A Characterization of LYM and Rank Logarithmically Concave Partially Ordered Sets and Its Applications

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The LYM property of a finite standard graded poset is one of the central notions in Sperner theory. It is known that the product of two finite standard graded posets satisfying the LYM properties may not have the LYM property again. In 1974, Harper proved that if two finite standard graded posets satisfying the LYM properties also satisfy rank logarithmic concavities, then their product also satisfies these two properties. However, Harper's proof is rather non-intuitive. Giving a natural proof of Harper's theorem is one of the goals of this thesis.

The main new result of this thesis is a characterization of rank-finite standard graded LYM posets that satisfy rank logarithmic concavities. With this characterization theorem, we are able to give a new, natural proof of Harper's theorem. In fact, we prove a strengthened version of Harper's theorem by weakening the finiteness condition to the rank-finiteness condition. We present some interesting applications of the main characterization theorem. We also give a brief history of Sperner theory, and summarize all the ingredients we need for the main theorem and its applications, including a new equivalent condition for the LYM property that is a key for proving our main theorem.

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Chapter 1

Introduction

Sperner theory is a branch of extremal set theory that, roughly speaking, seeks for the extremal size of certain sets (usually involving antichains and chains) in a partially ordered set. The motivation of this theory is perhaps the famous theorem proven by Sperner [25] in 1928, which states that for any n-element set S, the maximum cardinality of a collection $\mathscr{A}$ of subsets of S, such that no member of $\mathscr{A}$ is contained in any other member of $\mathscr{A}$, is the binomial coefficient $\max\{\binom{n}{i} : 1 \leq i \leq n\}$. Since the first proof by Sperner, many different proofs of Sperner's theorem have been given by different authors, including perhaps the shortest known proof by Lubell [20], who used the Lubell-Yamamoto-Meshalkin inequality in his proof. The Lubell-Yamamoto-Meshalkin inequality (also commonly known as the LYM inequality) was, as its name suggests, discovered and proven independently by Yamamoto [32], Meshalkin [21], and Lubell [20] in the 1950s and 1960s. More details about the LYM inequality and its connection with Sperner's theorem can be found in Section 2.6. The LYM inequality and Sperner's theorem were later generalized to the LYM properties and the (strong) Sperner properties for rank-finite graded partially ordered sets (posets), which are two central concepts in Sperner theory. For more about the history of Sperner theory and a rather detailed list of reference, we refer readers to [6].

It is natural to ask whether the product of two standard graded posets satisfying the LYM properties again has the LYM property. The answer to this question is negative: it is false in general that the product of two LYM standard graded posets has the LYM property, as we will see from the simple counterexample given in Chapter 6. However, Harper [14] proved in a paper published in 1974 that if two (finite) standard graded posets having the LYM properties also satisfy rank logarithmic concavities, then their product also has these two properties. We shall refer to this theorem as Harper's product theorem (though in this thesis we will mainly focus on the stronger case with the finiteness condition in the hypothesis replaced by the weaker rank-finiteness condition). Harper used the notion of flow morphisms of graphs in his proof, which is not intuitive, and it is not clear from his proof why rank logarithmic concavity plays a special role in his product theorem. Hsieh and Kleitman [15] later gave a property such that if two LYM partially ordered sets satisfy this property then their product also has the LYM property, and used this fact to reprove Harper's product theorem. They used the notion of duality from linear programming in their proofs, which again did not tell much about the combinatorial insight of a LYM poset that satisfies rank logarithmic concavity.

To seek out what simple properties a standard graded poset satisfying the LYM property and rank logarithmic concavity (we will call such a poset a LYM and RLC poset for short) may have, and to provide a natural proof of Harper's product theorem, is the motivation of this thesis. As the main new result of this thesis, we prove that the LYM property and the rank logarithmic concavity of a standard graded poset $(P, \leq)$ are equivalent to the Peck properties of the strands of the conjunction of $(P, \leq)$ with its dual. The Peck property of a graded poset is a well-studied property that has been given different characterizations both combinatorially and algebraically by Griggs ([10], [11]) , Stanley [28] , and Proctor [23] in the 1980s. Since the Peck property is a well-studied property, our characterization allows us to derive more properties of LYM and RLC posets more easily, including giving a natural proof of Harper's product theorem. The structure of this thesis can be summarized with the following outline.

In Chapter 2, we first introduce some basic definitions, combinatorial notions and algebraic notions we need throughout the thesis. We then introduce the LYM inequality and Sperner's theorem (which are motivations of Sperner theory and some later chapters), and define the (strong) Sperner property.

In Chapter 3, we prove some basic properties of a LYM poset, and show that the strong Sperner property is a consequence of the LYM property. We then summarize and prove several equivalent conditions for the LYM property due to Kleitman [18], and give some of their applications. We also prove several new equivalent conditions for the LYM property that we use to prove our main theorem.

In Chapter 4, we introduce the Peck property and prove some basic properties of Peck posets. We then summarize and prove several equivalent conditions for the Peck property due to Griggs [11], Stanley [28], and Proctor [23], and give an application.

In Chapter 5, we define the notion of conjunction, which is a poset constructed from two posets. We then focus on the conjunctions of standard graded posets with their duals, and state and prove our main characterization of LYM and RLC posets using conjunctions of this form.

In Chapter 6, we present some applications and consequences of our main theorem. In particular, we first give a simple proof of Harper's product theorem using our main result. We then present a rank-preserving LYM and RLC extension to the partition lattice, which is known to be not LYM for sufficiently large order, using our main theorem. Finally, we derive some identities on the rank sizes of upsets and downsets in a standard graded LYM and RLC poset.

Chapter 2

Preliminaries

2.1 Basic Definitions

A *partially ordered set* (or *poset* for short) is a pair $(P, \leq)$ such that for any $a, b, c \in P$, the following conditions are satisfied:

(a) $a \leq a$ (*reflexivity*);

(b) if $a \leq b$ and $b \leq a$ then $a = b$ (*antisymmetry*);

(c) if $a \leq b$ and $b \leq c$ then $a \leq c$ (*transitivity*).

Whenever the *order relation* $\leq$ is clear from the context, we may simply refer to the ground set P as the poset. In the remainder of this paragraph, all elements are in P. We write $a < b$ if $a \leq b$ and $a \neq b$. We say that a is *covered* by b and denote it by $a \lessdot b$, if $a < b$ and no c satisfies $a < c < b$. While the *covering relation* $\lessdot$ is clearly determined by the order relation $\leq$, we remark that given a covering relation $\lessdot$ on P, one can always obtain a corresponding (partial) order relation $\leq$ on P by taking the *reflexive transitive closure* of $\lessdot$, that is, by letting $a \leq b$ if and only if $a = b$ or there is a sequence $c_1, \cdots, c_n$ such that $a = c_1 \lessdot \cdots \lessdot c_n = b$. We say that two elements a, b are *comparable* if $a \leq b$ or $b \leq a$, and say that two elements are *incomparable* if they are not comparable. A *chain* in P is a set $C \subseteq P$ in which any two elements are comparable. A chain is said to be *maximal* if it is not a proper subset of another chain. An *antichain* in P is a set $A \subseteq P$ in which any two distinct elements are incomparable. A set U is called an *upset* if $y \in U$ and $y < x$ implies that $x \in U$, for all elements x, y. Dually, a set D is called a *downset* if $y \in D$ and $x < y$ implies that $x \in D$, for all elements x, y.

For any poset $(P, \leq)$, the *dual* of $(P, \leq)$ is the poset $(P^*, \preceq)$ with $P^* = P$, such that $p \preceq q$ in P^* if and only if $q \leq p$ in P. An *extension* of a poset $(P, \leq)$ is a poset $(Q, \preceq)$ with $Q = P$, such that if $p \leq q$ in P then $p \preceq q$ in Q. The *Cartesian product* of two posets $X = (P, \leq_P)$ and $Y = (Q, \leq_Q)$, denoted by $X \times Y$, is the poset with underlying set $\{(p, q) : p \in P, q \in Q\}$, with order relation $\leq$ satisfying $(p_1, q_1) \leq (p_2, q_2)$ in $X \times Y$ if and only if $p_1 \leq_P p_2$ in P and $q_1 \leq_Q q_2$ in Q. From time to time, we may write the above product as $P \times Q$ instead of $X \times Y$ when there is no chance of confusion with the Cartesian product of sets. Given a collection $\{(P_i, \leq_i)\}_{i \in I}$ of pairwise disjoint posets, their *order union* is the poset $(\bigcup_{i \in I} P_i, \leq)$, with $p \leq q$ if and only if $p, q \in P_j$ for some j and $p \leq_j q$. Given two posets $(P, \leq)$ and $(Q, \preceq)$, a map $f : P \to Q$ is called an *(order) isomorphism* if the following two conditions are satisfied:

(a) f is bijective.

(b) $f(x) \preceq f(y)$ if and only if $x \leq y$ for all $x, y \in P$.

The posets $X = (P, \leq)$ and $Y = (Q, \preceq)$ are said to be *isomorphic*, denoted by $X \cong Y$, if there is an isomorphism from P to Q.

For any positive integer c, a *c-grading* of a poset $(P, \leq)$ is a function $r : P \to \mathbb{Z}$ such that for all $x, y \in P$, $r(x) \equiv r(y) \pmod{c}$, and $x \lessdot y$ implies that $r(y) = r(x) + c$. A poset is called a *c-graded poset* if it has a c-grading r, in which case $r(x)$ is called the *rank* of x. If the value of c is not important, we simplify the terms above to a *grading* and a *graded poset*, respectively. For any graded poset $(P, \leq)$ with grading r, define $r_{\min}(P) := \inf\{r(p) : p \in P\}$ and $r_{\max}(P) := \sup\{r(p) : p \in P\}$. Note that these can be infinite. In the future we may write $r_{\min}$ (resp. $r_{\max}$) instead of $r_{\min}(P)$ (resp. $r_{\max}(P)$) whenever the poset is clear from the context. A *standard grading* of $(P, \leq)$ is a 1-grading r of $(P, \leq)$ satisfying $r_{\min}(P) = 0$. A *central grading* of $(P, \leq)$ is a 2-grading r of P satisfying $r_{\min}(P) = -r_{\max}(P)$. A poset is called a *standard graded poset* (a *centrally graded poset*, respectively) if it has a standard grading (a central grading, respectively).

For any graded poset $(P, \leq)$ with grading r and any integer k, we call the set $P_k := \{p \in P : r(p) = k\}$ the k-th *rank* of $(P, \leq)$. In this case we call the number k the *rank number* of

P_k. $(P, \leq)$ is called *rank-finite* if all of its ranks are finite. If $(P, \leq)$ is rank-finite, we call the number $W_k(P) := |P_k|$ the k-th *Whitney number*. As usual, we write W_k instead of $W_k(P)$ whenever P is clear from the context. For any graded poset $(P, \leq)$ and integers a, b, denote by $P[a, b]$ the union of all ranks P_k with $a \leq k \leq b$. We can think of $P[a, b]$ as a graded poset with the inherited order relation and grading from $(P, \leq)$. Clearly, if $(P, \leq)$ is rank-finite then $P[a, b]$ is finite. In this thesis, we will focus on rank-finite posets and will make the rank-finite hypothesis tacitly. Given a subset S of the k-th rank P_k of a graded poset $(P, \leq)$, we define the *shade* of S to be the set

$$\nabla S := \{p \in P : s \lessdot p \text{ for some } s \in S\},$$

so the shade of S is the set of elements that cover some element of S. Note that in a c-graded poset, the shade of a subset of P_k is a subset of P_{k+c}.

Let k and c be integers with $c > 0$, and let $\{a_{k+ic}\}_{\alpha \leq i \leq \beta, i \in \mathbb{Z}}$ be a sequence of real numbers. We do not exclude the case in which $\alpha = -\infty$ or $\beta = \infty$, so that the sequence could be infinite. The sequence $\{a_{k+ic}\}_{\alpha \leq i \leq \beta}$ is said to be *unimodal* if there exists $n \in \mathbb{Z}$ with $\alpha \leq n \leq \beta$ such that if i and j are integers satisfying $i \leq j \leq n$ or $n \leq j \leq i$ then $a_{k+ic} \leq a_{k+jc}$ (so unimodal sequences are precisely those sequences that first weakly increase and then weakly decrease). The sequence $\{a_{k+ic}\}_{\alpha \leq i \leq \beta}$ is said to be *logarithmically concave* if $a_j^2 \geq a_{j-c}a_{j+c}$ for all $k + \alpha c < j < k + \beta c$. The *rank sequence* of a c-graded poset is the finite sequence $(W_{r_{\min}}, W_{r_{\min}+c}, W_{r_{\min}+2c}, \cdots, W_{r_{\max}})$ if both $r_{\min}$ and $r_{\max}$ are finite. This definition extends naturally to the case where at least one of $r_{\min}$ and $r_{\max}$ is infinite, in which case the rank sequence is an infinite sequence. A c-graded poset $(P, \leq)$ is said to be *rank-logarithmically concave* (or *RLC* for short) if its rank sequence is logarithmically concave, and is said to be *rank-unimodal* if its rank sequence is unimodal. A centrally graded poset is called *rank-symmetric* if $W_{-k} = W_k$ for all k.

For any graded poset $(P, \leq)$ with grading r, we associate to its dual $(P^*, \preceq)$ the *dual grading* $r^* = -r$. A graded extension $(Q, \preceq)$ of a graded poset $(P, \leq)$ is said to be *rank preserving* if $r = l$, where r and l are the gradings of $(P, \leq)$ and $(Q, \preceq)$, respectively. If $X = (P, \leq)$ and $Y = (Q, \preceq)$ are c-graded with gradings r_1 and r_2, respectively, then the function $r : P \times Q \to \mathbb{Z}$ defined by $r(p, q) = r_1(p) + r_2(q)$ is a c-grading on $X \times Y$ (it is well-defined indeed since $r(p_1, q_1) = r_1(p_1) + r_2(q_1) \equiv r_1(p_2) + r_2(q_2) = r(p_2, q_2) \pmod c$ for all $(p_1, q_1), (p_2, q_2) \in P \times Q$). If $\{(P_i, \leq_i)\}_{i \in I}$ is a collection of pairwise disjoint c-graded posets with grading r_i for $(P_i, \leq_i)$, such that $r_i(p) \equiv r_j(q) \pmod{c}$ for all $i, j \in I$, $p \in P_i$ and $q \in P_j$, then their order union is a c-graded poset with grading r defined by $r(p) = r_i(p)$ for $p \in P_i$, for all $i \in I$ (we call this grading the *union grading* for $\bigcup_{i \in I} P_i$ with respect to $\{r_i\}_{i \in I}$).

A *graph* is a pair (V, E) where V is a non-empty set and E is a set of 2-subsets of V (which could be empty). A graph (V, E) is called finite if V is finite. The elements of V are called the *vertices* of the graph, and the elements of E are called the *edges* of the graph. We refer to the elements of an edge as the *ends* of that edge. If $\{u, v\} \in E$ then we say that u and v are *adjacent*. Given a graph (V, E), the *degree* of a vertex v is the integer $|\{e \in E : v \in e\}|$. A *path* in (V, E) is a non-empty sequence $(v_1, v_2, \cdots, v_n)$ of distinct vertices in V such that v_i and v_{i+1} are adjacent for each $i \in \{1, 2, \cdots, n-1\}$; the vertices v_1 and v_n are called the *ends* of the path. An *induced subgraph* $G[V']$ (induced by $V' \subseteq V$) is the graph (V', E') such that for all $u, w \in V'$, $\{u, w\} \in E'$ if and only if $\{u, w\} \in E$. A graph (V, E) is called connected if, for any two distinct vertices v, w in V, there is a path with ends v and w. An induced subgraph $G[V']$ of a graph $G = (V, E)$ is called a *connected component* of G if $G[V']$ is connected, and $G[W]$ is not connected for any $V' \subsetneq W \subseteq V$. A *bipartite graph* is a graph $(V_1 \cup V_2, E)$ with $V_1 \cap V_2 = \emptyset$, such that any edge in E has one end in V_1 and the other in V_2; the pair (V_1, V_2) is called a *bipartition* of $(V_1 \cup V_2, E)$. A *directed graph* is a pair (N, A) where N is a set and A is subset of $N \times N$ such that no element of A has two coordinates being the same. The elements of N are called the *nodes* of the directed graph, and the elements of A are called the *arcs* of the directed graph. If $a = (u, v)$ is an arc then v is called the *head* of a and u is called the *tail* of a. The *underlying graph* of a directed graph (N, A) is the graph (N, E) such that $\{u, v\} \in E$ if and only if $(u, v) \in A$ or $(v, u) \in A$. Given a graph $G = (V, E)$, the *line graph* $L(G)$ of G is the graph having E as the vertex set, with two vertices of $L(G)$ being adjacent if and only if they have exactly one common end in V. A *matching* for a graph $G = (V, E)$ is a subset $M \subseteq E$ such that the intersection of any two distinct elements in M is empty. The set of all matchings for a graph G is denoted by $\mathcal{M}(G)$. A matching M for a graph is called a *perfect matching* if every vertex of the graph is contained in some element of M.

Given a poset $(P, \leq)$, the *Hasse diagram* $H(P)$ of P is the directed graph (P, A) with $(p, q) \in A$ if and only if $p \lessdot q$ in P. From time to time, we may consider $H(P)$ as its underlying graph instead if the direction of the covering relation is not important.

We end this section with two remarks. First, and once again, for brevity in the later sections we may frequently refer to the underlying set as the poset; for example, we will refer to P^* as the dual of the poset P (instead of saying that $(P^*, \preceq)$ is the dual of $(P, \leq)$). Secondly, when we discuss graded posets and related notions like Whitney numbers, we will usually fix a grading without explicitly mentioning it when the particular choice of the grading is not important or is obvious.

2.2 Hall's Theorem

Hall's theorem, also known Hall's marriage theorem, is a fundamental theorem in combinatorics proven by Philip Hall in 1935. The original proof by Hall can be found in [12]. There are different formulations of Hall's theorem. The main version of the theorem we state and prove in this section will be based on [2], which uses the notion of systems of distinct representatives. We then derive a graph-theoretic version of Hall's theorem as a corollary.

Given a ground set X and a nonempty collection $\mathcal{S} = \{S_1, S_2, ..., S_n\}$ of subsets of X, a *system of distinct representatives* (or *SDR* for short) for $\mathcal{S}$ is an injective function $f : \mathcal{S} \to X$ that satisfies $f(S_i) \in S_i$ for $i = 1, 2, ..., n$. For simplicity, if a SDR f for $\mathcal{S}$ exists, we will sometimes represent f as a finite sequence $\{x_i\}_{i=1}^n$, where $x_i = f(S_i)$.

Proposition 2.1. (Hall's Theorem) *Let $\mathcal{S} = \{S_1, S_2, ..., S_n\}$ be a nonempty collection of subsets of a set X. Then $\mathcal{S}$ possesses a system of distinct representatives if and only if, for any $\mathcal{T} \subseteq \mathcal{S}$, we have*

$$|\mathcal{T}| \leq |\bigcup \mathcal{T}|,$$

where $\bigcup \mathcal{T} := \bigcup_{T \in \mathcal{T}} T$.

Proof. Suppose that some $\mathcal{T} \subseteq \mathcal{S}$ is such that $|\mathcal{T}| > |\bigcup \mathcal{T}|$. If $f : \mathcal{S} \to X$ is a function satisfying $f(S_i) \in S_i$ for all i, then the restriction of f to $\mathcal{T}$ has its images lying in $\bigcup \mathcal{T}$. Such a restriction $f|_{\mathcal{T}} : \mathcal{T} \to \bigcup \mathcal{T}$ cannot be injective since $|\mathcal{T}| > |\bigcup \mathcal{T}|$. Consequently, f is not injective and thus $\mathcal{S}$ does not have a SDR.

To show the converse, we proceed by induction on $n := |\mathcal{S}|$. The base case (in which $n = 1$) is obvious. Suppose that our result is true for $n = 1, 2, ..., k$. Assume that $n = k + 1$ (so that $\mathcal{S} = \{S_1, S_2, ..., S_{k+1}\}$) and that $|\mathcal{T}| \leq |\bigcup \mathcal{T}|$ for every $\mathcal{T} \subseteq \mathcal{S}$ (so that each S_i is nonempty). There are two possibilities.

(i) Suppose that for all $r \leq k$, the union of any r sets of the collection S contains at least $r+1$ elements of X. Pick x_1 to be any element in S_1. Then by assumption the union of any r sets of $\{S_2, ..., S_{k+1}\}$ contains at least r elements of $X \setminus \{x_1\}$. By the induction hypothesis, $\{S_2, ..., S_{k+1}\}$ has a SDR (with ground set $X \setminus \{x_1\}$) , say $\{x_2, ..., x_{k+1}\}$. The sequence $\{x_1, x_2, ..., x_{k+1}\}$ is then a SDR for $\mathcal{S}$.

(ii) Suppose that for some $r \leq k$, the union of the sets in some r-subcollection of $\mathcal{S}$ contains at most r elements of X. This union then contains precisely r elements of X, by assumption. Without loss of generality, we can label these r sets as $S_1, S_2, ..., S_r$. By the induction hypothesis, $\{S_1, S_2, ..., S_r\}$ has a SDR, say $x_1, x_2, ..., x_r$. Now note that the union of any t sets in $\{S_{r+1}, ..., S_{k+1}\}$ contains at least t elements of $X \setminus \{x_1, ..., x_r\}$. Indeed, if $\{S_{i_1}, ..., S_{i_t}\}$ is a t-subcollection of $\{S_{r+1}, ..., S_{k+1}\}$ whose union contains at most $t-1$ elements of $X \setminus \{x_1, ..., x_r\}$, then the union of $S_1, ..., S_r, S_{i_1}, ..., S_{i_t}$ contains at most $r+t-1$ elements of X, a contradiction. Hence by the induction hypothesis, $S_{r+1}, ..., S_{k+1}$ has a SDR $\{x_{r+1}, ..., x_{k+1}\}$ with ground set $X \setminus \{x_1, ..., x_r\}$. We can then conclude that $\{x_1, x_2, ..., x_{k+1}\}$ is a SDR of $\mathcal{S}$. The proposition then follows by induction. ■

As mentioned at the beginning of the section, Hall's theorem can be translated using graph theoretic terms. It provides a necessary and sufficient condition for a bipartite graph with equal-sized bipartition to have a perfect matching:

Corollary 2.2. (Hall's Theorem for Graph Theory) *Let $G = (V, E)$ be a finite bipartite graph with bipartition (V_1, V_2) satisfying $|V_1| = |V_2|$. Then G has a perfect matching if and only if for every $S \subseteq V_1$, we have*

$$|S| \leq |N_G(S)|$$

where $N_G(S) = \{v \in V_2 : \{u, v\} \in E$ for some $u \in S\}$.

Proof. Write V_1 as $\{v_1, v_2, \cdots, v_n\}$, and let $S_i := N_G(\{v_i\})$. For $W \subseteq V$, we call a matching M of G a *W-saturating matching* if every $w \in W$ is the end of some edge in M. Since $|V_1| = |V_2|$, M is a perfect matching of G if and only if M is a V_1-saturating matching of G. The latter is just equivalent to saying that with ground set V_2, the collection $\{S_1, S_2, \cdots, S_n\}$ has a system of distinct representatives, which happens to be the case if and only if $|S| \leq |N_G(S)|$ for every $S \subseteq V_1$, by Proposition 2.1. ■

Other formulations and more applications of Hall's theorem can be found in [29].

2.3 Network Flows

A *network* is a pair $(\mathscr{N}, c)$, where $\mathscr{N} = (N, A)$ is a directed graph with two special nodes $s, t \in N$ called the *source* and *sink*, respectively, such that $\mathscr{N}$ contains no arc that has s as its head and no arc that has t as its tail, and $c : A \to \mathbb{R}_{\geq}$ is a nonnegative real-valued function that represents the *capacities* of the arcs of $\mathscr{N}$. A *flow* on a network $((N, A), c)$ is a function $f : A \to \mathbb{R}_{\geq}$ satisfying the following two conditions:

(a) $f(u, v) \leq c(u, v)$, for all $(u, v) \in A$.

(b) $\sum_{u \in N:(u,v) \in A} f(u, v) = \sum_{w \in N:(v,w) \in A} f(v, w)$, for all $v \in N \setminus \{s, t\}$.

The *value* of a flow f is defined to be $\sum_{v \in N:(s,v) \in A} f(s, v)$. A flow is called a *maximum flow* if its value is the largest possible. A *cut* is a set $S \subseteq N$ such that $s \in S$ and $t \notin S$. The *cut-set* of a cut $S \subseteq N$ is the set $\delta(S) := \{(u, v) \in A : u \in S, v \notin S\}$. The capacity of a cut-set $\delta(S)$ is defined to be $c(\delta(S)) := \sum_{a \in \delta(S)} c(a)$. The capacity of a cut S is defined to be $c(\delta(S))$. A cut is called a *minimum cut* if its capacity is the smallest possible. A powerful and fundamental theorem in the theory of network flow is the following max-flow min-cut theorem.

Theorem 2.3. (Max-Flow Min-Cut Theorem) *For any network, the value of a maximum flow is equal to the capacity of a minimum cut.*

Probably the most famous proof of the max-flow min-cut theorem uses the idea of augmenting paths; we will omit the details here. Readers that are interested in the theory of network flow may find [5] (Chapter 3) and [19] (Chapter 8) useful. The following theorems can be proven using essentially the same idea of augmenting paths.

Theorem 2.4. (Integrity Theorem) *If a network has integer-valued arc capacities, then it has an integer-valued maximum flow.*

Theorem 2.5. *If S is a cut of the network $((N, A), c)$ with source s and sink t, then every flow of the network $((N, A \setminus \delta(S)), c)$ with source s and sink t has value* 0.

2.4 Tensor Products

Let V, W be finite dimensional vector spaces over a field F. Choose a basis $B(V) := \{v_1, v_2, ..., v_m\}$ for V and a basis $B(W) := \{w_1, w_2, ..., w_n\}$ for W, so that we can uniquely write the elements of V and W as $\sum_{i=1}^{m} a_i v_i$ and $\sum_{j=1}^{n} b_j w_j$, respectively. Denote by $V \otimes W$ the F-vector space whose basis is $\{e_{v,w} : (v, w) \in B(V) \times B(W)\}$, which is called the *tensor product* of V and W. Define a mapping $\otimes : V \times W \to V \otimes W$ by

$$\otimes(\sum_{i=1}^{m} a_i v_i, \sum_{j=1}^{n} b_j w_j) = \sum_{i,j} a_i b_j e_{i,j},$$

where $e_{i,j}$ is the basis element of $V \otimes W$ indexed by (v_i, w_j). We will write $v \otimes w$ for $\otimes(v, w)$ and will refer to this as the *outer product* of v and w. It follows immediately from the definition that the outer product satisfies the following conditions:

(a) $(v + v') \otimes w = v \otimes w + v' \otimes w$ for all $v, v' \in V$ and $w \in W$.

(b) $v \otimes (w + w') = v \otimes w + v \otimes w'$ for all $v \in V$ and $w, w' \in W$.

(c) $(\alpha v) \otimes w = v \otimes (\alpha w) = \alpha(v \otimes w)$ for all $v \in V$, $w \in W$, and $\alpha \in F$.

That is, the outer product is bilinear.

Note that the basis of $V \otimes W$ can now be rewritten as $\{v \otimes w : (v, w) \in B(V) \times B(W)\}$. If $X : V \to V$ and $Y : W \to W$ are linear transformations, the *tensor product* of X and Y is the linear transformation $X \otimes Y : V \otimes W \to V \otimes W$ defined by

$$X \otimes Y(v \otimes w) = X(v) \otimes Y(w)$$

for every basis element $v \otimes w$. Note that once we define the function on the basis of $V \otimes W$, it follows from the linearity that $X \otimes Y(x \otimes y) = X(x) \otimes Y(y)$ still holds for arbitrary elements $x \in V$ and $y \in W$. With this in mind, the following proposition follows immediately.

Proposition 2.6. *If X_1, X_2 are linear transformations on V and Y_1, Y_2 are linear transformations on W, then*

$$(X_1 \otimes Y_1)(X_2 \otimes Y_2) = (X_1 X_2) \otimes (Y_1 Y_2).$$

Proof. For any basis element $v \otimes w$, we have

$$\begin{aligned}(X_1 \otimes Y_1)(X_2 \otimes Y_2)(v \otimes w) &= X_1 \otimes Y_1(X_2(v) \otimes Y_2(w)) \\ &= X_1X_2(v) \otimes Y_1Y_2(w) \\ &= X_1X_2 \otimes Y_1Y_2(v \otimes w).\end{aligned}$$

Therefore $(X_1 \otimes Y_1)(X_2 \otimes Y_2) = (X_1X_2) \otimes (Y_1Y_2)$ as they agree on the basis. ∎

2.5 Representations of $\mathfrak{sl}(2, \mathbb{C})$

In this section, we briefly introduce some notions of representations of Lie algebras we need for Chapter 4. For more detailed treatment on theory of Lie algebras and representations, readers can check out [7] and [16], for example.

A *Lie algebra* over a field F is a finite dimensional vector space L over F, together with a binary operation $[\cdot,\cdot]$ (usually called the *bracket operation*) on L that has the following properties:

(a) $[\cdot,\cdot]$ is bilinear.

(b) $[x, y] = -[y, x]$ for all $x, y \in L$. (*skew symmetry*)

(c) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$. (*Jacobi identity*)

Denote by $\mathfrak{sl}(n, \mathbb{C})$ the set of all $n \times n$ trace-zero complex matrices equipped with the bracket operation defined by $[x, y] = xy - yx$. Straightforward verification shows that $\mathfrak{sl}(n, \mathbb{C})$ is a Lie algebra over $\mathbb{C}$. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ has dimension 3 as a vector space, and has a basis $\{x, y, h\}$ where

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

From now on, we will take $\{x, y, h\}$ as the default basis for $\mathfrak{sl}(2, \mathbb{C})$. In fact, the relation

$$[x, y] = h, \quad [h, x] = 2x, \quad [h, y] = -2y$$

completely determines the bracket operation of $\mathfrak{sl}(2,\mathbb{C})$ by the bilinearity and skew symmetry.

A *representation* of a Lie algebra L (over F) is a pair (V, T), where V is a nonzero finite dimensional F-vector space and $T : L \to \mathrm{End}(V)$ is a linear transformation satisfying $T([x,y]) = T(x)T(y) - T(y)T(x)$ for all $x, y \in L$. Here $\mathrm{End}(V)$ is the set of endomorphisms of V (that is, the set of linear transformations from V to itself), which is also a F-vector space. For brevity, instead of the pair (V, T), sometimes we may refer to V as a representation of L when the linear transformation T is understood or is not important.

Given a c-graded poset P, we associate to it an abstract complex vector space whose basis is $\{[p] : p \in P\}$, and denote this vector space by $\mathbb{C}P$. Similarly, we denote by $\mathbb{C}P_k$ the subspace of $\mathbb{C}P$ that is spanned by $\{[p] : p \in P_k\}$. Note that $\mathbb{C}P \cong \bigoplus_{j\in\mathbb{Z}} \mathbb{C}P_j$ if we define $\mathbb{C}P_j$ to be the zero vector space whenever $P_j = \emptyset$. If P is 2-graded and X, Y, and $H := XY - YX$ are linear operators from $\mathbb{C}P$ to $\mathbb{C}P$ satisfying

(a) $X(\mathbb{C}P_j) \subseteq \mathbb{C}P_{j+2}$ for each $j \in \mathbb{Z}$;

(b) $Y(\mathbb{C}P_j) \subseteq \mathbb{C}P_{j-2}$ for each $j \in \mathbb{Z}$;

(c) $\mathbb{C}P_j$ is the eigenspace for H with eigenvalue j for each $j \in \mathbb{Z}$ with $P_j \neq \emptyset$,

then straightforward calculations show that

$$H = XY - YX, \quad 2X = HX - XH, \quad -2Y = HY - YH.$$

That is, $\mathbb{C}P$, together with the linear transformation $T : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(\mathbb{C}P)$ defined by

$$T(x) = X, \quad T(y) = Y, \quad T(h) = H,$$

form a representation of $\mathfrak{sl}(2,\mathbb{C})$. In this thesis, for any 2-graded poset P, we say that the operators X, Y, and H span a representation of $\mathfrak{sl}(2,\mathbb{C})$ acting on $\mathbb{C}P$ if X, Y and H satisfy the three conditions above.

2.6 LYM inequality and Sperner's Theorem

In this section, we introduce Sperner's theorem, the starting point of Sperner theory. We also introduce the Lubell-Yamamoto-Meshalkin inequality, which we will use to give a short proof of Sperner's theorem. A collection $\mathscr{A}$ of subsets of a set S such that no member of $\mathscr{A}$ is contained in any other member of $\mathscr{A}$ is called a *Sperner family* of S.

Theorem 2.7. *Let $\mathscr{A}$ be a Sperner family of an n-element set S, and let a_i be the number of i-element sets in $\mathscr{A}$. Then*

$$\sum_{i=0}^{n} \frac{a_i}{\binom{n}{i}} \leq 1. \tag{2.1}$$

Proof. For simplicity, we may assume that $S = \{1, 2, \cdots, n\}$. For $X \in \mathscr{A}$, let $\pi(X)$ be the set $\{\sigma \in S_n : \sigma(\{1, 2, \cdots, |X|\}) = X\}$, where S_n is the symmetric group of order n. Note that $|\pi(X)| = |X|!(n - |X|)!$ for any $X \in \mathscr{A}$. Clearly, $\bigcup_{X \in \mathscr{A}} \pi(X) \subseteq S_n$. On the other hand, the fact that $\mathscr{A}$ is a Sperner family implies that $\pi(X) \cap \pi(Y) = \emptyset$ for $X \neq Y$. Of course, this is because if $\pi(X) \cap \pi(Y) \neq \emptyset$ then either $X \subseteq Y$ or $Y \subseteq X$, which is impossible as $\mathscr{A}$ is a Sperner family. It follows that if A_i denotes the collection of all i-element sets in $\mathscr{A}$ then we have

$$\sum_{X \in \mathscr{A}} |\pi(X)| = |\bigcup_{X \in \mathscr{A}} \pi(X)| \leq |S_n| = n!,$$

so that

$$n! \geq \sum_{X \in \mathscr{A}} |\pi(X)| = \sum_{i=0}^{n} \sum_{X \in A_i} |X|!(n - |X|)! = \sum_{i=0}^{n} a_i i!(n - i)!.$$

Dividing both sides by $n!$ yields

$$1 \geq \sum_{i=0}^{n} a_i \frac{i!(n-i)!}{n!} = \sum_{i=0}^{n} \frac{a_i}{\binom{n}{i}},$$

as desired. ∎

Inequality (2.1) is called the LYM inequality. Three of the earliest proofs of the LYM inequality can be found in [20], [21] and [32]. The proof given above is due to Lubell [20] in 1966. The LYM inequality gives perhaps the shortest known proof of Sperner's theorem, which is now commonly stated as follows:

Theorem 2.8. (Sperner's theorem) *Let S be an n-element set. Then*

$$\max\{|\mathscr{A}| : \mathscr{A} \textit{ is a Sperner family of } S\} = \max\{\binom{n}{i} : 0 \leq i \leq n\}.$$

Proof. Clearly, $\max\{\binom{n}{i} : 0 \leq i \leq n\} = \binom{n}{m}$, where $m = \lfloor n/2 \rfloor$. Since the collection of m-element subsets of S is a Sperner family of S, we have

$$\max\{|\mathscr{A}| : \mathscr{A} \text{ is a Sperner family of } S\} \geq \binom{n}{m}.$$

On the other hand, given any Sperner family $\mathscr{A}$ of S, if a_i denotes the number of i-element sets in $\mathscr{A}$ then the LYM inequality implies that

$$1 \geq \sum_{i=0}^{n} \frac{a_i}{\binom{n}{i}} \geq \frac{1}{\binom{n}{m}} \sum_{i=0}^{n} a_i = \frac{|\mathscr{A}|}{\binom{n}{m}}$$

or equivalently, $|\mathscr{A}| \leq \binom{n}{m}$, proving the desired equality. ∎

Sperner's theorem was first proven by Sperner in [25]. In fact, he proved a stronger result in the same paper, namely that any Sperner family of an n-element set S having the maximum cardinality must be the collection of all m-element subsets of S, where $\binom{n}{m}$ is a largest binomial coefficient. Another proof using symmetric chain partitions due to Greene and Kleitman can be found in [9]. We next see that the notion of Sperner family and Sperner's theorem can be generalized using the language of partially ordered sets.

Let $(P, \leq)$ be a graded poset, and let k be a positive integer. A set $B \subseteq P$ is called a *k-family* of $(P, \leq)$ if B does not contain any $(k+1)$-element chain in $(P, \leq)$. Note that the 1-families of $(P, \leq)$ are precisely the antichains in $(P, \leq)$. If P is finite, we say that $(P, \leq)$ is *k-Sperner* if the cardinality of any maximum-sized k-family of $(P, \leq)$ equals the sum of any k largest Whitney numbers of $(P, \leq)$. Note that any k largest ranks of $(P, \leq)$ form a k-family. For this reason, to show that $(P, \leq)$ is k-Sperner, it is always sufficient to only show that any k-family of $(P, \leq)$ has cardinality no larger than the sum of any k largest Whitney numbers of $(P, \leq)$. If $(P, \leq)$ is 1-Sperner then we simply say that $(P, \leq)$ is *Sperner*. If $(P, \leq)$ is k-Sperner for every positive integer k then we say that $(P, \leq)$ is *strongly Sperner* (or that $(P, \leq)$ has the *strong Sperner property*). If P is infinite but rank-finite then we say that $(P, \leq)$ is *k-Sperner* (respectively, *strongly Sperner*) if $P[a, b]$ is k-Sperner (respectively, strongly Sperner) for all integers a and b with $r_{\min} \leq a \leq b \leq r_{\max}$. The *Boolean lattice* B_n of order n is the set of all subsets of the set $\{1, 2, \cdots, n\}$ partially ordered by set inclusion. By associating the standard grading $x \mapsto |x|$ to B_n, one can always think of Boolean lattices as being standard graded posets. With these definitions, Sperner's theorem is equivalent to saying that B_n is

Sperner. In Chapter 3 we will see that Boolean lattices are in fact strongly Sperner. Other posets that have the strong Sperner properties include subspace lattices (posets of subspaces of finite dimensional vector spaces over finite fields) and finite products of chains; we will see these in Chapter 3 and Chapter 6. A more detailed treatment on Sperner theory can be found in [6].

We end this chapter by making one useful remark. If a c-graded poset $(P, \leq)$ with grading r is Sperner then W_i is nonempty for each i with $r_{\min} \leq i \leq r_{\max}$ and $i \equiv r_{\max} \pmod{c}$. The idea of this is that if some intermediate rank is empty then it is easy to see (by the definition of a grading) that all elements of any maximum-sized rank are incomparable with some element outside this maximum-sized rank, creating an antichain that has size greater than the maximum rank size and thus contradicting the Sperner property of $(P, \leq)$. Since we will almost exclusively work on Sperner posets, it is useful to keep this remark in mind.

Chapter 3

LYM Posets

3.1 Basic Properties

A rank-finite graded poset P is said to be *LYM* (or to have the *LYM property*) if, for every antichain A in P, we have the inequality

$$\sum_{p \in A} \frac{1}{W_{r(p)}} \leq 1.$$

Note that if P is infinite (but rank-finite), then the above sum could have infinitely many terms. Also notice that the LYM inequality can be rewritten as

$$\sum_{k} \frac{|A \cap P_k|}{W_k} \leq 1,$$

where k runs over all the numbers for which W_k is nonzero. With this generalized LYM inequality, Theorem 2.7 is equivalent to saying that Boolean lattices are LYM. We see that the LYM property is preserved by rank-preserving extensions.

Proposition 3.1. *Let P be a graded poset, and let Q be a rank-preserving extension of P. If P is LYM then Q is LYM.*

Proof. Let r be the grading of P and Q. Since Q is an extension of P, any antichain in Q is also an antichain in P. Moreover, since P and Q have the same grading, we have

$W_k(P) = W_k(Q)$ for all k. Therefore, if A is an antichain in Q, we have

$$\sum_k \frac{|A \cap Q_k|}{W_k} = \sum_k \frac{|A \cap P_k|}{W_k} \leq 1.$$

as P is LYM. This implies that Q is LYM. ∎

Recall that for a graded poset $(P, \leq)$ and a positive integer k, a set $B \subseteq P$ is called a *k-family* of $(P, \leq)$ if B does not contain any $(k+1)$-element chain in $(P, \leq)$. In fact, we can visualize a k-family of $(P, \leq)$ as the union of antichains, as the following proposition states.

Proposition 3.2. *A set $B \subseteq P$ is a k-family of a graded poset $(P, \leq)$ if and only if B can be expressed as the union of at most k antichains in $(P, \leq)$.*

Proof. If B can be expressed as the union of at most k antichains in P, then clearly any chain contained in B has length at most k since the intersection of a chain with an antichain has at most one element, proving that B is a k-family.

Conversely, we show by induction on k that any k-family of P can be written as the union of at most k antichains in P. By definition, any 1-family of P is an antichain in P. Suppose that for some $n > 1$, any $(n-1)$-family of P can be written as the union of at most $n-1$ antichains in P. Let B be an n-family of P. Let A be the set

$$\{p \in B : \text{ there does not exist } q \in B \text{ such that } q < p\}.$$

Clearly A is an antichain in P. The set $B' := B \setminus A$ is then an $(n-1)$-family. Indeed, if B' contains an n-element chain $\{p_1, p_2, \cdots, p_n\}$ with $p_1 < p_2 < \cdots < p_n$ then since $p_1 \notin A$, there is some element $p_0 \in B$ such that $p_0 < p_1$, implying that $\{p_0, p_1, \cdots, p_n\}$ is an $(n+1)$-element chain in B, contradicting that B is an n-family of P. So B' is an $(n-1)$-family of P. By the induction hypothesis, B' can be written as the union of at most $n-1$ antichains in P, so $B = B' \cup A$ can be written as the union of at most n antichains in P, proving the desired statement by induction. ∎

Although a LYM poset may not be finite in general, it turns out that the LYM property of an infinite poset is equivalent to those of its finite restrictions. The following proposition makes this more precise.

Proposition 3.3. *A rank-finite graded poset P is LYM if and only if $P[a,b]$ is LYM for all integers a and b with $r_{\min} \le a \le b \le r_{\max}$.*

Proof. Suppose that P is LYM. Let a and b be integers satisfying $r_{\min} \le a \le b \le r_{\max}$, and let A be an antichain in $P[a,b]$. Clearly, A is also an antichain in P. By definition, $W'_k = W_k$ for every k with $a \le k \le b$, where W'_k represents the k-th Whitney number of $P[a,b]$. Hence we have

$$\sum_{p \in A} \frac{1}{W'_{r(p)}} = \sum_{p \in A} \frac{1}{W_{r(p)}} \le 1,$$

implying that $P[a,b]$ is LYM.

Conversely, suppose that $P[a,b]$ is LYM for all integers a and b with $r_{\min} \le a \le b \le r_{\max}$. Let A be an antichain in P. If A is finite, there exist integers a and b with $r_{\min} \le a \le b \le r_{\max}$ such that

$$\sum_{p \in A} \frac{1}{W_{r(p)}} = \sum_{p \in A \cap P[a,b]} \frac{1}{W_{r(p)}} \le 1,$$

where the last inequality follows from the fact that $A \cap P[a,b]$ is an antichain in $P[a,b]$. If A is infinite, let $A_i := A \cap P[k-ic, k+ic]$ for $i \in \mathbb{N}$, where $k \in \mathbb{Z}$ is such that $A \cap P_k \neq \emptyset$. Consider the sequence $\{a_i\}_{i \in \mathbb{N}}$, where

$$a_i = \sum_{p \in A_i} \frac{1}{W_{r(p)}}.$$

This sequence is clearly nondecreasing, with each a_i being at most 1 by assumption, so must be convergent as a bounded monotonic sequence of real numbers, say $\lim_{i \to \infty} a_i = L$. Clearly $L \le 1$. On the other hand, we have

$$\sum_{p \in A} \frac{1}{W_{r(p)}} = \lim_{i \to \infty} \sum_{p \in A_i} \frac{1}{W_r(p)} = \lim_{i \to \infty} a_i = L,$$

and so

$$\sum_{p \in A} \frac{1}{W_{r(p)}} \le 1,$$

implying that P is LYM. ■

Now we see that the strong Sperner property is just a consequence of the LYM property.

Proposition 3.4. *Let P be a rank-finite graded poset. If P is LYM then P is strongly Sperner.*

Proof. In view of Proposition 3.3 and the definition of the strong Sperner property, it suffices to show the desired statement for the the case in which P is finite. Suppose that P is a finite graded LYM poset with grading r, with W_{i_1},W_{i_2},...,W_{i_k} being the k largest Whitney numbers of P (the choice of the indices i_j may not be unique but is not important). Let

$$F := P_{i_1} \cup P_{i_2} \cup ... \cup P_{i_k}$$

and let

$$n := |F| = \sum_{j=1}^{k} W_{i_j}.$$

The proof proceeds by showing the following two claims:

(a) If B is a k-family then $\nu(B) \leq k$, where $\nu(S) := \sum_{p \in S} \frac{1}{W_{r(p)}}$;

(b) If $S \subseteq P$ and $|S| > n$ then $\nu(S) > k$.

These two facts together imply that any subset of P having more than n elements cannot be a k-family. In other words, any k-family has at most n elements, which implies that P is strongly Sperner, as desired. Now we show (a) and (b).

(a) Let B be a k-family. Then, by Proposition 3.2, there exist pairwise disjoint antichains $A_1, A_2, \cdots, A_k$ in P such that $B = A_1 \cup ... \cup A_k$, so that we have

$$\nu(B) = \nu(\bigcup_{j=1}^{k} A_j) \leq \sum_{j=1}^{k} \nu(A_j) \leq k,$$

where the last inequality follows from the fact P is LYM.

(b) Let $\mathcal{T} := \{T \subseteq P : |T| > n\}$. Then the collection

$$\mathcal{S} := \{S \in \mathcal{T} : \nu(S) \leq \nu(T) \text{ for all } T \in \mathcal{T}\}$$

is nonempty by the finiteness of $\mathcal{T}$. Let S' be an element of $\mathcal{S}$ having the largest intersection with F. I will show that S' is a superset of F. Suppose that S' is not a superset of F. Then there exists an element a in $F \setminus S'$. On the other hand, since $|S'| > n = |F|$, there exists an element b in $S' \setminus F$. Since $a \in F$ and $b \notin F$, we have $W_{r(a)} \geq W_{r(b)}$, so that

$$\frac{1}{W_{r(a)}} \leq \frac{1}{W_{r(b)}}.$$

Let $S^* := (S' \cup \{a\}) \setminus \{b\}$. Then

$$\nu(S^*) = \nu(S') + \frac{1}{W_{r(a)}} - \frac{1}{W_{r(b)}} \leq \nu(S'),$$

meaning that $S^* \in \mathcal{S}$. But by construction, the intersection of S^* with F has one more element than the intersection of S' with F, contradicting the choice of S'. It follows that S' is a superset of F. Moreover, S' contains F properly since $|S'| > |F|$. Therefore

$$\nu(S') > \nu(F) = \nu(\bigcup_{j=1}^{k} P_{i_j}) = \sum_{j=1}^{k} \nu(P_{i_j}) = k.$$

Thus, (a) and (b) are proven and the proof is completed. ∎

The proof of Proposition 3.4 above is due to Wagner, who learned the idea from Kleitman via personal communication. Since we have already seen in Section 2.6 that Boolean lattices are LYM, Proposition 3.4 immediately implies that Boolean lattices are strongly Sperner.

We have just seen that the LYM property implies the strong Sperner property. In fact, the LYM property is a strictly stronger property. Indeed, consider the poset $(P = \{1, 2, 3\}, \leq)$ having $1 < 3$ and the other pairs of distinct elements being incomparable. The Hasse diagram of this poset is shown in Figure 3.1. If we consider the standard grading r on P with $r(1) = r(2) = 0$ and $r(3) = 1$ then it is easy to check that $(P, \leq)$ is strongly Sperner (all antichains have size at most 2 and all k-families have size at most 3 for $k \geq 2$). $(P, \leq)$ is not LYM, however, since $\{2, 3\}$ is an antichain with

$$\sum_{p \in \{2,3\}} \frac{1}{W_{r(p)}} = 1 + \frac{1}{2} > 1.$$

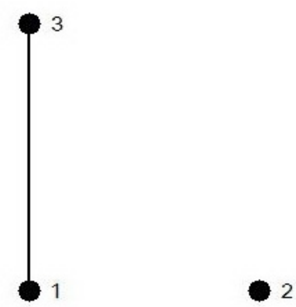

Figure 3.1: The Hasse diagram of a poset that is strongly Sperner but not LYM.

3.2 Characterization of LYM Posets

A rank-finite c-graded poset P is said to have the *normalized matching property* if, for any $S \subseteq P_k$ with $r_{\min} \leq k < r_{\max}$ and $k \equiv r_{\max} \pmod{c}$, the following inequality holds:

$$\frac{|S|}{W_k} \leq \frac{|\nabla S|}{W_{k+c}}$$

For a c-graded poset P and $r_{\min} < k \leq r_{\max}$ with $k \equiv r_{\max} \pmod{c}$, let $G_k(P)$ be the bipartite graph with bipartition $(P_k \times P_{k-c}, P_{k-c} \times P_k)$, with $\{(p, p'), (q, q')\}$ an edge in $G_k(P)$ if and only if $q \lessdot p$ in P or $p \lessdot q$ in P. The following proposition gives a characterization of LYM posets.

Proposition 3.5. *Let P be a c-graded poset. The following are equivalent:*

(a) P is LYM.

(b) P has the normalized matching property.

(c) For every $r_{\min} < k \leq r_{\max}$, the graph $G_k(P)$ has a perfect matching.

(d) P^* has the normalized matching property.

(e) P^* is LYM.

Proof. We first show that (a) and (e) are equivalent. Suppose that P is LYM, and let A be an antichain in P^*. Since two elements are comparable in P if and only if they are comparable in P^*, it follows that A is also an antichain in P. Let P_k^* and W_k^* denote the k-th rank and the k-th Whitney number of P^*, respectively. Then

$$\sum_j \frac{|A \cap P_j^*|}{W_j^*} = \sum_i \frac{|A \cap P_{-i}^*|}{W_{-i}^*} = \sum_i \frac{|A \cap P_i|}{W_i} \leq 1,$$

where the last inequality follows from the fact that P is LYM (here the index j in the first sum runs over all the integers from $-r_{\max}$ to $-r_{\min}$ for which W_j^* is nonzero, and the index i in the second sum runs over all the numbers from $r_{\min}$ to $r_{\max}$ for which W_i is nonzero). So P^* is LYM. Dually, if P^* is LYM then P is LYM. So (a) and (e) are equivalent.

To complete the proof, it suffices to show that (a), (b), and (c) are equivalent, in which case the equivalence between (d) and (e) would follow from the equivalence between (a) and (b).

(a) ⇒ (b): Let $r_{\min} \leq k < r_{\max}$ with P_k nonempty, and let S be a subset of P_k. Since each rank of P forms an antichain in P, it is easy to see that the set $S \cup (P_{k+c} \setminus \nabla S)$ is also an antichain in P; let us denote this antichain by A. Now we have

$$1 \geq \sum_i \frac{|A \cap P_i|}{W_i} = \frac{|S|}{W_k} + \frac{W_{k+c} - |\nabla S|}{W_{k+c}} = \frac{|S|}{W_k} - \frac{|\nabla S|}{W_{k+c}} + 1,$$

where the inequality holds since P is LYM. Rearranging the inequality above yields

$$\frac{|S|}{W_k} \leq \frac{|\nabla S|}{W_{k+c}}.$$

That is, P has the normalized matching property.

(b) ⇒ (c): Assume that P has the normalized matching property. In view of Hall's theorem (Corollary 2.2), to show that $G_k(P)$ has a perfect matching, it suffices to show that the neighbourhood of any set of m vertices in $P_{k-c} \times P_k$ has size at least m. Let S be an m-subset of $P_{k-c} \times P_k$. Then the projection of S onto P_{k-c} has at least m/W_k elements, that is,

$$|S'| := |\{x \in P_{k-c} : (x, y) \in S \text{ for some } y \in P_k\}| \geq \frac{m}{W_k}.$$

Since P has the normalized matching property, we have

$$\frac{|\nabla(S')|}{W_k} \geq \frac{|S'|}{W_{k-c}}.$$

Combining the two inequalities above yields

$$|\nabla(S')| \geq \frac{|S'|}{W_{k-c}} W_k \geq \frac{m}{W_k} \frac{W_k}{W_{k-c}} = \frac{m}{W_{k-c}}.$$

Since the adjacency relation in $G_k(P)$ depends only on the covering relation of the first coordinates of the vertices (in P), the number of vertices in $P_k \times P_{k-c}$ adjacent to at least one vertex in S is

$$|\nabla(S')||P_{k-c}| \geq \frac{m}{W_{k-c}} W_{k-c} = m.$$

Therefore $G_k(P)$ has a perfect matching, by Hall's theorem (Corollary 2.2).

(c) $\Rightarrow$ (b): Suppose that $G_k(P)$ has a perfect matching for all $r_{\min} < k \leq r_{\max}$. Let S be a subset of P_{k-c}. Since $G_k(P)$ has a perfect matching, $S \times P_k$ can be matched into $\nabla S \times P_{k-c}$. More precisely, there is an injective function $M : S \times P_k \to \nabla S \times P_{k-c}$ such $\{M(x,y),(x,y)\}$ is an edge in $G_k(P)$. Therefore

$$|S| \cdot W_k \leq |\nabla S| \cdot W_{k-c}$$

or equivalently,

$$\frac{|S|}{W_{k-c}} \leq \frac{|\nabla S|}{W_k}.$$

That is, P has the normalized matching property.

(b) $\Rightarrow$ (a): Suppose that P has the normalized matching property. In view of Proposition 3.3, we may assume without loss of generality that P is finite. For any antichain A in P, define $n(A) := \min_{p \in A} r(p)$. We show that

$$\sum_{p \in A} \frac{1}{W_{r(p)}} \leq 1 \tag{3.1}$$

for every antichain A in P, by induction on $n(A)$ (from $r_{\max}$ down to $r_{\min}$). If $n(A) = r_{\max}$, then $A \subseteq P_{r_{\max}}$ and the result is obvious. Suppose that for some $k \leq r_{\max}$, Inequality (3.1) holds for any antichain A in P with $n(A) = k$. We want to show that Inequality (3.1) holds for any antichain A in P with $n(A) = k-c$; let A' be such an antichain, and let $A_{k-c} := A' \cap P_{k-c}$. Clearly $A^* := (A' \cup \nabla A_{k-c}) \setminus A_{k-c}$ is also an antichain in P, with $n(A^*) = \min_{p \in A^*} r(P) = k$. Now we have

$$\begin{aligned}\sum_{p\in A'} \frac{1}{W_{r(p)}} &= \frac{|A_{k-c}|}{W_{k-c}} + \sum_{p\in A'\setminus A_{k-c}} \frac{1}{W_{r(p)}} \\ &\leq \frac{|\nabla A_{k-c}|}{W_k} + \sum_{p\in A'\setminus A_{k-c}} \frac{1}{W_{r(p)}} = \sum_{p\in A^*} \frac{1}{W_{r(p)}} \leq 1,\end{aligned}$$

where the first inequality follows from the normalized matching property of P, and the second inequality follows from the induction hypothesis. Inequality (3.1) thus holds for any antichain A in P, implying that P is LYM. This completes the proof of the equivalence of (a), (b), and (c). ■

The notion of normalized matching property was first introduced by Graham and Harper in [8]. The equivalence between condition (a) and condition (b) is due to Kleitman [18]. The equivalence between condition (b) and condition (c) is observed and kindly pointed out to me by Wagner, with proofs independently done by me. This latter equivalence is an essential tool we use to prove our main theorem in Chapter 5. The equivalence between condition (a) and condition (e) is proved mainly as a lemma for our main theorem.

We now show one application of Kleitman's characterization in Proposition 3.5. Let $T(b)$ be the set of all finite strings from the set $\{1, 2, \cdots, b\}$. Consider the poset $(T(b), \leq)$ such that $\alpha \leq \beta$ if and only if there exists $\gamma \in T(b)$ such that $\beta = \alpha\gamma$ (that is, β can be obtained by concatenating γ to the right of α). It is straightforward to check that the function that maps each string to its string length is a standard grading on $(T(b), \leq)$, in which case the i-th rank of $(T(b), \leq)$ consists of all the strings of length i. Clearly, $W_i = b^i$ for any nonnegative integer i. In fact, it is very easy to show that the poset $(T(b), \leq)$ is LYM since verifying the normalized matching property in this case is almost immediate.

Proposition 3.6. *The poset $(T(b), \leq)$ is LYM.*

Proof. Let i be a non-negative integer, and let S be a subset of W_i. Since $|\nabla(\{\alpha\})| = b$ for each $\alpha \in S$ and $\nabla(\{\alpha\}) \cap \nabla(\{\beta\}) = \emptyset$ for $\alpha \neq \beta$ in S, we have $|\nabla(S)| = b \cdot |S|$, so that

$$\frac{|\nabla(S)|}{W_{i+1}} = \frac{b \cdot |S|}{b^{i+1}} = \frac{|S|}{b^i} = \frac{|S|}{W_i},$$

verifying the normalized matching property of $(T(b), \leq)$. By Proposition 3.5, $(T(b), \leq)$ is LYM. ■

For finite posets, Kleitman [18] proved another equivalent condition for the LYM properties using regular chain covers. For clarity, we only consider standard graded posets here; the proof is essentially the same for any finite c-graded poset. A *maximal chain* in a graded poset P is a chain that intersects all the non-empty ranks of P. A *regular chain cover* of a finite standard graded poset P with grading r is a non-empty collection $\mathscr{C}$ of (not necessarily distinct) maximal chains in P such that, for each k with $0 \leq k \leq r_{\max}$, every element of P_k is in the same number of chains in $\mathscr{C}$.

Proposition 3.7. *For a finite standard graded poset P with grading r, the following are equivalent:*

(a) P is LYM.

(b) There is a regular chain cover of P.

Proof. Assume that P is LYM. Then P has the normalized matching property by Proposition 3.5. For each $p \in P$, we define Q_p to be

$$P_0 \times P_1 \times \cdots \times P_{k-1} \times P_{k+1} \times \cdots \times P_{r_{\max}-1} \times P_{r_{\max}}$$

where $k = r(p)$. Let P' be the set $\bigcup_{p \in P} \{p\} \times Q_p$ with covering relation $\lessdot$ such that $(p, q) \lessdot (p', q')$ in P' if and only if p is covered by p' in P. Informally speaking, for each k with $P_k \neq \emptyset$, we make $\prod_{i \neq k, W_i \neq 0} W_i$ copies of each element of P_k in P', and say that a copy of p is covered by a copy of p' in P' if and only if p is covered by p' in P. We will show that for each k with $0 \leq k < r_{\max}$, there is a perfect matching between $P_k \times Q_k$ and $P_{k+1} \times Q_{k+1}$ (both of which have size $\prod_{0 \leq i \leq r_{\max}} W_i$) in the Hasse diagram of P', where $Q_k = Q_x$ for some $x \in P$ with $r(x) = k$. Let us fix k and let S be an m-subset of $P_k \times Q_k$. Then

$$|S'| := |\{x \in P_k : (x, y) \in S \text{ for some } y \in Q_k\}| \geq \frac{m}{|Q_k|} \geq \frac{m}{\prod_{i \neq k} W_i}.$$

By the normalized matching property, we have

$$|\nabla(S')| \geq \frac{W_{k+1}}{W_k}|S'| \geq \frac{W_{k+1}}{W_k}\frac{m}{\prod_{i\neq k} W_i} = \frac{m}{\prod_{i\neq k+1} W_i},$$

so

$$|\nabla(S') \times Q_{k+1}| = |\nabla(S')| \cdot |Q_{k+1}| \geq \frac{m}{\prod_{i\neq k+1} W_i} \prod_{i\neq k+1} W_i = m.$$

Since every element of $\nabla(S') \times Q_{k+1}$ is comparable with every element of S in P (because the order relation of P' depends only on the first coordinates of elements), by Hall's theorem (Corollary 2.2) there is a perfect matching between $P_k \times Q_k$ and $P_{k+1} \times Q_{k+1}$ in the Hasse diagram of P'. Putting all the matchings betweens ranks together yields a collection of chains from $P_0 \times Q_0$ to $P_{r_{\max}} \times Q_{r_{\max}}$ in P'. By restricting to the first coordinate, we have a collection of maximal chains in P such that each element of P_k is contained in $\prod_{i\neq k} W_i$ chains, which is a regular chain cover of P.

Conversely, let $\mathscr{C}$ be a regular chain cover of P. Since every maximal chain must contain exactly one element from each non-empty rank of P, if $p \in P$ is such that $r(p) = k$ then $|\mathscr{C}_p| = |\mathscr{C}|/W_k$, where $\mathscr{C}_p$ is the subcollection $\{C \in \mathscr{C} : p \in C\}$ of $\mathscr{C}$. For $S \subseteq P$, let $\mathscr{C}_S$ be the collection of chains in $\mathscr{C}$ that contain at least one element of S. Let A be an antichain in P. Then

$$\mathscr{C}_A = \bigcup_{a\in A} \mathscr{C}_a = \bigcup_{k:0\leq k\leq r_{\max}} \bigcup_{a\in A\cap P_k} \mathscr{C}_a.$$

Since A is an antichain, $\mathscr{C}_a \cap \mathscr{C}_b = \emptyset$ if a and b are distinct elements of A. Hence

$$\begin{aligned} |\mathscr{C}_A| &= \sum_k \sum_{a\in A\cap P_k} |\mathscr{C}_a| \\ &= \sum_k \sum_{a\in A\cap P_k} \frac{|\mathscr{C}|}{W_k} = \sum_k |A\cap P_k|\frac{|\mathscr{C}|}{W_k} = |\mathscr{C}| \sum_k \frac{|A\cap P_k|}{W_k}. \end{aligned}$$

Clearly $|\mathscr{C}_A| \leq |\mathscr{C}|$, so

$$|\mathscr{C}| \sum_k \frac{|A\cap P_k|}{W_k} \leq |\mathscr{C}|,$$

implying that

$$\sum_k \frac{|A\cap P_k|}{W_k} \leq 1.$$

That is, P has the LYM property. ■

As an application of Proposition 3.7, we show that subspace lattices are LYM, hence strongly Sperner. Given a non-negative integer n and a finite field $GF(q)$ of q elements, the subspace lattice $V_n(q)$ is the set of all subspaces of the n-dimensional vector space over $GF(q)$ partially ordered by inclusion. Clearly, the function that sends each element of $V_n(q)$ to its dimension is a standard grading on $V_n(q)$, so we can think of $V_n(q)$ as a standard graded poset whose i-th rank consists of all i-dimensional subspaces. It is well-known that the number of i-dimensional subspaces of the m-dimensional vector space over $GF(q)$ is given by the Gaussian binomial coefficient $\begin{bmatrix} m \\ i \end{bmatrix}_q$ (for example, see [26, §1.3]). This is a key fact we need to derive the LYM property of $V_n(q)$.

Proposition 3.8. *The subspace lattice $V_n(q)$ is LYM.*

Proof. Let $\mathscr{C}$ be the collection of all distinct maximal chains in $V_n(q)$. We show that $\mathscr{C}$ is a regular chain cover of $V_n(q)$. From linear algebra, we have for any i-dimensional subspace W of the maximal vector space V with $0 \leq i \leq n$, the number of $(i+1)$-dimensional subspaces of V containing W is $\begin{bmatrix} n-i \\ 1 \end{bmatrix}_q$, and the number of $(i-1)$-dimensional subspaces of V containing W is $\begin{bmatrix} i \\ i-1 \end{bmatrix}_q$. In other words, for any i-th rank element W of $V_n(q)$, the number of elements covering W and the number of elements W covers depend only on i. Note that this implies that the number of distinct maximal chains in $V_n(q)$ containing an i-th rank element depends only on i (if u_i and d_i denote the number of elements covering an i-th rank element and the number of elements an i-th rank element covers, respectively, then the number of distinct maximal chains in $V_n(q)$ containing an i-th rank element is given by the number $\prod_{j=i}^{n-1} u_j \cdot \prod_{j=1}^{i} d_j$), so $\mathscr{C}$ is indeed a regular chain cover of $V_n(q)$. By Proposition 3.7, $V_n(q)$ is LYM. ■

Immediately, we can conclude that subspace lattices are strongly Sperner, as we mentioned in Section 2.6.

Corollary 3.9. *The subspace lattice $V_n(q)$ is strongly Sperner.*

Proof. The statement follows immediately from Proposition 3.4 and Proposition 3.8. ■

Chapter 4

Peck Posets

4.1 Basic Properties

A finite centrally graded poset is *Peck* if it is rank-symmetric, rank-unimodal, and strongly Sperner. The Peck property is a natural property in the sense that it is preserved under many common poset operations. Some of the special cases are formalized below.

Proposition 4.1. *Let P be a centrally graded poset, and let Q be a rank-preserving extension of P. If P is Peck then Q is Peck.*

Proof. Suppose that P is a Peck poset. Since any rank-preserving extension of a poset preserves ranks and thus Whitney numbers, Q is also rank-symmetric and rank-unimodal by the same properties of P. Let k be a positive integer. Since Q is an extension of P, any k-family of Q is also a k-family of P, hence has size no larger than the cardinality of a largest k-family of P, which is the sum of the k largest Whitney numbers as P is k-Sperner. Therefore Q is k-Sperner for any positive integer k, *i.e.*, Q is strongly Sperner and hence is Peck. ■

A Peck poset P is called *even* if the nonempty ranks of P all have even rank numbers, and is called *odd* if the nonempty ranks of P all have odd rank numbers. This property of being even or odd for a Peck poset P is called the *parity* of P.

Proposition 4.2. *If $\{P_i\}_{i\in I}$ is a finite collection of disjoint Peck posets having the same parity then their order union $\bigcup_{i\in I} P_i$ is Peck.*

Proof. Let r_i be the grading of P_i for each $i \in I$. Since all the P_i have the same parity, the union grading r for $\bigcup_{i \in I} P_i$ with respect to $\{r_i\}_{i \in I}$ is well-defined. Furthermore it is a central grading because each r_i is. With the gradings above, let W_j be the j-th Whitney number of $\bigcup_{i \in I} P_i$, and let $W_{i,j}$ be the j-th Whitney number of P_i. Then we have

$$W_j = \sum_{i \in I} W_{i,j} \tag{4.1}$$

for all $j \in \mathbb{Z}$. The rank symmetry and the rank unimodality of all the P_i then immediately implies that $\bigcup_{i \in I} P_i$ is rank-symmetric and rank-unimodal with respect to the union grading. If B is a k-family in $\bigcup_{i \in I} P_i$, then the restriction B_i of B to P_i is a k-family in P_i, for each $i \in I$. Each B_i has size no bigger than the sum of the k largest Whitney numbers for P_i by the strong Sperner property of P_i, so the sum of the size of all the B_i, being the size of B, is no larger than the the sum of the k largest Whitney numbers for $\bigcup_{i \in I} P_i$ by equation (4.1), proving that $\bigcup_{i \in I} P_i$ is strongly Sperner. Therefore $\bigcup_{i \in I} P_i$ is Peck. ∎

We end this section by remarking that the term Peck poset was first defined in reference to the author G.W. Peck, whose name was derived from the initials of six authors, Graham, West, Purdy, Erdös, Chung, and Kleitman.

4.2 Characterization of Finite Peck Posets

Before proceeding to the characterization, we first define some concepts that are important in this section. Let $(P, \leq)$ be a c-graded poset, and let m, n be integers such that $r_{\min} \leq m \leq n \leq r_{\max}$ and $r_{\min} \equiv m \equiv n \pmod{c}$. A *symmetric chain matching* from P_m to P_n is a collection $\mathcal{C}$ of pairwise disjoint chains in P that satisfies:

(a) every element in P_m is in exactly one chain in $\mathcal{C}$;

(b) every element in P_n is in exactly one chain in $\mathcal{C}$;

(c) every chain in $\mathcal{C}$ contains exactly one element of each P_k for $m \leq k \leq n$ with $k \equiv m \pmod{c}$, and does not contain any elements of P_k for $k > n$ or $k < m$.

It follows immediately from the definition that a necessary condition for the existence of such a matching is that $W_m = W_n$ and $P[m, n]$ is rank-unimodal.

Recall that we denote by $\mathbb{C}P$ the abstract complex vector space whose basis is $\{[p] : p \in P\}$, and we denote by $\mathbb{C}P_k$ the subspace of $\mathbb{C}P$ that is spanned by $\{[p] : p \in P_k\}$. If P is c-graded then a linear map $X : \mathbb{C}P \to \mathbb{C}P$ is called a *raising operator* on P if

$$X(\mathbb{C}P_k) \subseteq \mathbb{C}P_{k+c}, \text{ for all } k \in \mathbb{Z},$$

and is called a *lowering operator* on P if

$$X(\mathbb{C}P_k) \subseteq \mathbb{C}P_{k-c}, \text{ for all } k \in \mathbb{Z}.$$

A linear map $X : \mathbb{C}P \to \mathbb{C}P$ is called an *order raising operator* on P if for every $p \in P$, we have

$$X([p]) = \sum_{p \lessdot q} x(p,q)[q],$$

where $x(p,q)$ is a complex scalar depending on p and q. Here we define the empty sum to be the zero vector. Clearly, every order raising operator on P is a raising operator on P.

As we mentioned in the introduction chapter, several authors have given different equivalent conditions for the Peck property. They are unified in the following proposition.

Proposition 4.3. *For a centrally graded poset P with $r_{\max} = r$ (where $r \in \mathbb{Z}$), the following are equivalent:*

(a) P is Peck.

(b) For each $0 \leq i \leq r$ with $i \equiv r \pmod 2$, there is a symmetric chain matching from P_{-i} to P_i.

(c) There exists an order raising operator $R : \mathbb{C}P \to \mathbb{C}P$ on P such that for each $0 \leq i \leq r$ with $i \equiv r \pmod 2$, the map $R^i|_{\mathbb{C}P_{-i}} : \mathbb{C}P_{-i} \to \mathbb{C}P_i$ is invertible.

(d) There exists an order raising operator R and a lowering operator L on P such that R, L, and $H := RL - LR$ span a representation of $\mathfrak{sl}(2,\mathbb{C})$ acting on $\mathbb{C}P$.

Proof. **(a) $\Rightarrow$ (b):** Suppose that P is Peck. Fix an i such that $0 \leq i \leq r$ with $i \equiv r \pmod 2$. We want to show that there exist W_i disjoint chains such that each chain passes through every P_j with $-i \leq j \leq i$ and $j \equiv r \pmod 2$. Let

$$P' := P_{-i} \cup P_{-i+2} \cup ... \cup P_{i-2} \cup P_i,$$

and consider the network $((N, A), c)$ defined as follows:

	Nodes N
s	source
t	sink
u_x	(for all $x \in P'$)
v_x	(for all $x \in P'$)

	Arcs A	Capacities c
(s, u_x)	(for all $x \in P_{-i}$)	L
(v_x, t)	(for all $x \in P_i$)	L
(u_x, v_x)	(for all $x \in P'$)	1
(v_x, u_y)	(for all $x, y \in P'$ such that $x \lessdot y$)	L

where L is a large integer whose choice will be explained below.

By design, if we let $S := \{s\} \cup \{u_x : x \in P_{-i}\}$ then $\delta(S) = \{(u_x, v_x) : x \in P_{-i}\}$ is a cut-set with capacity W_i $(= W_{-i})$. By the max-flow min-cut theorem (Theorem 2.3), the maximum value of a flow is at most W_i. On the other hand, if we set the value of L in the definition above sufficiently large, say $L = W_i + 1$, then any cut-set with minimum capacity must be a subset of $\{(u_x, v_x) : x \in P'\}$ (otherwise the cut-set would have capacity at least $L = W_i + 1$ which is not minimum). Let C be a cut-set with minimum capacity, and let $C' := \{x \in P' : (u_x, v_x) \in C\}$. The fact that C is a cut-set implies that no chain in $P' \setminus C'$ intersects all ranks $P_{-i}, P_{-i+2}, ..., P_{i-2}, P_i$. Indeed, if this is not true, let $\{x_{-i}, x_{-i+2}, \cdots, x_i\}$ be such a chain with $x_j \in P_j$. This chain yields the set of arcs

$$\{(s, u_{x,-i}), (u_{x,-i}, v_{x,-i}), (v_{x,-i}, u_{x,-i+2}), \cdots, (v_{x,i-2}, u_{x,i}), (u_{x,i}, v_{x,i}), (v_{x,i}, t)\}$$

in the network. The function that has value 1 on all the arcs in the above set and has value 0 on every other arc in $A \setminus C$ is a flow of value 1 on the network $((N, A \setminus C), c)$. But this contradicts Theorem 2.5 since C is a cut-set. So no chain in $P' \setminus C'$ intersects all ranks $P_{-i}, P_{-i+2}, ..., P_{i-2}, P_i$. Hence $P' \setminus C'$ is an i-family of P. Since P is strongly Sperner, $|P' \setminus C'|$

is at most the sum of the i largest Whitney numbers, which equals $|P'| - W_i$ by the rank unimodality and rank symmetry of P. Consequently, $|C'| \geq W_i$. Since $|C| = |C'|$, by the max flow-min cut theorem again, the maximum value of a flow is at least W_i, and hence precisely W_i.

Since the arcs in A have integral capacities, by Theorem 2.4, there is an integer-valued maximum flow f on $((N, A), c)$. By the fact that $c(u_x, v_x) = 1$ for all $x \in P'$, there are W_i sets of arcs of the forms

$$\{(s, u_{x,-i}), (u_{x,-i}, v_{x,-i}), (v_{x,-i}, u_{x,-i+2}), \cdots, (v_{x,i-2}, u_{x,i}), (u_{x,i}, v_{x,i}), (v_{x,i}, t)\}$$

in which every arc has f-value 1 (here $u_{x,j}$ is some u_x with $x \in P_j$, and $v_{x,j}$ is some v_x with $x \in P_j$). Furthermore, any two of these sets of arcs are disjoint since each (u_x, v_x) has capacity only 1. Therefore, these sets of arcs correspond naturally to W_i disjoint chains from P_{-i} to P_i that satisfy the desired condition.

(b) $\Rightarrow$ (a): The rank-symmetry and rank-unimodality of P follow immediately from the existence of a symmetric chain matching from P_{-i} to P_i for each i by definition. It thus suffices to show that P is strongly Sperner to conclude what we want. If r is even, then by the rank-symmetry and rank-unimodality of P we have

$$W_0 \geq W_{-2} = W_2 \geq W_{-4} = W_4 \geq \ldots$$

Similarly, if r is odd, we have

$$W_{-1} = W_1 \geq W_{-3} = W_3 \geq W_{-5} = W_5 \geq \ldots$$

By the m largest ranks of P, we mean the ranks of P corresponding to the first m Whitney numbers in the orders given above . Intuitively, this means that whenever there is a tie, we always take the rank with negative index first. More precisely, we take the m largest ranks of P as $\{P_{a_n} : 1 \leq n \leq m\}$, where $a_n = (-1)^{n+1}2\lfloor(1/2)n\rfloor$ if r is even and $a_n = (-1)^n 2\lceil(1/2)n\rceil - 1$ if r is odd. Let A be any k-family of maximum size, and let $-l$ be the smallest index j such that P_j is one of the k largest ranks. If A is the union of the k largest ranks of P then we are done, so we may assume that some $a \in A$ is not in any of the k largest ranks. This means that $A \cap P_i$ is nonempty for some $i < -l$ or $i \geq -l + 2k$. We first assume that $A \cap P_i \neq \emptyset$ for some $i < -l$.

Let i' be $\min\{i : A \cap P_i \neq \emptyset\}$, and let $\mathcal{M}_{i'}$ be a symmetric chain matching from $P_{i'}$ to $P_{-i'}$. For each element $a \in A \cap P_{i'}$, there is a chain $C_a \in \mathcal{M}_{i'}$ that contains a. Let $U(a)$ be the element in $C_a \setminus A$ that has the smallest rank. Note that $U(a)$ exists for each $a \in A \cap P_{i'}$ since C_a has at least $k+2$ elements and A is a k-family. Consider the set A' obtained by replacing $A \cap P_{i'}$ with $\{U(a) : a \in A \cap P_{i'}\}$ in A. That is,

$$A' = (A \setminus P_{i'}) \cup \{U(a) : a \in A \cap P_{i'}\}.$$

Clearly $|A'| = |A|$. In fact, A' is also a k-family. To see this, we show that any subset of A' that is a chain in P has size at most k. Let $C \subseteq A'$ be a chain in P. If $C \subseteq A$ then what we want to show is obvious since A is a k-family. Assume that C is not a subset of A. Then $C \setminus A \neq \emptyset$. Let x be an element in $C \setminus A$ with maximum rank, say $r(x) = M$. Since $x \in A' \setminus A$, there exists $a \in P_{i'}$ such that $x = U(a)$. By the choice of $U(a)$, C_a (the chain in $\mathcal{M}_{i'}$ containing a) contains x and intersects A at each rank $i', i'+2, \cdots, M-2$, so that

$$|\{p \in C_a \cap A : p < x\}| = \frac{M - i'}{2},$$

from which we can conclude

$$|\{p \in C \cap A : p > x\}| \leq k - \frac{M - i'}{2},$$

since otherwise the union of the above two sets, as a subset of the k-family A, is a chain in P of size at least $k+1$, which is impossible. Since x was chosen in such a way that it has the maximum rank in $C \setminus A$, we have

$$|\{p \in C : p > x\}| = |\{p \in C \cap A : p > x\}| \leq k - \frac{M - i'}{2}.$$

Since any element of A' has rank strictly greater than i' by construction, and C is a subset of A', we can conclude that

$$|\{p \in C : p \leq x\}| \leq \frac{M - i'}{2}.$$

Adding up the above two inequalities yields $|C| \leq k$, which is what we want. Therefore A' is a k-family.

The above shows that whenever A has some element whose rank is less than $-l$, we can always construct a k-family of the same size whose minimum rank is 2 larger than the minimum rank of A. Applying this procedure finitely many times, one can eventually obtain a k-family that has the same size as A and has no element whose rank is less than $-l$.

Now suppose that $A \cap P_i \neq \emptyset$ for some $i \geq -l+2k$. Let i' be $\max\{i : A \cap P_i \neq \emptyset\}$ and consider a symmetric chain matching $\mathcal{M}_{i'}$ from $P_{-i'}$ to $P_{i'}$ again. For each element $a \in A \cap P_{i'}$, define C_a to be the chain in $\mathcal{M}_{i'}$ that contains a. Let $D(a)$ be the element in $C_a \setminus A$ that has the biggest rank. Now C_a may have fewer elements than it did in the previous case, but still has at least $k+1$ elements, so the existence of $D(a)$ is still guaranteed. Now applying the procedure we used in the first case in its dual form, one can once again obtain a k-family of the same size that has no element whose rank is at least $-l+2k$.

By applying both procedures above if necessary, any k-family of P that is not a subset of the union of the k largest ranks can always be transformed into a k-family of P of the same size that is a subset of the union of the k largest ranks. That is, P is strongly Sperner and hence is Peck.

(c) $\Rightarrow$ (b): For convenience, we can visualize the vector space $\mathbb{C}P$ as $\mathbb{C}^n$ where $n = |P|$. That is, we can view the elements of $\mathbb{C}P$ as n-dimensional vectors indexed by P, such that any basis vector $[p]$ is just the standard basis vector whose only nonzero entry is at the index corresponding to p. In this way, any linear transformation from $\mathbb{C}P_j$ to $\mathbb{C}P_k$ can be visualized as a linear transformation from $\mathbb{C}^L$ to $\mathbb{C}^N$ where $L = |P_j|$ and $N = |P_k|$, and thus can be represented by an $N \times L$ matrix (whose rows and columns are indexed by P_k and P_j, respectively) with complex entries.

Let $R : \mathbb{C}P \to \mathbb{C}P$ be an order raising operator satisfying the condition in (c). We want to construct for each i a symmetric chain matching from P_{-i} to P_i. Let i be such that $0 \leq i \leq r$ with $i \equiv r \pmod 2$. Note that

$$R^i|_{\mathbb{C}P_{-i}} = R|_{\mathbb{C}P_{i-2}} \cdot R|_{\mathbb{C}P_{i-4}} \cdot \ldots \cdot R|_{\mathbb{C}P_{-i+2}} \cdot R|_{\mathbb{C}P_{-i}},$$

where the multiplication here represents the function composition. Let M be the matrix representing the map $R^i|_{\mathbb{C}P_{-i}}$, and let M_j be the matrix representing $R|_{\mathbb{C}P_j}$. Then we have from the previous equation

$$M = M_{i-2}M_{i-4}\ldots M_{-i+2}M_{-i}.$$

Given any matrix A whose rows and columns are indexed by S and T, respectively, and for any $S' \subseteq S$ and $T' \subseteq T$, we define $A[S', T']$ to be the submatrix of A whose rows and columns are indexed by S' and T', respectively. Then by the Cauchy-Binet formula (see [1, §36]) we have

$$\det M = \sum \det(M_{i-2}[Q_i, Q_{i-2}]) \det(M_{i-4}[Q_{i-2}, Q_{i-4}]) ... \det(M_{-i}[Q_{-i+2}, Q_{-i}]),$$

where the sum is taken over all the Q_j satisfying $Q_j \subseteq P_j$ with $|Q_j| = |P_i|$ for all j. Note that this sum is always nonempty. Indeed, the fact that M is invertible implies that $|P_{-i}| = |P_i|$. Furthermore, since the dimension of the range can never be greater than the dimension of the domain under a linear map, we can further conclude that

$$|P_{-i}| = |P_i| \leq |P_{-i+2}| = |P_{i-2}| \leq |P_{-i+4}| = |P_{i-4}| \leq \cdots,$$

so that the sum always has at least one term. Since M is invertible, $\det M$ is nonzero, so that some term in the sum is nonzero. This term is the product of the determinants of some $|P_i| \times |P_i|$ matrices, all of which must be invertible as the term is nonzero.

Consider a nonzero term in the sum, and assume that it is defined by $Q'_{-i}, Q'_{-i+2}, ..., Q'_i$. For any Q'_j and Q'_{j+2}, notice that Hall's condition is satisfied, that is, for any $Q \subseteq Q'_j$, we have

$$|\{p \in Q'_{j+2} : q \lessdot p \text{ for some } q \in Q\}| \geq |Q|. \tag{4.2}$$

Indeed, if this were not the case then it would mean that the dimension of the subspace of P_j spanned by the vectors indexed by Q is greater than the dimension of its image under the map $R|_{\mathbb{C}Q}$ (since R is an order raising operator), which is impossible as $M_j[Q'_{j+2}, Q'_j]$ is invertible. Since Hall's condition (4.2) is satisfied, by Hall's theorem (Corollary 2.2) there is a perfect matching from Q'_j to Q'_{j+2} in the Hasse diagram $H(P)$. In other words, there is a one-to-one correspondence between Q'_j and Q'_{j+2} such that any pair of corresponding elements are comparable in P. Given an element x of $P_{-i} = Q'_{-i}$, using this correspondence we can construct a chain C_x containing x and elements from each rank $P_{-i+2}, P_{-i+4}, \cdots, P_i = Q'_i$. Putting together the chains C_x constructed in this way for all $x \in P_{-i}$ yields a symmetric chain matching from P_{-i} to P_i.

(b) $\Rightarrow$ (c): Suppose that there is a symmetric chain matching $\mathcal{M}_i$ from P_{-i} to P_i. Consider a linear map $S_i : \mathbb{C}P \to \mathbb{C}P$ satisfying the following: for any element $[p]$ in the basis of $\mathbb{C}P$, if there is some $q \in P$ such that $p \lessdot q$ and $\{p, q\}$ is a subset of a chain in $\mathcal{M}_i$, then $S_i([p]) = [q]$; otherwise, $S_i([p]) = 0$. Clearly, S_i is an order raising operator. Moreover, the existence of a symmetric chain matching from P_{-i} to P_i guarantees that $S_i^i|_{\mathbb{C}P_{-i}} : \mathbb{C}P_{-i} \to \mathbb{C}P_i$ is invertible by the construction of S_i.

Now we construct an order raising operator satisfying the desired requirement. If X is an order raising operator on $\mathbb{C}P$, what does the matrix $M(X)$ representing the map X look like? By definition $M(X)$ must be an $n \times n$ matrix (where $n = |P|$) whose (p, q)-entry is zero if q is not covered by p in P. Let us think of the (p, q)-entries of $M(X)$ for which $q \lessdot p$ in P as indeterminants $x_{p,q}$. Our claim is that if we obtain an order raising operator R by setting all these $x_{p,q}$ to be complex numbers that are algebraically independent over $\mathbb{Q}$ then R satisfies the desired requirement. First of all, it is possible to obtain R this way since the transcendence degree of $\mathbb{C}$ over $\mathbb{Q}$ is infinite (or more precisely, equals the cardinality of $\mathbb{R}$). In particular, there is always a set of algebraically independent complex numbers (over $\mathbb{Q}$) of size N for any positive integer N. Readers can find out more about algebraic independence and transcendence degree in Chapter 5 of [22] and Chapter 6 of [17]. Now to see the claim, we know from elementary linear algebra that the determinant of a matrix is a polynomial in its entries with integer coefficients. In particular, if $M_i(X)$ is the matrix representing the map $X^i|_{\mathbb{C}P_{-i}} : \mathbb{C}P_{-i} \to \mathbb{C}P_i$, then just as in the previous part we can write $\det(M_i(X))$ as the sum of determinants of some submatrices of $M(X)$ using the Cauchy-Binet formula, so that $\det(M_i(X))$ is a polynomial in $x_{p,q}$ with integer coefficients. By the first paragraph of this part, for an arbitrary fixed i, there exists an order raising operator S_i on $\mathbb{C}P$ such that $S_i^i|_{\mathbb{C}P_{-i}} : \mathbb{C}P_{-i} \to \mathbb{C}P_i$ is invertible, so $\det(M_i(S_i)) \neq 0$. This means that $\det(M_i(X))$ cannot be the zero polynomial. Consequently, by the way we set the values for $x_{p,q}$ in R, $\det(M_i(R)) \neq 0$, so that $R^i|_{\mathbb{C}P_{-i}} : \mathbb{C}P_{-i} \to \mathbb{C}P_i$ is invertible. This works for all $0 \leq i \leq r$ with $i \equiv r \pmod 2$, so the desired statement is proven.

(c) $\Rightarrow$ (d): Let $R : \mathbb{C}P \to \mathbb{C}P$ be an order raising operator satisfying the condition in (c). We first construct a new basis of $\mathbb{C}P$. Let v_1 be any nonzero vector in $\mathbb{C}P_{-r}$, and let $V_0 := \emptyset$. Then $v_1, Rv_1, \cdots, R^r v_1$ are nonzero and linearly independent (here we write Rv instead of $R(v)$). Let $V_1 := \{v_1, Rv_1, \cdots, R^r v_1\}$. Let $-s$ be the smallest integer such that $\mathbb{C}P_{-s}$ is not contained in $\mathrm{span}(V_1)$, and let v_2 be a vector in $\mathbb{C}P_{-s} \setminus \mathrm{span}(V_1)$. Note that $v_2, Rv_2, \cdots, R^s v_2$ are linearly independent and outside of $\mathrm{span}(V_1)$. To see this, we first simplify the notation by letting w_{-r+2m} be $R^m v_1$ for $m \in \{0, 1, \cdots, r\}$ (so that now V_1 can be rewritten as $\{w_{-r}, w_{-r+2}, \cdots, w_{r-2}, w_r\}$) and letting u_{-s+2n} be $R^n v_2$ for

$n \in \{0, 1, \cdots, s\}$. Then by design $w_{-r+2m} \in \mathbb{C}P_{-r+2m}$ and $u_{-s+2n} \in \mathbb{C}P_{-s+2n}$. Now suppose that $u_{-s+2a} := R^a v_2 \in \operatorname{span}(V_1)$ for some $a \in \{0, 1, \cdots, s\}$, and consider such an a with the smallest value possible. Then $a \geq 1$ since $v_2 \notin \operatorname{span}(V_1)$ by the choice of v_2. There is exactly one element in $\mathbb{C}P_{-s+2a} \cap V_1$, namely w_{-s+2a}, so $\operatorname{span}(\{u_{-s+2a}\}) = \operatorname{span}(\{w_{-s+2a}\})$. Evidently $\operatorname{span}(\{u_{-s+2n}\}) = \operatorname{span}(\{w_{-s+2n}\})$ for all $n \geq a$. If $-s + 2a \leq 0$ then $u_{-s+2(a-1)} \notin \operatorname{span}(\{w_{-s+2(a-1)}\})$ by design, so that $u_{-s+2(a-1)}$ and $w_{-s+2(a-1)}$ are linearly independent; but $\operatorname{span}(\{R^{2(a-1)-s} u_{-s+2(a-1)}\}) = \operatorname{span}(\{R^{2(a-1)-s} w_{-s+2(a-1)}\})$, contradicting the assumption that the map $R^{2(a-1)-s}$ is invertible. Similarly, if $-s + 2a > 0$ then u_{s-2a} and w_{s-2a} are linearly independent, with $\operatorname{span}(\{R^{-s+2a} u_{s-2a}\}) = \operatorname{span}(\{R^{-s+2a} w_{s-2a}\})$, which again contradicts the assumption that the map R^{-s+2a} is invertible. So $v_2, Rv_2, \cdots, R^s v_2$ are not in $\operatorname{span}(V_1)$. Let $V_2 := V_1 \cup \{v_2, Rv_2, ..., R^s v_2\}$. In general, if $\operatorname{span}(V_j) \neq \mathbb{C}P$ then we pick the smallest integer $-t$ such that $\mathbb{C}P_{-t}$ is not contained in $\operatorname{span}(V_j)$, and pick $v_{j+1} \in \mathbb{C}P_{-t} \setminus \operatorname{span}(V_j)$. Then $v_{j+1}, Rv_{j+1}, \cdots, R^t v_{j+1}$ are linearly independent and outside of $\operatorname{span}(V_j)$ by the invertibility condition in (c). Let $V_{j+1} := V_j \cup \{v_{j+1}, Rv_{j+1}, \cdots, R^t v_{j+1}\}$. Notice that the way we choose $-t$ guarantees that $-t$ is always non-positive due to the rank symmetry implied by (c). Repeat this procedure until $\operatorname{span}(V_{j'}) = \mathbb{C}P$. $V_{j'}$ is then a new basis for $\mathbb{C}P$.

Now consider the lowering operator $L : \mathbb{C}P \to \mathbb{C}P$ defined on the new basis by

$$L(R^j v_k) = \begin{cases} -j(r(v_k) + j - 1)R^{j-1} v_k, & \text{if } j \geq 1 \\ 0, & \text{if } j = 0, \end{cases}$$

where $r(x)$ denotes the rank of x. Now we show that R, L, and $H := RL - LR$ span a representation of $\mathfrak{sl}(2, \mathbb{C})$ acting on $\mathbb{C}P$. By the definition in Section 2.5, it suffices to show that $\mathbb{C}P_i$ is the eigenspace for H with eigenvalue i, for each $-r \leq i \leq r$ with $i \equiv r \pmod 2$. Assume that $R^j v_k$ is an element in the new basis we constructed above. If $j \neq 0$ and $R^{j+1} v_k$ is a new basis element then

$$\begin{aligned}
(RL - LR)(R^j v_k) &= RL(R^j v_k) - LR^{j+1} v_k \\
&= R(-j(r(v_k) + j - 1)R^{j-1} v_k) + (j + 1)(r(v_k) + j)R^j v_k \\
&= (-jr(v_k) - j^2 + j)R^j v_k + (jr(v_k) + j^2 + r(v_k) + j)R^j v_k \\
&= (r(v_k) + 2j)R^j v_k \\
&= r(R^j v_k)R^j v_k.
\end{aligned}$$

If $j = 0$ then the term $RL(R^j v_k)$ vanishes and we have

$$(RL - LR)(R^j v_k) = -L(Rv_k) = r(v_k)R^0 v_k = r(v_k)v_k.$$

If $R^{j+1}v_k$ is not a new basis element then $R^{j+1}v_k = 0$ and $j = r(R^j v_k) = -r(v_k)$. In this case, we have

$$(RL - LR)(R^j v_k) = RL(R^j v_k) = (-jr(v_k) - j^2 + j)R^j v_k = r(R^j v_k)R^j v_k.$$

This shows that $\mathbb{C}P_i$ is the eigenspace for H with eigenvalue i, for each $-r \leq i \leq r$ with $i \equiv r \pmod 2$, so that the operators R, L and H indeed span a representation of $\mathfrak{sl}(2, \mathbb{C})$ acting on $\mathbb{C}P$.

(d) $\Rightarrow$ (c): Suppose that R, L, and H are operators spanning a representation of $\mathfrak{sl}(2, \mathbb{C})$ acting on $\mathbb{C}P$ that also satisfy the assumption in (d). By Weyl's theorem ([16], p. 28), $\mathbb{C}P$ can be written as a direct sum of irreducible representations. Furthermore, since $\mathbb{C}P$ can be decomposed as a direct sum of eigenspaces of H, each of these irreducible representations has a basis consisting of eigenvectors of H. For any of these irreducible representations (of dimension $d + 1$), say W, if we pick $v_0 \in W$ to be one of the eigenvectors of H with the smallest eigenvalue (say $-\lambda$) and define $v_i := R^i v_0$, then we have

$$\begin{aligned}
Hv_i &= (-\lambda + 2i)v_i, && \text{for } i \geq 0; \\
Rv_i &= v_{i+1}, && \text{for } i \geq 0; \\
Lv_i &= i(\lambda - i + 1)v_{i-1}, && \text{for } i \geq 1.
\end{aligned}$$

Indeed, $Rv_i = v_{i+1}$ follows directly from the definition of v_i and $Hv_i = (-\lambda + 2i)v_i$ follows from the second equality and the fact $R(\mathbb{C}P_j) \subseteq \mathbb{C}P_{j+2}$. To show the third equality, we use induction on i. If $i = 1$ then since $Lv_0 = 0$,

$$Lv_1 = L(Rv_0) = RLv_0 - Hv_0 = 0 - (-\lambda v_0) = \lambda v_0,$$

so the base case holds. If $i > 1$ then by the induction hypothesis and the other two equalities, we have

$$\begin{aligned} Lv_i = L(Rv_{i-1}) &= RLv_{i-1} - Hv_{i-1} \\ &= R((i-1)(\lambda - i + 2)v_{i-2}) + (\lambda - 2(i-1))v_{i-1} \\ &= (i-1)(\lambda - i + 2)v_{i-1} + (\lambda - 2(i-1))v_{i-1} \\ &= i(\lambda - i + 1)v_{i-1}. \end{aligned}$$

Thus the third equality follows by induction.

A consequence of the irreducibility of W is that $v_i \neq 0$ for $0 \leq i \leq d$ and $v_i = 0$ for all $i > d$ (more details in [16], p. 32), so $v_0, v_1, \cdots, v_d$ form a basis for W because they are linearly independent. Moreover, since $v_{d+1} = 0$, Lv_{d+1} must be zero and by the third equality above we must have $0 = (d+1)(\lambda - d)v_d$, implying that $\lambda = d$ since $v_d \neq 0$. This means that W has a basis $\{v_0, v_1, \cdots, v_d\}$ satisfying

$$\begin{aligned} Hv_i &= (-d + 2i)v_i, && \text{for } 0 \leq i \leq d; \\ Rv_i &= v_{i+1}, && \text{for } 0 \leq i \leq d-1; \\ Lv_i &= i(d - i + 1)v_{i-1}, && \text{for } 1 \leq i \leq d. \end{aligned}$$

For each of the irreducible representations in the direct sum of $\mathbb{C}P$, we can obtain a basis satisfying the above conditions. These bases collectively form a new basis of $\mathbb{C}P$. Moreover, if an irreducible representation has dimension $d+1$ then the elements of the basis constructed in the above manner correspond naturally to the middle $d+1$ ranks of P (because $Hv_i = (-d+2i)v_i$), so that the set of all new basis vectors corresponding to a rank P_i form a new basis for $\mathbb{C}P_i$. Since any irreducible representation has a basis vector corresponding to P_i if and only if it has a basis vector corresponding to P_{-i}, and since $v_i := R^i v_0$, $R^i|_{\mathbb{C}P_{-i}} : \mathbb{C}P_{-i} \to \mathbb{C}P_i$ is invertible. ■

The equivalence between conditions (a) and (b) is due to Griggs; he gave two proofs in [10] and [11]. The equivalence between conditions (b) and (c) is due to Stanley [28]. The equivalence between conditions (c) and (d) is due to Proctor [23].

The Peck property is again preserved by taking products. Proposition 4.3 can be used to give a relatively straightforward proof of this statement.

Proposition 4.4. *If P and Q are Peck then $P \times Q$ is Peck.*

Proof. Let R', L' and $H' := R'L' - L'R'$ span a representation of $\mathfrak{sl}(2, \mathbb{C})$ acting on $\mathbb{C}P$, and let R'', L'' and $H'' := R''L'' - L''R''$ span a representation of $\mathfrak{sl}(2, \mathbb{C})$ acting on $\mathbb{C}Q$, with R', R'' order raising operators and L', L'' lowering operators. Such operators exist by Proposition 4.3. Define

$$R := R' \otimes I'' + I' \otimes R'' \quad \text{and} \quad L := L' \otimes I'' + I' \otimes L'',$$

where I' and I'' are the identity maps on $\mathbb{C}P$ and $\mathbb{C}Q$, respectively. Let $H := RL - LR$. By Proposition 2.6, we have

$$LR = L'R' \otimes I'' + L' \otimes R'' + R' \otimes L'' + I' \otimes L''R''$$

and

$$RL = R'L' \otimes I'' + R' \otimes L'' + L' \otimes R'' + I' \otimes R''L'',$$

so that

$$\begin{aligned} H &= RL - LR \\ &= (R'L' - L'R') \otimes I'' + I' \otimes (R''L'' - L''R'') \\ &= H' \otimes I'' + I' \otimes H''. \end{aligned}$$

If r, l are the gradings on P and Q, respectively, then for any $(p, q) \in P \times Q$,

$$\begin{aligned} H([p] \otimes [q]) &= H' \otimes I''([p] \otimes [q]) + I' \otimes H''([p] \otimes [q]) \\ &= H'([p]) \otimes [q] + [p] \otimes H''([q]) \\ &= r(p)[p] \otimes [q] + [p] \otimes l(q)[q] \\ &= (r(p) + l(q))([p] \otimes [q]) \\ &= \lambda(p, q)([p] \otimes [q]). \end{aligned}$$

Thus R, L, and H span a representation of $\mathfrak{sl}(2, \mathbb{C})$ acting on $\mathbb{C}P \otimes \mathbb{C}Q = \mathbb{C}(P \times Q)$. Moreover, R is an order raising operator on $\mathbb{C}(P \times Q)$. Indeed, for any $(p', q') \in P \times Q$,

$$\begin{aligned} R([p'], [q']) &= R' \otimes I''([p'], [q']) + I' \otimes R''([p'], [q']) \\ &= R'([p']) \otimes [q'] + [p'] \otimes R''([q']), \end{aligned}$$

so the (p,q)-entry of $R([p'],[q'])$ is nonzero only if the (p,q)-entry of $R'([p'])\otimes[q']$ is nonzero or the (p,q)-entry of $[p']\otimes R''([q'])$ is nonzero. The former is true only if $p' \lessdot p$ in P and $q'=q$, while the latter is true only if $q' \lessdot q$ in Q and $p'=p$ (since R', R'' are raising operators). In either case, $(p',q') \lessdot (p,q)$ in $P\times Q$, implying that R is an order raising operator on $\mathbb{C}(P\times Q)$. Similarly, L is a lowering operator on $\mathbb{C}(P\times Q)$. By Proposition 4.3 again, $P\times Q$ is Peck. ■

The proof of Proposition 4.4 above is due to Proctor [23].

We close this chapter by remarking on two important classes of Peck posets. The Peck property can be defined more standardly for a finite standard graded poset $(P,\leq)$ with maximum rank n having rank sequence $\{W_i\}_{i=0}^n$ as follows: $(P,\leq)$ is Peck if it is strongly Sperner and its rank sequence is unimodal and symmetric (that is, $W_i = W_{n-i}$ for all i with $0\leq i\leq n$). Given a non-negative integer n, it is well-known that the sequence of binomial coefficients $\{\binom{n}{i}\}_{i=0}^n$ is symmetric (that is, $\binom{n}{i}=\binom{n}{n-i}$ for all i with $0\leq i\leq n$) and unimodal. A less obvious fact is that the sequence of Gaussian binomial coefficients is also symmetric and unimodal (see [27], for example). These sequences are respectively the rank sequences of the Boolean lattice B_n and the subspace lattice $V_n(q)$, which are strongly Sperner as we have seen. This means that Boolean lattices and subspace lattices are Peck with the definition of Peck property above for finite standard graded posets.

Chapter 5

Main Results

5.1 Conjunctions of Posets

Given posets $(P, \leq_P)$ and $(Q, \leq_Q)$, we define a new poset $P \,\&\, Q$ as follows:

(a) the underlying set of $P \,\&\, Q$ is $\{(p, q) : p \in P, q \in Q\}$;

(b) (p, q) is covered by (p', q') in $P \,\&\, Q$ if and only if p is covered by p' in P and q is covered by q' in Q.

We then associate to $P \,\&\, Q$ the reflexive transitive closure of the covering relation defined above as its (partial) order relation. We refer to the poset $P \,\&\, Q$ as the *conjunction* of $(P, \leq_P)$ and $(Q, \leq_Q)$ (or simply the *conjunction* of P and Q). Again, sometimes we may informally use for simplicity the symbol $P \,\&\, Q$ to mean the underlying set $\{(p, q) : p \in P, q \in Q\}$ instead.

If both $(P, \leq_P)$ and $(Q, \leq_Q)$ are rank-finite 1-graded posets with gradings r and l, respectively, we define the function $\sigma : P \,\&\, Q \to \mathbb{Z}$ by $\sigma(p, q) = l(q) - r(p)$, and call the set

$$S_n := \{(p, q) \in P \,\&\, Q : \sigma(p, q) = n\}$$

the n-th *strand* of $P \,\&\, Q$ for any integer n. Note that the function σ is constant on each connected component of the Hasse diagram of $P \,\&\, Q$. If $P \,\&\, Q$ is non-empty then the strand S_n is non-empty precisely when $l_{\min}(Q) - r_{\max}(P) \leq n \leq l_{\max}(Q) - r_{\min}(P)$ (note that we do not require either of P and Q to have a maximum or a minimum rank, in which case n

is unbounded). Consider the function $\lambda : P \mathbin{\&} Q \to \mathbb{Z}$ defined by $\lambda(p,q) = r(p) + l(q)$. Note that although λ may not be a 2-grading on $P \mathbin{\&} Q$ (since the λ-values of different elements of $P \mathbin{\&} Q$ may have different parity), λ does restrict to a 2-grading on each strand.

Proposition 5.1. *Let $(S_n, \leq)$ be a strand of $P \mathbin{\&} Q$ with the inherited order relation. Then $\lambda|_{S_n}$ is a 2-grading on $(S_n, \leq)$.*

Proof. Let r and l be the 1-gradings of P and Q, respectively. Clearly, if $(p,q) \lessdot (p',q')$ then $\lambda(p',q') = r(p') + l(q') = r(p) + 1 + l(q) + 1 = \lambda(p,q) + 2$. If $(p,q) \in S_n$ with $\lambda(p,q) = k$ then we have $l(q) - r(p) = n$ and $l(q) + r(p) = k$ or equivalently,

$$r(p) = \frac{k-n}{2} \quad \text{and} \quad l(q) = \frac{k+n}{2}, \tag{5.1}$$

so that k must always have the same parity as n. In other words, $\lambda(p,q) \equiv \lambda(p',q') \pmod 2$ for any elements (p,q) and (p',q') of S_n, so $\lambda|_{S_n}$ is a 2-grading on $(S_n, \leq)$. ■

Proposition 5.1 allows us to view each strand of $P \mathbin{\&} Q$ as a 2-graded poset. In fact, Proposition 5.1 shows more, namely that each strand $(S_n, \leq)$ is rank-finite with the grading $\lambda|_{S_n}$. Indeed, if $S_{n,k}$ denotes the (non-empty) k-th rank of the strand $(S_n, \leq)$ then the proof of Proposition 5.1 shows that

$$\begin{aligned} S_{n,k} &= \{(p,q) \in P \mathbin{\&} Q : r(p) = \frac{k-n}{2} \text{ and } l(q) = \frac{k+n}{2}\} \\ &= P_{(k-n)/2} \times Q_{(k+n)/2}. \end{aligned}$$

Since both $P_{(k-n)/2}$ and $Q_{(k+n)/2}$ are finite by the rank-finiteness of P and Q, $S_{n,k}$ is finite. In the rest of the thesis, whenever we view a strand $(S_n, \leq)$ as a 2-graded poset, the function $\lambda|_{S_n}$ defined above will be the equipped grading.

One of the key properties of the conjunction operation is that the LYM properties are preserved (on the strands) under this operation, as the next theorem states.

Theorem 5.2. *Let P and Q be rank-finite 1-graded posets with gradings r and l, respectively. If P and Q are LYM then every strand of $P \mathbin{\&} Q$ is LYM.*

Proof. Let S_n be the (non-empty) n-th strand of $P \mathbin{\&} Q$. The ranks of S_n are $P_j \times Q_{j+n}$ where j is an integer. Assume that $P_i \times Q_{i+n}$ is not the lowest rank of S_n. Since P and Q

are LYM, by Proposition 3.5 the graphs $G_i(P)$ and $G_{i+n}(Q)$ have perfect matchings (since $i > r_{\min}$ and $i+n > l_{\min}$ by the choice of i). That is, there exist bijections

$$\varphi : P_i \times P_{i-1} \to P_{i-1} \times P_i$$

and

$$\psi : Q_{i+n} \times Q_{i+n-1} \to Q_{i+n-1} \times Q_{i+n}$$

such that if $\varphi(p, p') = (x, x')$ then $x \lessdot p$ in P and if $\psi(q, q') = (y, y')$ then $y \lessdot q$ in Q. Now consider the function

$$\theta : (P_i \times Q_{i+n}) \times (P_{i-1} \times Q_{i+n-1}) \to (P_{i-1} \times Q_{i+n-1}) \times (P_i \times Q_{i+n})$$

defined by

$$((p, q), (p', q')) \mapsto ((x, y), (x', y')),$$

where $(x, x') = \varphi(p, p')$ and $(y, y') = \psi(q, q')$. Then $(x, y) \lessdot (p, q)$ in $P \,\&\, Q$ by the choice of φ and ψ. Moreover, the bijectivity of θ follows directly from the bijectivity of φ and ψ. But this is just another way of saying that the graph $G_i(S_n)$ has a perfect matching for each i for which $G_i(S_n)$ is defined. By Proposition 3.5 again, S_n is LYM. ■

In the next proposition and the main theorem, we will make several statements about rank unimodality. However, the rank unimodality is not a property of a poset that can be easily shown. Fortunately, the rank unimodality is a consequence of the rank logarithmic concavity, and rank logarithmic concavity can be dealt with much more easily. We make this statement as a lemma for the later results.

Lemma 5.3. *Any logarithmically concave sequence of positive real numbers is unimodal.*

Proof. We prove this for infinite sequences: the proof for the finite case is essentially the same. Let $\{a_{k+ic}\}_{i\in\mathbb{Z}}$ be a logarithmically concave sequence of positive real numbers. Then for all $j \equiv k \pmod{c}$, we have $a_j^2 \geq a_{j-c}a_{j+c}$. Rewriting this inequality yields

$$\frac{a_j}{a_{j-c}} \geq \frac{a_{j+c}}{a_j}.$$

Since the entries are positive reals, the above implies that if $a_j < a_{j-c}$ then $a_{j+c} < a_j$. If

$\{a_{k+ic}\}_i$ is non-increasing or non-decreasing then we are done. Otherwise, from the above observation, there exist integers $m \equiv k \pmod{c}$ and n (which may be zeros) such that

$$a_{m-c} < a_m = a_{m+c} = \cdots = a_{m+nc} > a_{m+(n+1)c}.$$

From the same observation, we can further conclude that

$$\cdots < a_{m-2c} < a_{m-c} < a_m = a_{m+c} = \cdots = a_{m+nc} > a_{m+(n+1)c} > a_{m+(n+2)c} > \cdots,$$

so that $\{a_{k+ic}\}_i$ is unimodal. ■

Let P be a rank-finite standard graded poset, and let S_n be a strand of the poset $P^* \,\&\, P$. Then $(p', p) \in S_n$ if and only if $r(p) + r(p') = r(p) - r^*(p') = n$. This means that each strand of $P^* \,\&\, P$ is finite since P is rank-finite and standard graded. The poset $P^* \,\&\, P$ is a central object in the characterization theorem in the next section, and we shall see that there are some nice connections between the strands of $P^* \,\&\, P$ and the poset P.

Proposition 5.4. *Let P be a rank-finite standard graded poset. Then the following statements hold:*

(a) Every strand of $P^* \,\&\, P$ is rank-symmetric and hence centrally graded.

(b) P is rank logarithmically concave if and only if every strand of $P^* \,\&\, P$ is rank-unimodal.

Proof. (a) Let $(S_n, \leq)$ be a strand of $P^* \,\&\, P$ with the inherited order relation. Again we denote by $S_{n,k}$ the k-th rank of S_n. By using Equation (5.1) obtained in Proposition 5.1, we have $(p', p) \in S_{n,k}$ if and only if

$$r(p) = \frac{n+k}{2} \quad \text{and} \quad r(p') = \frac{n-k}{2},$$

so that we have $S_{n,k} = P_{(n-k)/2} \times P_{(n+k)/2}$ for any integer k (here we define P_i to be the empty set if i is not an integer). But this also means that $S_{n,-k} = P_{(n+k)/2} \times P_{(n-k)/2}$, from which we can conclude $|S_{n,k}| = |P_{(n-k)/2}| \cdot |P_{(n+k)/2}| = |S_{n,-k}|$, proving the rank symmetry for $(S_n, \leq)$. This means that $(S_n, \leq)$ is centrally graded since it is 2-graded.

(b) Suppose that P is RLC. In view of Lemma 5.3, it suffices to show the rank logarithmic concavity of a strand of $P^* \,\&\, P$ to conclude the rank unimodality of the strand. Let $(S_n, \leq)$

be a nonempty strand of $P^* \mathbin{\&} P$. If we denote the cardinality of $S_{n,k}$ by $W_{n,k}$ then we have from part (a) that

$$W_{n,k} = W_{(n-k)/2} W_{(n+k)/2}, \tag{5.2}$$

where $W_i = |P_i|$. Now we want to show that $W_{n,k}^2 \geq W_{n,k-2} W_{n,k+2}$. Indeed,

$$\begin{aligned} W_{n,k}^2 &= W_{(n-k)/2}^2 W_{(n+k)/2}^2 \\ &\geq W_{((n-k)/2)-1} W_{((n-k)/2)+1} W_{((n+k)/2)-1} W_{((n+k)/2)+1} \\ &= W_{(n-k-2)/2} W_{(n-k+2)/2} W_{(n+k-2)/2} W_{(n+k+2)/2} \\ &= (W_{(n-(k-2))/2} W_{(n+(k-2))/2})(W_{(n-(k+2))/2} W_{(n+(k+2))/2}) \\ &= W_{n,k-2} W_{n,k+2}. \end{aligned}$$

The inequality above follows from the rank logarithmic concavity of P. Thus every strand of $P^* \mathbin{\&} P$ is RLC and hence rank-unimodal.

Conversely, suppose that P is not RLC. Then there exists some i with $0 < i < r_{\max}(P)$ such that

$$W_i^2 < W_{i-1} W_{i+1}. \tag{5.3}$$

On the other hand, by equation (5.2) we have

$$W_{2i,0} = W_{(2i-0)/2} W_{(2i+0)/2} = W_i^2$$

and

$$W_{2i,-2} = W_{2i,2} = W_{(2i-2)/2} W_{(2i+2)/2} = W_{i-1} W_{i+1},$$

which in conjunction with inequality (5.3) implies that

$$W_{2i,0} < W_{2i,-2} \quad \text{and} \quad W_{2i,0} < W_{2i,2},$$

so that the strand $(S_{2i}, \leq)$ is not rank-unimodal. Therefore, if every strand of $P^* \mathbin{\&} P$ is rank-unimodal then P is RLC. ■

5.2 Main Theorem

For a standard graded poset P, we have just seen from the last section that the rank logarithmic concavity of P is equivalent to two of the three conditions of the Peck property, namely the rank unimodalities and the rank symmetries of the strands of $P^* \,\&\, P$. In fact, the LYM property and the rank logarithmic concavity of P are equivalent to the Peck properties of the strands of $P^* \,\&\, P$, as our main theorem states:

Theorem 5.5. *Let P be a rank-finite standard graded poset. The following are equivalent:*

(a) P is LYM and rank logarithmically concave.

(b) Every strand of $P^* \,\&\, P$ is rank-unimodal and LYM.

(c) Every strand of $P^* \,\&\, P$ is Peck.

Proof. **(a) $\Rightarrow$ (b):** Suppose that P is LYM and rank logarithmically concave. Then P^* is also LYM by Proposition 3.5. It then follows from Theorem 5.2 and Proposition 5.4 that every strand of $P^* \,\&\, P$ is LYM and rank-unimodal.

(b) $\Rightarrow$ (c): Suppose that every strand of $P^* \,\&\, P$ is rank-unimodal and LYM. To show that every strand of $P^* \,\&\, P$ is Peck, it remains to show that every strand of $P^* \,\&\, P$ is rank-symmetric and strongly Sperner. But the rank-symmetry is guaranteed by Proposition 5.4 (a) and the strong Sperner property is just a consequence of the LYM property by Proposition 3.4, so the desired statement follows.

(c) $\Rightarrow$ (a): Suppose that every strand of $P^* \,\&\, P$ is Peck. Then the rank logarithmic concavity of P follows from the rank-unimodality of the strands of $P^* \,\&\, P$ by Proposition 5.4 (b). It thus suffices to show that P is LYM. Let $0 < k \leq r_{\max}(P)$ and consider the graph $G_k(P)$. The vertices of $G_k(P)$ are precisely the elements of the set

$$(P_k \times P_{k-1}) \cup (P_{k-1} \times P_k),$$

and thus can be viewed as elements of the poset $P^* \,\&\, P$. Clearly, $\sigma(x) = 2k - 1$ for all vertices x in $G_k(P)$, with $P_k \times P_{k-1}$ and $P_{k-1} \times P_k$ lying in the (-1)-st rank and the 1-st rank of $P^* \,\&\, P$, respectively. Moreover, any element in the strand S_{2k-1} that is in the (-1)-st rank (the 1-st rank, respectively) of $P^* \,\&\, P$ must be an element of $P_k \times P_{k-1}$ (an element of

$P_{k-1} \times P_k$, respectively), so that if we view the strand S_{2k-1} as a poset, it has $P_k \times P_{k-1}$ and $P_{k-1} \times P_k$ as its (-1)-st rank and 1-st rank, respectively. Since every strand of $P^* \& P$ is Peck, by Proposition 4.3, there is a symmetric chain matching from $P_k \times P_{k-1}$ to $P_{k-1} \times P_k$. This symmetric chain matching is also a perfect matching for the graph $G_k(P)$. By Proposition 3.5, P is LYM. ■

Theorem 5.5 has some interesting applications, which we will see in the next chapter.

Chapter 6

Applications

6.1 Harper's Product Theorem

As we mentioned in the introduction, it is false in general that the product of two standard graded LYM posets also has the LYM property, which can be shown by the following counterexample. Let P be the two-element chain with $0 < 1$, and let Q be the set $\{x, y, z, w, v\}$ with covering relations given by $x \lessdot z$, $y \lessdot z$, $z \lessdot w$ and $z \lessdot v$. Associate to P and Q the standard gradings. Hasse diagrams of P and Q are shown in Figures 6.1. By verifying the normalized matching properties, it is easy to check that P and Q are both LYM. However, the poset $P \times Q$, whose Hasse diagram is also shown in Figure 6.1, does not satisfy the LYM property. Of course, the largest Whitney number of the poset $P \times Q$ is 3, but $\{(0, w), (0, v), (1, x), (1, y)\}$ is a 4-element antichain in $P \times Q$, so $P \times Q$ is not Sperner, and hence cannot be LYM by Proposition 3.4. Notice that the poset Q in this example is not rank logarithmically concave (and is not even rank-unimodal). However, if two standard graded LYM posets are both rank logarithmically concave, then their product also has these two properties. This result, which we will call Harper's product theorem, was first proven by Harper ([14]) and then by Hsieh and Kleitman ([15]) as mentioned in the introduction. In this section we give another proof of Harper's product theorem, and we will see that Harper's product theorem is a natural consequence of Theorem 5.5.

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be standard graded posets (so their order product $P \times Q$ is also standard graded), and let $(S, \leq)$ be a strand of $(P^* \times Q^*) \,\&\, (P \times Q)$ with the inherited order

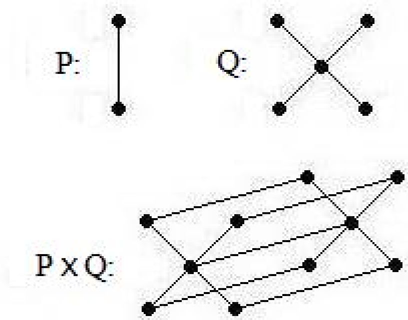

Figure 6.1: Two LYM posets P, Q and their product $P \times Q$ which is not LYM.

relation and the usual 2-grading for a strand. Define $(\overline{S}, \preceq)$ to be the 2-graded poset whose underlying set $\overline{S}$ is $\{(p', p, q', q) : (p', q', p, q) \in S\}$, such that $(p', p, q', q) \preceq (x', x, y', y)$ in $\overline{S}$ if and only if $(p', q', p, q) \leq (x', y', x, y)$ in S, with the rank of an element being again the sum of the ranks of all the components. We call $\preceq$ the *canonical order* of $\overline{S}$. Intuitively speaking, we define a new poset by interchanging the middle two coordinates of the elements of the strand while keeping the ordering and grading. It follows immediately that if $(S, \leq)$ is Peck then $(\overline{S}, \preceq)$ is also Peck. If $(S, \leq)$ is a strand of $(P^* \times Q^*) \,\&\, (P \times Q)$ then $\overline{S}$ is a subset of the underlying set of $(P^* \,\&\, P) \times (Q^* \,\&\, Q)$. In fact, if $\preceq_0$ is the order relation on $\overline{S}$ inherited from $(P^* \,\&\, P) \times (Q^* \,\&\, Q)$, then $(\overline{S}, \preceq)$ is an order extension of $(\overline{S}, \preceq_0)$.

Lemma 6.1. *Using the above notations, if $(P, \leq_P)$ and $(Q, \leq_Q)$ are standard graded posets and $(S, \leq)$ is a strand of $(P^* \times Q^*) \,\&\, (P \times Q)$ with inherited order relation $\leq$, then $(\overline{S}, \preceq)$ is a rank-preserving order extension of $(\overline{S}, \preceq_0)$.*

Proof. If $(p', p, q', q) \lessdot (x', x, y', y)$ in $(\overline{S}, \preceq_0)$ then one of the following two cases must be true:

(1) $(p', p) \lessdot (x', x)$ in $P^* \,\&\, P$ and $(q', q) = (y', y)$.

(2) $(q', q) \lessdot (y', y)$ in $Q^* \,\&\, Q$ and $(p', p) = (x', x)$.

If (1) is true then $p' \lessdot x'$ in P^*, $p \lessdot x$ in P, $q' = y'$, and $q = y$, which means that $(p', q') \lessdot (x', y')$ in $P^* \times Q^*$ and $(p, q) \lessdot (x, y)$ in $P \times Q$, implying that $(p', q', p, q) \lessdot (x', y', x, y)$ in $(P^* \times Q^*) \,\&\, (P \times Q)$ (and hence in $(S, \leq)$). This holds similarly if (2) is the case. Hence $(p', p, q', q) \lessdot (x', x, y', y)$ in $(\overline{S}, \preceq)$, implying that $(\overline{S}, \preceq)$ is an extension of $(\overline{S}, \preceq_0)$. This extension is clearly rank-preserving. ■

Lemma 6.1 says that $(P^* \times Q^*) \& (P \times Q)$ corresponds to a rank-preserving order extension of $(P^* \& P) \times (Q^* \& Q)$ in a natural way.

Theorem 6.2. (Harper's Product Theorem) *Let P and Q be rank-finite standard graded posets. If P and Q are LYM and RLC then so is $P \times Q$.*

Proof. Since $P \times Q$ is also standard graded, by Theorem 5.5, it suffices to show that every strand of $(P \times Q)^* \& (P \times Q) = (P^* \times Q^*) \& (P \times Q)$ is Peck. Let r, l be the gradings of P and Q, respectively, and let $(S_n, \leq)$ be a strand of $(P \times Q)^* \& (P \times Q)$. Then

$$\begin{aligned}\overline{S_n} &= \{(p', p, q', q) : r(p) + l(q) - r^*(p') - l^*(q') = n\} \\ &= \{(p', p, q', q) : r(p') + r(p) + l(q') + l(q) = n\} \\ &= \bigcup_{a+b=n} S_a^P \times S_b^Q,\end{aligned}$$

where S_a^P is the a-th strand of $P^* \& P$ and S_b^Q is the b-th strand of $Q^* \& Q$. Here we view the strands as sets without any order relations. If we associate to $\overline{S_n} = \bigcup_{a+b=n} S_a^P \times S_b^Q$ the order $\preceq_0$ inherited from $(P^* \& P) \times (Q^* \& Q)$ then since P and Q are both LYM and RLC, each S_a^P and S_b^Q is Peck by Theorem 5.5. By Proposition 4.4, each $S_a^P \times S_b^Q$ is Peck. Furthermore, with the constraint that $a + b = n$, it is easy to check that the ranks of $S_a^P \times S_b^Q$ have the same parity for different values of a and b (which must be the same as the parity of n), so that the order union $\bigcup_{a+b=n} S_a^P \times S_b^Q$ is Peck by Proposition 4.2. Since $(\overline{S_n}, \preceq_0)$ is a rank-preserving extension of this order union, $(\overline{S_n}, \preceq_0)$ is Peck by Proposition 4.1. Since $(\overline{S_n}, \preceq)$ with the canonical order relation is a rank-preserving order extension of $(\overline{S_n}, \preceq_0)$ by Lemma 6.1, the set $(\overline{S_n}, \preceq)$ is Peck by Proposition 4.1 again, implying that $(S_n, \leq)$ is Peck. Since S_n is a strand of $(P \times Q)^* \& (P \times Q)$, it follows from Theorem 5.5 that $P \times Q$ is LYM and RLC. ■

6.2 Stable Set Posets

Let $G = (V, E)$ be a finite graph. A *stable set* in G is a subset $A \subseteq V$ of vertices such that no edge of G has both ends in A. Let $S := S(G)$ be the set of all stable sets in G. Note that $(S, \subseteq)$ is a downset which contains every singleton set in the set of all subsets of V partially ordered by inclusion (which is called the *Boolean lattice* of V).

Consider the covering relation on S defined by $A \lessdot B$ if and only if $A \triangle B$ induces a path in G with both ends in B, where $A \triangle B$ denotes the symmetric difference of A and B (which is defined to be the set $(A \cup B) \setminus (A \cap B)$), and let $\leq$ be the reflexive transitive closure of $\lessdot$. Note that if A is covered by B in $(S, \subseteq)$ then $A \lessdot B$ in $(S, \leq)$, so the latter is an extension of the former. Furthermore, if $A \triangle B$ induces a path with both ends in B, then the fact that both A and B are stable sets implies that $|B| = |A| + 1$, so that this extension is rank-preserving.

The *claw* $K_{1,3}$ is the graph (V, E) with $|V| = 4$ and $|E| = 3$ such that one vertex has degree 3 and all other vertices have degree 1. A graph is called *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. For a graph $G = (V, E)$, let a_i be the number of i-element stable sets in G, and let m be the maximum cardinality of a stable set in G. Hamidoune ([13]) showed that if G is a claw-free graph then the sequence $\{a_i\}_{0 \leq i \leq m}$ is logarithmically concave. Wagner ([30]) gave another proof by observing the relation between logarithmically concave finite sequences and representations of $\mathfrak{sl}(2, \mathbb{C})$. In fact, with the representations Wagner constructed, Theorem 5.5 shows the following stronger result.

Theorem 6.3. *If G is a claw-free graph then the poset $(S(G), \leq)$ is LYM and rank -logarithmically concave.*

Proof. Let G be a claw-free graph. For $A, B \in S = S(G)$, let $M(A, B)$ be the set of connected components of the graph $G[A \cup B]$ induced by $A \cup B$ that are paths with both ends in $A \setminus B$, and let $W(A, B)$ be the set of connected components of the graph $G[A \cup B]$ induced by $A \cup B$ that are paths with both ends in $B \setminus A$. Wagner ([30]) shows that the linear operators R, L and $H : \mathbb{C}(S^* \,\&\, S) \to \mathbb{C}(S^* \,\&\, S)$ defined by

$$R([A] \otimes [B]) := \sum_{P \in M(A,B)} [A \triangle V(P)] \otimes [B \triangle V(P)]$$

and

$$L([A] \otimes [B]) := \sum_{P \in W(A,B)} [A \triangle V(P)] \otimes [B \triangle V(P)],$$

with $H := RL - LR$ span a representation of $\mathfrak{sl}(2, \mathbb{C})$ acting on $\mathbb{C}(S^* \,\&\, S)$, such that R is a raising operator and L is a lowering operator. In fact, R is an order raising operator. Of course, if $(A, B) \in S^* \,\&\, S$ and $P \in M(A, B)$ then $(A \triangle V(P)) \triangle A = V(P)$, which induces the path P whose ends are both in A by the definition of $M(A, B)$, meaning that $A \lessdot A \triangle V(P)$ in S^* (since $A \triangle V(P)$ is a stable set by the definition of $M(A, B)$). Similarly,

$B\triangle(B\triangle V(P)) = V(P)$, which induces the path P whose ends are both in $(B\triangle V(P))$, meaning that $B \lessdot (B\triangle V(P))$ in S (again, $B\triangle V(P)$ is a stable set). Therefore R is an order raising operator. By restricting the representation spanned by R, L and H to strands, Proposition 4.3 implies that every strand of S^* & S is Peck. By Theorem 5.5, $(S(G), \leq)$ is LYM and RLC. ■

Since $(S(G), \leq)$ is a rank-preserving extension of $(S(G), \subseteq)$, the logarithmic concavity of the sequence $\{a_i\}_{0\leq i\leq m}$ follows immediately.

Corollary 6.4. *If G is a claw-free graph then the sequence $\{a_i\}_{0\leq i\leq m}$ is logarithmically concave.* ■

If a line graph $L(G)$ has a vertex v with degree at least 3 then for any three distinct vertices adjacent to v, two of these vertices must be adjacent. In other words, all line graphs are claw-free. We will see that this observation along with Theorem 6.3 in fact gives us a way to construct a rank-preserving LYM extension to the partition lattice, which is known to be not LYM for large order.

A partition of a nonempty set A is a collection of nonempty disjoint subsets of A (called *blocks*) whose union equals A. The partition lattice $(\Pi_n, \preceq)$ of order n is a poset whose elements are the partitions of the set $\{0, 1, \cdots, n-1\}$, with two partitions $\pi \preceq \sigma$ if and only if each block in π is contained in some block in σ (for example, $\{\{0\},\{1\},\{2\}\} \preceq \{\{0,1\},\{2\}\} \preceq \{\{0,1,2\}\}$ in the partition lattice of order 3). Our convention of representing a partition is to write each block as an increasing sequence. For example, we represent the partition $\{\{2,1\},\{0,4\},\{3\}\} \in \Pi_5$ as $(0,4)(1,2)(3)$ (we usually also arrange the blocks so that the smallest numbers of the blocks are in increasing order, but this is not important). Because of this convention, we call a number a *right-most number* in π if it is the largest number in its block in π. Given a partition lattice $(\Pi_n, \preceq)$, it is easy to check that the function that maps each partition $\pi \in \Pi_n$ to $n - |\pi|$, where $|\pi|$ is the number of blocks π contains, is a standard grading on $(\Pi_n, \preceq)$. In this case, the rank sequence of $(\Pi_n, \preceq)$ consists of the famous Stirling numbers of the second kind, which is known to be logarithmically concave (see [31, p.138]). However, Spencer ([24]) has shown that for sufficiently large n, the partition lattice $(\Pi_n, \preceq)$ is not LYM (in fact, Canfield [4] showed that the partition lattice of order n is not even Sperner for sufficiently large n). We will use Theorem 6.3 to construct a rank-preserving LYM extension to a given partition lattice.

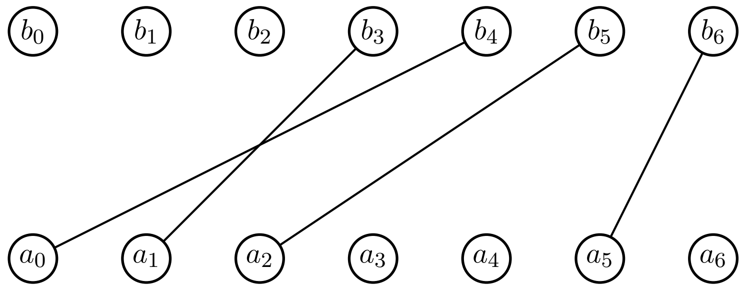

Figure 6.2: The graph isomorphic to $(V(G_7), \varphi(\pi))$ where $\pi = (0,4)(1,3)(2,5,6)$, with a_i, b_j representing $(0,i)$ and $(1,j)$, respectively.

For each positive integer n, let G_n be the graph having vertex set $[2] \times [n]$ (where $[i] = \{0, 1, \cdots, i-1\}$), with (u_1, u_2) and (v_1, v_2) being adjacent if and only if $u_1 < v_1$ and $u_2 < v_2$, or $v_1 < u_1$ and $v_2 < u_2$.

Proposition 6.5. *There is a one-to-one correspondence between* Π_n *and* $\mathcal{M}(G_n)$*.*

Proof. Consider the function $\varphi : \Pi_n \to \mathcal{M}(G_n)$ defined by the following rule. Given a partition π of $\{0, 1, \cdots, n-1\}$, we first represent it using our convention. $\varphi(\pi)$ is then defined to be the set of edges of G_n satisfying the following: if a number λ is not a right-most number in π then the edge $\{(0, \lambda), (1, \vec{\lambda})\}$ is in $\varphi(\pi)$, where $\vec{\lambda}$ is the number immediately to the right of λ in its block; no edges of other forms are contained in $\varphi(\pi)$. For example, if $\pi = (0,4)(1,3)(2,5,6) \in \Pi_7$ then Figure 6.2 shows the graph $(V(G_7), \varphi(\pi))$. Since all the blocks in a partition must be disjoint, and each number has a unique number on its left and a unique number on its right (if any), $\varphi(\pi)$ is indeed a matching for G_n so that the function φ is well-defined. Now consider the function $\psi : \mathcal{M}(G_n) \to \Pi_n$ defined by the following rule. For a given matching $M \in \mathcal{M}(G_n)$, we consider the smallest number λ such that $(0, \lambda)$ is an end of some edge in M, and consider the set $\{\lambda\}$ as a block; if $(1, \mu)$ is the other end of this edge then we put μ in the block containing λ. Now, if $\{(0, \mu), (1, \nu)\}$ is an edge in M then we put ν in the block containing μ, and then continue this procedure on ν; if no edge in M has this form then we "close" the block and start a new block containing the smallest number κ not contained in any previous block such that $(0, \kappa)$ is an end of some edge in M. We then build this block in the way described previously. Continue creating and building blocks in the above manner until we have considered all the edges in M. If all the edges in

M have been considered and there is still some number in $\{0, 1, \cdots, n-1\}$ that is not in any block, we consider each of these numbers itself as a one-element block. We then define $\psi(M)$ to be the collection of all the blocks defined above. The union of the blocks in $\psi(M)$ is clearly $\{0, 1, \cdots, n-1\}$ by construction, and the blocks are disjoint by the fact that M is a matching, which means that ψ is well-defined.

We then show that φ and ψ are mutual inverses. Given any partition $\pi \in \Pi_n$, two numbers a and b with $a < b$ are in the same block of π if and only if $\varphi(\pi)$ has a subset of the form

$$\Big\{\{(0,a),(1,c_1)\},\{(0,c_1),(1,c_2)\},\cdots,\{(0,c_t),(1,b)\}\Big\}$$

by the definition of φ. By the way we define the function ψ, a matching M has a subset of this form if and only if a and b are in the same block of $\psi(M)$. That is, two numbers are in the same block of π if and only if they are in the same block of $\psi(\varphi(\pi))$, which is equivalent to saying that $\psi(\varphi(\pi)) = \pi$. Similarly, $\varphi(\psi(M)) = M$ for all $M \in \mathcal{M}(G_n)$. Therefore φ and ψ are mutual inverses, which give the desired one-to-one correspondence. ■

By definition, for any graph G, the set of edges of G is the set of vertices of the line graph $L(G)$. Furthermore, a set of edges of G is a matching for G if and only if it is a stable set for $L(G)$, so the set of matchings for G coincides with the set $S(L(G))$ of stable sets in $L(G)$. With this in mind, Proposition 6.5 says that there is a one-to-one correspondence between the set Π_n of all partitions of $\{0, 1, \cdots, n-1\}$ and the set $S(L(G_n))$. In terms of posets, π_1 is covered by π_2 in $(\Pi_n, \preceq)$ if and only if every block in π_1 is contained in some block in π_2, with π_1 having exactly one more block than π_2. This is equivalent to saying that every right-most number in π_2 is a right-most number in π_1, and exactly one non-right-most number in π_2 is a right-most number in π_1. Using the function φ defined in Proposition 6.5, this is equivalent to saying that $\varphi(\pi_1) \subseteq \varphi(\pi_2)$ with $|\varphi(\pi_1)| = |\varphi(\pi_2)| - 1$ or in other words, $\varphi(\pi_1)$ is covered by $\varphi(\pi_2)$ in $(S(L(G_n)), \subseteq)$. Since any line graph is claw-free, $(S(L(G_n)), \leq)$ is a rank-preserving extension to $(S(L(G_n)), \subseteq)$ that is both LYM and RLC by Theorem 6.3, which corresponds naturally to a rank-preserving extension to $(\Pi_n, \preceq)$ that is both LYM and RLC. Note that the existence of such a LYM and RLC rank-preserving extension to the partition lattice of order n reproves the logarithmic concavity of the sequence of Stirling numbers of the second kind of order n.

6.3 Upsets and Downsets

In this section, we first prove an inequality using Theorem 5.5, and then derive some results about downsets and upsets in a standard graded LYM and RLC poset from this inequality.

Theorem 6.6. *Let P be a standard graded LYM and RLC poset with grading r. Let Δ be a downset in P, and let U be an upset in P. For each $0 \leq i \leq r_{\max}$, let $f_i := |\Delta_i| = |\Delta \cap P_i|$ and let $u_i := |U_i| = |U \cap P_i|$. Then for all $0 \leq j < k \leq r_{\max}$,*

$$f_k u_j \leq f_{k-1} u_{j+1}.$$

Proof. Let $0 \leq j < k \leq r_{\max}$. If $f_k = 0$ or $u_j = 0$ then the result is obvious, so we may assume that $f_k \neq 0$ and $u_j \neq 0$, so that Δ_k and U_j are both non-empty. Since P is a standard graded LYM and RLC poset, the strand $(S_{j+k}, \leq)$ of $P^* \mathbin{\&} P$ is Peck by Theorem 5.5, so there exists a symmetric chain matching from the $(j-k)$-th rank to the $(k-j)$-th rank in this strand by Proposition 4.3. Restricting this symmetric chain matching to the $(j-k)$-th rank and the $(j-k+2)$-th rank of the strand gives an injective function φ from $P_k \times P_j$ to $P_{k-1} \times P_{j+1}$ such that $x \lessdot \varphi(x)$ in the strand S_{j+k} for every $x \in P_k \times P_j$, which implies that $\varphi(\Delta_k \times U_j) \subseteq \Delta_{k-1} \times U_{j+1}$. Since φ is injective, we have

$$f_k u_j = |\Delta_k \times U_j| \leq |\Delta_{k-1} \times U_{j+1}| = f_{k-1} u_{j+1},$$

as desired. ■

More properties of upsets and downsets in a standard graded LYM and RLC poset can be derived from Theorem 6.6. We show them as a sequence of corollaries.

Corollary 6.7. *Let P be a standard graded LYM and RLC poset with grading r. Let Δ be a downset in P, and let U be an upset in P. For each $0 \leq i \leq r_{\max}$, let $f_i := |\Delta \cap P_i|$ and let $u_i := |U \cap P_i|$. Then the sequence $\{p_i\} := \{f_i/W_i\}_{i=0}^{r_{\max}}$ is non-increasing, and the sequence $\{q_i\} := \{u_i/W_i\}_{i=0}^{r_{\max}}$ is non-decreasing.*

Proof. Since P is LYM (hence Sperner), by the remark we made at the end of Chapter 2 $W_l \neq 0$ for all l with $0 \leq l \leq r_{\max}$, so the sequences $\{p_i\}$ and $\{q_i\}$ are well-defined. Let i be such that $0 \leq i < r_{\max}$. By considering the downset Δ and the whole poset P as an upset, and by setting $j = i$ and $k = i + 1$, Theorem 6.6 implies that $f_{i+1}W_i \leq f_i W_{i+1}$ or equivalently,

$$\frac{f_{i+1}}{W_{i+1}} \leq \frac{f_i}{W_i}, \tag{6.1}$$

so that the sequence $\{p_i\}$ is non-increasing. Similarly, by considering an upset and the whole poset P as a downset, the same argument shows that

$$\frac{u_{i+1}}{W_{i+1}} \geq \frac{u_i}{W_i}, \tag{6.2}$$

so that the sequence $\{q_i\}$ is non-decreasing. ■

Note that Corollary 6.7 implies that if $d_i \neq 0$ then $d_j \neq 0$ for all j with $0 \leq j \leq i$, and if $u_i \neq 0$ then $u_j \neq 0$ for all j with $i \leq j \leq r_{\max}$, although this is in fact a consequence of the normalized matching property of P independent of its rank logarithmic concavity.

Corollary 6.8. *Let P be a standard graded LYM and RLC poset with grading r. Let Δ be a downset in P, and let U be an upset in P. For each $0 \leq i \leq r_{\max}$, let $f_i := |\Delta \cap P_i|$ and let $u_i := |U \cap P_i|$. Then for all $0 \leq j < k \leq r_{\max}$ such that $f_{k-1} > 0$ and $u_j > 0$, we have*

$$\frac{f_k}{f_{k-1}} \leq \frac{W_k}{W_{k-1}} \leq \frac{W_{k-1}}{W_{k-2}} \leq \cdots \leq \frac{W_{j+1}}{W_j} \leq \frac{u_{j+1}}{u_j}.$$

Proof. Let j and k be such that $0 \leq j < k \leq r_{\max}$ with $f_{k-1} > 0$ and $u_j > 0$. Rearranging inequalities (6.1) and (6.2) yields

$$\frac{f_k}{f_{k-1}} \leq \frac{W_k}{W_{k-1}} \quad \text{and} \quad \frac{W_{j+1}}{W_j} \leq \frac{u_{j+1}}{u_j}.$$

Furthermore, it follows from the rank logarithmic concavity of P that

$$\frac{W_k}{W_{k-1}} \leq \frac{W_{k-1}}{W_{k-2}} \leq \cdots \leq \frac{W_{j+1}}{W_j},$$

from which the desired inequality follows by transitivity. ■

In addition to the properties of upsets and downsets in a general standard graded LYM and RLC poset above, Theorem 6.6 also provides us some information about downsets in a product of chains. For any positive integer n, let us denote by $[n]$ (and $[\infty]$, respectively) the graded poset $\{0, 1, ..., n-1\}$ (and $\mathbb{N} = \{0, 1, \cdots\}$, respectively) with the usual order of integers, such that the rank of each integer s is just s itself. We use the notation $[\infty]^m$ for the order product $[\infty] \times \cdots \times [\infty]$ of m copies of $[\infty]$. It follows immediately from the definitions that $[n]$ and $[\infty]$ are both rank-finite standard graded LYM and RLC posets. By Harper's product theorem (Theorem 6.2) and induction, the product of finitely many such chains is again LYM and RLC (so any finite product of chains is in particular strongly Sperner, as mentioned in Section 2.6).

Corollary 6.9. *Let $P := [n_1] \times \cdots \times [n_t]$ be the graded poset with the usual product grading ρ. Let Δ be a downset in P, and let $f_i := |\Delta \cap P_i|$ for all $0 \le i \le \rho_{\max}$. If $k + j > \rho_{\max}$ with $1 \le k \le \rho_{\max}$ and $1 \le j \le \rho_{\max}$ then $f_k f_j \le f_{k-1} f_{j-1}$. Furthermore, the sequence $\{f_i\}$ is non-increasing for $i \ge \rho_{\max}/2$.*

Proof. For each $1 \le l \le t$, consider the function $\psi_l : [n_l] \to [n_l]$ defined by $x \mapsto n_l - 1 - x$, and let $\psi : P \to P$ be the function defined by $(x_1, \cdots, x_t) \to (\psi_1(x_1), \cdots, \psi_t(x_t))$. Note that ψ is bijective. Let $U := \psi(\Delta)$ and let $u_i := |U \cap P_i|$. If $(\psi_1(x_1), \cdots, \psi_t(x_t)) \in U$ and $(\psi_1(x_1), \cdots, \psi_t(x_t)) \le (\psi_1(y_1), \cdots, \psi_t(y_t))$ in P then $(x_1, \cdots, x_t) \in \Delta$ and $(y_1, \cdots, y_t) \le (x_1, \cdots, x_t)$ in P, so $(y_1, \cdots, y_t) \in \Delta$ since Δ is a downset, meaning that $(\psi_1(y_1), \cdots, \psi_t(y_t)) \in U$. So U is an upset. Furthermore, if we let $n := \rho_{\max} = n_1 + \cdots + n_t - t$ then $x_1 + \cdots + x_t = n - i$ if and only if

$$\begin{aligned}\psi_1(x_1) + \cdots + \psi_t(x_t) &= n_1 - 1 - x_1 + \cdots + n_t - 1 - x_t \\ &= n_1 + \cdots + n_t - t - (x_1 + \cdots + x_t) = n - (n - i) = i,\end{aligned}$$

so that $\rho(x) = n - i$ if and only if $\rho(\psi(x)) = i$. Since $x \in \Delta$ if and only if $\psi(x) \in U$, we have for all i with $0 \le i \le n$, $x \in \Delta \cap P_{n-i}$ if and only if $\psi(x) \in U \cap P_i$, so that $u_i = f_{n-i}$ by the bijectivity of ψ.

For fixed $k, j \in \{1, 2, \cdots, n\}$ with $k + j > \rho_{\max} = n$, we let $i := n - j$. Then $i \in \{0, 1, \cdots, n-1\}$. Since $k + j > n$, it follows that $0 \le i = n - j < k \le n$, so that

$$f_k f_j = f_k f_{n-i} = f_k u_i \le f_{k-1} u_{i+1} = f_{k-1} f_{n-i-1} = f_{k-1} f_{j-1}$$

by Theorem 6.6. If $i > \rho_{\max}/2$, applying the inequality above with $k = j = i$ yields $f_i^2 \le f_{i-1}^2$, so that $f_i \le f_{i-1}$ as they are non-negative integers, proving that $\{f_i\}$ is non-increasing for

$i \geq \rho_{\max}/2$. ∎

Corollary 6.10. *Let $P := [n_1] \times \cdots \times [n_t] \times [\infty]^m$ be the graded poset with the usual product grading ρ. Let Δ be a downset in P, and let $f_i := |\Delta \cap P_i|$ for all $i \geq 0$. Then for any non-negative integer i, we have*

$$f_{i+1}^2 - f_i f_{i+2} \leq f_{i+1} W_{i+1} - f_{i+2} W_j.$$

Proof. Let $U := P \setminus \Delta$, and let $u_i := |U \cap P_i|$ for each $i \geq 0$. Evidently $u_i = W_i - f_i$. Clearly U is an upset in P, for otherwise there exist $p, q \in P$ with $p \leq q$ such that $p \in U$ and $q \notin U$, implying that $q \in \Delta$ and $p \notin \Delta$, which is impossible as Δ a downset in P. For any non-negative integer i, the inequality we want to prove is equivalent to the inequality

$$f_{i+2}(f_i - W_i) \geq f_{i+1}(f_{i+1} - W_{i+1}),$$

which is equivalent to

$$f_{i+2}(W_i - f_i) \leq f_{i+1}(W_{i+1} - f_{i+1}).$$

But this is true since

$$f_{i+2}(W_i - f_i) = f_{i+2} u_i \leq f_{i+1} u_{i+1} = f_{i+1}(W_{i+1} - f_{i+1}),$$

where the inequality follows from Theorem 6.6 with $j := i$ and $k := i + 2$. This proves the desired inequality. ∎

Note that the right-hand side of the inequality in Corollary 6.10 is non-negative by Theorem 6.6. Unfortunately, we are not able to conclude whether the sequence $\{f_i\}$ is logarithmically concave from Corollary 6.10 since it provides an upper bound instead. Of course, any attempts of showing that $\{f_i\}$ is logarithmically concave will not succeed, as Björner [3] showed that the sequence need not be even unimodal in general.

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