Stress Intensity Factors for Elliptical and Semi-Elliptical cracks subjected to an arbitrary mode I loading.

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Fatigue durability, damage tolerance and strength evaluations of cracked structural components require accurate determination of *stress intensity factors (SIF)*. Most practical crack configurations are embedded and surface breaking planar cracks subjected to complex two-dimensional stress fields.
The only cracked body configuration which has been studied analytically for all types of applied stress fields is a circular crack in an infinite elastic solid. However, this model is suitable only for a narrow class of practical applications. Much wider class of practical problems can be solved using the model of an elliptical crack. The exact analytical SIF solutions for an elliptical crack were obtained only for some particular cases of polynomial applied stress fields. In the present work the exact analytical SIF solution has been obtained for an elliptical crack embedded in an infinite elastic body and subjected to an arbitrary applied normal stress field (Mode I).
The most effective method of evaluating the stress intensity factor induced by an applied stress field is by using *the weight function* for a given cracked body. The weight function represents the SIF induced by a unit concentrated load. The only exact analytical weight function for a planar crack was obtained for a circular one. In the present research the exact analytical weight function has been derived for an elliptical crack embedded in an infinite elastic solid.
The weight function for an elliptical crack was subsequently employed in the alternating method to obtain the unique SIF solution for a surface breaking semi-elliptical crack in a semi-infinite body subjected to an arbitrary applied stress field.
The solutions obtained in the present work can be used in various practical applications, such as cracks in pressure vessels, welded structures and mechanical engineering components subjected to cyclic loading.

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Dedication

This thesis is dedicated to my husband Sinisa Vergas.

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Nomenclature

E	modulus of elasticity,
ν	Poisson's ratio,
(x, y, z)	Cartesian system of coordinates,
u, v, w or	
$u(x, y, z), v(x, y, z), w(x, y, z)$	displacement field,
$\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$	three-dimensional Laplace operator,
$\Theta = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}$	divergence of the displacement field,
$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{zx}$ or	
$\sigma_{xx}(x, y, z), \sigma_{yy}(x, y, z), \sigma_{zz}(x, y, z),$	
$\tau_{xy}(x, y, z), \tau_{yz}(x, y, z), \tau_{zx}(x, y, z)$	components of elastic stress tensor,
$G = E/(2(1 + \nu))$	shear modulus,
S	crack domain,
$p(x, y)$	stress field applied to the crack,
S_1	open domain in the crack plane outside of the crack,
$\sigma(x, y)$	normalized stress field applied to the crack,
$\Delta_{xy} = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$	two-dimensional Laplace operator,
ϕ_1, ϕ_2, ϕ_3	harmonic functions,

Q', $Q'(x', y', 0)$, $Q'(x', y')$	an arbitrary point on the crack front,
n, τ, b	local system of coordinates with
	origin Q' on the crack front,
σ_n, σ_τ, σ_b, σ_{nb}	stress tensor components
	in the local system of coordinates (n, τ, b),
Q, $Q(x, y, z)$	an arbitrary point,
ρ	distance from the point Q to the crack front,
ϑ	angle between QQ' and n,
K, $K(Q')$, $K(\varphi)$	stress intensity factor (SIF),
u_n, u_τ, u_b	components of the displacement field in coordinates (n, τ, b),
$w(x, y, 0)$	crack opening displacement,
Q_0, $Q_0(x_0, y_0, 0)$, $Q_0(x_0, y_0)$	an arbitrary point inside the crack domain,
	where the point load is applied,
$\delta(x - x_0)$, $\delta(y - y_0)$	delta-functions,
$P = 1$	magnitude of the applied force,
$W(Q', Q_0)$	weight function,
a	major semi-axis of an elliptical crack,
b	minor semi-axis of an elliptical crack,

r, θ	elliptical coordinates associated with the elliptical crack or polar coordinates associated with the circular crack,
$\sigma(r,\theta)$	normalized applied load in elliptical coordinates,
$\sigma_n^c(r)$, $\sigma_n^s(r)$	cos- and sin- components of the Fourier expansion of function $\sigma(r,\theta)$,
(ξ,η), (p_1,p_2)	auxiliary Cartesian systems of coordinates,
(R,ψ)	auxiliary elliptical system of coordinates,
$F(p_1,p_2)$, $F(R,\psi)$	inverse Fourier transform of the crack opening displacement,
$F_n^c(R)$, $F_n^s(R)$	cos- and sin- components of the Fourier expansion of function $F(R,\psi)$,
$J_n(Rr)$	Bessel function of the first kind of order n,

$$\alpha_{nm}^c = \int\limits_0^{2\pi} \sqrt{\frac{b^2}{a^2}\cos^2\psi + \sin^2\psi}\, \cos n\psi \cos m\psi \, d\psi,$$
$$\alpha_{nm}^s = \int\limits_0^{2\pi} \sqrt{\frac{b^2}{a^2}\cos^2\psi + \sin^2\psi}\, \sin n\psi \sin m\psi \, d\psi,$$

$A, B, \alpha, \beta, \varrho$	parameters of Weber-Schafheitlin integral
A_{kn}^c, A_{kn}^s	coefficients of expansion of $F_n^c(R)$, $F_n^s(R)$ into the system of Bessel functions,
C_{nmki}^c, C_{nmki}^s	matrices of coefficients α_{nm}^c and α_{nm}^s respectively,
B_{ni}^c, B_{ni}^s	coefficients associated with $\sigma_n^c(r)$, $\sigma_n^s(r)$ respectively,
$\mathfrak{J}_i(n+\frac{3}{2}, n+1, r^2)$	Jacobi Polynomial of degree i in r^2,

${}_2F_1$	hypergeometric function,
$p_n^c(r)$, $p_n^s(r)$	cos- and sin- components of the Fourier expansion of function $p(r,\theta)$,
$\mathcal{B}_{ni}^c$, $\mathcal{B}_{ni}^s$	coefficients associated with $p_n^c(r)$, $p_n^s(r)$ respectively,
Γ	gamma function,
$w(r,\theta)$	crack opening displacement in elliptical coordinates,
$\mathcal{A}_{kn}^c$, $\mathcal{A}_{kn}^s$	coefficients of expansion of crack opening displacement $w(r,\theta)$ and stress intensity factor into the Fourier series,
φ	parametric angle of the point $Q'(x' = a\cos\varphi, y' = b\sin\varphi)$,
N	any fixed number,
$K^N(\varphi)$	Nth approximation of the stress intensity factor $K(\varphi)$,
$\sigma_{zz,n}^c(r)$, $\sigma_{zz,n}^s(r)$	cos- and sin- terms of the Fourier expansion of stress tensor component $\sigma_{zz}(r,\theta)$,
$f_n^c(r)$, $f_n^s(r)$	coefficients of Fourier expansion of applied stress field $p(r,\theta)$ in polar coordinates,
σ_0, σ_2, p_{20}, p_{02}	magnitudes of the applied stress fields,
$k = \dfrac{\sqrt{a^2-b^2}}{a}$, $k' = \dfrac{b}{a}$	
$E(k)$	complete elliptic integral of the second kind,
$K(k)$	complete elliptic integral of the first kind,

C_0, C_2	coefficients of Fourier expansion of SIF induced by an even quadratic stress field,
A_{nk}	coefficients of applied polynomial stress field,
$Q_0(r_0, \theta_0)$	location of the applied force in elliptical coordinates,
ε_0, ε_1, ε_2, ε, ϵ	small parameters,
X_{2k}^N	even coefficients of Fourier expansion of SIF $K^N(\varphi)$ induced by a constant pressure distributed over a small elliptical region at the crack center,
D_{2k}, D_{2k+1}	matrices of coefficients $\alpha^c_{2i\,2j}$, $\alpha^c_{2i+1\,2j+1}$ respectively,
Δ_{2k}, Δ_{2k+1}	determinants of matrices D_{2k}, D_{2k+1},
$\delta_0^0 = 1$, δ_{2k}^{2i}	cofactor obtained by removing the first row and $i+1$ column of matrix Δ_{2k} for $i \leq k$, δ_{2k}^{2i}=0 for $i > k$,
$b^c_{n,i}$, $b^s_{n,i}$	normalized coefficients $\mathcal{B}^c_{ni}$ or $\mathcal{B}^s_{ni}$ respectively,
$W(\varphi; x_0, y_0)$	weight function for an elliptical (circular) crack in elliptical (polar) coordinates,
$W^N(Q', Q_0)$, $W^N(\varphi; r_0, \theta_0)$	Nth approximation of the weight function $W(\varphi; r_0, \theta_0)$,
$\mathcal{X}^N_m(r_0)$, $\mathcal{Y}^N_m(r_0)$	sin- and cos- coefficients of Fourier expansion of the weight function $W^N(\varphi; r_0, \theta_0)$ respectively,
$W^N(x; x_0, y_0)$	weight function for a tunnel crack,

Ω small region,

f_n^c sum of coefficients $(-1)^i b_{n,i}^c$,

f_n^s sum of coefficients $(-1)^i b_{n,i}^s$,

$N1 = N$, if N is even, $N1 = N - 1$ if N is odd,

$N2 = N - 1$, if N is even, $N2 = N$ if N is odd,

$\delta_{2k}^{2i,2j}$ cofactor obtained by removing the $i + 1$ row

and $j + 1$ column of matrix Δ_{2k} for $i, j \leq k$, $\delta_{2k}^{2i,2j}$=0 for $i, j > k$,

$\delta_{2k+1}^{2i+1,2j+1}$ cofactor obtained by removing the $i + 1$ row

and $j + 1$ column of matrix Δ_{2k+1} for $i, j \leq k$,

$\delta_{2k+1}^{2i+1,2j+1}$=0 for $i, j > k$,

$W^{approx}(Q', Q_0)$ approximate weight function,

$\rho_{Q'Q_0}$ the distance between points Q' and Q_0,

$\Gamma(Q_0)$ the length of the crack contour inverted with respect to the point Q_0,

$\sigma_r(x')$ residual stress in a weldment,

d half of width of the tensile residual stress region in a weldment,

d_0, ψ_0 parameters characterizing the location of a crack,

R_i, R_o inner and outer radii of the thick walled cylinder,

P_0 inner pressure in the thick walled cylinder,

R, ϕ polar coordinates associated with the cross section of the cylinder,

σ_R, σ_ϕ	principal stresses in the wall of the cylinder,
α, β, γ	non-dimensional parameters of the stress field in the wall of the cylinder,
κ	singularity order,
$w_0(r)$	component of the displacement induced by stress field $p(r,\theta) = \dfrac{1}{\sqrt{1-r^2}}$,
$V_1(S)$, $V_2(S)$, $V_3(\partial S)$	functional spaces,
∂S	crack front,
$H_{1/2}(S)$, $H_{-1/2}(S)$	Sobolev spaces,
$p^{N^\star}(x,y)$	applied polynomial stress field,
$K^\star(\varphi)$	stress intensity factor induced by the polynomial stress field,
A_{nk}	coefficients of the polynomial stress field,
Σ	domain of a semi-elliptical crack,
$P(x,y)$	stress field, applied over a semi-elliptical crack,
Σ_1	open domain in the plane $z=0$ outside of a semi-elliptical crack,
(ξ,η,ζ)	auxiliary Cartesian system of coordinates,
$K^{(0)}(\varphi)$, $K^{(1)}(\varphi)$,..	successive approximations of the stress intensity factor $K(\varphi)$ for a semi-elliptical crack,

$P^{(0)}(x,y),\ P^{(0)}(r,\theta)$	stress field applied over an elliptical crack
	in the zeroth iteration of the alternating method,
$p^{(0)}(x,y),\ p^{(0)}(r,\theta)$	stress field applied over a semi-elliptical crack
	in the zeroth iteration of the alternating method,
$p_n^{(0),c}(r)$	coefficients of the Fourier expansion
	of the applied stress field $p^{(0)}(r,\theta)$,
$\mathcal{A}_{nk}^{(0),c}$	coefficients of the Fourier expansion
	of the stress intensity factor $K^{(0)}(\varphi)$,
$w^{(0)}(x,y),\ w^{(0)}(r,\theta)$	crack opening displacement
	in the zeroth iteration of the alternating method,

$\mathbf{r} = \sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}$,

$\sigma_{yy}^{(0)},\ \sigma_{zz}^{(0)},\ \tau_{zx}^{(0)},\ \sigma_{zy}^{(0)}$	stress field components in the zeroth approximation of $K(\varphi)$,

$\mathbf{R} = \sqrt{(x-\xi)^2 + y^2 + (z-\zeta)^2}$, $\Phi = y\ln(\mathbf{R}+y)$,

$\mu = \dfrac{E}{2(1+\nu)}$, $\lambda = \dfrac{\nu E}{(1+\nu)(1-2\nu)}$,

$A_{13},\ A_{23},\ A_{33}$	kernel functions in the solution of Newmann problem for a plane,
$\sigma_{zz}^{(0)}(x,y)$	additional stress in the crack plane due to the free surface
	applied over a semi-elliptical crack,
	in the first iteration of the alternating method,

$\tilde{\sigma}_{zz}^{(0)}(x,y)$	additional stress in the crack plane due to the free surface applied over an elliptical crack, in the first iteration of the alternating method
$P^{(1)}(x,y),\ P^{(1)}(r,\theta)$	stress field applied over an elliptical crack in the first iteration of the alternating method,
$p^{(1)}(x,y),\ p^{(1)}(r,\theta)$	stress field applied over a semi-elliptical crack in the first iteration of the alternating method,
$M_1,\ f_\varphi,\ g$	components of the SIF expression for a semi-elliptical crack,
$Q^\star$	approximate form of the complete elliptical integral of the second kind.

Chapter 1

Introduction

The strength and durability of structures may be compromised by the presence of cracks. Due to the high stress concentration in the vicinity of a crack tip it can result in the failure of the structure. The Fracture Mechanics theory can be used to analyze structures and machine components with cracks and to obtain an efficient design.

The basic principles of Fracture Mechanics developed from studies of Inglis [7], Griffith [1], Westergaard [21] and others [29], are based on the concepts of linear elasticity. In 1913, Inglis [7] developed the stress solution for an elliptical hole in a semi-infinite plate subjected to remote uniform tension. This solution was used later by Griffith to formulate the energy criteria for fracture analysis based on the energy release rate concept [1].

In 1939, Westergaard [21] derived the general linear elastic solution for the stress field around a crack tip using complex stress functions. Irwin [19], in 1957, proposed the description of the stress field ahead of a crack tip (front) by means of only one parameter, the so called *stress intensity factor (SIF).* Irwin has also shown that the stress intensity factor is uniquely related to the energy release rate, proposed by Griffith, and therefore can be used to describe fracture phenomenon [14].

The stress intensity factor has become the fundamental concept in Linear Elastic Fracture Mechanics (LEFM). Various criteria and models of fracture and crack growth have been formulated in terms of the SIF ([42]). A very important material property, so called *fracture toughness*, is introduced in terms of the SIF. Fracture toughness expresses the material's resistance to brittle fracture and is widely used in design applications.

In order to make the concept of the SIF applicable to a particular problem, the SIF has to be obtained for a given cracked body and applied loading. Thus, the crack problem is formulated as a boundary value problem of linear elasticity, and solved in terms of the displacement field and/or the stress field. The SIF can be subsequently obtained from Irwin's expansion of the stress field (or the displacement field) in the vicinity of a crack tip.

Numerous one-dimensional crack problems have already been solved analytically and numerically [60], including cracks in anisotropic and orthotropic materials (see for example [56], [54]) and the interaction of multiple cracks [55]. However, most practical crack configurations are embedded and surface breaking planar cracks subjected to complex two-dimensional stress fields. Unfortunately, exact SIF-solutions for most planar cracks and applied loadings are not available. Note, that we are interested in obtaining the analytical solutions because they are the most general and ready to use in various engineering applications.

The first planar crack which was studied analytically was a circular crack embedded in an infinite body. Sneddon [23] has analyzed this problem for a simplified case when the applied loading was symmetric with respect to the center of the crack. The mixed boundary conditions in the crack plane can be transformed into a pair of dual integral equations for the unknown crack opening displacement. For such a system of equations an analytical solution exists. After the crack opening displacement has been found, the stress field and the displacement field are derived in the form of special integral representations.

Sneddon's approach was further expanded by Kassir and Sih [36] for the problem of a circular crack under arbitrary applied normal loading. The modified method was based on the Fourier expansion of the applied stress field, the unknown crack opening displacement and the SIF in polar coordinates. A set of dual integral equations was obtained and solved analytically for each term of the SIF expansion.

Fabrikant [52] has proposed another method for the problem of a circular crack. His method is based on a special integral representation of the distance between two arbitrary points and subsequent integral transforms.

Analytical SIF-solutions for planar cracks with more complicated geometries have been obtained only for certain special cases of the applied stress fields. The first contribution concerning the analysis of a stress field near an elliptical crack was made by Galin [32] who derived the solution for a contact problem for a punch of elliptical cross-section acting against a semi-infinite elastic body. His solution was adopted later for an inverse problem of a pressurized crack which is presently known as Galin's theorem. It states that if an elliptical crack is opened up by the applied pressure in the form of a polynomial, the form of the crack opening displacement also includes a polynomial of the same type.

Kassir and Sih [36], Shah and Kobayshi [43], Nishioka and Atluri [50] and Martin [41] obtained SIF-solutions for an elliptical crack subjected to various polynomial stress fields. Kassir and Sih [36], Shah and Kobayshi [43] and Nishioka and Atluri [50] used the same method based on the representation of the displacement field and the stress field around an elliptical crack by means of one harmonic function. This function satisfies the mixed boundary conditions in the plane of the crack and can be represented by a system of potential harmonic functions in ellipsoidal coordinates. If the applied pressure has the form of a polynomial, then the boundary condition in the crack domain can be transformed into the system of algebraic equations with unknown coefficients. The main difficulty of this method is the calculation of the derivatives in ellipsoidal coordinates. Also, this approach cannot be used for an arbitrary applied stress field.

Martin in [41] used the method of expansion of the applied stress field and the unknown crack opening displacement into the system of orthogonal Gegenbauer polynomials. The boundary condition in the crack domain is transformed into a system of linear algebraic equations. In general it is a system with an infinite number of equations, but it can be truncated in the case of a polynomial applied stress field. The SIF-solutions were obtained for constant, linear and quadratic stress fields.

However, the analytical SIF-solution for an elliptical crack embedded in an infinite body and subjected to an arbitrary applied stress field is still absent in the literature.

In the present research, the method of simultaneous dual integral equations, used previously by Shah and Kobayshi [43] for a problem of a circular crack, is adopted to the problem of an elliptical crack. The method allows the treatment of a wide class of applied stress distributions.

Crack problems formulated and solved for an infinite elastic solid are based on the assumption that the crack is sufficiently far away from the boundaries of the body. However, in practical applications the effect of free boundaries may significantly affect the stress distribution near the crack front. In order to account for the free boundary effect, the alternating method can be used. The (Schwarz-Newmann) alternating method was introduces by Kantorovich and Krylov in [34]. The method includes the successive, iterative superposition of two solutions in order to satisfy the boundary conditions. The first solution is the stress intensity factor and the stress field in an infinite body, containing a crack. The second solution is for the stress field in a semi-infinite or finite body subjected to the stress field applied to the boundary plane. This solution for a semi-infinite body has been derived analytically (see for example, Love [2], Kupradze [51]). For finite bodies the stress field can be obtained numerically using the finite element method (see for example, Smith [15], Nisioka and Atluri [50]).

Shah and Kobayashi in [45], Kassir and Sih in [36] have solved the problems of a penny-shaped crack and an elliptical crack near the boundary of a semi-infinite elastic solid using the alternating method. The main difficulty in using this method for a problem of an elliptical crack was that the first solution - the SIF and the stress field for an elliptical crack in an infinite body - was restricted by the case of a polynomial applied pressure. Therefore, Shah and Kobayashi [45] and Kassir and Sih [36] used the polynomial approximation of the correcting stress over the crack domain due to a free surface, which lead to the certain inaccuracy in results.

A wide class of crack problems are problems of surface breaking planar cracks in semi-infinite and finite bodies.

Hartranft and Sih [48] analyzed the problem of a semi-circular crack in a semi-infinite body by the alternating method. The SIF and the stress field solutions were obtained in the integral form with Bessel kernels. The stress field solution in a semi-infinite body loaded on the boundary plane was taken from [2].

Smith and Sorensen in [15] analyzed the semi-elliptical crack in a plate under uniform tension using the alternating method. The stress field due to the free surface was approximated by the polynomial of the third degree and Shah and Kobayashi's solution [43] was used to obtained the corresponding SIF and the stress field. The polynomial representation of the applied stress field lead to certain underestimation of the SIF-solution.

Nishioka and Atluri [50] used the alternating method with the finite element stress field solution for a problem of a semi-elliptical crack in a plate under remote tension and bending. A fifth degree polynomial was used to fit the stress field over the crack surface due to the free surface. That allowed to increase the accuracy of the results in comparison with previous works. Using the finite element solution for a stress field enables to analyze the finite bodies of an arbitrary shape.

Most practical crack configurations are embedded and surface breaking planar cracks in finite bodies. These problems cannot be solved analytically due to their complexity. Raju and Newman [25] used the finite element method to calculate stress intensity factors for semi-elliptical cracks in finite thickness plates subjected to tension. Later they fitted their results for elliptical, semi-elliptical and quarter-elliptical cracks into an empirical equation [27] for the SIF as a function of the parametric angle, crack depth and length, plate thickness and width for tension and bending loads. Their solution is applicable to a wide range of crack and plate parameters and is widely used in engineering applications.

Several variational and potential methods concerning investigation of existence and uniqueness of crack problem solutions [6],[22] have also been developed during the last 30 years. P.Ciarlet [8] proposed to use Sobolev spaces for the investigation of the displacement-traction problem in three-dimensional linear elasticity. Kupradze [51] has introduced a special form of elastic displacements by means of single-layer and double-layer potentials with unknown distributed densities. Costabel and Stephan [35] transformed the variational Dirihlet and Neumann problems to a system of boundary equations and proved the existence, uniqueness and stability of solutions in a special weighted Sobolev space.

1.1 The weight function method

The weight function concept proposed by Bueckner [20] and Rice [46] has proven to be the most efficient method for calculating SIFs. The weight function (also called a Green's function) represents the SIF for a crack subjected to a unit force applied to the crack surface. If the weight function for a given cracked body configuration is known, the SIF due to an arbitrary stress field can be determined by integrating the product of the weight function and the applied stress field over the crack domain.

The only exact analytical weight function for a planar crack is that one obtained for a circular crack in an infinite body ([36], [52], and others).

Some work has been done to obtain the exact analytical weight function for an elliptical crack. Roy and Saha [4] have transformed the governing integral equation for an elliptical crack in an infinite body subjected to a point load into a system of Fredholm integral equations of the second kind. This system can be further transformed into a system of linear algebraic equations. The weight function is sought in the form of the Fourier expansion. Roy and Saha [4] restricted their weight function to calculating only the first few terms, sufficient for obtaining SIFs for the constant, linear and quadratic applied stress fields.

In the work discussed below the unique exact analytical weight function for an elliptical crack in an infinite elastic body has been obtained.

Approximate weight functions are also widely used in engineering applications. Shen and Glinka [16] developed the general form of the approximate weight function for a semi-elliptical crack with three unknown coefficients which can be found from the reference SIF-solutions.

Wang, Lambert and Glinka [58] analyzed the weight functions for a circular and a plane cracks and proposed the approximate form of the weight function for an elliptical crack. The proposed weight function was validated to be appropriate for polynomial applied stress fields.

Glinka and Reinhardt [18] modified the weight function proposed by Oore and Burns in [37] for planar cracks of arbitrary shape.

Other approximate weight functions for elliptical, semi-elliptical and quarter-elliptical cracks have been derived, for example, in [59], [26], [3], [39] and others.

Planar cracks of other geometries, such as a plane crack, a plane crack with a wavy front, inverse external elliptical crack, a tunnel crack were considered in [30], [31], [61], [40] respectively.

The main purpose of the proposed research is to obtain the analytical solution for SIFs for an elliptical crack subjected to an arbitrary normal pressure, and to also derive the solution for the concentrated force load, i.e the weight function.

The SIF solution for an elliptical crack was used together with the alternating method to obtain the SIF-solution for a semi-elliptical crack.

The thesis is organized as follows.

The problem definition and the purpose of the research work is formulated in chapter 1.

The general mathematical formulation of a planar crack problem, the stress intensity factor concept and the weight function concept are explained in chapter 2.

In chapter 3 the exact analytical SIF-solution is derived for an embedded elliptical crack subjected to an arbitrary Mode I loading, including the particular case of a point load, i.e. the weight function.

In chapter 4 the SIF is derived for a surface semi-elliptical crack subjected to an arbitrary Mode I stress field.

A summary of all results and solutions as well as possible future work are discussed in chapter 5.

Results obtained in the present work were published in references [9], [10], [11], [12] and presented on 7th International Conference on Fatigue Damage of Structural Materials (Hyannis, 2008) and 12th International Conference on Fracture (Ottawa, 2010).

Chapter 2

Preliminaries

2.1 General formulation of an embedded planar crack problem

In order to analyze the stress field in a cracked body, the following boundary crack problem is formulated. A three-dimensional infinite isotropic elastic body (with material constants E and ν) is considered. The state of deformation is characterized by a displacement field

$$u = u(x, y, z), \quad v = v(x, y, z), \quad w = w(x, y, z)$$

in Cartesian coordinates (x, y, z) connected with the considered body. In the absence of mass forces, functions u, v, w satisfy the following equations of elastic equilibrium (Lame equations)[53]:

$$\begin{aligned}
\Delta u + \frac{1}{1-2\nu}\frac{\partial \Theta}{\partial x} &= 0, \\
\Delta v + \frac{1}{1-2\nu}\frac{\partial \Theta}{\partial y} &= 0, \\
\Delta w + \frac{1}{1-2\nu}\frac{\partial \Theta}{\partial z} &= 0,
\end{aligned} \tag{2.1}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \qquad \Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

The components

$$\sigma_{xx} = \sigma_{xx}(x,y,z), \quad \sigma_{yy} = \sigma_{yy}(x,y,z), \quad \sigma_{zz} = \sigma_{zz}(x,y,z),$$

$$\tau_{xy} = \tau_{xy}(x,y,z), \quad \tau_{yz} = \tau_{yz}(x,y,z), \quad \tau_{zx} = \tau_{zx}(x,y,z)$$

of the elastic stress tensor are determined on the basis of Hooke's Law and the Cauchy relationships as [53]

$$\begin{aligned}
\sigma_{xx} &= 2G\left(\frac{\partial u}{\partial x} + \frac{\nu}{1-2\nu}\Theta\right), \quad \tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \\
\sigma_{yy} &= 2G\left(\frac{\partial v}{\partial y} + \frac{\nu}{1-2\nu}\Theta\right), \quad \tau_{yz} = G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \\
\sigma_{zz} &= 2G\left(\frac{\partial w}{\partial z} + \frac{\nu}{1-2\nu}\Theta\right), \quad \tau_{zx} = G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right),
\end{aligned} \tag{2.2}$$

where $G = E/(2(1+\nu))$. It is also assumed that stresses and displacements vanish at the infinity.

A planar crack occupies an open domain S (Fig.2.1) in the plane $z = 0$. The crack is opened up by an applied normal pressure $p\,(x,y)$ symmetric with respect to the crack plane. Due to the symmetry of the applied load, it is sufficient to analyze only the upper half-space $z > 0$. The following boundary conditions are imposed.

$$\begin{aligned} \tau_{zx} = \tau_{zy} = 0, &\qquad \text{for} \qquad z = 0, \\ \sigma_{zz}(x, y, 0) = -p\,(x, y), &\qquad \text{for} \qquad (x, y) \in S, \\ w(x, y, 0) = 0, &\qquad \text{for} \qquad (x, y) \in S_1, \end{aligned} \tag{2.3}$$

where S_1 is an open domain in the plane $z = 0$ outside a crack.

Following Panasyuk [53], in order to satisfy the equations of equilibrium (2.1), the displacement field (u, v, w) is sought in the following form:

$$u = \phi_1 - \frac{z}{2(1-\nu)}\frac{\partial \phi_3}{\partial x}, \qquad v = \phi_2 - \frac{z}{2(1-\nu)}\frac{\partial \phi_3}{\partial y}, \qquad w = \phi_3 - \frac{z}{2(1-\nu)}\frac{\partial \phi_3}{\partial z}, \tag{2.4}$$

where $\phi_1 = \phi_1(x, y, z)$, $\phi_2 = \phi_2(x, y, z)$, $\phi_3 = \phi_3(x, y, z)$ are harmonic functions connected by the equations:

$$\frac{\partial \phi_1}{\partial z} = -\frac{1-2\nu}{2(1-\nu)}\frac{\partial \phi_3}{\partial x}, \qquad \frac{\partial \phi_2}{\partial z} = -\frac{1-2\nu}{2(1-\nu)}\frac{\partial \phi_3}{\partial y} \tag{2.5}$$

The stress field components are expressed using eq.(2.4) and (2.2) as

$$\frac{2(1-\nu^2)}{E}\sigma_{zz} = \frac{\partial \phi_3}{\partial z} - z\frac{\partial^2 \phi_3}{\partial z^2} \tag{2.6}$$

$$
\begin{aligned}
\frac{2(1-\nu^2)}{E}\sigma_{yy} &= 2\nu\frac{\partial\phi_3}{\partial z} + 2(1-\nu)\frac{\partial\phi_2}{\partial y} - z\frac{\partial^2\phi_3}{\partial y^2}, \\
\frac{2(1-\nu^2)}{E}\sigma_{xx} &= 2\nu\frac{\partial\phi_3}{\partial z} + 2(1-\nu)\frac{\partial\phi_1}{\partial x} - z\frac{\partial^2\phi_3}{\partial x^2}, \\
\frac{2(1-\nu^2)}{E}\tau_{zx} &= -z\frac{\partial^2\phi_3}{\partial x\partial z}; \quad \frac{2(1-\nu^2)}{E}\tau_{zy} = -z\frac{\partial^2\phi_3}{\partial y\partial z} \\
\frac{2(1-\nu^2)}{E}\tau_{xy} &= (1-\nu)\left(\frac{\partial\phi_1}{\partial y} + \frac{\partial\phi_2}{\partial x}\right) - z\frac{\partial^2\phi_3}{\partial x\partial y}.
\end{aligned}
\tag{2.7}
$$

As it follows from eq.(2.7), (2.7), (2.5), (2.4) the first condition of system (2.3) is satisfied. The stress field component σ_{zz} and the displacement w in the plane $z = 0$ are expressed through harmonic function ϕ_3 as

$$
\frac{\partial\phi_3}{\partial z}(x,y,0) = \frac{2(1-\nu^2)}{E}\sigma_{zz}(x,y,0), \qquad \phi_3(x,y,0) = w(x,y,0) \tag{2.8}
$$

Hence the problem (2.3) is reduced to finding one function ϕ_3, harmonic in the half-space $z > 0$ and satisfying the following conditions in the plane $z = 0$:

$$
\begin{aligned}
\frac{\partial\phi_3}{\partial z}(x,y,0) &= -\frac{2(1-\nu^2)}{E}p(x,y) \quad \text{for} \quad (x,y) \in S \\
\phi_3(x,y,0) &= 0 \quad \text{for} \quad (x,y) \in S_1
\end{aligned}
\tag{2.9}
$$

The function ϕ_3 is given in the form of the single-layer potential as

$$
\phi_3(x,y,z) = -\frac{1}{2\pi}\frac{\partial}{\partial z}\iint\limits_{\mathbb{R}^2} \frac{\phi_3(\xi,\eta,0+)d\xi\, d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}} \tag{2.10}
$$

After differentiating eq.(2.10) and taking into account that the integral in (2.10) is the harmonic function and considering eq.(2.8), the following governing integral equation for the unknown crack opening displacement in the region S is obtained:

$$\Delta_{xy} \iint\limits_S \frac{w(\xi, \eta, 0)\, d\xi\, d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} = -\sigma(x, y), \tag{2.11}$$

where

$$\sigma(x, y) = \frac{4\pi(1-\nu^2)}{E}\, p\,(x, y) \tag{2.12}$$

and Δ_{xy} is the two-dimensional Laplace operator:

$$\Delta_{xy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The eq.(2.11) is a particular case of the general equation derived in [35] based on the Potential Theory of Elasticity. The general equation [35] is suitable for any two-dimensional crack with a smooth crack surface S and a smooth crack front, subjected to the arbitrary stress fields applied to the crack surfaces. The eq.(2.11) is obtained under assumptions that the crack is planar and the applied stress field is symmetric with respect to the crack plane.

The solution of eq.(2.11) cannot be obtained in the general form. It has been solved analytically only for the cases of a circular crack (see for example [36]) and for an elliptical crack subjected to the specific polynomial stress fields ([36],[43],[41]). In the present thesis the method of simultaneous dual integral equations is employed to solve eq.(2.11) for the case of an elliptical crack under an arbitrary applied stress field.

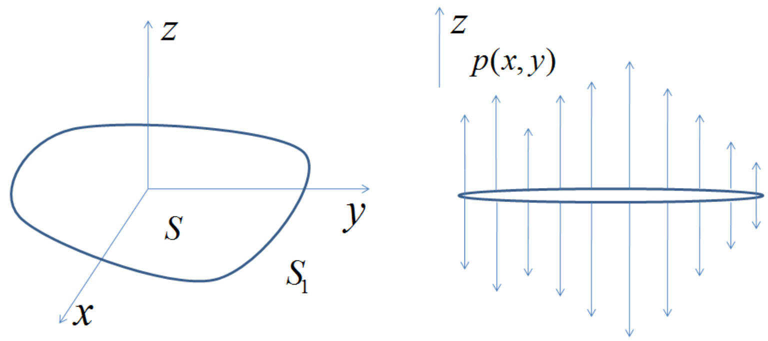

Figure 2.1: Planar crack under applied normal pressure.

2.2 The Stress Intensity Factors concept

Let us introduce a local system of coordinates (Fig.2.2) with the origin $Q' = Q'(x', y', 0)$ on the crack front and with the axes n and τ normal and tangential respectively to the crack front. The axis b is normal to the plane of the crack. In the coordinate system defined above, the stress tensor components σ_n, σ_τ, σ_b, σ_{nb} at the point $Q = Q(x, y, z)$ on the (n, b) coordinate plane admit the following decomposition [29] in the vicinity of the crack front:

$$\begin{aligned}
\sigma_n &= \frac{K}{\sqrt{2\pi\rho}} \cos\frac{\vartheta}{2}\left(1 - \sin\frac{\vartheta}{2}\sin\frac{3\vartheta}{2}\right) + O(1) \\
\sigma_b &= \frac{K}{\sqrt{2\pi\rho}} \cos\frac{\vartheta}{2}\left(1 + \sin\frac{\vartheta}{2}\sin\frac{3\vartheta}{2}\right) + O(1) \\
\sigma_\tau &= \frac{2\nu\, K}{\sqrt{2\pi\rho}} \cos\frac{\vartheta}{2} + O(1) \\
\sigma_{nb} &= \frac{K}{\sqrt{2\pi\rho}} \sin\frac{\vartheta}{2}\cos\frac{\vartheta}{2}\cos\frac{3\vartheta}{2} + O(1).
\end{aligned} \tag{2.13}$$

The radius ρ is the distance from point $Q(x, y, z)$ to point $Q'(x', y', 0)$ $(\rho \to 0)$, ϑ is the angle between $Q'Q$ and the normal n. The parameter K is called *the stress intensity factor (SIF)*. The SIF K is a function of the position of point Q' on the crack front ($K = K(Q')$), the crack geometry and the applied pressure.

Irwin [19] observed first that the stress field near the crack front contains a $\frac{1}{\sqrt{\rho}}$ singularity and can be quantitatively described by only one parameter - the stress intensity factor K. The displacement field u_n, u_τ, u_b at the point $Q(x, y, z)$ in the coordinates (ρ, ϑ) is given by Hartranft and Sih [47] as:

$$\begin{aligned} u_n &= \frac{1+\nu}{E}\sqrt{\frac{2\rho}{\pi}}K\cos\frac{\vartheta}{2}\left[(1-2\nu)+\sin^2\frac{\vartheta}{2}\right]+O(\rho) \\ u_b &= \frac{1+\nu}{E}\sqrt{\frac{2\rho}{\pi}}K\sin\frac{\vartheta}{2}\left[2(1-\nu)-\cos^2\frac{\vartheta}{2}\right]+O(\rho) \\ u_\tau &= O(\rho). \end{aligned} \tag{2.14}$$

The SIF for a given crack configuration can be obtained in two alternative ways. First, substituting $\vartheta = 0$, $\sigma_b = \sigma_{zz}$ in eq.(2.13) and taking the limit as $\rho \to 0$, $K(Q')$ is expressed as:

$$K(Q') = \lim_{\rho\to 0}\sqrt{2\pi\rho}\,\sigma_{zz}(x, y, 0) \tag{2.15}$$

Second, substituting $\vartheta = \pi$, $u_b = w$ in eq.(2.14) and taking a limit as $\rho \to 0$, $K(Q')$ is expressed as:

$$K(Q') = \frac{E}{2(1-\nu^2)}\sqrt{\frac{\pi}{2}}\lim_{\rho\to 0}\frac{w(x, y, 0)}{\sqrt{\rho}}. \tag{2.16}$$

The first approach is commonly used in numerical methods, while the second one is more convenient for the analytical procedure for deriving the SIF. The crack opening displacement $w(x, y, 0)$ is obtained from eq.(2.11) and subsequently eq.(2.16) is used to express the SIF $K(Q')$.

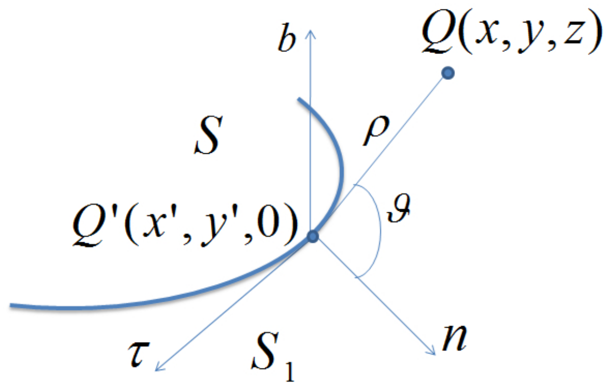

Figure 2.2: Local system of coordinates.

2.3 The Weight Function concept

The weight function method for calculating the stress intensity factors was first introduced by Bueckner [20] and Rice [46]. The weight function for a given cracked body configuration represents the SIF for a crack subjected to a unit concentrated load (Fig.2.3). The following notations are used. The point load is applied at an arbitrary point $Q_0 = Q_0(x_0, y_0, 0)$ inside the crack domain. The function $p(x, y)$ in this case can be expressed by the two-dimensional delta function:

$$p(x, y) = P\delta(x - x_0)\delta(y - y_0). \tag{2.17}$$

The magnitude of the applied force P can be taken equal to 1, or the corresponding SIF should be normalized by P. The SIF $K(Q')$ in the case of the applied force is a function of two points Q' and Q_0:

$$K(Q') = W(Q', Q_0) \tag{2.18}$$

Hence, the important feature of the weight function is that it depends only on the geometry of the cracked body.

If the weight function $W(Q', Q_0)$ for a given geometrical configuration is known, the SIF can be determined for any arbitrary applied stress $p(x, y)$. This is achieved by integrating the product of the weight function and the stress field applied over the crack domain:

$$K(Q') = \iint\limits_S W(Q', Q_0)p(x_0, y_0)dx_0dy_0 \tag{2.19}$$

The main difficulty in this approach is obtaining the weight function for a given cracked body. The exact analytical weight function has only been obtained for a circular crack in an infinite body ([43]). In this thesis the exact analytical weight function has been derived for an elliptical crack in an infinite body.

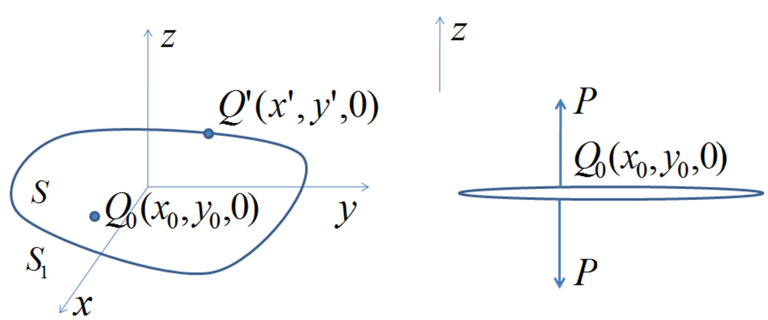

Figure 2.3: Planar crack under a pair of splitting concentrated forces.

Chapter 3

Stress Intensity Factors for an Embedded Elliptical Crack subjected to an arbitrary normal loading.

3.1 The method of simultaneous dual integral equations for an elliptical crack

The method of simultaneous dual integral equations, developed by Sneddon [23] and Kassir and Sih [36] for the problem of a circular crack is applied to solve the equation (2.11) for an elliptical crack. Let the crack have dimensions a, b $(a > b)$ (Fig.3.1). The crack domain S is defined in Cartesian coordinates (x, y) as $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} < 1$. For convenience Cartesian coordinates (x, y) are changed to the elliptical system of coordinates (r, θ) such that $x = ar\cos\theta, y = br\sin\theta$. Two domains S and S_1 are defined as

$$S : r < 1,\ 0 \leq \theta \leq 2\pi \qquad S_1 : r > 1,\ 0 \leq \theta \leq 2\pi. \tag{3.1}$$

The load is expanded, in the next step, into the Fourier series:

$$\sigma(x, y) = \sigma(r, \theta) = \frac{\sigma_0^c(r)}{2} + \sum_{n=1}^{\infty} (\sigma_n^c(r)\cos\, n\theta + \sigma_n^s(r)\sin\, n\theta) \tag{3.2}$$

The inverse Fourier transform

$$\frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} = \frac{1}{2\pi} \iint\limits_{\mathbb{R}^2} \frac{1}{\sqrt{p_1^2 + p_2^2}} e^{-\boldsymbol{i}(x\,p_1 + y\,p_2)} e^{\boldsymbol{i}(\xi\,p_1 + \eta\,p_2)} dp_1 dp_2 \tag{3.3}$$

(where $\boldsymbol{i} = \sqrt{-1}$) is applied to the eq.(2.11) and the new unknown function is introduced:

$$F(p_1, p_2) = \frac{1}{2\pi} \iint\limits_{S} e^{\boldsymbol{i}(\xi\,p_1 + \eta\,p_2)} w(\xi, \eta, 0)\, d\xi\, d\eta \tag{3.4}$$

Further, function $F(p_1, p_2)$ is expanded in elliptical coordinates $(ap_1 = R\cos\psi, bp_2 = R\sin\psi)$ into the Fourier series:

$$F(R, \psi) = \sum_{m=0}^{\infty} (F_m^c(R)\cos m\psi + F_m^s(R)\sin m\psi). \tag{3.5}$$

Using the expansion:

$$e^{-\boldsymbol{i}\,R\,r\cos(\theta-\psi)} = J_0(Rr) + \sum_{n=1}^{\infty} 2(-\boldsymbol{i})^n J_n(Rr)\cos n(\theta - \psi), \tag{3.6}$$

where $J_n(Rr)$ is a Bessel function of the first kind of order n, eq.(2.11) and the last displacement boundary condition in (2.3) can be transformed into a system of dual integral equations for every component $F_n^c(R)$:

$$\begin{aligned} \int\limits_0^\infty \sum_{m=0}^\infty \alpha_{nm}^c F_m^c(R) J_n(Rr) R^2 dR &= \frac{a\, b^2 \sigma_n^c(r)}{2(-\boldsymbol{i})^n}, \quad 0 \le r < 1 \\ \int\limits_0^\infty F_n^c(R) J_n(rR) R\, dR &= 0, \qquad\qquad r > 1, \end{aligned} \tag{3.7}$$

where

$$\alpha_{nm}^c = \int\limits_0^{2\pi} \sqrt{\frac{b^2}{a^2}\cos^2\psi + \sin^2\psi}\, \cos n\psi \cos m\psi\, d\psi. \tag{3.8}$$

An analogous system can be constructed for components $F_n^c(R)$ with parameters:

$$\alpha_{nm}^s = \int\limits_0^{2\pi} \sqrt{\frac{b^2}{a^2}\cos^2\psi + \sin^2\psi}\, \sin n\psi \sin m\psi\, d\psi. \tag{3.9}$$

Following Sneddon [24], the solution of (3.7) is sought in the form

$$F_n^c(R) R = R^{-1/2} \sum_{k=0}^\infty A_{kn}^c J_{n+2k+3/2}(R), \tag{3.10}$$

where $J_{n+2k+3/2}(R)$ is a Bessel function of the first kind of order $n+2k+3/2$.

Substituting (3.10) into the eq.(3.7) and making use of the Weber-Schafheitlin integral (3.11) [57]

$$\begin{aligned} \int\limits_0^\infty t^{-\varrho} J_\alpha(At) J_\beta(Bt) dt &= \frac{A^\alpha 2^{-\varrho} B^{\varrho-\alpha-1} \Gamma\left(\dfrac{1+\beta+\alpha-\varrho}{2}\right)}{\Gamma\left(\dfrac{1+\beta+\varrho-\alpha}{2}\right)\Gamma(1+\alpha)} \\ &\times {}_2F_1\left(\frac{1+\beta+\alpha-\varrho}{2}, \frac{1+\alpha-\beta-\varrho}{2}; \alpha+1; \frac{A^2}{B^2}\right), \end{aligned} \tag{3.11}$$

$$\mathrm{Re}(\beta+\alpha-\varrho+1)>1, \quad \mathrm{Re}\,\varrho>-1, \quad 0<A<B \tag{3.12}$$

the second equation of (3.7) is satisfied automatically and the first equation of (3.7) can be transformed into the system of linear equations with unknown coefficients A^c_{km}:

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} C^c_{nmki} A^c_{km} = B^c_{ni} \tag{3.13}$$

$$C^c_{nmki} = \begin{cases} \alpha^c_{nm}, & m+2\,k = n+2\,i \\ 0, & \text{otherwise}, \end{cases} \tag{3.14}$$

$$B^c_{ni} = \frac{2(n+2i+\frac{3}{2})\Gamma(n+i+1)}{(-\boldsymbol{i})^n\Gamma(1+n)\Gamma(i+3/2)} \int_0^1 \sigma^c_n(r) r^{n+1}\sqrt{1-r^2}\,\mathfrak{F}_i(n+\tfrac{3}{2}, n+1, r^2)dr, \tag{3.15}$$

where $\mathfrak{F}_i(n+\frac{3}{2}, n+1, r^2)$ is Jacobi Polynomial of degree i in r^2, defined by the hypergeometric function:

$$\mathfrak{F}_i(n+\tfrac{3}{2}, n+1, r^2) = {}_2F_1(-i, i+n+3/2, n+1, r^2). \tag{3.16}$$

In a similar manner, the analogous system for the unknown A^s_{km} can be obtained:

$$\sum_{m=1}^{\infty}\sum_{k=0}^{\infty} C^s_{nmki} A^s_{km} = B^s_{ni} \tag{3.17}$$

Next it is convenient to rewrite the systems (3.13), (3.17) in the following form with respect to the original applied stress field $p(x,y)$:

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} C^c_{nmki} \mathcal{A}^c_{km} = \mathcal{B}^c_{ni}, \tag{3.18}$$

$$\sum_{m=1}^{\infty}\sum_{k=0}^{\infty} C^s_{nmki}\mathcal{A}^s_{km} = \mathcal{B}^s_{ni}, \tag{3.19}$$

where

$$\mathcal{B}^c_{ni} = \frac{2(n+2i+\frac{3}{2})\Gamma(n+i+1)}{(-\boldsymbol{i})^n\Gamma(1+n)\Gamma(i+3/2)}\int_0^1 p^c_n(r)r^{n+1}\sqrt{1-r^2}\,\mathfrak{F}_i(n+\tfrac{3}{2},n+1,r^2)dr, \tag{3.20}$$

$$\mathcal{B}^s_{ni} = \frac{2(n+2i+\frac{3}{2})\Gamma(n+i+1)}{(-\boldsymbol{i})^n\Gamma(1+n)\Gamma(i+3/2)}\int_0^1 p^s_n(r)r^{n+1}\sqrt{1-r^2}\,\mathfrak{F}_i(n+\tfrac{3}{2},n+1,r^2)dr, \tag{3.21}$$

and

$$p(x,y) = p(r,\theta) = \frac{p_0(r)}{2} + \sum_{n=1}^{\infty}(p^c_n(r)\cos\, n\theta + p^s_n(r)\sin\, n\theta) \tag{3.22}$$

After solving (3.18), (3.19), the crack opening displacement can be obtained from the second equation of system (3.7) for $r<1$ and eqs.(3.10), (3.11) as:

$$\begin{aligned} w(r,\theta) = \frac{2(1-\nu^2)}{E}\frac{\pi\, b}{2}\sqrt{1-r^2}\sum_{n=0}^{\infty}\frac{(-\boldsymbol{i})^n\, r^n}{\Gamma(n+1)}\sum_{k=0}^{\infty}\frac{\Gamma(n+k+1)}{\Gamma(k+\frac{3}{2})}\mathfrak{F}_k(n+\tfrac{3}{2},n+1,r^2) \\ \times\,[\mathcal{A}^c_{kn}\cos\, n\theta + \mathcal{A}^s_{kn}\sin\, n\theta] \end{aligned} \tag{3.23}$$

In order to obtain limiting values of the crack opening displacement according to eq.(2.16), the following asymptotic expansion in the vicinity of a point $Q'(x'=a\cos\varphi, y'=b\sin\varphi)$ (i.e. $r=1$) is used (Fig.3.2):

$$\begin{aligned} x = ar\cos\theta = a\cos\varphi - \frac{\rho\, b\cos\varphi}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}} \\ y = br\sin\theta = b\sin\varphi - \frac{\rho\, a\sin\varphi}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}} \end{aligned} \tag{3.24}$$

$$1 - r^2 = 2\rho \frac{1}{b}\sqrt{\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi} + O(\rho^2) \tag{3.25}$$

The SIF can subsequently be obtained as:

$$K(\varphi) = \pi\sqrt{b}\left(\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi\right)^{1/4}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-\boldsymbol{i})^n(-1)^k[\mathcal{A}^c_{kn}\cos n\varphi + \mathcal{A}^s_{kn}\sin n\varphi] \tag{3.26}$$

The stress $\sigma_{zz}(x, y, 0)$ in the plane $z = 0$ outside of the crack ($r > 1$) can be obtained in the form of the Fourier expansion:

$$\sigma_{zz}(x, y, 0) = \sigma_{zz}(r, \theta) = \frac{\sigma^c_{zz,0}(r)}{2} + \sum_{n=1}^{\infty}(\sigma^c_{zz,n}(r)\cos\, n\theta + \sigma^s_{zz,n}(r)\sin\, n\theta), \qquad r > 1 \tag{3.27}$$

where each term $\sigma_{zz,n}(r)$ is derived using the first equation in (3.7):

$$\begin{aligned}\sigma^c_{zz,n}(r) = \frac{-(-\boldsymbol{i})^n}{\sqrt{r^2-1}}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}&\frac{\alpha^c_{nm}\,\mathcal{A}^c_{mk}\,r^{-m-2k-2}\,\Gamma(\frac{n+m+2k+3}{2})}{\Gamma(\frac{n-m-2k-1}{2})\Gamma(m+2k+5/2)}\\ &\times\ {}_2F_1\left(\tfrac{n+m+2k+2}{2}, \tfrac{-n+m+2k+2}{2}, m+2k+5/2, \tfrac{1}{r^2}\right)\end{aligned} \tag{3.28}$$

Terms $\sigma^s_{zz,n}(r)$ are obtained by substituting $\mathcal{A}^s_{mk}$ for $\mathcal{A}^c_{mk}$ in (3.28). Eq.(3.26) for the SIF can also be derived using eq.(3.28) and definition (2.15).

3.1.1 Special case of a circular crack

In the case of a circular crack ($a = b$), the system of equations (3.7) reduces to:

$$\begin{aligned}&\int_0^{\infty} F^c_n(R)J_n(Rr)R^2dR = \frac{a^3\sigma^c_n(r)}{2(-\boldsymbol{i})^n\alpha^c_{nn}}, \quad 0 \le r < 1\\ &\int_0^{\infty} F^c_n(R)J_n(rR)R\,dR = 0, \qquad\qquad r > 1,\end{aligned} \tag{3.29}$$

where $\alpha_{00}^c = 2\pi$ and $\alpha_{nn}^c = \pi$ for any $n \geq 1$.

According to [5], the system of equations (3.29) has an analytical solution:

$$F_n^c(R)R = \sqrt{\frac{2}{\pi}} \frac{a^3 R^{1/2}}{\alpha_{nn}^c (-\boldsymbol{i})^n} \int_0^1 t^{-n+1/2} J_{n+1/2}(tR)dt \int_0^t \frac{\sigma_n^c(\zeta)\zeta^{n+1}d\zeta}{\sqrt{t^2 - \zeta^2}} \tag{3.30}$$

Analogous system of equations and its solution hold for every $F_n^s(R)$ with $\alpha_{nn}^s = \pi$ for any $n \geq 1$. The crack opening displacement can be obtained as:

$$\begin{aligned} w(x,y,0) &= w(r,\theta) \\ &= \frac{2(1-\nu^2)}{E} \frac{2a}{\pi} \sum_{n=0}^{\infty} \int_r^1 \frac{r^n t^{-2n} dt}{\sqrt{t^2 - r^2}} \int_0^t \frac{(f_n^c(\zeta) \cos n\theta + f_n^s(\zeta) \sin n\theta)\zeta^{n+1} d\zeta}{\sqrt{t^2 - \zeta^2}}, \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} f_0^c(r) &= \frac{1}{2\pi} \int_0^{2\pi} p(r,\theta) d\theta = \frac{p_0^c(r)}{2}, \\ f_n^c(r) &= \frac{1}{\pi} \int_0^{2\pi} p(r,\theta) \cos n\theta \, d\theta = p_n^c(r), \\ f_n^s(r) &= \frac{1}{\pi} \int_0^{2\pi} p(r,\theta) \sin n\theta \, d\theta = p_n^s(r). \end{aligned} \tag{3.32}$$

By taking the limit according to eq.(2.16), the SIF is obtained in the form of a series:

$$K(\varphi) = 2\sqrt{\frac{a}{\pi}} \sum_{n=0}^{\infty} \int_0^1 \frac{r^{n+1}}{\sqrt{1-r^2}} (f_n^c(r) \cos n\varphi + f_n^s(r) \sin n\varphi) dr \tag{3.33}$$

The SIF for a circular crack subjected to an arbitrary applied stress field was previously obtained in the form of eq.(3.33) by Kassir and Sih [36].

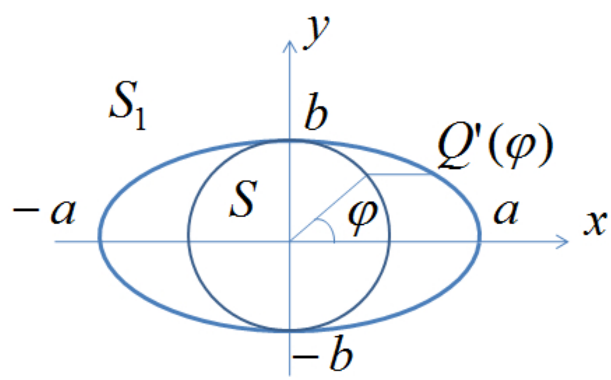

Figure 3.1: Elliptical crack.

3.2 Numerical results in comparison with existing SIF-solutions

The method developed in section 3.1 was used to obtain SIF-solutions for an elliptical crack subjected to five different polynomial applied stress fields. The results were then compared with the analytical expressions obtained by Kassir and Sih [36],Martin [41], Shah and Kobayshi [43].

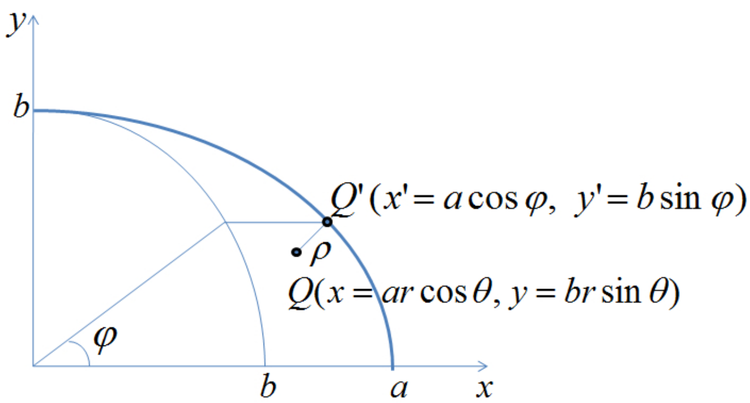

Figure 3.2: Distance to the crack front.

3.2.1 An elliptical crack under uniform pressure, applied over the entire crack area

In the case of an elliptical crack subjected to a uniform pressure, the applied stress field is represented (Fig.3.3) by the constant function:

$$p(x,y) = \sigma_0 \quad \text{or} \quad p(r,\theta) = \sigma_0 \quad \text{with} \quad p_0^c(r) = 2\sigma_0. \tag{3.34}$$

There is only one non-zero coefficient in the series (21):

$$\mathcal{A}_{00}^c = \frac{\mathcal{B}_{00}^c}{\alpha_{00}^c} = \frac{\sigma_0}{\sqrt{\pi}E(k)}, \tag{3.35}$$

where $E(k)$ is the elliptic integral of the second kind with $k = \dfrac{\sqrt{a^2 - b^2}}{a}$.

The SIF in such a case is:

$$K(\varphi) = \frac{\sqrt{\pi b}\,\sigma_0}{E(k)} \left(\frac{b^2}{a^2} \cos^2\varphi + \sin^2\varphi \right)^{1/4}. \tag{3.36}$$

This solution coincides with the SIF first obtained by Irwin [19]. The variation of the SIF along the contour for elliptical cracks having various b/a ratios and subjected to a uniform pressure σ_0 is shown in Fig.3.2.5.

The crack opening displacement in such a case is:

$$w(r,\theta) = \frac{2(1-\nu^2)}{E} \frac{b\,\sigma_0}{E(k)} \sqrt{1-r^2}. \tag{3.37}$$

This result coincides with Panasyuk's solution [53].

The stress σ_{zz} in the plane $z = 0$ outside of the crack ($r > 1$) can be obtained in the form of a series:

$$\begin{aligned} \sigma_{zz}(r,\theta) &= \frac{\sigma_{zz,0}(r)}{2} + \sum_{j=1}^{\infty} \sigma_{zz,2j}(r) \cos 2j\theta \\ &= \frac{2}{\pi}\sigma_0 \left(\frac{1}{\sqrt{r^2-1}} - \arcsin\frac{1}{r} \right) - \frac{4\sigma_0}{3\pi\, E(k)} \frac{r^{-2}}{\sqrt{r^2-1}} \\ &\times \sum_{j=1}^{\infty} (-1)^j \alpha_{2j,0} \frac{\Gamma(j+\frac{3}{2})}{\Gamma(j-\frac{1}{2})}\, {}_2F_1(j+1, -j+1, \tfrac{5}{2}, \tfrac{1}{r^2}) \cos 2j\theta \end{aligned} \tag{3.38}$$

Note that in a case of a circular crack ($a = b$) eq.(3.38) reduces to

$$\sigma_{zz}(r,\theta) = \frac{2}{\pi}\sigma_0 \left(\frac{1}{\sqrt{r^2-1}} - \arcsin\frac{1}{r} \right), \tag{3.39}$$

what coincides with the well known result, obtained in [51], [53] and others.

3.2.2 An elliptical crack under linear stress field dependent on coordinate x

In the case of an elliptical crack subjected to a linear stress field that varies in x, the applied stress field is represented (Fig.3.5) by the function:

$$p(x,y) = \sigma_0 \frac{x}{a} \quad \text{or} \quad p(r,\theta) = \sigma_0 r \cos\theta \qquad \text{with} \quad p_1^c(r) = \sigma_0 r. \tag{3.40}$$

The appropriate solution for $\mathcal{A}_{01}^c$ in eq.(3.13) is

$$\mathcal{A}_{01}^c = \frac{\mathcal{B}_{10}^c}{\alpha_{11}^c} = \frac{4\,\sigma_0}{3(-\boldsymbol{i})\sqrt{\pi}\alpha_{11}^c} \tag{3.41}$$

and

$$\alpha_{11}^c = -\frac{4}{3}\frac{(1-2\,k^2)E(k) - k'^2 K(k)}{k^2}, \tag{3.42}$$

where $K(k)$ is the complete elliptic integral of the first kind and $k' = \dfrac{b}{a}$.

The SIF in such a case is

$$K(\varphi) = -\frac{\sqrt{\pi\, b}\,\sigma_0\, k^2 \cos\varphi}{(1-2\,k^2)E(k) - k'^2 K(k)} \left(\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi\right)^{1/4}. \tag{3.43}$$

Solution (3.43) coincides exactly with the result of Shah and Kobayashi [43]. The variation of the SIF along the contour for elliptical cracks having various ratios b/a and subjected to linear pressure dependent on coordinate x is shown in Fig.3.6.

3.2.3 An elliptical crack under linear stress field dependent on coordinate y

In the case of an elliptical crack subjected to a linear stress field that varies in y, the applied stress field is represented (Fig.3.7) by the function:

$$p(x,y) = \sigma_0 \frac{y}{b} \quad or \quad p(r,\theta) = \sigma_0 r \sin\theta \qquad \text{with} \quad p_1^s(r) = \sigma_0 r. \tag{3.44}$$

Similarly to the previous case it can be obtained:

$$\mathcal{A}_{01}^s = \frac{\mathcal{B}_{10}^s}{\alpha_{11}^s} = \frac{4\,\sigma_0}{3(-\boldsymbol{i})\sqrt{\pi}\alpha_{11}^s} \tag{3.45}$$

and

$$\alpha_{11}^s = \frac{4}{3}\frac{(1+k^2)E(k) - k'^2K(k)}{k^2}. \tag{3.46}$$

The SIF in such a case is

$$K(\varphi) = \frac{\sqrt{\pi\,b}\,\sigma_0\,k^2 \sin\varphi}{(1+\,k^2)E(k) - k'^2K(k)}\left(\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi\right)^{1/4}. \tag{3.47}$$

This solution also coincides with the result of Shah and Kobayashi [43].The variation of the SIF along the contour for elliptical cracks having various ratios b/a and subjected to linear pressure dependent on coordinate y is shown in Fig.3.8.

3.2.4 An elliptical crack under quadratic stress field which is an odd function in x and y

In the case of an elliptical crack subjected to the quadratic stress field which is an odd function in x and y, the applied stress field is represented (Fig.3.9) by the function:

$$p(x, y) = \sigma_0 \left(\frac{x\, y}{a\, b} \right) \quad \text{or} \quad p(r, \theta) = \frac{\sigma_0\, r^2}{2} \sin 2\theta \qquad \text{with} \quad p_2^s(r) = \frac{\sigma_0\, r^2}{2}. \tag{3.48}$$

The appropriate coefficient in eq.(3.17) is

$$\mathcal{A}_{02}^s = \frac{\mathcal{B}_{20}^s}{\alpha_{22}^s} = -\frac{8\,\sigma_0}{15\,\sqrt{\pi}\,\alpha_{22}^s}, \tag{3.49}$$

where

$$\alpha_{22}^s = \frac{16}{15} \frac{E(k)(2k'^4 - 2k'^2 + 2) - K(k)(k'^4 + k'^2)}{k^4}. \tag{3.50}$$

Subsequently, the SIF is obtained in the form of

$$K(\varphi) = \frac{\sqrt{\pi\, b}\,\sigma_0\, k^4 \sin 2\varphi}{2(E(k)(2k'^4 - 2k'^2 + 2) - K(k)(k'^4 + k'^2))} \left(\frac{b^2}{a^2} \cos^2 \varphi + \sin^2 \varphi \right)^{1/4}. \tag{3.51}$$

After some manipulations eq.(3.51) coincides with the result obtained by Kassir and Sih [36]. The variation of the SIF along the contour for elliptical cracks having various ratios b/a subjected to quadratic pressure which is odd in x and y is shown in Fig.3.10.

3.2.5 An elliptical crack under quadratic stress field which is an even function in x and y

In the case of an elliptical crack subjected to the quadratic stress field which is an even function in x and y, the applied stress field is represented (Fig.3.11) by the function:

$$p(x,y) = p_{20}\left(\frac{x}{a}\right)^2 + p_{02}\left(\frac{y}{b}\right)^2 \quad \text{or} \quad p(r,\theta) = r^2(\sigma_0 + \sigma_2 \cos 2\theta) \tag{3.52}$$

with

$$\sigma_0 = \frac{p_{20}+p_{02}}{2}, \qquad \sigma_2 = \frac{p_{20}-p_{02}}{2} \tag{3.53}$$

$$p_0^c(r) = 2\,r^2\sigma_0, \qquad p_2^c(r) = r^2\sigma_2. \tag{3.54}$$

Solving the system, the following coefficients are obtained:

$$\mathcal{A}_{00}^c = \frac{8\sigma_0}{5\sqrt{\pi}\,\alpha_{00}^c}, \tag{3.55}$$

$$\mathcal{A}_{10}^c = -\frac{16}{15\sqrt{\pi}}\,\frac{\alpha_{22}^c\sigma_0 - \alpha_{02}^c\sigma_2}{\alpha_{00}^c\alpha_{22}^c - \alpha_{02}^{c\,2}}, \tag{3.56}$$

$$\mathcal{A}_{02}^c = -\frac{16}{15\sqrt{\pi}}\,\frac{\alpha_{00}^c\sigma_2 - \alpha_{02}^c\sigma_0}{\alpha_{00}^c\alpha_{22}^c - \alpha_{02}^{c\,2}}. \tag{3.57}$$

The SIF is

$$\begin{aligned} K(\varphi) &= \pi\sqrt{b}\left(\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi\right)^{1/4}\left[\mathcal{A}_{00}^c - \mathcal{A}_{10}^c - \mathcal{A}_{02}\cos 2\varphi\right] \\ &= \frac{8\sqrt{\pi\,b}}{15}\left(\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi\right)^{1/4}\left[C_0 + C_2\cos 2\varphi\right], \end{aligned} \tag{3.58}$$

where

$$C_0 = \sigma_0 \left(\frac{3}{\alpha^c_{00}} + \frac{2\alpha^c_{22}}{\alpha^c_{00}\alpha^c_{22} - \alpha^{c\,2}_{02}} \right) - \sigma_2 \, \frac{2\,\alpha^c_{02}}{\alpha^c_{00}\alpha^c_{22} - \alpha^{c\,2}_{02}}, \tag{3.59}$$

$$C_2 = 2\,\frac{\alpha^c_{00}\,\sigma_2 - \alpha^c_{02}\,\sigma_0}{\alpha^c_{00}\alpha^c_{22} - \alpha^{c\,2}_{02}}. \tag{3.60}$$

This result coincides with Martin's result [41]. The variation of the SIF along the contour for elliptical cracks of various ratios b/a, subjected to quadratic pressure even in x and y with $\sigma_2 = \dfrac{\sigma_0}{2}$ is shown in Fig.3.12.

In all examples above the SIF solutions were represented by the first few terms of eq.(3.26). This observation can be generalized for any polynomial stress field. If the applied stress field is given by

$$p(x, y) = \sum_{n=0}^{p} \sum_{k=0}^{q} A_{nk} x^n y^k, \tag{3.61}$$

the stress intensity factor is obtained as

$$K(\varphi) = \pi\sqrt{b} \left(\frac{b^2}{a^2} \cos^2\varphi + \sin^2\varphi \right)^{1/4} \sum_{n=0}^{p+q} \sum_{k=0}^{p+q-n} (-\boldsymbol{i})^n (-1)^k [\mathcal{A}^c_{kn} \cos n\varphi + \mathcal{A}^s_{kn} \sin n\varphi], \tag{3.62}$$

i.e. the series is truncated.

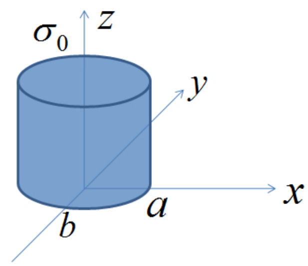

Figure 3.3: Elliptical crack under uniform pressure.

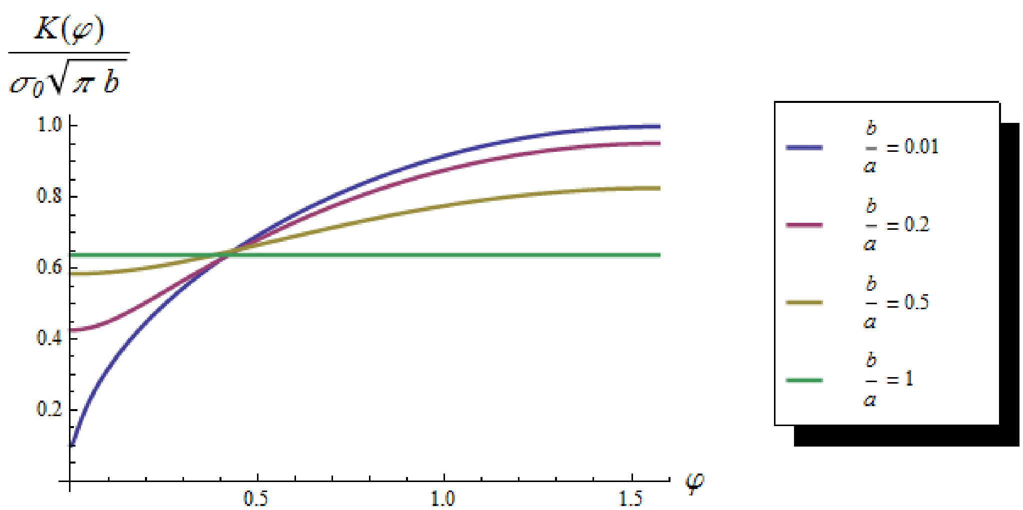

Figure 3.4: The variation of SIF along contours of elliptical cracks of various ratios b/a subjected to uniform pressure σ_0.

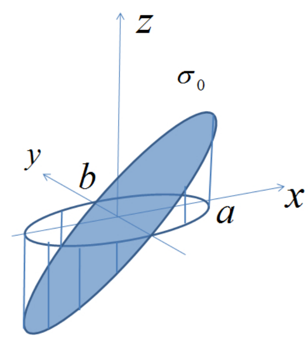

Figure 3.5: Elliptical crack under linearly applied stress dependent on x.

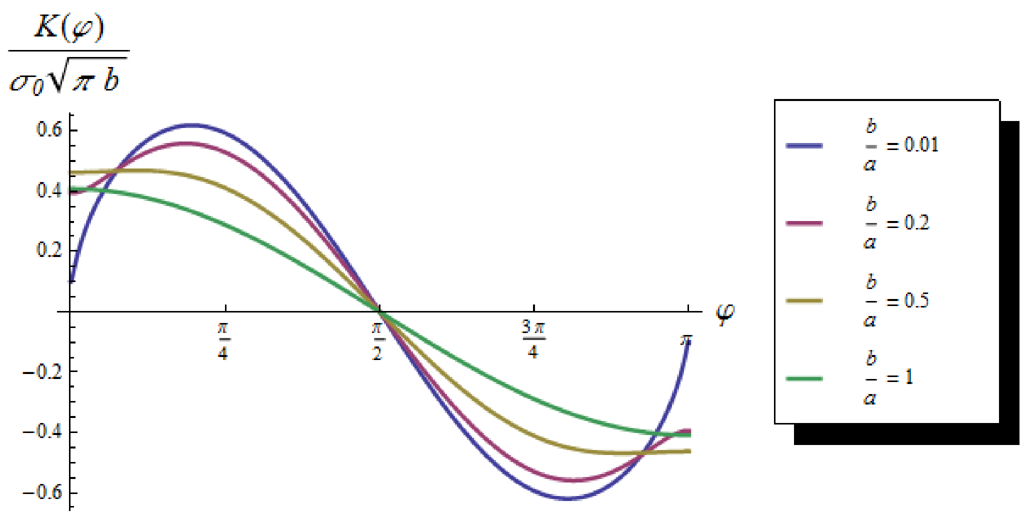

Figure 3.6: The variation of SIF along contours of elliptical cracks of various ratios b/a subjected to linear pressure dependent on x.

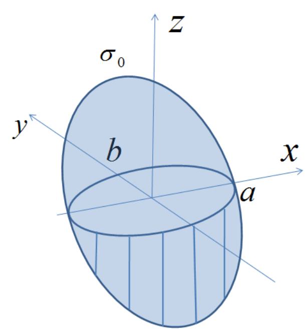

Figure 3.7: Elliptical crack under linearly applied stress dependent on y.

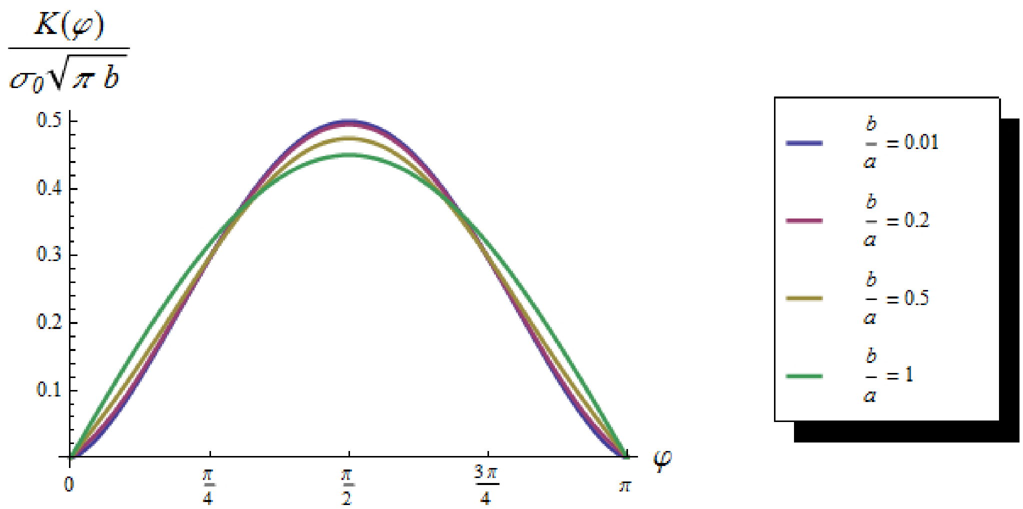

Figure 3.8: The variation of SIF along contours of elliptical cracks of various ratios b/a subjected to linear pressure dependent on coordinate y.

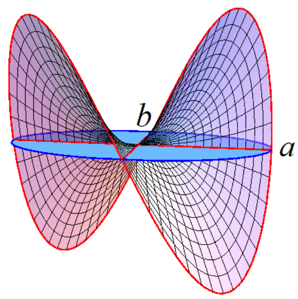

Figure 3.9: Elliptical crack under quadratic applied stress being an odd function in x and y.

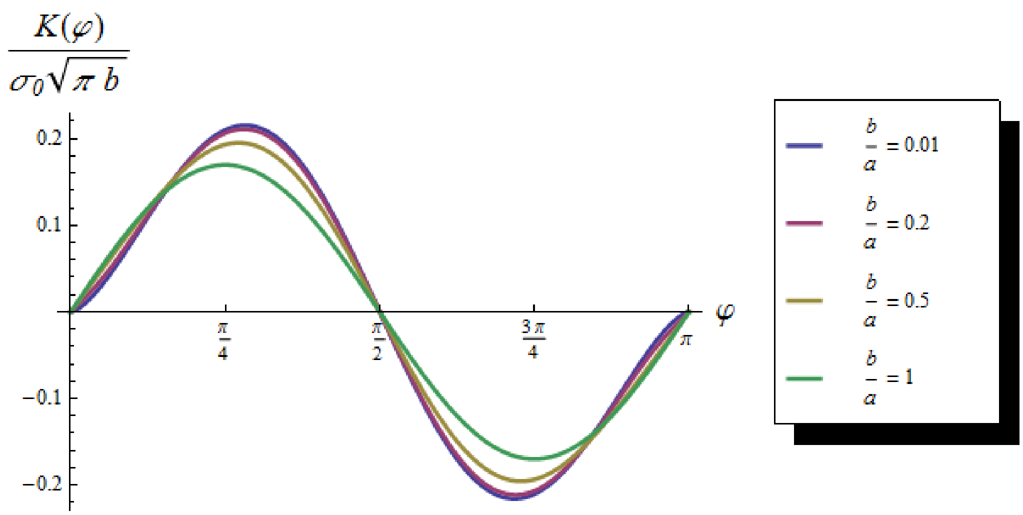

Figure 3.10: The variation of SIF along contours of elliptical cracks of various ratios b/a subjected to quadratic stress odd in x and y.

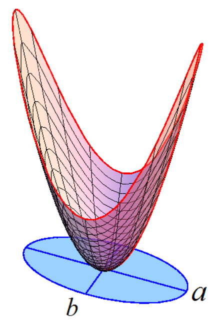

Figure 3.11: Elliptical crack under quadratic applied stress being an even function in x and y.

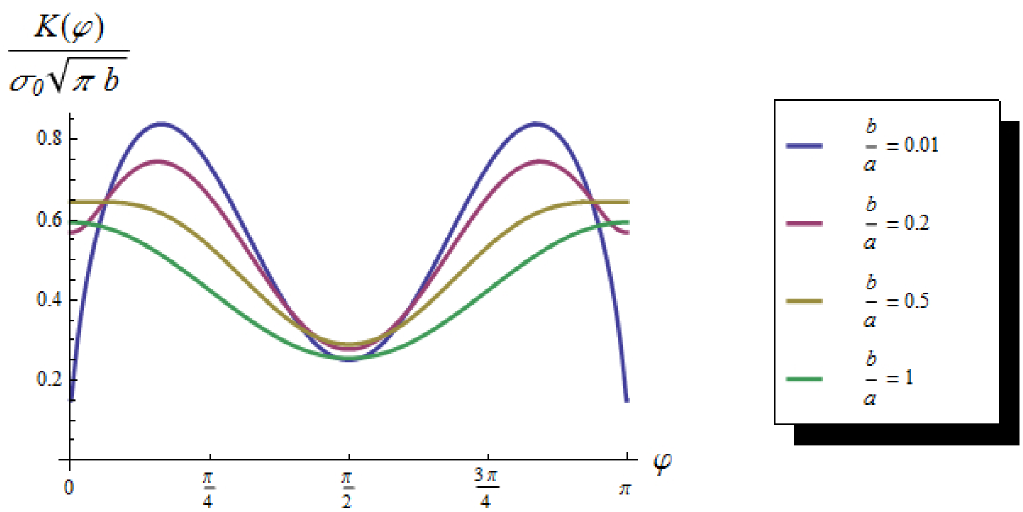

Figure 3.12: The variation of SIF along contours of elliptical cracks of various ratios b/a subjected to quadratic applied stress even in x and y.

3.3 The weight function for an elliptical crack

3.3.1 An Elliptical Crack Subjected to a Point Load Applied at the Crack Center.

In order to obtain the weight function $W(Q', Q_0) = W(\varphi; 0, 0)$ for an elliptical crack for the particular case of a point load applied at the center of the crack $Q_0(0, 0)$ (Fig.3.13), the SIF induced by a constant pressure σ_0 distributed over the small elliptical region (Fig.3.14) is derived first.

The applied stress field is given in the form:

$$p(r, \theta) = \frac{p_0^c(r)}{2} = \begin{cases} \sigma_0, & r \leq \varepsilon_0 \\ 0, & \varepsilon_0 < r < 1, \end{cases} \tag{3.63}$$

where $\varepsilon_0 << 1$ is a small parameter.

Then the coefficients $\mathcal{B}_{0i}^c$ from eq.(3.20) are

$$\mathcal{B}_{0i}^c = \sigma_0 \frac{(4i+3)\Gamma(i+1)}{\Gamma(i+3/2)} \varepsilon_0^2 \, {}_2F_1(-\tfrac{1}{2} - i, 1+i, 2, \varepsilon_0^2) \tag{3.64}$$

and for $n > 0$ $\mathcal{B}_{ni}^c = 0, \mathcal{B}_{ni}^s = 0$. The following equation is also employed:

$$\sum_{i=0}^{\infty} (-1)^i \mathcal{B}_{0i}^c = \frac{4\sigma_0}{\sqrt{\pi}} \left(1 - \sqrt{1 - \varepsilon_0^2}\right) \tag{3.65}$$

For any fixed number of terms N ($N = 0, 2, 4...$), the Nth approximation of the SIF can be presented in the form of:

$$K^N(\varphi) = \pi\sqrt{b}\left(\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi\right)^{1/4}\sum_{k=0}^{N/2} X_{2k}^N \cos 2k\varphi. \tag{3.66}$$

The following notations are also used:

$$D_{2k} = \begin{pmatrix} \alpha_{00}^c & \alpha_{02}^c & \alpha_{04}^c & ... & \alpha_{0,2k}^c \\ \alpha_{20}^c & \alpha_{22}^c & \alpha_{24}^c & ... & \alpha_{2,2k}^c \\ \alpha_{40}^c & \alpha_{42}^c & \alpha_{44}^c & ... & \alpha_{4,2k}^c \\ ... & ... & ... & ... & ... \\ \alpha_{2k,0}^c & \alpha_{2k,2}^c & \alpha_{2k,4}^c & ... & \alpha_{2k,2k}^c \end{pmatrix}, \ k = 0, 1, 2, ...\tfrac{N}{2}, \tag{3.67}$$

$\Delta_{2k} = \det D_{2k}$, $\delta_0^0 = 1$, δ_{2k}^{2i} is the cofactor obtained by removing the first row and $i+1$ column from the matrix D_{2k} for $i \leq k$ and $\delta_{2k}^{2i} = 0$ for $i > k$. Then the coefficients X_{2k}^N in eq.(3.66) are given by

$$X_{2k}^N = \frac{\delta_N^{2k}}{\Delta_N}\frac{4\,\sigma_0}{\sqrt{\pi}}\left(1 - \sqrt{1-\varepsilon_0^2}\right) + \sum_{j=0}^{N/2-1}\left(\frac{\delta_{2j}^{2k}}{\Delta_{2j}} - \frac{\delta_N^{2k}}{\Delta_N}\right)(-1)^j \mathcal{B}_{0j}^c \tag{3.68}$$

In order to derive the weight function corresponding to a point load applied at the center of the crack, the applied pressure is given in the form of a constant force distributed over the area of the small central ellipse:

$$\sigma_0 = \frac{P}{\pi\,a\,b\,\varepsilon_0^2}, \tag{3.69}$$

where $P = 1$ is the magnitude of the applied force. After taking the limit of eq.(3.64) as $\varepsilon_0 \to 0$, the coefficients $\mathcal{B}^c_{0i}$ in (3.68) reduce to

$$\mathcal{B}^c_{0i} = \frac{P}{\pi\, a\, b}\, b^c_{0,i}, \qquad b^c_{0,i} = \frac{(4i+3)\Gamma(i+1)}{\Gamma(i+3/2)} \tag{3.70}$$

The N-th order approximation $W^N(\varphi; 0, 0)$ of the weight function $W(\varphi; 0, 0)$ is derived from the SIF solution (3.66) as

$$W^N(\varphi; 0, 0) = \frac{P}{a\sqrt{b}} \left(\frac{b^2}{a^2} \cos^2\varphi + \sin^2\varphi \right)^{1/4} \sum_{k=0}^{N/2} \mathcal{X}^N_{2k} \cos 2k\varphi, \tag{3.71}$$

where

$$\mathcal{X}^N_{2k} = \frac{\delta^{2k}_N}{\Delta_N} \frac{2}{\sqrt{\pi}} + \sum_{j=0}^{N/2-1} \left(\frac{\delta^{2k}_{2j}}{\Delta_{2j}} - \frac{\delta^{2k}_N}{\Delta_N} \right) (-1)^j\, b^c_{0,j} \tag{3.72}$$

The variation of the weight function $W^N(\varphi; 0, 0)$ along the contour for the ellipse with dimensions $a = 1$, $b = 0.5$ is shown in Fig.3.15 for various number of terms N being considered.

For the case when $a = b$ eq.(3.71) reduces to:

$$W^N(\varphi; 0, 0) = \frac{P}{(\pi\, b)^{3/2}} \quad \text{for any } N, \tag{3.73}$$

hence

$$W(\varphi; 0, 0) = \frac{P}{(\pi\, b)^{3/2}} \tag{3.74}$$

Solution (3.74) coincides with the well known SIF-solution for a circular crack under a point load applied at the center of the crack [24].

Let us consider the zero order approximation of (3.71), i.e. taking only one term in the complete solution:

$$W^0(\varphi;0,0) = \frac{P}{2\,a\sqrt{\pi\,b}\,E(k)}\left(\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi\right)^{1/4}, \tag{3.75}$$

where $k^2 = (a^2 - b^2)/a^2$. This solution has been given in [33] as the SIF solution for an elliptical crack under a point load applied at the center of the crack. But this is only the zeroth term of the exact solution (3.71) and eq.(3.75) tends to zero as the crack tends to a tunnel crack, i.e. when $a/b \ll 1$. It means, that the correct weight function for the tunnel crack requires using higher order terms of eq.(3.71).

Let us consider the second order approximation of eq.(3.71) for the case when $a = b(1+\varepsilon)$, $\varepsilon \ll 1$, i.e. two first terms in eq.(3.71). In this case the weight function (3.71) is:

$$W^2(\varphi;0,0) = \frac{P}{(\pi\,b)^{3/2}}\left(1 - \frac{\varepsilon}{4}(3 + 5\cos 2\varphi)\right), \tag{3.76}$$

and

$$W(\varphi;0,0) = W^2(\varphi;0,0) + O(\varepsilon^2). \tag{3.77}$$

This coincides with Borodachev's solution [39] for an elliptical crack close to a circle crack and subjected to a point load at the crack center.

The numerical results for certain particular cases of the cracks eccentricity may now be compared with the boundary element data [13]. The variation of the weight function along the contours of three different elliptical cracks is shown in Fig.3.16, and the analytical solution (3.71) is in very good agreement with the numerical data. For the ellipse with $a = 1$, $b = 0.7$ the maximum difference δ between two solutions is 2.22%, for the ellipse with $a = 1$, $b = 0.5$ - $\delta = 4.07\%$, for the ellipse with $a = 1$, $b = 0.5$ - $\delta = 6.64\%$. In all results below, the difference δ between the analytical $W^N(\varphi; 0, 0)$ and numerical $W^{num}(\varphi; 0, 0)$ solution is calculated as:

$$\delta = \max_{0 \le \varphi \le 2\pi} \left| \frac{W^N(\varphi; 0, 0) - W^{num}(\varphi; 0, 0)}{W^N(\varphi; 0, 0)} \right|. \tag{3.78}$$

In all results below, the number of terms N in eq.(3.71) is chosen such that the deviation between $W^N(\varphi; 0, 0)$ and $W^{N+2}(\varphi; 0, 0)$ does not exceed 3%.

Let us consider a tunnel crack. A tunnel crack can be modeled as an elliptical crack with the eccentricity ratio $b/a \ll 1$ (Fig.3.17). Variation of the weight function $W^N(\varphi; 0, 0)$ (eq.(3.71)) along the contours of elliptical cracks with with fixed $b = 0.7$ and increasing a is shown in Fig.3.18. The weight function $W^N(\varphi; 0, 0)$ for small ratio b/a can be rewritten as a function of coordinate x by substituting $\varphi = \arccos \frac{x}{a}$ in eq.(3.71). This operation returns the weight function $W^N(x; x_0, y_0) = W^N(x; 0, 0)$ for a tunnel crack. A comparison of the derived weight function for a tunnel crack with the numerical data [13] is shown in Fig.3.19. An ellipse with an aspect ratio of $b/a = 0.1$ is used to model a tunnel crack, and the number of terms $N = 60$ is used in eq.(3.71). The maximum difference δ between analytical and numerical solutions is 12.14%.

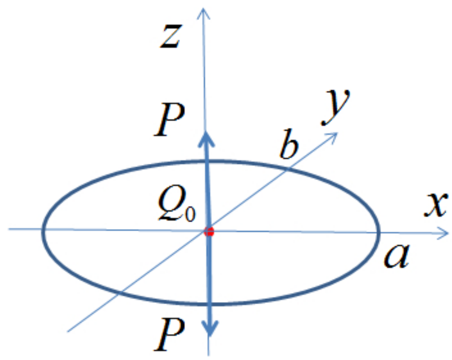

Figure 3.13: Elliptical crack subjected to the point load at the crack center.

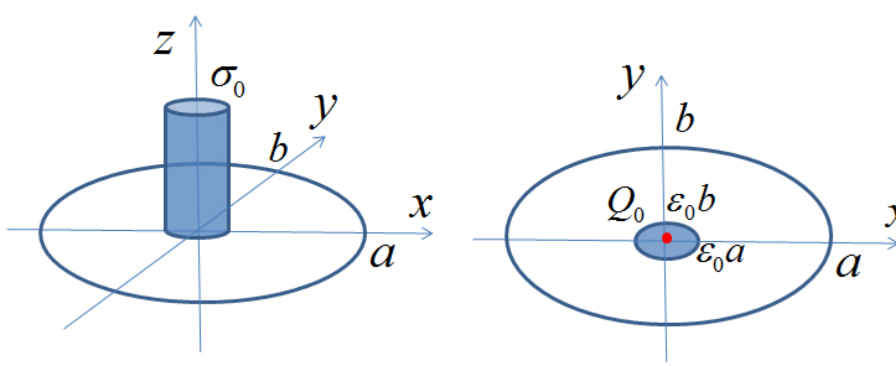

Figure 3.14: Normal pressure distributed over a small elliptical region around the center of the crack.

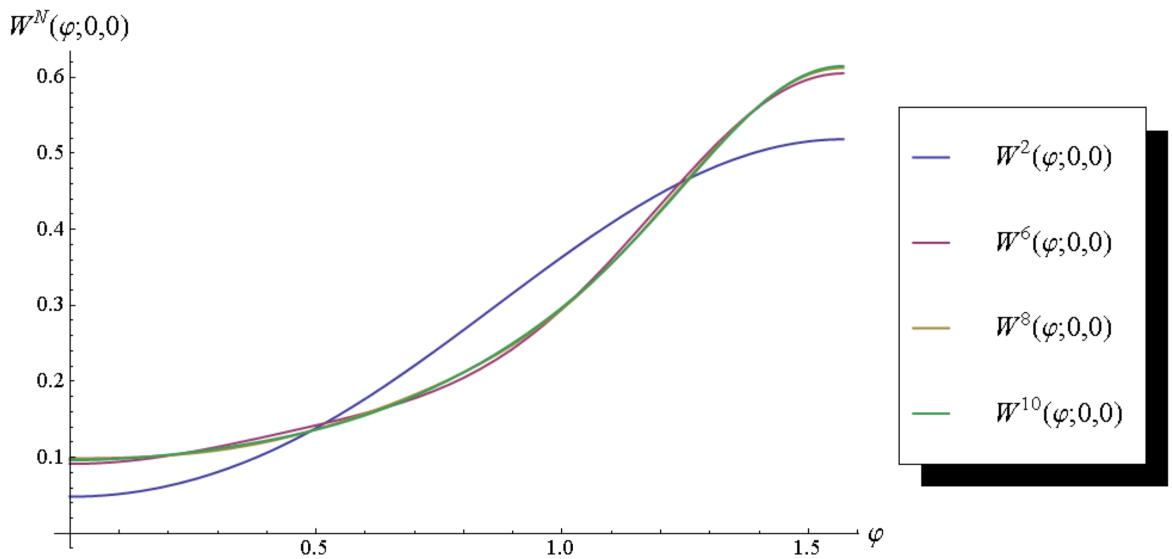

Figure 3.15: The variation of the weight function $W^N(\varphi; 0, 0)$ along the contour of the ellipse with $a = 1$, $b = 0.5$ obtained for various number of terms.

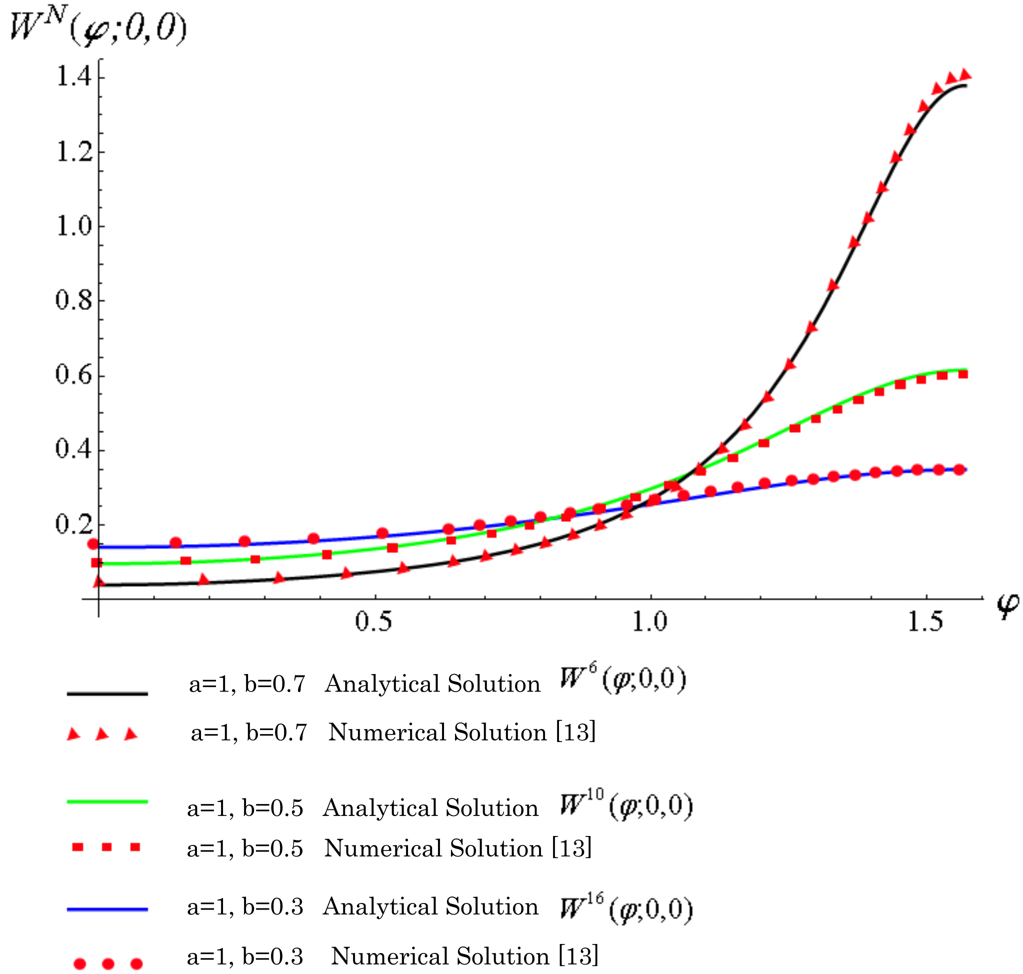

Figure 3.16: The variation of the weight function $W^N(\varphi; 0, 0)$ along the contour for elliptical cracks with a=1 and b=0.7, b=0.5, b=0.3.

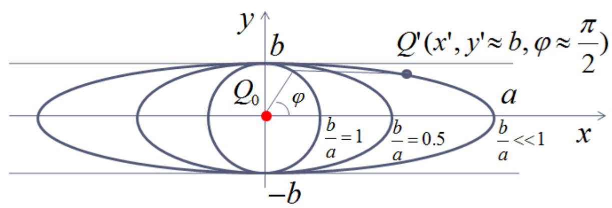

Figure 3.17: Tunnel crack approximated by an elliptical crack with $b/a \ll 1$.

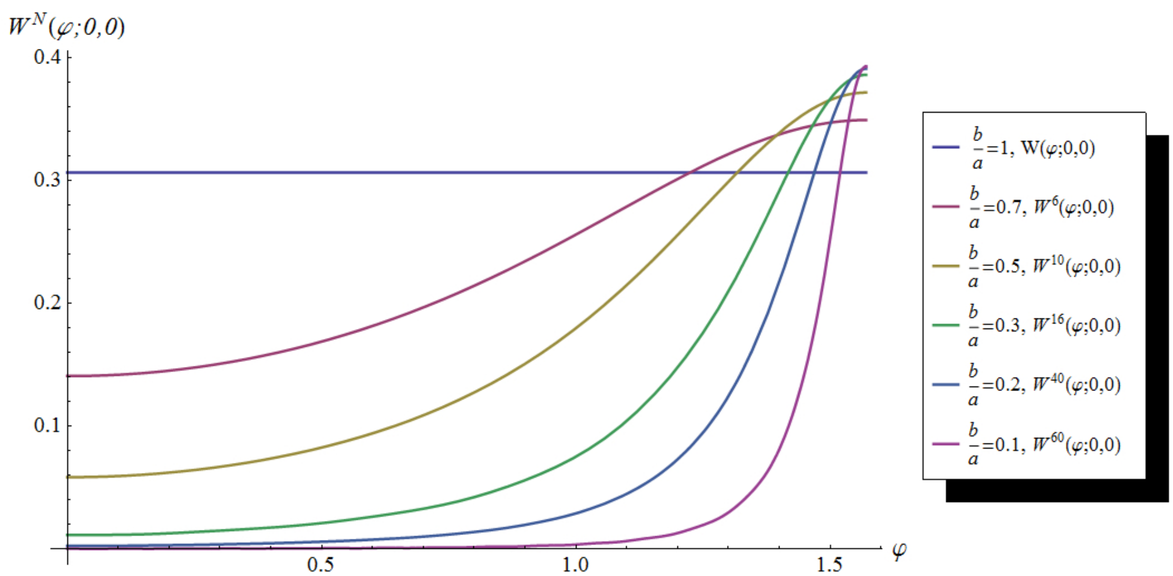

Figure 3.18: The variation of the weight function $W^N(\varphi;0,0)$ along the contour for the elliptical cracks with b=0.7 and increasing dimension 'a'.

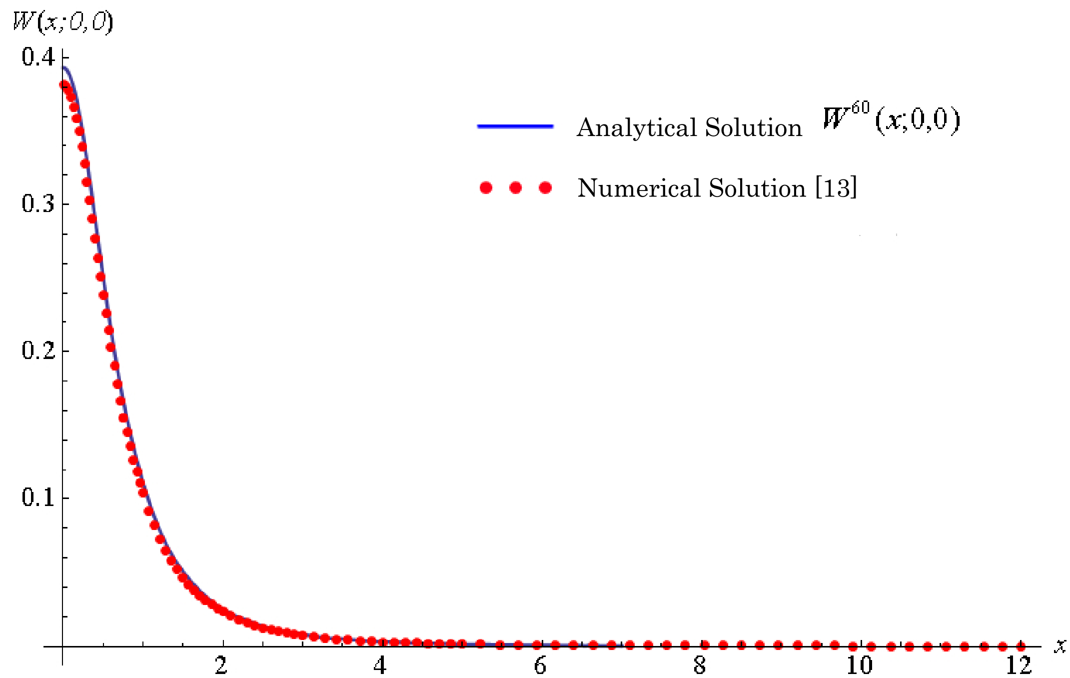

Figure 3.19: The variation of the weight function $W^N(x;0,0)$ along the contour for the tunnel crack with b=0.7 in comparison with the numerical data.

3.3.2 General Weight Function for an Elliptical Crack.

A general weight function $W(Q', Q_0)$ for a point load applied at arbitrary location $Q_0(x_0, y_0)$ (Fig.3.20) can now be derived. In order to derive the weight function corresponding to a point load, the SIF-solution corresponding to a constant pressure distributed over a small region (Fig.3.21) is considered first.

Let $x_0 = a\,r_0 \cos\theta_0$ and $y_0 = b\,r_0 \sin\theta_0$. The applied stress is taken in the form:

$$p(r,\theta) = \begin{cases} \sigma_0, & (r,\theta) \in \Omega \\ 0, & (r,\theta) \in S \backslash \Omega, \end{cases} \tag{3.79}$$

where

$$\Omega = \begin{cases} r_0 - \varepsilon_1 \le r \le r_0 + \varepsilon_1, \\ \theta_0 - \varepsilon_2 \le \theta \le \theta_0 + \varepsilon_2 \end{cases} \tag{3.80}$$

and

$$\sigma_0 = \frac{P}{4\,a\,b\,\varepsilon_1 \varepsilon_2\, r_0}. \tag{3.81}$$

The small parameters $\varepsilon_1 << 1$ and $\varepsilon_2 << 1$ are used to describe the small domain Ω around the point Q_0 where the load is applied.

The applied stress (3.79) is then expanded into the Fourier series, according to eq.(3.22), with terms:

$$p_n^c(r) = \begin{cases} \dfrac{2\,\varepsilon_2 \sigma_0}{\pi} \cos n\theta_0, & r_0 - \varepsilon_1 \le r \le r_0 + \varepsilon_1 \\ 0, & r_0 + \varepsilon_1 < r < 1 \text{ or } 0 \le r < r_0 - \varepsilon_1 \end{cases} \tag{3.82}$$

and

$$
p_n^s(r) = \begin{cases} \dfrac{2\,\varepsilon_2\sigma_0}{\pi}\sin n\theta_0, & r_0-\varepsilon_1 \le r \le r_0+\varepsilon_1 \\ 0, \qquad r_0+\varepsilon_1 < r < 1 \;\text{ or }\; 0 \le r < \; r_0-\varepsilon_1 \end{cases} \tag{3.83}
$$

Coefficients $\mathcal{B}_{ni}^c$ in (3.20) take the form:

$$
\mathcal{B}_{ni}^c = p_n^c(r)\frac{4\Gamma(n+i+1)(n+2i+3/2)}{\Gamma(n+1)\Gamma(i+3/2)(-\boldsymbol{i})^n} r_0^{n+1}\sqrt{1-r_0^2}\mathfrak{F}_i(n+\tfrac{3}{2},n+1,r_0^2)\varepsilon_1 + O(\varepsilon_1^2), \tag{3.84}
$$

or in non-dimensional notation:

$$
\mathcal{B}_{ni}^c = \frac{P}{\pi\,a\,b}\,b_{n,i}^c + O(\varepsilon_1^2), \tag{3.85}
$$

$$
b_{n,i}^c = \frac{2\Gamma(n+i+1)(n+2i+3/2)}{\Gamma(n+1)\Gamma(i+3/2)(-\boldsymbol{i})^n} r_0^n\sqrt{1-r_0^2}\mathfrak{F}_i(n+\tfrac{3}{2},n+1,r_0^2)\cos n\theta_0. \tag{3.86}
$$

Next, the following observation is used:

$$
\begin{aligned}
\sum_{i=0}^{\infty}\frac{2\Gamma(n+i+1)(n+2i+3/2)}{\Gamma(n+1)\Gamma(i+3/2)}\int r^{n+1}\sqrt{1-r^2}\mathfrak{F}_i(n+\tfrac{3}{2},n+1,r^2)dr \\
= \frac{2}{\sqrt{\pi}}\int\frac{r^{n+1}}{\sqrt{1-r^2}}dr,
\end{aligned} \tag{3.87}
$$

but in the sense of equality of derivatives, i.e.

$$
\sum_{i=0}^{\infty}(-1)^i\,b_{n,i}^c = \frac{2}{(-\boldsymbol{i})^n\sqrt{\pi}}\frac{r_0^n\cos n\theta_0}{\sqrt{1-r_0^2}}, \tag{3.88}
$$

or

$$
\sum_{i=0}^{\infty}(-1)^i\,b_{n,i}^c = f_n^c, \qquad f_n^c = \frac{2}{(-\boldsymbol{i})^n\sqrt{\pi}}\frac{r_0^n\cos n\theta_0}{\sqrt{1-r_0^2}}. \tag{3.89}
$$

The corresponding weight function $W^N(Q', Q_0) = W^N(\varphi; r_0, \theta_0)$ is written in the following form based on eq.(3.26:

$$W^N(\varphi; r_0, \theta_0) = \frac{P}{a\sqrt{b}} \left(\frac{b^2}{a^2} \cos^2 \varphi + \sin^2 \varphi \right)^{1/4} \sum_{m=0}^{N} \left(\mathcal{X}_m^N(r_0, \theta_0) \cos m\varphi \right. \\ \left. + \mathcal{Y}_m^N(r_0, \theta_0) \sin m\varphi \right) . \tag{3.90}$$

The following notations are subsequently introduced. For any fixed N

$$N1 = \begin{cases} N, & N \text{ is even;} \\ N-1, & N \text{ is odd.} \end{cases} \tag{3.91}$$

$$N2 = \begin{cases} N-1, & N \text{ is even;} \\ N, & N \text{ is odd.} \end{cases} \tag{3.92}$$

The matrix D_{2k} is defined by eq.(3.67) for $k = 0, 1, 2 \ldots \frac{N1}{2}$. The parameter $\delta_{2k}^{2i,2j}$ is a cofactor obtained by removing the $i+1$ row and $j+1$ column from matrix D_{2k} if $i, j \leq k$ and 0 if $i, j > k$. Hence,

$$\mathcal{X}_{2s}^N(r_0, \theta_0) = \sum_{i=0}^{N1/2} \left[\sum_{k=i}^{N1/2} (-1)^k \frac{\delta_{2k}^{2i,2s}}{\Delta_{2k}} b_{2i,k-i}^c + (-1)^i \frac{\delta_{N1}^{2i,2s}}{\Delta_{N1}} \left(f_{2i}^c - \sum_{l=0}^{N1/2-i} (-1)^l b_{2i,l}^c \right) \right] . \tag{3.93}$$

The odd coefficients $\mathcal{X}_m^N(r_0, \theta_0)$ in expression (3.90) can be written in a similar way. Let

$$D_{2k+1} = \begin{pmatrix} \alpha_{11}^c & \alpha_{13}^c & \alpha_{15}^c & ... & \alpha_{1,2k+1}^c \\ \alpha_{31}^c & \alpha_{33}^c & \alpha_{35}^c & .. & \alpha_{3,2k+1}^c \\ \alpha_{51}^c & \alpha_{53}^c & \alpha_{55}^c & ... & \alpha_{5,2k+1}^c \\ ... & ... & ... & ... & ... \\ \alpha_{2k+1,1}^c & \alpha_{2k+1,3}^c & \alpha_{2k+1,5}^c & ... & \alpha_{2k+1,2k+1}^c \end{pmatrix}, \ k = 0, 1, 2, ... \tfrac{N2-1}{2}, \qquad (3.94)$$

and $\delta_{2k+1}^{2i+1,2s+1}$ is a cofactor obtained by removing the $i+1$ row and $s+1$ column from matrix D_{2k+1}. Then

$$\begin{aligned} \mathcal{X}_{2s+1}^N(r_0, \theta_0) = (-\boldsymbol{i}) \sum_{i=0}^{(N2-1)/2} \Bigg[\sum_{k=i}^{(N2-1)/2} (-1)^k \frac{\delta_{2k+1}^{2i+1,2s+1}}{\Delta_{2k+1}} \, b_{2i+1,k-i}^c \\ + (-1)^i \frac{\delta_{N2}^{2i+1,2s+1}}{\Delta_{N2}} \left(f_{2i+1}^c - \sum_{l=0}^{(N2-1)/2-i} (-1)^l \, b_{2i+1,l}^c \right) \Bigg], \end{aligned} \qquad (3.95)$$

Coefficients $\mathcal{Y}_m^N(r_0, \theta_0)$ are obtained by changing in equations (3.93), (3.95) parameters α_{nm}^c into α_{nm}^s and $\cos n\theta_0$ into $\sin n\theta_0$.

After the coefficients $\mathcal{X}_m^N(r_0, \theta_0)$ and $\mathcal{Y}_m^N(r_0, \theta_0)$ have been derived and substituted into eq.(3.90), eq.(3.90) represents the weight function for an elliptical crack.

The variation of the weight function $W^N(\varphi; r_0, \theta_0)$ along the contour of the ellipse with $b = 0.5$, $a = 1$ is shown in Fig.3.22 for various number of terms. The point load is applied at the point $Q_0(x_0 = 0.4124, y_0 = 0.289688)$ or in elliptical coordinates $Q_0(r_0 = 0.711161, \theta_0 = 0.952195)$. The weight function $W^{30}(\varphi; 0.711161, 0.952195)$ is compared in

Fig.3.23 with the numerical data obtained from reference [13]. The maximum difference δ between the analytical and numerical weight functions is 8.20%. In all results below, the number of terms N in eq.(3.90) is chosen such that the deviation between $W^N(\varphi; r_0, \theta_0)$ and $W^{N+2}(\varphi; r_0, \theta_0)$ does not exceed 3%.

The weight functions $W^0(\varphi; r_0, \theta_0)$, $W^1(\varphi; r_0, \theta_0)$, $W^2(\varphi; r_0, \theta_0)$ are given in the analytical form in appendix A.

In the case when $a = b$ the WF (3.90) reduces to

$$W^N(\varphi; r_0, \theta_0) = \frac{P}{(\pi\, b)^{3/2}\sqrt{1 - r_0^2}} \left(1 + 2\sum_{m=1}^{N} r_0^m \cos\, m(\varphi - \theta_0)\right), \tag{3.96}$$

which for $N \to \infty$ coincides with the weight function for a circular crack, obtained by Kassir and Sih in [36], and can be represented in the closed form as:

$$W(\varphi; r_0, \theta_0) = \frac{\sqrt{1 - r_0^2}}{(\pi\, b)^{3/2}[1 - 2\, r_0 \cos(\varphi - \theta_0) + r_0^2]}. \tag{3.97}$$

For the case when $y_0 = 0$, the WF-solution (3.90) can be simplified. Substituting $\theta_0 = 0$ into (3.90), the equation (3.90) reduces to

$$W^N(\varphi; r_0, 0) = \frac{P}{a\sqrt{b}} \left(\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi\right)^{1/4} \sum_{m=0}^{N} \mathcal{X}_m^N(r_0, 0) \cos m\varphi \tag{3.98}$$

The variation of the weight function (3.98) along the contour of the elliptical crack with $a = 1$, $b = 0.5$ is shown in Fig.3.24 in comparison with the numerical data [13]. The point load is applied at $x_0 = 0.5$, $y_0 = 0$. The maximum difference δ between the analytical and numerical weight functions is 4.98%.

In the case when $x_0 = 0$, the WF-solution (3.90) can be simplified. Substituting $\theta_0 = \dfrac{\pi}{2}$ in (3.90), the expression reduces to

$$W^N(\varphi; r_0, \frac{\pi}{2}) = \frac{P}{a\sqrt{b}} \left(\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi\right)^{1/4} \sum_{s=0}^{N1/2} (-1)^s \left(\mathcal{X}_{2s}^N(r_0, \frac{\pi}{2}) \cos 2s\varphi \right.$$
$$\left. + \mathcal{Y}_{2s+1}^N(r_0, \frac{\pi}{2}) \sin (2s+1)\varphi\right). \tag{3.99}$$

The variation of the weight function (3.99) along the contour of the elliptical crack with $a = 1$, $b = 0.5$ is shown in Fig.3.25 in comparison with numerical data [13]. The point load is applied at $x_0 = 0$, $y_0 = 0.25$. The maximum difference δ between the analytical and numerical weight functions is 4.36%.

Next, a tunnel crack is modeled as an ellipse with aspect ratio $b/a = 0.1$ and eq.(3.99) is used to obtain the corresponding weight function $W^N(\varphi; r_0; \frac{\pi}{2})$. After changing the coordinates to $\varphi = \arccos(x/a)$ for the upper front and $\varphi = -\arccos(x/a)$ for the lower front, the weight function $W^N(x; 0, y_0) = W^N(\varphi; r_0; \frac{\pi}{2})$ for a tunnel crack is derived.

The variation of the weight function $W^{80}(x; 0, 0.1)$ along the upper and lower fronts of the tunnel crack with $b = 0.5$ is shown in Fig.3.26 in comparison with the numerical data [13]. The maximum difference δ between the analytical and numerical solutions is 12.45% and 14.46% for the upper and lower fronts respectively. The unit load is applied at the point $x_0 = 0,\ y_0 = 0.1$.

In the case when the point load is applied at the center of a crack, i.e. $x_0 = 0, y_0 = 0$, the weight function $W^N(\varphi; 0, 0)$ of eq.(3.90) reduces to eq.(3.71).

The stress intensity factor $K^N(\varphi)$ for an elliptical crack induced by the applied stress field $p(r_0, \theta_0)$ can be subsequently obtained as:

$$K^N(\varphi) = \int_0^{2\pi} \int_0^1 W^N(\varphi; r_0, \theta_0) p(r_0, \theta_0) a\, b\, r_0 dr_0 d\theta_0, \tag{3.100}$$

where the weight function $W^N(\varphi; r_0, \theta_0)$ is given by eq.(3.90).

In the case of a circular crack $(a = b)$ eq.(3.100) reduces to

$$K(\varphi) = \int_0^{2\pi} \int_0^1 W(\varphi; r_0, \theta_0) p(r_0, \theta_0) a^2\, r_0 dr_0 d\theta_0, \tag{3.101}$$

where the weight function $W(\varphi; r_0, \theta_0)$ is given by eq.(3.97).

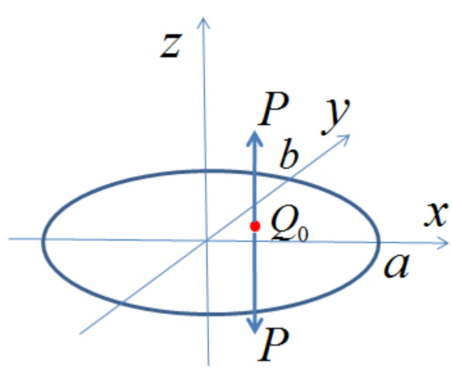

Figure 3.20: Elliptical crack subjected to the point load at the arbitrary location inside the crack domain.

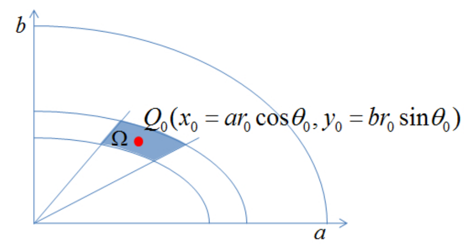

Figure 3.21: A small region around the location of the applied force.

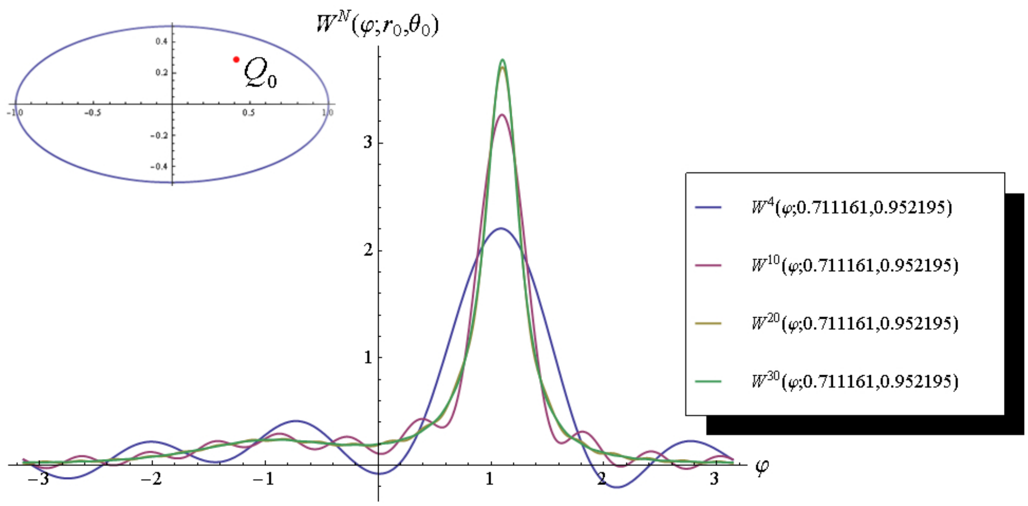

Figure 3.22: The variation of the weight function along the contour ofthe elliptical crack with a=1, b=0.5 for various number of terms. The point load is applied at $x_0 = 0.4124$, $y_0 = 0.289688$ ($r_0 = 0.711161$, $\theta_0 = 0.952195$).

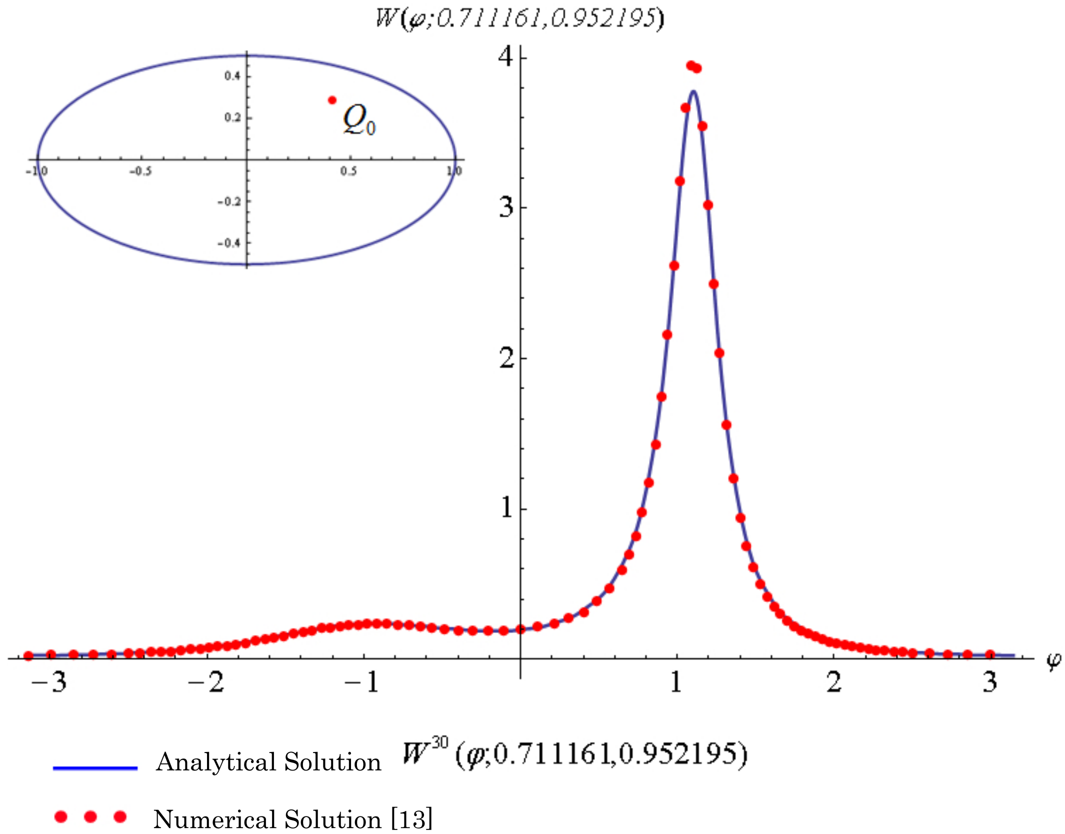

Figure 3.23: The variation of the weight function along the contour of the elliptical crack with a=1, b=0.5. The point load is applied at $x_0 = 0.4124$, $y_0 = 0.289688$.

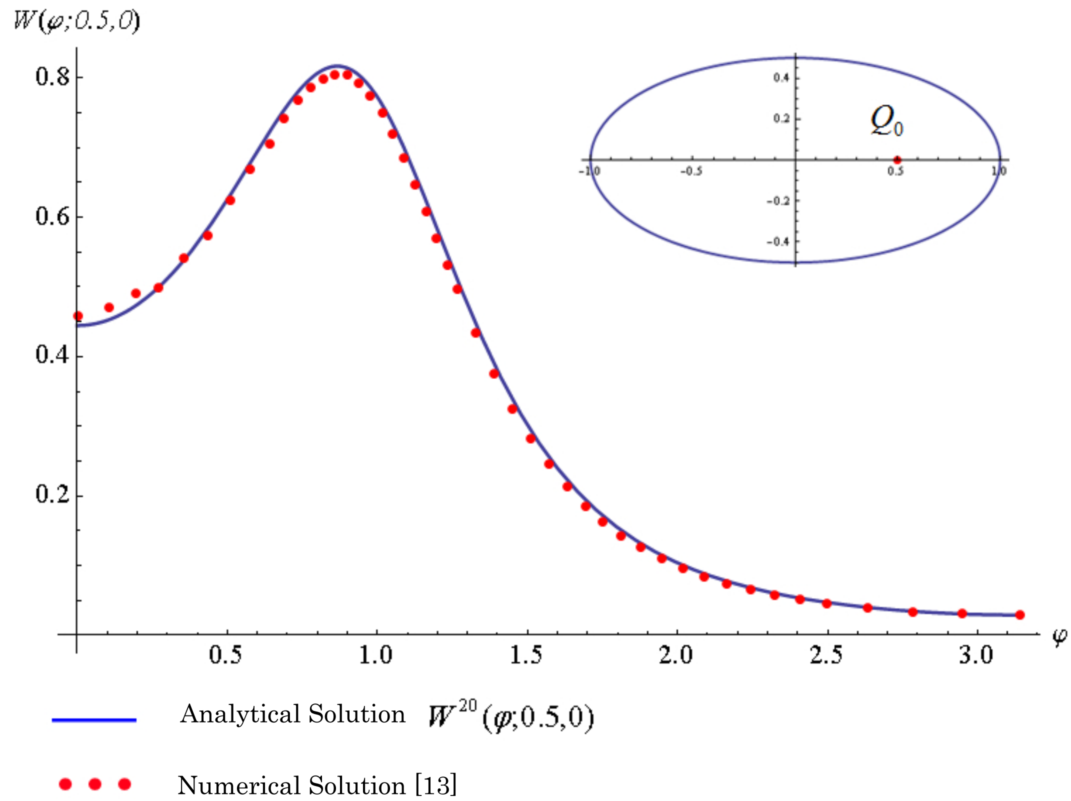

Figure 3.24: The variation of the weight function along the contour of the elliptical crack with a=1, b=0.5. The point load is applied at $x_0 = 0.5$, $y_0 = 0$.

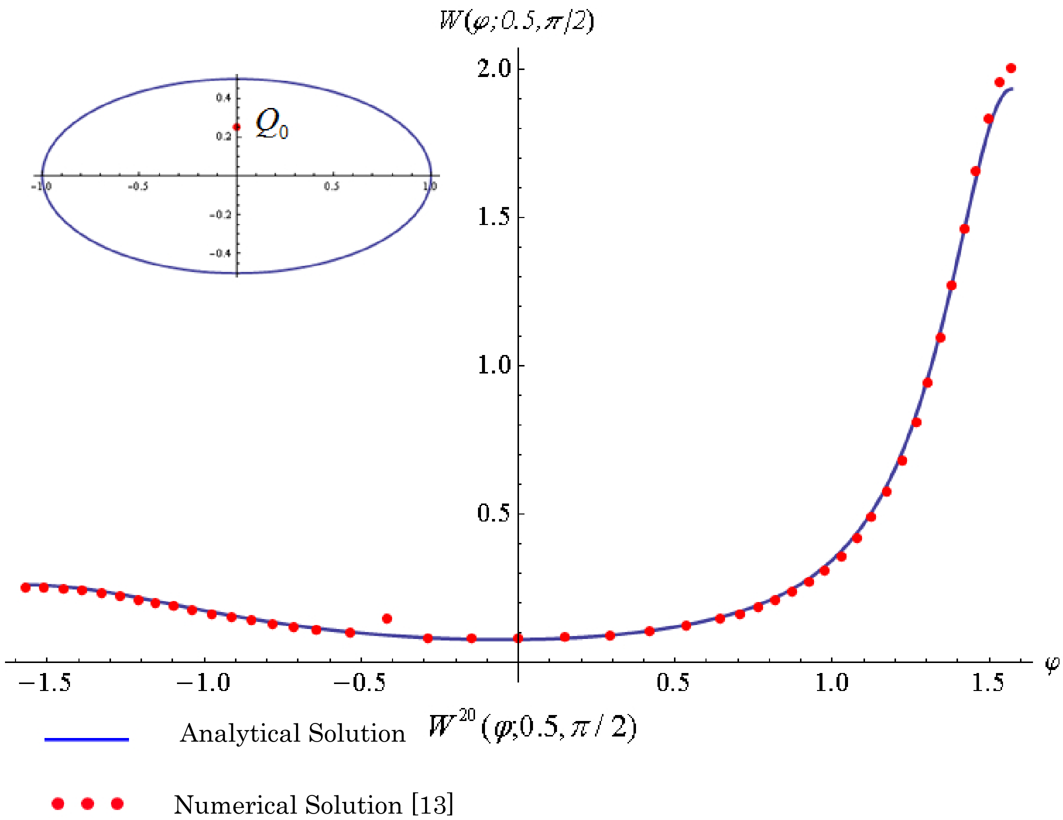

Figure 3.25: The variation of the weight function along the contour of the elliptical crack with a=1, b=0.5. The point load is applied at $x_0 = 0$, $y_0 = 0.25$.

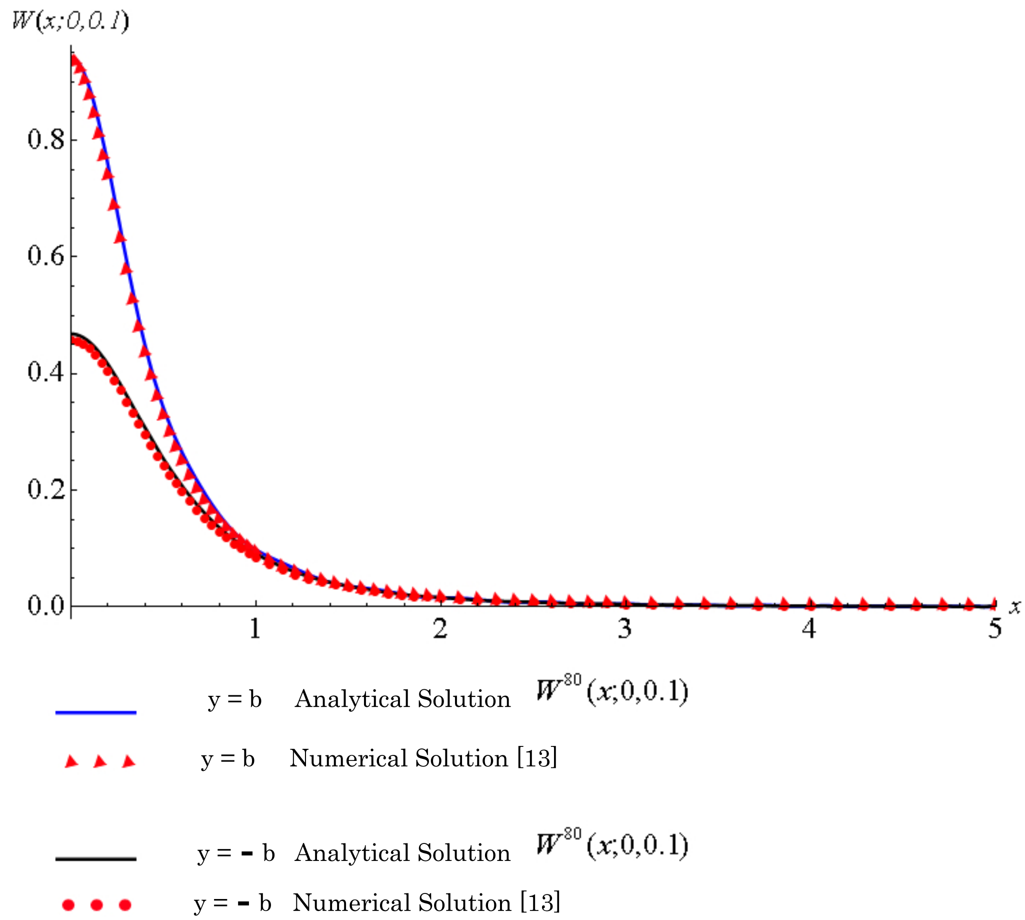

Figure 3.26: The variation of the weight function along the fronts $y = b$ and $y = -b$ of the tunnel crack with b=0.5. Point load is applied at $x_0 = 0$, $y_0 = 0.1$.

3.3.3 Comparison of the exact and the approximate weight functions.

In the present section the exact weight function $W^N(Q', Q_0)$ defined by eq.(3.90) is compared with the approximate weight function $W^{approx}(Q', Q_0)$ proposed by Oore and Burns in [37] and modified by Glinka and Reinhardt in [18]:

$$W^{approx}(Q', Q_0) = \frac{\sqrt{2}}{\pi\, l^2_{Q'Q_0} \sqrt{\Gamma(Q_0)}}, \tag{3.102}$$

where $l_{Q'Q_0}$ is the distance between points Q' and Q_0 and $\Gamma(Q_0)$ is the length of the crack contour inverted with respect to point Q_0.

In the case of a circular crack, when $a = b$, the approximate weight function $W^{approx}(Q', Q_0)$ of eq.(3.102) reduces to eq.(3.97) i.e. coincides with the exact weight function for a circular crack.

In the case of an elliptical crack the difference between the exact and the approximate weight functions depends on the aspect ratio b/a of an ellipse and the position of the point Q_0. For example, when $a = 1$, $b = 0.5$ and the point load is applied at point Q_0 with $x_0 = 0.7$, $y_0 = 0.3$, the max difference between the solutions is 37.9% according to eq.(3.103).

$$\left| \frac{W^{40}(Q', Q_0) - W^{approx}(Q', Q_0)}{W^{40}(Q', Q_0)} \right| \leq 37.9\% \tag{3.103}$$

Variation of the exact weight function $W^{40}(Q', Q_0)$ and the approximate weight function $W^{approx}(Q', Q_0)$ along the contour of the ellipse with $a = 1$, $b = 0.7$ is shown in the Fig.3.27. In the case when the ellipse has dimensions $a = 1$, $b = 0.4$ and the point load is applied at point Q_0 with $x_0 = 0.35$, $y_0 = 0.1$, the max difference between the solutions is 141.9%. Variation of the exact weight function $W^{60}(Q', Q_0)$ and the approximate weight function $W^{approx}(Q', Q_0)$ along the contour of the ellipse with $a = 1$, $b = 0.4$ is shown in the Fig.3.28.

The accuracy of the SIF-solution obtained by using the approximate weight function depends significantly on the applied stress field. For example, the weight function proposed in [58] was validated for polynomial applied stress distributions up to second degree and the difference between approximate SIF-solutions and the exact ones (obtained in section 3.2) reaches 10% for aspect ratio of the ellipse $b/a = 0.2$.

Therefore, using the exact weight function (3.90) in engineering applications can improve significantly the accuracy of results.

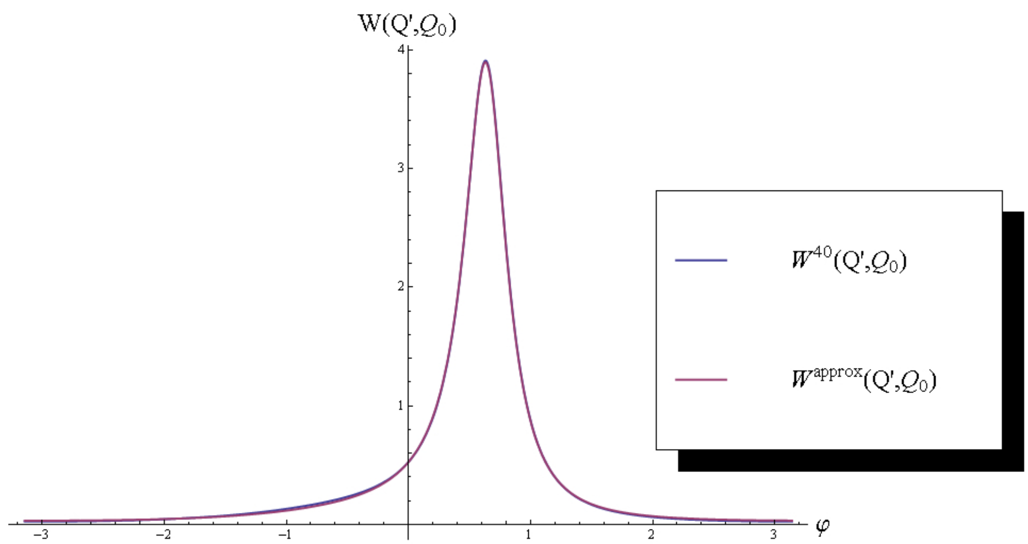

Figure 3.27: The exact weight function in comparison with the approximate weight function for the ellipse with $a = 1, b = 0.7$. The point load is applied at point Q_0 with $x_0 = 0.7$, $y_0 = 0.3$.

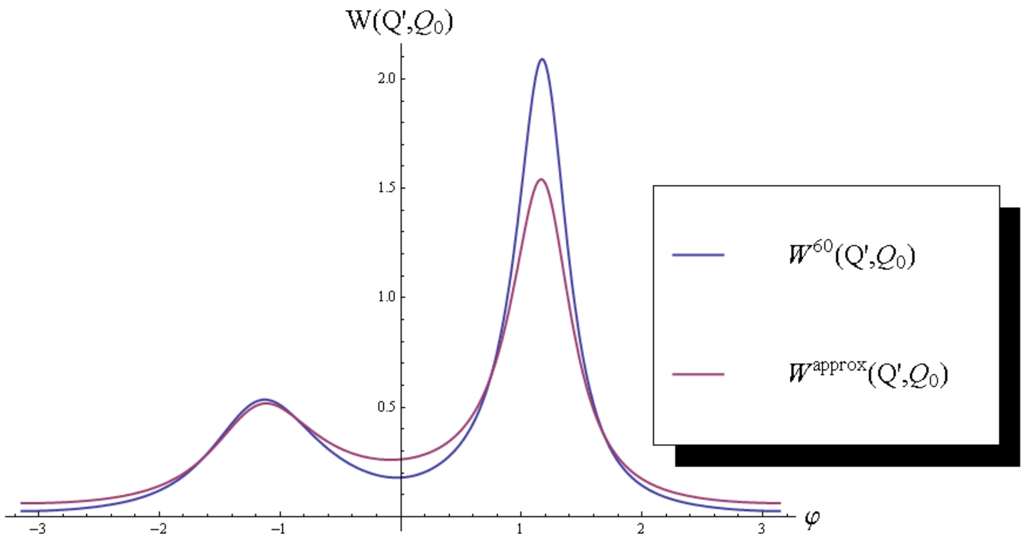

Figure 3.28: The exact weight function in comparison with the approximate weight function for the ellipse with $a = 1, b = 0.4$. The point load is applied at point Q_0 with $x_0 = 0.35$, $y_0 = 0.1$.

3.3.4 Application of the weight function

In this section it will be show how the proposed weight function may be used to determine the stress intensity factors illustrated by two different examples.

In the first example, a welded pipe with a crack is considered (Fig.3.29). In the system of coordinates (x', y', z') associated with the pipe, the longitudinal residual stress $\sigma_r(x')$ in a weldment is given by the following equation [28]:

$$\sigma_r(x') = \sigma_0 \left[1 - \left(\frac{x'}{d}\right)^2\right] e^{-\frac{1}{2}\left(\frac{x'}{d}\right)^2}, \tag{3.104}$$

where d is a half of width of of the tensile residual stress region (Fig.3.30).

The elliptical crack (with dimensions a, b) is located inside the wall of the pipe in the plane $z' = 0$. The crack is shifted by distance d_0 from the center of the weldment and rotated by angle ψ_0 from axis x' (Fig.3.31). In order to make the weight function for an infinite body applicable to this problem, it is assumed that a, b, d_0 are at least three times less than the thickness of the wall of the pipe. Coordinates (x, y) associated with the elliptical crack are connected with the coordinates (x', y') by the following relationships:

$$\begin{aligned} x' &= x \cos\psi_0 + y \sin\psi_0 + d_0 \\ y' &= -x \sin\psi_0 + y \cos\psi_0. \end{aligned} \tag{3.105}$$

The stress field $p(x, y)$ applied to the crack surface is obtained from eqs.(3.104),(3.105) as

(Fig.3.32):

$$p(x,y) = \sigma_0 \left[1 - \left(\frac{x\cos\psi_0 + y\sin\psi_0 + d_0}{d}\right)^2\right] e^{-\frac{1}{2}\left(\frac{x\cos\psi_0 + y\sin\psi_0 + d_0}{d}\right)^2} \tag{3.106}$$

The stress intensity factor is obtained from eqs.(3.100), (3.106) with $N = 4$. The variation of the stress intensity factor $K(\varphi) = K^4(\varphi)$ along the crack contour (for $a = 1$, $b = 0.5$) is shown in Fig.3.33 for $d = \frac{a}{2}$, $\psi_0 = \frac{\pi}{3}$, $d_0 = \frac{a}{4}$.

In the second example, a thick walled cylinder loaded by internal pressure P_0 is considered (Fig.3.34), where R_i is the internal radius, R_o is the external radius of the cylinder. A system of Cartesian coordinates (x', y', z') is associated with the cylinder, and the system of polar coordinates (R, ϕ) is associated with the cross-section of the cylinder. An elliptical crack (with dimensions a, b) is located in the wall of the cylinder in the plane $z' = 0$ at the distance d_0 from the center of the cylinder. In order to make the model of an elliptical crack in an infinite body applicable to this problem, it is assumed that a, b are at least three times less than the thickness of the wall of the cylinder and distances to the inner and outer surfaces of the cylinder, i.e. $a, b << R_o - R_i, R_0 - d_0, d_0 - R_i$. In polar coordinates R, ϕ principal stresses are given by the following equation [49]

$$\sigma_R = \frac{P_0 R_i^2}{R_o^2 - R_i^2}\left[1 - \frac{R_o^2}{R^2}\right], \qquad \sigma_\phi = \frac{P_0 R_i^2}{R_o^2 - R_i^2}\left[1 + \frac{R_o^2}{R^2}\right] \tag{3.107}$$

It is assumed that the crack is located in the plane $\phi = 0$ and is rotated by angle ϕ_0. The coordinates (x, y) associated with the crack are connected with the coordinates (x', y') by relationship (3.105). Then the stress field applied over the crack domain is obtained by substituting eq.(3.105) in the σ_ϕ of eq.(3.107):

$$p(x,y) = \sigma_0 \left(1 + \frac{R_o^2}{(x\cos\psi_0 + y\sin\psi_0 + d_0)^2}\right), \qquad \sigma_0 = \frac{P_0 R_i^2}{R_o^2 - R_i^2}. \tag{3.108}$$

The stress intensity factor induced by stress field (3.108) is obtained as:

$$K(\varphi) = K_1(\varphi) + K_2(\varphi), \tag{3.109}$$

where $K_1(\varphi)$ corresponds to the constant applied stress field σ_0 and it is obtained from eq.(3.36). The component $K_2(\varphi)$ corresponds to the stress field given in the non-dimensional form by eq.(3.110) (Fig.(3.35)).

$$p(x,y) = \frac{\sigma_0}{(\alpha(x/a) + \beta(y/b) + \gamma)^2},. \tag{3.110}$$

where $\alpha = \dfrac{a\cos\phi_0}{R_o}$, $\beta = \dfrac{b\sin\phi_0}{R_o}$, $\gamma = \dfrac{d_0}{R_o}$. The stress intensity factor $K_2(\varphi)$ is obtained from eq.(3.100) with $N = 6$. The variation of the stress intensity factor $K_2(\varphi) = K^6(\varphi)$ along the crack contour (for $a = 1$, $b = 0.4$, $\alpha = 1.1$, $\beta = 0.2$, $\gamma = 2$) is shown in Fig.3.36.

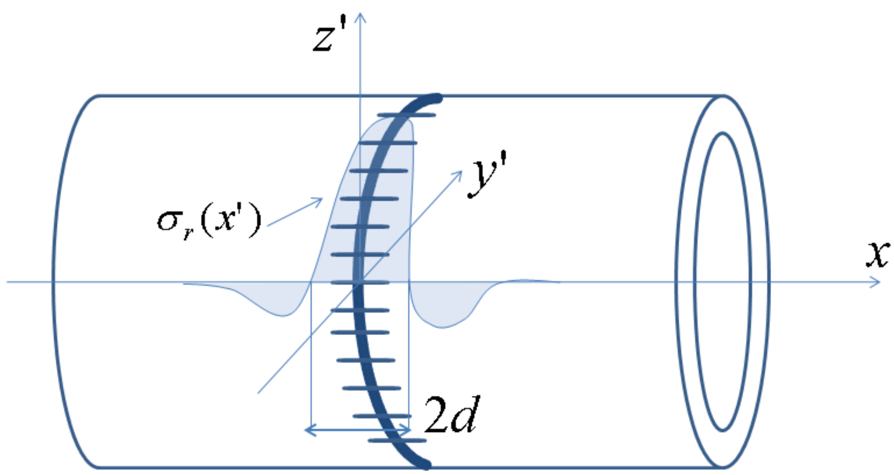

Figure 3.29: Welded pipe.

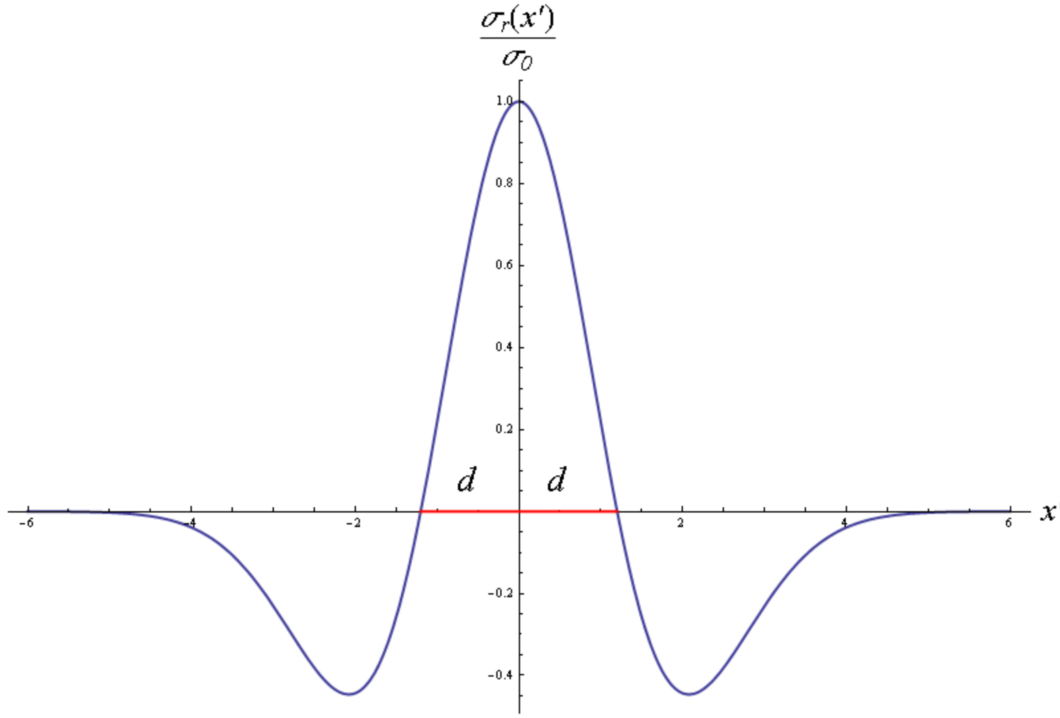

Figure 3.30: The residual stress field given by eq.(3.104).

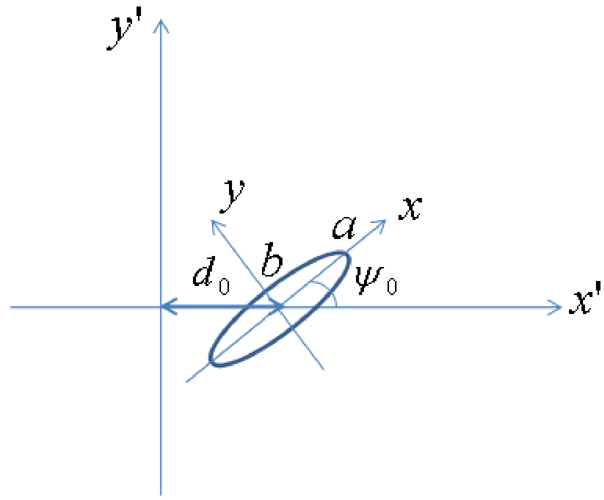

Figure 3.31: Location of the crack inside the pipe.

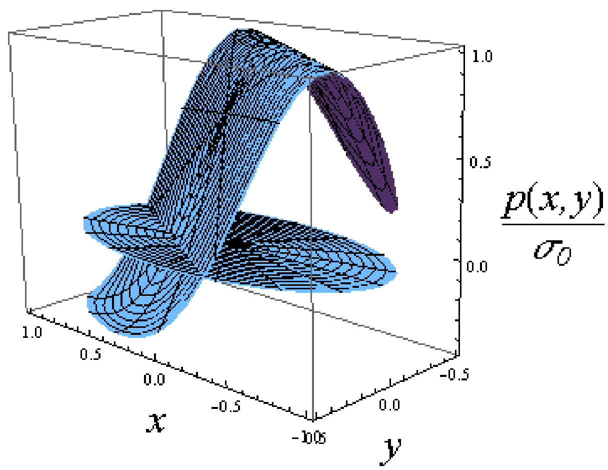

Figure 3.32: The elliptical crack subjected to the stress field given by eq.(3.106).

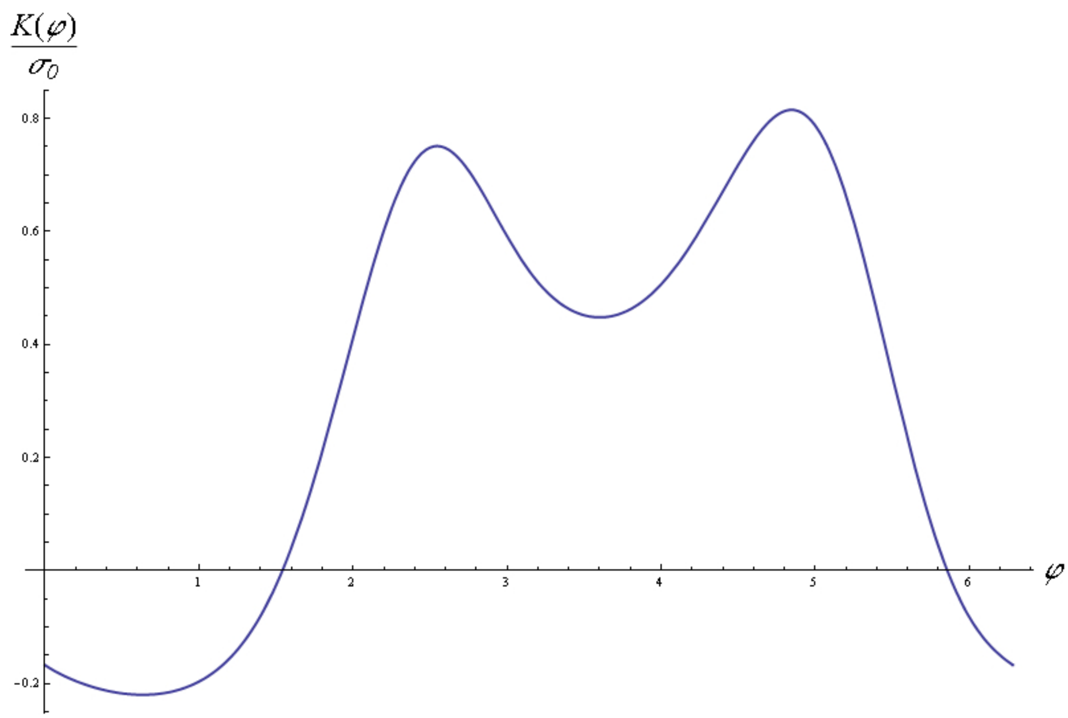

Figure 3.33: The variation of the SIF along the contour of an elliptical crack with $a = 1$, $b = 0.5$ subjected to the stress field given by eq.(3.106).

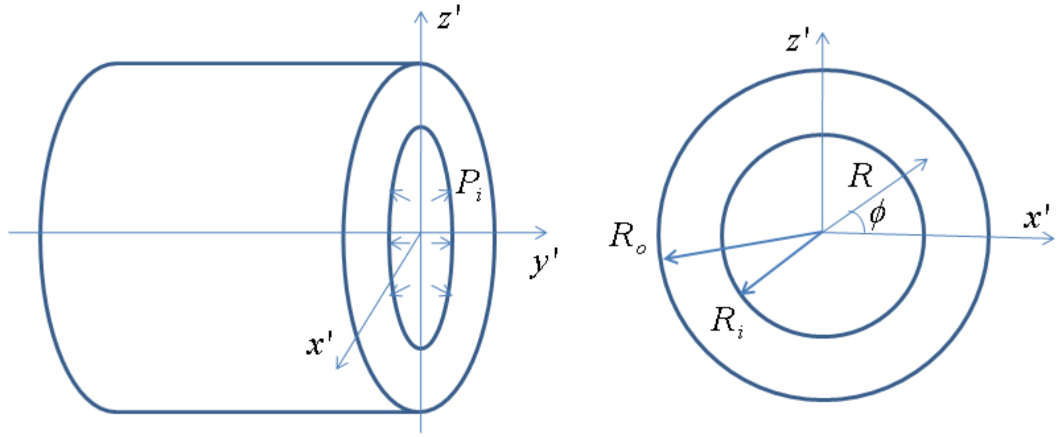

Figure 3.34: A thick walled cylinder loaded by an inner pressure.

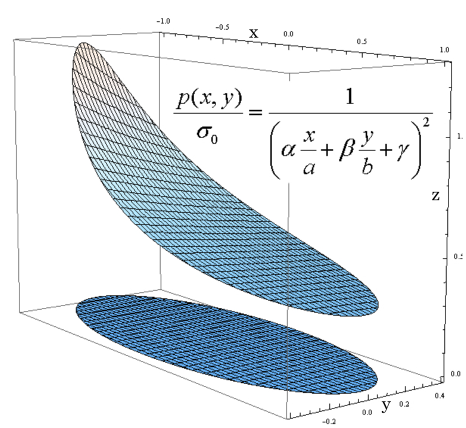

Figure 3.35: Stress field in a thick walled cylinder, given by eq. (3.110).

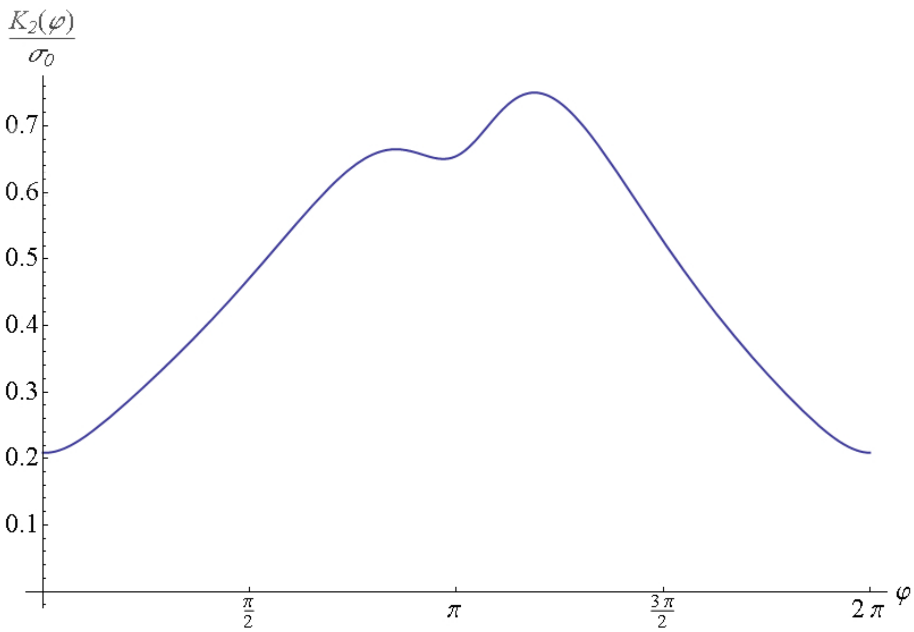

Figure 3.36: The variation of the SIF along the contour of an elliptical crack with $a = 1$, $b = 0.4$ subjected to the stress field given by eq.(3.110).

3.4 Existence of SIF-solutions

The solvability of crack problems have been studied in [22], [6], [35], [11] and others, using the variational and potential methods and suitable Sobolev spaces. Crack problems were formulated in terms of displacements and the existence, uniqueness and stability of solutions were proved for specific classes of applied stress functions.

However, the existence of displacements for a particular applied stress does not guarantee the existence of the corresponding stress intensity factor. This is demonstrated in the following example. Let a circular crack (Fig.3.37) be subjected to the stress field (3.111) given in polar coordinates as

$$p(r, \theta) = \frac{1}{\sqrt{1 - r^2}}. \tag{3.111}$$

This stress field is singular at the crack front. The corresponding crack opening displacement can be found from eq.(3.31) in the form of

$$w(r, \theta) = \frac{2(1 - \nu^2)}{E} \frac{2\,a}{\pi}\, w_0(r), \tag{3.112}$$

where $w_0(r)$ is given numerically in Fig.(3.38). As it can be seen in Fig.(3.38) the crack opening displacement at the distance ρ from the boundary does not have the order $\sqrt{\rho}$, as it is stated in eq.(2.14). The SIF induced by the applied pressure (3.111) tends to infinity everywhere along the crack front, as obtained from eq.(3.101). This finding can be generalized in the following way. Any stress field having the singularity of $\dfrac{1}{(1-r)^\kappa}$ with

$\kappa \geq 1/2$ results in an infinite stress intensity factor. This behavior (and all other results discussed in this chapter) holds for both, circular and elliptical crack, according to the properties of their weight functions (3.97), (3.90).

The applied stress field may also induce an infinite stress intensity factor if it contains a singularity located inside the crack domain. For example, the stress field given by the equation

$$p(r,\theta) = \frac{1}{r^{\kappa}}, \tag{3.113}$$

where $\kappa \geq 2$ has a singularity at the center of the crack and induces an infinite SIF, according to eq.(3.101). A unit circular crack subjected to the stress field 3.113 with $\kappa = 2$ is shown for example in Fig.3.39. The crack opening displacement according to eq.(3.31) also does not exist (i.e. tends to infinity).

However, such stress field can be posed only mathematically because it is not feasible in practice. From a physical point of view, the applied stress field represents a force distributed over an area. Mathematically it means, that the integral of the stress field over the crack plane must be finite. Hence, the equation:

$$|\iint\limits_{S} p(x,y)dx\,dy| < \infty \tag{3.114}$$

represents the necessary condition for the applied stress field $p(x,y)$ in order to assure the existence of the crack opening displacement and the stress intensity factor. Note, that in example 3.111 the condition 3.114 is satisfied, but the stress intensity factor does not exist.

Hence, the main result of section 3.4 is as follows. *The stress intensity factor does not exist for any stress field which does not satisfy the necessary condition 3.114 or has a singularity of certain order at the crack front.*

Another important for practical applications conclusion comes from the fact, that the integral of the weight function over the crack domain is finite at any point along the crack front.

The stress intensity factor induced by a bounded applied stress field is finite at any point of the crack front.

However, a rigorous proof of existence of SIF solutions is done in the following way. First, functional spaces $V_1(S)$, $V_2(S)$, $V_3(\partial S)$ are chosen for the applied stress field $p(x,y)$, crack opening displacement $w(x,y)$ and stress intensity factor $K(\varphi)$ respectively, i.e.

$$p(x,y) \in V_1(S), \qquad w(x,y) \in V_2(S), \qquad K(\varphi) \in V_3(\partial S), \tag{3.115}$$

where ∂S is the crack contour. Second, the left hand side of the equation (2.11) is considered as an operator $V_2(S) \to V_1(S)$ and the existence, uniqueness and stability of its solution $w(x,y)$ is proved based on the properties of this operator. Third, it is proved that the crack opening displacement $w(x,y)$ near the crack front has the form:

$$w(x,y) \sim K(\varphi)\sqrt{\rho}, \tag{3.116}$$

where the stress intensity factor $K(\varphi) \in V_3(\partial S)$.

Sobolev spaces are commonly used in the Theory of Elasticity to investigate crack problems. For example, in [35], [11], [22] the existence of displacement solutions of crack problems is proved in spaces $V_1(S) = H_{-1/2}(S)$, $V_2(S) = H_{1/2}(S)$, where $H_{-1/2}(S)$, $H_{1/2}(S)$ are Sobolev spaces (see [22] for definitions). However, as seen from example 3.111 the space $H_{-1/2}(S)$ has to be narrowed down in order to assure that eq.(3.116) is satisfied, i.e. the existence of SIF.

There are also certain practical aspects with using Sobolev spaces to solve crack problems. Namely the application of Sobolev spaces in the theory of elasticity requires using certain integral norms. An integral norm represents an "average" of a function and does not describe the behavior of this function at any particular point. However, if the magnitude of the stress intensity factor even at one particular point is significantly higher than its "average" along the crack front, fracture may occur at this point.

Another obstacle in using Sobolev spaces in engineering practice comes from the fact, that the Sobolev norms are defined in a rather complicated way. In order to verify that a function belongs to the Sobolev space with non-integer index, its Fourier transform must be taken and subsequently integrated with some additional factor over the infinite plane. Therefore often it is easier to confirm the existence of the stress intensity factor by direct integration of the product of the applied stress field and the proposed weight function over the crack domain.

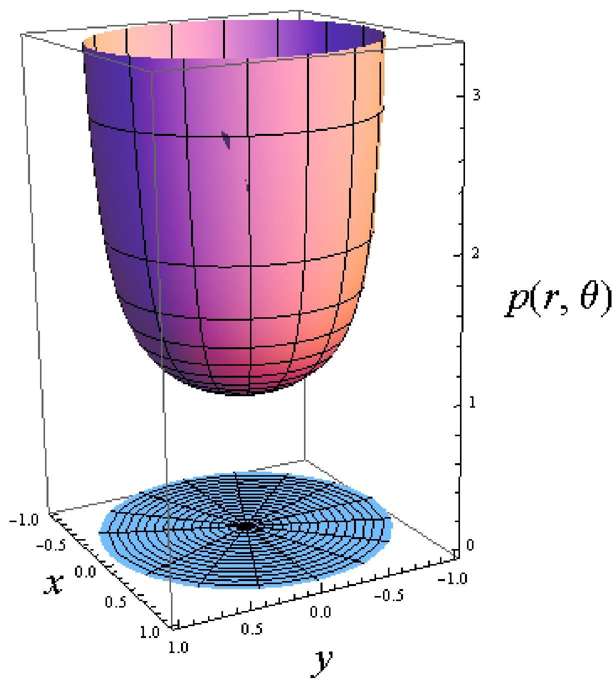

Figure 3.37: A unit circular crack subjected to the stress field (3.113) singular at the crack front.

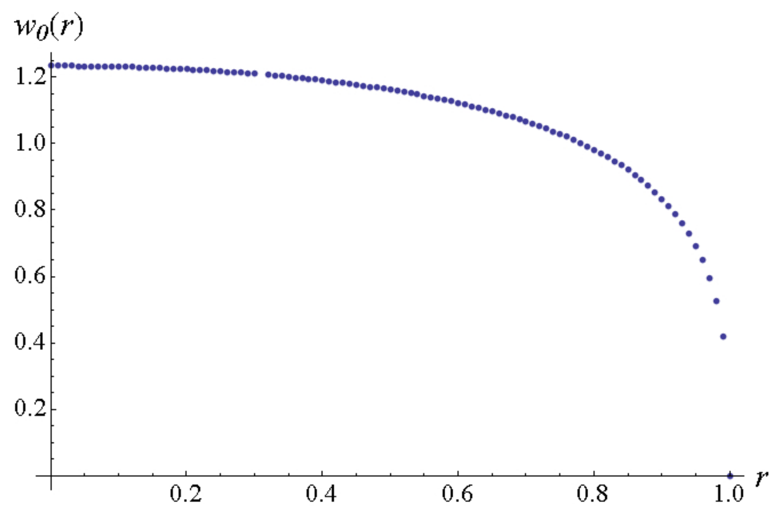

Figure 3.38: The component $w_0(r)$ of the crack opening displacement induced by stress field given by eq.(3.111)

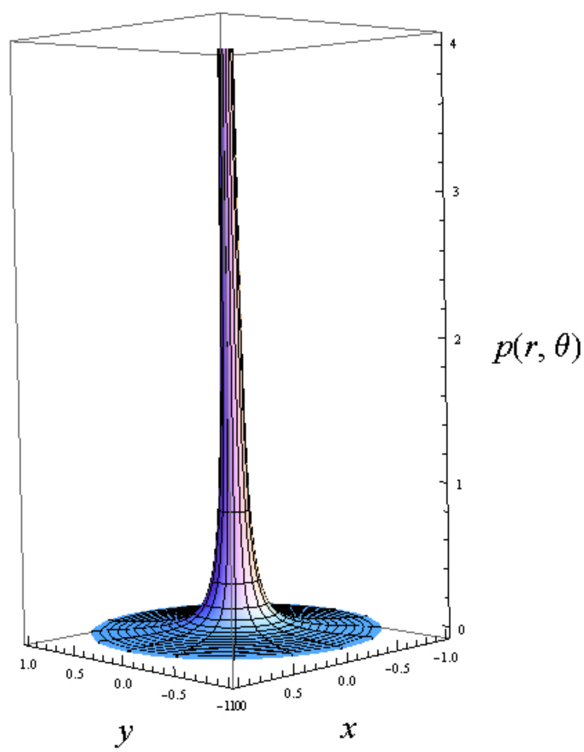

Figure 3.39: A unit circular crack subjected to the stress field (3.113) singular at the crack center.

3.5 Convergence of the weight function and the SIF-solutions.

It has been found, based on the analysis of several examples presented above, that the number of terms in the weight function $W^N(\varphi; r_0, \theta_0)$ required to achieve sufficient accuracy depends strongly on the aspect ratio b/a of the ellipse and the position of the load point $Q_0(r_0, \theta_0)$. The required number of terms (N) increases as the point Q_0 (Fig.2.3) tends to the crack front (i.e. when $r_0 \to 1$), or as the aspect ratio b/a tends to 0. However, the convergence of the stress intensity factor $K^N(\varphi)$ is the most important factor from the engineering applications point of view. The stress intensity factor $K^N(\varphi)$ is obtained from eq.(3.100) and it strongly depends on the applied stress field $p(x, y)$.

The rigorous study of the convergence of the SIF-solutions is impossible without choosing suitable functional spaces and proving first the existence of the solution, as described in section 3.4. However, the accuracy and the convergence tendency of the SIF solutions are illustrated by the example problems solved in due course. Numerous stress intensity factors have been obtained for a variety of stress fields typical in engineering practice. In all cases, the number of terms N required to achieve sufficient accuracy was relatively small, for instance in the case of examples discussed in section 3.3.4, the number of terms N was 4 and 6.

The important for practical applications property can be deduced from the form of the general weight function for an elliptical crack (eq.(3.90)). If the stress field is given by an $N^{\star}$th order polynomial in the form of eq.(3.117):

$$p^{N^{\star}}(x,y) = \sum_{n=0}^{N^{\star}} \sum_{k=0}^{N^{\star}-n} A_{n\,k} x^n y^k, \tag{3.117}$$

the corresponding stress intensity factor $K^{\star}(\varphi)$ is obtained as

$$\begin{aligned} K^{\star}(\varphi) &= \int_0^{2\pi}\int_0^1 W(\varphi; r_0, \theta_0) p^{N^{\star}}(r_0,\theta_0) a\, b\, r_0 dr_0 d\theta_0 \\ &= \int_0^{2\pi}\int_0^1 W^{N^{\star}}(\varphi; r_0, \theta_0) p^{N^{\star}}(r_0,\theta_0) a\, b\, r_0 dr_0 d\theta_0, \end{aligned} \tag{3.118}$$

i.e. it means that it is enough to take $N = N^{\star}$ terms in the weight function (3.90) to obtain the exact SIF-solution for a polynomial stress field of order $N^{\star}$.

The stress field $p(x,y)$ in many engineering situations can be approximated by a polynomial $p^{N^{\star}}(x,y)$ of the order $N^{\star}$ with such an accuracy that the error does exceed the magnitude of parameter ϵ. In mathematical terms it means that the applied stress field is given by a function $p(x,y)$ such that there exists a small parameter ϵ, such that at any point $Q(x,y)$ inside the crack domain the following inequality is satisfied.

$$|p(x,y) - p^{N^{\star}}(x,y)| < \epsilon. \tag{3.119}$$

Multiplying eq.(3.119) first by the exact weight function $W(\varphi; r_0, \theta_0)$ and second by the approximate weight function $W^{N^{\star}}(\varphi; r_0, \theta_0)$ and integrating both products over the crack domain the following estimate can be obtained:

$$|K(\varphi) - K^{N^\star}(\varphi)| < 2\epsilon \frac{\sqrt{\pi b}}{E(k)}, \tag{3.120}$$

where $K(\varphi)$ is the exact stress intensity factor at any point $Q'(\varphi)$ on the crack front induced by the applied stress field $p(x, y)$ and $K^{N^\star}(\varphi)$ is the $N^\star$th approximation of $K(\varphi)$ given by expression

$$K^{N^\star}(\varphi) = \int_0^{2\pi}\int_0^1 W^{N^\star}(\varphi; r_0, \theta_0)p(r_0, \theta_0)a\, b\, r_0 dr_0 d\theta_0 \tag{3.121}$$

As an example, let consider the stress field $p(x, y)$ given by eq.(3.110). The result of the approximation of function $p(x, y)$ by polynomials $p^{N^\star}(x, y)$ for $N^\star = 2,\ 4,\ 6$ are given in table 3.1. The stress intensity factors $K^2(\varphi)$, $K^4(\varphi)$ and $K^6(\varphi)$ have been subsequently calculated from eq.(3.100). It can be shown that differences between subsequent approximations are:

$$\begin{aligned} &|K^4(\varphi) - K^2(\varphi)| \leq 0.025817 \qquad \text{or} \qquad \left|\frac{K^4(\varphi) - K^2(\varphi)}{K^4(\varphi)}\right| \leq 31.4\%, \\ &|K^6(\varphi) - K^4(\varphi)| \leq 0.00351229 \qquad \text{or} \qquad \left|\frac{K^6(\varphi) - K^4(\varphi)}{K^6(\varphi)}\right| \leq 2.7\%. \end{aligned} \tag{3.122}$$

As it is seen in the Table 3.1 and eq.(3.122) the stress field (3.110) can be approximated by the polynomial of 6th degree with the accuracy 0.7%, and the deviation between stress intensity factors $K^4(\varphi)$ and $K^6(\varphi)$ is less than 2.7%, therefore $K^6(\varphi)$ can be considered as the sufficiently accurate SIF-solution for this problem.

Hence, the main conclusion of section 3.5 is as follows: *The number of terms N required*

$N^\star$	ϵ	$\lvert(p(x,y) - p^{N^\star}(x,y))/p(x,y)\rvert$
2	0.0774551	33.88%
4	0.0261332	2.24%
6	0.000891069	0.70%

Table 3.1: Approximation of the stress field (3.110) by polynomials of various orders $N^\star$.

to achieve sufficient accuracy of $K^N(\varphi)$ for a particular stress field $p(x,y)$ depends on how accurate this stress field can be approximated by the polynomial of order N.

3.6 Conclusions

The following results have been obtained in chapter 3:

- The problem of an elliptical crack embedded in an infinite elastic body and subjected to an arbitrary applied stress field was solved analytically in terms of the stress intensity factor. The SIF solution is unique and applicable for a wide class of applied normal stress fields.
- It was shown that the stress intensity factors derived for some particular cases of polynomial stress fields coincide with solutions already available in the literature.
- The weight function for an elliptical crack embedded in an infinite body has been also derived.
- For some particular cases of ellipse eccentricity the weight function reduces to the known solutions.
- The weight function has been compared with numerical boundary element data and good agreement has been achieved.
- Various applications of the weight function were presented.
- The limitations of the weight function and admissible stress fields have been discussed.

Chapter 4

Stress Intensity Factors for a Surface Semi-Elliptical Crack subjected to an arbitrary normal loading

Surface cracks are the most common crack configurations in engineering structures. In the present section the problem of a surface semi-elliptical crack (also known as ”perturbation problem”) in a semi-infinite solid and subjected to an arbitrary normal applied stress field is considered. The problem is solved analytically by using the alternating method. In order to use the alternating method, the weight function for an embedded elliptical crack in an infinite body is required. Thus, the exact analytical weight function derived in Chapter 3 enables one to obtain accurate SIF-solutions for a semi-elliptical crack.

4.1 Formulation of the problem for a surface semi-elliptical crack in a semi-infinite solid.

The boundary value problem for a surface semi-elliptical crack is formulated as follows. Suppose that a semi-elliptical crack (with semi-axes a and b, $a \geq b$) is embedded in a three-dimensional semi-infinite ($y > 0$) elastic body and occupies the open domain Σ : $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} < 1, y > 0$ in the plane $z = 0$ (Fig.4.1). The plane $y = 0$ represents a free surface (Fig.4.2). The crack is opened up by an arbitrary applied normal stress $P\,(x, y)$, symmetric with respect to the crack plane. Together with the standard equations of the elastic equilibrium (2.1) the following boundary conditions are imposed

$$\begin{aligned} \tau_{zx} = \tau_{zy} = 0, \qquad & \text{for} \qquad z = 0 \\ \sigma_{zz}(x, y, 0) = -P\,(x, y), \qquad & \text{for} \qquad (x, y) \in \Sigma \\ w(x, y, 0) = 0, \qquad & \text{for} \qquad (x, y) \in \Sigma_1 \\ \tau_{yx} = \sigma_{yy} = \tau_{yz} = 0, \qquad & \text{for} \qquad y = 0 \end{aligned} \tag{4.1}$$

where $\Sigma_1 : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} > 1, y > 0$ is an open domain in the plane $z = 0$ outside of the crack, τ_{zx}, τ_{zy}, $\sigma_{zz}(x, y, 0)$, τ_{yx}, τ_{yz}, σ_{yy} are the components of the elastic stress tensor.

The corresponding embedded elliptical crack problem is formulated as follows. The crack occupies the domain $S : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} < 1$ in the plane $z = 0$ and the stress field $P(x, y)$ is applied symmetrically with respect to the plane $y = 0$. The boundary conditions in this case are partially the same as eq.(4.1) (i.e.the first three conditions):

$$\begin{aligned} &\tau_{zx} = \tau_{zy} = 0, \qquad \text{for} \qquad z = 0 \\ &\sigma_{zz}(x, y, 0) = -p\,(x, y), \qquad \text{for} \qquad (x, y) \in S \\ &w(x, y, 0) = 0, \qquad \text{for} \qquad (x, y) \in S_1 \end{aligned} \tag{4.2}$$

where $S_1 : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} > 1$ is an open domain in the plane $z = 0$ outside of the crack and

$$p(x, y) = \begin{cases} P(x, y), & y > 0 \\ P(x, -y), & y < 0 \end{cases} \tag{4.3}$$

If the problem defined by eq.(4.2) is considered instead of the problem of eq.(4.1), the first three boundary conditions in eq.(4.1) are satisfied by definition. The conditions $\tau_{yx} = \tau_{yz} = 0$ on the plane $y = 0$ are satisfied due to the symmetry of the applied load $p(x, y)$. In order to eliminate the normal stress σ_{yy} on the plane $y = 0$ the alternating method is employed.

The following notations are used. Domains Σ and Σ_1 in elliptical coordinates ($x = a\,r\cos\theta, y = b\,r\sin\theta$) are described by

$$\Sigma : \quad 0 < r < 1, \qquad 0 < \theta < \pi, \qquad \Sigma_1 : \quad r > 1, \qquad 0 < \theta < \pi. \tag{4.4}$$

An arbitrary point $Q'(x', y') = Q'(\varphi)$ on the crack contour is described by a parametric angle φ such that:

$$x' = a\cos\varphi, \qquad y' = b\sin\varphi, \tag{4.5}$$

where for a semi-elliptical crack $0 < \varphi < \pi$ and for the corresponding elliptical crack $0 \leq \varphi \leq 2\pi$.

Together with the coordinates (x, y, z) the system (ξ, η, ζ) will be used as variables of integration.

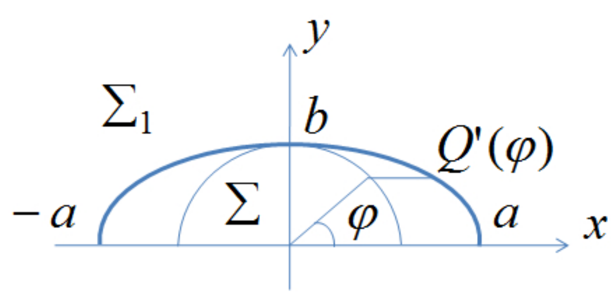

Figure 4.1: Semi-elliptical crack.

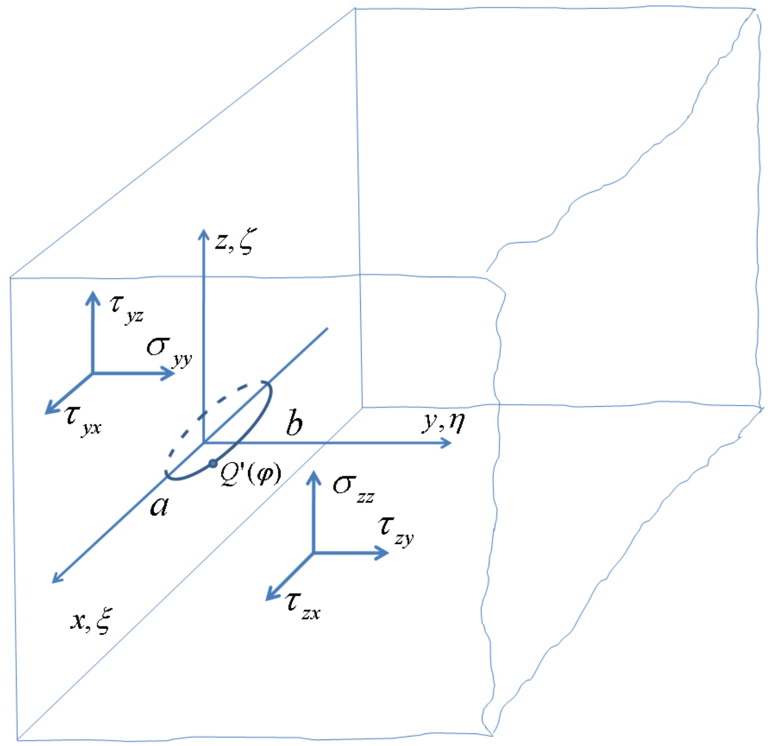

Figure 4.2: Coordinate systems associated with a surface semi-elliptical crack in a semi-infinite domain.

4.2 The alternating method applied to the problem of a semi-elliptical crack

In order to use the alternating method to solve the problem of a semi-elliptical crack, two following solutions are employed.

Solution 1 is that one for the stress intensity factor and the stress field component σ_{yy}

obtained for an infinite body containing an elliptical crack. The stress intensity factor $K(\varphi)$ for an elliptical crack can be obtained using either eq.(3.26) or eq.(3.100) with sufficiently large number of terms N. The stress field component σ_{yy} is obtained using the analytical solution presented in reference [53].

Solution 2 is that one for the stress field components $(\tau_{zx}, \tau_{zy}, \sigma_{zz})$ in the un-cracked semi-infinite body, loaded on the boundary plane. Solution 2 has been given in the analytical form in reference [51].

Next, the alternating method includes the successive approximations $K^{(0)}(\varphi), K^{(1)}(\varphi), ...$ of the SIF-solution $K(\varphi)$ for a semi-elliptical crack. The iterating process was organized as follows.

In order to obtain the zeroth approximation $K^{(0)}(\varphi)$ of the stress intensity factor, the problem defined by eq.(4.2) was considered instead of the problem of eq.(4.1). It was also convenient to use the following notations:

$$P(x, y) = P^{(0)}(x, y), \qquad p(x, y) = p^{(0)}(x, y). \tag{4.6}$$

Due to the symmetry (eqs.(4.3),(4.6)), the applied loading $p^{(0)}(x, y)$ can be expanded into the cosine Fourier series in elliptical coordinates (r, θ) as follows:

$$p^{(0)}(x, y) = p^{(0)}(r, \theta) = \frac{p_0^{(0)}(r)}{2} + \sum_{n=1}^{\infty} p_n^{(0),c}(r) \cos n\theta. \tag{4.7}$$

The stress intensity $K^{(0)}(\varphi)$ was subsequently obtained using eq.(3.26) as

$$K^{(0)}(\varphi) = \pi\sqrt{b}\left(\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi\right)^{1/4}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-i)^n(-1)^k\mathcal{A}_{kn}^{(0),c}\cos n\varphi, \qquad (4.8)$$

where $\mathcal{A}_{kn}^{(0),c}$ are coefficients obtained from the system (3.18) with the stress field $p^{(0)}(x,y)$.

The stress intensity factor $K^{(0)}(\varphi)$ can also be obtained from eq.(3.100) as

$$K^{(0)}(\varphi) = \int_0^{2\pi}\int_0^1 W(\varphi; r_0, \theta_0)p^{(0)}(r_0, \theta_0)a\, b\, r_0 dr_0 d\theta_0, \qquad (4.9)$$

where $W(\varphi; r_0, \theta_0) = W^N(\varphi; r_0, \theta_0)$ with sufficiently large number of terms N (which depends on the applied stress field $p(r,\theta)$).

The crack opening displacement $w(x,y) = w^{(0)}(x,y) = w(r,\theta) = w^{(0)}(r,\theta)$ was obtained using eq.(3.23) as

$$\begin{aligned} w^{(0)}(r,\theta) = {} & \frac{2(1-\nu^2)}{E}\frac{\pi\, b}{2}\sqrt{1-r^2} \\ & \times \sum_{n=0}^{\infty}\frac{(-i)^n\, r^n}{\Gamma(n+1)}\sum_{k=0}^{\infty}\frac{\Gamma(n+k+1)}{\Gamma(k+\frac{3}{2})}\mathfrak{F}_k(n+\tfrac{3}{2}, n+1, r^2)\mathcal{A}_{kn}^{(0),c}\cos n\theta. \end{aligned} \qquad (4.10)$$

The stress field component $\sigma_{yy} = \sigma_{yy}^{(0)}$ is given in reference [53] as

$$\frac{2(1-\nu^2)}{E}\sigma_{yy}^{(0)} = \frac{1}{2\pi}\iint_S\left[-2\nu\frac{\partial^2}{\partial z^2} + (1-2\nu)\frac{\partial^2}{\partial y^2} + z\frac{\partial^3}{\partial y^2\partial z}\right]\frac{1}{\mathbf{r}}\, w^{(0)}(\xi,\eta)d\xi\, d\eta \qquad (4.11)$$

where

$$\mathbf{r} = \sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}. \qquad (4.12)$$

In order to eliminate the normal stress on the free boundary, the Newman boundary value problem is considered, where the stress vector

$$(\tau_{yx}, \sigma_{yy}, \tau_{yz}) = (0, -\sigma_{yy}^{(0)}, 0) \tag{4.13}$$

is prescribed on the plane $y = 0$. The stress field solution $\sigma_{zz} = \sigma_{zz}^{(0)}$ to this problem is given in [51] as

$$\sigma_{zz}^{(0)} = -\frac{1}{2\pi} \iint\limits_{\mathbb{R}^2} \left[(\lambda + 2\mu) \frac{\partial A_{13}}{\partial z} + \lambda \frac{\partial A_{23}}{\partial x} + \lambda \frac{\partial A_{33}}{\partial y} \right] \sigma_{yy}^{(0)}(\zeta, \xi) d\zeta \, d\xi, \tag{4.14}$$

where the following notations are used:

$$\begin{aligned}
A_{13} &= \frac{1}{2\mu} \frac{\partial^2 \mathbf{R}}{\partial z \partial y} + \frac{1}{2(\lambda + \mu)} \frac{\partial^2 \Phi}{\partial z \partial y}, \qquad A_{23} = \frac{1}{2\mu} \frac{\partial^2 \mathbf{R}}{\partial x \partial y} + \frac{1}{2(\lambda + \mu)} \frac{\partial^2 \Phi}{\partial x \partial y}, \\
A_{33} &= -\frac{2\lambda + 3\mu}{2\mu(\lambda + \mu)} \frac{1}{\mathbf{R}} + \frac{1}{2\mu} \frac{\partial^2 \mathbf{R}}{\partial y^3}, \\
\mathbf{R} &= \sqrt{(x - \xi)^2 + y^2 + (z - \zeta)^2}, \qquad \Phi = y \ln(\mathbf{R} + y) - \mathbf{R}, \\
\mu &= \frac{E}{2(1 + \nu)}, \qquad \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}.
\end{aligned} \tag{4.15}$$

After substituting $z = 0$ in eq.(4.14) the additional stress due to the free boundary $\sigma_{zz}^{(0)} = \sigma_{zz}^{(0)}(x, y)$, $y > 0$ is obtained over the domain of the semi-elliptical crack. Due to the symmetry of the problem, shear stresses $\tau_{zx}^{(0)}$ and $\tau_{zy}^{(0)}$ vanish in the crack plane, i.e. $\tau_{zx}^{(0)} = 0$ and $\tau_{zy}^{(0)} = 0$ in the plane $z = 0$.

Next, the stress field $P^{(1)}(x, y) = P^{(0)}(x, y) - \sigma_{zz}^{(0)}(x, y)$ is applied over the crack domain and the first approximation for the SIF $K(\varphi)$ is obtained as

$$K^{(1)}(\varphi) = \int_0^{2\pi}\int_0^1 W(\varphi; r_0, \theta_0) p^{(1)}(r_0, \theta_0) a\, b\, r_0\, dr_0\, d\theta_0, \tag{4.16}$$

where the stress field $p^{(1)}(r_0, \theta_0)$ is given in Cartesian coordinates by

$$p^{(1)}(x, y) = \begin{cases} P^{(1)}(x, y), & y > 0 \\ P^{(1)}(x, -y), & y < 0, \end{cases} \tag{4.17}$$

or

$$K^{(1)}(\varphi) = K^{(0)}(\varphi) - \int_0^{2\pi}\int_0^1 W(\varphi; r_0, \theta_0) \tilde{\sigma}_{zz}^{(0)}(r_0, \theta_0) a\, b\, r_0\, dr_0\, d\theta_0, \tag{4.18}$$

where the stress field $\tilde{\sigma}_{zz}^{(0)}$ is given in Cartesian coordinates by

$$\tilde{\sigma}_{zz}^{(0)}(x, y) = \begin{cases} \sigma_{zz}^{(0)}(x, y), & y > 0 \\ \sigma_{zz}^{(0)}(x, -y), & y < 0. \end{cases} \tag{4.19}$$

The process is repeated until the sequence $K^{(0)}(\varphi), K^{(1)}(\varphi), \ldots, K^{(n)}(\varphi)$ converges.

4.2.1 Numerical procedure of obtaining the SIF-solution by the alternating method.

In order to obtain the stress field component σ_{yy} normal to the free boundary, induced by the stress field $P(x, y)$ applied to the surface of the elliptical crack, the following numerical procedure was used. The region $[0, 2a] \times [0, 2a]$ in the plane xz was divided into two subregions $[0, a] \times [0, 2a]$ and $[a, 2a] \times [0, 2a]$ excluding the interval $x \in [0, a]$ (Fig.4.3). Next, the integral (4.11) was calculated for 20×20 points in each subregion. Numerical integration was performed using the software Mathematica 6. Obtained values were projected symmetrically to the regions $[-2a, 0] \times [0, 2a]$ and $[-2a, 2a] \times [-2a, 0]$. It was observed that the stress σ_{yy} decreases rapidly away from the crack, hence using the regions greater than $[0, 2a] \times [0, 2a]$ contributed insignificantly into the final solution. It was also observed that increasing the number of points where the integral (4.11) was calculated did not improve the SIF-solution. Note, that the full stress field (4.11) was used everywhere in the plane xz what enables us to obtain the results of better accuracy than results obtained by other authors who used the asymptotic expansions (2.13) near the points $(a, 0)$ and $(-a, 0)$ (see for instance, [48], [17]).

In order to obtain the stress field component σ_{zz} normal to the crack plane, induced by the stress field $-\sigma_{yy}$, applied to the free surface, the following numerical procedure was used. The region $[0, a] \times [0, b]$ in the plane xy was divided into three subregions $[0, a] \times [b/10, b]$, $[0, a - b/10] \times [0, b/10]$ and $[a - b/10, a] \times [0, b/10]$ (Fig.4.4). Next, the integral (4.14) was calculated for 10×10 points in each subregion. Numerical integration was performed using the software Mathematica 6. Obtained values were projected symmetrically with respect to axes x and y. It was also observed that increasing the number of points where the integral (4.14) was calculated did not improve the SIF-solution.

The weight function (3.90) was used for obtaining the SIF-solution for an elliptical crack.

Note, that the SIF-solutions used previously ([44], [50]) in conjunction with the alternating method were limited by the polynomial applied stress field, what lead to the certain inaccuracy of the final results. However, since the stress field σ_{zz} has a singularity at the point $(x = \pm a, y = 0)$, a large number of terms in eq.(3.90) is required to achieve the sufficient accuracy of the final solution.

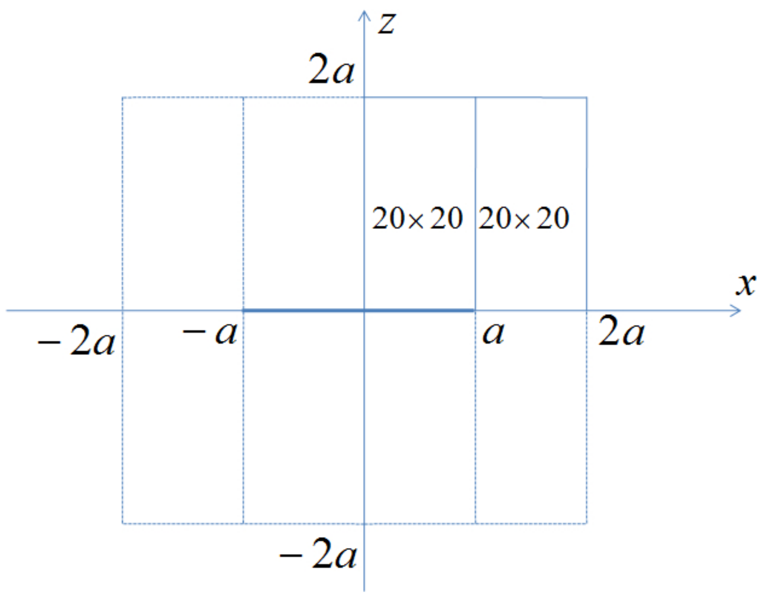

Figure 4.3: Plane xz and the regions for numerical evaluation of stress σ_{yy}.

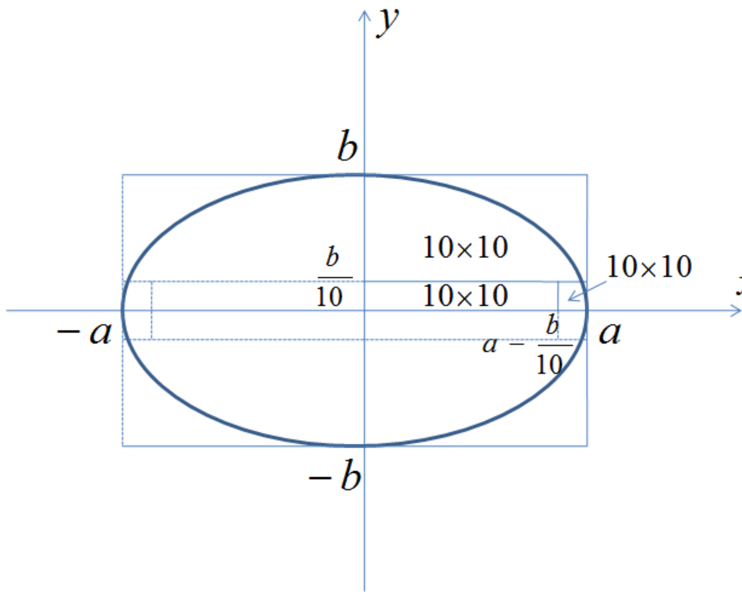

Figure 4.4: Plane xy and the regions for numerical evaluation of stress σ_{zz}.

4.3 Numerical SIF data compared with existing literature solutions.

The SIF solution for a semi-elliptical crack in a semi-infinite body subjected to a uniform pressure has been determined by using the alternating method and compared with Raju-Newman solution [27] and Lena Nilsson's data [38]. Raju-Newman SIF solution [27] was obtained by using the finite element method for a semi-elliptical crack in a finite thickness plate. Numerical data was subsequently fitted into an empirical equation. This equation in the case of a semi-infinite solid is given as

$$K(\varphi) = \sigma_0 \sqrt{\pi \frac{b}{Q^\star}} M_1 \, f_\varphi \, g, \tag{4.20}$$

where

$$\begin{aligned} M_1 &= 1.13 - 0.09 \frac{b}{a}, \\ f_\varphi &= \left[\left(\frac{b}{a} \right)^2 \cos^2 \varphi + \sin^2 \varphi \right]^{1/4}, \\ g &= 1 + 0.1(1 - \sin \varphi)^2, \end{aligned} \tag{4.21}$$

and $Q^\star$ is the approximation of the complete elliptical integral of the second kind $E(k)$ [27] given by

$$Q^\star = 1 + 1.464 \left(\frac{b}{a} \right)^{1.65}. \tag{4.22}$$

Lena Nilsson's solution [38] was also obtained by using the finite element method for a semi-elliptical crack in a finite thickness plate. Numerical data used for comparison is for the case when the crack depth is 10 times smaller than the plate width.

The variation of the first four approximations $K^{(0)}(\varphi)$, $K^{(1)}(\varphi)$, $K^{(2)}(\varphi)$, $K^{(3)}(\varphi)$ of the stress intensity factor $K(\varphi)$ along the contour of a semi-elliptical crack with $a = 1$, $b = 0.5$ subjected to the uniform tension $p(x, y) = \sigma_0$ obtained by the alternating method and the SIF data from ref.[27] and ref.[38] are shown in Fig.4.5. Number of terms $N = 30$ was used in eq.(3.100).

Next, the SIF was obtained for a semi-elliptical crack subjected to the quadratic stress field

$$p(x,y) = \sigma_0 \left(\frac{y}{b}\right)^2. \tag{4.23}$$

The variation of the first three approximations $K^{(0)}(\varphi)$, $K^{(1)}(\varphi)$, $K^{(2)}(\varphi)$ of the stress intensity factor $K(\varphi)$ along the contour of a semi-elliptical crack with $a = 1$, $b = 0.5$ subjected to the applied stress field (4.23) is shown in Fig.4.6 in comparison with Lena Nilsson solution [38]. Number of terms $N = 20$ was used in eq.(3.100). As seen from Fig.4.6 Nilsson's solution coincides with the zeroth approximation of the exact SIF-solution $K(\varphi)$ and does not take into account the effect of the free boundary.

4.4 Conclusions

The following results have been obtained in Chapter 4.

- The problem of a surface breaking semi-elliptical crack in a semi-infinite elastic solid subjected to an arbitrary mode I stress field was formulated and solved.

- The SIF solution for a semi-elliptical crack was obtained by using the alternating method together with the weight function for an elliptical crack in an infinite body, obtained in chapter 3.

- The SIF solutions for a semi-elliptical crack subjected to a uniform tension and

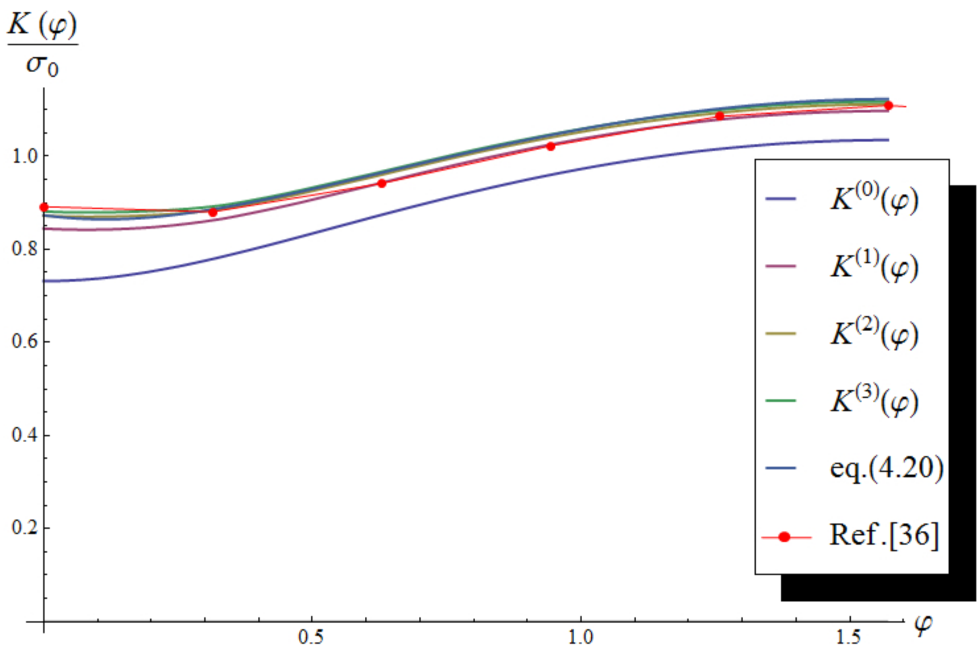

Figure 4.5: The variation of the SIF along the contour of a semi-elliptical crack subjected to a uniform tension.

quadratic stress field was compared with available literature solutions and a good agreement of results was shown.

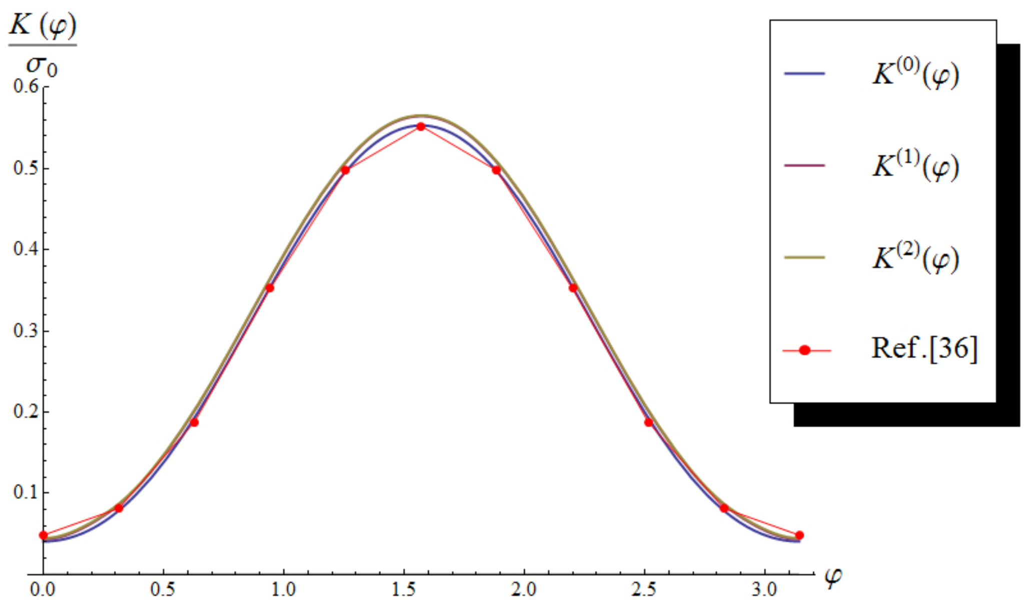

Figure 4.6: The variation of the SIF along the contour of a semi-elliptical crack subjected to a quadratic stress field.

Chapter 5

Summary of the results and recommendations for future work

The above work investigated the problem of an elliptical crack embedded in an infinite elastic solid, subjected to an arbitrary applied normal stress field. The problem was formulated as a boundary value problem of the Theory of Elasticity, and the governing system of equations was transformed into an integral equation for the unknown crack opening displacement. This equation was solved using the method of simultaneous dual integral equations. Subsequently, expansions of the displacement field near the crack front were used to extract the stress intensity factor. The SIF solution represents the unique analytical expression applicable for a wide class of applied stress fields. The SIF solution reduces to the known expressions for some particular cases of applied polynomial stress fields.

The exact analytical weight function for an elliptical crack was obtained as a particular case of the SIF induced by an applied concentrated force. The weight function was obtained in the convenient form of a Fourier Series. The weight function reduces to the known expressions for some particular cases of ellipse eccentricity and the position of the point load. The weight function was compared with the boundary elements data and a good agreement of results was obtained.

Tunnel cracks were modeled as low aspect ratio elliptical cracks. The weight function for a tunnel crack was derived from the weight function for an elliptical crack.

The weight function enables the calculation of SIF due to an arbitrary two-dimensional stress field applied to the crack surface. The applications of the weight function were demonstrated by showing how it could be used to calculate SIFs for cracks in pressure vessels and welded structures.

The limitations of the weight function and the SIF solutions were discussed. The necessary conditions for the applied stress in order to assure the existence of the stress intensity factor were introduced.

The rate of convergence of the SIF solutions was investigated for various applied stress fields. It was shown that for stress fields close to polynomials only the first few terms of the full expression are required to obtain an accurate approximation of SIF.

The SIF solution (or the weight function) for an elliptical crack was subsequently used to solve the problem of a surface breaking semi-elliptical crack embedded in a semi-infinite elastic solid and subjected to an arbitrary applied normal stress field. In order to eliminate stresses on the free boundary, the alternating method was employed. The SIF solution for a semi-elliptical crack was compared with data available in literature for the cases of applied uniform tension and quadratic stress field and a good agreement of results was shown.

The SIF solution (or the weight function) for an elliptical crack embedded in an infinite body can also be used in conjunction with the alternating method to solve some other elliptical crack problems with the free boundaries, such as the problem of an elliptical crack near a free boundary or the problem of a corner quarter-elliptical crack.

Using the weight function with the finite element alternating method enables to solve elliptical crack problems for finite bodies.

The SIF solutions for an elliptical or semi-elliptical crack may be used in various practical applications, including cracks in mechanical components subjected to cyclic loading.

APPENDICES

Appendix A

First Terms of the Weight Function for an elliptical crack

An elliptical crack with dimensions $(a, b, a \geq b)$ occupies the domain $x^2/a^2 + y^2/b^2 < 1$ in the plane $z = 0$ in Cartesian coordinates (x, y, z). The crack is subjected to the point load applied at an arbitrary location $Q_0(x_0, y_0)$ inside the crack domain (Fig.A.1). The elliptical coordinates (r, θ) are introduced as follows:

$$
\begin{aligned}
x &= a\,r\cos\theta, \qquad y = b\,r\sin\theta, \\
x_0 &= a\,r_0\cos\theta_0, \qquad y_0 = b\,r_0\sin\theta_0.
\end{aligned}
\tag{A.1}
$$

For any fixed number N, the Nth approximation $W^N(\varphi; r_0, \theta_0)$ of the weight function $W(\varphi; r_0, \theta_0)$ is given by:

$$W^N(\varphi; r_0, \theta_0) = \frac{P}{a\sqrt{b}} \left(\frac{b^2}{a^2} \cos^2 \varphi + \sin^2 \varphi \right)^{1/4} \sum_{m=0}^{N} \big(\mathcal{X}_m^N(r_0, \theta_0) \cos\, m\varphi \qquad \text{(A.2)}$$

$$+\, \mathcal{Y}_m^N(r_0, \theta_0) \sin\, m\varphi \big)\,,$$

where φ is the parametric angle defining the point $Q'(a\cos\varphi, b\sin\varphi)$ on the crack contour, $P = 1$ is the magnitude of the applied force.

The following notations are used:

$$\begin{aligned} \alpha_{nm}^c &= \int_0^{2\pi} \sqrt{\frac{b^2}{a^2}\cos^2\phi + \sin^2\phi}\; \cos n\phi \cos m\phi \, d\phi, \\ \alpha_{nm}^s &= \int_0^{2\pi} \sqrt{\frac{b^2}{a^2}\cos^2\phi + \sin^2\phi}\; \sin n\phi \sin m\phi \, d\phi, \end{aligned} \qquad \text{(A.3)}$$

$$\begin{aligned} f_n^c &= \frac{2}{\sqrt{\pi}(-\boldsymbol{i})^n} \frac{r_0^n \cos n\theta_0}{\sqrt{1-r_0^2}}, \qquad f_n^s = \frac{2}{\sqrt{\pi}(-\boldsymbol{i})^n} \frac{r_0^n \sin n\theta_0}{\sqrt{1-r_0^2}} \\ b_{0,0}^c &= \frac{6\sqrt{1-r_0^2}}{\sqrt{\pi}}, \qquad b_{0,1}^c = \frac{14\,(2-5r_0^2)\,\sqrt{1-r_0^2}}{3\sqrt{\pi}}, \\ b_{2,0}^c &= -\frac{14\, r_0^2 \sqrt{1-r_0^2}}{\sqrt{\pi}} \cos 2\theta_0, \end{aligned} \qquad \text{(A.4)}$$

The zeroth approximation $W^0(\varphi; r_0, \theta_0)$ is given by:

$$W^0(\varphi; r_0, \theta_0) = \frac{P}{a\sqrt{b}} \left(\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi \right)^{1/4} \mathcal{X}_0^0(r_0, \theta_0), \qquad \text{(A.5)}$$

where

$$\mathcal{X}_0^0(r_0, \theta_0) = \frac{1}{\alpha_{00}^c} f_0^c = \frac{1}{4\,E(k)} \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{1-r_0^2}} \qquad \text{(A.6)}$$

The weight function (A.5) is suitable for obtaining the stress intensity factor $K(\varphi)$ induced by a constant applied stress field, i.e. if the stress field is given by

$$p(x, y) = p(r, \theta) = \sigma_0, \tag{A.7}$$

the stress intensity factor is obtained as

$$\begin{aligned} K(\varphi) &= \int_0^{2\pi}\int_0^1 W^0(\varphi; r_0, \theta_0) p(r_0, \theta_0) a\, b\, r_0 dr_0 d\theta_0 \\ &= \frac{\sqrt{\pi\, b}\,\sigma_0}{E(k)} \left(\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi\right)^{1/4}, \end{aligned} \tag{A.8}$$

where $k^2 = (a^2 - b^2)/a^2$.

The first approximation $W^1(\varphi; r_0, \theta_0)$ is given by

$$\begin{aligned} W^1(\varphi; r_0, \theta_0) = \frac{P}{a\sqrt{b}} \left(\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi\right)^{1/4} (&\mathcal{X}_0^1(r_0, \theta_0) + \mathcal{X}_1^1(r_0, \theta_0)\cos\varphi \\ &+ \mathcal{Y}_1^1(r_0, \theta_0)\sin\varphi), \end{aligned} \tag{A.9}$$

where

$$\begin{aligned} \mathcal{X}_0^1(r_0, \theta_0) &= \frac{1}{\alpha_{00}^c} f_0^c = \frac{1}{4\, E(k)} \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{1 - r_0^2}}, \\ \mathcal{X}_1^1(r_0, \theta_0) &= \frac{-\boldsymbol{i}}{\alpha_{11}^c} f_1^c = -\frac{3\, k^2}{4((1 - 2k^2)E(k) - k'^2 K(k))} \frac{2}{\sqrt{\pi}} \frac{r_0\cos\theta_0}{\sqrt{1 - r_0^2}}, \\ \mathcal{Y}_1^1(r_0, \theta_0) &= \frac{-\boldsymbol{i}}{\alpha_{11}^s} f_1^s = \frac{3k^2}{4((1 + k^2)E(k) - k'^2 K(k))} \frac{2}{\sqrt{\pi}} \frac{r_0 \sin\theta_0}{\sqrt{1 - r_0^2}} \end{aligned} \tag{A.10}$$

and $k' = b/a$.

The weight function (A.10) is suitable for obtaining the stress intensity factor $K(\varphi)$ induced by a liner applied stress field, i.e. if the stress field is given by:

$$p(x,y) = \sigma_0 + \sigma_1 \frac{x}{a} + \sigma_2 \frac{y}{b} \qquad \text{or} \qquad p(r,\theta) = \sigma_0 + \sigma_1 r\cos\theta + \sigma_2 r\sin\theta, \tag{A.11}$$

the stress intensity factor is obtained as:

$$\begin{aligned} K(\varphi) &= \int_0^{2\pi}\int_0^1 W^1(\varphi; r_0, \theta_0) p(r_0,\theta_0) a\, b\, r_0 dr_0 d\theta_0 \\ &= \frac{\sqrt{\pi\, b}}{E(k)} \left(\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi\right)^{1/4} \\ &\times \left(\sigma_0 + \frac{-k^2 E(k)\sigma_1 \cos\varphi}{((1-2k^2)E(k) - k'^2 K(k))} + \frac{k^2 E(k)\sigma_2 \sin\varphi}{(1+k^2)E(k) - k'^2 K(k)}\right). \end{aligned} \tag{A.12}$$

The second approximation $W^1(\varphi; r_0, \theta_0)$ is given by:

$$\begin{aligned} W^2(\varphi; r_0, \theta_0) = \frac{P}{a\sqrt{b}} \left(\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi\right)^{1/4} & (\mathcal{X}_0^2(r_0,\theta_0) \\ & + \mathcal{X}_1^2(r_0,\theta_0)\cos\varphi + \mathcal{X}_2^2(r_0,\theta_0)\cos 2\varphi \\ & + \mathcal{Y}_1^1(r_0,\theta_0)\sin\varphi + \mathcal{Y}_2^2(r_0,\theta_0)\sin 2\varphi), \end{aligned} \tag{A.13}$$

where

$$\begin{aligned} \mathcal{X}_0^2(r_0,\theta_0) &= \frac{1}{\alpha_{00}^c} b_{0,0}^c - \left(\frac{\delta_2^{0,0}}{\Delta_2} b_{0,1}^c + \frac{\delta_2^{2,0}}{\Delta_2} b_{2,0}^c\right) + \frac{\delta_2^{0,0}}{\Delta_2}(f_0^c - b_{0,0}^c + b_{0,1}^c) - \frac{\delta_2^{2,0}}{\Delta_2}(f_2^c - b_{2,0}^c), \\ \mathcal{X}_1^2(r_0,\theta_0) &= \frac{-\boldsymbol{i}}{\alpha_{11}^c} f_1^c = -\frac{3\,k^2}{4((1-2k^2)E(k) - k'^2 K(k))} \frac{2}{\sqrt{\pi}} \frac{r_0\cos\theta_0}{\sqrt{1-r_0^2}}, \\ \mathcal{X}_2^2(r_0,\theta_0) &= -\left(\frac{\delta_2^{0,2}}{\Delta_2} b_{0,1}^c + \frac{\delta_2^{2,2}}{\Delta_2} b_{2,0}^c\right) + \frac{\delta_2^{0,2}}{\Delta_2}(f_0^c - b_{0,0}^c + b_{0,1}^c) - \frac{\delta_2^{2,2}}{\Delta_2}(f_2^c - b_{2,0}^c), \end{aligned} \tag{A.14}$$

$$
\begin{aligned}
\mathcal{Y}_1^2(r_0,\theta_0) &= \frac{-\boldsymbol{i}}{\alpha_{11}^s} f_1^s = \frac{3k^2}{4((1+k^2)E(k) - k'^2K(k)} \frac{2}{\sqrt{\pi}} \frac{r_0 \sin\theta_0}{\sqrt{1-r_0^2}}, \\
\mathcal{Y}_2^2(r_0,\theta_0) &= \frac{-1}{\alpha_{22}^s} f_2^s = \frac{16}{15} \frac{k^4}{E(k)(2k'^4 - 2k'^2 + 2) - K(k)(k'^4 + k'^2)} \frac{2}{\sqrt{\pi}} \frac{r_0^2 \sin 2\theta_0}{\sqrt{1-r_0^2}},
\end{aligned} \tag{A.15}
$$

$$
\Delta_2 = \alpha_{00}^c \alpha_{22}^c - \alpha_{02}^{c\,2}, \qquad \delta_2^{0,0} = \alpha_{22}^c, \qquad \delta_2^{0,2} = \delta_2^{2,0} = -\alpha_{20}^c, \qquad \delta_2^{2,2} = \alpha_{00}^c. \tag{A.16}
$$

The weight function $W^2(\varphi; r_0, \theta_0)$ is suitable for obtaining the stress intensity factor $K(\varphi)$ induced by a quadratic applied stress field, i.e. if the applied stress field is given by

$$
\begin{aligned}
p(x,y) &= \sigma_0 + \sigma_1 \frac{x}{a} + \sigma_2 \frac{y}{b} + \sigma_{20} \frac{x^2}{a^2} + \sigma_{02} \frac{y^2}{b^2} + \sigma_{11} \frac{xy}{ab}, \\
&\text{or} \\
p(r,\theta) &= \left(\sigma_0 + \frac{\sigma_{20} r^2}{2} + \frac{\sigma_{02} r^2}{2} \right) + \sigma_1 r \cos\theta + \sigma_2 r \sin\theta \\
&\quad + \left(\frac{\sigma_{20} r^2}{2} - \frac{\sigma_{02} r^2}{2} \right) \cos\, 2\theta + \frac{\sigma_{11} r^2}{2} \sin 2\theta,
\end{aligned} \tag{A.17}
$$

the stress intensity factor is obtained as

$$
\begin{aligned}
K(\varphi) &= \int_0^{2\pi} \int_0^1 W^2(\varphi; r_0, \theta_0) p(r_0, \theta_0) a\, b\, r_0 dr_0 d\theta_0 \\
&= \frac{\sqrt{\pi}\, b}{E(k)} \left(\frac{b^2}{a^2} \cos^2\varphi + \sin^2\varphi \right)^{1/4} \\
&\times \Bigg(\sigma_0 + \frac{8\, E(k)}{15} C_0 + \frac{-k^2 E(k) \sigma_1 \cos\varphi}{((1-2k^2)E(k) - k'^2 K(k))} + \frac{k^2 E(k) \sigma_2 \sin\varphi}{(1+k^2)E(k) - k'^2 K(k)} \\
&+ \frac{8\, E(k)}{15} C_2 \cos 2\varphi + \frac{k^4 E(k) \sigma_{11} \sin 2\varphi}{2(E(k)(2k'^{\,4} - 2k'^{\,2} + 2) - K(k)(k'^{\,4} + k'^{\,2}))} \Bigg),
\end{aligned} \tag{A.18}
$$

where

$$C_0 = \sigma_0^\star \left(\frac{3}{\alpha_{00}^c} + \frac{2\alpha_{22}^c}{\alpha_{00}^c \alpha_{22}^c - \alpha_{02}^{c\,2}} \right) - \sigma_2^\star \, \frac{2\,\alpha_{02}^c}{\alpha_{00}^c \alpha_{22}^c - \alpha_{02}^{c\,2}},$$
$$C_2 = 2\, \frac{\alpha_{00}^c \, \sigma_2^\star - \alpha_{02}^c \, \sigma_0^\star}{\alpha_{00}^c \alpha_{22}^c - \alpha_{02}^{c\,2}}, \tag{A.19}$$
$$\sigma_0^\star = \frac{\sigma_{20} + \sigma_{02}}{2}, \qquad \sigma_2^\star = \frac{\sigma_{20} - \sigma_{02}}{2}.$$

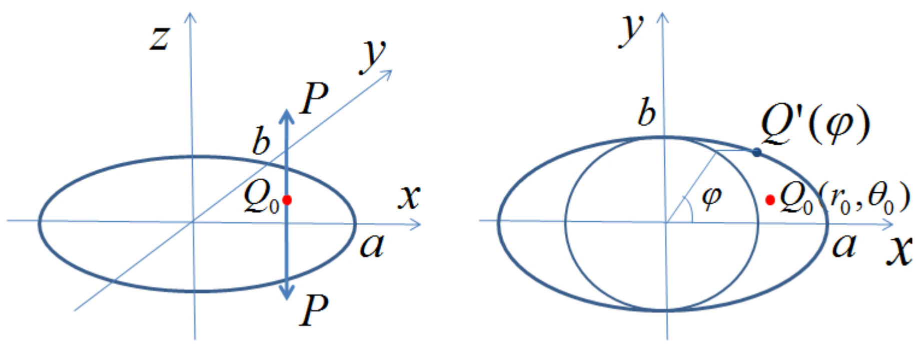

Figure A.1: An elliptical crack in an infinite solid subjected to a point load.

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