will be called an *interior triangle* if at least one of x, y or z is an interior vertex and if they determine an inner face of G .

Definition 2.2.4. Let x, y, z be an interior triangle of G . We will label the angle of xyz at x with l , where l is the index of the order in which x is greater than y and z . We will call this labelling the *order labelling*. In the following Lemmas, we will show some properties of the order labelling.

Lemma 2.2.5. Let \leq_1, \leq_2, \leq_3 be a standard 3-dimensional representation of V and let G be the plane triangulation induced by this representation. If xyz is an interior triangle of G , then the labels 1, 2 and 3 given by the order labelling appear in counterclockwise order.

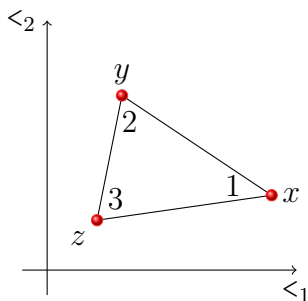


Figure 2.2: The labels appear in counterclockwise order.

Proof. Let x, y, z be an interior triangle of G and assume that x, y and z are maximal with respect to \leq_1, \leq_2, \leq_3 respectively, if this is not the case we may just relabel the vertices. Since x is maximal with respect to the first order and y is maximal with respect to second order then z cannot lie above y or to the right of x . Now, using the Claim proved in the proof of Proposition 2.1.4—that this embedding does not map any vertex to the triangle delimited by $(x_1, x_2), (x_1, y_2)$ and (y_1, y_2) —it follows that z must lie below the line segment joining x and y as shown in Figure 2.2. So the labels appear in counterclockwise order. □

Lemma 2.2.6. Let \leq_1, \leq_2, \leq_3 be a standard 3-dimensional representation of V and let G be the plane triangulation induced by this representation. If xy is an interior edge of G and xyw_1 and xyw_2 are the two interior triangles to which x and y belong, then the order label of w_1 in xyw_1 is different from the order label of w_2 in xyw_2 .

Proof. We will proceed by contradiction. Let ijk denote a cyclic permutation of 1 2 3. Assume that the labels of w_1 and w_2 are the same, say i . We can also assume that the order of the vertices in each of the triangles in counterclockwise order is xyw_1 and xw_2y , otherwise, we may just relabel x and y . It follows from Lemma 2.2.5 that the label of the angle at x in xyw_1 is j . Similarly the label of the angle at y in xw_2y is j , as shown

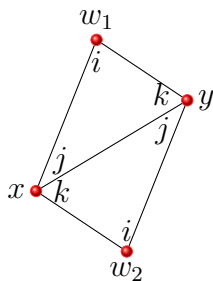


Figure 2.3: A configuration that is not possible.

in Figure 2.3. This implies, in particular, that $x \geq_j y$. But we can also infer that $x \leq_j y$. So $x = y$, a contradiction. \square

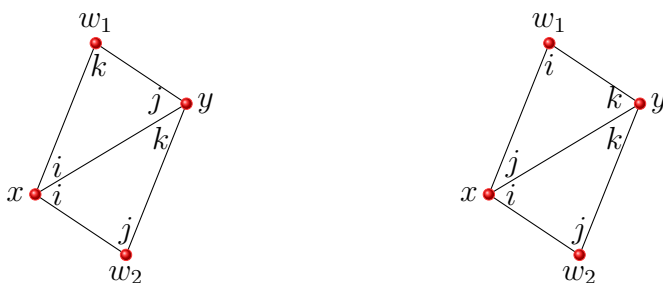


Figure 2.4: Labels at x stay the same or they change from i to j in counterclockwise order.

We now note an immediate consequence of the previous lemma. Consider a configuration as in the hypothesis of Lemma 2.2.6. The label of x at both triangles is the same or it changes from i to j in counterclockwise order, where ijk denotes a cyclic permutation of $1\ 2\ 3$. These two possibilities are illustrated in Figure 2.4. Note that there cannot be a transition in counterclockwise order from j to i , since this would be equivalent to the configuration shown in Figure 2.3.

2.3 Normal labelling

In this section we begin by introducing the definition of normal labelling of a planar triangulation. Then we will show how a 3-dimensional representation of a graph G induces a normal labelling of G by means of the order labelling.

Definition 2.3.1. Let G be a plane triangulation. A *normal labelling* of G is a labelling of the angles of the inner triangles of G so that the following two properties hold.

- Each inner triangle has an angle labelled 1, 2 and 3, the labels appearing in counterclockwise order.

- At each interior vertex, the angles labelled i form a consecutive block. The blocks 1, 2 and 3 at each vertex appear in counterclockwise order.

Lemma 2.3.2. *Let \leq_1, \leq_2, \leq_3 be a standard 3-dimensional representation of V and let G be the plane triangulation induced by this representation. The order labelling of G defines a normal labelling.*

Proof. It follows from Lemma 2.2.5 that each interior angle of G gets a label and the order of these labels in each triangle is counterclockwise.

We are only left to prove that at each interior vertex, there is a unique block of each type and that these blocks appear in counterclockwise order. Let v be an interior vertex and let ijk be a cyclic permutation of 1 2 3.

We can see that $\deg(v) \geq 3$, otherwise G would not be a plane triangulation. We have seen that the transition of the labels at v must be from i to j , from j to k or from k to i in counterclockwise order. We are only left to prove that there is only one block of each kind. Equivalently, we may prove that there are only three transitions in the labels of the angles at v . Note that there is a one to one correspondence between a transition from label j to k and an outgoing edge from v in $D_i = (V, A_i)$. This is because if there is an outgoing edge from v to w in D_i , then it corresponds to a label change in the angles at v from j to k since $w <_i^* v$. Conversely, if there is a label change in v , say from j to k , through the edge vw this implies that the labels at w are i , and so $vw \in A_i$. Since the out degree of v in D_i is one, it follows that this type of transition is unique. Similarly, the other two possible transitions are unique. Hence there are only three transitions, one of each kind. \square

In the following two lemmas we show some properties of normal labellings.

Lemma 2.3.3 (Schnyder, [33]). *Let G be a plane triangulation with a normal labelling, let xy be an interior edge of G and let ijk be a cyclic permutation of 1 2 3. The only possible labellings of the angles of the two adjacent triangular faces having xy in the boundary are the ones shown in Figure 2.5, up to a renaming of x and y .*

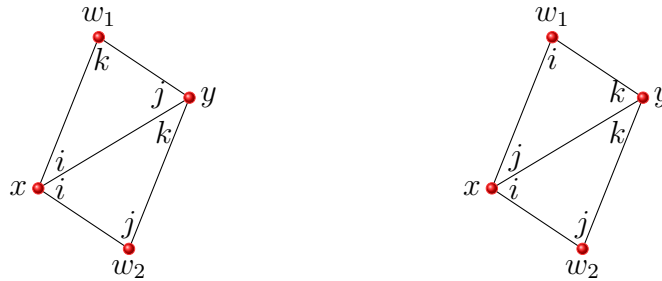


Figure 2.5: Only possible labellings of two adjacent triangular faces.

Proof. We first observe that any labelling different from the ones shown in Figure 2.5 has w_1 and w_2 labelled the same in both triangles. So assume that w_1 and w_2 are both labelled the same, say i , in the two triangular faces. From this, it follows that the label of x in xyw_1 is j . We also have that the label of x in xw_2y is k . This contradicts the definition of normal labelling. \square

Let G be a plane triangulation with a normal labelling and let C be a cycle of G . We say a vertex $c \in V_C$ is of type i , if all the angles at c of the triangular faces of G enclosed by C are labelled i .

Lemma 2.3.4 (Schnyder, [33]). *Let G be a plane triangulation and let C be a cycle in G . For any normal labelling of G , there is a vertex of each type in C .*

Proof. We will proceed by induction on the number of triangular faces that C encloses. We can see the statement holds when the cycle encloses only one triangular face by definition of normal labelling.

Now, assume that the statement holds for all cycles enclosing at most n triangular faces. Let $C = c_0c_1 \dots c_{k-1}c_k c_0$ be a cycle in G enclosing $n + 1$ triangular faces. We will consider two possible cases. First we consider the case when there is an edge in G between two nonadjacent vertices of C , say c_r and c_s . Assume, by contradiction, that C does not have a vertex of type i . We observe that the two cycles C' and C'' shown in Figure 2.6 enclose at most n triangular faces of G . So by the induction hypothesis, these two cycles have a vertex of type i . The vertex of type i in C' and C'' must be one of c_r or c_s in both cases, otherwise C would have a vertex of type i . Let us assume, by contradiction, that the vertex of type i in C' is different from the vertex of type i in C'' . We may assume without loss of generality that the vertices of type i in C' and C'' are c_r and c_s respectively. Now if we observe the interior edge $c_r c_s$, we can see that the assumption that the vertices of type i are not the same leads to a contradiction of Lemma 2.3.3, as the vertex labeled i in the two triangular faces having $c_r c_s$ in its boundary are opposed to each other. Therefore both vertices of type i in C' and C'' are the same, hence C has a vertex of type i , a contradiction.

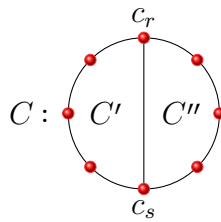


Figure 2.6: The vertices of type i in C' and C'' must be the same.

Let us now consider the case where there is no edge between two non adjacent vertices in C . Let $C = c_0c_1 \dots c_k c_0$ and, as illustrated in Figure 2.7, let d_l be the common neighbor of $c_l c_{l+1}$ so that $c_l d_l c_{l+1}$ defines the triangular face enclosed by C , where the indices are taken modulo k . We will call each of the triangular faces $c_l d_l c_{l+1}$ T_l . As done in the previous case, we will proceed by contradiction, so assume that C does not contain a vertex of type i .

We will first show that there exists a triangle T_l so that the label of d_l in T_l is i . We will proceed by contradiction. Assume that there is no such triangle. Since no vertex d_l is labelled i in each triangle T_l , then the vertex labelled i in each of these triangles is c_l or c_{l+1} .

Now, let us observe that for any pair of consecutive triangles T_l and T_{l+1} , we cannot have that c_{l+1} is labelled i in both of the triangles as this would imply c_{l+1} is a vertex of type i in C . So, we must have that for all triangles T_l either c_l or c_{l+1} is labelled i . This would imply that d_l is labelled k or j in all the triangles T_l , where ijk denotes a cyclic permutation of 1 2 3. We may assume that the label of d_l is j in all T_l , as the other possible case is proved similarly. This is illustrated in Figure 2.7. As we can observe in

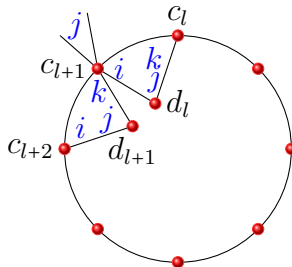


Figure 2.7: The block labelled j at each vertex of C lies outside the cycle.

Figure 2.7, the block labelled j at each vertex of C lies in the exterior of the cycle. If we consider the cycle $c_0 d_0 c_1 c_2 \dots c_k c_0$, that encloses at most n triangular faces, we see that this cycle does not contain a vertex of type j . This contradicts the induction hypothesis, so there must be a triangular face T_l for which the label of d_l in such triangle is i .

Now let us consider the cycle $C' = c_0 c_1 \dots c_l d_l c_{l+1} \dots c_k c_0$. This cycle encloses at most n triangular faces, so it must contain a vertex of type i . By assumption this vertex cannot be any of $c_0, \dots, c_{l-1}, c_{l+2}, \dots, c_k$. Also, this vertex cannot be d_l , as this contradicts the definition of normal labelling. Finally, if the vertex of type i in C' is c_l or c_{l+1} this would contradict Lemma 2.3.3. So there must be a vertex of type i . We may use this argument with $i = 1, 2, 3$ and the result follows. □

2.4 Every plane triangulation admits a normal labelling

The main result of this section will show that if G is an embedded plane triangulation, then G admits a normal labelling. The result is proved using induction on the number of vertices. We will first prove a lemma that will be useful.

Lemma 2.4.1 (Kampen,[21]). *Let G be an embedded plane triangulation with at least four vertices, and let f denote the outer face of G . If $y \in V_G$ is in the boundary of f , then there exists a neighbor of y not in the boundary f , say w , so that y and w only have two common neighbors.*

Proof. We will proceed by induction on the number of vertices. We can see that this clearly holds for the unique plane triangulation on 4 vertices, the unique interior vertex has exactly two common neighbors with any vertex in the boundary of the exterior face.

Now, assume that the statement holds for graphs with at most n vertices. Let G be a plane triangulation with $n+1$ vertices. Let $y \in V_G$ and let w_1, w_2 and w_3 be three neighbors of y in counterclockwise order so that w_2 is not in the boundary of the exterior face. Since G is a plane triangulation, yw_1w_2 and yw_2w_3 determine 2 faces of G . If the only two common neighbors that w_2 and y have are w_1 and w_3 then there is nothing to prove. So, suppose that there is another common neighbor of y and w_2 , say $z \in V \setminus \{w_1, w_3\}$. Since y, w_2 and z form a triangle, we can consider the subgraph H of G induced by y, w_2, z , and all the vertices lying inside this triangle.

Claim. The graph H defined above is a plane triangulation.

Proof (Claim). Assume, by contradiction, that H is not a plane triangulation. Hence there exist two non adjacent vertices u and $v \in V_H$ so that $H + uv$ is planar. Since H is induced by a subset of vertices of G , this implies $uv \notin E_G$. Finally, we observe that $G + uv$ is also planar, a contradiction. \square

The graph H defined above has at most n vertices, since at least one of w_1 and w_3 is not a vertex of H . By the induction hypothesis, there is a vertex not in the boundary of the exterior face of H , say w , so that w and y have exactly two common neighbors in H . Since w is not in the boundary of the exterior face of H , then w is not in the boundary of the exterior face of G . Consequently $N_G(w) = N_H(w)$, so y and w have exactly two common neighbors. \square

Proposition 2.4.2 (Schnyder, [33]). *Let G be a plane triangulation. The graph G has a normal labelling.*

Proof. We proceed by induction on the number of vertices. The statement clearly holds for the plane triangulation on 3 vertices.

Assume that the statement holds for plane triangulations with at most n vertices. Let G be a plane triangulation with $n + 1$ vertices and let y be a vertex in the boundary of the exterior face of G .

By Lemma 2.4.1 there exists an interior vertex $w \in V_G$ so that y and w have exactly two common neighbors, say x and z . Consider the graph H obtained from G by removing w and by adding the edges $yw_i, w_i \in N(w) \setminus \{x, z\}$; that is, H is obtained from G by contracting the edge yw . By the induction hypothesis, H has a normal labelling.

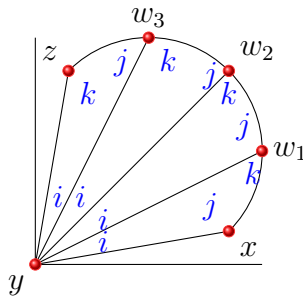


Figure 2.8: Labelling in H .

Note that it follows from Lemma 2.3.4 that all angles at y have the same label in each triangular face. This is because y and the other two vertices in the boundary of the exterior face of H form a cycle. Assume that all angles at y are labelled i in H . The labels of the triangular faces in H formed by two neighbors of w and y are shown in Figure 2.8.

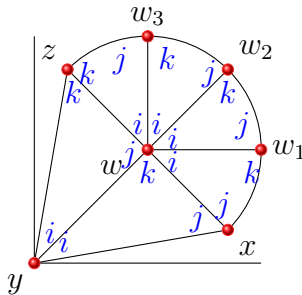


Figure 2.9: Labelling in G .

Now, in G label each triangular face of G not containing w in its boundary the same as in H . All triangular faces containing w in its boundary will be labelled as shown in Figure 2.9.

We will now prove that the labelling defined above is a normal labelling of G . We can see that the first condition of Definition 2.3.1 is satisfied for every interior triangle not having w in its boundary since these labels were taken from the normal labelling of H . As illustrated in Figure 2.9 this condition is satisfied for triangular faces having w in their boundary.

The second condition of Definition 2.3.1 is also satisfied for all interior vertices $v \in V \setminus \{w, x, z\}$, since the labels at each angle did not change. This condition is also satisfied by w as we can see in Figure 2.9. Finally, x and z also satisfy this condition since the only difference is that one of the angles was subdivided, and both angles are labelled the same as the original one. So this is a normal labelling of G . □

2.5 Every planar graph has dimension at most 3

In this section we will show that the dimension of the incidence poset of a planar graph is at most 3. We will prove this statement for plane triangulations. The main result will follow, since planar graphs are subgraphs of plane triangulations and the dimension of a graph can only increase if we add more edges. We explain why this assertion holds. Let G and H be graphs so that $G \subseteq H$ and let R be a d -representation of H that satisfies the edge property for E_H . We can see that in particular R satisfies the edge property for E_G , and so $\dim(G) \leq \dim(H)$. In other words, if we add edges to G then its poset dimension can only increase.

We will assume that the plane triangulation is embedded in the plane and that we are given a normal labelling of the plane triangulation. We will define three digraphs D_l ,

$l = 1, 2, 3$, in terms of the normal labelling. The digraphs D_l will be useful to prove that 3 particular linear orders, defined in terms of the normal labelling, define a 3 dimensional representation of G . In what follows in this section, ijk will denote a cyclic permutation of $1\ 2\ 3$.

Definition 2.5.1. Let G be an embedded plane triangulation with a normal labelling. For a given edge $xy \in E_G$, f_{xy} and g_{xy} will denote the two triangular faces of G having xy in their boundary. Let us consider the digraph $D_i = (V, A_i)$, where

$$A_i = \{(x, y) : xy \in E_G \text{ and the label of the angles at } y \text{ in } f_{xy} \text{ and } g_{xy} \text{ is } i\}.$$

It follows from Lemma 2.3.3 that $\{A_i\}_{i=1,2,3}$ induces a partition on the set of interior arcs of G . That is, each interior arc will belong to one A_i . We can see that the exterior vertices have outdegree 0. Now observe that the outdegree of each interior vertex of G is exactly one. This follows from the fact that, in a normal labelling, at each vertex x there is a block of ones followed by a block of twos followed by a block of threes. In the following lemma we will prove that these digraphs are acyclic.

Lemma 2.5.2. *Let G be a plane triangulation with a normal labelling and let D_i be a digraph as defined above. The digraph D_i does not contain cycles.*

Proof. We will proceed by contradiction, so assume that there is a cycle in D_i . First note that the cycle has to be directed. This follows from the fact that the outdegree of each exterior vertex is 0 and the outdegree of each interior vertex is one. So assume that there is a directed cycle $C = c_0c_1 \dots c_kc_0$ in D_i . Since each arc (c_l, c_{l+1}) is an element of D_i , it follows that the labels of the angles at c_{l+1} in $f_{c_l c_{l+1}}$ and $g_{c_l c_{l+1}}$ are i . But this contradicts Lemma 2.3.4, since this cycle would not contain a vertex of type k . \square

We know each digraph D_i is acyclic. We can also see that for each interior vertex x , there is a path in D_i going from x to one exterior vertex. We will call the outgoing path from x in D_i $P_i(x)$ and will call the exterior vertex at which this path ends a_i . We can see, that in D_i , the indegree of a_j and a_k is 0. This follows from the fact that the labels of the angles at each exterior vertex are the same. For a given interior vertex, these paths do not share vertices, except for the initial one. This is stated in the following Lemma.

Lemma 2.5.3. *Let G be a plane triangulation and let x be an interior vertex. The paths $P_i(x)$ and $P_j(x)$ defined above have no vertices in common except for x .*

Proof. We proceed by contradiction. Assume that the paths $P_i(x)$ and $P_j(x)$ have a common vertex, say y . Hence there is a path from x to y in D_i , say $x = p_0p_1 \dots p_l = y$. Also, there is a path from x to y in D_j , say $x = q_0q_1 \dots q_m = y$. Note that each of the arcs $(p_n, p_{n+1}) \in D_i$ and $(q_n, q_{n+1}) \in D_j$. This implies that the label of p_{n+1} in each of these arcs is i and the label of q_{n+1} in each of these arcs is j . As a consequence, the cycle induced by these two paths does not have a vertex of type k which contradicts Lemma 2.3.4. \square

We have seen that given an interior vertex x of a plane triangulation, the paths $P_i(x)$, $P_j(x)$ and $P_k(x)$ only have x in common. So if we consider these three paths together with the three exterior edges we can see this partitions the set of interior faces of G into three disjoint sets as shown in Figure 2.10.

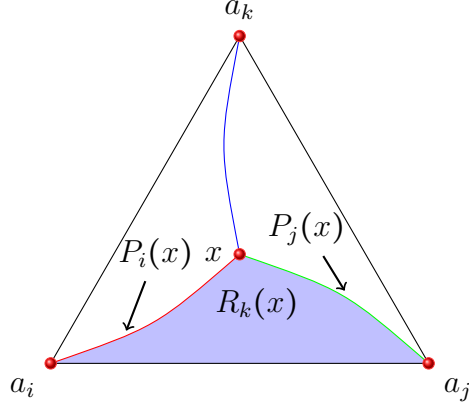


Figure 2.10: Three regions induced by the outgoing paths at a given vertex

Definition 2.5.4. As illustrated in Figure 2.10, the region enclosed by $P_i(x)$, $P_j(x)$ and $a_i a_j$, including both paths and the edge, will be called $R_k(x)$.

Note that the regions $R_i(x)$ do not partition the set of vertices or edges, since there are vertices and edges that belong to two different regions. We can also see that any interior vertex $y \in V_G \setminus \{x\}$ is either contained inside a region or it is in the boundary between two adjacent regions. Furthermore, if we consider an edge, we see that the two endpoints of the edge must both be contained in some region. Otherwise, this would contradict the planarity of G .

Now let us consider three relations on V , \lesssim_i , $i = 1, 2, 3$, where $a \lesssim_i b$ if and only if $a = b$ or a is adjacent to b and b is labelled i in f_{ab} or g_{ab} . Let \leq_i denote the transitive closure of \lesssim_i .

Lemma 2.5.5. *Let G be a plane triangulation and let \leq_i be the relation on V_G as defined above. The relation \leq_i defines a partial order on V_G .*

Proof. We can see that the relation is reflexive and transitive by definition. We are only left to prove it is antisymmetric. Assume, by contradiction, that there exist $a, b \in V_G$, $a \neq b$, so that $a \leq_i b$ and $b \leq_i a$. This implies there is a sequence $\{a_l\}_{l=1}^p$, where $a_1 = a$, $a_p = a$ and $a_q = b$, $1 < q \leq p$, so that $a_1 \lesssim_i a_2 \lesssim_i \dots \lesssim_i a_p$. Observe that the vertices a_1, \dots, a_p , induce a subgraph H of G which contains at least one cycle. Let $C = c_0 c_1 \dots c_r c_0$ be the shortest cycle in H . We may assume that $c_0 \lesssim_i c_1 \lesssim_i \dots \lesssim_i c_r \lesssim_i c_0$, otherwise $c_0 \lesssim_i c_r \lesssim_i \dots \lesssim_i c_1 \lesssim_i c_0$ would hold and we could relabel the vertices so this condition holds. Since G is embedded, we now note that $c_0 c_1 \dots c_r$ might be in counterclockwise order or in clockwise order. We will show that neither of those two possibilities can occur. Assume that the vertices of C are in counterclockwise order. It follows that C does not have a vertex of type j . This is because at each vertex c_{l+1} , the angle at c_{l+1} in $f_{c_l c_{l+1}}$ or $g_{c_l c_{l+1}}$ is labelled i . In each of these two cases, by definition of normal labelling, the label of c_{l+1} in the triangular face enclosed by C is either k or i . This is illustrated in Figure 2.11. This contradicts Lemma 2.3.4. So it is not possible.

For the case when the vertices of C are in clockwise order, we use a similar argument. In this case, C will not have a vertex of type k —a contradiction again. So such a cycle

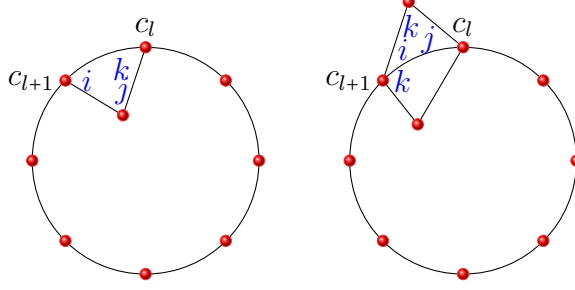


Figure 2.11: C does not contain a vertex of type j

cannot exist. Hence $a = b$ and so the relation is antisymmetric and \leq defines a partial order on V . \square

We now define \leq_i to be any linear extension of \leq_i . In the following Proposition we will prove that these three linear orders define a 3-dimensional representation of V satisfying the edge property for the edges in E_G .

Proposition 2.5.6. *Let G be a plane triangulation and let \leq_i be the three orders defined above. The orders \leq_i define a 3-dimensional representation of G satisfying the edge property for all $e \in E_G$.*

Proof. We will first prove that the orders \leq_i satisfy the vertex property. Let $x, y \in V_G$. We know there is i so that $y \in R_i(x)$. Since G is planar and $y \in R_i(x)$, then $P_i(y)$ must have a common vertex with $P_j(x)$ or $P_k(x)$. Let y^* be such a vertex and assume that $y^* \in P_j(x)$; the case when $y^* \in P_k(x)$ is proved in a similar way. Let $P = y_1 \dots y_p$ be the subpath of $P_i(y)$ going from y to y^* and let $Q = x_1 \dots x_q$ be the subpath of $P_j(x)$ going from x to y^* . We can observe that the labels of the angles at y_{l+1} in $f_{y_l y_{l+1}}$ and $g_{y_l y_{l+1}}$ are both i , since P is a subpath of $P_i(y)$. Now, observe that exactly one of the labels at x_l is i in one of the triangular faces having $x_l x_{l+1}$ in the boundary. This holds because Q is a subpath of $P_j(x)$. Since \leq_i is the transitive closure of \leq_i , we have $y \leq_i x$. Finally, since \leq_i is a linear extension of \leq_i , it follows that $y \leq_i x$. We can prove in a similar way that $y \leq_j x$ or $y \leq_k x$, depending on whether $x \in R_j(y)$ or $x \in R_k(y)$, since $x \in R_i(y)$ is not possible. So the vertex property holds.

We will now show these orders satisfy the edge property for all $e \in E_G$. Let xy be an edge of G and let z be a vertex of G different from x and y . As we observed before, it follows from the planarity of G , that there must be a region $R_i(z)$ so that both vertices x and $y \in R_i(z)$. And as argued before, we can show that $x \leq_i z$ and $y \leq_i z$. Hence the edge property holds and so the total orders \leq_i define a 3 dimensional representation. So by Proposition 2.1.2, the dimension of G is at most 3. \square

Chapter 3

Applications of Schnyder's Theorem

As we saw in the previous chapter, a straightline embedding can be obtained from the 3-dimensional representation of a planar graph. In this chapter we will present a barycentric embedding of a planar graph. The graph will be embedded in the plane $x + y + z = 2n - 5$, where n is the number of vertices of the graph. Each vertex is assigned a point on the mentioned plane with integer coordinates. In the second section of this chapter we introduce the concept of canonical ordering. We then show that any of the three linear orders in a 3-dimensional representation is a canonical ordering of $V(G[R])$. We then outline an algorithm, shown in [13], that receives as input a plane triangulation T and a canonical ordering of its vertices and outputs a straightline embedding of T in an $\mathcal{O}(n) \times \mathcal{O}(n)$ grid. In the third section of this chapter we consider the problem of tree decomposition, and we prove that a plane triangulation can be decomposed into three edge disjoint trees.

3.1 Barycentric Embedding

In this section we introduce the concept of barycentric embedding. We show that a barycentric embedding is a straightline embedding in the plane $x + y + z = 1$. Finally, we show a particular kind of barycentric embedding for plane triangulations.

In this section we will use the notation f_{xy} and g_{xy} for the faces that have xy in its boundary and the digraphs D_i from Definition 2.5.1. We will also use the regions $R_i(x)$ from Definition 2.5.4. The embedding that will be shown in this section, will be described in terms of the number of triangular faces enclosed by the regions $R_i(x)$, denoted by $|R_i(x)|$.

Definition 3.1.1. Let G be a graph. A *barycentric embedding* of G is an injective function $f : V(G) \rightarrow \mathbb{R}^3$, $v \mapsto (v_1, v_2, v_3)$, so that

1. The image of each vertex is in the plane $x + y + z = 1$, that is, $v_1 + v_2 + v_3 = 1$ for all $v \in V(G)$.
2. For each edge $uv \in E(G)$ and $w \in V(G) \setminus \{u, v\}$ there is a coordinate k so that $u_k, v_k < w_k$.

We now show that each barycentric embedding is a straightline embedding in the plane $x + y + z = 1$.

Proposition 3.1.2. *Let G be a graph. If f is a barycentric embedding of G , the f is a straightline embedding of G in the plane $x + y + z = 1$.*

Proof. We have by definition of barycentric embedding that f is injective and that each vertex will be mapped to a point in the plane $x + y + z = 1$. Now, observe this implies that each line segment $\overline{f(u)f(v)}$ joining the images of u and v lies in the plane $x + y + z = 1$.

Let $uv, xy \in E(G)$. We will first show that if uv and xy are non adjacent edges, then $\overline{f(u)f(v)} \cap \overline{f(x)f(y)} = \emptyset$. From the second condition of Definition 3.1.1, we obtain the existence of i, j, k , and l so that

$$x_i, y_i < u_i \quad x_j, y_j < v_j \quad u_k, v_k < x_k \quad u_l, v_l < y_l.$$

Note that $i \neq k, l$ and $j \neq k, l$. Now, since $\{i, j, k, l\} \subseteq \{1, 2, 3\}$ we must have $i = j$ or $k = l$. We may assume, without loss of generality, that $i = j$. So $x_i, y_i < u_i, v_i$. Observe that each point $z \in \overline{f(u)f(v)}$ satisfies $z_i \geq \min\{u_i, v_i\}$ and each point $w \in \overline{f(x)f(y)}$ satisfies $w_i \leq \max\{x_i, y_i\}$. Therefore, by transitivity we get $w_i < z_i$ for any pair of points $w \in \overline{f(x)f(y)}$ and $z \in \overline{f(u)f(v)}$. So

$$\overline{f(u)f(v)} \cap \overline{f(x)f(y)} = \emptyset.$$

Let us now consider the case where the two edges are adjacent. So assume the edges are uv and vw . We will show that

$$\overline{f(u)f(v)} \cap \overline{f(v)f(w)} = \{f(v)\}.$$

The only possible way there could be more than one point in the intersection of these two line segments, is if $f(u), f(v), f(w)$ are collinear and $f(u) \in \overline{f(v)f(w)}$, or $f(w) \in \overline{f(v)f(u)}$. So assume this occurs and let us see what happens. We may assume without loss of generality that $f(u) \in \overline{f(v)f(w)}$. If this is the case, then there exists $\lambda \in (0, 1)$ so that

$$\begin{aligned} (u_1, u_2, u_3) &= \lambda(v_1, v_2, v_3) + (1 - \lambda)(w_1, w_2, w_3) \\ &= (\lambda v_1 + (1 - \lambda)w_1, \lambda v_2 + (1 - \lambda)w_2, \lambda v_3 + (1 - \lambda)w_3). \end{aligned} \quad (3.1)$$

Now, by Definition 3.1.1, there is i so that $v_i, w_i < u_i$. This implies $\lambda v_i < \lambda u_i$ and $(1 - \lambda)w_i < (1 - \lambda)u_i$. Therefore $\lambda v_i + (1 - \lambda)w_i < \lambda u_i + (1 - \lambda)u_i = u_i$, contradicting equation (3.1). So $\overline{f(u)f(v)} \cap \overline{f(v)f(w)} = \{f(v)\}$, as desired. \square

Let us consider the digraphs D_i , the paths $P_i(u)$, and the regions $R_i(v)$ defined in section 2.5. Recall $|R_i(v)|$ denotes the number of triangular faces enclosed by $R_i(v)$. We will now prove some properties about the regions $R_i(v)$. These lemmas will provide us with the necessary tools to be able to show that given an edge uv of a plane triangulation T and a vertex w , there is k so that $|R_k(u)|, |R_k(v)| < |R_k(w)|$. Once we prove this, we will define a barycentric embedding in terms of the number of triangular faces in each region.

Lemma 3.1.3. *If $P_i(u) \cap P_i(v) \neq \emptyset$ and w is the first vertex in $P_i(v)$ while traversing $P_i(u)$, then $P_i(u) \cap P_i(v) = P_i(w)$.*

Proof. Since w is the first vertex of $P_i(u)$ appearing in $P_i(v)$, all the following vertices in $P_i(u)$ and $P_i(v)$ are determined. This follows from the fact that at each interior vertex there is exactly one outgoing edge along path i . Therefore all the remaining vertices in $P_i(u)$ and $P_i(v)$ must be the vertices along $P_i(w)$. \square

Lemma 3.1.4. *For every vertex $u \in R_k(v)$ and $i \neq k$, $P_i(u) \subseteq R_k(v)$.*

Proof. We can observe that the statement is clearly true for vertices along $P_i(v)$ from Lemma 3.1.3.

Now let us consider the case where $u \in P_j(v)$. It suffices to prove that the neighbor of u , say w , along $P_i(u)$ belongs to $R_k(v)$. If we assume this is not true, then the angles of $f_{u,w}$ and $g_{u,w}$ at u would be j and k in counterclockwise order. Let w^- be the predecessor of u in $P_j(v)$ and let w^+ be the successor of u in $P_j(v)$. Observe that the labels of the angles of $f_{w^-,u}$ and $g_{w^-,u}$ at u are both j , since (w^-, u) is an edge of D_j . Also note that the angles of f_{u,w^+} and g_{u,w^+} at u are k and i in counterclockwise order. Now, since we are assuming w is outside $R_k(v)$, this implies we have two intervals of angles labelled k at u , which is a contradiction to the definition of normal labelling.

We will now show that the statement holds for any other vertex u in $R_k(v)$ not along $P_i(v)$ or $P_j(v)$. Let us assume, by contradiction, that xy is the first edge along $P_i(u)$ not in $R_k(v)$. This implies $x \in P_i(v)$ or $x \in P_j(v)$, as these two paths bound $R_k(v)$. Hence x is a vertex $x \in P_i(v) \cup P_j(v)$ having an edge along $P_i(x)$ outside $R_k(v)$, which contradicts what we proved earlier. \square

Lemma 3.1.5. *Let $u \in R_k(v)$, then $v \notin R_k(u)$.*

Proof. Since $u \in R_k(v)$, by Lemma 3.1.4 we have that $P_i(u), P_j(u) \subseteq R_k(v)$, as illustrated in Figure 3.1. As the figure may suggest, we have that $v \notin R_k(u)$. Indeed, as this would

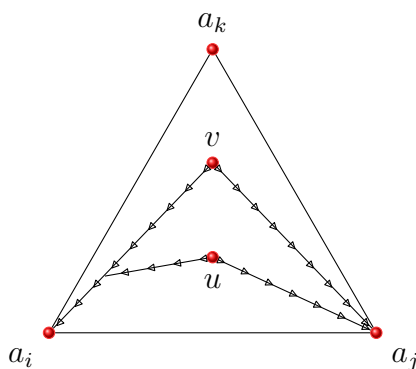


Figure 3.1: $P_i(u), P_j(u) \subseteq R_k(v)$

imply $P_i(v), P_j(v) \subseteq R_k(u)$. But this is not possible, unless $u = v$. \square

Lemma 3.1.6. *If $u \in R_k(v)$, then $R_k(u) \subset R_k(v)$.*

Proof. Let u be in $R_k(v)$. By Lemma 3.1.4 we can conclude that $P_i(u)$ and $P_j(u)$ are both contained in $R_k(v)$. Hence $R_k(u) \subset R_k(v)$.

Now, by Lemma 3.1.5 we get $R_k(u) \subset R_k(v)$. □

Proposition 3.1.7. *Let T be a plane triangulation. If $uv \in E(T)$ and $w \in V(T) \setminus \{u, v\}$, then $|R_k(u)|, |R_k(v)| < |R_k(w)|$ for some $k \in \{1, 2, 3\}$.*

Proof. As we observed in Section 2.5, given a vertex w , each edge uv belongs to a region $R_k(w)$. It is not possible for uv to have its endpoints properly contained in two different regions, as this would contradict the planarity of T .

By Lemma 3.1.6 we have $R_k(u), R_k(v) \subset R_k(w)$. Since $w \notin R_k(u), R_k(v)$, it follows that $R_k(u)$ and $R_k(v)$ do not contain any triangular face containing w . Therefore $|R_k(u)|, |R_k(v)| < |R_k(w)|$ as desired. □

Theorem 3.1.8. *Let T be a plane triangulation with n vertices. Define $f : V(G) \rightarrow \mathbb{R}^3$,*

$$v \mapsto \frac{1}{(2n-5)} (|R_1(v)|, |R_2(v)|, |R_3(v)|)$$

if v is an interior vertex and

$$a_1 \mapsto (1, 0, 0) \quad a_2 \mapsto (0, 1, 0) \quad a_3 \mapsto (0, 0, 1)$$

for the exterior vertices. Then f is a barycentric embedding of T .

Proof. Since T is a plane triangulation it has $3n - 6$ edges, and it follows from Euler's formula that T has $2n - 4$ triangular faces. Of these $2n - 4$ triangular faces only one is unbounded, so there are $2n - 5$ interior faces.

First let us prove that f is injective. We can see that the images of two exterior vertices are different. Now, if we consider an interior vertex and an exterior vertex, we can see they are mapped to two different points, as there is no region $R_i(v)$ of an interior vertex v containing all the $2n - 5$ interior triangular faces. Finally, let $u, v \in V(G)$ be two different interior vertices. We must have $u \in R_k(v)$ for some k . From Lemma 3.1.6 and Lemma 3.1.5 we get that $|R_k(u)| < |R_k(v)|$. So $f(u) \neq f(v)$, thus f is an injective mapping.

It is clear that every exterior vertex is mapped to a point on the plane $x + y + z = 1$. For an interior vertex v we have the following

$$|R_1(v)| + |R_2(v)| + |R_3(v)| = 2n - 5,$$

as the total number of interior faces of T is $2n - 5$. So each interior vertex is also mapped to a point on the plane $x + y + z = 1$.

Let us now prove the second condition of Definition 3.1.1. Let $uv \in E(T)$ and let $w \in V(T) \setminus \{u, v\}$. It follows from Proposition 3.1.7 that there is k so that

$$|R_k(u)|, |R_k(v)| < |R_k(w)|.$$

So the k -th coordinate of $f(w)$ satisfies the required condition. □

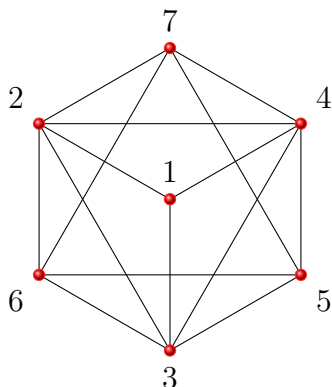


Figure 3.2: The graph induced by the linear orders.

We will now illustrate, with an example, how these embeddings look. Let us consider the following 3-dimensional representation of $\{1, 2, 3, 4, 5, 6, 7\}$

$$\begin{aligned} \leq_1: & 7624135 \\ \leq_2: & 5743126 \\ \leq_3: & 6532147. \end{aligned}$$

One can check that these linear orders induce the graph shown in Figure 3.2.

The embedding from the proof Proposition 2.1.4 is illustrated in Figure 3.3.

Finally, when considering the barycentric embedding for the same graph, it looks as shown in Figure 3.4.

3.2 Canonical Ordering

In this section the concept of canonical ordering, from [13], will be introduced. We will see the relation between a 3-dimensional representation and canonical orderings of the vertex set.

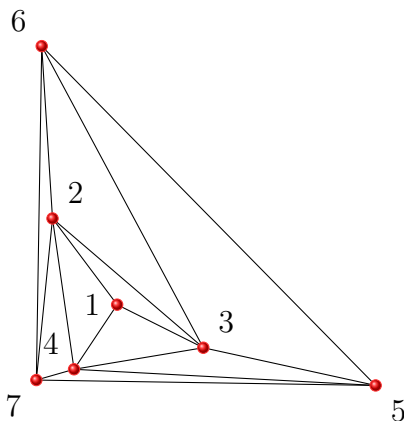


Figure 3.3: Straightline embedding from Chapter 1.

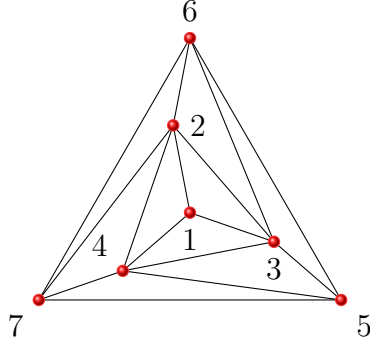


Figure 3.4: Barycentric Embedding.

Definition 3.2.1. A *canonical ordering* of the vertices of a plane triangulation G with exterior face a_1, a_2, a_3 is a total order $\leq: v_1 v_2 v_3 \cdots v_n$, where $v_1 \stackrel{\text{def}}{=} a_1$, $v_2 \stackrel{\text{def}}{=} a_2$ and $v_n \stackrel{\text{def}}{=} a_3$, of $V(G)$ so that the following two conditions hold for $4 \leq k \leq n$,

1. $G_{k-1} \stackrel{\text{def}}{=} G[\{v_1, \dots, v_{k-1}\}]$ is 2-connected and the boundary of its exterior face is a cycle C_{k-1} containing the edge $a_1 a_2$.
2. The vertex v_k is in the exterior face of G_{k-1} and its neighbors form a subinterval with at least two elements of the path $C_{k-1} - a_1 a_2$.

Now, we establish a result that relates 3-dimensional representations with canonical orderings.

Proposition 3.2.2. If $R = \{\leq_1, \leq_2, \leq_3\}$ is a standard 3-dimensional representation, then \leq_3 is a canonical ordering of $G[R]$.

Proof. We will proceed as in the proof of the “Canonical representation lemma for plane graphs” [13], by reverse induction.

Let $R = \{\leq_1, \leq_2, \leq_3\}$ be a standard 3-dimensional representation. We may consider the embedding used in the proof of Proposition 2.1.4. This embedding has a_1, a_2, a_3 as exterior face. If $G[R]$ only has 3 vertices, we can see that the result holds, so assume $|V(G[R])| \geq 4$.

Since $G[R]$ is a plane triangulation with at least four vertices, $G[R]$ is 3-connected, therefore G_{n-1} is 2-connected. Observe that the exterior cycle C_{n-1} contains the edge $a_1 a_2$. It is also clear that a_3 lies in the exterior face of G_{n-1} . Now, the neighbors of a_3 form a subinterval of $C_{n-1} - a_1 a_2$, in fact the neighbors of a_3 in $G[R]$ are precisely the vertices of the path $C_{n-1} - a_1 a_2$. Another property of G_{n-1} that will be useful, is that the region enclosed by C_{n-1} is triangulated. So by adding a vertex u in the exterior face of G_{n-1} and joining it to each vertex of the exterior face we get a plane triangulation, in this particular case, the plane triangulation obtained is isomorphic to $G[R]$.

Assume the requirements from Definition 3.2.1 conditions are satisfied and that the region enclosed by the exterior cycle is triangulated for all $k > i$. First let us prove that v_i is in the exterior face of G_{i-1} . Assume v_i is in an interior face of G_{i-1} and let us see what happens. Observe that none of the edges incident to v_i have been removed, otherwise v_i

would already be in the exterior face of G_{i-1} because all the previously removed vertices v_l have been in the exterior face of G_{l-1} . So, consider the edge corresponding to the outgoing arc from v_i in the digraph D_3 , from Definition 2.5.1, say $v_i w$. This implies that $v_i <_3 w$. But this is not possible, as this means w would have been removed before v_i , this is $w \notin V(G_{i-1})$.

Let us show that the neighbors of v_i form a subinterval of $C_{i-1} - a_1 a_2$ of at least 2 elements. It is clear that v_i has at least two neighbors, otherwise G_i would have not been 2-connected. Now, all the neighbors of v_i in G_{i-1} form a subinterval of $C_{i-1} - a_1 a_2$, otherwise $G[\mathbb{R}]$ would have contained faces that were not triangles.

If $i = 4$, then G_{i-1} is a cycle of length three and hence 2-connected and it is easy to see that the region enclosed by C_{i-1} is triangulated, as C_{i-1} is a triangle. If $i > 4$, we proceed as follows. Note that the region enclosed by C_i is triangulated by induction hypothesis and that v_i is in the boundary of the exterior face of C_i . Since the region enclosed by C_i is triangulated, it follows that the region enclosed by C_{i-1} is also triangulated as it was obtained by removing a vertex in the boundary of C_i .

Now, as illustrated in Figure 3.5, \hat{G}_{i-1} , obtained by adding a vertex to the exterior face of G_{i-1} and joining it to each vertex of C_{i-1} , is a plane triangulation with at least four vertices and therefore 3-connected, so G_{i-1} is 2-connected. It is also clear that the bounding cycle of G_{i-1} always contains $a_1 a_2$.

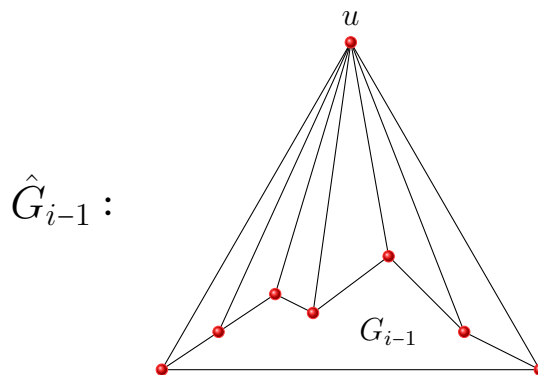


Figure 3.5: The graph \hat{G}_{i-1} is a plane triangulation.

□

Observe that, in the proof of Proposition 3.2.2, we did not make any special assumption about considering \leq_3 . We may have as well chosen \leq_1 or \leq_2 from the representation. This tells us that a standard 3-dimensional representation provides us with 3 canonical orderings of the graph it induces.

In the article [13], the canonical order is given as input to an algorithm that produces a straightline embedding of a graph. A publication of Kant [20], includes a nice pseudocode description of such algorithm, which we will now show.

Algorithm 3.2.3 (Straightline embedding).

Require: Graph G and a canonical ordering of $V(G)$, $\leq: v_1 v_2 \cdots v_n$

```

1:  $P(v_1) \stackrel{\text{def}}{=} (0, 0)$ 
2:  $L(v_1) \stackrel{\text{def}}{=} \{v_1\}$ 
3:  $P(v_2) \stackrel{\text{def}}{=} (2, 0)$ 
4:  $L(v_2) \stackrel{\text{def}}{=} \{v_2\}$ 
5:  $P(v_3) \stackrel{\text{def}}{=} (1, 1)$ 
6:  $L(v_3) \stackrel{\text{def}}{=} \{v_3\}$ 
7: for  $k = 4 \dots n$  do
8:   for  $v \in \bigcup_{i=l+1}^{r-1} L(c_i)$  do
9:      $x(v) \stackrel{\text{def}}{=} x(v) + 1$ 
10:   end for
11:   for  $v \in \bigcup_{i=r}^q L(c_i)$  do
12:      $x(v) \stackrel{\text{def}}{=} x(v) + 2$ 
13:   end for
14:    $P(v_k) \stackrel{\text{def}}{=} \text{Point of intersection of line of slope 1 going through } P(c_l) \text{ and line of}$ 
    $\text{slope } -1 \text{ going through } P(c_r).$ 
15:    $L(v_k) \stackrel{\text{def}}{=} \bigcup_{i=l+1}^{r-1} L(c_i)$ 
16: end for
17: return  $P$ 

```

In the pseudocode shown above, $c_1 c_2 \dots c_l \dots c_r \dots c_q$ denotes the path C_{k-1} from Definition 3.2.1. The vertices c_i are ordered from left to right, according to their position $P(c_i)$. The indices l and r correspond to the first and last vertices of the path to which v_k is adjacent respectively. The idea of the algorithm is to maintain a straightline embedding of the vertices that have been added so far. We will now briefly describe how the algorithm works. Start by placing three of the vertices, as indicated in lines 1,3 and 5. It is clear that so far the positions induce a straightline embedding of G . In lines 8–13, the vertices in $\bigcup_{i=l+1}^{r-1} L(c_i)$ are being shifted one unit to the right while elements in $\bigcup_{i=r}^q L(c_i)$ are being shifted two units to the right. After this operation, the positions previously assigned still define a straightline embedding. The shifting took place so that v_k could be placed at the point of intersection of the line with slope 1 going through $P(c_l)$ and the line of slope -1 going through $P(c_r)$. Finally, $L(v_k)$ is defined to be $\bigcup_{i=l+1}^{r-1} L(c_i)$, so that whenever v_k needs to be shifted to the right, all the elements of $L(v_k)$ must be shifted to guarantee that the embedding remains valid.

We now show how this embedding looks. For this, we consider the graph from Figure 3.2, and $\leq_2: 5743126$ from the orders that induce such graph. The embedding obtained by the algorithm is shown in Figure 3.6.

We have seen how to get canonical orderings from a 3-dimensional representation. We now show how to get a normal labelling of a plane triangulation from a canonical ordering. Follow Algorithm 3.2.3. Label the angles of the triangle $v_1 v_2 v_3$ with 1, 2, and 3 in counterclockwise order. Let f_1, \dots, f_l be the triangular faces generated when adding v_k . Note that each of f_i has a v_k in its boundary, label each of these angles 3. Assign the labels 1 and 2 in counterclockwise order to the remaining angles of each f_i , as illustrated in Figure 3.7.

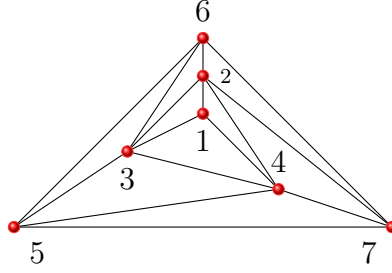


Figure 3.6: Embedding obtained from canonical ordering.

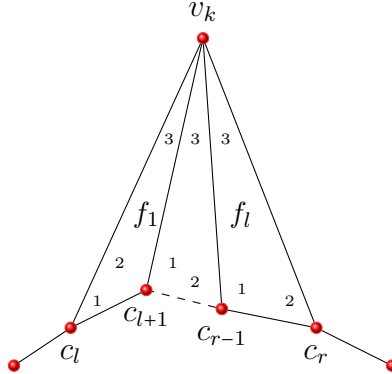


Figure 3.7: Labelling of faces f_1, \dots, f_l .

Proposition 3.2.4. *The labelling obtained by following the procedure described above is a normal labelling.*

Proof. It is clear that each angle of an interior face will get assigned a label of 1 2 or 3 in counterclockwise order.

Let us now show that at each interior vertex the angles labeled i form a consecutive block and that the blocks appear in counterclockwise order. Let v_j be an interior vertex of the graph. We can observe that the only moment at which angles at v_j will be labelled 3 is when we add this vertex. This proves uniqueness and consecutiveness of the block of angles labelled 3. Let k be the position, from left to right, of v_j in the path $c_1 c_2 \dots c_q$, that is $v_j = c_k$ at subsequent iterations where v_j is still in the exterior cycle of the graph.

Let w be the vertex to be added. Since G is a plane triangulation, four possible cases may occur. Each of these cases, except for the first one, is illustrated in Figure 3.8.

1. The vertex w is not a neighbor of c_k . If this is the case, then this does not change the number of angles labelled at c_k .
2. The vertices w, c_{k-1} , and c_k are in the boundary of a triangular face, but w, c_k , and c_{k+1} are not in the boundary of a triangular face. If this is the case, we can see that c_k is the rightmost neighbor of w . The label that the angle at c_k will be assigned will be 2.

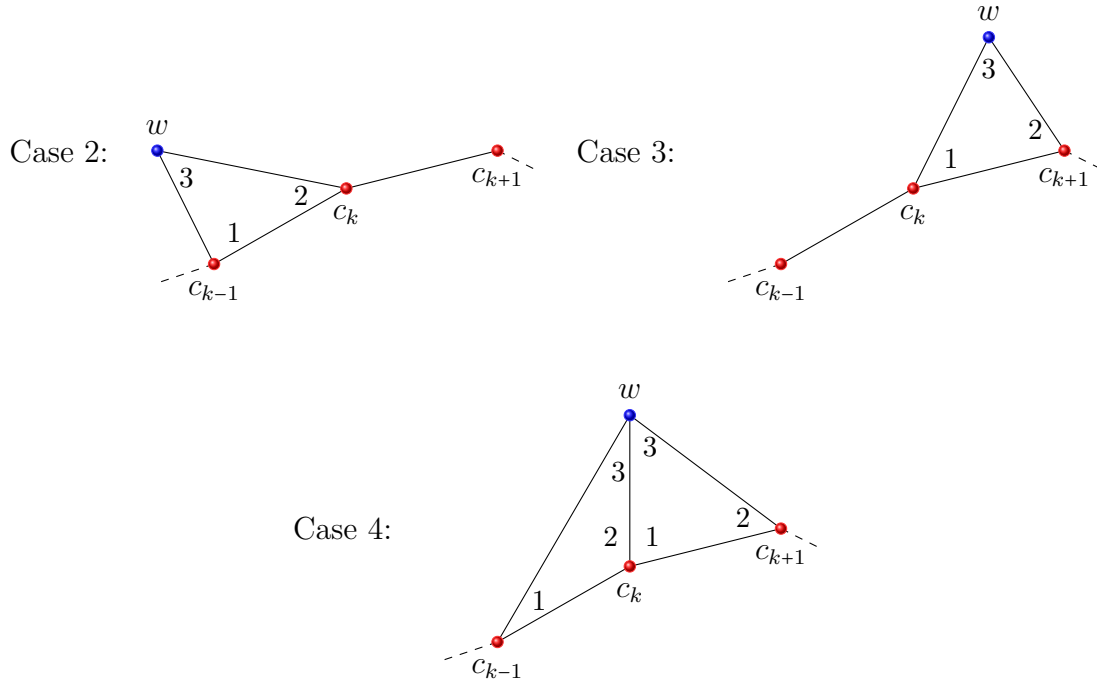


Figure 3.8: Possibilities for the new vertex w .

3. The vertices w, c_{k-1} , and c_k are not in the boundary of a triangular face and w, c_k , and c_{k+1} are in the boundary of a triangular face. In this case, c_k is the leftmost neighbor of w . Since w will be assigned a position to the right of c_k , the label at c_k will be 1.
4. The vertices w, c_{k-1} , and c_k and w, c_k , and c_{k+1} are in the boundary of a triangular face. If this is the case, then we can see that the label of the angle at c_k in w, c_{k-1}, c_k will be 2 and the label of the angle at c_k in w, c_k, c_{k+1} will be 1.

Note that 1, 2 and 3, may occur several times. As 2 occurs, there will be a block of angles labelled 2 occurring to the left of c_k . While 3 occurs, there will be a block of angles labelled 1 occurring at the right of c_k . This implies that while 1, 2 or 3 occur, there will be no label 1 or 2 appearing to the left or right of c_k respectively. We consider case 4 at last because after this occurs, v_j will no longer be among $c_1 \cdots c_q$. Also observe that case 4 must occur at some point, otherwise G would have a face delimited by at least four edges, $c_{k-1}c_k$ and $c_k c_{k+1}$ among them. Let us note that after 4 occurs there will be no more edges incident to v_j added to G , as this would contradict the definition of canonical ordering. When case 4 occurs we still assign a label 1 to the right of v_j and a label 2 to the left of v_j . Note that these two labels are adjacent and delimit the block of angles labelled 1 and 2. The blocks of angles labelled 1, 2, and 3 are unique and they appear in counterclockwise order, as desired.

3.3 Tree Decomposition

In this section we will introduce the concept of tree decomposition of a graph. We will also show, as a consequence of Schnyder's theorem, that the edge set of every plane triangulation can be decomposed into 3 edge disjoint trees. This is a particular case of the more general problem of tree decomposition. If a given graph represents a communication network, a tree decomposition of such a graph into k trees becomes useful when we try to find k edge disjoint paths between any pair of vertices. Each of these paths can be found efficiently, one in each tree.

A *tree decomposition* of a graph G , with n vertices, is a partition $\{E_1, \dots, E_k\}$ of its set of edges $E(G)$, so that $G[E_i]$ is a tree.

Let G be a graph and let P be a partition of $V(G)$. We define the multigraph $G_P = (V_P, E_P)$ as follows. The set of vertices will be the partition, that is $V_P = P$, and E_P will contain an edge joining p_1 and p_2 for each edge of G with an endpoint in p_1 and an endpoint in p_2 .

The problem of decompositions of graphs into spanning trees has been studied by Nash-Williams and Tutte. They independently proved the following result.

Theorem 3.3.1 (Nash-Williams [29], Tutte[39]). *A graph G contains k edge disjoint spanning trees if and only if for every partition P of $V(G)$, there are at least $k(|P| - 1)$ edges in E_P .*

One consequence of Schnyder's Theorem is that the set of edges of every planar triangulation with n vertices can be partitioned into three sets, each of which induces a tree that has $n - 1$ vertices. We state this result in the following theorem.

Theorem 3.3.2. *Let G be a plane triangulation with n vertices. The edge set $E(G)$ can be partitioned into three edge disjoint trees, each with $n - 1$ vertices.*

Proof. Let us recall the digraphs D_i from Definition 2.5.1. It follows from Lemma 2.3.3 that the set of arcs of each digraph D_i induces a partition of the set of edges of G . We observed that in each D_i the exterior vertices have outdegree 0. It is also true that the interior vertices have outdegree 1, since from the definition of normal labelling there is a unique block of labels of each type at each vertex. Also, we have proved in Lemma 2.5.2 that each D_i is acyclic. So, each D_i induces a tree T'_i containing each interior vertex of G and an exterior vertex, a_i . Finally, we may add the vertex a_{i+1} and the edge $a_i a_{i+1}$ to T'_i to obtain T_i . Each T_i is a tree, and it contains $n - 1$ vertices. \square

We now show an example of the tree partition obtained from the digraphs D_i . Let us consider the graph G from our previous examples in this chapter. The left graph of Figure 3.9 exhibits the digraphs D_1 in green, D_2 in blue and D_3 in red. The right graph of Figure 3.9 shows the tree partition induced by the digraphs D_i .

There is also a result from Ringel [32] which proves that every maximal planar bipartite graph can be decomposed into two trees.

More recently, a Theorem of Yuan and Cai gives necessary and sufficient conditions for a graph on n vertices to admit a tree decomposition in which every tree has $n - 1$ vertices.

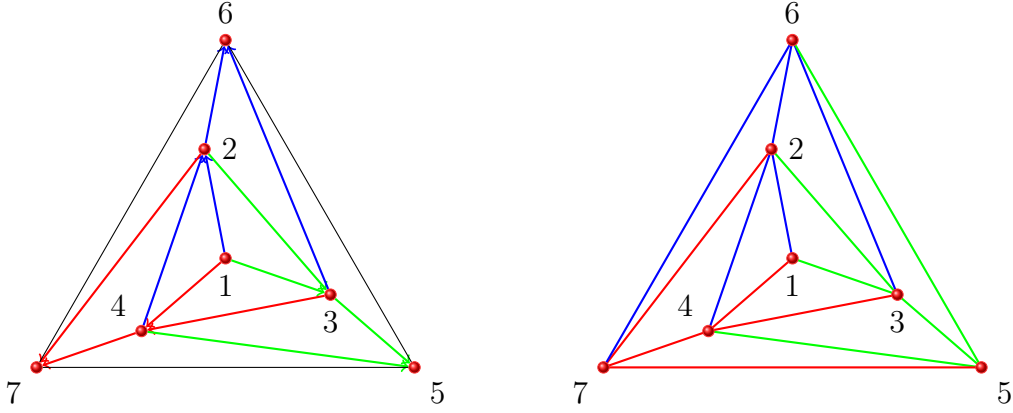


Figure 3.9: Tree partition obtained by the digraphs induced by the normal labelling.

Theorem 3.3.3 (Yuan & Cai [43]). *A connected graph G with $n \geq 3$ vertices and $k(n-2)$ edges has a tree decomposition $\{E_1, \dots, E_k\}$ if and only if for each subset of vertices X , so that $|X| \geq 2$ and $G[X]$ is connected the following inequality holds*

$$E(G[X]) \leq k(|X| - 1) - \sum_{i=1}^k |X \cap \{u_i\}|,$$

where u_i is the unique vertex in $V(G) \setminus V(G[E_i])$.

□

Chapter 4

Schnyder's Theorem: An Alternate Proof

Our aim for this chapter will be to present an alternate proof of Schnyder's theorem. We begin by showing that a graph induced by a standard 3-dimensional representation is planar. In the following section we prove that a plane triangulation has poset dimension at most 3.

Let us recall that a 3-dimensional representation R of a finite set V is a sequence of three linear orders \leq_1, \leq_2, \leq_3 of V so that

$$\bigcap_{i \in [3]} \leq_i = \emptyset. \quad (4.1)$$

We call equation (4.1) the *vertex property*. We will usually denote the maximum element of the i -th order by a_i , that is,

$$a_i = \max_{\leq_i} V.$$

The representation is said to be *standard* if a_i is among the first two elements of \leq_j , for all $i \in \{1, 2, 3\}$ and $j \neq i$. We will call a_1, a_2 , and a_3 the *exterior elements* of R . The elements of V that are not exterior will be called *interior elements* of R . A pair of elements $u, v \in V$ are said to satisfy the *edge property* in R if for every $z \in V \setminus \{u, v\}$ there is an order $\leq_i \in R$ so that $u, v \leq_i z$.

4.1 Three Dimensional Representations Induce Planar Graphs

Our aim for this section will be to prove that standard 3-dimensional representations induce plane triangulations.

Given a 3-dimensional representation R of V we define the graph induced by R and denote it by $G[R] = (V, E)$, where

$$E = \{uv \mid u, v \in V, \text{ and } u \text{ and } v \text{ satisfy the edge property in } R\}.$$

In this same context, we call a_1, a_2 , and a_3 the *exterior vertices* of $G[R]$. The vertices of $G[R]$ that are not exterior will be called *interior vertices*.

Lemma 4.1.1. *Let $R = \{\leq_1, \leq_2, \leq_3\}$ be a 3-dimensional representation of $V \stackrel{\text{def}}{=} \{v_1, \dots, v_n\}$. Assume $a_1 = v_i$ and*

$$\leq_2: v_1 v_2 \cdots v_n.$$

If

$$\leq'_2: a_1 v_1 \cdots v_{i-1} v_{i+1} \cdots v_n,$$

then $G[R] \subseteq G[R']$, where $R' \stackrel{\text{def}}{=} \{\leq_1, \leq'_2, \leq_3\}$.

Proof. Observe that the vertex property holds for R' . Proving that $E(G[R]) \subseteq E(G[R'])$ will suffice. Let $uv \in E(G[R])$. We will prove that the edge property holds for uv in R' . Let $z \in V \setminus \{u, v\}$. Note that if $z = a_1$ then the edge property holds in \leq_1 . So assume $z \neq a_1$. First, let us consider the case $a_1 \in \{u, v\}$. We know, by the edge property in R , that $u, v <_2 z$ or $u, v <_3 z$. Hence in R' we have $u, v <'_2 z$ or $u, v <_3 z$, as the only element that changed its position was a_1 and it became the minimum element of \leq_2 . Now, the case $a_1 \notin \{u, v\}$ can be handled with a similar argument. In this case, the relative positions of u, v and z were not changed from R to R' . Therefore there is an order in R' in which z is greater than u and v . \square

From the previous lemma, we may observe that given a 3-dimensional representation R we may obtain a standard 3-dimensional representation R so that $G[R] \subseteq G[R]$. We obtain R from R by “translating” each a_i in \leq_j to be among the first 2 elements in \leq_j , $i \neq j$. Informally speaking, we may “standardize” any 3-dimensional representation. For this reason, we will now derive some properties of standard 3-dimensional representations.

Lemma 4.1.2. *Let R be a standard 3-dimensional representation. If*

$$w_0 \leq_2 w_1 \leq_2 \cdots \leq_2 w_m \leq_2 w_{m+1}$$

are the neighbors of a_1 in $G[R]$, then

$$w_{m+1} \leq_3 w_m \leq_3 \cdots \leq_3 w_1 \leq_3 w_0.$$

Furthermore, $w_0 = a_3$ and $w_{m+1} = a_2$.

Proof. To prove the first part, it suffices to prove $w_i \leq_3 w_j$ whenever $j \leq i$. Let $i, j \in \{0, 1, \dots, m+1\}$ so that $j \leq i$. It follows from the edge property that there is an order $\leq_k \in R$ so that $a_1, w_i \leq_k w_j$. We can observe $k \neq 1$, since $w_j \leq_1 a_1$. We also have, by assumption, that $k \neq 2$, as $w_j \leq_2 w_i$. So $k = 3$, as desired.

To prove $w_0 = a_3$ and $w_{m+1} = a_2$ we will first show that $a_3 a_1, a_2 a_1 \in E(G[R])$. Let $z \in V \setminus \{a_1, a_3\}$. It is clear that $a_1, a_3 \leq_2 z$, since R is standard. A similar argument proves $a_1 a_2 \in E(G[R])$. Since w_{m+1} is the largest element in \leq_2 that is a neighbor of a_1 , then $a_2 = w_{m+1}$. Similarly, since w_0 is the largest element in \leq_3 that is a neighbor of a_1 , it follows that $w_0 = a_3$. \square

Lemma 4.1.3. *Let R be a standard 3-dimensional representation. Let*

$$w_0 \leq_2 w_1 \leq_2 \cdots \leq_2 w_m \leq_2 w_{m+1}$$

be the neighbors of a_1 in $G[R]$. The set

$$S_i = \{z \in V \mid w_i <_2 z <_2 w_{i+1} \text{ and } w_{i+1} <_3 z <_3 w_i\}$$

is empty.

Proof. Let us assume there is S_i , so that $S_i \neq \emptyset$. Let $y_\star = \min_{\leq_2} S_i$. If we prove $y_\star a_1 \in E(G[R])$ this would be a contradiction, as $y \neq w_j$, for all j . Assume, by contradiction, that $y_\star a_1$ is not an edge of $G[R]$. Then, there exists a vertex z which does not satisfy the edge property for $y_\star a_1$. This implies in particular that $z <_2 y_\star$ and $z <_3 y_\star$. Since the edge property must be satisfied for $w_i a_1$ and z , we must have $w_i <_2 z$. Similarly we have $w_{i+1} <_3 z$. But this contradicts the minimality of y_\star . So $y_\star a_1 \in E(G[R])$, again, a contradiction. So $S_i = \emptyset$. \square

Lemma 4.1.4. *Let R be a standard 3-dimensional representation. If*

$$w_0 \leq_2 w_1 \leq_2 \cdots \leq_2 w_m \leq_2 w_{m+1}$$

are the neighbors of a_1 in $G[R]$, then $w_i w_{i+1} \in E(G[R])$.

Proof. Assume $w_i w_{i+1} \notin E(G[R])$ and let us see what happens. Hence there exists $y \in V \setminus \{w_i w_{i+1}\}$ so that for each order \leq_k of R either $y <_k w_i$ or $y <_k w_{i+1}$. This implies, using Lemma 4.1.2, in particular that $y <_2 w_{i+1}$ and $y <_3 w_i$. Now, since the edge property must hold for $a_1 w_i$ and $a_1 w_{i+1}$ we must have $w_i <_2 y$ and $w_{i+1} <_3 y$. But this contradicts Lemma 4.1.3, as $y \in S_i$. So $w_i w_{i+1} \in E(G[R])$. \square

Lemma 4.1.5. *Let $R =_{\leq_1, \leq_2, \leq_3}$ be a standard 3-dimensional representation of V , $|V| > 3$. If $b = \max_{\leq_1} V \setminus \{a_1\}$, then $a_1 b \in E(G[R])$ and a_1 and b have exactly two common neighbors.*

Proof. We will first show that $a_1 b \in E(G)$, for this we will prove that the edge property is satisfied for $a_1 b$. Let $x \in V(G) \setminus \{a_1, b\}$. By the vertex property there is an order in R , say \leq_j , so that $b <_j x$. From the definition of b we can infer that $j \neq 1$. Since the representation is standard we must have $a_1 <_j b$. So $a_1 <_j b <_j x$ as desired.

Now let us prove that $|N(a_1) \cap N(b)| = 2$. Observe that b is an interior vertex of $G[R]$. It follows from Lemma 4.1.4 that a_1 and b have at least two common neighbors.

We will now show that $|N(a_1) \cap N(b)| < 3$. Let us assume, by contradiction, that $|N(a_1) \cap N(b)| \geq 3$. Let z_1, z_2 , and z_3 be common neighbors of a_1 , and b . We may assume, without loss of generality, that $z_1 <_2 z_2 <_2 z_3$. By applying Lemma 4.1.2 we obtain $e_1 <_3 z_3 <_3 z_2 <_3 z_1$. Two possibilities arise now.

Case 1) $b <_2 z_2$. It follows from the edge property, applied to $z_2 a_1$ and b , that $a_1, z_2 <_3 b$. This contradicts the edge property for the edge $b z_3$ and z_2 .

Case 2) $z_2 <_2 b$. By the edge property we have $z_1, b <_3 z_2$. But this is contradicts $z_2 <_3 z_1$.

In either case we get a contradiction. So $|N(a_1) \cap N(b)| = 2$. □

Lemma 4.1.6. *Let $R =_{\leq_1, \leq_2, \leq_3}$ be a standard 3-dimensional representation of V , $|V| > 4$. Let*

$$w_0 \leq_2 w_1 \leq_2 \cdots \leq_2 w_m \leq_2 w_{m+1}$$

be the neighbors of a_1 in $G[R]$. If $b = \max_{\leq_1} V \setminus \{a_1\}$, then $b = w_i$ for some i , $1 \leq i \leq n$, and

$$w_i \leq_2 z \leq_2 w_{i+1} \text{ and } w_i \leq_3 z \leq_3 w_{i-1}$$

for all $z \in N_{G[R]}(b)$.

Proof. It follows from Lemma 4.1.5 that $b = w_i$. Since $|V| > 4$ it follows that $i > 0$.

Now, let z be a neighbor of w_i , we will prove that $w_i \leq_2 z \leq_2 w_{i+1}$. Assume that $z <_2 w_i$ and let us see what happens. By Lemma 4.1.4 we have that $w_i w_{i-1}$ is an edge of $G[R]$. Since the edge property must hold for $w_i w_{i-1}$ and z , we must have $w_i <_3 w_{i-1} <_3 z$. But this contradicts the edge property for $w_i z$ and w_{i-1} , as there would not be an order in which $z, w_i < w_{i-1}$, recall $b = w_i$. We also note that $w_{i+1} <_2 z$ would contradict the edge property for $w_i z$ and w_{i+1} . It can be proved using similar arguments that $w_i \leq_3 z \leq_3 w_{i-1}$. □

Lemma 4.1.7. *Let $R =_{\leq_1, \leq_2, \leq_3}$ be a standard 3-dimensional representation of V , $|V| > 3$, let $b = \max_{\leq_1} V \setminus \{a_1\}$, and let $\leq'_i = \leq_i|_{V \setminus \{a\}}$. If $R' =_{\leq'_1, \leq'_2, \leq'_3}$, then R' is a standard 3-dimensional representation, and the graph G' obtained by contracting the edge $a_1 b$ in $G[R]$ satisfies $G' = G[R']$.*

Proof. Note that the vertex property holds for R' . This is because the relative orders of the elements is preserved. If there was a pair of elements $u, v \in V \setminus \{b\}$ for which the vertex property does not hold in R' , then it would not hold in R which is a contradiction. It is clear that R' is standard, since the relative positions of the vertices is preserved. In particular a_1, a_2 , and a_3 are still the exterior vertices and each of them is among the first 2 elements in the orders in which they are not maximum.

We are only left to show that G' is induced by R' . We are going to show that the edge property holds for every edge of G' . Let $st \in E(G')$ and $z \in V(G')$. Two possible cases arise.

- Case 1) If $st \in E(G)$, then there is an order, say \leq_j , so that $s, t <_j z$. So it follows that $s, t <'_j z$.
- Case 2) If $st \notin E(G)$, then $st = a_1 u$, where $u \in N_G(b)$. Since the edge property holds for bu and z in R , there is an order \leq_j so that $b, u <_j z$. If $z = a_1$ then $b, u \leq'_1 z$. Otherwise note that $j \neq 1$, as there is no element $z \neq a_1$ which satisfies $b <_j z$. Since the representation is standard, $a_1 <_j b$ if $j \neq 1$. So we have that $a_1, u <'_j z$.

In either case the edge property is satisfied, as desired. This proves $E(G') \subseteq E(G[R])$. Let us now prove $E(G[R']) \subseteq E(G')$. Let $st \in E(G[R'])$. If $st \in E(G[R])$, then $s, t \neq b$ so $st \in E(G')$. Now assume $st \notin E(G[R])$. It follows that b is not above both s and t in any order. This implies $a_1 \in \{s, t\}$. Let us assume, without loss of generality, that $s = a_1$.

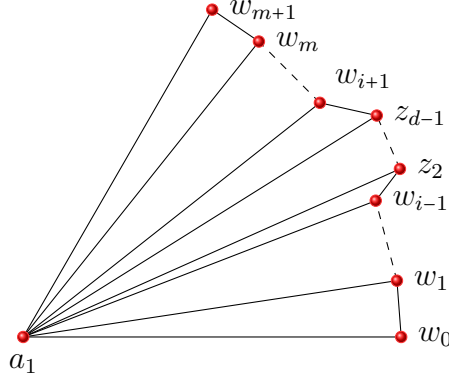


Figure 4.1: Layout of neighbors of a_1 in $G[\mathbf{R}']$.

Note that it suffices to prove that $t \in N_{G[\mathbf{R}]}(b)$, as this would imply $st \in E(G')$. Let us prove that the edge property is satisfied for bt in \mathbf{R} . Let $y \in V \setminus \{b, t\}$. If $y = a_1$, then we have $b, t <_1 y$. Now, let us consider the case $y \neq a_1$. Using the fact that $a_1 t \in E(G[\mathbf{R}'])$ we have that $a_1, t <'_2 y$ or $a_1, t <'_3 y$. This implies $a_1, t <_2 y$ or $a_1, t <_3 y$. Since $a_1 t \notin E(G[\mathbf{R}])$ the only vertex that does not satisfy the edge property for $a_1 t$ in \mathbf{R} is b . This implies $b <_2 t$ and $b <_3 t$. And so $b, t <_2 y$ or $b, t <_3 y$. So $bt \in E(G[\mathbf{R}])$, as desired. \square

Proposition 4.1.8. *Let \mathbf{R} be a standard 3-dimensional representation of $|V|$. The graph induced by \mathbf{R} is a plane triangulation is a plane triangulation having $a_1 a_2 a_3$ as a face. Furthermore, if*

$$w_0 \leq_2 \cdots \leq_2 w_{m+1}$$

are the neighbors of a_1 then $a_1 w_i w_{i+1}$ denote the rest of the triangular faces of $G[\mathbf{R}]$ having a_i in its boundary.

Proof. We will proceed by induction on $|V|$. It can be checked by hand that the statement holds for $|V| = 3, 4$, and 5 .

Assume the statement is true for all sets with at most n elements. Let V be a set with $n + 1$ elements and let $\mathbf{R} = \{\leq_1, \leq_2, \leq_3\}$ be a standard 3-dimensional representation of V . Let w_i be the second largest element of \leq_1 . By Lemma 4.1.2 and Lemma 4.1.6 we have that

$$w_0 \leq_2 \cdots \leq_2 w_{i-1} = z_1 \leq_2 w_i \leq_2 z_2 \cdots \leq_2 z_{d-1} = w_{i+1} \leq_2 w_{m+1}$$

and

$$w_{m+1} \leq_3 \cdots \leq_3 w_{i+1} = z_{d-1} \leq_3 w_i \leq_3 z_{d-2} \cdots \leq_3 z_1 = w_{i-1} \leq_3 w_0,$$

where each w_j is a neighbor of a_1 and $\{z_1, \dots, z_{d-1}\} = N_{G[\mathbf{R}]}(w_i) \setminus \{a_1\}$.

Let $\mathbf{R}' = \{\leq'_1, \leq'_2, \leq'_3\}$, where the order $\leq'_i = \leq_i \upharpoonright_{V \setminus \{b\}}$. By induction hypothesis, $G[\mathbf{R}']$ is a plane triangulation having $a_1 a_2 a_3$ as a face, $a_1 w_j w_{j+1}$ are triangular faces for $0 \leq j \leq i - 1$ and $i + 1 \leq j \leq m$, and $a_1 z_j z_{j+1}$ are triangular faces for $1 \leq j \leq d - 2$. We may now consider an embedding of $G[\mathbf{R}']$ having $a_1 a_2 a_3$ as exterior face and observe that the layout of the neighborhood of a_1 is as shown in Figure 4.1. Since $G[\mathbf{R}']$ results from contracting $a_i w_i$ in $G[\mathbf{R}]$, we can see that $G[\mathbf{R}]$ is also planar. We can see this by removing the edges $a_1 z_j$, $2 \leq j \leq d - 2$, placing b inside the region bounded by the cycle $C = a_1 z_1 \cdots z_{d-1}$ and joining

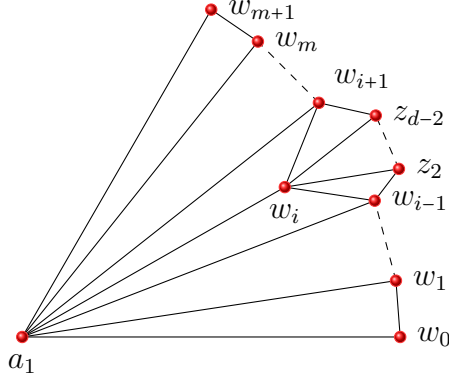


Figure 4.2: Layout of neighbors of a_1 in $G[\mathbf{R}]$.

it to a_1 and z_j , $1 \leq j \leq d-1$. By doing that, we have not introduced any edge crossings, as the only neighbors of w_i are the vertices of C , as illustrated in Figure 4.2. We now conclude that $G[\mathbf{R}]$ is a plane triangulation, as it has $n+1$ vertices and $3(n+1)-6$ edges. We can also see that $a_1a_2a_3$ is a face of this triangulation and all other faces are of the form $a_1w_jw_{j+1}$, as desired. □

Theorem 4.1.9. *If \mathbf{R} is a standard 3-dimensional representation, then $G[\mathbf{R}]$ is a plane triangulation.*

Proof. This is an immediate consequence of Proposition 4.1.8. □

Now, we can see that if R is a 3-dimensional representation, then $G[R]$ is a planar graph. This follows from the fact that the “standardization” of R induces a supergraph of $G[R]$ which is a plane triangulation.

4.2 Plane Triangulations Have Poset Dimension at Most 3

Our aim for this section is to prove the converse of Proposition 4.1.8.

Let R be a 3-dimensional representation of V . We define the complex induced by R to be the pair $\Sigma(R) = (V, \mathcal{F})$, where $\mathcal{F} \subseteq 2^V$ and for each $f \in \mathcal{F}$ and $v \in V$ there is an order of R in which v is greater than or equal to each element of f .

Lemma 4.2.1. *Let \mathbf{R} be a standard 3-dimensional representation. Then $\Sigma(\mathbf{R})$ contains every triangular face of $G[\mathbf{R}]$ having a_1 in its boundary, except for $a_1a_2a_3$.*

Proof. We will assume that the neighbors of a_1 are ordered with respect to \leq_2 , that is, $w_0 \leq_2 w_1 \leq_2 \dots \leq_2 w_{m+1}$. It follows from Lemma 4.1.4 that every triangular face having a_1 in its boundary must be of the form $a_1w_iw_{i+1}$. We can see that if $z \in \{a_1, w_i, w_{i+1}\}$ then $a_1, w_i, w_{i+1} \leq_j z$, where $j = 1$ if $z = a_1$, $j = 2$ if $z = w_{i+1}$ or $j = 3$ if $z = w_i$. Let $z \in V \setminus \{a_1, w_i, w_{i+1}\}$. Since w_ia_1 and $w_{i+1}a_1 \in E(G[\mathbf{R}])$ we must have that $w_i <_k z$ and

$w_{i+1} <_j z$. If $j = 2$ or $k = 3$ then there is nothing to prove, as in each of these cases there will be an order in which z is above a_1, w_i , and w_{i+1} . So assume $j = 3$ and $k = 2$. If this is the case, we must have $w_{i+1} \leq_2 z$ or $w_i \leq_3 z$. Otherwise this would imply $w_i <_2 z <_2 w_{i+1}$ and $w_{i+1} <_3 z <_3 w_i$. But this is not possible, as this would contradict Lemma 4.1.3. \square

Lemma 4.2.2. *Let R be a standard 3-dimensional representation. Let $w_0 \leq_2 w_1 \leq_2 \cdots \leq_2 w_{m+1}$ be the neighbors of a_1 in $G[R]$. The neighbors of a_1 in \leq_2 are ordered so that $w_i w_{i+1} a_1$ and $w_{i-1} w_i a_1$ are triangular faces of $G[R]$ induced by R .*

Proof. This result is an immediate consequence of Lemma 4.2.1. \square

Lemma 4.2.3. *Let R be a standard 3-dimensional representation of V . If $a_1 w_j w_{j+1}$ are the interior triangular faces of $G[R]$, then we must have*

$$w_0 \leq_2 \cdots \leq_2 w_{m+1}$$

or

$$w_{m+1} \leq_2 \cdots \leq_2 w_0.$$

Proof. Assume that $w_0 <_2 w_1$. Since $a_1 w_0 w_1 \in \Sigma(R)$, by Lemma 4.2.1 and $w_0, w_1 <_1 a_1$, $a_1, w_0 <_2 w_1$, then $a_1, w_1 <_3 w_0$, otherwise the edge property would not be satisfied for $a_1 w_1$ and w_0 . Also note that we must have $w_1 <_2 w_2$, because if we assume $w_2 <_2 w_1$, we then have $w_1 <_3 w_2$. Since $a_1 w_1 w_2 \in \Sigma(R)$ we must have $a_1, w_1, w_2 \leq_3 w_0$. Also, since $a_1 w_0 w_1 \in \Sigma(R)$ we must have $a_1, w_0, w_1 \leq_3 w_2$, a contradiction. So $w_1 <_2 w_2$. A similar argument can be used to show that $w_i <_2 w_{i+1}$. So we get

$$w_0 \leq_2 w_1 \leq_2 \cdots \leq_2 w_{m+1},$$

as desired. The case $w_1 <_2 w_0$ can be handled by observing that this implies $w_0 <_3 w_1$ and now using a similar argument but applied to \leq_3 . This would yield

$$w_0 \leq_3 w_1 \leq_3 \cdots \leq_3 w_{m+1},$$

and then by applying Lemma 4.1.2 the result follows. \square

Proposition 4.2.4. *Let T be a plane triangulation, and let $a_1 a_2 a_3$ be a triangular face of T . There exists a standard 3-dimensional representation R of $V(T)$ having a_1, a_2 , and a_3 as exterior vertices so that $T = G[R]$.*

Proof. We will proceed by induction on the number of vertices. The statement is clearly true for $n = 3$. We provide illustrations for the case $n = 4$ and $n = 5$ in Figure 4.3.

Let T be a plane triangulation. We may assume T is embedded in the plane, so that the outer face is $a_1 a_2 a_3$. Let $w_0 \stackrel{\text{def}}{=} a_2, w_1, w_2, \dots, w_m, w_{m+1} \stackrel{\text{def}}{=} a_3$ be the neighbors of a_1 indexed so that $a_1 w_i w_{i+1}$ and $a_1 w_{i+1} w_{i+2}$ are adjacent triangular faces. By Lemma 2.4.1, there exists a vertex $w_i \in N(a_1) \setminus \{a_2, a_3\}$ so that $|N_T(a_1) \cap N_T(w_i)| = 2$. Let d be the degree of w_i , and let x_1 and x_{d-1} be the common neighbors of w_i and a_1 . Since each edge of T lies in the boundary of two faces of T then $a_1 w_i x_1$ and $a_1 w_i x_{d-1}$ are two interior faces of T . We may assume, without loss of generality, that the neighbors of w_i , in

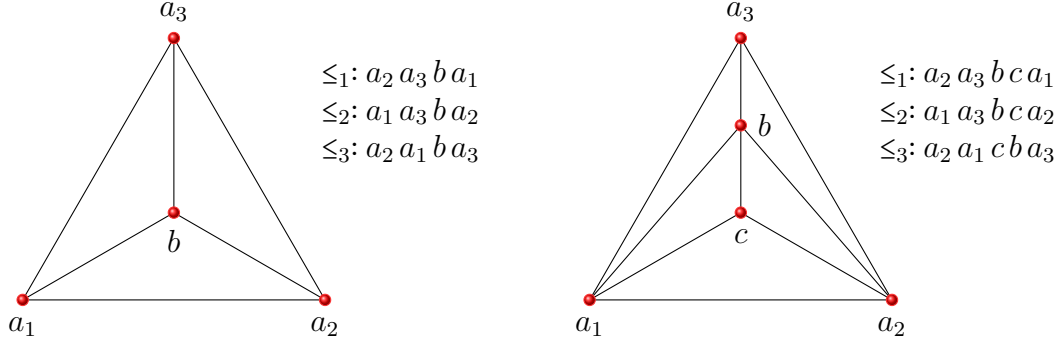


Figure 4.3: Cases $n = 4$ and $n = 5$.

counterclockwise order are $a_1, x_1, x_2, \dots, x_{d-1}$, as depicted in Figure 4.4. We observe that $x_j x_{j+1} \in E(T)$, for $j = 1, \dots, d-2$, as there is a triangular face having two counterclockwise consecutive incident edges to w_i in its boundary.

Let T' be the graph obtained from T after contracting the edge $a_1 w_i$ and let α_1 be the vertex resulting from the contraction. We can observe that T' is planar. Furthermore, T' is a plane triangulation, as it has $3(n-1) - 6$ edges. By induction hypothesis, there is a standard 3-dimensional representation $\mathbf{R}' = \{\leq'_1, \leq'_2, \leq'_3\}$ of $V(T')$ that has α_1, a_2 , and a_3 as exterior vertices. By Lemma 4.2.3 and without loss of generality we may assume that

$$w_0 \leq'_2 \cdots \leq'_2 w_{i-1} = x_1 \leq'_2 \cdots \leq'_2 x_{d-1} = w_{i+1} \leq'_2 \cdots \leq'_2 w_{m+1}.$$

An immediate consequence of applying Lemma 4.1.2 is that

$$w_{m+1} \leq'_3 \cdots \leq'_3 w_{i+1} = x_{d-1} \leq'_3 \cdots \leq'_3 x_1 = w_{i-1} \leq'_3 \cdots \leq'_3 w_0.$$

Now, we will define three linear orders on V and show that they induce T . Let \leq_1 be the linear order in which $\text{ord}_{\leq_1}(u) = \text{ord}_{\leq'_1}(u)$ for $v \in V \setminus \{a_1, w_i\}$, $\text{ord}_{\leq_1}(w_i) = n$ and $\text{ord}_{\leq_1}(a_1) = n + 1$. We define \leq_2 as the order in which $\text{ord}_{\leq_2}(u) = \text{ord}_{\leq'_2}(u)$ for all $u \leq'_2 x_1$, $\text{ord}_{\leq_2}(w_i) = \text{ord}_{\leq'_2}(x_1) + 1$, and $\text{ord}_{\leq_2}(u) = \text{ord}_{\leq'_2}(u) + 1$ for all $u >'_2 x_1$. We define \leq_3 in a similar way as \leq_2 , but using x_{d-1} instead of x_1 . Informally speaking, \leq_1 is the order

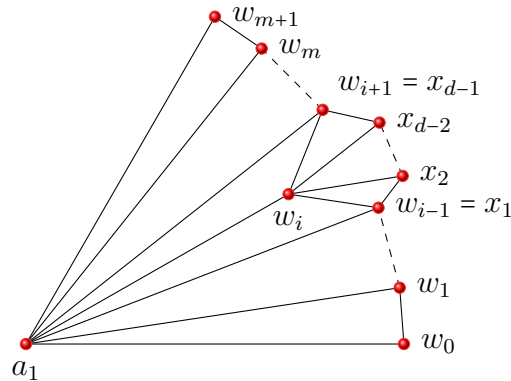


Figure 4.4: Layout of neighbors of a_1 in $G[\mathbf{R}]$.

resulting after “inserting” w_i below a_1, \leq_2 is the order obtained by “inserting” w_i above x_1 and \leq_3 is the order obtained by “inserting” w_i above x_{d-1} .

Let $R = \{\leq_1, \leq_2, \leq_3\}$. We will now show that R is a standard 3-dimensional representation. We will first show that the vertex property is satisfied. For this it is enough to show that $u <_j v$ and $v <_k u$ for any $u, v \in V(T)$. Let $u, v \in V(T)$. It is clear this is satisfied if $u, v \neq w_i$, as the relative positions of u and v from R' are preserved in R . So let us assume that $u = w_i$. If $v = a_1$, then $w_i <_1 a_1$ and $a_1 <_2 w_i$. So assume $v \neq a_1$. It is clear, by the way \leq_1 was constructed, that $v \leq_1 w_i$. It follows from the edge property in R' applied to $a_1 x_1, v$ that $a_1, x_1 \leq'_2 v$ or $a_1, x_1 \leq'_3 v$. This implies $w_i \leq_2 v$ or $w_i \leq_3 v$, as w_i was “inserted” above x_1 in \leq'_2 in the former case. In the latter case we see that $x_{d-1} <_3 x_1$ which implies $w_i <_3 x_1$ and so $w_i <_3 v$, as desired. The representation R is standard, as the vertices a_1, a_2 , and a_3 are among the first two elements in the orders in which they are not the maximum element.

We will now prove that R induces T . Let $uv \in E(T)$ and let $z \in V(T) \setminus \{u, v\}$. Two possible cases arise now.

Case 1) $uv \in E(G[R'])$. Observe that if $z \neq w_i$, then the edge property is satisfied. This is because the relative positions of u, v , and z will be preserved in R . Let us now consider the case $z = w_i$. If $u, v \neq a_1$, then $u, v <_1 w_i$. So assume $u = a_1$, and as a consequence $v = w_j, j \neq i$. It follows from Lemma 4.2.2, that the neighbors of a_1 in T' satisfy

$$w_0 \leq'_2 w_1 \leq'_2 \cdots \leq'_2 w_{i-1} = x_1 \leq'_2 x_2 \leq'_2 \cdots \leq'_2 x_{d-1} = w_{i+1} \leq'_2 \cdots \leq'_2 w_{m+1}.$$

So, it follows that in \leq_2 we have

$$w_0 \leq_2 w_1 \leq_2 \cdots \leq_2 w_{i-1} \leq_2 w_i \leq_2 x_2 \leq_2 \cdots \leq_2 w_{i+1} \leq_2 \cdots \leq_2 w_{m+1}. \quad (4.2)$$

Applying Lemma 4.1.2 to \leq'_3 and the definition of \leq_3 we get

$$w_{m+1} \leq_3 \cdots \leq_3 w_{i+1} \leq_3 w_i \leq_3 x_2 \leq_3 \cdots \leq_3 w_{i-1} \leq_3 \cdots \leq_3 w_0. \quad (4.3)$$

So, as we can see $a_1, w_j <_2 w_i$ if $j < i$ and $a_1, w_j <_3 w_i$ if $j > i$. So the edge property holds.

Case 2) $uv \notin E(G[R'])$. In this case we must have $uv = a_1 w_i$ or $uv = w_i x_j$. First assume $uv = a_1 w_i$. It follows from the edge property that $a_1, w_{i-1} <'_k z$, $k = 2$ or 3 . If $k = 2$, then $a_1, w_i <_2 z$ as w_i was *inserted* right above w_{i-1} . If $k = 3$ we have $a_1, w_i <_3 z$ as $w_i <_3 w_{i-1}$.

Finally, let us consider the case when $uv = w_i x_j$. Since $a_1 x_j$ is an edge in T' we have that $a_1 x_j <'_k z$. Using the fact that $k \neq 1$ and inequalities (4.2) or (4.3) we get $w_i, x_j <_k z$.

In conclusion R is a standard 3-dimensional representation that induces T and has a_1, a_2 and a_3 as exterior vertices. \square

Theorem 4.2.5. *Let G be a graph on n vertices. The dimension of G is at most 3 if and only if G is planar.*

Proof. Let G be a planar graph.

First let us assume $\dim(G) \leq 3$. Let R be a 3-dimensional representation of V so that G is induced by R . As we mentioned earlier in this chapter, we may obtain a standard 3-dimensional representation \mathbf{R} from R that induces a supergraph H of G . It will be enough to prove that H is planar. Since \mathbf{R} is a standard 3-dimensional representation, it follows from Proposition 4.1.8 that H is a plane triangulation. Hence $G \subseteq H$ is planar.

Conversely, assume G is planar. In Chapter 2 we have observed that if H and G are graphs so that $G \subseteq H$, then $\dim(G) \leq \dim(H)$. So let H be a plane triangulation that has G as a subgraph. We may obtain H from G by adding as many edges as possible while maintaining the graph planar. It follows from Proposition 4.2.4 that $\dim(H) \leq 3$. Hence $\dim(G) \leq 3$, as desired. □

Chapter 5

Simplicial Complexes

In Chapter 2 we have seen the proof of Schnyder's theorem. It proves that a graph is planar if and only if its poset dimension is at most 3. In this chapter we intend to study what occurs in a more general context. For this, we will consider structures called abstract simplicial complexes. We present a theorem of Ossona de Mendez that proves that any abstract simplicial complex with poset dimension d has a geometric realization in a hyperplane $H \subseteq \mathbb{R}^n$ of dimension $d-1$. This defines a natural order labelling, analogous to the normal labelling defined in Chapter 2, for such simplicial complexes. We study some of the properties of the order labelling in Section 5.3.

5.1 Abstract simplicial complexes

We now introduce some definitions about abstract simplicial complexes.

Definition 5.1.1. An *abstract simplicial complex* is a pair $\Delta = (V, \mathcal{F})$ where V is a set and \mathcal{F} is a collection of subsets of V so that if $F \in \mathcal{F}$ and $G \subseteq F$, then $G \in \mathcal{F}$. The set V will be called the *vertex set* of Δ . For our purposes, we will only consider the abstract simplicial complexes with finite vertex set. The elements of \mathcal{F} will be called *faces* of Δ . For a given face $F \in \mathcal{F}$ we call $\dim(F) \stackrel{\text{def}}{=} |F| - 1$ the *dimension* of F . The *dimension* of Δ is defined as the maximum dimension of its faces, that is

$$\dim(\Delta) = \max_{F \in \mathcal{F}} \dim(F).$$

An abstract simplicial complex Δ is said to be *pure*, if the dimensions of all its maximal faces are the same, in other words, if $F \in \mathcal{F}$ is such that F is not properly contained in another face of Δ then $\dim(F) = \dim(\Delta)$.

Example 5.1.2. Consider the abstract simplicial complex $\Delta = (\{1, 2, \dots, 7\}, \mathcal{F})$, where

$$\mathcal{F} \stackrel{\text{def}}{=} \{\{1\}, \{2\}, \dots, \{7\}, \{1, 2\}, \{1, 3\}, \dots, \{1, 7\}, \{2, 3\}, \{5, 6\}, \{6, 7\}, \{1, 2, 3\}, \{1, 5, 6\}\}.$$

We can depict Δ as shown in Figure 5.1. We can observe that Δ is not pure, since $\{1, 7\}$ is a maximal face that has dimension 1, whereas $\dim(\Delta) = 2$.

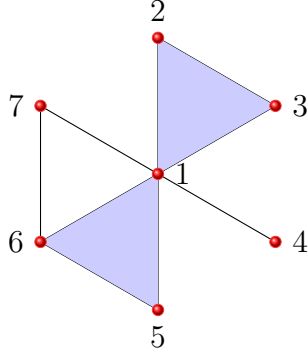


Figure 5.1: An abstract simplicial complex

We now recall that in Chapter 1, given a graph, we defined the incidence poset of the graph. We will now define the incidence poset of an abstract simplicial complex.

Definition 5.1.3. Given an abstract simplicial complex $\Delta = (V, \mathcal{F})$, we define the incidence poset $I = (\mathcal{F}, \leq_{\Delta})$ of Δ . In this case we consider the inclusion order for the set of faces, that is, for F and $G \in \mathcal{F}$ $F \leq_{\Delta} G$ if and only if $F \subseteq G$. We say that Δ has *poset dimension* d if $\dim(I) = d$.

When we were considering graphs, we defined a k -dimensional representation, we will now define a k -dimensional representation for abstract simplicial complexes.

Definition 5.1.4. Let $R = \leq_1, \dots, \leq_k$ be a sequence of total orders of V . We say that R is a k -dimensional representation of V if

$$\bigcap_i \leq_i = \emptyset. \quad (5.1)$$

In a similar fashion as in the graph case, we will call equation 5.1 the *vertex property*. Similarly to the graph case, if a_i denotes the maximum element with respect to \leq_i , we say the representation is *standard* if a_i is among the first $k - 1$ elements in each of \leq_j for $i \neq j$. The elements a_i will be called *exterior vertices* and all other elements of V will be called *interior vertices*. Given a subset X of V , we define the *supremum section* of X with respect to R , $S_R(X)$, as the subset of X consisting of the elements that are maximum with respect to some order in R . When the context is clear we will only denote it by $S(X)$.

Given an abstract simplicial complex $\Delta = (V, \mathcal{F})$, and a k -dimensional representation R of V , we say R satisfies the *face property* if for every face F of Δ and every vertex $x \notin F$, there is an order $\leq \in R$ so that $v \leq x$ for all $v \in F$, or equivalently, if $x \in S(F \cup \{x\})$.

In the following result we show how a k -dimensional representation is related to the poset dimension of an abstract simplicial complex.

Proposition 5.1.5. *An abstract simplicial complex $\Delta = (V, \mathcal{F})$ has poset dimension at most k if and only if there is k -dimensional representation of V satisfying the face property for \mathcal{F} .*

Proof. Let $\Delta = (V, \mathcal{F})$ be an abstract simplicial complex of poset dimension at most k . Let \leq_1, \dots, \leq_k be k linear orders of V so that $\bigcap_i \leq_i = \leq_\Delta$. Define $\leq_i \stackrel{\text{def}}{=} \leq_i|_V$, that is, \leq_i is the restriction of \leq_i to V .

We can see that $R \stackrel{\text{def}}{=} \leq_1, \dots, \leq_k$ satisfies the vertex property. Otherwise, there would be two vertices x and y so that $x \leq_i y$ for all i , and so $x \leq y$ for all i , a contradiction, since $x \not\leq_\Delta y$.

Now, to prove that R satisfies the face property we will proceed by contradiction. Assume there is a face $F \in \mathcal{F}$ and a vertex $x \notin F$ so that $x \notin S(F \cup \{x\})$. Since $x \notin S(F \cup \{x\})$ it follows that in each order \leq_i there is a vertex $v_i \in F$ so that $x \leq_i v_i$. Also observe that $v_i \leq_i F$ for all i . Hence, by transitivity, $x \leq_i F$ and so $x \leq_\Delta F$, a contradiction.

Conversely, assume that there is a k -dimensional representation $R = \leq_1, \dots, \leq_k$ of V satisfying the face property for the elements in \mathcal{F} . We will construct k linear orders so that their intersection is \leq_Δ . We will proceed in a similar fashion as we did in Proposition 2.1.2. Consider the first order \leq_1 . We first extend it to \leq'_1 by inserting each of the faces of dimension 1 right after the maximum of its two elements, that is, the face $\{x, y\}$, would be inserted right after the largest of x and y . Then we extend \leq'_1 to \leq''_2 by inserting the faces of dimension 2, so a face $\{x, y, z\}$ would be inserted after the maximum of $\{x, y\}$, $\{x, z\}$ and $\{y, z\}$. Note that by transitivity, the vertices of each of the faces will lie below each of the faces they belong to. Continuing in this fashion we obtain a linear order \leq_1 of \mathcal{F} . We apply the same procedure to each of \leq_j to obtain k linear orders \leq_j of \mathcal{F} . We will now prove that $\bigcap_i \leq_i = \leq_\Delta$.

It suffices to prove that in $\leq^* \stackrel{\text{def}}{=} \bigcap_i \leq_i$, $F \leq^* G$ if and only if $F \subseteq G \in \mathcal{F}$. We can see, by construction, that if $F \subseteq G$ then $F \leq_i G$ for all i . Consequently $F \leq^* G$. To prove that $F \leq^* G$ implies $F \subseteq G \in \mathcal{F}$ we will proceed by contradiction. Assume that $F \not\subseteq G$ and $F \leq^* G$, this is, $F \leq_i G$ for all i . Since $F \not\subseteq G$, there exists a vertex x so that $x \in F \setminus G$. In particular $x \notin G$, and since R satisfies the face property, $x \in S(G \cup \{x\})$, that is, there is an order \leq_j so that $u \leq_j x$ for all $u \in G$. By construction of \leq_j , $G \leq_j x$. We have, by construction of \leq_j , that $x \leq_j F$ and by transitivity $G \leq_j F$, a contradiction. So $F \leq^* G$ implies $F \subseteq G$, as desired. \square

5.2 Geometric realization of an abstract simplicial complex

The aim of this section will be to prove that an abstract simplicial complex can be represented geometrically. In order to achieve our goal we will first introduce geometric simplicial complexes. Then we show that every abstract simplicial complex of dimension d can be represented by a geometric simplicial complex in \mathbb{R}^{2d+1} . We conclude this section by proving a theorem of Ossona de Mendez. This result states that if the poset dimension of an abstract simplicial complex is d , then it is realizable in a hyperplane of dimension $d - 1$. For the case of graphs, this gives a geometric realization of an abstract simplicial complex consisting of the vertices, the edges, and the interior faces of the graph.

A set of points $\{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$ is said to be *affinely independent* if the only solution to $\sum_i \lambda_i v_i = 0$ so that $\sum_i \lambda_i = 0$ is $\lambda_i = 0$. Assume that $\{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$ is affinely

independent. Hence

$$\left(-\sum_{i=2}^n \lambda_i\right)v_1 + \sum_{i=2}^n \lambda_i v_i = 0$$

only if $\lambda_i = 0$ for all i and observing that $\lambda_1 = -\sum_{i=2}^k \lambda_i$. By rewriting the previous equation we get

$$\sum_{i=2}^n \lambda_i (v_i - v_1) = 0$$

only if $\lambda_i = 0$ for all i . So we can see that if $\{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$ is an affinely independent set then $\{v_2 - v_1, \dots, v_k - v_1\}$ is a linearly independent set.

Conversely, if $\{v_2 - v_1, \dots, v_k - v_1\}$ is linearly independent, then

$$\sum_{i=2}^n \lambda_i (v_i - v_1) = 0$$

only if $\lambda_i = 0$ for all i . Let $\lambda_1 \stackrel{\text{def}}{=} -\sum_{i=2}^n \lambda_i$ and note that $\lambda_1 = 0$. We can now rewrite the previous equation as $\sum_{i=1}^n \lambda_i v_i = 0$, we also get that $\sum_{i=1}^n \lambda_i = 0$ and all this will only hold if $\lambda_i = 0$. So, if $\{v_2 - v_1, \dots, v_k - v_1\}$ is linearly independent then $\{v_1, v_2, \dots, v_k\}$ is affinely independent. Together with what we proved in the previous paragraph, we get that $\{v_1, v_2, \dots, v_k\}$ is affinely independent if and only if $\{v_2 - v_1, \dots, v_k - v_1\}$ is linearly independent.

Given a set of points $S = \{v_1, \dots, v_k\}$, we define the set

$$\text{conv}(S) \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^k \lambda_i v_i \mid \sum_{i=1}^k \lambda_i = 1 \text{ and } \lambda_i \geq 0 \text{ for all } i \right\},$$

called the *convex hull* of S , that is, $\text{conv}(S)$ is the set of all linear combinations of points in S so that the coefficients are non negative and add up to 1.

We define a *simplex* σ as the convex hull of a finite affinely independent set S . Now, if $F \subseteq S$ we call $\text{conv}(F)$ a face of σ .

Definition 5.2.1. A *geometric simplicial complex* is a non empty collection of simplices Λ so that the following two conditions hold.

1. For each simplex $\sigma \in \Lambda$, if f is a face of σ , then $f \in \Lambda$.
2. Given any two simplices $\sigma_1, \sigma_2 \in \Lambda$, their intersection $\sigma_1 \cap \sigma_2$ is a face of each of σ_1 and σ_2 .

Now, we will give the definition of a geometric realization of an abstract simplicial complex. The following definition is in terms of a geometric simplicial complex.

Definition 5.2.2. Let $\Delta = (V, \mathcal{F})$ be an abstract simplicial complex. A *geometric realization* of Δ in \mathbb{R}^n is an injective function $f : V \rightarrow \mathbb{R}^n$ so that $\Lambda_f = \{\text{conv}(f(F)) : F \in \mathcal{F}\}$ is a geometric simplicial complex, where $f(F) \stackrel{\text{def}}{=} \{f(v) : v \in F\}$. An abstract simplicial complex Δ is said to be *realizable* in \mathbb{R}^n if there exists a geometric realization of Δ in \mathbb{R}^n .

We will now prove some lemmas which will be useful to show that an abstract simplicial complex of dimension d can be realized in \mathbb{R}^{2d+1} .

Lemma 5.2.3 (Ossoona de Mendez, [30]). *Let $\Delta = (V, \mathcal{F})$ be an abstract simplicial complex. If $f : V \rightarrow \mathbb{R}^n$ is a injective mapping so that for every pair of faces F and G of Δ we have $\text{conv}(f(F)) \cap \text{conv}(f(G)) = \text{conv}(f(F) \cap f(G))$, then f is a geometric realization of Δ .*

Proof. We begin by proving that for every face F of Δ , $f(F)$ is an affinely independent set, this guarantees that the elements of Λ_f are simplices. Let $F = \{u_1, \dots, u_k\}$ and let $f_j \stackrel{\text{def}}{=} f(u_j)$. Assume, by contradiction, that $\{f_1, \dots, f_k\}$ is an affinely dependent set. Then, there is a non trivial solution to

$$\sum_{i=1}^k \lambda_i f_i = 0 \quad \sum_{i=1}^k \lambda_i = 0. \quad (5.2)$$

Furthermore, we may assume, for the sake of the argument, that $\{u_1, \dots, u_k\}$ are labelled according to $\lambda_1, \dots, \lambda_k$, so that $\lambda_1, \dots, \lambda_d > 0$ and $\lambda_{d+1}, \lambda_{d+2}, \dots, \lambda_k \leq 0$. This is possible since the solution to equation (5.2) is non trivial, so there must be at least one coefficient that is positive. Now, we may rewrite equation (5.2) as

$$\sum_{i=1}^d \lambda_i f_i = - \sum_{i=d+1}^k \lambda_i f_i \quad \sum_{i=1}^d \lambda_i = - \sum_{i=d+1}^k \lambda_i. \quad (5.3)$$

Since all λ_i , $i = 1, \dots, d$ are positive, then $\mu \stackrel{\text{def}}{=} \sum_{i=1}^d \lambda_i > 0$. So from equation (5.3) we get

$$\frac{\lambda_1}{\mu} f_1 + \dots + \frac{\lambda_d}{\mu} f_d = - \frac{\lambda_{d+1}}{\mu} f_{d+1} - \dots - \frac{\lambda_k}{\mu} f_k, \quad (5.4)$$

where $\sum_{i=1}^d \lambda_i / \mu = 1$ and $\lambda_i > 0$, $i = 1, \dots, d$, and $\sum_{i=d+1}^k -\lambda_i / \mu = 1$ and $-\lambda_i \geq 0$, for $i = d+1, \dots, k$. Thus $\text{conv}(\{f_1, \dots, f_d\}) \cap \text{conv}(\{f_{d+1}, \dots, f_k\}) \neq \emptyset$.

On the other hand, we have that

$$\text{conv}(\{f_1, \dots, f_d\}) \cap \text{conv}(\{f_{d+1}, \dots, f_k\}) = \text{conv}(\{f_1, \dots, f_d\} \cap \{f_{d+1}, \dots, f_k\}).$$

And since f is injective, we get that $\{f_1, \dots, f_d\} \cap \{f_{d+1}, \dots, f_k\} = \emptyset$, so

$$\text{conv}(\{f_1, \dots, f_d\}) \cap \text{conv}(\{f_{d+1}, \dots, f_k\}) = \emptyset,$$

a contradiction. So $\{f_1, \dots, f_k\}$ must be an affinely independent set.

We can now see that Λ_f satisfies the first condition of Definition 5.2.1, since Δ is an abstract simplicial complex.

Finally, the second condition of Definition 5.2.1 is satisfied. This is because if $\text{conv}(f(F))$ and $\text{conv}(f(G)) \in \Lambda_f$ then

$$\text{conv}(f(F)) \cap \text{conv}(f(G)) = \text{conv}(f(F) \cap f(G)) = \text{conv}(f(F \cap G)),$$

as desired. □

An abstract simplicial complex of dimension d can be realized in \mathbb{R}^{2d+1} . To achieve this we will consider an injective mapping from the vertex set to the *moment curve* in \mathbb{R}^{2d+1} , defined as $\{(t, t^2, \dots, t^{2d+1}) | t \in \mathbb{R}\}$.

Lemma 5.2.4 (Matoušek,[27]). *Let $\Delta = (V, \mathcal{F})$ be an abstract simplicial complex. If $f : V \rightarrow \mathbb{R}^d$ is an injective mapping so that $f(F \cup G)$ is affinely independent for every $F, G \in \mathcal{F}$ then f is a geometric realization of Δ .*

Proof. We will use the previous lemma to prove the result. It is enough to show that $\text{conv}(f(F)) \cap \text{conv}(f(G)) = \text{conv}(f(F) \cap f(G))$ for all $F, G \in \mathcal{F}$. It is clear that if $x \in \text{conv}(f(F) \cap f(G))$ then $x \in \text{conv}(f(F)) \cap \text{conv}(f(G))$.

Let $x \in \text{conv}(f(F)) \cap \text{conv}(f(G))$. We have

$$x = \sum_{u \in F} \lambda_u f(u) = \sum_{v \in G} \mu_v f(v),$$

where $\sum \lambda_u = 1$ and $\sum \mu_v = 1$. This implies

$$\sum_{u \in F} \lambda_u f(u) - \sum_{v \in G} \mu_v f(v) = 0.$$

But this can be rewritten as

$$\sum_{u \in F \setminus G} \lambda_u f(u) - \sum_{v \in G \setminus F} \mu_v f(v) + \sum_{u \in F \cap G} (\lambda_u - \mu_u) f(u) = 0. \quad (5.5)$$

Now, since $f(F \cup G)$ is affinely independent, $\lambda_u = 0$ for all $u \in F \setminus G$. Similarly $\mu_v = 0$ for all $v \in G \setminus F$. Since $\sum \lambda_u = 1$ and $\sum \mu_u = 1$ we now have that $\sum_{u \in F \cap G} \lambda_u = 1$ and $\sum_{u \in F \cap G} \mu_u = 1$. Now, it follows from equation (5.5) that $\lambda_u = \mu_u$ for $u \in F \cap G$. Therefore

$$x = \sum_{u \in F \cap G} \lambda_u f(u) \quad \text{with} \quad \sum_{u \in F \cap G} \lambda_u = 1,$$

so $x \in \text{conv}(f(F) \cap f(G))$ as desired. □

Lemma 5.2.5 (Matoušek,[27]). *The moment curve in \mathbb{R}^d is intersected by any hyperplane in at most d points. Every set of at most $d+1$ points on the moment curve in \mathbb{R}^d is affinely independent.*

Proof. Consider any hyperplane in \mathbb{R}^d , say $a_1 x_1 + \dots + a_d x_d = b$. We note any point of intersection of the moment curve and the hyperplane must satisfy $a_1 t + a_2 t^2 + \dots + a_d t^d = b$. In other words, such point of intersection is a root of the polynomial $a_1 t + a_2 t^2 + \dots + a_d t^d - b$. Since $a_i, b \in \mathbb{R}$ it follows from the fundamental theorem of algebra that this polynomial has at most d real roots. Hence the first statement holds.

Now, consider a set S of at most $d+1$ points on the moment curve. Construct a set S' with $d+1$ points by adding points on the moment curve to S , if necessary. We claim there is no hyperplane containing all the elements of S' . If there was such a hyperplane, say $a_1 x_1 + \dots + a_d x_d = b$, then each point of S' would satisfy this equation. That is, there would be $d+1$ values of t so that $a_1 t + a_2 t^2 + \dots + a_d t^d - b = 0$, which contradicts the fundamental theorem of algebra. So we have that S' is affinely independent. We now use the fact that if a set is affinely independent then any subset of this set is affinely independent. Hence S is affinely independent. □

Theorem 5.2.6 (Matoušek,[27]). *Let $\Delta = (V, \mathcal{F})$ be an abstract simplicial complex of dimension d with $V = \{v_1, \dots, v_k\}$. The function $f : V \rightarrow \mathbb{R}^{2d+1}$, $v_i \mapsto (i, i^2, \dots, i^{2d+1})$ is a geometric realization of Δ .*

Proof. We can see that f is an injective mapping. Now it suffices to prove that for all $F, G \in \mathcal{F}$ the set $f(F \cup G)$ is affinely independent. Let $F, G \in \mathcal{F}$. Since $\dim(\Delta) = d$ then $|F|, |G| \leq d + 1$, so $|F \cup G| \leq 2d + 2$. So it follows from Lemma 5.2.5 that $f(F \cup G)$ is affinely independent. Finally, Lemma 5.2.4 implies that f is a geometric realization of Δ . \square

5.2.1 Realization of abstract simplicial complexes with poset dimension d

We now show that if the poset dimension of an abstract simplicial complex Δ is d , then Δ is realizable in a hyperplane of \mathbb{R}^d . For this, we now present some lemmas that we will use later on.

Lemma 5.2.7 (Ossona de Mendez, [30]). *Let $\Delta = (V, \mathcal{F})$ be an abstract simplicial complex. If $f : V \rightarrow \mathbb{R}^n$ is a mapping so that for every pair of disjoint faces F and $G \in \mathcal{F}$ we have $\text{conv}(f(F)) \cap \text{conv}(f(G)) = \emptyset$, then f is injective and*

$$\text{conv}(f(F)) \cap \text{conv}(f(G)) = \text{conv}(f(F) \cap f(G))$$

for all faces $F, G \in \mathcal{F}$.

Proof. First we will show that f is injective. Let $x, y \in V$ be two different vertices. Since $\{x\}$ and $\{y\}$ are disjoint faces, then

$$\begin{aligned} \{f(x)\} \cap \{f(y)\} &= \text{conv}(\{f(x)\}) \cap \text{conv}(\{f(y)\}) \\ &= \text{conv}(f(\{x\})) \cap \text{conv}(f(\{y\})) \\ &= \emptyset, \end{aligned}$$

so $f(x) \neq f(y)$ desired.

We can see that $\text{conv}(f(F \cap G)) = \text{conv}(f(F) \cap f(G)) \subseteq \text{conv}(f(F)) \cap \text{conv}(f(G))$. Let us now prove that $\text{conv}(f(F)) \cap \text{conv}(f(G)) \subseteq \text{conv}(f(F) \cap f(G))$. This is clearly the case when $\text{conv}(f(F)) \cap \text{conv}(f(G)) = \emptyset$.

We now assume that $\text{conv}(f(F)) \cap \text{conv}(f(G)) \neq \emptyset$. Note that this implies that $F \cap G \neq \emptyset$ and so $f(F) \cap f(G) \neq \emptyset$. Let $p \in \text{conv}(f(F)) \cap \text{conv}(f(G))$, we then have

$$p = \sum_{u \in F} \lambda_u f(u) = \sum_{v \in G} \mu_v f(v) \tag{5.6}$$

with each $\lambda_u, \mu_v \geq 0$, $\sum_u \lambda_u = 1$ and $\sum_v \mu_v = 1$. By rewriting equation (5.6) we obtain

$$\sum_{u \in F \setminus G} \lambda_u f(u) + \sum_{u \in F \cap G} \lambda_u f(u) = \sum_{v \in G \setminus F} \mu_v f(v) + \sum_{v \in G \cap F} \mu_v f(v). \tag{5.7}$$

Let

$$\nu_u = \begin{cases} \min\{\lambda_u, \mu_u\} & \text{if } u \in F \cap G \\ 0 & \text{: if } u \in (F \setminus G) \cup (G \setminus F) \end{cases}$$

We will now prove that $\sum \nu_u = 1$. Since $0 \leq \sum \mu_u, \sum \lambda_u \leq 1$ for all u , then $0 \leq \sum \nu_u \leq 1$. Assume, by contradiction, that $\sum \nu_u < 1$. From equation 5.7 we get

$$\frac{1}{1 - \sum \nu_u} \left(\sum_{u \in F \setminus G} \lambda_u f(u) + \sum_{u \in F \cap G} (\lambda_u - \nu_u) f(u) \right) = \frac{1}{1 - \sum \nu_u} \left(\sum_{v \in G \setminus F} \mu_v f(v) + \sum_{v \in G \cap F} (\mu_v - \nu_v) f(v) \right). \quad (5.8)$$

We can observe that if $\lambda_u - \nu_u \neq 0$ then $\mu_u - \nu_u = 0$ and if $\mu_u - \nu_u \neq 0$ then $\lambda_u - \nu_u = 0$. This implies that all the non zero coefficients in equation (5.8) involve only elements $u \in F' \subseteq F$ and $v \in G' \subseteq G$, where $F' \cap G' = \emptyset$. Also note that these coefficients satisfy the condition of adding up to 1. Therefore $\text{conv}(f(F')) \cap \text{conv}(f(G')) \neq \emptyset$, which is a contradiction, since $F' \cap G' = \emptyset$. Hence $\sum \nu_u = 1$. Now, since $\sum_{u \in F \cap G} \lambda_u \geq \sum \nu_u = 1$, then $\sum_{u \in F \cap G} \lambda_u = 1$. An analogous argument shows that $\sum_{v \in G \cap F} \mu_v = 1$. Thus $p \in \text{conv}(f(F) \cap f(G))$. \square

We have now provided tools to prove the following known theorem.

Theorem 5.2.8. *Let $\Delta = (V, \mathcal{F})$ be an abstract simplicial complex and $f : V \rightarrow \mathbb{R}^n$. The function f is a geometric realization of Δ if and only if for every pair of disjoint faces $F, G \in \mathcal{F}$ we have $\text{conv}(f(F)) \cap \text{conv}(f(G)) = \emptyset$.*

Proof. If f is a geometric realization of Δ , then Λ_f is a geometric simplicial complex. Therefore if $F, G \in \mathcal{F}$ are disjoint, then $\text{conv}(f(F))$ and $\text{conv}(f(G)) \in \Lambda_f$ are disjoint.

Conversely, assume f is a mapping so that for every pair of disjoint faces $F, G \in \mathcal{F}$ $\text{conv}(f(F)) \cap \text{conv}(f(G)) = \emptyset$. It follows from Lemma 5.2.7 that

$$\text{conv}(f(F)) \cap \text{conv}(f(G)) = \text{conv}(f(F) \cap f(G))$$

for all faces $F, G \in \mathcal{F}$. Finally, by Lemma 5.2.3, it follows that f is a geometric realization of Δ . \square

Let $\Delta = (V, \mathcal{F})$ be an abstract simplicial complex of dimension d and let $R = \leq_1, \dots, \leq_d$ be a d -dimensional representation of V satisfying the face property for \mathcal{F} . We will consider a function $f : V \rightarrow \mathbb{R}^d$, $f(v) = (f_1(v), \dots, f_d(v))$, so that, for each $i = 1, \dots, d$, $f_i(u) > 0$ for all $u \in V$, and if $u >_i v$ then $f_i(u) \geq (d+1) f_i(v)$. For the rest of this section, if $u \in V$, then $u_i \stackrel{\text{def}}{=} f_i(u)$. Given a set of vertices $F \subseteq V$, we define $\sigma_i(F) \stackrel{\text{def}}{=} \max_{u \in F} \{u_i\}$ and $\Theta(F) \stackrel{\text{def}}{=} \{(x_1, \dots, x_d) \in \mathbb{R}^d | x_i \leq \sigma_i(F) \text{ for all } i\}$. Now, an immediate consequence of these definitions is the following lemma.

Lemma 5.2.9. *Let $\Delta = (V, \mathcal{F})$ be an abstract simplicial complex, R be a d -dimensional representation and $f : V \rightarrow \mathbb{R}^d$ be a function as defined above. If a subset $F \subset V$ is a face of Δ then the following two conditions hold.*

- For each $u \in F$, (u_1, \dots, u_d) lies in the boundary of $\Theta(F)$.
- If $v \in V \setminus F$, then $v \in \Theta(X)^c$.

Proof. Let $w \in F$. Since $F \in \mathcal{F}$, then $F \setminus \{w\} \in \mathcal{F}$. Since the face property is satisfied by R for \mathcal{F} , then there exists an order \leq_i so that $w >_i u$ for all $u \in F \setminus \{w\}$. So we have $u \geq_i w$ for all $u \in F$. Hence $w_i \geq u_i$ for all $u \in F$. And so $w_i = \sigma_i(F)$, as desired.

Now let $v \in V \setminus F$. Since the face property is satisfied for F , we have that there is an order \leq_j so that $v >_j u$ for all $u \in F$. So we get that $v_j > u_j$ for all $u \in F$. This implies $v_j > \sigma_j(F)$, as desired. \square

We now prove one lemma that will be useful to prove the main theorem of this subsection.

Lemma 5.2.10. *Let $\Delta = (V, \mathcal{F})$ be an abstract simplicial complex of poset dimension d and let R be a d -dimensional representation of V satisfying the face property for \mathcal{F} . If $f : V \rightarrow \mathbb{R}^d$, $f(v) = (f_1(v), \dots, f_d(v))$, is so that each f_i is a positive increasing function so that $u >_i v$ implies $f_i(u) \geq (d+1)f_i(v)$ then the following two conditions hold.*

- *If $u \in V$ is so that there exists i so that $u_i > \sigma_i(G)$ for some $G \in \mathcal{F}$, then we have $u_i/\sigma_i(G) \geq d+1$.*
- *If $u \in G \in \mathcal{F}$, then $\sum_k u_k/\sigma_k(G) < d+1$.*

Proof. To prove the first part we observe that if $u_i > \sigma_i(G)$ then $u >_i v$ for all $v \in G$. Hence $u_i \geq (d+1)v_i$ for all v . This would hold in particular for the v that maximizes v_i , so $u_i \geq (d+1)\sigma_i(G)$, and the result follows.

In order to prove the second condition we first observe that $u_k \leq \sigma_k(G)$ for all k . Hence $u_k/\sigma_k(G) \leq 1$ for all k , so $\sum_k u_k/\sigma_k(G) < d+1$ as desired. \square

We now present the main theorem of this section.

Theorem 5.2.11 (Ossona de Mendez,[30]). *Let $\Delta = (V, \mathcal{F})$ be an abstract simplicial complex of poset dimension d , let R be a d -dimensional representation of V satisfying the face property for \mathcal{F} , and let $f : V \rightarrow \mathbb{R}^d$, $f(v) = (f_1(v), \dots, f_d(v))$, so that each f_i is a positive increasing function so that $u >_i v$ implies $f_i(u) \geq (d+1)f_i(v)$. The function $\phi : V \rightarrow \mathbb{R}^d$, $\phi(v) = (\phi_1(v), \dots, \phi_d(v))$, where $\phi_i(v) = f_i(v)/(\sum_i f_i(v))$ is a geometric realization of Δ in a hyperplane of dimension $d-1$.*

Proof. We will proceed by applying Theorem 5.2.8. We can see that ϕ is injective because f_i is increasing for each i . It is enough to prove that for two disjoint faces F and $G \in \mathcal{F}$, $\text{conv}(f(F)) \cap \text{conv}(f(G)) = \emptyset$. To prove this, we will exhibit a hyperplane that separates the elements of $f(F)$ from the elements of $f(G)$. Define the sets $I = \{i : \text{there is } x \in f(F) \text{ with } x_i > \sigma_i(G)\}$ and $J = \{j : \text{there is } x \in f(G) \text{ with } x_j > \sigma_j(F)\}$. It is an immediate consequence of Lemma 5.2.9 that I and J are non empty sets. Now, let $a_I(x) = \sum_{i \in I} f_i(x)/\sigma_i(G)$ and $a_J(x) = \sum_{j \in J} f_j(x)/\sigma_j(F)$. It follows from Lemma 5.2.10 that $a_I(x) \geq d+1$ and $a_J(x) < d+1$ for all $x \in f(F)$. Likewise $a_J(x) \geq d+1$ and $a_I(x) < d+1$ for all $x \in f(G)$. Let $H(x) \stackrel{\text{def}}{=} a_I(x) - a_J(x)$. We can see that $H(x) > 0$ for $x \in f(F)$ and $H(x) < 0$ for $x \in f(G)$. So the hyperplane $H(x) = 0$ separates $f(F)$ and $f(G)$, hence, by Theorem 5.2.8, ϕ is a geometric realization of Δ in \mathbb{R}^d .

We now observe that $\sum_i \phi_i(u) = 1$, so every vertex is mapped to a point in the hyperplane $\sum_i x_i = 1$ of dimension $d-1$. For a face $F \in \mathcal{F}$, we can see that $\text{conv}(f(F))$ is contained in this same hyperplane, as desired. \square

5.3 The Order Labelling

In this section we consider a higher dimensional version of the concept of normal labelling introduced in Definition 2.3.1. The concept of order labelling for abstract simplicial complexes is introduced. We will then focus on abstract simplicial complexes of dimension 4 and derive some properties of the order labelling. The properties we will show will be analogous to the properties of a normal labelling. We begin by introducing a definition.

Definition 5.3.1. Given a k -dimensional representation R of a finite set V , we define the *complex* of R , denoted by $\Sigma(R)$, as

$$\Sigma(R) = \{F \subseteq V \mid x \in S(F \cup \{x\}) \text{ for all } x \in V\}.$$

As we will see in the following proposition $(V, \Sigma(R))$ is an abstract simplicial complex.

Proposition 5.3.2. *Let R be a k -dimensional representation of a finite set V . The pair $\Delta = (V, \Sigma(R))$ defines an abstract simplicial complex and $\dim(\Delta) \leq k - 1$.*

Proof. It is clear that $\Sigma(R)$ is a collection of subsets of V . Now, let $F \in \Sigma(R)$ and let $G \subseteq F$. We will show that $G \in \Sigma(R)$. Let $x \in V$. Since $F \in \Sigma(R)$, then $x \in S(F \cup \{x\})$, that is, there is an order \leq_i of R in which x is greater than or equal to all elements of F . This implies, in particular, that in that same order, \leq_i , x is greater than or equal to all elements of G . Hence $x \in S(G \cup \{x\})$, as desired.

Now, to prove $\dim(\Delta) \leq k - 1$ we will proceed by contradiction. Assume there is a face $F \in \Sigma(R)$ so that $\dim(F) > k - 1$. This implies $|F| > k$. By definition of $\Sigma(R)$, we must have that $x \in S(F \cup \{x\})$ for all $x \in V$. If we let $b_i = \max_{\leq_i} F$ we can see that $F \setminus \{b_1, \dots, b_k\}$ is not empty. For $y \in F \setminus \{b_1, \dots, b_k\}$ we can see that $y \notin S(F)$, a contradiction. □

For the purposes of the following results in what remains in this section, we will let a_i denote the maximum element in the i -th order of a representation.

Lemma 5.3.3. *Let $R =_{\leq_1, \dots, \leq_k}$ be k -dimensional representation of V . If we let*

$$R' =_{\leq_1, \dots, \leq_{i-1}, \leq_i, \leq_{i+1}, \dots, \leq_k}$$

be a k -dimensional representation of V , where $\leq_j = \{(a_i, u) \mid u \in V\} \cup \{(u, v) \mid u, v \in V \setminus \{a_i\}\}$, then $\Sigma(R) \subseteq \Sigma(R')$.

Proof. Let $F \in \Sigma(R)$. We will show that $F \in \Sigma(R')$ or equivalently that $x \in S_{R'}(F \cup \{x\})$ for all $x \in V$. So, let $x \in V$. We know that $x \in S_R(F \cup \{x\})$. We will consider two cases. First, if $x = a_i$, then it is clear that $x \in S_{R'}(F \cup \{x\})$ since x will remain the maximum element of the i -th order. Now, if $x \neq a_i$, let \leq_l be the order in which $x \geq_l u$ for all $u \in F \cup \{x\}$. If $l = i$ then the assertion still holds, since this order remained unchanged in R' . If $l \neq i$ we have $x \geq_l u$ for all $u \in F \cup \{x\}$ since the relative order of all elements but a_i is preserved; even in the case where $a_i \in F$ the assertion holds, because $1 = \text{ord}_{\leq_l}(a_i) < \text{ord}_{\leq_l}(x)$. So the result follows. □

We can now see that given a k -dimensional representation R of V , we may *standardize* it by repeatedly using the previous result on each of the maximum elements of each of the orders in R . We would then obtain a standard k -dimensional representation R satisfying $\Sigma(R) \subseteq \Sigma(R)$.

Lemma 5.3.4. *Let $R =_{\leq_1, \dots, \leq_k}$ be a standard k -dimensional representation of V and let $F \in \Sigma(R)$ so that $|F| < k$ and so that F contains at least one interior vertex. For a given order, say \leq_i , there is a face $F' \in \Sigma(R)$ so that F is properly contained in F' and $\max_{\leq_i} F = \max_{\leq_i} F'$.*

Proof. Let $x \in F$ be such that $x = \max_{\leq_i} F$, we will find $F' \in \Sigma(R)$ so that $F \subseteq F'$ and so that $x = \max_{\leq_i} F'$. Since $|F| < k$, it follows from the pigeon hole principle that there is at least one element $u \in F$ so that $u = \max_{\leq_j} F$ and $u = \max_{\leq_l} F$. Note that $u \neq a_n$ for all n , since F contains at least one interior vertex, say z , and $a_n \leq_m z$ for all $n \neq m$. Also note it is possible that $u = x$, in such case we will assume, without loss of generality, that $l = i$. We now observe that the set

$$U = \{v \in V \mid u <_j v, v <_n f \text{ for } n \neq j \text{ and some } f \in F\}$$

is not empty, as $a_j \in U$ and F contains at least one interior vertex. Let $w = \min_{\leq_j} U$, we will show that $F' = F \cup \{w\} \in \Sigma(R)$.

Let $z \in V$, we will prove that $z \in S(F' \cup \{z\})$. If $z \in F$ we have that $z \in S(F)$, this is z is the maximum element of F in some order, say \leq_m . Observe that if $z = u$ then z is the maximum element of F in \leq_l and if $z \neq u$ then z is the maximum element of F in \leq_m with $m \neq j$. In either case we have that $z \in S(F)$. Now, let us consider the case when $z \notin F$. Since $z \in S(F \cup \{z\})$, this means there is an order, say \leq_m in which z is above all elements of F . We now consider two possible cases. First we consider the case when there is an order $\leq_{m'} \neq \leq_j$ in which z is above all elements of F . In this particular case, we also have $w \leq_{m'} z$ by definition of w . Therefore $z \in S(F' \cup \{z\})$. We now consider the second case, this is the case when the only order in which z is above all elements of F is \leq_j . If we assume that $z \notin S(F' \cup \{z\})$ this implies $z \leq_j w$. At this point we observe that if $z = w$ then $z \in S(F' \cup \{z\})$. In the other hand, if $z \neq w$ then $z <_j w$. And since the only order in which z is above all elements of F is \leq_j it follows that $z \in U$. This contradicts the minimality of w , so we must have $z \in S(F' \cup \{z\})$, as desired. \square

An immediate consequence of the previous lemma is the following.

Corollary 5.3.5. *If $R =_{\leq_1, \dots, \leq_k}$ is a standard k -dimensional representation of V then $\Sigma(R)$ is pure of dimension $k - 1$.*

Proof. Let $F \in \Sigma(R)$. We will show that F is contained in a face $F' \in \Sigma(R)$ of dimension $k - 1$. The assertion is clearly true if $\dim(F) = k - 1$, so assume that $\dim(F) < k - 1$. This implies $|F| < k$.

We first consider the case when F only contains exterior vertices. We may assume, without loss of generality, that $F = \{a_1, \dots, a_l\}$ with $l < k$. Let $F' = \{a_1, \dots, a_l, w\}$ where $\text{ord}_{\leq_k}(w) = k$, we will show that $F' \in \Sigma(R)$. It is enough to show that $x \in S(F' \cup \{x\})$ for all $x \in V$. If x is an exterior vertex, say $x = a_n$, then $u \leq_n x$ for all u in $F' \cup \{x\}$. If x is

an interior vertex then $x \succeq_k u$ for all u in $F' \cup \{x\}$, since the R is standard. In either case $x \in S(F' \cup \{x\})$, so $F' \in \Sigma(R)$. So we obtain a face F' so that $F \subset F'$ and F' contains an interior vertex. If necessary, F' can be extended to a face of dimension k by repeatedly applying Lemma 5.3.4.

If F contains at least one interior vertex then we may repeatedly apply Lemma 5.3.4 until a face of dimension k is obtained.

Since, by Proposition 5.3.2, no face has dimension greater than $k - 1$ we can see $\Sigma(R)$ is pure. □

Definition 5.3.6. Given a standard k -dimensional representation $R = \leq_1, \dots, \leq_k$, and a face $F \in \Sigma(R)$ of dimension $k - 1$, we will label each vertex $u \in F$ as follows. The label of u in F is i , where i is the index of the unique order in R for which $u \succeq_i v$ for all $v \in F$. We call this labelling the *order labelling*. Observe that each vertex of F gets a unique label in F as no two elements of F can be maximum elements of F with respect to the same order.

We can see that for the case $k = 3$ this is the order labelling defined in Chapter 2. We will now focus our attention for the case $k = 4$. As we will see, some of the properties of the order labelling when $k = 3$ extend naturally for the case $k = 4$.

Proposition 5.3.7. *Let $R = \leq_1, \dots, \leq_4$ be a standard 4-dimensional representation. Let $T_s = \{s, x, y, z\}$ and $T_t = \{t, x, y, z\}$ be two different faces in $\Sigma(R)$. The order labels of s in T_s and t in T_t are different.*

Proof. We proceed by contradiction. Assume that the order labels of s and t are the same, say i . Since $T_s \in \Sigma(R)$, then $t \in S(T_s \cup \{t\})$. From the order labelling of T_t we can see that t is greater than x, y and z only in the i -th order, hence $t >_i s$. An analogous argument for s gives us $s >_i t$, a contradiction. So the order labels of s and t cannot be the same. □

Proposition 5.3.8. *Let $R = \leq_1, \dots, \leq_4$ be a standard 4-dimensional representation. Let $T_s = \{s, x, y, z\}$ and $T_t = \{t, x, y, z\}$ be two different faces of size 4 in $\Sigma(R)$ and let i and j be the order labels of $s \in T_s$ and $t \in T_t$ respectively. The vertex w labelled i in T_t has label j in T_s .*

Proof. We proceed by contradiction. First, we note that $w \in T_s \cap T_t$. Assume that w has label $l \neq i$ in T_s . Let u be the vertex labelled l in T_t . Observe that $u \in T_s \cap T_t$. This implies $w <_l u$, but also $u <_l w$, a contradiction. So $l = i$. □

Proposition 5.3.9. *Let $R = \leq_1, \dots, \leq_4$ be a standard 4-dimensional representation. Let $T_s = \{s, x, y, z\}$ and $T_t = \{t, x, y, z\}$ be two different faces in $\Sigma(R)$. There is a unique vertex having different labels in T_s and T_t .*

Proof. Let i and j be the labels of $s \in T_s$ and $t \in T_t$ respectively. From the previous proposition we know that the vertex labelled i in T_t has label j in T_s . Assume, by contradiction, that there is another vertex, say u , so that the label of u in T_s , say l_s , is different from the label of u in T_t , say l_t . Let v be the vertex labelled l_s in T_t . Observe that $u, v \in T_s \cap T_t$. From the labels of u in T_t and v in T_s we can conclude that $u >_{l_s} v$ and $v >_{l_s} u$ which is a contradiction. So $l_s = l_t$. □

The three previous propositions show some properties about the order labelling. These properties and their analogues for the 3 dimensional case are illustrated in Figure 5.2. In Figure 5.2, we can observe that the label of s and t are different, and that the only vertex that changes labels in the two tetrahedra (respectively triangles) is x .

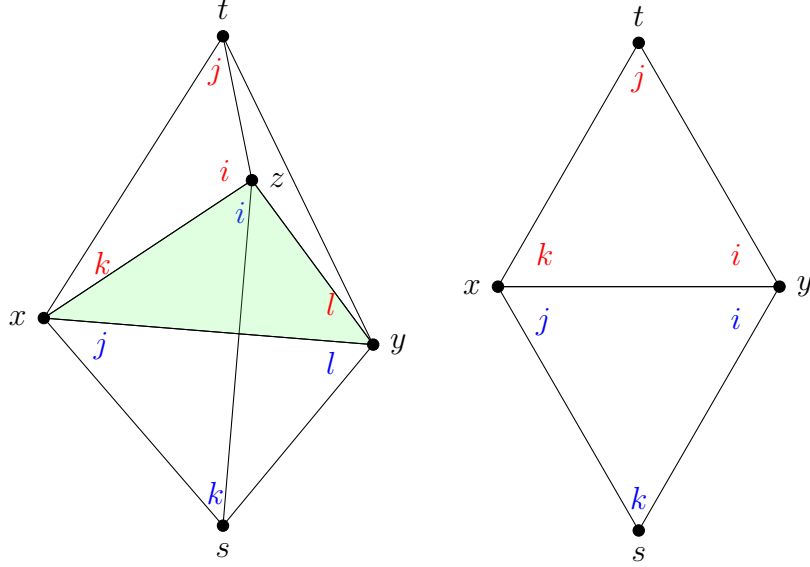


Figure 5.2: Order labelling of two adjacent tetrahedra and the analogous for two adjacent triangles

Given an interior triangular face of $\Sigma(R)$, we may assign one of $\binom{4}{2} = 6$ types to each of these faces. We will say that a triangular face F is of type (i, j) or (j, i) if the labels of the two vertices that have the same label in the two tetrahedra containing F are i and j . For example, the green triangular face in Figure 5.2 is of type (i, l) .

As we can see these types induce a partition of the set of interior triangular faces into 6 equivalence classes. We will establish a “connectedness” property about each of the equivalence classes, but first we prove a lemma which will be useful.

Lemma 5.3.10. *Let R be a standard 4-dimensional representation of V . For each interior vertex $u \in V$, there is an (i, j) triangle t so that $u \in t$, $u \leq_i x$ and $u \leq_j x$ for all $x \in t$.*

Proof. Let i, j, k, l be a permutation of $1, 2, 3, 4$ and let

$$U = \{y \in V \mid y <_i u, y <_j u, y <_k u \text{ and } u <_l y\}.$$

We can see U is not empty, as $a_l \in U$ since R is standard. Let $w = \min_{\leq_l} U$. We will show that $\{u, w\} \in \Sigma(R)$ by proving that $x \in S(\{u, w\} \cup \{x\})$ for all $x \in V$. This clearly holds for u and w . For $x \in V \setminus \{u, w\}$, we can see that there is at least one order in which $u <_{i'} x$. If $i' \neq l$ then $x \in S(\{u, w\} \cup \{x\})$. Now, if $u <_{i'} x$ only holds for $i' = l$, we must have $w <_{i'} x$, otherwise it would contradict the minimality of w , as $x \in U$. Hence $x \in S(\{u, w\} \cup \{x\})$ for all $x \in V$.

Now let

$$U' = \{y \in V \mid y <_i u, y <_j u, y <_l w \text{ and } u <_k y\}.$$

We can observe $a_k \in U'$. Let $w' = \min_{\leq_k} U'$. We will show that $\{u, w, w'\} \in \Sigma(R)$ by showing that $x \in S(\{u, w, w'\} \cup \{x\})$ for all $x \in V$. The statement clearly holds if $x \in \{u, w, w'\}$, so assume $x \in V \setminus \{u, w, w'\}$. By the vertex property, there is an order $\leq_{j'}$ so that $u <_{j'} x$. If $j' \neq k$, then $x \in S(\{u, w, w'\} \cup \{x\})$. For the case when $u <_{j'} x$ only holds for $j' = k$ we must have $w' <_{j'} x$, otherwise this would contradict the minimality of w' , since $x \in U'$. Therefore $\{u, w, w'\} \in \Sigma(R)$ and u is maximal in t with respect to \leq_i and \leq_j . \square

Proposition 5.3.11. *Let R be a standard 4-dimensional representation of V . For each interior vertex $u \in V$, there is a sequence of (i, j) triangles $t_1, \dots, t_k \in \Sigma(R)$ so that $t_i \cap t_{i+1} \neq \emptyset$ for $i = 1, \dots, k-1$, $u \in t_1$, and $a_i, a_j \in t_k$.*

Proof. Let $t_1 = \{u, v_1, w_1\}$ be the triangle obtained by applying Lemma 5.3.10 to u . In t_1 , we may assume, without loss of generality, that $x \leq_i v_1$ and $x \leq_j w_1$ for all $x \in t_1$. We continue in this way, applying Lemma 5.3.10 to v_i until this is no longer possible. We obtain a sequence t_1, \dots, t_q of triangles. This sequence must terminate at a given time, since each time we get a triangular face, the vertices v_p define an increasing sequence in \leq_i . Then we apply repeatedly Lemma 5.3.10 to w_i starting at w_q . Hence we obtain another sequence t_q, \dots, t_k . All the triangles in the sequence $t_1, \dots, t_q, \dots, t_k$ are of the same type by construction. Note also that $v_k = a_i$ and $w_q = a_j$. In fact, $v_q = \dots = v_k = a_i$ since we could no longer apply the lemma to v_q and it remained fixed since then. Finally, we observe that $v_r \in t_r \cap t_{r+1}$ for $r = 1, \dots, q-1$, and $w_r \in t_r \cap t_{r+1}$ for $r = q, \dots, k-1$. \square

Let $\Delta = (V, \mathcal{F})$ be an abstract simplicial complex. We say Δ is a triangulation of the sphere S^2 , if there is a geometric realization f of Δ , so that $\bigcup_{F \in \mathcal{F}} \text{conv}(f(F)) \cong S^2$, that is, if $\bigcup_{F \in \mathcal{F}} \text{conv}(f(F))$ is homeomorphic to S^2 .

We will now use the geometric realization from Theorem 5.2.11 to state the following proposition. We will derive an analogous result to Lemma 2.3.4. In this case, we will call a vertex u of type i in a triangulated sphere T , if all corners at u enclosed in the sphere T are labelled i .

Lemma 5.3.12. *Let $R =_{\leq_1, \leq_2, \leq_3, \leq_4}$ be a standard 4-dimensional representation of V , let $i \in \{1, 2, 3, 4\}$ and let $T \subseteq \Sigma(R)$ be a triangulation of the sphere. Let S be the simplicial complex induced by $V(T)$, and all vertices lying in the interior of T . There is $u \in V(T)$ so that all interior corners are labelled i .*

Proof. We proceed by contradiction. Assume that T is a triangulation of the sphere not satisfying the Lemma and enclosing as few 3 dimensional faces as possible. Two possible cases arise.

Case 1) There exist 3 vertices, $x, y, z \in V(T)$ so that $\{x, y\}, \{x, z\}, \{y, z\} \in T$ and the triplet $\{x, y, z\} \in \Sigma(R) \setminus T$. Observe that $\{x, y, z\}$ divides T into two sphere triangulations, T_1 and T_2 , each of which contain fewer 3-dimensional faces.

The sphere triangulations T_1 and T_2 must contain a vertex of type i . Observe that these vertices must be one of x, y , and z , otherwise T would have a vertex of type i .

Now, we can also see that these vertices cannot be the same for T_1 and T_2 , as this would imply T had a vertex of type i . We may assume without loss of generality that the vertices of type i in T_1 and T_2 are x and y respectively. Now, consider the two 3 dimensional faces containing $\{x, y, z\}$, say $T_s = \{s, x, y, z\}$ and $T_t = \{t, x, y, z\}$. We may assume without loss of generality that T_s is enclosed by T_1 and that T_t is enclosed by T_2 . This implies that the label of x is different in T_s from its label in T_t . Similarly, the labels of y in T_s and in T_t are different. This contradicts Proposition 5.3.9, as there should only be one vertex having different labels.

- Case 2) No 3-dimensional face separates T into two smaller triangulations of the sphere. In this case, we consider a 2-dimensional face $\{x, y, z\} \in T$. Let s be the common neighbor of x, y , and z lying within T . Note that

$$T' = T \setminus \{x, y, z\} \cup \{x, y, s\} \cup \{x, z, s\} \cup \{y, z, s\}$$

is a triangulation of the sphere enclosing fewer 3-dimensional faces. We can see that in T' , s cannot be the vertex of type i , as this implies s would have at most two different labels at its corners whereas it must have four different labels by Definition 5.3.6. Now, assume that the vertex labelled i in T' is one of x, y , and z . We may assume, without loss of generality, that vertex x is of type i in T' . Since x is not of type i in T , this implies the label of x in $\{x, y, z, s\}$ is j , $j \neq i$. By applying Proposition 5.3.7 to $\{x, y, s\}$ it follows that the label of z in $\{x, y, z, s\}$ is i . On the other hand, if we apply the same result, but now to $\{x, z, s\}$ we get that the label of y in $\{x, y, z, s\}$ is i , a contradiction. So T' does not contain a vertex labelled i , since if it were any other vertex, this would also be a vertex of type i in T . This contradicts the minimality of T .

In either case, we get that there is no such triangulation of the sphere T , as desired. \square

Chapter 6

Sage implementation

This chapter will be devoted to showing the source code of some functions that were implemented during the development of this work. These functions were implemented in Sage [35]. We will start by briefly describing how to use Sage to work with graphs. We then present a function that obtains a 3-dimensional representation of a plane triangulation, following the idea of Proposition 4.2.4. A function that obtains a barycentric embedding, described in Section 3.1, is also given. We also present an implementation of Algorithm 3.2.3, which produces a straightline embedding using a canonical ordering of the vertices of a plane triangulation. After this, we show functions that obtain the induced simplicial complex $\Sigma(R)$, Definition 5.3.1, of a d -representation. We also show a function that obtains a geometric realization of the simplicial complexes when $d = 4$ based on Theorem 5.2.11.

6.1 Graph Theory in Sage

Sage is an open-source mathematical software. For details about how to obtain, install and use Sage, the reader is referred to <http://www.sagemath.org/>. To try Sage online, without installing it, the reader can visit <http://www.sagenb.org/>.

The programming language Python is used as interface in Sage. One can define functions and objects as done in Python and use them in Sage. One of the built-in types of Python in Sage that we will use to emulate embeddings or labellings are dictionaries. We can think of Python dictionaries as arrays that are indexed by different objects. So, for an embedding, dictionaries will allow us to see the mapping as a list of points in \mathbb{R}^n that are indexed by the vertices.

In Sage, graphs are implemented as objects, so once we create a graph and assign it to a variable, say G , we can access all the methods using the `.` operator. This will become more clear as we explain how to perform very basic operations on graphs.

A possible way to create a graph and store it in the variable G is by issuing the command.

```
sage: G=Graph()
```

The previous command assigns to the variable G an empty graph, a graph with no vertices and no edges. By default graphs in Sage are simple graphs, that is, unless we

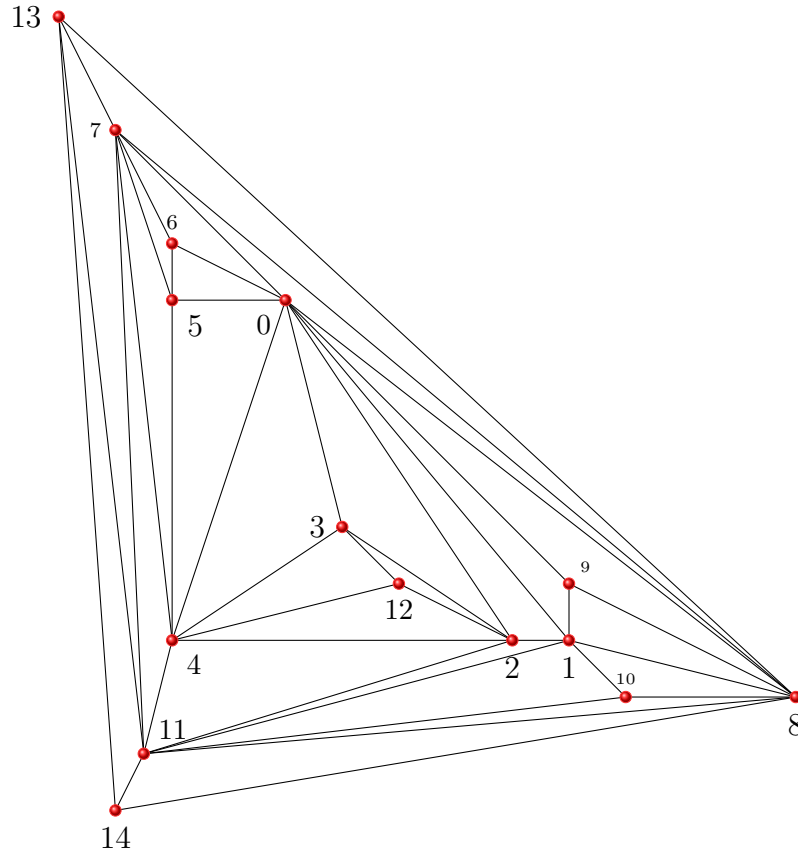


Figure 6.1: A plane triangulation.

indicate to Sage to allow loops or multiple edges, when we add a loop it will be discarded, likewise, when adding an edge more than once, only one edge will be preserved. Now, let us show how to add vertices and edges to G . Let us assume we want to create the graph shown in Figure 6.1.

In Sage, we can add edges to graphs even if some of the endpoints are not in the vertex set of the graph. If the vertex is not part of the graph, a new vertex will be created. So we can proceed to add the 9 edges to G . This can be achieved using the method `add_edges`, which can receive as argument a list of pairs, as shown below.

```
sage: G.add_edges([(0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6),
(0,7), (0, 8), (0, 9), (1, 2), (1, 8), (1, 9), (1, 10), (1, 11),
(2, 3), (2, 4), (2, 11), (2, 12), (3, 4), (3, 12), (4, 5),
(4, 7), (4, 11), (4, 12), (5, 6), (5, 7), (6, 7), (7, 8), (7, 11),
(7, 13), (8, 9), (8, 10), (8, 11), (8, 13), (8, 14), (10, 11),
(11, 13), (11, 14), (13, 14)])
```

There are several other methods for graphs that are available in Sage. For more details about these methods, we refer the reader to <http://www.sagemath.org/doc/reference/graphs.html>. Among these methods, we can find one that tests for planarity.

As we can see, the graph G that we had previously defined is planar. We can test for planarity in Sage using the method `is_planar`, as shown below.

```
sage: G.is_planar()
True
```

We can also obtain a combinatorial embedding, a description of the ordering of the neighbors of each vertex in clockwise order, of the graph in case the graph is planar. To achieve this, we can also use the method `is_planar` to set the embedding and then we may retrieve it with the `get_embedding` method.

```
sage: G.is_planar(set_embedding=True)
True
sage: G.get_embedding()
{0: [1, 9, 8, 7, 6, 5, 4, 3, 2], 1: [2, 11, 10, 8, 9, 0],
 2: [0, 3, 12, 4, 11, 1], 3: [0, 4, 12, 2],
 4: [0, 5, 7, 11, 2, 12, 3], 5: [0, 6, 7, 4], 6: [0, 7, 5],
 7: [0, 8, 13, 11, 4, 5, 6], 8: [0, 9, 1, 10, 11, 14, 13, 7],
 9: [0, 1, 8], 10: [1, 11, 8], 11: [1, 2, 4, 7, 13, 14, 8, 10],
 12: [2, 3, 4], 13: [7, 8, 14, 11], 14: [8, 11, 13]}
```

We now give an idea of how to interpret the embedding. As we can see, there is a description for each vertex. For example, for vertex 5 we have the description 5: [0, 6, 7, 4], which means that the order of the neighbors of 5 in clockwise order is 0, 6, 7, 4.

Another method that will result of interest is `trace_faces`. This method receives as argument a combinatorial embedding and outputs a list containing lists of edges that define the faces of the graph. In the example we have been following we get.

```
sage: G.trace_faces(G.get_embedding())
[[ (8, 13), (13, 14), (14, 8) ], [ (3, 0), (0, 2), (2, 3) ],
 [ (9, 8), (8, 1), (1, 9) ], [ (8, 0), (0, 7), (7, 8) ],
 [ (11, 14), (14, 13), (13, 11) ], [ (4, 7), (7, 5), (5, 4) ],
 [ (7, 13), (13, 8), (8, 7) ], [ (10, 11), (11, 1), (1, 10) ],
 [ (9, 1), (1, 0), (0, 9) ], [ (5, 6), (6, 0), (0, 5) ],
 [ (11, 2), (2, 1), (1, 11) ], [ (8, 9), (9, 0), (0, 8) ],
 [ (4, 12), (12, 2), (2, 4) ], [ (2, 12), (12, 3), (3, 2) ],
 [ (7, 11), (11, 13), (13, 7) ], [ (0, 3), (3, 4), (4, 0) ],
 [ (0, 1), (1, 2), (2, 0) ], [ (11, 4), (4, 2), (2, 11) ],
 [ (4, 3), (3, 12), (12, 4) ], [ (11, 7), (7, 4), (4, 11) ],
 [ (8, 14), (14, 11), (11, 8) ], [ (10, 8), (8, 11), (11, 10) ],
 [ (7, 0), (0, 6), (6, 7) ], [ (10, 1), (1, 8), (8, 10) ],
 [ (5, 0), (0, 4), (4, 5) ], [ (7, 6), (6, 5), (5, 7) ]]
```

We can observe that these faces correspond to the ones observed in Figure 6.1.

6.2 Obtaining 3-dimensional representations of plane triangulations

The aim of this section will be to present an implementation of a function that obtains a standard 3-dimensional representation of a plane triangulation.

We begin by showing a function that receives as input a graph G and two vertices of G , e and u . The output is the graph resulting from deleting the vertex u of G and joining the vertex e to all the former neighbors of u . We can see that if e and u are initially neighbors, then the output is the graph resulting from contracting the edge (u,v) .

```
01 def contract( G, e, u ):
02     H=Graph(G)
03     u_nbrs=G.neighbors(u)
04     H.delete_vertex(u)
05     H.add_edges( [ (e,v) for v in u_nbrs ] )
06     return H
```

We now present a function that receives as input a plane triangulation G and a face of G . The output of this function consists of a list of 3 lists of vertices, which we can think of as the 3 orders of a standard 3-dimensional representation.

```
01 def get_realizers( G, face ):
02     e1,e2,e3=face
03     if len(G) == 3:
04         return [[e3,e2,e1],
05                 [e1,e3,e2],
06                 [e1,e2,e3]]
07
08     e1_nbrs=set(G.neighbors(e1))
09
10     for u in e1_nbrs:
11         if u==e2 or u==e3:
12             continue
13         common_nbrs=e1_nbrs.intersection( G.neighbors( u ) )
14         if len( common_nbrs )==2:
15             break
16
17     r1,r2,r3 = get_realizers( contract(G,e1,u), face )
18
19     r1.insert(r1.index(e1),u)
20
21     c1,c2 = common_nbrs
22     r2.insert( min([r2.index(c1), r2.index(c2)])+1, u )
23     r3.insert( min([r3.index(c1), r3.index(c2)])+1, u )
```


24

```
25     return [r1,r2,r3]
```

The function `get_realizers` is recursive. Our aim is to obtain a standard 3-dimensional representation of G having the vertices of `face` as exterior vertices. We begin by naming the vertices of `face` `e1`, `e2` and `e3`. The base case is when the input is a 3-cycle, and in this case we return three lists, which we may consider as orders. These 3 orders define a standard 3-dimensional representation of G and are shown in lines 04–06. If it is the case that the graph has more than 3 vertices, then we proceed to find a neighbor of `e1`, which we call `u`, so that `u` is not a vertex in `face` and has exactly two neighbors in common with `e1`. This is done in lines 08–16. The recursive step is done in 17, we apply this same procedure to the graph resulting from contracting the edge `(e1,u)` in G with the same `face` as argument. The variables `r1`, `r2` and `r3` define a standard 3-dimensional representation of this graph, and as justified in Proposition 4.2.4, these linear orders are extended so that the resulting ones define a representation for the current graph.

We now refer to the graph G we defined in Section 6.1 to obtain a standard 3-dimensional representation of its vertex set. As exterior face we will use `[8,13,14]`. The output we get is

```
sage: get_realizers(G,[8,13,14])
[[14, 13, 11, 7, 4, 2, 12, 1, 10, 5, 6, 3, 0, 9, 8],
 [8, 14, 11, 10, 1, 9, 0, 2, 3, 12, 4, 5, 6, 7, 13],
 [8, 13, 7, 0, 9, 6, 5, 4, 3, 12, 2, 1, 10, 11, 14]]
```

6.3 Barycentric Embedding

In this section we show a function we implemented that obtains a barycentric embedding of a plane triangulation.

We start by showing the code of two functions that work together to obtain the barycentric coordinates.

```
01 def barycentric_coordinates( u, triangles, R ):
02     pos=[0,0,0]
03     for t in triangles:
04         for i in [0..2]:
05             o=R[i]
06             if o.index( u ) >= max([ o.index(v) for v in t ]):
07                 pos[ i ] += 1
08                 break
09     return tuple(pos)
10
11 def barycentric_position( G ):
12     if not G.is_planar( set_embedding=True ) or 3*len(G)-6!=G.num_edges():
13         return {}
14     triangles = faces( G )
15     R= get_realizers( G, triangles[0] )
```

```

16     triangles=triangles[1:]
17
18     barycentric_pos={}
19     for u in G:
20         barycentric_pos[ u ]= barycentric_coordinates( u, triangles, R
21     )
22     return barycentric_pos

```

The function `barycentric_coordinates` counts the number of triangles in each of the regions $R_i(u)$, to achieve this, we iterate over a the list of interior triangles and check in which order u is above all of the vertices of a given triangle. The function `barycentric_position` receives as input a graph G and checks if it is a planar triangulation. If it is, it proceeds to obtain the faces of the graph, for this we used the function `faces` written by D. Joyner and available to the public at http://trac.sagemath.org/sage_trac/ticket/6236. We then obtain a representation for the vertex set using the first face in the list of faces. After this is done, we discard the first face to obtain a list of interior faces. Finally, we iterate over the set of vertices calling the function `barycentric_coordinates` to obtain the coordinates of each vertex.

Now, we can use these functions to obtain a drawing of G using these coordinates.

```

01 def plot_triangulation( G ):
02     pos = barycentric_position( G )
03
04     vt= [(0,0), ( 1,0), (1/2,sqrt(3)/2)]
05     convex_comb = lambda c: tuple( [ sum([ c[j]*vt[j][i] for j in [0..2]])
06 for i in [0..1] ] )
07     cartesian_pos = dict( [( v, convex_comb( pos[v] )) for v in G ] )
08     return G.plot( pos = cartesian_pos )

```

The function `plot_triangulation` can be used to indicate Sage to produce the drawing shown in Figure 6.2. The drawn graph G that is drawn is the same we defined in Section 6.1.

6.4 Embedding From Canonical Ordering

We now show the code for an implementation of Algorithm 3.2.3. We call this function `canonical_positions`. The input in this case is a plane triangulation G and a canonical ordering of its vertices. The output consists of a dictionary which gives a coordinate for each of the vertices of G .

```

01 def canonical_positions( G, o ):
02     pos={}
03     pos[o[0]]=( 0, 0 )
04     pos[o[1]]=( 2, 0 )
05     pos[o[2]]=( 1, 1 )

```

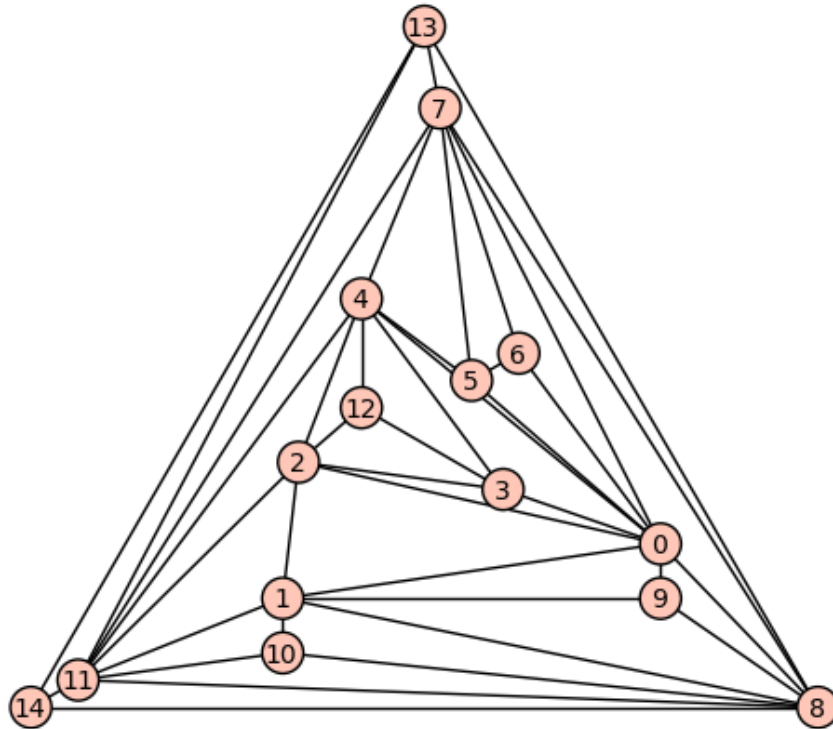


Figure 6.2: Embedding using barycentric coordinates.

```

06 L={}
07 L[ o[0] ]=set([o[0]])
08 L[ o[1] ] =set([o[1]])
09 L[ o[2] ] =set([o[2]])
10
11 P=[ o[0], o[2], o[1] ]
12
13 for k in [3..len(o)-1]:
14     w=o[ k ]
15     l= min( [P.index( u ) for u in G.neighbors(w) if u in P ] )
16     r= max( [P.index( u ) for u in G.neighbors(w) if u in P ] )
17
18     for v in Union( [ L[ u ] for u in P[l+1:r] ] ):
19         x,y=pos[v]
20         pos[v] = (x+1,y)
21     for v in Union( [ L[ u ] for u in P[r:] ] ):
22         x,y=pos[v]
23         pos[v] = (x+2,y)
24     u=P[l]
25     v=P[r]
26     xl,yl= pos[u]

```

```

27     xr,yr= pos[v]
28     pos[ w ] = ( (xl+xr+yr-yl)/2,(xr-xl+yr+yl)/2 )
29     Lw=[set( [ w ] )]
30     Lw.extend( [ L[ u ] for u in P[l+1:r] ] )
31     L[w] = Union( Lw )
32     newP=P[:l+1]
33
34     newP.append( w )
35     newP.extend(P[r:])
36     P=newP
37     return pos

```

We know from Proposition 3.2.2 that any order of a standard 3-dimensional representation is a canonical ordering of the vertices of G . So, as an example, we will show the output obtained by using the graph G defined in Section 6.1 and

$$[8, 14, 11, 10, 1, 9, 0, 2, 3, 12, 4, 5, 6, 7, 13],$$

which is the second order in the output we got from the example from Section 6.2.

```

sage: canonical_positions(G, [8, 14, 11, 10, 1, 9, 0, 2, 3, 12, 4, 5,
6, 7, 13])
{0: (7, 5), 1: (19, 3), 2: (17, 6), 3: (14, 7), 4: (15, 9), 5: (14,
10), 6: (13, 11), 7: (13, 12), 8: (0, 0), 9: (7, 4), 10: (19, 2), 11:
(24, 1), 12: (15, 8), 13: (13, 13), 14: (26, 0)}

```

We can request from Sage a drawing of G using these positions by issuing the following command.

```

sage: G.plot( pos=canonical_positions(G, [8, 14, 11, 10, 1, 9, 0, 2,
3, 12, 4, 5, 6, 7, 13]) )

```

The output given by Sage is shown in Figure 6.3.

6.5 Simplicial Complexes in Sage

The aim in this section is to present several functions related to some of the results obtained in Chapter 5. Simplicial complexes are implemented in Sage, and they can be constructed given a list of maximal faces.

As we did before, we will use a list of lists to represent d -dimensional representations. We begin by showing a function that obtains the complex $\Sigma(R)$ for a k -dimensional representation R .

```

01 def complex( R ):
02     V=Set( R[0] )
03     d=len(R)

```

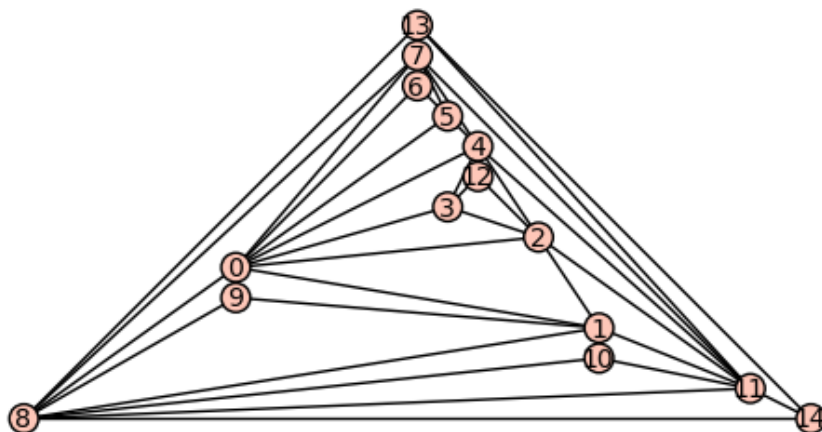


Figure 6.3: Embedding using canonical ordering.

```

04  facets=[]
05
06  for X in V.subsets( d ):
07      flag=False
08      for v in V:
09          if v not in supremum_section( R, X.union( Set([v]) ) ):
10              flag=True
11              break
12  if flag:
13      continue
14  facets.append( list(X) )
15  return SimplicialComplex( facets )

```

In order to define the complex of R it suffices to obtain a list of its maximal faces. We obtain such list by iterating over each d -subset of the vertex set and check if it satisfies the condition from Definition 5.3.1. Note that we use the fact that a d -representation does not induce a face of dimension d . We can also observe that a function called `supremum_section` is used. Such a function can be easily implemented in Sage given the k -dimensional representation R and the subset X . We can just iterate over R and at each step find which element of X is above the rest.

We have also implemented a function that obtains a geometric realization of the complex, as described in Theorem 5.2.11. Since the position of the vertices only depends on the representation, the function `realizer_pos` only takes receives one argument.

```

01 def component( R, v, factor, index ):
02     denom=sum([factor^R[ii].index(v) for ii in [0..len(R)-1]])
03     return float(factor^R[ index ].index( v )/denom)
04
05 def realizer_pos( R ):

```

```

06     d= len(R)
07     pos = dict( [ (v, tuple([component(R, v,d+1,i) for i in [0..d-1] ]
) ) for v in R[ 0 ] ] )
08     return pos

```

Another function that was implemented is one that obtains the order labelling of the complex induced by a d -representation, given the complex and the representation itself. This function calls an independent function, called `get_labels`, which gets as input a maximal face of the complex and the representation, it outputs a. By using this function, `get_order_labelling` just iterates over each maximal face

```

01 def get_labels( facet, R ):
02     labels={}
03     for i in [0..len(R)-1]:
04         o=R[ i ]
05         maximal=o[ max( [o.index( v ) for v in facet ] ) ]
06         labels[ maximal ] = i
07     return labels
08
09 def get_order_labelling( C, R ):
10     order_labelling={}
11     for facet in C.facets():
12         order_labelling[ facet ] = get_labels( facet, R )
13     return order_labelling

```

Now that we have a function that can assign each vertex of the complex a position, we may use Sage's plotting functions to obtain some insight into the case where $d = 4$. This was implemented in the function `plot_complex_dim4`.

```

01 def plot_complex_dim4( R ):
02     C=complex(R)
03     pos = realizer_pos( R )
04     d=len(R)
05     vt= [(0,0,1), (0.943,0,0.333), (-0.471,0.816,0.3333), (-0.471, -0.816,
0.3333)]
06     convex_comb = lambda c: tuple( [ sum([ c[j]*vt[j][i] for j in [0..3]])
for i in [0..2] ] )
07     cartesian_pos = dict( [( v, convex_comb( pos[v] )) for v in R[0] ]
)
08
09     P=Graphics()
10     for triangle in C.n_faces( 2 ):
11         P = P + polygon3d( [ cartesian_pos[v] for v in triangle ], alpha=0.1,
aspect_ratio=[1,1,1], color='green' )
12     for edge in C.n_faces( 1 ):

```

```

13     P = P + line3d( [ cartesian_pos[ v ] for v in edge ], aspect_ratio=[1,1,1],
color='black',thickness=2 )
14     for vertex in C.vertices():
15         P = P + point3d( cartesian_pos[ vertex], size= 8, aspect_ratio=[1,1,1],
color='red' )
16     return P

```

As an example, we issued the following commands.

```

sage: R = [[2, 7, 6, 5, 3, 4, 1], \
.....:      [2, 7, 1, 4, 3, 5, 6], \
.....:      [2, 6, 1, 3, 4, 5, 7], \
.....:      [7, 6, 1, 3, 5, 4, 2]]
sage: plot_complex_dim4( R )

```

This produces the illustration shown in Figure 6.4

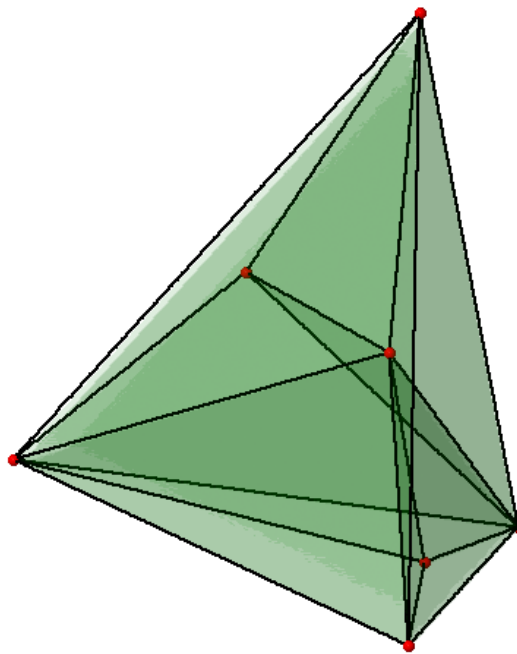


Figure 6.4: The complex induced by R.

Since we have an implementation of a function that obtains the order labelling of the complex induced by a representation, we may also observe the sequence of triangles whose existence was proved in Proposition 5.3.11. As we mentioned before, the set of triangles is partitioned into 6 equivalence classes, and this is illustrated in Figure 6.5. This family of plots was obtained with a modified version of the function `plot_complex_dim4`.

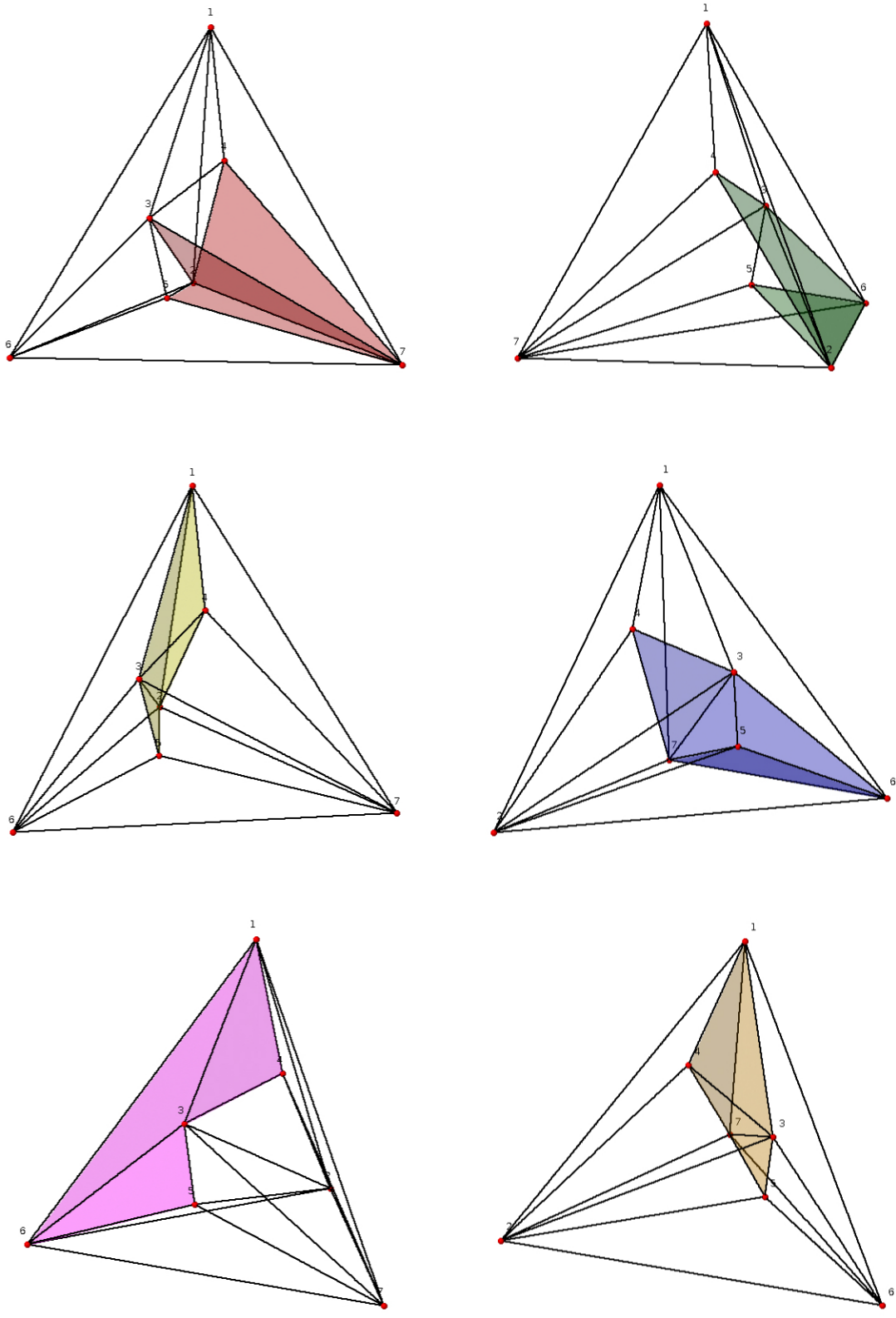


Figure 6.5: The complex induced with triangles colored by type.

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