

# On Transcendence of Irrationals with Non-eventually Periodic $b$ -adic Expansions

by

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## **Authors Declaration**

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

## Abstract

It is known that almost all numbers are transcendental in the sense of Lebesgue measure. However there is no simple rule to separate transcendental numbers from algebraic numbers. Today research in this direction is about establishing new transcendence criteria for new families of transcendental numbers.

By applying a recent refinement of Subspace Theorem, Boris Adamczewski and Yann Bugeaud determined new transcendence criteria for real numbers which we shall present in this thesis. Published only three years ago, their articles explore combinatorial, algorithmic and dynamic approaches in discussing the notion of complexity of both continued fraction and  $b$ -adic expansions of a certain class of real numbers. The condition on the expansions are those of being stammering and non-eventually periodic. Taking together these articles give a well-structured picture of the interrelationships between sequence characteristics of expansion (i.e. complexity, periodicity, type of generator) and algebraic characteristics of number itself (i.e. class, transcendency).

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## Dedication

This thesis is dedicated to the memory of my grandmother.

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# Chapter 1

## Introduction

It is known that almost all numbers are transcendental in the sense of Lebesgue measure. However there is no simple rule to separate transcendental numbers from algebraic numbers. Today's research in this direction is about establishing new transcendence criteria for new families of transcendental numbers.

I have been studying the most recent results of application of Schmidt's Subspace Theorem to establish new transcendence criteria of real numbers made by Boris Adamczewski and Yann Bugeaud. Published only three years ago, their articles explore combinatorial, algorithmic and dynamic approaches in discussing the notion of complexity of both continued fraction and  $p$ -adic expansions of real numbers. The condition on the expansions are those of being stammering and non-eventually periodic. These articles give a well-structured picture of the relationships between sequence characteristics of expansion (e.g. complexity, periodicity, type of generator) and algebraic characteristics of the number itself (e.g. class, transcendency).

Despite the fact that all of the articles share the same goal, to separate algebraic and transcendental numbers, they may be distinguished. The first approach [AB07] involves combinatorial and quantitative criteria to state that irrational automatic numbers and irrational numbers with sublinear complexity are transcendental. The second work [AB05] shows that the continued fraction expansion of an algebraic number of degree  $\geq 3$  can't be generated by recurrent or binary morphisms. The third paper [AB06] is devoted to palindromic numbers as one of the cases when real and  $p$ -adic numbers sharing the same expansion are rational if and only if they are algebraic. Due to Adamczewski and Bugeaud we can include in the list of such numbers stammering, morphic and automatic (numbers whose continued fraction or  $b$ -adic expansion is generated by stammering sequences, recurrent morphism or finite automata).



The main aim of my thesis is to construct a dialogue between different applications of Schmidt's Subspace Theorem in establishing new transcendence criteria. The thesis is organized as follows. In the first chapter we give an overview of the pre-history and evolution of Schmidt's Subspace Theorem as well as its application to study Norm Form equations and recurrence sequences. Three different approaches to complexity notion are discussed in the second chapter. Each part recalls some definitions and states the main results from the above mentioned articles. The third Chapter is devoted to technical details of applications of the Subspace Theorem.

# Chapter 2

## The Schmidt Subspace Theorem: pre-history and following evolution

### 2.1 The Schmidt Subspace Theorem

The history of the fabulous Schmidt Subspace Theorem [Sch72] goes back to the year 1955 when Roth [Rot55] established his original result on diophantine approximation. In a series of papers published between 1965 and 1972, Schmidt worked on an extension of Roth's techniques until he proved his theorem and obtained a simultaneous approximation result.

**Theorem. 1 (Roth)**

*Suppose  $\alpha$  is real and algebraic of degree  $d \geq 2$  then for each  $\delta > 0$  the inequality*

$$\left| \alpha - \frac{p}{q} \right| < q^{-(2+\delta)}$$

*has only finitely many solutions in rationals  $p/q$ .*

This result asserts that real algebraic numbers cannot be too well approximated by rational numbers.

An effective generalization of this theorem could be done in at least three directions: instead of considering an ordinary absolute value of the real number field we can take any absolute value of  $\mathbb{Q}(\alpha)$ ; we can choose the field of approximants  $\mathbb{Q}(\alpha)$  to be any number field; or we can involve different valuations in the approximation process of set of points by an element of  $\mathbb{Q}(\alpha)$ .

Ridout [Rid58] gave a generalization of Roth's theorem that yields the same result for real algebraic numbers approximated by rationals with respect to different valuations.

**Theorem. (Ridout)**

Let  $\{p_1, \dots, p_l\}$ ,  $\{q_1, \dots, q_m\}$  be two disjoint sets of prime numbers  $0 < \lambda, \mu < 1$ . Let  $P = P(p_1, \dots, p_l, \lambda)$ ,  $Q = Q(q_1, \dots, q_m, \mu)$  be the sets of integers defined by

$$P = \{p = p_1^{a_1} \dots p_l^{a_l} p^* : p^* \leq p^{1-\lambda}\},$$

$$Q = \{q = q_1^{b_1} \dots q_m^{b_m} q^* : q^* \leq q^{1-\mu}\}.$$

Then for every real algebraic number  $\alpha \neq 0$ , every finite set  $v_1, \dots, v_n$  of ultrametric places of  $\mathbb{Q}(\alpha)$  and every  $\epsilon > 0$ , the inequality

$$\left| \alpha - \frac{p}{q} \right|_{\prod_{i=1}^n v_i} < q^{-(2+\lambda+\mu-\epsilon)}$$

has only finitely many solutions.

While Roth's Theorem considers rational approximations to a given algebraic point on the line, the Subspace Theorem deals with approximations to given hyperplanes in higher dimensional space, defined over the field of algebraic numbers, by means of rational points in that space. So one can say that the Subspace Theorem is a higher dimensional generalization of Roth's theorem:

Let us denote the norm of  $\mathbf{x}=(x_1, \dots, x_n)$  as  $\|\mathbf{x}\| := \max |x_i| : i = 1, \dots, n$ .

**Theorem. 2 (Schmidt Subspace Theorem)**

Suppose  $L_1(\mathbf{x}), \dots, L_n(\mathbf{x})$  are linearly independent linear forms in  $\mathbf{x}$  with algebraic coefficients. Given  $\delta > 0$ , there are finitely many proper linear subspaces  $T_1, \dots, T_w$  of  $\mathbb{R}^n$  such that every integer point  $\mathbf{x} \neq 0$  with

$$|L_1(\mathbf{x}) \dots L_n(\mathbf{x})| < \|\mathbf{x}\|^{-\delta}$$

lies in one of these subspaces.

Roth's theorem is the special case of the Subspace Theorem when  $n = 2$ ,  $\mathbf{x} = (q, p)$  and  $\|\mathbf{x}\|$  is the ordinary absolute value on  $\mathbb{R}^2$ . Take  $L_1(q, p) = q$  and  $L_2(q, p) = q\alpha - p$ , then  $|L_1(\mathbf{x})L_2(\mathbf{x})| < \|\mathbf{x}\|^{-\delta}$  corresponds to  $|q| |q\alpha - p| < (p^2 + q^2)^{-\delta/2}$ , or  $\left| \alpha - \frac{p}{q} \right| < (p^2 + q^2)^{-\delta/2} q^{-2} < q^{-\delta-2}$ . By the Subspace Theorem the solution  $\mathbf{x}$  lies in a finite

number of proper linear subspaces of  $\mathbb{R}^2$ , i.e., rational subspaces of dimension 1. Thus the set  $p/q$  is finite and Roth's theorem follows.

However, one significant drawback of this theorem is that it is ineffective; the proof of the theorem does not enable us to determine the subspaces  $T_1, \dots, T_w$ . The next significant step in the evolution of Subspace Theorem was done by Schmidt in 1989, when he gave an explicit upper bound for the number  $w$  of subspaces. The quantitative version involves the notion of heights and normalized absolute values on number fields, which needs to be introduced before stating the result.

Let  $K$  be an algebraic number field. Denote its ring of integers by  $O_K$  and its collection of places (equivalent classes of absolute values) by  $M_K$ . For  $v \in M_K, x \in K$  we define the absolute value  $|x|_v$  by

- (i)  $|x|_v = |\sigma(x)|^{1/[K:\mathbb{Q}]}$  if  $v$  corresponds to the embedding  $\sigma : K \hookrightarrow \mathbb{R}$ ;
- (ii)  $|x|_v = |\sigma(x)|^{2/[K:\mathbb{Q}]} = |\bar{\sigma}(x)|^{2/[K:\mathbb{Q}]}$  if  $v$  corresponds to the pair of conjugate complex embeddings  $\sigma, \bar{\sigma} : K \hookrightarrow \mathbb{C}$ ;
- (iii)  $|x|_v = (N_\varphi)^{-\text{ord}_\varphi(x)/[K:\mathbb{Q}]}$  if  $v$  corresponds to the prime ideal  $\varphi$  of  $O_K$ .

For any set  $X$  let  $|X|$  denote the cardinality of  $X$ . Then  $N_\varphi = |(O_K/\varphi)|$  is the norm of  $\varphi$  and  $\text{ord}_\varphi(x)$  is the exponent of  $\varphi$  in the prime ideal decomposition of  $(x)$ , with  $\text{ord}_\varphi(x) := \infty$ . In case (i) or (ii) we call  $v$  real infinite or complex infinite, respectively; in the case (iii) we call  $v$  finite. The absolute values satisfy the **Product formula**

$$\prod_{v \in M_K} |x|_v = 1 \text{ for } x \in K \setminus \{0\}$$

**Definition** We define the  $K$ -**height** of  $x \in K$  to be

$$\mathfrak{N}_K(x) = \prod_{v \in M_K} \max\{1, |x|_v\}$$

Observe that  $\mathfrak{N}_\mathbb{Q}(x) = |x|$  (the usual absolute value) for  $x \in \mathbb{Z}$  and that

$$\mathfrak{N}_L(x) = \mathfrak{N}_K(x)^{[L:K]}$$

for  $x \in K$  and for a finite extension  $L$  of  $K$ .

**Definition** The **height** of  $\mathbf{x}=(x_1, \dots, x_n) \in K^n$  with  $\mathbf{x} \neq 0$  is defined as follows

$$|\mathbf{x}|_v = \max |x_i|_v, v \in M_K.$$

Now define

$$\mathfrak{N}(\mathbf{x}) = \prod_{v \in M_K} \max\{1, |\mathbf{x}|_v\}.$$

Again, in the special case  $\mathbf{x} \in \mathbb{Z}^n$ , we have  $\mathfrak{N}(\mathbf{x}) = \|\mathbf{x}\|$ . Moreover, we define another height  $H$  by taking Euclidean norms at the infinite places, namely

$$H(\mathbf{x}) = \prod_{v \in M_K} \max\{1, |\mathbf{x}|_{v,2}\},$$

$$\text{where } |\mathbf{x}|_{v,2} = \left( \left( \sum_{i=1}^n |x_i|_v^2 \right)^{\frac{1}{2}} \right)^{d(v)}$$

for  $v$  infinite,  $|\mathbf{x}|_{v,2} = |\mathbf{x}|_v$  for  $v$  finite, and  $d(v) = \frac{1}{[K:\mathbb{Q}]}$  or  $\frac{2}{[K:\mathbb{Q}]}$  depending on whether  $v$  is real infinite or complex infinite, respectively. Note that for infinite places  $v$ ,  $|\cdot|_v$  is a power of the Euclidean norm.

Now if  $L = a_1x_1 + \dots + a_nx_n$  is a nonzero linear form with algebraic coefficients we put  $H(L) = H(\mathbf{a})$ ,  $\mathbf{a} = (a_1, \dots, a_n)$ .

The quantitative version of Schmidt's Subspace Theorem [Sch89] is as follows

**Theorem. 3**

*Suppose  $L_1(\mathbf{x}), \dots, L_n(\mathbf{x})$  are linearly independent linear forms in  $\mathbf{x} = (x_1, \dots, x_n)$  with algebraic coefficients in some algebraic number field  $K$  of degree  $d = [K : \mathbb{Q}]$ . Consider the inequality*

$$|L_1(\mathbf{x}) \dots L_n(\mathbf{x})| < |\det(L_1, \dots, L_n)| |\mathbf{x}|^{-\delta} \tag{2.1}$$

where  $0 < \delta < 1$ .

*Then the set of solutions of inequality (2.1) with*

$$\mathbf{x} \in \mathbb{Z}^n, \|\mathbf{x}\| \gg \max \left\{ (n!)^{8/\delta}, H(L_1), \dots, H(L_n) \right\},$$

*is contained in the union of at most  $w = (2d)^{2^{26n\delta-2}}$  proper linear subspaces of  $\mathbb{R}^n$ .*

In terms of giving the upper bound for the number of linear proper subspaces, the next result after Schmidt's is due to Vojta [Voj89]. Essentially, it says that apart from finitely many extensions, which are dependent on  $\delta$ , the solutions of (2.1) are

in the union of finitely many proper linear subspaces of  $\mathbb{R}^n$  which are independent of  $\delta$ . Moreover, Ru and Vojta formulated the Subspace Theorem with “moving targets”, where the linear forms  $L_1, \dots, L_n$  from (2.5) vary in a small range.

One of the most significant contributions to the development of the Subspace Theorem was done by Schlickewei. In 1977 Schlickewei [Sch77b] extended Subspace Theorem of 1972 to the  $p$ -adic case and to number fields. In 1990, he generalized Schmidt’s quantitative version of the Subspace Theorem to the  $p$ -adic case over  $\mathbb{Q}$ , and later in 1992, to number fields. The version of the Subspace Theorem due to Schlickewei is the one that was mainly used by Adamczewski and Bugeaud, it will be stated in Chapter 3.

It should be mentioned that the best version (considering the dependency on the parameters) of the quantitative Subspace Theorem is due to Evertse [Eve96]. Recently, Evertse and Schlickewei proved a much more general result involving all algebraic numbers and not only those lying in a fixed number field [ES99, ES02].

The powerful tool - Subspace Theorem - and its generalizations and extensions can be adapted to numerous branches of Number Theory. It’s application to Diophantine inequalities started with papers “Simultaneous approximation of algebraic numbers by rationals” [Sch70, Sch91] and “Approximation of algebraic numbers by algebraic numbers of bounded degree” [Wir69]. Application of Subspace Theorem to Diophantine equations gave finiteness results for  $S$ -unit equations [DR76, Sch77b, vdPS82]. Some explicit upper bounds for the number of solutions of linear equations in variables which lie in a multiplicative group derived in [ESS02]. Considering applications of Subspace Theorem to norm form equations and decomposable form equations, numerous results could be found in [Sch72, Gy1, Eve95, EGGR02], important research papers in linear recurrence sequences are [vdPS91, Sch99, CZ98]. We concentrate a little on norm form equations and linear recurrence sequences in the following pages.

## 2.2 Norm form equations

For any set  $\alpha_1 = 1, \alpha_2, \dots, \alpha_m$  of linearly independent algebraic numbers over  $\mathbb{Q}$  let us define the algebraic field extension  $K$  by  $K = \mathbb{Q}(\alpha_1, \dots, \alpha_m)$  with degree  $n = [K : \mathbb{Q}]$ ,  $m \leq n$ , and denote  $\mathbb{Q}$ -isomorphisms of  $K \hookrightarrow \mathbb{C}$  as  $\sigma_1 = id, \sigma_2, \dots, \sigma_n$  be the . For any element  $\alpha \in K$ , denote the action  $\sigma_i(\alpha) = \alpha^{(i)}$ . Consider the linear forms  $L^{(i)}(\mathbf{x}) = \alpha_1^{(i)}x_1 + \dots + \alpha_m^{(i)}x_m$  for  $i = 1, \dots, n$ . Then exists a non-zero rational integer  $a_0$  such that the form

$$F(\mathbf{x}) = a_0 N_{K/\mathbb{Q}}(\alpha_1 x_1 + \dots + \alpha_m x_m) = a_0 \prod_{i=1}^n L^{(i)}(\mathbf{x}) = a_0 \aleph(L(\mathbf{x}))$$

will be contained in  $\mathbb{Q}[x]$ .

**Definition** A **norm form** is any form  $F$  of the type  $F(\mathbf{x}) = a_0 \aleph(L(\mathbf{x}))$  for some  $L$  as above and  $a_0 \in \mathbb{Q}^\times$ .

Let  $p_1, \dots, p_s$  be rational primes with  $s \geq 1$ , and  $b \in \mathbb{Z} \setminus \{0\}$  a rational integer which is relatively prime to  $p_1, \dots, p_s$  if  $s > 1$ .

**Definition** The equation

$$F(\mathbf{x}) = bp_1^{z_1} \dots p_s^{z_s} \tag{2.2}$$

in  $\mathbf{x} \in \mathbb{Z}^m$ , and  $z_1, \dots, z_s \in \mathbb{Z}_{\geq 0}$  with  $(x_1, \dots, x_m, p_1, \dots, p_s) = 1$  if  $s > 1$  is called a **norm form equation** or more precisely a norm form equation of Mahler type where  $F$  is a norm form.

While  $\mathbf{x}$  runs through the integer points in  $\mathbb{Z}^m$  and  $\alpha_1, \dots, \alpha_m$  are fixed elements of an algebraic number field  $K$ , the linear expression  $L(\mathbf{x})$  runs through a  $\mathbb{Z}$ -module  $M$  of rank  $m$  with basis  $\alpha_1, \dots, \alpha_m$ . So we could rewrite the norm form equation (2.2) as

$$a_0 \aleph(\mu) = bp_1^{z_1} \dots p_s^{z_s},$$

where  $\mu \in M$ .

Let  $\mathbb{Q}M$  be the set of products  $q\mu$  with  $q \in \mathbb{Q}, \mu \in M$ . Then  $\mathbb{Q}M$  consists of  $\alpha_1 x_1 + \dots + \alpha_m x_m$  with  $x_i \in \mathbb{Q}$  ( $i = 1, \dots, m$ ). Let  $E$  be a subfield of  $K$  and let  $M^E$  be the set of  $\mu$  such that  $\lambda\mu \in \mathbb{Q}M$  for every  $\lambda \in E$ . Then  $M^E$  is a submodule of  $M$ . If  $E \subseteq E'$ , then  $M^{E'} \subseteq M^E$  and we have  $M^{\mathbb{Q}} = M$ .

**Definition** The  $\mathbb{Z}$ -module  $M$  is called **non-degenerate** if  $K$  has no subfield  $E \neq \mathbb{Q}$  and  $E$  not imaginary quadratic. Otherwise  $M$  is called **degenerate**.

Degeneracy is important, for if  $M$  is degenerate there is a non-zero integer  $b$  for which (2.2) has infinitely many solutions for  $s = 0$ . Before justifying this last remark, we will consider the simplest case which has infinitely many solutions.

Suppose  $\alpha_1, \dots, \alpha_m$  form an integral basis for  $K$  and put  $n = m$ . Then the norm form equation  $\aleph(L(\mathbf{x})) = 1$  turns into the equation  $\aleph(\epsilon) = 1$  where  $\epsilon = \alpha_1 x_1 + \dots + \alpha_n x_n$ . Thus  $\epsilon$  is a unit and from the Dirichlet's Unit Theorem it follows that we have infinitely many solutions unless  $K$  is  $\mathbb{Q}$  or imaginary quadratic.

In general, suppose that there exists a subfield  $E$  with  $M^E \neq 0$ . In this case it is known that the set  $D_M^E$  of elements  $\nu \in E$  such that  $\nu\mu \in M$ , for every  $\mu \in M$  is an order in  $E$ . That is,  $D_M^E$  is in fact a subring of  $E$  which contains 1. The ring  $D_M^E$  is called a coefficient ring of  $\aleph$ . The group  $\epsilon_M^E$  of units of  $D_M^E$  of norm  $\aleph_{E/\mathbb{Q}}(\epsilon) = 1$  is a multiplicative group which is a subgroup  $C_M$  of index 1 or 2 (depending on whether  $\aleph(\epsilon) = -1$ ) in the group of all units of  $D_M^E$ . Hence, by the generalization of Dirichlet's Unit Theorem to orders,  $C_M$  is infinite unless  $E = \mathbb{Q}$  or  $E$  is imaginary quadratic.

Now let  $\mu$  be a solution of (2.2) and denote the set of elements  $\mu\epsilon$  where  $\epsilon \in C_M$  as  $\mu C_M$ . Then for every  $\epsilon \in C_M$  we have that  $\aleph(\mu\epsilon) = \aleph(\mu)$  and  $\mu\epsilon \in M^E$ . Thus every element of the coset  $\mu C_M$  is again a solution of (2.2). We call  $\mu C_M$  a **family of solutions** of (2.2).

Thus the norm form equation  $\aleph(\mu) = b$ ,  $\mu \in M^E$  has infinitely many solutions if  $M$  is degenerate. So the condition of non-degeneracy is necessary in establishing the finiteness of the number of solutions.

An application of his Subspace Theorem gave Schmidt [Sch71] the finiteness of numbers of solutions of (2.2) in the case  $s = 0$  if  $M$  is non-degenerate. The generalization of this result to the case  $s > 0$  is due to Schlickewei [Sch77a]. After Schmidt [Sch72] derived a finiteness of solutions of (2.2) when  $s = 0$  in terms of families of solutions, Schlickewei [Sch77a] again extended his result to equation (2.2). Further generalization of this theorem to an arbitrary algebraic number fields was done by Laurent [Lau84].

We remark that these results are ineffective. An effective finiteness results for norm form equations where  $\alpha_k$  has degree at least 3 over the field  $\mathbb{Q}(\alpha_1, \dots, \alpha_{k-1})$  for  $k = 1, \dots, m$  were established by Györy and Papp [Gy0, Gy1]. These results were proved with assumption of  $x_m \neq 0$  and under condition that  $M$  is degenerate. Thus, it could not be derived from results of Schmidt and Schlickewei. Under the same assumption of  $x_m \neq 0$  of Györy's theorem Evertse and Györy [EG85] proved that the number of solutions of (2.2) is bounded above by

$$(4 \cdot 7^{g(2s+2\omega(b)+3)})^{m-1}$$



where  $g$  is the degree of the normal closure of  $K$  over  $\mathbb{Q}$ , and  $\omega(b)$  counts the number of distinct prime factors of  $b$ .

Considering the case of non-degenerate  $M$  and assumption of  $s = 0$  Schmidt applied his quantitative version of the Subspace Theorem [Sch90] to derive a uniform upper bound for the number of solutions of (2.2). An upper bound of family of solutions of equation (2.2) is a nice result of Györy [Gy3], appeared as a clever consequence of a more general theorem in decomposable form equations. For the case  $b = 1$  the bound  $(2^{33}n^2)^{m^3(s+1)}$  is due to Evertse [Eve95] (he used an assumption that  $\alpha_k$  has degree at least 3 over the field  $\mathbb{Q}(\alpha_1, \dots, \alpha_{k-1})$  for  $k = 1, \dots, m$ ) Under this assumption Evertse and Györy together [EG97] established the bound

$$(2^{33}n^2)^{e(m)(s+1)} \cdot \psi_m(b)$$

for the number of solutions, where  $e(m) = \frac{1}{3}m(m+1)(2m+1) - 2$  and

$$\psi_m(b) = \binom{n}{m-1}^{\omega(b)} \prod_{p|b, \text{prime}} \binom{p^{(b)} + m - 1}{m-1}$$

In the non-degenerate case the upper bound will be  $(2^{33}n^2)^{e(m)(s+\omega(b)+1)}$ .

## 2.3 Linear recurrence sequences

**Definition Linear recurrence sequence** of order  $t$  is a sequence  $\{u_k\}_{k \in \mathbb{Z}_{\geq 0}}$  of complex numbers where each term is a linear combination of the  $t$  preceding terms with fixed coefficients  $c_0 \dots c_{t-1}$  not all equal to zero:

$$u_{k+t} = c_{t-1}u_{k+t-1} + \dots + c_0u_k, \quad (k \in \mathbb{Z}_{\geq 0}) \quad (2.3)$$

Let us assume that for a given sequence  $\{u_k\}_{k \in \mathbb{Z}_{\geq 0}}$  we have fixed initial values  $u_0 \dots u_{t-1}$  not all equal to zero. Suppose that  $t$  is minimal. The companion polynomial of relation (2.3) is defined as

$$P(\mathbf{x}) = \mathbf{x}^t - c_{t-1}\mathbf{x}^{t-1} - \dots - c_0 = \prod_{i=1}^r (\mathbf{x} - \alpha_i)^{d_i} \quad (2.4)$$

where zeros  $\alpha_1, \dots, \alpha_r$  are all distinct.

Given (2.3) and (2.4) and initial values  $u_0 \dots u_{t-1}$ , the sequence  $\{u_k\}_{k \in \mathbb{Z}_{\geq 0}}$  can be represented as the sum of polynomials  $p_i$  of degree  $d_i - 1$  ( $i = 1, \dots, r$ )

$$u_k = \sum_{i=1}^r p_i(k) \alpha_i^k$$

Solving the equation

$$u_k = 0, \quad (k \in \mathbb{Z}_{\geq 0}) \quad (2.5)$$

is a well-known problem. Similarly, in the case of the norm form equation, the problem of giving an algorithm to determine explicitly the solutions  $k$  of (2.5) seems to be currently out of reach.

### **Theorem. 4 (Skolem-Mahler-Lech)**

*The set of solutions  $k$  of equation (2.5) is the union of finitely many arithmetic progressions and a finite set (possibly empty).*

Recall that a nondegenerate sequence  $\{u_k\}_{k \in \mathbb{Z}_{\geq 0}}$  is a sequence, such that for the roots  $\alpha_i$  of the companion polynomial  $P(\mathbf{x})$  in (2.4) none of the quotients  $\alpha_i/\alpha_j$  ( $i \neq j$ ) is a root of unity. By the condition of nondegeneracy the set of solutions of (2.5) cannot contain an arithmetic progression thus finiteness of equation (2.5) is it is an easy consequence of the Skolem-Mahler-Lech theorem.

Schmidt not only gave an effective bound for the number of solutions of (2.5), but extends the theorem of Skolem-Mahler-Lech to the level of quantitative independency

of any assumption on nondegeneracy. Moreover, the upper bound from Schmidt's result [Sch99] is completely uniform, in other words does not depend on initial conditions  $u_0, \dots, u_{t-1}$  of a sequence  $\{u_k\}_{k \in \mathbb{Z}_{\geq 0}}$  or coefficients  $c_0, \dots, c_{t-1}$  from recurrence relation (2.3).

**Theorem. 5**

*Let  $\{u_k\}_{k \in \mathbb{Z}_{\geq 0}}$  be a linear recurrence sequence of order  $t$  (possibly degenerate). Then the set  $M$  of solutions  $k \in \mathbb{Z}_{\geq 0}$  of equation (2.5), i.e., of  $u_k = 0$ , consists of finitely many arithmetic progressions  $P_1, \dots, P_{t_i}$  and of a finite set  $M_1$  such that*

$$c(t) = t_i + (M_1) \leq \exp(\exp(\exp(20t))).$$

*In particular for any non-degenerate sequence  $\{u_k\}_{k \in \mathbb{Z}_{\geq 0}}$  equation (2.5) does not have more than  $c(t)$  solutions.*

This result was improved by Patrick Brodie Allen in his Master's [All07] in 2006. In particular, he showed that one may take  $c(t) \leq (\exp(\exp(t^{\sqrt{11}t})))$ .

# Chapter 3

## Notions of complexity. Transcendence criteria

### 3.1 Combinatorial approach

#### 3.1.1 Combinatorial transcendence criteria

In two papers, the first [Bor09] published in 1909 and the second [Bor50] in 1950, Borel studied the  $b$ -ary expansion of real numbers, where  $b > 2$  is a positive integer. By his conjecture it is expected that every irrational algebraic number is normal in base  $b$ . In particular that could mean that if the  $b$ -ary expansion of an irrational number is, in some sense, “simple”, then it is either quadratic or transcendental. The term “simple” has many interpretations. It may denote real numbers whose expansions have some regularity, or can be produced by a simple algorithm (see 3.2 Chapter 3), or arise from a simple dynamical system.(see 3.3 Chapter 3)

In this chapter, we shall focus on the combinatorial properties of the  $b$ -adic expansions of irrational algebraic numbers. The first step towards a proof that these numbers are normal in base  $b$ , if true, would be the following conjecture.

#### **Conjecture. 6**

*If a real irrational number  $\alpha$  is clearly abnormal, in the sense that for some  $b \geq 2$ , its  $b$ -ary expansion strongly differs from a normal sequence, then it is transcendental.*

This principle uses the notion of normality which in the combinatorial area usually is associated with the following notion of complexity.

The usual measure of the complexity of an infinite word  $\mathbf{u} = u_1u_2\dots$  defined on a finite alphabet is obtained by counting the number  $p(n)$  of distinct blocks of length  $n$  occurring in the word  $\mathbf{u}$ . In particular, if a real number is **normal** in base  $b \geq 2$  then its  $b$ -adic expansion on the alphabet  $\{0, 1, \dots, b - 1\}$  satisfies  $p(n) = b^n$  for any positive integer  $n$ . Normality of the number means that each one of the  $b^n$  blocks of length  $n$  occurs in the  $b$ -adic expansion of the number occurs with the frequency  $1/b^n$ . The function  $p$  is commonly called the **complexity function**. Periodic sequences have bounded (finite) complexity, while non-periodic have infinite complexity. For example quadratic numbers have a periodic continued fraction expansion whose complexity is bounded. Also the complexity of the decimal expansion of rationals is finite as their expansion is eventually periodic.

Although Borel's conjecture is still open, some partial results have been achieved. First the transcendence of the real numbers whose  $b$ -adic expansion behaves as a Sturmian sequence was established by Ferenczi and Mauduit [FM97] from a clever reformulation of a theorem of Ridout [Rid57].

**Definition** A **Sturmian sequence** (on a binary alphabet) is a non eventually periodic sequence of minimal complexity, i.e. that satisfies  $p(n) = n + 1$  for every  $n \geq 1$ . More generally, the sequence  $\mathbf{u}$  is **Sturmian** on  $l$  letters if  $p(n) = n + l - 1$ .

In other words on the alphabet of cardinality  $l$  for a Sturmian sequence  $p(1) = l$ ,  $p(n + 1) - p(n) = 1$  for any  $n \geq 1$ , which means the existence of one word of length  $n$  that is a prefix of two different words of length  $n + 1$  and each of the other words of length  $n$  is a prefix of one and only one word of length  $n + 1$ .

The method that was proposed by Ferenczi and Mauduit [FM97] relies on a combinatorial translation of a result of Ridout [Rid57] which claims that if the  $b$ -ary expansion of a number contains infinitely many  $(2+\epsilon)$ -times repeated concatenations of words (that is, a word followed by itself and then by its beginning of relative length at least  $\epsilon$ ), at distances from the beginning which are comparable with lengths of the considered words, then it is transcendental. Then the transcendence of the number with the Sturmian-behaved expansion is a consequence of this criterion and of the combinatorial properties of Sturmian sequences.

Before stating this properties, we need to introduce some notation from combinatorics on words.

Let  $A$  be a given set, not necessarily finite. The length of a word  $W$  on the alphabet  $A$ , that is, the number of letters making up  $W$ , is denoted by  $|W|$ . For any positive integer  $l$ , we write  $W^l$  for the word  $W \dots W$  ( $l$  times repeated concatenation of the word

$W$ ). More generally, for any positive rational number  $x$ , we denote by  $W^x$  the word  $W^{\lceil x \rceil} W^{\lceil x \rceil - x}$ ,  $W^{\lceil x \rceil}$  is the prefix of  $W$  of length  $\lceil (x - [x])|W| \rceil$ . Here, and in all that follows,  $[y]$  and  $\lceil y \rceil$  denote, respectively, the integer part and the upper integer part of the real number  $y$ .

Let  $k \geq 2$  be an integer. To any sequence  $\mathbf{u}$  on any finite alphabet  $A = \{0, \dots, k-1\}$  we associate the real number whose expansion in base  $k$  is  $0.u_0u_1 \dots u_n \dots$ , namely

$$S_k(\mathbf{u}) := \sum_{n=0}^{+\infty} \frac{u_n}{k^{n+1}}$$

**Theorem. (Ferenczi and Mauduit)**

*If  $\theta$  is an irrational number and for every  $n \in \mathbb{N}$  the expansion of  $\theta$  in base  $k$  begins with  $0.U_nV_nV_nV'_n$ , where  $U_n$  is a possibly empty and  $V_n$  is a nonempty word on an alphabet  $A = \{0, \dots, k-1\}$ ,  $V'_n$  is a prefix of  $V_n$ ,  $|V_n| \rightarrow +\infty$ ,  $\limsup(|U_n|/|V_n|) < +\infty$  and  $\liminf(|V'_n|/|V_n|) > 0$ , then  $\theta$  is a transcendental number.*

*Proof.* Let  $r_n = |U_n|$ ,  $s_n = |V_n|$ , and choose  $0 < \epsilon < \liminf(|V'_n|/|V_n|)$ . Let  $t_n$  be the rational number whose expansion in base  $k$  is eventually periodic and repeats prefix of expansion of  $\theta$ , that is  $0.U_nV_n \dots V_n \dots$ , then

$$t_n = \frac{a_n}{k^{r_n}} + \frac{b_n}{k^{s_n}} + \frac{b_n}{k^{s_n^2}} + \frac{b_n}{k^{s_n^3}} + \dots = \frac{a_n}{k^{r_n}} + \frac{b_n(k^{-s_n})}{1 - k^{-s_n}} = \frac{a_n}{k^{r_n}} + \frac{b_n}{k^{s_n} - 1}$$

Thus,

$$t_n = \frac{p_n}{k^{r_n}(k^{s_n} - 1)}$$

for some integer  $p_n$ . For  $n$  large enough the approximation of  $\theta$  by rationals  $t_n$  is evaluated as

$$|\theta - t_n| \leq \frac{1}{k^{r_n + (\epsilon + 2)s_n}}$$

Now, suppose  $\theta$  were an algebraic irrational. Then, from a theorem of Ridout [Rid57], if there exist infinitely many rational numbers  $P_n/Q_n$ , with  $Q_n = k^{m_n}Q'_n$  (the numbers  $k$ ,  $m_n$  and  $Q'_n$  being integers), such that

$$\left| \frac{P_n}{Q_n} - \theta \right| < c_1(Q_n)^{-\rho}$$

and

$$Q'_n < c_2(Q_n)^\mu,$$

where  $c_1$  and  $c_2$  are positive constants, then  $\rho \leq 1 + \mu$ . Since  $\liminf \frac{s_n}{r_n + s_n} = \frac{1}{\limsup(r_n/s_n) + 1} > 0$ , and up to restricting  $n$  to a strictly increasing sequence of integers, one can suppose that  $\frac{s_n}{r_n + s_n} \rightarrow \eta > 0$ . In particular, there exist two numbers  $\rho$  and  $\mu$  such that, for all  $n$  in some infinite set,

$$1 + \frac{s_n}{r_n + s_n} < 1 + \mu < \rho < 1 + (1 + \epsilon) \frac{s_n}{r_n + s_n}.$$

This choice of  $\rho$  and  $\mu$  together with the choice  $P_n = p_n, Q_n = k^{r_n}(k^{s_n} - 1), m_n = r_n$ , and  $Q'_n = k^{s_n} - 1$  gives us the desired contradiction. Hence  $\theta$  is transcendental.

*Q.E.D.*

The transcendence in the Sturmian case is then a consequence of this criterion and of the combinatorial properties of Sturmian sequences.

**Theorem. (Ferenczi, Mauduit)**

*If there exists  $k$  such that the expansion of  $\theta$  in base  $k$  is a Sturmian sequence, then  $\theta$  is a transcendental number.*

*Proof.* We begin with two lemmas that we will not prove.

**Lemma 1** *If  $\mathbf{u}$  is Sturmian on  $l$  letters and not recurrent,  $\mathbf{u}$  is ultimately equal to a Sturmian recurrent sequence on  $l' < l$  letters.*

**Lemma 2** *If  $\mathbf{u}$  is a recurrent Sturmian sequence and there exist two words  $W_0$  and  $W_1$ , and a sequence of integers  $a_n \geq 1, n \geq 1$ , such that if the words  $W_n, n \in \mathbb{N}$ , are given by the recursion formulas*

$$W_{n+1} = W_n^{a_n} W_{n-1}$$

*for  $n \geq 1$ , then, for any  $N \geq 1$  and  $n \geq 1$ , the word  $u_0 u_1 \dots u_{N-1}$  is of the form  $X_0 X_1 \dots X_k$ , where  $X_1, X_2, \dots, X_{k-1}$  are equal either to  $W_n$  or to  $W_{n+1}$ ,  $X_0$  is a (possibly empty) suffix of either  $W_n$  or  $W_{n+1}$ ,  $X_k$  is a (possibly empty) prefix of either  $W_n$  or  $W_{n+1}$ . This decomposition, which is not unique, is independent of  $N$  for fixed  $n$ .*

As the transcendence does not depend on the initial values of  $\mathbf{u}$ , it is enough, because of Lemma 1, to prove our claim if  $\theta = S_k(\mathbf{u})$  for a recurrent Sturmian sequence  $\mathbf{u}$ . Let then  $a_n$  and  $W_n$  be as in Lemma 2.

Then, for each  $n$ , a suitable initial segment of  $\mathbf{u}$  is  $X_0X_1\dots X_{k-1}$  as in Lemma 2;  $X_0$  is either a suffix of  $W_n$ , denoted by  $T_n$ , or a suffix of  $W_{n+1}$ , which may be a suffix of  $W_{n-1}$ , denoted again by  $T_n$ , or is of the form  $T_nW_n^{c_n}W_{n-1}$  with  $T_n$  a suffix of  $W_n$  and  $0 \leq c_n \leq a_n$  an integer (every considered suffix may be empty). Then the first  $b_n$  words among  $X_1, \dots, X_{k-1}$  are  $W_n$  for some integer  $b_n \geq 0$ , and then comes one  $W_{n+1}$  (if not,  $\mathbf{u}$  would be ultimately periodic).

Hence, for every  $n$ ,  $\mathbf{u}$  begins by either

(1) the word  $T_nW_n^{b_n+a_n}W_{n-1}$

(2) the word  $T_nW_n^{c_n}W_{n-1}$ , where  $T_n$  a suffix of  $W_n$  or of  $W_{n-1}$  and  $b_n$  and  $c_n$  are non-negative integers. Let  $q_n$  be the length of  $W_n$ , satisfying  $q_{n+1} = q_n a_n + q_{n-1}$ . Then:

- if, for infinitely many  $n$ , the case (2) occurs with  $c_n \geq 3$ , last combinatorial criterion applied for this sequence with  $U_n = T_n$  and  $V_n = V'_n = W_n$  yields the transcendence of  $S_k(\mathbf{u})$ ;

- if not, we take  $U_n$  to be  $T_n$  in case (1) and  $T_nW_n^{c_n}W_{n-1}$  in case (2), so we have ultimately (i.e., for each  $n$  large enough)  $|U_n| \leq 5q_n$ . And

If  $a_n + b_n \geq 3$  for infinitely many  $n$ , we take  $V_n = W_n$  and apply criterion of Ferenczi and Mauduit with  $V'_n = V_n$ , which yields the result;

If  $a_n + b_n \leq 2$  ultimately but  $a_n + b_n = 2$  infinitely often, then  $q_{n-1} \geq q_n/3$  ultimately, and last combinatorial criterion with  $V_n = W_n$  and  $V'_n = W_{n-1}$  yields the result;

Finally, in the remaining case we must have  $b_n = 0$  and  $a_n = 1$  ultimately. In this case, where the reader will recognize the Fibonacci recursion, we have also  $W_nW_{n-1} = W_{n-1}W_{n-1}W_{n-4}W_{n-3}$ , and we apply combinatorial criterion of Ferenczi and Mauduit with  $V_n = W_{n-1}$  and  $V'_n = W_{n-4}$ , as  $|V_n| = q_{n-1}$  is then larger than  $|U_n|/10$  and smaller than  $8|V'_n|$ .

*Q.E.D.*

Later the following combinatorial criterion given in [FM97] has been used to establish some new examples of transcendental numbers with low complexity [AC03, AZ98, AZ00, RZ00].

**Criterion. 7**

*Let  $b \geq 2$  be an integer. The complexity function of the  $b$ -adic expansion of every irrational algebraic number satisfies*

$$\liminf_{n \rightarrow \infty} (p(n) - n) = +\infty. \tag{3.1}$$



The first result from [AB07] due to Adamczewski and Bugeaud is a considerable improvement of (3.1).

**Criterion. 8**

*Let  $b \geq 2$  be an integer. The complexity function of the  $b$ -adic expansion of every irrational algebraic number satisfies*

$$\liminf_{n \rightarrow \infty} \frac{p(n)}{n} = +\infty.$$

A straightforward conclusion from Criterion 8 is that if the  $b$ -adic expansion of an irrational real number has sublinear complexity (i.e., such that  $p(n) = O(n)$ ) then it is transcendental.

Actually, we are able to deal also, under some conditions, with non-integer bases. Given real  $\beta > 1$ , we can expand in base  $\beta$  every real  $\xi = a_n \dots a_0.a_{-1} \dots a_{-m} \dots$  as due to the  $\beta$ -expansion of  $\xi$ , introduced by Rényi [R57]

$$\xi = \beta^n a_n + \beta^{n-1} a_{n-1} + \dots + a_0 + \beta^{-1} a_{-1} + \dots + \beta^{-m} a_{-m} + \dots$$

**Definition** A **Pisot** (resp. Salem) number is a real algebraic integer bigger than 1, whose conjugates lie inside the open unit disc (resp. inside the closed unit disc, with at least one of them on the unit circle).

For example, any integer  $b \geq 2$  is a Pisot number. In the same article [AB07] by Adamczewski and Bugeaud we see the following result.

**Theorem. 9**

*Let  $\beta > 1$  be a Pisot or a Salem number. The complexity function of the  $\beta$ -expansion of every algebraic number in  $(0, 1) \setminus \mathbb{Q}(\beta)$  satisfies*

$$\liminf_{n \rightarrow \infty} \frac{p(n)}{n} = +\infty.$$

### 3.1.2 Transcendence criterion for stammering sequences

Different variations of transcendence criterion for stammering sequences are used in all 3 papers as the main tool to prove general results.

Let  $\mathbf{a} = \{a_k\}_{k \geq 1}$  be a sequence of elements from  $A$ , that we identify with the infinite word  $a_1 a_2 \dots a_l \dots$ . Let  $w$  be a rational number with  $w > 1$ .

**Definition** We say that  $\mathbf{a}$  satisfies **Condition  $(*)_w$**  if  $\mathbf{a}$  is not eventually periodic and if there exists a sequence of finite words  $\{V_n\}_{n \geq 1}$  such that:

- (i) For any  $n \geq 1$ , the word  $V_n^w$  is a prefix of the word  $\mathbf{a}$ ;
- (ii) The sequence  $\{|V_n|\}_{n \geq 1}$  is increasing.

In other words,  $\mathbf{a}$  satisfies **Condition  $(*)_w$**  if  $\mathbf{a}$  is not eventually periodic and if there exist infinitely many 'non-trivial' repetitions (the size of which is measured by  $w$ ) at the beginning of the infinite word  $\mathbf{a}$ .

Thus here we introduce a class of sequences with a particular combinatorial property: many repetitions close to the beginning of the word.

**Definition** A sequence satisfying **Condition  $(*)_w$**  for some  $w > 1$  is a **stammering sequence**.

In [AB07] one can find a transcendence criterion for the real number  $\alpha$  whose  $\beta$ -adic expansion forms a stammering sequence. This criterion is used further in the paper as a key fact to establish the main result.

#### **Theorem. 10**

*Let  $\beta > 1$  be a Pisot or a Salem number. Let  $\mathbf{a} = \{a_k\}_{k \geq 1}$  be a bounded sequence of rational integers. If there exists a real number  $w > 1$  such that  $\mathbf{a}$  satisfies **Condition  $(*)_w$** , then the real number  $\alpha := \sum_{k=1}^{+\infty} \frac{a_k}{\beta^k}$  either belongs to  $\mathbb{Q}(\beta)$ , or is transcendental.*

The proof of Theorem 10 rests on the Schmidt Subspace Theorem [Sch72], and more precisely on a  $p$ -adic generalization due to Schlickewei [Sch92] and Evertse [Eve96]. The particular case when  $\beta \geq 2$  is an integer was proved in [ABL04] by Adamczewski, Bugeaud and Luca. Also Adamczewski [Ada04] proved that, under a stronger assumption on the sequence  $\mathbf{a} = \{a_k\}_{k \geq 1}$ , that the number  $\beta$  defined in the statement of Theorem 10 is transcendental.

Theorem 10 is considerably stronger than the criterion of Ferenczi and Mauduit [FM97]: here the assumption  $w > 1$  replaces their assumption  $w > 2$ . The conclusion

of Theorem 10 also holds if the sequence  $\mathbf{a}$  is an unbounded sequence of integers that does not increase too rapidly. However, the Mahler method, when applicable, gives the transcendence of the infinite series  $\sum_{k=1}^{+\infty} a_k \beta^{-k}$  for every algebraic number  $\beta$  such that this series converges. It is possible to get a transcendence criterion for an algebraic number  $\beta$  which is neither a Pisot nor a Salem number using the approach followed for proving Theorem 10.

In the [AB05] Adamczewski and Bugeaud are concerned with the same question that was conjectured by Borel, but in the case when the  $b$ -adic expansion of  $\alpha$  is replaced by its sequence of partial quotients. Recall that the eventually periodic continued fraction expansion of an irrational number  $\alpha$  could be only in case if  $\alpha$  is a quadratic irrational. However, still not much known about bounds on growth of partial quotients of algebraic numbers with degree strictly greater than 2. The strong belief that sequence of partial quotients of such a number is unbounded was based on some generalizations from numerous examples and a conjecture that these numbers share behavior with most numbers in this respect. The first question about the existence of a limit on the growth of the sequence of partial quotients of the continued fraction expansion of all real non-quadratic irrational algebraic numbers was considered by Khintchin [Khi49]. Despite the fact that since that time no example is known, the condition of sequence bounding takes an important place in stating transcendence criterion for continued fractions in all past results [ADQZ01, Dav02, Bax04], in which, roughly speaking, the size  $w$  of the repetition is required to be all the more large than the partial quotients are big. Unlike these results, the new transcendence criterion established by Adamczewski and Bugeaud can be easily applied even if  $\alpha$  has unbounded partial quotients.

In 1844 Liouville first applied transcendence theory to studying numbers given by their continued fraction expansions are due to Liouville in 1844. He constructed an infinite class of real transcendental numbers using continued fractions whose sequence of partial quotients grows too fast for the number to be algebraic. Later, in the 1850s, he gave a necessary condition for a number to be algebraic, and thus a sufficient condition for a number to be transcendental.

Liouville's criterion essentially says that algebraic numbers cannot be very well approximated by rational numbers. In other words if a number can be very well approximated by rational numbers then it occurs to be transcendental, which relates to the certain exponent of approximation. He showed that if  $\alpha$  is an algebraic number of degree  $d \geq 2$  and  $\epsilon > 0$  is any number, then the expression

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{d+\epsilon}}$$

can be satisfied by only finitely many rational numbers  $p/q$ .

In the twentieth century this topic was extensively developed by Maillet [Mai06]. He showed in 1906, amongst other things, that there exist transcendental numbers whose continued fractions have bounded partial quotients.

Deeper results could be found in the work by Thue [Thu09], Siegel [Sie21], and Roth [DR55] who reduced the exponent in Liouville's work from  $d + \epsilon$  to  $d/2 + 1 + \epsilon$ , and finally, in 1955, to  $2 + \epsilon$  respectively. This result, known as the Thue-Siegel-Roth theorem, is best possible, since if the exponent  $2 + \epsilon$  is replaced by just 2 then the result is no longer true.

Roth's work effectively ended the work started by Liouville, and his theorem allowed mathematicians to prove the transcendence of many more numbers, such as the Champernowne constant. The theorem is still not strong enough to detect all transcendental numbers, though, and many famous constants including  $e$  and  $\pi$  either are not or are not known to be very well approximable in the above sense [Mah53].

Further investigations in the approximation of algebraic irrationals by rationals are due to Baker in 1962 and 1964 [Bak64] where he showed that  $\sqrt[3]{2}$  cannot be too well approximated. The approximation results on real numbers by quadratic numbers due to Schmidt in 1967 [Sch67] are a main tool for the next steps by Davison in 1989 [Dav89], and by Queffelec [Que98] who established in 1998 the transcendence of the Prouhet-Thue-Morse continued fraction. In 2001, Allouche, Davison, Queffelec and Zamboni [ADQZ01] proved the transcendence of Sturmian continued fractions, continued fractions whose sequence of partial quotients is Sturmian. The transcendence of the Rudin-Shapiro and of the Baum-Sweet continued fractions was proved in 2006 by Adamczewski, Bugeaud and Davison [ABD06].

Finally in 2005, Adamczewski and Bugeaud showed that the continued fraction expansion of an algebraic number of degree at least three cannot be generated by a binary morphism. The main results of [AB05] are two new combinatorial transcendence criterion, which considerably improve upon previous results.

The transcendence criterion for "purely" stammering continued fractions was stated in [AB05] as follows.

**Theorem. 11**

*Let  $\mathbf{a} = \{a_l\}_{l \geq 1}$  be a sequence of positive integers. Let  $\{p_l/q_l\}_{l \geq 1}$  denote the sequence of convergents to the real number  $\alpha := [0; a_1, a_2, \dots, a_l, \dots]$ .*

*If there exists a rational number  $w \geq 2$  such that  $\mathbf{a}$  satisfies Condition  $(*)_w$ , then  $\alpha$  is transcendental.*

If there exists a rational number  $w > 1$  such that  $\mathbf{a}$  satisfies Condition  $(*)_w$ , and if the sequence  $\{q_l^{1/l}\}_{l \geq 1}$  is bounded (which is in particular the case when the sequence  $\mathbf{a}$  is bounded), then  $\alpha$  is transcendental.

Theorem 11 consists of two statements. The first has no condition on the growth of the sequence  $\{q_l^{1/l}\}_{l \geq 1}$ . The second statement improves upon Theorem 4 from [ADQZ01], which requires, together with some extra rather constraining hypotheses, the stronger assumption  $w > 3/2$ .

To avoid situations when repetitions are restricted to appear at the very beginning of the sequence we can use results from [Dav02], where shifts are allowed, however the length must be controlled in terms of the size of the repetitions.

**Definition** Let  $w$  and  $w'$  be nonnegative rational numbers with  $w > 1$ . We say that  $\mathbf{a}$  **satisfies Condition  $(*)_{w,w'}$**  if  $\mathbf{a}$  is not eventually periodic and if there exist two sequences of finite words  $\{U_n\}_{n \geq 1}$ ,  $\{V_n\}_{n \geq 1}$  such that:

- (i) For any  $n \geq 1$ , the word  $U_n V_n^w$  is a prefix of the word  $\mathbf{a}$ ;
- (ii) The sequence  $\left\{ \frac{|U_n|}{|V_n|} \right\}_{n \geq 1}$  is bounded from above by  $w'$ ;
- (iii) The sequence  $\{|V_n|\}_{n \geq 1}$  is increasing.

We are now ready to state a transcendence criterion for (general) stammering continued fractions from the [AB05] due to Adamczewski and Bugeaud.

**Theorem. 12**

Let  $\mathbf{a} = \{a_l\}_{l \geq 1}$  be a sequence of positive integers. Let  $\{p_l/q_l\}_{l \geq 1}$  denote the sequence of convergents to the real number  $\alpha := [0; a_1, a_2, \dots, a_l, \dots]$ . Assume that the sequence  $\{q_l^{1/l}\}_{l \geq 1}$  is bounded and set  $M = \limsup_{l \rightarrow +\infty} q_l^{1/l}$  and  $m = \liminf_{l \rightarrow +\infty} q_l^{1/l}$ . Let  $w$  and  $w'$  be non-negative real numbers with

$$w > (2w' + 1) \frac{\log M}{\log m} - w'. \tag{3.2}$$

If  $\mathbf{a}$  satisfies Condition  $(*)_{w,w'}$ , then  $\alpha$  is transcendental.

An immediate consequence of Theorem 12 is the following.

**Corollary. 13**

Let  $\mathbf{a} = \{a_l\}_{l \geq 1}$  be a sequence of positive integers. Let  $\{p_l/q_l\}_{l \geq 1}$  denote the sequence of convergents to the real number

$$\alpha := [0; a_1, a_2, \dots, a_l, \dots].$$

Assume that the sequence  $\{q_l^{1/l}\}_{l \geq 1}$  converges. Let  $w$  and  $w'$  be non-negative real numbers with  $w > w' + 1$ . If  $\mathbf{a}$  satisfies Condition  $(*)_{w, w'}$ , then  $\alpha$  is transcendental.

It is not hard to check that Theorem 12 improves Theorem 6.3 of Davison [Dav02]. Indeed, to apply his transcendence criterion,  $w$  and  $w'$  must satisfy

$$w > (2w' + \frac{3}{2}) \frac{\log M}{\log m}.$$

which is stronger condition than (3.2).

Theorems 11 and 12 yield many new results that could not be obtained with the earlier transcendence criterion.

**Theorem. 14**

Let  $b \geq 2$  be an integer. Let  $\mathbf{a} = \{a_l\}_{l \geq 1}$  be a sequence of integers in  $\{0, \dots, b-1\}$ . Let  $w$  and  $w'$  be non-negative rational numbers with  $w > 1$ . If  $\mathbf{a}$  satisfies Condition  $(*)_{w, w'}$ , then the real number  $\sum_{l \geq 1} a_l/b^l$  is transcendental.

Theorem 14 works for both “purely” and general stammering sequences, i.e., cases when the repetitions do not occur too far away from the beginning of the infinite word. Again, the main tool for the proofs of Theorems 11 and 12 is the Schmidt Subspace Theorem. This (more precisely, a  $p$ -adic version of it) is also the key auxiliary result for establishing Theorem 14.

### 3.1.3 Transcendence criterion for palindromic sequences

In this section we introduce a class of sequences, called palindromic sequences, enjoying another combinatorial property: the appearing of infinitely many symmetric patterns not far from the beginning of the sequence. This combinatorial property is described via the notion of reversal. Note that continued fractions involving similar sequences were previously considered in [ABL04]. The following theorem published in 2004 gave a new way of identifying transcendental numbers.

**Theorem. (Adamczewski, Bugeaud, Luca)**

*Let  $\alpha$  be irrational, and suppose its expansion on the alphabet  $\{0, 1, \dots, 9\}$  satisfies condition  $(*)_{w,w'}$ . Then,  $\alpha$  is transcendental.*

A strong connection between palindromic and stammering sequences is proved in [AB06], where Adamczewski and Bugeaud presented how initially palindromic sequences under the certain assumption that the prefix of word has an excess of repetitions, thus, actually behaves as stammering sequence.

As above, we use the terminology from combinatorics on words. Let  $A$  be a finite set. The length of a finite word  $W$  on the alphabet  $A$ , that is, the number of letters composing  $W$ , is denoted by  $|W|$ . The reversal (or the mirror image) of  $W := a_1 \cdots a_n$  is the word  $\overline{W} := a_n \cdots a_1$ . In particular,  $W$  is a **palindrome** if and only if  $W = \overline{W}$ . We identify any sequence  $\mathbf{a} = \{a_n\}_{n \geq 1}$  of elements from  $A$  with the infinite word  $a_1 a_2 \cdots a_n \cdots$ .

**Definition** An infinite sequence  $\mathbf{a}$  is called a **palindromic sequence** if there exist real numbers  $w, w'$  and three sequences of finite words  $\{U_n\}_{n \geq 1}$ ,  $\{V_n\}_{n \geq 1}$ , and  $\{W_n\}_{n \geq 1}$  such that

- (i) for any  $n \geq 1$ , the word  $W_n U_n V_n \overline{U_n}$  is a prefix of the word  $\mathbf{a}$ ;
- (ii) the sequence  $\left\{ \frac{|V_n|}{|U_n|} \right\}_{n \geq 1}$  is bounded from above by  $w$ ;
- (iii) the sequence  $\left\{ \frac{|W_n|}{|U_n|} \right\}_{n \geq 1}$  is bounded from above by  $w'$ ;
- (iv) the sequence  $\{|U_n|\}_{n \geq 1}$  is increasing.

In other words, a palindromic sequence has the property that infinitely many symmetric patterns (i.e., the words  $U_n V_n \overline{U_n}$ ) occur not too far from its beginning.

When the word  $W_n$  from the definition of a palindromic sequence is empty for any  $n \geq 1$ , i.e. if  $w' = 0$  we call this type of palindromic sequence as initial palindromic sequence.

**Definition** An infinite sequence  $\mathbf{a}$  is called an **initially palindromic sequence** if there exist a real number  $w$  and two sequences of finite words  $\{U_n\}_{n \geq 1}, \{V_n\}_{n \geq 1}$  such that

- (i) for any  $n \geq 1$ , the word  $U_n V_n \overline{U_n}$  is a prefix of the word  $\mathbf{a}$ ;
- (ii) the sequence  $\left\{ \frac{|V_n|}{|U_n|} \right\}_{n \geq 1}$  is bounded from above by  $w$ ;
- (iii) for any  $n \geq 1$ ,  $|U_{n+1}| \geq (w + 2) |U_n|$  ;

Numerous examples of classical sequences in word combinatorics, such as Sturmian sequences, the Thue-Morse sequence, or paperfolding sequences, turn out to be palindromic. Palindromic sequences should be compared with stammering sequences introduced in section 2.1.2: in the first we have some excess of symmetry, while in second we have some excess of periodicity. We say that a real (resp.,  $p$ -adic) number is a palindromic number if its expansion in some base  $b \geq 2$  (resp., its Hensel expansion) is a palindromic sequence.

The main result from the [AB06] due to Adamczewski and Bugeaud is a new transcendence criterion for palindromic sequences.

**Theorem. 15**

*Let  $p$  be a prime number and let  $\mathbf{a} = \{a_k\}_{k \geq 1}$  be a palindromic sequence on the alphabet  $(0, 1, \dots, p - 1)$ . Then, the numbers*

$$\alpha := \sum_{k=1}^{+\infty} \frac{a_k}{p^k}, \quad \alpha_p := \sum_{k=1}^{+\infty} a_k p^k \tag{3.3}$$

*are both algebraic if and only if both are rationals.*

In transcendence theory, many results assert that at least one number among some finite list is transcendental. Apparently, Theorem 15 is the first result of this type which deals with an Archimedean number and a non-Archimedean one.



## 3.2 Algorithmic approach

In this section we discuss how the complexity of expansion of real numbers can be interpreted in an algorithmic way. Here the notion of complexity of sequences will be related to the notion of complexity of the algorithm implemented on a certain type of machine chosen to act as a sequence generator. In computational complexity theory the main criterion in classifying computational problems in terms of complexity is their inherent difficulty. To measure the difficulty of solving the problem, we focus on the running time (or the number of steps) that the machine spends to complete the problem of size  $n$  (by any way of measuring of amount of input information). We do not focus on the particular algorithm from the beginning. Among all possible of them we choose the fastest first; its running time will give the complexity of the problem. Thus the time required to solve the problem (i.e., the relationship between size of input and total steps of solving) can be expressed as a function  $T(n)$  of the size of input, that is our **complexity function**. Since the time taken on different inputs of the same size can be different, the worst-case time complexity  $T(n)$  is defined to be the maximum time taken over all inputs of size  $n$ .

In 1965, Hartmanis and Stearns [HS65] proposed an approach for the notion of complexity of real numbers, by involving the quantitative aspect of the notion of calculability introduced by Turing [Tur37].

**Definition** A real number is said to be **computable in time  $T(n)$**  if there exists a multitape Turing machine which gives the first  $n$  terms of its binary expansion in (at most)  $T(n)$  operations. The simpler real numbers in that sense, that is, the numbers for which one can choose  $T(n) = O(n)$ , are said to be **computable in real time**, in other words they have sublinear order (of  $n$  time) complexity. Rational numbers have this property.

The problem of Hartmanis and Stearns about the existence of irrational algebraic numbers which are computable in real time intuitively seems likely to have a negative answer if ever solved. After Cobham's [Cob68] restriction of 1968 of this problem to a particular class of Turing machines (finite automata), there were several attempts by Loxton and van der Poorten [LvdP82] in 1982, who finally proved [LvdP88] in 1988 that the restricted problem is solved. In particular, Cobham, Loxton and van der Poorten proved that the  $b$ -adic expansion of every irrational algebraic number cannot be generated by a finite automaton. Later in his paper [Bec94] Becker explained a serious gap that was found in the solution for the restricted problem [LvdP88], that was based on Mahler's method [Mah29, Mah30a, Mah30b].

### 3.2.1 Transcendence criterion for $k$ -automatic sequences

Finite automata take the bottom place in the hierarchy of Turing machines as one of the most basic simulators of machine program computation. As was mentioned above these kind of machines output “real time” produced expansions of sequences, so the complexity function is sublinear. The notion of a sequence generated by a finite automaton, (or more precisely a finite automaton with output function, i.e. a “uniform tag system”) has been introduced and studied by Cobham in 1972 (see [Cob80, Cob68]).

**Theorem. (Cobham)**

*Automatic sequences have complexity  $p(n) = O(n)$ .*

In 1980, Christol, Kamae, Mendes France and Rauzy [CKFR80] proved that a sequence with values in a finite field is automatic if and only if the related formal power series is algebraic over the rational functions with coefficients in this field. This was the starting point of numerous results linking automata theory, combinatorics and number theory.

**Definition** An infinite sequence  $\mathbf{a} = \{a_n\}_{n \geq 0}$  is said to be **generated by a  $k$ -automaton** if  $a_n$  is a finite-state function of the base- $k$  representation of  $n$ . This means that there exists a finite automaton starting with the  $k$ -ary expansion of  $n$  as input and producing the term  $a_n$  as output.

Recall some definitions from automata theory and combinatorics on words.

**Definition** Let  $k$  be an integer with  $k \geq 2$ . We denote by  $\sum_k$  the set  $\{0, 1, \dots, k-1\}$ . A  **$k$ -automaton** is defined as a 6-tuple

$$A = (Q, \sum_k, \delta, q_0, \Delta, \tau) ,$$

where  $Q$  is a finite set of states,  $\sum_k$  is the input alphabet,  $\delta : Q \times \sum_k \rightarrow Q$  is the transition function,  $q_0$  is the initial state,  $\Delta$  is the output alphabet and  $\tau : Q \rightarrow \Delta$  is the output function.

For a state  $q$  in  $Q$  and for a finite word  $W = w_1 w_2 \dots w_n$  on the alphabet  $\sum_k$ , we define recursively  $\delta(q, W)$  by  $\delta(q, W) = \delta(\delta(q, w_1 w_2 \dots w_{n-1}), w_n)$ . Let  $n \geq 0$  be an integer and let  $w_r w_{r-1} \dots w_1 w_0$  in  $(\sum_k)^{r+1}$  be the  $k$ -ary expansion of  $n$ ; thus,  $n = \sum_{i=0}^r w_i k^i$ .

**Definition** We denote by  $W_r$  the word  $w_0 w_1 \dots w_r$ . Then, a sequence  $\mathbf{a} = \{a_n\}_{n \geq 0}$  is said to be  **$k$ -automatic** if there exists a  $k$ -automaton  $A$  such that  $a_n = \tau(\delta(q_0, W_n))$  for all  $n \geq 0$ .

**Example** Due to Thue we know that any binary sequence of length 4 must contain a square, i.e., two consecutive identical blocks. He was interested in existence of infinite sequence on three letters without squares. Also he was interested in determining whether it is possible to find an infinite binary sequence that contains no cube, i.e., no three consecutive identical blocks, or even no overlap. The answer to all three questions is given by a well-known classical example of a binary automatic sequence - the Thue-Morse sequence.

This sequence begins with  $\mathbf{a} = \{a_n\}_{n \geq 0} = 0110100110010 \dots$ . It can be defined by the following rule:  $a_n$  is equal to 0 (resp. to 1) if the sum of the digits in the binary expansion of  $n$  is even (resp. is odd).

It is easy to check that this sequence can be generated by the 2-automaton

$$A = (\{q_0, q_1\}, \{0, 1\}, \delta, q_0, \{0, 1\}, \tau),$$

where  $\delta(q_0, 0) = \delta(q_1, 1) = q_0$ ,  $\delta(q_0, 1) = \delta(q_1, 0) = q_1$ , and  $\tau(q_0) = 0$ ,  $\tau(q_1) = 1$ .

**Example** A classical example of a binary automatic number is given by

$$\sum_{n=1}^{+\infty} \frac{1}{2^{2^n}}$$

which is transcendental, as proved by Kempner [Kem16].

In 1994 Becker [Bec94] established that, for any given non-eventually periodic automatic sequence  $\mathbf{u} = u_1 u_2 \dots$ , the real number  $\sum_{k \geq 1} u_k b^{-k}$  is transcendental. Criterion 8 confirms straightforwardly this conjecture, since any automatic sequence has a sub-linear complexity function. The following theorem due to Adamczewski and Bugeaud, stated as their main result and proved in [AB07], is a direct consequence of Criterion 8.

**Theorem. 16**

*Let  $b \geq 2$  be an integer. The  $b$ -adic expansion of any irrational algebraic number cannot be generated by a finite automaton. That is, irrational automatic numbers are transcendental.*

### 3.2.2 Transcendence criterion for morphic sequences

#### Theorem. 17 (Cobham)

*A sequence can be generated by a finite automaton if and only if it is generated by a uniform morphism.*

Thanks to Cobham [Cob80], we know that sequences generated by finite automata can be characterized in terms of iterations of morphisms of free monoids generated by finite sets.

**Definition** For a finite set  $A$ , we denote by  $A^*$  the free monoid generated by  $A$ . The empty word is the neutral element of  $A^*$ . Let  $A$  and  $B$  be two finite sets. A function from  $A$  to  $B^*$  can be uniquely extended to an homomorphism between the free monoids  $A^*$  and  $B^*$ . We call such a homomorphism a **morphism from  $A$  to  $B$** .

The morphism is **uniform** if all words in the image of  $A$  have the same length. Let  $\phi$  be a morphism from  $A$  into itself. Then  $\phi$  is said to be **prolongable** if there exists a letter  $a$  such that  $\phi(a) = aW$ , where  $W$  is a non-empty word such that  $\phi^k(W)$  is a nonempty word for every  $k \geq 0$ . In that case, the sequence of finite words  $\{\phi^k(a)\}_{k \geq 1}$  converges in  $A^{\mathbb{N}}$  (endowed with the product topology of the discrete topology on each copy of  $A$ ) to an infinite word  $\mathbf{a} = \lim_{k \rightarrow +\infty} \phi^k(a)$ . This infinite word is clearly a fixed point for  $\phi$  and we say that  **$\mathbf{a}$  is generated by the morphism  $\phi$** . Moreover, if every letter occurring in  $\mathbf{a}$  occurs at least twice, then we say that  $\mathbf{a}$  is generated by a recurrent morphism. If the alphabet  $A$  has two letters, then we say that  $\mathbf{a}$  is generated by a binary morphism. More generally, an infinite sequence  $\mathbf{a}$  in  $A^{\mathbb{N}}$  is said to be **morphic** if there exist a sequence  $\mathbf{u}$  generated by a morphism defined over an alphabet  $B$  and a morphism from  $B$  to  $A$  such that  $\mathbf{a} = \phi(\mathbf{u})$ .

**Example** The Fibonacci morphism  $\delta$  defined from the alphabet  $0, 1$  into itself by  $\delta(0) = 01$  and  $\delta(1) = 0$  is a binary, recurrent morphism which generates the Fibonacci infinite word.

$$\mathbf{a} = \lim_{n \rightarrow +\infty} \delta^n(0) = 010010100100101001 \dots$$

This infinite word is the most famous example of a Sturmian sequence [MH40].

**Lemma. 3 (Morse, Hedlund).** *For all  $n \geq 0$ , the Fibonacci word has exactly  $n + 1$  distinct factors of length  $n$ .*

**Definition** We say that a real number  $\alpha$  is **automatic** (respectively, generated by a morphism, generated by a recurrent morphism, or morphic) if there exists an integer  $b \geq 2$  such that the  $b$ -adic expansion of  $\alpha$  is automatic (respectively, generated by a morphism, generated by a recurrent morphism, or morphic).

Theorem 16 establishes a particular case of the following widely believed conjecture.

**Conjecture 18** *Irrational morphic numbers are transcendental.*

The method introduced in [AB07] allows one to confirm this conjecture for a wide class of morphisms.

**Theorem. 19**

*The binary expansion of algebraic irrational numbers cannot be generated by a morphism.*

Theorem 19 is much more general than that of [FM97] combined with a result of Berstel and Seebold [BS93] that binary irrational numbers which are a fixed point of a primitive morphism or of a morphism of constant length  $\geq 2$  are transcendental, as observed by Allouche and Zamboni [AZ98].

For  $b$ -adic expansions with  $b \geq 3$ , Adamczewski and Bugeaud obtained a similar result as in Theorem 19, but using an additional assumption.

**Theorem. 20**

*Let  $b \geq 3$  be an integer. The  $b$ -adic expansion of an algebraic irrational number cannot be generated by a recurrent morphism.*

However, it is not proved yet that ternary algebraic numbers (expansion on base 3 alphabet) cannot be generated by a morphism. Consider for instance the fixed point

$$\mathbf{u} = 01212212221222212222212222221222 \dots$$

of the morphism defined by  $0 \rightarrow 012, 1 \rightarrow 12, 2 \rightarrow 2$ , and set  $\alpha = \sum_{k \geq 1} u_k 3^{-k}$ . The method introduced in [AB07] by Adamczewski and Bugeaud does not apply to show the transcendence of  $\alpha$ . The transcendence of  $\alpha$  follows as a consequence of deep transcendence results proved in [Ber97] and in [DNNS96], concerning the values of theta series at algebraic points.

The main contribution from the [AB05] toward questions with morphic sequences is the following result, which fully solved a question studied by Queffelec [Que00].

**Theorem. 21**

*The continued fraction expansion of an algebraic number of degree at least three cannot be generated by a recurrent morphism.*

The following corollary is a direct consequence of Theorem 21.

**Corollary. 22**

*The continued fraction expansion of an algebraic number of degree at least three cannot be generated by a binary morphism.*

### 3.3 Dynamical approach

In this Section, we will discuss the notion of symbolic complexity of sequences from a dynamical point of view.

**Definition** Let  $A$  be a given set, finite or not. A **subshift** on  $A$  is a symbolic dynamical system  $(X, S)$ , where  $S$  is the classical shift transformation defined from  $A^{\mathbb{Z}_{\geq 1}}$  into itself by  $S(\{a_n\}_{n \geq 1}) = \{a_n\}_{n \geq 2}$  and  $X$  is a subset of  $A^{\mathbb{Z}_{\geq 1}}$  such that  $S(X) \subset X$ . With an infinite sequence  $\mathbf{a}$  in  $A^{\mathbb{Z}_{\geq 1}}$ , we associate the subshift  $\mathbf{Im}_{\mathbf{a}} = (X, S)$ , where  $X := \overline{O(\mathbf{a})}$  denotes the closure of the orbit of the sequence  $\mathbf{a}$  under the action of  $S$ . The **complexity function**  $p_{\mathbf{a}}$  of a sequence  $\mathbf{a}$  associates with any positive integer  $n$  the number  $p_{\mathbf{a}}(n)$  of distinct blocks of  $n$  consecutive letters occurring in it. More generally, the complexity function  $p_{\mathfrak{S}}$  of a subshift  $\mathfrak{S} = (X, S)$  associates with any positive integer  $n$  the number  $p_{\mathfrak{S}}(n)$  of distinct blocks of  $n$  consecutive letters occurring in at least one element of  $X$ .

With a subshift  $\mathfrak{S} = (X, S)$  on  $\mathbb{Z}_{\geq 1}$  one can associate the set  $C_{\mathfrak{S}}$  defined by

$$C_{\mathfrak{S}} = \{\alpha \in (0, 1), \alpha = [0; a_1, a_2, \dots] \text{ such that } \{a_n\}_{n \geq 1} \in \mathfrak{S}\}.$$

In particular, if a real number  $\alpha$  lies in  $C_{\mathfrak{S}}$ , then this is also the case for any  $\alpha$  in  $C_{\alpha} := \overline{\{T_n(\alpha)\}_{n \geq 0}}$ , where  $T$  denotes the Gauss map, as truncating shift operator for the continued fractions defined from  $(0, 1)$  into itself by  $T(x) = 1/x - [1/x]$  so that  $T([0; a_1, a_2, \dots]) = [0; a_2, a_3, \dots]$ .

One manner in which to examine the complexity of the continued fraction expansion of  $\alpha$  is to study the behavior of the sequence  $\{T_n(\alpha)\}_{n \geq 0}$ , precisely, we are interested in the structure of the underlying dynamical system  $(C_{\alpha}, T)$ . We can intuitively assume that there exist a strong connection between the size of  $C_{\alpha}$  and complexity is the continued fraction expansion of  $\alpha$ , in other words the complexity increases with the growth of size of  $C_{\alpha}$ .

Therefore, this assumption gives a conclusion that in case of too simple structure of the symbolic dynamical system  $\mathfrak{S}$  we can expect that no algebraic number of degree at least three appears in  $C_{\mathfrak{S}}$ .

Adamczewski and Bugeaud stated the following main question in transcendence dynamic theory: “Let  $\mathfrak{S}$  be a subshift on  $\mathbb{Z}_{\geq 1}$  with sublinear complexity, that is, whose complexity function satisfies  $p_{\mathfrak{S}}(n) \leq Mn$  for some absolute constant  $M$  and any positive integer  $n$ . Does the set  $C_{\mathfrak{S}}$  only contain quadratic or transcendental numbers?”

This question opens a wild area to research, and it is not much done yet after Morse and Hedlund [MH40] claimed that a subshift  $\mathfrak{S}$  whose complexity function satisfies  $p_{\mathfrak{S}}(n) \leq n$  for some positive integer  $n$  is periodic. In that case, it follows that  $C_{\mathfrak{S}}$  is a finite set composed only of quadratic numbers. Moreover, it is shown by Allouche in [ADQZ01] that for a Sturmian subshift  $\mathfrak{S}$  (a subshift with complexity  $p_{\mathfrak{S}}(n) = n + 1$  for every  $n \geq 1$ ), the set  $C_{\mathfrak{S}}$  is an uncountable set completely made up by transcendental numbers. The following result of Adamczewski and Bugeaud from the [AB05] slightly improves this result.

**Theorem. 23**

*Let  $\mathfrak{S}$  be a subshift on  $\mathbb{Z}_{\geq 1}$ . If the set  $C_{\mathfrak{S}}$  contains a real algebraic number of degree at least three, then the complexity function of  $\mathfrak{S}$  satisfies*

$$\lim_{n \rightarrow +\infty} p_{\mathfrak{S}}(n) - n = +\infty.$$

Classical example of subshifts with low complexity is linearly recurrent subshifts. Let  $\mathfrak{S} = (X, S)$  be a subshift and  $W$  be a finite word. The **cylinder** associated with  $\langle W \rangle$  is, by definition, the subset  $\langle W \rangle$  of  $X$  formed by the sequences that begin in the word  $W$ . A minimal subshift  $(X, S)$  is **linearly recurrent** if there exists a positive constant  $c$  such that for each cylinder  $\langle W \rangle$  the return time to  $\langle W \rangle$  under  $S$  is bounded by  $c|W|$ . The following theorem by Adamczewski and Bugeaud [AB05] solve the question stated above to this specific class of subshifts with low complexity.

**Theorem. 24**

*Let  $\mathfrak{S}$  be a linearly recurrent subshift on  $\mathbb{Z}_{\geq 1}$ . Then, the set  $C_{\mathfrak{S}}$  is composed only of quadratic or transcendental numbers.*



# Chapter 4

## Application of Schmidt Subspace Theorem

As was mentioned above, the proof of the transcendence criterion (Theorem 10) for stammering sequence rests on an improvement of Theorem 3, which is the  $p$ -adic generalization of the quantitative version of the Subspace Theorem [Lau84]. This criterion is of great importance as, after the Subspace Theorem, it plays the most important role in the proof of the general results achieved in [AB07], [AB05] and [AB06]. The main point of this chapter is to illustrate the proof of Theorem 10 given originally in [AB07]. Later in the subsequent papers [AB05] and [AB06], Adamczewski and Bugeaud gave modified criteria for stammering sequences with additional conditions on bounds  $w$  and  $w'$  for sequences from Condition  $(*)_{w,w'}$ . Proofs of those criteria repeat the same approach that is demonstrated below.

### Proof of Theorem 10

*Proof.* Let us fix the parameter  $w > 1$  and keep the conditions from Theorem 10, i.e. take sequences  $\{U_n\}_{n \geq 1}$  and  $\{V_n\}_{n \geq 1}$  denoted in Condition  $(*)_w$ . Also set  $r_n = |U_n|$ ,  $s_n = |V_n|$  for any  $n \geq 1$ . We will show that the real number

$$\alpha := \sum_{k=1}^{+\infty} \frac{a_k}{\beta^k}$$

either is in  $\mathbb{Q}(\beta)$ , or is transcendental. The general line of proof is to consider approximations to the real number  $\alpha$  in the number field  $\mathbb{Q}(\beta)$  infinitely many times by rationals obtained from  $\alpha$  by truncating its expansion and completing it by periodicity. Precisely, for any positive integer  $n$ , we define the sequence  $\{b_k^{(n)}\}_{k \geq 1}$  by

$$b_h^{(n)} = a_h \text{ for } 1 \leq h \leq r_n + s_n,$$

$$b_{r_n+h+js_n}^{(n)} = a_{r_n+h} \text{ for } 1 \leq h \leq s_n \text{ and } j \geq 0.$$

Observe that the sequence  $\{b_k^{(n)}\}_{k \geq 1}$  is eventually periodic, with pre-period  $U_n$  and with period  $V_n$ . Set our rational approximations  $\alpha_n$  as

$$\alpha_n = \sum_{k=1}^{+\infty} \frac{b_k^{(n)}}{\beta^k} \quad (4.1)$$

The difference is

$$\alpha - \alpha_n = \sum_{k=r_n+\lceil ws_n \rceil+1}^{+\infty} \frac{a_k - b_k^{(n)}}{\beta^k}$$

Note that  $|U_n V_n^w| = r_n + s_n \lfloor w \rfloor + \lceil s_n(w - \lfloor w \rfloor) \rceil = r_n + s_n(\lfloor w \rfloor + \lceil w - \lfloor w \rfloor \rceil)$  where  $(\lfloor w \rfloor + \lceil w - \lfloor w \rfloor \rceil) = \lceil w \rceil$  is exactly the part that we repeat in  $\alpha_n$ .

**Lemma 3** *For any integer  $n$ , there exists an integer polynomial  $P_n(X)$  of degree at most  $r_n + s_n - 1$  such that*

$$\alpha_n = \frac{P_n(\beta)}{\beta^{r_n}(\beta^{s_n} - 1)}.$$

Moreover, the coefficients of  $P_n(X)$  are bounded in absolute value by  $2 \max_{k \geq 1} |a_k|$ .

*Proof.* From (4.1) we have

$$\begin{aligned} \alpha_n &= \sum_{k=1}^{r_n} \frac{a_k}{\beta^k} + \sum_{k=r_n+1}^{+\infty} \frac{b_k^{(n)}}{\beta^k} = \sum_{k=1}^{r_n} \frac{a_k}{\beta^k} + \frac{1}{\beta^{r_n}} \sum_{k=1}^{+\infty} \frac{b_{r_n+k}^{(n)}}{\beta^k} \\ &= \sum_{k=1}^{r_n} \frac{a_k}{\beta^k} + \frac{1}{\beta^{r_n}} \sum_{k=1}^{s_n} \frac{a_{r_n+k}}{\beta^k} \left( \sum_{j=0}^{+\infty} \frac{1}{\beta^{js_n}} \right) = \sum_{k=1}^{r_n} \frac{a_k}{\beta^k} + \sum_{k=1}^{s_n} \frac{a_{r_n+k}}{\beta^{k+r_n-s_n}(\beta^{s_n} - 1)} \\ &= \frac{P_n(\beta)}{\beta^{r_n}(\beta^{s_n} - 1)}, \end{aligned}$$

where we define

$$P_n(X) = \sum_{k=1}^{r_n} a_k X^{r_n-k} (X^{s_n} - 1) + \sum_{k=1}^{s_n} a_k X^{s_n-k}.$$

*Q.E.D.*

Let  $K = \mathbb{Q}(\beta)$ ,  $d = [K : \mathbb{Q}]$ . We assume that  $\alpha$  is algebraic, and we consider the following linear forms of three variables with algebraic coefficients. For the place  $v$  corresponding to the embedding of  $K$  in  $\mathbb{R}$  defined by  $\beta \mapsto \beta$  we choose the continuation of  $|\cdot|_v$  to  $\overline{\mathbb{Q}}$  defined by  $|x|_v = |x|^{1/d}$ . Set  $L_{1,v}(x, y, z) = x$ ,  $L_{2,v}(x, y, z) = y$ , and  $L_{3,v}(x, y, z) = \alpha x + \alpha y + z$ . From (4.1) and Lemma 3 we conclude that

$$\begin{aligned} & |L_{3,v}(\beta^{r_n+s_n}, -\beta^{r_n}, -P_n(\beta))|_v = |\alpha(\beta^{r_n}(\beta^{s_n} - 1)) - P_n(\beta)|^{1/d} \\ & = |\beta^{r_n}(\beta^{s_n} - 1)(\alpha - \alpha_n)|^{1/d} < \left| \beta^{r_n+s_n} \frac{1}{\beta^{r_n+\lceil ws_n \rceil}} \sum_{k=1}^{+\infty} \frac{a_{k+r_n+\lceil ws_n \rceil} - b_{k+r_n+\lceil ws_n \rceil}^{(n)}}{\beta^k} \right|^{1/d} \\ & \ll (\beta^{s_n - \lceil ws_n \rceil})^{1/d} < \frac{1}{\beta^{(w-1)s_n/d}} \end{aligned} \quad (4.2)$$

Here and later the constants implied by the Vinogradov symbol depend (at most) on  $\alpha, \beta$  and  $\max_{k \geq 1} |a_k|$ , but are independent of  $n$ . Let us denote by  $S_\infty$  the set of all other infinite places on  $K$  and by  $S_0$  the set of all finite places on  $K$  dividing  $\beta$ . Note that in our case  $S_0$  is empty because  $\beta$  is an algebraic unit. For any  $v$  in  $S = S_0 \cup S_\infty$ , set  $L_{1,v}(x, y, z) = x$ ,  $L_{2,v}(x, y, z) = y$ , and  $L_{3,v}(x, y, z) = z$ . Clearly, for any  $v$  in  $S$ , the forms  $L_{1,v}, L_{2,v}$  and  $L_{3,v}$  are linearly independent. Let us take  $x_n = (\beta^{r_n+s_n}, -\beta^{r_n}, -P_n(\beta))$  and estimate the product

$$\prod := \prod_{v \in S} \prod_{i=1}^3 \frac{|L_{i,v}(x_n)|_v}{|x_n|_v} = \prod_{v \in S} |\beta^{r_n+s_n}|_v |\beta^{r_n}|_v \frac{|L_{3,v}(x_n)|_v}{|x_n|_v^3}$$

from above. By the product formula and the definition of  $S$ , we immediately get that

$$\prod = \prod_{v \in S} \frac{|L_{3,v}(x_n)|_v}{|x_n|_v^3} \quad (4.3)$$

Since the polynomial  $P_n(X)$  has integer coefficients and since  $\beta$  is an algebraic integer, we have  $|L_{3,v}(x_n)|_v = |P_n(\beta)|_v \leq 1$  for any integer and for any place  $v$  in  $S_0$ . Moreover, as the conjugates of  $\beta$  have moduli at most 1, we have for any infinite place  $v$  in  $S_\infty$

$$|L_{3,v}(x_n)|_v \ll (r_n + s_n)^{d_v/d},$$

where  $d_v = 1$  or  $2$  according as  $v$  is real infinite or complex infinite, respectively. Together with (4.2) and (4.3), this gives

$$\begin{aligned} \prod &\ll (r_n + s_n)^{d-1/d} \beta^{-(w-1)s_n/d} \prod_{v \in S} |x_n|_v^{-3} \\ &\ll (r_n + s_n)^{d-1/d} \beta^{-(w-1)s_n/d} \prod_{v \in S} H(x_n)^{-3}, \end{aligned}$$

since  $|x_n|_v = 1$  if  $v$  does not belong to  $S$ . By Lemma 3, and from the fact that the moduli of the complex conjugates of  $\beta$  are at most 1, we see that

$$H(x_n) \ll (r_n + s_n)^d \beta^{(r_n + s_n)/d}.$$

It follows from Condition  $(*)_w$  that

$$\prod_{v \in S} \prod_{i=1}^3 \frac{|L_{i,v}(x_n)|_v}{|x_n|_v} \ll (r_n + s_n)^{dw} H(x_n)^{-(w-1)s_n/(r_n + s_n)} H(x_n)^{-3} \ll H(x_n)^{-3-\epsilon},$$

for some positive real number  $\epsilon$ .

Now we apply the quantitative version of the Subspace Theorem to conclude that the points  $x_n$  lie in a finite number of proper subspaces of  $K^3$ . So there exist a non-zero triple  $(x_0, y_0, z_0)$  in  $K^3$  and infinitely many integers  $n$  such that

$$x_0 - y_0 \frac{\beta^{r_n}}{\beta^{r_n + s_n}} - z_0 \frac{P_n(\beta)}{\beta^{r_n + s_n}} = 0.$$

After taking the limit along this subsequence of integers and noting that  $\{s_n\}_{n \geq 1}$  tends to infinity, we get that  $x_0 = z_0 \alpha$ . Thus,  $\alpha$  belongs to  $K = \mathbb{Q}(\beta)$ .

*Q.E.D.*

### Proof of Criterion 8

*Proof.* Let  $\alpha$  be an irrational number. Without loss of generality, we assume that  $\alpha$  is in  $(0, 1)$  and we denote by  $0.u_1 u_2 \dots u_k \dots$  its  $b$ -adic expansion. The sequence  $\{u_k\}_{k \geq 1}$  takes its values in  $\{0, 1, \dots, b-1\}$  and is not eventually periodic. We assume that there exists an integer  $c \geq 2$  such that the complexity function  $p$  of  $\{u_k\}_{k \geq 1}$  satisfies

$$p(n) \leq cn \text{ for infinitely many integers } n \geq 1.$$

We shall derive that Condition  $(*)_w$  is then fulfilled by the sequence  $\{u_k\}_{k \geq 1}$  for a suitable  $w > 1$ . By Theorem 10, this will imply that  $\alpha$  is transcendental.

Let  $n_k$  be an integer with  $p(n_k) \leq cn_k$ . Denote by  $U(l)$  the prefix of  $\mathbf{u} := u_1u_2\dots$  of length  $l$ . By the pigeonhole principle, there exists (at least) one word  $M_k$  of length  $n_k$  which has (at least) two occurrences in  $U((c+1)n_k)$ . Thus, there are (possibly empty) words  $A_k, B_k, C_k$  and  $D_k$ , such that

$$U((c+1)n_k) = A_kM_kC_kD_k = A_kB_kM_kD_k \text{ and } |B_k| \geq 1$$

We observe that  $|A_k| \leq cn_k$ . We have to distinguish three cases:

- (i)  $|B_k| > |M_k|$ ;
- (ii)  $\lceil |M_k|/3 \rceil \leq |B_k| \leq |M_k|$ ;
- (iii)  $1 \leq |B_k| \leq \lceil |M_k|/3 \rceil$

Case (i). Under this assumption, there exists a word  $E_k$  such that

$$U((c+1)n_k) = A_kM_kE_kM_kD_k.$$

Since  $|E_k| \leq (c-1)|M_k|$ , the word  $A_k(M_kE_k)^s$  with  $s = 1 + 1/c$  is a prefix of  $\mathbf{u}$ . Moreover, we observe that

$$|M_kE_k| \geq |M_k| \geq \frac{|A_k|}{c}.$$

Case (ii). Under this assumption, there exist two words  $E_k$  and  $F_k$  such that

$$U((c+1)n_k) = A_kM_k^{1/3}E_kM_k^{1/3}E_kF_k.$$

Thus, the word  $A_k(M_k^{1/3}E_k)^2$  is a prefix of  $\mathbf{u}$ . Moreover, we observe that

$$\left| M_k^{1/3}E_k \right| \geq \frac{|M_k|}{3} \geq \frac{|A_k|}{3c}$$

Case (iii). In this case,  $B_k$  is clearly a prefix of  $M_k$ , and we infer from  $B_kM_k = M_kC_k$  that  $B_k^t$  is a prefix of  $M_k$ , where  $t$  is the integer part of  $|M_k|/|B_k|$ . Observe that  $t \geq 3$ . Setting  $s = \lfloor t/2 \rfloor$ , we see that  $A_k(B_k^s)^2$  is a prefix of  $\mathbf{u}$  and

$$|B_k^s| \geq \frac{|M_k|}{4} \geq \frac{|A_k|}{ck}$$

In each of the three cases above, we have proved that there are finite words  $U_k, V_k$  such that  $U_k V_k^{1+1/c}$  is a prefix of  $\mathbf{u}$  and:

- $|U_k| \leq cn_k$ ;
- $|V_k| \geq n_k/4$ ;
- $w \geq 1 + 1/c > 1$ .

Consequently, the sequence  $\{|U_k|/|V_k|\}_{k \geq 1}$  is bounded from above by  $4c$ . Moreover, it follows from the lower bound  $|V_k| \geq n_k/4$  that we may assume that the sequence  $\{|V_k|\}_{k \geq 1}$  is strictly increasing. This implies that the sequence  $\mathbf{u}$  satisfies Condition  $(*)_{1+1/c}$ . By applying Theorem 10 with  $\beta = b$ , we conclude that  $\alpha$  is transcendental.

*Q.E.D.*

### Proof of Theorem 15

*Proof.* Theorem 15 is the consequence of the following more general result by Adamczewski and Bugeaud from the same paper [AB06].

#### Theorem. 25

Let  $b \geq 2$  be an integer and let  $\mathbf{a} = \{a_k\}_{k \geq 1}$  be a palindromic sequence on the alphabet  $\{0, 1, \dots, b-1\}$ , with parameters  $w, w'$ . Assume that  $\mathbf{a}$  is not ultimately periodic. Let  $p$  be a prime divisor of  $b$  and let  $p^u$  be the greatest power of  $p$  dividing  $b$ . Assume that

$$\frac{\log p^u}{\log b} > \frac{w'}{1 + w'}.$$

Then at least one of the numbers  $\alpha := \sum_{k=1}^{+\infty} \frac{a_k}{p^k}$ ,  $\alpha_p := \sum_{k=1}^{+\infty} a_k p^k$  is transcendental.

*Proof.* Consider a palindromic sequence  $\mathbf{a} = \{a_k\}_{k \geq 1}$  from the condition of theorem and fix the parameters  $w, w'$ . Let sequences  $\{U_n\}_{n \geq 1}$ ,  $\{V_n\}_{n \geq 1}$  and  $\{W_n\}_{n \geq 1}$  be as denoted in the definition for palindromic sequences. Also set  $r_n = |U_n|$ ,  $s_n = |V_n|$  and  $t_n = |W_n|$  for any  $n \geq 1$ .

Then the sequence  $\mathbf{a}$  represented as  $\mathbf{a} = W_n U_n V_n \overline{U_n} \dots$  acts as expansion for the numbers  $\alpha, \alpha_p$ .

Let assume that  $\alpha$  and  $\alpha_p$  are algebraic. Our aim is to show that they are rationals. Here again we do the same approximations of  $\alpha$  by a rational construction

of an appropriate polynomial whose coefficients repeat the beginning of palindromic sequence.

Define the integer polynomial  $P_n(X)$  for any positive integer  $n$  by

$$P_n(X) = \sum_{k=1}^{t_n} a_k X^k + \sum_{k=t_n+1}^{2r_n+2t_n+s_n} a_{2r_n+2t_n+s_n-k+1} X^k.$$

Then we see that the rational number  $P_n(b)/b^{2r_n+2t_n+s_n+1}$  has the  $b$ -adic expansion  $0.W_n U_n V_n \overline{U_n W_n}$  and we evaluate the approximation of  $\alpha$  as

$$\left| \alpha - \frac{P_n(b)}{b^{2r_n+2t_n+s_n+1}} \right| \leq \frac{1}{b^{2r_n+t_n+s_n+1}}$$

At the same time

$$P_n(X) = \sum_{k=1}^{r_n+t_n} a_k X^k + \sum_{k=r_n+t_n+1}^{2r_n+2t_n+s_n} a_{2r_n+2t_n+s_n-k+1} X^k.$$

Consequently, for any prime divisor  $p$  of  $b$ , we have

$$|\alpha_p - P_n(b)|_p \leq |b|_p^{r_n+t_n+1}$$

We consider the linearly independent linear forms with algebraic coefficients

$$L_{1,\infty} = \alpha x - y, \quad L_{2,\infty} = x, \quad L_{3,\infty} = z,$$

$$L_{1,p} = x, \quad L_{2,p} = y - \alpha_p z, \quad L_{3,p} = z,$$

and for every prime number  $p' \neq p$  dividing  $b$ , we set

$$L_{1,p'} = x, \quad L_{2,p'} = y, \quad L_{3,p'} = z.$$

We evaluate the product of the norms of these linear forms at the integer points

$$x_n = (b^{2t_n+2r_n+s_n+1}, P_n(b), 1).$$

Let us evaluate products first without dividing by norm of  $x_n$ . For the set of infinite places

$$\begin{aligned} \Pi_\infty &= \prod_{i=1}^3 |L_{i,\infty}|_\infty = |\alpha b^{2t_n+2r_n+s_n+1} - P_n(b)| b^{2t_n+2r_n+s_n+1} \\ &= \left| \alpha - \frac{P_n(b)}{b^{2r_n+2t_n+s_n+1}} \right| b^{2(2t_n+2r_n+s_n+1)} \leq b^{2(2t_n+2r_n+s_n+1)-(2r_n+t_n+s_n)} = b^{3t_n+2r_n+s_n+2} \end{aligned}$$

For any prime divisor  $p$  of  $b$

$$\begin{aligned}\Pi_p &= \prod_{i=1}^3 |L_{i,p}|_p = |b|_p^{2t_n+2r_n+s_n+1} |P_n(b) - \alpha_p|_p \leq |b|_p^{2t_n+2r_n+s_n+1+(r_n+t_n+1)} \\ &= |b|_p^{3t_n+3r_n+s_n+2} = p^{-u(3t_n+3r_n+s_n+2)}\end{aligned}$$

Finally product for all prime numbers  $p' \neq p$  dividing  $b$

$$\Pi_{p'|b} = \prod_{i=1}^3 |L_{i,p'}|_{p'} = \prod_{p'|b} |b|_{p'}^{2t_n+2r_n+s_n+1} |P_n(b)|_{p'} \leq \prod_{p'|b} |b|_{p'}^{2t_n+2r_n+s_n+1}$$

Since the polynomial  $P_n(b)$  has integer coefficients and since  $b$  is an algebraic integer, we have  $|P_n(\beta)|_{p'} \leq 1$  for any integer  $p'$ .

Note that  $b = p^u k$ , where  $k$  could be represented as  $k^{-1} = \prod_{p'|b} |b|_{p'}$ . Then

$$\prod_{p'|b} |b|_{p'}^{2t_n+2r_n+s_n+1} = k^{-(2t_n+2r_n+s_n+1)}$$

Collecting all products and dividing by  $H(x_n)$  we obtain

$$\begin{aligned}\prod_n &\ll H(x_n)^{-3} \Pi_\infty \Pi_p \Pi_{p'|b} = H(x_n)^{-3} [p^u k]^{3t_n+2r_n+s_n+2} p^{-u(3t_n+3r_n+s_n+2)} k^{-(2t_n+2r_n+s_n+1)} \\ &= H(x_n)^{-3} p^{-u(r_n)} k^{t_n} = H(x_n)^{-3} p^{-u(r_n)} \left(\frac{b}{p^u}\right)^{t_n} = H(x_n)^{-3} b^{t_n} p^{-u(r_n+t_n)}\end{aligned}$$

Recall that, by assumption, there is a real number  $w'$  such that  $t_n \leq w' r_n$  for any  $n \geq 1$ . Consequently, we infer that

$$\prod_n \ll H(x_n)^{-3} b^{t_n} p^{-u(r_n+t_n)} \leq H(x_n)^{-3} (b^{w'} p^{-u(1+w')})^{r_n}.$$

It follows from condition of theorem that there exists a positive real number  $C < 1$  satisfying  $\prod_n \ll H(x_n)^{-3} C^{r_n}$ . Thus, by Conditions (ii) and (iii) from the definition of an initially palindromic sequence, we get that

$$\prod_n \ll H(x_n)^{-\epsilon}$$

for some positive real number  $\epsilon$ . We then apply Subspace Theorem and conclude that there exists a nonzero integer triple  $(z_1, z_2, z_3)$  and an infinite set of distinct positive integers  $N_1$  such that

$$z_1 b^{2t_n+2r_n+s_n+1} + z_2 P_n(b) + z_3 = 0,$$

for any  $n$  in  $N_1$ . Dividing this equation by  $b^{2t_n+2r_n+s_n+1}$  and letting  $n$  tend to infinity along  $N_1$ , we get that  $z_1 + z_2 \alpha = 0$ , thus  $\alpha$  is rational. Consequently, the sequence  $\mathbf{a}$  is ultimately periodic and each  $\alpha_p$  is rational.



*Q.E.D.*

*Proof of Theorem 15.* Since the condition of Theorem 25 is satisfied whenever  $b$  is a prime number, Theorem 15 is just a case of Theorem 25. Indeed, in the case of  $b$  prime the only divisor of  $b$  is  $b$  itself, so  $p^u = b$  and the condition of Theorem 25 corresponds to

$$\frac{\log b}{\log b} = 1 > \frac{w'}{1 + w'}$$

which is always true since  $1 + w' > w'$ .

*Q.E.D.*

### **Proof of Theorem 16**

*Proof.* Suppose that we have a non-eventually periodic automatic sequence  $\{a_k\}_{k \geq 1}$  defined on a finite alphabet  $A$ . By a result of Cobham [Cob80] there exist a morphism  $\varphi$  from an alphabet  $B = \{1, 2, \dots, r\}$  to the alphabet  $A$  and a uniform morphism  $\phi$  from  $B$  into itself such that  $\mathbf{a} = \varphi(\mathbf{u})$ , where  $\mathbf{u}$  is a fixed point for  $\phi$ . It follows by a result of Cobham that if the sequence  $\mathbf{u}$  satisfies Condition  $(*)_w$  and if  $\varphi$  is a non-erasing morphism (the image length of any letter under  $\varphi$  is at least 1) then  $\mathbf{a} = \varphi(\mathbf{u})$  satisfies Condition  $(*)_w$  as well. By the pigeonhole principle, the prefix of length  $r + 1$  of  $\mathbf{u}$  can be written under the form  $W_1 u W_2 u W_3$ , where  $u$  is a letter and  $W_1, W_2, W_3$  are (not necessary non-empty) finite words.

Let us define the sequences  $\{U_n\}_{n \geq 1}$  and  $\{V_n\}_{n \geq 1}$  as  $U_n = \phi^n(W_1)$  and  $V_n = \phi^n(u W_2)$  for any  $n \geq 1$ . Now we need to check that the assumptions of Theorem 10 are satisfied by  $\mathbf{u}$ . Remember that  $\phi$  is a morphism of constant length, so

$$\frac{|U_n|}{|V_n|} \leq \frac{|W_1|}{|W_2| + 1} \leq r - 1$$

thus, bounded above. On the other hand,  $\phi^n(u)$  is a prefix of  $V_n$  of length at least  $|V_n|/r$ . It follows that Condition  $(*)_{1+1/r}$  is satisfied by the sequence  $\mathbf{u}$ , and also by the sequence  $\mathbf{a}$ . Let  $b \geq 2$  be an integer. Applying Theorem 10 with  $\beta = b$ , we conclude that the automatic number  $\sum_{k=1}^{+\infty} a_k b^{-k}$  is transcendental.

*Q.E.D.*

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