Interference Management in Non-cooperative Networks

by

Seyed Abolfazl Motahari

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AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Seyed Abolfazl Motahari

Abstract

Spectrum sharing is known as a key solution to accommodate the increasing number of users and the growing demand for throughput in wireless networks. While spectrum sharing improves the data rate in sparse networks, it suffers from interference of concurrent links in dense networks. In fact, interference is the primary barrier to enhance the overall throughput of the network, especially in the medium and high signal-to-noise ratios (SNRs). Managing interference to overcome this barrier has emerged as a crucial step in developing efficient wireless networks. This thesis deals with optimum and sub-optimum interference management-cancelation in non-cooperative networks.

Several techniques for interference management including novel strategies such as interference alignment and structural coding are investigated. These methods are applied to obtain optimum and sub-optimum coding strategies in such networks. It is shown that a single strategy is not able to achieve the maximum throughput in all possible scenarios and in fact a careful design is required to fully exploit all available resources in each realization of the system.

This thesis begins with a complete investigation of the capacity region of the two-user Gaussian interference channel. This channel models the basic interaction between two users sharing the same spectrum for data communication. New outer bounds outperforming known bounds are derived using Genie-aided techniques. It is proved that these outer bounds meet the known inner bounds in some special cases, revealing the sum capacity of this channel over a certain range of parameters which has not been known in the past.

A novel coding scheme applicable in networks with single antenna nodes is proposed next. This scheme converts a single antenna system to an equivalent Multiple Input Multiple Output (MIMO) system with fractional dimensions. Interference can be aligned along these dimensions and higher multiplexing gains can be achieved. Tools from the field of Diophantine approximation in number theory are used to show that the proposed coding scheme in fact mimics the traditional schemes used in MIMO systems where each data stream is sent along a direction and alignment happens when several streams are received along the same direction. Two types of constellation are proposed for the encoding part, namely the single layer constellation and the multi-layer constellation. Using single layer constellations, the coding scheme is applied to the two-user X channel. It is proved that the total Degrees-of-Freedom (DOF), i.e. $\frac{4}{3}$, of the channel is achievable almost surely. This is the first example in which it is shown that a time invariant single antenna system does not fall short of achieving this known upper bound on the DOF. Using multi-layer constellations, the coding scheme is applied to the symmetric three-user GIC. Achievable DOFs are derived for all channel gains. It is observed that the DOF is everywhere discontinuous (as a function of the channel gain). In particular, it is proved that for the irrational channel gains the achievable DOF meets the upper bound of $\frac{3}{2}$. For the rational gains, the achievable DOF has a gap to the known upper bounds. By allowing carry over from multiple layers, however, it is shown that higher DOFs can be achieved for the latter.

The K-user single-antenna Gaussian Interference Channel (GIC) is considered, where the channel coefficients are NOT necessarily time-variant or frequency selective. It is proved that the total DOF of this channel is $\frac{K}{2}$ almost surely, i.e. each user enjoys half of its maximum DOF. Indeed, we prove that the static time-invariant interference channels are rich enough to allow simultaneous interference alignment at all receivers. To derive this result, we show that single-antenna interference channels can be treated as *pseudo multiple-antenna systems* with infinitely-many antennas. Such machinery enables us to prove that the real or complex $M \times M$ MIMO GIC achieves its total DOF, i.e., $\frac{MK}{2}$, $M \geq 1$. The pseudo multiple-antenna systems are developed based on a recent result in the field of Diophantine approximation which states that the convergence part of the Khintchine-Groshev theorem holds for points on non-degenerate manifolds. As a byproduct of the scheme, the total DOFs of the $K \times M$ X channel and the uplink of cellular systems are derived.

Interference alignment requires perfect knowledge of channel state information at all nodes. This requirement is sometimes infeasible and users invoke random coding to communicate with their corresponding receivers. Alternative interference management needs to be implemented and this problem is addressed in the last part of the thesis. A coding scheme for a single user communicating in a shared medium is proposed. Moreover, polynomial time algorithms are proposed to obtain best achievable rates in the system. Successive rate allocation for a K-user interference channel is performed using polynomial time algorithms.

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To my parents,
Seyedeh Saboora and Seyed Alimohammad Motahari,

and

To my beloved wife,
Elham Monifi.

Contents

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List of Abbreviations

IC	Interference Channel
GIC	Gaussian Interference Channel
MAC	Multiple Access Channel
BC	Broadcast Channel
FDMA	Frequency Division Multiple Access
TDMA	Time Division Multiple Access
HK	Han and Kobayashi
DOF	Degrees Of Freedom
MIMO	Multiple-Input Multiple-Output
MISO	Multiple-Input Single-Output
SIMO	Single-Input Multiple-Output
SNR	Signal to Noise Ratio
SINR	Signal to Interference plus Noise Ratio
MG	Multiplexing Gain
PDF	Probability Density Function
AWGN	Additive White Gaussian Noise
TD	Time Division
FD	Frequency Division
ETW	Etkin, Tse, and Wang

Notation

Boldface Upper-Case Letters	Matrices
Boldface Lower-Case Letters	Vectors
A^t	Transpose of A
$\|A\|$	Determinant of the matrix A/ Cardinality of the set A
$tr\{A\}$	Trace of A
$A \succeq 0$	Matrix A is positive semi-definite
$U \preceq V$	$V - U$ is a positive semi-definite matrix
$\gamma(x)$	$0.5 \log_2(1 + x)$
I	The identity matrix
K	Number of users
E	The set $\{1, 2, \ldots, K\}$
2^E	The power set of the set E
$\mathbf{x}(S)$	$\sum_{i \in S} x_i$
$\mathbf{x}_S$	$[x_i]_{i \in S}$
$\mathbf{x}_{-i}$	$\mathbf{x}_{\overline{\{i\}}} = [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_K]$
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^n$	The n-dimensional Euclidean space
$\mathbb{Q}$	The set of rational numbers
$\mathbb{N}$	The set of nonnegative integers
$\lfloor x \rfloor$	The greatest integer less than x
$E[X]$	The expectation of the random variable X
(m, n)	The greatest common divisor of integers m and n
$U \cup V$	The union of two sets U and V
$U \cap V$	The intersection of two sets U and V
$U + V$	The Minkowski's sum of two sets U and V
$U \backslash V$	The difference of two sets U and V
$\overline{U}$	The complement of a set U
$(a, b)_{\mathbb{Z}}$	The set of integers between a and b.

Chapter 1

Introduction

Interference management plays a crucial role in future wireless systems as the number of users sharing the same spectrum is growing rapidly. In fact, an increase in the number of users results in an increase in the amount of interference in the system. This interference may cause a severe degradation in the system's performance.

The study of interaction between non-cooperative users sharing the same channel goes back to Shannon's work on the two-way channel in [1]. His work was followed by several researchers and the two-user interference channel emerged as a fundamental problem regarding interaction between users causing interference in the networks. In this channel, two senders transmit independent messages to their corresponding receivers via a common channel. The characterization of the channel's capacity region which reveals the acceptable rates in the system has been an open problem for more than 40 years.

There are some special cases where the exact capacity region has been characterized. These cases include the strong and very strong interference channels and a class of deterministic interference channels [2, 3, 4]. These examples revealed that the coding scheme achieving the capacity region of each case differs from other schemes and there is no universal coding applicable to all cases.

A limiting expression for the capacity region was obtained in [5] (see also [6]). Due to excessive computational complexity, this expression can not be used directly to fully characterize the capacity region. To show this, Cheng and Verdú proved that for the Gaussian Multiple Access Channel (MAC), which can be considered as a special case of the GIC,[1] the limiting expression fails to fully characterize the capacity region by relying only on

[1] In a special case of the Gaussian IC, the received signals at both receivers are statistically equivalent. The capacity region of this channel is equivalent to that of the Gaussian MAC observed from one of the receivers.

the Gaussian distributions [7]. There are, however, some special cases where the limiting expressions can be optimized. For example, the sum capacity of the Gaussian MAC can be achieved by relying on the simple scheme of Frequency/Time Division Multiple Access (FDMA/TDMA) [8].

The achievablity of rates in the limiting expression comes from simple encoding and decoding strategies. Each sender encodes data by using a random codebook, and each receiver decodes data by treating the interference as noise. In contrast, using more sophisticated encoders and decoders may result in collapsing the limiting expression into a single letter formula.

The idea of superposition coding originally developed by Cover in [9] was first applied to the IC by Carleial [10]. He used superposition coding to split data at the senders and successive decoding to decode data at the receivers. Incorporating joint typical decoding in the receivers, Han and Kobayashi (HK) proposed an achievable rate region which is still the best inner bound for the capacity region [11].

The HK scheme can be directly applied to the Gaussian IC. Nonetheless, there are two sources of difficulties in characterizing the full HK achievable rate region. First, the optimal distributions are unknown. Second, even if we confine the distributions to be Gaussian, computation of the full HK region under the Gaussian distribution is still difficult due to numerous degrees of freedom involved in the problem formulation. The main cause of this complexity is the cardinality of the time-sharing parameter. Recently, Chong *et al.* [12] presented a simpler expression with less inequalities for the HK achievable region. Although the new expression reduces the cardinality of the time-sharing parameter, it is still prohibitively complex to find the full HK achievable region.

Among all interference channels, the two-user GIC is the most applicable and important one. In fact, the capacity region of this channel has been open for several decades and still researchers are trying to close the gap between the inner bounds and outer bounds known for this channel. Although, the HK achievable scheme is still the best for this special case, many outer bounds are derived based on some characteristics of Gaussian distributions. Among numerous outer bounds, three of them are of special interest.

The first one obtained by Sato [13] was originally derived for the degraded Gaussian IC. Sato showed that the capacity region of the degraded Gaussian IC is outer bounded by a certain degraded broadcast channel whose capacity region is fully characterized. In [14], Costa proved that the capacity region of the degraded Gaussian broadcast channel is equivalent to that of the one-sided weak Gaussian IC. Hence, Sato outer bound can be used for the one-sided Gaussian IC as well.

The second outer bound obtained for the weak Gaussian IC is due to Kramer [15].

Kramer's outer bound is based on the fact that removing one of the interfering links enlarges the capacity region. Therefore, the capacity region of the two-user Gaussian IC is inside the intersection of the capacity regions of the underlying one-sided Gaussian ICs. For the case of weak Gaussian IC, the underlying one-sided IC is weak, for which the capacity region is unknown. However, Kramer used the outer bound obtained by Sato to derive an outer bound for the weak Gaussian IC.

The third outer bound due to Etkin, Tse, and Wang (ETW) is based on the Genie aided technique [16]. A genie that provides some extra information to the receivers can only enlarge the capacity region. The genie in the ETW scheme provides information about the intended signal to the receiver. They showed that the proposed outer bound outperforms other bounds over certain ranges of parameters. Moreover, using a similar method, they presented an outer bound for the mixed Gaussian IC. Using these new outer bounds and simple HK achievable scheme, they characterized the capacity region of this channel within 1 bit.

1.1 Interference Alignment

In contrary to the major research activities on the interference channels, c.f. [16, 17, 18, 19], the problem of characterizing the capacity region of Gaussian Interference Channels (GIC) is still open. As a major step, in [20], it is shown that in the two-user GIC, the Han-Kobayashi (HK) scheme [11] achieves within one bit of the capacity region, as long as the interference from the private message in the HK scheme is designed to be below the noise level.

The result of [16] has provided a clear understanding about the behavior of the two-user GIC. However, it turns out that moving from the two-user scenario to a larger number of users is a challenging task. Indeed, for K-user GIC ($K > 2$), the Han-Kobayashi approach of managing the interference is not enough and we need to incorporate a new approach of interference management known as *Interference Alignment*.

Interference Alignment is a solution for making the interference less severe at receivers by merging the communication dimensions occupied by the interfering signal. In [21], Maddah-Ali, Motahari, and Khandani introduced the concept of Interference Alignment and showed its capability in achieving the full Degrees-Of-Freedom (DOF) for certain classes of two-user X channels. Being simple and at the same time powerful, interference alignment provided the spur for further research. Interference alignment is not only usable for lowering the harmful effect of the interference, but it can also be applied to provide security in networks as proposed in [22].

Interference Alignment in n-dimensional Euclidean spaces for $n \geq 2$ is studied by several researchers, c.f. [21, 23, 24, 25]. In this method, at each receiver a subspace is dedicated to interference, then the signaling is designed such that all the interfering signals are squeezed in the interference sub-space. Such an approach saves some dimensions for communicating desired signals, rather than wasting it due to the interference. Using this method, Cadambe and Jafar showed that, contrary to the popular belief, a K-user Gaussian interference channel with varying channel gains can achieve its total DOF which is $\frac{K}{2}$. Later, in [26], it is shown that the same result can be achieved using a simple approach based on a particular pairing of the channel matrices. The assumption of varying channel gains, particularly noting that all the gains should be known at the transmitters' sides, is unrealistic which limits the application of these important theoretical results in practice.

In [27], followed by [28, 29], the application of Interference Alignment is extended from two or more spatial/temporal/frequency dimensions to one dimension, but at the signal level. In [27], it is shown that lattice codes, rather than random Gaussian codes, are essential parts of signaling for three-user time-invariant GICs. In [28], after aligning interference using lattice codes the aggregated signal is decoded and its effect is subtracted from the received signal. In fact, [28] shows that the very-strong interference region of the K-user GIC is strictly larger than the corresponding region when alignment is not applied. In their scheme, to make the interference less severe, transmitters use lattice codes to reduce the code-rate of the interference which guarantees decodability of the interference at the receiver. In [29], Sridharan et al. showed that the DOF of a class of 3-user GICs with fixed channel gains can be greater than one. This result was obtained using layered lattice codes along with successive decoding at the receiver.

In [30] and [31], the results from the field of Diophantine approximation in Number Theory are used to show that interference can be aligned using properties of rational and irrational numbers and their relations. They showed that the total DOF of some classes of time-invariant single antenna interference channels can be achieved. In particular, Etkin and Ordentlich in [30] proposed an upper bound on the total DOF which maintains the properties of channel gains with respect to being rational or irrational. Using this upper bound, surprisingly, they proved that the DOF is everywhere discontinuous for the class of channels under investigation.

1.2 Summary of Dissertation and Main Contributions

This dissertation is about optimal/suboptimal coding designs for non-cooperative networks in order to increase the system's throughput. Four problems are considered. First, the capacity region of the classic two-user Gaussian interference channel is studied. Second, interference alignment in one-dimensional spaces is investigated. Third, the DOF of the K-user GIC is considered. Fourth, random coding schemes for non-cooperative network is analyzed. A chapter is dedicated to each of these problems. In what follows, the main contributions of this dissertation is presented.

Chapter 2: Two-user Gaussian Interference Channel

Chapter two is dedicated to characterization of the capacity region of the two-user Gaussian IC. It also includes a comprehensive survey of known results for this channel. By introducing the notion of admissible ICs, a new outer bounding technique for the two-user Gaussian IC is proposed. The proposed technique relies on an extremal inequality recently proved by Liu and Viswanath [32]. It is shown that by using this scheme, one can obtain tighter outer bounds for both weak and mixed Gaussian ICs. More importantly, the sum capacity of the Gaussian weak IC for a certain range of the channel parameters is derived. The summary of results presented in this chapter is as follows.

- **Weak Gaussian IC**

  1. The sum capacity of this channel is derived for a certain range of parameters (it is called very weak interference regime). This is the first result obtained for the Weak Gaussian IC in more than 30 years.
  2. A new outer bound on the capacity region is obtained. It is proved that this bound is tighter than previously known outer bounds.
  3. It is proved that enlarging the simple HK achievable region using either FD/TD or the time-sharing parameter results in the same region. This fact considerably reduces the number of free parameters in the HK region.

- **One-sided Weak Gaussian IC**

  1. A new proof for Sato's outer bound on the capacity region is presented.
  2. It is shown that similar to the weak Gaussian IC, enlarging by FD/TD or time-sharing results in the same region. Hence, an explicit formula for the HK region is obtained in this case.

- **Mixed Gaussian IC**

  1. The sum capacity of the mixed Gaussian IC is derived for all ranges of channel parameters.

2. A new outer bound on the capacity region is obtained. This bound outperforms other existing bounds over all ranges of underlying parameters.
3. It is shown that FD/TD and time-sharing may result in different enlargement for the HK region. Moreover, an explicit expression for the simple HK achievable rate region is derived.

Chapter 3: Interference Alignment in One Dimension

Chapter 3 studies interference alignment in real line by using structural codes. Several important results are obtained by proposing a novel coding scheme. The main tool in proving these results comes from the filed of Diophantine approximation in number theory. The summary of the results presented in Chapter 3 is as follows.

- **A Novel Coding Scheme**

  1. The scheme converts a single antenna system to an equivalent MIMO system with fractional dimensions.
  2. Two types of constellation are proposed for the encoding part, namely the single layer constellation and the multi-layer constellation.

- **The Two-user X Channel**

  1. It is proved that the degrees-of-freedom $\frac{4}{3}$ is achievable almost surely. In other words, the set of channel parameters that this DOF may not be feasible has measure zero. This is an important result as this is a first example showing a time varying channel is not needed to achieve the total DOF of the system.
  2. It is shown that the DOF of the three-user GIC is greater than $\frac{4}{3}$ by using the proposed coding scheme.

- **The Symmetric Three-user Gaussian IC**

  1. It is proved that for all irrational channel gains, the total DOF of $\frac{3}{2}$ is achievable. This is a the first example of the fully connected three-user Gaussian IC where the total DOF is achieved without relying on the variations of the channel.
  2. For rational gains, a new coding strategy is proposed where signal points are selected from rational numbers represented to an appropriate base. By allowing carry over, it is proved that higher DOF is achievable and this DOF only is related to numerator or denominator of the channel gain.

Chapter 4: K-user Gaussian Interference Channel

Chapter 4 analyzes the total DOF of the K-user GIC. By extending the coding scheme proposed in Chapter 3, it is shown that interference alignment is possible simultaneously at several receivers. This fact relies a recent result in the field of Diophantine approximation which states that the convergence part of the Khintchine-Groshev theorem holds for points on non-degenerate manifolds. The summary of the results presented in Chapter 4 is as follows.

- **A Novel Coding Scheme**
  1. It is proved that a single antenna transceiver can behave as a multiple antenna node in high SNR regimes.
  2. It is shown that simultaneous interference alignment at several receivers is possible in single antenna systems.
- **The Total DOF of The K-user GIC**
  1. The DOF of the K-user GIC is derived for the case where channel is fixed over time/frequency.
  2. As a byproduct, the DOF of MIMO GIC is also derived.
- **Some Extensions**
  1. The total DOF of the $K \times M$ X channel is derived and it is proved that $\frac{KM}{K+M-1}$ is achievable almost surely.
  2. The total DOF of the uplink in a cellular system with M active users within each cell is derived. It is shown that the DOF per cell equals $\frac{M}{M+1}$ which means in a dense network all cells achieve one DOF and there is no need for frequency reuse in the system.

Chapter 5: Random Coding and Interference Management

Chapter 5 deals with communication networks where users invoke random codes to transmit their messages to the corresponding receivers. Receivers are allowed to decode the other users' messages to increase their own data rate. Using tools from Combinatorial Optimization, several algorithms are proposed to find decodable users and allocate appropriate rates to them. The summary of the results in Chapter 5 is as follows.

- **Polynomial Time Algorithms**

  1. An efficient algorithm is proposed by which a receiver can find the maximum decodable subset of active transmitters in a system.
  2. Focusing on a single user, an algorithm is proposed to allocate an achievable rate considering the fact that the receiver either decodes the interference or treats it as noise.
  3. An algorithm for successive rate allocation in a channel with K users is provided. Given some ordering on users, this algorithm allocated rates to users based on their priority.

Chapter 2

Two-user Gaussian Interference Channel

In this chapter, the capacity region of the two-user Gaussian Interference Channel (GIC) is studied. Three classes of channels are considered: weak, one-sided, and mixed GICs. For the weak GIC, a new outer bound on the capacity region is obtained that outperforms previously known outer bounds. The sum capacity for a certain range of channel parameters is derived. For this range, it is proved that using Gaussian codebooks and treating interference as noise are optimal. It is shown that when Gaussian codebooks are used, the full Han-Kobayashi achievable rate region can be obtained by using the naive Han-Kobayashi achievable scheme over three frequency bands (equivalently, three subspaces). For the one-sided GIC, an alternative proof for the Sato's outer bound is presented. We derive the full Han-Kobayashi achievable rate region when Gaussian codebooks are utilized. For the mixed GIC, a new outer bound is obtained that outperforms previously known outer bounds. For this case, the sum capacity for the entire range of channel parameters is derived. It is proved that the full Han-Kobayashi achievable rate region using Gaussian codebooks is equivalent to that of the one-sided GIC for a particular range of channel parameters.

This chapter is organized as follows. In Section 2.1, we present some basic definitions and review the HK achievable region when Gaussian codebooks are used. We study the time-sharing and the concavification methods as means to enlarge the basic HK achievable region. We investigate conditions for which the two regions obtained from time-sharing and concavification coincide. Finally, we consider an optimization problem based on an extremal inequality and compute its optimal solution.

In Section 2.2, the notion of an admissible IC is introduced. Some classes of admissible ICs for the two-user Gaussian case is studied and outer bounds on the capacity regions of these classes are computed. We also obtain the sum capacity of a specific class of admissible IC where it is shown that using Gaussian codebooks and treating interference as noise is optimal.

In Section 2.3, we study the capacity region of the weak GIC. We first derive the sum capacity of this channel for a certain range of parameters where it is proved that users should treat the interference as noise and transmit at their highest possible rates. We then derive an outer bound on the capacity region which outperforms the known results. We finally prove that the basic HK achievable region results in the same enlarged region by using either time-sharing or concavification. This reduces the complexity of the characterization of the full HK achievable region when Gaussian codebooks are used.

In Section 2.4, we study the capacity region of the one-sided GIC. We present a new proof for the Sato outer bound using the extremal inequality. Then, we present methods to simplify the HK achievable region such that the full region can be characterized.

In Section 2.5, we study the capacity region of the mixed GIC. We first obtain the sum capacity of this channel and then derive an outer bound which outperforms other known results. Finally, by investigating the HK achievable region for different cases, we prove that for a certain range of channel parameters, the full HK achievable rate region using Gaussian codebooks is equivalent to that of the one-sided IC. Finally, in Section 2.6, we conclude the chapter.

2.1 Preliminaries

2.1.1 The Two-user Interference Channel

Definition 1 (two-user IC)**.** *A two-user discrete memoryless IC consists of two finite sets $\mathscr{X}_1$ and $\mathscr{X}_2$ as input alphabets and two finite sets $\mathscr{Y}_1$ and $\mathscr{Y}_2$ as the corresponding output alphabets. The channel is governed by conditional probability distributions $\omega(y_1, y_2|x_1, x_2)$ where $(x_1, x_2) \in \mathscr{X}_1 \times \mathscr{X}_2$ and $(y_1, y_2) \in \mathscr{Y}_1 \times \mathscr{Y}_2$.*

Definition 2 (capacity region of the two-user IC)**.** *A code $(2^{nR_1}, 2^{nR_2}, n, \lambda_1^n, \lambda_2^n)$ for the two-user IC consists of the following components for User $i \in \{1, 2\}$:*

1) A uniform distributed message set $\mathcal{M}_i \in [1, 2, ..., 2^{nR_i}]$.

2) A codebook $\mathcal{X}_i = \{\boldsymbol{x}_i(1), \boldsymbol{x}_i(2), ..., \boldsymbol{x}_i(2^{nR_i})\}$ where $\boldsymbol{x}_i(\cdot) \in \mathscr{X}_i^n$.

3) An encoding function $F_i : [1, 2, ..., 2^{nR_i}] \to \mathcal{X}_i$.

4) A decoding function $G_i : \boldsymbol{y}_i \to [1, 2, ..., 2^{nR_i}]$.

5) The average probability of error $\lambda_i^n = \mathbb{P}(G_i(\boldsymbol{y}_i) \neq \mathcal{M}_i)$.

A rate pair (R_1, R_2) *is achievable if there is a sequence of codes* $(2^{nR_1}, 2^{nR_2}, n, \lambda_1^n, \lambda_2^n)$ *with vanishing average error probabilities. The capacity region of the IC is defined to be the closure of the set of achievable rates.*

Let $\mathscr{C}_{IC}$ denote the capacity region of the two-user IC. The limiting expression for $\mathscr{C}_{IC}$ can be stated as [6]

$$\mathscr{C}_{IC} = \lim_{n\to\infty} closure \left(\bigcup_{\mathbb{P}(\mathbf{X}_1^n)\mathbb{P}(\mathbf{X}_2^n)} \left\{ (R_1, R_2) \mid \begin{array}{l} R_1 \leq \frac{1}{n}\mathbf{I}\left(\mathbf{X}_1^n; \mathbf{Y}_1^n\right) \\ R_2 \leq \frac{1}{n}\mathbf{I}\left(\mathbf{X}_2^n; \mathbf{Y}_2^n\right) \end{array} \right\} \right). \tag{2.1}$$

In this chapter, we focus on the two-user GIC which can be represented in standard form as [10, 33]

$$\begin{aligned} y_1 &= x_1 + \sqrt{a}x_2 + z_1, \\ y_2 &= \sqrt{b}x_1 + x_2 + z_2, \end{aligned} \tag{2.2}$$

where x_i and y_i denote the input and output alphabets of User $i \in \{1, 2\}$, respectively, and $z_1 \sim \mathcal{N}(0, 1)$, $z_2 \sim \mathcal{N}(0, 1)$ are standard Gaussian random variables. Constants $a \geq 0$ and $b \geq 0$ represent the gains of the interference links. Furthermore, Transmitter i, $i \in \{1, 2\}$, is subject to the power constraint P_i. Achievable rates and the capacity region of the GIC can be defined in a similar fashion as that of the general IC with the condition that the codewords must satisfy their corresponding power constraints. The capacity region of the two-user GIC is denoted by $\mathscr{C}$. Clearly, $\mathscr{C}$ is a function of the parameters P_1, P_2, a, and b. To emphasize this relationship, we may write $\mathscr{C}$ as $\mathscr{C}(P_1, P_2, a, b)$ as needed.

Remark 1. *Since the capacity region of the general IC depends only on the marginal distributions [33], the ICs can be classified into equivalent classes in which channels within a class have the same capacity region. In particular, for the GIC given in (2.2), the choice of joint distributions for the pair* (z_1, z_2) *does not affect the capacity region as long as the marginal distributions remain Gaussian with zero mean and unit variance. Hence, without any loss of generality, the random variables* z_1 *and* z_2 *can be assumed to be un-correlated.*

Depending on the values of a and b, the two-user GIC is classified into weak, strong, mixed, one-sided, and degraded GIC. In Figure 2.1, regions in ab-plane together with their associated names are shown. Briefly, if $0 < a < 1$ and $0 < b < 1$, then the channel is called *weak GIC*. If $1 \leq a$ and $1 \leq b$, then the channel is called *strong GIC*. If either $a = 0$ or $b = 0$, the channel is called *one-sided GIC*. If $ab = 1$, then the channel is called *degraded GIC*. If either $0 < a < 1$ and $1 \leq b$, or $0 < b < 1$ and $1 \leq a$, then the channel is called *mixed GIC*. Finally, the *symmetric GIC* (used throughout the chapter for illustration purposes) corresponds to $a = b$ and $P_1 = P_2$.

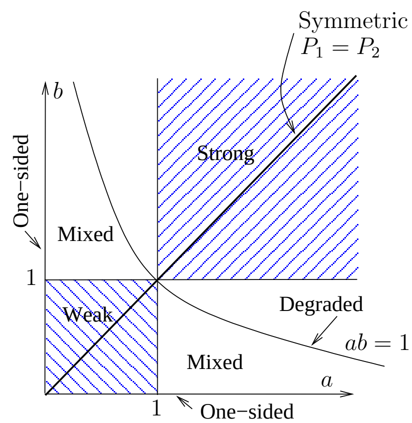

Figure 2.1: Classes of the two-user ICs.

Among all classes shown in Figure 2.1, the capacity region of the strong GIC is fully characterized [3, 2]. In this case, the capacity region can be stated as the collection of all rate pairs (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\leq \gamma(P_1), \\ R_2 &\leq \gamma(P_2), \\ R_1 + R_2 &\leq \min\left\{\gamma(P_1 + aP_2), \gamma(bP_1 + P_2)\right\}. \end{aligned}$$

2.1.2 Support Functions

Throughout this chapter, we use the following facts from convex analysis. There is a one to one correspondence between any closed convex set and its support function [34]. The support function of any set $D \subset \mathbb{R}^m$ is a function $\sigma_D : \mathbb{R}^m \to \mathbb{R}$ defined as

$$\sigma_D(\mathbf{c}) = \sup\{\mathbf{c}^t\mathbf{R}|\mathbf{R} \in D\}. \tag{2.3}$$

We observe that σ_D is a convex function, since it is the pointwise supremum of a family of linear functions. Clearly, if the set D is nonempty compact, then the sup is attained and can be replaced by max. In this case, the solutions of (2.3) correspond to the boundary points of D [34]. The following relation is the dual of (2.3) and holds when D is closed and convex

$$D = \{\mathbf{R}|\mathbf{c}^t\mathbf{R} \leq \sigma_D(\mathbf{c}), \forall\ \mathbf{c}\}. \tag{2.4}$$

From (2.3), it is easy to show that if $D \subseteq D'$ then $\sigma_D \leq \sigma_{D'}$. The converse also holds when D and D' are closed and convex. In fact, by using (2.4) one can easily prove that if $\sigma_D \leq \sigma_{D'}$ then $D \subseteq D'$.

2.1.3 Han-Kobayashi Achievable Region

The best inner bound for the two-user GIC is the full HK achievable region denoted by $\mathscr{C}_{HK}$ [11]. Despite having a single letter formula, $\mathscr{C}_{HK}$ is not fully characterized yet. In fact, finding the optimum distributions achieving boundary points of $\mathscr{C}_{HK}$ is still an open problem. We define $\mathscr{G}$ as a subset of $\mathscr{C}_{HK}$ where Gaussian distributions are used for codebook generation. Using a shorter description of $\mathscr{C}_{HK}$ obtained in [12], $\mathscr{G}$ can be described as follows.

Let us first define $\mathscr{G}_0$ as the collection of all rate pairs $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying

$$\begin{aligned}
R_1 &\leq \psi_1 = \gamma\left(\frac{P_1}{1+a\beta P_2}\right), && (2.5)\\
R_2 &\leq \psi_2 = \gamma\left(\frac{P_2}{1+b\alpha P_1}\right), && (2.6)\\
R_1+R_2 &\leq \psi_3 = \min\left\{\psi_{31}, \psi_{32}, \psi_{33}\right\}, && (2.7)\\
2R_1+R_2 &\leq \psi_4 = \gamma\left(\frac{P_1+a(1-\beta)P_2}{1+a\beta P_2}\right)+\gamma\left(\frac{\alpha P_1}{1+a\beta P_2}\right)+\gamma\left(\frac{\beta P_2+b(1-\alpha)P_1}{1+b\alpha P_1}\right), && (2.8)\\
R_1+2R_2 &\leq \psi_5 = \gamma\left(\frac{\beta P_2}{1+b\alpha P_1}\right)+\gamma\left(\frac{P_2+b(1-\alpha)P_1}{1+b\alpha P_1}\right)+\gamma\left(\frac{\alpha P_1+a(1-\beta)P_2}{1+a\beta P_2}\right), && (2.9)
\end{aligned}$$

for fixed $\alpha \in [0,1]$ and $\beta \in [0,1]$,[1] and

$$\begin{aligned}
\psi_{31} &= \gamma\left(\frac{P_1+a(1-\beta)P_2}{1+a\beta P_2}\right)+\gamma\left(\frac{\beta P_2}{1+b\alpha P_1}\right), && (2.10)\\
\psi_{32} &= \gamma\left(\frac{\alpha P_1}{1+a\beta P_2}\right)+\gamma\left(\frac{P_2+b(1-\alpha)P_1}{1+b\alpha P_1}\right), && (2.11)\\
\psi_{33} &= \gamma\left(\frac{\alpha P_1+a(1-\beta)P_2}{1+a\beta P_2}\right)+\gamma\left(\frac{\beta P_2+b(1-\alpha)P_1}{1+b\alpha P_1}\right). && (2.12)
\end{aligned}$$

The region $\mathscr{G}_0$ is a polytope and a function of four variables P_1, P_2, α, and β. To emphasize this relation, we may write $\mathscr{G}_0(P_1, P_2, \alpha, \beta)$ as needed. It is convenient to represent $\mathscr{G}_0$ in a matrix form as $\mathscr{G}_0 = \left\{\mathbf{R} \in \mathbb{R}_+^2 | A\mathbf{R} \leq \Psi(P_1, P_2, \alpha, \beta)\right\}$ where $\mathbf{R} = (R_1, R_2)^t$, $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)^t$, and

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}^t.$$

Equivalently, $\mathscr{G}_0$ can be represented as the convex hull of its extreme points, i.e., $\mathscr{G}_0(P_1, P_2, \alpha, \beta) = \text{conv}\left\{r_1, r_2, \ldots, r_K\right\}$, where it is assumed that $\mathscr{G}_0$ has K extreme points. It is easy to show that $K \leq 7$.

[1] In the HK scheme, two independent messages are encoded at each transmitter, namely the *common message* and the *private message*. α and β are the parameters that determine the amount of power allocated to the common and private messages for the two users, i.e., αP_1, βP_2 and $(1-\alpha)P_1$, $(1-\beta)P_2$ of the total power is used for the transmission of the private/common messages to the first/second users, respectively.

Now, $\mathscr{G}$ can be defined as a region obtained from enlarging $\mathscr{G}_0$ by making use of the time-sharing parameter. By incorporating the time sharing parameter into the achievable rate region, the feasible region of Ψ can be enlarged to its convex hull. In fact, $\mathscr{G}$ is the collection of all rate pairs $\mathbf{R} = (R_1, R_2)^t \in \mathbb{R}_+^2$ satisfying

$$A\mathbf{R} \leq \sum_{i=1}^{q} \lambda_i \Psi(P_{1i}, P_{2i}, \alpha_i, \beta_i), \tag{2.13}$$

where $q \in \mathbb{N}$ and

$$\sum_{i=1}^{q} \lambda_i P_{1i} \leq P_1, \tag{2.14}$$

$$\sum_{i=1}^{q} \lambda_i P_{2i} \leq P_2, \tag{2.15}$$

$$\sum_{i=1}^{q} \lambda_i = 1, \tag{2.16}$$

$$\lambda_i \geq 0,\ (\alpha_i, \beta_i) \in [0,1]^2;\ \forall i \in \{1, \ldots, q\}. \tag{2.17}$$

It is easy to show that $\mathscr{G}$ is a closed, bounded and convex region. In fact, the capacity region $\mathscr{C}$ which contains $\mathscr{G}$ is inside the rectangle defined by inequalities $R_1 \leq \gamma(P_1)$ and $R_2 \leq \gamma(P_2)$. Moreover, $(0,0)$, $(\gamma(P_1), 0)$, and $(0, \gamma(P_2))$ are extreme points of both $\mathscr{C}$ and $\mathscr{G}$. Hence, to characterize $\mathscr{G}$, we need to obtain all extreme points of $\mathscr{G}$ that are in the interior of the first quadrant (the same argument holds for $\mathscr{C}$). In other words, we need to obtain $\sigma_{\mathscr{G}}(c_1, c_2)$, the support function of $\mathscr{G}$, either when $1 \leq c_1$ and $c_2 = 1$ or when $c_1 = 1$ and $1 \leq c_2$.

We also define $\mathscr{G}_1$ and $\mathscr{G}_2$ obtained by enlarging $\mathscr{G}_0$ in two different manners. The region $\mathscr{G}_1$ is defined as

$$\mathscr{G}_1(P_1, P_2) = \bigcup_{(\alpha,\beta)\in[0,1]^2} \mathscr{G}_0(P_1, P_2, \alpha, \beta). \tag{2.18}$$

The region $\mathscr{G}_1$ is not necessarily a convex region. Hence, it can be further enlarged by the convex hull operation. The region $\mathscr{G}_2$ is defined as the collection of all rate pairs $\mathbf{R} = (R_1, R_2)^t \mathbb{R}_+^2$ satisfying

$$\mathbf{R} = \sum_{i=1}^{q'} \lambda_i \mathbf{R}_i \tag{2.19}$$

where $q' \in \mathbb{N}$ and

$$A\mathbf{R}_i \leq \Psi(P_{1i}, P_{2i}, \alpha_i, \beta_i), \tag{2.20}$$
$$\sum_{i=1}^{q'} \lambda_i P_{1i} \leq P_1, \tag{2.21}$$
$$\sum_{i=1}^{q'} \lambda_i P_{2i} \leq P_2, \tag{2.22}$$
$$\sum_{i=1}^{q'} \lambda_i = 1, \tag{2.23}$$
$$\lambda_i \geq 0,\ (\alpha_i, \beta_i) \in [0,1]^2;\ \forall i \in \{1, \ldots, q'\}. \tag{2.24}$$

It is easy to show that $\mathscr{G}_2$ is a closed, bounded and convex region. In fact, $\mathscr{G}_2$ is obtained by using the simple method of TD/FD. To see this, let us divide the available frequency band into q' sub-bands where λ_i is the fraction of the width of the i'th band from the total available frequency band and $\sum_{i=1}^{q'} \lambda_i = 1$. User 1 and 2 allocate P_{1i} and P_{2i} in the i'th sub-band, respectively. Therefore, all rate pairs in $\mathscr{G}_0(P_{1i}, P_{2i}, \alpha_i, \beta_i)$ are achievable in the i'th sub-band for fixed $(\alpha_i, \beta_i) \in [0,1]^2$. Hence, all rate pairs in $\sum_{i=1}^{q'} \lambda_i \mathscr{G}_0(P_{1i}, P_{2i}, \alpha_i, \beta_i)$ are achievable provided that $\sum_{i=1}^{q'} \lambda_i P_{1i} \leq P_1$ and $\sum_{i=1}^{q'} \lambda_i P_{2i} \leq P_2$.

Clearly, the chain of inclusions $\mathscr{G}_0 \subseteq \mathscr{G}_1 \subseteq \mathscr{G}_2 \subseteq \mathscr{G} \subseteq \mathscr{C}_{HK} \subseteq \mathscr{C}$ always holds.

2.1.4 Concavification Versus Time-Sharing

The goals of this subsection are two-folded. First, we aim at providing some necessary conditions such that $\mathscr{G}_2 = \mathscr{G}$. Second, we bound q and q' which are the number of parameters involved in the descriptions of $\mathscr{G}$ and $\mathscr{G}_2$, respectively. However, we derive the required conditions for the more general case where there are M users in the system. To this end, assume an achievable scheme for an M-user channel where the power constraint $\mathbf{P} = [P_1, P_2, \ldots, P_M]$ is given. The corresponding achievable region can be represented as

$$D_0(\mathbf{P}, \Theta) = \{\mathbf{R} | A\mathbf{R} \leq \Psi(\mathbf{P}, \Theta)\}, \tag{2.25}$$

where A is a $K \times M$ matrix and $\Theta \in [0,1]^M$. The region D_0 is a polyhedron in general, but for the purpose of this chapter, it suffices to assume that it is a polytope. Since D_0 is a convex region, the convex hull operation has no effect. However, it is possible to enlarge D_0 by using two different methods which are explained next. The first method is based on using the time-sharing parameter. Let us denote the corresponding region as D which can be written as

$$D = \left\{ \mathbf{R} | A\mathbf{R} \leq \sum_{i=1}^{q} \lambda_i \Psi(\mathbf{P}_i, \Theta_i), \sum_{i=1}^{q} \lambda_i \mathbf{P}_i \leq \mathbf{P}, \sum_{i=1}^{q} \lambda_i = 1, \lambda_i \geq 0, \Theta_i \in [0,1]^M \ \forall i \right\}, \quad (2.26)$$

where $q \in \mathbb{N}$.

In the second method, we use TD/FD to enlarge the achievable rate region. This results in an achievable region D_2 represented as

$$D_2 = \left\{ \mathbf{R} = \sum_{i=1}^{q'} \lambda_i \mathbf{R}_i | A\mathbf{R_i} \leq \Psi(\mathbf{P}_i, \Theta_i), \sum_{i=1}^{q'} \lambda_i \mathbf{P}_i \leq \mathbf{P}, \sum_{i=1}^{q'} \lambda_i = 1, \lambda_i \geq 0, \Theta_i \in [0,1]^M \ \forall i \right\}, \quad (2.27)$$

where $q' \in \mathbb{N}$. We refer to this method as concavification. It can be readily shown that D and D_2 are closed and convex, and $D_2 \subseteq D$. We are interested in situations where the inverse inclusion holds.

The support function of D_0 is a function of $\mathbf{P}$, Θ, and $\mathbf{c}$. Hence, we have

$$\sigma_{D_0}(\mathbf{c}, \mathbf{P}, \Theta) = \max\{\mathbf{c}^t \mathbf{R} | A\mathbf{R} \leq \Psi(\mathbf{P}, \Theta)\}. \quad (2.28)$$

For fixed $\mathbf{P}$ and Θ, (2.28) is a linear program. This problem is feasible because $\mathbf{R} = \mathbf{0}$ satisfies all constraints. Therefore, the strong duality in linear programming holds for this problem [34, Problem 5.23]. Hence, we obtain [34, page 225]

$$\sigma_{D_0}(\mathbf{c}, \mathbf{P}, \Theta) = \min\{\mathbf{y}^t \Psi(\mathbf{P}, \Theta) | A^t \mathbf{y} = \mathbf{c}, \mathbf{y} \geq 0\}. \quad (2.29)$$

In general, $\hat{\mathbf{y}}$, the minimizer of (2.29), is a function of $\mathbf{P}$, Θ, and $\mathbf{c}$. We say D_0 possesses *the unique minimizer property* if $\hat{\mathbf{y}}$ merely depends on $\mathbf{c}$, for all $\mathbf{c}$. In this case, we have

$$\sigma_{D_0}(\mathbf{c}, \mathbf{P}, \Theta) = \hat{\mathbf{y}}^t(\mathbf{c}) \Psi(\mathbf{P}, \Theta), \quad (2.30)$$

where $A^t \hat{\mathbf{y}} = \mathbf{c}$. This condition means that for any $\mathbf{c}$ the extreme point of D_0 maximizing the objective $\mathbf{c}^t\mathbf{R}$ is an extreme point obtained by intersecting a set of specific hyperplanes. A necessary condition for D_0 to possess the unique minimizer property is that each inequality in describing D_0 is either redundant or active for all $\mathbf{P}$ and Θ.

Theorem 1. *If D_0 possesses the unique minimizer property, then $D = D_2$.*

Proof. Since $D_2 \subseteq D$ always holds, we need to show $D \subseteq D_2$ which can be equivalently verified by showing $\sigma_D \leq \sigma_{D_2}$. The support function of D can be written as

$$\sigma_D(\mathbf{c}, \mathbf{P}) = \max\left\{\mathbf{c}^t \mathbf{R} | \mathbf{R} \in D\right\}. \quad (2.31)$$

By fixing $\mathbf{P}$, $\mathbf{P}_i$'s, Θ_i's, and λ_i's, the above maximization becomes a linear program. Hence, relying on weak duality of linear programming, we obtain

$$\sigma_D(\mathbf{c},\mathbf{P}) \leq \min_{A^t\mathbf{y}=\mathbf{c},\mathbf{y}\geq 0} \mathbf{y}^t \sum_{i=1}^{q} \lambda_i \Psi(\mathbf{P}_i,\Theta_i). \tag{2.32}$$

Clearly, $\hat{\mathbf{y}}(\mathbf{c})$, the solution of (2.29), is a feasible point for (2.32) and we have

$$\sigma_D(\mathbf{c},\mathbf{P}) \leq \hat{\mathbf{y}}^t(\mathbf{c}) \sum_{i=1}^{q} \lambda_i \Psi(\mathbf{P}_i,\Theta_i). \tag{2.33}$$

Using (2.30), we obtain

$$\sigma_D(\mathbf{c},\mathbf{P}) \leq \sum_{i=1}^{q} \lambda_i \sigma_{D_0}(\mathbf{c},\mathbf{P}_i,\Theta_i). \tag{2.34}$$

Let us assume $\hat{\mathbf{R}}_i$ is the maximizer of (2.28). In this case, we have

$$\sigma_D(\mathbf{c},\mathbf{P}) \leq \sum_{i=1}^{q} \lambda_i \mathbf{c}^t \hat{\mathbf{R}}_i. \tag{2.35}$$

Hence, we have

$$\sigma_D(\mathbf{c},\mathbf{P}) \leq \mathbf{c}^t \sum_{i=1}^{q} \lambda_i \hat{\mathbf{R}}_i. \tag{2.36}$$

By definition, $\sum_{i=1}^{q} \lambda_i \hat{\mathbf{R}}_i$ is a point in D_2. Therefore, we conclude

$$\sigma_D(\mathbf{c},\mathbf{P}) \leq \sigma_{D_2}(\mathbf{c},\mathbf{P}). \tag{2.37}$$

This completes the proof. □

Corollary 1 (Han [35])**.** *If D_0 is a polymatroid, then D=D_2.*

Proof. It is easy to show that D_0 possesses the unique minimizer property. In fact, for given $\mathbf{c}$, $\hat{\mathbf{y}}$ can be obtained in a greedy fashion independent of $\mathbf{P}$ and Θ. □

In what follows, we upper bound q and q'.

Theorem 2. *The cardinality of the time-sharing parameter q in (2.26) is less than M+K+1, where M and K are the dimensions of $\mathbf{P}$ and $\Psi(\mathbf{P})$, respectively. Moreover, if $\Psi(\mathbf{P})$ is a continuous function of $\mathbf{P}$, then $q \leq M + K$.*

Proof. Let us define E as

$$E = \left\{ \sum_{i=1}^{q} \lambda_i \Psi(\mathbf{P}_i, \Theta_i) | \sum_{i=1}^{q} \lambda_i \mathbf{P}_i \leq \mathbf{P}, \sum_{i=1}^{q} \lambda_i = 1, \lambda_i \geq 0, \Theta_i \in [0,1]^M \ \forall i \right\}. \tag{2.38}$$

In fact, E is the collection of all possible bounds for D. To prove $q \leq M + K + 1$, we define another region E_1 as

$$E_1 = \{(\mathbf{P}', \mathbf{S}') | 0 \leq \mathbf{P}', \mathbf{S}' = \Psi(\mathbf{P}', \Theta'), \Theta' \in [0,1]^M\}. \tag{2.39}$$

As a direct consequence of Caratheodory's theorem [36], the convex hull of E_1 denoted by conv E_1 can be obtained by convex combinations of no more than $M + K + 1$ points in E_1. Moreover, if $\Psi(\mathbf{P}', \Theta')$ is continuous, then $M + K$ points are sufficient due to the extension of Caratheodory's theorem [36]. Now, we define the region $\hat{E}$ as

$$\hat{E} = \{\mathbf{S}' | (\mathbf{P}', \mathbf{S}') \in \text{conv } E_1, \mathbf{P}' \leq \mathbf{P}\}. \tag{2.40}$$

Clearly, $\hat{E} \subseteq E$. To show the other inclusion, let us consider a point in E, say $S = \sum_{i=1}^{q} \lambda_i \Psi(\mathbf{P}_i, \Theta_i)$. Since $(\mathbf{P}_i, \Psi(\mathbf{P}_i, \Theta_i))$ is a point in E_1, $\sum_{i=1}^{q} \lambda_i (\mathbf{P}_i, \Psi(\mathbf{P}_i, \Theta_i))$ belongs to conv E_1. Having $\sum_{i=1}^{q} \lambda_i \mathbf{P}_i \leq \mathbf{P}$, we conclude $\sum_{i=1}^{q} \lambda_i \Psi(\mathbf{P}_i, \Theta) \in \hat{E}$. Hence, $E \subseteq \hat{E}$. This completes the proof. □

Corollary 2 (Etkin, Parakh, and Tse [37])**.** *For the M-user GIC where users use Gaussian codebooks for data transmission and treat the interference as noise, the cardinality of the time-sharing parameter is less than $2M$.*

Proof. In this case, $D_0 = \{\mathbf{R} | \mathbf{R} \leq \Psi(\mathbf{P})\}$ where both $\mathbf{P}$ and $\Psi(\mathbf{P})$ have dimension M and $\Psi(\mathbf{P})$ is a continuous function of $\mathbf{P}$. Applying Theorem 2 yields the desired result. □

In the following theorem, we obtain an upper bound on q'.

Theorem 3. *To characterize boundary points of D_2, it suffices to set $q' \leq M + 1$.*

Proof. Let us assume $\hat{\mathbf{R}}$ is a boundary point of D_2. Hence, there exists $\mathbf{c}$ such that

$$\sigma_{D_2}(\mathbf{c}, \mathbf{P}) = \max_{\mathbf{R} \in D_2} \mathbf{c}^t \mathbf{R} = \mathbf{c}^t \hat{\mathbf{R}}, \tag{2.41}$$

where $\hat{\mathbf{R}} = \sum_{i=1}^{q'} \hat{\lambda}_i \hat{\mathbf{R}}_i$ and the optimum is achieved for the set of parameters $\hat{\Theta}_i$, $\hat{\lambda}_i$, and $\hat{\mathbf{P}}_i$. The optimization problem in (2.41) can be written as

$$\begin{aligned} \sigma_{D_2}(\mathbf{c}, \mathbf{P}) = &\max \quad \sum_{i=1}^{q'} \lambda_i g(\mathbf{c}, \mathbf{P}_i) \\ &\text{subject to: } \sum_{i=1}^{q'} \lambda_i = 1, \ \sum_{i=1}^{q'} \lambda_i \mathbf{P}_i \leq \mathbf{P}, \\ &\qquad 0 \leq \lambda_i, 0 \leq \mathbf{P}_i, \ \forall i \in \{1, 2, \ldots, q'\}, \end{aligned} \tag{2.42}$$

where $g(\mathbf{c},\mathbf{P})$ is defined as

$$g(\mathbf{c},\mathbf{P}) = \max\ \mathbf{c}^t\mathbf{R} \qquad (2.43)$$
$$\text{subject to: } A\mathbf{R} \leq \Psi(\mathbf{P},\Theta),\ 0 \leq \Theta \leq 1.$$

In fact, $\sigma_{D_2}(\mathbf{c},\mathbf{P})$ in (2.42) can be viewed as the result of the concavification of $g(\mathbf{c},\mathbf{P})$ [36]. Hence, using Theorem 2.16 in [36], we conclude that $q' \leq M+1$. □

Remarkable point about Theorem 3 is that the upper bound on q' is independent of the number of inequalities involved in the description of the achievable rate region.

Corollary 3. *For the M-user GIC where users use Gaussian codebooks and treat the interference as noise, we have $D_2 = D$ and $q = q' = M+1$.*

Proof. As of Corollary 2, $D_0 = \{\mathbf{R}|\mathbf{R} \leq \Psi(\mathbf{P})\}$ where both $\mathbf{P}$ and $\Psi(\mathbf{P})$ have dimension M. It is easy to show that D_0 possesses the unique minimizer property. Hence, $D = D_2$. Applying Theorem 3 yields the desired result. □

2.1.5 Extremal Inequality

In [32], the following optimization problem is studied:

$$W = \max_{Q_{\mathbf{X}} \preceq S} h(\mathbf{X}+\mathbf{Z}_1) - \mu h(\mathbf{X}+\mathbf{Z}_2), \qquad (2.44)$$

where $h(\mathbf{Y})$ is the differential entropy of $\mathbf{Y}$. $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are n-dimensional Gaussian random vectors with the strictly positive definite covariance matrices $Q_{\mathbf{Z}_1}$ and $Q_{\mathbf{Z}_2}$, respectively. The optimization is over all random vectors $\mathbf{X}$ independent of $\mathbf{Z}_1$ and $\mathbf{Z}_2$. $\mathbf{X}$ is also subject to the covariance matrix constraint $Q_{\mathbf{X}} \preceq S$, where S is a positive definite matrix. In [32], it is shown that for all $\mu \geq 1$, this optimization problem has a Gaussian optimal solution for all positive definite matrices $Q_{\mathbf{Z}_1}$ and $Q_{\mathbf{Z}_2}$. However, for $0 \leq \mu < 1$ this optimization problem has a Gaussian optimal solution provided $Q_{\mathbf{Z}_1} \preceq Q_{\mathbf{Z}_2}$, i.e., $Q_{\mathbf{Z}_2} - Q_{\mathbf{Z}_1}$ is a positive semi-definite matrix. It is worth noting that for $\mu = 1$ this problem when $Q_{\mathbf{Z}_1} \preceq Q_{\mathbf{Z}_2}$ is studied under the name of the worse additive noise [38, 39].

In this chapter, we consider a special case of (2.44) where $\mathbf{Z}_1$ and $\mathbf{Z}_2$ have the covariance matrices $N_1 I$ and $N_2 I$, respectively, and the trace constraint is considered, i.e.,

$$W = \max_{tr\{Q_{\mathbf{X}}\} \leq nP} h(\mathbf{X}+\mathbf{Z}_1) - \mu h(\mathbf{X}+\mathbf{Z}_2). \qquad (2.45)$$

In the following lemma, we provide the optimal solution for the above optimization problem when $N_1 \leq N_2$.

Lemma 1. *If $N_1 \le N_2$, the optimal solution of (2.45) is i.i.d. Gaussian for all $0 \le \mu$ and we have*

1. *For $0 \le \mu \le \frac{N_2+P}{N_1+P}$, the optimum covariance matrix is PI and the optimum value is*
$$W = \frac{n}{2}\log\left[(2\pi e)(P+N_1)\right] - \frac{\mu n}{2}\log\left[(2\pi e)(P+N_2)\right]. \tag{2.46}$$

2. *For $\frac{N_2+P}{N_1+P} < \mu \le \frac{N_2}{N_1}$, the optimum covariance matrix is $\frac{N_2-\mu N_1}{\mu-1}I$ and the optimum value is*
$$W = \frac{n}{2}\log\left[(2\pi e)\frac{N_2-N_1}{\mu-1}\right] - \frac{\mu n}{2}\log\left[\frac{\mu(2\pi e)(N_2-N_1)}{\mu-1}\right]. \tag{2.47}$$

3. *For $\frac{N_2}{N_1} < \mu$, the optimum covariance matrix is 0 and the optimum value is*
$$W = \frac{n}{2}\log(2\pi e N_1) - \frac{\mu n}{2}\log(2\pi e N_2). \tag{2.48}$$

Proof. From the general result for (2.44), we know that the optimum input distribution is Gaussian. Hence, we need to solve the following maximization problem:
$$\begin{aligned} W =&\max \frac{1}{2}\log\left((2\pi e)^n|Q_{\mathbf{X}}+N_1I|\right) - \frac{\mu}{2}\log\left((2\pi e)^n|Q_{\mathbf{X}}+N_2I|\right) \\ &\text{subject to: } 0 \preceq Q_{\mathbf{X}},\ tr\{Q_{\mathbf{X}}\} \le nP. \end{aligned} \tag{2.49}$$

Since $Q_{\mathbf{X}}$ is a positive semi-definite matrix, it can be decomposed as $Q_{\mathbf{X}} = U\Lambda U^t$, where Λ is a diagonal matrix with nonnegative entries and U is a unitary matrix, i.e., $UU^t = I$. Substituting $Q_{\mathbf{X}} = U\Lambda U^t$ in (2.49) and using the identities $tr\{AB\} = tr\{BA\}$ and $|AB+I| = |BA+I|$, we obtain
$$\begin{aligned} W =&\max \frac{1}{2}\log\left((2\pi e)^n|\Lambda+N_1I|\right) - \frac{\mu}{2}\log\left((2\pi e)^n|\Lambda+N_2I|\right) \\ &\text{subject to: } 0 \preceq \Lambda,\ tr\{\Lambda\} \le nP. \end{aligned} \tag{2.50}$$

This optimization problem can be simplified as
$$W =\max \frac{n}{2}\sum_{i=1}^{n}\left[\log(2\pi e)(\lambda_i+N_1) - \mu\log(2\pi e)(\lambda_i+N_2)\right] \tag{2.51}$$
$$\text{subject to: } 0 \le \lambda_i\ \forall i,\ \sum_{i=1}^{n}\lambda_i \le nP.$$

By introducing Lagrange multipliers ψ and $\Phi = \{\phi_1, \phi_2, \dots, \phi_n\}$, we obtain
$$\begin{aligned} L(\Lambda,\psi,\Phi) = \max&\frac{n}{2}\sum_{i=1}^{n}\left[\log(2\pi e)(\lambda_i+N_1) - \mu\log(2\pi e)(\lambda_i+N_2)\right] \\ &+ \psi\left(nP - \sum_{i=1}^{n}\lambda_i\right) + \sum_{i=1}^{n}\phi_i\lambda_i. \end{aligned} \tag{2.52}$$

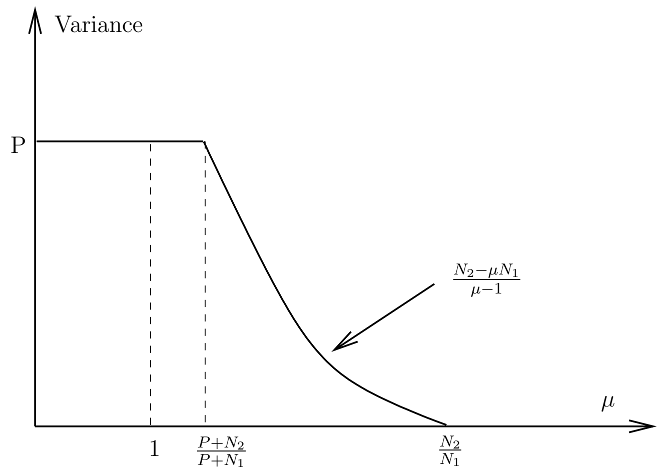

Figure 2.2: Optimum variance versus μ.

The first order KKT necessary conditions for the optimum solution of (2.52) can be written as

$$\frac{1}{\lambda_i+N_1}-\frac{\mu}{\lambda_i+N_2}-\psi+\phi_i=0,\ \forall i\in\{1,2,\ldots,n\}, \tag{2.53}$$

$$\psi\left(nP-\sum_{i=1}^{n}\lambda_i\right)=0, \tag{2.54}$$

$$\phi_i\lambda_i=0,\ \forall i\in\{1,2,\ldots,n\}. \tag{2.55}$$

It is easy to show that when $N_1\le N_2$, $\lambda=\lambda_1=\ldots=\lambda_n$ and the only solution for λ is

$$\lambda=\begin{cases} P, & \text{if} \quad 0 \le \mu \le \frac{N_2+P}{N_1+P} \\ \frac{N_2-\mu N_1}{\mu-1}, & \text{if} \quad \frac{N_2+P}{N_1+P} < \mu \le \frac{N_2}{N_1} \\ 0, & \text{if} \quad \frac{N_2}{N_1} < \mu \end{cases} \tag{2.56}$$

Substituting λ into the objective function yields the desired result. $\square$

In Figure 2.2, the optimum variance as a function of μ is plotted. This figure shows that for any value of $\mu\le\frac{P+N_2}{P+N_1}$, we need to use the maximum power to optimize the objective function, whereas for $\mu>\frac{P+N_2}{P+N_1}$, we use less power than what is permissible.

Lemma 2. *If $N_1>N_2$, the optimal solution of (2.45) is i.i.d. Gaussian for all $1\le\mu$. In this case, the optimum variance is 0 and the optimum W is*

$$W=\frac{n}{2}\log(2\pi eN_1)-\frac{\mu n}{2}\log(2\pi eN_2). \tag{2.57}$$

Proof. The proof is similar to that of Lemma 1 and we omit it here. $\square$

Corollary 4. *For $\mu = 1$, the optimal solution of (2.45) is i.i.d. Gaussian and the optimum W is*

$$W = \begin{cases} \frac{n}{2}\log\left(\frac{P+N_1}{P+N_2}\right), & \text{if } N_1 \leq N_2 \\ \frac{n}{2}\log\left(\frac{N_1}{N_2}\right), & \text{if } N_1 > N_2. \end{cases} \tag{2.58}$$

We frequently apply the following optimization problem in the rest of the chapter:

$$f_h(P, N_1, N_2, a, \mu) = \max_{tr\{Q_{\mathbf{X}}\} \leq nP} h(\mathbf{X} + \mathbf{Z}_1) - \mu h(\sqrt{a}\mathbf{X} + \mathbf{Z}_2), \tag{2.59}$$

where $N_1 \leq N_2/a$. Using the identity $h(A\mathbf{X}) = \log(|A|) + h(\mathbf{X})$, (2.59) can be written as

$$f_h(P, N_1, N_2, a, \mu) = \frac{\mu n}{2}\log a + \max_{tr\{Q_{\mathbf{X}}\} \leq nP} h(\mathbf{X} + \mathbf{Z}_1) - \mu h(\mathbf{X} + \frac{\mathbf{Z}_2}{\sqrt{a}}). \tag{2.60}$$

Now, Lemma 1 can be applied to obtain

$$f_h(P, N_1, N_2, a, \mu) = \begin{cases} \frac{1}{2}\log\left[(2\pi e)(P+N_1)\right] - \frac{\mu}{2}\log\left[(2\pi e)(aP+N_2)\right] & \text{if } 0 \leq \mu \leq \frac{P+N_2/a}{P+N_1} \\ \frac{1}{2}\log\left[(2\pi e)\frac{N_2/a-N_1}{\mu-1}\right] - \frac{\mu}{2}\log\left[\frac{a\mu(2\pi e)(N_2/a-N_1)}{\mu-1}\right] & \text{if } \frac{P+N_2/a}{P+N_1} < \mu \leq \frac{N_2}{aN_1} \\ \frac{1}{2}\log(2\pi e N_1) - \frac{\mu}{2}\log(2\pi e N_2) & \text{if } \frac{N_2}{aN_1} < \mu \end{cases} \tag{2.61}$$

2.2 Admissible Channels

In this section, we aim at building ICs whose capacity regions contain the capacity region of the two-user GIC, i.e., $\mathscr{C}$. Since we ultimately use these to outer bound $\mathscr{C}$, these ICs need to have a tractable expression (or a tractable outer bound) for their capacity regions.

Let us consider an IC with the same input letters as that of $\mathscr{C}$ and the output letters $\tilde{y}_1$ and $\tilde{y}_2$ for Users 1 and 2, respectively. The capacity region of this channel, say $\mathscr{C}'$, contains $\mathscr{C}$ if

$$I(x_1^n; y_1^n) \leq I(x_1^n; \tilde{y}_1^n), \tag{2.62}$$

$$I(x_2^n; y_2^n) \leq I(x_2^n; \tilde{y}_2^n), \tag{2.63}$$

for all $p(x_1^n)p(x_2^n)$ and for all $n \in \mathbb{N}$.

One way to satisfy (2.62) and (2.63) is to provide some extra information to either one or to both receivers. This technique is known as *Genie aided outer bounding*. In [15], Kramer has used such a genie to provide some extra information to both receivers such that they can decode both users' messages. Since the capacity region of this new interference channel is equivalent to that of the *Compound Multiple Access Channel* whose capacity region is known,

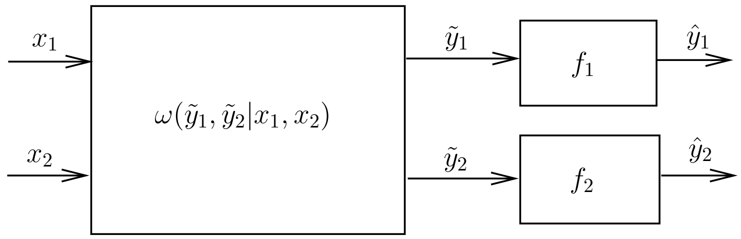

Figure 2.3: An admissible channel. f_1 and f_2 are deterministic functions.

reference [15] obtains an outer bound on the capacity region. To obtain a tighter outer bound, reference [15] further uses the fact that if a genie provides the exact information about the interfering signal to one of the receivers, then the new channel becomes the one-sided GIC. Although the capacity region of the one-sided GIC is unknown for all ranges of parameters, there exists an outer bound for it due to Sato and Costa [40, 14] that can be applied to the original channel. In [20], Etkin *et al.* use a different genie that provides some extra information about the intended signal. Even though at first glance their proposed method appears to be far from achieving a tight bound, they have shown that the corresponding bound is tighter than the one due to Kramer for certain ranges of parameters.

Next, we introduce the notion of admissible channels to satisfy (2.62) and (2.63).

Definition 3 (Admissible Channel)**.** *An IC $\mathscr{C}'$ with input letter x_i and output letter $\tilde{y}_i$ for User $i \in \{1, 2\}$ is an admissible channel if there exist two deterministic functions $\hat{y}_1^n = f_1(\tilde{y}_1^n)$ and $\hat{y}_2^n = f_2(\tilde{y}_2^n)$ such that*

$$I(x_1^n; y_1^n) \leq I(x_1^n; \hat{y}_1^n), \tag{2.64}$$

$$I(x_2^n; y_2^n) \leq I(x_2^n; \hat{y}_2^n) \tag{2.65}$$

hold for all $p(x_1^n)p(x_2^n)$ and for all $n \in \mathbb{N}$. $\mathscr{E}$ denotes the collection of all admissible channels (see Figure 2.3).

Due to the data processing inequality, (2.64) and (2.65) imply (2.62) and (2.63), respectively. Hence, the capacity region of an admissible channel is an outer bound to the original IC.

Remark 2. *Genie aided channels are among admissible channels. To see this, let us assume a genie provides s_1 and s_2 as side information for User 1 and 2, respectively. In this case, $\tilde{y}_i = (y_i, s_i)$ for $i \in \{1, 2\}$. By choosing $f_i(y_i, s_i) = y_i$, we observe that $\hat{y}_i = y_i$, and hence, (2.64) and (2.65) trivially hold.*

To obtain the tightest outer bound, we need to find the intersection of the capacity regions of all admissible channels. Nonetheless, it may happen that finding the capacity region of an admissible channel is as hard as that of the original one (in fact, based on the definition, the channel itself is one of its admissible channels). Hence, we need to find classes of admissible channels, say $\mathscr{F}$, which possess two important properties. First, their capacity regions are close to $\mathscr{C}$. Second, either their exact capacity regions are computable or there exist good outer bounds for them. Since $\mathscr{F} \subseteq \mathscr{E}$, we have

$$\mathscr{C} \subseteq \bigcap_{\mathscr{F}} \mathscr{C}'. \tag{2.66}$$

Recall that there is a one to one correspondence between a closed convex set and its support function. Since $\mathscr{C}$ is closed and convex, there is a one to one correspondence between $\mathscr{C}$ and $\sigma_{\mathscr{C}}$. In fact, boundary points of $\mathscr{C}$ correspond to the solutions of the following optimization problem

$$\sigma_{\mathscr{C}}(c_1, c_2) = \max_{(R_1,R_2)\in\mathscr{C}} c_1R_1 + c_2R_2. \tag{2.67}$$

Since we are interested in the boundary points excluding the R_1 and R_2 axes, it suffices to consider $0 \leq c_1$ and $0 \leq c_2$ where $c_1 + c_2 = 1$.

Since $\mathscr{C} \subseteq \mathscr{C}'$, we have

$$\sigma_{\mathscr{C}}(c_1, c_2) \leq \sigma_{\mathscr{C}'}(c_1, c_2). \tag{2.68}$$

Taking the minimum of the right hand side, we obtain

$$\sigma_{\mathscr{C}}(c_1, c_2) \leq \min_{\mathscr{C}'\in\mathscr{F}} \sigma_{\mathscr{C}'}(c_1, c_2), \tag{2.69}$$

which can be written as

$$\sigma_{\mathscr{C}}(c_1, c_2) \leq \min_{\mathscr{C}'\in\mathscr{F}} \max_{(R_1,R_2)\in\mathscr{C}'} c_1R_1 + c_2R_2. \tag{2.70}$$

For convenience, we use the following two optimization problems

$$\sigma_{\mathscr{C}}(\mu, 1) = \max_{(R_1,R_2)\in\mathscr{C}} \mu R_1 + R_2, \tag{2.71}$$

$$\sigma_{\mathscr{C}}(1, \mu) = \max_{(R_1,R_2)\in\mathscr{C}} R_1 + \mu R_2, \tag{2.72}$$

where $1 \leq \mu$. It is easy to show that the solutions of (2.71) and (2.72) correspond to the boundary points of the capacity region.

In the rest of this section, we introduce classes of admissible channels and obtain upper bounds on $\sigma_{\mathscr{C}'}(\mu, 1)$ and $\sigma_{\mathscr{C}'}(1, \mu)$.

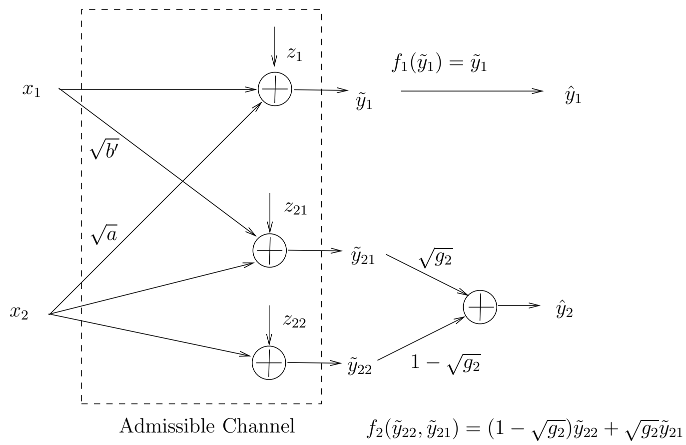

Figure 2.4: Class A1 admissible channels.

2.2.1 Classes of Admissible Channels

Class A1

This class is designed to obtain an upper bound on $\sigma_{\mathscr{C}}(\mu, 1)$. Therefore, we need to find a tight upper bound on $\sigma_{\mathscr{C}'}(\mu, 1)$. A member of this class is a channel in which User 1 has one transmit and one receive antenna whereas User 2 has one transmit antenna and two receive antennas (see Figure 2.4). The channel model can be written as

$$\begin{aligned} \tilde{y}_1 &= \quad x_1 + \sqrt{a}x_2 + z_1, \\ \tilde{y}_{21} &= \quad x_2 + \sqrt{b'}x_1 + z_{21}, \\ \tilde{y}_{22} &= \quad x_2 + z_{22}, \end{aligned} \tag{2.73}$$

where $\tilde{y}_1$ is the signal at the first receiver, $\tilde{y}_{21}$ and $\tilde{y}_{22}$ are the signals at the second receiver, z_1 is additive Gaussian noise with unit variance, z_{21} and z_{22} are additive Gaussian noise with variances N_{21} and N_{22}, respectively. Transmitters 1 and 2 are subject to the power constraints of P_1 and P_2, respectively.

To investigate admissibility conditions in (2.64) and (2.65), we introduce two deterministic functions f_1 and f_2 as follows (see Figure 2.4)

$$f_1(\tilde{y}_1^n) = \tilde{y}_1^n, \tag{2.74}$$

$$f_2(\tilde{y}_{22}^n, \tilde{y}_{21}^n) = (1-\sqrt{g_2})\tilde{y}_{22}^n + \sqrt{g_2}\tilde{y}_{21}^n, \tag{2.75}$$

where $0 \leq g_2$. For $g_2 = 0$, the channel can be converted to the one-sided GIC by letting $N_{21} \to \infty$ and $N_{22} = 1$. Hence, Class A1 contains the one-sided GIC obtained by removing the link between Transmitter 1 and Receiver 2. Using f_1 and f_2, we obtain

$$\hat{y}_1^n = x_1^n + \sqrt{a} x_2^n + z_1^n, \tag{2.76}$$

$$\hat{y}_2^n = \sqrt{b' g_2} x_1^n + x_2^n + (1 - \sqrt{g_2}) z_{22}^n + \sqrt{g_2} z_{21}^n. \tag{2.77}$$

Hence, this channel is admissible if the corresponding parameters satisfy

$$\begin{aligned} b' g_2 &= b, \\ (1 - \sqrt{g_2})^2 N_{22} + g_2 N_{21} &= 1. \end{aligned} \tag{2.78}$$

We further add the following constraints to the conditions of the channels in Class A1:

$$\begin{aligned} b' &\leq N_{21}, \\ a N_{22} &\leq 1. \end{aligned} \tag{2.79}$$

Although these additional conditions reduce the number of admissible channels within the class, they are needed to get a closed form formula for an upper bound on $\sigma_{\mathscr{C}'}(\mu, 1)$. In the following lemma, we obtain the required upper bound.

Lemma 3. *For the channels modeled by (2.73) and satisfying (2.79), we have*

$$\begin{aligned} \sigma_{\mathscr{C}'}(\mu, 1) \leq & \min \frac{\mu_1}{2} \log\left[2\pi e (P_1 + a P_2 + 1)\right] + \frac{1}{2} \log\left(\frac{N_{21}}{N_{22}} + \frac{b' P_1}{N_{22}} + \frac{P_2}{P_2 + N_{22}}\right) \\ & - \frac{\mu_2}{2} \log(2\pi e) + \mu_2 f_h\left(P_1, 1, N_{21}, b', \frac{1}{\mu_2}\right) + f_h(P_2, N_{22}, 1, a, \mu_1) \\ & \textit{subject to: } \mu_1 + \mu_2 = \mu, \ \mu_1, \mu_2 \geq 0. \end{aligned} \tag{2.80}$$

Proof. Let us assume R_1 and R_2 are achievable rates for User 1 and 2, respectively. Furthermore, we split μ into $\mu_1 \geq 0$ and $\mu_2 \geq 0$ such that $\mu = \mu_1 + \mu_2$. Using Fano's inequality, we obtain

$$\begin{aligned} n(\mu R_1 + R_2) \leq & \mu I(x_1^n; \tilde{y}_1^n) + I(x_2^n; \tilde{y}_{22}^n, \tilde{y}_{21}^n) + n\epsilon_n \\ = & \mu_1 I(x_1^n; \tilde{y}_1^n) + \mu_2 I(x_1^n; \tilde{y}_1^n) + I(x_2^n; \tilde{y}_{22}^n, \tilde{y}_{21}^n) + n\epsilon_n \\ \overset{(a)}{\leq} & \mu_1 I(x_1^n; \tilde{y}_1^n) + \mu_2 I(x_1^n; \tilde{y}_1^n | x_2^n) + I(x_2^n; \tilde{y}_{22}^n, \tilde{y}_{21}^n) + n\epsilon_n \\ = & \mu_1 I(x_1^n; \tilde{y}_1^n) + \mu_2 I(x_1^n; \tilde{y}_1^n | x_2^n) + I(x_2^n; \tilde{y}_{21}^n | \tilde{y}_{22}^n) + I(x_2^n; \tilde{y}_{22}^n) + n\epsilon_n \\ = & \mu_1 h(\tilde{y}_1^n) - \mu_1 h(\tilde{y}_1^n | x_1^n) + \mu_2 h(\tilde{y}_1^n | x_2^n) - \mu_2 h(\tilde{y}_1^n | x_1^n, x_2^n) \\ & + h(\tilde{y}_{21}^n | \tilde{y}_{22}^n) - h(\tilde{y}_{21}^n | x_2^n, \tilde{y}_{22}^n) + h(\tilde{y}_{22}^n) - h(\tilde{y}_{22}^n | x_2^n) + n\epsilon_n \\ = & \left[\mu_1 h(\tilde{y}_1^n) - \mu_2 h(\tilde{y}_1^n | x_1^n, x_2^n)\right] + \left[\mu_2 h(\tilde{y}_1^n | x_2^n) - h(\tilde{y}_{21}^n | x_2^n, \tilde{y}_{22}^n)\right] \\ & + \left[h(\tilde{y}_{21}^n | \tilde{y}_{22}^n) - h(\tilde{y}_{22}^n | x_2^n)\right] + \left[h(\tilde{y}_{22}^n) - \mu_1 h(\tilde{y}_1^n | x_1^n)\right] + n\epsilon_n, \end{aligned} \tag{2.81}$$

where (a) follows from the fact that x_1^n and x_2^n are independent. Now, we separately upper bound the terms within each bracket in (2.81).

To maximize the terms within the first bracket, we use the fact that Gaussian distribution maximizes the differential entropy subject to a constraint on the covariance matrix. Hence, we have

$$\begin{aligned}\mu_1 h(\tilde{y}_1^n) - \mu_2 h(\tilde{y}_1^n|x_1^n, x_2^n) &= \mu_1 h(x_1^n + \sqrt{a}x_2^n + z_1^n) - \mu_2 h(z_1^n)\\ &\leq \frac{\mu_1 n}{2}\log\left[2\pi e(P_1 + aP_2 + 1)\right] - \frac{\mu_2 n}{2}\log(2\pi e).\end{aligned} \tag{2.82}$$

Since $b' \leq N_{21}$, we can make use of Lemma 1 to upper bound the second bracket. In this case, we have

$$\begin{aligned}\mu_2 h(\tilde{y}_1^n|x_2^n) - h(\tilde{y}_{21}^n|x_2^n, \tilde{y}_{22}^n) &= \mu_2\left(h(x_1^n + z_1^n) - \frac{1}{\mu_2}h(\sqrt{b'}x_1^n + z_{21}^n)\right)\\ &\leq \mu_2 n f_h\left(P_1, 1, N_{21}, b', \frac{1}{\mu_2}\right),\end{aligned} \tag{2.83}$$

where f_h is defined in (2.61).

We upper bound the terms within the third bracket as follows [20]:

$$\begin{aligned}h(\tilde{y}_{21}^n|\tilde{y}_{22}^n) - h(\tilde{y}_{22}^n|x_2^n) &\overset{(a)}{\leq} \sum_{i=1}^n h(\tilde{y}_{21}[i]|\tilde{y}_{22}[i]) - h(z_{22}^n)\\ &\overset{(b)}{\leq} \sum_{i=1}^n \frac{1}{2}\log\left[2\pi e\left(N_{21} + b'P_1[i] + \frac{P_2[i]N_{22}}{P_2[i] + N_{22}}\right)\right] - \frac{n}{2}\log\left(2\pi e N_{22}\right)\\ &\overset{(c)}{\leq} \frac{n}{2}\log\left[2\pi e\left(N_{21} + \frac{1}{n}\sum_{i=1}^n b'P_1[i] + \frac{\frac{1}{n}\sum_{i=1}^n P_2[i]N_{22}}{\frac{1}{n}\sum_{i=1}^n P_2[i] + N_{22}}\right)\right]\\ &\quad - \frac{n}{2}\log\left(2\pi e N_{22}\right)\\ &\leq \frac{n}{2}\log\left[2\pi e\left(N_{21} + b'P_1 + \frac{P_2 N_{22}}{P_2 + N_{22}}\right)\right] - \frac{n}{2}\log\left(2\pi e N_{22}\right)\\ &\leq \frac{n}{2}\log\left(\frac{N_{21}}{N_{22}} + \frac{b'P_1}{N_{22}} + \frac{P_2}{P_2 + N_{22}}\right),\end{aligned} \tag{2.84}$$

where (a) follows from the chain rule and the fact that removing independent conditions does not decrease differential entropy, (b) follows from the fact that Gaussian distribution maximizes the conditional entropy for a given covariance matrix, and (c) follows from Jensen's inequality.

For the last bracket, we again make use of the definition of f_h. In fact, since $aN_{22} \leq 1$, we have

$$\begin{aligned}h(\tilde{y}_{22}^n) - \mu_1 h(\tilde{y}_1^n|x_1^n) &= h(x_2^n + z_{22}^n) - \mu_1 h(\sqrt{a}x_2^n + z_1^n)\\ &\leq n f_h(P_2, N_{22}, 1, a, \mu_1).\end{aligned} \tag{2.85}$$

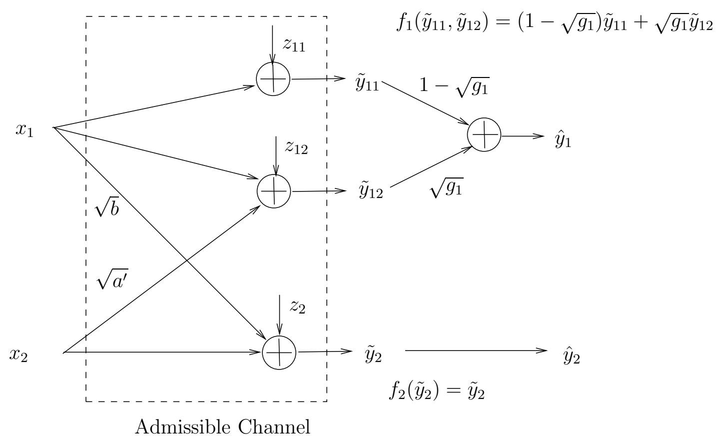

Figure 2.5: Class A2 admissible channels.

Adding all inequalities, we obtain

$$\begin{aligned}\mu R_1 + R_2 \leq & \frac{\mu_1}{2}\log\left[2\pi e(P_1 + aP_2 + 1)\right] - \frac{\mu_2}{2}\log(2\pi e) + \frac{1}{2}\log\left(\frac{N_{21}}{N_{22}} + \frac{b'P_1}{N_{22}} + \frac{P_2}{P_2 + N_{22}}\right)\\ &+\mu_2 f_h\left(P_1, 1, N_{21}, b', \frac{1}{\mu_2}\right) + f_h(P_2, N_{22}, 1, a, \mu_1),\end{aligned} \tag{2.86}$$

where the fact that $\epsilon_n \to 0$ as $n \to \infty$ is used to eliminate ϵ_n form the right hand side of the inequality. Now, by taking the minimum of the right hand side of (4.16) over all μ_1 and μ_2, we obtain the desired result. This completes the proof. □

Class A2

This class is the complement of Class A1 in the sense that we use it to upper bound $\sigma_{\mathscr{C}}(1, \mu)$. A member of this class is a channel in which User 1 is equipped with one transmit and two receive antennas, whereas User 2 is equipped with one antenna at both transmitter and receiver sides (see Figure 2.5). The channel model can be written as

$$\begin{aligned}\tilde{y}_{11} =& \quad x_1 + z_{11},\\ \tilde{y}_{12} =& \quad x_1 + \sqrt{a'}x_2 + z_{12},\\ \tilde{y}_2 =& \quad x_2 + \sqrt{b}x_1 + z_2,\end{aligned} \tag{2.87}$$

where $\tilde{y}_{11}$ and $\tilde{y}_{12}$ are the signals at the first receiver, $\tilde{y}_2$ is the signal at the second receiver, z_2 is additive Gaussian noise with unit variance, z_{11} and z_{12} are additive Gaussian noise with variances N_{11} and N_{12}, respectively. Transmitter 1 and 2 are subject to the power constraints P_1 and P_2, respectively.

For this class, we consider two linear functions f_1 and f_2 as follows (see Figure 2.5):

$$f_1(\tilde{y}_{11}^n, \tilde{y}_{12}^n) = (1 - \sqrt{g_1})\tilde{y}_{11}^n + \sqrt{g_1}\tilde{y}_{12}^n, \tag{2.88}$$
$$f_2(\tilde{y}_2^n) = \tilde{y}_2^n. \tag{2.89}$$

Similar to Class A1, when $g_1 = 0$, the admissible channels in Class A2 become the one-sided GIC by letting $N_{12} \to \infty$ and $N_{11} = 1$. Therefore, we have

$$\hat{y}_1^n = x_1^n + \sqrt{a' g_1} x_2^n + (1 - \sqrt{g_1}) z_{11}^n + \sqrt{g_1} z_{12}^n, \tag{2.90}$$
$$\hat{y}_2^n = \sqrt{b} x_1^n + x_2^n + z_2^n. \tag{2.91}$$

We conclude that the channel modeled by (2.87) is admissible if the corresponding parameters satisfy

$$\begin{aligned} a' g_1 &= a, \\ (1 - \sqrt{g_1})^2 N_{11} + g_1 N_{12} &= 1. \end{aligned} \tag{2.92}$$

Similar to Class A1, we further add the following constraints to the conditions of Class A2 channels:

$$\begin{aligned} a' &\leq N_{12}, \\ b N_{11} &\leq 1. \end{aligned} \tag{2.93}$$

In the following lemma, we obtain the required upper bound.

Lemma 4. *For the channels modeled by (2.87) and satisfying (2.93), we have*

$$\begin{aligned} \sigma_{\mathscr{C}'}(1, \mu) \leq \min \frac{\mu_1}{2} \log\left[2\pi e(bP_1 + P_2 + 1)\right] + \frac{1}{2} \log\left(\frac{N_{12}}{N_{11}} + \frac{a' P_2}{N_{11}} + \frac{P_1}{P_1 + N_{11}}\right) \\ - \frac{\mu_2}{2} \log(2\pi e) + \mu_2 f_h\left(P_2, 1, N_{12}, a', \frac{1}{\mu_2}\right) + f_h(P_1, N_{11}, 1, b, \mu_1) \\ \textit{subject to: } \mu_1 + \mu_2 = \mu, \ \mu_1, \mu_2 \geq 0. \end{aligned} \tag{2.94}$$

Proof. The proof is similar to that of Lemma 3 and we omit it here. □

Class B

A member of this class is a channel with one transmit antenna and two receive antennas for each user modeled by (see Figure 2.6)

$$\begin{aligned} \tilde{y}_{11} &= x_1 + z_{11}, \\ \tilde{y}_{12} &= x_1 + \sqrt{a'} x_2 + z_{12}, \\ \tilde{y}_{21} &= x_2 + \sqrt{b'} x_1 + z_{21}, \\ \tilde{y}_{22} &= x_2 + z_{22}, \end{aligned} \tag{2.95}$$

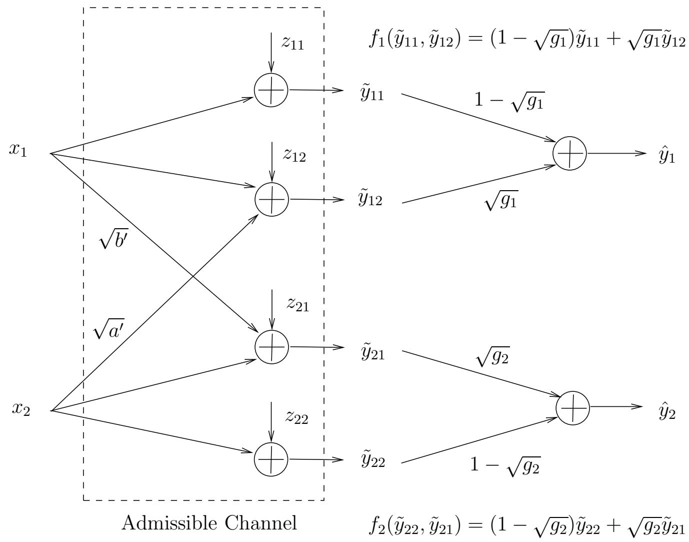

Figure 2.6: Class B admissible channels.

where $\tilde{y}_{11}$ and $\tilde{y}_{12}$ are the signals at the first receiver, $\tilde{y}_{21}$ and $\tilde{y}_{22}$ are the signals at the second receiver, and z_{ij} is additive Gaussian noise with variance N_{ij} for $i,j \in \{1,2\}$. Transmitter 1 and 2 are subject to the power constraints P_1 and P_2, respectively. In fact, this channel is designed to upper bound both $\sigma_{\mathscr{C}}(\mu,1)$ and $\sigma_{\mathscr{C}}(1,\mu)$.

Next, we investigate admissibility of this channel and the conditions that must be imposed on the underlying parameters. Let us consider two linear deterministic functions f_1 and f_2 with parameters $0 \leq g_1$ and $0 \leq g_2$, respectively, as follows (see Figure 2.6)

$$f_1(\tilde{y}_{11}^n, \tilde{y}_{12}^n) = (1-\sqrt{g_1})\tilde{y}_{11}^n + \sqrt{g_1}\tilde{y}_{12}^n, \tag{2.96}$$

$$f_2(\tilde{y}_{22}^n, \tilde{y}_{21}^n) = (1-\sqrt{g_2})\tilde{y}_{22}^n + \sqrt{g_2}\tilde{y}_{21}^n. \tag{2.97}$$

Therefore, we have

$$\hat{y}_1^n = x_1^n + \sqrt{a'g_1}x_2^n + (1-\sqrt{g_1})z_{11}^n + \sqrt{g_1}z_{12}^n, \tag{2.98}$$

$$\hat{y}_2^n = \sqrt{b'g_2}x_1^n + x_2^n + (1-\sqrt{g_2})z_{22}^n + \sqrt{g_2}z_{21}^n. \tag{2.99}$$

To satisfy (2.64) and (2.65), it suffices to have

$$\begin{aligned} a'g_1 &= a, \\ b'g_2 &= b, \\ (1-\sqrt{g_1})^2N_{11} + g_1N_{12} &= 1, \\ (1-\sqrt{g_2})^2N_{22} + g_2N_{21} &= 1. \end{aligned} \tag{2.100}$$

Hence, a channel modeled by (2.95) is admissible if there exist two nonnegative numbers g_1 and g_2 such that the equalities in (2.100) are satisfied. We further add the following two constraints to the equality conditions in (2.100):

$$\begin{aligned} b'N_{11} &\leq N_{21}, \\ a'N_{22} &\leq N_{12}. \end{aligned} \tag{2.101}$$

Although adding more constraints reduces the number of the admissible channels, it enables us to compute an outer bound on $\sigma_{\mathscr{C}'}(\mu,1)$ and $\sigma_{\mathscr{C}'}(1,\mu)$.

Lemma 5. *For the channels modeled by (2.95) and satisfying (2.101), we have*

$$\begin{aligned} \sigma_{\mathscr{C}'}(\mu,1) \leq & \mu\gamma\left(\frac{P_1}{N_{11}}+\frac{P_1}{a'P_2+N_{12}}\right)+\gamma\left(\frac{P_2}{N_{22}}+\frac{P_2}{b'P_1+N_{21}}\right) \\ & +f_h(P_2,N_{22},N_{12},a',\mu)+\frac{\mu}{2}\log((2\pi e)(a'P_2+N_{12})) \\ & -\frac{1}{2}\log((2\pi e)(P_2+N_{22})), \end{aligned} \tag{2.102}$$

$$\begin{aligned} \sigma_{\mathscr{C}'}(1,\mu) \leq & \gamma\left(\frac{P_1}{N_{11}}+\frac{P_1}{a'P_2+N_{12}}\right)+\mu\gamma\left(\frac{P_2}{N_{22}}+\frac{P_2}{b'P_1+N_{21}}\right) \\ & +f_h(P_1,N_{11},N_{21},b',\mu)+\frac{\mu}{2}\log((2\pi e)(b'P_1+N_{21})) \\ & -\frac{1}{2}\log((2\pi e)(P_1+N_{11})). \end{aligned} \tag{2.103}$$

Proof. We only upper bound $\sigma_{\mathscr{C}'}(\mu,1)$ and an upper bound on $\sigma_{\mathscr{C}'}(1,\mu)$ can be similarly obtained. Let us assume R_1 and R_2 are achievable rates for User 1 and User 2, respectively. Using Fano's inequality, we obtain

$$\begin{aligned} n(\mu R_1+R_2) \leq & \mu I(x_1^n;\tilde{y}_{11}^n,\tilde{y}_{12}^n)+I(x_2^n;\tilde{y}_{22}^n,\tilde{y}_{21}^n)+n\epsilon_n \\ = & \mu I(x_1^n;\tilde{y}_{12}^n|\tilde{y}_{11}^n)+\mu I(x_1^n;\tilde{y}_{11}^n) \\ & +I(x_2^n;\tilde{y}_{21}^n|\tilde{y}_{22}^n,)+I(x_2^n;\tilde{y}_{22}^n)+n\epsilon_n \\ = & \mu h(\tilde{y}_{12}^n|\tilde{y}_{11}^n)-\mu h(\tilde{y}_{12}^n|x_1^n,\tilde{y}_{11}^n)+\mu h(\tilde{y}_{11}^n)-\mu h(\tilde{y}_{11}^n|x_1^n) \\ & +h(\tilde{y}_{21}^n|\tilde{y}_{22}^n)-h(\tilde{y}_{21}^n|x_2^n,\tilde{y}_{22}^n)+h(\tilde{y}_{22}^n)-h(\tilde{y}_{22}^n|x_2^n)+n\epsilon_n \\ = & \left[\mu h(\tilde{y}_{12}^n|\tilde{y}_{11}^n)-\mu h(\tilde{y}_{11}^n|x_1^n)\right]+\left[h(\tilde{y}_{21}^n|\tilde{y}_{22}^n)-h(\tilde{y}_{22}^n|x_2^n)\right] \\ & +\left[\mu h(\tilde{y}_{11}^n)-h(\tilde{y}_{21}^n|x_2^n,\tilde{y}_{22}^n)\right]+\left[h(\tilde{y}_{22}^n)-\mu h(\tilde{y}_{12}^n|x_1^n,\tilde{y}_{11}^n)\right]+n\epsilon_n. \end{aligned} \tag{2.104}$$

Next, we upper bound the terms within each bracket in (2.104) separately. For the first bracket, we have

$$\begin{aligned}
\mu h(\tilde{y}_{12}^n|\tilde{y}_{11}^n) - \mu h(\tilde{y}_{11}^n|x_1^n) &\overset{(a)}{\leq} \mu \sum_{i=1}^{n} h(\tilde{y}_{12}[i]|\tilde{y}_{11}[i]) - \frac{\mu n}{2}\log(2\pi e N_{11}) \\
&\overset{(b)}{\leq} \mu \sum_{i=1}^{n} \frac{1}{2}\log\left[2\pi e\left(N_{12} + a'P_2[i] + \frac{P_1[i]N_{11}}{P_1[i]+N_{11}}\right)\right] - \frac{\mu n}{2}\log(2\pi e N_{11}) \\
&\overset{(c)}{\leq} \frac{\mu n}{2}\log\left[2\pi e\left(N_{12} + \frac{1}{n}\sum_{i=1}^{n} a'P_2[i] + \frac{\frac{1}{n}\sum_{i=1}^{n} P_1[i]N_{11}}{\frac{1}{n}\sum_{i=1}^{n} P_1[i]+N_{11}}\right)\right] \\
&\quad - \frac{\mu n}{2}\log(2\pi e N_{11}) \\
&\leq \frac{\mu n}{2}\log\left[2\pi e\left(N_{12} + a'P_2 + \frac{P_1N_{11}}{P_1+N_{11}}\right)\right] - \frac{\mu n}{2}\log(2\pi e N_{11}) \\
&= \frac{\mu n}{2}\log\left(\frac{N_{12}}{N_{11}} + \frac{a'P_2}{N_{11}} + \frac{P_1}{P_1+N_{11}}\right),
\end{aligned} \tag{2.105}$$

where (a) follows from the chain rule and the fact that removing independent conditions increases differential entropy, (b) follows from the fact that Gaussian distribution optimizes conditional entropy for a given covariance matrix, and (c) follows form Jenson's inequality.

Similarly, the terms within the second bracket can be upper bounded as

$$h(\tilde{y}_{21}^n|\tilde{y}_{22}^n) - h(\tilde{y}_{22}^n|x_2^n) \leq \frac{n}{2}\log\left(\frac{N_{21}}{N_{22}} + \frac{b'P_1}{N_{22}} + \frac{P_2}{P_2+N_{22}}\right). \tag{2.106}$$

Using Lemma 1 and the fact that $N_{11} \leq N_{21}/b'$, the terms within the third bracket can be upper bounded as

$$\begin{aligned}
\mu h(\tilde{y}_{11}^n) - h(\tilde{y}_{21}^n|x_2^n, \tilde{y}_{22}^n) &= \mu\left(h(x_1^n + z_{11}^n) - \frac{1}{\mu}h(\sqrt{b'}x_1^n + z_{21}^n)\right) \\
&\leq \mu n f_h\left(P_1, N_{11}, N_{21}, b', \frac{1}{\mu}\right).
\end{aligned} \tag{2.107}$$

Since $1 \leq \mu$, from (2.61) we obtain

$$\mu h(\tilde{y}_{11}^n) - h(\tilde{y}_{21}^n|x_2^n, \tilde{y}_{22}^n) \leq \frac{\mu n}{2}\log((2\pi e)(P_1+N_{11})) - \frac{n}{2}\log((2\pi e)(b'P_1+N_{21})). \tag{2.108}$$

For the last bracket, again we use Lemma 1 to obtain

$$\begin{aligned}
h(\tilde{y}_{22}^n) - \mu h(\tilde{y}_{12}^n|x_1^n, \tilde{y}_{11}^n) &= h(x_2^n + z_{22}^n) - \mu h(\sqrt{a'}x_2^n + z_{12}^n) \\
&\leq n f_h(P_2, N_{22}, N_{12}, a', \mu).
\end{aligned} \tag{2.109}$$

Adding all inequalities, we have

$$\begin{aligned}
\mu R_1 + R_2 \leq& \frac{\mu}{2}\log\left(\frac{N_{12}}{N_{11}} + \frac{a'P_2}{N_{11}} + \frac{P_1}{P_1+N_{11}}\right) + \frac{1}{2}\log\left(\frac{N_{21}}{N_{22}} + \frac{b'P_1}{N_{22}} + \frac{P_2}{P_2+N_{22}}\right) \\
&+ \frac{\mu}{2}\log((2\pi e)(P_1+N_{11})) - \frac{1}{2}\log((2\pi e)(b'P_1+N_{21})) \\
&+ f_h(P_2, N_{22}, N_{12}, a', \mu),
\end{aligned} \tag{2.110}$$

where the fact that $\epsilon_n \to 0$ as $n \to \infty$ is used to eliminate ϵ_n from the right hand side of the inequality. By rearranging the terms, we obtain

$$\mu R_1 + R_2 \leq \mu\gamma\left(\frac{P_1}{N_{11}} + \frac{P_1}{a'P_2 + N_{12}}\right) + \gamma\left(\frac{P_2}{N_{22}} + \frac{P_2}{b'P_1 + N_{21}}\right)$$
$$+ f_h(P_2, N_{22}, N_{12}, a', \mu) + \frac{\mu}{2}\log((2\pi e)(a'P_2 + N_{12})) - \frac{1}{2}\log((2\pi e)(P_2 + N_{22})).$$

This completes the proof. □

A unique feature of the channels within Class B is that for $1 \leq \mu \leq \frac{P_2 + N_{12}/a'}{P_2 + N_{22}}$ and $1 \leq \mu \leq \frac{P_1 + N_{21}/b'}{P_1 + N_{11}}$, the upper bounds in (2.102) and (2.103) become, respectively,

$$\mu R_1 + R_2 \leq \mu\gamma\left(\frac{P_1}{N_{11}} + \frac{P_1}{a'P_2 + N_{12}}\right) + \gamma\left(\frac{P_2}{N_{22}} + \frac{P_2}{b'P_1 + N_{21}}\right) \tag{2.111}$$

and

$$R_1 + \mu R_2 \leq \gamma\left(\frac{P_1}{N_{11}} + \frac{P_1}{a'P_2 + N_{12}}\right) + \mu\gamma\left(\frac{P_2}{N_{22}} + \frac{P_2}{b'P_1 + N_{21}}\right). \tag{2.112}$$

On the other hand, if the receivers treat the interference as noise, it can be shown that

$$R_1 = \gamma\left(\frac{P_1}{N_{11}} + \frac{P_1}{a'P_2 + N_{12}}\right) \tag{2.113}$$

and

$$R_2 = \gamma\left(\frac{P_2}{N_{22}} + \frac{P_2}{b'P_1 + N_{21}}\right) \tag{2.114}$$

are achievable. Comparing upper bounds and achievable rates, we conclude that the upper bounds are indeed tight. In fact, this property is first observed by Etkin *et al.* in [20]. We summarize this result in the following theorem:

Theorem 4. *The sum capacity in Class B is attained when transmitters use Gaussian codebooks and receivers treat the interference as noise. In this case, the sum capacity is*

$$\mathscr{C}'_{sum} = \gamma\left(\frac{P_1}{N_{11}} + \frac{P_1}{a'P_2 + N_{12}}\right) + \gamma\left(\frac{P_2}{N_{22}} + \frac{P_2}{b'P_1 + N_{21}}\right). \tag{2.115}$$

Proof. By substituting $\mu = 1$ in (3.15), we obtain the desired result. □

Class C

Class C is designed to upper bound $\sigma_{\mathscr{C}}(\mu, 1)$ for the mixed GIC where $1 \leq b$. Class C is similar to Class A1 (see Figure 2.4), however we impose different constraints on the parameters of the channels within Class C. These constraints assist us in providing upper bounds by using the fact that at one of the receivers both signals are decodable.

For channels in Class C, we use the same model that is given in (2.73). Therefore, similar to channels in Class A1, this channel is admissible if the corresponding parameters satisfy

$$\begin{aligned} b'g_2 &= b, \\ (1-\sqrt{g_2})^2 N_{22} + g_2 N_{21} &= 1. \end{aligned} \tag{2.116}$$

Next, we change the constraints in (2.79) as

$$\begin{aligned} b' &\geq N_{21}, \\ aN_{22} &\leq 1. \end{aligned} \tag{2.117}$$

Through this change of constraints, the second receiver after decoding its own signal will have a less noisy version of the first user's signal, and consequently, it is able to decode the signal of the first user as well as its own signal. Relying on this observation, we have the following lemma.

Lemma 6. *For a channel in Class C, we have*

$$\begin{aligned} \sigma_{\mathscr{C}'}(\mu, 1) \leq & \frac{\mu-1}{2}\log\left(2\pi e(P_1 + aP_2 + 1)\right) + \frac{1}{2}\log\left(2\pi e\left(\frac{P_2 N_{22}}{P_2 + N_{22}} + b'P_1 + N_{21}\right)\right) \\ & - \frac{1}{2}\log(2\pi e N_{21}) - \frac{1}{2}\log(2\pi e N_{22}) + f_h(P_2, N_{22}, 1, a, \mu - 1). \end{aligned} \tag{2.118}$$

Proof. Since the second user is able to decode both users' messages, we have

$$R_1 \leq \frac{1}{n} I(x_1^n; \tilde{y}_1^n), \tag{2.119}$$

$$R_1 \leq \frac{1}{n} I(x_1^n; \tilde{y}_{21}^n, \tilde{y}_{22}^n | x_2^n), \tag{2.120}$$

$$R_2 \leq \frac{1}{n} I(x_2^n; \tilde{y}_{21}^n, \tilde{y}_{22}^n | x_1^n), \tag{2.121}$$

$$R_1 + R_2 \leq \frac{1}{n} I(x_1^n, x_2^n; \tilde{y}_{21}^n, \tilde{y}_{22}^n). \tag{2.122}$$

From $aN_{22} \leq 1$, we have $I(x_1^n; \tilde{y}_1^n) \leq I(x_1^n; \tilde{y}_{21}^n | x_2^n) = I(x_1^n; \tilde{y}_{21}^n, \tilde{y}_{22}^n | x_2^n)$. Hence, (2.120) is redundant. It can be shown that

$$\mu R_1 + R_2 \leq \frac{\mu - 1}{n} I(x_1^n; \tilde{y}_1^n) + \frac{1}{n} I(x_1^n, x_2^n; \tilde{y}_{21}^n, \tilde{y}_{22}^n). \tag{2.123}$$

Hence, we have

$$\begin{aligned}\mu R_1 + R_2 &\leq \frac{\mu-1}{n}h(\tilde{y}_1^n) - \frac{\mu-1}{n}h(\tilde{y}_1^n|x_1^n) + \frac{1}{n}h(\tilde{y}_{21}^n,\tilde{y}_{22}^n) - \frac{1}{n}h(\tilde{y}_{21}^n,\tilde{y}_{22}^n|x_1^n,x_2^n)\\ &= \frac{\mu-1}{n}h(\tilde{y}_1^n) + \frac{1}{n}h(\tilde{y}_{21}^n|\tilde{y}_{22}^n) - \frac{1}{n}h(\tilde{y}_{21}^n,\tilde{y}_{22}^n|x_1^n,x_2^n)\\ &\quad + \left[\frac{1}{n}h(\tilde{y}_{22}^n) - \frac{\mu-1}{n}h(\tilde{y}_1^n|x_1^n)\right].\end{aligned} \tag{2.124}$$

Next, we bound the different terms in (2.124). For the first term, we have

$$\frac{\mu-1}{n}h(\tilde{y}_1^n) \leq \frac{\mu-1}{2}\log\left(2\pi e(P_1 + aP_2 + 1)\right). \tag{2.125}$$

The second term can be bounded as

$$\frac{1}{n}h(\tilde{y}_{21}^n|\tilde{y}_{22}^n) \leq \frac{1}{2}\log\left(2\pi e\left(\frac{P_2N_{22}}{P_2+N_{22}} + b'P_1 + N_{21}\right)\right). \tag{2.126}$$

The third term can be bounded as

$$\frac{1}{n}h(\tilde{y}_{21}^n,\tilde{y}_{22}^n|x_1^n,x_2^n) = \frac{1}{2}\log(2\pi eN_{21}) + \frac{1}{2}\log(2\pi eN_{22}). \tag{2.127}$$

The last terms can be bounded as

$$\frac{1}{n}h(\tilde{y}_{22}^n) - \frac{\mu-1}{n}h(\tilde{y}_1^n|x_1^n) = \frac{1}{n}h(x_2^n + z_{22}^n) - \frac{\mu-1}{n}h(\sqrt{a}x_2^n + z_1) \tag{2.128}$$

$$\leq f_h(P_2, N_{22}, 1, a, \mu-1). \tag{2.129}$$

Adding all inequalities, we obtain the desired result. □

2.3 Weak Gaussian Interference Channel

In this section, we focus on the weak GIC. We first obtain the sum capacity of this channel for a certain range of parameters. Then, we obtain an outer bound on the capacity region which is tighter than the previously known outer bounds. Finally, we show that time-sharing and concavification result in the same achievable region for Gaussian codebooks.

2.3.1 Sum Capacity

In this subsection, we use the Class B channels to obtain the sum capacity of the weak IC for a certain range of parameters. To this end, let us consider the following minimization problem:

$$
\begin{aligned}
W =& \min \gamma \left(\frac{P_1}{N_{11}} + \frac{P_1}{a' P_2 + N_{12}} \right) + \gamma \left(\frac{P_2}{N_{22}} + \frac{P_2}{b' P_1 + N_{21}} \right) \\
& \text{subject to:} \\
& \quad a' g_1 = a \\
& \quad b' g_2 = b \\
& \quad b' N_{11} \leq N_{21} \\
& \quad a' N_{22} \leq N_{12} \\
& \quad (1 - \sqrt{g_1})^2 N_{11} + g_1 N_{12} = 1 \\
& \quad (1 - \sqrt{g_2})^2 N_{22} + g_2 N_{21} = 1 \\
& \quad 0 \leq [a', b', g_1, g_2, N_{11}, N_{12}, N_{22}, N_{21}].
\end{aligned}
\tag{2.130}
$$

The objective function in (2.130) is the sum capacity of Class B channels obtained in Theorem 4. The constraints are the combination of (2.100) and (2.101) where applied to confirm the admissibility of the channel and to validate the sum capacity result. Since every channel in the class is admissible, we have $\mathscr{C}_{sum} \leq W$. Substituting $S_1 = g_1 N_{12}$ and $S_2 = g_2 N_{21}$, we have

$$
\begin{aligned}
W =& \min \gamma \left(\frac{(1 - \sqrt{g_1})^2 P_1}{1 - S_1} + \frac{g_1 P_1}{a P_2 + S_1} \right) + \gamma \left(\frac{(1 - \sqrt{g_2})^2 P_2}{1 - S_2} + \frac{g_2 P_2}{b P_1 + S_2} \right) \\
& \text{subject to:} \\
& \quad \frac{b(1 - S_1)}{(1 - \sqrt{g_1})^2} \leq S_2 < 1 \\
& \quad \frac{a(1 - S_2)}{(1 - \sqrt{g_2})^2} \leq S_1 < 1 \\
& \quad 0 < [g_1, g_2].
\end{aligned}
\tag{2.131}
$$

By first minimizing with respect to g_1 and g_2, the optimization problem (2.131) can be decomposed as

$$
\begin{aligned}
W =& \min W_1 + W_2 \\
& \text{subject to: } 0 < S_1 < 1,\ 0 < S_2 < 1,
\end{aligned}
\tag{2.132}
$$

where W_1 is defined as

$$
\begin{aligned}
W_1 =& \min_{g_1} \gamma \left(\frac{(1 - \sqrt{g_1})^2 P_1}{1 - S_1} + \frac{g_1 P_1}{a P_2 + S_1} \right) \\
& \text{subject to: } \frac{b(1 - S_1)}{S_2} \leq (1 - \sqrt{g_1})^2,\ 0 < g_1.
\end{aligned}
\tag{2.133}
$$

Similarly, W_2 is defined as

$$\begin{aligned} W_2 =& \min_{g_2} \gamma \left(\frac{(1-\sqrt{g_2})^2 P_2}{1-S_2} + \frac{g_2 P_2}{bP_1 + S_2} \right) \\ & \text{subject to: } \frac{a(1-S_2)}{S_1} \leq (1-\sqrt{g_2})^2,\ 0 < g_2. \end{aligned} \tag{2.134}$$

The optimization problems (2.133) and (2.134) are easy to solve. In fact, we have

$$W_1 = \begin{cases} \gamma\left(\frac{P_1}{1+aP_2}\right) & \text{if } \sqrt{b}(1+aP_2) \leq \sqrt{S_2(1-S_1)} \\ \gamma\left(\frac{bP_1}{S_2} + \frac{(1-\sqrt{b(1-S_1)/S_2})^2 P_1}{aP_2+S_1}\right) & \text{Otherwise} \end{cases} \tag{2.135}$$

$$W_2 = \begin{cases} \gamma\left(\frac{P_2}{1+bP_1}\right) & \text{if } \sqrt{a}(1+bP_1) \leq \sqrt{S_1(1-S_2)} \\ \gamma\left(\frac{aP_2}{S_1} + \frac{(1-\sqrt{a(1-S_2)/S_1})^2 P_2}{bP_1+S_2}\right) & \text{Otherwise.} \end{cases} \tag{2.136}$$

From (2.135) and (2.136), we observe that for S_1 and S_2 satisfying $\sqrt{b}(1+aP_2) \leq \sqrt{S_2(1-S_1)}$ and $\sqrt{a}(1+bP_1) \leq \sqrt{S_1(1-S_2)}$, the objective function becomes independent of S_1 and S_2. In this case, we have

$$W = \gamma\left(\frac{P_1}{1+aP_2}\right) + \gamma\left(\frac{P_2}{1+bP_1}\right), \tag{2.137}$$

which is achievable by treating interference as noise. In the following theorem, we prove that it is possible to find a certain range of parameters such that there exist S_1 and S_2 yielding (2.137).

Theorem 5. *The sum capacity of the two-user GIC is*

$$\mathscr{C}_{sum} = \gamma\left(\frac{P_1}{1+aP_2}\right) + \gamma\left(\frac{P_2}{1+bP_1}\right), \tag{2.138}$$

for the range of parameters satisfying

$$\sqrt{b}P_1 + \sqrt{a}P_2 \leq \frac{1-\sqrt{a}-\sqrt{b}}{\sqrt{ab}}. \tag{2.139}$$

Proof. Let us fix a and b, and define D as

$$D = \left\{ (P_1, P_2) | P_1 \leq \frac{\sqrt{S_1(1-S_2)}}{b\sqrt{a}} - \frac{1}{b}, P_2 \leq \frac{\sqrt{S_2(1-S_1)}}{a\sqrt{b}} - \frac{1}{a}, 0 < S_1 < 1, 0 < S_2 < 1 \right\}. \tag{2.140}$$

In fact, if D is feasible then there exist $0 < S_1 < 1$ and $0 < S_2 < 1$ satisfying $\sqrt{b}(1 + aP_2) \leq \sqrt{S_2(1 - S_1)}$ and $\sqrt{a}(1 + bP_1) \leq \sqrt{S_1(1 - S_2)}$. Therefore, the sum capacity of the channel for all feasible points is attained due to (2.137).

We claim that $D = D'$, where D' is defined as

$$D' = \left\{(P_1, P_2)|\sqrt{b}P_1 + \sqrt{a}P_2 \leq \frac{1 - \sqrt{a} - \sqrt{b}}{\sqrt{ab}}\right\}. \tag{2.141}$$

To show $D' \subseteq D$, we set $S_1 = 1 - S_2$ in (2.140) to get

$$\left\{(P_1, P_2)|P_1 \leq \frac{S_1}{b\sqrt{a}} - \frac{1}{b}, P_2 \leq \frac{1 - S_1}{a\sqrt{b}} - \frac{1}{a}, 0 < S_1 < 1\right\} \subseteq D. \tag{2.142}$$

It is easy to show that the left hand side of the above equation is another representation of the region D'. Hence, we have $D' \subseteq D$.

To show $D \subseteq D'$, it suffices to prove that for any $(P_1, P_2) \in D$, $\sqrt{b}P_1 + \sqrt{a}P_2 \leq \frac{1-\sqrt{a}-\sqrt{b}}{\sqrt{ab}}$ holds. To this end, we introduce the following maximization problem:

$$J = \max_{(P_1,P_2)\in D} \sqrt{b}P_1 + \sqrt{a}P_2, \tag{2.143}$$

which can be written as

$$J = \max_{(S_1,S_2)\in(0,1)^2} \frac{\sqrt{S_1(1 - S_2)} + \sqrt{S_2(1 - S_1)}}{\sqrt{ab}} - \frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}. \tag{2.144}$$

It is easy to show that the solution to the above optimization problem is

$$J = \frac{1}{\sqrt{ab}} - \frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}. \tag{2.145}$$

Hence, we deduce that $D \subseteq D'$. This completes the proof. □

Remark 3. *The above sum capacity result for the weak GIC (see also [41]) has been established independently in [18] and [19].*

As an example, let us consider the symmetric GIC. In this case, the constraint in (2.139) becomes

$$P \leq \frac{1 - 2\sqrt{a}}{2a\sqrt{a}}. \tag{2.146}$$

In Figure 2.7, the admissible region for P, where treating interference as noise is optimal, versus $\sqrt{a}$ is plotted. For a fixed P and all $0 \leq a \leq 1$, the upper bound in (2.130) and the lower bound when receivers treat the interference as noise are plotted in Figure 2.8. We observe that up to a certain value of a, the upper bound coincides with the lower bound.

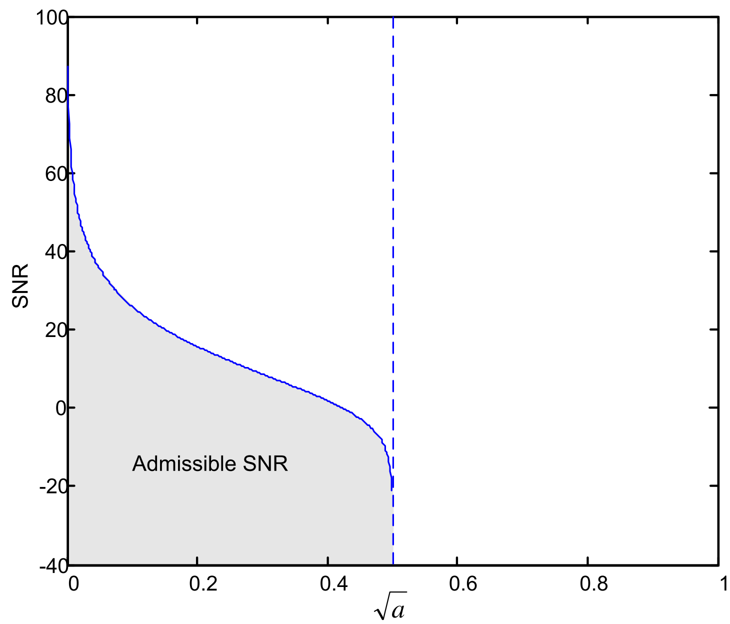

Figure 2.7: The shaded area is the region where treating interference as noise is optimal for obtaining the sum capacity of the symmetric GIC.

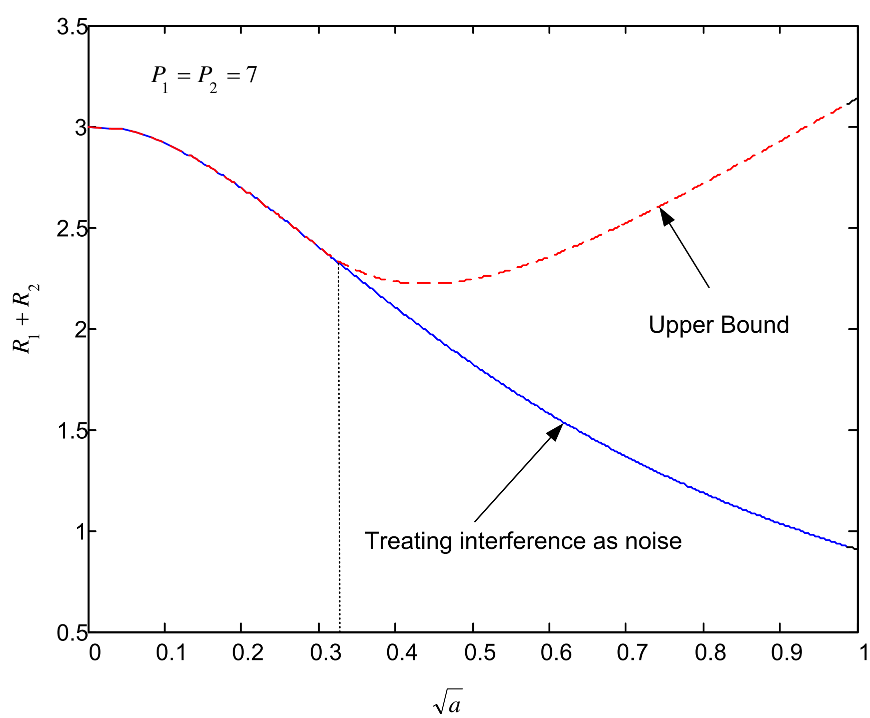

Figure 2.8: The upper bound obtained by solving (2.130). The lower bound is obtained by treating the interference as noise.

2.3.2 New Outer Bound

For the weak GIC, there are two outer bounds that are tighter than the other known bounds. The first one, due to Kramer [15], is obtained by relying on the fact that the capacity region of the GIC is inside the capacity regions of the two underlying one-sided GICs. Even though the capacity region of the one-sided GIC is unknown, there exists an outer bound for this channel that can be used instead. Kramers' outer bound is the intersection of two regions E_1 and E_2. E_1 is the collection of all rate pairs (R_1, R_2) satisfying

$$R_1 \leq \gamma\left(\frac{(1-\beta)P'}{\beta P' + 1/a}\right), \tag{2.147}$$
$$R_2 \leq \gamma(\beta P'), \tag{2.148}$$

for all $\beta \in [0, \beta_{\max}]$, where $P' = P_1/a + P_2$ and $\beta_{\max} = \frac{P_2}{P'(1+P_1)}$. Similarly, E_2 is the collection of all rate pairs (R_1, R_2) satisfying

$$R_1 \leq \gamma(\alpha P''), \tag{2.149}$$
$$R_2 \leq \gamma\left(\frac{(1-\alpha)P''}{\alpha P'' + 1/b}\right), \tag{2.150}$$

for all $\alpha \in [0, \alpha_{\max}]$, where $P'' = P_1 + P_2/b$ and $\alpha_{\max} = \frac{P_1}{P''(1+P_2)}$.

The second outer bound, due to Etkin *et al.* [20], is obtained by using Genie aided technique to upper bound different linear combinations of rates that appear in the HK achievable region. Their outer bound is the union of all rate pairs (R_1, R_2) satisfying

$$R_1 \leq \gamma(P_1), \tag{2.151}$$
$$R_2 \leq \gamma(P_2), \tag{2.152}$$
$$R_1 + R_2 \leq \gamma(P_1) + \gamma\left(\frac{P_2}{1+bP_1}\right), \tag{2.153}$$
$$R_1 + R_2 \leq \gamma(P_2) + \gamma\left(\frac{P_1}{1+aP_2}\right), \tag{2.154}$$
$$R_1 + R_2 \leq \gamma\left(aP_2 + \frac{P_1}{1+bP_1}\right) + \gamma\left(bP_1 + \frac{P_2}{1+aP_2}\right), \tag{2.155}$$
$$2R_1 + R_2 \leq \gamma(P_1 + aP_2) + \gamma\left(bP_1 + \frac{P_2}{1+aP_2}\right) + 0.5\log\left(\frac{1+P_1}{1+bP_1}\right), \tag{2.156}$$
$$R_1 + 2R_2 \leq \gamma(bP_1 + P_2) + \gamma\left(aP_2 + \frac{P_1}{1+bP_1}\right) + 0.5\log\left(\frac{1+P_2}{1+aP_2}\right). \tag{2.157}$$

In the outer bound proposed here, we derive an upper bound on all linear combinations of the rates. Recall that to obtain the boundary points of the capacity region $\mathscr{C}$, it suffices to calculate $\sigma_{\mathscr{C}}(\mu, 1)$ and $\sigma_{\mathscr{C}}(1, \mu)$ for all $1 \leq \mu$. To this end, we make use of channels in

A1 and B classes and channels in A2 and B classes to obtain upper bounds on $\sigma_{\mathscr{C}}(\mu,1)$ and $\sigma_{\mathscr{C}}(1,\mu)$, respectively.

In order to obtain an upper bound on $\sigma_{\mathscr{C}}(\mu,1)$, we introduce two optimization problems as follows. The first optimization problem is written as

$$\begin{aligned}W_1(\mu) =&\min \frac{\mu_1}{2}\log\left[2\pi e(P_1+aP_2+1)\right] - \frac{\mu_2}{2}\log(2\pi e) + \mu_2 f_h\left(P_1, 1, N_{21}, b', \frac{1}{\mu_2}\right) \\ &+ \frac{1}{2}\log\left(\frac{N_{21}}{N_{22}} + \frac{b'P_1}{N_{22}} + \frac{P_2}{P_2+N_{22}}\right) + f_h(P_2, N_{22}, 1, a, \mu_1) \\ &\text{subject to:} \\ &\quad \mu_1+\mu_2=\mu \\ &\quad b'g_2 = b \\ &\quad b' \le N_{21} \\ &\quad aN_{22} \le 1 \\ &\quad (1-\sqrt{g_2})^2 N_{22} + g_2 N_{21} = 1 \\ &\quad 0 \le [\mu_1, \mu_2, b', g_2, N_{22}, N_{21}].\end{aligned} \tag{2.158}$$

In fact, the objective of the above minimization problem is an upper bound on the support function of a channel within Class A1 which is obtained in Lemma 3. The constraints are the combination of (2.78) and (2.79) which are applied to guarantee the admissibility of the channel and to validate the upper bound obtained in Lemma 3. Hence, $\sigma_{\mathscr{C}}(\mu,1) \le W_1(\mu)$. By using a new variable $S=(1-\sqrt{g_2})^2 N_{22}$, we obtain

$$\begin{aligned}W_1(\mu) =&\min \frac{\mu_1}{2}\log\left[2\pi e(P_1+aP_2+1)\right] + \mu_2 f_h\left(P_1, 1, \frac{1-S}{g_2}, \frac{b}{g_2}, \frac{1}{\mu_2}\right) \\ &+ \frac{1}{2}\log\left[(1-\sqrt{g_2})^2\left(\frac{1-S+bP_1}{g_2 S} + \frac{P_2}{(1-\sqrt{g_2})^2P_2+S}\right)\right] - \frac{\mu_2}{2}\log(2\pi e) \\ &+ f_h\left(P_2, \frac{S}{(1-\sqrt{g_2})^2}, 1, a, \mu_1\right) \\ &\text{subject to:} \\ &\quad \mu_1+\mu_2=\mu \\ &\quad S \le 1-b \\ &\quad S \le \frac{(1-\sqrt{g_2})^2}{a} \\ &\quad 0 \le [\mu_1, \mu_2, S, g_2].\end{aligned} \tag{2.159}$$

The second optimization problem is written as

$$
\begin{aligned}
W_2(\mu) = &\min \mu\gamma \left(\frac{P_1}{N_{11}} + \frac{P_1}{a'P_2 + N_{12}}\right) + \gamma\left(\frac{P_2}{N_{22}} + \frac{P_2}{b'P_1 + N_{21}}\right) \\
&+ \frac{\mu}{2}\log((2\pi e)(a'P_2 + N_{12})) - \frac{1}{2}\log((2\pi e)(P_2 + N_{22})) \\
&+ f_h(P_2, N_{22}, N_{12}, a', \mu) \\
&\text{subject to:} \\
&\quad a'g_1 = a \\
&\quad b'g_2 = b \\
&\quad b'N_{11} \leq N_{21} \\
&\quad a'N_{22} \leq N_{12} \\
&\quad (1-\sqrt{g_1})^2 N_{11} + g_1 N_{12} = 1 \\
&\quad (1-\sqrt{g_2})^2 N_{22} + g_2 N_{21} = 1 \\
&\quad 0 \leq [a', b', g_1, g_2, N_{11}, N_{12}, N_{22}, N_{21}].
\end{aligned}
\tag{2.160}
$$

For this problem, Class B channels are used. In fact, the objective value is the upper bound on the support function of channels within the class obtained in Lemma 5 and the constraints are defined to obtain the closed form formula for the upper bound and to confirm that the channels are admissible. Hence, we deduce $\sigma_{\mathscr{C}}(\mu, 1) \leq W_2(\mu)$. By using new variables $S_1 = g_1 N_{12}$ and $S_2 = g_2 N_{21}$, we obtain

$$
\begin{aligned}
W_2(\mu) = &\min \mu\gamma \left(\frac{(1-\sqrt{g_1})^2 P_1}{1-S_1} + \frac{g_1 P_1}{aP_2 + S_1}\right) + \gamma\left(\frac{(1-\sqrt{g_2})^2 P_2}{1-S_2} + \frac{g_2 P_2}{bP_1 + S_2}\right) \\
&+ f_h\left(P_2, \frac{1-S_2}{(1-\sqrt{g_2})^2}, \frac{S_1}{g_1}, \frac{a}{g_1}, \mu\right) + \frac{\mu}{2}\log\left((2\pi e)(\frac{aP_2 + S_1}{g_1})\right) \\
&- \frac{1}{2}\log\left((2\pi e)(P_2 + \frac{1-S_2}{(1-\sqrt{g_2})^2})\right) \\
&\text{subject to:} \\
&\quad \frac{b(1-S_1)}{(1-\sqrt{g_1})^2} \leq S_2 < 1 \\
&\quad \frac{a(1-S_2)}{(1-\sqrt{g_2})^2} \leq S_1 < 1 \\
&\quad 0 < [g_1, g_2].
\end{aligned}
\tag{2.161}
$$

In a similar fashion, one can introduce two other optimization problems, say $\tilde{W}_1(\mu)$ and $\tilde{W}_2(\mu)$, to obtain upper bounds on $\sigma_{\mathscr{C}}(1, \mu)$ by using the upper bounds on the support functions of channels in Class A2 and Class B.

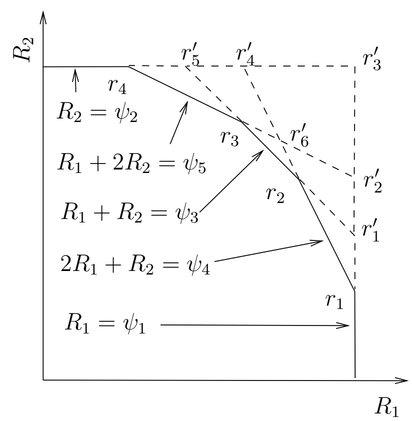

Figure 2.9: $\mathscr{G}_0$ for the weak GIC. r_1, r_2, r_3, and r_4 are extreme points of $\mathscr{G}_0$ in the interior of the first quadrant.

Theorem 6 (New Outer Bound)**.** *For any rate pair* (R_1, R_2) *achievable for the two-user weak GIC, the inequalities*

$$\mu_1 R_1 + R_2 \leq W(\mu_1) = \min\{W_1(\mu_1), W_2(\mu_1)\}, \tag{2.162}$$

$$R_1 + \mu_2 R_2 \leq \tilde{W}(\mu_2) = \min\{\tilde{W}_1(\mu_2), \tilde{W}_2(\mu_2)\}, \tag{2.163}$$

hold for all $1 \leq \mu_1, \mu_2$.

To obtain an upper bound on the sum rate, we can apply the following inequality:

$$\mathscr{C}_{\text{sum}} \leq \min_{1 \leq \mu_1, \mu_2} \frac{(\mu_2 - 1)W(\mu_1) + (\mu_1 - 1)\tilde{W}(\mu_2)}{\mu_1 \mu_2 - 1}. \tag{2.164}$$

2.3.3 Han-Kobayashi Achievable region

In this sub-section, we aim at characterizing $\mathscr{G}$ for the weak GIC. To this end, we first investigate some properties of $\mathscr{G}_0(P_1, P_2, \alpha, \beta)$. First of all, we show that none of the inequalities in describing $\mathscr{G}_0$ is redundant. In Figure 2.9, all possible extreme points are shown. It is easy to prove that $r'_i \notin \mathscr{G}_0$ for $i \in \{1, 2, \ldots, 6\}$. For instance, we consider $r'_6 = \left(\frac{2\psi_4 - \psi_5}{3}, \frac{2\psi_5 - \psi_4}{3}\right)$. Since $\psi_{31} + \psi_{32} + \psi_{33} = \psi_4 + \psi_5$ (see Section II.C), we have

$$\begin{aligned}
\psi_3 &= \min\{\psi_{31}, \psi_{32}, \psi_{33}\} \\
&\leq \frac{1}{3}(\psi_{31} + \psi_{32} + \psi_{33}) \\
&= \frac{1}{3}(\psi_4 + \psi_5).
\end{aligned}$$

However, $\frac{1}{3}(\psi_4+\psi_5)$ is the sum of the components of r'_6. Therefore, r'_6 violates (2.7) in the definition of the HK achievable region. Hence, $r'_6 \notin \mathscr{G}_0$. As another example, let us consider $r'_1 = (\psi_1, \psi_3 - \psi_1)$. We claim that r'_1 violates (2.8). To this end, we need to show that $\psi_4 \leq \psi_3 + \psi_1$. However, it is easy to see that $\psi_4 \leq \psi_{31} + \psi_1$, $\psi_4 \leq \psi_{32} + \psi_1$, and $\psi_4 \leq \psi_{33} + \psi_1$ reduce to $0 \leq (1-\alpha)(1-b+\beta(1-ab)P_2)$, $0 \leq (1-\beta)(1-a+(1-ab)P_1)$, and $0 \leq (1-\alpha)(1-\beta)aP_2$, respectively. Therefore, $r'_1 \notin \mathscr{G}_0$.

We conclude that $\mathscr{G}$ has four extreme points in the interior of the first quadrant, namely

$$r_1 = (\psi_1, \psi_4 - 2\psi_1), \tag{2.165}$$
$$r_2 = (\psi_4 - \psi_3, 2\psi_3 - \psi_4), \tag{2.166}$$
$$r_3 = (2\psi_3 - \psi_5, \psi_5 - \psi_3), \tag{2.167}$$
$$r_4 = (\psi_5 - 2\psi_2, \psi_2). \tag{2.168}$$

Most importantly, $\mathscr{G}_0$ possesses the unique minimizer property. To prove this, we need to show that $\hat{\mathbf{y}}$, the minimizer of the optimization problem

$$\begin{aligned}\sigma_{D_0}(c_1, c_2, P_1, P_2, \alpha, \beta) &= \max\{c_1R_1 + c_2R_2 | A\mathbf{R} \leq \Psi(P_1, P_2, \alpha, \beta)\} \\ &= \min\{\mathbf{y}^t\Psi(P_1, P_2, \alpha, \beta) | A^t\mathbf{y} = (c_1, c_2)^t, \mathbf{y} \geq 0\},\end{aligned} \tag{2.169}$$

is independent of the parameters P_1, P_2, α, and β and only depends on c_1 and c_2. We first consider the case $(c_1, c_2) = (\mu, 1)$ for all $1 \leq \mu$. It can be shown that for $2 < \mu$, the maximum of (3.18) is attained at r_1 regardless of P_1, P_2, α, and β. Therefore, the dual program has the minimizer $\hat{\mathbf{y}} = (\mu - 2, 0, 0, 1, 0)^t$ which is clearly independent of P_1, P_2, α, and β. In this case, we have

$$\sigma_{D_0}(\mu, 1, P_1, P_2, \alpha, \beta) = (\mu - 2)\psi_1 + \psi_4, \; 2 < \mu. \tag{2.170}$$

For $1 \leq \mu \leq 2$, one can show that r_2 and $\hat{\mathbf{y}} = (0, 0, 2-\mu, \mu-1, 0)^t$ are the maximizer and the minimizer of (3.18), respectively. In this case, we have

$$\sigma_{D_0}(\mu, 1, P_1, P_2, \alpha, \beta) = (2-\mu)\psi_3 + (\mu - 1)\psi_4, \; 1 \leq \mu \leq 2. \tag{2.171}$$

Next, we consider the case $(c_1, c_2) = (1, \mu)$ for all $1 \leq \mu$. Again, it can be shown that for $2 < \mu$ and $1 \leq \mu \leq 2$, $\hat{\mathbf{y}} = (0, \mu - 2, 0, 0, 1)^t$ and $\hat{\mathbf{y}} = (0, 0, 2 - \mu, 0, \mu - 1)^t$ minimizes (3.18), respectively. Hence, we have

$$\sigma_{D_0}(1, \mu, P_1, P_2, \alpha, \beta) = (\mu - 2)\psi_2 + \psi_5, \quad \text{if } 2 < \mu, \tag{2.172}$$
$$\sigma_{D_0}(1, \mu, P_1, P_2, \alpha, \beta) = (2-\mu)\psi_3 + (\mu - 1)\psi_5, \text{ if } 1 \leq \mu \leq 2. \tag{2.173}$$

We conclude that the solutions of the dual program are always independent of P_1, P_2, α, and β. Hence, $\mathscr{G}_0$ possesses the unique minimizer property.

Theorem 7. *For the two-user weak GIC, time-sharing and concavification result in the same region. In other words, $\mathscr{G}$ can be fully characterized by using TD/FD and allocating power over three different dimensions.*

Proof. Since $\mathscr{G}_0$ possesses the unique minimizer property, from Theorem 1, we deduce that $\mathscr{G} = \mathscr{G}_2$. Moreover, using Theorem 3, the number of frequency bands is at most three. $\square$

To obtain the support function of $\mathscr{G}_2$, we need to obtain $g(c_1, c_2, P_1, P_2, \alpha, \beta)$ defined in (2.43). Since $\mathscr{G}_0$ possesses the unique minimizer property, (2.43) can be simplified. Let us consider the case where $(c_1, c_2) = (\mu, 1)$ for $\mu > 2$. It can be shown that for this case

$$g = \max_{(\alpha,\beta)\in[0,1]^2} (\mu - 2)\psi_1(P_1, P_2, \alpha, \beta) + \psi_4(P_1, P_2, \alpha, \beta). \tag{2.174}$$

Substituting into (2.42), we obtain

$$\begin{aligned}
\sigma_{\mathscr{G}_2}(\mu, 1, P_1, P_2) = &\max \sum_{i=1}^{3} \lambda_i \left[(\mu - 2)\psi_1(P_{1i}, P_{2i}, \alpha_i, \beta_i) + \psi_4(P_{1i}, P_{2i}, \alpha_i, \beta_i)\right] \qquad (2.175)\\
&\text{subject to:}\\
&\qquad \sum_{i=1}^{3} \lambda_i = 1\\
&\qquad \sum_{i=1}^{3} \lambda_i P_{1i} \leq P_1\\
&\qquad \sum_{i=1}^{3} \lambda_i P_{2i} \leq P_2\\
&\qquad 0 \leq \lambda_i, 0 \leq P_{1i}, 0 \leq P_{2i},\ \forall i \in \{1, 2, 3\}\\
&\qquad 0 \leq \alpha_i \leq 1, 0 \leq \beta_i \leq 1,\ \forall i \in \{1, 2, 3\}.
\end{aligned}$$

For other ranges of (c_1, c_2), a similar optimization problem can be formed. It is worth noting that even though the number of parameters in characterizing $\mathscr{G}$ is reduced, it is still prohibitively difficult to characterize boundary points of $\mathscr{G}$. In Figures (2.10) and (2.11), different bounds for the symmetric weak GIC are plotted. As shown in these figures, the new outer bound is tighter than the previously known bounds.

2.4 One-sided Gaussian Interference Channels

Throughout this section, we consider the one-sided GIC obtained by setting $b = 0$, i.e, the second receiver incurs no interference from the first transmitter. One can further split the

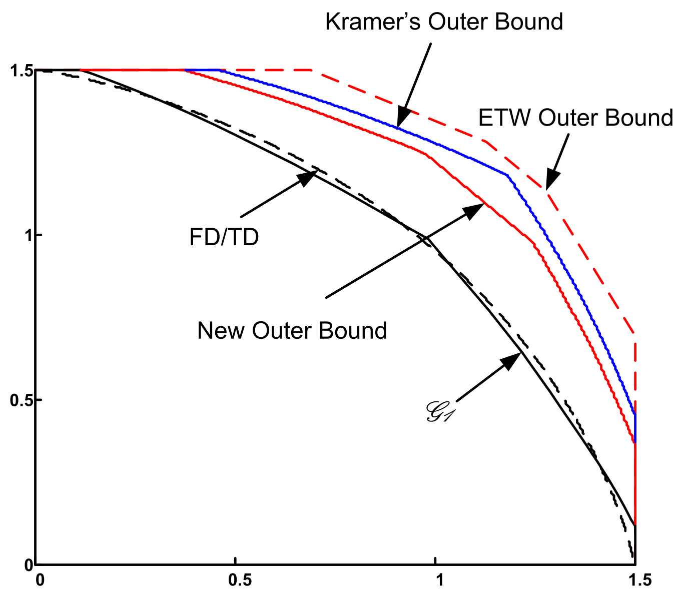

Figure 2.10: Comparison between different bounds for the symmetric weak GIC when $P = 7$ and $a = 0.2$.

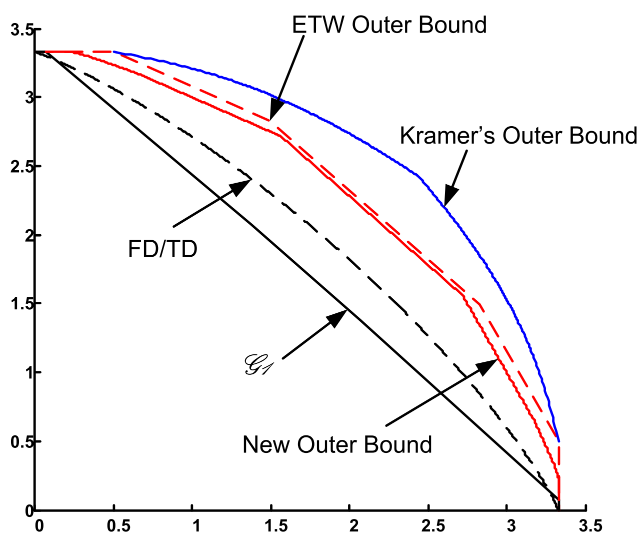

Figure 2.11: Comparison between different bounds for the symmetric weak GIC when $P = 100$ and $a = 0.1$.

class of one-sided ICs into two subclasses: the *strong one-sided IC* and the *weak one-sided IC*. For the former, $a \geq 1$ and the capacity region is fully characterized [33]. In this case, the capacity region is the union of all rate pairs (R_1, R_2) satisfying

$$\begin{aligned} R_1 &\leq \gamma(P_1), \\ R_2 &\leq \gamma(P_2), \\ R_1 + R_2 &\leq \gamma(P_1 + aP_2). \end{aligned}$$

For the latter, $a < 1$ and the full characterization of the capacity region is still an open problem. Therefore, we always assume $a < 1$. Three important results were proved for this channel. The first one, proved by Costa in [14], states that the capacity region of the weak one-sided IC is equivalent to that of the degraded IC with an appropriate change of parameters. The second one, proved by Sato in [13], states that the capacity region of the degraded GIC is outer bounded by the capacity region of a certain degraded broadcast channel. The third one, proved by Sason in [33], characterizes the sum capacity by combining Costa's and Sato's results.

In this section, we provide an alternative proof for the outer bound obtained by Sato. We then characterize the full HK achievable region where Gaussian codebooks are used, i.e., $\mathscr{G}$.

2.4.1 Sum Capacity

For the sake of completeness, we first state the sum capacity result obtained by Sason.

Theorem 8 (Sason)**.** *The rate pair* $\left(\gamma\left(\frac{P_1}{1+aP_2}\right), \gamma(P_2)\right)$ *is an extreme point of the capacity region of the one-sided GIC. Moreover, the sum capacity of the channel is attained at this point.*

2.4.2 Outer Bound

In [13], Sato derived an outer bound on the capacity of the degraded IC. This outer bound can be used for the weak one-sided IC as well. This is due to Costa's result which states that the capacity region of the degraded GIC is equivalent to that of the weak one-sided IC with an appropriate change of parameters.

Theorem 9 (Sato)**.** *If the rate pair* (R_1, R_2) *belongs to the capacity region of the weak one-sided IC, then it satisfies*

$$\begin{aligned} R_1 &\leq \gamma\left(\frac{(1-\beta)P}{1/a+\beta P}\right), \\ R_2 &\leq \gamma(\beta P), \end{aligned} \tag{2.176}$$

for all $\beta \in [0,1]$ *where* $P = P_1/a + P_2$.

Proof. Since the sum capacity is attained at the point where User 2 transmits at its maximum rate $R_2 = \gamma(P_2)$, other boundary points of the capacity region can be obtained by characterizing the solutions of $\sigma_{\mathscr{C}}(\mu, 1) = \max\{\mu R_1 + R_2 | (R_1, R_2) \in \mathscr{C}\}$ for all $1 \leq \mu$. Using Fano's inequality, we have

$$\begin{aligned} n(\mu R_1 + R_2) \leq & \mu I(x_1^n; y_1^n) + I(x_2^n; y_2^n) + n\epsilon_n \\ = & \mu h(y_1^n) - \mu h(y_1^n | x_1^n) + h(y_2^n) - h(y_2^n | x_2^n) + n\epsilon_n \\ = & [\mu h(x_1^n + \sqrt{a} x_2^n + z_1^n) - h(z_2^n)] + [h(x_2^n + z_2^n) - \mu h(\sqrt{a} x_2^n + z_1^n)] + n\epsilon_n \\ \stackrel{(a)}{\leq} & \frac{\mu n}{2} \log\left[2\pi e(P_1 + aP_2 + 1)\right] - \frac{n}{2}\log(2\pi e) + [h(x_2^n + z_2^n) - \mu h(\sqrt{a} x_2^n + z_1^n)] \\ & + n\epsilon_n \\ \stackrel{(b)}{\leq} & \frac{\mu n}{2} \log\left[2\pi e(P_1 + aP_2 + 1)\right] - \frac{n}{2}\log(2\pi e) + n f_h(P_2, 1, 1, a, \mu) + n\epsilon_n, \end{aligned}$$

where (a) follows from the fact that Gaussian distribution maximizes the differential entropy for a given constraint on the covariance matrix and (b) follows from the definition of f_h in (2.59).

Depending on the value of μ, we consider the following two cases:

1- For $1 \leq \mu \leq \frac{P_2 + 1/a}{P_2 + 1}$, we have

$$\mu R_1 + R_2 \leq \mu \gamma\left(\frac{P_1}{1 + aP_2}\right) + \gamma(P_2). \tag{2.177}$$

In fact, the point $\left(\gamma\left(\frac{P_1}{1+aP_2}\right), \gamma(P_2)\right)$ which is achievable by treating interference as noise at Receiver 1, satisfies (2.177) with equality. Therefore, it belongs to the capacity region. Moreover, by setting $\mu = 1$, we deduce that this point corresponds to the sum capacity of the one-sided GIC. This is in fact an alternative proof for Sason's result.

2- For $\frac{P_2 + 1/a}{P_2 + 1} < \mu \leq \frac{1}{a}$, we have

$$\mu R_1 + R_2 \leq \frac{\mu}{2}\log(P_1 + aP_2 + 1) + \frac{1}{2}\log\left(\frac{1/a - 1}{\mu - 1}\right) - \frac{\mu}{2}\log\left(\frac{a\mu(1/a - 1)}{\mu - 1}\right). \tag{2.178}$$

Equivalently, we have

$$\mu R_1 + R_2 \leq \frac{\mu}{2}\log\left(\frac{(aP + 1)(\mu - 1)}{\mu(1 - a)}\right) + \frac{1}{2}\log\left(\frac{1/a - 1}{\mu - 1}\right), \tag{2.179}$$

where $P = P_1/a + P_2$. Let us define E_1 as the set of all rate pairs (R_1, R_2) satisfying (2.179), i.e.

$$\begin{aligned} E_1 = \{(R_1, R_2) | \mu R_1 + R_2 \leq \frac{\mu}{2}\log\left(\frac{(aP + 1)(\mu - 1)}{\mu(1 - a)}\right) + \frac{1}{2}\log\left(\frac{1/a - 1}{\mu - 1}\right), \\ \forall \mu : \ \frac{P_2 + 1/a}{P_2 + 1} < \mu \leq \frac{1}{a}\}. \end{aligned} \tag{2.180}$$

We claim that E_1 is the dual representation of the region defined in the statement of the theorem, see (2.4). To this end, we define E_2 as

$$E_2 = \left\{(R_1, R_2)|R_1 \leq \gamma\left(\frac{(1-\beta)P}{1/a+\beta P}\right), R_2 \leq \gamma(\beta P),\ \forall\beta \in [0,1]\right\}. \tag{2.181}$$

We evaluate the support function of E_2 as

$$\sigma_{E_2}(\mu, 1) = \max\left\{\mu R_1 + R_2|(R_1, R_2) \in E_2\right\}. \tag{2.182}$$

It is easy to show that $\beta = \frac{1/a-1}{P(\mu-1)}$ maximizes the above optimization problem. Therefore, we have

$$\sigma_{E_2}(\mu, 1) = \frac{\mu}{2}\log\left(\frac{(aP+1)(\mu-1)}{\mu(1-a)}\right) + \frac{1}{2}\log\left(\frac{1/a-1}{\mu-1}\right). \tag{2.183}$$

Since E_2 is a closed convex set, we can use (2.4) to obtain its dual representation which is indeed equivalent to (4.21). This completes the proof. □

2.4.3 Han-Kobayashi Achievable Region

In this subsection, we characterize $\mathscr{G}_0$, $\mathscr{G}_1$, $\mathscr{G}_2$, and $\mathscr{G}$ for the weak one-sided GIC. $\mathscr{G}_0$ can be characterized as follows. Since there is no link between Transmitter 1 and Receiver 2, User 1's message in the HK achievable region is only the private message, i.e., $\alpha = 1$. In this case, we have

$$\psi_1 = \gamma\left(\frac{P_1}{1+a\beta P_2}\right), \tag{2.184}$$

$$\psi_2 = \gamma(P_2), \tag{2.185}$$

$$\psi_{31} = \gamma\left(\frac{P_1 + a(1-\beta)P_2}{1+a\beta P_2}\right) + \gamma(\beta P_2), \tag{2.186}$$

$$\psi_{32} = \gamma\left(\frac{P_1}{1+a\beta P_2}\right) + \gamma(P_2), \tag{2.187}$$

$$\psi_{33} = \gamma\left(\frac{P_1 + a(1-\beta)P_2}{1+a\beta P_2}\right) + \gamma(\beta P_2), \tag{2.188}$$

$$\psi_4 = \gamma\left(\frac{P_1 + a(1-\beta)P_2}{1+a\beta P_2}\right) + \gamma\left(\frac{P_1}{1+a\beta P_2}\right) + \gamma(\beta P_2), \tag{2.189}$$

$$\psi_5 = \gamma(\beta P_2) + \gamma(P_2) + \gamma\left(\frac{P_1 + a(1-\beta)P_2}{1+a\beta P_2}\right), \tag{2.190}$$

It is easy to show that $\psi_3 = \min\{\psi_{31}, \psi_{32}, \psi_{33}\} = \psi_{31}$, $\psi_{31} + \psi_1 = \psi_4$, $\psi_{31} + \psi_2 = \psi_5$. Hence, $\mathscr{G}_0$ can be represented as all rate pairs (R_1, R_2) satisfying

$$R_1 \leq \gamma\left(\frac{P_1}{1 + a\beta P_2}\right), \tag{2.191}$$

$$R_2 \leq \gamma(P_2), \tag{2.192}$$

$$R_1 + R_2 \leq \gamma\left(\frac{P_1 + a(1-\beta)P_2}{1 + a\beta P_2}\right) + \gamma(\beta P_2). \tag{2.193}$$

We claim that $\mathscr{G}_2 = \mathscr{G}$. To prove this, we need to show that $\mathscr{G}_0$ possesses the unique minimizer property. $\mathscr{G}_0$ is a pentagon with two extreme points in the interior of the first quadrant, namely r_1 and r_2 where

$$r_1 = \left(\gamma\left(\frac{P_1}{1 + a\beta P_2}\right), \gamma\left(\frac{(1-\beta)aP_2}{1 + P_1 + \beta a P_2}\right) + \gamma(\beta P_2)\right), \tag{2.194}$$

$$r_2 = \left(\gamma\left(\frac{P_1 + a(1-\beta)P_2}{1 + a\beta P_2}\right) + \gamma(\beta P_2) - \gamma(P_2), \gamma(P_2)\right). \tag{2.195}$$

Using the above, it can be verified that $\mathscr{G}_0$ possesses the unique minimizer property.

Next, we can use the optimization problem in (2.42) to obtain the support function of $\mathscr{G}$. However, we only need to consider $(c_1, c_2) = (\mu, 1)$ for $\mu > 1$. Therefore, we have

$$g(\mu, 1, P_1, P_2, \beta) = \max_{0 \leq \beta \leq 1} \mu\gamma\left(\frac{P_1}{1 + \beta a P_2}\right) + \gamma(\beta P_2) + \gamma\left(\frac{(1-\beta)aP_2}{1 + P_1 + \beta a P_2}\right). \tag{2.196}$$

Substituting into (2.42), we conclude that boundary points of $\mathscr{G}$ can be characterized by solving the following optimization problem:

$$W = \max \sum_{i=1}^{3} \lambda_i \left[\mu\gamma\left(\frac{P_{1i}}{1 + \beta_i a P_{2i}}\right) + \gamma(\beta_i P_{2i}) + \gamma\left(\frac{(1-\beta_i)aP_{2i}}{1 + P_{1i} + \beta_i a P_{2i}}\right)\right] \tag{2.197}$$

subject to:

$$\sum_{i=1}^{3} \lambda_i = 1$$

$$\sum_{i=1}^{3} \lambda_i P_{1i} \leq P_1$$

$$\sum_{i=1}^{3} \lambda_i P_{2i} \leq P_2$$

$$0 \leq \beta_i \leq 1, \ \forall i \in \{1, 2, 3\}$$

$$0 \leq [P_{1i}, P_{2i}, \lambda_i], \ \forall i \in \{1, 2, 3\}.$$

For the sake of completeness, we provide a simple description for $\mathscr{G}_1$ in the next lemma.

Lemma 7. *The region $\mathscr{G}_1$ can be represented as the collection of all rate pairs (R_1, R_2) satisfying*

$$R_1 \leq \gamma\left(\frac{P_1}{1 + a\beta' P_2}\right), \tag{2.198}$$

$$R_2 \leq \gamma(\beta' P_2) + \gamma\left(\frac{a(1-\beta')P_2}{1 + P_1 + a\beta' P_2}\right), \tag{2.199}$$

for all $\beta' \in [0,1]$. Moreover, $\mathscr{G}_1$ is convex and any point that lies on its boundary can be achieved by using superposition coding and successive decoding.

Proof. Let E denote the set defined in the above lemma. It is easy to show that E is convex and $E \subseteq \mathscr{G}_1$. To prove the reverse inclusion, it suffices to show that the extreme points of $\mathscr{G}_0$, r_1 and r_2 (see (2.194) and (2.195)) are inside E for all $\beta \in [0,1]$. By setting $\beta' = \beta$, we see that $r_1 \in E$. To prove $r_2 \in E$, we set $\beta' = 1$. We conclude that $r_2 \in E$ if the following inequality holds

$$\gamma\left(\frac{P_1 + a(1-\beta)P_2}{1 + a\beta P_2}\right) + \gamma(\beta P_2) - \gamma(P_2) \leq \gamma\left(\frac{P_1}{1 + aP_2}\right), \tag{2.200}$$

for all $\beta \in [0,1]$. However, (2.200) reduces to $0 \leq (1-a)(1-\beta)P_2$ which holds for all $\beta \in [0,1]$. Hence, $\mathscr{G}_1 \subseteq E$. Using these facts, it is straightforward to show that the boundary points $\mathscr{G}_1$ are achievable by using superposition coding and successive decoding. □

Figure 2.12 compares different bounds for the one-sided GIC.

2.5 Mixed Gaussian Interference Channels

In this section, we focus on the mixed Gaussian Interference channel. We first characterize the sum capacity of this channel. Then, we provide an outer bound on the capacity region. Finally, we investigate the HK achievable region. Without loss of generality, we assume $a < 1$ and $b \geq 1$.

2.5.1 Sum Capacity

Theorem 10. *The sum capacity of the mixed GIC with $a < 1$ and $b \geq 1$ can be stated as*

$$\mathscr{C}_{sum} = \gamma(P_2) + \min\left\{\gamma\left(\frac{P_1}{1 + aP_2}\right), \gamma\left(\frac{bP_1}{1 + P_2}\right)\right\}. \tag{2.201}$$

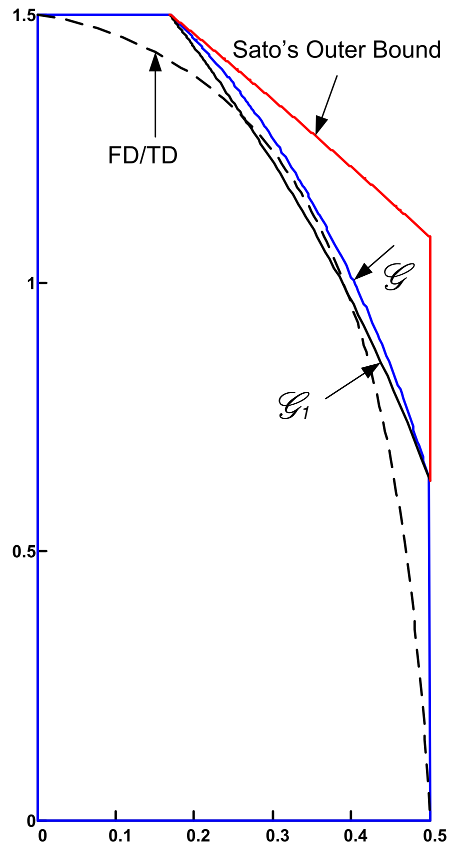

Figure 2.12: Comparison between different bounds for the one-sided GIC when $P_1 = 1$, $P_2 = 7$, and $a = 0.4$.

Proof. We need to prove the achievablity and converse for the theorem.

Achievablity part: Transmitter 1 sends a common message to both receivers, while the first user's signal is considered as noise at both receivers. In this case, the rate

$$R_1 = \min\left\{\gamma\left(\frac{P_1}{1+aP_2}\right), \gamma\left(\frac{bP_1}{1+P_2}\right)\right\} \tag{2.202}$$

is achievable. At Receiver 2, the signal from Transmitter 1 can be decoded and removed. Therefore, User 2 is left with a channel without interference and it can communicate at its maximum rate which is

$$R_2 = \gamma(P_2). \tag{2.203}$$

By adding (2.202) and (2.203), we obtain the desired result.

Converse part: The sum capacity of the GIC is upper bounded by that of the two underlying one-sided GICs. Hence, we can obtain two upper bounds on the sum rate. We first remove the interfering link between Transmitter 1 and Receiver 2. In this case, we have a one-sided GIC with weak interference. The sum capacity of this channel is known [33]. Hence, we have

$$\mathscr{C}_{sum} \leq \gamma(P_2) + \gamma\left(\frac{P_1}{1+aP_2}\right). \tag{2.204}$$

By removing the interfering link between Transmitter 2 and Receiver 1, we obtain a one-sided GIC with strong interference. The sum capacity of this channel is known. Hence, we have

$$\mathscr{C}_{sum} \leq \gamma\left(bP_1 + P_2\right), \tag{2.205}$$

which equivalently can be written as

$$\mathscr{C}_{sum} \leq \gamma(P_2) + \gamma\left(\frac{bP_1}{1+P_2}\right). \tag{2.206}$$

By taking the minimum of the right hand sides of (2.204) and (2.206), we obtain

$$\mathscr{C}_{sum} \leq \gamma\left(P_2\right) + \min\left\{\gamma\left(\frac{P_1}{1+aP_2}\right), \gamma\left(\frac{bP_1}{1+P_2}\right)\right\}. \tag{2.207}$$

This completes the proof. □

Remark 4. *In an independent work [18], the sum capacity of the mixed GIC is obtained for a certain range of parameters, whereas in the above theorem, we characterize the sum capacity of this channel for the entire range of its parameters (see also [41]).*

By comparing $\gamma\left(\frac{P_1}{1+aP_2}\right)$ with $\gamma\left(\frac{bP_1}{1+P_2}\right)$, we observe that if $1+P_2 \leq b+abP_2$, then the sum capacity corresponds to the sum capacity of the one-sided weak GIC, whereas if $1+P_2 > b+abP_2$, then the sum capacity corresponds to the sum capacity of the one-sided strong IC. Similar to the one-sided GIC, since the sum capacity is attained at the point where User 2 transmits at its maximum rate $R_2 = \gamma(P_2)$, other boundary points of the capacity region can be obtained by characterizing the solutions of $\sigma_{\mathscr{C}}(\mu, 1) = \max\{\mu R_1 + R_2 | (R_1, R_2) \in \mathscr{C}\}$ for all $1 \leq \mu$.

2.5.2 New Outer Bound

The Genie aided technique is used by Etkin *et al.* in [20] to obtain an outer bound on the capacity of the mixed GIC. This bound is the union of all rate pairs (R_1, R_2) satisfying

$$\begin{aligned}
R_1 &\leq \gamma(P_1), && (2.208)\\
R_2 &\leq \gamma(P_2), && (2.209)\\
R_1 + R_2 &\leq \gamma(P_2) + \gamma\left(\frac{P_1}{1+aP_2}\right), && (2.210)\\
R_1 + R_2 &\leq \gamma(P_2 + bP_1), && (2.211)\\
2R_1 + R_2 &\leq \gamma(P_1 + aP_2) + \gamma\left(bP_1 + \frac{P_2}{1+aP_2}\right) + \gamma\left(\frac{P_1}{1+bP_1}\right). && (2.212)
\end{aligned}$$

The capacity region of the mixed GIC is inside the intersection of the capacity regions of the two underlying one-sided GICs. Removing the link between Transmitter 1 and Receiver 2 results in a weak one-sided GIC whose outer bound E_1 is the collection of all rate pairs (R_1, R_2) satisfying

$$\begin{aligned}
R_1 &\leq \gamma\left(\frac{(1-\beta)P'}{\beta P' + 1/a}\right), && (2.213)\\
R_2 &\leq \gamma(\beta P'), && (2.214)
\end{aligned}$$

for all $\beta \in [0, \beta_{\max}]$, where $P' = P_1/a + P_2$ and $\beta_{\max} = \frac{P_2}{P'(1+P_1)}$. On the other hand, removing the link between Transmitter 2 and Receiver 1 results in a strong one-sided GIC whose capacity region E_2 is fully characterized as the collection of all rate pairs (R_1, R_2) satisfying

$$\begin{aligned}
R_1 &\leq \gamma(bP_1), && (2.215)\\
R_2 &\leq \gamma(P_2), && (2.216)\\
R_1 + R_2 &\leq \gamma(bP_1 + P_2). && (2.217)
\end{aligned}$$

Using the channels in Class C, we upper bound $\sigma_{\mathscr{C}}(\mu, 1)$ based on the following optimization problem:

$$\begin{aligned} W(\mu) =&\min \frac{\mu-1}{2}\log\left(2\pi e(P_1+aP_2+1)\right) - \frac{1}{2}\log(2\pi e N_{21}) - \frac{1}{2}\log(2\pi e N_{22}) \\ &+ \frac{1}{2}\log\left(2\pi e\left(\frac{P_2N_{22}}{P_2+N_{22}} + b'P_1 + N_{21}\right)\right) + f_h(P_2, N_{22}, 1, a, \mu-1) \\ &\text{subject to:} \\ &\quad b'g_2 = b \\ &\quad b' \geq N_{21} \\ &\quad aN_{22} \leq 1 \\ &\quad (1-\sqrt{g_2})^2 N_{22} + g_2 N_{21} = 1 \\ &\quad 0 \leq [b', g_2, N_{22}, N_{21}]. \end{aligned} \tag{2.218}$$

By substituting $S = g_2 N_{21}$, we obtain

$$\begin{aligned} W(\mu) =&\min \frac{\mu-1}{2}\log\left(2\pi e(P_1+aP_2+1)\right) - \frac{1}{2}\log\left(\frac{2\pi e(1-S)}{(1-\sqrt{g_2})^2}\right) \\ &+ \frac{1}{2}\log\left(2\pi e\left(\frac{P_2(1-S)}{(1-\sqrt{g_2})^2P_2+1-S} + \frac{bP_1+S}{g_2}\right)\right) \\ &+ f_h\left(P_2, \frac{1-S}{(1-\sqrt{g_2})^2}, 1, a, \mu-1\right) - \frac{1}{2}\log\left(\frac{2\pi e S}{g_2}\right) \\ &\text{subject to:} \\ &\quad S < 1 \\ &\quad a(1-S) \leq (1-\sqrt{g_2})^2 \\ &\quad 0 \leq [S, g_2]. \end{aligned} \tag{2.219}$$

Hence, we have the following theorem that provides an outer bound on the capacity region of the mixed GIC.

Theorem 11. *For any rate pair (R_1, R_2) achievable for the two-user mixed GIC, $(R_1, R_2) \in E_1 \bigcap E_2$. Moreover, the inequality*

$$\mu R_1 + R_2 \leq W(\mu) \tag{2.220}$$

holds for all $1 \leq \mu$.

2.5.3 Han-Kobayashi Achievable Region

In this subsection, we study the HK achievable region for the mixed GIC. Receiver 2 after decoding its own signal will have a less noisy version of the first user's signal, and consequently, it is able to decode the signal of the first user as well as its own signal. Hence, User

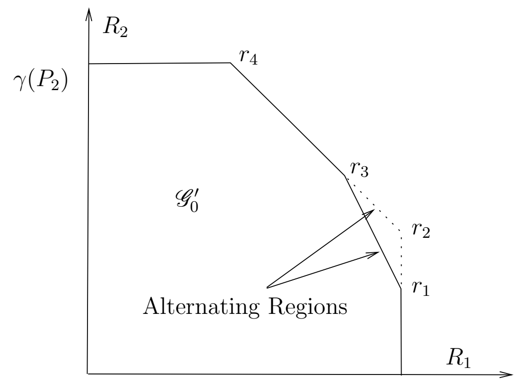

Figure 2.13: The new region $\mathscr{G}_0'$ which is obtained by enlarging $\mathscr{G}_0$.

1 associates all its power to the common message. User 2, on the other hand, allocates βP_2 and $(1-\beta)P_2$ of its total power to its private and common messages, respectively, where $\beta \in [0,1]$. Therefore, we have

$$\psi_1 = \gamma\left(\frac{P_1}{1+a\beta P_2}\right), \tag{2.221}$$
$$\psi_2 = \gamma(P_2), \tag{2.222}$$
$$\psi_{31} = \gamma\left(\frac{P_1+a(1-\beta)P_2}{1+a\beta P_2}\right)+\gamma(\beta P_2), \tag{2.223}$$
$$\psi_{32} = \gamma(P_2+bP_1), \tag{2.224}$$
$$\psi_{33} = \gamma\left(\frac{a(1-\beta)P_2}{1+a\beta P_2}\right)+\gamma(\beta P_2+bP_1), \tag{2.225}$$
$$\psi_4 = \gamma\left(\frac{P_1+a(1-\beta)P_2}{1+a\beta P_2}\right)+\gamma(\beta P_2+bP_1), \tag{2.226}$$
$$\psi_5 = \gamma(\beta P_2)+\gamma(P_2+bP_1)+\gamma\left(\frac{a(1-\beta)P_2}{1+a\beta P_2}\right). \tag{2.227}$$

Due to the fact that the sum capacity is attained at the point where the second user transmits at its maximum rate, the last inequality in the description of the HK achievable region can be removed. Although the point $r_5' = (\psi_3-\gamma(P_2),\gamma(P_1))$ in Figure 2.9 may not be in $\mathscr{G}_0$, this point is always achievable due to the sum capacity result. Hence, we can enlarge $\mathscr{G}_0$ by removing r_3 and r_4. Let us denote the resulting region as $\mathscr{G}_0'$. Moreover, one can show that r_2', r_3', r_4', and r_6' are still outside $\mathscr{G}_0'$. However, for the mixed GIC, it is possible that r_1' belongs to $\mathscr{G}_0'$. In Figure 2.13, two alternative cases for the region $\mathscr{G}_0'$ along with the new labeling of its extreme points are plotted. The new extreme points can be written as

$$\begin{aligned} r_1 &= (\psi_1, \psi_4 - 2\psi_1), \\ r_2 &= (\psi_1, \psi_3 - \psi_1), \\ r_3 &= (\psi_4 - \psi_3, 2\psi_3 - \psi_4), \\ r_4 &= (\psi_3 - \psi_2, \psi_2). \end{aligned}$$

In fact, we have either $\mathscr{G}_0' = \mathrm{conv}\{r_1, r_3, r_4\}$ or $\mathscr{G}_0' = \mathrm{conv}\{r_2, r_4\}$.

To simplify the characterization of $\mathscr{G}_1$, we consider three cases:

Case I: $1 + P_2 \leq b + abP_2$.

Case II: $1 + P_2 > b + abP_2$ and $1 - a \leq abP_1$.

Case III: $1 + P_2 > b + abP_2$ and $1 - a > abP_1$.

Case I ($1 + P_2 \leq b + abP_2$): In this case, $\psi_3 = \psi_{31}$. Moreover, it is easy to verify that $\psi_{31} + \psi_1 \leq \psi_4$ which means (2.8) is redundant for the entire range of parameters. Hence, $\mathscr{G}_0' = \mathrm{conv}\{r_2, r_4\}$ consists of all rate pairs (R_1, R_2) satisfying

$$R_1 \leq \gamma\left(\frac{P_1}{1 + a\beta P_2}\right), \tag{2.228}$$

$$R_2 \leq \gamma\left(P_2\right), \tag{2.229}$$

$$R_1 + R_2 \leq \gamma\left(\frac{P_1 + a(1-\beta)P_2}{1 + a\beta P_2}\right) + \gamma(\beta P_2), \tag{2.230}$$

where $\beta \in [0, 1]$. Using a reasoning similar to the one used to express boundary points of $\mathscr{G}_1$ for the one-sided GIC, we can express boundary points of $\mathscr{G}_1$ as

$$R_1 \leq \gamma\left(\frac{P_1}{1 + a\beta P_2}\right), \tag{2.231}$$

$$R_2 \leq \gamma(\beta P_2) + \gamma\left(\frac{a(1-\beta)P_2}{1 + P_1 + a\beta P_2}\right), \tag{2.232}$$

for all $\beta \in [0, 1]$.

Theorem 12. *For the mixed GIC satisfying $1 \leq ab$, region $\mathscr{G}$ is equivalent to that of the one sided GIC obtained from removing the interfering link between Transmitter 1 and Receiver 2.*

Proof. If $1 \leq ab$, then $1 + P_2 \leq b + abP_2$ holds for all P_1 and P_2. Hence, $\mathscr{G}_0'(P_1, P_2, \beta)$ is a pentagon defined by (2.228), (2.229), and (2.229). Comparing with the corresponding region for the one-sided GIC, we see that $\mathscr{G}_0'$ is equivalent to $\mathscr{G}_0$ obtained for the one-sided GIC. This directly implies that $\mathscr{G}$ is the same for both channels. □

Case II ($1+P_2 > b+abP_2$ and $1-a \le abP_1$): In this case, $\psi_3 = \min\{\psi_{31}, \psi_{32}\}$. It can be shown that $\mathscr{G}_1$ is the union of three regions E_1, E_2, and E_3, i.e, $\mathscr{G}_0 = E_1 \bigcup E_2 \bigcup E_3$. Region E_1 is the union of all rate pairs (R_1, R_2) satisfying

$$R_1 \le \gamma\left(\frac{P_1}{1+a\beta P_2}\right), \tag{2.233}$$

$$R_2 \le \gamma(\beta P_2) + \gamma\left(\frac{a(1-\beta)P_2}{1+P_1+a\beta P_2}\right). \tag{2.234}$$

for all $\beta \in [0, \frac{b-1}{(1-ab)P_2}]$. Region E_2 is the union of all rate pairs (R_1, R_2) satisfying

$$R_1 \le \gamma\left(\frac{bP_1}{1+\beta P_2}\right), \tag{2.235}$$

$$R_2 \le \gamma\left(\frac{P_1+a(1-\beta)P_2}{1+a\beta P_2}\right) + \gamma(\beta P_2) - \gamma\left(\frac{bP_1}{1+\beta P_2}\right). \tag{2.236}$$

for all $\beta \in [\frac{b-1}{(1-ab)P_2}, \frac{(b-1)P_1+(1-a)P_2}{(1-ab)P_1P_2+(1-a)P_2}]$. Region E_3 is the union of all rate pairs (R_1, R_2) satisfying

$$R_1 \le \gamma\left(\frac{bP_1(1+\frac{(1-ab)P_1}{1-a})}{1+bP_1+P_2}\right), \tag{2.237}$$

$$R_2 \le \gamma\left(P_2\right), \tag{2.238}$$

$$R_1 + R_2 \le \gamma(bP_1+P_2). \tag{2.239}$$

Case III ($1+P_2 > b+abP_2$ and $1-a > abP_1$): In this case, $\psi_3 = \min\{\psi_{31}, \psi_{32}\}$. Similar to Case II, we have $\mathscr{G}_1 = E_1 \bigcup E_2 \bigcup E_3$, where regions E_1, E_2, and E_3 are defined as follows. Region E_1 is the union of all rate pairs (R_1, R_2) satisfying

$$R_1 \le \gamma\left(\frac{P_1}{1+a\beta P_2}\right), \tag{2.240}$$

$$R_2 \le \gamma(\beta P_2) + \gamma\left(\frac{a(1-\beta)P_2}{1+P_1+a\beta P_2}\right). \tag{2.241}$$

for all $\beta \in [0, \frac{b-1}{(1-ab)P_2}]$. Region E_2 is the union of all rate pairs (R_1, R_2) satisfying

$$R_1 \le \gamma\left(\frac{P_1}{1+a\beta P_2}\right), \tag{2.242}$$

$$R_2 \le \gamma\left(\frac{a(1-\beta)P_2}{1+P_1+a\beta P_2}\right) + \gamma(\beta P_2 + bP_1) - \gamma\left(\frac{P_1}{1+a\beta P_2}\right). \tag{2.243}$$

for all $\beta \in [\frac{b-1}{(1-ab)P_2}, 1]$. Region E_3 is the union of all rate pairs (R_1, R_2) satisfying

$$R_1 \le \gamma\left(\frac{P_1}{1+aP_2}\right), \tag{2.244}$$

$$R_2 \le \gamma\left(P_2\right), \tag{2.245}$$

$$R_1 + R_2 \le \gamma(bP_1+P_2). \tag{2.246}$$

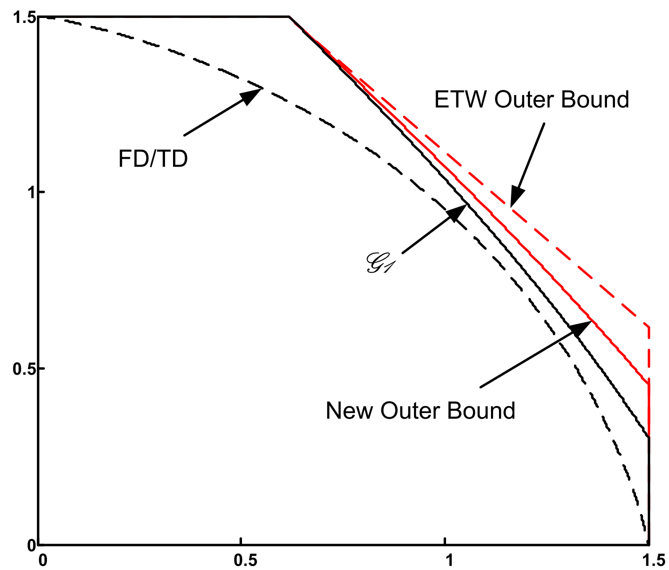

Figure 2.14: Comparison between different bounds for the mixed GIC when $1+P_2 \leq b+abP_2$ (Case I) for $P_1 = 7$, $P_2 = 7$, $a = 0.6$, and $b = 2$.

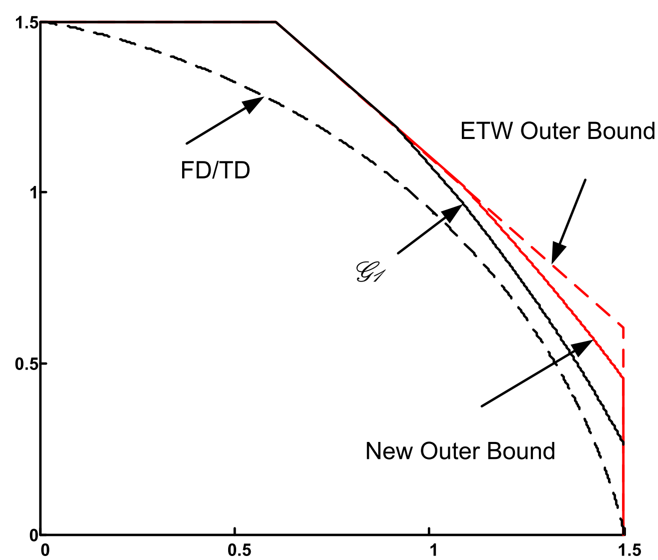

Figure 2.15: Comparison between different bounds for the mixed GIC when $1+P_2 > b+abP_2$ and $1 - a \leq abP_1$ (Case II) for $P_1 = 7$, $P_2 = 7$, $a = 0.4$, and $b = 1.5$.

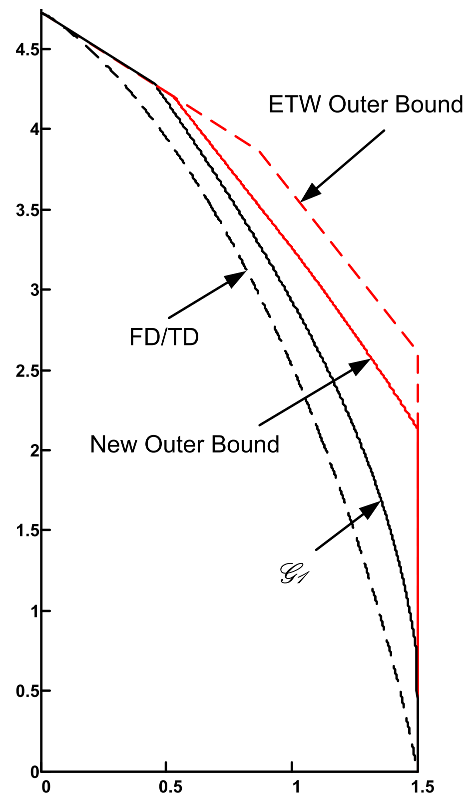

Figure 2.16: Comparison between different bounds for the mixed GIC when $1+P_2 > b+abP_2$ and $1-a > abP_1$ (Case III) for $P_1 = 7$, $P_2 = 700$, $a = 0.01$, and $b = 1.5$.

Remark 5. *Region E_3 in Case II and Case III represents a facet that belongs to the capacity region of the mixed GIC. It is important to note that, surprisingly, this facet is achievable when the second transmitter uses both the common message and the private message. In fact, this is the first GIC where both common and private messages are used to achieve points on the boundary of the capacity region.*

Different bounds are compared for the mixed GIC for Cases I, II, and III in Figures 2.14, 2.15, and 2.16, respectively.

2.6 Conclusion

We have studied the capacity region of the two-user GIC. The sum capacities, inner bounds, and outer bounds have been considered for three classes of channels: weak, one-sided, and mixed GIC. We have used admissible channels as the main tool for deriving outer bounds on the capacity regions.

For the weak GIC, we have derived the sum capacity for a certain range of channel parameters. In this range, the sum capacity is attained when Gaussian codebooks are used and interference is treated as noise. Moreover, we have derived a new outer bound on the capacity region. This outer bound is tighter than the Kramer's bound and the ETW's bound. Regarding inner bounds, we have reduced the computational complexity of the HK achievable region. In fact, we have shown that when Gaussian codebooks are used, the full HK achievable region can be obtained by using the naive HK achievable scheme over three frequency bands.

For the one-sided GIC, we have presented an alternative proof for the Sato's outer bound. We have also derived the full HK achievable region when Gaussian codebooks are used.

For the mixed GIC, we have derived the sum capacity for the entire range of its parameters. Moreover, we have presented a new outer bound on the capacity region that outperforms ETW's bound. We have proved that the full HK achievable region using Gaussian codebooks is equivalent to that of the one-sided GIC for a particular range of channel gains. We have also derived a facet that belongs to the capacity region for a certain range of parameters. Surprisingly, this facet is obtainable when one of the transmitters uses both the common message and the private message.

Chapter 3

Interference Alignment in One Dimension

The first examples of interference alignment in one-dimensional spaces are reported in [30] and [31] where the results from the field of Diophantine approximation in number theory are used to show that interference can be aligned using properties of rational and irrational numbers and their relations. They showed that the total DOF of some classes of time-invariant single antenna interference channels can be achieved. In particular, Etkin and Ordentlich in [30] proposed an upper bound on the total DOF which maintains the properties of channel gains with respect to being rational or irrational. Using this upper bound, surprisingly, they proved that the DOF is everywhere discontinuous. This chapter broadens the applications of interference alignment. In fact, we will show that it is possible to perform alignment in single dimensional systems such as time-invariant networks equipped with single antennas at all nodes.

The organization of this chapter is as follows. In Section II, we summarize the main contributions of this chapter. In Section III, we propose a novel coding scheme in which data streams are encoded using constellation points from integers and transmitted in the directions of irrational numbers. Two types of constellation designs are considered, namely the single layer and the multi-layer constellations. It is shown that the coding provides sufficient tools to accomplish interference alignment in one-dimensional spaces.

Throughout Section V, the single layer constellation is incorporated in the coding scheme. First, the performance of a decoder is analyzed using the Khintchine-Groshev theorem in number theory. It is shown that under some regularity conditions data streams can carry data with fractional multiplexing gains. The two-user X channel is considered as the first example in which the single layer constellation is incorporated in the coding scheme. It is proved that for this channel the total DOF of $\frac{4}{3}$ is attainable almost surely. For the K-user GIC, achievable DOFs are characterized for some classes of channels. Finally, it is proved that the DOF of $\frac{4}{3}$ is achievable for the three-user GIC almost surely.

Throughout Section V, the multi-layer constellation is incorporated in the coding scheme. The channel under investigation is the symmetric three-user GIC. An achievable DOF is derived for all channel gains. Viewed as a function of the channel gain, this achievable DOF is everywhere discontinuous. It is shown that the total DOF of $\frac{3}{2}$ is achievable for all irrational gains. For rational gains, the achievable rate has a gap to the available upper bounds. In Section VII, we conclude the chapter.

3.1 Main Contributions

In this chapter, we are primarily interested in characterizing the total DOF of the two-user X channel and the K-user GIC. Let $\mathcal{C}$ denote the capacity region of the K-user GIC (a similar argument can be used for the X channel). The DOF region denoted by $\mathcal{R}$ associated with the channel is in fact the shape of $\mathcal{C}$ in the high SNR regime scaled by $\log \mathrm{SNR}$. All extreme points of $\mathcal{R}$ can be identified by solving the following optimization problem:

$$r_{\boldsymbol{\lambda}} = \lim_{\mathrm{SNR}\to\infty} \max_{\mathbf{R}\in\mathcal{C}} \frac{\boldsymbol{\lambda}^t \mathbf{R}}{\log \mathrm{SNR}}. \tag{3.1}$$

The total DOF refers to the case where $\boldsymbol{\lambda} = \{1, 1, \ldots, 1\}$, i.e., the sum-rate is concerned. Throughout this chapter, r_{sum} denotes the total DOF of the system. In what follows, we summarize main contributions of this chapter regarding the total DOF of the X channel and the K-user GIC.

3.1.1 Bringing Another Dimension to the Picture: Rational Dimension

Proposed in [21], the first example of interference alignment is done in Euclidean spaces. Briefly, the n-dimensional Euclidean space ($n \geq 2$) available at a receiver is partitioned into two subspaces. A subspace is dedicated to interference and all interfering users are forced to respect this constraint. The major technique is to reduce the dimension of this subspace so that the available dimension in the signal subspace allows higher data rate for the intended users. Alignment using structural codes is also considered by several researchers [27, 29]. Structural interference alignment is used to make the interference caused by users less severe by reducing the number of possible codewords at receivers. Even though useable in one-dimensional spaces, this technique does not allow transmission of different data streams as there is only one dimension available for transmission.

In this chapter, we show that there exist available dimensions (called rational dimensions) in one-dimensional spaces which open new ways of transmitting several data streams from a transmitter and interference alignment at the receiver. A coding scheme that provides sufficient tools to incorporate the rational dimensions in transmission is proposed. This coding scheme relies on the fact that irrational numbers can play the role of directions in Euclidean spaces and data can be sent by using rational numbers. This fact is proved by using the results of Hurwitz, Khintchine, and Groshev obtained in the field of Diophantine Approximation. In the encoding part, two types of constellation are used to modulate data streams. Type I or single layer constellation refers to the case where all integer points in an interval are chosen as constellation points. Despite its simplicity, it is shown that the single layer constellation is capable of achieving the total DOF of several channels. Type II or multi-layer constellation refers to the case that a subset of integer points in an interval is chosen as constellation points. Being able of achieving the total DOF of some channels, this constellation is more useful when all channel gains are rational.

3.1.2 Breaking the Ice: Alignment in One dimension

Obtained results regarding the total DOF of networks are based on interference alignment in n-dimensional Euclidean spaces where $n \geq 2$, c.f. [21, 23, 24, 42, 43, 44]. For example in [24], the total DOF of the K-user Gaussian interference channel is derived when each transmitter and receiver is equipped with a single antenna. In order to be able to align the interference, however, it is assumed that the channel is varying. This in fact means that nodes are equipped with multiple antennas and channel coefficients are diagonal matrices.

Recently, [30] and [31] independently reported that the total DOF of some classes of fully connected GICs can be achieved. Although being time invariant, these classes have measure zero with respect to Lebesque measure. In this chapter, we prove that the total DOF of time invariant two-user X channel which is $\frac{4}{3}$ can be attained almost surely. In other words, the set of channels that this DOF can not be achieved has measure zero. This is done by incorporating rational dimensions in transmission. In fact, two independent data streams from each transmitter are send, while at each receiver two interfering streams are aligned. This achieves the multiplexing gain of $\frac{1}{3}$ per data streams and the total of $\frac{4}{3}$ for the system. We also prove that the same DOF can be achieved for the three-user GIC. However, for this case there is a gap between the available upper bound, i.e. $\frac{3}{2}$, and the achievable DOF.

3.1.3 K-user GICs: Channel Gains May Help

In [30], it is shown that the total DOF of a K-user GIC interference channel can be achieved almost surely when all the cross links have rational gains while the direct links have irrational gains. This result is generalized by introducing the concept of rational dimensions. The rational dimension of a set of numbers is defined as the dimension of numbers over the field of rational numbers. For example, if all numbers are rational then the dimension is one. We show that if the cross links arriving at a receiver has rational dimension m or less and it is the case for all receivers then the total DOF of $\frac{K}{m+1}$ is achievable. In special case where $m = 1$, it collapses to the result of Etkin and Ordentlich.

3.1.4 Strange Behavior: Discontinuity of DOF

To highlight some important features of the three-user GIC, the symmetric case in which the channel is governed by a single channel gain is considered. First, it is proved that when the channel gain is irrational then the total DOF of the channel can be achieved. This is obtained by using multi-layer constellations in encoding together with Hurwitz's theorem in the analysis. There is, however, a subtle difference between this result and the one obtained for the K-user GIC. Here, we prove that the result holds for all irrational numbers while in the K-user case we prove that it holds for almost all real numbers. In fact, there may be some irrational numbers not satisfying the requirements of the K-user case.

When the channel gain is rational then more sophisticated multi-layer constellation design is required to achieve higher performance. The reason is that interference and data are sharing the same dimension and splitting them requires more structure in constellations. We propose a multi-layer constellation in which besides satisfying the requirement of splitting interference and data, points are packed efficiently in the real line. This is accomplished by allowing carry over from different levels. Being much simpler in design, avoiding carry over, however, results in lower DOF. We show that the DOF is roughly related to the maximum of numerator and denominator, but it is always less than $\frac{3}{2}$.

Viewing the total DOF of the channel as a function of the channel gain, we observe that this function is everywhere discontinuous which means it is discontinuous at all points. This is a strange behavior as in all previous results the DOF is a continuous function almost everywhere. Although this is only achievable, the result of Etkin an Ordentlich in [30] confirms that this is in fact the case.

3.2 Coding Scheme

In this section, a coding scheme for data transmission in a shared medium is proposed. It is assumed that the channel is real, additive, and time invariant. The Additive White Gaussian Noise (AWGN) with variance σ^2 is added to the received signals at all receivers. Moreover, transmitters are subject to the power constraint P. The Signal to Noise Ratio (SNR) is defined as $\text{SNR} = \frac{P}{\sigma^2}$.

The proposed coding is rather general and can be applied to several communication systems as it will be explored in details in the following sections. In what follows, the encoding and decoding parts of the scheme are explained. The important features unique to the scheme are also investigated.

3.2.1 Encoding

A transmitter limits its input symbols to a finite set which is called the transmit constellation. Even though it has access to the continuum of real numbers, restriction to a finite set has the benefit of easy and feasible interference management. Having a set of finite points as input symbols, however, does not rule out transmission of multiple data streams from a single transmitter. In fact, there are situations where a transmitter wishes to send data to several receivers (such as the X channel) or having multiple data streams intended for a single receiver increases the throughput of the system (such as the interference channel). In what follows, it is shown how a finite set of points can accommodate different data streams.

Let us first explain the encoding of a single data stream. The transmitter selects a constellation $\mathcal{U}_i$ to send the data stream i. The constellation points are chosen from integer points, i.e., $\mathcal{U}_i \subset \mathbb{Z}$. It is assumed that $\mathcal{U}_i$ is a bounded set. Hence, there is a constant Q_i such that $\mathcal{U}_i \subset [-Q_i, Q_i]$. The cardinality of $\mathcal{U}_i$ which limits the rate of data stream i is denoted by $|\mathcal{U}_i|$.

Two choices for the constellation $\mathcal{U}_i$ are considered. The first one, referred to as Type I or single layer constellation, corresponds to the case where all integers between $-Q_i$ and Q_i are selected. This is a simple choice yet capable of achieving the total DOF of several channels.

In the second one, referred to as Type II or multi-layer constellation, constellation points are represented to a base $W \in \mathbb{N}$. In other words, a point in the constellation can be written as

$$u_i(\mathbf{b}) = \sum_{k=0}^{L-1} b_l W^l, \tag{3.2}$$

where $b_l \in \{0, 1, \ldots, a-1\}$ and $l \in \{1, 2, \ldots, L-1\}$. $\mathbf{b} = (b_0, \ldots, b_{L-1})$ is in fact another way of expressing u_i in W-array representation. a is the upper limit on the digits and clearly $a < W$. In fact, if $a = W$ then Type II constellation renders itself as Type I constellation which is not of interest. Each constellation point can be expressed by L digits and each digit carries independent message. Each of these digits is referred to as a layer of data. In other words, Type II constellation carries L layers of information.

Having formed the constellation, the transmitter constructs a random codebook for data stream i with rate R_i. This can be accomplished by choosing a probability distribution on the input alphabets. The uniform distribution is the first candidate and it is selected here for the sake of brevity.

In general, the transmitter wishes to send L data streams to one or several receivers. It first constructs L data streams using the above procedure. Then, it combines them using a linear combination of all data streams. The transmit signal can be represented by

$$u = T_1 u_1 + T_2 u_2 + \ldots + T_L u_L, \tag{3.3}$$

where $u_i \in \mathcal{U}_i$ carries information for data stream i. T_i is a constant real number that functions as a separator splitting data stream i from the transmit signal. In fact, one can make an analogy between single and multiple antenna systems by regarding that the data stream i is in fact transmitted in the direction T_i.

T_i's are rationally independent, i.e., the equation $T_1 x_1 + T_2 x_2 + \ldots + T_L x_L = 0$ has no rational solutions. This independence is due to the fact that a unique map from constellation points to the message sets is required. By relying on this independence, any real number u belonging to the set of constellation points is uniquely decomposable as $u = \sum_{i=1}^{L} T_i u_i$. Observe that if there is another possible decomposition $u = \sum_{i=1}^{L} T_i u_i'$ then it forces T_i's to be rationally dependent.

To adjust the power, the transmitter multiplies the signal by a constant A, i.e., the transmit signal is $x = Au$.

3.2.2 Received Signal and Interference Alignment

A receiver in the system may observe a signal which is a linear combination of several data streams and AWGN. The received signal in its general form can be represented as

$$y = g_0 u_0 + \underbrace{g_1 u_1 + \ldots + g_M u_M}_{I} + z, \tag{3.4}$$

where u_i is the received signal corresponding to the data stream i and z is the AWGN with covariance σ^2. g_i is a constant which encapsulates several multiplicative factors from a transmitter to the receiver. Without loss of generality, it is assumed that the receiver wishes to decode the first data stream u_0 which is encoded with rate R_0. The rest of data streams is the interference for the intended data stream and is denoted by I.

The proposed encoding scheme is not optimal in general. However, it provides sufficient tools to accomplish interference alignment in the network which in turn maximizes the throughput of the system. In n-dimensional Euclidean spaces ($n \geq 2$), two interfering signals are aligned when they receive in the same direction at the receiver. In general, m signals are aligned at a receiver if they span a subspace with dimension less than m. We claim that, surprisingly, similar arguments can be applied in one-dimensional spaces. The definition of aligned data streams is needed first.

Definition 4 (Aligned Data Streams)**.** *Two data streams u_i and u_j are said to be aligned at a receiver if the receiver observes a rational combination of them.*

As it will be shown in the following sections, if two streams are aligned, then their effect at the receiver is similar to a single data stream at high SNR regimes. This is due to the fact that rational numbers form a field and therefore the sum of constellations is again a constellation from $\mathbb{Q}$ with enlarged cardinality.

To increase R_0, it is desirable to align data streams in the interference part of the signal, i.e. I. The interference alignment in its simplest form happens when several data streams arrive at the receiver with similar coefficients, e.g. $I = gu_1 + gu_2 + \ldots + gu_M$. In this case, the data streams can be bundled to a single stream with the same coefficient. It is possible to extend this simple case of interference alignment to more general cases. First, the following definition is needed.

Definition 5 (Rational Dimension)**.** *The rational dimension of a set of real numbers $\{h_1, h_2, \ldots, h_M\}$ is m if there exists a set of real numbers $\{H_1, H_2, \ldots, H_m\}$ such that each h_i can be represented as a rational combination of H_j's, i.e., $h_i = \alpha_{i1}H_1 + \alpha_{i2}H_2 + \ldots + \alpha_{im}H_m$ where $\alpha_{ik} \in \mathbb{Q}$ for all $k \in \{1, 2, \ldots, m\}$. In particular, $\{h_1, h_2, \ldots, h_M\}$ are rationally independent if the rational dimension is M, i.e., none of the numbers can be represented as the rational combination of other numbers.*

Remark 6. *In the above definition, one can replace the set of rational numbers with integers as multiplication of irrational numbers with integers results in irrational numbers. Therefore, two alternative definitions are used in this chapter.*

In fact, the rational dimension is the effective dimension seen at the receiver. To see this, suppose that the coefficients in the interference part of the signal $I = g_1u_1 + g_2u_2 + \ldots + g_Mu_M$

has rational dimension m with bases $\{G_1, \ldots, G_m\}$. Therefore, each g_i for $i \in \{1, 2, \ldots, M\}$ can be written as $g_i = \alpha_{i1}G_1 + \alpha_{i2}G_2 + \ldots + \alpha_{im}G_m$ where α_{ik} is an integer. Plugging into the equation, it is easy to see that I can be represented as $I = G_1 I_1 + G_2 I_2 + \ldots + G_m I_m$ where I_k is a linear combination of data streams with integer coefficient. In fact, if the coefficients have dimension m then the interference part of the signal occupies m rational dimensions and one dimension is available for the signal. On the other hand, since the dimension is one, it can be concluded that the multiplexing gain of the intended data stream is $\frac{1}{m+1}$. In one extreme case the rational dimension is one and all coefficients are an integer multiple of a real number and $m = 1$.

3.2.3 Decoding

After rearranging the interference part of the signal, the received signal can be represented as

$$y = G_0 u_0 + G_1 I_1 + \ldots + G_m I_m + z, \tag{3.5}$$

where $G_0 = g_0$ to unify the notation. In what follows, the decoding scheme used to decode u_1 from y is explained. It is worth noting that if the receiver is interested in more than one data stream, then it performs the same decoding procedure for each data stream.

At the receiver, the received signal is first passed through a hard decoder. The hard decoder looks at the received constellation $\mathcal{U}_r = G_0\mathcal{U}_0 + G_1\mathcal{I}_1 + \ldots + G_m\mathcal{I}_m$ and maps the received signal to the nearest point in the constellation. This changes the continuous channel to a discrete one in which the input symbols are from the transmit constellation $\mathcal{U}_1$ and the output symbols are from the received constellation.

Remark 7. *$\mathcal{I}_j$ is the constellation due to single or multiple data streams. Since it is assumed that in the latter case it is a linear combination of multiple data streams with integer coefficients, it can be concluded that $\mathcal{I}_j \subset \mathbb{Z}$ for $j \in \{1, 2, \ldots, m\}$.*

To bound the performance of the decoder, it is assumed that the received constellation has the property that there is a many-to-one map from $\mathcal{U}_r$ to $\mathcal{U}_0$. This in fact implies that if there is no additive noise in the channel then the receiver can decode the data stream with zero error probability. This property is called property Γ. It is assumed that this property holds for all received constellations. To satisfy this requirement at all receivers, usually a careful transmit constellation design is needed at all transmitters.

Let $d_{\min}$ denote the minimum distance in the received constellation. Having Property Γ, the receiver passes the output of the hard decoder through the many-to-one map from $\mathcal{U}_r$ to $\mathcal{U}_0$. The output is called $\hat{u}_1$. Now, a joint-typical decoder can be used to decode the data stream from a block of $\hat{u}_0$s. To calculate the achievable rate of this scheme, the error probability of transmitting a symbol from $\mathcal{U}_0$ and receiving another symbol, i.e. $P_e = Pr\{\hat{U}_0 \neq U_0\}$ is bounded as:

$$P_e \leq Q\left(\frac{d_{\min}}{2\sigma}\right) \leq \exp\left(-\frac{d_{\min}^2}{8\sigma^2}\right). \tag{3.6}$$

Now, P_e can be used to lower bound the rate achievable for the data stream. In [30], Etkin and Ordentlich used Fano's inequality to obtain a lower bound on the achievable rate which is tight in high SNR regimes. Following similar steps, one can obtain

$$\begin{aligned} R_0 &= I(\hat{U}_0, U_0) \\ &= H(U_0) - H(U_0|\hat{U}_0) \\ &\overset{a}{\geq} H(U_0) - 1 - P_e \log|\mathcal{U}_0| \\ &\overset{b}{\geq} \log|\mathcal{U}_0| - 1 - P_e \log|\mathcal{U}_0| \end{aligned} \tag{3.7}$$

where (a) follows from Fano's inequality and (b) follows from the fact that U_1 has the uniform distribution. To have multiplexing gain of at least r_0, $|U_1|$ needs to scale as SNR^{r_0}. Moreover, if P_e scales as $\exp\left(SNR^{-\epsilon}\right)$ for an $\epsilon > 0$, then it can be shown that $\frac{R_0}{\log \mathrm{SNR}}$ approaches r_0 at high SNR regimes.

Remark 8. *After interference alignment the interference term has no longer the uniform distribution. However, the lower bound on the achievable rate given in (4.12) is independent of the probability distributions of the interference terms. It is possible to obtain better performance provided the distribution of the interference is exploited.*

3.3 Single Layer Constellation

In this section, the single layer constellation is used to modulate all data streams at all transmitters. Even though it is the simplest form of constellation, it is powerful enough to provide interference alignment which in turn increases the throughput of the system. Before deriving important results regarding DOF of the X and interference channels using this constellation, the performance of a typical decoder is analyzed. The attempt is to make the analysis universal and applicable to both channels.

3.3.1 Performance Analysis: The Khintchine-Groshev Theorem

The decoding scheme proposed in the previous section is used to decode the data stream u_0 from the received signal in (3.5). To satisfy Property Γ, it is assumed that $\{G_0, G_1, \ldots, G_m\}$

are independent over rational numbers. Due to this independence, any point in the received constellation has a unique representation in the bases $\{G_0, G_1, \ldots, G_m\}$ and therefore Property Γ holds in this case.

Remark 9. *In a random environment, it is easy to show that the set of* $\{G_0, G_1, \ldots, G_m\}$ *being dependent has measure zero (with respect to Lebesgue measure). Hence, in this section it is assumed that Property* Γ *holds unless otherwise stated.*

To use the lower bound on the data rate given in (4.12), one needs to calculate the minimum distance between points in the received constellation. Let us assume each stream in (3.5) is bounded (as it is the case since transmit constellations are bounded by the assumption). In particular, $\mathcal{U}_0 = [-Q_0, Q_0]$ and $\mathcal{I}_j = [-Q_j, Q_j]$ for all $j \in \{1, 2, \ldots, m\}$. Since points in the received constellation are irregular, finding $d_{\min}$ is not easy in general. Thanks to the theorems of Khintchine and Groshev, however, it is possible to lower bound the minimum distance. As it will be shown later, using this lower bound at high SNR regimes is asymptotically optimum. We digress here and explain some background needed for stating the theorem of Khintchine and Groshev.

The field of Diophantine approximation in number theory deals with approximation of real numbers with rational numbers. The reader is referred to [45, 46] and the references therein. The Khintchine theorem is one of the cornerstones in this field. It gives a criteria for a given function $\psi : \mathbb{N} \to \mathbb{R}_+$ and real number α such that $|p + \alpha q| < \psi(|q|)$ has either infinitely many solutions or at most finitely many solutions for $(p, q) \in \mathbb{Z}^2$. Let $\mathcal{A}(\psi)$ denote the set of real numbers α such that $|p+\alpha q| < \psi(|q|)$ has infinitely many solutions in integers. The theorem has two parts. The first part is the convergent part and states that if $\psi(|q|)$ is convergent, i.e.,

$$\sum_{q=1}^{\infty} \psi(q) < \infty$$

then $\mathcal{A}(\psi)$ has measure zero with respect to Lebesque measure. This part can be rephrased in more convenient way as follows. For almost all real numbers, $|p + \alpha q| > \psi(|q|)$ holds for all $(p, q) \in \mathbb{Z}^2$ except for finitely many of them. Since the number of integers violating the inequality is finite, one can find a constant κ such that

$$|p + \alpha q| > \kappa\psi(|q|)$$

holds for all integers p and q almost surely. The divergent part of the theorem states that $\mathcal{A}(\psi)$ has the full measure, i.e. the set $\mathbb{R} - \mathcal{A}(\psi)$ has measure zero, provided ψ is decreasing and $\psi(|q|)$ is divergent, i.e.,

$$\sum_{q=1}^{\infty} \psi(q) = \infty.$$

There is an extension to Khintchine's theorem which regards the approximation of linear forms. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_m)$ denote an m-tuple in $\mathbb{R}^m$ and $\mathbb{Z}^m$, respectively. Let $\mathcal{A}_m(\psi)$ denote the set of m-tuple real numbers $\boldsymbol{\alpha}$ such that

$$|p + \alpha_1 q_1 + \alpha_2 q_2 + \ldots + \alpha_m q_m| < \psi(|\mathbf{q}|_\infty) \tag{3.8}$$

has infinitely many solutions for $p \in \mathbb{Z}$ and $\mathbf{q} \in \mathbb{Z}^m$. $|\mathbf{q}|_\infty$ is the supremum norm of $\mathbf{q}$ defined as $\max_i |q_i|$. The following theorem gives the Lebesque measure of the set $\mathcal{A}_m(\psi)$.

Theorem 13 (Khintchine-Groshev)**.** *Let $\psi : \mathbb{N} \to \mathbb{R}^+$. Then the set $\mathcal{A}_m(\psi)$ has measure zero provided*

$$\sum_{q=1}^{\infty} q^{m-1}\psi(q) < \infty, \tag{3.9}$$

and has the full measure if

$$\sum_{q=1}^{\infty} q^{m-1}\psi(q) = \infty \qquad \text{and } \psi \text{ is monotonic.} \tag{3.10}$$

In this chapter, the convergent part of the theorem is used. Moreover, given an arbitrary $\epsilon > 0$ the function $\psi(q) = \frac{1}{q^{m+\epsilon}}$ satisfies (4.2). In fact, the convergent part of the theorem used in this chapter can be stated as follows. For almost all m-tuple real numbers there exists a constant κ such that

$$|p + \alpha_1 q_1 + \alpha_2 q_2 + \ldots + \alpha_m q_m| > \frac{\kappa}{(\max_i |q_i|)^{m+\epsilon}} \tag{3.11}$$

holds for all $p \in \mathbb{Z}$ and $\mathbf{q} \in \mathbb{Z}^m$.

The Khintchine-Groshev theorem can be used to bound the minimum distance of points in the received constellation. In fact, a point in the received constellation has a linear form, i.e., $u_r = G_0 u_0 + G_1 I_1 + \ldots + G_m I_m$. Dividing by G_0 and using (4.5), one can conclude that

$$d_{\min} > \frac{\kappa G_0}{(\max_{i\in\{1,\ldots,m\}} Q_i)^{m+\epsilon}} \tag{3.12}$$

The probability of error in hard decoding, see (4.11), can be bounded as

$$P_e < \exp\left(-\frac{(\kappa G_0)^2}{8\sigma^2(\max_{i\in\{1,\ldots,m\}} Q_i)^{2m+2\epsilon}}\right). \tag{3.13}$$

Let us assume Q_i for $i \in \{0, 1, \ldots, m\}$ is $\lfloor \gamma_i P^{\frac{1-\epsilon}{2(m+1+\epsilon)}} \rfloor$ where γ_i is a constant. Moreover, ϵ is the constant appeared in (4.5). We also assume that $G_0 = \gamma P^{\frac{m+2\epsilon}{2(m+1+\epsilon)}}$. As it will be shown later, these assumptions are realistic and can be applied to the coding schemes proposed in this chapter. It is worth mentioning that in this chapter it is assumed that each data stream carries the same rate in the asymptotic case of high SNR, i.e., they have the same multiplexing gain. However, in more general cases one may consider different multiplexing gains for different data streams. Substituting in (4.16) yields

$$P_e < \exp\left(-\delta P^{\epsilon}\right), \tag{3.14}$$

where δ is a constant and a function of γ, κ, σ, and γ_i's. The lower bound obtained in (4.12) for the achievable rate becomes

$$\begin{aligned} R_0 &> (1 - P_e)\log|\mathcal{U}_0| - 1 \\ &\stackrel{a}{=} (1-\exp(-\delta P^{\epsilon}))\log(2\lfloor \gamma_i P^{\frac{1-\epsilon}{2(m+1+\epsilon)}} \rfloor) - 1 \\ &> \frac{(1-\epsilon)(1-\exp(-\delta P^{\epsilon}))}{2(m+1+\epsilon)}(\log(P)+\vartheta) - 1 \end{aligned} \tag{3.15}$$

where (a) follows from the fact that $|\mathcal{U}_\prime| = 2Q_0$ and ϑ is a constant. The multiplexing gain of the data stream u_0 can be computed using (3.15) as follows

$$\begin{aligned} r_0 &= \lim_{P\to\infty} \frac{R_0}{0.5\log(P)} \\ &> \frac{1-\epsilon}{m+1+\epsilon}. \end{aligned} \tag{3.16}$$

Since ϵ can be made arbitrarily small, we can conclude that $r = \frac{1}{m+1}$ is indeed achievable. In the following theorem, this result and its required conditions are summarized.

Theorem 14. *A receiver can reliably decode the data stream u_0 with multiplexing gain $\frac{1}{m+1}$ from the received signal $y = G_0u_0 + G_1I_1 + \ldots + G_mI_m + z$ if the following regularity conditions are satisfied:*

1. *$G_0 = \gamma P^{\frac{m+2\epsilon}{2(m+1+\epsilon)}}$ where γ is a constant.*
2. *$u_0 \in [-Q_0, Q_0]$ where $Q_0 = \lfloor \gamma_0 P^{\frac{1-\epsilon}{2(m+1+\epsilon)}} \rfloor$ and γ_0 is a constant. Moreover, the uniform distribution is used to construct the random codebook.*
3. *For $i \in \{1, 2, \ldots, m\}$, $I_i \in [-Q_i, Q_i]$ where $Q_i = \lfloor \gamma_i P^{\frac{1-\epsilon}{2(m+1+\epsilon)}} \rfloor$ and γ_i is a constant.*
4. *G_is for $i \in \{0, 1, \ldots, _m\}$ are independent over rational numbers.*
5. *$\{\frac{G_1}{G_0}, \frac{G_2}{G_0}, \ldots, \frac{G_m}{G_0}\}$ is among m-tuples that satisfy (4.5).*

Moreover, the last two conditions hold almost surely.

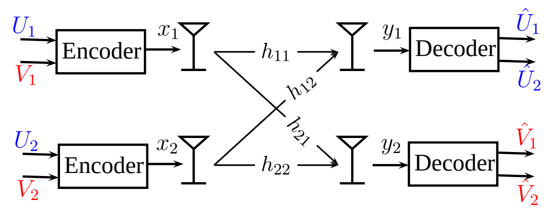

Figure 3.1: The two-user X channel: Transmitter 1 sends data streams U_1 and V_1 to Receiver 1 and 2, respectively. Similarly, Transmitter 2 sends data streams U_2 and V_2 to Receiver 1 and 2, respectively.

3.3.2 Two-user X channel: DOF $= \frac{4}{3}$ is Achievable Almost Surely

The proposed coding scheme using the single layer constellation is applied to the two-user X channel as the first example. The two-user X channel is introduced in [21] where the first explicit interference alignment is used to achieve the total DOF of a class of MIMO X channels. In this channel, see Figure 3.1, there are two transmitters and two receivers. Transmitter 1 wishes to send data streams U_1 and V_1 to Receivers 1 and 2, respectively. Similarly, Transmitter 1 wishes to send data streams U_2 and V_2 to Receivers 1 and 2, respectively. The input-output relation of the channel can be stated as

$$y_1 = h_{11}x_1 + h_{12}x_2 + z_1,$$
$$y_2 = h_{21}x_1 + h_{22}x_2 + z_2,$$

where z_1 and z_2 are AWGN with variance σ^2. x_1 and x_2 are input symbols of Transmitter 1 and 2, respectively. Input signals are subject to the power constraint P. h_{ij} is the channel gain from Transmitter j to Receiver i. Moreover, channel gains are assumed to be constant over time. y_1 and y_2 are received signals at Receiver 1 and 2, respectively.

In [23], an upper bound on the DOF of the channel is obtained. This upper bound for the single antenna case is $\frac{4}{3}$. We will show that this upper bound is in fact achievable. If each data stream occupies $\frac{1}{3}$ of DOF then the total DOF becomes $\frac{4}{3}$. Therefore, it is assumed that all data streams, i.e. U_1, U_2, V_1 and V_2, use the same constellation with integer points from interval $[-Q, Q]$ with $Q = \lfloor \gamma P^{\frac{1-\epsilon}{2(3+\epsilon)}} \rfloor$ where γ and ϵ are two arbitrary constants. Transmitter 1 (respectively 2) encodes the data streams U_1 and V_1 (respectively U_2 and V_2) utilizing the encoding scheme proposed in the previous section. The following linear combinations are used to send the data streams through the channel.

$$x_1 = G(h_{22}u_1 + h_{12}v_1), \tag{3.17}$$

$$x_2 = G(h_{21}u_2 + h_{11}v_2), \tag{3.18}$$

where G is the normalizing factor. To find G, one needs to calculate the transmit power of User 1 and 2. It is easy to show that there exists a constant γ' such that $G = \gamma' P^{\frac{2+2\epsilon}{2(3+\epsilon)}}$ normalizes the transmit power to be less than P at both receivers.

After rearranging, the received signal can be written as

$$\begin{aligned} y_1 &= Gh_{11}h_{22}u_1 + Gh_{12}h_{21}u_2 + Gh_{11}h_{12}(\underbrace{v_1 + v_2}_{I_1}) + z_1, \\ y_2 &= Gh_{21}h_{22}(\underbrace{u_1 + u_2}_{I_2}) + Gh_{12}h_{21}v_1 + Gh_{11}h_{22}v_2 + z_2. \end{aligned}$$

Now, it becomes clear why the linear combinations in (3.17) and (3.18) are used to combine the data streams at the transmitters. In fact, the data streams V_1 and V_2 not intended for Receiver 1 arrive with the same coefficients at Receiver 1. In other words, they are aligned at the receiver and hence their effect can be regarded as a single data stream. Let I_1 denote the sum $v_1 + v_2$. Clearly, I_1 is an integer and belongs to $[-2Q\ 2Q]$. Receiver 1 wishes to decode U_1 and U_2. As proposed in the previous section, each data stream is decode separately at the receiver. Therefore, decoding of the data stream U_1 is first considered. It is easy to see that all regularity conditions given in Theorem 14 are satisfied with $m = 2$. Hence, Receiver 1 can reliably decode U_1 which has the multiplexing gain of $\frac{1}{3}$. Similarly, Receiver 2 can decode U_2 which has the multiplexing gain of $\frac{1}{3}$. A similar phenomenon happens in the second receiver. Therefore, we have proved the following theorem.

Theorem 15. *The DOF of the two-user X channel is $\frac{4}{3}$ almost surely.*

3.3.3 K-user Gaussian Interference Channel: Special Cases

The K-user GIC models a network in which K transmitter-receiver pairs (users) sharing a common bandwidth wish to have reliable communication at maximum rate. The channel's input-output relation can be stated as, see Figure 4.1,

$$\begin{aligned} y_1 &= h_{11}x_1 + h_{12}x_2 + \ldots + h_{1K}x_K + z_1, \\ y_2 &= h_{21}x_1 + h_{22}x_2 + \ldots + h_{2K}x_K + z_2, \\ \vdots &= \ \vdots \qquad\quad \vdots \qquad\quad \ddots \qquad \vdots \\ y_K &= h_{K1}x_1 + h_{K2}x_2 + \ldots + h_{KK}x_K + z_K, \end{aligned} \tag{3.19}$$

where x_i and y_i are input and output symbols of User i for $i \in \{1, 2, \ldots, K\}$, respectively. z_i is AWGN with variance σ^2 for $i \in \{1, 2, \ldots, K\}$. Transmitters are subject to the power constraint P.

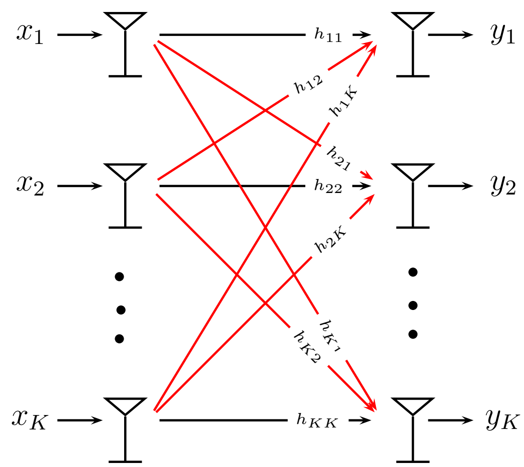

Figure 3.2: The K-user GIC. User i for $i \in \{1, 2, \ldots, K\}$ wishes to communicate with its corresponding receiver while receiving interference from other users.

An upper bound on the DOF of this channel is obtained in [24]. The upper bound states that the total DOF of the channel is less than $\frac{K}{2}$ which means each user can at most use one half of its maximum DOF. This upper bound can be achieved by using single layer constellation in special case where all cross gains are rational numbers [30]. This is due to the fact that these coefficients lie on a single rational dimensional space and therefore the effect of the interference caused by several transmitters behaves as that of interference caused by a single transmitter. Using a single data stream, one can deduce that the multiplexing gain of $\frac{1}{2}$ is achievable for each user.

Restriction to transmission of single data streams is not optimal in general. As an example showing this fact, in the next subsection, it is proved that by having multiple data streams one can obtain higher DOF. However, using single data streams has the advantage of simple analysis. We are interested in the DOF of the system when each user employs a single data stream. The following theorem states the result. This in fact generalizes the result obtained in [30].

Theorem 16. *The DOF of $\frac{K}{m+1}$ is achievable for the K-user Gaussian interference channel using the single data stream transmission scheme provided the set of cross gains at each receiver has the rational dimension of at most m.*

Proof. To communicate with its corresponding receiver, each transmitter transmits one data stream modulated with single layer constellation. It is assumed that all users use the same constellation, i.e., $\mathcal{U}_i = [-Q \ Q]$ for $i \in \{1, 2, \ldots, K\}$. We claim that under the conditions assumed in the theorem each transmitter can achieve the multiplexing gain of $\frac{1}{m+1}$. To accommodate this data rate, Q is set to $\lfloor P^{\frac{1-\epsilon}{2(m+1+\epsilon)}} \rfloor$. The transmit signal from Transmitter i

is $x_i = Gu_i$ for $i \in \{1, 2, \ldots, k\}$ where G is the normalizing factor and equals $\gamma P^{\frac{m+2\epsilon}{2(m+1+\epsilon)}}$ and γ is a constant. Due to the symmetry obtained by proposed coding scheme, it is sufficient to analyze the performance of the first user. The received signal at Receiver 1 can be represented as

$$y_1 = G(h_{11}u_1 + h_{1K}u_2 + \ldots + h_{1K}u_K) + z_1. \tag{3.20}$$

Let us assume the rational dimension of $(h_{12}, h_{13}, \ldots, h_{1K})$ is less than m. Hence, there exists a set of real numbers $(g_1, g_2, \ldots, g_m)$ such that each h_{1j} can be represented as

$$h_{1j} = \sum_{l=1}^{m} \alpha_{jl} g_l, \tag{3.21}$$

where $\alpha_{jl} \in \mathbb{Z}$ for $j \in \{2, \ldots, K\}$ and $l \in \{1, 2, \ldots, m\}$. Substituting in (3.20) and rearranging yields

$$y_1 = G(h_{11}u_1 + g_1 I_1 + \ldots + g_m I_m) + z_1. \tag{3.22}$$

where $I_l \in \mathbb{Z}$ for $l \in \{1, 2, \ldots, m\}$ and

$$I_l = \sum_{j=2}^{K} \alpha_{jl} u_j. \tag{3.23}$$

It is easy to prove that there is a constant γ_l such that $I_l \in [-Q_l \ Q_l]$ for $l \in \{1, 2, \ldots, m\}$ where $Q_l = \lfloor \gamma_l P^{\frac{1-\epsilon}{2(m+1+\epsilon)}} \rfloor$. Receiver 1 decodes its corresponding data stream from received signal in (3.22) using the decoding scheme proposed in the previous section. By one-to-one correspondence with regularity conditions in Theorem 14, one can deduce that Receiver one is able to decode the data stream u_1 and in fact the multiplexing gain of $\frac{1}{m+1}$ is achievable almost surely. Due to the symmetry, we can conclude that the DOF of $\frac{K}{m+1}$ is achievable for the system. This completes the proof. □

3.3.4 Three-user Gaussian Interference Channel: DOF $= \frac{4}{3}$ is Achievable Almost Surely

In this subsection, we consider the three-user GIC. First, the following model is defined as the standard model for the channel.

Definition 6. *The three-user interference channel is called standard if it can be represented as*

$$\begin{aligned} y_1 &= G_1 x_1 + x_2 + x_3 + z_1 \\ y_2 &= G_2 x_2 + x_1 + x_3 + z_2 \\ y_3 &= G_3 x_3 + x_1 + G_0 x_2 + z_3, \end{aligned} \tag{3.24}$$

where x_i for User i is subject to the power constraint P. z_i at Receiver i is AWGN with variance σ^2.

In the following lemma, it is proved that in fact characterizing the DOF of the standard channel causes no harm on the generalization of the problem.

Lemma 8. *For every three-user GIC there exists a standard channel with the same DOF.*

Proof. The channel model is the special case of that of K-user GIC in (4.19) where $K = 3$, i.e., the input-output relation can be written as

$$\begin{aligned} y_1 &= h_{11}x_1 + h_{12}x_2 + h_{13}x_3 + z_1 \\ y_2 &= h_{21}x_1 + h_{22}x_2 + h_{23}x_3 + z_2 \\ y_3 &= h_{31}x_1 + h_{32}x_2 + h_{33}x_3 + z_3. \end{aligned} \tag{3.25}$$

Clearly, linear operations at transmitters and receivers do not affect the capacity region of the channel. Hence, we adopt the following linear operations:

1. Transmitter 1 sends $x_1 = \frac{h_{23}h_{12}}{h_{21}}\tilde{x}_1$ to the channel and Receiver 1 divides the received signal by $h_{12}h_{13}$.

2. Transmitter 2 sends $x_2 = h_{13}\tilde{x}_2$ to the channel and Receiver 2 divides the received signal by $h_{12}h_{23}$.

3. Transmitter 3 sends $x_3 = h_{12}\tilde{x}_3$ to the channel and Receiver 3 divides the received signal by $\frac{h_{21}}{h_{12}h_{23}h_{31}}$.

If $\tilde{y}_i$ for $i \in \{1, 2, 3\}$ denotes the output of Receiver i after above operations then it is easy to see that from input $\tilde{x}_i$ to output $\tilde{y}_i$ the channel behaves as (4.21), i.e., it can be written as

$$\begin{aligned} \tilde{y}_1 &= G_1\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 + \tilde{z}_1 \\ \tilde{y}_2 &= G_2\tilde{x}_2 + \tilde{x}_1 + \tilde{x}_3 + \tilde{z}_2 \\ \tilde{y}_3 &= G_3\tilde{x}_3 + \tilde{x}_1 + G_0\tilde{x}_2 + \tilde{z}_3, \end{aligned} \tag{3.26}$$

where $\tilde{z}_i$ is the Gaussian noise at Receiver i for $i \in \{1, 2, 3\}$ with variance $\sigma_i^2 = \delta_i\sigma^2$ where δ_i is constant depending on the channel coefficients. Similarly, the input power constraint of Transmitter i for $i \in \{1, 2, 3\}$ becomes $P_i = \gamma_i P$ where γ_i is constant depending on the channel coefficients. Moreover, the channel coefficients can be written as

$$\begin{aligned}
G_0 &= \frac{h_{13}h_{21}h_{32}}{h_{12}h_{23}h_{31}},\\
G_1 &= \frac{h_{11}h_{12}h_{23}}{h_{12}h_{21}h_{13}},\\
G_2 &= \frac{h_{22}h_{13}}{h_{12}h_{23}},\\
G_3 &= \frac{h_{33}h_{12}h_{21}}{h_{12}h_{23}h_{31}}.
\end{aligned}$$

Since the above operations change the input powers as well as the noise variances, the completion of the theorem requires additional steps to make the power constraints as well as noise variances all equal. Notice that increasing (resp. decreasing) the power and decreasing (resp. increasing) the noise variance enlarges (resp. shrinks) the capacity region of the channel. Therefore, two channels are defined as follows. In the first channel with the same input-output relation as of (3.26) the power constraints at all transmitters and the noise variances at all receivers are set to $\max\{P_1, P_2, P_3\}$ and $\min\{\sigma_1^2, \sigma_2^2, \sigma_3^2\}$, respectively. Similarly, in the second channel the power constraints and noise variances are set to $\max\{P_1, P_2, P_3\}$ and $\min\{\sigma_1^2, \sigma_2^2, \sigma_3^2\}$, respectively. The capacity region of the channel is sandwiched between that of these two channels. Moreover, at high power regimes the SNRs of these two channel differ by a constant multiplicative factor. Hence, they share the same DOF and either of them can be used as the desired channel. This completes the proof. □

Having the standard model, a special case that the total DOF of the channel can be achieved is identified in the following theorem.

Theorem 17. *If the channel gain G_0 in (4.21) is rational then the DOF of $\frac{3}{2}$ is achievable almost surely.*

Proof. If G_0 is rational, then the set of cross gains at each receiver takes up one rational dimension. Applying Theorem 16 with $m = 1$ gives the desired result. □

In general, the event of having rational G_0 has probability zero. The following theorem concerns the general case.

Theorem 18. *The DOF of $\frac{4}{3}$ is achievable for the three-user GIC almost surely.*

Proof. The encoding used to prove this theorem is asymmetrical. User 1 encodes two data streams while User 2 and 3 encode only one data stream. In fact, the transmit constellation of Users 1,2, and 3 are $\mathcal{U}_1 + G_0\mathcal{U}_1'$, $\mathcal{U}_2$, and $\mathcal{U}_2$, respectively. It is assumed that $\mathcal{U}_1$, $\mathcal{U}_1'$, $\mathcal{U}_2$,

$\mathcal{U}_3$ are single layer constellation with points in $[-Q\ Q]$. We claim that each data stream can carry data with multiplexing gain of $\frac{1}{3}$, and since there are four data streams, the DOF of $\frac{4}{3}$ is achievable. To accommodate such rate $Q = \lfloor \gamma P^{\frac{1-\epsilon}{2(3+\epsilon)}} \rfloor$ where γ and ϵ are two arbitrary constants. The input signals from Transmitters 1, 2, and 3 are $x_1 = A(u_1 + G_0 u_1')$, $x_2 = Au_2$, and $x_3 = Au_3$, respectively. A is the normalizing factor which controls the output power of all transmitters. It can be readily shown that there exists a constant γ' such that $A = \gamma' P^{\frac{2+2\epsilon}{2(3+\epsilon)}}$.

The decoding at Receivers are performed differently. The received signal at Receiver 1 can be represented as

$$y_1 = A(G_1 u_1 + G_1 G_0 u_1' + I_1) + z_1, \tag{3.27}$$

where $I_1 = u_2 + u_3$ is the interference caused by Users 2 and 3. Clearly $I_1 \in [-2Q\ 2Q]$. Receiver 1 is interested in both u_1 and u_1' and performs the proposed decoding scheme for each of them separately. By applying Theorem 14, one can deduce that each of data streams u_1 and u_1' can accommodate $\frac{1}{2}$ of multiplexing gain.

The received signal at Receiver 2 can be represented as

$$y_2 = A(G_2 u_2 + I_2 + G_0 u_1') + z_2, \tag{3.28}$$

where $I_2 = u_1 + u_3$ is the aligned part of the interference caused by Users 2 and 3 and $I_2 \in [-2Q\ 2Q]$. Receiver 2 is interested in u_2 while I_2 and u_1' are interference. An application of Theorem 14 shows that the multiplexing gain of $\frac{1}{3}$ is achievable for data stream u_2.

Finally, the received signal at Receiver 3 can be represented as

$$y_3 = A(G_3 u_3 + u_1 + G_0 I_3) + z_2, \tag{3.29}$$

where $I_3 = u_1' + u_2$ is the aligned part of the interference caused by Users 2 and 3 and $I_3 \in [-2Q\ 2Q]$. Receiver 3 is interested in u_3 while I_3 and u_1 are interference. Again by using Theorem 14, one can deduce that the multiplexing gain of $\frac{1}{3}$ is achievable for data stream u_3. This completes the proof. $\square$

3.4 multi-layer Constellation

In this section, multi-layer constellations are incorporated in the encoding scheme. Here, the focus would be on the symmetric three-user GIC. This channel is modeled by:

$$\begin{aligned} y_1 &= x_1 + h(x_2 + x_3) + z_1 \\ y_2 &= x_2 + h(x_3 + x_1) + z_2 \\ y_3 &= x_3 + h(x_1 + x_2) + z_3 \end{aligned} \tag{3.30}$$

where x_i and y_i are the transmit and the received signals of User i, respectively. The additive noise z_i for $i \in \{1,2,3\}$ is Gaussian distributed with zero mean and variance σ^2. Users are subject to the power constraints P.

This channel is among channels satisfying conditions of Theorem 17. Hence, one can deduce that the total DOF of $\frac{3}{2}$ is achievable for this channel almost surely. The reason for considering the symmetric case is to reveal some aspects of multi-layer constellations. In this section, we obtain an achievable DOF for all channel gains. For example, it will be shown the multi-layer constellation is capable of achieving the total DOF of $\frac{3}{2}$ for all irrational gains.

As pointed out in Section 3.2, in multi-layer constellations, constellation points are selected from points represented in the base $W \in \mathbb{N}$. Since the channel is symmetric, all transmitters use the same constellation $\mathcal{U}$ in which a point can be represented as

$$u(\mathbf{b}) = \sum_{k=0}^{L-1} b_l W^l, \tag{3.31}$$

where $b_l \in \{0,1,\ldots,a-1\}$ for all $l \in \{0,2,\ldots,L-1\}$. $\mathbf{b}$ represents the vector $(b_0, b_1, \ldots, b_{L-1})$. a is the factor which controls the number of constellation points. We assume $a < W$. Therefore, all constellation points in (3.31) are distinct and the size of the constellation is $|\mathcal{U}| = a^L$. Hence, the maximum rate possible for this data stream is bounded by $L \log a$.

A random codebook is generated by randomly choosing points form $\mathcal{C}$ using the uniform distribution. This can be accomplished by imposing a uniform distribution on each b_l. The signal transmitted by User 1,2, and 3 are respectively $x_1 = Au(\mathbf{b})$, $x_2 = Au(\mathbf{b}')$, and $x_3 = Au(\mathbf{b}'')$. A is the normalizing factor and controls the output power.

Remark 10. *The multi-layer constellation used in this chapter has DC component. In fact, this component needs to be removed at all transmitters. However, it only duplicates the achievable rate and has no effect as far as the DOF is concerned.*

To obtain A, one needs to compute the input power. Since b_l and b_j are independent for $l \neq j$, we have the following chain of inequalities

$$\begin{aligned}
E[X_1^2] &= A^2 W^{2(L-1)} \sum_{l=0}^{L-1} E\left[b_l^2\right] W^{-2l} \\
&\leq A^2 W^{2(L-1)} \frac{(a-1)(2a-1)}{6} \sum_{l=0}^{\infty} W^{-2l} \\
&\leq A^2 W^{2(L-1)} \frac{a^2}{3} \times \frac{1}{1-W^{-2}} \\
&\leq \frac{A^2 a^2 W^{2L}}{W^2-1}.
\end{aligned}$$

Hence, if $A = \frac{\sqrt{(W^2-1)P}}{aW^L}$ then $E\left[X_i^2\right] \leq P$ which is the desired power constraint.

Due to the symmetry of the system, it suffices to analyze the first user's performance. The received constellation signal at Receiver 1 can be written as

$$y_1 = A\sum_{l=0}^{L-1}\Big(b_l + hI_l\Big)W^l + z_1, \tag{3.32}$$

where $I_l = b'_l + b''_l$ is the interference caused by Transmitters 1 and 2. Clearly, the interference is aligned and $I_l \in \{0, 1, \ldots, 2(a-1)\}$. A point in the received constellation $\mathcal{U}_r$ can be represented as

$$u_r(\mathbf{b}, \mathbf{I}) = A\sum_{l=0}^{L-1}\Big(b_l + hI_l\Big)W^l, \tag{3.33}$$

where $\mathbf{I}$ represents the vector $(I_0, I_1, \ldots, I_{L-1})$. As pointed out before the received constellation needs to satisfy Property Γ. Here, Property Γ translates into the following relation:

$$\Gamma: \quad u_r(\mathbf{b}, \mathbf{I}) \neq u_r(\tilde{\mathbf{b}}, \tilde{\mathbf{I}}) \text{ iff } (\mathbf{b}, \mathbf{I}) \neq (\tilde{\mathbf{b}}, \tilde{\mathbf{I}}),$$

which means that the receiver is able to extract both $\mathbf{b}_1$ and $\mathbf{I}_1$ from the received constellation.

Using (4.12) to bound the achievable rate, the total DOF of the channel can be written as

$$\begin{aligned} r_{\text{sum}} &= \lim_{P\to\infty} \frac{3R_1}{0.5\log P} \\ &\geq \lim_{P\to\infty} \frac{3\left(\log|\mathcal{U}| - 1 - P_e\log|\mathcal{U}|\right)}{0.5\log P} \\ &= \lim_{P\to\infty} \frac{3L(1-P_e)\log a}{0.5\log P}, \end{aligned} \tag{3.34}$$

where P_e depends on the minimum distance in the received constellation $d_{\min}$ as of (4.11). In fact, to obtain the maximum rate we need to select the design parameters a, W, and L. Selection of these parameters needs to provide 1) Property Γ in the received constellation, 2) exponential decrease in P_e as P goes to infinity, 3) maximum achievable DOF of the system. In the following, we investigate the relation between these factors for rational and irrational channel gains separately.

3.4.1 Rational Channel Gains

In this subsection, we prove the following theorem which provides an achievable DOF for the symmetric three-user GIC with rational gains.

Theorem 19. *The following DOF is achievable for the symmetric three-user GIC where the channel gain is rational, i.e.* $h = \frac{n}{m}$*:*

$$r_{sum} = \begin{cases} \frac{3\log(n)}{\log(n(2n-1))} & \text{if } 2n \geq m, \\ \frac{3\log(s+1)}{\log((s+1)(2s+1))} & \text{if } 2n < m \text{ and } m = 2s+1, \\ \frac{3\log(s)}{\log(2s^2-n)} & \text{if } 2n < m \text{ and } m = 2s. \end{cases}$$

Since h is rational, it can be represented as $h = \frac{n}{m}$ where $(m, n) = 1$. In this case, Equation (3.33) can be written as

$$u_r(\mathbf{b}, \mathbf{I}) = \frac{A}{m} \sum_{l=0}^{L-1} \Big(mb_l + nI_l\Big) W^l. \tag{3.35}$$

The theorem is proved by partitioning the set of rational numbers in three subsets and analyzing the performance of the system in each of them. Let us first assume that Property Γ holds for given W and a. To obtain the total DOF of the system, one needs to derive the minimum distance in the received constellation. It is also easy to show that $d_{\min} = \frac{A}{m}$. Using (4.11), the bound on the error probability is

$$P_e < \exp\left(-\frac{(W^2-1)P}{8(am\sigma)^2 W^{2L}}\right).$$

Let L be set as

$$L = \lfloor \frac{\log\left(P^{0.5-\epsilon}\right)}{\log(W)} \rfloor, \tag{3.36}$$

where $\epsilon > 0$ is an arbitrary constant. Clearly, with this choice of K, $P_2 \leq \exp\left(-\gamma P^{2\epsilon}\right)$ where γ is a constant. This results in $P_e \to 0$ as SNR $\to \infty$. By using (3.34), the DOF of the system can be derived as

$$\begin{aligned} r_{\text{sum}} &= \lim_{P\to\infty} \frac{3L(1-P_e)\log a}{0.5\log P} \\ &= \lim_{P\to\infty} \frac{3L\log(a)}{0.5\log P} \\ &= \lim_{P\to\infty} \frac{\lfloor \frac{\log\left(P^{0.5-\epsilon}\right)}{\log W} \rfloor \log a}{0.5\log P} \\ &= \frac{\log a}{\log W}(1-2\epsilon). \end{aligned} \tag{3.37}$$

Since ϵ can be chosen arbitrarily small, the DOF of the system can be written as

$$r_{\text{sum}} = \frac{3\log a}{\log W}. \tag{3.38}$$

Table 3.1: Relation between a and W to satisfy Property Γ.

	$h = n/m$	a	W
Case I	$2n \geq m$	n	$n(2n-1)$
Case II	$2n < m$ and $m = 2s+1$	$s+1$	$(s+1)(2s+1)$
Case III	$2n < m$ and $m = 2s$	s	$2s^2 - n$

From (3.38), one can deduce that in order to maximize the total DOF of the system one needs to maximize a and minimize W while respecting Property Γ. In fact, if it is possible to have $W = a^2$ then the upper bound of $\frac{3}{2}$ can be touched. However, it is not possible in this case. The above theorem states that W and a can have the relation given in Table 3.1. Even though the relation is quadratic for all cases, the achievable DOF is always below the upper bound.

To complete the proof of Theorem 19, it is sufficient to prove that Property Γ holds for the cases given in Table 3.1.

Lemma 9. *Property* Γ *holds for all cases shown in Table 3.1.*

Proof. This lemma is proved by induction on L. To show that the lemma holds for $L = 0$, it is sufficient to prove that the equation

$$m(b_0 - \tilde{b}_0) + n(I_0 - \tilde{I}_0) = 0 \tag{3.39}$$

has no nontrivial solution when $b_0, \tilde{b}_0 \in \{0, 1, \ldots, a-1\}$, and $I_0, \tilde{I}_0 \in \{0, 1, \ldots, 2(a-1)\}$. In fact, two necessary conditions for the equation (3.39) to have a solution are $I_0 - \tilde{I}_0$ is divisible by m and $b_0 - \tilde{b}_0$ is divisible by n. We can prove that this equation has no solution if one of the two conditions does not hold. We consider each case separately.

Case I: In this case $a = n$. Using the fact that $-(n-1) \leq b_0 - \tilde{b}_0 \leq n-1$, one can deduce that $n \nmid (b_0 - \tilde{b}_0)$.

Case II: In this case $a = s+1$ where $m = 2s+1$. Using the fact that $-2s \leq I_0 - \tilde{I}_0 \leq 2s$, one can deduce that $m \nmid (I_0 - \tilde{I}_0)$.

Case III: In this case $a = s$ where $m = 2s$. Using the fact that $-2(s-1) \leq I_0 - \tilde{I}_0 \leq 2(s-1)$, one can deduce that $m \nmid (I_0 - \tilde{I}_0)$.

Now, it is assumed that the statement of the lemma holds for $L - 1$. To show it also holds for L, one needs to prove the equation

$$\frac{A}{m} \sum_{l=0}^{L} \Big(m(b_l - \tilde{b}_l) + n(I_l - \tilde{I}_l) \Big) W^l = 0 \tag{3.40}$$

has no nontrivial solution. Equivalently, (3.40) can be written as

$$\begin{aligned} & m(b_0 - \tilde{b}_0) + n(I_0 - \tilde{I}_0) \\ & = W \left(\sum_{l=0}^{L-1} \Big(m(b_{l+1} - \tilde{b}_{l+1}) + n(I_{l+1} - \tilde{I}_{l+1}) \Big) W^l \right). \end{aligned} \tag{3.41}$$

In two steps, we prove that the above equation has no solution. First, it is assumed that the right hand side of (3.41) is zero. Due to inductive assumption, it results in $b_l = \tilde{b}_l$ and $I_l = \tilde{I}_l$ for all $l \in \{1, 2, \ldots, L-1\}$. In addition, (3.41) reduces to

$$m(b_0 - \tilde{b}_0) + n(I_0 - \tilde{I}_0) = 0. \tag{3.42}$$

It was already shown that the above equation has no solution except the trivial one $b_0 = \tilde{b}_0$ and $I_0 = \tilde{I}_0$. Notice that this step holds for all three cases.

Second, it is assumed that the right hand side of (3.41) is non-zero. Now, (3.41) can be written as

$$m(b_0 - \tilde{b}_0) + n(I_0 - \tilde{I}_0) = cW, \tag{3.43}$$

where $c \in \mathbb{Z}$ and $c \neq 0$. We prove that (3.43) has no nontrivial solution in each three cases.

Case I: Since $W = n(2n-1)$ in this case, n divides $n(I_0 - \tilde{I}_0)$ as well as cW, but it can not divide $m(b_0 - \tilde{b}_0)$ because $(m, n) = 1$ and $-(n-1) \leq b_0 - \tilde{b}_0 \leq n-1$. Hence, (3.43) has a solution if $b_0 = \tilde{b}_0$ which contradicts the fact that $n|I_0 - \tilde{I}_0| < |c|W$.

Case II: In this case $W = (s+1)(2s+1)$ and $m = 2s+1$. Hence, $2s+1$ divides both $m(b_0 - \tilde{b}_0)$ and cW whereas it can not divide $n(I_0 - \tilde{I}_0)$. This is due to the fact that $(2n, m = 2s+1) = 1$ and $-2s \leq I_0 - \tilde{I}_0 \leq 2s$. Hence, (3.43) has a solution if $I_0 = \tilde{I}_0$ which contradicts the fact that $m|b_0 - \tilde{b}_0| < |c|W$.

Case III: In this case $W = 2s^2 - n$ and $m = 2s$. Due to the symmetry and the fact that

$$\left| m(b_0 - \tilde{b}_0) + n(I_0 - \tilde{I}_0) \right| < 2W, \tag{3.44}$$

it suffices to assume $l = 1$. Substituting $W = 2s^2 - n$, Equation (3.43) can equivalently be written as

$$2s(b_0 - \tilde{b}_0) + n(I_0 - \tilde{I}_0 + 1) = 2s^2. \tag{3.45}$$

It is easy to observe that $2s$ divides $2s(b_0 - \tilde{b}_0)$ as well as $2s^2$, but it can not divide $n(I_0 - \tilde{I}_0 + 1)$ because $(2s, n) = 1$ and $-(2s-1) \leq I_0 - \tilde{I}_0 \leq 2s-1$. Hence, (3.43) has a solution if $I_0 + 1 = \tilde{I}_0$ which is impossible because $2s|b_0 - \tilde{b}_0| < 2s^2$. This completes the proof. □

3.4.2 Irrational Channel Gains

In this subsection, it is shown that when the symmetric channel gain is irrational then the total DOF of the system is achievable, i.e., $r_{\text{sum}} = \frac{3}{2}$. This result relies on a theorem in the field of Diophantine approximation due to Hurwitz. The theorem states as follows.

Theorem 20 (Hurwitz [46])**.** *There exist infinitely many solutions in integers m and n to the Diophantine inequality*

$$\mid \frac{n}{m} - h \mid < \frac{1}{m^2\sqrt{5}}, \tag{3.46}$$

for a given irrational h.

Hurwitz's theorem approximates an irrational number by a rational one and the goodness of the approximation is measured by the size of the denominator.

Theorem 21. *The total DOF of $\frac{3}{2}$ for the symmetric three-user GIC is achievable for all irrational channel gains.*

Remark 11. *This result can be readily extended to the symmetric K-user GIC. In fact, it is easy to show that if the symmetric channel gain is irrational, then $\frac{K}{2}$ is an achievable DOF.*

For an irrational channel gain h, let us assume m and n are two integers satisfying (3.46). Therefore, $h = \frac{n}{m} + \delta$ where $|\delta| < \frac{1}{m^2\sqrt{5}}$. To transmit data, W is chosen as

$$W = \left\lceil \frac{2(1+2h)(a-1)}{\frac{1}{m} - 4(a-1)|\delta|} \right\rceil + 1, \tag{3.47}$$

where $a = \lfloor \frac{m^{1-\epsilon}\sqrt{5}}{4} \rfloor$ and ϵ is an arbitrary positive number. The following chain of inequalities shows that W is positive.

$$\begin{aligned} 4(a-1)|\delta| &\leq \frac{4(a-1)}{m^2\sqrt{5}} \\ &\leq \frac{4a}{m^2\sqrt{5}} \\ &\leq \frac{m^{1-\epsilon}}{m^2} \\ &\leq \frac{1}{m}. \end{aligned}$$

In the following lemma, it is proved that the received constellation possesses Property Γ.

Lemma 10. *The received constellation in (3.33) possesses Property Γ.*

Proof. Suppose there are $(\mathbf{b}, \mathbf{I})$ and $(\tilde{\mathbf{b}}, \tilde{\mathbf{I}})$ such that their corresponding constellation points are the same. Hence, we have

$$h = -\frac{m\sum_{l=0}^{L-1}(b_l - \tilde{b}_l)W^l}{n\sum_{k=0}^{K}(I_l - \tilde{I}_l)W^l}, \tag{3.48}$$

which is a contradiction, because the right hand side is a rational number whereas the left hand side is an irrational number. This completes the proof. □

To characterize the total DOF of the system, we need to derive the minimum distance of points in the received constellation. In the following lemma, the minimum distance is obtained.

Lemma 11. *The minimum distance among the received constellation points with L levels of coding is lower-bounded as $d_{min} \geq A\left(\frac{1}{m} - 4(a-1)|\delta|\right)$.*

Proof. This lemma is also proved by induction on L. In order to emphasize that the minimum distance is a function of L, we may write $d_{\min}(L)$. For $L = 0$, we have

$$d_{\min}(0) = \min_{\Omega}\ A|\hat{b}_0 - h\hat{I}_0|, \tag{3.49}$$

where $\hat{b}_0 = b_0 - \tilde{b}_0$, $\hat{I}_0 = \tilde{I}_0 - I_0$, and Ω is defined as

$$\Omega = \{(\hat{b}_0, \hat{I}_0) : |\hat{b}_0| \leq 2(a-1), |\hat{I}_0| \leq 4(a-1)\}.$$

Since $h = \frac{n}{m} + \delta$, we have

$$\begin{aligned} d_{\min}(0) &= \min_{\Omega}\ A\left|\hat{b}_0 - \frac{n}{m}\hat{I}_0 - \delta\hat{I}_0\right| & (3.50)\\ &\geq \min_{\Omega}\ A\left|\hat{b}_0 + \frac{n}{m}\hat{I}_0\right| - \max_{\Omega}\ A|\delta\hat{I}_0|. & (3.51)\end{aligned}$$

Since $|\hat{I}_0| \leq 4(a-1)$, we have

$$d_{\min}(0) \geq A\left(\frac{1}{m} - 4(a-1)|\delta|\right), \tag{3.52}$$

which is the desired result.

Now, it is assumed that the statement in the lemma holds for any $L-1$ level code. We need to show it also holds for L level codes. The difference between two distinct constellation points is written as

$$\Delta = AW\sum_{l=0}^{L-1}(\hat{b}_{l+1} - h\hat{I}_{l+1})W^l + A(\hat{b}_0 - h\hat{I}_0). \tag{3.53}$$

Let us assume the first term in (3.53) is zero. In this case, the minimum distance can be lower-bounded as

$$d_{\min}(L) \geq \min_{\Omega} \; A\left|\hat{b}_0 - h\hat{I}_0\right|. \tag{3.54}$$

The minimization problem is equivalent to that of case $L = 0$. Hence,

$$d_{\min}(L) \geq A\left(\frac{1}{m} - 4(a-1)|\delta|\right), \tag{3.55}$$

which is the desired result. If the first term in (3.53) is non-zero, then its absolute value is at least $d_{\min}(L-1)$. By the assumption of induction, we have

$$d_{\min}(L-1) \geq A\left(\frac{1}{m} - 4(a-1)|\delta|\right). \tag{3.56}$$

Therefore, we can obtain the following chain of inequalities

$$\begin{aligned}
d_{\min}(K) &= \min|\Delta| \\
&\geq W d_{\min}(K-1) - \max A\left|\hat{b}_0 - h\hat{I}_0\right| \\
&\geq AW(\frac{1}{m} - 4(a-1)|\delta|) \\
&\qquad\qquad - 2A(1+2h)(a-1) \\
&\geq A(\frac{1}{m} - 4(a-1)|\delta|)\times \\
&\qquad\qquad \left(W - \frac{2(1+2h)(a-1)}{\frac{1}{m} - 4(a-1)|\delta|}\right) \\
&\geq A(\frac{1}{m} - 4(a-1)|\delta|).
\end{aligned}$$

This completes the proof. □

Having a lower bound on the minimum distance, we can derive an upper bound for the error probability as follows

$$\begin{aligned}
P_e &< \exp\left(\frac{d_{\min}^2}{8\sigma^2}\right) \\
&\leq \exp\left(-\frac{A^2(\frac{1}{m} - 4(a-1)|\delta|)^2}{8\sigma^2}\right).
\end{aligned} \tag{3.57}$$

Due to Hurwitz's theorem, there are infinitely many solutions for (3.46), i.e., there is a sequence of m converging to infinity and satisfying (3.46). Therefore, there exists a sequence of P's converging to infinity and satisfying $m = \lfloor \log(P) \rfloor$. We take the limit in (4.12) with respect to this sequence. L is again chosen as

$$L = \lfloor \frac{\log\left(P^{0.5-\epsilon}\right)}{\log(W)} \rfloor, \tag{3.58}$$

To show that P_e decays exponentially with respect to P, we consider the following chain of inequalities

$$\begin{aligned} P_e &\leq \exp\left(-\frac{(W^2-1)P}{8a^2\sigma^2 W^{2L}}(\frac{1}{m}-4(a-1)|\delta|)^2\right) \\ &\leq \exp\left(-\frac{W^2-1}{8a^2\sigma^2}(\frac{1}{m}-4(a-1)|\delta|)^2 P^{2\epsilon}\right) \\ &\overset{(a)}{\simeq} \exp\left(-\gamma P^{2\epsilon}\right) \to 0 \text{ as } P \to \infty \end{aligned}$$

where (a) comes from the fact that $\frac{W^2-1}{8a^2\sigma^2}(\frac{1}{m}-4(a-1)|\delta|)^2$ approaches a constant, say γ, as $P \to \infty$. The total DOF can be calculated using (4.12) as follows

$$\begin{aligned} r_{\text{sum}} &= \lim_{P\to\infty} \frac{3L\log(a)}{0.5\log P} \\ &= \lim_{P\to\infty} \frac{3\log(a)}{\log(W)}(1-2\epsilon) \\ &= \frac{3}{2}(1-\epsilon)(1-2\epsilon). \end{aligned}$$

Since ϵ can be chosen arbitrarily small, $r_{\text{sum}} = \frac{3}{2}$ is achievable.

3.5 Conclusion

We proposed a novel coding scheme in which data is modulated using constellation carved from rational points and directed by multiplying by irrational numbers. Using tools from the field of Diophantine approximation in number theory, in particular the Khintchine-Groshev and Hurwitz theorems, we proved that the proposed coding scheme achieves the total DOF of several channels. We considered the single layer and multi-layer constellations for the encoding part.

Using the single layer constellation, we proved that the time-invariant two-user X channel and three-user GIC achieve the DOF of $\frac{4}{3}$ alike. However, for the former it meets the upper bound which means that the total DOF of the two-user X channel is established. This is the first example in which it is shown that a time invariant single antenna system does not fall short of achieving its total DOF.

Using the multi-layer constellation, we derived an achievable DOF for the symmetric three-user GIC. We showed that this achievable DOF is an everywhere discontinuous function with respect to the channel gain. In particular, we proved that for the irrational channel gains the achievable DOF meets the upper bound $\frac{3}{2}$ and for the rational gains, even by allowing carry over from multiple layers, the achievable DOF has a gap to the available upper bounds.

Chapter 4

K-user Gaussian Interference Channel

The first example of interference alignment in one-dimensional spaces which mimics that of n-dimensional spaces ($n > 2$) is presented in Chapter 3. Using irrational numbers as transmit directions and applying the Khintchine-Groshev theorem, we showed that the two-user X channel achieves its total DOF. This is the first channel in which no variations in coefficients over time or frequency and no multiple antennas are required to achieve the total DOF. This is because rational dimensions in one-dimensional spaces can play the role of real dimensions in more-than-two dimensional spaces. In this paper, we take one step forward and prove that the total DOF of the K-user GIC can be achieved without the need for channel variation over time/frequency/space, i.e., it is shown that the total DOF of this channel is $\frac{K}{2}$ and each user can enjoys half of its maximum DOF. Indeed, we prove that the static time-invariant interference channels are rich enough which allow simultaneous interference alignment at all receivers. To derive this result, we show that single-antenna interference channels can be treated as *pseudo multiple-antenna systems* with infinitely-many antennas, as many as rationally-independent irrational numbers. Such machinery enables us to prove that the real or complex $M \times M$ Multiple Input Multiple Output (MIMO) GIC achieves its total DOF, i.e., $\frac{MK}{2}$, $M \geq 1$. The pseudo multiple-antenna systems are developed based on a recent result in the field of Diophantine approximation which states that the convergence part of the Khintchine-Groshev theorem holds for points on non-degenerate manifolds.

This chapter is organized as follows: In Section 4.1, the main theorem of this paper is stated and some discussions are followed. In Section 4.2, some background on the field of Diophantine approximation and in particular the Khintchine-Groshev type theorems are presented. Section 4.3 describes the coding scheme used to prove the main theorem. Moreover, the performance analysis based on recent results in the field of Diophantine approximation is presented. In Section 4.4, the main theorem of the paper is proved. In Section 4.5, we obtain the total DOFs of the $K \times M$ X channel as well as the uplink communication scenario in cellular systems. Finally, Section 4.6 concludes the paper.

4.1 Main Contributions and Discussions

The main theorem of this paper concerns the total DOF of the K-user GIC and stated as follows:

Theorem 22. *The total DOF of the K-user GIC with real and time invariant channel coefficients is $\frac{K}{2}$ for almost all cases.*

Using similar approaches the following theorems are also proved in this chapter.

Theorem 23. *The total DOF of the $K \times M$ X channel with real and time invariant channel coefficients is $\frac{KM}{K+M-1}$ for almost all cases.*

Theorem 24. *The total DOF of the uplink of a cellular systems with K cells and M users within each cell is $\frac{KM}{M+1}$. In other words, achievable DOF per cell is $\frac{M}{M+1}$ which approaches one as the number of active users approaches to infinity.*

4.1.1 Pseudo Multiple-Antenna Systems

It has been known that the multiple-antenna, time-varying, and/or frequency-selective channels provide enough freedom which allows us to choose appropriate signaling directions to maximize the channel gains, and avoiding or aligning interference. In contrary, it was commonly believed that time-invariant frequency-flat single-antenna channels are restrictive in the sense that it prevents us to play with directions. Here, we develop a machinery that transforms the single-antenna systems to pseudo multiple-antenna systems with infinite-many antennas. Indeed the number of available dimensions in the resultant pseudo multiple-antenna systems is as many as rationally-independent irrational numbers. We see that the pseudo multiple-antenna channels is rich enough in a sense that it mimics the behavior of real multi-dimensional systems (in time/frequency/space) and for example allows us to simultaneously align interference at all receivers of static single-antenna channels.

Time, frequency, and space are known as the basic dimensions for communications. In [21, 23, 24, 25], the freedom provided by these dimensions are utilized to align the interferences and provide interference-free links for signals. In [27], it is shown that we are not restricted to time/frequency/space dimensions for interference alignment. In fact, in [27], it is shown that if the favorite signals and interfering signals are received in different *power level*, and form a nested lattice at the receiver, then we can decode the signal without decoding the interference. In this paper, we propose a signaling scheme which is distinguished from the aforementioned schemes in the following senses: (i) Unlike [21, 23, 24, 25], in this scheme, signal and interference are not separated in time/frequency/space dimensions. In fact, similar to the scheme of [27], both signal and interference are received in a single communication dimension. (ii) On the other hand, unlike [27], the signal and interference are not separated based on the received power level. Indeed, in the proposed scheme in some cases, both signal and interference have a comparable power. Roughly speaking, in this scheme, signal and interference are modulated over different irrational numbers which are separable at the receiver. We will show that irrational numbers in the field of real numbers are rich enough to achieve the full DOF of time-invariant interference channels.

4.1.2 Almost All vs All Cases

In the statement of the theorem, it is emphasized that $\frac{K}{2}$ is achievable for almost all cases. It means the collection of all possible $\mathbf{h}$ in which the total DOF $\frac{K}{2}$ may not be achieved has measure zero. In other words, if all channel gains are drawn independently from a random distribution then almost surely all of them are irrational and satisfy properties required for achieving the total DOF of the channel.

In other extreme, if all channel gains are rational then the total DOF is strictly less than $\frac{k}{2}$. This is due to the recent upper bound on the total DOF obtained by Etkin and Ordentlich in [30]. This result together with Theorem 22 implies that the total DOF of the channel is everywhere discontinuous with respect to channel coefficients. This is due to the fact that for any set of channel gains one can find a set of rational numbers arbitrarily close to it. This behavior is unique to this channel (or related networks with single antennas). In fact, almost all total DOFs obtained for MIMO systems are discontinuous at a point or on a set of measure zero. However, none of them is everywhere discontinuous.

It cannot be concluded that for all cases where the theorem is silent about the total DOF of $\frac{K}{2}$ is not achievable. In fact, it is proved that there are some cases where the total DOF can be achieved and those cases are out of the scope of the theorem, c.f., [30, 31, 47]. As an example, the total DOF can be achieved by using a single layer constellation at transmitters in the special case where all cross gains are rational numbers and all direct gains are algebraic irrationals (this is the case for almost all irrationals)[30]. This is due to the fact that cross gains lie on a single rational dimension and therefore the effect of the interference caused by several transmitters behaves as that of interference caused by a single transmitter. Using a single data stream, one can deduce that the multiplexing gain of $\frac{1}{2}$ is achievable for each user.

4.1.3 Time Varying versus Time-Invariant Channels

Cadambe and Jafar in their paper [24] proved that the total DOF of the time-varying K-user GIC can be achieved. This interesting result reveals that in a non-cooperative network each user can enjoy half of its maximum possible multiplexing gain. It is shown that the variation of the channel in time, if it is fast enough to be assumed independent, provides enough freedom to align the interference. However, such an assumption about variation of wireless channels is not practically realistic. Moreover, it imposes inadmissible delay to the system, specially when we note that wireless channels are changing slowly.

Here, we propose a signaling scheme which achieves the full DOF in almost all realizations of the channel, without imposing any delay to the system or requiring channel variation. Indeed, the channel can be static over time and still it is possible to achieve the total DOF of the channel.

4.1.4 MIMO and Complex Coefficients Cases

Let us consider the K-user MIMO GIC where each node in the network is equipped with M antennas. The upper bound on the total DOF states that at most $\frac{MK}{2}$ is achievable for this channel. Except for the three-user case where Cadambe and Jafar in [24] through explicit interference alignment showed that $\frac{3M}{2}$ is achievable, the total DOF of K-user MIMO GIC with static channel states is not considered in the literature. Again if we assume time-variant channels, however, this upper bound can be achieved, see [24].

The applicability of the theorem is not restricted to the single antenna case. In fact, we can also show that for the K-user MIMO GIC the total DOF of the channel can be achieved for almost all cases. This can be proved by simply viewing a single user as M virtual users in which a transmit antenna is paired with a receive antenna. Using separate encoding (resp. decoding) at all transmit (resp. receive) antennas, the channel becomes a MK-user single antenna GIC. Applying the theorem to this channel, we conclude that the total of $\frac{MK}{2}$ is achievable and this meets the upper bound.

Needless to say that the result is also applicable to channels (either single or multiple antennas) with complex coefficients. In fact, the real and imaginary parts of the input and the output can be paired. This converts the channel to $2K$ virtual users. Therefore, the total DOF of the channel can be achieved by a simple application of the theorem.

It is worth noting that joint processing between all antennas and/or real-imaginary parts at a transmitter increases the achievable sum rate of the channel. However, at high SNR regimes this increase vanishes and the total DOF of the channel can be achieved by separate coding over all available dimensions.

4.2 Diophantine Approximation: Khintchine-Groshev Type Theorems

In number theory, the field of Diophantine approximation deals with approximation of real numbers with rational numbers. The reader is referred to [45, 46] and the references therein. The Khintchine theorem is one of the cornerstones in this field. It gives a criterion for a given function $\psi : \mathbb{N} \to \mathbb{R}_+$ and real number v such that $|p+vq| < \psi(|q|)$ has either infinitely many solutions or at most finitely many solutions for $(p,q) \in \mathbb{Z}^2$. Let $\mathcal{A}(\psi)$ denote the set of real numbers such that $|p+vq| < \psi(|v|)$ has infinitely many solutions in integers. The theorem has two parts. The first part is the convergence part and states that if $\psi(|q|)$ is convergent, i.e.,

$$\sum_{q=1}^{\infty} \psi(q) < \infty$$

then $\mathcal{A}(\psi)$ has measure zero with respect to Lebesque measure. This part can be rephrased in more convenient way as follows. For almost all real numbers, $|p+vq| > \psi(|q|)$ holds for all $(p,q) \in \mathbb{Z}^2$ except for finitely many of them. Since the number of integers violating the inequality is finite, one can find a constant κ such that

$$|p+vq| > \kappa\psi(|q|)$$

holds for all integers p and q almost surely. The divergence part of the theorem states that $\mathcal{A}(\psi)$ has the full measure, i.e. the set $\mathbb{R} - \mathcal{A}(\psi)$ has measure zero, provided ψ is decreasing and $\psi(|q|)$ is divergent, i.e.,

$$\sum_{q=1}^{\infty} \psi(q) = \infty.$$

There is an extension to Khintchine's theorem due to Groshev which regards approximation of linear forms. Let $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_m)$ denote an m-tuple in $\mathbb{R}^m$ and $\mathbb{Z}^m$, respectively. Let $\mathcal{A}_m(\psi)$ denote the set of m-tuple real numbers $\boldsymbol{g}$ such that

$$|p + \mathbf{v}\cdot\mathbf{q}| < \psi(|\mathbf{q}|_\infty) \tag{4.1}$$

has infinitely many solutions for $p \in \mathbb{Z}$ and $\mathbf{q} \in \mathbb{Z}^m$. $|\mathbf{q}|_\infty$ is the supremum norm of $\mathbf{q}$ defined as $\max_i |q_i|$. The following theorem gives the Lebesque measure of the set $\mathcal{A}_m(\psi)$.

Theorem 25 (Khintchine-Groshev)**.** *Let $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$. Then the set $\mathcal{A}_m(\psi)$ has measure zero provided*

$$\sum_{q=1}^{\infty} q^{m-1}\psi(q) < \infty, \tag{4.2}$$

and has the full measure if

$$\sum_{q=1}^{\infty} q^{m-1}\psi(q) = \infty \qquad \textit{and } \psi \textit{ is monotonic.} \tag{4.3}$$

In [47], Theorem 25 is used to prove that the total DOF of the two-user X channel can be achieved using a simple coding scheme. It is also proved that the three-user GIC can achieve the DOF of $\frac{4}{3}$ almost surely. Note that Theorem 25 does not include the case where elements of $\mathbf{v}$ are related. It turned out that such a shortcoming in this theorem prevented us to prove the achievablity of $\frac{3}{2}$ for three-user GIC. Let us assume $\boldsymbol{v}$ lies on a manifold with dimension less than m in $\mathbb{R}^m$. In this case, the theorem may not be correct as the measure of the manifold is zero with respect to Lebesque measure. Recently, [48] and [49] independently extended the convergence part of the theorem to the class of non-degenerate manifolds. However, a subclass of non-degenerate manifolds is sufficient for the proofs of the results in this paper. Therefore, in the following theorem we state the theorem in its simplest form by limiting the scope of it.

Theorem 26 ([48] and [49])**.** *Let $n \leq m$, $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, and $g_1, g_2, \ldots, g_m$ be functions from $\mathbb{R}^n$ to $\mathbb{R}$ with the following conditions:*

1. *g_i for $i \in \{1, 2, \ldots, m\}$ is analytic,*
2. *$1, g_1, g_2, \ldots, g_m$ are linearly independent over $\mathbb{R}$.*

For any monotonic function $\psi : \mathbb{N} \rightarrow \mathbb{R}_+$ such that $\sum_{q=1}^{\infty} q^{m-1}\psi(q) < \infty$ the inequality

$$|p + q_1 g_1(\mathbf{v}) + q_2 g_2(\mathbf{v}) + \ldots + q_m g_m(\mathbf{v})| < \psi(|\mathbf{q}|_\infty) \tag{4.4}$$

has at most finitely many solutions $(p, \mathbf{q}) \in \mathbb{Z} \times \mathbb{Z}^m$ for almost all $\mathbf{v} \in \mathbb{R}^n$.

Throughout this paper, the function $\psi(q)$ is chosen as $\frac{1}{q^{m+\epsilon}}$ for an arbitrary $\epsilon > 0$. Clearly, this function satisfies (4.2) and is an appropriate candidate for the theorem. If all conditions of the theorem hold then one can find a constant κ such that for almost all $\mathbf{v} \in \mathbb{R}^n$

$$|p + q_1 g_1(\mathbf{v}) + q_2 g_2(\mathbf{v}) + \ldots + q_m g_m(\mathbf{v})| > \frac{\kappa}{(\max_i |q_i|)^{m+\epsilon}} \tag{4.5}$$

holds for all $p \in \mathbb{Z}$ and $\mathbf{q} \in \mathbb{Z}^m$.

One class of functions satisfying the conditions in Theorem 26 is of special interest. Let $\mathcal{G}(\mathbf{v})$ denote the set of all monomials with variables from the set $\mathbf{v} = \{v_1, v_2, \ldots, v_n\}$. In other words, a function g belongs to $\mathcal{G}(\mathbf{v})$ if it can be represented as $g = v_1^{s_1} v_2^{s_2} \cdots v_n^{s_m}$ for some nonnegative integers $s_1, s_2, \ldots, s_n$. It is easy to show that any collection of functions from $\mathcal{G}(\mathbf{v})$ satisfies the conditions of Theorem 26. More specifically, all functions belonging to $\mathcal{G}(\mathbf{v})$ are analytic. Moreover, a set of monomials are independent over $\mathbb{R}$ as long as they are distinct. As a special case when set $\mathbf{v}$ has only one member, i.e. $\mathbf{v} = \{v\}$, then we have $\mathcal{G}(v) = \{1, v, v^2, v^3, \ldots\}$.

4.3 Coding Scheme and Performance Analysis

Remember in the multiple-antenna systems, the transmitted signal is a linear combination of some vectors (or directions), where data is embedded in the coefficients of the linear combinations. Independency of the vectors allows us to decode the data streams transmitted in each direction and to avoid interference from data streams transmitted in other directions. Note that the vectors or directions are chosen as a function of channel parameters.

Roughly speaking here in the proposed signaling scheme, the transmitted signal is a linear combination of some irrational numbers, where data is embedded in the coefficients of the linear combination. Mimicking the terminology of multiple-antenna systems, we call each of these irrational numbers as a *direction*. Here again, these directions are independent, in a sense that any of them cannot be written as a *rational combination* of the others. We will show that this independency keeps the different data streams separated at the receivers, as long as the coefficients of the linear combination are selected from the *rational* or equivalently the *integer* numbers. Like multiple-antenna systems, these directions are functions of channel coefficients. Initially, we proposed this signaling scheme in [47].

In what follows, we formally describe the signaling and coding scheme.

Encoding: Let us assume Transmitter i for $i \in \{1, 2, \ldots, K\}$ wishes to send L_i data streams to its corresponding receiver. Moreover, each stream carries data with multiplexing gain of approximately $\frac{1}{m}$ for a constant $m \in \mathbb{N}$. Notice that m is independent of i's and L_i's. In other words, we assume that all data streams in the system have the same multiplexing gain.

Let us first explain the encoding of a single data stream. The transmitter selects the constellation $\mathcal{C} = (-Q, Q)_{\mathbb{Z}}$ as the set of input symbols. Even though it has access to the continuum of real numbers, restriction to a finite set has the benefit of easy and feasible interference alignment. Let us assume $Q = \gamma P^{\frac{1-\epsilon}{2(m+\epsilon)}}$ where γ is a constant. Notice that since the number of input symbols are bounded by $2Q-1$, the data stream modulated by $\mathcal{C}$ can at most provide $\frac{1-\epsilon}{m+\epsilon}$ DOF. We will show that at high SNR regimes this DOF can be achieved.

Having formed the constellation, Transmitter i for $i \in \{1, 2, \ldots, K\}$ constructs a random codebook for data stream l for $l \in \{1, 2, \ldots, L_i\}$ with rate R_{il}. This can be accomplished by choosing a probability distribution on the input alphabets. The uniform distribution is the first candidate and it is selected for the sake of simplicity. Note that since the constellation is symmetrical by assumption, the expectation of the uniform distribution is zero and the transmit signal has no DC component. The power consumed by the data stream l can be loosely upper-bounded as Q^2.

To send L_i data streams, Transmitter i first constructs L_i independent single data streams by following the above procedure for each data stream. Then, it combines them using a linear combination of all data streams. The transmit signal from Transmitter i can be represented as

$$x_i = A \sum_{l=0}^{L_i-1} T_{il} u_{il}, \tag{4.6}$$

where $u_{il} \in \mathcal{U}$ carries information for l's data stream of User i. T_{il} is a constant real number which plays as the role of a vector that the data stream l is transmitted in that direction, see [47]. T_{il}'s are functions of channel coefficients. We will choose T_{il}'s as monomials with variables from channel coefficients, i.e., $T_{il} \in \mathcal{G}(\mathbf{h})$ for all $i \in \{1, 2, \ldots, K\}$ and $l \in \{0, 1, \ldots, L_i - 1\}$. T_{il}'s are also chosen to be independent over rational numbers , i.e., the equation $T_{i1}w_1 + T_{i2}w_2 + \cdots + T_{iL_i}w_{L_i} = 0$ has no rational solutions. This independency is provides a one-to-one mapping from constellation points u_{il}'s and transmit signal x_i. In other words, a transmit sinal x_i is uniquely decomposable as $u = A\sum_{l=0}^{L_i-1} T_{il}u_{il}$. Observe that if there is another possible decomposition $x_i = A\sum_{i=0}^{L_i-1} T_{il}u'_{il}$ then it forces T_{il}'s to be dependent. The parameter A controls the input power of all users. In what follows, we show how to choose a unique A for all transmitters, independent of L_i's, by calculating the upper-bound of the input power of all users. We start with the following chain of inequalities

$$\begin{aligned} E[x_i^2] &\overset{(a)}{=} A^2 \sum_{l=0}^{L_i-1} T_{il}^2 E\left[u_{il}^2\right] \\ &\overset{(b)}{\leq} A^2 Q^2 \left(\sum_{l=0}^{L_i-1} T_{il}^2 \right) \\ &= A^2 Q^2 \lambda_i^2 \end{aligned}$$

where (a) follows from the fact that all data streams are independent and (b) follows from the fact that $u_{il}^2 \leq Q^2$ for all $i \in \{1, 2, \ldots, K\}$ and $l \in \{0, 1, \ldots, L_i - 1\}$. We use a shorthand notation λ_i as $\lambda_i = \sum_{l=0}^{L_i} T_{il}^2$. Since each T_{il} depends only on channel coefficients which are constants, λ_i for $i \in \{1, 2, \ldots, K\}$ is a constant. To satisfy the power constraint, it is required that

$$A \leq \frac{P^{\frac{1}{2}}}{Q\lambda_i}$$

for all $i \in \{1, 2, \ldots, K\}$. Clearly, it is sufficient to choose

$$A = \frac{\zeta P^{\frac{1}{2}}}{Q}$$

where $\zeta = \min_i \frac{1}{\lambda_i}$. By assumption $Q = \gamma P^{\frac{1-\epsilon}{2(m+\epsilon)}}$. Hence, we have

$$A = \xi P^{\frac{m-1+2\epsilon}{2(m+\epsilon)}}, \tag{4.7}$$

where $\xi = \frac{\zeta}{\gamma}$.

In fact, A and Q are two important design parameters in the encoding. Q controls the cardinality of the input constellation which in turn provides the maximum achievable rate for individual data streams. Here, the cardinality of the constellation grows roughly with $P^{\frac{1}{2m}}$. On the other hand, A controls the minimum distance in the received constellation which in turn affects the performance. Our calculation reveals that no matter how many data streams each transmitter is intended to send, Q and A only depend on m which is the reciprocal of the multiplexing gain of each data streams.

Received Signal and Interference Alignment: The received signal at Receiver j can be represented as

$$y_j = A\Big(\sum_{l=0}^{L_j-1} h_{jj}T_{jl}u_{jl} + \underbrace{\sum_{i=1\&i\neq j}^{K}\sum_{l=0}^{L_i-1} h_{ji}T_{il}u_{il}}_{I_j}\Big) + z_j, \tag{4.8}$$

where I_j is the aggregated interference caused by all users. Since $T_{il} \in \mathcal{G}(\mathbf{h})$, one can conclude that the received direction for data stream u_{il} is again a member of $\mathcal{G}(\mathbf{h})$, i.e., $h_{ji}T_{il} \in \mathcal{G}(\mathbf{h})$. The maximum number of received directions in I_j is $\sum_{i=1\&i\neq j}^{K} L_i$. However, it is possible that some of the directions becomes equivalent which results in reduction in the number of received directions. In fact, the design in the transmit directions aims at reducing the number of received directions in all I_j's and the more the merrier. If a number of data streams arrives at the same direction then we say that they are aligned. As it will be shown later, the behavior of aligned data streams mimics that of a single data stream as far as the DOF is concerned. Let us assume, the total number of received directions in I_j is L'_j , i.e., we have

$$I_j = \sum_{l=0}^{L'_j-1} T'_{jl}u'_{jl}, \tag{4.9}$$

where T'_{jl}'s are received directions and u'_{jl} is the sum of data streams arriving at direction T'_{jl}. If f_{jl} data streams arrive at the direction T'_{jl} then $u'_{jl} \in (-f_{jl}Q, f_{jl}Q)_{\mathbb{Z}}$. To have a uniform bound, let us define $f = \max_{(j,l)} f_{jl}$ and $\mathcal{U}' = (-fQ, fQ)_{\mathbb{Z}}$. Clearly, $u'_{jl} \in \mathcal{U}'$ for all $j \in \{1, 2, \ldots, K\}$ and $l \in \{0, 1, \ldots, L'_j - 1\}$.

Decoding: After rearranging the interference part of the signal, the received signal at Receiver j can be represented as

$$y_j = A\left(\sum_{l=0}^{L_j-1} h_{jj}T_{jl}u_{jl} + \sum_{l=0}^{L'_j-1} T'_{jl}u'_{jl}\right) + z_j. \tag{4.10}$$

We assume that $L_j + L'_j \leq m$ for all $j \in \{1, 2, \ldots, K\}$. Receiver j is interested in data streams u_{jl} for all $l \in \{0, 1, \ldots, L_j - 1\}$.

The data stream u_{jl} for a given l is decoded as follows. The received signal is first passed through a hard decoder. The hard decoder looks at the received constellation

$$\mathcal{V}_j = A\left(\sum_{l=0}^{L_j-1} h_{jj}T_{jl}\mathcal{U} + \sum_{l=0}^{L'_j-1} T'_{jl}\mathcal{U}'\right)$$

and maps the received signal to the nearest point in the constellation. This changes the continuous channel to a discrete one in which the input symbols are from the transmit constellation $\mathcal{U}$ and the output symbols are from the received constellation $\mathcal{V}_j$.

It is assumed that the received constellation has the property that there is a many-to-one map from $\mathcal{V}_j$ to $\mathcal{U}_j = \sum_{l=0}^{L_j-1} h_{jj}T_{jl}\mathcal{U}$. Recall that the transmit directions are chosen in such a way that all u_{jl}'s can be recovered uniquely from $\mathcal{U}_j$. This in fact implies that if there is no additive noise in the channel then the receiver can decode all intended data streams with zero error probability. This property holds, for example, when $h_{jj}T_{jl}$'s and T'_{jl} are all distinct and linearly independent over rational numbers. Throughout this chapter, we always design the transmit directions in such a way that this condition holds.

The equivalent channel between u_{jl} and the output of the hard decoder $\hat{u}_{jl}$ becomes a discrete channel and a joint-typical decoder can be used to decode the data stream from a block of $\hat{u}_{jl}$'s. To decode another data stream, Receiver j performs the same procedure used for decoding u_{jl}. In fact, joint-decoding is not used to decode all intended data streams.

Performance Analysis: Let $d_{j_{\min}}$ denote the minimum distance in the received constellation $\mathcal{V}_j$. The average error probability in the equivalent discrete channel from input u_{jl} to output $\hat{u}_{jl}$, i.e. $P_e = Pr\{\hat{u}_{jl} \neq u_{jl}\}$ is bounded as:

$$P_e \leq Q\left(\frac{d_{j_{\min}}}{2}\right) \leq \exp\left(-\frac{d^2_{j_{\min}}}{8}\right). \tag{4.11}$$

P_e can be used to lower bound the rate achievable for the data stream u_{jl}. In [30], Etkin and Ordentlich used Fano's inequality to obtain a lower bound on the achievable rate which is tight in high SNR regimes. Following similar steps, one can obtain

$$\begin{aligned} R_{jl} &= I(\hat{u}_{jl}, u_{jl}) \\ &= H(u_{jl}) - H(u_{jl}|\hat{u}_{jl}) \\ &\overset{(a)}{\geq} H(u_{jl}) - 1 - P_e \log |\mathcal{U}| \\ &\overset{(b)}{=} (1 - P_e) \log |\mathcal{U}| - 1 \\ &\overset{(c)}{=} (1 - P_e) \log(2Q - 1) - 1, \end{aligned} \tag{4.12}$$

where (a) follows from Fano's inequality, (b) follows from the fact that u_{jl} has a uniform distribution on its range, and (c) follows from the fact that $|\mathcal{U}|$ which is the number of integers in the interval $[-Q, Q]$ is bounded by $2Q-1$. Let us assume that $P_e \to 0$ as $P \to \infty$. Under this condition, the achievable multiplexing gain from data stream u_{jl} can be obtained as follows:

$$\begin{aligned} r_{jl} &= \lim_{P \to \infty} \frac{R_{jl}}{0.5 \log P} \\ &\geq \lim_{P \to \infty} \frac{\log Q}{0.5 \log P} \\ &\overset{(a)}{=} \frac{1 - \epsilon}{m + \epsilon}, \end{aligned} \tag{4.13}$$

where (a) follows from the fact that $Q = \gamma P^{\frac{1-\epsilon}{2(m+\epsilon)}}$. Since $\epsilon > 0$ is an arbitrary constant, the multiplexing gain of $\frac{1}{m}$ is achievable for the data stream u_{jl}.

Provided that all intended data streams can be successfully decoded at all receivers, the achievable DOF of User i can be written as $\frac{L_i}{m}$. However, it is achievable under the condition that $P_e \to 0$ as $P \to \infty$ and it needs to be shown. To this end, one requires to calculate the minimum distance between points in the received constellation.

Recall that $L_j + L'_j \leq m$ and $h_{jj}T_{jl}$'s and T'_{jl}'s are all distinct and monomials with variables from the channel coefficients. Theorem 26 can be applied to obtain a lower bound on the minimum distance. Let us assume that one of the directions in $h_{jj}T_{jl}$'s or T'_{jl}' is 1. Then a point in $\mathcal{V}_j$ can be represented as

$$v = A \left(v_0 + \sum_{l=1}^{L_j + L_j - 1} \hat{T}_l v_l \right), \tag{4.14}$$

where $\hat{T}_l$'s are all distinct monomials at receiver j. Moreover, v_l for all $l \in \{0, 1, \ldots, L_j + L'_j - 1\}$ are bounded by $(-fQ, fQ)_{\mathbb{Z}}$. Therefore, the difference between any two point in the received constellation $\mathcal{V}_j$ can be bounded using (4.5) as follows:

$$d_{j_{\min}} > \frac{\kappa A}{(2fQ)^{L_j+L'_j-1+\epsilon}}.$$

Since $L_j + L'_j \leq m$, we have

$$d_{j_{\min}} > \frac{\kappa A}{(2fQ)^{m-1+\epsilon}}. \tag{4.15}$$

The probability of error in hard decoding (see (4.11)) can be bounded as

$$P_e < \exp\left(-\eta\left(\frac{A}{Q^{m-1+\epsilon}}\right)^2\right), \tag{4.16}$$

where η is a constant and a function of γ, κ, σ, and γ_is.

Substituting A and Q in (4.16) yields

$$P_e < \exp\left(-\eta P^{\epsilon}\right), \tag{4.17}$$

which shows that P_e has the desired property.

The following theorem summarizes the conditions needed to achieve the multiplexing gain of $\frac{1}{m}$ per data stream.

Theorem 27. *Consider K-user GICs parameterized by the channel coefficient vector $\mathbf{h}$. Transmitter i sends L_i data stream along directions $\mathcal{T}_i = \{T_{i0}, T_{i2}, \ldots, T_{i(L_i-1)}\}$ for all $i \in \{1, 2, \ldots, K\}$. Moreover, the interference part of the received signal at Receiver i has L'_i effective data streams with received directions $\mathcal{T}'_i = \{T'_{i0}, T'_{i2}, \ldots, T'_{i(L'_i-1)}\}$ for all $i \in \{1, 2, \ldots, K\}$. Let the following conditions for all $i \in \{1, 2, \ldots, K\}$ hold:*

C1 *Components of $\mathcal{T}_i$ are distinct member of $\mathcal{G}(\mathbf{h})$ and linearly independent over the field of rational numbers.*

C2 *Components of $h_{ii}\mathcal{T}_i$ and $\mathcal{T}'_i$ are all distinct.*

C3 *One of the elements of either $h_{ii}\mathcal{T}_i$ or $\mathcal{T}'_i$ is 1.*

Then, by encoding each data stream using the constellation $\mathcal{U} = (-Q, Q)_{\mathbb{Z}}$ where $Q = \gamma P^{\frac{1-\epsilon}{2(m+\epsilon)}}$ and γ is a constant, the following DOF is achievable for almost all channels:

$$r_{sum} = \frac{L_1 + L_2 + \cdots + L_K}{m}, \tag{4.18}$$

where m is the maximum received directions among all receivers, i.e., $m = \max_i L_i + L'_i$.

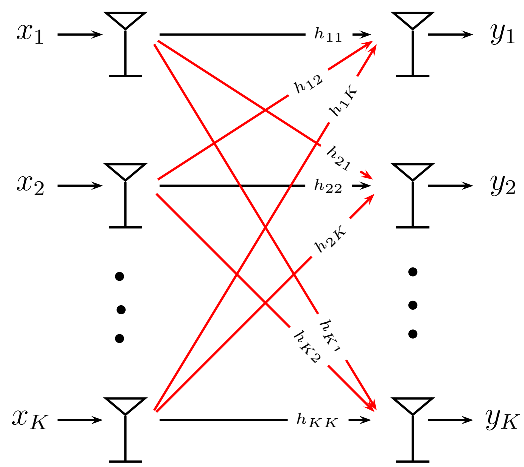

Figure 4.1: The K-user GIC. User i for $i \in \{1, 2, \ldots, K\}$ wishes to communicate with its corresponding receiver while receiving interference from other users.

Remark 12. *If C2 holds then the measure of the event "components of $h_{ii}\mathcal{T}_i$ and $\mathcal{T}'_i$ are dependent over the field of rational numbers" is zero.*

Remark 13. *If C3 does not hold then by adding a virtual data stream in the direction 1 at the receiver, one can conclude that $\frac{1}{m+1}$ is achievable for all data streams.*

Theorem 27 implies that the most difficult part of the design is the selection of transmit directions for all users. This is due to the fact that random selection results in $m = \sum_{i=1}^{K} L_i$ received directions which in turn provides 1 DOF for the channel. A careful design is needed to reduce the number of received directions at all users. In the following sections, we provide such a design for the K-user GIC.

4.4 K-user Gaussian Interference Channel

4.4.1 System Model

The K-user GIC models a network in which K transmitter-receiver pairs (users) sharing a common bandwidth wish to have reliable communication at their maximum rates. The channel's input-output relation can be stated as follows, see Figure 4.1,

$$\begin{aligned} y_1 &= h_{11}x_1 + h_{12}x_2 + \ldots + h_{1K}x_K + z_1, \\ y_2 &= h_{21}x_1 + h_{22}x_2 + \ldots + h_{2K}x_K + z_2, \\ \vdots &= \quad \vdots \qquad\quad \vdots \qquad\quad \ddots \qquad \vdots \\ y_K &= h_{K1}x_1 + h_{K2}x_2 + \ldots + h_{KK}x_K + z_K, \end{aligned} \tag{4.19}$$

where x_i and y_i are input and output symbols of User i for $i \in \{1, 2, \dots, K\}$, respectively. z_i is Additive White Gaussian Noise (AWGN) with unit variance for $i \in \{1, 2, \dots, K\}$. Transmitters are subject to the power constraint P. h_{ji} represents the channel gain between Transmitter i and Receiver j. It is assumed that all channel gains are real and time invariant. The set of all channel gains is denoted by $\mathbf{h}$, i.e., $\mathbf{h} = \{h_{11}, \dots, h_{1K}, h_{21}, \dots, h_{2K}, \dots, h_{K1}, \dots, h_{KK}\}$. Since the noise variances are normalized, the Signal to Noise Ratio (SNR) is equivalent to the input power P. Hence, we use them interchangeably throughout this chapter.

In this chapter, we are primarily interested in characterizing the total DOF of the K-user GIC. Let $\mathcal{C}$ denote the capacity region of this channel. The DOF region associated with the channel is in fact the shape of $\mathcal{C}$ in high SNR regimes scaled by $\log \mathrm{SNR}$. Let us denote the DOF region by $\mathcal{R}$. All extreme points of $\mathcal{R}$ can be identified by solving the following optimization problem:

$$r_{\boldsymbol{\lambda}} = \lim_{\mathrm{SNR}\to\infty} \max_{\mathbf{R}\in\mathcal{C}} \frac{\boldsymbol{\lambda}^t \mathbf{R}}{\log \mathrm{SNR}}. \tag{4.20}$$

The total DOF refers to the case where $\boldsymbol{\lambda} = \{1, 1, \dots, 1\}$, i.e., the sum-rate is concerned. Throughout this chapter, r_{sum} denotes the total DOF of the system.

An upper bound on the DOF of this channel is obtained in [24]. The upper bound states that the total DOF of the channel is less than $\frac{K}{2}$ which means each user can at most enjoy one half of its maximum DOF.

4.4.2 Three-user Gaussian Interference Channel: DOF $= \frac{3}{2}$ is Achievable

In this section, we consider the three-user GIC and explain in detail that by an appropriate selection of transmit directions, the DOF of $\frac{3}{2}$ is achievable for almost all cases. We will explain in more detail that by an appropriate selection of transmit directions, this DOF can be achieved.

In [47], we defined the standard model of the three-user GIC. The definition is as follows:

Definition 7. *The three-user interference channel is called standard if it can be represented as*

$$\begin{aligned} y_1 &= G_1 x_1 + x_2 + x_3 + z_1 \\ y_2 &= G_2 x_2 + x_1 + x_3 + z_2 \\ y_3 &= G_3 x_3 + x_1 + G_0 x_2 + z_3, \end{aligned} \tag{4.21}$$

where x_i for User i is subject to the power constraint P. z_i at Receiver i is AWGN with unit variance.

In [47], it is also proved that every three-user GIC has an equivalent standard channel as far as the DOF is concerned. The parameters in the standard channel is related to the parameters of the original one thorough the following equations.

$$
\begin{aligned}
G_0 &= \frac{h_{13}h_{21}h_{32}}{h_{12}h_{23}h_{31}}, \\
G_1 &= \frac{h_{11}h_{12}h_{23}}{h_{12}h_{21}h_{13}}, \\
G_2 &= \frac{h_{22}h_{13}}{h_{12}h_{23}}, \\
G_3 &= \frac{h_{33}h_{12}h_{21}}{h_{12}h_{23}h_{31}}.
\end{aligned}
$$

As mentioned in the previous section, transmit directions are monomials with variables from channel coefficients. For the three user case, we only use G_O as the generator of transmit directions. Therefore, transmit directions are selected from the set $\mathcal{G}(G_0)$ which is a subset of $\mathcal{G}(G_0, G_1, G_2, G_3)$. Clearly, $\mathcal{G}(G_0) = \{1, G_0, G_0^2, G_0^3, \cdots\}$.

We consider two different cases based on the value of G_0 being algebraic or transcendental. Although the measure of being algebraic is zero, we prove that for each case the total DOF can be achieved if the transmit and receive directions satisfy the conditions of Theorem 27. We start with the case where G_0 is algebraic.

Case I: G_0 is algebraic

By definition, if G_0 is algebraic then it is a root of a polynomial with integer coefficients. Let us assume G_0 satisfies

$$a_d G_0^d + a_{d-1} G_0^{d-1} + \ldots + a_1 G_0 + a_0 = 0, \tag{4.22}$$

where $a_d, a_{d-1}, \ldots, a_0$ are integers. In other words, the set $\mathcal{T} = \{1, G_0, G_0^2, \ldots, G_0^{d-1}\}$ is a basis for $\mathcal{G}(G_0)$ over rational numbers. Therefore, as the transmit directions need to be independent over the field of rational numbers, the transmitters are restricted to choose their transmit directions among numbers in $\mathcal{T}$. We assume that all transmitters transmit along all directions in $\mathcal{T}$, i.e., $\mathcal{T}_i = \mathcal{T}$ for all $i \in \{1, 2, 3\}$. By this selection, C1 in Theorem 27 holds for all transmitters.

In this case, Transmitter i sends $L_i = d$ data streams as follows

$$x_i = A \sum_{j=0}^{d-1} G_0^j u_{ij}, \tag{4.23}$$

for all $i \in \{1, 2, 3\}$. The received signal at Receiver 1 can be written as

$$y_1 = A\left(\sum_{j=0}^{d-1} G_1 G_0^j u_{1j} + \sum_{j=0}^{d-1} G_0^j u'_{1j}\right) + z_1, \tag{4.24}$$

where $u'_{1j} = u_{2j} + u_{3j}$ for all $j \in \{0, 1, \ldots, d-1\}$. The signals from Transmitters 2 and 3 are aligned and the number of received directions is $L'_1 = d$. Moreover C2 and C3 in Theorem 27 hold for this receiver. Since the received signal at Receiver 2 is similar to that of Receiver 1, we can deduce that $L'_2 = d$ and C2 and C3 hold.

The received signal at Receiver 3 can be written as

$$y_3 = A\left(\sum_{j=0}^{d-1} G_3 G_0^j u_{3j} + \sum_{j=0}^{d} G_0^j u'_{3j}\right) + z_3, \tag{4.25}$$

where $u'_{3j} = u_{2j} + u_{1(j-1)}$ for $j \in \{1, 2, \ldots, d-1\}$, $u'_{30} = u_{20}$, and $u'_{3d} = u_{1d}$. The number of received directions from interfering users is $d+1$. However, they are not independent over the filed of rational numbers. Using (4.22), G_0^d can be represented as a linear combination of $\{1, G_0, G_0^2, \ldots, G_0^{d-1}\}$ with rational coefficients. Multiplying both sides of (4.25) by a_d, we have

$$\tilde{y}_3 = A\left(\sum_{j=0}^{d-1} a_d G_3 G_0^j u_{3j} + \sum_{j=0}^{d-1} G_0^j a_d u'_{3j} + a_d G_0^d u'_{3d}\right) + \tilde{z}_3, \tag{4.26}$$

where $\tilde{y}_3 = a_d y_3$ and $\tilde{z}_3 = a_d z_3$. Substituting form (4.25), we obtain

$$\tilde{y}_3 = A\left(\sum_{j=0}^{d-1} a_d G_3 G_0^j u_{3j} + \sum_{j=0}^{d-1} G_0^j \underbrace{(a_d u'_{3j} - a_j u'_{3d})}_{u''_j}\right) + \tilde{z}_3. \tag{4.27}$$

Clearly, $L'_3 = d$ and C2 and C3 hold for this receiver as well.

The maximum number received directions at all receivers is $m = 2d$. Since C1, C2, and C3 hold at all receivers, by applying Theorem 27 we conclude that the total DOF of $\frac{3}{2}$ is achievable for almost all cases.

Remark 14. *In a special case, $d = 1$ in (4.22). In other words, G_0 is a rational number. This case is considered in [30] and it is proved that it can achieve the total DOF of the channel.*

Case II: G_0 is transcendental

If G_0 is transcendental then all members of $\mathcal{G}(G_0)$ are linearly independent over the filed of rational numbers. Hence, we are not limited to any subset of $\mathcal{G}(G_0)$ as far as the independence of transmit directions is concerned. We will show that $\frac{3n+1}{2n+1}$ is an achievable DOF for any $n \in \mathbb{N}$. To this end, we propose a design which is not symmetrical.

Transmitter 1 uses the set of directions $\mathcal{T}_1 = \{1, G_0, G_0^2, \dots, G_0^n\}$ to transmit $L_1 = n + 1$ to its corresponding receiver. Clearly $\mathcal{T}_1$ satisfies C1. The transmit signal form User 1 can be written as

$$x_1 = A \sum_{j=0}^{n} G_0^j u_{1j}.$$

Transmitters 2 and 3 transmit in $L_2 = L_3 = n$ directions using $\mathcal{T}_2 = \mathcal{T}_3 = \{1, G_0, G_0^2, \dots, G_0^{n-1}\}$. Clearly both $\mathcal{T}_2$ and $\mathcal{T}_3$ satisfy C1. The transmit signals can be expressed as

$$x_2 = A \sum_{j=0}^{n-1} G_0^j u_{2j}$$

and

$$x_3 = A \sum_{j=0}^{n-1} G_0^j u_{3j}.$$

The received signal at Receiver 1 can be expressed as:

$$y_1 = A \left(\sum_{j=0}^{n} G_1 G_0^j u_{1j} + \sum_{j=0}^{n-1} G_0^j u'_{1j} \right) + z_1, \tag{4.28}$$

where $u'_{1j} = u_{2j} + u_{3j}$. In fact, transmit signals from Users 2 and 3 are aligned at Receiver 1. This is due to the fact that out of $2n$ possible received directions only n directions are effective, i.e., $L'_1 = n$. One can also confirm that C2 and C3 hold at Receiver 1.

The received signal at Receiver 2 can be expressed as:

$$y_2 = A \left(\sum_{j=0}^{n-1} G_2 G_0^j u_{2j} + \sum_{j=0}^{n} G_0^j u'_{2j} \right) + z_2, \tag{4.29}$$

where $u'_{2j} = u_{1j} + u_{3j}$ for all $j \in \{0, 1, \dots, n-1\}$ and $u'_{2n} = u_{1n}$. At Receiver 2, transmitted signals from Users 1 and 3 are aligned and the number of effective received directions is $L'_2 = n + 1$. Moreover, it can be easily seen that C2 and C3 hold at Receiver 2.

The received signal at Receiver 3 can be expressed as:

$$y_3 = A \left(\sum_{j=0}^{n-1} G_3 G_0^j u_{3j} + \sum_{j=0}^{n} G_0^j u'_{3j} \right) + z_3, \tag{4.30}$$

where $u'_{3j} = u_{1j} + u_{2j}$ for all $j \in \{1, 2, \dots, n\}$ and $u'_{30} = u_{10}$. At Receiver 3, transmitted signals from Users 1 and 2 are aligned and the number of effective received directions is $L'_2 = n + 1$. Clearly, C2 and C3 hold for Receiver 3.

Since C1, C2, and C3 hold at all users, we only need to obtain the number of maximum received directions at all receivers. To this end, we observe that

$$m = \max\{L_1 + L_1', L_2 + L_2', L_3 + L_3'\} = 2n + 1$$

. Therefore, an application of Theorem 27 reveals that the following DOF is achievable.

$$\begin{aligned} r_{\text{sum}} &= \frac{L_1 + L_2 + L_3}{m} \\ &= \frac{3n+1}{2n+1}. \end{aligned} \tag{4.31}$$

Since n is an arbitrary integer, one can conclude that $\frac{3}{2}$ is achievable for the three-user GIC almost surely.

4.4.3 K-user Gaussian Interference Channel: DOF $= \frac{K}{2}$ is Achievable

In this section, we prove the main theorem of this chapter, i.e., the DOF of $\frac{K}{2}$ is achievable for the K-user GIC. As pointed out in Section 4.3, we need to design the transit directions of all transmitters in such a way that they satisfy the conditions of Theorem 27. Recall that all transmit directions are monomials with variables in $\mathbf{h}$. We reserve the direct gains and do not use them as generating variables. The reason is that C2 in Theorem 27 requires that all received directions be distinct. By setting aside the direct gains, a transmit direction from the intended user is multiplied by the direct gain and therefore it is distinct from all other transmit directions (by C1 all transmit directions from a user are distinct).

We assume that all channel gains are transcendental. In one hand, since the measure of being algebraic is zero, this assumption is innocuous. On the other hand, as we learned from the three-user case algebraic gains are beneficial as they reduce the number of transmit directions required to achieve the total DOF of the channel.

We start with selecting the transmit directions for User i. A direction $T \in \mathcal{G}(\mathbf{h})$ is chosen as the transmit direction for User i if it can be represented as

$$T = \prod_{j=1}^{K} \prod_{l=1}^{K} h_{jl}^{s_{jl}}, \tag{4.32}$$

where s_{jl}'s are integers satisfying

$$\begin{cases} s_{jj} = 0 & \forall\ j \in \{1, 2, \ldots, K\} \\ 0 \leq s_{ji} \leq n-1 & \forall\ j \in \{1, 2, \ldots, K\}\ \&\ j \neq i \\ 0 \leq s_{jl} \leq n & \text{Otherwise.} \end{cases}$$

The set of all transmit directions is denoted by $\mathcal{T}_i$. It is easy to show that the cardinality of this set is

$$L_i = n^{K-1}(n+1)^{(K-1)^2}. \tag{4.33}$$

Clearly, $\mathcal{T}_i$ satisfies C1 for all $i \in \{1, 2, \ldots, K\}$.

To compute L'_i (the number of independent received directions due to interference), we investigate the effect of Transmitter k on Receiver i. Let us first define $\mathcal{T}_r$ as the set of directions represented by (4.32) and satisfying

$$\begin{cases} s_{jj} = 0 & \forall\ j \in \{1, 2, \ldots, K\} \\ 0 \le s_{jl} \le n & \text{Otherwise.} \end{cases} \tag{4.34}$$

We claim that $\mathcal{T}_{ik}$, the set of received directions at Receiver i due to Transmitter k, is a subset of $\mathcal{T}_r$. In fact, all transmit directions of Transmitter k arrive at Receiver i multiplied by h_{ik}. Based on the selection of transmit directions, however, the maximum power of h_{ik} in all members of $\mathcal{T}_{ik}$ is $n-1$. Therefore, none of the received directions violates the condition (4.44) and this proves the claim.

Since $\mathcal{T}_r$ is not related to User k, one can conclude that $\mathcal{T}_{ik} \subseteq \mathcal{T}_r$ for all $k \in \{1, 2, \ldots, K\}$ and $k \neq i$. Hence, we deduce that all interfering users are aligned in the directions of $\mathcal{T}_r$. Now, L'_i can be obtained by counting the members of $\mathcal{T}_r$. It is easy to show that

$$L'_i = (n+1)^{K(K-1)}. \tag{4.35}$$

The received directions at Receiver i are members of $h_{ii}\mathcal{T}_i$ and $\mathcal{T}_r$. Since h_{ii} does not appear in members of $\mathcal{T}_r$, the members of $h_{ii}\mathcal{T}_i$ and $\mathcal{T}_r$ are distinct. Therefore, C2 holds at Receiver i. Since all the received directions are irrationals, C3 does not hold at Receiver i.

Since C_1 and C_2 hold for all users, we can apply Theorem 27 to obtain the DOF of the channel. We have

$$\begin{aligned} r_{\text{sum}} &= \frac{L_1 + L_2 + \ldots + L_K}{m+1} \\ &= \frac{Kn^{K-1}(n+1)^{(K-1)^2}}{m+1} \end{aligned} \tag{4.36}$$

where m is

$$\begin{aligned} m &= \max_i L_i + L'_i \\ &= n^{K-1}(n+1)^{(K-1)^2} + (n+1)^{K(K-1)}. \end{aligned} \tag{4.37}$$

Combining the two equations, we obtain

$$r_{\text{sum}} = \frac{K}{1 + (\frac{n+1}{n})^{K-1} + \frac{1}{n^{K-1}(n+1)^{(K-1)^2}}}. \tag{4.38}$$

Since n can be arbitrary large, we conclude that $\frac{K}{2}$ is achievable for the K-user GIC.

4.5 Some Extensions

In this section, we use the proposed coding scheme to characterize the total DOF of the uplink communication scenario in cellular systems and the $K \times M$ X channel.

4.5.1 Cellular Systems: Uplink

System Model

In a cellular network, an area is partitioned into several cells and within each cell there is a base station serving users inside the cell. There are two modes of operation. In the uplink mode, users within a cell transmit independent messages to the base station in the cell; whereas in the downlink mode, the base station broadcast independent messages to all users inside the cell. In this section, we only consider the uplink mode. Information theoretically, the uplink mode corresponds to a network in which several Multiple Access Channels (MAC) share the same spectrum for data transmission. Let us assume there exist M users in each MAC and there are K MACs in the network. The received signal at the base station in Cell k can be represented as

$$y_k = \underbrace{\sum_{l=1}^{M} h_{k(kl)} x_{kl}}_{\text{users within the cell}} + \underbrace{\sum_{i=1 \& i\neq k}^{K} I_{ki}}_{\text{intra cell interference}} + z_k \tag{4.39}$$

where I_{ki} is the aggregate interference from all users in Cell i, i.e.,

$$I_{ki} = \sum_{l=1}^{M} h_{k(il)} x_{il}. \tag{4.40}$$

Let $\mathcal{C}_{\text{up}}$ denote the capacity region of this channel. The DOF region associated with the channel can be defined as the shape of the region in high SNR regimes scaled by $\log \text{SNR}$. Let us denote the DOF region by $\mathcal{R}_{\text{up}}$. We are primarily interested in the main facet of the DOF region defined as:

$$r_{\text{up}} = \lim_{\text{SNR}\to\infty} \max_{\mathbf{R}\in\mathcal{C}_{\text{up}}} \frac{\sum_{k=1}^{K}\sum_{l=1}^{M} R_{kl}}{\log \text{SNR}}, \tag{4.41}$$

where R_{kl} is an achievable rate for the l'th user in Cell k.

The Total DOF of $\frac{KM}{M+1}$ is Achievable

To obtain an upper bound on the total DOF of this channel, we assume that all users within a cell can cooperate. This cooperation converts the uplink mode to a MISO K-user GIC with M antennas at the transmitters and one antenna at the receives. An upper bound on the DOF of the MISO K-user GIC is obtained in [44]. The upper bound states that the total DOF of the channel is less than $\frac{KM}{M+1}$. We will show that this DOF is achievable.

We start with selecting the transmit directions of the m'th user in Cell k. A direction $T \in \mathcal{G}(\mathbf{H})$ ($\mathbf{H}$ is the set of all channel gains) is chosen as the transmit direction for this user if it can be represented as

$$T = \prod_{j=1}^{K} \prod_{i=1}^{K} \prod_{l=1}^{M} h_{j(il)}^{s_{j(il)}}, \tag{4.42}$$

where $s_{j(il)}$'s are integers satisfying

$$\begin{cases} s_{j(jl)} = 0 & \forall\ j \in \{1,2,\ldots,K\}\ \&\ l \in \{1,2,\ldots,M\} \\ 0 \le s_{j(km)} \le n-1 & \forall\ j \in \{1,2,\ldots,K\}\ \&\ j \neq k \\ 0 \le s_{j(il)} \le n & \text{Otherwise.} \end{cases}$$

The set of all transmit directions is denoted by $\mathcal{T}_{km}$. It is easy to show that the cardinality of this set is

$$L_{km} = n^{K-1}(n+1)^{(KM-1)(K-1)}. \tag{4.43}$$

Clearly, $\mathcal{T}_{km}$ satisfies C1.

We claim that all signals from non-intended cells are aligned at all base stations. In order to prove the claim, we introduce $\mathcal{T}_i$ as the set of received direction due to interference at the i'th base stations. Clearly,

$$\mathcal{T}_i = \bigcup_{k=1 \& k\neq i}^{K} \bigcup_{m=1}^{M} (h_{i(km)}\mathcal{T}_{km}).$$

Let us define $\mathcal{T}$ as the set of directions represented by (4.42) and satisfying

$$\begin{cases} s_{j(jl)} = 0 & \forall\ j \in \{1,2,\ldots,K\}\ \&\ l \in \{1,2,\ldots,M\} \\ 0 \le s_{j(il)} \le n & \text{Otherwise.} \end{cases} \tag{4.44}$$

We claim that $\mathcal{T}_i \subseteq \mathcal{T}$. In fact, all transmit directions of the m'th user in Cell k arrive at Receiver i multiplied by $h_{i(km)}$. Based on the selection of transmit directions, however, the maximum power of $h_{i(km)}$ in all members of $\mathcal{T}_{km}$ is $n-1$. Therefore, none of the received directions violates the condition (4.44) and this proves the claim.

Since $\mathcal{T}$ is not related to the i's base station, one can conclude that $\mathcal{T}_i \subseteq \mathcal{T}$ for all $i \in \{1,2,\ldots,K\}$. Hence, we deduce that all interfering users are aligned in the directions of $\mathcal{T}$. Now, L_i' can be obtained by counting the members of $\mathcal{T}_r$. It is easy to show that

$$L_i' = (n+1)^{MK(K-1)}. \tag{4.45}$$

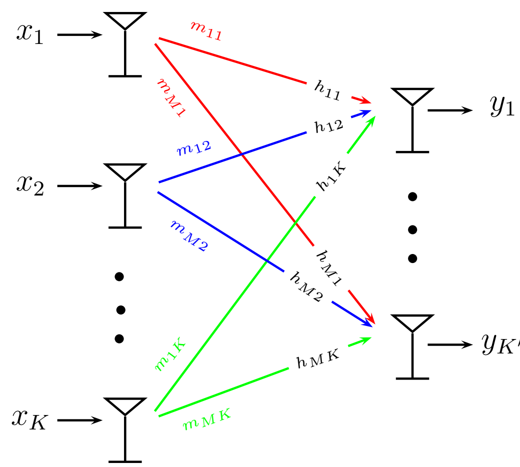

Figure 4.2: The $K \times M$ X Channel. User i for $i \in \{1, 2, \ldots, K\}$ wishes to transmit an independent message m_{ji} to Receiver j for all $j \in \{1, 2, \ldots, M\}$.

The total number of received directions at the i'th base stations is $\sum_{l=1}^{M} L_{il} + L_i'$. Since C_1 and C_2 hold at all base stations, we can to obtain the total DOF of the channel as

$$\begin{aligned} r_{\text{sum}} &= \frac{\sum_{k=1}^{K}\sum_{m=1}^{M} L_{km}}{Mn^{K-1}(n+1)^{(KM-1)(K-1)} + (n+1)^{MK(K-1)} + 1} \\ &= \frac{MKn^{K-1}(n+1)^{(KM-1)(K-1)}}{Mn^{K-1}(n+1)^{(KM-1)(K-1)} + (n+1)^{MK(K-1)} + 1} \\ &= \frac{MK}{M + \left(\frac{n+1}{n}\right)^{K-1} + \frac{1}{n^{K-1}(n+1)^{(KM-1)(K-1)}}}. \end{aligned} \tag{4.46}$$

Since n can be arbitrary large, we conclude that $\frac{MK}{M+1}$ is achievable for the uplink of a cellular system.

4.5.2 $K \times M$ X Channel

System Model

The $K \times M$ X channel models a network in which K transmitters wish to communicate with M receivers. Unlike the interference channel, each transmitter has a messages for each receiver. In other words, Transmitter i for all $i \in \{1, 2, \ldots, K\}$ wishes to transmit an independent message to Receiver j for all $j \in \{1, 2, \ldots, M\}$. The message transmitted by Transmitter i and intended for Receiver j is denoted by m_{ji}. The channel's input-output relation can be stated as follows, see Figure 4.2,

$$\begin{aligned} y_1 &= h_{11}x_1 + h_{12}x_2 + \ldots + h_{1K}x_K + z_1, \\ y_2 &= h_{21}x_1 + h_{22}x_2 + \ldots + h_{2K}x_K + z_2, \\ \vdots &= \quad \vdots \qquad \vdots \qquad \ddots \qquad \vdots \\ y_M &= h_{M1}x_1 + h_{M2}x_2 + \ldots + h_{MK}x_K + z_M, \end{aligned} \tag{4.47}$$

where x_i and y_i are input and output symbols of User i for $i \in \{1, 2, \ldots, K\}$, respectively. z_i is Additive White Gaussian Noise (AWGN) with unit variance for $i \in \{1, 2, \ldots, K\}$. Transmitters are subject to the power constraint P. h_{ji} represents the channel gain between Transmitter i and Receiver j. It is assumed that all channel gains are real and time invariant.

Let $\mathcal{C}_X$ denote the capacity region of this channel. The DOF region associated with the channel can be defined as the shape of the region in high SNR regimes scaled by $\log \mathrm{SNR}$. Let us denote the DOF region by $\mathcal{R}_X$. We are primarily interested in the main facet of the DOF region defined as:

$$r_{\mathrm{Xsum}} = \lim_{\mathrm{SNR}\to\infty} \max_{\mathbf{R}\in\mathcal{C}_X} \frac{\sum_{i=1}^{K}\sum_{j=1}^{M} R_{ij}}{\log \mathrm{SNR}}, \tag{4.48}$$

where R_{ij} is an achievable rate for the message m_{ij} and $\mathbf{R}$ is the set of all achievable rates. The DOF achievable by the message m_{ij} is denoted by r_{ij}.

The Total DOF of $\frac{KM}{K+M-1}$ is Achievable

An upper bound on the DOF of this channel is obtained in [25]. The upper bound states that the total DOF of the channel is less than $\frac{KM}{K+M-1}$ which means each message can at most achieve $\frac{1}{K+M-1}$ of DOF. We will show that this DOF is achievable. To this end, Transmitter i for all $i \in \{1, 2, \ldots, K\}$ transmits M signals along M directions as follows:

$$x_i = \sum_{j=1}^{M} h_{ji}x_{ji}, \tag{4.49}$$

where x_{ji} is the signal carrying the message m_{ji}. Let us focus on the signals intended for Receiver 1, i.e., $x_{11}, x_{12}, \ldots, x_{1K}$. The received signals due to these transmit signals can be written as

$$\begin{aligned} \tilde{y}_1 &= h_{11}^2 x_{11} + h_{12}^2 x_{12} + \ldots + h_{1K}^2 x_{1K} \\ I_{21} &= (h_{21}h_{11})x_{11} + (h_{22}h_{12})x_{12} + \ldots + (h_{2K}h_{1K})x_{1K} \\ \vdots &= \quad \vdots \qquad \vdots \qquad \ddots \qquad \vdots \\ I_{M1} &= (h_{M1}h_{11})x_{11} + (h_{M2}h_{12})x_{12} + \ldots + (h_{M1}h_{1K})x_{1K}. \end{aligned} \tag{4.50}$$

Since $x_{11}, x_{12}, \ldots, x_{1K}$ are not intended for Receiver j for all $j \in \{2, 3, \ldots, M\}$, I_{j1} is a part of interference at Receiver j. We claim that we can align all interfering signals $x_{11}, x_{12}, \ldots, x_{1K}$ at all Receivers $j \in \{2, 3, \ldots, M\}$.

Let $\mathbf{H}_1$ denote the set of all coefficients appeared in $I_{21}, I_{31}, \ldots, I_{M1}$, i.e., $\mathbf{H}_1 = \{(h_{21}h_{11}),$ $(h_{22}h_{12}), \ldots, (h_{M2}h_{12}), h_{M1}h_{1K})\}$. $\mathbf{H}_1$ has $(M-1)K$ members. The set of all monomials with variables in $\mathbf{H}_1$ is denoted by $\mathcal{G}(\mathbf{H}_1)$. Let $\mathcal{T}_1$ denote a subset of $\mathcal{G}(\mathbf{H}_1)$ consisting of monomials represented by

$$T = \prod_{i=1}^{K} \prod_{j=1}^{M} (h_{ji}h_{1i})^{s_{ji}}, \tag{4.51}$$

where

$$\begin{cases} s_{1i} = 0 & \forall\ i \in \{1, 2, \ldots, K\} \\ 0 \leq s_{ji} \leq n & \text{Otherwise.} \end{cases}$$

Clearly, $\mathcal{T}_1$ has $(n+1)^{(M-1)K}$ members.

The message m_{1i} for $i \in \{1, 2, \ldots, K\}$ is transmitted along directions in $\mathcal{T}_{1i}$ where $\mathcal{T}_{1i} \subset \mathcal{T}_1$. A direction T in $\mathcal{T}_{1i}$ can be represented as

$$T = \prod_{l=1}^{K} \prod_{j=1}^{M} (h_{jl}h_{1l})^{s_{jl}}, \tag{4.52}$$

where

$$\begin{cases} s_{1l} = 0 & \forall\ l \in \{1, 2, \ldots, K\} \\ 0 \leq s_{ji} \leq n-1 & \forall\ j \in \{1, 2, \ldots, M\}\ \&\ j \neq 1 \\ 0 \leq s_{jl} \leq n & \text{Otherwise.} \end{cases}$$

It is easy to show that the cardinality of $\mathcal{T}_{1i}$ is $n^{M-1}(n+1)^{(M-1)(K-1)}$. The received directions due to x_{1i} at all receivers belong to $\mathcal{T}_1$. In fact, x_{1i} arrives at receiver j multiplied by $(h_{ji}h_{1i})$ and since the power of $(h_{ji}h_{1i})$ in all directions in x_{1i} is less than n we conclude that the received directions are all in $\mathcal{T}_1$. Therefore, all transmit signals are aligned and the total number of directions in I_{j1} for all $j \in \{2, 3, \ldots, M\}$ is $(n+1)^{(M-1)K}$.

A similar argument can be applied for signals intended for Receiver j for all $j \in \{2, 3, \ldots, M\}$. Therefore, the received signals can be represented as

$$\begin{aligned} y_1 &= \tilde{y}_1 + I_{12} + I_{13} + \ldots + I_{1M} + z_1, \\ y_2 &= \tilde{y}_2 + I_{21} + I_{23} + \ldots + I_{2M} + z_2, \\ \vdots &= \ \vdots \qquad \vdots \qquad \qquad \ddots \qquad \vdots \\ y_M &= \tilde{y}_M + I_{M1} + I_{M2} + \ldots + I_{(M-1)M} + z_1, \end{aligned} \tag{4.53}$$

where I_{ji} is the part of interference caused by all messages intended for Receiver i at Receiver j. Due to symmetry, we only consider the received directions at Receiver 1. At Receiver 1, there are M_1 interfering signals each of which consisting of at most $(n+1)^{(M-1)K}$ directions. Therefore, the total number of interfering directions is $L_1' = (M-1)(n+1)^{(M-1)K}$. On the other hand, $\tilde{y}_1$ consists of $Kn^{M-1}(n+1)^{(M-1)(K-1)}$ directions. This is due to the fact that $\tilde{y}_1 = h_{11}^2 x_{11} + h_{12}^2 x_{12} + \ldots + h_{1K}^2 x_{1K}$ and x_{1i} for all $i \in \{1, 2, \ldots, K\}$ consists of $n^{M-1}(n+1)^{(M-1)(K-1)}$ directions. Therefore, the total number of received directions is

$$L = (M-1)(n+1)^{(M-1)K} + Kn^{M-1}(n+1)^{(M-1)(K-1)}.$$

Using Theorem 27, we can conclude that

$$r_{\text{Xsum}} \geq \frac{KMn^{M-1}(n+1)^{(M-1)(K-1)}}{Kn^{M-1}(n+1)^{(M-1)(K-1)} + (M-1)(n+1)^{(M-1)K} + 1} \tag{4.54}$$

is achievable for the X channel. By rearranging, we obtain

$$r_{\text{Xsum}} \geq \frac{KM}{K + (M-1)\left(\frac{n+1}{n}\right)^{M-1} + \frac{1}{n^{M-1}(n+1)^{(M-1)(K-1)}}}. \tag{4.55}$$

Since (4.55) holds for all n, we obtain

$$r_{\text{Xsum}} = \frac{KM}{K+M-1}, \tag{4.56}$$

which is the desired result. In a special case, $M = K$ and the total DOF is $\frac{K^2}{2K-1}$. This shows that as the number of transmitter and receivers increases the DOFs of X and GIC behave similarly.

4.6 Conclusion

In this chapter, we have considered the K-user Gaussian Interference Channel (GIC). We have proved that the total DOF of the system can be achieved with a static channel. This result is obtained by proposing a new coding scheme in which several fractional dimensions are imbedded into a single real line. These fractional dimensions play the role of integral dimensions in Euclidean spaces. This fact is supported by a recent extension of the Khintchine-Groshev theorem for the non-degenerate manifolds. The total DOF of the MIMO case as well as the complex case is also achieved by a simple application of the main result.

Chapter 5

Random Coding and Interference Management

In this chapter, point-to-point communication in an environment where several users sharing the same channel is studied. First, the interaction between users is ignored and assumed that all users except the desired user are transmitting using known coding schemes. The intended user who incurs interference from a number of interfering users tries to maximize its achievable rate. It is also assumed that interfering users use single codebooks, to be defined later, for data transmission. These codebooks are generated randomly and independent of each other. Therefore, interference alignment is not possible as it requires joint design for all users' coding schemes. Having information about the rates and codebooks of interfering users, the receiver is allowed to decode interfering messages. This in turn means that the signal transmitted from any interfering user is either decoded or considered as noise.

We propose the following method to obtain an achievable rate for the channel. Assuming its own data is decoded successfully, the receiver finds the maximum decodable subset of interfering users. By a maximum decodable subset, we mean a set of users that are decodable at the receiver, regarding the rest as noise and any decodable set is a proper subset of it. It is shown that this task can be accomplished by using a polynomial time algorithm. Once the receiver obtains the maximum decodable subset, it can partition the interfering users into two disjoint subsets, namely *decodable users* and *non-decodable users*. Then, the transmitter's rate is chosen such that the intended signal can be jointly decoded with the set of decodable users. We also propose a polynomial time algorithm to find the maximum achievable rate obtainable by this method.

To obtain the maximum achievable rate, one needs to find the maximum decodable subset of interfering users. Due to the large number of possible choices, having efficient algorithms that find the set of decodable users with maximum cardinality is desired. To this end, an algorithm that enables the receiver to accomplish this task in polynomial time is proposed in this chapter.

It must be noted that the model described above can also be used as a suitable model for the cognitive radio that is defined as a radio aware of its surroundings, c.f., [50] and [51]. In this case, the intended user can be considered as a secondary user and other interfering users as primary users, refer to [50] for basic definitions. To satisfy the assumption of cognitive radios that no secondary user should harm the primary users' communications, we assume that the effect of the secondary user on the primary users is negligible. This assumption is realistic when the secondary user is equipped with a low power transmitter or not allowed to transmit higher than some certain power level. Hence, the secondary user tries to communicate at the maximum rate, while its receiver knows the codebooks and rates of primary users.

As an application, this model is used in successive rate allocation for the K-user Gaussian IC. A polynomial time algorithm is proposed for such rate allocation. In fact, given an ordering on users one requires to characterize achievable rate region for the K-user IC where each transmitter is allowed to transmit data by using a single codebook and each receiver is allowed to decode any subset of interfering users. However, as this task is difficult to accomplish in general, other criteria other than priority on users are considered in the literature. The state of the art work for deriving achievable rate vectors treats interfering users as noise [52, 37, 53, 54, 55, 56, 57, 58, 59]. For example, in [37] the K-user Gaussian IC is studied where transmitters are allowed to allocate different powers in different bandwidths and receivers treat interference as noise. Recently, in [60, 61], successive interference cancelation is studied. For example, in [60] the optimal order of decoding that maximizes the minimum rate among all users is obtained.

Throughout this chapter, several converse theorems are proved. It is worth mentioning that these proofs are only true in a loose sense, i.e., when it is assumed that random codes are used by all users and there is no cooperation in coding design between users. In fact, structural codes may perform in regimes outside those that a converse theorem is established.

The organization of this chapter is as follows. In Section 5.1, the system model and some background materials are introduced. In Section 5.2, a discrete memoryless channel consisting of K transmitters and one receiver is considered. It is assumed that the users' rate vector is not necessarily inside the capacity region of the Multiple Access Channel (MAC) seen at the receiver side which results in failure of the receiver to reliably decode all the data streams. The receiver's task, however, is to find a maximum decodable subset of transmitters so that their data can be decoded from the received signal. A polynomial-time algorithm which finds the maximum decodable subset of users is proposed.

In Section 5.3, single-user data transmission over a channel with $K-1$ interfering users is considered. First a lower bound and an upper bound on the capacity of this channel is obtained. Then, a method that characterizes an achievable rate for the channel is proposed. This achievable rate is a function of other users' rates. It is proved that this function is piecewise linear.

In Section 5.4, additive channels where the interference caused by other users is Gaussian are considered. It is proved that for this case, the Gaussian codebook achieves the capacity where each interfering user is either decoded or treated as noise by the receiver.

In Section 5.5, applications of the proposed algorithms to the K-user Gaussian IC are investigated. First a polynomial time algorithm that characterizes points obtainable from successive maximization of users' rates is developed. Then the notion of one-sided Gaussian ICs to the K-user case is generalized. A point on the boundary of the capacity region of this channel is characterized. Finally the capacity of the strong one-sided K-user Gaussian IC is obtained. In Section 5.6, the chapter is concluded.

5.1 Preliminaries

5.1.1 System Model

We consider single-user data transmission over a channel $\mathscr{S}$ with $K-1$ interfering users. $\mathscr{S}$ is specified by the transition probability function $\omega(y_1|x_1, x_2, \ldots, x_K)$ where $x_i \in \mathscr{X}_i$ is the input letter to the channel from the i'th user and $y_1 \in \mathscr{Y}_1$ is the output letter received by the receiver, see Figure 5.1. The set of users' indices is denoted by E. x_1 is the input letter from the intended user and x_i for $i = 2, 3, \ldots, K$ are input letters from interfering users. We assume that the interfering users transmit data at the rate vector $\mathbf{R}_{-1} = [R_2, R_3, \ldots, R_K]$ by using single codebooks generated randomly from the joint probability distribution $p_{X_2}(x_2)p_{X_3}(x_3)\cdots p_{X_K}(x_K)$. We are interested in characterizing the capacity of this channel.

We also consider the continuous Gaussian case modeled by

$$y_1 = x_1 + x_2 + \cdots + x_K + z, \tag{5.1}$$

where x_1 and y_1 denote transmitted and received symbols, respectively. x_i, $i = 2, 3, \ldots, K$, is the input symbol corresponding to the i'th interfering user that uses a single Gaussian codebook with power P_i and rate R_i. z is the additive white Gaussian noise with variance N.

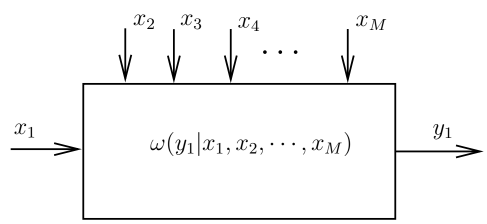

Figure 5.1: Single user in an interfering medium. x_1 is the input letter from the intended user and x_i, $i = 2, 3, \ldots, K$, is the input symbol corresponding to the i'th interfering user.

The transmitter is subject to the average power P_1 and tries to send data at the maximum rate R_1.

5.1.2 Submodular Functions

Definition 8. *Let E be a finite nonempty set. A function $f : 2^E \to \mathbb{R}$ is called a submodular function if it satisfies*

$$f(V \cup U) + f(V \cap U) \leq f(V) + f(U), \tag{5.2}$$

for any $V, U \subseteq E$. A function f is called supermodular if $-f$ is submodular. A modular function is a function which is both submodular and supermodular.

Submodular functions are one of the most important objects in discrete optimization. In fact, they play the same role in discrete optimization as convex functions do in the continuous case [62]. Besides having a polynomial-time algorithm based on the ellipsoid method [63], there are combinatorial algorithms for minimizing submodular functions in strongly polynomial time, c.f. [62] and [64].

If a submodular function is nondecreasing, i.e. $f(U) \leq f(V)$ if $U \subseteq V$, and $f(\varnothing) = 0$, then the associated polyhedron

$$\mathcal{B}(f) = \{\mathbf{x} | \mathbf{x}(U) \leq f(U),\ \forall U \subseteq E, \mathbf{x} \geq 0\}, \tag{5.3}$$

is a polymatroid. Likewise, if a supermodular function is nondecreasing and $f(\varnothing) = 0$, then the associated polyhedron

$$\mathcal{G}(f) = \{\mathbf{x} | \mathbf{x}(U) \geq f(U),\ \forall U \subseteq E\}, \tag{5.4}$$

is a contra-polymatroid.

5.1.3 Properties of Mutual Information for Independent Random Variables

In this subsection, we review some important equalities and inequalities in Information Theory. We consider K independent random variables $X_1, X_2, \dots, X_K$. Moreover, let $E = \{1, 2, \dots, K\}$ denote the set of random variables' indices. For any random variable Y, we have the following properties:

1) *Chain Rule*: For any disjoint subsets U and V, we have the following inequality:

$$I(\mathbf{X}_{U\cup V}; Y) = I(\mathbf{X}_V; Y|\mathbf{X}_U) + I(\mathbf{X}_U; Y). \tag{5.5}$$

2) *Independent Conditioning Inequality*: For any disjoint subsets U and V, the following inequality holds:

$$I(\mathbf{X}_U; Y) \leq I(\mathbf{X}_U; Y|\mathbf{X}_V). \tag{5.6}$$

3) *Polymatroidal Property*: In [35], it is shown that the set function $\sigma(U) = I(\mathbf{X}_U; Y|\mathbf{X}_{\overline{U}})$ is submodular and nondecreasing, i.e.,

$$\sigma(U \cup V) + \sigma(U \cap V) \leq \sigma(U) + \sigma(V),\ \forall U, V \subseteq E. \tag{5.7}$$

Hence, its associated polyhedron is a polymatroid.

4) *Contra-polymatroidal Property*: We claim that the set function ρ defined as $\rho(U) = I(\mathbf{X}_U; Y)$ is a supermodular function. To this end, fix any arbitrarily subsets U and V. Let $S = U \cap V$. From the chain rule, we have

$$I(\mathbf{X}_{U\cup V}; Y) = I(\mathbf{X}_U; Y) + I(\mathbf{X}_{V\setminus U}; Y|\mathbf{X}_U), \tag{5.8}$$

which can equivalently be written as

$$\rho(U \cup V) = \rho(U) + I(\mathbf{X}_{V\setminus S}; Y|\mathbf{X}_{U\setminus S}, \mathbf{X}_S). \tag{5.9}$$

From Independent Conditioning Property, we have $I(\mathbf{X}_{V\setminus S}; Y|\mathbf{X}_S) \leq I(\mathbf{X}_{V\setminus S}; Y|\mathbf{X}_{U\setminus S}, \mathbf{X}_S)$. Hence,

$$\rho(U \cup V) \geq \rho(U) + I(\mathbf{X}_{V\setminus S}; Y|\mathbf{X}_S). \tag{5.10}$$

Adding $\rho(U \cap V) = \rho(S)$ to both sides, we obtain

$$\rho(U \cup V) + \rho(U \cap V) \geq \rho(U) + I(\mathbf{X}_{V\setminus S}; Y|\mathbf{X}_S) + I(\mathbf{X}_S; Y). \tag{5.11}$$

Since $I(\mathbf{X}_{V\setminus S}; Y|\mathbf{X}_S) + I(\mathbf{X}_S; Y) = I(\mathbf{X}_V; Y)$, we have

$$\rho(U \cup V) + \rho(U \cap V) \geq \rho(U) + \rho(V), \tag{5.12}$$

as claimed. It is easy to show that ρ is nondecreasing and hence its associated polyhedron is a contra-polymatroid.

5.1.4 Multiple Access Capacity Region

One of the most important results in Information Theory is the characterization of the capacity region of the MAC [65, 66]. The capacity region of a MAC can be represented as follows. We define $\mathscr{P}$ as the collection of all probability distributions which can be written as $\mathbb{P}(x_1, x_2, \dots, x_K, y) = p(x_1)p(x_2)\cdots p(x_K)\omega(y|x_1, x_2, \dots, x_K)$, where $\omega(y|x_1, x_2, \dots, x_K)$ is the channel transition probability function. Now, the capacity region of a MAC is

$$\mathcal{C}_{\mathrm{MAC}} = \mathrm{conv}\left(\bigcup_{\mathbb{P}\in\mathscr{P}} \mathcal{C}_{\mathrm{MAC}}(\mathbb{P})\right), \tag{5.13}$$

where conv$(\cdot)$ denotes convex hull operation, and $\mathcal{C}_{\mathrm{MAC}}(\mathbb{P})$ is defined as

$$\mathcal{C}_{\mathrm{MAC}}(\mathbb{P}) = \{\mathbf{R}|\mathbf{R}(U) \le I(\mathbf{X}_U; Y|\mathbf{X}_{\overline{U}}),\ \forall\, U \subseteq E\}. \tag{5.14}$$

Using the polymatroidal property of the mutual information, it is easy to show that $\mathcal{C}_{\mathrm{MAC}}(\mathbb{P})$ is a polymatroid. It is worth noting that even though $\mathcal{C}_{\mathrm{MAC}}$ is the union of polymatroids, it is not necessarily a polymatroid. However, $\mathcal{C}_{\mathrm{MAC}}$ is a polymatroid for the K-user Gaussian MAC modeled by

$$y = x_1 + x_2 + \cdots + x_K + z, \tag{5.15}$$

where y is the received symbol, x_i is the transmitted symbol of user i, and z is additive white Gaussian noise with zero mean and variance N. User i is also subject to an average power constraint P_i. The capacity region of the K-user Gaussian MAC can be stated as

$$\mathcal{C}_{\mathrm{GMAC}} = \{\mathbf{R}|\mathbf{R}(U) \le \gamma\left(\frac{\mathbf{P}(U)}{N}\right),\ \forall\, U \subseteq E\}, \tag{5.16}$$

where $\gamma(x) = 0.5\log_2(1 + x)$.

5.2 Maximum Decodable Subset

In this section, we consider a discrete memoryless channel consisting of K transmitters with input alphabet $\mathscr{X}_i$ for the ith transmitter and one receiver with output alphabet $\mathscr{Y}$ where each transmitter uses a single codebook for data transmission. This channel is specified by the transition probability function $\omega(y|x_1, x_2, \dots, x_K)$ where $x_i \in \mathscr{X}_i$ is the input letter to the channel from the ith transmitter and $y \in \mathscr{Y}$ is the output letter received by the receiver, see Figure 5.2. The random codebooks used for data transmission at the rate vector $\mathbf{R} = [R_1, R_2, \dots, R_K]$ are generated by using the joint probability distribution $p_{X_1}(x_1)p_{X_2}(x_2)\cdots p_{X_K}(x_K)$ for random variables $X_1, X_2, \dots, X_K$.

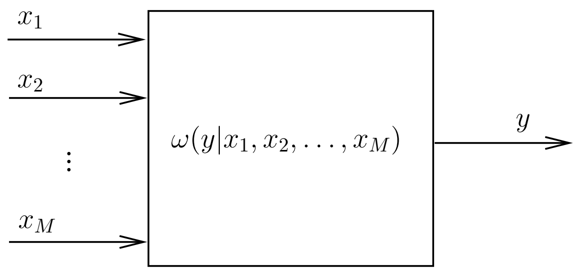

Figure 5.2: Transmitter i uses a random codebook for data transmission at rate R_i. Receiver's task is to find the maximum decodable subset of users.

The rate vector $\mathbf{R}$ may fall outside of the capacity region of the MAC seen at the receiver side which results in failure to reliably decode all data streams. The receiver's task, however, is to find a decodable subset of transmitters so that their data can be decoded from the received signal. To this end, the receiver partitions the set of transmitters into two disjoint parts and tries to jointly decode the data sent by the transmitters within the first partition, while considering the signals of transmitters in the second partition as noise.

In what follows, we compute the complexity of finding a decodable subset of transmitters by an exhaustive search. Let $E = \{1, 2, \ldots, K\}$ denote the set of transmitters' indices. There are 2^K ways to partition E into two subsets; and to verify that a subset V with cardinality k is decodable, $2^k - 1$ inequalities must be verified due to (5.14). Hence, in general, the total number of inequalities to be checked is

$$\sum_{k=0}^{K} \binom{K}{k} (2^k - 1) = 3^K - 2^K,$$

which is exponential in the number of users.

Definition 9 (Maximum decodable subset)**.** *A set of transmitters is a maximal decodable subset if all transmitters in the subset are jointly decodable by the receiver, and is not a proper subset of any other decodable subset. If the maximal decodable subset is unique, we call it the* maximum decodable subset*.*

Lemma 12. *For any channel, there is a maximum decodable subset.*

Proof. Suppose the receiver is able to decode two subsets of transmitters, namely U and V, such that none of them is a subset of the other. U and V are proper subsets of their union $U \cup V$. Besides, their union is decodable by the receiver. This contradicts the fact that both subsets are maximal. $\square$

We first describe some properties of the maximum decodable subset. There are two cases of special interest. The first case occurs when all transmitters are decodable by the receiver, i.e., the maximum decodable subset is the set E. In this case, the transmitters' rates must satisfy the inequalities given in (5.14). In the second case, however, none of the transmitters is decodable by the receiver, i.e., the maximum decodable subset is empty. The following Lemma shows that for the second case the rate vector $\mathbf{R}$ must be in a certain contra-polymatroid.

Lemma 13. *None of the signals is decodable by the receiver if and only if transmitters' rates satisfy*

$$\mathbf{R}(U) > I(\mathbf{X}_U; Y), \ \forall\, U \subseteq E. \tag{5.17}$$

Moreover, the region of the rate vectors satisfying above inequalities forms a contra-polymatroid.

Proof. We first prove that if a rate vector $\mathbf{R}$ satisfies (5.17), then none of the signals are decodable. To this end, we assume that V is the maximum decodable subset and $V \neq \emptyset$. Since V is a decodable subset, we have the following constraints on the rates of the members of V.

$$\mathbf{R}(T) \leq I(\mathbf{X}_T; Y|\mathbf{X}_{V\setminus T}), \ \forall\, T \subseteq V. \tag{5.18}$$

By substituting $T = V$ in the above equation, we have

$$\mathbf{R}(V) \leq I(\mathbf{X}_V; Y), \tag{5.19}$$

which is a contradiction and this completes the "if" part of the proof.

Next, we need to prove that if the inequalities in (5.17) are not satisfied, there is at least a transmitter which is decodable. Suppose there are some subsets that do not satisfy (5.17). Assume W has the minimum cardinality among all and satisfies

$$\mathbf{R}(W) \leq I(\mathbf{X}_W; Y). \tag{5.20}$$

If $|W| = 1$, then the transmitter in W is decodable by considering everything else as noise which is the desired result. Hence, we assume $|W| > 1$. If all members of W are jointly decodable, then we have found a decodable subset. Otherwise, there must be a subset of W, say V, satisfying

$$\mathbf{R}(V) > I(\mathbf{X}_V; Y|\mathbf{X}_{W\setminus V}). \tag{5.21}$$

By decomposing the mutual information in (5.20), we obtain

$$\mathbf{R}(W) \leq I(\mathbf{X}_V; Y|\mathbf{X}_{W\setminus V}) + I(\mathbf{X}_{W\setminus V}; Y). \tag{5.22}$$

From the minimality of $|W|$, we have

$$\mathbf{R}(W\backslash V) > I(\mathbf{X}_{W\backslash V};Y). \tag{5.23}$$

By combining the two inequalities (5.21) and (5.23) and considering the fact that $\mathbf{R}(W) = \mathbf{R}(W\backslash V) + \mathbf{R}(V)$, we conclude that

$$\begin{aligned}\mathbf{R}(W) &> I(\mathbf{X}_{W\backslash V};Y) + I(\mathbf{X}_V;Y|\mathbf{X}_{W\backslash V}), &(5.24)\\ &> I(\mathbf{X}_W;Y), &(5.25)\end{aligned}$$

which is a contradiction. This completes the "only if" part of the proof.

It is easy to see that the function on the right hand side of (5.17) is a supermodular function and monotone, hence the region formed by rates satisfying (5.17) is a contra-polymatroid. □

In the following theorem, the characterization of the maximum decodable subset is presented.

Theorem 28. *A subset $S \subseteq E$ is a maximum decodable subset if and only if the transmitters' rates satisfy the following inequalities*

$$\begin{aligned}\mathbf{R}(V) &\leq I(\mathbf{X}_V;Y|\mathbf{X}_{S\backslash V}),\ \forall\ V \subseteq S, &(5.26)\\ \mathbf{R}(U) &> I(\mathbf{X}_U;Y|\mathbf{X}_S),\quad \forall\ U \subseteq \overline{S}. &(5.27)\end{aligned}$$

Proof. Inequality (5.26) corresponds to the capacity region of the MAC for members of S considering members of $\overline{S}$ as noise. Hence, the members of S are decodable iff the inequalities in (5.26) are satisfied. The set S is a maximum decodable subset if no other transmitters in $\overline{S}$ is decodable by the receiver. Now, by applying Lemma 13 and considering that all members of S are decoded, we conclude that none of the transmitters in $\overline{S}$ is decodable iff the inequalities in (5.27) are satisfied. This completes the proof. □

For a given maximum decodable subset $S \subseteq E$, we define D^S as

$$\begin{aligned}D^S = \{\mathbf{R}|\mathbf{R}(T) \leq I(\mathbf{X}_V;Y|\mathbf{X}_{S\backslash V}),\ \forall\ V \subseteq S,&\\ \mathbf{R}(U) > I(\mathbf{X}_U;Y|\mathbf{X}_S),\quad \forall\ U \subseteq \overline{S}\}.& &(5.28)\end{aligned}$$

D^S is a polyhedron because it is the intersection of finitely many half spaces. By Theorem 28, D^S consists of all rate vectors with the same maximum decodable subset S. Since for any rate vector there is an associated maximum decodable subset, we have $\cup_{S\subseteq E} D^S = \mathbb{R}_+^K$. This means that $\mathbb{R}_+^K$ is represented as the union of finitely many polyhedral sets. An example for the case of the additive two-user Gaussian channel is given in Figure 5.3.

The result of this section can be directly extended to continuous channels. The most applicable class of continuous channels is the additive Gaussian channel defined by

$$y = x_1 + x_2 + \cdots + x_K + z, \tag{5.29}$$

where z is an additive Gaussian noise with zero mean and variance N. We assume users transmit at rates $\mathbf{R} = [R_1, \ldots, R_K]$ using Gaussian codebooks with average powers $\mathbf{P} = [P_1, \ldots, P_K]$. In the following example, we apply the result of Theorem 28 to a two-user additive Gaussian channel.

Example 1. *Consider a two-user additive Gaussian channel where the received signal can be written as $y = x_1 + x_2 + z$. In this case, E has four subsets, namely $S_1 = \{1, 2\}$, $S_2 = \{1\}$, $S_3 = \{2\}$, and $S_4 = \emptyset$. By applying Theorem 28, we obtain the following conditions for the subsets of E to be the maximum decodable subset.*

1. *S_1 is the maximum decodable subset. In this case, the conditions $R_1 \leq \gamma(\frac{P_1}{N})$, $R_2 \leq \gamma(\frac{P_2}{N})$, and $R_1 + R_2 \leq \gamma(\frac{P_1+P_2}{N})$ must be satisfied.*

2. *S_2 is the maximum decodable subset. In this case, the conditions $R_1 \leq \gamma\left(\frac{P_1}{P_2+N}\right)$ and $R_2 > \gamma\left(\frac{P_2}{N}\right)$ must be satisfied.*

3. *S_3 is the maximum decodable subset. In this case, the conditions $R_2 \leq \gamma\left(\frac{P_2}{P_1+N}\right)$ and $R_1 > \gamma\left(\frac{P_1}{N}\right)$ must be satisfied.*

4. *S_4 is the maximum decodable subset. In this case, the conditions $R_1 > \gamma(\frac{P_1}{P_2+N})$, $R_2 > \gamma(\frac{P_2}{P_1+N})$, and $R_1 + R_2 > \gamma(\frac{P_1+P_2}{N})$ must be satisfied.*

The set of conditions described above partitions $\mathbb{R}_+^2$ into four regions, as illustrated in Figure 5.3. It can be seen from the figure that $D^{\{1,2\}}$ is a polymatroid corresponding to the capacity region of a two-user MAC and $D^{\emptyset}$ is a contra-polymatroid according to Lemma 13.

The above example shows that finding the maximum decodable subset is equivalent to finding the region where the transmitters' rate vector belongs to. Since the number of regions grows exponentially with the number of transmitters, finding a polynomial-time algorithm for solving the problem is desired. To this end, we first define the function $f : 2^E \to \mathbb{R}$ as follows

$$f(V) = I(\mathbf{X}_V; Y|\mathbf{X}_{\overline{V}}) - \mathbf{R}(V), \tag{5.30}$$

where $V \subseteq E$.

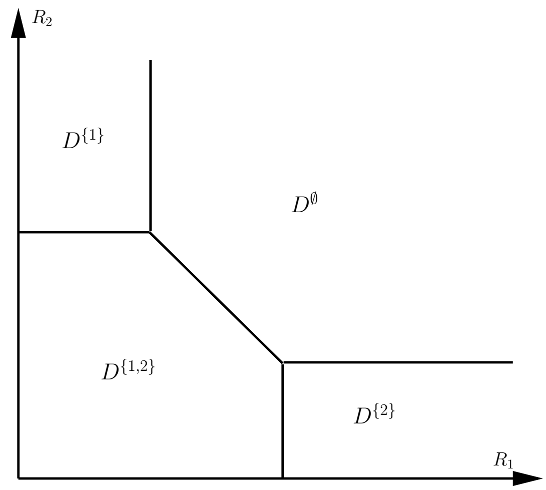

Figure 5.3: Decision regions used for determining the maximum decodable subset for a two-user additive Gaussian Channel. For any rate in $D^{\{1,2\}}$, the receiver can decode both signals. For rates in $D^{\{1\}}$ and $D^{\{2\}}$, the receiver is able to decode transmitters 1 and 2, respectively. Finally, the receiver can decode neither 1 nor 2 for any rate in $D^{\emptyset}$.

Lemma 14. *The function f defined in (5.30) is a submodular function.*

Proof. The result directly follows from the modularity of $\mathbf{R}$ and the submodularity of mutual information. □

Since there are polynomial-time algorithms for minimizing any submodular functions, c.f., [62] and [64], the following optimization problem can be solved efficiently:

$$f(W) = \min_{V \subseteq E} f(V). \tag{5.31}$$

If the minimum of f in (5.31) is zero, then all transmitters are decodable by the receiver due to (5.14). Otherwise, there is at least one transmitter in the set E which is not decodable. In the following theorem, we prove that indeed all members of the minimizer of f are not decodable, and they need to be considered as noise.

Theorem 29. *No member of the subset W that minimizes f in (5.31) is decodable by the receiver, provided that the minimum in (5.31) is not zero and the minimum cardinal minimizer is used. In fact, all members of W must be considered as noise, i.e., if S is the maximum decodable subset then $W \cap S = \varnothing$.*

Proof. We first partition the minimizer subset W into two disjoint sets U and T where $U = W \cap S$ and $T = W \backslash S$, see Figure 5.4. We need to show that $U = \emptyset$. Suppose U is

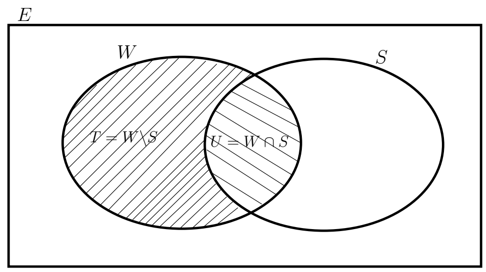

Figure 5.4: E is the ground set. S is the maximum decodable subset. W is the minimizer of f in (5.31).

nonempty. Hence $|U| \geq 1$ and $|T| < |W|$. Since U is a subset of S, from (5.26), we have

$$\mathbf{R}(U) \leq I(\mathbf{X}_U; Y|\mathbf{X}_{S\backslash U}). \tag{5.32}$$

The inclusion $S\backslash U \subseteq \overline{W}$ and independence of random variables imply $I(\mathbf{X}_U; Y|\mathbf{X}_{S\backslash U}) \leq I(\mathbf{X}_U; Y|\mathbf{X}_{\overline{W}})$. Hence,

$$\mathbf{R}(U) \leq I(\mathbf{X}_U; Y|\mathbf{X}_{\overline{W}}). \tag{5.33}$$

From the definition of f in (5.30), we have

$$f(W) = I(\mathbf{X}_W; Y|\mathbf{X}_{\overline{W}}) - \mathbf{R}(W). \tag{5.34}$$

From the chain rule and the fact that T and U partition W into two disjoint subsets, we have the following equation

$$\begin{aligned} I(\mathbf{X}_W; Y|\mathbf{X}_{\overline{W}}) &= I(\mathbf{X}_T; Y|\mathbf{X}_{\overline{W}}, \mathbf{X}_U) + I(\mathbf{X}_U; Y|\mathbf{X}_{\overline{W}}), \\ &= I(\mathbf{X}_T; Y|\mathbf{X}_{\overline{T}}) + I(\mathbf{X}_U; Y|\mathbf{X}_{\overline{W}}). \end{aligned} \tag{5.35}$$

Substituting (5.35) into (5.34) and using $\mathbf{R}(W) = \mathbf{R}(T) + \mathbf{R}(U)$, we obtain

$$f(W) = f(T) + I(\mathbf{X}_U; Y|\mathbf{X}_{\overline{W}}) - \mathbf{R}(U). \tag{5.36}$$

Using the inequality (5.33), we conclude that

$$f(T) \leq f(W). \tag{5.37}$$

If $f(T) < f(W)$, then it contradicts the optimality of W, and if $f(T) = f(W)$, then it contradicts the fact that $|W|$ has minimum cardinality among all minimizers. This completes the proof. $\square$

By applying Theorem 29 and using the well-known submodular function minimization algorithms as a subroutine, c.f. [64] and [62], we propose the following polynomial-time algorithm for finding the maximum decodable subset.

Algorithm 1 (Finding the maximum decodable subset)**.**

1. *Set* $S = E$*.*
2. *Find* W *such that*
$$f(W) = \min_{V \subseteq S} f(V),$$
where f *is*
$$f(V) = I(\mathbf{X}_V; Y|\mathbf{X}_{S\backslash V}) - \mathbf{R}(V). \tag{5.38}$$
3. *If* $W = \varnothing$ *STOP.* S *is the maximal decodable subset. Otherwise,* $S \backslash W \longrightarrow S$*.*
4. *If* $S = \varnothing$ *STOP. No subset of* E *is decodable. Otherwise, GO TO step 2.*

Theorem 30. *Algorithm 1 converges to the maximum decodable subset in polynomial time.*

Proof. Since in each iteration W is a nonempty set (otherwise, the algorithm stops), this algorithm converges at most in $|E|$ iterations. Furthermore, in each iteration, we need to minimize a submodular function which can be done in polynomial time [62]. Hence, the total running time of the algorithm is polynomial in time. □

5.3 An Achievable Rate

In this section, we propose a method to obtain an achievable rate for the channel $\mathscr{S}$. We also provide a polynomial time algorithm to characterize this achievable rate. A lower bound for the capacity of $\mathscr{S}$ can be obtained by considering interfering users in E as noise and optimizing over all input distributions. Hence, we have

$$\max_{p(x_1)} I(X_1; Y_1) \leq C, \tag{5.39}$$

where C denotes the capacity of $\mathscr{S}$. Now, we assume that regardless of the input distribution, the receiver is able to decode all interfering users considering its own signal as noise. By this assumption, an upper bound on the capacity can be obtained as follows

$$C \leq \max_{p(x_1)} I(X_1; Y_1|\mathbf{X}_{E\backslash 1}). \tag{5.40}$$

Let us assume that the transmitter uses $p_{X_1}(x_1)$ to generate a single random codebook. We need to find the maximum achievable rate R_1. If R_1 is an achievable rate, then the receiver can successfully decode its intended data. After decoding its own signal, the receiver can search in the set $E\backslash\{1\}$ for the maximum decodable subset $S \subseteq E\backslash\{1\}$. This procedure can be done efficiently using Algorithm 1. Let us define $V = E\backslash(S \cup \{1\})$. V is the set of users that receiver treats as noise. From (5.26), we have

$$\mathbf{R}(U) \leq I(\mathbf{X}_U; Y_1|\mathbf{X}_{S\cup\{1\}\backslash U}),\ \forall U \subseteq S. \tag{5.41}$$

To find R_1, we consider the MAC consisting of user 1 and the users in S, while the users in V are considered as noise. From (5.14), the rate vector $\mathbf{R}$ is achievable if

$$\mathbf{R}(U) \leq I(\mathbf{X}_U; Y_1|\mathbf{X}_{S\cup\{1\}\backslash U}),\ \forall U \subseteq S \cup \{1\}. \tag{5.42}$$

Since half of the inequalities in (5.42) are satisfied by (5.41) and the only unknown parameter is R_1, we can maximize the user's rate based on the following optimization problem:

$$R_1(\mathbf{R_{-1}}) = \min_{U\subseteq S} I(X_1, \mathbf{X}_U; Y_1|\mathbf{X}_{S\backslash U}) - \mathbf{R}(U). \tag{5.43}$$

The optimization problem (5.43) is again a submodular function minimization and can be solved by polynomial-time algorithms.

In the following, we summarize the above procedure.

Algorithm 2 (finding an achievable rate)**.**

1. *Given $p(x_1)$, find the maximum decodable subset S among interfering users by using Algorithm 1 and assuming that the user's data is decoded.*

2. *Solve the submodular function minimization in (5.43).*

As a by-product of the above algorithm, we can find the subset of interfering users that can be first decoded at the receiver and its effect can be removed.

Proposition 1. *If U minimizes (5.43), then the receiver is capable of decoding all users in $W = S\backslash U$ by considering everything else as noise.*

Proof. At the first step, one needs to decode W. This requires,

$$\mathbf{R}(T) \leq I(\mathbf{X}_T; Y_1|\mathbf{X}_{W\backslash T}),\ \forall T \subseteq W. \tag{5.44}$$

Suppose there is a subset $T^\star$ that does not satisfy (5.44), that is,

$$\mathbf{R}(T^\star) > I(\mathbf{X}_{T^\star}; Y_1|\mathbf{X}_{W\backslash T^\star}). \tag{5.45}$$

Hence,

$$\begin{aligned}
\tilde{R}_1 &\stackrel{\triangle}{=} I(X_1, \mathbf{X}_{U\cup T^\star}; Y_1|\mathbf{X}_{S\backslash(U\cup T^\star)}) - \mathbf{R}(U\cup T^\star) \\
&\stackrel{(a)}{=} I(X_1, \mathbf{X}_U; Y_1|\mathbf{X}_{S\backslash U}) - \mathbf{R}(U) \\
&\quad + I(\mathbf{X}_{T^\star}; Y_1|\mathbf{X}_{W\backslash T^\star}) - \mathbf{R}(T^\star) \\
&\stackrel{(b)}{=} R_1(\mathbf{R}_{-1}) + I(\mathbf{X}_{T^\star}; Y_1|\mathbf{X}_{W\backslash T^\star}) - \mathbf{R}(T^\star) \\
&\stackrel{(c)}{<} R_1(\mathbf{R}_{-1}),
\end{aligned} \tag{5.46}$$

where (a) follows from the chain rule and the fact that $(S\backslash(U\cup T^\star))\cup T^\star = S\backslash U$ and $S\backslash(U\cup T^\star) = W\backslash T^\star$, (b) follows from the definition of $R_1(\mathbf{R}_{-1})$ and minimality of U, and (c) follows form (5.45). The last inequality contradicts the fact that U is the solution for the minimization problem in (5.43). This completes the proof. □

In light of Proposition 1, the set E is decomposable into three disjoint subsets, namely V, $U\cup\{1\}$, and W. V is the complement of $S\cup\{1\}$, namely the union of the maximum decodable subset S and the intended user. Therefore, the receiver is not able to decode the interfering users in V and considers them as noise. W is the part of S that the receiver can decode by considering everything else as noise. $U\cup\{1\}$ is the subset of users that need to be decoded jointly after removing the effect of W.

As indicated in (5.43), the achievable rate is a function of interfering users' rates. In order to derive some properties of this function, we need the following definition.

Definition 10 (piecewise linear functions [36]). *A function $f:\mathbb{R}^K\to\mathbb{R}$ is piecewise linear if firstly its domain can be represented as the union of finitely many polyhedral sets, and secondly f is "affine" within each polyhedral set, i.e., $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x}+b$ for some vector $\mathbf{a}$ and scalar b.*

In the following theorem, we summarize some properties of R_1 as a function of $\mathbf{R}_{-1}$.

Theorem 31. *The function $R_1(\mathbf{R}_{-1})$ defined in (5.43) is piecewise linear. More precisely, $R_1(\mathbf{R}_{-1})$ consists of at most 3^{K-1} collection of affine functions.*

Proof. Likewise (5.28), let us define the region D^S as

$$\begin{aligned}
D^S = \{\mathbf{R}_{-1}|\mathbf{R}(T) &\le I(\mathbf{X}_T; Y_1|\mathbf{X}_{S\backslash T}, X_1),\ \forall\, T\subseteq S, \\
\mathbf{R}(U) &> I(\mathbf{X}_U; Y_1|\mathbf{X}_S, X_1),\ \forall\, U\subseteq V\},
\end{aligned} \tag{5.47}$$

where $V = E\backslash(S\cup\{1\})$. Due to (5.43), the function $R_1(\mathbf{R}_{-1})$ is defined as the pointwise minimum of $2^{|S|}$ affine functions over the polyhedral set D^S. As a result, $R_1(\mathbf{R}_{-1})$ is piecewise linear, continuous, and concave over D^S, c.f., Theorem 2.49 in [36].

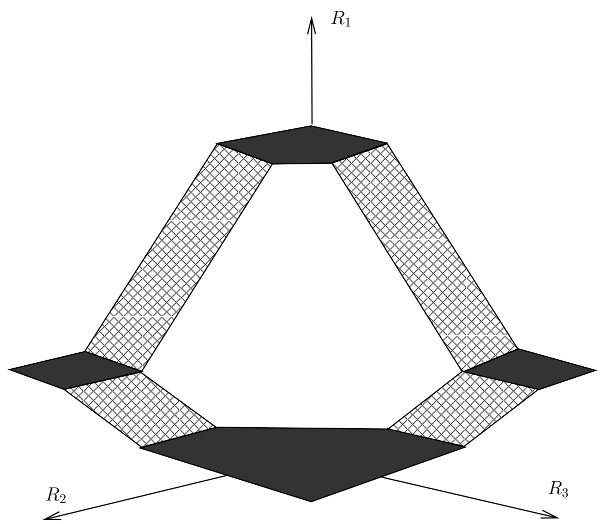

Figure 5.5: The function $R_1(\mathbf{R}_{-1})$ for a channel with two interfering users

Since $R_1(\mathbf{R}_{-1})$ is a piecewise linear function over each D^S and $\cup_{S\subseteq E\setminus\{1\}} D^S = \mathbb{R}_+^{K-1}$, it is a piecewise linear function over $\mathbb{R}_+^{K-1}$. Moreover, each polyhedron D^S is divided into at most $2^{|S|}$ sub-polyhedra in each of which R_1 is an affine function. Hence, the total number of components is not more than

$$\sum_{|S|=0}^{K-1} 2^{|S|} \binom{K-1}{|S|} = 3^{K-1}. \tag{5.48}$$

This completes the proof.

□

It is worth noting that, although R_1 is a concave function over each D^S, it is not a concave function over $\mathbb{R}_+^{K-1}$.

Example 2. *Consider an additive channel $y_1 = x_1 + x_2 + x_3 + z_1$ where all users use Gaussian codebooks for data transmission. In this case, the maximum decodable subset of interfering users is a subset of $\{2, 3\}$. Hence, there are four regions $D^{\emptyset}$, $D^{\{2\}}$, $D^{\{3\}}$, and $D^{\{2,3\}}$ where $R_1(R_2, R_3)$ is a concave function over each of them. For instance, $R_1(R_2, R_3) = \gamma\left(\frac{P_1}{P_2+P_3+N_1}\right)$ over $D^{\emptyset}$ and $R_1(R_2, R_3) = \gamma\left(\frac{P_1}{P_2+N_1}\right) - g(R_3)$ over $D^{\{3\}}$ where $g(R_3)$ is either R_3 or 0. In Fig. 5.5, an example of the function $R_1(R_2, R_3)$ for this channel is illustrated. As depicted in the figure, $R_1(R_2, R_3)$ is a piecewise linear and continuous function. It also consists of 9 components, i.e., 3^{K-1} for $K = 3$.*

Example 3. *In this example, we consider binary adder channel with $K-1$ interfering users. The channel model can be written as $y_1 = x_1 \oplus x_2 \oplus \ldots \oplus x_K$. We further assume that users'* *codebooks are randomly chosen from Bernouli sequences with* $p(0) = p(1) = 0.5$. *In this case, it is easy to show that*

$$R_1(\mathbf{R}_{-1}) = [1 - \mathbf{R}(E\backslash\{1\})]^+, \tag{5.49}$$

where $[a]^+ = a$ *if* $a \geq 0$ *and 0 otherwise. This reflects the fact that the function* $R_1(\mathbf{R}_{-1})$ *may have less than* 3^{K-1} *components.*

5.4 Channel's Capacity for the Gaussian Case

In this section, we prove that provided using Gaussian distribution for codebook generation, the achievable rate obtained in the previous section is indeed the capacity (in the loose sense) of the additive channel with Gaussian noise and $K-1$ Gaussian interfering users.

To show that any rate above C (the output of Algorithm 2 where $p(x_1)$ is Gaussian) is not achievable, we construct a degraded broadcast channel and show that if a rate $R_1 > C$ is achievable, then one can communicate reliably outside the capacity region of this channel which is a contradiction. The following lemma assists us in constructing such a degraded channel.

Lemma 15. *For any set of independent Gaussian codebooks with power vector* $\mathbf{P} = [P_1, P_2, \ldots, P_K]$ *and rate vector* $\mathbf{R} = [R_1, R_2, \ldots, R_K]$, *there is a* K*-user Gaussian broadcast channel with the following properties:*

1. *The transmitter's total power is* $\mathbf{P}(E)$.
2. *There are* L *noise levels:* $N_1 < N_2 < \ldots < N_L$.
3. *Users are partitioned into* L *disjoint subsets, that is,* $E = \bigcup_{i=1}^{L} U_i$. *All users in* U_i *have the same noise level* N_i, *for* $i = \{1, 2, \ldots, L\}$.
4. *The rate vector* $\mathbf{R}$ *lies on the boundary of the capacity region.* $\mathbf{R}$ *is achievable using Gaussian codebooks with powers in one to one correspondence with the components of* $\mathbf{P}$.

Proof. We aim at building a Gaussian broadcast channel with x as input and $\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_L$ as outputs, where $\hat{y}_i = x + n_i$ and n_i is additive white Gaussian noise with variance N_i. To this end, we first construct a K-user Gaussian MAC with noise level N and transmit power vector $\mathbf{P}$ with the property that the rate vector $\mathbf{R}$ is achievable. Hence,

$$\mathbf{R}(T) \leq \gamma\left(\frac{\mathbf{P}(T)}{N}\right), \ \forall T \subseteq E. \tag{5.50}$$

By monotonicity of γ, it is always possible to find an N such that the rate vector $\mathbf{R}$ is achievable. Indeed, $\mathbf{R}$ is achievable for any $N \in [0, N_1]$, where N_1 corresponds to the case that for any noise above N_1 at least one of the inequalities in (5.50) turns to equality. Let U_1 denote the set of users for which the corresponding inequality in (5.50) turns to equality with noise level N_1 (in case of having more than one equality we choose the maximum cardinal subset), i.e.,

$$\mathbf{R}(U_1) = \gamma\left(\frac{\mathbf{P}(U_1)}{N_1}\right). \tag{5.51}$$

Now, we correspond users in U_1 to the output of the Gaussian channel $\hat{y}_1 = x + n_1$, where n_1 is additive Gaussian noise with variance N_1.

Let us consider a set $T \subseteq E \backslash U_1$. Using (5.50) for the set of users in $T \cup U_1$ yields

$$\mathbf{R}(T \cup U_1) < \gamma\left(\frac{\mathbf{P}(T \cup U_1)}{N_1}\right). \tag{5.52}$$

By plugging (5.51), we obtain

$$\mathbf{R}(T) < \gamma\left(\frac{\mathbf{P}(T)}{N_1 + \mathbf{P}(U_1)}\right), \ \forall T \subseteq E \backslash U_1. \tag{5.53}$$

We can apply the same procedure to (5.53), i.e., we increase N_1 until one of the inequalities turns to equality. Let N_2 denote the maximum noise level satisfying (5.53) with equality. Clearly, $N_1 < N_2$. If U_2 denotes the set of users satisfying (5.53) with equality, then we have

$$\mathbf{R}(U_2) = \gamma\left(\frac{\mathbf{P}(U_2)}{N_2 + \mathbf{P}(U_1)}\right). \tag{5.54}$$

By plugging in (5.53), we obtain

$$\mathbf{R}(T) < \gamma\left(\frac{\mathbf{P}(T)}{N_1 + \mathbf{P}(U_1) + \mathbf{P}(U_2)}\right), \ \forall T \subseteq E \backslash U_1 \cup U_2. \tag{5.55}$$

Now, we correspond users in U_2 to the output of the Gaussian channel $\hat{y}_2 = x + n_2$, where n_2 is additive Gaussian noise with variance N_2.

By repeating the above procedure, we can construct a set of channels with noise levels $N_1 < N_2 < \ldots < N_L$ and associate set of users $U_1, U_2, \ldots, U_L$ with $E = \bigcup_{j=1}^{L} U_j$ such that

$$\mathbf{R}(U_i) = \gamma\left(\frac{\mathbf{P}(U_i)}{N_i + \mathbf{P}(\bigcup_{j=1}^{i-1} U_j)}\right), \tag{5.56}$$

$$\mathbf{R}(T) \leq \gamma\left(\frac{\mathbf{P}(T)}{N_i + \mathbf{P}(\bigcup_{j=1}^{i-1} U_j)}\right), \ \forall T \subseteq U_i, \tag{5.57}$$

$$\mathbf{R}(T) < \gamma\left(\frac{\mathbf{P}(T)}{N_i + \mathbf{P}(\bigcup_{j=1}^{i} U_j)}\right), \ \forall T \subseteq \bigcup_{j=i+1}^{L} U_j. \tag{5.58}$$

Now, assume that the transmitter with total power $\mathbf{P}(E)$ uses K-level Gaussian codebooks for data broadcasting. The transmitted signal can be written as $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_K$, where $\mathbf{x}_l$ is a Gaussian codeword with power P_l and rate R_l and contains information for l'th user. The received signal at noise level N_i can be written as $\hat{\mathbf{y}}_i = \mathbf{x} + \mathbf{n}_i$. The set of inequalities in (5.58) implies that all users at noise level N_i can decode data streams of users in $\bigcup_{j=i+1}^{L} U_j$ considering everything else as noise. By removing the effect of users in $\bigcup_{j=i+1}^{L} U_j$ from the received signal, the set of inequalities in (5.57) implies that all users in U_i can decode their own signal considering users in $\bigcup_{j=1}^{i-1} U_j$ as noise. In other words, all users at the same level of noise can decode their signals by first decoding the users at upper levels and removing their effect and considering users at lower levels as Gaussian noise. Hence, we obtain a Gaussian broadcast channel in which the rate vector $\mathbf{R}$ is achievable and Gaussian codebooks are constructed according to the power vector $\mathbf{P}$. It remains to show that $\mathbf{R}$ is on the boundary of the capacity region. The capacity region of the Gaussian broadcast channel is fully characterized and there is an explicit expression for boundary points, c.f. [8]. The equalities in (5.56) guarantee that the rate vector $\mathbf{R}$ lies on the boundary of the capacity region. This completes the proof. □

Theorem 32. *The rate C, the output of Algorithm 2, is the capacity of the channel described in (5.1).*

Proof. We rewrite the achievable rate given in Algorithm 2 by using the Gaussian distribution as codebook generator. As discussed earlier, the set of users can be partitioned into three subsets V, $U \cup \{1\}$, and W.

W is the subset of interfering users that the receiver can decode considering everything else as noise. Since the Gaussian noise is the worst noise for additive channels, c.f. [39] and [38], and W is decodable when other users are considered as Gaussian noise, W is decodable for any arbitrary distribution for intended user. As a result, interfering users in W can be completely eliminated regardless of the input codebook.

V is the complement of the maximum decodable subset and must be considered as noise. From (5.27), we have

$$\mathbf{R}(T) > \gamma\left(\frac{\mathbf{P}(T)}{N + \mathbf{P}(V \backslash T)}\right), \ \forall T \subseteq V. \tag{5.59}$$

U is the solution to the minimization problem in (5.43). Hence, we have

$$C + \mathbf{R}(U) = \gamma\left(\frac{P_1 + \mathbf{P}(U)}{N + \mathbf{P}(V)}\right). \tag{5.60}$$

We apply Lemma 15 to the set of users in V with associated power vector $\mathbf{P}(V)$ and rate vector $\mathbf{R}(V)$. Let $N_1 < N_2 < \ldots < N_L$ denote the noise levels and $U_1, U_2, \ldots, U_L$ denote the collection of subsets of users associated to each level of noise for the Gaussian broadcast channel with the properties given in Lemma 15. We claim that $N_L < N$. To verify this, we substitute U_L into (5.56) and (5.59). Hence, we obtain

$$\gamma\left(\frac{\mathbf{P}(U_L)}{N_L + \mathbf{P}(V \backslash U_L)}\right) > \gamma\left(\frac{\mathbf{P}(U_L)}{N + \mathbf{P}(V \backslash U_L)}\right) \tag{5.61}$$

which results in $N_L < N$.

Next, we add $U_{L+1} = U \cup \{1\}$ as a set of new users to the Gaussian broadcast channel with noise level $N_{L+1} = N$ and increase the transmitter's total power by $P_1 + \mathbf{P}(U)$. It is easy to verify that the conditions in (5.56), (5.57), and (5.58) are still satisfied with new broadcast channel. Consequently, the rate vector lies on the boundary of the capacity region. Therefore, reliable data transmission at any rate above C results in reliable data transmission outside the capacity region which is a contradiction. This completes the proof. □

5.5 Applications for the K-user Gaussian IC

In this section, we apply the proposed algorithms to the K-user Gaussian IC modeled by

$$y_i = \sum_{i=1}^{K} h_{ij} x_j + z_i, \tag{5.62}$$

where x_j is the transmitted symbol of user j and h_{ij} denotes the link's gain between the j'th transmitter and the i'th receiver. z_i is additive white Gaussian noise with zero mean and variance N_i. User i is also subject to an average power constraint P_i. The capacity region of this channel is denoted by $\mathscr{C}_{GIC}$.

It is more convenient to write the system model in matrix form as

$$\mathbf{y} = H\mathbf{x} + \mathbf{z}, \tag{5.63}$$

where $\mathbf{y} = [y_1, y_2, \ldots, y_K]^T$ and $\mathbf{x} = [x_1, x_2, \ldots, x_K]^T$ denote the output and input vectors, respectively. $H = [h_{ij}]$ is the matrix of links' gains, and $\mathbf{z} = [z_1, z_2, \ldots, z_K]^T$ is the Gaussian noise vector which has a diagonal covariance matrix. By scaling transformations, it is possible to write the channel model (5.63) in standard form where the noise variances and diagonal elements of H are one [10].

Let us assume each transmitter is allowed to transmit data by using a single Gaussian codebook and each receiver is allowed to decode any subset of interfering users. Let Ψ denote the set of decoding strategies. By a decoding strategy $\psi = \{S_1, S_2, \ldots, S_K\} \in \Psi$, we mean that the receiver i tries to decode all users' data in S_i. Clearly, $S_i \subseteq E$ and $i \in S_i$. Since there are 2^{K-1} possible choices for each S_i, we have $2^{K(K-1)}$ possible strategies in total. Hence, $|\Psi| = 2^{K(K-1)}$.

Given a strategy, a rate vector $\mathbf{R}$ is achievable with respect to that strategy if every receiver can reliably decode its associated users. Therefore, an achievable rate region $\mathscr{C}_\psi$ can be defined as a set of all rate vectors that are achievable with respect to the strategy ψ. Let $\mathscr{C}_o = \bigcup_{\psi \in \Psi} \mathscr{C}_\psi$. Clearly, $\mathscr{C}_o \subseteq \mathscr{C}_{GIC}$ and it can be shown that $\mathscr{C}_o$ is not convex in general.

5.5.1 Some Extreme Points of $\mathscr{C}_o$

Given an ordering π of users, we aim at maximizing users' rates in accordance with π. In general, there are $K!$ orderings of users which result in $K!$ not necessarily distinct achievable rates in the capacity region. Due to the polymatroidal property of the capacity region of the Gaussian MAC, every permutation leads to a distinct achievable rate vector; whereas, $\mathscr{C}_o$ is not a polymatroid and hence there may be some permutations that lead to the same achievable rate vector. Without loss of generality, we may assume the order is the same as that of users' indices, i.e., permutation matrix is identity.

Setting the first user's rate to its maximum value $R_1 = \gamma\left(P_1\right)$ imposes some constraints on the other user's rates as they must be decoded by the first receiver. The reason is that R_1 is achievable if the first receiver can decode all the interfering users by considering its own signal as noise and eliminating their effects from the received signal.

Maximization of the second user's rate is more delicate, since its transmission should not affect the first user's data rate. However, we have the choice of lowering other users' rates as much as needed. Hence, we assume users in the set $\{3, 4, \ldots, K\}$ are decoded at the first and second receivers by considering everything else as noise and their effects are removed. R_2 must be chosen such that both receivers can decode it. The maximum decodable subset at the first receiver is $\{1\}$ by the assumption. For the second user, we can find the maximum decodable subset of interfering users which in this case is either $\emptyset$ or $\{1\}$. Now, we can run Algorithm 2 at both receivers to find an achievable rate for each receiver. Clearly, the minimum of the two achievable rates are achievable and we set R_2 to this value. Besides, we obtain the strategy $\psi^{(2)} = \{S_1^{(2)}, S_2^{(2)}\}$ in which R_1 and R_2 are achievable, where $S_1^{(2)}, S_2^{(2)} \subseteq E^{(2)}$ and $E^{(i)} = \{1, 2, \ldots, i\}$.

To maximize the rate of user i, we proceed as follows. We treat users above index i as they do not exist, i.e., we put constraints on their rates in such a way that all the receivers with indices in $E^{(i)}$ can decode them first and remove their effects. From maximization of users' rates in the previous steps, we have $\mathbf{R}_{E^{(i-1)}}$ and its corresponding achievable strategy $\psi^{(i-1)} = \{S_1^{(i-1)}, \ldots, S_{i-1}^{(i-1)}\}$, where $S_j^{(i-1)} \subseteq E^{(i-1)},\ \forall j \in E^{(i-1)}$. R_i must be chosen such that all receivers in $E^{(i)}$ can decode it. The maximum decodable subset of interfering users is given by $\psi^{(i-1)}$ for all receivers in $E^{(i-1)}$. We can also find the maximum decodable subset of interfering users at receiver i by running Algorithm 1. Let $S_i^{(i-1)}$ denote this subset. From (5.43), R_i is achievable at the receiver $j \in E^{(i)}$, if it is less than R_{ij} which is defined as

$$R_{ij} = \min_{U \subseteq S_j^{(i-1)}} \gamma \left(\frac{h_{ji}^2 P_i + \sum_{k \in U} h_{jk}^2 P_k}{1 + \sum_{k \in E^{(i-1)} \backslash S_j^{(i-1)}} h_{jk}^2 P_k} \right) - \mathbf{R}(U). \tag{5.64}$$

Hence, R_i can be chosen as the minimum of all R_{ij}s. For the next step, we need a new achievable strategy. It is easy to see that $\psi^{(i)} = \{S_1^{(i-1)} \cup \{i\}, \ldots, S_{i-1}^{(i-1)} \cup \{i\}, S_i^{(i-1)} \cup \{i\}\}$ is the proper strategy at step i. Now, we can iterate until the last user.

Algorithm 3 (successive maximization of users' rates)**.**

1. *Set* $R_1 = \gamma(P_1)$ *and* $S_1^{(1)} = \{1\}$.
2. *For* $i = 2 : K$, *do:*
   (a) *Find the maximum decodable subset of interfering users* $S_i^{(i-1)}$ *in the set* $E^{(i-1)}$ *for receiver* i *assuming that users in the set* $E \backslash E^{(i-1)}$ *are decoded and their effects are removed.*
   (b) *Solve the following optimization problem*

   $$R_i = \min_{j \in E^{(i)}} R_{ij}, \tag{5.65}$$

   where R_{ij} *is defined in (5.64).*
   (c) $S_j^{(i)} = S_j^{(i-1)} \cup \{i\}$, *for all* $j \in E^{(i)}$.

For the sake of completeness, in the following theorem, we state that the above algorithm finishes in polynomial time.

Theorem 33. *Algorithm 3 converges to an extreme point of* $\mathscr{C}_o$ *in polynomial time.*

Proof. At the i'th iteration, we need to solve i submodular function minimizations. Hence, in total, a submodular function minimization subroutine is invoked for $K(K+1)/2$ times. Moreover, at each step, we need to find the maximum decodable subset which can be accomplished in polynomial time based on Theorem 28. Hence, Algorithm 3 converges to an extreme point of $\mathscr{C}_o$ in polynomial time. $\square$

It is worth noting that for the two-user Gaussian IC in the case of strong and very strong interference [33], the output of Algorithm 3 is a point on the boundary of the capacity region. In the case of weak interference, however, the output of Algorithm 3 coincides with Costa's result in [14]. Unfortunately, the optimality of the result claimed by Costa has not been proved yet [33]. As a result, proving the optimality of extreme points obtained from Algorithm 3 has at least the same level of difficulty as that of the two-user case.

5.5.2 Generalized One-sided Gaussian IC

Parallel to the definition of the one-sided Gaussian IC [14], we define the generalized one-sided Gaussian IC as one in which the channel matrix H can be represented as a triangular matrix by row permutations. For the sake of simplicity, we always assume that H is lower triangular. Hence, the first user incurs no interference from other users, i.e., $y_1 = x_1 + z_1$, the second user incurs interference only form the first user, i.e., $y_2 = h_{21}x_1 + x_2 + z_2$, and in general, user i incurs interference from preceding users, i.e., $y_i = h_{i1}x_1 + \ldots + h_{i(i-1)}x_{i-1} + x_i + z_i$.

The capacity region of strong and very strong two-user Gaussian ICs is known and corresponds to the capacity of the corresponding compound MAC where both receivers decode both users' messages [3] [2]. Therefore, for the K-user case, it is interesting to find similar situations where the capacity is achievable when all receivers decode all messages sent by all transmitters. However, by a counter example, it is easy to show that having the condition $h_{ij}^2 \geq 1,\ \forall i, j \in E$, is not sufficient to establish similar results. To find similar situations, we define the strong generalized one-sided Gaussian IC as the channel with triangular channel matrix H with the property that $h_{ik}^2 \geq h_{jk}^2$ whenever $i \geq j$. In the following theorem, we prove that the capacity region of the strong generalized one-sided Gaussian IC can be fully characterized.

Theorem 34. *The capacity region of the strong generalized Z Gaussian IC is* $\bigcap_{i \in E} \mathcal{C}_{MAC}(i)$, *where* $\mathcal{C}_{MAC}(i)$ *denotes the capacity region of the MAC seen at the ith receiver.*

Proof. This theorem can be also proved by induction on the number of users. For a single user, it is trivial. We assume that for a channel with $m-1$ users and a triangular channel matrix, the capacity region is $\bigcap_{i \in E \setminus \{m\}} \mathcal{C}_{MAC}(i)$. Now, we add a new user which does not interfere with other users and only receives interference from all other users. Let $\mathcal{C}_{GIC}$ denote the capacity region of K-user Gaussian IC. It suffices to show that for any rate vector $\mathbf{R} = [\mathbf{R}_{-m}, R_m]$ in $\mathcal{C}_{GIC}(m)$, receiver m is able to decode all users' messages. The idea that we use here is similar to the idea of Han and Kobayashi for proving the capacity region of strong and very strong two-user Gaussian ICs [11]. Since $\mathbf{R}_{-m}$ is achievable and there is no interference from user m, we have $\mathbf{R}_{-m} \in \bigcap_{i \in E\setminus\{m\}} \mathcal{C}_{MAC}(i)$. In particular, $\mathbf{R}_{-m} \in \mathcal{C}_{MAC}(m-1)$. Hence, receiver $m-1$ which has $y_{m-1} = h_{(m-1)1}x_1 + \cdots + h_{(m-1)(m-2)}x_{m-2} + x_{m-1} + z_{m-1}$ as the received signal can jointly decode all users in the set $E\setminus\{m\}$. Since R_m is decodable by the mth receiver, it can be removed from the received signal $y_m = h_{m1}x_1 + \cdots + h_{K(m-1)}x_{m-1} + x_m + z_m$. Now, receiver m can try to decode other users' data from $\tilde{y}_m = h_{m1}x_1 + \cdots + h_{m(m-1)}x_{m-1} + z_m$. Let $\tilde{\mathcal{C}}_{MAC}(m-1)$ denote the capacity of this MAC. By hypothesis, $h_{ik}^2 \geq h_{jk}^2$ whenever $i \geq j$. Therefore, $\tilde{\mathcal{C}}_{MAC}(m-1) \subseteq \mathcal{C}_{MAC}(m-1)$. Hence, receiver m is able to decode the rate vector $\mathbf{R}_{-m}$. This completes the proof. $\square$

5.6 Conclusion

We investigated data transmission over a channel with $K-1$ interfering users. By establishing certain properties of the maximum decodable subset, we proposed a polynomial time algorithm that separates the interfering users into two disjoint parts: the users that the receiver is able to jointly decode their messages and its complement. We introduced an optimization problem that gives an achievable rate for this channel. We proposed a polynomial time algorithm for solving this optimization problem. We also established the capacity of the additive Gaussian channel with Gaussian interfering users and showed that the Gaussian distribution is optimal and the proposed achievable rate is the capacity of this channel.

As an application of this method, we investigated data transmission for the case of K-user interference channel when transmitters use single codebooks for data transmission, and receivers are allowed to decode other users' messages. We then introduced an achievable rate region $\mathscr{C}_o$. We obtained some extreme points of $\mathscr{C}_o$ by successive maximization of users' rates. Finally, we obtained the capacity region of the strong generalized one-sided Gaussian IC.

Chapter 6

Future Research Directions

In this chapter, some interesting problems that emerge from this dissertation are discussed. These problems can provide the spur to further research.

6.1 Interference Channels

6.1.1 The Two-user Case

The capacity region of the two-user Gaussian IC is far from being fully characterized. Even though the best achievable scheme is due to HK, the best input distributions are not known. In fact, Gaussian distributions may not be optimal for all channel parameters. The outer bounds presented in the thesis are generally optimal when the sum capacity is concerned. It is interesting to see if it can provide sufficient tools to derive other points on the boundary of the capacity region.

6.1.2 The Three-user Case

The DOF of the three-user Gaussian channel is not known when all channel gains are rational. In fact, the coding scheme presented for the symmetric case may not be optimal as there is no tight upper bound for this case. It would be a very interesting problem to see if there is a universal coding scheme that gives the best DOF for this case. In order to prove such statement, one needs to obtain a sufficiently tight upper bound on the sum capacity so as in the high SNR regimes provides a tight bound on the DOF.

6.1.3 The K-user Case

The coding scheme used for the three-user with rational coefficient case can be brought to the K-user Gaussian IC. In fact, in wireless systems channel estimation is always performed with finite precision and therefore it is rational. Hence, as the case of three-user, a careful design is needed to achieve higher multiplexing gains in the channel. It is also interesting to obtain the relation between the channel coefficients and achievable DOFs.

6.2 Interference Alignment

With the advent of interference alignment, new directions in interference management came to existance as Interference alignment emerged as a promising method to mitigate the effect of interference in a network.

The major drawback regarding interference alignment is that it needs full channel state information to realize its full potential. Therefore, practical applications are only possible when efficient feedback strategies are designed and carefully analyzed.

The concept of relaying in wireless networks has recently attracted many researchers in the areas of communications, networking and information theory. It is demonstrated that employing relays improves the coverage and reliability in a wireless network. One promising research direction would be analyzing the interaction between relay networks and interference channels. In future wireless networks, a node can operate as a sender, a receiver and/or a relay. In order to increase the throughput of these networks, therefore, traditional and new schemes should be combined in an efficient way. Relay assisted interference alignment could be a potential path to advanced interference management.

Providing a secure communication network is of fundamental importance in future. A secure system can be obtained by sacrificing available resources. But as the resources are scarce this results in tremendous loss in the throughput of the system. Interference alignment, however, can be used in a different fashion to provide security and performance at the same time. This time, interference caused by several users can be accumulated for the eavesdropper while it can be aligned for the intended users to increase the available Degrees-of-Freedom.

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