Multigraphs with High Chromatic Index

by

Jessica M. McDonald

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2009

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In this thesis we take a specialized approach to edge-colouring by focusing exclusively on multigraphs with high chromatic index. The bulk of our results can be classified into three categories. First, we prove results which aim to characterize those multigraphs achieving known upper bounds. For example, Goldberg's Theorem says that $\chi' \leq \Delta + 1 + \frac{\Delta-2}{g_o+1}$ (where χ' denotes chromatic index, Δ denotes maximum degree, and g_o denotes odd girth). We characterize this bound by proving that for a connected multigraph G, $\chi'(G) = \Delta + 1 + \frac{\Delta-2}{g_o+1}$ if and only if $G = \mu C_{g_o}$ and $(g_o + 1)|2(\mu - 1)$ (where μ denotes maximum edge-multiplicity).

Our second category of results are new upper bounds for chromatic index in multigraphs, and accompanying polynomial-time edge-colouring algorithms. Our bounds are all approximations to the famous Seymour-Goldberg Conjecture, which asserts that $\chi' \leq \max\{\lceil \rho \rceil, \Delta + 1\}$ (where $\rho = \max\{\frac{2|E[S]|}{|S|-1} : S \subseteq V, |S| \geq 3 \text{ and odd}\}$). For example, we refine Goldberg's classical Theorem by proving that $\chi' \leq \max\{\lceil \rho \rceil, \Delta + 1 + \frac{\Delta-3}{g_o+3}\}$.

Our third category of results are characterizations of high chromatic index in general, with particular focus on our approximation results. For example, we completely characterize those multigraphs with $\chi' > \Delta + 1 + \frac{\Delta-3}{g_o+3}$.

The primary method we use to prove results in this thesis is the method of Tashkinov trees. We first solidify the theory behind this method, and then provide general edge-colouring results depending on Tashkinov trees. We also explore the limits of this method, including the possibility of vertex-colouring graphs which are not line graphs of multigraphs, and the importance of Tashkinov trees with regard to the Seymour-Goldberg Conjecture.

Acknowledgements

I would first and foremost like to thank Penny Haxell, my doctoral supervisor, for her expert guidance and support throughout my entire time at Waterloo. I would also like to thank the other members of my committee — Bruce Richter, Nick Wormald, Therese Biedl, and Hal Kierstead — for their helpful suggestions regarding the final draft of this thesis.

Without making a huge list of names, I really would like to thank everyone in my life, here and away, who supported me throughout my graduate studies. Thanks especially to all the great friends that I met in Waterloo, for helping make my time here so enjoyable, and to my family, who are always there for me.

Contents

List of Figures

Chapter 1

Introduction

An *edge-colouring* of a multigraph G is an assignment of colours to the edges of G such that adjacent edges receive different colours. The central problem in this area is to determine the minimum number of colours needed for an edge colouring - this is called the *chromatic index* of the multigraph, and denoted $\chi' := \chi'(G)$. It is easy to see that the chromatic index of a multigraph G must always be at least its maximum degree $\Delta := \Delta(G)$. But how high can chromatic index be?

If G is a simple graph, then the answer to this question is very easy. Due to a fundamental theorem of Vizing [44] from the 1960's, the chromatic index of G must be equal to either Δ or $\Delta + 1$. A great amount of research has gone into trying to distinguish these two chromatic classes, however in general the problem is known to be NP-hard [16]. In multigraphs, on the other hand, there is no such dichotomy. For example, the multigraph $2K_3$ has $\Delta = 4$ but its chromatic index is clearly 6. In general, a multigraph may have chromatic index which greatly exceeds $\Delta + 1$. Such multigraphs are the subject of this thesis.

There are limits to the value of chromatic index for multigraphs. Famous such upper bounds by Vizing [44] and Goldberg [11] will be of central importance in this thesis. These bounds combine Δ and other graph parameters, namely maximum edge-multiplicity $\mu := \mu(G)$ and odd-girth $g_o := g_o(G)$ (the length of the shortest odd cycle in G), respectively. We will introduce and contextualize these two theorems in Chapter 2, where we also present the rest of the background information necessary for our work here. Paramount among this is a discussion of the celebrated Seymour-Goldberg Conjecture, and an introduction to the method of Tashkinov trees.

The influence of the Seymour-Goldberg Conjecture on this thesis cannot be overstated. The conjecture, posed independently by Seymour [37] and by Goldberg [13] in the 1970's, asserts that any multigraph G must have

$$\chi'(G) \le \max\{\lceil\rho(G)\rceil,\ \ \Delta+1\}$$

where

$$\rho(G) := \max\left\{\frac{2|E[S]|}{|S|-1} : S \subseteq V(G), |S| \ge 3 \text{ and odd }\right\}.$$

In our study we will support this conjecture in a number of ways, proving that multigraphs with highest chromatic index do obey this conjecture, and have chromatic index determined by a dense odd subgraph. In order to properly appreciate these results, it is essential that the reader keep the Seymour-Goldberg Conjecture in mind, beginning right from Chapter 2.

The method of Tashkinov trees is the primary method we use to prove results in this thesis. This is a structural technique, developed by Tashkinov [42] in 2000, which generalizes the earlier notion of Kierstead paths [21], and in turn the classical alternating path argument of König [24]. As such, Tashkinov trees lie at the heart of most major results in edge-colouring theory. There are however, some gaps in the English literature concerning this method. Hence, in Chapter 2, in addition to introducing Tashkinov trees, we provide both a proof and an algorithm to solidify the theory.

In Chapter 3 we begin our study of multigraphs with high chromatic index by asking a natural question: given an upper bound on chromatic index, what type of multigraphs actually achieve this bound? That is, if we know that $\chi'(G) \le a$ for all multigraphs G, can we characterize the chromatic class $\chi'(G) = a$? This chapter, which forms the basis of the paper [28], addresses this question for Goldberg's bound and for Vizing's bound. In the case of Goldberg's Theorem, we are able to get the complete characterization we seek (Theorem 3.2.2). In the case of Vizing's Theorem however, we must settle for proving some necessary conditions of the chromatic class (Theorems 3.3.3 – 3.3.5).

In addition to studying known upper bounds, in this thesis we also want to provide new upper bounds for chromatic index in multigraphs. In Chapter 4 we show how Tashkinov trees can, in general, be used to prove upper bounds of the form

$$\chi' \le \max\{\lceil\rho\rceil,\ \ \Delta+t\}$$

for various values of t (Theorem 4.1.3). That is, we show how Tashkinov trees can be used to prove bounds which approximate the Seymour-Goldberg Conjecture. Of the particular bounds that we show using this method, Theorems 4.2.1 and 4.2.2, involving odd-girth g_o and girth g (the length of the shortest cycle in the underlying graph of G), respectively, actually prove that the Conjecture holds for a number of new classes of multigraphs (Theorems 4.2.3 and 4.2.4). Significantly, the bound involving g_o also refines the classical upper bound of Goldberg. Our proofs in this chapter are algorithmic in nature, and we take care to provide an accompanying polynomial-time colouring algorithm for each of our specific bounds (Theorems 4.3.1 – 4.3.4).

Another way to interpret a result of the form

$$\chi' \leq \max\{\lceil\rho\rceil, \ \ \Delta + t\},$$

is to think of it as

$$\chi' > \Delta + t \ \ \Rightarrow \ \ \chi' = \lceil\rho\rceil.$$

This is equivalent because $\lceil\rho\rceil$ is actually a lower bound for chromatic index. To see this, note that for a multigraph G, given any odd set $S \subseteq V(G)$ with $|S| \geq 3$, the maximum size of a colour class in $G[S]$ is $(|S|-1)/2$. Hence, since chromatic index is an integer,

$$\chi'(G[S]) \geq \left\lceil \frac{2|E[S]|}{|S|-1} \right\rceil.$$

We will use the fact that $\lceil\rho\rceil$ is a lower bound for chromatic index repeatedly in this thesis — in particular, we will use it whenever we want to establish the chromatic index of a particular multigraph.

When we view our edge-colouring bounds in the form

$$\chi' > \Delta + t \ \ \Rightarrow \ \ \chi' = \lceil\rho\rceil,$$

it is natural to ask if we can characterize the chromatic class $\chi' > \Delta + t$. We discuss this problem in Chapter 5 of this thesis. We first provide some general techniques for characterization, including a generalization of methods used in Chapter 3 (Propositions 5.1.1 – 5.1.4). Then, we give a complete characterization for our g_o result from the previous chapter (Theorem 5.2.1). As the Chapter 4 result refined Goldberg's bound, this characterization result extends the previous characterization of Goldberg's bound that we found in Chapter 3. Here in Chapter 5, we are also able to apply our general characterization techniques to a different result from Chapter 4, and get more information about Vizing's upper bound. More precisely, we are able to characterize large multiples of simple graphs that achieve Vizing's bound – a result which is best-possible (Theorem 5.3.1).

Edge-colouring a multigraph is equivalent to vertex-colouring its line graph, and in Chapter 6 we recognize this equivalence. We try to understand vertex-colouring results as they relate to edge-colouring, and vice versa. As all of our work in this thesis is based on the method of Tashkinov trees, we try to extend this method so that it can be used to vertex-colour graphs that are not line graphs of multigraphs. However, our efforts seem to indicate that such an extension is not possible.

To conclude this thesis, we reflect on both the results that we have presented, and the methods we have used. This includes our contributions to understanding multigraphs with high chromatic index with respect to Δ, μ, g, and especially g_o. In this last chapter, we also look at the big picture and ask about the importance of Tashkinov trees in proving the Seymour-Goldberg Conjecture. We discuss other conjectures and problems in this area, how they are related to our work here, and future possibilities for study. As we shall see, the results of this thesis make significant headway in understanding multigraphs with high chromatic index - however they also raise a number of interesting questions.

Chapter 2

Edge-colourings, alternating paths and Tashkinov trees

In this chapter we introduce the background material necessary for the rest of this thesis. First, in Section 2.1, we describe the major known results in the area of edge-colouring. In Section 2.2, we define Tashkinov trees and state Tashkinov's Theorem. We show how many of the results in edge-colouring have been, or can be, attained using some form of Tashkinov tree. The third and final section of this chapter is devoted to establishing the proof of Tashkinov's Theorem and the algorithm implied by the proof. This is important, as we are aware of only one proof in English and it contains a flaw (not found in Tashkinov's original Russian publication). Also, while the algorithm implied by the proof has been alluded to by other authors, it has never been described explicitly, and we will need to build on this algorithm to get our colouring algorithms of Section 4.3.

2.1 Central results in edge-colouring

Edge-colouring first appeared in graph theory literature in the 1880's, courtesy of P. G. Tait ([40],[41], see also [10]). Tait was attempting to prove the Four Colour Theorem, and had shown an equivalence between 4-colourablity of planar maps, and 3-edge-colourability of cubic planar maps. Unfortunately, Tait's attempt to prove that every planar cubic map is 3-edge-colourable was seriously flawed. Given this rocky start, we might say that edge-colouring theory really began with the first correct result published — König's Theorem of 1916.

Theorem 2.1.1. [24] (König's Theorem) *Let G be a bipartite multigraph. Then,*

$$\chi'(G) = \Delta.$$

König's Theorem is significant because it exhibits a large family of multigraphs which achieve the canonical lower bound for chromatic index. It took another thirty years for the first meaningful upper bound on chromatic index to be published. Now known as Shannon's Theorem, this result emerged through C. E. Shannon's [38] study of electrical networks.

Theorem 2.1.2. [38] (Shannon's Theorem) *Let G be a multigraph. Then,*

$$\chi'(G) \leq \frac{3\Delta}{2}.$$

A great breakthrough in edge-colouring theory — arguably *the* great breakthrough when it comes to simple graphs — was made by V. G. Vizing [44] in the 1960's, with the following now-famous theorem.

Theorem 2.1.3. [44] (Vizing's Theorem) *Let G be a multigraph. Then,*

$$\chi'(G) \leq \Delta + \mu.$$

If G is a simple graph, then $\mu = 1$ and Vizing's Theorem gives just the two possibilities for chromatic index – Δ and $\Delta + 1$. A great amount of the work that has been done on edge-colouring has been to try to distinguish graphs of *class one* (chromatic index Δ) and *class two* (chromatic index $\Delta + 1$). However, in 1981, Holyer [16] proved that this is an NP-hard decision problem. Keeping this in mind, in this thesis we will not make any attempt to distinguish between these two lowest possible values for chromatic index in multigraphs — our interest lies in multigraphs with high chromatic index.

The innovation of Vizing's Theorem led to increased research in edge-colouring. Two important results which have emerged since are a refinement of Shannon's bound, due to Goldberg [11], and a refinement of Vizing's bound, due to Steffen [39]. These results use the concepts of odd-girth g_o and girth g, which we should note are only defined in multigraphs which contain an odd cycle, or contain a cycle, respectively. Restricting ourselves to such multigraphs is not a major assumption however, as we already know that all bipartite multigraphs have chromatic index exactly Δ.

Theorem 2.1.4. [11] (Goldberg's Theorem) *Let G be a multigraph containing an odd cycle. Then,*

$$\chi'(G) \leq \Delta + 1 + \frac{\Delta - 2}{g_o - 1}.$$

Theorem 2.1.5. [39] (Steffen's Theorem) *Let G be a multigraph containing a cycle. Then,*

$$\chi'(G) \leq \Delta + \left\lceil \frac{\mu}{\lfloor g/2 \rfloor} \right\rceil.$$

Since girth and odd-girth are both at least three, it is easy to see how Goldberg's bound refines Shannon's bound, and how Steffen's bound refines Vizing's bound. Another way of refining Vizing's bound is the following early theorem of Ore [30]. Here, rather than using the global parameters of maximum degree and maximum edge-multiplicity, we have the local parameters $d(v)$, the degree of a vertex v, and $\mu(v)$, the maximum multiplicity of an edge incident to v.

Theorem 2.1.6. [30] (Ore's Theorem) *Let G be a multigraph. Then,*

$$\chi'(G) \le \max\{d(v) + \mu(v) | v \in V(G)\}.$$

In addition to the above upper bounds, the last forty years has also seen a number of conjectures emerge, each purporting to explain edge-colouring in more depth. The most significant of these was proposed independently by Seymour [37] and by Goldberg [13] in the 1970's, and is now referred to as the Seymour-Goldberg Conjecture. This conjecture, already discussed briefly in our introduction, can be stated in the following three forms:

$$\chi' \le \max\{\lceil \rho \rceil, \;\; \Delta + 1\},$$

$$\chi' > \Delta + 1 \quad \Rightarrow \quad \chi' = \lceil \rho \rceil,$$

and

$$\chi' \in \{\lceil \rho \rceil, \;\; \Delta, \;\; \Delta + 1\}.$$

This last version highlights the great strength of the Seymour-Goldberg Conjecture: it generalizes Vizing's Theorem for simple graphs by showing that there are exactly three possible values for the chromatic index of a multigraph. Moreover, while we know that it is NP-hard to decide between chromatic index Δ and $\Delta + 1$, if the Seymour-Goldberg Conjecture is true, then deciding whether or not $\chi'(G) > \Delta + 1$ (and determining chromatic index exactly in this case) is polynomial-time solvable. This is because Edmonds' Matching Polytope Theorem [9] implies that $\max\{\rho(G), \;\; \Delta\}$ is equal to the *fractional chromatic index* of G, a quantity that can be computed in polynomial time. (See, for example, [2] for more on fractional chromatic index).

Not only is the Seymour-Goldberg Conjecture the most important conjecture in edge-colouring, but it is strong enough to imply many other open conjectures in the area. This list includes both the Critical Multigraph Conjecture the Weak Critical Graph Conjecture, which we will discuss later in this thesis. One fact which has been firmly established however, is that the Seymour-Goldberg Conjecture is true asymptotically. That is, Kahn [20] has shown that for a multigraph G,

$$\chi'(G) \sim \max\{\rho(G), \Delta\} \quad \text{as} \quad \max\{\Delta, \rho(G)\} \to \infty.$$

The conjecture is also known to be true for all multigraphs that do not contain a K_5^--minor (Marcotte [27]). In addition, there is family of approximation results, starting in the 1970's, which prove that

$$\chi' \leq \max\left\{\lceil\rho\rceil,\ \ \Delta + 1 + \frac{\Delta - 2}{m - 1}\right\}$$

for certain values of m. Such results have been proved by Goldberg [11][12] ($m = 9$), Nishizeki and Kashiwagi [29] and independently Tashkinov [42] ($m = 11$), Stiebitz, Favrholt and Toft [8] ($m = 13$), and Scheide [35] ($m = 15$). Since

$$\frac{\Delta - 2}{14} < 1 \quad \Leftrightarrow \quad \Delta \leq 15,$$

the best of these results show that the Seymour-Goldberg Conjecture holds when maximum degree is at most 15.

One may note that the name Tashkinov appears in the above paragraph, and this is not a coincidence. The 2000 citation given is indeed where Tashkinov trees were first introduced. However, as we shall see in the next section, the majority of results mentioned here used the method of Tashkinov trees in their proofs – even going back to König's Theorem — whether the authors were aware of it or not.

2.2 From alternating paths to Tashkinov trees

The proof of König's Theorem is based on the idea of building a colouring one edge at a time, using alternating paths. Suppose we have a partial edge-coloring of a multigraph (with at least Δ colours), but there is an edge $e = xy$ still uncoloured. There must be at least one colour α not used on any edge incident to x, and at least one colour β not used on any edge incident to y. If $\alpha = \beta$, then we can clearly colour e, and otherwise we consider the maximal (α, β)-alternating path beginning at x. We can swap the two colours along this path if we like, without affecting the fact that the colouring is proper. As long as the path does not end at y, such a swap would allow us to extend the colouring to e with the colour β. In the case of a bipartite graph, the path certainly cannot end at y, because this would create an odd cycle. This proves that Δ colours are sufficient to edge-colour a bipartite graph, that is, this proves König's Theorem.

While the idea of using alternating paths to "augment" partial edge-colourings seems very basic, it is really at the heart of many results in the edge-colouring literature. To better understand this, we need the much more general method of Tashkinov trees.

Let G be a multigraph and let ϕ be a partial edge colouring of G. We say that $T = (p_0, e_0, p_1, \ldots, p_{n-1}, e_{n-1}, p_n)$ is a ϕ-*Tashkinov tree* in G if

(T1.) $p_0, \ldots, p_n$ are distinct vertices in G and, for each $i \in \{0, \ldots, n-1\}$, $e_i \in E(G)$ and has ends p_{i+1} and p_k for some $k \in \{0, \ldots, i\}$, and

(T2.) e_0 is uncoloured by ϕ and for each $i \in \{1, \ldots, n-1\}$,

$$\phi(e_i) \in \bigcup_{j \leq i} \phi(p_j),$$

where $\phi(p_j)$ is the set of colours not appearing on any edge incident to p_j. We will also refer to $\phi(p_j)$ as the set of colours that are not used at p_i, that are not seen at p_i, or, most commonly, that are missing at p_i. Note that (T1) merely requires T to be a tree — the weight of the definition is in (T2), where we restrict the colour of each edge in the tree.

We say that a ϕ-Tashkinov tree $T = (p_0, e_0, p_1, \ldots, p_n)$ is ϕ-*elementary* if all the colours missing at its vertices are distinct, i.e.,

$$\phi(p_i) \cap \phi(p_j) = \emptyset$$

for all $0 \leq i < j \leq n$. It will also sometimes be convenient to refer to a set of vertices W as ϕ-elementary — this means $\phi(w_1) \cap \phi(w_2) = \emptyset$ for all distinct $w_1, w_2 \in W$.

Clearly, an alternating path is always a Tashkinov tree. Moreover, an alternating path used to augment a colouring ϕ must have two vertices with a common missing colour (the two ends of the path), and so it is not ϕ-elementary. The following theorem describes this "augmenting" in the more general setting — that is, it provides the framework for the method of Tashkinov trees. Note that given a partial edge-colouring ϕ, *dom*(ϕ) denotes the domain of ϕ, that is, the set of edges that are coloured by ϕ.

Theorem 2.2.1. [42] (Tashkinov's Theorem) *Let G be a multigraph and let ϕ be a partial $(\Delta + s)$-edge-colouring of G, with $s \geq 1$. Suppose that there exists a ϕ-Tashkinov tree $T = (p_0, e_0, p_1, \ldots, p_n)$ in G which is not ϕ-elementary. Then, there exists a $(\Delta + s)$-edge colouring ψ of dom$(\phi) \cup \{e_0\}$.*

We can now see how the method of Tashkinov's Theorem generalizes the alternating path technique of König. In either case we have a partial colouring, and a structure (alternating path or Tashkinov tree) with one uncoloured edge. We are able to modify the existing colouring so that it may be extended to the uncoloured edge, provided the structure has two vertices with a common missing colour.

Despite the similarities between alternating paths and Tashkinov trees, it should be emphasized that the jump from one to the other is quite large. In fact, there is no question that this jump would not have happened without one important intermediary step: Kierstead paths.

The original proof of Vizing's Theorem involves building a fan structure where the first edge is uncoloured, and then uses a series of alternating paths to recolour and extend to the first edge. The fan structure here is not a Tashkinov tree however, and the recolouring process is somewhat complicated. In the 1980's, H. A. Kierstead gave a new proof of Vizing's Theorem. Within his proof, Kierstead defined what we now refer to as *Kierstead paths*, which have the same definition as Tashkinov trees, except that in (T1), k is required to be i, making the structure a path. Kierstead proved Theorem 2.2.1 in the case that T is a path — that is, he proved the following theorem.

Theorem 2.2.2. [21] (Kierstead's Theorem) *Let G be a multigraph and let ϕ be a partial $(\Delta + s)$-edge-colouring of G, with $s \geq 1$. Suppose that there exists a ϕ-Kierstead path $P = (p_0, e_0, p_1, \ldots, p_n)$ in G which is not ϕ-elementary. Then, there exists a $(\Delta + s)$-edge colouring ψ of dom$(\phi) \cup \{e_0\}$.*

Theorem 2.2.2 is really the most difficult part of Kierstead's proof of Vizing's Theorem. Assuming this result, we can start the proof by assigning a maximum domain $(\chi' - 1)$-edge colouring ϕ to G. If $\chi' \leq \Delta + 1$ then Vizing's Theorem certainly holds. So, we may assume that $(\chi' - 1) \geq \Delta + 1$. Hence, we may apply Kierstead's Theorem. Choose any uncoloured edge e_0 with ends p_0 and p_1. Note that there must be some colour α missing at p_0, and that there must be an α-edge incident to p_1, which we may label as e_1. We continue to build this structure as far as we can, ending up with a maximal ϕ-Kierstead path $P = (p_0, e_0, p_1, \ldots, p_n)$ in G which has $n \geq 2$. Kierstead's Theorem tells us that P must be ϕ-elementary, since ϕ has maximum domain. So in particular, the last vertex in the path, p_n, must see every colour missing at every previous vertex in the path. Since there are $\chi' - 1$ colours being used, and each vertex may see Δ coloured edges (except for p_0 and p_1, which see at most $\Delta - 1$ each), we get that the number of colours seen by p_n is at least

$$\left| \bigcup_{i=0}^{n-1} \phi(p_i) \right| \geq n(\chi' - 1 - \Delta) + 2.$$

On the other hand, since P is maximal, each edge incident to p_n that is coloured with one of these colours must be between p_n and $p_0, \ldots, p_{n-1}$, as otherwise it could be used to extend P (See Figure 2.1). There are at most $n\mu$ such edges, so we must

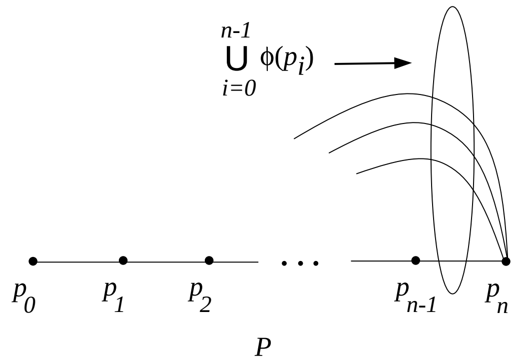

Figure 2.1: The last vertex in a maximal ϕ-Kierstead path P

have

$$\begin{aligned} & n(\chi' - 1 - \Delta) + 2 \leq n\mu \\ \Rightarrow\ & \chi' - 1 - \Delta \leq \mu - \frac{2}{n} \\ \Rightarrow\ & \chi' \leq \Delta + \mu, \end{aligned}$$

proving Vizing's Theorem. Shannon's Theorem is even easier to obtain - just note that p_n has degree at most Δ, and get

$$\begin{aligned} & n(\chi' - 1 - \Delta) + 2 \leq \Delta \\ \Rightarrow\ & 2(\chi' - 1 - \Delta) + 2 \leq \Delta \\ \Rightarrow\ & \chi' \leq \frac{3\Delta}{2}. \end{aligned}$$

Of the other upper bounds in the previous section, Goldberg's bound was proved using the alternating paths technique that is generalized by both Kierstead paths and Tashkinov trees. Steffen used Kierstead paths explicitly in establishing his bound. Ore's bound can be easily proved by using a Tashkinov tree that is a fan (as noted by Favrholt, Stiebitz and Toft [8]). Of the results in the previous section towards the Seymour-Goldberg Conjecture, Kahn used the probabilistic method and Marcotte used a structural analysis — but the others also used a Tashkinov tree method, in some form or another.

We know that Tashkinov trees generalize Kierstead paths completely; however it is sometimes helpful to use Kierstead paths in particular, rather than the more general version. There are really two reasons for this. The first is that while a Kierstead path is always a Tashkinov tree, a *maximal* Kierstead path is not necessarily a *maximal* Tashkinov tree. In proofs, it will always be important that the structure in question is as large as possible, in the sense that no other edge could be added. It may sometimes serve us to know that all (or part of) the Tashkinov tree we have constructed is actually an alternating path, a path, a fan, or some other specific structure. Note that in the case of the proofs of Vizing and Shannon's Theorems above, we did not need to appeal to such structure, and hence the arguments would have been identical had we constructed a maximal Tashkinov tree, instead of a maximal Kierstead path. However, there is another reason to use Kierstead paths instead of Tashkinov trees when possible — they hide much less difficulty. We have already stated that the jump from alternating paths to Tashkinov trees is very large. As we shall see in the next section, however, the majority of this challenge lies in moving from Kierstead paths to Tashkinov trees.

2.3 The proof of Tashkinov's Theorem

We divide our work on the proof of Tashkinov's Theorem into two subsections. First, we concentrate on establishing the proof in English. Rather than presenting the entire (very lengthy) argument here, we refer the reader to [8], where a nearly complete proof is given, and simply endeavor to fill the gap.

The (partial) proof that we present in our first subsection is based on a sequence of colour-swaps along alternating paths. In fact, the entire proof of Tashkinov's Theorem is like this, and naturally lends itself to an algorithm. While other authors have realized this (eg. [8], [35]), the algorithm has not yet been explicitly written out or analyzed - we provide these details in our second subsection.

2.3.1 Filling the gap

Tashkinov trees are generalizations of Kierstead paths and, as we have already mentioned, Tashkinov's Theorem is an extension of Kierstead's Theorem. Kierstead's original proof of his Theorem is easy to follow, and fits comfortably on a single page. The proof uses an induction on j and $j-i$, where we begin by assuming that $\phi(p_i) \cap \phi(p_j) \neq \emptyset$ for some $0 \leq i < j \leq n$. In Tashkinov's Theorem, we use an induction where we hope for trees to be as path-like as possible, so that we may eventually apply Kierstead's Theorem.

Proof. *(Theorem 2.2.1, Tashkinov's Theorem)* Let G be a multigraph and let $s \geq 1$, as in the statement of Theorem 2.2.1. However, suppose that there exists a counterexample to this Theorem. That is, suppose that there exists a partial $(\Delta+s)$-edge colouring ϕ of G, and a ϕ-Tashkinov tree $T = (p_0, e_0, p_1, \ldots, p_n)$ in G that is

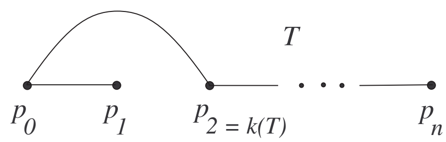

Figure 2.2: The case $k(T)$=2

not ϕ-elementary, but such that $dom(\phi) \cup \{e_0\}$ is not $(\Delta + s)$-edge colourable. Note that there exists $k \in \{0, \dots, n\}$ such that

$$T^k := (p_k, \dots, p_n)$$

is a path; define $k(T)$ to be the smallest such k for T. Among all counterexamples, choose (T, ϕ) such that

1. $k(T)$ is as small as possible
2. $|V(T)|$ is as small as possible, subject to (1).

Consider the value $k(T)$. Note that since $e_0 = (p_0, p_1)$, it is impossible to have $k(T) = 1$. If $k(T) = 0$, then T is a ϕ-Kierstead path. If $k(T) = 2$, then $(p_1, e_0, p_0, e_1, p_2, \dots, p_n)$ is a ϕ-Kierstead path (See Figure 2.2). In either case, since T is not ϕ-elementary, Kierstead's Theorem (Theorem 2.2.2) says that $dom(\phi) \cup \{e_0\}$ is $(\Delta + s)$-edge colourable, contradicting the fact that (T, ϕ) is a counterexample. So, we may assume that $k(T) \geq 3$.

If $k(T) < n$, then a correct argument to conclude this proof may be found in [8]. We will deal with the remaining case, that is, when $k(T) = n$.

We are searching for a contradiction to our counterexample (T, ϕ). Note that any truncation of T is still a ϕ-Tashkinov tree. That is, if we define

$$T_j := (p_0, e_0, p_1, \dots, p_j),$$

then we know that T_j is a ϕ-Tashkinov tree, for all $j \in \{1, \dots, n\}$. Moreover, $k(T_j) \leq k(T)$ and $|V(T_j)| < |V(T)|$ for all $j < n$. So, it must be the case that every T_j is ϕ-elementary, otherwise we would not have chosen (T, ϕ) as our counterexample. In particular, this implies that T_{n-1} is ϕ-elementary. Since T is not ϕ-elementary, this means that we must have $\phi(p_i) \cap \phi(p_n) \neq \emptyset$ for some $i \in \{0, \dots, n-1\}$.

In addition to choosing T carefully for our counterexample, we also choose ϕ carefully. While T is a ϕ-Tashkinov tree, it may also be a ψ-Tashkinov tree, for some other partial $(\Delta + s)$-edge colouring ψ of G. Let $\mathcal{C}^T$ denote the set of all colourings ψ such that (T, ψ) is a counterexample. Clearly, $\phi \in \mathcal{C}^T$. Just as we know that T_{n-1} is ϕ-elementary, we also know that T_{n-1} is ψ-elementary, for every $\psi \in \mathcal{C}^T$. So, since (T, ψ) is a counterexample for all $\psi \in \mathcal{C}^T$, there must always be some $i \in \{0, \ldots, n-1\}$ such that $\psi(p_i) \cap \psi(p_n) \neq \emptyset$.

We will proceed by proving the following series of statements:

(A). There exists $\psi \in \mathcal{C}^T$ such that $\psi(p_j) \cap \psi(p_n) \neq \emptyset$ for some $j \neq n-1$, and some colour $\beta \in \psi(p_j) \cap \psi(p_n)$ is not used on any edge of T.

(B). There exists $\psi \in \mathcal{C}^T$ such that $\psi(e_{n-1})$ is seen by p_{n-1}.

(C). There exists $\psi \in \mathcal{C}^T$ such that $\psi(e_{n-1})$ is seen by p_{n-1}, and $\psi(p_j) \cap \psi(p_n) \neq \emptyset$ for some $j \neq n-1$.

Statement (A) will help us prove statement (B), and statement (B) will help us prove statement (C). Once we establish statement (C), we will be able to deduce a contradiction.

The following two claims will aid us greatly in our arguments. We include both proofs as we will want to refer to them when we discuss the algorithm implied by this theorem.

Claim 1. [8] *Let $\psi \in \mathcal{C}^T$. If $j \in \{1, \ldots, n-1\}$, then there are at least four colours in $\cup_{i=0}^{j} \psi(p_i)$ that are unused on the edges of T_j.*

Proof of Claim. Since $|\psi(p_0)|, |\psi(p_1)| \geq s+1$ and $|\psi(p_i)| \geq s$ for all $i \in \{2, \ldots, j\}$, and since T_j is ψ-elementary, we know that

$$\left| \bigcup_{i=0}^{j} \psi(p_i) \right| \geq (j+1)s + 2 \geq j + 3.$$

On the other hand, T_j consists of exactly j edges, $j-1$ of which are coloured. Since

$$(j+3) - (j-1) = 4,$$

we get our desired result. □

Claim 2. [8] *Let $\psi \in \mathcal{C}^T$. Suppose that $\alpha \in \psi(p_i), \beta \in \psi(p_j)$, and α is unused on the edges of T_j, for some $0 \leq i < j \leq n-1$. Then, $\alpha \neq \beta$, and there is a maximal (α, β)-alternating path Q between p_i and p_j (with respect to ψ). Moreover, if $\psi' = \psi(i, j, \alpha, \beta)$ is the colouring obtained from ψ by switching α and β along Q, then $\psi' \in \mathcal{C}^T$.*

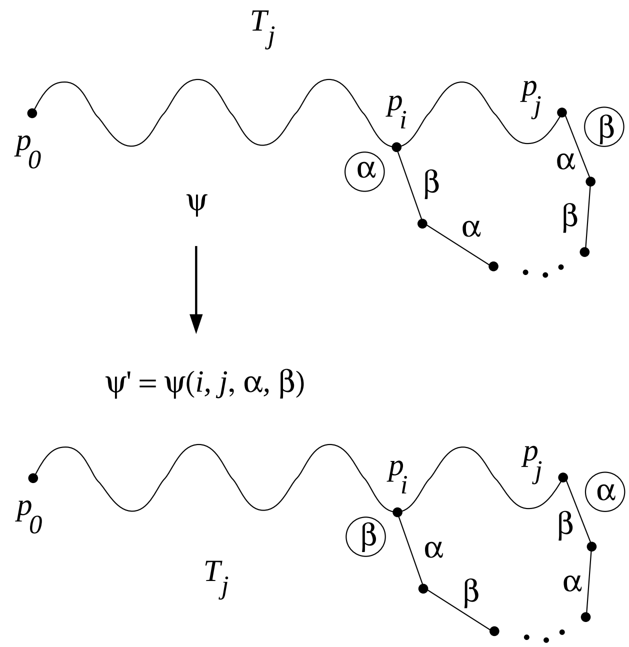

Figure 2.3: Claim 2

Proof of Claim. It is clear that $\alpha \neq \beta$, because we know that T_{n-1} must be ψ-elementary. Let Q be the maximal (α, β)-alternating path starting at p_j. Since T_j is ψ-elementary, if Q ends at a vertex on T_j, then it must end at p_i. Moreover, the fact that T_j is a ψ-elementary ψ-Tashkinov tree implies that β cannot occur on an edge of T_j. So, since α was chosen to be unused on the edges of T_j, we have that $E(Q) \cap E(T_j) = \emptyset$. Because of this, if we define ψ' from ψ by swapping α and β along Q, we know that T_j is ψ'-elementary. If Q does not end at p_i, then $\alpha \in \psi'(p_i) \cap \psi'(p_j)$, so T_j is not ψ'-elementary, which contradicts our choice of (T, ψ). So, Q must end at p_i, and hence $\beta \in \psi'(p_i)$ and $\alpha \in \psi'(p_j)$. Since T_j has no α or β edges under ψ' (or ψ), this is enough to tell us that T is a ψ'-Tashkinov tree. Moreover, the fact that T is not ψ-elementary implies that T is not ψ'-elementary, so $\psi' \in \mathcal{C}^T$. $\square$

Figure 2.3 depicts the colour change described in Claim 2, and it may be helpful to refer to this diagram when the claim is applied. Note that in Figure 2.3, and from here on in this thesis, a circle around a colour name means that the colour is missing at a particular vertex. Also, we often draw a wavy line for a tree, to indicate that while it may help to think of the tree as a path, it may in fact have a more complicated structure.

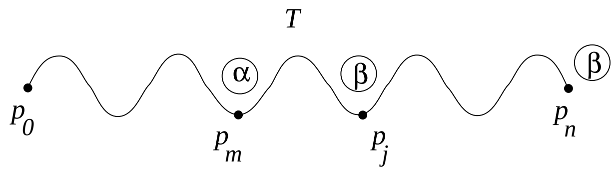

Figure 2.4: Working to establish (A)

We now work to establish (A). To this end, let ψ be any colouring in $\mathcal{C}^T$. We know that there exists $\beta \in \psi(p_j) \cap \psi(p_n)$ for some $j \in \{0, \ldots, n-1\}$, by our discussion above. Claim 1 tells us that there are at least 4 colours in $\cup_{i=0}^{n-2} \psi(p_i)$ that are unused on the edges of T_{n-2} (since $n = k(T) \geq 3$). There are only two edges in T that are not in T_{n-2}, so we may in fact choose a colour $\alpha \in \cup_{i=0}^{n-2} \psi(p_i)$ that is unused on T, and that also satisfies $\alpha \neq \beta$. Say $\alpha \in \psi(p_m)$. We now have the situation depicted roughly in Figure 2.4, and will proceed according to the value of j.

Suppose first that $j = n-1$. We claim that in this case, β is unused on T. To see this, first note that $\beta \in \psi(p_n)$ implies that β is not used on the last edge of T. On the other hand, if β is used on an edge of T_{n-1}, then it must be missing at some vertex in T_{n-2} (by definition of a Tashkinov tree). Since we are assuming that $\beta \in \psi(p_{n-1})$ already, this would mean that T_{n-1} is not ψ-elementary, a contradiction. So, indeed, β is not used on any edge of T. Since α is unused on T, we can modify ψ to get $\psi' = \psi(m, j, \alpha, \beta) \in \mathcal{C}^T$, as described in Claim 2. Then, $\beta \in \psi'(p_m) \cap \psi'(p_n)$. However, since neither α nor β were used on T under ϕ, we also get that β is not used on T under ψ'. Hence, ψ' satisfies (A) in this case.

Suppose now that $j \leq n-2$ (refer again to Figure 2.4). If (A) is not satisfied, then it must mean that β is used on T. We may assume that α is seen by p_n, since if not, the fact that α is also missing at p_m and that α is unused on T immediately implies that ψ satisfies (A). So, we may consider the maximum (α, β)-alternating path Q starting at p_n. If Q contains a vertex on T_{n-2}, then let Q' be the segment of Q between p_n and the first vertex of Q on T_{n-2}, and define

$$T' = (p_0, e_0, p_1, \ldots, p_{n-2}, Q', p_n).$$

Then, T' is a ψ-Tashkinov tree (since α and β are missing at vertices in T_{n-2}), and T' is not ψ-elementary, (since $\psi(p_j) \cap \psi(p_n) \neq \emptyset$). However, $k(T') \leq n-1$, so (T', ψ) contradicts our choice of (T, ϕ). Hence, we know that Q cannot contain any vertices from T_{n-2}. This implies that Q does not contain any edges of T either, since both e_{n-1} and e_{n-2} have one end in T_{n-2} (for e_{n-1} this is because $k(T) = n$). With this in mind, let ψ' be the colouring obtained by switching the colours β and α along Q.

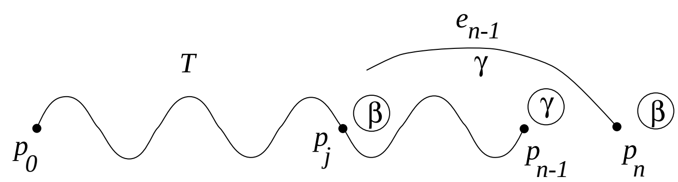

Figure 2.5: Working to establish (B)

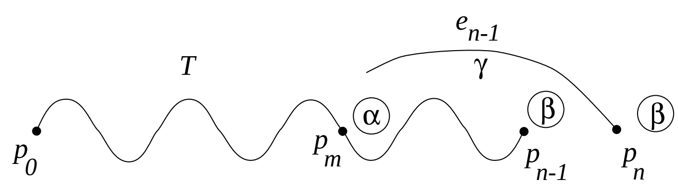

Figure 2.6: Working to establish (C)

After this switch, we get $\alpha \in \psi'(p_m) \cap \psi'(p_n)$, and α is unused on T. So, ψ' satisfies (A).

We have now established (A), and want to work to prove (B). Let $\psi \in \mathcal{C}^T$ be a colouring satisfying (A), i.e., with $\psi(p_j) \cap \psi(p_n) \neq \emptyset$ for some $j \neq n-1$, and some colour $\beta \in \psi(p_j) \cap \psi(p_n)$ not used on any edge of T. If ψ does not satisfy (B), then it means that $\psi(e_{n-1}) \in \psi(p_{n-1})$. For ease of notation, let $\gamma := \psi(e_{n-1})$. We have the situation depicted by Figure 2.5. Since β is unused on T, we may define $\psi' = \psi(j, n-1, \beta, \gamma)$ as in Claim 2, and be assured that $\psi' \in \mathcal{C}^T$. However now, p_{n-1} sees γ under ψ', and we know that $\psi'(e_{n-1}) = \gamma$ (since $\beta \in \psi(p_n)$). So, we have succeeded in finding a colouring in $\mathcal{C}^T$ satisfying (B).

Suppose now that there exists $\psi \in \mathcal{C}^T$ satisfying (B) but not (C). This means that while $\gamma := \psi(e_{n-1})$ is seen by p_{n-1}, we must have $\psi(p_i) \cap \psi(p_n) = \emptyset$ for all $i \neq n-1$. So, since T is not ψ-elementary, there must exist some $\beta \in \psi(p_{n-1}) \cap \psi(p_n)$. Claim 1 tells us that there are at least 4 colours in $\cup_{i=0}^{n-2} \psi(p_i)$ which are unused on the edges of T_{n-2} (since $n = k(T) \geq 3$). So, we may choose one of these colours α, say $\alpha \in \psi(p_m)$, that is also unused on T. Since γ is used on T, clearly $\alpha \neq \gamma$. We also know that $\alpha \neq \beta$, because otherwise T_{n-1} would not be ψ-elementary. We have the situation depicted in Figure 2.6. Now we may define a new colouring $\psi' = \psi(m, n-1, \alpha, \beta)$, as described in Claim 2, and be assured that $\psi' \in \mathcal{C}^T$. Since $\gamma \neq \alpha, \beta$, we know that ψ' still satisfies (B). However, ψ' also satisfies (C), since $\beta \in \psi'(p_m) \cap \psi'(p_n)$.

We have now succeeded in finding a colouring $\psi \in \mathcal{C}^T$ satisfying (C), and we will use this ψ to get a contradiction. To this end, we define

$$T' = (p_0, e_0, p_1, \ldots, p_{n-2}, e_{n-1}, p_n).$$

We claim that T' is a ψ-Tashkinov tree. Certainly, since $k(T) = n$, we know that $e_{n-1} = (p_i, p_n)$ for some $i \leq n-2$. Also, T is a ψ-Tashkinov tree, so since $\psi(e_{n-1})$ is seen by p_{n-1} (by (C)), this colour must be missing at some p_l for $l \leq n-2$. Hence, T' satisfies the two properties of being a ψ-Tashkinov tree. However, by (c), T' is not ψ-elementary, so, (T', ψ) is a counterexample. Since $k(T') < k(T)$, this contradicts our choice of (T, ϕ), and hence completes our proof. □

2.3.2 As an algorithm

We shall refer to the algorithm implied by Tashkinov's Theorem as *Tashkinov's algorithm*. Here, we will not only detail Tashkinov's algorithm, but we will prove that in general, it runs in polynomial time. In fact, we shall see that we can modify the algorithm slightly so that the number of recolourings required only depends on Δ. Before we can even start to discuss the specifics of Tashkinov's algorithm however, we need to discuss *Kierstead's algorithm*.

Although we have not seen all of the proof of Tashkinov's Theorem, we did see that Kierstead's Theorem is an essential part of the proof. So, it should not be a surprise that Tashkinov's algorithm is dependent on the algorithm implied by Kierstead's proof, which we call *Kierstead's algorithm*. This much simpler algorithm has the following input and output.

Kierstead's Algorithm

INPUT:

- multigraph G
- partial $(\Delta + s)$-edge colouring ϕ of G $(s \geq 1)$
- ϕ-Kierstead path $P = (p_0, e_0, p_1, \ldots, p_n)$ that is **not** ϕ-elementary

OUTPUT:

- $(\Delta + s)$-edge colouring ϕ^* of G with $dom(\phi^*) = dom(\phi) \cup \{e_0\}$

Kierstead's algorithm has never been explicitly detailed; however it is an important part of Tashkinov's algorithm, and hence an important part of our algorithms later in this thesis. So, we take the time to provide the details now. Our description follows Kierstead's proof exactly, and hence the reader may refer to [21] for additional clarifications as necessary.

To begin Kierstead's algorithm, initialize $i < j$ such that $\phi(p_i) \cap \phi(p_j) \neq \emptyset$. Then, the algorithm is iterative. In each iteration we modify ϕ and/or P so that we can choose new i, j with $\phi(p_i) \cap \phi(p_j) \neq \emptyset$, and j strictly smaller than it previously was, or $j - i$ strictly smaller than it previously was. When we get to $i = 0$ and $j = 1$, then we can define ϕ^*.

At the start of an iteration, we have ϕ, a ϕ-Kierstead path $(p_0, e_0, p_1, \ldots, p_N)$ (where $N \leq n$), and values for $i < j$ such that $\phi(p_i) \cap \phi(p_j) \neq \emptyset$. We begin the iteration by doing the following:

0. Choose $\alpha \in \phi(p_i) \cap \phi(p_j)$.

We then proceed according to which of the following three cases we are in:

Case 1: $j = 1$

Case 2: $j > 1$ and $j - i = 1$

Case 3: $j > 1$ and $j - i > 1$

Case 1 is the easiest to resolve. Here, there is only one step we need to do:

1. Let ϕ^* be the (partial) edge-colouring obtained from ϕ by colouring e_0 with α.

Once we have done this, we are not only finished with the iteration, but we have completed the algorithm. We stop and output ϕ^*. Case 2 is only slightly more complicated. Here, we proceed as follows:

1. Find p_m such that $\beta := \phi(e_{j-1}) \in \phi(p_m)$. (*Such an* $m < j - 1$ *exists by definition of Kierstead path*).
2. Modify ϕ by recolouring edge e_{j-1} with α, and modify P by truncating after p_{j-1}. (P *is still clearly a* ϕ*-Kierstead path*).
3. Set $i := m$ and $j := j - 1$, and proceed to the next iteration, having decreased the value of j. (*This is allowed because now,* $\beta \in \phi(p_m) \cap \phi(p_{j-1})$).

Case 3 is the most involved case for Kierstead's algorithm. We proceed as follows:

1. Choose $\delta \in \phi(p_{i+1})$. (*This is possible because* $s \geq 1$).
2. If $\delta = \alpha$, set $j := i + 1$ (*which decreases* $j - i$), and then proceed to the next iteration.
3. Consider C, the maximal (α, δ)-alternating path starting at p_{i+1}.
   (a) If C ends at a vertex p_k with $k < i$, then set $i := k$ and $j := i$ or $i + 1$ (depending on whether α or δ is missing at p_k). Proceed to the next iteration, having decreased the value of j.
   (b) If C contains an edge e_k with $k \leq i$, then find a vertex p_l with $l < k$ and $\phi(e_k) \in \phi(p_l)$ (*such an* l *exits by definition of Kierstead path*). Set $i := l$, $j := i$ or $i + 1$, (depending on whether $\phi(e_k)$ is α or δ). Proceed to the next iteration, having decreased the value of j.
   (c) Otherwise, modify ϕ by swapping α and δ along C. (*Note that this makes* $\alpha \in \phi(p_{i+1})$).
      i. If C ends at p_i, then set $i := i + 1$ and leave j unchanged. Proceed to the next iteration, having decreased the value of $j - i$.
      ii. If C does not end at p_i, then modify P by truncating after p_{i+1}. Then, leave i unchanged, but set $j := i + 1$. Proceed to the next iteration, having decreased the value of j.

The important fact to realize about our description of Kierstead's algorithm is that each iteration does terminate, and hence the algorithm will terminate. Moreover, note that each iteration consists of at most one re-colouring. The worst-case scenario for this algorithm is that we start with $(i, j) = (0, n)$, and then it takes n iterations to get to $(n - 1, n)$, then one iteration to get to $(n - 2, n - 1)$, then $n - 1$ iterations to get to $(n - 2, n - 1)$, then one iteration to get to $(n - 3, n - 2)$, and so on, until we get to $(0, 1)$. In total, this would mean

$$\frac{n(n+1)}{2} + n$$

iterations. Since $n \leq |V(T)| - 1$, we clearly need only a polynomial number of recolourings (depending on $|V(G)|$). In fact, we can ensure that the number of recolourings depends only on Δ, if we add in one preliminary step. The preliminary step is to truncate P after $p_{\Delta-1}$. The reason we can do this, and still have our algorithm work (that is, still be able to pick α in Step 0), is as follows. If P is any ϕ-elementary ϕ-Kierstead path, for ϕ a partial colouring with $\Delta + s$ colours, then the union of colours missing on P has size at most $\Delta + s$. So,

$$\begin{aligned} & |V(P)|(\Delta + s - \Delta) + 2 \leq \Delta + s \\ \Rightarrow \quad & s|V(P)| + 2 \leq \Delta + s \\ \Rightarrow \quad & s(|V(P)| - 1) \leq \Delta - 2 \\ \Rightarrow \quad & |V(P)| \leq \frac{\Delta - 2}{s} + 1. \end{aligned}$$

Since $s \geq 1$, this means that $|V(P)| \leq \Delta - 1$. So, every ϕ-Kierstead path with at least Δ vertices is not ϕ-elementary. Hence, the preliminary step works as desired. Note that, although the number of recolourings depends only on Δ, many of these recolourings involve swapping an alternating path, which could involve all the vertices of the multigraph. Hence, overall, the algorithm is polynomial depending on $|V(G)|$ and Δ.

Now that we have seen Kierstead' algorithm, we are ready to detail Tashkinov's algorithm. Like Kierstead's algorithm, Tashkinov's algorithm is iterative, however the induction parameters are quite different. Just as we saw in the proof of Tashkinov's Theorem, the goal now is to make our Tashkinov tree as 'path-like' as possible, and then apply Kierstead's algorithm. In each iteration, we modify T and/or ϕ so that either $k(T)$ decreases (where $k(T)$ is defined as in the proof of Tashkinov's Theorem), or $k(T)$ remains the same and $|V(T)|$ decreases. The input and output of the algorithm are as follows.

Tashkinov's Algorithm

INPUT:

- multigraph G
- partial $(\Delta + s)$-edge colouring ϕ of G $(s \geq 1)$
- ϕ-Tashkinov tree $T = (p_0, e_0, p_1, \ldots, p_n)$ that is **not** ϕ-elementary

OUTPUT:

- $(\Delta + s)$-edge colouring ϕ^* of G with $dom(\phi^*) = dom(\phi) \cup \{e_0\}$

Suppose that we start an iteration of Tashkinov's Algorithm with the tree $T = (p_0, \ldots, p_N)$. (Note that if this is not our first iteration, then N may not be equal to n). We first check our two induction parameters.

0. (a) If $k(T) \leq 2$, then either T is a ϕ-Kierstead path or can be reordered as a ϕ-Kierstead path (*see Figure 2.2*). So, apply Kierstead's algorithm to get ϕ^*. This completes not only the iteration, but the entire algorithm, so stop and output ϕ^*.
   (b) If T_{N-1} is not ϕ-elementary, then replace T with T_{N-1}, and proceed to the next iteration. ($T_{N-1} := (p_0, \ldots, p_{N-1})$*, as defined in the proof of Tashkinov's Theorem*).

With this preliminary step taken care of, we proceed with the iteration depending on which of the following three cases T falls into.

Case 1: $k(T) = N$

Case 2: $k(T) < N$ and $\exists\ i \in \{k(T), \ldots, N-1\}$ such that $\phi(p_i) \cap \phi(p_N) \neq \emptyset$

Case 3: $k(T) < N$ and $\phi(p_i) \cap \phi(p_N) = \emptyset\ \forall\ i \in \{k(T), \ldots, N-1\}$

Unlike Kierstead's algorithm, where only one case involved swapping colours along an alternating path, all three cases here are potentially complex. Since we have already seen the proof for Case 1, it should be the easiest to follow. However, there is an important issue to note regarding Claims 1 and 2 from the proof of Tashkinov's Theorem.

Looking at the short proof of Claim 1 that we presented, it is easy to see that it will hold at every point in our algorithm. However, Claim 2 may not hold at all times during the algorithm. Certainly, if we have the set-up of the claim with T and ϕ, and if Q is the maximal (α, β)-alternating path starting at p_j, then we can define $\phi' = \phi(i, j, \alpha, \beta)$ by swapping α and β along Q. Moreover, because of our inclusion of Step 0(b), we may assume that T_{n-1} is ϕ-elementary, which is a key element of our proof. The only potential problem with the proof of Claim 2 then, is that it is possible that T_j is not ϕ'-elementary. This was not possible in our proof of Claim 2, because there, T and ϕ had already been chosen as the best possible counterexample — which is essentially what we are striving for in this algorithm. So, this "problem" actually represents another desirable outcome for us, where we could immediately proceed to the next iteration with T and ϕ replaced by T_j and ϕ'. So, in our algorithm, each time we want to replace ϕ with $\phi(i, j, \alpha, \beta)$, we either get that T_j is not elementary with respect to the new colouring (and we can immediately complete the iteration), or we get that Claim 2 holds.

Now we return to our description of an iteration of Tashkinov's algorithm. If we are in Case 1, then we proceed as follows:

1. Choose j and β such that $\beta \in \phi(p_j) \cap \phi(p_N)$.

2. We want to ensure that ϕ and β satisfy condition (A) from the proof, namely

$$j \neq N-1, \text{ and } \beta \text{ is not used on any edge of } T.$$

If this holds, then go to the next step. If not, choose $m \leq N-2$ and $\alpha \in \phi(p_m)$ such that α is not used on the edges of T_{N-2} (*possible by Claim 1*). If $\alpha \in \phi(p_N)$, swap the names of α and β, set $j := m$, and proceed to the next step. Otherwise, do as follows:

   (a) If $j = N-1$, replace ϕ by $\phi(m, j, \alpha, \beta)$. If T_j is not ϕ-elementary with respect to this new colouring, then replace T with T_j and proceed to the next iteration. Otherwise, set $j := m$ and proceed to the next step, knowing that (A) is satisfied.

   (b) If $j \leq N-2$ but β is used on T, let Q be the maximal (α, β)-alternating path starting at p_N.

      i. If Q contains a vertex on T_{N-2}, then let Q' be the segment of Q between p_N and the first vertex of Q on T_{N-2}, and define $T' = (p_0, \ldots, p_{N-2}, Q', p_N)$. Proceed to the next iteration with T' in place of T. (*Possible because* $k(T') < N-1$).

      ii. Otherwise, modify ϕ by switching the colours α and β along Q. Then switch the names of α and β throughout G, to get that ϕ satisfies (A).

3. We now want to ensure that ϕ satisfies condition (B) from the proof, namely

$$\gamma := \phi(e_{N-1}) \text{ is seen by } p_{N-1}.$$

If this holds, then go directly to the next step. If not, replace ϕ by $\phi(j, N-1, \beta, \gamma)$. If T_{N-1} is not ϕ-elementary, then replace T with T_{N-1} and proceed to the next iteration. Otherwise, ϕ satisfies (B).

4. We now want to ensure that ϕ satisfies condition (C) from the proof, namely

$$\gamma = \phi(e_{N-1}) \text{ is seen by } p_{N-1}, \text{ and } \phi(p_{j'}) \cap \phi(p_N) \neq \emptyset \text{ for some } j' \neq N-1.$$

(*Note that this first condition already holds.*) Pick $\beta' \in \phi(p_{j'}) \cap \phi(p_N)$ for some j' (*possible since T_{N-1} is ϕ-elementary*). If $j' \neq N-1$ then proceed to the next step. If $j' = N-1$, choose $\alpha' \in \phi(p_{m'})$ that is unused on T, for some $m' \leq N-2$ (*possible by Claim 1*). Then, replace ϕ with $\phi(m', N-1, \alpha', \beta')$. If $T_{m'}$ is not ϕ-elementary with respect to this new colouring, then replace T with $T_{m'}$ and proceed to the next iteration. Otherwise, proceed to the next step, knowing that ϕ satisfies (C).

5. Replace T by $(p_0, \ldots, p_{N-2}, e_{N-1}, p_N)$ and proceed to the next iteration (*this is allowed because $k(T)$ has decreased*).

Note that by going through Case 1 once, we are performing at most 3 recolourings. Note also that after we've completed an iteration through Case 1, we always get that $k(T)$ has decreased by at least one. This will be important for our analysis of the algorithm.

If we are in Case 2, then the iteration is actually much simpler to describe than in Case 1:

1. Choose $i \geq k(T)$ such that $\phi(p_i) \cap \phi(p_N) \neq \emptyset$, and choose $\beta \in \phi(p_i) \cap \phi(p_N)$.
   (a) If $i \neq N-1$, choose $\gamma \in \phi(p_{i+1})$. Then, replace ϕ by $\phi(i, i+1, \beta, \gamma)$. If T_{i+1} is not ϕ-elementary with respect to this new colouring, then replace T by T_{i+1} and proceed to the next iteration. Otherwise, set $i := i+1$ (*possible because $\beta \in \phi(p_{i+1}) \cap \phi(p_N)$*). Repeat this recolouring action until $i = N-1$.
   (b) If $i = N-1$, choose $\beta \in \phi(p_{N-1}) \cap \phi(p_N)$ and let $\gamma = \phi(e_{N-1})$. Also choose $j < N-1$ such that $\gamma \in \phi(p_j)$. Then, modify ϕ by recolouring e_{N-1} with β, and replace T by T_{N-1}. Proceed to the next iteration.

Note that by going through Case 2 once, we are performing at most $N-4$ recolourings (because $i \geq k(T) \geq 3$). If we've completed an iteration through Case 2, then $|V(T)|$ has definitely decreased, but it is possible that $k(T)$ has stayed this same. As we will see, this is one reason that Case 2 is not as desirable as Case 1, from a complexity standpoint.

Case 3 is just as complicated as Case 1, and since we have not seen the proof for this case, it may be more difficult to follow. Nevertheless, we provide the steps here.

1. Choose i such that $\phi(p_i) \cap \phi(p_N) \neq \emptyset$, and choose $\beta \in \phi(p_i) \cap \phi(p_N)$.

2. If $i = k(T) - 1$, then choose $\alpha \in \phi(p_m)$ (for some $m \leq k(T) - 2$) such that α is not used on $T_{k(T)-2}$ *(possible by Claim 1)*. Replace ϕ by $\phi(m, k(T)-1, \alpha, \beta)$. If $T_{k(T)-1}$ is not ϕ-elementary with respect to this new colouring, then replace T by $T_{k(T)-1}$ and proceed to the next iteration. Otherwise, set $i := m$ and continue to the next step.

3. Choose $\gamma \in \phi(p_{k(T)})$, and choose $\varepsilon \in \phi(p_{m'})$ (for some $m' \leq k(T) - 2$) that is unused on $T_{k(T)}$ and with $\varepsilon \neq \gamma$ (*possible by Claim 1*).

4. If $\varepsilon \in \phi(p_N)$, then replace ϕ by $\phi(m, k(T), \varepsilon, \gamma)$. If $T_{k(T)}$ is not ϕ-elementary with respect to this new colouring, then replace T with $T_{k(T)}$ and proceed to the next iteration. Otherwise, leave T unchanged, and continue this iteration at the beginning of Case 2 (*since now,* $\varepsilon \in \phi(p_{k(T)}) \cap \phi(p_N)$).

5. Define R as the maximal (ε, β)-alternating path starting at p_N. (*Note that* $\beta \in \phi(p_N)$ *and by Step 4,* ε *is seen by* p_N.)

   (a) If $V(R) \cap V(T_{k(T)-1}) \neq \emptyset$, then choose an index $h \leq k(T) - 1$ such that $p_h \in R$ and the subpath of R, R', between p_h and p_n does not contain any other vertices of $V(T_{k(T)-1})$. If $h < k(T) - 1$, then replace T by $(p_0, \ldots, p_{k(T)-2}, R', p_N)$, and proceed to the next iteration. Otherwise, replace T by $(p_0, \ldots, p_{k(T)-1}, R', p_N)$ and proceed to the next iteration.

   (b) Otherwise, modify ϕ by swapping β and ε along R. Then, replace this colouring by $\phi(m', k(T), \varepsilon, \gamma)$. If $T_{k(T)}$ is not ϕ-elementary with respect to this new colouring, then replace T with $T_{k(T)}$ and proceed to the next iteration. Otherwise, leave T unchanged, and continue this iteration at the beginning of Case 2 (*since now,* $\varepsilon \in \phi(p_{k(T)}) \cap \phi(p_N)$).

Note that a single iteration may consist of the steps of Case 3, followed by those of Case 2. While going through the steps of Case 3 once only requires at most 3 recolourings, if this is followed by an application of Case 2, it could mean up to $N-1$ recolourings in one iteration. Since Case 2 always terminates an iteration however, this is not something that causes a problem with termination. If an iteration does terminate in Case 3 however, there are a number of possibilities for how our induction criteria may be met. It is possible that: both $|V(T)|$ and $k(T)$ have decreased by at least one (eg. if we terminate in Step 2, or possible if we terminate in Step 4, Step 5(a) or Step 5(b)); $|V(T)|$ has decreased but $k(T)$ has not (possible if we terminate in Step 4 or Step 5(b)), or ; $k(T)$ has decreased but $|V(T)|$ has stayed the same or even increased (possible if we terminate in Step 5(a)).

Overall, Tashkinov's algorithm will run until $k(T) \leq 2$, at which point we can apply Kierstead's algorithm to terminate. Since $k(T)$ need not decrease after every iteration, there is something to be said here about the algorithm terminating in general. If $k(T)$ does not decrease after an iteration, then we at least get that $|V(T)|$ decreases. Moreover, while $|V(T)|$ may increase during our algorithm (via Step 2(b)(i) of Case 1, or Step 5(a) of Case 3), each time this happens, $k(T)$ does decrease. So, Tashkinov's algorithm does terminate.

We may include a preliminary step in Tashkinov's algorithm to truncate our tree T to Δ vertices, just as we did with Kierstead's algorithm. Since an iteration may actually increase the size of T however, it is not enough to truncate once, at the start of the algorithm. Instead, we must add this truncation step at the start of each iteration, before Step 0. Then, we are assured that N is bounded by $\Delta - 1$ in any iteration.

The last iteration of Tashkinov's algorithm is Kierstead's algorithm, and from our previous discussion, we know that for $T = (p_0, \ldots, p_N)$ this requires at most

$$\frac{N(N+1)}{2} + N \quad \leq \quad \frac{(\Delta - 1)\Delta}{2} + \Delta - 1$$

recolourings. Other iterations require at most 3, $N-4$ or $N-1$ recolourings (depending on whether we are in Case 1, Case 2 or Case 3), but in any case this means at most $\Delta - 2$ recolourings. To give a bound on the total number of recolourings needed for the algorithm, recall that $k(T) \leq n$ at the outset, and the last iteration occurs when $k(T) \leq 2$. While $k(T)$ never increases, it may not decrease after every iteration. The only Case which guarantees a decrease in $k(T)$ is Case 1 ($k(T) = N$). However, we do know that if $k(T)$ doesn't decrease in a step, then N does (in particular, it can't increase). So, at any point in the algorithm, it takes at most $N - k(T) \leq (\Delta - 1) - 3 = \Delta - 4$ iterations to get a decrease in $k(T)$. Hence, an application of Tashkinov's algorithm requires at most

$$(n-2)(\Delta - 4) + 1$$

iterations, and at most

$$(n-2)\left[3 + (\Delta - 5) \cdot (\Delta - 2)\right] + \left(\frac{(\Delta - 1)\Delta}{2} + \Delta - 1\right)$$

recolourings. Although we may replace n is these expressions with Δ (because of our initial truncating step), some of these recolourings may involve the entire vertex set of G. So overall, Tashkinov's algorithm is polynomial in $|V(G)|$ and Δ.

Chapter 3

Achieving maximum chromatic index

In the previous chapter we saw a number of upper bounds for chromatic index, and we saw how these can be proved using the method of Tashkinov trees. This method can do more however - by analyzing the structure of the trees, we can gain information about those multigraphs which achieve maximimum chromatic index. Ideally, given a bound $\chi'(G) \leq a$, we would like to characterize the chromatic class $\chi'(G) = a$.

In the first section of this chapter, we discuss the easy direction when it comes to characterizing upper bounds — the sufficiency direction. That is, we provide a number of canonical examples which achieve famous upper bounds. In Section 3.2, we focus on Goldberg's bound in particular, and are able to prove a complete characterization. We will see that this result is a generalization of a characterization of Shannon's bound, due to Vizing — the only characterization which had been previously known. The third and final section of this chapter focuses on Vizing's bound. Here, we are not able to deduce a characterization, but we do provide a number of necessary conditions.

The results of this chapter form the body of the paper [28].

3.1 Canonical examples

In the introduction of this thesis we used $2K_3$ as an example of a multigraph with chromatic index higher than $\Delta + 1$. In fact, multiples of triangles can be thought of as *the* canonical examples of high chromatic index. Clearly, every edge in a μK_3

must receive a different colour, so $\chi'(\mu K_3) = 3\mu$. More formally, since μK_3 has 3 vertices, we could apply the $\lceil\rho\rceil$-lower bound to its entire vertex set, yielding

$$\chi'(\mu K_3) \geq \lceil \rho(\mu K_3) \rceil \geq \left\lceil \frac{2|E(\mu K_3)|}{|V(\mu K_3)| - 1} \right\rceil = \left\lceil \frac{2(3\mu)}{3-1} \right\rceil = 3\mu.$$

Since $\Delta = 2\mu$, we can also express 3μ as $3\Delta/2$, which is perhaps a more recognizable value. In fact, $3\Delta/2$ is the exact number given by Shannon's classical upper bound. So, we have just proved that

$$\chi'(\mu K_3) = \frac{3\Delta}{2},$$

and hence all multiples of triangles achieve Shannon's upper bound.

A triangle is both an odd clique and an odd cycle, and this suggests a generalization of our above argument to get two families of canonical examples.

Theorem 3.1.1. *Let μK_{d+1} be a multiple of an odd clique. Then,*

$$\chi'(\mu K_{d+1}) = \mu(d+1) = \Delta + \mu.$$

Proof. Since μK_{d+1} has $d + 1 \geq 3$ vertices, and $d + 1$ is odd, we may apply the $\lceil\rho\rceil$-lower bound to the entire vertex set. This yields

$$\chi'(\mu K_{d+1}) \quad \geq \quad \lceil \rho(\mu K_{d+1}) \rceil \geq \left\lceil \frac{2[\mu d(d+1)/2]}{d} \right\rceil = \mu(d+1) = \Delta + \mu.$$

Since Vizing's bound tells us that $\chi' \leq \Delta + \mu$, we have determined chromatic index exactly for all multiples of odd cliques. □

Theorem 3.1.2. *Let μC_k be a multiple of an odd cycle. Then,*

$$\chi'(\mu C_k) = \left\lceil \frac{\Delta k}{k-1} \right\rceil = \Delta + 1 + \left\lfloor \frac{\Delta - 2}{k-1} \right\rfloor.$$

Proof. Since μC_k has $k \geq 3$ vertices, and k is odd, we may apply the $\lceil\rho\rceil$-lower bound to the entire vertex set. This yields

$$\chi'(\mu C_k) \quad \geq \quad \lceil \rho(\mu C_k) \rceil \geq \left\lceil \frac{2[\mu k]}{k-1} \right\rceil = \left\lceil \frac{\Delta k}{k-1} \right\rceil.$$

Since we are concerned only with multigraphs having chromatic index strictly greater than $\Delta + 1$, it may be helpful to rewrite this last expression as

$$\Delta + \left\lceil \frac{\Delta}{k-1} \right\rceil = \Delta + \left\lfloor \frac{\Delta + k - 2}{k-1} \right\rfloor = \Delta + 1 + \left\lfloor \frac{\Delta - 1}{k-1} \right\rfloor = \Delta + 1 + \left\lfloor \frac{\Delta - 2}{k-1} \right\rfloor,$$

where the last equality is due to the fact that $k-1$ is even and $\Delta - 1$ is odd. Goldberg's bound tells us that $\chi' \leq \Delta + 1 + \frac{\Delta-2}{k-1}$. However, since chromatic index is always an integer, this further implies

$$\chi' \leq \Delta + 1 + \left\lfloor \frac{\Delta - 2}{k-1} \right\rfloor .$$

Hence, we have determined chromatic index exactly for all multiples of odd cycles. □

We now have a class of examples which we know achieve Shannon's bound (multiples of triangles), a class of examples which we know achieve Vizing bound (multiples of odd cliques), and a class of examples which we know achieve Goldberg's bound (multiples of odd cycles). In the next section, we will see the sufficiency of these examples for Shannon's bound and for Goldberg's bound.

3.2 Shannon's bound and Goldberg's bound

Vizing characterized Shannon's upper bound in his 1968 doctoral dissertation, and the characterization is as follows.

Theorem 3.2.1. [43] *Let G be a connected multigraph. Then, $\chi'(G) = \frac{3\Delta}{2}$ if and only if $G = \mu K_3$.*

Note that connectivity is assumed in the above result, and we will almost always make that assumption in this thesis. We can always focus on colouring components of a disconnected multigraph separately, and it makes sense to do so.

Theorem 3.2.1 was actually the only known characterization of an upper bound on chromatic index, until the following characterization of Goldberg's upper bound.

Theorem 3.2.2. *Let G be a connected multigraph containing an odd cycle. Then, $\chi'(G) = \Delta + 1 + \frac{\Delta-2}{g_o-1}$ if and only if $G = \mu C_{g_o}$ and $(g_o - 1) \mid 2(\mu - 1)$.*

As Goldberg's bound generalizes Shannon's upper bound, so does our Theorem 3.2.2 generalize Theorem 3.2.1. Thus, within the following proof, the case $g_o = 3$ serves as justification for Vizing's characterization.

Proof. *(Theorem 3.2.2)* First suppose that $G = \mu C_{g_o}$ and $(g_o - 1) \mid 2(\mu - 1)$. By Theorem 3.1.2, we know that

$$\chi'(G) = \Delta + 1 + \left\lfloor \frac{\Delta - 2}{g_o - 1} \right\rfloor ,$$

and since $\Delta = 2\mu$, our divisibility assumption says that we may remove the floor signs from this expression. Hence, G achieves Goldberg's upper bound of $\Delta + 1 + \frac{\Delta-2}{g_o-1}$, establishing the backwards implication of our statement.

Suppose now that the forwards implication of our statement is false. Then, there exists a connected multigraph G that contains an odd cycle and that has $\chi'(G) = \Delta + 1 + \frac{\Delta-2}{g_o-1}$ but either $G \neq \mu C_{g_o}$ or $G = \mu C_{g_o}$ and $(g_o - 1) \nmid 2(\mu - 1)$. In fact, since $\frac{\Delta-2}{g_o-1}$ is an integer (because the chromatic index is an integer), $G = \mu C_{g_o}$ would imply $(g_o - 1) \mid 2(\mu - 1)$. So, it must be the case that $G \neq \mu C_{g_o}$.

Since G contains an odd cycle, we know that $\Delta \geq 2$. If $\Delta = 2$, then, since G is connected, it must be the case that $G = \mu C_{g_o}$ with $\mu = 1$, which is a contradiction. So we may assume that $\Delta \geq 3$. Note that this implies that $\frac{\Delta-2}{g_o-1} > 0$, and hence $\chi'(G) = \Delta + 1 + \frac{\Delta-2}{g_o-1} > \Delta + 1$.

Let ϕ be any partial $(\Delta + \frac{\Delta-2}{g_o-1})$-edge-colouring of G having maximum domain. We know that there is at least one edge, say e_0, that is uncoloured by ϕ. Let p_0 be one end of e_0 and p_1 the other end. Choose $\alpha \in \phi(p_0)$ and $\beta \in \phi(p_1)$. Since ϕ has maximum domain, $\alpha \neq \beta$ and there exists an α, β alternating path $p_1, \ldots, p_m, p_0$ of even length joining p_1 and p_0. Together with e_0, this forms an odd cycle of length $m + 1$, so that $m + 1 \geq g_o$.

Let $P = (p_0, e_0, p_1, \ldots, p_m)$. Note that P is both a ϕ-Kierstead path and a ϕ-Tashkinov tree. Extend P to a maximal ϕ-Tashkinov tree

$$T = (p_0, e_0, p_1, \ldots, p_m, \ldots, p_n).$$

Theorem 2.2.1 tells us that T is ϕ-elementary. Note that since ϕ has $\Delta + \frac{\Delta-2}{g_o-1}$ colours,

$$|\phi(p_0)|, |\phi(p_1)| \geq \frac{\Delta - 2}{g_o - 1} + 1,$$

and for $j \in \{2, \ldots, n\}$,

$$|\phi(p_j)| \geq \frac{\Delta - 2}{g_o - 1}.$$

So, the fact that T is ϕ-elementary implies that, for $j \in \{0, 1\}$,

$$\left| \bigcup_{i=0}^{n} \phi(p_i) \setminus \phi(p_j) \right| \geq n \left(\frac{\Delta - 2}{g_o - 1} \right) + 1,$$

and for $j \in \{2, \ldots, n\}$,

$$\left| \bigcup_{i=0}^{n} \phi(p_i) \setminus \phi(p_j) \right| \geq n \left(\frac{\Delta - 2}{g_o - 1} \right) + 2.$$

Suppose that $n = m = g_o - 1$. So, $T = P$, and T has exactly g_o vertices. Since T is ϕ-elementary, we know that, for every $j \in \{0, \ldots, n\}$, p_j must see every colour in $\cup_{i=0}^{n} \phi(p_i) \setminus \phi(p_j)$. Moreover, since T is maximal, all these colours must be on edges induced by $V(T)$. So the number of coloured edges incident to p_j in $G[V(T)]$ is at least

$$n\left(\frac{\Delta - 2}{g_o - 1}\right) + 1 = \Delta - 1$$

for $j \in \{0, 1\}$, and at least

$$n\left(\frac{\Delta - 2}{g_o - 1}\right) + 2 = \Delta$$

for $j \in \{2, \ldots, n\}$. Of course, since e_o is uncoloured, we know that there are at least Δ edges incident to each vertex in $G[V(T)]$. In fact, all these edges must occur between consecutive vertices of the cycle $(p_0, \ldots, p_{g_o-1})$, since it has length g_o and hence must be chordless. Since G is connected, this tells us that the underlying graph of G is C_{g_o} and that G is Δ-regular. Hence, $G = \mu C_{g_o}$, which is a contradiction.

We may now assume that $n \geq g_o$. Note that

$$\left| \bigcup_{i=0}^{n} \phi(p_i) \right| \geq= (n+1)\left(\frac{\Delta - 2}{g_o - 1}\right) + 2.$$

So, since ϕ has $\Delta + \frac{\Delta-2}{g_o-1}$ colours, we must have

$$(n+1)\left(\frac{\Delta - 2}{g_o - 1}\right) + 2 \leq \Delta + \frac{\Delta - 2}{g_o - 1},$$

so

$$\left(\frac{g_o}{g_o - 1}\right)(\Delta - 2) \leq \Delta - 2.$$

This is a contradiction, since $\Delta > 2$. □

While only a specific family of multiples of odd cycles achieve Goldberg's upper bound, we saw in the first section of this chapter that *all* multiples of odd cycles achieve the *floor* of Goldberg's upper bound. We are not yet able to completely characterize those multigraphs achieving the floor of the bound; however we will have more to say later in this thesis about multigraphs which are *close* to achieving Goldberg's upper bound.

3.3 Vizing's bound

In the first section of this chapter we showed that all multiples of odd cliques achieve Vizing's upper bound. Recall from Section 2.1 that characterizing those multigraphs with $\chi' = \Delta + 1$ is an NP-hard problem, hence multiples of odd cliques cannot be the only multigraphs to achieve $\Delta + \mu$ in general. Unfortunately, even when we assume $\mu \geq 2$ we cannot hope for sufficiency in this. For example, the multigraph $2K_7 - e$ ($2K_7$ with one edge removed) achieves Vizing's upper bound, as the following simple computation demonstrates:

$$\chi'(2K_7 - e) \geq \lceil \rho(2K_7 - e) \rceil \geq \left\lceil \frac{2(40)}{6} \right\rceil = 14 = 12 + 2 = \Delta + \mu.$$

Characterizing those multigraphs (with $\mu \geq 2$) that achieve Vizing's upper bound is still an open problem, and it appears to be a very difficult problem. However, there was one necessary condition proved in the 1980's, and we will provide some additional necessary conditions here.

The result from the 1980's was due to H. A. Kierstead [21]. Kierstead's alternate proof of Vizing's Theorem, which we have already discussed in Section 2.2, actually proved a strengthened version of the theorem. His strengthening involves a *k-sided triangle*, which is defined to be a multigraph on k edges whose underlying graph is a triangle.

Theorem 3.3.1. [21] *Let G be a multigraph with $\mu \geq 2$. If $\chi'(G) = \Delta + \mu$, then G contains a 2μ-sided triangle as a subgraph.*

Thinking back to Kierstead's proof of Vizing's Theorem from Section 2.2, we can see roughly how this additional result was obtained. Given a maximal Kierstead path, the last vertex in the path has many edges going to the earlier vertices in the path (see Figure 2.1), which points to the existence of dense triangles.

Theorem 3.3.1 says that only multigraphs with dense triangles can have chromatic index $\Delta + \mu$, which is a very nice result. However, since there are many multigraphs containing 2μ-sided triangles which do not have chromatic index $\Delta + \mu$ (for example, when G is itself a 2μ-sided triangle), there is much room for improvement. Here, we extend Theorem 3.3.1 in three ways, each time replacing "2μ-sided triangle" with a different multigraph on five vertices.

The main difficulty in using Kierstead paths (or Tashkinov trees) to extend Theorem 3.3.1, is that it is not in general possible to build large such structures: we may get a triangle on many edges containing e_0 and e_1, and then not be able extend the path (or tree) any further. Of course, if we are to extend Theorem 3.3.1, we must allow the existence of these dense triangles. In the following lemma we argue "past the triangle" to get a Kierstead path of length four. Note that it is significant here that we get a Kierstead path and not just a general Tashkinov tree: we want to know what our structure looks like.

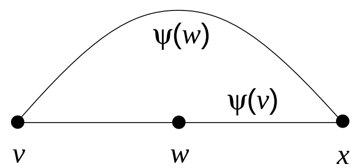

Figure 3.1: The location of the colours $\psi(w)$ and $\psi(v)$

Lemma 3.3.2. *Let $G \neq \mu K_3$ be a connected multigraph with $\chi'(G) = \Delta + \mu$ and $\mu \geq 2$. Then there exists a partial $(\chi'-1)$-edge-colouring ϕ of G which has maximum domain, and a ϕ-Kierstead path $P = (p_0, e_0, p_1, \ldots, p_4)$ in G.*

Proof. Let ψ be any partial $(\Delta + \mu - 1)$-edge-colouring of G having maximum domain. We know that there is at least one edge, say e, that is uncoloured by ψ. Suppose that the end-vertices of e are v and w, respectively. Define $e_0(\psi) = e$, $p_0(\psi) = v$, and $p_1(\psi) = w$. Note that $(p_0(\psi), e_0(\psi), p_1(\psi))$ is a ψ-Kierstead path. There must exist some edge f incident to w (with other end $x \neq v$, say) extending this ψ-Kierstead path, since otherwise v and w have a common missing colour (which could be used to extend ψ to e). Let $e_1(\psi) = f$ and $p_2(\psi) = x$. Suppose that the ψ-Kierstead path $(p_0(\psi), e_0(\psi), p_1(\psi), e_1(\psi), p_2(\psi))$ cannot be extended further.

We know that under ψ, x must see each of the colours in $\psi(v) \cup \psi(w)$ (by Theorem 2.2.2), and $|\psi(v) \cup \psi(w)| \geq 2[\Delta + \mu - 1 - (\Delta - 1)] = 2\mu$. So, since the ψ-Kierstead path cannot be extended, these 2μ colours must appear on edges between x and v and edges between x and w under ψ. Since there are at most 2μ such edges, we know that there are exactly 2μ such edges. Also, due to the nature of the colours we are placing, we know that: $|\psi(v)| = |\psi(w)| = \mu$, and under ψ the μ edges between w and x are coloured with the colours $\psi(v)$, and the μ edges between v and x are coloured with the colours $\psi(w)$ (see Figure 3.1).

Suppose that there exists an edge incident to v or w that leaves the triangle (v, w, x) and is coloured with a colour $d \in \psi(x)$ under ψ. Without loss of generality, suppose that the edge is incident to v. In this situation, we define a new partial colouring φ from ψ by removing the colour $\psi(f)$ from f and assigning it instead to edge e (φ is proper because $f \in \psi(v)$). Define $p_0(\varphi) = x$, $e_0(\varphi) = f$, $p_1(\varphi) = w$, $e_1(\varphi) = e$ and $p_2(\varphi) = v$. Now, we have $d \in \varphi(x)$. Suppose that g is this d-coloured edge and it has ends v and y. Define $e_2(\varphi) = g$ and $p_3(\varphi) = y$. Then, $(p_0(\varphi), e_0(\varphi), p_1(\varphi), e_1(\varphi), p_2(\varphi), e_2(\varphi), p_3(\varphi))$ is a φ-Kierstead path in G.

Suppose now that no such d-coloured edge exists under ψ. Then, since $\psi(x)$ must be entirely disjoint from $\psi(v)$ and $\psi(w)$ (by Theorem 2.2.2), we know that all the colours $\psi(x)$ must appear on edges between v and w in ψ. Since $|\psi(x)| \geq$

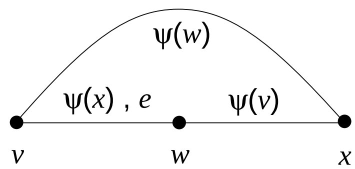

Figure 3.2: Edges out of the triangle are limited in colour

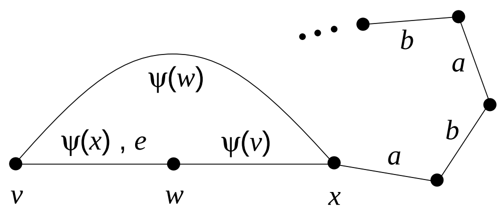

Figure 3.3: The path $\widetilde{P}$

$(\Delta+\mu-1)-\Delta=\mu-1$, it must be the case that $|\psi(x)|=\mu-1$ (since e is uncoloured under ψ). Hence, v,w,x form a μK_3 and each of v,w,x has degree Δ in G, with every incident edge (except for e) coloured under ψ. We observe that the edges out of the triangle cannot be coloured with any of the colours $\psi(v),\psi(w),\psi(x)$ under ψ (see Figure 3.2). So, there are only $(\Delta+\mu-1)-(3\mu-1)=(\Delta-2\mu)$ colours with which to colour these edges in ψ. However, since there are exactly $\Delta-2\mu$ edges coming out of each of the three vertices, we know that the 3 edge sets out of the triangle must all be coloured with this same $(\Delta-2\mu)$-colour set under ψ, call it S. Since $G\neq\mu K_3$, S is nonempty and we can choose a colour $a\in S$. Also, choose a colour $b\in\psi(x)$. Let G' be the multigraph obtained from G by deleting all the edges of the triangle v,w,x. Consider $\widetilde{P}$, the maximal (a,b)-alternating path in G' starting at x, with respect to ψ (see Figure 3.3).

Define a colouring γ on G' to be the same as ψ on G', except for the colours a and b swapped along $\widetilde{P}$. Regardless of where $\widetilde{P}$ ends, the three colour sets out of our triangle are not identical under γ — either one of v,w,x is incident to a b-edge while the other two are incident to an a-edge, or one of them is incident to an a-edge and the other are incident to b-edges. We extend our definition of γ as follows: we place the colour (a or b) that is incident to only one of v,w,x on an edge of the triangle opposite to the vertex it is incident to, along with $\psi(x)-\{b\}$, and let the last uncoloured edge here be $e_0(\gamma)$. Use $\psi(v)$ and $\psi(w)$ to colour the other 2μ edges of the triangle. Now, define $p_0(\gamma)$, $p_1(\gamma)$, $e_1(\gamma)$ and $p_2(\gamma)$ to coincide with our new choice of $e_0(\gamma)$. Note that under γ, $p_2(\gamma)$ is missing a colour d (either a or b) which

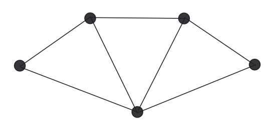

Figure 3.4: The graph H

goes out of both $p_0(\gamma)$ and $p_1(\gamma)$. Hence, we can argue as above to get a Kierstead path of length three.

We may now assume that we have constructed a ϕ-Kierstead path $P = (p_0, e_0, p_1, \ldots, p_3)$ in G, where ϕ is a $(\Delta + \mu - 1)$-edge colouring of G that has maximum domain. Let c be the colour of edge e_1 on P. So, $c \in \phi(p_0)$. We know, by Theorem 2.2.2, that c must be seen by p_3. However, it certainly is not on (p_0, p_3), nor can it be on (p_2, p_3) or (p_1, p_3). Therefore we can choose to extend our ϕ-Kierstead path to p_4 via an edge coloured c. □

For all $k \geq 2$, define a *k-pyramid* to be a multigraph consisting of a path Q of length three and one additional vertex v, such that there are $4k-5$ edges between v and Q. Let H be the graph depicted in Figure 3.4. For all $k \geq 7$, define a *k-sided H* to be a multigraph on k edges whose underlying graph is H. We can now state our three necessity results for achieving Vizing's bound. Note that all our results assume that $G \neq \mu K_3$ — obviously this multigraph does achieve Vizing's bound, so if we do not discount it, we would never be able to guarantee subgraphs on five vertices.

Theorem 3.3.3. *Let $G \neq \mu K_3$ be a connected multigraph with $\mu \geq 2$. If $\chi'(G) = \Delta + \mu$, then G contains a μ-pyramid as a subgraph.*

Proof. Suppose, for a contradiction, that there exists a connected multigraph $G \neq \mu K_3$ with $\mu \geq 2$ and $\chi'(G) = \Delta + \mu$, that does not contain a μ-pyramid as a subgraph.

Lemma 3.3.2 tells us that there exists a partial $(\chi' - 1)$-edge colouring ϕ of G that has maximum domain, and a ϕ-Kierstead path $(p_0, e_0, p_1, \ldots, p_4)$ in G. We build a maximal ϕ-Kierstead path starting with $p_0, \ldots, p_4$. Let $P = (p_0, \ldots, p_n)$ be the path that we build. Theorem 2.2.2 tells us that P is ϕ-elementary.

Note that $|\phi(p_0)|, |\phi(p_1)| \geq \mu$, and for $2 \leq i \leq n$, $|\phi(p_i)| \geq \mu - 1$. Since P is ϕ-elementary, this means that

$$\left| \bigcup_{i=0}^{n-1} \phi(p_i) \right| \geq \mu + \mu + (n-2)(\mu - 1) = n\mu - n + 2.$$

The fact that P is ϕ-elementary also implies that p_n must see every colour in $\cup_{i=0}^{n-1} \phi(p_i)$. Moreover, since P cannot be extended any further, every colour in

$\cup_{i=0}^{n-1} \phi(p_i)$ must appear on an edge between p_n and $(p_0, \ldots, p_{n-1})$. Note that we may partition $(p_0, \ldots, p_{n-1})$ into $\lfloor \frac{n}{4} \rfloor$ paths on four vertices, and one path on $n - 4 \lfloor \frac{n}{4} \rfloor$ vertices. So, since G has no μ-pyramid as a subgraph, the number of edges between p_n and $(p_0, \ldots, p_{n-1})$ is at most

$$\left\lfloor \frac{n}{4} \right\rfloor (4\mu - 6) + \left(n - 4 \left\lfloor \frac{n}{4} \right\rfloor\right) \mu = n\mu - 6 \left\lfloor \frac{n}{4} \right\rfloor.$$

So, we must have

$$n\mu - 6 \left\lfloor \frac{n}{4} \right\rfloor \geq n\mu - n + 2,$$

which implies that

$$\left\lfloor \frac{n}{4} \right\rfloor \leq \frac{n-2}{6}. \tag{3.1}$$

Since $\frac{n-3}{4} \leq \lfloor \frac{n}{4} \rfloor$, this inequality immediately tells us that $\frac{n-3}{4} \leq \frac{n-2}{6}$, which means that $n \leq 5$. However, we already know that $n \geq 4$, so in fact we must have $n = 4$ or $n = 5$. Neither of these values satisfy inequality (3.1), so we get our desired contradiction. □

Theorem 3.3.4. *Let $G \neq \mu K_3$ be a connected multigraph with $\mu \geq 6$. If $\chi'(G) = \Delta + \mu$, then G contains a $(4\mu - 2)$-sided H as a subgraph.*

Proof. Note that since a μ-pyramid has $4\mu - 5$ edges between a vertex and a 3-path, any μ-pyramid with $\mu \geq 6$ has H as its underlying graph. A μ-pyramid has $4\mu - 5 + 3 = 4\mu - 2$ edges in total. □

Theorem 3.3.5. *Let $G \neq \mu K_3$ be a connected multigraph with $\mu \geq 2$ and $\Delta \leq \mu^2$. If $\chi'(G) = \Delta + \mu$, then G contains K_5 as a subgraph.*

Proof. Suppose, for a contradiction, that there exists a connected multigraph $G \neq \mu K_3$ with $\mu \geq 2$, $\Delta \leq \mu^2$, and $\chi'(G) = \Delta + \mu$, that does not contain K_5 as a subgraph.

Lemma 3.3.2 tells us that there exists a partial $(\chi' - 1)$-edge-colouring ϕ that has maximum domain, and a ϕ-Kierstead path $P = (p_0, e_0, p_1, \ldots, p_4)$ in G. Note that P is also a ϕ-Tashkinov tree. Extend P to a maximal ϕ-Tashkinov tree $T = (p_0, \ldots, p_n)$. Theorem 2.2.1 tells us that T is ϕ-elementary.

Note that $|\phi(p_0)|, |\phi(p_1)| \geq \mu$, and for $2 \leq i \leq n$, $|\phi(p_i)| \geq \mu - 1$. Since T is ϕ-elementary, this means that, for $2 \leq j \leq n$,

$$\left| \bigcup_{i=0}^{n} \phi(p_i) \setminus \phi(p_j) \right| \geq 2\mu + (n-2)(\mu - 1) = n\mu - (n-2),$$

and for $0 \le j \le 1$,

$$\left| \bigcup_{i=0}^{n} \phi(p_i) \setminus \phi(p_j) \right| \ge \mu + (n-1)(\mu - 1) = n\mu - (n-1).$$

The fact that T is ϕ-elementary also implies that, for every $j \in \{0, \dots, n\}$, p_j must see every colour in $\cup_{i=0}^{n} \phi(p_i) \setminus \phi(p_j)$. Moreover, since T is maximal, all these colours must be on edges induced by $V(T)$. So each vertex of T must see at least $n\mu - (n-1)$ colours on edges which do not leave the tree. If $\mu \ge n$, this means that every vertex in T must be adjacent to every other vertex in T. Therefore G contains a copy of K_{n+1}. Since $n \ge 4$, this means that G must contain K_5, which is a contradiction. So it must be the case that $n \ge \mu + 1$.

Note that since T is ϕ-elementary,

$$\left| \bigcup_{i=0}^{n} \phi(p_i) \right| \ge 2\mu + (n-1)(\mu - 1) \ge 2\mu + (\mu)(\mu - 1) = \mu^2 + \mu.$$

Since ϕ has $(\Delta + \mu - 1)$ colours, we must have

$$\mu^2 + \mu \le \Delta + \mu - 1 \quad \Rightarrow \quad \mu^2 < \Delta,$$

which is a contradiction. □

Note that by Kuratowski's Theorem, Theorem 3.3.5 has the following corollary.

Corollary 3.3.6. *Let $G \neq \mu K_3$ be a connected multigraph with $\mu \ge 2$ and $\Delta \le \mu^2$. If $\chi'(G) = \Delta + \mu$, then G is nonplanar.*

These partial results are all that we know towards characterizing Vizing's bound in general. However, we will have more to say about the special case of multiples of simple graphs later in this thesis, after we introduce some more advanced methods.

Chapter 4

Bounding chromatic index

The Seymour-Goldberg Conjecture asserts that

$$\chi'(G) \leq \max\{\lceil \rho(G) \rceil, \Delta + 1\}$$

for any multigraph G. In this chapter we work towards this conjecture by proving results of the form

$$\chi'(G) \leq \max\{\lceil \rho(G) \rceil, \Delta + t\}$$

for various values of t. For each such result that we prove, we will also provide a polynomial-time algorithm which will edge-colour any multigraph G with at most

$$\max\{\lceil \rho(G) \rceil, \Delta + t\}$$

colours. Moreover, each algorithm will detect if $\lceil \rho(G) \rceil > \Delta + t$, and in this case, will provide a "certificate" of chromatic index. That is, the algorithm will find a set $S \subseteq V(G)$, $|S| \geq 3$ and odd, such that

$$\chi'(G) = \lceil \rho(G) \rceil = \left\lceil \frac{2|E(G[S])|}{|S| - 1} \right\rceil = \chi'(G[S]).$$

As in the previous chapter, our work here all depends on the method of Tashkinov trees. In Section 4.1, we will see how Tashkinov trees are naturally suited to prove results towards the Seymour-Goldberg Conjecture. In fact, we will prove a general result of the form

$$\chi'(G) \leq \max\{\lceil \rho(G) \rceil, \Delta + t\},$$

given an assumption about Tashkinov trees in G. Then, in Section 4.2, we will use this general result to get specific new results of this desired form. The last section of this chapter, Section 4.3, is devoted to providing the corresponding algorithms that we have promised.

4.1 A general result

We have already learned a great deal about Tashkinov trees. At this point however, we need to make a few key observations in order to move forward. In the discussion that follows, G is a multigraph and ϕ is a partial $(\Delta + s)$-edge-colouring of G, for some $s \geq 1$. Let $T = (p_0, e_0, p_1, \dots, p_n)$ be a maximal ϕ-Tashkinov tree in G that is ϕ-elementary. We can think of the colours of ϕ being divided into two categories — those missing at a vertex in T, and those that are not. Our first observation is about the former.

Observation 1. *If $\alpha \in \phi(p_i)$, then α induces a perfect matching of $V(T) \setminus p_i$.*

To see this, note that since T is ϕ-elementary, $\alpha \in \phi(p_i)$ must be seen by every vertex in $V(T) \setminus p_i$. However, there cannot be any α-coloured edge leaving $V(T)$, because if there were, T could be extended via this edge, contradicting the maximality of T.

Observation 2. *$|V(T)|$ is odd and $|V(T)| \geq 3$.*

The fact that $|V(T)|$ is odd is immediate from Observation 1. The fact that $|V(T)| \geq 3$ is something that we have already mentioned — since T is ϕ-elementary, two ends of an uncoloured edge in G cannot have a common missing colour.

Observation 1 tells us about colours in

$$\mathcal{M}_{T,\phi} := \bigcup_{i=0}^{n} \phi(p_i).$$

We can also make an observation about the other colours in ϕ, $\overline{\mathcal{M}}_{T,\phi}$, as follows.

Observation 3. *If $\beta \in \overline{\mathcal{M}}_{T,\phi}$, then there is an odd number of β edges leaving $V(T)$.*

This observation simply comes from the fact that a colour $\beta \in \overline{\mathcal{M}}_{T,\phi}$ must be seen by every vertex in T and, by Observation 2, $|V(T)|$ is odd .

Analyzing the set of edges leaving $V(T)$ is a very important component of our work in this chapter. Clearly, any coloured edge leaving $V(T)$ must have a colour in $\overline{\mathcal{M}}_{T,\phi}$. Moreover, by Observation 3, every colour from this set must occur on at least one edge leaving $V(T)$. If a colour occurs on more than one edge leaving $V(T)$, then we say that the colour is *defective* (with respect to T and ϕ). The following proposition formalizes observations made by a number of other authors (see eg. [8], [35]).

Proposition 4.1.1. *Let G be a multigraph and let ϕ be a partial $(\Delta + s)$-edge-colouring of G, for some $s \geq 1$. Let T be a maximal ϕ-Tashkinov tree in G which is ϕ-elementary, and suppose that T has no defective colours. Then,*

$$\lceil \rho(G) \rceil \geq \Delta + s + 1.$$

If $\Delta + s = \chi'(G) - 1$, then moreover,

$$\chi'(G) = \lceil \rho(G) \rceil = \chi'(G[V(T)]).$$

Proof. By Observation 2, we know that $|V(T)|$ is odd, and $|V(T)| \geq 3$. Hence, by definition of ρ,

$$\rho(G) \geq \frac{2|E(G[V(T)])|}{|V(T)| - 1}. \tag{4.1}$$

Observation 1 tells us that every colour in $\mathcal{M}_{T,\phi}$ occurs on exactly $(|V(T)| - 1)/2$ edges in $G[V(T)]$. Since T has no defective colours, Observation 3 tells us that every colour in $\overline{\mathcal{M}}_{T,\phi}$ occurs on exactly one edge leaving $V(T)$. Since none of these colours are missing at any vertex in $V(T)$, this means that each occurs on exactly $(|V(T)|-1)/2$ edges in $G[V(T)]$ as well. In addition to the coloured edges in $G[V(T)]$, we also know that T contains one uncoloured edge. Hence,

$$|E(G[V(T])| \geq (\Delta + s) \left(\frac{|V(T)| - 1}{2} \right) + 1. \tag{4.2}$$

Now, combining equations (4.1) and (4.2), we get that

$$\begin{aligned} \lceil \rho(G) \rceil &\geq \left\lceil \frac{2 \left[(\Delta + s) \left(\frac{|V(T)|-1}{2} \right) + 1 \right]}{|V(T)| - 1} \right\rceil \\ &\geq \Delta + s + \left\lceil \frac{2}{|V(T)| - 1} \right\rceil . \\ &\geq \Delta + s + 1, \end{aligned}$$

as desired. If $\Delta + s = \chi'(G) - 1$, then this inequality becomes

$$\lceil \rho(G) \rceil \geq \chi'(G).$$

Since $\lceil \rho(G) \rceil$ is also a lower bound for chromatic index, this actually tells us that the two values are equal. Hence, the chromatic index of $G[V(T)]$ determines the chromatic index of the entire multigraph. □

Proposition 4.1.1 already gives a great indication of why Tashkinov trees are naturally suited to proving results towards the Seymour-Goldberg Conjecture. Of course, if our tree T does have defective colours, then we will have to do further work.

We have already used the language of *extending* T, by which we mean the process of adding an edge e_n and a vertex p_{n+1} such that $(p_0, e_0, p_1, \ldots, p_n, e_n, p_{n+1})$ is still a ϕ-Tashkinov tree, and possibly a repetition of this process. We have also already used the concept of a maximal ϕ-Tashkinov tree, where T cannot be extended any further. However, note that besides ϕ there may be other partial edge colourings ψ, with the same domain (and the same number of colours) as ϕ, such that T is also a ψ-Tashkinov tree. Call this set of colourings $\mathcal{D}$. If T is a maximal ψ-Tashkinov tree for all colourings $\psi \in \mathcal{D}$, then we say that T is a *domain-maximal* ϕ-Tashkinov tree. Let $\mathcal{S}$ be the set of colourings in $\mathcal{D}$ which can be obtained from ϕ via a sequence of colour-swaps along (maximal) alternating paths. If T is a maximal ψ-Tashkinov tree for all colourings $\psi \in \mathcal{S} \subset \mathcal{D}$, then we say that T is a *swap-maximal* ϕ-Tashkinov tree. The concept of swap-maximality will be of more practical use to us in this chapter; however, domain-maximality is an important idea as well.

When we are working with Tashkinov trees and want to perform a colour-swap along an alternating path in the multigraph, there is a set of colours that we need to be particularly aware of. Define $\mathcal{U}_{T,\phi}$ to be the set of all colours that are *used* on the edges of T under ϕ, i.e.,

$$\mathcal{U}_{T,\phi} := \bigcup_{i=1}^{n-1} \phi(e_i).$$

Note that property (T2) of Tashkinov trees implies that $\mathcal{U}_{T,\phi} \subseteq \mathcal{M}_{T,\phi}$. Moreover, since $|\mathcal{U}_{T,\phi}| \leq |V(T)| - 2$ and $|\mathcal{M}_{T,\phi}| \geq |V(T)|s + 2 \geq |V(T)| + 2$, we get that

$$\mathcal{U}_{T,\phi} \subset \mathcal{M}_{T,\phi}$$

(see also Claim 1 in Section 2.3.1). While we have already seen the importance of the set $\mathcal{M}_{T,\phi}$, there is a sense in which the colours of $\mathcal{U}_{T,\phi}$ are the only colours that matter for T. For example, suppose we do want to modify ϕ by swapping two colours along a (maximal) alternating path. If we know that the alternating path does not intersect T, then of course, we can make this swap and T will remain a ϕ-Tashkinov tree. However even if the alternating path does intersect T, as long as the two alternating colours are not in $\mathcal{U}_{T,\phi}$, this will still work. That is, T will still remain a ϕ-Tashkinov tree after the swap, because property (T2) does not depend on the two colours of the alternating path.

The idea of making colour-swaps without affecting a Tashkinov tree is at the core of the following lemma of Favrholt, Stiebitz and Toft [8], which gives us a great deal of information in the case that a Tashkinov tree has a defective colour. This result is a combination of Proposition 9.3 and a special case of Proposition 9.7 from [8], all stated here in a more general manner. The restriction to a special case simplifies the argument significantly, and for completeness, we include a proof here.

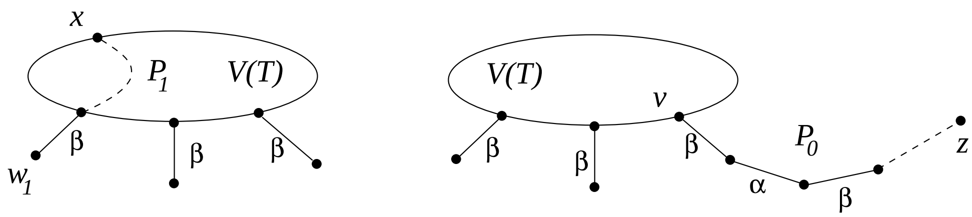

Figure 4.1: The paths P_0 and P_1

Proposition 4.1.2. [8] *Let G be a multigraph and let ϕ be a partial $(\Delta + s)$-edge-colouring of G, for some $s \geq 1$. Let T be a maximal ϕ-Tashkinov tree in G that is ϕ-elementary, and has a defective colour β. Let $\alpha \in \phi(x)$ where $x \in V(T)$ and $\alpha \notin \mathcal{U}_{T,\phi}$, and let P be the maximal (α, β)-alternating path beginning at x. Then, provided T is swap-maximal, all of the following statements hold:*

1. *P contains every β-edge that leaves T.*
2. *The last vertex v of P that is also on T is such that*
$$\phi(v) \subseteq \mathcal{U}_{T,\phi}.$$
3. *The first two vertices w_1, w_2 of P are such that*
$$V(T) \cup \{w_1, w_2\}$$
is ϕ-elementary.

Proof. If P does not contain every β edge that leaves T, then define ϕ' by swapping α and β along P. Note that since β is a defective colour and T is maximal, $\beta \notin \mathcal{M}_{T,\phi}$, and in particular $\beta \notin \mathcal{U}_{T,\phi}$. So, since $\alpha, \beta \notin \mathcal{U}_{T,\phi}$, we know that T is a ϕ'-Tashkinov tree. However now, $\beta \in \phi'(x)$, and T can be extended to a larger ϕ'-Tashkinov tree via a β-edge. This contradicts the fact that T is swap-maximal. So, P must in fact contain every β edge that leaves T, establishing (1).

Note that, since the number of β edges leaving $V(T)$ is odd (by Observation 3), (1) implies that P ends at some vertex $z \in V(G) \setminus V(T)$. Let P_0 be the reverse direction subpath of P that begins at z and ends at the first vertex v in this subpath that is on T. Let P_1 be the subpath of P that begins at x and ends at the first vertex w_1 in this subpath that is not in T. See Figure 4.1.

Suppose, for a contradiction to (2), that $\phi(v) \nsubseteq \mathcal{U}_{T,\phi}$. We want to define a new colouring ϕ' from ϕ where α and β are swapped along P_0. If $v = x$, then $\alpha \in \phi(v)$, so we can just define ϕ' to be identical to ϕ except for this swap, and it will be proper. If $v \neq x$, then choose some $\gamma \in \phi(v) \setminus \mathcal{U}_{T,\phi}$. Since T is ϕ-elementary, we know that $\alpha \neq \gamma$, so we may consider H, the (α, γ) subgraph induced by $V(T)$. Note that by Observation 1, there are no α or γ edges leaving $V(T)$. So, we can define ϕ' to be the proper colouring obtained from ϕ by swapping α and β along P_0, and by swapping α and γ on H. Now, regardless of which definition of ϕ' applies, we know that T is a ϕ'-Tashkinov tree, since $\alpha, \beta, \gamma \notin \mathcal{U}_{T,\phi}$. Moreover, ϕ' has the same domain and the same number of colours as ϕ. However, $\beta \in \phi'(v)$, and there are at least two β-edges leaving $V(T)$ under ϕ' (the fact that β was defective meant that there were at least three β edge leaving $V(T)$ under ϕ, by Observation 3). So, T can be extended to a larger ϕ'-Tashkinov tree, contradicting the fact that T is swap-maximal.

It remains now to prove (3). We start by proving the following:

(a). $V(T) \cup \{w_1\}$ is ϕ-elementary.

Since $V(T)$ is ϕ-elementary, if this does not hold then there exists $\varepsilon \in \phi(w_1) \cap \phi(y)$ for some $y \in V(T)$. Clearly, $\varepsilon \neq \alpha, \beta$, since w_1 is incident to both an α-edge and a β-edge. Let Q be the maximal (α, ε)-alternating path beginning at w_1. Since $\varepsilon, \alpha \in \mathcal{M}_{T,\phi}$ and since T is maximal, $V(Q) \cap V(T) = \emptyset$. So, if we define ϕ_1 by swapping ε and α along Q, we get that T is a ϕ_1-Tashkinov tree. However now, P_1 is the maximal (α, β)-alternating path starting from x. Since P_1 stops at w_1, and hence does not include all β-edges leaving T, this contradicts (1) applied to ϕ_1. So, we have established (a).

We now prove the following second intermediate statement:

(b). $\{w_1\} \cup \{w_2\}$ is ϕ-elementary.

If (b) is not true, then there exists some $\varepsilon \in \phi(w_1) \cap \phi(w_2)$. In this case, define ϕ_2 by recolouring the α-edge joining w_1 and w_2 with ε. However now, $\alpha \in \phi_2(w_1)$, so we get a contradiction to (a) applied to ϕ_2. Hence, (b) is established.

Now, by (a) and (b), the only way that (1) does not hold is if there exists some $\varepsilon \in \phi(w_2) \cap \phi(y)$ for some $y \in V(T)$. Clearly, $\varepsilon \neq \alpha, \beta$, because w_2 is incident to both an α-edge and a β-edge. We may assume that $\varepsilon \notin \mathcal{U}_{T,\phi}$. This is because if not, then we may choose a colour $\varepsilon' \in \mathcal{M}_{T,\phi} \setminus \mathcal{U}_{T,\phi}$ with $\varepsilon' \neq \alpha$ (since $|\mathcal{U}_{T,\phi}| \leq |V(T)| - 2$). We then define ϕ_3 by swapping ε and ε' along the maximal $(\varepsilon, \varepsilon')$-alternating path beginning at w_2. Since $\varepsilon, \varepsilon' \in \mathcal{M}_{T,\phi}$, and T is maximal with respect to ϕ, this path does not intersect $V(T)$, and hence T is a ϕ_3-Taskinov tree. However now, we may use $\varepsilon' \notin \mathcal{U}_{T,\phi}$ in place of ε. Hence, our assumption is valid.

Now, choose $\tau \in \phi(w_1)$. Clearly, $\tau \neq \alpha, \beta$, and by (b) we know that $\tau \neq \varepsilon$. By (a) we also know that $\tau \notin \mathcal{M}_{T,\phi}$, and in particular, $\tau \notin \mathcal{U}_{T,\phi}$. Let R be the maximal (ε, τ)-alternating path beginning at w_1, and define ϕ_4 by swapping τ and ε along R. Since $\tau, \varepsilon \notin \mathcal{U}_{T,\phi}$, T is a ϕ_4-Tashkinov tree. If R does not end at w_2, then $\varepsilon \in \phi_4(w_1) \cap \phi_4(w_2)$, contradicting (b). So, it must be the case that R ends at w_2. However, this means that $\varepsilon \in \phi_4(w_1) \cap \phi_4(y)$, contradicting (a). Hence, we have established our desired result. □

Together, propositions 4.1.1 and 4.1.2 enable us to prove the following extremely useful theorem.

Theorem 4.1.3. *Let G be a multigraph, and let $\tau \geq 3$ be an integer. Suppose that there exists a ϕ-elementary ϕ-Tashkinov tree with at least τ vertices, for ϕ some partial $(\chi' - 1)$-edge-colouring of G. Then,*

$$\chi'(G) \leq \max\left\{\lceil \rho \rceil,\ \ \Delta + 1 + \frac{\Delta - 3}{\tau + 1}\right\}.$$

Proof. We first resolve the case $(\chi' - 1) < \Delta + 1$. Here, the theorem holds immediately unless $\frac{\Delta-3}{\tau+1} < 0$, that is, unless $\Delta \leq 2$. If $\Delta = 1$, then G is a matching, so $\chi'(G) = \Delta = 1$. Since $\Delta+1+\frac{\Delta-3}{\tau+1} \geq 1+1+\frac{-2}{4} \geq 1$, the theorem does hold in this case. If, on the other hand, $\Delta = 2$, then either $\chi'(G) = \Delta = 2$, or G contains an odd cycle and $\chi'(G) = 3$. If $\chi'(G) = \Delta = 2$, then $\Delta+1+\frac{\Delta-3}{g_0+1} \geq 2+1+\frac{-1}{4} \geq 2$, so the theorem holds. If G contains an odd cycle on g_o vertices, then $\lceil \rho(G) \rceil \geq \lceil 2(g_0)/(g_0 - 1) \rceil = 3$, so the theorem also holds in this case.

We may now assume that $(\chi' - 1) \geq \Delta + 1$. Let T be a maximal ϕ-elementary ϕ-Tashkinov tree in G with at least τ vertices, where ϕ is a partial $(\chi' - 1)$-edge-colouring of G. If T is not swap-maximal, then modify ϕ via colour-swaps and keep extending T until it is swap-maximal. If T has no defective colours, then Proposition 4.1.1 immediately tells us that $\chi'(G) = \lceil \rho(G) \rceil$. So, we may assume that there exists a defective colour $\beta \notin \mathcal{M}_{T,\phi}$.

Since $|\mathcal{U}_{T,\phi}| < |\mathcal{M}_{T,\phi}|$, we may choose $\alpha \in \phi(x)$ for some $x \in V(T)$ such that $\alpha \notin \mathcal{U}_{T,\phi}$. Let P be the maximal (α, β)-alternating path starting at x. Proposition 4.1.2 says that P must contain all the β-edges leaving T, and that the first two vertices on P not on T (say w_1, w_2), have the property that

$$V(T) \cup \{w_1, w_2\}$$

is ϕ-elementary. Note that $\beta \notin \phi(w_1) \cup \phi(w_2)$, since both vertices are incident to β-edges. So, we get that

$$|\mathcal{M}_{T,\phi}| + |\phi(w_1)| + |\phi(w_2)| + 1 \leq \chi' - 1. \tag{4.3}$$

Every vertex in T has at least $(\chi'-1)-\Delta$ missing colours. Since the first two vertices on T are adjacent to an uncoloured edge, they have at least one more missing colour each. So, Equation (4.3) implies that

$$\begin{aligned} & \big[|V(T)|(\chi'-1-\Delta)+2\big]+2(\chi'-1-\Delta)+1 \le \chi'-1 \\ \Rightarrow\quad & (|V(T)|+1)(\chi'-1-\Delta)+(\chi'-1-\Delta)+3 \le \chi'-1 \\ \Rightarrow\quad & (|V(T)|+1)(\chi'-1-\Delta) \le \Delta-3 \\ \Rightarrow\quad & \chi' \le \Delta+1+\frac{\Delta-3}{|V(T)|+1}, \end{aligned}$$

as desired. □

4.2 Specific new results

Using Theorem 4.1.3, we can see that we "only" need to build large Tashkinov trees in order to get results towards the Seymour-Goldberg Conjecture. In this section, we demonstrate four ways to guarantee the size of a Tashkinov tree: using odd girth, using girth, using extremal graph theory, and by controlling the colours used on the tree.

We have already seen a Tashkinov tree construction using odd girth — in particular, we used it in Theorem 3.2.2 to characterize Goldberg's upper bound of

$$\chi'(G) \le \Delta+1+\frac{\Delta-2}{g_o-1},$$

which applies to all multigraphs G containing an odd cycle. Since we found that a connected G achieves this upper bound if and only if $G=\mu C_{g_o}$ and $(g_o-1)|(2\mu-2)$, and we know from the proof of Theorem 3.1.2 that $\chi'(\mu C_{g_o}) = \lceil \rho(C_{g_o})\rceil$, Theorem 3.2.2 actually implies that

$$\chi'(G) \le \max\left\{\lceil \rho(G)\rceil,\ \ \Delta+1+\frac{\Delta-3}{g_o-1}\right\}.$$

Now, by using Theorem 4.1.3, and by tweaking our previous construction slightly, we are able to improve this denominator and get the following result.

Theorem 4.2.1. *Let G be a multigraph which contains an odd cycle. Then,*

$$\chi'(G) \le \max\left\{\lceil \rho(G)\rceil,\ \ \Delta+1+\frac{\Delta-3}{g_o+3}\right\}.$$

Proof. Note first that if $(\chi' - 1) < \Delta + 1$, then the theorem holds immediately unless $\frac{\Delta-3}{g_o+3} < 0$, that is, unless $\Delta \leq 2$. As we saw in the proof of Theorem 4.1.3, if $\Delta = 1$ then G is a matching, so $\chi'(G) = \Delta = 1$, and the theorem holds. If $\Delta = 2$, then either $\chi'(G) = \Delta = 2$, or G contains an odd cycle and $\chi'(G) = 3$. In both cases, the same reasoning used in the general proof shows that our theorem holds.

Now we may assume that $(\chi' - 1) \geq \Delta + 1$. Let ϕ be a partial $(\chi' - 1)$-edge colouring of G with maximum domain, and let e_0 be an uncoloured edge in G with ends p_0 and p_1. Choose $\alpha \in \phi(p_0)$ and $\beta \in \phi(p_1)$. Since ϕ has maximum domain, $\alpha \neq \beta$ and there exists an (α, β)- alternating path joining p_1 and p_0. Together with e_0, this forms an odd cycle with at least g_0 vertices. By eliminating any one of the edges in this cycle (except for e_0), we create a ϕ-Tashkinov tree T with at least g_0 vertices. By Tashkinov's Theorem, we know that T must be ϕ-elementary, because ϕ has maximum domain.

Suppose first that that T is swap-maximal. Note that by Proposition 4.1.1, we may also assume that T has a defective colour. So, since $|\mathcal{U}_{T,\phi}| < |\mathcal{M}_{T,\phi}|$, Proposition 4.1.2(2) implies that there exists $v \in V(T)$ such that $\phi(v) \subseteq \mathcal{U}_{T,\phi}$. However, by our choice of T, we know that $\mathcal{U}_{T,\phi} = \{\alpha, \beta\}$. We know that $(\chi' - 1) \geq \Delta + 1$, so every vertex in T has at least $(\chi' - 1) - \Delta \geq 1$ missing colours. Since p_0 and p_1 are incident to e_0, they have at least 2 missing colours each. However, since T is ϕ-elementary, α and β are only missing at p_0 and p_1, respectively, contradicting the existence of v.

We may now assume that T is not swap-maximal. So, modify ϕ via colour-swaps and keep extending T until it is swap-maximal. Clearly, since the domain of ϕ did not change, Tashkinov's Theorem still assures us that T is ϕ-elementary. By definition of swap-maximal though, we know that we have increased the size of $V(T)$. Observation 2 tells us that $|V(T)|$ must be odd, so we know that $|V(T)| \geq g_0 + 2$. Hence, our desired result follows by an application of Theorem 4.1.3. □

Note that as Theorem 3.2.2 characterizes those multigraphs with

$$\chi'(G) > \Delta + 1 + \frac{\Delta - 3}{g_o - 1},$$

we will be able to characterize those multigraphs with

$$\chi'(G) > \Delta + 1 + \frac{\Delta - 3}{g_o + 3}.$$

However, such a characterization is not immediate from our proof above, and indeed we will use some other techniques in order to obtain it. This characterization and other similar results are the main content of the next chapter.

We now move on to our next Tashkinov tree construction, using girth. This construction actually begins the same way as the construction we just saw, but we are essentially able to use two cycles for our argument instead of just one.

Theorem 4.2.2. *Let G be a multigraph which contains a cycle. Then,*

$$\chi'(G) \leq \max\left\{\lceil \rho(G) \rceil,\ \ \Delta + 1 + \frac{\Delta - 3}{\lfloor 3g/2 \rfloor + 2}\right\}.$$

Proof. Note first that if $(\chi' - 1) < \Delta + 1$, then the theorem holds immediately unless $\frac{\Delta-3}{\lfloor 3g/2\rfloor+1} < 0$, that is, unless $\Delta \leq 2$. As we saw in the proof of Theorem 4.1.3, if $\Delta = 1$ then G is a matching, so $\chi'(G) = \Delta = 1$, and the theorem holds. If $\Delta = 2$, then either $\chi'(G) = \Delta = 2$, or G contains an odd cycle and $\chi'(G) = 3$. In both cases, the same reasoning used in the general proof shows that our theorem holds.

We may now assume that $(\chi' - 1) \geq \Delta + 1$. Let ϕ be a partial $(\chi' - 1)$-edge colouring of G with maximum domain, and let e_0 be an uncoloured edge in G with ends p_0 and p_1. As we saw above, we may choose $\alpha \in \phi(p_1)$ and $\beta \in \phi(p_2)$ such that there exists an (α, β)-alternating path joining p_1 and p_0. Let C_0 be the cycle formed by this path, together with e_0. Let T be one of the ϕ-elementary ϕ-Tashkinov trees created by deleting one edge (not e_0) from C_0.

If T is swap-maximal, then we can use the same argument presented in the previous proof to get a contradiction. Hence, we may assume that T is not swap-maximal. So, we are able to modify ϕ via colour-swaps until we get that T is not maximal, i.e., that T can be extended. At this point, this means that there exists a colour γ that is missing at a vertex on T, but that appears on an edge leaving the set $V(T)$. Let x_1 be a vertex on T incident to such an edge. Choose $\varepsilon \in \phi(x_1)$, and let P be the maximal (γ, ε)-alternating path beginning at x_1.

If $V(P) \cap V(T) = \{x_1\}$, then we can clearly add P to T and get a ϕ-Tashkinov tree. However, the last vertex in P must be missing either ε or γ. So, this tree would not be ϕ-elementary, contradicting the fact that ϕ has maximum domain (by Tashkinov's Theorem). So, there exists a subpath $P' \subseteq P$ that starts at x_1 and ends at a vertex x_2 on T. In particular, this means that we get a ϕ-elementary ϕ-Tashkinov tree by adding P', less one edge, to T. The vertex set of this new tree is $V(C_0) \cup V(P')$. It remains only to find the cardinality of this set.

Note that the two ends of P' are at distance at most $(|C_0| - 1)/2$ on C_0, since C_0 is an odd cycle (as in the previous proof). So, P' must have length at least $g - (|C_0| - 1)/2$, in order not to create a cycle of order less than g. Hence,

$$|V(C_0 \cup P')| \geq |C_0| + g - \left(\frac{|C_0| - 1}{2}\right) = \frac{|C_0| + 1 + 2g}{2}.$$

We know that $|C_0| \geq g_0 \geq g$. Of course, if g is even, we know that $|C_0| \geq g + 1$. So, the above equation tells us that

$$|V(C_0 \cup P')| \geq \left\lfloor \frac{3g}{2} \right\rfloor + 1.$$

Applying Theorem 4.1.3, we get our desired result. □

The proofs of Theorem 4.2.1 and 4.2.2 have several nice corollaries that are worth presenting before we go on. First, we have the following.

Corollary 4.2.3. *Let G be a multigraph (that is not bipartite). Then, the Seymour-Goldberg Conjecture is true for G, provided that either of the following conditions hold.*

1. $g_o \geq \Delta - 5$
2. $g \geq \frac{2\Delta-7}{3}$

Proof. To see (1), we look to Theorem 4.2.1. This says that the Seymour-Goldberg Conjecture holds for G, provided that $\frac{\Delta-3}{g_o+3} < 1$, or equivalently, $g_o \geq \Delta - 5$. Similarly, to see (2), we look to Theorem 4.2.2. This says that the Seymour-Goldberg Conjecture holds for G, provided that $\frac{\Delta-3}{\lfloor 3g/2 \rfloor+2} < 1$. This condition is equivalent to $\lfloor 3g/2 \rfloor + 2 > \Delta - 3$, i.e., $\lfloor 3g/2 \rfloor \geq \Delta - 4$. If g is even, this condition translates to $3g/2 \geq \Delta - 4$, or equivalently, $g \geq \frac{2\Delta-8}{3}$. On the other hand, if g is odd, this condition translates to $\frac{3g-1}{2} \geq \Delta - 4$, or equivalently, $g \geq \frac{2\Delta-7}{3}$. Clearly, by assuming this second inequality, we get that both inequalities hold, so we have established (2). □

We already saw that Theorem 4.2.1 is an improvement of Goldberg's bound in one sense. Now we can also note that while Goldberg's bound implies that the Seymour-Goldberg Conjecture holds for $g_o \geq \Delta$, we have made an improvement on this condition with Corollary 4.2.3 (1).

Recall that Steffen's Theorem says that, for multigraphs containing a cycle,

$$\chi' \leq \Delta + \left\lceil \frac{\mu}{\lfloor g/2 \rfloor} \right\rceil .$$

So if $\frac{\mu}{\lfloor g/2 \rfloor} \leq 1$, then we know that $\chi' \leq \Delta + 1$. If g is even, this condition translates to $g \geq 2\mu$, and if g is odd this condition translates to $g \geq 2\mu + 1$. So, overall, this condition is equivalent to $g \geq 2\mu$. Hence, Steffen's Theorem implies that the Seymour-Goldberg Conjecture is true when $g \geq 2\mu$. Condition (2) is a strict improvement of this if and only if $\frac{2\Delta-7}{3} < 2\mu$, or equivalently, $\Delta \leq 3\mu + 3$. If G is a multiple of a simple graph, then this only applies when G is a 2-multiple or a 3-multiple. However, if G is not a multiple of a simple graph, then there are many examples where $\Delta \leq 3\mu + 3$.

There is another family of multigraphs for which the Seymour-Goldberg Conjecture is known to hold, but which we have not mentioned yet in this thesis -

multigraphs on a small number of vertices. In 1997, Plantholt and Tipnis [31] completely determined the chromatic index of all multigraphs on at most 10 vertices, and showed that all multigraphs of this size satisfied the Seymour-Goldberg Conjecture. Plantholt and Tipnis' approach was a direct case analysis which did not involve Tashkinov trees. However, the proofs of Theorems 4.2.1 and 4.2.2 also have implications for multigraphs with few vertices. Namely, we can prove the following.

Proposition 4.2.4. *Let G be a multigraph (that is not bipartite). Then, the Seymour-Goldberg Conjecture is true for G, provided that either of the following conditions hold.*

1. $|V(G)| \leq g_o + 4$
2. $|V(G)| \leq \lfloor 3g/2 \rfloor + 3$

Proof. Suppose G is a counterexample to the Seymour-Goldberg Conjecture. Then, obviously, $\chi'(G) > \Delta+1$ and $\chi'(G) \neq \lceil \rho(G) \rceil$. However, with these two assumptions, the proof of Theorem 4.2.1 shows that there exists ϕ a partial $(\chi'-1)$-edge colouring of G with maximum domain, and T a ϕ-elementary ϕ-Tashkinov tree with $|V(T)| \geq g_o + 2$. If this tree is not swap-maximal, then make the necessary colour swaps and extensions to ϕ and T so that T is swap-maximal. Since $\chi'(G) \neq \lceil \rho(G) \rceil$, Proposition 4.1.1 tells us that T must have a defective colour. This means, by Observation 3, that there are at least three vertices in $V(G) \setminus V(T)$. Hence, $|V(G)| \geq g_o + 5$.

To see (2), we again assume that G is a counterexample to the Seymour-Goldberg Conjecture. Then, the proof of Theorem 4.2.1 tells us that there exists ϕ a partial $(\chi' - 1)$-edge colouring of G with maximum domain, and T a ϕ-elementary ϕ-Tashkinov tree with $|V(T)| \geq \lfloor 3g/2 \rfloor + 1$. Now, we can repeat the argument that we have just presented, to get that $|V(G)| \geq |V(T)| + 3 \geq \lfloor 3g/2 \rfloor + 4$, as desired. □

We now examine a dramatically different method of constructing large Tashkinov trees. This technique does not involve girth or odd-girth, but instead involves the colours used on the edges of a tree. Before we say more, there is an important observation we should make about maximal Tashkinov trees: given a multigraph G, a partial $(\Delta + s)$-edge colouring ϕ of G (with $s \geq 1$), and an edge e_0 that is uncoloured by ϕ, every maximal ϕ-Tashkinov tree starting with e_0 has the same vertex set. The order of such a tree may differ, but the vertex set will always be the same. To see this, think about building a maximal ϕ-Tashkinov tree one edge at a time. If we have built T and are looking to extend it, our options for extension depend only on the vertex set of T (including their set of missing colours), not on the edges of T. So, no matter how we choose the edges, we end up with the same vertex set. The following lemma tells us that there is always a way to choose our edges so that a single colour appears on every second edge.

Lemma 4.2.5. *Let G be a multigraph and let ϕ be a partial $(\Delta + s)$-edge colouring of G, where $s \geq 1$. Let T be a maximal ϕ-Tashkinov tree that is ϕ-elementary. Suppose that the colour α is used on the first coloured edge of T. Then, there exists a ϕ-Tashkinov tree T' with $V(T') = V(T)$ and $(|V(T)| - 1)/2$ of its edges coloured α, so*

$$|\mathcal{U}_{T',\phi}| \leq \frac{|V(T)| - 1}{2}.$$

Proof. Let $T = (p_0, e_0, p_1, \ldots, p_n)$, with $\alpha = \phi(e_1)$. We build T' one edge at a time, starting with $T' = (p_0, e_0, p_1, e_1, p_2)$. We only add edges from $G[V(T)]$ to T', so as to maintain $V(T') \subseteq V(T)$. If $V(T') \subset V(T)$ then we know, by the uniqueness of the vertex set of a maximal Tashkinov tree, that T' can be extended in this way. Moreover, if $|V(T')|$ is even, then every colour in $\mathcal{M}_{T',\phi}$ must appear between $V(T')$ and $V(T) \setminus V(T')$ (by Observation 1). Hence, every second time we add an edge to T', we can choose an edge coloured α. In the end, T' will have $|V(T')| - 2$ coloured edges, with $|V(T')|$ odd. Since e_1 is coloured α, this gives us our desired result. □

The following lemma shows how Lemma 4.2.5 can be used to build a large Tashkinov tree.

Lemma 4.2.6. *Let G be a multigraph and let ϕ be a partial $(\Delta + s)$-edge-colouring of G, for some $s \geq 1$. Then, any swap-maximal ϕ-elementary ϕ-Tashkinov tree in G, that has a defective colour, must have at least $2s + 3$ vertices.*

Proof. Let T_0 be a swap-maximal ϕ-elementary ϕ-Tashkinov tree in G that has a defective colour. By Lemma 4.2.5, we know that we can find a ϕ-Tashkinov tree $T = (p_0, e_0, p_1, \ldots, p_n)$ in G with $|\mathcal{U}_{T,\phi}| \leq (|V(T)| - 1)/2$ and with $V(T) = V(T_0)$. Note that this last property ensures that T is also a swap-maximal ϕ-elementary ϕ-Tashkinov tree, and has a defective colour.

Since $|\mathcal{U}_{T,\phi}| < |\mathcal{M}_{T,\phi}|$, Proposition 4.1.2 (2) implies us that there exists $v \in V(T)$ such that $\phi(v) \subseteq \mathcal{U}_{T,\phi}$. If v is p_0 or p_1, then we know that it is incident to at least one uncoloured edge, e_0, and hence

$$|\phi(v)| \geq (\Delta + s) - (\Delta - 1) = s + 1.$$

Since $|\mathcal{U}_{T,\phi}| \leq (|V(T)| - 1)/2$, this implies that

$$s + 1 \leq \frac{|V(T)| - 1}{2}. \tag{4.4}$$

If, on the other hand, v is not p_0 or p_1, then we can only say that $|\phi(v)| \geq s$. However, we know that e_1 must be coloured with a colour $\alpha \in \phi(p_0)$. Since T is ϕ-elementary, this means that $\alpha \notin \phi(v)$. So, there us at least one colour in $\mathcal{U}_{T,\phi}$ which is not in $\phi(v)$. Hence, we get that

$$s \leq \frac{|V(T)| - 1}{2} - 1.$$

So, in either case, we know that inequality (4.4) holds. Rearranging, we get that

$$2s + 3 \leq |V(T)|,$$

as desired. □

Note the following immediate corollary to Lemma 4.2.6. This corollary was independently proved by Scheide in [35].

Corollary 4.2.7. [35] *Let G be a multigraph and let ϕ be a partial $(\chi' - 1)$-edge-colouring of G. Then, any swap-maximal ϕ-elementary ϕ-Tashkinov tree in G, that has a defective colour, must have at least $2(\chi' - \Delta) + 1$ vertices.*

Scheide's proof of Corollary 4.2.7 is only slightly different than that which we have presented here — rather than using a tree with many edges of the same colour, like we have, he used a tree which, after a certain point, had coloured edges appearing in pairs. Scheide used Corollary 4.2.7 to prove that

$$\chi' \leq \max\left\{\lceil \rho \rceil, \;\; \Delta + \sqrt{\frac{\Delta}{2}}\right\}.$$

In fact, when we apply Theorem 4.1.3 to Lemma 4.2.6, we get the following very small improvement of this result (which was also independently noted by Scheide [36]).

Theorem 4.2.8. *Let G be a multigraph. Then,*

$$\chi'(G) \leq \max\left\{\lceil \rho(G) \rceil, \;\; \Delta + \sqrt{\frac{\Delta - 1}{2}}\right\}.$$

Proof. Note first that if $(\chi' - 1) < \Delta + 1$, then the theorem holds immediately unless $\sqrt{(\Delta - 1)/2} < 1$, that is, unless $\Delta \leq 2$. As we saw in the proof of Theorem 4.1.3, if $\Delta = 1$ then G is a matching with $\chi'(G) = \Delta = 1$, and if $\Delta = 2$, then either $\chi'(G) = \Delta = 2$, or G contains an odd cycle and $\chi'(G) = 3$. The $\Delta = 1$ case is resolved because here, $\Delta + \sqrt{(\Delta - 1)/2} = 1$, and the $\chi'(G) = \Delta = 2$ case is resolved because here, $\Delta + \sqrt{(\Delta - 1)/2} \geq 2$. If G contains an odd cycle, then, as we saw in the proof of Theorem 4.1.3, $\lceil \rho(G) \rceil \geq 3$, so the theorem holds in this case as well.

We may now assume that $(\chi' - 1) \geq \Delta + 1$. Let T be any swap-maximal ϕ-Tashkinov tree, where ϕ is a partial $(\chi' - 1)$-edge colouring of G with maximum domain. By Tashkinov's Theorem, we know that T is ϕ-elementary. Moreover, by Proposition 4.1.1, we may assume that T has a defective colour. Hence, by Corollary 4.2.7, we know that $|V(T)| \geq 2(\chi' - \Delta) + 1$. Applying Theorem 4.1.3, we get that

$$\chi' \leq \max\left\{\lceil \rho(G) \rceil, \;\; \Delta + 1 + \frac{\Delta - 3}{2(\chi' - \Delta) + 2}\right\}.$$

Of course,

$$\begin{aligned} \chi' \leq \Delta + 1 + \frac{\Delta - 3}{2(\chi' - \Delta) + 2} \;\; &\Leftrightarrow \;\; (\chi' - \Delta) - 1 \leq \frac{\Delta - 3}{2(\chi' - \Delta) + 2} \\ &\Leftrightarrow \;\; 2(\chi' - \Delta)^2 - 2 \leq \Delta - 3 \\ &\Leftrightarrow \;\; 2\chi'^2 - 4\chi'\Delta + (2\Delta^2 - \Delta + 1) \leq 0. \end{aligned}$$

When we use the quadratic formula on this last expression, we get

$$\begin{aligned} \chi' &\leq \frac{4\Delta + \sqrt{(4\Delta)^2 - 4(2)(2\Delta^2 - \Delta + 1)}}{2(2)} \\ &= \Delta + \sqrt{\frac{16\Delta^2 - 16\Delta^2 + 8\Delta - 8}{16}} \\ &= \Delta + \sqrt{\frac{\Delta - 1}{2}}, \end{aligned}$$

as desired. □

Our fourth and final method of constructing large Tashkinov trees relies on a result from extremal graph theory. In fact, we also need to appeal to Lemma 4.2.7 to deal with one case of our construction. As a result of this, we get the following more complicated-looking bound — however its result is strong. The bound says that if $\chi' \neq \lceil \rho \rceil$, then chromatic index is extremely low in terms of g, or on the order of $\Delta + \sqrt{\mu}$, which is very low in terms of Vizing's Theorem.

Theorem 4.2.9. *Let G be a multigraph which contains a cycle. Then,*

$$\chi'(G) \leq \max\left\{\lceil \rho(G) \rceil, \;\; \Delta + \frac{1}{2} + \sqrt{\mu - \frac{3}{4}}, \;\; \Delta + 1 + \frac{\Delta - 3}{f(g)}\right\}$$

where

$$f(g) = \begin{cases} 3 \cdot 2^{\lfloor g/2 \rfloor} - 1 & \text{if } g \text{ is odd,} \\ 2^{\lfloor g/2 \rfloor} & \text{if } g \text{ is even.} \end{cases}$$

Proof. Assume that

$$\chi'(G) > \max\left\{\Delta + \frac{1}{2} + \sqrt{\mu - \frac{3}{4}},\ \ \Delta + 1 + \frac{\Delta - 3}{f(g)}\right\}.$$

Note that since $\mu \geq 1$,

$$\chi'(G) > \Delta + \frac{1}{2} + \sqrt{\mu - \frac{3}{4}} \geq \Delta + \frac{1}{2} + \sqrt{\frac{1}{4}} = \Delta + 1.$$

Let T be a swap-maximal ϕ-Tashkinov tree, where ϕ is a $(\chi' - 1)$-edge colouring of maximum domain. By Tashkinov's Theorem, we know that T must be ϕ-elementary.

We examine the underlying graph of $G[V(T)]$. If this underlying graph has minimum degree at least three, then by a result in extremal graph theory (see Bollobás [5], pg. 105), we know that

$$|V(T)| \geq \begin{cases} 3 \cdot 2^{\lfloor g/2 \rfloor} - 2 & \text{if } g \text{ is odd,} \\ 2^{\lfloor g/2 \rfloor} - 1 & \text{if } g \text{ is even.} \end{cases}$$

So, in this case, we get that $|V(T)| \geq f(g) - 1$. Since we have assumed that

$$\chi'(G) > \Delta + 1 + \frac{\Delta - 3}{f(g)},$$

Theorem 4.1.3 tells us that $\chi'(G) = \lceil \rho(G) \rceil$, as desired.

Now, suppose that the minimum degree of $G[V(T)]$ is at most 2. Since T is ϕ-elementary, this means that there is some vertex $v \in V(T)$ that has only two neighbours v_1, v_2 in $V(T)$, but is incident to at least $|\mathcal{M}_{T,\phi} \setminus \phi(v)|$ edges in T. This means that

$$|\mathcal{M}_{T,\phi} \setminus \phi(v)| \leq 2\mu,$$

and so

$$(|V(T)| - 1)(\chi' - \Delta - 1) + 2 \leq 2\mu,$$

and so

$$\chi' \leq \Delta + 1 + \frac{2(\mu - 1)}{|V(T)| - 1}.$$

We may assume that T has a defective colour, by Proposition 4.1.1. So, Lemma 4.2.7 tells us that $|V(T)| \geq 2(\chi' - \Delta) + 1$. Substituting this in to the above inequality, we get that:

$$\begin{aligned}
&\chi' \leq \Delta + 1 + \frac{2(\mu - 1)}{2(\chi' - \Delta)}\\
\Rightarrow\ &\chi'(\chi' - \Delta) \leq (\Delta + 1)(\chi' - \Delta) + (\mu - 1)\\
\Rightarrow\ &(\chi')^2 + \chi'(-2\Delta - 1) + (\Delta^2 + \Delta - \mu + 1) \leq 0\\
\Rightarrow\ &\chi' \leq \frac{2\Delta + 1 + \sqrt{(-2\Delta - 1)^2 - 4(1)(\Delta^2 + \Delta - \mu + 1)}}{2(1)}\\
\Rightarrow\ &\chi' \leq \Delta + \frac{1}{2} + \sqrt{\frac{4\Delta^2 + 4\Delta + 1 - 4(\Delta^2 + \Delta - \mu + 1)}{4}}\\
\Rightarrow\ &\chi' \leq \Delta + \frac{1}{2} + \sqrt{\mu - \frac{3}{4}}.
\end{aligned}$$

This violates our original assumption about the value of $\chi'(G)$, hence we have completed our proof. □

4.3 Corresponding colouring algorithms

When Favrholt, Steibitz and Toft proved

$$\chi' \leq \max\left\{\lceil\rho\rceil,\ \ \Delta + 1 + \frac{\Delta - 2}{m - 1}\right\}$$

for $m = 13$, and Scheide proved this for $m = 15$, they provided corresponding polynomial-time $\max\left\{\lceil\rho\rceil,\ \ \Delta + 1 + \frac{\Delta-2}{m-1}\right\}$-edge-colouring algorithms. Moreover, when applied to a multigraph G, their algorithms detected if

$$\lceil\rho(G)\rceil > \Delta + 1 + \frac{\Delta - 2}{m - 1},$$

and in this case, provided a set $S \subseteq V(G)$, $|S| \geq 3$ and odd, such that

$$\chi'(G) = \lceil\rho(G)\rceil = \left\lceil\frac{2E(G[S])}{|S| - 1}\right\rceil = \chi'(G[S]).$$

We now adhere to this standard by presenting such an algorithm for each of our four main results in the previous section.

We proved all of our results in Section 4.2 using Theorem 4.1.3. So now, it makes sense to provide the general framework for an algorithm to accompany Theorem 4.1.3, and then fill in the details for each of our 4 specific results — Theorems 4.2.1, 4.2.2, 4.2.8, and 4.2.9. The input and output of the general algorithm is as follows. (Note that we assume our input multigraph has $\Delta \geq 3$ — we will discuss the cases $\Delta = 1$ and 2 separately.)

A general $\max\left\{\lceil\rho\rceil, \Delta + 1 + \frac{\Delta-3}{\tau+1}\right\}$**-Edge Colouring Algorithm**

INPUT:

- A multigraph G with $\Delta \geq 3$

OUTPUT:

- A $\max\left\{\lceil\rho\rceil, \Delta + 1 + \left\lfloor\frac{\Delta-3}{\tau+1}\right\rfloor\right\}$-edge colouring of G.
- The answer to "Is $\lceil\rho(G)\rceil > \Delta + 1 + \left\lfloor\frac{\Delta-3}{\tau+1}\right\rfloor$?"
  If the answer is yes, then we get a set $S \subseteq V(G)$, $|S| \geq 3$ and odd, such that
  $$\chi' = \lceil\rho\rceil = \left\lceil\frac{2E(G[S])}{|S|-1}\right\rceil = \chi'(G[S]).$$

Of course, we cannot provide an algorithm with the above input and output for every value of τ. We will give the basic steps of the algorithm for a general τ, and then later fill in the specific details corresponding to the bounds of Theorems 4.2.1, 4.2.2, 4.2.8, and 4.2.9.

We start the general algorithm by initializing:

$S := \emptyset$,
$k := \Delta + 1 + \left\lfloor\frac{\Delta-3}{\tau+1}\right\rfloor$, and
ϕ a partial k-edge colouring of G (which may be an empty colouring).

Then, the algorithm is iterative, where in each iteration we colour one more edge, until the whole multigraph is coloured. An iteration proceeds as follows. (Note that that the occurrence of the word *somehow* below is what will need to be justified to get each of the four specific algorithms we seek).

1. If ϕ colours all of G, then stop and output ϕ, S. Otherwise, choose $e \in E(G)$ that is uncoloured by ϕ.

2. Define T as a maximal ϕ-Tashkinov tree starting with the uncoloured edge e.

(a) If T is not ϕ-elementary, then apply Tashkinov's algorithm to get a k-edge colouring of $dom(\phi) \cup \{e\}$. Rename this new colouring ϕ, and proceed to the next iteration.

(b) If T does not have any defective colours, then set $S := V(T)$, and colour e with a new colour. Rename this new colouring ϕ, increase k by one, and proceed to the next iteration.

(c) Otherwise, *somehow* find colour swap(s) which, when applied to ϕ, make it possible to extend T to a larger ϕ-Tashkinov tree. Do the swap(s), and then extend T until it is a maximal ϕ-Tashkinov tree. Go back to Step 2(a) and continue with this iteration.

Note that if we start the algorithm with an empty colouring, then in our first iteration, the ϕ-Tashkinov tree T will certainly not be ϕ-elementary (in fact, it will just consist of a single uncoloured edge), so we will proceed via (a), using Tashkinov's algorithm. We have already discussed Tashkinov's algorithm in great detail in Section 2.3.2. So, we know that given a ϕ-Tashkinov tree on $n+1$ vertices which is not ϕ-elementary, the algorithm provides a larger domain colouring after performing at most

$$(n-2)\left[3+(\Delta-5)\cdot(\Delta-2)\right]+\left(\frac{(\Delta-1)\Delta}{2}+\Delta-1\right)$$

recolourings (where we may assume $n \leq \Delta - 1$ by an initial truncating step). So, whenever (a) applies, we simply apply Tashkinov's algorithm, and after at most

$$(\Delta-3)\left[3+(\Delta-5)\cdot(\Delta-2)\right]+\left(\frac{(\Delta-1)\Delta}{2}+\Delta-1\right)$$

recolourings, we will succeed in colouring one more edge, completing the iteration.

If Step (b) of the algorithm applies, then we know that T is a maximal ϕ-elementary ϕ-Tashkinov tree with no defective colour. So, Proposition 4.1.1 implies that

$$\chi'(G) \geq \lceil \rho(G) \rceil \geq \left\lceil \frac{2|E(G[V(T)])|}{|V(T)|-1} \right\rceil \geq k+1.$$

Hence, we know we may increase the number of colours we are using by at least one, and we may take $S := V(T)$ as our certificate that G is not k-edge colourable. Step (b) instructs us to use this new colour to colour the edge e, which completes the iteration.

If neither Step (a) or Step (b) applies in a given iteration, then we know that T is a maximal ϕ-elementary ϕ-Tashkinov tree which has a defective colour. Consider the following potential actions for Step (c). Choose a defective colour β, and a colour $\alpha \in \mathcal{M}_{T,\phi} \setminus \mathcal{U}_{T,\phi}$. So, $\alpha \in \phi(x)$ for some $x \in V(T)$. Let P be the maximal (α, β)-alternating path beginning at x.

(S1.) If P does't contain every β-edge which leaves T, then perform the swap indicated in the proof of Proposition 4.1.2(1). Then, T can be extended.

(S2.) If P does contain every β-edge which leaves T, but the last vertex v of P which is on T is such that $\phi(v) \not\subseteq \mathcal{U}_{T,\phi}$, then perform the swap(s) indicated in the proof of Proposition 4.1.2(2). (There will be at most 2 such swaps). Then, T can be extended.

(S3.) If P does contain every β-edge which leaves T, but the first two vertices w_1, w_2 of P not on T are such that $V(T) \cup \{w_1, w_2\}$ is not ϕ-elementary, then perform the swap(s) indicated in the proof of Proposition 4.1.2(3). (There will be at most 2 such swaps). Then, T can be extended.

For each specific instance of the general colouring algorithm that we wish to establish, we will show that in Step (c) one of (S1), (S2), or (S3) above always applies.

In the general colouring algorithm, we may go through Step (c) multiple times in a given iteration, but each time our tree T becomes strictly larger. Since our graph is finite this cannot continue indefinitely. So, each iteration terminates (i.e., succeeds in colouring one additional edge), and hence the algorithm terminates. Moreover, from our above discussion, if Υ is the maximum number of times we go through Step (c) in a single iteration, then the algorithm terminates after doing at most

$$|E|\left(2\cdot\Upsilon+(\Delta-3)\left[3+(\Delta-5)\cdot(\Delta-2)\right]+\left(\frac{(\Delta-1)\Delta}{2}+\Delta-1\right)\right) \qquad (4.5)$$

recolourings. Of course, not only is Υ finite, but we know that $\Upsilon \leq \Delta - 3$ since no ϕ-Tashkinov tree with at least Δ vertices is ϕ-elementary (see Section 2.3.2) and every maximal ϕ-Tashkinov tree has at least 3 vertices (see Observation 2). Alternatively, we also know that $\Upsilon \leq \tau - 1$ for whatever particular value of τ we are using. This is because we know that if Step (c) applies in an iteration, then T is ϕ-elementary and has a defective colour, so

$$\begin{aligned}
& |\mathcal{M}_{T,\phi}| + 1 \leq k \\
\Rightarrow\quad & |V(T)|(k-\Delta) + 2 + 1 \leq k \\
\Rightarrow\quad & (|V(T)|-1)(k-\Delta) + 3 + (k-\Delta) \leq k \\
\Rightarrow\quad & (|V(T)|-1)(k-\Delta) \leq \Delta - 3 \\
\Rightarrow\quad & k \leq \Delta + \frac{\Delta-3}{|V(T)|-1} \\
\Rightarrow\quad & k+1 \leq \Delta + 1 + \frac{\Delta-3}{|V(T)|-1}.
\end{aligned}$$

Since k was initialized to be $\Delta + 1 + \lfloor \frac{\Delta-3}{\tau+1} \rfloor$ and never decreases, this implies that $|V(T)| \leq \tau + 1$. Hence, $\Upsilon \leq \tau + 2 - 3 = \tau - 1$. However we choose to bound Υ, the important point to note is that (4.5) is a polynomial in $|E|$ and Δ. Since some recolourings may involve the entire vertex-set of the multigraph, this tells us that the general algorithm is polynomial in $|E|$, Δ, and $|V|$.

When the general algorithm terminates, it is clear that every edge of G will be coloured. There is something to be said however, about the number of colours that appear on the edges of G. There are two distinct possibilities here. The first possibility is that in the course of the algorithm, no iteration terminated via (b) — in this case, the final colouring will use only $\Delta + 1 + \left\lfloor \frac{\Delta-3}{\tau+1} \right\rfloor$ colours, the same number of colours that we started with. The second possibility is that some iteration terminated via (b) — in this case, the algorithm will output a nonempty S along with the colouring, and this is our certificate that the colouring uses exactly $\chi' = \lceil \rho \rceil$ colours. To see this, note that S was defined the last time we increased k in the algorithm, via (b). Hence, S is the vertex-set of some Tashkinov tree that had no defective colours. This means that

$$k = \lceil \rho \rceil = \left\lceil \frac{2|E(G[S])|}{|S|-1} \right\rceil = \chi'(G[S]) = \chi'(G),$$

by Proposition 4.1.1.

We have now completed the justification of our general algorithm. Of course, the word algorithm should really be in quotations here, because it remains to show how we can find appropriate swap(s) in Step (c). To this end, for each of the four specific instances of the algorithm that we wish to establish, we shall prove that one of (S1), (S2) or (S3) always applies in Step (c).

Theorem 4.3.1. *Let G be a multigraph. Then, there exists a polynomial-time algorithm to edge-colour G with at most*

$$\max \left\{ \lceil \rho(G) \rceil, \Delta + 1 + \left\lfloor \frac{\Delta - 3}{g_o + 3} \right\rfloor \right\}$$

colours. Moreover, if $\lceil \rho(G) \rceil > \Delta + 1 + \left\lfloor \frac{\Delta-3}{g_o+3} \right\rfloor$, then the algorithm provides a set $S \subseteq V(G)$, $|S| \geq 3$ and odd, such that

$$\chi'(G) = \lceil \rho(G) \rceil = \left\lceil \frac{2|E(G[S])|}{|S|-1} \right\rceil = \chi'(G[S]).$$

Proof. Suppose first that $\Delta \leq 2$. We have already shown, in proof of Theorem 4.2.1, that the bound holds in this case. However, we can also note that these very special multigraphs can be χ'-edge coloured in polynomial-time. Moreover, in the case that G has an odd cycle, we provide this cycle as the required certificate of chromatic index.

Now assume that $\Delta \geq 3$. We run our general algorithm with $\tau = g_o + 2$. To do this end, it remains only to detail Step (c). So, suppose we are in Step (c), in some iteration of the algorithm. This means that we have a maximal ϕ-elementary ϕ-Tashkinov tree T that has a defective colour, where ϕ is a partial k-edge-colouring of G . The action we take depends on the size of $V(T)$.

If $|V(T)| \leq g_o$, then choose colours γ and ε missing at the two ends of e, say p_0 and p_1, respectively. Build the maximal (γ, ε)-alternating path P starting from p_1. We know that $p_0 \cup V(P) \subseteq V(T)$, by the uniqueness of the vertex-sets of maximal Tashkinov trees. In particular, this means that $p_0 \cup V(P)$ is ϕ-elementary. So, P must end at p_0. Let T' be a ϕ-Tashkinov tree given by $\{e\} \cup P$, less any one edge of P. (Note that we must have $|V(T')| = |V(T)| = g_o$ because P forms an odd cycle with the edge e). Since $\mathcal{U}_{T',\phi} = \{\gamma, \varepsilon\}$, and T' is ϕ'-elementary, there is no vertex v on T' with $\phi(v) \subseteq \mathcal{U}_{T',\phi}$ (see the proof of Theorem 4.2.1). Hence, we can make colour swap(s) according to either (S1) or (S2) so that T' can be extended. Of course, these same colour swaps alow us to extend T. This is the required action of Step (c).

If $|V(T)| > g_o$, then it is not possible that there are two vertices w_1 and w_2 outside of $V(T)$ such that $V(T) \cup \{w_1, w_2\}$ is ϕ-elementary. This is because, if such vertices existed, then

$$|\mathcal{M}_{T,\phi}| + 1 + 2(k - \Delta) \leq k,$$

and since $|V(T)|$ is odd, we know that $|V(T)| \geq g_o + 2$, so

$$\begin{aligned}
&\Rightarrow \quad (g_o + 2)(k - \Delta) + 2 + 1 + 2(k - \Delta) \leq k \\
&\Rightarrow \quad (g_o + 3)(k - \Delta) + 3 + (k - \Delta) \leq k \\
&\Rightarrow \quad (g_o + 3)(k - \Delta) \leq \Delta - 3 \\
&\Rightarrow \quad k \leq \Delta + \frac{\Delta - 3}{g_o + 3} \\
&\Rightarrow \quad k + 1 \leq \Delta + 1 + \frac{\Delta - 3}{g_o + 3}.
\end{aligned}$$

However, this is a contradiction, since k was initialized to be $\Delta + 1 + \lfloor \frac{\Delta - 3}{g_o + 1} \rfloor$. So, since vertices w_1 and w_2 do not exist, we can make colour-swaps according to (S1) or (S2). This will allow T to be extended, and hence is the required action of Step (c). $\square$

Theorem 4.3.2. *Let G be a multigraph. Then, there exists a polynomial-time algorithm to edge-colour G with at most*

$$\max \left\{ \lceil \rho(G) \rceil, \Delta + 1 + \left\lfloor \frac{\Delta - 3}{\lfloor 3g/2 \rfloor + 2} \right\rfloor \right\}$$

colours. Moreover, if $\lceil \rho(G) \rceil > \Delta + 1 + \left\lfloor \frac{\Delta - 3}{\lfloor 3g/2 \rfloor + 2} \right\rfloor$*, then the algorithm provides a set* $S \subseteq V(G)$*,* $|S| \geq 3$ *and odd, such that*

$$\chi'(G) = \lceil \rho(G) \rceil = \left\lceil \frac{2|E(G[S])|}{|S| - 1} \right\rceil = \chi'(G[S]).$$

Proof. Suppose first that $\Delta \leq 2$. We have already shown, in proof of Theorem 4.2.2, that the bound holds in this case. However, we can also note that these very special multigraphs can be χ'-edge coloured in polynomial-time. Moreover, in the case that G has an odd cycle, we provide this cycle as the required certificate of chromatic index

Now assume that $\Delta \geq 3$. We run our general algorithm with $\tau = \lfloor 3g/2 \rfloor + 1$. To this end, it remains only to detail Step (c). So, suppose we are in Step (c), in some iteration of the algorithm. This means that we have a maximal ϕ-elementary ϕ-Tashkinov tree T that has a defective colour, where ϕ is a partial k-edge-colouring of G . The action we take depends on the size of $V(T)$.

If $|V(T)| \leq \lfloor 3g/2 \rfloor$, then proceed as follows. Choose colours γ and ε missing at the two ends of e, say p_0 and p_1, respectively. Build the maximal (γ, ε)-alternating path P starting from p_1. We know that $p_0 \cup V(P) \subseteq V(T)$, by the uniqueness of the vertex-sets of maximal Tashkinov trees, and we know that P ends at p_0 since T is ϕ-elementary. Let T' be the ϕ-Tashkinov tree defined by $\{e\} \cup P$, less one edge of P. If $V(T') = V(T)$, then T' is maximal, and since $\mathcal{U}_{T',\phi} = \{\gamma, \varepsilon\}$, and T' is ϕ'-elementary, there is no vertex v on T' with $\phi(v) \subseteq \mathcal{U}_{T',\phi}$ (see the proof of Theorem 4.2.1). Hence, we can make colour swap(s) according to (S1) or (S2) so that T' can be extended. Of course, these same colour swaps alow us to extend T. This satisfies the required action of Step (c).

We claim that $|V(T')| < |V(T)|$ is actually not possible in this case. To see this, note that if this is true, then T' is not maximal. So, we can choose a colour $\delta \in \mathcal{M}_{T',\phi}$ which also occurs on an edge leaving $V(T')$. Let x_1 be a vertex on T' incident to such an edge. Choose $\lambda \in \phi(x_1)$, and let P' be the maximal (δ, λ)-alternating path beginning at x_1. We know that $V(T') \cup V(P') \subseteq V(T)$. This means, in particular, that $V(T') \cup V(P')$ is ϕ-elementary, so we know that the last vertex of P' must be on T. Hence, P' (or some sub-path of P') forms a cycle with T'. By the same counting argument we used at the end of the proof of Theorem 4.2.2, we know that

$$|V(T)| \geq |V(T') \cup V(P')| \geq \lfloor 3g/2 \rfloor + 1,$$

which we assumed was not the case here. So, our prior actions are sufficient to cover the case $|V(T)| \leq \lfloor 3g/2 \rfloor$.

If $|V(T)| \geq \lfloor 3g/2 \rfloor + 1$, then it is not possible that there are two vertices w_1 and w_2 outside of $V(T)$ such that $V(T) \cup \{w_1, w_2\}$ is ϕ-elementary. This is because, if such vertices existed, then

$$\begin{aligned}
& |\mathcal{M}_{T,\phi}| + 1 + 2(k - \Delta) \leq k \\
\Rightarrow\ & (\lfloor 3g/2 \rfloor + 1)(k - \Delta) + 2 + 1 + 2(k - \Delta) \leq k \\
\Rightarrow\ & (\lfloor 3g/2 \rfloor + 2)(k - \Delta) + 3 + (k - \Delta) \leq k \\
\Rightarrow\ & (\lfloor 3g/2 \rfloor + 2)(k - \Delta) \leq \Delta - 3 \\
\Rightarrow\ & k \leq \Delta + \frac{\Delta - 3}{\lfloor 3g/2 \rfloor + 2} \\
\Rightarrow\ & k + 1 \leq \Delta + 1 + \frac{\Delta - 3}{\lfloor 3g/2 \rfloor + 2}.
\end{aligned}$$

However, this is a contradiction, since k was initialized to be $\Delta + 1 + \left\lfloor \frac{\Delta - 3}{\lfloor 3g/2 \rfloor + 2} \right\rfloor$. So, since vertices w_1 and w_2 do not exist, we can make swap(s) according to (S1) or (S3). This will allow T to be extended, which is the required action of Step (c). $\square$

In contrast to Theorems 4.2.1 and 4.2.2, the tree constructions of Theorems 4.2.8 and 4.2.9 depend on the number of colours in ϕ, because they both rely on Lemma 4.2.7. Since we now want the flexibility of working with potentially less than $(\chi' - 1)$ colours, we will have to rely on the more general Lemma 4.2.6 instead of Lemma 4.2.7 (which was used in Section 4.2).

Note that the following result has been proved independently by Scheide [36].

Theorem 4.3.3. *Let G be a multigraph. Then, there exists a polynomial-time algorithm to edge-colour G with at most*

$$\max\left\{ \lceil \rho(G) \rceil, \Delta + \sqrt{\frac{\Delta - 1}{2}} \right\}$$

colours. Moreover, if $\lceil \rho(G) \rceil > \Delta + \sqrt{\frac{\Delta-1}{2}}$, then the algorithm provides a set $S \subseteq V(G)$, $|S| \geq 3$ and odd, such that

$$\chi'(G) = \lceil \rho(G) \rceil = \left\lceil \frac{2E(G[S])}{|S| - 1} \right\rceil = \chi'(G[S]).$$

Proof. The case $\Delta \leq 2$ is resolved by the same reasoning given in the proof of Theorem 4.3.1. So, we may assume that $\Delta \geq 3$.

Run the general algorithm with k initialialized to be $\Delta + \sqrt{(\Delta - 1)/2}$. Suppose that we are in Step (c) in some iteration, where $k = \Delta + s$ for some value of s. Since

T is maximal, ϕ-elementary, and has a defective colour, we know that T has at least $2s+3$ vertices, by Lemma 4.2.6. We claim that there cannot exist any vertices w_1, w_2 outside of $V(T)$ such that $V(T) \cup \{w_1, w_2\}$ is ϕ-elementary. To, see this, note that if this is true, then

$$\begin{aligned}
& [|V(T)|(k-\Delta)+2] + 2(k-\Delta) + 1 \leq k \\
\Rightarrow\quad & k+1 \leq \Delta + 1 + \frac{\Delta - 3}{|V(T)|+1} \\
\Rightarrow\quad & \Delta + s + 1 \leq \Delta + 1 + \frac{\Delta - 3}{2s+4} \\
\Rightarrow\quad & 2s^2 + 4s + (3-\Delta) \leq 0 \\
\Rightarrow\quad & s \leq \frac{-4 + \sqrt{4^2 - 4(2)(3-\Delta)}}{2(2)} \\
\Rightarrow\quad & s \leq -1 + \sqrt{\frac{16 - 24 + 8\Delta}{16}} \\
\Rightarrow\quad & s \leq -1 + \sqrt{\frac{\Delta - 1}{2}} \\
\Rightarrow\quad & k \leq \Delta - 1 + \sqrt{\frac{\Delta - 1}{2}}.
\end{aligned}$$

Since we started the algorithm with $k = \Delta + \sqrt{(\Delta-1)/2}$, and k never decreased, this is a contradiction. So we can make swap(s) according to (S1) or (S3). This will allow T to be extended, which is the required action of Step (c). □

Theorem 4.3.4. *Let G be a multigraph. Then, there exists a polynomial-time algorithm to edge-colour G with at most*

$$\max\left\{\lceil \rho(G) \rceil, \Delta + \frac{1}{2} + \sqrt{\mu - \frac{3}{4}},\ \ \Delta + 1 + \left\lfloor \frac{\Delta - 3}{f(g)} \right\rfloor\right\}$$

colours (where $f(g)$ is as defined in Theorem 4.2.9). Moreover, if $\lceil \rho(G) \rceil > \max\left\{\Delta + \frac{1}{2} + \sqrt{\mu - \frac{3}{4}},\ \ \Delta + 1 + \left\lfloor \frac{\Delta-3}{f(g)} \right\rfloor\right\}$, *then the algorithm provides a set $S \subseteq V(G)$, $|S| \geq 3$ and odd, such that*

$$\chi'(G) = \lceil \rho(G) \rceil = \left\lceil \frac{2|E(G[S])|}{|S|-1} \right\rceil = \chi'(G[S]).$$

Proof. Again, the case $\Delta \leq 2$ is easily resolved, so we may assume that $\Delta \geq 3$. As in our previous proof, we run the general algorithm, initializing

$$k = \max\left\{\Delta + \frac{1}{2} + \sqrt{\mu - \frac{3}{4}}, \Delta + 1 + \left\lfloor \frac{\Delta - 3}{f(g)} \right\rfloor\right\}.$$

Suppose that we are in Step (c) of some iteration, with $k = \Delta + s$ for some s.

We claim that the minimum degree of $G[V(T)]$ must be at least 3. To see this, note that if the minimum degree is 2, then since T is ϕ-elementary and maximal, then for any $v \in V(T)$,

$$\begin{aligned} & |\mathcal{M}_{T,\phi} \setminus \phi(v)| \leq 2\mu \\ \Rightarrow \quad & (|V(T)| - 1)(s) + 2 \leq 2\mu \\ \Rightarrow \quad & s \leq \frac{2(\mu - 1)}{|V(T)| - 1} \end{aligned}$$

However, since T has a defective colour, Lemma 4.2.6 tells us that $|V(T)| \geq 2s + 3$. So, we know that

$$\begin{aligned} & s \leq \frac{2(\mu - 1)}{2s + 2} \\ \Rightarrow \quad & 2s^2 + 2s + (2 - 2\mu) \leq 0 \\ \Rightarrow \quad & s \leq \frac{-2 + \sqrt{2^2 - 4(2)(2 - 2\mu)}}{2(2)} \\ \Rightarrow \quad & s \leq -\frac{1}{2} + \sqrt{\frac{4 - 16 + 16\mu}{16}} \\ \Rightarrow \quad & s \leq -\frac{1}{2} + \sqrt{\mu - \frac{3}{4}} \\ \Rightarrow \quad & k \leq \Delta - \frac{1}{2} + \sqrt{\mu - \frac{3}{4}}. \end{aligned}$$

Since we started the algorithm with $k \geq \Delta + \frac{1}{2} + \sqrt{\mu - \frac{3}{4}}$, and k never decreases, this is a contradiction. So, the minimum degree of $G[V(T)]$ cannot be 2.

Since the minimmum degree of $G[V(T)]$ is at least 3, then by the extremal result that we quoted in the proof of Theorem 4.2.9, $|V(T)| \geq f(g) - 1$. We claim that because of this, there cannot be two vertices w_1, w_2 outside of $V(T)$ such that $V(T) \cup \{w_1, w_2\}$ is ϕ-elementary. This is because , if such vertices did exist, then (since T is ϕ-elementary and has a defective colour),

$$\begin{aligned} & [|V(T)|(k - \Delta) + 2] + 2(k - \Delta) + 1 \leq k \\ \Rightarrow \quad & k + 1 \leq \Delta + 1 + \frac{\Delta - 3}{|V(T)| + 1} \\ \Rightarrow \quad & k + 1 \leq \Delta + 1 + \frac{\Delta - 3}{f(g)}. \end{aligned}$$

However, since $k \geq \Delta + 1 + \frac{\Delta - 3}{f(g)}$, this is a contradiction. So, we can make swaps according to (S1) or (S3). This will allow T to be extended, which is the required action of Step (c). $\square$

Our algorithms provide not only the colourings that we seek, but give us valuable information about multigraphs with high chromatic index. That is, for those multigraphs with highest chromatic index, our algorithms provide a vertex-set S with $\chi'(G) = \chi'(G[S])$. These vertex sets are always the vertex-sets of Tashkinov trees, and moreover, it is possible to analyze both their size and the structure. In the following chapter we study such sets S in detail, and examine what it means for a multigraph to have high chromatic index.

Chapter 5

Characterizing high chromatic index

The previous chapter was all about finding $\max\{\lceil\rho\rceil, \Delta + t\}$-edge colourings, for various values of t. Of course, a result that says

$$\chi'(G) \le \max\{\lceil\rho(G)\rceil, \Delta + t\}$$

can also be interpreted as

$$\chi'(G) > \Delta + t \quad \Rightarrow \quad \chi'(G) = \lceil\rho(G)\rceil.$$

From the definition of ρ, this means that

$$\begin{gathered}\chi'(G) > \Delta + t \\ \Downarrow \\ \exists\ S \subseteq V(G),\ |S| \ge 3 \text{ and odd, such that } \chi'(G) = \left\lceil \frac{2|E[S]|}{|S|-1} \right\rceil.\end{gathered}$$

Going one step further, we can view this as a characterization:

$$\begin{gathered}\chi'(G) > \Delta + t \\ \Updownarrow \\ \exists\ S \subseteq V(G),\ |S| \ge 3 \text{ and odd, such that } \chi'(G) = \left\lceil \frac{2|E[S]|}{|S|-1} \right\rceil > \Delta + t.\end{gathered}$$

Of course, such a characterization is not very satisfying. It tells us that there is a subgraph $G[S]$ that determines the chromatic index of G, but we don't even know the size of S, let alone anything about the structure of $G[S]$ (apart from the number of edges). In this chapter, we search for meaningful characterizations of multigraphs with high chromatic index. In Section 5.1 we discuss this problem in general. As we have already proved one characterization result in this thesis (Theorem 3.2.2), this includes a discussion of how the methods used there can be generalized. In Section 5.2, we extend Theorem 3.2.2 by characterizing those multigraphs with high chromatic index with respect to our refinement of Goldberg's bound (Theorem 4.2.1). In Section 5.3, we use Theorem 4.2.8 to obtain a best-possible result characterizing large multiples of simple graphs achieving Vizing's upper bound.

5.1 General characterization techniques

In Section 3.2, we characterized Goldberg's upper bound, showing that for a connected multigraph G,

$$\chi'(G) = \Delta + 1 + \frac{\Delta - 2}{g_o - 1} \quad \Leftrightarrow \quad G = \mu C_{g_o} \text{ and } (g_o - 1)|2(\mu - 1).$$

Now that we know more about the method of Tashkinov trees, we can view this proof as consisting of two parts:

1. Proving that
$$\chi'(G) > \Delta + 1 + \frac{\Delta - 3}{g_o - 1} \quad \Rightarrow \quad \chi'(G) = \lceil \rho(G) \rceil,$$
by finding a Tashkinov tree T which has $\chi' = \left\lceil \frac{2|E(G[V(T)])|}{|V(T)|-1} \right\rceil$.
2. Analyzing the Tashkinov tree T to show that $G[V(T)] = G = \mu C_{g_o}$.

Chapter 4 was about proving results of type (1); this chapter is about results which mimic (2). Recall that to prove (1) in Section 3.2, we built a maximal ϕ-elementary ϕ-Tashkinov tree, where ϕ is a $(\chi' - 1)$-edge colouring of maximum domain. Our proof of part (2) can be generalized to the following.

Proposition 5.1.1. *Let G be a connected multigraph with $\chi'(G) > \Delta + 1$. Suppose that T is a ϕ-elementary, maximal ϕ-Tashkinov tree in G, where ϕ is a $(\chi' - 1)$-edge colouring of maximum domain. Then,*

$$|V(T)| \le \frac{\Delta - 2}{\chi' - \Delta - 1} + 1,$$

and moreover, $|V(T)| = \frac{\Delta-2}{\chi'-\Delta-1} + 1$ if and only if $G[V(T)] = G$ and G is Δ-regular.

Proof. Let v be any vertex in T. Since T is ϕ-elementary and maximal, v sees every colour of $\mathcal{M}_{T,\phi} \setminus \phi(v)$ within $G[V(T)]$. Every $w \in V(T)$ has $|\phi(w)| \geq (\chi' - 1 - \Delta)$; if w is one of the first two vertices on T then it has at least one more missing colour, and is adjacent to at least one uncoloured edge. So, in general, v must be incident to at least $(|V(T)|-1)(\chi'-1-\Delta)+2$ edges in $G[V(T)]$. Since v has degree at most Δ, this tells us that

$$\begin{aligned} &(|V(T)| - 1)(\chi' - 1 - \Delta) + 2 \leq \Delta \\ \Rightarrow \quad &|V(T)| \leq \frac{\Delta - 2}{\chi' - \Delta - 1} + 1, \end{aligned}$$

as desired. Note that this upper bound can only be achieved if

$$(|V(T)| - 1)(\chi' - 1 - \Delta) + 2 = \Delta,$$

which means that the vertex v has degree Δ, and all of its neighbours lie in $G[V(T)]$. Since v was chosen arbitrarily and G is connected, this means that $G = G[V(T)]$ and G is Δ-regular. □

In the case of Theorem 3.2.2, we were analyzing $\chi' = \Delta + 1 + \frac{\Delta - 2}{g_o - 1}$, and so the bound of Proposition 5.1.1 becomes $|V(T)| \leq (g_o - 1) + 1 = g_o$. Of course, as we have now seen repeatedly, a maximal Tashkinov tree must have at least g_o vertices. So, the bound is achieved in this case, and hence $G = G[V(T)]$ is a Δ-regular graph on g_o vertices. In other words, $G = \mu C_{g_o}$, the desired result of Theorem 3.2.2.

We proved our results in Chapter 4 by building ϕ-Tashkinov trees that are not only maximal and ϕ-elementary, but that have no defective colours. With this additional assumption we can get a lower bound to complement the upper bound of Proposition 5.1.1

Proposition 5.1.2. *Let G be a connected multigraph with $\chi'(G) > \Delta + 1$. Suppose that T is a ϕ-elementary, maximal ϕ-Tashkinov tree in G that has no defective colours, where ϕ is a $(\chi' - 1)$-edge colouring of maximum domain. Then,*

$$|V(T)| \geq \frac{\chi'}{\mu},$$

and moreover $|V(T)| = \frac{\chi'}{\mu}$ if and only if $G[V(T)]$ is a copy of $\mu K_{|V(T)|}$ with fewer than $(|V(T)| - 1)/2$ edges missing.

Proof. Since T has no defective colours, every colour in ϕ occurs on exactly $(|V(T)|-1)/2$ edges of $G[V(T)]$. Since we know that $G[V(T)]$ has at least one edge that is uncoloured by ϕ (the first edge of the tree), this tells us that

$$|E(G[V(T)])| > (\chi' - 1)\left(\frac{|V(T)| - 1}{2}\right).$$

On the other hand,

$$|E(G[V(T)])| \leq \frac{\mu|V(T)|(|V(T)|-1)}{2}.$$

So, we get that

$$\begin{aligned} & (\chi'-1)\left(\frac{|V(T)|-1}{2}\right) < \frac{\mu|V(T)|(|V(T)|-1)}{2} \\ \Rightarrow \quad & \chi'-1 < \mu|V(T)| \\ \Rightarrow \quad & \frac{\chi'}{\mu} \leq |V(T)|, \end{aligned}$$

as desired. Note that if $G[V(T)]$ is not a copy of $\mu K_{|V(T)|}$ with fewer than $(|V(T)|-1)/2$ edges missing, then

$$|E(G[V(T)])| \leq \frac{\mu|V(T)|(|V(T)|-1)}{2} - \frac{|V(T)|-1}{2} = \frac{(\mu|V(T)|-1)(|V(T)|-1)}{2}.$$

However, this means that

$$\begin{aligned} & (\chi'-1)\left(\frac{|V(T)|-1}{2}\right) < \frac{(\mu|V(T)|-1)(|V(T)|-1)}{2} \\ \Rightarrow \quad & \chi'-1 < \mu|V(T)|-1 \\ \Rightarrow \quad & \frac{\chi'}{\mu} < |V(T)|. \end{aligned}$$

So, $|V(T)|$ cannot achieve the lower bound in this case. □

The theory of Tashkinov trees is used very little in the above proof. In fact, the proof only requires the fact that the number of edges in $G[V(T)]$ is at least

$$(\chi'-1)\left(\frac{|V(T)|-1}{2}\right)+1.$$

So, the results of Proposition 5.1.2 for $V(T)$ should also be true for any S where $G[S]$ has at least

$$(\chi'-1)\left(\frac{|S|-1}{2}\right)+1$$

edges, provided $|S| \geq 3$ and odd. Of course, it is not hard to see that for such an S,

$$\begin{aligned} & |E[S]| \geq (\chi'-1)\left(\frac{|S|-1}{2}\right)+1 \\ \Leftrightarrow \quad & |E[S]| > (\chi'-1)\left(\frac{|S|-1}{2}\right) \\ \Leftrightarrow \quad & \frac{2|E[S]|}{|S|-1} > \chi'-1 \\ \Leftrightarrow \quad & \left\lceil \frac{2|E[S]|}{|S|-1} \right\rceil = \lceil \rho \rceil = \chi'. \end{aligned}$$

So, the results of Proposition 5.1.2 for $V(T)$ actually hold for any set which determines chromatic index, that is, for any $S \subseteq V(G)$, $|S| \geq 3$ and odd with $\chi' = \lceil \rho \rceil = \left\lceil \frac{2|E[S]|}{|S|-1} \right\rceil$. In fact, Proposition 5.1.1 also holds for any such set, and hence we get the following result.

Proposition 5.1.3. *Let G be a connected multigraph with $\chi'(G) > \Delta + 1$. Then, any $S \subseteq V(G)$, $|S| \geq 3$ and odd with $\chi' = \lceil \rho \rceil = \left\lceil \frac{2|E[S]|}{|S|-1} \right\rceil$ satisfies*

$$\frac{\chi'}{\mu} \leq |S| \leq \frac{\Delta - 2}{\chi' - \Delta - 1} + 1.$$

Moreover, $|S| = \frac{\Delta-2}{\chi'-\Delta-1} + 1$ if and only if $G[S] = G$ and G is Δ-regular, and $|S| = \frac{\chi'}{\mu}$ if and only if $G[S]$ is a copy of $\mu K_{|S|}$ with fewer than $(|S|-1)/2$ edges missing.

Proof. We have already justified the lower bound and corresponding characterization; it remains only to prove the upper bound and its characterization. To this end, we know that

$$\chi' = \left\lceil \frac{2|E[S]|}{|S| - 1} \right\rceil,$$

or equivalently,

$$\frac{2|E[S]|}{|S| - 1} > \chi' - 1. \tag{5.1}$$

We apply the standard bound of $|E[S]| \leq \frac{\Delta|S|}{2}$ to (5.1). This gives us

$$\begin{aligned} \frac{\Delta|S|}{|S|-1} > \chi' - 1 \quad &\Rightarrow \quad \frac{\Delta}{|S|-1} > \chi' - \Delta - 1 \\ &\Rightarrow \quad \Delta - 1 \geq (\chi' - \Delta - 1)(|S| - 1) \\ &\Rightarrow \quad \frac{\Delta - 1}{\chi' - \Delta - 1} + 1 \geq |S|. \end{aligned}$$

This is very close to the upper bound we desire, but not exactly - we must eliminate the case $|S| = \frac{\Delta-1}{\chi'-\Delta-1} + 1$.

Suppose that we do have $|S| = \frac{\Delta-1}{\chi'-\Delta-1} + 1$. By rearranging this expression, we get

$$\chi' = \Delta + 1 + \frac{\Delta - 1}{|S| - 1}.$$

In particular, this means that $\frac{\Delta-1}{|S|-1}$ is an integer, since chromatic index is an integer. We know that $(|S|-1)$ is even, so this implies that Δ is odd. However, this means that it is not possible to have $|E[S]| = \frac{\Delta|S|}{2}$, again since $|S|$ is odd. So, $|E[S]| \leq \frac{\Delta|S|-1}{2}$, and hence

$$\begin{aligned}\frac{2|E[S]|}{|S|-1} &\leq \frac{2\left(\frac{\Delta|S|-1}{2}\right)}{|S|-1} = \frac{\Delta|S|-1}{|S|-1} = \Delta + \frac{\Delta-1}{|S|-1} \\ &= \Delta + \frac{\Delta-1}{\left(\frac{\Delta-1}{\chi'-\Delta-1}\right)} = \chi' - 1.\end{aligned}$$

This is a contradiction to (5.1). Hence, we must have $|S| < \frac{\Delta-1}{\chi'-\Delta-1} + 1$, that is, $|S| \leq \frac{\Delta-2}{\chi'-\Delta-1} + 1$, as desired.

Now, suppose that $|S| = \frac{\Delta-2}{\chi'-\Delta-1} + 1$, but that $G[S]$ is not Δ-regular. By rearranging our expression for S, we get that

$$\chi' = \Delta + 1 + \frac{\Delta-2}{|S|-1}.$$

Since this means that $\frac{\Delta-2}{|S|-1}$ is an integer, and we know that $(|S|-1)$ is even, this implies that Δ is even. Hence,

$$|E[S]| < \frac{\Delta|S|}{2} \quad \Rightarrow \quad |E[S]| \leq \frac{\Delta|S|}{2} - 1.$$

So, we get that

$$\begin{aligned}\frac{2|E[S]|}{|S|-1} &\leq \frac{2\left(\frac{\Delta|S|}{2}-1\right)}{|S|-1} = \frac{\Delta|S|-2}{|S|-1} = \Delta + \frac{\Delta-2}{|S|-1} \\ &= \Delta + \frac{\Delta-2}{\left(\frac{\Delta-2}{\chi'-\Delta-1}\right)} = \chi' - 1.\end{aligned}$$

This contradicts (5.1). So, we know that $G[S]$ is Δ-regular. Hence, since G is connected, it must also be the case that $G[S] = G$. □

Proposition 5.1.3 tells us that Propositions 5.1.1 and 5.1.2 can be translated from results about Tashkinov trees to results about any set that determines $\chi' = \lceil \rho \rceil > \Delta + 1$. Note that in the proof of Theorem 3.2.2, we did also need that fact that $|V(T)| \geq g_o$ for the Tashkinov tree T in question, a bound which we justified using the theory of Tashkinov trees. However, if we have any set $S \subseteq V(G)$, $|S| \geq 3$ and odd with

$$\chi'(G) = \lceil \rho(G) \rceil = \left\lceil \frac{2|E[S]|}{|S|-1} \right\rceil > \Delta + 1,$$

we immediately do know that

$$|S| \geq g_o.$$

This is because $\chi'(G) = \chi'(G[S]) > \Delta + 1$, while all bipartite multigraphs have chromatic index exactly Δ by König's Theorem. So, we truly "lose nothing" when we move from analyzing Tashkinov trees that determine chromatic index, to general sets S that determine chromatic index. For the remainder of this chapter, we will choose this latter approach, and thus not need to mention Tashkinov trees again.

Before we begin using Proposition 5.1.3 to prove specific characterizations, as we will in Sections 5.2 and 5.3, we further our discussion about characterizing high chromatic index in general. That is, we prove an additional result about the structure of $G[S]$ in the case when G has a *critical edge*. By this we mean an edge e such that $\chi'(G \setminus e) < \chi'(G)$. This proposition extends the idea that if S determines chromatic index, then $G[S]$ must contain an odd cycle.

Proposition 5.1.4. *Let G be a connected multigraph with $\chi'(G) > \Delta + 1$, that has a critical edge e. Let S be any subset of the vertex set, $|S| \geq 3$ and odd, such that $\chi' = \lceil \rho(G) \rceil = \left\lceil \frac{2|E[S]|}{|S|-1} \right\rceil$. Then $G[S]$ is the disjoint union of e and exactly $(\chi' - 1)$ matchings of size $(|S| - 1)/2$. Moreover,*

$$G[S] = P_0 \cup P_1 \cup \cdots \cup P_k$$

for some $k \geq 1$, where P_0 is an odd multi-cycle and for all $i \in \{0, \ldots, k\}$, P_{i+1} is either an odd multi-cycle intersecting $P_0 \cup \cdots \cup P_i$ in exactly one vertex, or is an odd-length multi-path intersecting $P_0 \cup \cdots \cup P_i$ in exactly two vertices.

In order to prove Proposition 5.1.4, we must rely on a commonly known structural result from matching theory. In this result, which follows, a graph G is called *hypomatchable* if $G \setminus v$ admits a perfect matching for all $v \in V(G)$.

Lemma 5.1.5. (See e.g. [26], p. 59) *Let G be a connected simple graph that is hypomatchable. Then,*

$$G = P_0 \cup P_1 \cup \cdots \cup P_k$$

for some $k \geq 1$, where P_0 is an odd cycle and for all $i \in \{0, \ldots, k\}$, P_{i+1} is either an odd cycle intersecting $P_0 \cup \cdots \cup P_i$ in exactly one vertex, or is an odd-length path intersecting $P_0 \cup \cdots \cup P_i$ in exactly two vertices.

Given this result, we can provide our proof about multigraphs with critical edges, as follows.

Proof. *(Proposition 5.1.4)* Since $\chi'(G[S]) = \chi'(G)$, we immediately know that the critical edge e must be in $G[S]$. Let ϕ be a $(\chi' - 1)$-edge colouring of $G \setminus e$. There are $\chi' - 1$ colours in total, and in $G[S] \setminus e$, each colour class has size at most $(|S| - 1)/2$. So,

$$|E[S]| \leq (\chi' - 1)\left(\frac{|S| - 1}{2}\right) + 1.$$

On the other hand,

$$\chi'(G) = \lceil \rho(G) \rceil = \left\lceil \frac{2|E[S]|}{|S|-1} \right\rceil \quad \Rightarrow \quad \frac{2|E[S]|}{|S|-1} > \chi' - 1$$
$$\Rightarrow \quad |E[S]| > (\chi' - 1)\left(\frac{|S|-1}{2}\right).$$

So, we know that $G[S]$ is the disjoint union of e and exactly $(\chi' - 1)$ matchings, each of size $(|S| - 1)/2$.

On $G[S]$, ϕ consists of exactly $(\chi' - 1)$ colour classes, each of size $(|S| - 1)/2$. Since $(\chi' - 1) \geq \Delta + 1$, no vertex in S can be incident to every one of these colour classes. Hence, $G[S]$ is hypomatchable - or rather, the underlying graph of $G[S]$ is hypomatchable. So, we apply Proposition 5.1.5 to the underlying graph of $G[S]$. By allowing multiples edges in the structure obtained, we get our desired result. □

If every edge in a multigraph G is a critical edge, then we say that G is *edge-critical*. Clearly, if G is edge-critical, then in Proposition 5.1.4, $G[S] = G$. So, according to this Proposition, the Seymour-Goldberg Conjecture implies that all critical multigraphs with $\chi' > \Delta + 1$ have a very specific structure.

For critical multigraphs G one implication of the structure given by Proposition 5.1.4 is that

$$|E(G)| = (\chi' - 1)\left(\frac{|V(G)| - 1}{2}\right) + 1$$

exactly. It is worth noting that this equality actually defines another version of the Seymour-Goldberg Conjecture. That is, the following statement is equivalent to the conjecture:

Every critical multigraph with $\chi' > \Delta + 1$ *has exactly*

$$(\chi' - 1)\left(\frac{|V| - 1}{2}\right) + 1 \quad \textit{edges.}$$

This version is less well-known, but has been previously mentioned in the literature by authors such as Hilton and Jackson [15]. To see the equivalence, suppose that we have a multigraph G with $\chi'(G) > \Delta + 1$. Remove edges from G until we get G' that is edge-critical and $\chi'(G') = \chi'(G)$. The Seymour-Goldberg Conjecture is true for G if and only if $\chi' = \lceil \rho(G) \rceil$, and true for G' if and only if $\chi' = \lceil \rho(G') \rceil$. If $\chi' = \lceil \rho(G') \rceil$, then any set S which defines ρ is such that $G'[S] \subseteq G[S]$. So, if the Seymour-Goldberg Conjecture is true for G', then it is true for G as well. However, note that the opposite also holds. This is because, since G' is edge-critical, we cannot have $G'[S] \subset G[S]$, as $\chi'(G'[S]) = \chi'(G[S]) = \chi'$. So, indeed the Seymour-Goldberg

Conjecture holds for G if and only if it holds for G'. Now let us look more closely at G'. Since G' is edge-critical, the Seymour-Goldberg Conjecture is true for G' if and only if $|V|$ is odd with

$$\chi' = \left\lceil \frac{2|E|}{|V|-1} \right\rceil .$$

We have already seen that this is equivalent to

$$|E| \geq (\chi'-1)\left(\frac{|V|-1}{2}\right)+1.$$

However, since G' is edge-critical, and a colour-class in $|V|$ has maximum size $(|V|-1)/2$, we know that

$$|E| \leq (\chi'-1)\left(\frac{|V|-1}{2}\right)+1.$$

Hence, we get the desired equivalence.

At first glance, the "number of edges" version of the Seymour-Goldberg Conjecture may look much more approachable than the version we have been working with. However, note that a critical multigraph with $\chi' > \Delta+1$ has not even been shown to always have an odd number of vertices — let alone an exact number of edges on an odd number of vertices. The assertion that all critical multigraphs with $\chi' > \Delta+1$ have an odd number of vertices is actually a famous conjecture in its own right — the *Critical Multigraph Conjecture* ([17], see also [19]). In fact, there is even a *Weak Critical Graph Conjecture*, which claims that there exists a constant $c > 2$, such that critical multigraphs G with $|V(G)| \leq c \cdot \Delta$ satisfy the Critical Multigraph Conjecture (see [19]). Even this much weaker conjecture has not been established.

We can see that studying critical multigraphs is an important and special part of studying multigraphs with high chromatic index. However, we still aim to prove results characterizing all multigraphs with high chromatic index. If G does not have a critical edge, then an S that determines chromatic index still has

$$|E[S]| \geq (\chi'-1)\left(\frac{|S|-1}{2}\right)+1,$$

but this is not necessarily an equality. Moreover, even if it is an equality, it is possible that $G[S]$ is not a union of $(\chi'-1)$ near-perfect matchings, plus one edge, but rather $G[S]$ could be a union of $(\chi'-1)$ matchings, at least one of which is not near-perfect, and then at least two other edges. In this way, $G[S]$ is not necessarily hypomatchable, and so the structure we had above does not apply. Still, the idea of this structure gives us some idea of what we are looking for in general. This is interesting to keep in mind with respect to the following section, where we focus on odd-girth. Here, in graphs of highest chromatic index, we find dense odd cycles, or structures that are very similar to dense odd cycles.

5.2 High chromatic index with respect to g_o

Our main result in this section is the following.

Theorem 5.2.1. *Let G be a connected multigraph containing an odd cycle. Then,*

$$\chi'(G) > \Delta + 1 + \frac{\Delta - 3}{g_o + 3}$$

if and only if G contains a k-vertex subgraph with at least $\left(\Delta + 1 + \left\lfloor \frac{\Delta-3}{g_o+3} \right\rfloor\right)\left(\frac{k-1}{2}\right)+1$ edges, where

$$k \in \{g_o, g_o + 2, g_o + 4\}.$$

Moreover:

1. *If $k = g_o + 4$ occurs, then $\chi' = \Delta + 1 + \frac{\Delta-2}{g_o+3}$ and G is Δ-regular on $g_o + 4$ vertices;*
2. *If $k = g_o + 2$ occurs, then $\chi'(G) \leq \Delta + 1 + \frac{\Delta-2}{g_0+1}$.*

Proof. Suppose that G contains a k-vertex subgraph that has at least $\left(\Delta + 1 + \left\lfloor \frac{\Delta-3}{g_0+3} \right\rfloor\right)\left(\frac{k-1}{2}\right) + 1$ edges, and k is odd and at least three. Then

$$\begin{aligned}
\chi'(G) &\geq \lceil \rho(G) \rceil \geq \left\lceil \frac{2\left[\left(\Delta + 1 + \left\lfloor \frac{\Delta-3}{g_o+3} \right\rfloor\right)\left(\frac{k-1}{2}\right) + 1\right]}{k-1} \right\rceil \\
&= \Delta + 1 + \left\lfloor \frac{\Delta - 3}{g_o + 3} \right\rfloor + \left\lceil \frac{2}{k-1} \right\rceil = \Delta + 1 + \left(\left\lfloor \frac{\Delta - 3}{g_o + 3} \right\rfloor + 1\right) \\
&> \Delta + 1 + \frac{\Delta - 3}{g_o + 3},
\end{aligned}$$

as desired.

Now, suppose that $\chi'(G) > \Delta + 1 + \frac{\Delta-3}{g_o+3}$. Theorem 4.2.1 tell us that in this case, $\chi' = \lceil \rho \rceil$. So, there exists $S \subseteq V(G)$, $|S| \geq 3$ and odd with $\chi' = \lceil \rho(G) \rceil = \left\lceil \frac{2|E[S]|}{|S|-1} \right\rceil$. By Proposition 5.1.3, we know that

$$|S| \leq \frac{\Delta - 2}{\chi' - \Delta - 1} + 1. \tag{5.2}$$

Note that $\chi' - \Delta - 1 > \frac{\Delta-3}{g_o+3}$ can also be expressed as

$$\chi' - \Delta - 1 \geq \frac{\Delta - 2}{g_o + 3}.$$

Substituting this into our above bound, we get that

$$|S| \leq (g_o + 3) + 1 = g_o + 4. \tag{5.3}$$

Let $k = |S|$. We know that $k \geq g_o$, by our comments following Proposition 5.1.3. So, since k is odd, $k \in \{g_o, g_o + 2, g_o + 4\}$. Note that

$$\begin{aligned}
&\chi' = \lceil \rho(G) \rceil = \left\lceil \frac{2|E[S]|}{|S| - 1} \right\rceil \\
\Rightarrow\quad & \frac{2|E[S]|}{k-1} > \chi' - 1 \\
\Rightarrow\quad & |E[S]| > (\chi' - 1)\left(\frac{k-1}{2}\right) \\
\Rightarrow\quad & |E[S]| \geq (\chi' - 1)\left(\frac{k-1}{2}\right) + 1,
\end{aligned}$$

since k is odd. So, to get our desired lower bound on the number of edges induced by S, we need only show that

$$\begin{aligned}
&(\chi' - 1) \geq \Delta + 1 + \left\lfloor \frac{\Delta - 3}{g_0 + 3} \right\rfloor \\
\Leftarrow\quad & \chi' \geq \Delta + 1 + \left(\left\lfloor \frac{\Delta - 3}{g_o + 3} \right\rfloor + 1 \right) \\
\Leftarrow\quad & \chi' > \Delta + 1 + \frac{\Delta - 3}{g_o + 3}.
\end{aligned}$$

Since this last line is an assumption, we indeed have the number of edges we are looking for.

Suppose now that we are in the case $k = g_o + 2$. Here, inequality (5.2) (with $|S| = k = g_o + 2$) says that

$$g_o + 2 \leq \frac{\Delta - 2}{\chi' - \Delta - 1} + 1$$

Rearranging this expression, we get that

$$\chi' \leq \Delta + 1 + \frac{\Delta - 2}{g_o + 1}.$$

Finally, suppose that we are in the case $k = g_o + 4$. Now, when we substitute $|S| = k = g_o + 4$ in (5.2), we know that we get an equality (see (5.3)). That is, we get

$$g_o + 4 = \frac{\Delta - 2}{\chi' - \Delta - 1} + 1.$$

Rearranging, this says that we must have

$$\chi' = \Delta + 1 + \frac{\Delta - 2}{g_o + 3}$$

exactly. Moreover, since we do get equality in (5.2), Proposition 5.1.3 tells us that $G = G[S]$ and G is Δ-regular. □

Note that Theorem 5.2.1 makes sense in the context of Theorem 3.2.2. If $\chi'(G) = \Delta + 1 + \frac{\Delta-2}{g_o-1}$, then Theorem 5.2.1 says that G must contain a dense subgraph on $k = g_o$ vertices. Of course, it does not tell us that $G = \mu K_{g_o}$ in this case, so it is not a complete generalization of the earlier Theorem. However, by repeating the argument of Theorem 5.2.1 with slight modifications, we can choose to characterize multigraphs with even higher values of chromatic index. When we replace the denominator g_o+3 by g_o-1, we get Theorem 3.2.2, as described at the beginning of the previous section. Alternatively, by replacing g_o+3 by g_o+1, we get the following intermediate result.

Theorem 5.2.2. *Let G be a connected multigraph containing an odd cycle. Then,*

$$\chi'(G) > \Delta + 1 + \frac{\Delta - 3}{g_o + 1}$$

if and only if G contains a k-vertex subgraph with at least $\left(\Delta + 1 + \left\lfloor \frac{\Delta-3}{g_o+1} \right\rfloor\right) \left(\frac{k-1}{2}\right) + 1$ edges, where

$$k \in \{g_o, g_o + 2\}.$$

Moreover, if $k = g_o + 2$ occurs, then $\chi' = \Delta + 1 + \frac{\Delta-2}{g_o+1}$ and G is Δ-regular on $g_o + 2$ vertices.

Proof. We repeat the argument of the above proof, except with $\frac{\Delta-3}{g_o+1}$ replacing $\frac{\Delta-3}{g_o+3}$. The backwards direction is again straightforward. In the forwards direction, the bound of

$$|S| \leq \frac{\Delta - 2}{\chi' - \Delta - 1} + 1$$

now yields

$$|S| \leq (g_o + 1) + 1 = g_o + 2.$$

So, if we let $k = |S|$, then we get that $k \in \{g_o, g_o + 2\}$. The argument to show that

$$|E[S]| \geq \left(\Delta + 1 + \left\lfloor \frac{\Delta - 3}{g_o + 1} \right\rfloor\right) \left(\frac{k - 1}{2}\right) + 1$$

is completely identical to the analogous argument above, except with g_o+1 replacing $g_o + 3$.

If we are in the case $k = g_o + 2$, then we achieve equality in the upper bound on $|S|$. Namely,

$$g_o + 2 = \frac{\Delta - 2}{\chi' - \Delta - 1} + 1.$$

Rearranging, this tells us that we must have

$$\chi' = \Delta + 1 + \frac{\Delta - 2}{g_o + 1}.$$

Moreover, since we achieve equality in the upper bound, Proposition 5.1.3 tells us that $G = G[S]$ and G is Δ-regular. □

Together, Theorems 3.2.2, 5.2.1 and 5.2.2 give us a very complete picture of what it means to be a multigraph with $\chi' > \Delta+1+\frac{\Delta-3}{g_o+3}$. Moreover, they point to a pattern which we conjecture governs all multigraphs with high chromatic index with respect to g_o. At the highest extreme for chromatic index, the only possibility is a multiple of C_{g_o}; as chromatic index gets lower, there are possibilities on increasingly larger sets of vertices, and the multigraph may look less and less like C_{g_o}. However, at any point, there is a trade-off between a larger vertex set for our characterizing subgraph, and the density of this subgraph. For example, if the characterizing subgraph has maximum size, then it must be Δ-regular, and hence be the entire multigraph. More formally, we conjecture the following extension of Theorems 3.2.2, 5.2.1 and 5.2.2.

Conjecture 5.2.3. *Let G be a connected multigraph containing an odd cycle, and let $t \in \{-1, 1, 3, 5, \ldots\}$. Then,*

$$\chi'(G) > \Delta + 1 + \frac{\Delta - 3}{g_o + t}$$

if and only if G contains a k-vertex subgraph with at least $\left(\Delta + 1 + \left\lfloor \frac{\Delta-3}{g_o+t} \right\rfloor\right)\left(\frac{k-1}{2}\right)+1$ edges, and

$$k \in \{g_o, g_o + 2, \ldots, g_o + t + 1\}.$$

Moreover:

1. *If $k = g_o + t + 1$ occurs, then $\chi' = \Delta + 1 + \frac{\Delta-2}{g_o+t}$ and G is Δ-regular on $g_o + t + 1$ vertices;*
2. *For all $i \in \{2, 4, \ldots, t-1\}$, if $k = g_o + i$ occurs, then $\chi' \leq \Delta + 1 + \frac{\Delta-2}{g_o+i-1}$.*

Using the same arguments that we used above, it is easy to see that Conjecture 5.2.3 will hold for a particular value of t, provided that

$$\chi' > \Delta + 1 + \frac{\Delta - 3}{g_o + t} \quad \Rightarrow \quad \chi' = \lceil \rho \rceil$$

holds for all multigraphs. In order to get such a result for $t > 3$ however, we would need to improve our argument from the proof of Theorem 4.2.1 - which would likely require additional theory about Tashkinov trees. Of course, if the Seymour-Goldberg Conjecture is true, then it would imply such results for all t.

5.3 Multiples of simple graphs and Vizing's bound

We saw in Section 3.1 that multiples of odd cliques achieve Vizing's upper bound. If multiples of odd cliques were the only multigraphs to achieve Vizing's upper bound, then this would be a beautiful result — but we know that this is not the case. Now, with our additional knowledge, we are able to prove that if we restrict ourselves to the realm of larger multiples of simple graphs, this simple characterization can work.

Theorem 5.3.1. *Let G be a simple, connected graph with maximum degree d and let $t > d/2$ be any integer. Then, $\chi'(tG) = t(d+1)$ if and only if $G = K_{d+1}$ and d is even.*

The requirement for the multiple to be large, $t > d/2$, is absolutely necessary here. In fact, we shall show that if t is any smaller, then there are examples of non-clique multiples of simple graphs which achieve Vizing's upper bound. First however, let us prove Theorem 5.3.1.

Proof. *(Theorem 5.3.1)* As we know already that multiples of odd cliques do achieve Vizing's upper bound (Theorem 3.1.1), we have only to prove the forward direction of the theorem. To this end, suppose that $\chi'(tG) = t(d+1)$. Theorem 4.2.8 says that

$$\chi'(tG) > td + \sqrt{\frac{td-1}{2}} \quad \Rightarrow \quad \chi'(tG) = \lceil \rho(tG) \rceil.$$

Since $d < 2t$, we know that

$$td + \sqrt{\frac{td-1}{2}} < td + \sqrt{t^2 - \frac{1}{2}} < td + t = t(d+1) = \chi'(tG).$$

So, we get that

$$\chi'(tG) = t(d+1) \quad \Rightarrow \quad \chi'(tG) = \lceil \rho(tG) \rceil.$$

This means that there exists $S \subseteq V(G)$, with $|S| \geq 3$ and odd, such that

$$\left\lceil \frac{2t|E(G[S])|}{|S|-1} \right\rceil = td + t,$$

or equivalently,

$$\frac{2t|E(G[S])|}{|S|-1} > td + t - 1.$$

Proposition 5.1.3 tells us that

$$|S| \geq \frac{td+t}{t} = d+1,$$

and

$$|S| \leq \frac{td-2}{td+t-td-1} + 1 = \frac{td-2}{t-1} + 1 = d+1+\frac{d-2}{t-1}.$$

Since $t > d/2$, this last inequality implies that

$$|S| < d+1+\frac{d-2}{d/2-1} = d+1+\frac{2(d-2)}{d-2} = d+3.$$

So, we know that $|S| = d+1$ or $|S| = d+2$.

We now have two cases to explore. Suppose first that $|S| = d+2$, which means that d is odd. Then, we know that

$$\begin{aligned} & \frac{2t|E(G[S])|}{d+1} > td+t-1 \\ \Rightarrow \quad & 2|E(G[S])| > \left(d+1-\frac{1}{t}\right)(d+1) \\ \Rightarrow \quad & 2|E(G[S])| > (d+2)d+1-\left(\frac{d+1}{t}\right). \end{aligned}$$

Since $t \geq \frac{d+1}{2}$, we know that $\left(\frac{d+1}{t}\right) \leq 2$. Hence, we get that

$$2|E(G[S])| > (d+2)d-1,$$

or equivalently,

$$2|E(G[S])| \geq (d+2)d.$$

Since $|S| = d+2$, we also know that $2|E(G[S])| \leq d(d+2)$. So $G[S]$ must be a d-regular graph on $d+2$ vertices. However, this is not possible, since $d(d+2)$ is odd.

Now we know that we must have $|S| = d+1$ and d is even. This means that $|S|$ achieves the lower bound provided by Proposition 5.1.3. So, $G[S]$ must be a copy of μK_{d+1} with fewer than $d/2$ edges missing. Since $t > d/2$, and since G is a t-multiple of a simple graph, this implies that $G[S] = \mu K_{d+1} = G$. Hence, we have our desired result. □

We now show that Theorem 5.3.1 is best possible. To this end, we provide two classes of examples - one class of graphs where maximum degree is even, and one class of graphs where maximum degree is odd.

Proposition 5.3.2. *Let t be any positive integer, and let d be any even integer such that $d \geq 2t$. Let G be the complement of any 2-regular graph on $d+3$ vertices. Then the maximum degree of G is d, and $\chi'(tG) = t(d+1)$.*

Proof. Since $\overline{G}$ is 2-regular and $|V(G)| = d+3$, G is d-regular and $|E(G)| = d(d+3)/2$. Since $|V(G)| = d+3$ is odd, we get that

$$\begin{aligned} \chi'(tG) &\geq \lceil \rho(tG) \rceil \geq \left\lceil \frac{2t[d(d+3)/2]}{d+2} \right\rceil = \left\lceil \frac{td^2+3td}{d+2} \right\rceil \\ &= td + \left\lceil \frac{td}{d+2} \right\rceil = td + \left\lceil t - \frac{2t}{d+2} \right\rceil. \end{aligned}$$

Since $d \geq 2t$, this implies that

$$\chi'(tG) \geq td + t,$$

and hence $\chi'(tG) = t(d+1)$. □

Proposition 5.3.3. *Let t be any positive integer, and let d be any odd integer such that $d \geq 2t+1$. Define G to be the complement of $(d-1)/2$ single edges and one path of length two. Then the maximum degree of G is d, and $\chi'(tG) = t(d+1)$.*

Proof. Note that $|V(G)| = d+2$, and every vertex in G has degree exactly d, except for one vertex of degree $d-1$. So, $|E(G)| = (d(d+2)-1)/2$. Since $|V(G)| = d+2$ is odd, we get that

$$\begin{aligned} \chi'(tG) &\geq \lceil \rho(tG) \rceil \geq \left\lceil \frac{2t[(d(d+2)-1)/2]}{d+1} \right\rceil = \left\lceil \frac{td^2+2td-t}{d+1} \right\rceil \\ &= td + \left\lceil \frac{td-t}{d+1} \right\rceil = td + \left\lceil t - \frac{2t}{d+1} \right\rceil. \end{aligned}$$

Since $d \geq 2t+1$, this implies that

$$\chi'(tG) \geq td + t,$$

and hence $\chi'(tG) = t(d+1)$. □

Theorems 5.3.1, 5.3.2 and 5.3.3 provide a threshold for multiples of simple graphs achieving Vizing's theorem. For any pair of positive integers t, d with $t > \frac{d}{2}$, the t-multiple of a simple graph G (where G has maximum degree d) achieves Vizing's upper bound if and only if G is an odd clique. On the other hand, for every pair of positive integers t, d with $t \leq \frac{d}{2}$, there exists a simple graph G with maximum degree d, that is not an odd clique, such that tG achieves Vizing's upper bound.

Chapter 6

Vertex-colouring

Throughout this thesis, we have been discussing edge-colourings of multigraphs. Of course, an edge-colouring of a multigraph G is equivalent to a vertex-colouring of the line graph of G, that is,

$$\chi'(G) = \chi(L(G)).$$

One side of this equivalency tells us that all known vertex-colouring results can be translated into edge-colouring results, and we discuss this in the first section of this chapter. Since line graphs of multigraphs are such a special class, general vertex-colouring results, such as Brooks' Theorem, tend to be quite weak when translated into the realm of edge-colouring. One possible exception to this norm is the bound provided by Reed's Conjecture, and we present some results here discussing what high chromatic index could mean with respect to this bound. Section 6.2 is devoted to the other half of the equivalence — translating edge-colouring results into vertex-colouring results. Ideally, such vertex-colouring results would be shown to hold for more than just line graphs of multigraphs. In particular, given that we have used Tashkinov trees as our main edge-colouring technique in this thesis, it would be nice to extend this method to vertex-colour some graphs that are not line graphs of multigraphs. We explore this possibility in detail in Section 6.3; however, it appears that the method of Tashkinov trees does not work beyond edge-colouring.

6.1 Vertex-colouring results as edge-colouring results

Given a graph G, the standard upper and lower bounds for chromatic number are

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1,$$

where $\omega := \omega(G)$ denotes clique number (the size of the largest clique in G). Although both of these bounds are easy to see, they are each best-possible in the sense that there are examples of graphs which achieve each extreme. Hence, it is worth seeing what this range implies in terms of edge-colouring.

Let G be any multigraph. Then, the above bounds imply that

$$\omega(L(G)) \leq \chi'(G) \leq \Delta(L(G)) + 1.$$

Note that $\Delta(L(G)) = \Delta_e(G)$, where we define $\Delta_e(G)$ to be the maximum *edge-degree* of G, that is, the maximum number of edges adjacent to any edge in G. The clique number of $L(G)$ is more difficult to translate, since a clique in $L(G)$ can come from any set of pairwise adjacent edges in G. This means that

$$\omega(L(G)) = \max\left\{\Delta(G), t(G)\right\},$$

where we define $t(G)$ to be the *triangle thickness* of G, that is, the maximum number of edges in any single triangle of G. So, in general, the canonical vertex-colouring bounds translate to

$$\max\left\{\Delta(G), t(G)\right\} \leq \chi'(G) \leq \Delta_e(G) + 1$$

for edge-colouring. The following Theorem of Brooks from 1941 provides some additional information (see e.g. [7]).

Theorem 6.1.1. (Brooks' Theorem) *Let G be a simple connected graph which is not an odd cycle and not complete. Then,*

$$\chi(G) \leq \Delta(G).$$

Think about replacing the graph in the Brooks' Theorem with $L(G)$, where G is a multigraph. We must ensure that $L(G)$ is not an odd cycle - however since $L(G)$ is connected, this happens if and only if the multigraph G is an odd cycle, or if it is a copy of $K_{1,3}$ (possible if $L(G)$ is a triangle). Note that $K_{1,3}$ is an example of a *multi-star* - a connected multigraph where every edge is incident to a common vertex. Also, note that $L(G)$ is a complete graph if and only if the multigraph G is a multi-star, or if it is a triangle (possible if $L(G)$ is a triangle). So, Brooks' Theorem as it applies to line graphs of multigraphs can be restated as follows:

Let G be a connected multigraph which is not an odd cycle and not a multi-star. Then,

$$\chi'(G) \leq \Delta_e(G).$$

This version of the theorem makes sense in terms of what we already know about edge-colourings - at least we can easily see that if G is an odd cycle, then $\chi'(G) =$

$3 > \Delta_e(G) = 2$, and if G is a multi-star, then $\chi'(G) = |E(G)| > \Delta_e(G) = |E(G)| - 1$. As we have not seen any edge-colouring results using the parameter Δ_e however, it is difficult to immediately compare such an upper bound with what we already know. In general,

$$\Delta - 1 \leq \Delta_e \leq 2\Delta - 1,$$

so Brooks' Theorem does imply a bound of

$$\chi' \leq 2\Delta - 1.$$

This bound is extremely weak however - it is even weaker than Shannon's bound of $\frac{3\Delta}{2}$, for instance.

One vertex-colouring bound that may be more meaningful for edge-colouring is the bound suggested by Reed's Conjecture. The conjecture combines the natural lower bound ω with the natural upper bound $\Delta + 1$.

Conjecture 6.1.2. [34] (Reed's Conjecture) *Let G be a simple graph. Then,*

$$\chi(G) \leq \left\lceil \frac{\Delta + 1 + \omega(G)}{2} \right\rceil.$$

Reed's Conjecture has been proved for line graphs of multigraphs, by King, Reed and Vetta [23]. (In fact, King and Reed have recently extended this proof to include all claw-free graphs). King, Reed and Vetta's result, translated into the language of edge-colouring, reads as follows.

Theorem 6.1.3. [23] *Let G be a multigraph. Then,*

$$\chi'(G) \leq \left\lceil \frac{\Delta_e + 1 + \omega(L(G))}{2} \right\rceil.$$

From our discussion above about the parameter ω, we know that

$$\left\lceil \frac{\Delta_e + 1 + \Delta(G)}{2} \right\rceil \leq \left\lceil \frac{\Delta_e + 1 + \omega(L(G))}{2} \right\rceil.$$

Moreover, we know that if this equality is strict, then it is because $\omega(L(G))$ comes from $t(G)$ instead of $\Delta(G)$. Hence, we get the following proposition.

Proposition 6.1.4. *Let G be a multigraph. Suppose that*

$$\chi'(G) = \left\lceil \frac{\Delta_e + \Delta + 1}{2} \right\rceil + k$$

for some $k \geq 1$. Then, G contains a $(\Delta + 2k - 1)$-sided triangle.

Proof. Note that

$$\chi'(G) = \left\lceil \frac{\Delta_e + \Delta + 1}{2} \right\rceil + k = \left\lceil \frac{\Delta_e + 1 + (\Delta + 2k)}{2} \right\rceil.$$

Theorem 6.1.3 tells us that we must have

$$\left\lceil \frac{\Delta_e + 1 + (\Delta + 2k)}{2} \right\rceil \leq \left\lceil \frac{\Delta_e + 1 + \omega(L(G))}{2} \right\rceil,$$

which means that we must have

$$\omega(L(G)) \geq \Delta + 2k - 1.$$

Since $\omega(L(G)) > \Delta(G)$, it must be the case that

$$\omega(G) = t(G) \geq \Delta + 2k - 1.$$

So, G contains a $(\Delta + 2k - 1)$-sided triangle. □

Proposition 6.1.4 is somewhat reminiscent of Theorem 3.3.1, where Kierstead proved that if $\chi'(G) = \Delta + \mu$, then G must contain a 2μ-sided triangle. It is difficult to compare Theorem 6.1.3 and Vizing's Theorem however, because of the possible variation of Δ_e. On one extreme, if $\Delta_e = 2\Delta - 1$, then

$$\left\lceil \frac{\Delta_e + \Delta + 1}{2} \right\rceil = \left\lceil \frac{3\Delta}{2} \right\rceil,$$

so even the stronger version of Theorem 6.1.3 (with Δ in place of ω) is weaker than Shannon's Theorem. At the other extreme end, the only way to get $\Delta_e = \Delta - 1$ is if G is a multi-star, in which case

$$\chi'(G) = \Delta = \left\lceil \frac{\Delta_e + \Delta + 1}{2} \right\rceil.$$

There are a larger variety of examples of multigraphs with Δ_e equal to Δ or $\Delta + 1$, and in these cases

$$\left\lceil \frac{\Delta_e + \Delta + 1}{2} \right\rceil = \Delta + 1,$$

which in our viewpoint is as low a bound as one would hope to get for chromatic index. So, the range of the bound that we are interested in, in terms of studying multigraphs with high chromatic index, is when $\Delta_e \geq \Delta + 2$. With this provision, we can say something about those multigraphs that achieve $\lceil \frac{\Delta_e+\Delta+1}{2} \rceil$.

Theorem 6.1.5. *Let G be a multigraph with $\Delta_e \geq \Delta + 2$, and suppose that*

$$\chi'(G) = \left\lceil \frac{\Delta_e + \Delta + 1}{2} \right\rceil.$$

Then, G must contain one of the following:

1. *a 3-vertex subgraph in which one pair of vertices is adjacent, and there are $2\left\lceil \frac{\Delta_e - \Delta - 1}{2} \right\rceil + 1$ edges between the third vertex and the first two vertices; or*
2. *a 4-vertex subgraph which consists of a path of length two, and $3\left\lceil \frac{\Delta_e - \Delta - 1}{2} \right\rceil + 2$ edges between the fourth vertex and the path.*

Proof. Let ϕ be a partial $(\chi' - 1)$-edge colouring of G with maximum domain. Note that since

$$\chi' - 1 = \left\lceil \frac{\Delta_e + \Delta + 1}{2} \right\rceil - 1 \geq \left\lceil \frac{2\Delta + 3}{2} \right\rceil - 1 \geq \Delta + 1,$$

we may apply Kierstead's Theorem. In fact, choose any uncoloured edge e_0, and let $P = (p_0, e_0, p_1, \ldots, p_n)$ be a maximal ϕ-Kierstead path beginning with e_0. Since ϕ has maximum domain, we know that P must be ϕ-elementary (by Kierstead's Theorem), and we also know that $n \geq 2$.

Since P is ϕ-elementary, we know that

$$\left| \bigcup_{i=0}^{n-1} \phi(p_i) \right| \geq n\left(\chi' - \Delta - 1\right) + 2 \geq n \left\lceil \frac{\Delta_e - \Delta - 1}{2} \right\rceil + 2.$$

Of course, the fact that P is ϕ-elementary also implies that p_n must see every colour in $|\cup_{i=0}^{n-1} \phi(p_i)|$. Since P is maximal, all these colours must occur on edges between p_n and $(p_0, \ldots, p_{n-1})$.

Suppose first that n is even. Then, we can view $(p_0, \ldots, p_{n-1})$ as $n/2$ pairs of adjacent vertices, (p_i, p_{i+1}), where $i \in \{0, \ldots, n-1\}$. So, there are at least

$$\begin{aligned} \left\lceil \frac{2}{n} \cdot \left(n \left\lceil \frac{\Delta_e - \Delta - 1}{2} \right\rceil + 2 \right) \right\rceil &= 2 \left\lceil \frac{\Delta_e - \Delta - 1}{2} \right\rceil + \left\lceil \frac{4}{n} \right\rceil \\ &\geq 2 \left\lceil \frac{\Delta_e - \Delta - 1}{2} \right\rceil + 1 \end{aligned}$$

edges between one of these pairs and p_n. Hence, we get our desired 3-vertex subgraph.

We may now assume that n is odd. We may also assume that G does not contain our desired 3-vertex subgraph. Hence, there are at most

$$\frac{n-3}{2} \cdot 2 \left\lceil \frac{\Delta_e - \Delta - 1}{2} \right\rceil = (n-3) \left\lceil \frac{\Delta_e - \Delta - 1}{2} \right\rceil$$

edges between p_n and $(p_3, \ldots, p_{n-1})$. This means that between p_n and the path (p_0, p_1, p_2), there are at least

$$n\left\lceil\frac{\Delta_e-\Delta-1}{2}\right\rceil+2-(n-3)\left\lceil\frac{\Delta_e-\Delta-1}{2}\right\rceil=3\left\lceil\frac{\Delta_e-\Delta-1}{2}\right\rceil+2$$

edges. Hence, we get our desired 4-vertex subgraph. □

The above result and its proof is very reminiscent of our results in Section 3.3 regarding Vizing's upper bound, and again Kierstead's Theorem 3.3.1. For example, one similarity is that we are again using Kierstead paths instead of the more general Tashkinov trees, so that we will know more about the specific structure of the subgraph we find. One notable difference between Theorem 6.1.5 and our Theorems 3.3.3, 3.3.4 and 3.3.5 is that we have no analogue to Lemma 3.3.2, and hence we cannot assume that P has more than three vertices. So, we may only be able to get a 3-vertex subgraph. Theorem 3.3.1 also had this constraint, but the main difference between Theorem 3.3.1 and Theorem 6.1.5 is that while Kierstead was able to prove his result for a single necessary subgraph, we require a second possibility. The reason we need two subgraphs for our above proof is to contend with the case when n is odd. If n is odd and we are talking about Vizing's upper bound, we can always bound the number of edges between p_n and p_{n-1} with μ. This is reasonable (and in the case of Theorem 3.3.1, enough), because $\chi - 1 - \Delta = \mu - 1$. Unfortunately, we do not have the parameter μ to work with above. While we could guarantee that there are no more than $\frac{\Delta_e}{2}$ edges between p_n and some other p_i, our value of $\chi' - 1 - \Delta$ may be significantly lower than $\frac{\Delta_e}{2}$. So, including the second possible necessary subgraph appears to be the only reasonable solution.

6.2 Edge-colouring results as vertex-colouring results

We now turn to the other side of the line graph equivalence: viewing edge-colouring results as vertex-colouring results. In order to properly state edge-colouring results as vertex-colouring results, we need to understand how to describe the class of line graphs of multigraphs. In 1968, Beineke [3] famously characterized line graphs of simple graphs, proving the following theorem.

Theorem 6.2.1. [3] (Beineke's Theorem) *Let G be a simple graph. G is the line graph of a simple graph if and only if none of $G_1, \ldots, G_9$ occur as an induced subgraph of G (see Fig. 6.1).*

In the 1970's, Bermond and Meyer [4] (see also Hemmiger [14]) proved the following multigraph analogue of Beineke's Theorem.

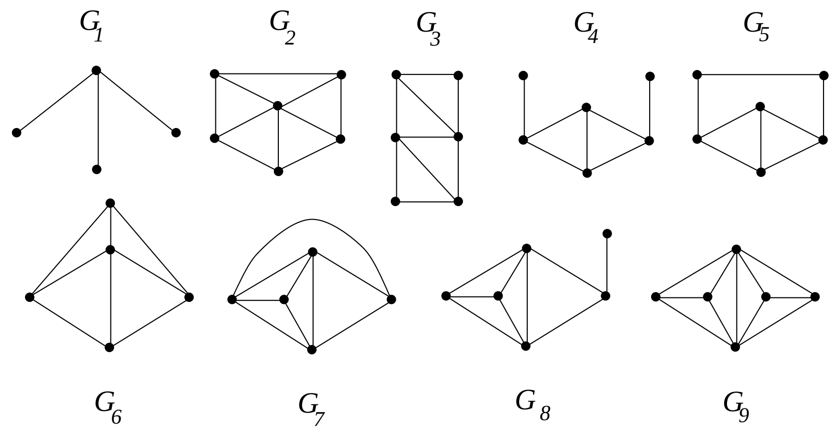

Figure 6.1: The forbidden subgraphs of Beineke's Theorem

Theorem 6.2.2. [4] *Let G be a simple graph. G is the line graph of a multigraph if and only if none of $F_1, \ldots, F_7$ occur as induced subgraph of G (see Fig. 6.2).*

Note that in Beineke's characterization of line graphs of simple graphs, there are nine forbidden subgraphs, $G_1, \ldots, G_9$ - and $F_1, \ldots, F_7$ is not a subset of this list. In fact, the two lists share only three graphs in common: $F_1 = G_1$ (the claw $K_{1,3}$), $F_2 = G_2$ and $F_3 = G_3$.

It is now possible for us to restate any edge-colouring result as a vertex colouring result — or at least try to. We can replace "Let G be a multigraph" with "Let G be a graph that does not induce any of $F_1, \ldots F_7$", and then replace $\chi'(G)$ with $\chi(G)$. The difficulty comes in re-expressing the parameters of the multigraph as parameters of the line graph of the multigraph.

We saw in the previous section that for any multigraph G,

$$\Delta(L(G)) = \Delta_e(G)$$

and

$$\omega(L(G)) = \max\{\Delta(G), t(G)\}.$$

So now, if we have a chromatic index bound which involves $\Delta(G)$, we can replace this with $\omega(L(G))$ for the vertex version, but the resulting statement might be weaker. For example, in general, Shannon's Theorem implies the following.

Theorem 6.2.3. (Vertex version of Shannon's Theorem) *Let G be a simple graph that does not induce any of $F_1, \ldots, F_7$. Then,*

$$\chi(G) \leq \frac{3\omega}{2}.$$

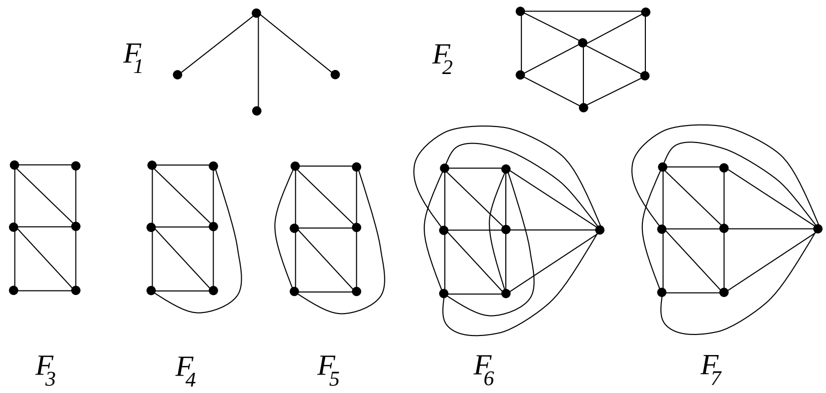

Figure 6.2: The forbidden subgraphs of Theorem 6.2.2

If G is the line graph of a multigraph which has a triangle containing at least $\Delta + 1$ edges, then Shannon's Theorem actually implies that

$$\chi(G) < \frac{3\omega}{2}.$$

Still, Theorem 6.2.3 gives us a reasonable bound for the chromatic number of line graphs of multigraphs. It is natural to wonder whether this bound actually holds for a larger class of simple graphs. For example, would Shannon's result still hold if one or more of $F_1, \ldots, F_7$ were removed from the list? The answer to this question is not known, however similar questions have been asked and answered affirmatively.

As we wrote Shannon's Theorem in terms of vertex-colouring, we can also write Vizing's Theorem for simple graphs as follows.

Theorem 6.2.4. (Vertex version of Vizing's Theorem for simple graphs) *Let G be a graph that does not induce any of $G_1, \ldots, G_9$. Then,*

$$\chi(G) \leq \omega + 1.$$

Javdekar [18] conjectured that the result of Theorem 6.2.4 would still hold if only G_1 or G_7 ($K_{1,3}$ or K_5^-) were forbidden as induced subgraphs. Kierstead and Schmerl [22] proved that Javdekar's Conjecture was equivalent to the following statement: "If G is a multigraph with $\mu = 2$, which does not contain a 4-sided triangle, then $\chi'(G) \leq \Delta + 1$." Of course, Kierstead's [21] Theorem 3.3.1 does prove this, and hence Javdekar's Conjecture is now the following theorem.

Theorem 6.2.5. [21] *Let G be a graph that does not induce G_1 or G_7. Then,*

$$\chi(G) \leq \omega + 1.$$

In fact, this result has since been extended to an even larger class by Randerath [32] (see [33]).

It is not entirely straightforward to state a vertex version of Vizing's Theorem for multigraphs, because of the μ parameter. Certainly though, since a multiedge corresponds to a clique in its line graph, Vizing's Theorem implies the following.

Theorem 6.2.6. *Let G be a graph that does not induce any of $F_1, \ldots, F_7$. Then,*

$$\chi(G) \leq 2\omega.$$

Although Theorem 6.2.6 is even weaker than Theorem 6.2.3, it is noteworthy because it has been extended to claw-free graphs by Seymour and Chudnovsky [6]. (By claw-free, we mean containing no induced $K_{1,3} = F_1 = G_1$). Note that in their result, which follows, $\alpha(G)$ is the independence number of G, i.e., the size of the largest independent set in G.

Theorem 6.2.7. [6] *Let G be a claw-free graph with $\alpha(G) \geq 3$. Then,*

$$\chi(G) \leq 2\omega.$$

In this thesis, all of our edge-colouring bounds were proved using the method of Tashkinov trees. So, in the next section, we explore the possibility of using this technique to vertex-colour graphs which are not line graphs of multigraphs.

6.3 Vertex-Tashkinov trees?

So far we have only seen one characterization of line graph of multigraphs, and using this characterization alone, it is not at all clear how one could define Tashkinov trees in a line graph - let alone prove a vertex version of Tashkinov's Theorem. However, there is a second characterization of line graphs of multigraphs which is much more natural in this situation. As Theorem 6.2.2 is a multigraph analogue of Beineke's Theorem, this second characterization is a multigraph analogue of Krausz's Theorem [25] — a theorem dating from the 1940's, and is based on the idea of a clique-partition. The multigraph analogue, which follows, was again proved by Bermond and Meyer [4] (see also Hemmiger [14]) in the 1970's.

Theorem 6.3.1. [4] *Let G be a simple graph. G is the line graph of a multigraph if and only if there exists a family of subgraphs $\mathcal{W} = \{W_i | i \in I\}$ of G such that:*

(L1.) every edge of G occurs in at least one W_i,

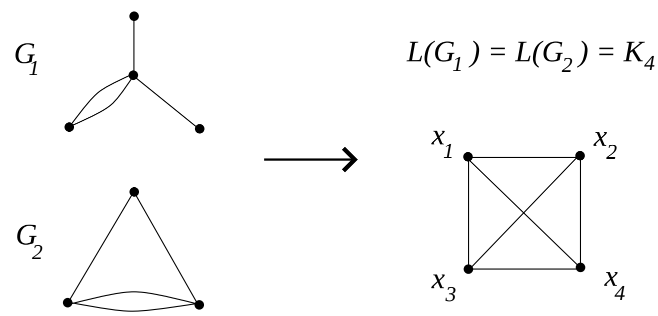

Figure 6.3: Two multigraphs with the same line graph.

(L2.) every vertex of G *occurs in exactly two* W_i*, and*

(L3.) each W_i *is a clique.*

It is not hard to understand why Theorem 6.3.1 works - each W_v corresponds to a vertex v in the multigraph, so that the vertices of the clique represent the edges incident to v in the multigraph. Each vertex of the line graph occurs in exactly two cliques of $\mathcal{W}$, because the corresponding edge in the multigraph has exactly two end vertices. The difference between Theorem 6.3.1 and Krausz's Theorem is that for simple graphs, the (L1) condition says that an edge occurs in exactly one clique. In both cases, it is clear why we would want each edge to occur in at least one clique - we want $\mathcal{W}$ to give us all the information about adjacencies in G. To see what the difference between the two theorems means, consider a pair of adjacent vertices x and y in the multigraph. These vertices will correspond to two cliques W_x and W_y in the line graph, and each edge between x and y in the multigraph will correspond to a vertex in $W_x \cap W_y$. If there is only one edge between x and y there will not be any edges in $W_x \cap W_y$, however we will get edges in this intersection if there are parallel edges between x and y.

Given two multigraphs $G_1 \neq G_2$, it is possible that $L(G_1) = L(G_2)$. For example, a multi-star with $|E|$ edges and triangle with $|E|$ edges both give $K_{|E|}$ as a line graph. However, note that when we pair a simple graph G with a set $\mathcal{W}$ satisfying (L1), (L2) and (L3), we do get a one-to-one relationship between the set of all multigraphs and the set of all line graphs of multigraphs. Figure 6.3 gives an example of this: pair K_4 with either $\mathcal{W}_1 = \{(x_2), (x_4), (x_1, x_3), (x_1, x_2, x_3, x_4)\}$ (corresponding to G_1) or $\mathcal{W}_2 = \{(x_1, x_3, x_4), (x_2, x_3, x_4), (x_1, x_2)\}$ (corresponding to G_2). This correspondence between a multigraph and its line graph is another reason that Theorem 6.3.1 is better suited to our work in this section than the other characterization theorem we have seen, Theorem 6.2.2.

Given the equivalence provided by Theorem 6.3.1, there is a natural way for us to define *vertex-Tashkinov trees.* Let G be a graph and suppose that there exists a family of subgraphs $\mathcal{W} = \{W_i | i \in I\}$ of G such that (L1), (L2) and (L3) hold. Let ϕ be a partial vertex-colouring of G. We say that $T = (v_0, .., v_{n-1})$ is a *ϕ-vertex-Tashkinov tree* in G if

(V1.) $v_0, \ldots, v_{n-1}$ are distinct vertices in G and there exist distinct $W_0, \ldots, W_n \in \mathcal{W}$ such that, for all $i \in \{0, \ldots, n-1\}$,

$$v_i \in W_k \cap W_{i+1}$$

for some $k \in \{0, \ldots, i\}$, and

(V2.) v_0 is uncoloured by ϕ and for each $i \in \{1, \ldots, n-1\}$,

$$\phi(v_i) \in \bigcup_{j \leq i} \overline{\phi}(W_j),$$

where $\overline{\phi}(W_j)$ is the set of colours not used at any vertex on W_j. Note that this last bit of notation differs slightly from the notation we have been using for edge-colouring. The reason for this change is simply because of the fact when ϕ is an edge-colouring and v is a vertex, $\phi(v)$ has no inherent meaning, while when ϕ is a vertex-colouring and W is a clique, $\phi(W)$ is already understood to mean the set of colours used on W. The rest of our notation and terminology stays the same in this chapter - for example, a ϕ-vertex Tashkinov tree T is called *ϕ-elementary* if

$$\overline{\phi}(W_i) \cap \overline{\phi}(W_j) = \emptyset$$

for all $0 \leq i < j \leq n$.

In light of Theorem 6.3.1, ϕ-Tashkinov trees in multigraphs correspond exactly to ϕ-vertex Tashkinov trees in line graphs of multigraphs. So, we can restate Tashkinov's Theorem from this vertex-colouring viewpoint, with an eye towards possible extensions.

Theorem 6.3.2. (Tashkinov's Theorem for vertex-colouring) *Let G be a simple graph with $\mathcal{W} = \{W_i | i \in I\}$ a family of subgraphs of G such that (L1), (L2) and (L3) hold. Let ϕ be a partial $(\omega + s)$-vertex colouring of G, with $s \geq 1$ and $\omega \geq |W_i|$ for all $i \in I$. Suppose that there exists a ϕ-vertex-Tashkinov tree $T = (v_0, \ldots, v_{n-1})$ in G that is not ϕ-elementary. Then, there exists a $(\omega + s)$-vertex colouring ψ of $dom(\phi) \cup \{v_0\}$.*

If Theorem 6.3.2 still held with one of the conditions (L1), (L2) or (L3) missing or relaxed, then the Tashkinov tree method could be used to vertex-colour graphs which are not line graphs of simple graphs. Unfortunately, this does not appear to be the case.

We saw the definition of Kierstead paths in Section 2.2. Now, it is possible to define a *ϕ-vertex-Kierstead path* as a ϕ-vertex-Tashkinov tree with the property that in (V1), we must always have $k = i$. So, the following Theorem is direct corollary of Theorem 6.3.2. We provide the proof (which follows the proof of Kierstead's proof of Theorem 2.2.2 closely) so as to highlight all the places where properties (L1), (L2) and (L3) are used.

Theorem 6.3.3. (Kierstead's Theorem for vertex-colouring) *Let G be a simple graph with $\mathcal{W} = \{W_i | i \in I\}$ a family of subgraphs of G such that (L1), (L2) and (L3) hold. Let ϕ be a partial $(\omega + s)$-vertex colouring of G with $s \geq 1$ and suppose that there exists a ϕ-vertex-Kierstead path $P = (v_0, \dots, v_{n-1})$ in G that is not ϕ-elementary. Then, there exists a $(\omega + s)$-edge colouring ψ of $dom(\phi) \cup \{v_0\}$.*

Proof. Since P is not ϕ-elementary, we know that there exists $\alpha \in \overline{\phi}(W_i) \cap \overline{\phi}(W_j)$ for some $0 \leq i < j \leq n$. Our argument is by induction on j.

If $j = 1$, then $\alpha \in \overline{\phi}(W_0) \cap \overline{\phi}(W_1)$. By (L1), every neighbour of v_0 is in some W_k with v_0. By (L2), v_0 is only in W_0 and W_1. So, every neighbour of v_0 is in W_0 or W_1, meaning that v_0 has no neighbour coloured α. Hence, we can extend ϕ to v_0 simply by defining $\phi(v_0) = \alpha$.

We may now assume that $j \geq 2$ and proceed by a secondary induction on $j - i$. Suppose first that $j - i = 1$. Consider the vertex $v_{j-1} = v_i$, which has the property that $v_{j-1} \in W_i \cap W_j$. In particular, this means that $\gamma := \phi(v_{j-1})$ is a different colour than α. Define ϕ' from ϕ simply by recolouring v_{j-1} with α. Note that ϕ' is proper for the same reasons given in the base case $j = 1$ (i.e., by (L1) and (L2)). Moreover, since we only changed the colour of v_{j-1} (and this can only affect W_{j-1} and W_j from $\mathcal{W}$), $P' = (v_0, \dots, v_{j-2})$ is clearly a ϕ'-vertex-Kierstead path. Since P is a ϕ-vertex-Kierstead path, there exists $m \leq j - 2$ such that $\gamma \in \overline{\phi}(W_m)$. Since we removed the colour γ from a vertex in $W_{j-1} = W_i$, and since W_{j-1} is a clique (by (L3)), we must have $\gamma \in \overline{\phi'}(W_{j-1})$. The fact that P' has $\gamma \in \overline{\phi'}(W_m) \cap \overline{\phi'}(W_{j-1})$ is enough to satisfy our primary induction hypothesis, since $j - 1 < j$.

Suppose now that $j - i \geq 2$. Since $s \geq 1$, we know that $|\overline{\phi}(W_{i+1})| \geq (\omega + s) - \omega \geq 1$. So, we may choose $\beta \in \overline{\phi}(W_{i+1})$. By the minimality of $j - i$, we know that $\beta \neq \alpha$ and moreover that α occurs on a vertex of W_{i+1}. Consider the maximal (α, β)-alternating component C which includes this vertex. Let ϕ' be the colouring obtained from ϕ by swapping the colours α and β on C. Note that since W_{i+1} is a clique (by (L3)), it contains only one vertex coloured α under ϕ, and hence $\overline{\phi'}(W_{i+1}) = \overline{\phi}(W_{i+1}) - \beta + \alpha$.

We claim that we may assume $\phi'(v_k) = \phi(v_k)$ for all $k \leq i$. To see this, note that if v_k is coloured α or β, it means that $\phi(v_k) \in \overline{\phi}(W_l)$ for some $l < k$. Since

$\alpha \in \overline{\phi}(W_i)$ and $\beta \in \overline{\phi}(W_{i+1})$, this means that either $\overline{\phi}(W_l) \cap \overline{\phi}(W_i)$ or $\overline{\phi}(W_l) \cap \overline{\phi}(W_{i+1})$ is nonempty. Either way, since $i, i+1 < j$, this contradicts the minimality of j.

We also claim that we may assume $\overline{\phi'}(W_k) = \overline{\phi}(W_k)$ for all $k < i$. Of course this is true if W_k contains no vertices in C, or contains vertices from C of both colours. So, the only way that $\overline{\phi'}(W_k) \neq \overline{\phi}(W_k)$, is if $W_k \cap C$ contains only vertices of a single colour. Since W_k is a clique (by (L3)), this implies that one of α or β must be missing from W_k. But then, this means that either $\overline{\phi}(W_k) \cap \overline{\phi}(W_i)$ or $\overline{\phi}(W_k) \cap \overline{\phi}(W_{i+1})$ is nonempty. Either way, since $i, i+1 < j$, this contradicts the minimality of j.

By the above two claims, we know that $P' = (v_0, \ldots, v_i)$ is a ϕ'-vertex-Kierstead path. If $\overline{\phi'}(W_i) = \overline{\phi}(W_i)$, then the fact that P' has $\alpha \in \overline{\phi'}(W_i) \cap \overline{\phi'}(W_{i+1})$ is enough to satisfy our primary induction hypothesis, since $i + 1 < j$.

We may now assume that $\overline{\phi'}(W_i) \neq \overline{\phi}(W_i)$. Since $\alpha \in \overline{\phi}(W_i)$ and W_i is a clique (by (L3)), this means that there was a single β-coloured vertex in W_i whose colour changed to α when we moved from ϕ to ϕ'. Hence, $\overline{\phi'}(W_i) = \overline{\phi}(W_1) - \alpha + \beta$.

Let us examine the component C more closely. Given any vertex $x \in C$, (L1) and (L2) tell us that $x \in W' \cap W''$ for some $W', W'' \in \mathcal{W}$, and that neighbours of x in C must lie in one of W' or W''. Since W' and W'' are both cliques (by (L3)) and all vertices of C are coloured α or β under ϕ, C contains at most one other vertex from each of W' and W''. We already know that C contains exactly one vertex from W_{i+1} and exactly one vertex from W_i. So, this means that if x is any other vertex in C, then x has a neighbour from W' in C, and x has a neighbour from W'' in C. In particular, this implies that $C \cap W_j = \emptyset$. This is because $W_j \neq W_i, W_{i+1}$, but W_j cannot contain two vertices of C since $\alpha \in \overline{\phi}(W_j)$ and W_j is a clique (by (L3)). So, we know that $\alpha \in \overline{\phi'}(W_j)$.

We have now established that P has $\alpha \in \overline{\phi'}(W_{i+1}) \cap \overline{\phi'}(W_j)$, which is enough to satisfy our secondary induction hypothesis. Hence, we have completed our proof. □

Properties (L1) and (L2) appear completely essential to Theorem 6.3.3 - even the base case of $i = 0$ and $j = 1$ requires both of these properties. This base case does not require property (L3) however, causing us to question whether or not Theorem 6.3.3 could hold with (L3) relaxed.

The case $j - i = 1$ in the above proof can in fact be resolved without appealing to (L3). If $W_{j-1} = W_i$ has multiple vertices coloured β under ϕ, then take the maximal (α, β)-alternating component Q containing these vertices, and include in the definition of ϕ' the swapping of α and β on Q. The only issue then is whether or not $P' = (v_0, \ldots, v_{j-2})$ would still be a ϕ'-vertex-Kierstead path. However, we may assume that $\phi'(v_k) = \phi(v_k)$ for all $k \leq i$ and $\overline{\phi'}(W_k) = \overline{\phi}(W_k)$ for all $k < i$, by the minimality of j, for the same reasons outlined in the inductive step above. Since $j-2=i-1$ in this case, this is more than enough.

Unfortunately, the inductive step of $j-i>1$ in the proof of Theorem 6.3.3 uses property (L3) in much more complex ways that the case $j-i=1$ does. In particular, without (L3), the component C might actually intersect W_j, which would cause major problems for our argument. Although it is possible that such issues could be resolved, it appears to us that Theorem 6.3.3 really requires all three of (L1), (L2) and (L3).

Since Theorem 6.3.3 needs (L1), (L2) and (L3) to hold, the proof of the vertex version of Tashkinov's Theorem definitely needs all three of these properties to hold. However, it is possible that there is another special case of Tashkinov's Theorem for which these assumptions could be relaxed. Apart from Kierstead paths, there is only one other special case of Tashkinov trees in the literature. *Multi-fans* were introduced by Favrholt, Stiebitz and Toft in [8], who noted that they were very well-suited to give short proofs of such classical results such a Shannon's Theorem and Ore's Theorem. As Kierstead paths are Tashkinov trees where in $k=i$ in (T1), multi-fans are Tashkinov trees where $k=0$ in (T1). Of course, we can also define *ϕ-vertex-multi-fans* in an analogous way. The following Theorem is the vertex-multi-fan analogue of Theorem 6.3.3.

Theorem 6.3.4. *Let G be a simple graph with $\mathcal{W}=\{W_i|i\in I\}$ a family of subgraphs of G such that (L1), (L2) and (L3) hold. Let ϕ be a partial $(\omega+s)$-vertex colouring of G with $s\geq 0$ and suppose that there exists a ϕ-vertex-multi-fan $F=(v_0,\dots,v_{n-1})$ in G that is not ϕ-elementary. Then, there exists a $(\omega+s)$-vertex colouring ψ of $dom(\phi)\cup\{v_0\}$.*

Theorem 6.3.4 is not exactly a special case of Theorem 6.3.2, as it holds for $s=0$ in addition to all $s\geq 1$. The reason for this will soon become apparent, as we will present a proof of Theorem 6.3.4, borrowing heavily from our proof of Theorem 6.3.3. Again, our aim is to highlight the usages of properties (L1), (L2) and (L3).

Proof. *(Theorem 6.3.4)* Since F is not ϕ-elementary, we know that there exists $\alpha\in\overline{\phi}(W_i)\cap\overline{\phi}(W_j)$ for some $0\leq i<j\leq n$. Our argument is by induction on j.

The case $j=1$ is word-for-word the same as our $j=1$ case in the proof of Theorem 6.3.3. So, we may assume that $j\geq 2$. Instead of a secondary induction on $j-i$, this time we proceed with a secondary induction on i.

The case $i=0$ is very similar to the case $j-i=1$ above, with a few key exceptions. We again consider the vertex v_{j-1}, which has the the property that $v_{j-1}\in W_0\cap W_j$, and $\phi(v_{j-1}):=\gamma\neq\alpha$. Since F is a ϕ-vertex-multi-fan, there exists $m\leq j-2$ such that $\gamma\in\overline{\phi}(W_m)$. We again define ϕ' from ϕ by recolouring

v_{j-1} with α, and again we know that ϕ' is proper by (L1) and (L2). We claim that $F' = (v_0, \ldots, v_{m-1})$ is a ϕ'-vertex-multi-fan. To see this, note that only the vertex v_{j-1} and W_0, W_j from $\mathcal{W}$ are affected by the colouring change, and $m - 1 < j - 1$. This is enough to show the claim, since while α is no longer missing from W_0, none of $v_1, \ldots, v_{m-1}$ are coloured α, because $v_1, \ldots, v_{m-1} \in W_0$, and $\alpha \in \overline{\phi}(W_0)$. Now, since W_0 is a clique (by (L3)) and we stripped the colour γ from a vertex in W_0, we must have $\gamma \in \phi'(W_0)$. The fact that $\gamma \in \phi'(W_0) \cap \phi'(W_m)$ in F' is enough to satisfy our primary induction hypothesis, since $m < j$.

Now suppose that $i \geq 1$. Since $s \geq 0$, we know that $|\phi(W_0)| \geq (\omega+s)-(\omega-1) \geq 1$ (because v_0 is uncoloured). So, we may choose $\beta \in \phi(W_0)$. By the minimality of i, we know that $\beta \neq \alpha$ and moreover that the colour α occurs on a vertex of W_0. Consider the maximal (α, β)-alternating component C which includes this vertex. Let ϕ' be the colouring obtained from ϕ by switching the colours α and β on Q. Note that, since W_0 is a clique (by (L3)), it contains only one vertex coloured α under ϕ, and hence $\overline{\phi'}(W_{i+1}) = \overline{\phi}(W_{i+1}) - \beta + \alpha$.

We claim that we may assume $\phi'(v_k) = \phi(v_k)$ for all $k \leq i$ and $\overline{\phi'}(W_k) = \overline{\phi}(W_k)$ for all $k < i$. These arguments are identical to those above in the proof of Theorem 6.3.3, except with W_0 playing the role of W_{i+1}. These claims are more than enough to imply that $F' = (v_0, \ldots, v_{i-1})$ is a ϕ'-vertex-multi-fan. If $\overline{\phi'}(W_i) = \overline{\phi}(W_i)$, then the fact that F' has $\alpha \in \overline{\phi'}(W_0) \cap \overline{\phi'}(W_i)$ is enough to satisfy our primary induction hypothesis, since $i < j$.

We may now assume that $\overline{\phi'}(W_i) \neq \overline{\phi}(W_i)$. Since $\alpha \in \overline{\phi}(W_i)$ and W_i is a clique (by (L3)), this means that there was a single β-coloured vertex in W_i whose colour changed to α when we moved from ϕ to ϕ'. Hence, $\overline{\phi'}(W_i) = \overline{\phi}(W_i) - \alpha + \beta$. Our analysis of the component C in the above proof still holds here, except that now, it is only the cliques W_i and W_0 which have exactly one vertex in C. Hence, as above, $\alpha \in \overline{\phi'}(W_j)$ by (L3). So, we have established that F has $\alpha \in \overline{\phi'}(W_0) \cap \overline{\phi'}(W_j)$, which is enough to satisfy our secondary induction hypothesis. Hence, we have completed our proof. □

Unfortunately, while the proof of Theorem 6.3.4 is somewhat different than that of Theorem 6.3.3, both proofs have the same dependencies on (L3). In particular, this proof again needs (L3) to show that $C \cap W_j = \emptyset$ in the inductive step, and without this the argument is in trouble.

While there may be a special case of Tashkinov's theorem which will hold without assuming all of (L1), (L2) and (L3), neither Kierstead paths nor multi-fans appear to fit the bill. After this investigation, it is our belief that Tashkinov trees are only useful for edge-colouring.

Chapter 7

Conclusion and future work

In this thesis we took a specialized approach to edge-colouring by focusing exclusively on multigraphs with high chromatic index. Now, we reflect on our contributions in this area, and discuss possibilities for future work.

One aspect of our work in this thesis was studying multigraphs which achieve known upper bounds, in particular Goldberg's and Vizing's classical upper bounds. Here, we were able to completely characterize those multigraphs achieving Goldberg's upper bound (Theorem 3.2.2), and provide some new necessary conditions for a multigraph to achieve Vizing's upper bound in general (Theorems 3.3.3 – 3.3.5). We also showed that if we restrict ourselves to large multiples of simple graphs, then those multigraphs achieving Vizing's upper bound can be characterized (Theorem 5.3.1).

Another aspect of our study was providing new upper bounds for chromatic index (Theorems 4.2.1, 4.2.2, 4.2.8 and 4.2.9), with accompanying edge-colouring algorithms (Theorems 4.3.1 – 4.3.4). Most significant among these are Theorems 4.2.1 and 4.2.2, which concern odd-girth and girth, respectively. These Theorems both imply that new cases of the Seymour-Goldberg Conjecture hold, as described by Theorem 4.2.3 and 4.2.4.

A third aspect of our work in this thesis was characterizing high chromatic index in general. We proved results about the size and structure of any set that determines high chromatic index via $\lceil \rho \rceil$ — that is, any set $S \subseteq V$, $|S| \geq 3$ and odd, such that

$$\chi' = \lceil \rho \rceil = \left\lceil \frac{2|E[S]|}{|S|-1} \right\rceil > \Delta + 1.$$

Using this analysis, we were able to get a very complete picture of high chromatic index with respect to odd-girth (Theorems 5.2.1 and 5.2.2). Our general results also enabled us to prove the characterization result for multiples of simple graphs that we have already mentioned.

In addition to the edge-colouring results of this thesis, we also contributed to the method of Tashkinov trees. We first solidified the theory, completing a correct English version of Theorem 2.2.1, and detailing the corresponding algorithm. In Chapter 4 we contributed to the theory by providing a general bound for edge-colouring depending on Tashkinov trees (Theorem 4.1.3), and in Chapter 6 we explored the limits of the method with our discussion and proof of Theorems 6.3.3 and 6.3.4. In Chapter 5, we saw that Tashkinov trees can be used in the characterization process (as in the proof of Theorem 3.2.2), but that we can also do the analysis by looking at more general sets S which determine high chromatic index.

Reflecting upon our results in this thesis, we can see that whenever we proved

$$\chi' = \lceil \rho \rceil > \Delta + 1,$$

we did so by finding a Tashkinov tree T with

$$\chi' = \lceil \rho \rceil = \left\lceil \frac{2|E[V(T)]|}{|V(T)| - 1} \right\rceil.$$

One major question that this brings to light is this the following.

Question 1. *Let G be a multigraph with $\chi' = \lceil \rho \rceil > \Delta + 1$. Then, must there exist a ϕ-Tashkinov tree T with*

$$\chi'(G) = \lceil \rho(G) \rceil = \left\lceil \frac{2|E[V(T)]|}{|V(T)| - 1} \right\rceil,$$

for some $(\chi' - 1)$-edge colouring ϕ of G with maximum domain?

What Question 1 is really getting at is the following question.

Question 2. *Can the Seymour-Goldberg Conjecture be proved using Tashkinov trees?*

Another variant of this can be worded as follows.

Question 3. *Let G be a multigraph with $\chi' > \Delta + 1$. Then, must there exist a maximal ϕ-elementary ϕ-Tashkinov tree in G that has no defective colours, for some $(\chi' - 1)$-edge colouring ϕ of G with maximum domain?*

An affirmative answer to Question 3 immediately implies the Seymour-Goldberg Conjecture, by Proposition 4.1.1. However, we aim to see the relationship between Tashkinov trees and the Seymour-Goldberg Conjecture with even more clarity. To this end, let us introduce to the discussion Andersen's Conjecture.

Andersen's Conjecture. [1] *Let G be a critical multigraph with $\chi' > \Delta + 1$. Then, there exists a $(\chi' - 1)$-edge colouring of $G - e$ for some e, such that no colour is missing at two different vertices.*

Andersen's Conjecture dates from 1977 and was shown to be equivalent to the Seymour-Goldberg Conjecture by Goldberg [11]. To see this equivalence, recall that we have already shown that the Seymour-Goldberg Conjecture is true for all multigraphs if and only if it is true for all critical multigraphs. Clearly, if Andersen's Conjecture is true, then we get the right number of edges for G to satisfy the "critical version" of the Seymour-Goldberg Conjecture that we saw in Section 5.1, that is,

$$|E| = (\chi' - 1)\left(\frac{|V| - 1}{2}\right) + 1.$$

Of course, having this number of edges in a critical multigraph implies that the edges are a union of $(\chi' - 1)$ near-perfect matchings, plus one edge. This immediately implies a $(\chi' - 1)$ edge colouring of $G - e$ (for any e). Such a colouring has the desired property of Andersen's Conjecture, because each colour class is a near-perfect matching of G. So, Andersen's Conjecture is indeed equivalent to the Seymour-Goldberg Conjecture. (Note that we have also shown that Andersen's Conjecture holds for one edge e if and only if it holds for every edge e).

We would like to relate our Question 3 to Andersen's Conjecture and thus we pose an equivalent question for critical multigraphs.

Question 4. *Let G be a critical multigraph with $\chi' > \Delta + 1$. Then, must there exist a ϕ-elementary ϕ-Tashkinov tree that spans G, for a $(\chi' - 1)$-edge colouring ϕ of $G - e$ for some e?*

Note that Questions 3 and 4 are equivalent because of our previous comments about critical multigraphs, and also because a Tashkinov tree that spans G clearly has no defective colours.

The similarity between Andersen's Conjecture and Question 4 is striking. The colouring required for Andersen's Conjecture consists of $(\chi' - 1)$ near perfect matchings of G, since no pair of vertices are allowed to have a common missing colour. This is exactly what is required of ϕ in Question 4, since the spanning Tashkinov tree is ϕ-elementary. The only difference between Andersen's Conjecture and Question 4 is that a Tashkinov tree structure is required in the latter case. Moreover, if these problems are equivalent, then that would imply that Question 1 is true, that Question 2 is true if the Seymour-Goldberg Conjecture can be proved at all, and that Question 3 is equivalent to the Seymour-Goldberg Conjecture.

If every colouring satisfying Andersen's Conjecture had a spanning Tashkinov tree, then this would certainly imply equivalence between Question 4 and Andersen's Conjecture. Unfortunately, the example of Figure 7.1 shows that this is not always the case. In the picture, the circles beside vertices denote that the colour in question is missing at that vertex. So, the colouring consists of exactly $(\chi' - 1) = \Delta + 1 = 10 + 1 = 11$ colours, and each colour is missing at exactly one vertex. However, any maximal Tashkinov tree starting with e_0 contains only the vertices above the dotted line, as none of the colours crossing this line are missing above it. So, this is an example of a colouring that satisfies Andersen's Conjecture, but does not contain a spanning Tashkinov tree.

Despite the example of Figure 7.1, Andersen's Conjecture and Question 4 would still be equivalent if, for every multigraph G satisfying Andersen's Conjecture, there exists *some* colouring satisfying the conjecture which also has a spanning Tashkinov tree. This immediately suggests the idea of somehow modifying one colouring satisfying Andersen's Conjecture to get another.

Let G be a multigraph, and let ϕ be a (possibly partial) edge-colouring of G. Given any two colours $\alpha \neq \beta$ in ϕ, note that α and β induce a subgraph of G whose components are all either double edges, alternating even cycles, or alternating paths. A modification to ϕ obtained by swapping α and β on one of these components is called an *interchange*. Note that if ϕ satisfies Andersen's Conjecture, then an interchange will not affect this.

It is unfortunately not true that every colouring satisfying Andersen's Conjecture has a sequence of interchanges that can be used to get a colouring which has a spanning Tashkinov tree. In fact, the colouring in our above example (Figure 7.1) has the property that every alternating component is either a double edge or an alternating path. So, there is no interchange which will actually modify this colouring (except to relabel the colour classes).

Despite what we have already said about our example in Figure 7.1, it does provide a glimmer of hope in terms of proving that Andersen's Conjecture and Question 4 are equivalent. This is because, while the colouring cannot be modified via interchanges to have a spanning Tashkinov tree, there is another way to make a satisfactory modification. This modified colouring is pictured in Figure 7.2, and the edges which have changed colour are in bold. Note that this modification caused no change in the set of colours missing at any vertex; however in the new colouring, every Tashkinov tree starting with e_0 is spanning.

We can view the bold edges in Figure 7.2 as a set of three alternating paths, each between the vertex missing yellow and the vertex missing orange. The modification made in moving from Figure 7.1 to Figure 7.2 can then be described as swapping

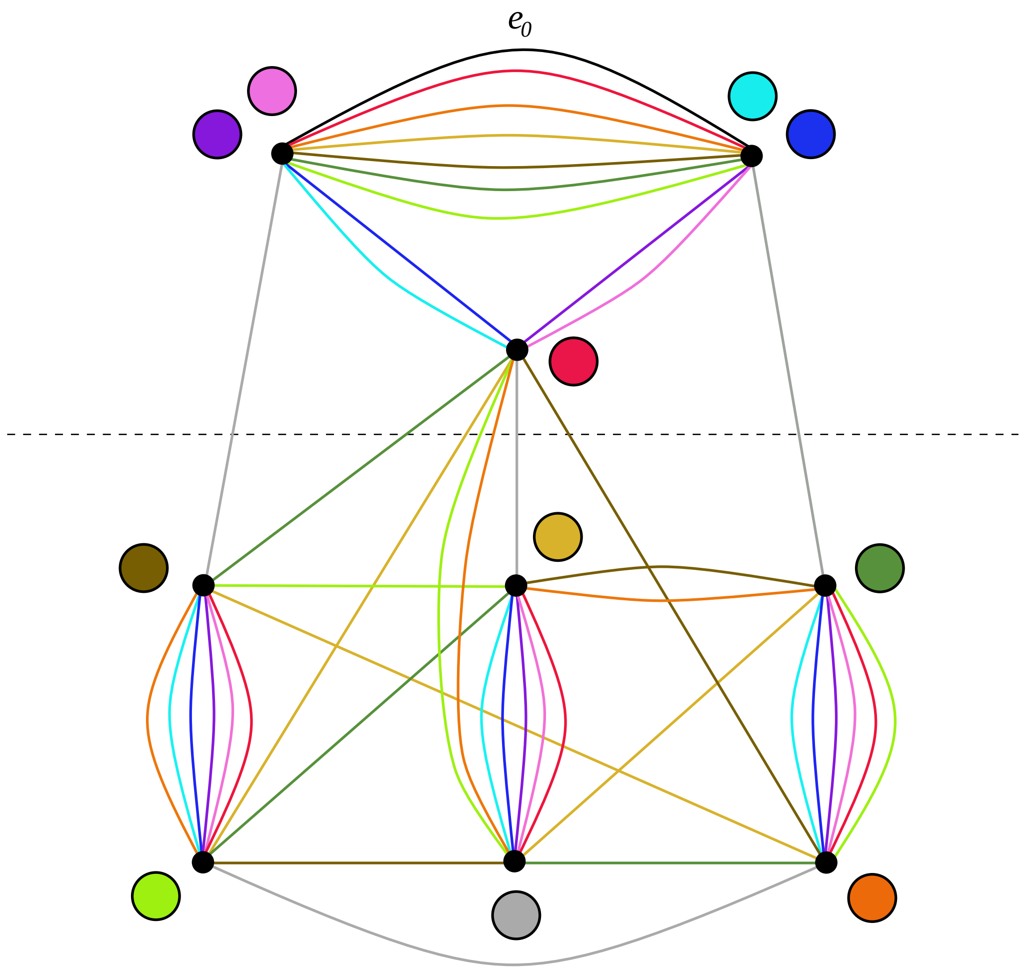

Figure 7.1: No maximal Tashkinov tree goes through the dotted line

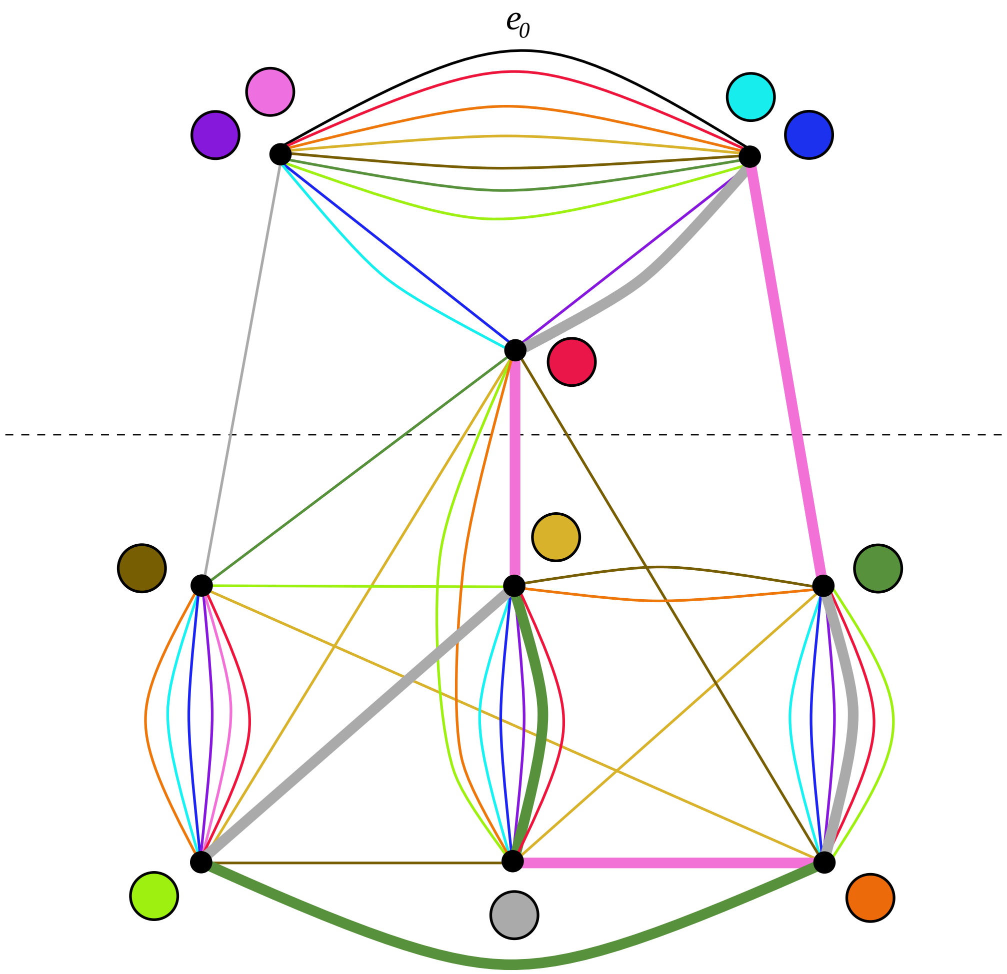

Figure 7.2: Every maximal Tashkinov tree is spanning

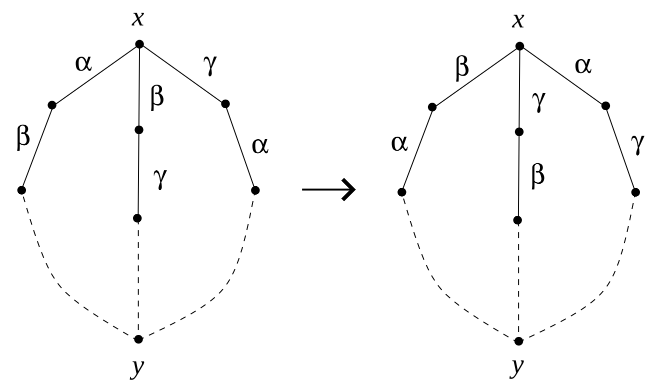

Figure 7.3: A colouring modification using three alternating paths

the colours along each of these three paths. In general, suppose we have a (partial) colouring ϕ of a multigraph G, with α, β, γ three colours of ϕ, and x and y two vertices of G. If x is incident to an edge of each of these colours, and there exists an (α, β)-alternating path, a (β, γ)-alternating path, and a (γ, α)-alternating path between x and y, then ϕ can be modified by swapping along all three alternating paths. (Note that these alternating paths do not need to be maximal). See Figure 7.3. It is difficult to see how this colouring modification scheme could be used in general, since the set-up is so specific. However, its existence in our example tells us that it is still possible for Andersen's Conjecture to be equivalent to Conjecture 4.

We would be remiss to talk about colouring modifications and interchanges without mentioning the following problem, posed by Vizing in 1965.

Question 5. ([43], see also [19]) *Let G be a multigraph, and let ϕ be a $(\chi' + k)$-edge-colouring of G. Is it always possible to obtain a χ'-edge-colouring of G from ϕ, by a sequence of interchanges?*

Note that in the above problem, the only way that the number of colours in ϕ is decreased via interchange, is if the component on which the interchange occurs is a single edge, and that edge is the only occurrence of that colour in the multigraph. Moreover, the only way to decrease the number of occurrences of a specific colour α in ϕ via interchange with β is for the component of interchange to be an (α, β)-alternating path P with both end-edges coloured α. There is a role, however, for alternating cycles to play in the interchange process. For example, doing an interchange of β and another colour γ on an alternating cycle might create the alternating path P.

Clearly, the possibilities for interchanges are complex. This may be the reason that Question 5 has received almost no attention in the literature. Jensen and Toft [19] included the problem in their 1995 book of graph colouring problems, but the only results they had to report were from Vizing himself. Namely, Vizing [43] had noted that if $\chi'(G) = \Delta + \mu$ or $\chi'(G) = 3\Delta/2$, then Question 5 has an affirmative answer. He also made a point to note that a general affirmative answer to his question would be guaranteed if it was true that, for any two different $(\chi' + k)$-edge-colourings of G using the same colours, there is a finite sequence of interchanges which translates one colouring into the other. However, Vizing provided a counterexample for this. While the colouring in our example of Figure 7.1 uses fewer than χ' colours, this whole problem seems highly related to our previous discussion.

In this thesis, rather than taking an arbitrary edge colouring with too many colours and trying to "work down" to χ', we use partial colourings with no more than χ' colours and try to "work up" to a full edge-colouring. Suppose ϕ is a partial χ'-edge-colouring. If there exists an uncoloured edge e with two ends missing a common colour (say α), then we can *extend* ϕ by colouring e with α. In light of Question 5, it thus makes sense to ask the following question.

Question 6. *Let G be a multigraph, and let ϕ be a partial χ'-edge-colouring of G. Is it always possible to obtain a χ'-edge-colouring of G from ϕ, by a sequence of interchanges and extensions?*

It is not difficult to see that an affirmative answer to Question 6 implies an affirmative answer to Question 5. Whether the two questions are equivalent though, or whether Question 6 is strictly harder, is not clear to us. However, we cannot help but think that if Tashkinov trees are enough to prove the Seymour-Goldberg Conjecture, then alternating paths must say nearly everything about chromatic index - or about high chromatic index at least.

Questions 1 - 6 are not the only direction in which to continue work from this thesis. In terms of characterizations, there is still very little known about Vizing's upper bound in general, and even more necessary conditions would be helpful. Moreover, recall that as Goldberg's bound generalizes Shannon's upper bound, Steffen's bound (Theorem 2.1.5) generalizes Vizing's upper bound. Apart from what is known about Vizing's upper bound, there is absolutely nothing known about the class of multigraphs achieving Steffen's upper bound, and this is a tantalizing open problem.

In terms of edge-colouring bounds, there is still a large distance between what is known, and the Seymour-Goldberg Conjecture. While there may be new ways to apply our general bound (Theorem 4.1.3), an even better approach might be to try to further build upon the method of Tashkinov trees, so that the general bound itself could be improved. Also, Lemma 4.2.5, which tells us that we can get a Tashkinov tree T where only at most $(|V(T)| - 1)/2$ different colours are used on the edges of

T, seems ripe for improvement. If this number of colours could be reduced, it would lead to a direct improvement of Theorem 4.2.8.

In this thesis we have made significant headway in the study of multigraphs with high chromatic index. However, all the specific problems mentioned here underscore the fact that a great deal remains to be known. With continued work, there is hope for many more interesting results in this area.

References

[1] I. D. Andersen. On edge-colourings of graphs, Math. Scand. 40 (1977) 161-175. 98

[2] D. Avis, C. De Simone, and B. Reed. On the fractional chromatic index of a graph and its complement, Operations Research Letters 33 (2005) 385-388. 7

[3] L. Beineke. Derived graphs and digraphs, Beiträge Zur Graphentheorie (H. Sachs, H. Voss, and H. Walter, eds.) Teubner, Leipzig, 1968, 17-33. 86

[4] J. C. Bermond and J. C. Meyer. Graphe Représentatif de arêtes d'un multigraphe, J. Math. pures et appl., 52 (1973) 299-308. 86, 87, 89

[5] B. Bollobás. Extremal Graph Theory. Courier Dover Publications, 2004. 53

[6] M. Chudnovsky and P. Seymour. The structure of claw-free graphs, Surveys in combinatorics, London Math. Soc. Lecture Note Ser., 327, Cambridge: Cambridge Univ. Press (2005) 153171. 89

[7] R. Diestel. Graph Theory, third edition. Springer, 2005. 82

[8] L. M. Favrholdt, M. Stiebitz, and B. Toft. Graph edge colouring: Vizing's Theorem and Goldberg's Conjecture, Preprints 2006 No. 20, IMADA, The University of Southern Denmark, 91 pages. 8, 11, 12, 13, 14, 39, 41, 42, 94

[9] J. Edmonds. Maximum Matching and a Polyhedron with 0,1-Vertices, J. of Research of the National Bureau of Standards (B) 69 (1965) 125-130. 7

[10] S. Fiorini and R. J. Wilson. Edge-colourings of graphs, Pitman, London, 1977. 5

[11] M. K. Goldberg. Edge-coloring of multigraphs: recolouring technique, J. Graph Theory 8 (1984) 123-137. 1, 6, 8, 98

[12] M. K. Goldberg. Structure of multigraphs with restrictions on the chromatic class, Metody Diskret. Analiz. 30 (1977) 3-12. [In Russian] 8

[13] M. K. Goldberg. On multigraphs with almost-maximal chromatic class, Diskret. Analiz. 23 (1973) 3-7. [In Russian] 1, 7

[14] R. L. Hemminger. Characterizations of the line graph of a multigraph, Research Report 12 (1972) Department of Mathematics, Vanderbilt University. 86, 89

[15] A. J. W. Hilton and B. Jackson. A Note Concerning the Chromatic Index of Multigraphs, J. Graph Th. 11(3) (1987) 267-272. 72

[16] I. Holyer. The NP-completeness of edge-colouring, SIAM J. Comput. 10(4) (1981) 718-720. 1, 6

[17] I. T. Jakobsen. On critical graphs with chromatic index 4, Disc. Math. 9 (1974) 265-276. 73

[18] M. Javdekar. Note on Choudum's chromatic lower bound for a class of graphs, J. Graph Th. 4 (1980) 265-268. 88

[19] T. Jensen, B. Toft. Graph colouring problems, Wiley-Interscience, New York, 1995. 73, 102, 103

[20] J. Kahn. Asymptotics of the chromatic index for multigraphs, J. Combin. Theory Ser. B 68 (1996) 223-254. 7

[21] H. A. Kierstead. On the chromatic index of multigraphs without large triangles, J. Combin. Theory Ser. B 36 (1984) 156-160. 2, 10, 19, 32, 88

[22] H. A. Kierstead and J. H. Schmerl. Some applications of Vizing's Theorem to vertex colourings of graphs, Disc. Math. 45 (1983) 277-285. 88

[23] A. D. King, B. A. Reed, and A. Vetta. An upper bound on the chromatic number of line graphs, Eur. J. Combin. 28 (2007) 2182-2187. 83

[24] D. König. Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre, Math. Ann. 77 (1916) 453-465. 2, 5

[25] J. Krausz. New proof of a theorem of Whitney on planar graphs, Mat. Fiz. Lapok, 50 (1943) 75-85. 89

[26] L. Lovász. Combinatorial Problems and Exercises, Second Edition, AMS Chelsea Publishing, Providence, 2007. 71

[27] O. Marcotte. On the chromatic index of multigraphs and a conjecture of Seymour, II. In: W. Cook and P. D. Seymour, eds., Polyhedral Combinatorics, Proc. DIMACS Workshop, Morristown, N.J., June 12-16, 1989, pgs. 245-279. American Mathematical Society, 1990. 8

[28] J. M. McDonald. Achieving maximum chromatic index in multigraphs, Disc. Math. 309 (2009) 2077-2084. 2, 27

[29] T. Nishizeki and K. Kashiwagi. On the 1.1 edge-coloring of multigraphs, SIAM J. Disc. Math. 3(3) (1990) 391-410. 8

[30] O. Ore. The Four Colour Problem, Academic Press, New York (1967) MR 36-74. 7

[31] M. J. Plantholt and S. K. Tipnis. The chromatic index of multigraphs of order at most 10, Disc. Math. 177 (1997) 185-193. 49

[32] B. Randerath. The Vizing bound for the chromatic number based on forbidden pairs, Ph.D. thesis, RWTH Aachen, Shaker Verlag 1998. 89

[33] B. Randerath and I. Schiermeyer. Vertex Colouring and Forbidden Subgraphs — A Survey, Graphs Combin. 20 (2004) 1-40. 89

[34] B. Reed. ω, Δ and χ, J. Graph Th. 27 (1998) 177-212. 83

[35] D. Scheide. On the 15/14 edge-colouring of multigraphs, Preprints 2007 No. 11, IMADA, The University of Southern Denmark, 49 pages. 8, 12, 39, 51

[36] D. Scheide. A polynomial-time $\Delta + \sqrt{\frac{\Delta-1}{2}}$-edge colouring algorithm, Preprints 2009 No. 4, IMADA, The University of Southern Denmark, 15 pages. 51, 61

[37] P. D. Seymour. On multi-colourings of cubic graphs, and conjectures of Fulkerson and Tutte, Proc. London Math. Soc. 3 (1979) 423-460. 1, 7

[38] C. E. Shannon. A theorem on colouring the lines of a network, J. Math. Phys. 28 (1949) 148-151. 6

[39] E. Steffen. A refinement of Vizing's Theorem, Disc. Math. 218 (2000) 289-291. 6

[40] P. G. Tait. Remarks on the colourings of maps, Proc. Royal Soc. Edin. 10 (1880) 501-503, 729. 5

[41] P. G. Tait. Note on a theorem in the geometry of position, Trans. Royal Soc. Edin. 29 (1898) 270-285. 5

[42] V. A. Tashkinov. On an algorithm to color the edges of a multigraph, Diskret. Analiz. 7 (2000) 72-85. [in Russian] 2, 8, 9

[43] V. G. Vizing. The chromatic class of a multigraph, Kibernetika (Kiev) 1 (1965) 29-39. [in Russian] English translation: Cybernetics 1 (1965) 32-41. 29, 102, 103

[44] V. G. Vizing. On an estimate of the chromatic class of a p-graph, Diskret. Analiz. 3 (1964), 25-30. 1, 6