A Generalization of the Discounted Penalty Function in Ruin Theory

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

As ruin theory evolves in recent years, there has been a variety of quantities pertaining to an insurer's bankruptcy at the centre of focus in the literature. Despite the fact that these quantities are distinct from each other, it was brought to our attention that many solution methods apply to nearly all ruin-related quantities. Such a peculiar similarity among their solution methods inspired us to search for a general form that reconciles those seemingly different ruin-related quantities.

The stochastic approach proposed in the thesis addresses such issues and contributes to the current literature in three major directions.

(1) It provides a new function that unifies many existing ruin-related quantities and that produces more new quantities of potential use in both practice and academia.

(2) It applies generally to a vast majority of risk processes and permits the consideration of combined effects of investment strategies, policy modifications, etc, which were either impossible or difficult tasks using traditional approaches.

(3) It gives a shortcut to the derivation of intermediate solution equations. In addition to the efficiency, the new approach also leads to a standardized procedure to cope with various situations.

The thesis covers a wide range of ruin-related and financial topics while developing the unifying stochastic approach. Not only does it attempt to provide insights into the unification of quantities in ruin theory, the thesis also seeks to extend its applications in other related areas.

Acknowledgements

Interest is the most powerful and lasting driving force behind everything we do. I became fascinated by ruin theory in my first course given by Professor Willmot. He demonstrated to us the beautiful mathematical consistency among various approaches. I was also lucky to be involved in many thought-intriguing projects with Professor Cai and hence developed strong interests for further exploring ruin theory. I am thankful to both Professor Cai and Professor Willmot for giving me the support and guidance to work on the topic which best suits my interest and research ability.

I would also like to thank my parents, Xiaoli, Xinping, Dian, Chunmei, Baojiang, Tingting and other friends in Waterloo and Montreal for their constant love and support. I am grateful to Professor Shiu, Professor Kolkiewicz, Professor Landriault, Professor Garrido and Professor Drekic for providing me with constructive criticisms and suggestions. They all contribute in one way or another to foster my interest in pursuing a happy researcher's life.

This thesis is not just a fulfillment of requirements for the doctoral degree, but also a chance for me to show my appreciation and share with potential readers the excitement of exploring the ruin theory.

獻給我親愛的爸爸媽媽

Dedicated to my dear parents

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Chapter 1

Introduction

It all starts with a simple and inspiring model proposed by the Swedish actuary Filip Lundberg in 1903. Those who have doubts about ruin theory might be surprised to find out that the theory actually outdated many disciplines of modern sciences and stood up to challenges over more than a century. As every scientific theory evolves, ruin theory has grown from a simple but thought-intriguing model to a specialized area which is nowadays equipped with state-of-art techniques developed alongside many other areas in applied probability.

This chapter is dedicated to the overview of classical topics of interest and techniques in the literature. The content of this chapter serves two main purposes.

In order to pave the way for the development of a unifying approach in later chapters, we need to review many classical approaches and techniques, particularly those developed in the past decade, for analyzing ruin-related quantities. A motivation for the construction of a new unifying tool will be discussed in the end as an implication of comparison among those well-studied quantities.

This chapter also intends to summarize the pros and cons of the classical approaches, which shall be compared with those of the unifying approach throughout the thesis. To set up the tone of future comparison, we name a few advantages and disadvantages of the classical approach. The classical analytical techniques are straightforward and require only basic understanding of infinitesimal arguments to construct basis for computation and derivation of ruin-related quantities. Although introduced under the framework of compound Poisson model in this chapter, the classical approaches are generally applicable to a great majority of risk models in the ruin literature. However, as they were used in general risk models and applied to solve more ruin-related quantities, the arguments often become unduely repetitive and tedious, particularly, in many practical models where interest rates are involved.

1.1 Mathematical Preliminaries

It is not until recent years that operator analysis has come to the attention of actuarial scientists. Despite of the limited research on this topic, the use of operators in recent literature has enormously reduced the amount of work in the analysis of ruin-related quantities.

In the section, we introduce a few important operators that would facilitate solving integral-differential equations in later chapters.

Definition 1.1.1. *For any integrable function $f(y)$ defined for $y \geq 0$ and a real number $s \geq 0$, the* Dickson-Hipp transform *of $f(y)$ is given by*

$$\mathcal{T}_s f(x) = e^{sx} \int_x^{\infty} e^{-sy} f(y)\, dy\ , \qquad x \geq 0.$$

And $\mathcal{T}_s$ is called a Dickson-Hipp operator.

The Dickson-Hipp operator appeared in Li and Garrido [38] in the context of Sparre Andersen model and was systematically exploited in Gerber and Shiu [24]. It has since become a major tool in analyzing defective renewal equations.

The Dickson-Hipp operator possesses a number of nice properties, among which three are of particular use to us in the next section. Hence we provide detailed proofs for these properties.

Lemma 1.1.1.

$$\mathcal{T}_{s_1}\mathcal{T}_{s_2}f(x) = \frac{\mathcal{T}_{s_2}f(x) - \mathcal{T}_{s_1}f(x)}{s_1 - s_2} \ .$$

Proof. By changing the order of integrations,

$$\begin{aligned}
\mathcal{T}_{s_1}\mathcal{T}_{s_2}f(x) &= e^{s_1 x}\int_x^\infty e^{-s_1 y}e^{s_2 y}\int_y^\infty e^{-s_2 u}f(u)\,du\,dy \\
&= e^{s_1 x}\int_x^\infty e^{-s_2 u}f(u)\int_x^u e^{-(s_1-s_2)y}\,dy\,du \\
&= \frac{1}{s_1-s_2}e^{s_1 x}\int_x^\infty e^{-s_2 u}f(u)[e^{-(s_1-s_2)x} - e^{-(s_1-s_2)u}]\,du \\
&= \frac{1}{s_1-s_2}\left[e^{s_2 x}\int_x^\infty e^{-s_2 u}f(u)du - e^{s_1 x}\int_x^\infty e^{-s_1 u}f(u)du\right].
\end{aligned}$$

□

A special case of the Dickson-Hipp transform that has been used frequently in theoretical derivation in ruin theory is the Laplace transform

$$\mathcal{L}f(s) = \mathcal{T}_s f(0) = \int_0^\infty e^{-sy}f(y)dy.$$

Following the conventions in ruin theory, the notation $\tilde{f}(s)$ is also used interchangeably with $\mathcal{L}f(s)$ in the thesis for the Laplace transform of $f(x)$.

For brevity, we also use the notation $f \star g$ to denote the convolution of $f(x)$ and $g(x)$,

$$f \star g(x) = \int_0^x f(x-y)g(y)dy = \int_0^x g(x-y)f(y)dy. \tag{1.1.1}$$

Lemma 1.1.2.

$$\mathcal{T}_s\{f \star g\}(x) = \tilde{g}(s)\cdot\mathcal{T}_s f(x) + \mathcal{T}_s g \star f(x) \ .$$

Proof. Consider the Laplace transform as a special case of the Dickson-Hipp operator as shown in (1.1.1) and apply the Lemma 1.1.1.

$$\begin{aligned}
\mathcal{L}\{\mathcal{T}_s\{f \star g\}\}(z) &= \frac{\mathcal{L}\{f \star g\}(s) - \mathcal{L}\{f \star g\}(z)}{z-s} \\
&= \frac{\mathcal{L}f(s)\cdot\mathcal{L}g(s) - \mathcal{L}f(z)\cdot\mathcal{L}g(z)}{z-s} \\
&= \frac{\mathcal{L}f(s) - \mathcal{L}f(z)}{z-s}\cdot\mathcal{L}g(s) + \frac{\mathcal{L}g(s)-\mathcal{L}g(z)}{z-s}\cdot\mathcal{L}f(z) \\
&= \mathcal{L}g(s)\cdot\mathcal{L}\mathcal{T}_s f(z) + \mathcal{L}\{\mathcal{T}_s g \star f\}(z)\ .
\end{aligned}$$

Observe that taking inverse Laplace transform with respect to z yields the desired equality.

□

Another very interesting discovery in Gerber and Shiu [24] is the left inverse of Dickson-Hipp operator. In what follows, we shall use the notation $\mathcal{I}$ for the identity operator and $\mathcal{D}$ for the differentiation operator with respect to the argument of the function on which the operator is performed.

Lemma 1.1.3.

$$(s\mathcal{I} - \mathcal{D})\mathcal{T}_s f(x) = f(x)\ .$$

Proof. It can be verified that

$$\begin{aligned}
& (s\mathcal{I} - \mathcal{D})\{e^{sx}\int_x^\infty e^{-sy} f(y)dy\} \\
= \ & se^{sx}\int_x^\infty e^{-sy} f(y)dy - se^{sx}\int_x^\infty e^{-sy} f(y)dy + f(x) = f(x).
\end{aligned}$$

□

Now we look at another operator that comes out of our need in solving integro-differential equations with Gamma distributed claim sizes. Although it was not usually treated as an operator in the past literature, we shall find it convenient to do so in order to facilitate our derivations in dealing with many differential equations to be seen in later chapters.

Definition 1.1.2. *For any integrable function $f(y)$ and $s \geq 0$, the* exponential convolution transform *of $f(y)$ is given by*

$$\mathcal{E}_s f(x) = e^{-sx} \int_0^x e^{sy} f(y) dy.$$

And $\mathcal{E}_s$ is called an exponential convolution operator.

The name comes from the fact that the operator yields a convolution of the integrable function and an exponential function. A property we shall use frequently with this operator is given by the next lemma.

Lemma 1.1.4.

$$(s\mathcal{I} + \mathcal{D})\mathcal{E}_s f(x) = f(x) \ .$$

Proof. It can be verified that

$$\begin{aligned} & (s\mathcal{I} + \mathcal{D})\{e^{-sx} \int_0^x e^{sy} f(y) dy\} \\ = \ & se^{-sx} \int_0^x e^{sy} f(y) dy + (-s) e^{-sx} \int_0^x e^{sy} f(y) dy + f(x) = f(x). \end{aligned}$$

□

In the analysis of integro-differential equations, it is generally the integral term that increases the level of difficulty in searching for solutions. As we shall see in later chapters, we often make certain assumption about the claim size distribution to find explicit solutions. For instance, in many cases the claim sizes are assumed to be exponentially distributed with the distribution function $Q(y) = 1 - e^{-\beta y}$. The integral term, which will appear frequently, involving the claim size distribution can now be written in terms of exponential convolution transforms,

$$\int_0^x m(x-y)dQ(y) = \beta \int_0^x m(x-y)e^{-\beta y}dy = \beta \mathcal{E}_\beta m(x).$$

If the claim sizes follow Gamma distribution

$$Q(y) = 1 - e^{-\beta y} \sum_{k=0}^{n-1} \frac{(\beta y)^{k-1}}{k!},$$

we can easily represent the integral term as a multiple fold exponential convolution transforms,

$$\int_0^x m(x-y)dQ(y) = \beta^n \mathcal{E}_\beta^n m(x).$$

Note that taking $(\beta\mathcal{I} + \mathcal{D})$ n times on both sides, we obtain

$$\{(\beta\mathcal{I} + \mathcal{D})^n\} \int_0^x m(x-y)dQ(y) = \beta^n m(x).$$

Similarly, if the claim sizes follow a mixture of n exponential distributions, i.e.

$$Q(y) = \theta_1(1 - e^{-\beta_1 y}) + \theta_2(1 - e^{-\beta_2 y}) + \cdots + \theta_n(1 - e^{-\beta_n y}),$$

then the integral term becomes

$$\int_0^x m(x-y)dQ(y) = \theta_1\beta_1\mathcal{E}_{\beta_1}m(x) + \theta_2\beta_2\mathcal{E}_{\beta_2}m(x) + \cdots + \theta_n\beta_n\mathcal{E}_{\beta_n}m(x).$$

It implies that

$$\{\prod_{i=1}^n(\beta_i\mathcal{I} + \mathcal{D})\} \int_0^x m(x-y)dQ(y) = \sum_{i=1}^n \prod_{j=1, j\neq i}^n (\beta_i\mathcal{I} + \mathcal{D})m(x).$$

The beauty of exponential convolution operator lies in the fact that such integral terms can be converted to derivative terms that are more mathematically tractable.

1.2 Classical Compound Poisson Risk Model

Typically, good mathematical models are based on relatively idealized assumptions, but also leave room for further refinement and more realistic considerations. They are always rich sources of inspiration for researchers in generations to come. The compound Poisson risk model introduced by Lundberg is beyond the shadow of a doubt one of such kind and thereby an ideal place to start our introduction to ruin theory.

We begin with the basic setup of the model. An insurer's asset consists of an initial investment x and continuous premium income collected at a constant rate of c per period, whereas its liability is to cover a sequence of insurance claims $\{Y_1, Y_2, \cdots\}$. The arrival of claims is modelled by a Poisson process $\{N(t), t \geq 0\}$ with intensity λ and the claim sizes are assumed to be mutually independent and identically distributed with common distribution $Q(y)$. Hence the aggregate claims up to time t is given by

$$Z(t) = \sum_{i=1}^{N(t)} Y_i.$$

The aggregate claim follows the compound Poisson distribution, hence the name of the model. Since the insurer's surplus is its assets of the initial investment and premium income less its liability of aggregate claims, we shall base our analysis on the surplus driven by the stochastic process $\{X(t), t \geq 0\}$ with

$$X(t) = x + ct - Z(t).$$

In this simple model, we focus on the insurer's ability to manage its surplus through the control of initial investment x. It is obvious in the interest of such an insurer with how large initial investment its surplus would remain solvent with a relatively large chance in long run.

This question gives rise to the study of probability of ruin, which is a measure to quantify the likelihood that an insurer's asset would eventually be insufficient to cover its liabilities in long run. In mathematical terms, the probability of ultimate ruin is defined by

$$\psi(x) = \mathbb{P}^x(\tau_0 < \infty),$$

where the measure $\mathbb{P}^x$ is defined for $X(t)$ starting off with an initial investment x and the time of ruin is given by

$$\tau_0 = \inf\{X(t) < 0\}$$

with the convention that $\inf \varnothing = \infty$.

The major task of ruin theory in its early stage was to search for solutions to the probability of ruin as an explicit function of the initial investment if available, or give reasonably accurate approximations or tight bounds otherwise. The probability of ruin has always been and still is a favorable quantity of interest in many fields of applied probability. It is often viewed as the first step leading towards the investigation of more sophisticated quantities.

1.2.1 Gerber-Shiu Function

Another historic contribution to ruin theory was made by actuarial scientists Hans U. Gerber and Elias S.W. Shiu in their seminal paper [22] published in 1998, where the expected discounted penalty function comes to light. As a measurement of economic costs resulted from an insurer's bankruptcy, the expected discounted penalty function (or called Gerber-Shiu function) is defined by

$$m(x) = \mathbb{E}^x[e^{-\delta\tau_0} w(X_{\tau_0 -}, |X_{\tau_0}|) I(\tau_0 < \infty)],$$

where $\delta \geq 0$ is the discounting force of interest and the bounded function $w(x, y)$ is often interpreted as a penalty imposed on the insurer's bankruptcy depending on the amount of surplus prior to ruin x and the amount of deficit at ruin y.

The purpose of studying such a quantity is multiple-fold. First of all, as $m(x)$ reduces to $\psi(x)$ by letting $\delta = 0$ and $w(x, y) = 1$, the expected discounted penalty function is clearly a generalization of the probability of ruin. Secondly, the function accommodates a wide variety of quantities pertaining to the insurer's financial conditions at the time of ruin. To name a few, we observe that

- when $w(x, y) = e^{-rx-sy}$, $m(x)$ as a function of (δ, r, s) is a tri-variate Laplace transform of the time of ruin τ_0, the surplus prior to ruin $X(\tau_0-)$ and the deficit at ruin $X(\tau_0)$.

- when $\delta = 0$ and $w(x, y) = I(x + y \leq z), m(x)$ as a function of z gives the distribution function of the claim causing ruin.

- when $w(x, y) = (K - x)_+, m(x)$ can be used to find the price of perpetual American put option with exercise value K.

For a complete account of the family of Gerber-Shiu functions, readers are referred to Gerber and Shiu [21, 22, 25, 26], etc.

The traditional approach to solve the Gerber-Shiu function $m(x)$ is through a series of probabilistic arguments as follows. Since the time until the first claim is exponentially distributed with mean $1/\lambda$, a claim occurs with probability density $\lambda e^{-\lambda t}$ at time t. The surplus immediately prior to the first claim would have accumulated to $x + ct$ as a result of continuously receiving premium income at a constant rate c. If the size of first claim y is larger than the current surplus level $x + ct$, ruin occurs and the penalty is exercised in an amount determined by the surplus prior to ruin $x + ct$ and the surplus at ruin $x + ct - y$. Otherwise, the surplus remains positive and the surplus process continues as if it starts again at $x + ct - y$. One should keep in mind that the Gerber-Shiu function takes account of the time value of money by definition, the nominal values at time t have to be discounted by the factor $e^{-\delta t}$. Translating into mathematical terms,

$$m(x) = \int_0^\infty e^{-\delta t} \left\{ \int_{x+ct}^\infty w(x + ct, y - x - ct) dQ(y) + \int_0^{x+ct} m(x + ct - y) dQ(y) \right\} \lambda e^{-\lambda t} dt.$$

A change of variable $z = x + ct$ results in

$$m(x) = \frac{\lambda}{c} \mathcal{T}_s \sigma(x), \tag{1.2.1}$$

where $s = (\lambda + \delta)/c$ and

$$\sigma(x) = \int_0^x m(x - y) dQ(y) + \int_x^\infty w(x, y - x) dQ(y).$$

For notational convenience, we let

$$\zeta(x) = \int_x^\infty w(x, y - x) dQ(y).$$

It should be noted that the differentiability and integrability are often implicitly assumed as situation warrants. We usually treat $Q(y)$ as a continuous distribution function with density function $q(y)$. However, most results in what follows can be generalized to include claim size distributions with a countable number of discontinuities.

Applying Lemma 1.1.3, we see that the Gerber-Shiu function satisfies the following integro-differential equation

$$m'(x) = \frac{\lambda+\delta}{c}m(x) - \frac{\lambda}{c}\left\{\int_0^x m(x-y)dQ(y) + \int_x^\infty w(x, y-x)dQ(y)\right\}. \qquad (1.2.2)$$

In an attempt to display as many commonly used techniques as possible, we shall employ three different approaches to solve this equation. As depicted by the proverb "All roads lead to Rome", it won't be long before one is amazed to realize the hidden mathematical consistency, which is truly, in the author's point of view, the beauty of ruin theory.

Operator Analysis

We shall first start with the method that relies on the operators introduced in the previous section. Inspired by the pioneering work on operator analysis in Gerber and Shiu [24], the method was recently formulated in Cai *et al.* [7].

Observe from Lemma 1.1.1 that the parameter of Dickson-Hipp operator can be shifted at the cost of having an extra term involving a second order Dickson-Hipp operator. Thus

$$\mathcal{T}_s\{m \star q + \zeta\}(x) = \mathcal{T}_\rho\{m \star q + \zeta\}(x) - (s-\rho)\mathcal{T}_\rho\mathcal{T}_s\{m \star q + \zeta\}(x). \qquad (1.2.3)$$

Note that

$$\mathcal{T}_\rho\{q \star m + \zeta\}(x) = \mathcal{T}_\rho q \star m(x) + \mathcal{T}_\rho\zeta(x) + \tilde{q}(\rho)\cdot\mathcal{T}_\rho m(x)$$

with the last equality from Lemma 1.1.2. Plugging it into (1.2.3), we have

$$m(x) = \frac{\lambda}{c}\big\{\mathcal{T}_\rho q \star m(x) + \mathcal{T}_\rho\zeta(x) + \tilde{q}(\rho)\cdot\mathcal{T}_\rho m(x) - (s-\rho)\mathcal{T}_\rho\mathcal{T}_s\{m \star q + \zeta\}(x)\big\}. \qquad (1.2.4)$$

Now we suppose the parameter ρ is chosen to be the non-negative root of the fundamental Lundberg equation

$$\frac{\lambda}{c}\tilde{q}(\rho) = s - \rho. \tag{1.2.5}$$

In view of (1.2.1) and the fact that constants can move through Dickson-Hipp operators, we obtain

$$\mathcal{T}_\rho\{\tilde{q}(\rho)\cdot m\}(x) - \mathcal{T}_\rho\{(s-\rho)\mathcal{T}_s\{m \star q + \zeta\}\}(x) = 0.$$

By canceling the last two terms, (1.2.4) reduces to

$$m(x) = \frac{\lambda}{c}\mathcal{T}_\rho q \star m(x) + \frac{\lambda}{c}\mathcal{T}_\rho \zeta(x),$$

which yields the beautiful defective renewal equation

$$m(x) = \frac{\lambda\pi}{c}\int_0^x m(x-y)q_1(y)dy + \frac{\lambda}{c}\mathcal{T}_\rho\zeta(x) \tag{1.2.6}$$

with $\pi = \int_0^\infty \mathcal{T}_\rho q(y)dy$ and the generalized equilibrium distribution

$$q_1(y) = \frac{1}{\pi}\mathcal{T}_\rho q(y).$$

Dickson-Hipp Transform Approach

For the lack of appropriate name to summarize the method, the name is chosen to indicate that this procedure introduced in Gerber and Shiu [22] resembles the construction of a Dickson-Hipp transform.

We multiply both sides of (1.2.2) by $e^{-\rho x}$ and let $m_\rho(x) = e^{-\rho x}m(x)$ for notational brevity.

$$m'_\rho(x) = (s-\rho)m_\rho(x) - \frac{\lambda}{c}\int_0^x m_\rho(x-y)e^{-\rho y}dQ(y) - \frac{\lambda}{c}e^{-\rho x}\zeta(x).$$

Integrating both sides from 0 to z yields

$$\frac{c}{\lambda}[m_\rho(z) - m_\rho(0)] = \frac{c}{\lambda}(s-\rho)\int_0^z m_\rho(x)dx - \int_0^z\int_0^x m_\rho(x-y)e^{-\rho y}dQ(y)dx - \int_0^z e^{-\rho x}\zeta(x)dx.$$

Recall from (1.2.5) that $c/\lambda(s-\rho) = \tilde{q}(\rho)$. Thus, we can rewrite the above equation as

$$\frac{c}{\lambda}[m_\rho(z) - m_\rho(0)] = \tilde{q}(\rho)\int_0^z m_\rho(x)dx - \int_0^z\int_0^x m_\rho(y)e^{-\rho(x-y)}q(x-y)dydx - \int_0^z e^{-\rho x}\zeta(x)dx \tag{1.2.7}$$

Interchanging the order of integration and changing the variable $x-y=t$ in the innermost integral gives

$$\begin{aligned}\int_0^z m_\rho(y)e^{-\rho(x-y)}q(x-y)dydx &= \int_0^z m_\rho(y)\int_y^z e^{-\rho(x-y)}q(x-y)dxdy \\ &= \int_0^z m_\rho(y)\int_0^{z-y} e^{-\rho t}q(t)dtdy.\end{aligned}$$

Therefore, we must have

$$\begin{aligned}\frac{c}{\lambda}[m_\rho(z) - m_\rho(0)] &= \tilde{q}(\rho)\int_0^z m_\rho(x)dx - \int_0^z m_\rho(x)\int_0^{z-x} e^{-\rho t}q(t)dtdx - \int_0^z e^{-\rho x}\zeta(x)dx \\ &= \int_0^z m_\rho(x)\int_{z-x}^\infty e^{-\rho t}q(t)dtdx - \int_0^z e^{-\rho x}\zeta(x)dx. \qquad (1.2.8)\end{aligned}$$

Since $w(x,y)$ is a bounded function, then there must exists an M such that

$$m_\rho(x) = e^{-\rho x}m(x) \le M\psi(x) \to 0, \qquad \text{as } x\to\infty.$$

Letting $z\to\infty$ in (1.2.7) and applying the bounded convergence theorem to the first integral on the right gives

$$\frac{c}{\lambda}m_\rho(0) = \int_0^\infty e^{-\rho x}\zeta(x)dx.$$

Substituting the expression for $m_\rho(0)$ in (1.2.8) and rearranging terms

$$m_\rho(z) = \frac{\lambda}{c}\int_0^z m_\rho(z-x)\int_x^\infty e^{-\rho t}q(t)dtdx + \frac{\lambda}{c}\int_0^z e^{-\rho x}\zeta(x)dx.$$

Multiplying both sides by $e^{\rho z}$,

$$\begin{aligned}m(z) &= \frac{\lambda}{c}e^{\rho z}\int_0^z e^{-\rho(z-x)}m(z-x)\int_x^\infty e^{-\rho y}q(y)dy + \frac{\lambda}{c}e^{\rho z}\int_z^\infty e^{-\rho x}\zeta(x)dx \\ &= \frac{\lambda}{c}\int_0^z m(z-x)e^{\rho x}\int_x^\infty e^{-\rho y}q(y)dy + \frac{\lambda}{c}e^{\rho z}\int_z^\infty e^{-\rho x}\zeta(x)dx,\end{aligned}$$

which is same as (1.2.6) upon rearrangement.

Laplace Transform Approach

The Laplace transform is another technique that is widely used in ruin theory. We now present the third derivation of the renewal equation using Laplace transforms. For more detailed and complete account of the derivation, readers should consult with Willmot [48].

Taking Laplace transforms on both sides of (1.2.1)

$$\begin{aligned}\tilde{m}(u) &= \frac{\lambda}{c}\frac{\tilde{\sigma}(s)-\tilde{\sigma}(u)}{u-s} \\ &= \frac{\lambda}{c}\frac{\tilde{\sigma}(s)-\tilde{\zeta}(u)-\tilde{m}(u)\tilde{q}(u)}{u-s}.\end{aligned}$$

Solving for $\tilde{m}(u)$ yields

$$\left\{u-s+\frac{\lambda}{c}\tilde{q}(u)\right\}\tilde{m}(u)=\frac{\lambda}{c}\tilde{\sigma}(s)-\frac{\lambda}{c}\tilde{\zeta}(u).$$

Note that the expression embraced by the brackets on the left hand side also appears in the Lundberg equation (1.2.5). Since $\tilde{m}(u)$ is finite for all $u \geq 0$, then by letting $u=\rho$ we must have $\tilde{\sigma}(s)=\tilde{\zeta}(\rho)$. Otherwise $\tilde{m}(\rho)$ is not well-defined.

With Lemma 1.1.1 in mind, we divide both sides by $u-\rho$ in order to construct the Laplace transform of a Dickson-Hipp operator.

$$\left\{\frac{u-s+(\lambda/c)\tilde{q}(u)}{u-\rho}\right\}\tilde{m}(u)=\frac{\lambda}{c}\frac{\tilde{\zeta}(\rho)-\tilde{\zeta}(u)}{u-\rho}. \tag{1.2.9}$$

We anticipate that this Laplace transform equation is that of a renewal equation, which implies that it can be written as

$$\{1-\tilde{h}(u)\}\tilde{m}(u)=\frac{\lambda}{c}\frac{\tilde{\zeta}(\rho)-\tilde{\zeta}(u)}{u-\rho}. \tag{1.2.10}$$

Then it remains to figure out what $h(x)$ is. We wish to manipulate the expression in the brackets on the left hand side of (1.2.9) so that $\tilde{h}(u)$ can be written as a recognizable Laplace transform of certain function.

$$\frac{u-s+(\lambda/s)\tilde{q}(u)}{u-\rho}=1-\frac{s-\rho-(\lambda/c)\tilde{q}(u)}{u-\rho}.$$

Therefore, we have

$$\begin{aligned}\tilde{h}(u) &= \frac{s-\rho-(\lambda/c)\tilde{q}(u)}{u-\rho} \\ &= \frac{\lambda}{c}\frac{\tilde{q}(\rho)-\tilde{q}(u)}{u-\rho}\end{aligned}$$

with the last equality resulted from the Lundberg equation (1.2.5).

Inserting the expression for $\tilde{h}(u)$ into (1.2.10) and taking the inverse Laplace transform on both sides gives

$$m(x) - \frac{\lambda}{c}\mathcal{T}_\rho q \star m(x) = \frac{\lambda}{c}\mathcal{T}_\rho \zeta(x),$$

which leads to the defective renewal equation (1.2.6) upon rearrangement.

The analysis of defective renewal equations can be found in Willmot and Lin [50]. The general solution to the equation (1.2.6) is given by

$$m(x) = \frac{\lambda}{c-\lambda\pi}\int_0^x \mathcal{T}_\rho\zeta(x-y)g(y)dy + \frac{\lambda}{c}\mathcal{T}_\rho\zeta(x),$$

where $g(y)$ is a compound geometric density function

$$g(y) = \sum_{n=1}^{\infty}\left(1-\frac{\lambda\pi}{c}\right)\left(\frac{\lambda\pi}{c}\right)^n q_1^{\star n}(y).$$

We now wrap up this section by giving a closed-form solution to the probability of ruin in a special case that is to be seen frequently throughout the thesis.

Example 1.2.1. *Special case: exponential claim size distribution*

As we see from (1.2.1), the probability of ultimate ruin satisfies

$$\psi(x) = \frac{\lambda}{c}\mathcal{T}_{\lambda/c}\sigma(x), \tag{1.2.11}$$

where

$$\sigma(x) = \int_0^x \psi(x-y)dQ(y) + 1 - Q(x).$$

It follows from Lemma 1.1.4 that

$$(\frac{\lambda}{c} - \mathcal{D})\psi(x) = \frac{\lambda}{c}\left\{\int_0^x \psi(x-y)dQ(y) + 1 - Q(x)\right\}. \tag{1.2.12}$$

As alluded to in the discussion of exponential convolution transform, the simplest case is to assume that the claim size follow an exponential distribution, i.e.

$$Q(y) = 1 - e^{-\beta y}, \qquad y \geq 0.$$

Hence, (1.2.12) can be represented as

$$(\frac{\lambda}{c} - \mathcal{D})\psi(x) = \frac{\lambda}{c}\{\beta\mathcal{E}_\beta\psi(x) + e^{-\beta x}\}.$$

Applying Lemma 1.1.4 gives

$$(\beta + \mathcal{D})(\frac{\lambda}{c} - \mathcal{D})\psi(x) = \frac{\lambda}{c}\left\{\beta\psi(x) + (\beta + \mathcal{D})e^{-\beta x}\right\},$$

which simplifies to

$$\psi''(x) + (\beta - \frac{\lambda}{c})\psi'(x) = 0, \qquad x \geq 0. \tag{1.2.13}$$

The general solution to (1.2.13) is given by

$$\psi(x) = Ae^{-(\beta-\lambda/c)x}, \qquad x \geq 0,$$

where A is a constant to be determined.

Substituting the general solution of $\psi(x)$ into (1.2.11) yields

$$Ae^{-(\beta-\lambda/c)x} = \frac{\lambda}{c}e^{(\lambda/c)x}\int_x^\infty \left[\frac{\beta c}{\lambda}Ae^{-\beta y} - \frac{\beta c}{\lambda}Ae^{-(\lambda/c+\beta)y} + e^{-(\lambda/c+\beta)y}\right]dy.$$

By matching the coefficients of the terms involving $e^{-(\lambda/c+\beta)y}$, we obtain immediately that

$$A = \frac{\lambda}{\beta c}.$$

Therefore, the probability of ultimate ruin in the classical compound Poisson model with exponential claim size distribution is given by

$$\psi(x) = \frac{\lambda}{\beta c}e^{-(\beta-\lambda/c)x}, \qquad x \geq 0.$$

1.2.2 Total Dividends Paid up to Ruin

As ruin theory evolves in recent years, there has been revived interests in dividend problems which dated back to 1957 by an Italian probabilist and actuary Bruno De Finetti. Rather than the penalty occurred at the time of bankruptcy, the focus of dividend problems is to investigate the payments of dividends paid out to an insurance company's shareholders throughout its life time up to the time of bankruptcy. Recent papers on the development of dividend problems in classical models can be found in Lin *et al.* [41], Lin and Pavlova [39], Gerber and Shiu [22, 23, 27, 28] etc.

A typical dividend problem in the framework of the compound Poisson model can be described as follows. It is assumed that the insurer has the obligation to pay out a constant rate α of dividends when its surplus exceeds a level b, commonly referred to as dividend threshold in ruin literature. Therefore, when the surplus runs below the dividend threshold, the dynamics of the surplus process remains the same as in the classical case where the growth of surplus is driven by the constant rate c of premium income and the surplus drops by insurance claims $Z(t)$. However, as the surplus reaches the dividend threshold b, the rate of growth in surplus reduces to $c - \alpha$ as a result of dividend payout. The interests of such a model is to study the expected total amount of dividends paid all the way until the time of ruin.

In the papers mentioned above, the stream of continuous dividend payments is often represented as a stochastic process by itself and the expected total dividends as its expectation. However, as we shall see in the next chapter, it is much more intuitive and constructive to have this quantity defined as follows,

$$V(x) \triangleq \mathbb{E}^x \left[\int_0^{\tau_0} e^{-\delta t} l(X_t) dt \right], \tag{1.2.14}$$

where

$$l(x) = \begin{cases} \alpha, & x \geq b; \\ 0, & 0 \leq x < b. \end{cases}$$

Note that the total dividends $V(x)$ is obviously not a special case of the Gerber-Shiu function defined in previous section. Nevertheless, we can apply similar probabilistic arguments to conduct analysis of the dividends.

We start with the relatively simple case where the initial investment x exceeds the dividend threshold b. As the time of first claim is exponentially distributed with mean $1/\lambda$, the surplus would reach $x+(c-\alpha)t$ by the time t at which an insurance claim occurs with a chance of $\lambda e^{-\lambda t}$. If the size of claim y is larger than the current surplus level prior to the claim $x+(c-\alpha)t$, ruin occurs immediately and no future dividend payments will be expected. Hence we shall only consider the possibility that the insurance claim is less than $x+(c-\alpha)t$ and the surplus process regenerates itself due to the strong Markov property. The current value of future dividends is the same as the total dividends generated by the process starting from the new surplus level $V(x+(c-\alpha)t-y)$. Regardless of whether ruin occurs or not, the shareholders would have already accumulated a stream of dividend payments by time t which resembles a continuous annuity $\alpha\bar{s}_{\overline{t}|}$. Using the law of total probability, we write

$$V(x) = \int_0^\infty e^{-\delta t}\left\{\int_0^{x+(c-\alpha)t} V(x+(c-\alpha)t-y)dQ(y) + \alpha\bar{s}_{\overline{t}|}\right\}\lambda e^{-\lambda t}dt, \qquad x \geq b.$$

A change of variable $z = x+(c-\alpha)t$ results in

$$V(x) = \frac{\lambda}{c-\alpha}\mathcal{T}_s\{V \star q\}(x) + \frac{\alpha}{\lambda+\delta}, \qquad x \geq b, \tag{1.2.15}$$

where $s = (\lambda+\delta)/(c-\alpha)$.

Multiplying both sides of (1.2.15) by the operator $s\mathcal{I} - \mathcal{D}$ yields

$$V'(x) = \frac{\lambda+\delta}{c-\alpha}V(x) - \frac{\lambda}{c-\alpha}\int_0^x V(x-y)dQ(y) - \frac{\alpha}{c-\alpha}, \qquad x \geq b. \tag{1.2.16}$$

As before, we could find at least the three approaches to turn (1.2.15) into a renewal equation. For brevity, we shall only use the relatively concise method of operator analysis to obtain the result.

$$\begin{aligned}V(x) &= \frac{\lambda}{c-\alpha}\left[\mathcal{T}_\rho\{V\star q\}(x) - (s-\rho)\mathcal{T}_\rho\mathcal{T}_s\{V\star q\}(x)\right] + \frac{\alpha}{\lambda+\delta}\\ &= \frac{\lambda}{c-\alpha}\left[\mathcal{T}_\rho q\star V(x) + \tilde{q}(\rho)\cdot\mathcal{T}_\rho V(x) - (s-\rho)\mathcal{T}_\rho\mathcal{T}_s\{V\star q\}(x)\right] + \frac{\alpha}{\lambda+\delta}\end{aligned}$$

In view of (1.2.15), we can expand

$$\begin{aligned}\tilde{q}(\rho)\cdot\mathcal{T}_\rho V(x) &= \tilde{q}(\rho)\cdot\mathcal{T}_\rho\left\{\frac{\lambda}{c-\alpha}\mathcal{T}_s\{V\star q\}(x) + \frac{\alpha}{\lambda+\delta}\right\}(x)\\ &= (s-\rho)\mathcal{T}_\rho\mathcal{T}_s\{V\star q\}(x) + \mathcal{T}_\rho\left\{\frac{\alpha\tilde{q}(\rho)}{\lambda+\delta}\right\}(x)\end{aligned}$$

with the last equality derived by using the Lundberg equation

$$\frac{\lambda}{c-\alpha}\tilde{q}(\rho) = s-\rho.$$

Since

$$\mathcal{T}_\rho\left\{\frac{\alpha\tilde{q}(\rho)}{\lambda+\delta}\right\}(x) = \frac{\alpha\tilde{q}(\rho)}{\rho(\lambda+\delta)},$$

we have

$$\begin{aligned}V(x) &= \frac{\lambda}{c-\alpha}\left[\mathcal{T}_\rho q\star V(x) + \frac{\alpha\tilde{q}(\rho)}{\rho(\lambda+\delta)}\right] + \frac{\alpha}{\lambda+\delta}\\ &= \frac{\lambda}{c-\alpha}\mathcal{T}_\rho q\star V(x) + \frac{\alpha(s-\rho)}{\rho(\lambda+\delta)} + \frac{\alpha}{\lambda+\delta}\\ &= \frac{\lambda}{c-\alpha}\mathcal{T}_\rho q\star V(x) + \frac{\alpha}{\rho(c-\alpha)}\end{aligned}$$

with the second last equality resulted from the Lundberg equation.

Therefore, the expected total dividends also satisfies a defective renewal equation

$$V(x) = \frac{\lambda}{c-\alpha}\int_0^x V(x-y)q_1(y)dy + \frac{\alpha}{\rho(c-\alpha)}, \qquad x\geq b.$$

When the initial investment x is lower than the dividend threshold b, we need to break down the possible scenarios into two cases: (1) If the first claim occurs before the surplus reaches b, i.e. $t<(b-x)/c$, the surplus process restarts at $x+ct-y$ if the size of claim y is smaller than the current surplus level $x + ct$. There has been no dividend payments by the time t in this case. (2) If the first claim occurs after the surplus attains b, i.e. $t \geq (b-x)/c$, the surplus process must have regenerated itself at the dividend threshold b. The current value at time $(b-x)/c$ of future dividends paid up to ruin is given by $V(b)$.

Applying the law of total probability, we obtain

$$V(x) = \int_0^{(b-x)/c} e^{-\delta t} \left\{ \int_0^{x+ct} V(x+ct-y)dQ(y) \right\} \lambda e^{-\lambda t} dt + e^{-\delta(b-x)/c} V(b) \int_{(b-x)/c}^{\infty} \lambda e^{-\lambda t} dt, \quad 0 \leq x < b.$$

Making changes of variables results in

$$V(x) = \frac{\lambda}{c} e^{s_1 x} \int_x^b e^{-s_1 z} V \star q(z) dz + e^{-s_1(b-x)} V(b), \tag{1.2.17}$$

where $s_1 = (\lambda+\delta)/c$ and $s_2 = (\lambda+\delta)/(c-\alpha)$. Rearranging terms yields

$$\begin{aligned} V(x) &= \frac{\lambda}{c}\mathcal{T}_{s_1}\{V \star q\}(x) - \frac{\lambda}{c} e^{s_1 x} \int_b^{\infty} e^{-s_1 z} V \star q(z) dz + e^{-s_1(b-x)} V(b) \\ &= \frac{\lambda}{c}\mathcal{T}_{s_1}\{V \star q\}(x) + e^{-s_1(b-x)}[V(b) - \frac{\lambda}{c}\mathcal{T}_{s_1}\{V \star q\}(b)]. \end{aligned}$$

Applying the operator $s_1\mathcal{I} - \mathcal{D}$ to both sides yields

$$V'(x) = \frac{\lambda+\delta}{c} V(x) - \frac{\lambda}{c}\int_0^x V(x-y)dQ(y), \qquad 0 \leq x < b. \tag{1.2.18}$$

We now give an explicit solution for the special case where the claim size distribution $Q(y) = 1 - e^{-\beta y}$ and $\delta > 0$.

Example 1.2.2. *Special case: exponential claim size distribution*

Multiplying both sides of (1.2.16) by the operator $\beta\mathcal{I} + \mathcal{D}$ gives

$$V''(x) + (\beta - \frac{\lambda+\delta}{c-\alpha})V'(x) - \frac{\delta\beta}{c-\alpha}V(x) + \frac{\alpha\beta}{c-\alpha} = 0, \qquad x \geq b.$$

The general solution to the ordinary differential equation is given by

$$V(x) = \frac{\alpha}{\delta} + A_1 e^{\gamma_1 x} + A_2 e^{\gamma_2 x},$$

where

$$\gamma_1 = \frac{(\lambda+\delta) - \beta(c-\alpha) - \sqrt{[\beta(c-\alpha) - (\lambda+\delta)]^2 + 4\delta\beta(c-\alpha)}}{2(c-\alpha)},$$
$$\gamma_2 = \frac{(\lambda+\delta) - \beta(c-\alpha) + \sqrt{[\beta(c-\alpha) - (\lambda+\delta)]^2 + 4\delta\beta(c-\alpha)}}{2(c-\alpha)}.$$

It follows from the definition of $V(x)$ in (1.2.14) that $V(x) \leq \alpha/\delta$. Hence the coefficient $A_2 = 0$ as γ_2 is strictly positive. Multiplying both sides of (1.2.18) by $\beta\mathcal{I} + \mathcal{D}$ gives

$$V''(x) + (\beta - \frac{\lambda+\delta}{c})V'(x) - \frac{\delta\beta}{c}V(x) = 0, \qquad 0 \leq x < b,$$

which admits the solution

$$V(x) = B_1 e^{\eta_1 x} + B_2 e^{\eta_2 x},$$

where

$$\eta_1 = \frac{(\lambda+\delta) - \beta c - \sqrt{[\beta c - (\lambda+\delta)]^2 + 4\delta\beta c}}{2c},$$
$$\eta_2 = \frac{(\lambda+\delta) - \beta c + \sqrt{[\beta c - (\lambda+\delta)]^2 + 4\delta\beta c}}{2c}.$$

Substituting the solution into (1.2.18) and equating the coefficients of terms involving $e^{-\beta x}$ to zero gives

$$\frac{B_1}{\beta+\eta_1} + \frac{B_2}{\beta+\eta_2} = 0.$$

By letting $x \to b$ in both (1.2.15) and (1.2.17), we can see that

$$V(b-) = V(b+),$$

which implies that

$$B_1 e^{\eta_1 b} + B_2 e^{\eta_2 b} = \frac{\alpha}{\delta} + A_1 e^{\gamma_1 b}.$$

Substituting the solutions into (1.2.16) and equating the coefficients of the terms involving $e^{-\beta x}$ to zero gives

$$\frac{B_1 e^{\eta_1 b}}{\beta+\eta_1} + \frac{B_2 e^{\eta_2 b}}{\beta+\eta_2} + \frac{A_1 e^{\gamma_1 b}}{\beta+\gamma_1} + \frac{\alpha}{\beta\delta} = 0.$$

Solving the system of equations and inserting the coefficients into the solution, we have

$$V(x) = \frac{\alpha\gamma_1}{\delta\beta}\frac{(\beta+\eta_1)e^{\eta_1 x}-(\beta+\eta_2)e^{\eta_2 x}}{(\eta_2-\gamma_1)e^{\eta_2 b}-(\eta_1-\gamma_1)e^{\eta_1 b}}, \qquad 0 \le x < b, \tag{1.2.19}$$

$$V(x) = \frac{\alpha}{\delta}[1-e^{\gamma_1(x-b)}]+V(b)e^{\gamma_1(x-b)}, \qquad x \ge b. \tag{1.2.20}$$

There are two interesting phenomena that attract our attention. (1) Even though the expected discounted penalty $m(x)$ and expected total dividends $V(x)$ are distinct quantities, it is peculiar to see that they all satisfy surprisingly similar homogeneous or inhomogeneous integro-differential equations given in (1.2.2), (1.2.16) and (1.2.18). The similarity among these equations may suggest that these quantities belong to the same solution system. (2) All the solution methods developed for the Gerber-Shiu function see their applications in solving the expected total dividends. It might be an indication that we have been dealing with different aspects of a more general form.

A question arises naturally - are they members of a larger family of functions?

1.2.3 Generalized Gerber-Shiu Function

We now give an affirmative answer to this question with a slightly heuristic argument. The purpose of this section is to show that a general form of function can be used to reconcile the Gerber-Shiu function and the total discounted dividends paid up to ruin. Readers will find rigorous proofs for more general underlying risk processes in later chapters.

Such a function will be called a *generalized Gerber-Shiu function* throughout the thesis. It is constructed on the basis of total discounted dividends as follows.

$$H(x) \triangleq \mathbb{E}^x\left[\int_0^{\tau_d} e^{-\delta t} l(X_t)dt\right], \qquad x \ge d, \tag{1.2.21}$$

where the constant $\delta \ge 0$ is the discounting force of interest, the stopping time τ_d of the real valued stochastic process $X = \{X(t); t \ge 0\}$ is given by

$$\tau_d = \inf\{t | X(t) < d\}, \qquad d \in \mathbb{R},$$

with the convention that $\inf \varnothing = \infty$ and $l(\cdot)$ is a $\mathcal{B}(\mathbb{R})$-measurable function. As it shall be clear in later chapters, we would refer to τ_d as the *time of default* to be distinguished from τ_0, the time of ruin. The measurable function $l(\cdot)$ will be called *cost function*, as it has a natural interpretation of representing business costs. Thereby, in the context of ruin problems, the generalized Gerber-Shiu function of the form (1.2.21) can be viewed as the expected total discounted business costs incurred up to the time of default.

The derivation of solution to a generalized Gerber-Shiu function always involves the infinitesimal generator of the underlying risk process X.

Definition 1.2.1. *The* infinitesimal generator *of a stochastic process X is an operator $\mathfrak{A}$, which is defined on a suitable function f by*

$$\mathfrak{A}f(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}^x[f(X_t)] - f(x)}{t}, \qquad x \in \mathbb{R}. \tag{1.2.22}$$

The set of functions f such that the limit exists for $x \in \mathbb{R}$ is denoted by $D(\mathfrak{A})$, called the domain *of the generator $\mathfrak{A}$.*

We can easily obtain the infinitesimal generator of the risk process described by the classical compound Poisson model. By the definition of compound Poisson process, the number of claims up to time t is given by

$$\mathbb{P}(N_t = n) = \frac{(\lambda t)^x e^{-\lambda t}}{n!}, \qquad n = 0, 1, 2, \cdots .$$

Therefore, there is no claim by the time t with the probability of $e^{-\lambda t} = 1 - \lambda t + o(t)$, one claim by time t with the probability of $\lambda t e^{-\lambda t} = \lambda t + o(t)$ and more than one claim by time t with the probability $1 - e^{-\lambda t} - \lambda t e^{-\lambda t} = o(t)$.

Had there been no claim, the surplus process would have been accumulated to $x + ct$ by time t. Otherwise, the surplus process would be at $x + ct - y$ by time t if the size of a single claim occurred before t is given by y. Combining all those infinitesimally small terms, we can write by the total law of probability

$$\mathbb{E}^x[f(X_t)] = (1-\lambda t)f(x+ct) + \lambda t \int_0^\infty f(x+ct-y)dQ(y) + o(t).$$

We are now ready to derive the infinitesimal generator of the classical compound Poisson model.

$$\begin{aligned} \mathfrak{A}f(x) &= \lim_{t\downarrow 0} \frac{(1-\lambda t)f(x+ct) + \lambda t \int_0^\infty f(x+ct-y)dQ(y) - f(x) + o(t)}{t} \\ &= \lim_{t\downarrow 0} \left\{ \frac{f(x+ct)-f(x)}{t} + \frac{\lambda t f(x+ct)}{t} + \frac{\lambda t \int_0^\infty f(x+ct-y)dQ(y)}{t} \right\}, \end{aligned}$$

which yields that

$$\mathfrak{A}f(x) = cf'(x) - \lambda f(x) + \lambda \int_0^\infty f(x-y)dQ(y). \tag{1.2.23}$$

Assume that $H \in D(\mathfrak{A})$. Since the generalized Gerber-Shiu function is defined on $[d, \infty)$, we must have

$$\mathfrak{A}H(x) = cH'(x) - \lambda H(x) + \lambda \int_0^{x-d} H(x-y)dQ(y), \qquad x \geq d. \tag{1.2.24}$$

On the other hand, with the specific form of the generalized Gerber-Shiu function, we see by definition and change of variables that

$$\mathbb{E}^x[H(X_t)] = \mathbb{E}^x\left[\int_0^{\tau_d - t} e^{-\delta s} l(X_{s+t})ds\right] = \mathbb{E}^x\left[\int_t^{\tau_d} e^{-\delta(u-t)} l(X_{u-t})du\right] = \mathbb{E}^x\left[e^{\delta t}\int_t^{\tau_d} e^{-\delta s} l(X_s)ds\right].$$

Assuming all necessary conditions are satisfied, we derive that

$$\begin{aligned} \lim_{t\downarrow 0} \frac{\mathbb{E}^x[H(X_t)] - H(x)}{t} &= \mathbb{E}^x\left[\lim_{t\downarrow 0} \frac{e^{\delta t}\int_t^{\tau_d} e^{-\delta s} l(X_s)ds - \int_0^{\tau_d} e^{-\delta s} l(X_s)ds}{t}\right] \\ &= \mathbb{E}^x\left[\frac{d}{dt}\left\{e^{\delta t}\int_t^{\tau_d} e^{-\delta s} l(X_s)ds\right\}\Bigg|_{t=0^+}\right] \\ &= \delta\mathbb{E}^x\left[\int_0^{\tau_d} e^{-\delta s} l(X_s)ds\right] - l(x) \\ &= \delta H(x) - l(x). \end{aligned} \tag{1.2.25}$$

In view of (1.2.22) and (1.2.25), we now arrive at the equation that is absolutely essential to finding the solution of the generalized Gerber-Shiu function,

$$\mathfrak{A}H(x) - \delta H(x) + l(x) = 0, \qquad x \geq d. \tag{1.2.26}$$

As we shall see in later chapters, the equation holds true consistently for a great variety of risk processes, which is, in the author's point of view, a further manifestation of the beauty of ruin theory.

In the classical compound Poisson model, we set the level of default $d = 0$. In view of (1.2.24) and (1.2.26), we conclude that any generalized Gerber-Shiu function would satisfy the integro-differential equation

$$cH'(x) - (\lambda + \delta)H(x) + \lambda \int_0^x H(x-y)dQ(y) + l(x) = 0, \qquad x \geq 0. \tag{1.2.27}$$

Comparing (1.2.2) and (1.2.27), one might think that the Gerber-Shiu function $m(x)$ is a special case of the generalized Gerber-Shiu function $H(x)$ where the cost function

$$l(x) = \lambda \int_x^\infty w(x, y-x)dQ(y).$$

Such a conjecture will be proven to be valid in Section 2.3.5.

In consideration of dividend threshold strategy, we can easily show that

$$\mathfrak{A}H(x) = \begin{cases} (c-\alpha)H'(x) - \lambda H(x) + \lambda \int_0^{x-d} H(x-y)dQ(y), & x \geq b \\ cH'(x) - \lambda H(x) + \lambda \int_0^{x-d} H(x-y)dQ(y), & 0 \leq x < b. \end{cases} \tag{1.2.28}$$

It follows immediately by comparing (1.2.14) and (1.2.21) that the dividends paid up to ruin $V(x)$ is obviously a special case of the generalized Gerber-Shiu function $H(x)$ by choosing

$$l(x) = \begin{cases} \alpha, & x \geq b; \\ 0, & 0 \leq x < b. \end{cases} \tag{1.2.29}$$

Substituting (1.2.28) and (1.2.29) into (1.2.26) reproduces the system integro-differential equations (1.2.16) and (1.2.18) satisfied by $V(x)$.

As we have seen so far, this new approach involving generalized Gerber-Shiu function provides a short-cut to the solution equations once the prior knowledge of infinitesimal generator and cost function is acquired. In fact, infinitesimal generators of the vast majority of risk processes are very well-studied and readily available on many standard textbooks. It only remains for us to find appropriate cost function for various specific cases of the generalized Gerber-Shiu function.

In the next few chapters, we will be looking at more general classes of risk processes, which are all essentially generalizations in one way or another of the classical compound Poisson risk model. Many of these existing risk models are well-studied using conventional approaches with the objectives of finding solutions to either the probability of ruin, Gerber-Shiu function or total dividends paid up to ruin. In an attempt to further develop the new tool of generalized Gerber-Shiu function, we shall prove the formula (1.2.26) for each individual class of underlying risk processes. We would also investigate cost functions for a wide range of traditional and new ruin-related quantities and demonstrate how the formula provides a fast-track to the solutions.

Chapter 2

Piecewise-deterministic Compound Poisson Risk Models

The first generalization of the compound Poisson model to be discussed in the thesis is introduced to meet the practical needs of incorporating in risk models interest return, dividend payments, etc. We will start with a heuristic motivation of the generalization from the classical compound Poisson model and then define a more general class of processes called the piecewise-deterministic compound Poisson processes, and the generalized Gerber-Shiu function in rigorous mathematical terms. As a special case, we shall revisit the classical compound Poisson model using the newly developed approach as opposed to applying the traditional approaches introduced in Chapter 1. Later on, we shall employ the new approach in a few more examples of piecewise-deterministic compound Poisson risk models, where the efficiency and versatility of the approach becomes more apparent.

2.1 Piecewise-deterministic Markov Process

The class of piecewise-deterministic Markov processes (PDMP) was introduced by Davis [14], and has ever since drawn increased interests from researchers from a great variety of areas in applied probability and engineering. Among its early natural applications, the PDMP risk models were first studied by Dassios and Embrechts [13] to take into account interests and inflation in the study of insurance surplus processes. Under the PDMP framework, many martingale tools were brought in to deal with ruin-related quantities in far more general settings than the classical Poisson risk model.

Despite its potential in application, there has been relatively sparse PDMP presence in actuarial literature. To make the thesis self-contained and our results comparable to those well-known in the actuarial literature, we shall restrict our attention to a small class of the PDMP model, namely the piecewise-deterministic compound Poisson process (PDCP for short) and restate some fundamental properties for future references. For a comprehensive introduction to PDMPs, readers are referred to Davis [14], [15] and Rolski *et al.* [44].

In the classical compound Poisson model, the dynamics of a surplus process $\{U(t), t \geq 0\}$ is given by

$$dU(t) = cdt - dZ(t) \ ,$$

where the insurer's initial surplus u and the premium income rate c are given and the aggregate claims $Z(t) = \sum_{i=1}^{N(t)} Y_i$ is a random sum of insurance claims defined as follows. The occurrence of insurance claims follow a Poisson process $\{N(t), t \geq 0\}$ with intensity rate λ. All claims $Y_1, Y_2, \cdots$ are mutually independent and identically distributed with the common distribution $Q(y)$ and mean κ. As shown in Figure 2.1, the geometric feature of this model is the linear growth in surplus in between any two consecutive claims.

To make the surplus process more adaptable to various realistic situations, we attempt to extend the risk models as far as we can while preserving the most essential Markov properties enjoyed by the classical compound Poisson model. Instead of assuming independent

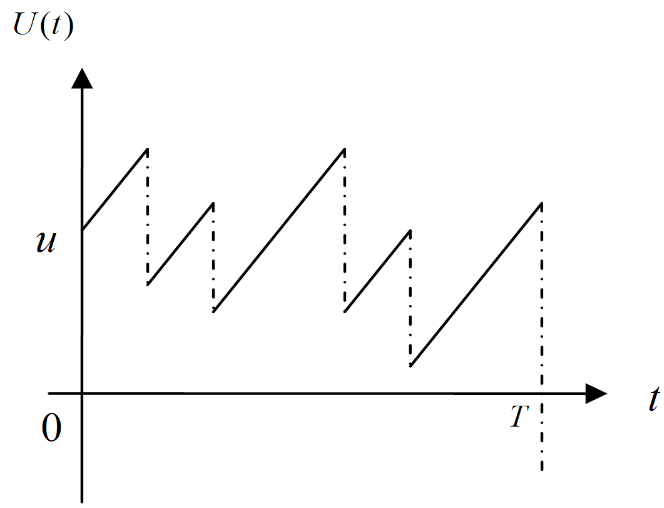

Figure 2.1: Sample path of classical compound Poisson model

structures, we want the claim sizes to have certain dependency with the actual value of surplus at the time of claim arrivals. There is also a need to allow for non-linear accumulation of surplus in the period between any two consecutive claims as long as some regularity conditions are imposed to ensure the Markov property. Combining these requests, we see that one of the candidate models that rise to the challenge is the piecewise-deterministic compound Poisson process.

We assume as given a probability space $(\Omega, \mathcal{F}, P)$ satisfying the usual hypothesis.

Definition 2.1.1. *A (standard, one-dimensional) Piecewise-deterministic Compound Poisson Process is a real-valued adapted càdlàg process* $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$*, defined on probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with the properties that*

1. $X(0) = x$*, a.s.;*

2. *Let* $T_0 = 0$ *and* $T_1, T_2, T_3, \cdots$ *denote the sequence of jump points. Then the counting process defined by* $N(t) = \sum_{i=1}^{\infty} I(T_i \leq t)$ *follows a homogeneous Poisson process with*

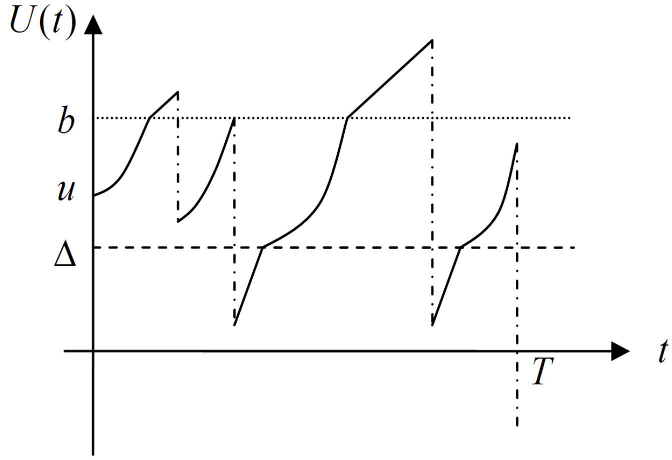

Figure 2.2: Sample path of a piecewise-deterministic compound Poisson model

intensity rate λ*;*

3. *The jump sizes* $\Delta X(T_k) = X(T_k) - X(T_k-)$ *for* $k = 1, 2, 3, \cdots$ *are determined by a transition measure* $Q\big(\cdot; X(T_k-)\big)$*;*

4. *The continuous pieces* $\{X_t; T_k \le t < T_{k+1}, k = 0, 1, 2, \cdots\}$ *are deterministically governed by a vector field* $\mathfrak{X}$*.*

The triplet $(\mathfrak{X}, \lambda, Q)$ are called the local characteristics of the PDCP. It should be noted that if X is a càdlàg process there exists a sequence of $\{T_n\}_{n=1}^{\infty}$ of stopping times of $\{\mathcal{F}_t\}$ which exhausts the jumps of X (c.f. Proposition 2.26 Karatzas and Shreve [31]). The second property that enumerates the sequence of jump points is well justified.

Albeit a rather abstract concept from differential geometry, the vector field appears naturally in many areas including ruin theory. For instance, in the compound Poisson model with investment, apart from the reduction caused by insurance claims, the instantaneous increase in the surplus process is attributable to the present surplus amount times the force of interest plus the instantaneous premium income,

$$dU(t) = [c + rU(t)]dt,$$

which means for any continuously differentiable function $f(x)$,

$$\frac{d}{dt} f\big(U(t)\big) = [c + rU(t)] \frac{d}{dx} f\big(U(t)\big) = \mathfrak{X} f\big(U(t)\big),$$

where the operator $\mathfrak{X} = (c + rx)d/dx$ is known as the vector field.

As one shall see in later sections, the vector field $\mathfrak{X}$ in the majority, if not all, of the applications in ruin theory met the following requirement. Hence we will assume throughout the chapter that the vector field $\mathfrak{X}$ in the definition of PDCP can always be represented as follows.

For a given finite partition $\Pi_n = \{b_0 = x, b_1, \cdots, b_n = \infty\}$ of $[x, \infty)$, $g(x)$ is Lipschitz continuous on each subinterval of the partition $[b_i, b_{i+1})$ for any $i = 0, 1, \cdots, n$. Then the ordinary differential equation

$$\frac{d}{dt} X(t) = g\big(X(t)\big), \qquad X(0) = x \in \mathbb{R}, \tag{2.1.1}$$

uniquely determines a deterministic process, known as a flow or integral curve and in convention denoted by $\phi(t, x)$ or $\phi_x(t)$ for brevity. Hence for any continuously differentiable function $f(x)$,

$$\frac{d}{dt} f\big(\phi_x(t)\big) = \mathfrak{X} f\big(\phi_x(t)\big), \tag{2.1.2}$$

where the corresponding vector field is given by

$$\mathfrak{X} f(x) = g(x) \frac{d}{dx} f(x). \tag{2.1.3}$$

In various applications, we are often given information regarding the flow $\phi(t, x)$. Then the corresponding vector field is obtainable from (2.1.1) or (2.1.3).

There are two properties of the flow that of particular interest to us.

1. The map $x \mapsto \phi(t,x)$ is one-to-one and onto; Its inverse with respect to x, $\phi^{-1}(t,x) = \phi(-t,x)$ for all $x \in \mathbb{R}$.

2. The family $\{\phi(t,x)\}_{t\in\mathbb{R}}$ is a group. i.e. for any $t,s \in \mathbb{R}$, $\phi(t+s,x) = \phi(t,\phi(s,x))$ for all $x \in \mathbb{R}$.

As we have seen in Chapter 1, it is absolutely essential in the analytical arguments that a process regenerates itself at a certain point, which in mathematical terms is the strong Markov property. We can not define a strong Markov property without properly defining a set of measures under which the process “restarts”.

Definition 2.1.2. *A Piecewise-deterministic Compound Poisson Family is a real-valued adapted càdlàg process* $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$*, defined on probability space* $(\Omega, \mathcal{F})$*, together with a family of probability measures* $\{\mathbb{P}^x\}_{x\in\mathbb{R}}$ *on* $(\Omega,\mathcal{F})$*, such that*

1. *for each* $A \subseteq \mathbb{R}$*, the mapping* $x \mapsto Q(A;x)$ *is measurable;*

2. $\mathbb{P}^x[X(0) = x] = 1$*, for any* $x \in \mathbb{R}$*;*

3. *under each* $\mathbb{P}^x$*, the process* X *is a piecewise-deterministic compound Poisson process starting at* x*.*

Since all quantities to be discussed are functionals of the PDCPs, our analysis heavily relies on the strong Markov property proved in Theorem 25.5 of Davis [15].

Theorem 2.1.1. *Let* X *be a piecewise-deterministic compound Poisson family,* τ *be a stopping time with respect to* $\{\mathcal{F}_t; t \geq 0\}$ *such that* $\tau < \infty$ *a.s. and* f *be a bounded measurable function. Then*

$$\mathbb{E}^x[f(X_\tau + s)|\mathcal{F}_\tau] = \mathbb{E}^{X_\tau}[f(X_s)], \qquad \textit{for all } s \geq 0.$$

As we alluded to in Chapter 1, many quantities of interest in ruin theory and financial mathematics can be solved via differential equations involving generators. However, as pointed out in Davis [15], it is rather difficult to characterize the domain of the infinitesimal generator defined in Definition 1.2.1 for PDMPs, but there are easily checked sufficient conditions for the domain of another type of generator, which also characterizes a stochastic process, called extended generator.

Definition 2.1.3. *Suppose there exists a measurable function h such that $t \mapsto h(X_t)$ is integrable $\mathbb{P}^x$-a.s. for each $x \in \mathbb{R}$ and the process*

$$f(X_t) - f(x) - \int_0^t h(X_s)ds$$

is a local martingale. Then we write $h = \mathfrak{A}f$ and $\mathfrak{A}$ is called the extended generator *of the process $X = \{X_t; t \geq 0\}$. The set of functions f such that the above property holds, denoted by $D(\mathfrak{A})$, is called the domain of extended generator $\mathfrak{A}$.*

Remark 2.1.1.

1. *It can be shown that if $f \in D(\hat{\mathfrak{A}})$ where $\hat{\mathfrak{A}}$ is the infinitesimal generator of X, then*

$$f(X_t) - f(x) - \int_0^t \hat{\mathfrak{A}}f(X_s)ds$$

is a martingale, hence is a local martingale. In other words, the extended generator $\mathfrak{A}$ is indeed an extension of $\hat{\mathfrak{A}}$ in that $D(\hat{\mathfrak{A}}) \subset D(\mathfrak{A})$ and $\hat{\mathfrak{A}}f = \mathfrak{A}f$ for all $f \in D(\hat{\mathfrak{A}})$.

2. *If f is continuously differentiable, then it follows from (2.1.2)*

$$f(\phi_x(t)) - f(x) - \int_0^t \mathfrak{X}f(\phi_x(s))ds = 0 \tag{2.1.4}$$

which is a trivial martingale. Thus $\mathfrak{X}$ is the extended generator of the deterministic process $\phi_x(t)$. In fact its domain $D(\mathfrak{X})$ is the set of all measurable functions f such that $t \mapsto f(\phi_x(t))$ is absolutely continuous.

We apply Theorem 26.14 in Davis [15] to give the sufficient conditions for checking the membership of $D(\mathfrak{A})$ and the extended generator for PDCPs.

Theorem 2.1.2. *Let $\{X_t; 0 \le t < \infty\}$ be a piecewise-deterministic compound Poisson process. Then the domain $D(\mathfrak{A})$ of the extended generator $\mathfrak{A}$ of $\{X_t\}$ consists of functions such that*

1. *The function $t \mapsto f\big(\phi_x(t)\big)$ is absolutely continuous for all initial values $x \in \mathbb{R}$;*
2. $\mathbb{E}^x\big[\sum_{k=1}^{n} |f(X(T_k)) - f(X(T_k-))|\big] < \infty$, *for* $n = 1, 2, ...$

And for each $f \in D(\mathfrak{A})$, $\mathfrak{A}f$ is given by

$$\mathfrak{A}f(x) = \mathfrak{X}f(x) - \lambda f(x) + \lambda\, \mathcal{Q}f(x), \tag{2.1.5}$$

where

$$\mathcal{Q}f(x) = \int_{-\infty}^{\infty} f(y)Q(dy; x).$$

Another process that arises frequently in ruin theory is the associated counting process defined by

$$N(t, A) = \sum_{i=1}^{\infty} I(T_i \le t)I(X(T_i) \in A), \qquad \text{for } A \in \mathcal{B}(\mathbb{R})\ .$$

This process records the frequency of the underlying piecewise-deterministic compound Poisson process being in A as a result of each jump by the time t. An appealing fact about the associated counting process is that its compensator can be written as

$$\tilde{N}(t, A) = \lambda \int_0^t Q(A; X_s)ds.$$

The following theorem elucidates the connection between the associated counting process and its compensator process.

Theorem 2.1.3. *For all nonnegative $\mathcal{F}_t$-adapted predictable process C_t,*

$$\mathbb{E}^x\left[\int_0^\infty \int_{-\infty}^\infty C(s)N(ds,dy)\right] = \mathbb{E}^x\left[\int_0^\infty \int_{-\infty}^\infty C(s)\tilde{N}(ds,dy)\right] .$$

The proof can be found in Brémaud [4], Chapter II Section 2.

2.2 Generalized Gerber-Shiu Functions

From this point on, we start to look at piecewise-deterministic compound Poisson risk models, where an insurer's surplus is driven by a real valued PDCP process $X = \{X_t, 0 \leq t < \infty\}$. The sequence of jump points $\{T_n, n = 1, 2, \cdots\}$ represents the arrivals of insurance claims, whereas the measure Q determines changes in surplus caused by claims or unexpected income. The initial investment, which is represented by the initial value of the PDCP, is set to be x.

The primary focus of the thesis is given to a *generalized Gerber-Shiu function* defined as follows,

$$H(x) = \mathbb{E}^x\left[\int_0^{\tau_d} e^{-\delta t} l(X_t)dt\right], \tag{2.2.1}$$

where $\delta \geq 0$, the *cost function* $l(\cdot)$ is $\mathcal{B}(\mathbb{R})$-measurable and the *time of default* τ_d is defined by

$$\tau_d = \inf\{t | X(t) < d\}, \qquad d \in \mathbb{R}$$

with the convention that $\inf \varnothing = \infty$. The intuitive interpretation of the generalized Gerber-Shiu function is the expected present value of all future business costs arising from maintaining the surplus process up to the time of default. Conventionally, when $d = 0$, τ_0 is called the *time of ruin*.

As it shall become clear shortly, the advantage of analyzing the generalized Gerber-Shiu function is that many ruin-related functionals of the surplus process can be accommodated in such a unified form, which can be exploited systematically from integro-differential equations associated with the extended generators. We use the arguments similar to the ones given in Theorem 32.2 of Davis [15] to prove the following major result.

Theorem 2.2.1. *Suppose $l(x)$ is continuous on $[d,\infty)$ except for a countable set of discontinuities D and that H defined in (2.2.1) is bounded, then H is continuous on $[d,\infty)$, differentiable on $[d,\infty)\backslash D\cup\{b_1,b_2,\cdots,b_n\}$ and satisfies*

$$\mathfrak{A}H(x)-\delta H(x)+l(x)=0,\qquad x\ge d, x\notin D\cup\{b_1,b_2,\cdots,b_n\}\,. \tag{2.2.2}$$

Proof. It is trivial to prove that $T_1\wedge t$ is an $\mathcal{F}_t$-stopping time. Let $y=X_{T_1\wedge t}$. Since the function $t\mapsto l(\phi_x(t))$ is integrable on the interval $[0,\epsilon]$, for any $t\in[0,\epsilon]$, we must have

$$H(x)=\mathbb{E}^x\left[\int_0^{T_1\wedge t}e^{-\delta s}l(X_s)ds\right]+\mathbb{E}^x\left[\int_{T_1\wedge t}^{\tau_d}e^{-\delta s}l(X_s)ds\right].$$

Recall from Theorem 2.1.1 that if we define $Y_s=X_{T_1\wedge t+s}$, then $Y=\{Y_s,0\le s<\infty\}$ is a PDCP starting at y adapted to $\{\mathcal{H}_s=\mathcal{F}_{T_1\wedge t+s},0\le s<\infty\}$. Define $\tau_d^Y=\inf\{t|Y(t)<d\}$. Since Y has the same distribution under $\mathbb{P}^y$ as X under $\mathbb{P}^x$, we must have $\tau_d^Y=\tau_d-T_1\wedge t$. Therefore,

$$\begin{aligned}
\mathbb{E}^x\left[\int_{T_1\wedge t}^{\tau_d}e^{-\delta s}l(X_s)ds\right] &= \mathbb{E}^x\left\{\mathbb{E}^y\left[\int_{T_1\wedge t}^{\tau_d}e^{-\delta s}l(X_s)ds|X_{T_1\wedge t}=y\right]\right\}\\
&= \mathbb{E}^x\left\{e^{-\delta(T_1\wedge t)}\mathbb{E}^y\left[\int_0^{\tau_d-T_1\wedge t}e^{-\delta s}l(Y_s)ds\right]\right\}\\
&= \mathbb{E}^x\left\{e^{-\delta(T_1\wedge t)}\mathbb{E}^y\left[\int_0^{\tau_d^Y}e^{-\delta s}l(Y_s)ds\right]\right\}=\mathbb{E}^x\left[e^{-\delta(T_1\wedge t)}H(X_{T_1\wedge t})\right].
\end{aligned}$$

Note that under $\mathbb{P}^x$, T_1 follows the exponential distribution with parameter λ and whenever $s < T_1$, we have the deterministic sample path $X_s = \phi_x(s)$. Therefore,

$$\begin{aligned}
\mathbb{E}^x\left[\int_0^{T_1\wedge t} e^{-\delta s} l(X_s)ds\right] &= \mathbb{E}^x\left[I(t<T_1)\int_0^t e^{-\delta s} l(X_s)ds\right] + \mathbb{E}^x\left[I(t\geq T_1)\int_0^{T_1} e^{-\delta s} l(X_s)ds\right] \\
&= e^{-\lambda t}\int_0^t e^{-\delta s} l\big(\phi_x(s)\big)ds + \int_0^t \lambda e^{-\lambda s}\int_0^s e^{-\delta u} l\big(\phi_x(u)\big)\,du\,ds \\
&= \int_0^t e^{-(\lambda+\delta)s} l\big(\phi_x(s)\big)\,ds
\end{aligned}$$

with the last equality from integration by parts. On the other hand,

$$\begin{aligned}
\mathbb{E}^x\left[e^{-\delta(T_1\wedge t)}H(X_{T_1\wedge t})\right] &= \mathbb{E}^x\left[I(t<T_1)e^{-\delta t}H(X_t)\right] + \mathbb{E}^x\left[I(t\geq T_1)e^{-\delta T_1}H(X_{T_1})\right] \\
&= \mathbb{E}^x\left[I(t<T_1)e^{-\delta t}H(X_t)\right] + \mathbb{E}^x\left[I(t\geq T_1)e^{-\delta T_1}\mathbb{E}^x\big[H(X_{T_1})|\mathcal{F}_{T_1-}\big]\right] \\
&= \mathbb{E}^x\left[I(t<T_1)e^{-\delta t}H(X_t)\right] + \mathbb{E}^x\left[I(t\geq T_1)e^{-\delta T_1}\mathbb{E}^x\big[H(X_{T_1})|X_{T_1-}\big]\right]
\end{aligned}$$

with the last equality from the strong Markov property. Since Q determines the jump mechanism, for any $z \geq d$,

$$\mathbb{E}^x\left[H(X_{T_1})|X_{T_1-}=z\right] = \int_{-\infty}^{\infty} H(u)Q(du;z).$$

Together with the fact that $z \mapsto Q(A;z)$ is $\mathcal{B}(\mathbb{R})$-measurable, we have

$$\mathbb{E}^x\left[I(t\geq T_1)e^{-\delta T_1}\mathbb{E}^x\big[H(X_{T_1})|X_{T_1-}\big]\right] = \mathbb{E}^x\left[I(t\geq T_1)e^{-\delta T_1}\mathcal{Q}H(X_{T_1-})\right].$$

Note that when $t < T_1$, the PDCP X remains in the deterministic piece. Thus we can simplify that

$$\mathbb{E}^x\left[I(t<T_1)e^{-\delta t}H(X_{T_1\wedge t})\right] = e^{-(\lambda+\delta)t}H\big(\phi_x(t)\big) + \int_0^t \lambda e^{-(\lambda+\delta)s}\mathcal{Q}H\big(\phi_x(s)\big)\,ds.$$

Therefore, putting all the pieces together, we have

$$H(x) = \int_0^t e^{-(\lambda+\delta)s} \left[l\big(\phi_x(s)\big) + \lambda\ \mathcal{Q}H\big(\phi_x(s)\big) \right] ds + e^{-(\lambda+\delta)t} H\big(\phi_x(t)\big). \tag{2.2.3}$$

Since all the elements involving $\phi_x(s), s \in [0,t]$ are deterministic with respect to the time argument s, we shall adopt a simpler notation $f^\star(s) = f\big(\phi_x(s)\big)$. Thus, (2.2.3) can be written as

$$H^\star(0) = \int_0^t e^{-(\lambda+\delta)s} \left[l^\star(s) + \lambda\ \mathcal{Q}H^\star(s) \right] ds + e^{-(\lambda+\delta)t} H^\star(t),$$

i.e.

$$H^\star(t) = \int_0^t e^{(\lambda+\delta)(t-s)} g^\star(s) ds + e^{(\lambda+\delta)t} H^\star(0), \tag{2.2.4}$$

where for notational convenience $g^\star(s) = -l^\star(s) - \lambda\ \mathcal{Q}H^\star(s)$. Note that

$$\begin{aligned}
\int_0^t (\lambda+\delta) H^\star(s) ds &= \int_0^t \int_0^s (\lambda+\delta) e^{(\lambda+\delta)(s-u)} g^\star(u)\, du\, ds + \int_0^t (\lambda+\delta) e^{(\lambda+\delta)s} H^\star(0)\, ds \\
&= \int_0^t g^\star(u) \int_u^t (\lambda+\delta) e^{(\lambda+\delta)(s-u)} ds\, du + e^{(\lambda+\delta)t} H^\star(0) - H^\star(0) \\
&= \int_0^t g^\star(u) e^{(\lambda+\delta)(t-u)} du - \int_0^t g^\star(u)\, du + e^{(\lambda+\delta)t} H^\star(0) - H^\star(0).
\end{aligned}$$

The boundedness of $H(x)$ allows the change of order of integrations. Substituting the expression for $H^\star(t)$ from (2.2.4) into above equation, we have hence shown that

$$H^\star(t) - H^\star(0) = \int_0^t (\lambda+\delta) H^\star(s)\, ds + \int_0^t g^\star(s)\, ds,$$

i.e.

$$H(\phi_x(t)) - H(x) = \int_0^t (\lambda+\delta) H(\phi_x(s))\, ds - \int_0^t \big[l(\phi_x(s)) + \lambda\ \mathcal{Q}H(\phi_x(s)) \big]\, ds. \tag{2.2.5}$$

Hence $H^{\star}(t)$ is absolutely continuous for all $x \geq d$, which in turn implies $H(x)$ is absolutely continuous for all $x \geq d$ as $\phi_x(t)$ is differentiable. Since there exists a real number M such that $H(x) < M$ for all x, then

$$\mathbb{E}^x \left[\sum_{k=1}^{n} |H(X_k) - H(X_k-)| \right] \leq 2Mn < \infty \text{ for } n = 1, 2, 3, \cdots$$

It follows from Theorem 2.1.2 that $H(x) \in D(\mathfrak{X})$.

For any $z \geq x$, there must be a $t \geq 0$ such that $z = \phi(t, x)$, which by the first property of the flow determines that $x = \phi(-t, z)$. By the second property of the flow, we must have for any $0 \leq s \leq t$,

$$\phi(s, x) = \phi(s, \phi(-t, z)) = \phi(s - t, z).$$

Hence we can write that

$$\begin{aligned} H(z) - H(\phi_z(-t)) &= \int_0^t (\lambda + \delta) H(\phi_z(s - t)) ds - \int_0^t \left[l(\phi_z(s - t)) + \lambda \, \mathcal{Q}H(\phi_z(s - t)) \right] ds \\ &= \int_{-t}^0 (\lambda + \delta) H(\phi_z(r)) dr - \int_{-t}^0 \left[l(\phi_z(r)) + \lambda \, \mathcal{Q}H(\phi_z(r)) \right] dr. \end{aligned}$$

Since x is chosen arbitrarily, we would have

$$H(x) - H(\phi_x(-t)) = \int_{-t}^0 (\lambda + \delta) H(\phi_x(r)) dr - \int_{-t}^0 \left[l(\phi_x(r)) + \lambda \, \mathcal{Q}H(\phi_x(r)) \right] dr. \tag{2.2.6}$$

It follows by (2.2.5) that

$$\lim_{t \to 0} \frac{H(\phi_x(t)) - H(x)}{t} = \lim_{t \to 0} (\lambda + \delta) H(\phi_x(t)) - l(\phi_x(t)) - \lambda \, \mathcal{Q}H(\phi_x(t)).$$

Similarly from (2.2.6) that

$$\lim_{t \to 0} \frac{H(x) - H(\phi_x(-t))}{t} = \lim_{t \to 0} (\lambda + \delta) H(\phi_x(-t)) - l(\phi_x(-t)) - \lambda \, \mathcal{Q}H(\phi_x(-t)).$$

Since

$$\begin{aligned}
\lim_{t\to 0}\frac{H(\phi_x(t))-H(x)}{t} &= \lim_{t\to 0}\frac{H(\phi_x(t))-H(x)}{\phi_x(t)-x}\lim_{t\to 0}\frac{\phi_x(t)-x}{t}\\
&= g(\phi_x(0+))\lim_{t\to 0}\frac{H(\phi_x(t))-H(x)}{\phi_x(t)-x},\\
\lim_{t\to 0}\frac{H(x)-H(\phi_x(-t))}{t} &= \lim_{t\to 0}\frac{H(x)-H(\phi_x(-t))}{x-\phi_x(-t)}\lim_{t\to 0}\frac{x-\phi_x(-t)}{t}\\
&= g(\phi_x(0-))\lim_{t\to 0}\frac{H(x)-H(\phi_x(-t))}{x-\phi_x(-t)},
\end{aligned}$$

it is obvious that $H(x)$ is differentiable where both $l(x)$ and $g(x)$ are continuous.

Recall from (2.1.4) that

$$H(\phi_x(t))-H(x)=\int_0^t \mathfrak{X}H(\phi_x(s))ds. \tag{2.2.7}$$

In view of (2.2.5) and (2.2.7), we obtain

$$\mathfrak{X}H(x)=(\lambda+\delta)H(x)-l(x)-\lambda\,\mathcal{Q}H(x),\qquad x\geq d, x\notin D\cup\{b_1,b_2,\cdots,b_n\},$$

which can be simplified as (2.2.2) according to (2.1.5). □

The theorem shows that by choosing specific cost functions we can immediately obtain integro-differential equations for ruin-related quantities for a great variety of processes determined by different settings of $(\mathfrak{X},\lambda,Q)$. Then it remains to solve the specific integro-differential equations subject to certain boundary conditions in order to obtain the quantities of interest.

Remark 2.2.1. *Note that by definition $H(x)=0$ if $x<d$. Therefore, we can see that*

$$\mathcal{Q}H(x)=\int_d^\infty H(y)Q(dy;x). \tag{2.2.8}$$

If we further assume that there would be only negative jumps due to insurance claims whose distribution is independent of the current surplus level, then with a slight abuse of notation we use Q as a point distribution function as opposed to the measure function originally defined.

$$\mathcal{Q}H(x) = \int_d^x H(y)dQ(x-y) = \int_0^{x-d} H(x-y)dQ(y) \tag{2.2.9}$$

with the last equality resulted from a change of variable.

2.3 Classical Compound Poisson Model

The shifted compound Poisson process is obviously a simple example of PDCP. We now revisit the classical compound Poisson risk model in the context of PDCP.

Recall that the sample path in between two consecutive claims is continuous and linearly determined by the insurance premium rate c, i.e.

$$\frac{d}{dt}X(t) = \frac{d}{dt}(x+ct) = c. \tag{2.3.1}$$

From (2.1.1), we must have $g(\cdot) = c$, which implies from (2.1.3) that the extended generator of the deterministic path is given by

$$\mathfrak{X} = c\frac{d}{dx}.$$

Note that the event of ruin occurs at the first time the surplus falls below zero. Thus the stopping time of interest to us is the time of ruin with the level of default set at $d = 0$. Therefore, the extended generator of the classical compound Poisson risk model is given by

$$\mathfrak{A}H(x) = cH'(x) + \lambda \int_0^x H(x-y)dQ(y) - \lambda H(x), \qquad x \geq 0. \tag{2.3.2}$$

2.3.1 Total Dividends Paid up to Ruin by Threshold

The dividend threshold strategy requires that once the surplus reaches the threshold level b, dividends should be paid out at the rate of α to the insurance company's shareholders, which means the deterministic path remains linear but with a reduced slope $c-\alpha$, where $\alpha \in (0,c)$. Therefore, when the dividend threshold is imposed, the extended generator of the deterministic process changes to

$$\mathfrak{X} = \begin{cases} c\, d/dx, & \text{if } 0 \le x < b, \\ (c-\alpha)d/dx, & \text{if } x \ge b. \end{cases}$$

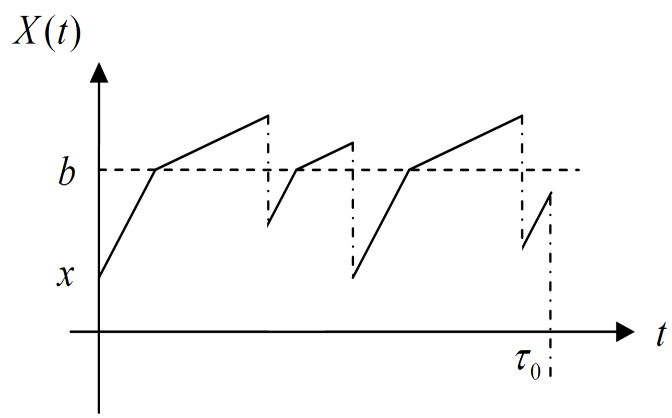

Figure 2.3: Sample path of compound Poisson model with dividend threshold

We are interested in the expected present value of dividends paid up to the time of ruin with the threshold strategy, defined by

$$V(x;b) = \mathbb{E}^x\left[\int_0^{\tau_0} e^{-\delta t} l\big(X(t)\big)dt\right], \qquad x \ge 0,$$

where $\delta > 0$ and the cost function is given by

$$l(x) = \begin{cases} \alpha, & \text{if } x \ge b. \\ 0, & \text{if } 0 \le x < b. \end{cases} \tag{2.3.3}$$

Since $l(x)$ is a bounded function, it is easy to see that $V(x;b)$ is also bounded. In view of (2.2.2), (2.3.2) and (2.3.3), we can quickly obtain the integro-differential equations for dividends paid up to ruin $V(x;b)$ expected in the classical model

$$cV'(x;b) - (\lambda+\delta)V(x;b) + \lambda\int_0^x V(x-y;b)dQ(y) = 0, \qquad 0 < x < b,$$

and

$$(c-\alpha)V'(x;b)-(\lambda+\delta)V(x;b)+\lambda\int_0^x V(x-y;b)dQ(y)+\alpha=0, \qquad x>b,$$

which are precisely the equation (1.2.16) and (1.2.18) in Chapter 1 obtained through traditional probabilistic arguments. Hence we have so far demonstrated the consistency between the traditional approach discussed in Chapter 1 and the newly proposed approach.

2.3.2 Total Dividends Paid up to Ruin by Barrier

With the dividend threshold strategy, an insurer has the responsibility to pay out a certain portion of its premium income as dividends once the surplus reaches the threshold level. Hence the dividend rate α takes value in $(0,c)$. We now consider the extreme case where the dividend rate α is set to be the premium rate c, which means any further premium income would be paid out completely and the surplus would be capped at the level where the dividend payment begins. Such a level is often referred to as *dividend barrier*, which we shall denote by b_0 to be distinguished from the dividend threshold. For more detailed discussion of dividend barrier strategies, readers are referred to Lin *et al.* [41] and Gerber and Shiu [27].

Hence the extended generator in this case becomes

$$\mathfrak{X}=\begin{cases} c\,d/dx, & \text{if } 0\le x<b_0,\\ 0, & \text{if } x=b_0.\end{cases}$$

Note that the second part of the generator uniquely determines the trivial deterministic process

$$X(t)=b_0,$$

given that $X(0)=b_0$, which corresponds to the sample path at the barrier level prior to an insurance claim.

It would be interesting to find out the expected present value of dividends paid up to the time of ruin with the barrier strategy, defined by

$$V(x) = \mathbb{E}^x \left[\int_0^{\tau_0} e^{-\delta t} l\big(X(t)\big) dt \right], \qquad 0 \le x \le b_0, \tag{2.3.4}$$

where $\delta > 0$ and the cost function is given by

$$l(x) = \begin{cases} c, & \text{if } x = b_0. \\ 0, & \text{if } 0 \le x < b_0. \end{cases}$$

Since $V(x) \le c/\delta$, it follows from Theorem 2.2.1 that

$$cV'(x) - (\lambda + \delta)V(x) + \lambda \int_0^x V(x-y)dQ(y) = 0, \qquad 0 < x < b_0. \tag{2.3.5}$$

To find explicit solutions to the above integro-differential equation, we often need an extra boundary condition to determine an unknown coefficient.

Corollary 2.3.1. *With the dividend barrier strategy, the function $V(x)$ defined in (2.3.4) satisfies the following boundary condition*

$$\mathfrak{A}V(b_0) - \delta V(b_0) + l(b_0) = 0. \tag{2.3.6}$$

Proof. Letting $x = b_0$ in the proof of Theorem 2.2.1, we can obtain (2.2.5) for the trivial integral curve $\phi_{b_0}(t) = b_0$. Hence

$$V(b_0) - V(b_0) = (\lambda + \delta)V(b_0)t - [l(b_0) + \lambda \mathcal{Q}V(b_0)]t.$$

Since $\mathfrak{A}V(b_0) = -\lambda V(b_0) + \lambda \mathcal{Q}V(b_0)$ in the case of dividend barrier strategy, (2.3.6) is obtained upon rearrangement. □

This boundary condition (2.3.6) is intentionally written in the form which would conform with those for other models in later chapters. As we have seen in the proof, in the classical compound Poisson model, (2.3.6) reduces to

$$V(b_0) = \frac{c}{\lambda+\delta} + \frac{\lambda}{\lambda+\delta}\int_0^{b_0} V(b_0-y)dQ(y).$$

Letting $x \to b_0$ in (2.3.5) and substituting in the boundary condition, we have

$$\begin{aligned} V'(b_0-) &= \frac{\lambda+\delta}{c}V(b_0) - \frac{\lambda}{c}\int_0^{b_0} V(b_0-y)dQ(y) \\ &= \frac{\lambda+\delta}{c}\left[\frac{c}{\lambda+\delta} - \frac{\lambda}{\lambda+\delta}\int_0^{b_0} V(b_0-y)dQ(y)\right] + \frac{\lambda}{c}\int_0^{b_0} V(b_0-y)dQ(y), \end{aligned}$$

from which we yield an alternative form of the boundary condition that

$$V'(b_0-) = 1.$$

The condition was derived through traditional probabilistic arguments in Bühlmann [5] and for more general models in Gerber *et al.* [20].

2.3.3 Insurer's Accumulated Utility

When a risk process is used to model and assess a line of insurance business, the insurer might be interested in a quantitative measure of the company's overall performance in maintaining its surplus reserve. In the context of microeconomics, the accumulated utility up to default provides such a tool to quantify an insurer's satisfaction gained from surplus at each moment throughout the life of the business. As an application, the accumulated utility of an insurer's surplus wealth conforms to the generalized Gerber-Shiu function,

$$U(x) = \mathbb{E}^x\left[\int_0^{\tau_d} u(X_t)dt\right],$$

where d is a pre-determined level of default for a particular line of business and $u(\cdot)$ is the utility function representing the insurer's attitude towards current surplus.

We consider the classical compound Poisson risk model with a general claim size distribution whose moment generating function is assumed to exist. In order to obtain closed-form solutions, we specify $u(x)$ to be the exponential utility function $-e^{-ax}/a$, which is commonly used in actuarial science and economics owing to its constant risk aversion property. In the classical model, $d = 0$. The safety loading condition $c > \lambda\kappa$ is imposed to ensure positive drift. For future references, we introduce a new notation

$$W(x) = \mathbb{E}^x \left[\int_0^{\tau_0} e^{-aX_t} dt \right], \tag{2.3.7}$$

where $a > 0$. Hence $U(x) = -W(x)/a$. We shall now focus on the properties and solutions to the accumulated exponential utility up to ruin $W(x)$.

Lemma 2.3.1. *For $x \geq 0$, W(x) is a bounded function.*

Proof. Construct an auxiliary function

$$f(s) = cs + \lambda[1 - M_Q(s)],$$

where $M_Q(s)$ is the moment generating function of claim size distribution $Q(x)$. Since $f(0) = 0$ and $\lambda M_Q'(s)|_{s=0+} = \lambda\kappa < c$, hence $f(s) > 0$ in a positive neighborhood of zero. In view of the fact that $f''(s) < 0$ and for all $s \geq 0$, there must exist a positive solution to $f(s) = 0$ denoted by R. Recall that

$$\mathbb{E}^x \left[e^{-aX_t}\right] = \mathbb{E}^x[e^{-a(x+ct-\sum_{i=1}^{N(t)} Y_i)}] = e^{-ax-act} e^{-\lambda t[1-M_Q(a)]} = e^{-ax} e^{-f(a)t}.$$

For $0 < a < R$, we must have $f(a) > 0$, then

$$\mathbb{E}^x \left[\int_0^{\tau_0} e^{-aX_t} dt \right] \leq \frac{1}{f(a)} e^{-ax} \leq \frac{1}{f(a)}.$$

Therefore,

$$\mathbb{E}^x \left[\int_0^{\tau_0} e^{-aX_t} dt \right] \leq \mathbb{E}^x \left[\int_0^{\infty} e^{-aX_t} dt \right] = \int_0^{\infty} \mathbb{E}^x \left[e^{-aX_t}\right] dt,$$

with the last equality from Fubini's theorem.

For $a \geq R$, we have $a > a_0$ where $0 < a_0 < R$. Since $X_t \geq 0$ on $[0, \tau_0]$, it follows that

$$\mathbb{E}^x \left[\int_0^{\tau_0} e^{-aX_t} dt \right] \leq \mathbb{E}^x \left[\int_0^{\tau_0} e^{-a_0 X_t} dt \right] \leq \frac{1}{f(a_0)} e^{-a_0 x} \leq \frac{1}{f(a_0)}.$$

$\square$

Hence it follows from Theorem 2.2.1 that $W(x)$ satisfies the following integro-differential equation

$$cW'(x) - \lambda W(x) + \lambda \int_0^x W(x-y) dQ(y) + e^{-ax} = 0, \qquad x > 0. \tag{2.3.8}$$

Corollary 2.3.2. *The solution to $W(x)$ defined in (2.3.7) is given by*

$$W(x) = \frac{e^{-ax}}{a(c - \lambda\kappa)} \int_0^x e^{ay} g(y) dy + \frac{1}{ac} e^{-ax}, \qquad x \geq 0, \tag{2.3.9}$$

where the associated compound geometric density function

$$g(x) = \sum_{n=1}^{\infty} (1 - \frac{\lambda\kappa}{c})(\frac{\lambda\kappa}{c})^n q_1^{\star n}(x)$$

and the equilibrium density function $q_1(x) = (1/\kappa)\overline{Q}(x)$.

Proof. We assume for simplicity the claim size distribution has the density $q(y) = Q'(y)$, but all of the following derivations can be extended to include discontinuous claim sizes.

In terms of operators, (2.3.8) can be written as

$$(\frac{\lambda}{c}\mathcal{I} - \mathcal{D})W(x) = \frac{\lambda}{c} W \star q(x) + \frac{\lambda}{c} h(x), \tag{2.3.10}$$

where $\mathcal{D}$ and $\mathcal{I}$ are the differentiation and identity operators respectively, $h(x) = (1/\lambda)e^{-ax}$ and the convolution operator is defined by

$$W \star q(x) = \int_0^x W(x-y) q(y) dy.$$

Taking the Dickson-Hipp operator $\mathcal{T}_{\lambda/c}$, which is the inverse operator of $(\lambda/c)\mathcal{I} - \mathcal{D}$, on both sides of (2.3.10) gives

$$W(x) = \frac{\lambda}{c}\mathcal{T}_{\lambda/c}\{W \star q + h\}(x), \tag{2.3.11}$$

where

$$\mathcal{T}_{\lambda/c} f(x) = e^{(\lambda/c)x} \int_x^{\infty} e^{-(\lambda/c)y} f(y)dy.$$

It is easy to prove that $\mathcal{T}_{\lambda/c}\{W \star q\}(x)$ and $\mathcal{T}_{\lambda/c}\{e^{-ax}\}$ exist as both $W(x)$ and e^{-ax} are bounded functions. Hence, we have

$$\begin{aligned} W(x) &= \frac{\lambda}{c}\left[\mathcal{T}_0\{W \star q + h\}(x) - \frac{\lambda}{c}\mathcal{T}_0\mathcal{T}_{\lambda/c}\{W \star q + h\}(x)\right] \\ &= \frac{\lambda}{c}\left[\mathcal{T}_0 q \star W(x) + \mathcal{T}_0 h(x) + \mathcal{T}_0 W(x) - \frac{\lambda}{c}\mathcal{T}_0\mathcal{T}_{\lambda/c}\{W \star q + h\}(x)\right]. \end{aligned}$$

The above two equalities can be easily proved by taking Laplace transforms.

It follows from (2.3.11) that

$$\mathcal{T}_0 W(x) = \frac{\lambda}{c}\mathcal{T}_0\mathcal{T}_{\lambda/c}\{W \star q + h\}(x).$$

Therefore, we arrive at the following defective renewal equation

$$W(x) = \frac{\lambda}{c} W \star \overline{Q}(x) + \frac{1}{ac}e^{-ax}, \tag{2.3.12}$$

which admits the desired solution (2.3.9). □

Remark 2.3.1. *The solution (2.3.9) is in fact the convolution of a compound geometric distribution with an exponential distribution.*

$$W(x) = \frac{1}{a(c-\lambda\kappa)} \int_{0+}^{x} e^{-a(x-y)} dG(y) = \frac{1}{a(c-\lambda\kappa)}\mathcal{E}_a g(x),$$

where in the Riemann-Stieltjes integral the compound geometric distribution is given by

$$G(y) = 1 - \frac{\lambda\kappa}{c} + \int_0^y g(t)dt, \qquad y \geq 0.$$

For more on compound geometric convolutions, readers are referred to Willmot and Cai [49].

As with many other ruin-related quantities, closed-form solutions can be found for the accumulated utility up to ruin in many special cases of claim size distributions. The simplest among these examples would be the exponential claim size which leads to the following result.

Corollary 2.3.3. *If the claim size distribution $Q(y)$ is exponential with mean $1/\beta$, $W(x)$ admits an explicit solution given by*

$$W(x) = \frac{\lambda}{ac^2(a-\beta+\lambda/c)} e^{-(\beta-\lambda/c)x} + \frac{a-\beta}{ac(a-\beta+\lambda/c)} e^{-ax}, \qquad x \geq 0. \tag{2.3.13}$$

Proof. With a few steps of substitution and differentiation, (2.3.8) simplifies to

$$cW''(x) + (c\beta - \lambda)W'(x) + (\beta - a)e^{-ax} = 0. \tag{2.3.14}$$

Therefore, the solution to $W(x)$ can be represented as

$$\begin{aligned} W(x) &= C_1 + \int_0^x e^{-(\beta-\lambda/c)y} \left(C_2 - \int_0^y \frac{\beta-a}{c} e^{-at} e^{(\beta-\lambda/c)t} dt \right) dy \\ &= C_1 + C_2 \int_0^x e^{-(\beta-\lambda/c)y} dy - \int_0^x e^{-(\beta-\lambda/c)y} \int_0^y \frac{\beta-a}{c} e^{-at} e^{-(\beta-\lambda/c)t} dt dy \\ &= C_1 + C_2 \frac{1-e^{-(\beta-\lambda/c)x}}{\beta-\lambda/c} - \frac{\beta-a}{c(a-\beta+\lambda/c)} \int_0^x e^{-(\beta-\lambda/c)y} \left[1 - e^{-(a-\beta+\lambda/c)y}\right] dy \\ &= C_1 + \frac{C_2}{\beta-\lambda/c} - \frac{\beta-a}{ac(\beta-\lambda/c)} + \left[\frac{\beta-a}{c(a-\beta+\lambda/c)(\beta-\lambda/c)} - \frac{C_2}{\beta-\lambda/c} \right] e^{-(\beta-\lambda/c)x} \\ &\quad - \frac{\beta-a}{ac(a-\beta+\lambda/c)} e^{-ax} \end{aligned} \tag{2.3.15}$$

where C_1 and C_2 are coefficients to be determined.

Substituting (2.3.15) for $W(x)$ in (2.3.8), we find out that on the left hand side, all the constant terms, terms with $e^{-(\beta-\lambda/c)x}$ and terms with e^{-ax} cancel out. Then equating terms with $e^{-\beta x}$ with zero gives us the first constraint on the coefficients,

$$\begin{aligned}\frac{1}{\beta}C_1+\frac{1}{\beta(\beta-\lambda/c)}C_2-\frac{\beta-a}{ac\beta(\beta-\lambda/c)}+\frac{\beta-a}{\lambda(a-\beta+\lambda/c)(\beta-\lambda/c)}\\ -\frac{c}{\lambda(\beta-\lambda/c)}C_2-\frac{1}{ac(a-\beta+\lambda/c)}=0\end{aligned} \tag{2.3.16}$$

It follows from (2.3.9) that for an arbitrary $\epsilon>0$,

$$\begin{aligned}W(x) &= \frac{1}{a(c-\lambda\kappa)}\left\{e^{-ax}\int_0^{x-\epsilon}e^{ay}dG(y)+e^{-ax}\int_{x-\epsilon}^{x}e^{ay}dG(y)\right\}+\frac{1}{ac}e^{-ax}\\ &\le \frac{1}{a(c-\lambda\kappa)}\left\{e^{-ax}e^{a(x-\epsilon)}+G(x)-G(x-\epsilon)\right\}+\frac{1}{ac}e^{-ax}.\end{aligned}$$

Hence we obtain

$$\lim_{x\to\infty}W(x)\le\frac{1}{a(c-\lambda\kappa)}\left\{e^{-a\epsilon}+\lim_{x\to\infty}[G(x)-G(x-\epsilon)]\right\}+\lim_{x\to\infty}\frac{1}{ac}e^{-ax}=\frac{1}{a(c-\lambda\kappa)}e^{-a\epsilon}.$$

Since the limit equals zero for any $\epsilon>0$, we can conclude that $\lim_{x\to\infty}W(x)=0$.

Letting $x\to\infty$ in (2.3.15) we have the second constraint on the coefficients for the case in which $c>\lambda/\beta$,

$$C_1+\frac{C_2}{\beta-\lambda/c}-\frac{\beta-a}{ac(\beta-\lambda/c)}=0. \tag{2.3.17}$$

Combining (2.3.16) and (2.3.17) we get

$$C_1=\frac{a^2-2a\beta+a\lambda/c+\beta^2-\lambda^2/c^2}{ac(\beta-\lambda/c)(a-\beta+\lambda/c)}$$

and

$$C_2=\frac{a\beta c-a^2c-\lambda\beta+\lambda^2/c}{ac^2(a-\beta+\lambda/c)}.$$

$\square$

2.3.4 Total Claim Costs up to Ruin

One of the main focuses in actuarial mathematics is to quantify the future liability of an insurance company by computing the expected amount which the company must hold in reserve for upcoming insurance claims. Applying the same idea to risk models for a business line of an insurance company, one would be interested in knowing the total amount of discounted claims to be expected up to the time of possible default. Hence we shall define such a quantity in this section and derive its connection to the generalized Gerber-Shiu function.

In practice every single insurance claim is accompanied by a certain amount of business cost resulted from claim appraisal, investigation, settlement negotiation, etc. The final costs to the insurer may be quite different from the actual size of claims. Hence we assume as given a bounded function $\varpi(x, y)$ that measures the cost of each claim depending on the surplus prior to the time of claim x and the resulting new surplus y. As in the classical model, we assume the line of business defaults when the surplus goes below zero and the safety loading $c > \lambda\kappa$ is satisfied. Since all claims arrive at the sequence of jump points $\{T_1, T_2, T_3, \cdots\}$, then the expected present value of total claim costs up to the time of ruin can be written as

$$K(x) = \mathbb{E}^x \big[\sum_{i=1}^{N} e^{-\delta T_i} \varpi(X_{T_i -}, X_{T_i}) \big], \tag{2.3.18}$$

where $N = \max\{n : T_n \leq \tau_0\}$ with the convention that $\max\{\mathbb{N}\} = \infty$ and $\delta > 0$.

Interestingly, we can express the total costs up to ruin as a special case of the generalized Gerber-Shiu function as follows. In terms of the associated counting process, (2.3.18) can also be written as

$$K(x) = \mathbb{E}^x \big[\int_0^{\tau_0} \int_{-\infty}^{\infty} e^{-\delta t} \varpi(X_{t-}, y) N(dt, dy) \big].$$

Note that $\{X_{t-}, 0 \leq t < \infty\}$ is the left-continuous modification of $\{X_t, 0 \leq t < \infty\}$ and hence is a predictable process. By Theorem 2.1.3, we must have

$$\begin{aligned}K(x) &= \mathbb{E}^x\Big[\int_0^{\tau_0}\int_{-\infty}^{\infty} e^{-\delta t}\varpi(X_{t-},y)\tilde{N}(dt,dy)\Big]\\ &= \mathbb{E}^x\Big[\int_0^{\tau_0} e^{-\delta t}\lambda\int_{-\infty}^{\infty}\varpi(X_{t-},X_{t-}-y)dQ(y)dt\Big]\\ &= \mathbb{E}^x\Big[\int_0^{\tau_0} e^{-\delta t}\lambda\int_{-\infty}^{\infty}\varpi(X_{t},X_{t}-y)dQ(y)dt\Big]. \qquad (2.3.19)\end{aligned}$$

The last equality can be explained as follows. Since the càdlàg process can only have countable discontinuities, then for each $\omega\in\Omega$, $\{t : X_{t-}(\omega)\neq X_t(\omega)\}$ is a countable set, and hence $\int_0^\tau\int_{-\infty}^\infty \varpi(X_t(\omega),X_t(\omega)-y)dQ(y)dt = \int_0^\tau\int_{-\infty}^\infty \varpi(X_{t-}(\omega),X_{t-}(\omega)-y)dQ(y)dt$. Therefore, in view of (2.2.1) and (2.3.19), the total claim costs can be recovered from the generalized Gerber-Shiu function with a special cost function

$$l(x) = \lambda\int_0^\infty \varpi(x,x-y)dQ(y). \qquad (2.3.20)$$

Corollary 2.3.4. *The solution to $K(x)$ defined in (2.3.18) is given by*

$$K(x) = \frac{c}{c-\lambda\pi}\int_0^x \hat{\zeta}(x-y)g(y)dy + \hat{\zeta}(x), \qquad x\geq 0, \qquad (2.3.21)$$

where

$$\pi = \int_0^\infty e^{\gamma x}\int_x^\infty e^{-\gamma y}dQ(y)dx,$$
$$\hat{\zeta}(z) = \frac{\lambda}{c}e^{\gamma z}\int_z^\infty e^{-\gamma x}\int_0^\infty \varpi(x,x-y)dQ(y)dx,$$

the associated compound geometric density function is given by

$$g(y) = \sum_{n=1}^\infty \big(1-\frac{\lambda\pi}{c}\big)\big(\frac{\lambda\pi}{c}\big)^n \hat{q}^{\star n}(y),$$

where the generalized equilibrium density function

$$\hat{q}(y) = \frac{1}{\pi}e^{\gamma y}\int_y^\infty e^{-\gamma x}dQ(x).$$

The constant γ is the unique positive root to the Lundberg fundamental equation

$$\frac{\lambda}{c}\tilde{q}(s) = \frac{\lambda+\delta}{c} - s. \tag{2.3.22}$$

Proof. Since $K(x)$ is evidently bounded, we obtain the following integro-differential equation by inserting (2.3.20) into (2.2.2),

$$cK'(x) - (\lambda+\delta)K(x) + \lambda\int_0^x K(x-y)dQ(y) + \lambda\int_0^\infty \varpi(x, x-y)dQ(y) = 0. \tag{2.3.23}$$

We can rewrite the equation in terms of operators,

$$(\frac{\lambda+\delta}{c}\mathcal{I} - \mathcal{D})K(x) = \frac{\lambda}{c}K \star q(x) + \frac{\lambda}{c}\zeta(x),$$

where

$$\zeta(x) = \int_0^\infty \varpi(x, x-y)dQ(y).$$

Since $K(x), q(x)$ and $\zeta(x)$ are all bounded functions, their corresponding Dickson-Hipp transforms exist. Using the arguments similar to those of Corollary 2.3.2, we have

$$\begin{aligned} K(x) &= \frac{\lambda}{c}\mathcal{T}_{(\lambda+\delta)/c}\{K \star q + \zeta\}(x) \qquad (2.3.24)\\ &= \frac{\lambda}{c}\mathcal{T}_\gamma\{K \star q + \zeta\}(x) - (\frac{\lambda+\delta}{c} - \gamma)\mathcal{T}_\gamma\mathcal{T}_{(\lambda+\delta)/c}\{K \star q + \zeta\}(x)\\ &= \frac{\lambda}{c}\{K \star \mathcal{T}_\gamma q(x) + \mathcal{T}_\gamma\zeta(x) + \tilde{q}(\gamma)\mathcal{T}_\gamma K(x)\} - (\frac{\lambda+\delta}{c} - \gamma)\mathcal{T}_\gamma\mathcal{T}_{(\lambda+\delta)/c}\{K \star q + \zeta\}(x), \end{aligned}$$

where the constant γ is the solution to (2.3.22) and the safety loading condition $c > \lambda\kappa$ ensures that it is a unique positive root.

In view of (2.3.22) and (2.3.24), we have

$$\frac{\lambda}{c}\tilde{q}(\gamma)K(x) - (\frac{\lambda+\delta}{c} - \gamma)\mathcal{T}_{(\lambda+\delta)/c}\{K \star q + \zeta\}(x).$$

Hence, it follows that

$$K(x) = \frac{\lambda}{c}\{K \star \mathcal{T}_\gamma q(x) + \mathcal{T}_\gamma \zeta(x)\},$$

i.e.

$$K(x) = \frac{\lambda\pi}{c}\int_0^x K(x-y)d\hat{Q}(y) + \hat{\zeta}(x), \tag{2.3.25}$$

which gives the desired solution. □

Remark 2.3.2. *The solution (2.3.21) is in fact another example of compound geometric convolution,*

$$K(x) = \frac{1}{c-\lambda\pi}\int_0^x \mathcal{T}_\gamma l(x-y)dG(y),$$

where $l(x)$ is given in (2.3.20) and the compound geometric distribution is given by

$$G(y) = 1 - \frac{\lambda\pi}{c} + \int_0^y g(t)dt. \tag{2.3.26}$$

A good example of the total claim costs up to ruin is the discounted aggregate claim with a policy limit of M, defined by

$$K_M(x) = \mathbb{E}^x\big[\sum_{i=1}^{N} e^{-\delta T_i}\left[(X_{T_i-} - X_{T_i}) \wedge M\right]\big].$$

Assume that claim sizes are exponentially distributed with mean $1/\beta$. Observe from (2.3.20) that $\varpi(x, x-y) = y \wedge M$ and

$$l(x) = \lambda\int_0^\infty (y\wedge M)dQ(y) = \lambda\int_0^M ydQ(y) + \lambda M\overline{Q}(M) = \frac{\lambda}{\beta}(1-e^{-\beta M}).$$

If we set the premium income $\overline{c} = \lambda E(Y_i \wedge M)$, then the expected present value of total premium income collected up to the time of ruin is given by

$$P_M(x) = \mathbb{E}^x\left[\int_0^{\tau_0} e^{-\delta t} l\big(X(t)\big)dt\right],$$

where

$$l(x) = \overline{c} = \frac{\lambda}{\beta}(1 - e^{-\beta M}).$$

Hence it is not surprising that the total claim costs $K_M(x)$ is essentially equivalent to the total premium income $P_M(x)$ in the sense that the insurer's asset matches its liability.

Corollary 2.3.5. *If $Q(y)$ is exponentially distributed with mean $1/\beta$, $K_M(x)$ admits an explicit solution given by*

$$K_M(x) = \frac{\lambda(1 - e^{-\beta M})}{\delta\beta}(1 - \frac{\rho + \beta}{\beta}e^{\rho x}), \qquad x \geq 0,$$

where ρ is the unique negative root to the Lundberg fundamental equation

$$cs^2 + (\beta c - \lambda - \delta)s - \delta\beta = 0. \tag{2.3.27}$$

Proof. In this case, we have in (2.3.21) that

$$\hat{\zeta}(x) = \frac{\lambda}{\beta c\gamma}(1 - e^{-\beta M}).$$

Hence $K_M(x)$ is apparently a non-decreasing function of x. Since $K_M(x)$ is a bounded non-decreasing function, there must exist a finite number K such that $\lim_{x\to\infty} K_M(x) = K$. Then taking limits on both sides of (2.3.25) gives

$$\begin{aligned} K &= \frac{\lambda}{c}\int_0^\infty \mathcal{T}_\gamma q(x)dx \cdot K + \frac{\lambda}{\beta c\gamma}(1 - e^{-\beta M}). \\ &= \frac{\lambda[1 - \tilde{q}(\gamma)]}{c\gamma}K + \frac{\lambda}{\beta c\gamma}(1 - e^{-\beta M}) \\ &= (1 - \frac{\delta}{c\gamma})K + \frac{\lambda}{\beta c\gamma}(1 - e^{-\beta M}) \end{aligned}$$

with the last equality from (2.3.22). Hence we obtain the boundary condition that

$$\lim_{x\to\infty} K_M(x) = \frac{\lambda}{\delta\beta}(1 - e^{-\beta M}).$$

After standard algebraic simplification, (2.3.23) turns into a second order differential equation

$$cK''_M(x) - (c\beta - \lambda - \delta)K'_M(x) - \delta\beta K_M(x) + \lambda(1 - e^{-\beta M}) = 0. \tag{2.3.28}$$

We first recall that the fundamental solutions to the corresponding homogeneous equation

$$cK''_M(x) - (c\beta - \lambda - \delta)K'_M(x) - \delta\beta K_M(x) = 0$$

can be represented as

$$C_1 e^{\rho x} + C_2 e^{\gamma x},$$

where C_1 and C_2 are to be determined, $-\beta < \rho < 0$ and $\gamma > 0$ are the two real roots of the characteristic equation (2.3.27), which corresponds the Lundberg fundamental equation (2.3.22) in the case of exponential claim size distribution. We also have a particular solution to (2.3.28) that $K_M(x) = \lambda(1 - e^{\beta M})/(\delta\beta)$. Therefore, the general solutions to $K_M(x)$ are given by

$$K_M(x) = C_1 e^{\rho x} + C_2 e^{\gamma x} + \frac{\lambda(1 - e^{\beta M})}{\delta\beta}. \tag{2.3.29}$$

Since $K_M(x)$ is bounded, we must have $C_2 = 0$. Substituting (2.3.29) for $K_M(x)$ in (2.3.28) yields

$$C_1 = -\frac{\lambda(1 - e^{-\beta M})}{\delta\beta}\frac{\rho + \beta}{\beta}.$$

Therefore, the desired result is obtained. $\square$

2.3.5 Gerber-Shiu Functions

As the name suggests, the famous Gerber-Shiu function can be deduced from its generalized version (2.2.1) with a special cost function. In doing so, we now amend the definition of $K(x)$ in previous section to construct a new quantity

$$m(x) = \mathbb{E}^x\big[\sum_{i=1}^{N} e^{-\delta T_i}\varpi(X_{T_i-}, X_{T_i})\big], \tag{2.3.30}$$

where $\delta \geq 0, N = \max\{n : T_n \leq \tau_0\}$ with the convention that $\max\{\mathbb{N}\} = \infty$ and

$$\varpi(x,y) = \begin{cases} 0, & \text{for } y \geq 0, \\ w(x,-y), & \text{for } y < 0, \end{cases}$$

with a bounded function $w(x,y)$.

We can adopt arguments almost identical to those in the previous section to convert $m(x)$ into a generalized Gerber-Shiu function. Hence, we also have

$$m(x) = \mathbb{E}^x\left[\int_0^{\tau_0} e^{-\delta t} l(X_t)dt\right],$$

where

$$l(x) = \lambda \int_x^{\infty} w(x, y-x)dQ(y). \tag{2.3.31}$$

Note that with this special choice of $\varpi(x,y)$, the function $m(x)$ defined in (2.3.30) can be represented as

$$m(x) = \mathbb{E}^x\left[e^{-\delta\tau_0}w(X_{\tau_0-}, |X_{\tau_0}|)I(\tau_0 < \infty)\right],$$

where $\delta \geq 0$. Hence we obtain the classical definition of *Gerber-Shiu function*. It should be noted that the indicator is an indispensable part of the representation. By the definition of $\varpi(x,y)$, in the event that $\tau_0 = \infty$, the value of the process $X_t \geq 0$ for all $t \geq 0$, then $\varpi(x,y) = 0$ and hence $m(x) = 0$.

Since $w(x,y)$ is bounded, there must exist B such that $w(x,y) \leq B$ for any $x, y \in \mathbb{R}$. We now see that

$$m(x) \leq \lambda B\mathbb{E}^x\left[\int_0^{\tau_0} e^{-\delta t}\overline{Q}(y)dy\right] \leq \lambda B\mathbb{E}^x\left[\int_0^{\infty} \overline{Q}(y)dy\right] = \lambda\kappa B.$$

By Theorem 2.2.1, $m(x)$ satisfies the corresponding integro-differential equation

$$\mathfrak{X}m(x) - (\lambda+\delta)m(x) + \lambda\int_0^x m(x-y)dQ(y) + \lambda\int_x^\infty w(x, y-x)dQ(y) = 0, \qquad x > 0. \tag{2.3.32}$$

The Gerber-Shiu functions have been extensively studied in a variety of risk models, many of which are essentially PDCPs or more generally PDMPs. With different choices of the extended generator of deterministic sample paths, we can obtain integro-differential equations to the Gerber-Shiu functions for a vast amount of PDCPs.

For instance, in view of (2.3.1) and (2.3.32), the Gerber-Shiu function in the classical compound Poisson model has to be the solution to the following equation,

$$cm'(x) - (\lambda+\delta)m(x) + \lambda\int_0^x m(x-y)dQ(y) + \lambda\int_x^\infty w(x, y-x)dQ(y) = 0, \qquad x > 0, \tag{2.3.33}$$

which is precisely equation (2.16) in Gerber and Shiu [22].

2.3.6 Insurer's Life Annuity

We now look at a life annuity of an insurance company, which is an annuity with continuous payments of one dollar per time unit payable up to the company's bankruptcy. It can be utilized to quantify the insurance company's continuous contribution to its employees pension funds until its bankruptcy if it occurs.

If the insurer's surplus is driven by a PDCP process $X = \{X_t, t > 0\}$ with the safety loading condition $c > \lambda\kappa$ satisfied and the annuity contributions are invested at a constant rate of return $\delta > 0$, the expected present value of such a life annuity from the perspective of annuity-holder can be determined by

$$\overline{a}(x) = \mathbb{E}^x\left[\int_0^{\tau_0} e^{-\delta t}dt\right] = \mathbb{E}^x\left[\frac{1-e^{-\delta\tau_0}}{\delta}\right]. \tag{2.3.34}$$

It is evident that such an annuity is bounded and a special of the generalized Gerber-Shiu function where $l(x) = 1$. Hence it satisfies the following integro-differential equation

$$c\overline{a}'(x) - (\lambda+\delta)\overline{a}(x) + \lambda\int_0^x \overline{a}(x-y)dQ(y) + 1 = 0, \qquad x \geq 0. \tag{2.3.35}$$

Corollary 2.3.6. *The solution to $\overline{a}(x)$ defined in (2.3.34) is given by*

$$\overline{a}(x) = \frac{1}{\gamma(c-\lambda\pi)}G(x), \qquad x \geq 0,$$

where the compound geometric distribution $G(x)$ is given in (2.3.26) and γ is the unique positive root to the Lundberg fundamental equation (2.3.22).

Proof. In terms of operators, (2.3.35) can be written as

$$(\frac{\lambda+\delta}{c}\mathcal{I} - \mathcal{D})\overline{a}(x) = \frac{\lambda}{c}\overline{a}\star q(x) + \frac{1}{c}.$$

Since both $\overline{a}(x)$ and $q(x)$ are bounded, their Dickson-Hipp transforms exist. Hence

$$\overline{a}(x) = \frac{\lambda}{c}\mathcal{T}_{(\lambda+\delta)/c}\{\overline{a}\star q\}(x) + \mathcal{T}_{(\lambda+\delta)/c}\{\frac{1}{c}\}(x).$$

Using the arguments similar to those in Corollary 2.3.4, we obtain

$$\overline{a}(x) = \frac{\lambda\pi}{c}\int_0^x \overline{a}(x-y)d\hat{Q}(y) + \frac{1}{\gamma c},$$

which yields the desired solution. □

We shall now focus on the special case where claim sizes are exponentially distributed to develop a life contingency type of formula.

Corollary 2.3.7. *If $Q(y)$ is exponentially distributed with mean $1/\beta$, $\overline{a}(x)$ admits an explicit solution given by*

$$\overline{a}(x) = \frac{1}{\delta} - \frac{1}{\delta}\frac{\rho+\beta}{\beta}e^{\rho x}, \qquad x \geq 0, \tag{2.3.36}$$

where ρ is the unique negative solution to the Lundberg fundamental equation (2.3.27).

Proof. Equation (2.3.35) reduces to

$$c\overline{a}''(x) + (c\beta - \lambda - \delta)\overline{a}'(x) - \delta\beta\overline{a}(x) + \beta = 0, \qquad x \geq 0.$$

Apparently, $\overline{a}(x) = 1/\delta$ is a particular solution to the differential equation. In view of the fact that $\overline{a}(x)$ is bounded, the solution is in the form of

$$\overline{a}(x) = \frac{1}{\delta} + ae^{\rho x}, \tag{2.3.37}$$

where a is the coefficient to be determined and ρ is the unique negative solution to the Lundberg equation (2.3.27). Inserting (2.3.37) into (2.3.35) yields that $a = -(\rho + \beta)/(\delta\beta)$.

□

We define a contingent claim of one dollar payable at the time of the insurance company's bankruptcy or the default of a certain business line,

$$\overline{A}(x) = \mathbb{E}^x\left[e^{-\delta\tau_0}\right],$$

which is indeed a special case of Gerber-Shiu function and could have been obtained from (2.3.33). However, to avoid repetitive derivations, we find the solution by using the result (2.5.5) for a more general model in the later section. Hence,

$$\overline{A}(x) = \frac{\rho + \beta}{\beta}e^{\rho x}, \qquad x \geq 0. \tag{2.3.38}$$

Comparing (2.3.36) and (2.3.38), we now arrive at a formula that is analogous with the famous life contingencies formula

$$1 = \delta\overline{a}(x) + \overline{A}(x), \qquad \text{for all } x \geq 0.$$

It has the interpretation that an initial loan of one dollar at present should be equal to the expected present value of a series of continuous payment of interest due δ dollar per time unit up until the insurer's bankruptcy and a final payment of one dollar to clear off the balance at the time of bankruptcy.

2.4 Compound Poisson Model with Constant Interest and Liquid Reserve

The idea of incorporating surplus investment with constant interest rate in a risk process was introduced in Sundt and Teugels [45], Embrechts and Schmidli [18], etc. It assumes that an insurer collects premiums at a constant rate c, and provides compensations to claims that arrive according to the compound Poisson process $Z(t)$. The insurer's surplus at any time is completely invested in a risk-free asset which earns interest at a constant rate r. In contrast with classical model where the growth of surplus appears to be linear, the surplus process now accumulates with compound interest in a fashion that can be easily characterized by a PDCP. In the absence of random claims, the deterministic path of the PDCP process is given by

$$\frac{d}{dt}X_t = \frac{d}{dt}(xe^{rt} + c\bar{s}_{\overline{t}|}) = rX_t + c, \qquad x \geq 0,$$

which means the generator

$$\mathfrak{X} = (rx + c)\frac{d}{dx}, \qquad x \geq 0. \tag{2.4.1}$$

Another threshold strategy that comes often with surplus investment is the so-called liquid reserve strategy, which requires a prudent insurer to keep the limited working capital liquid to deal with insurance claims when the surplus reserve is running relatively low. Hence we assume that the insurer sets a benchmark, *liquid reserve limit* Δ, below which the surplus as in classical model increases at the constant premium rate c and above which the excess of surplus would be invested in money market with the force of interest r. Accordingly we find that

$$\frac{d}{dt}X_t = \begin{cases} d/dt(\Delta + (x-\Delta)e^{rt} + c\bar{s}_{\overline{t}|}) = r(X_t - \Delta) + c, & x \geq \Delta \\ d/dt(x+ct) = c, & 0 \leq x < \Delta. \end{cases}$$

Hence the generator for the deterministic piece in the model with both constant interest and liquid reserve is given by

$$\mathfrak{X} = \begin{cases} [r(x-\Delta)+c]d/dx, & x \geq \Delta \\ cd/dx, & 0 \leq x < \Delta. \end{cases} \tag{2.4.2}$$

As a further generalization, we amend the above model with the inclusion of a dividend threshold and investment cap b. When $x \geq b$, the excess of surplus stops being invested in money market, instead a portion of the surplus will be paid out as dividends at a constant rate α. A sample path of such a process is given in Figure 2.2. Hence we must have

$$\frac{d}{dt}X_t = \frac{d}{dt}[b + r(b-\Delta)t + (c-\alpha)t] = r(b-\Delta) + c - \alpha, \qquad x \geq b.$$

Hence the generator for the deterministic part is given by

$$\mathfrak{X} = \begin{cases} [r(b-\Delta)+c-\alpha]d/dx, & x \geq b, \\ [r(x-\Delta)+c]\,d/dx, & \Delta \leq x < b, \\ cd/dx, & 0 \leq x < \Delta. \end{cases} \tag{2.4.3}$$

Traditional probabilistic derivations are given in Cai *et al.* [7].

In the compound Poisson model with constant interest (2.4.1), the corresponding Gerber-Shiu function $m(x)$ must satisfy

$$(rx+c)m'(x) - (\lambda+\delta)m(x) + \lambda\int_0^x m(x-y)dQ(y) + \lambda\int_x^\infty w(x,y-x)dQ(y) = 0, \qquad x > 0.$$

Taking $\delta = 0$ and $w(x,y) = 1$ would lead to the equation (1) in Sundt and Teugels [45], which is satisfied by the probability of ultimate ruin .

Following the same line of logic, by the substitution of the generator in the compound Poisson model with both constant interest and liquid reserve (2.4.2), we obtain the system of equations for the Gerber-Shiu function denoted by $m(x;\Delta)$ in Cai *et al.* [8],

$$[r(x-\Delta)+c]m'(x;\Delta)-(\lambda+\delta)m(x;\Delta)+\lambda\int_0^x m(x-y;\Delta)dQ(y)+\lambda\int_x^\infty w(x,y-x)dQ(y)=0,\quad x\geq\Delta,$$

$$cm'(x;\Delta)-(\lambda+\delta)m(x;\Delta)+\lambda\int_0^x m(x-y;\Delta)dQ(y)+\lambda\int_x^\infty w(x,y-x)dQ(y)=0,\quad 0<x<\Delta.$$

Interested readers are referred to Cai *et al.* [7] for the derivation of these integro-differential equations through traditional probabilistic arguments and detailed solutions to the Gerber-Shiu function.

In the compound Poisson model with constant interest, dividend and liquid reserve strategies (2.4.3), the parameter vector $\mathbf{b}=(\Delta,b)$ is employed to emphasize the dependency of ruin-related quantities on these parameters.

We can easily obtain the system of equations of the Gerber-Shiu function denoted by $m(x;\mathbf{b})$ by substitution of its corresponding generator.

$$[r(b-\Delta)+c-\alpha]m'(x;\mathbf{b})-(\lambda+\delta)m(x;\mathbf{b})+\lambda\int_0^x m(x-y;\mathbf{b})dQ(y)+\lambda\int_x^\infty w(x,y-x)dQ(y)=0,\quad x>b,$$

$$[r(x-\Delta)+c]\,m'(x;\mathbf{b})-(\lambda+\delta)m(x;\mathbf{b})+\lambda\int_0^x m(x-y;\mathbf{b})dQ(y)+\lambda\int_x^\infty w(x,y-x)dQ(y)=0,\quad \Delta\leq x<b,$$

$$cm'(x;\mathbf{b})-(\lambda+\delta)m(x;\mathbf{b})+\lambda\int_0^x m(x-y;\mathbf{b})dQ(y)+\lambda\int_x^\infty w(x,y-x)dQ(y)=0,\quad 0\leq x<\Delta.$$

Readers are referred to Cai *et al.* [7] for traditional probabilistic derivations and detailed solutions.

As another typical example of the generalized Gerber-Shiu function, we can also find the expected present value of dividends paid up to the time of ruin defined by

$$V(x;\mathbf{b}) = \mathbb{E}^x\left[\int_0^{\tau_0} e^{-\delta t} l\big(X(t)\big)dt\right],$$

where $\delta > 0$ and the cost function is given by

$$l(x) = \begin{cases} \alpha, & \text{if } x \geq b, \\ 0, & \text{if } 0 \leq x < b. \end{cases}$$

Therefore, by Theorem 2.2.1, we have the following system of integro-differential equations,

$$\begin{aligned} cV'(x;\mathbf{b}) - (\lambda+\delta)V(x;\mathbf{b}) + \lambda\int_0^x V(x-y;\mathbf{b})dQ(y) = 0, \qquad & 0 \leq x < \Delta, \\ [r(x-\Delta)+c]\,V'(x;\mathbf{b}) - (\lambda+\delta)V(x;\mathbf{b}) + \lambda\int_0^x V(x-y;\mathbf{b})dQ(y) = 0, \qquad & \Delta \leq x < b, \\ [r(\Delta-b)+(c-\alpha)]\,V'(x;\mathbf{b}) - (\lambda+\delta)V(x;\mathbf{b}) + \lambda\int_0^x V(x-y;\mathbf{b})dQ(y) + \alpha = 0, \qquad & x > b. \end{aligned}$$

These equations are obtained in Cai *et al.* [7] through lengthy traditional probabilistic arguments. Interested readers can find solutions to $V(x;\mathbf{b})$ in that paper.

Similarly, one can work out integro-differential equations satisfied by the generalized Gerber-Shiu function for all kinds of risk models with combinations of dividend barrier and investment strategies, such as the model with dividend barrier and constant interest in Yuen *et al.* [52].

2.5 Compound Poisson Model with Two-sided Jumps

Random jumps in surplus process are often assumed to be resulted from insurance claims. Hence it is considered to have only downward jumps in risk models by the nature of claims. However, for more general applications, one might need to incorporate upward jumps in surplus as well. For instance, Kennedy [32] considers the probability of ruin in a system of program trading. The net outcome of trades is modelled by a compound Poisson process with both positive and negative jumps, which represent increase or decrease of the total capital as a result of trading in various financial markets.

Since the compound Poisson process with two-sided jumps is another example of PDCP, it is natural to find the applications of generalized Gerber-Shiu functions in this type of risk models. Even though we refer to Kennedy's model merely for the purpose of giving a motivation for double-sided jumps, the generalized Gerber-Shiu can be used to reproduce the results given in Kennedy [32] obtained through probabilistic arguments.

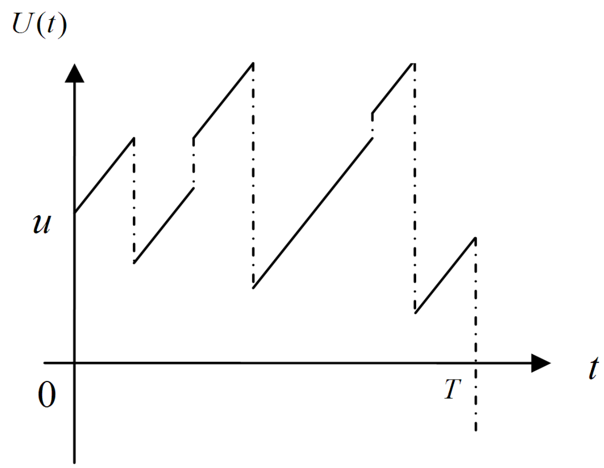

Figure 2.4: Sample path of compound Poisson model with two sided jumps

Assume that random events happen to an insurer in a Poisson process fashion. Each event turns out to be either a random insurance claim with common distribution $Q^{-}(y)$ or a random investment income (cash injection) with common distribution $Q^{+}(y)$. The probability of the event being an insurance claim is assumed to be π and thus the event happens to be an investment income with the chance $1-\pi$. Therefore, the jump size distribution is given by

$$Q(y) = \pi Q^{+}(y)I(y \geq 0) + (1-\pi)[1 - Q^{-}(-y)I(y < 0)].$$

When both $Q^{+}(y)$ and $Q^{-}(y)$ are differentiable with density function $q^{+}(y)$ and $q^{-}(y)$ respectively, the density function of the claim size distribution can be written as

$$p(y) = \pi q^+(y) I(y \geq 0) + (1-\pi) q^-(-y) I(y < 0).$$

If we are only interested in ruin-related quantities at or up to the time of ruin, then we set the level of default $d = 0$ and the operator $\mathcal{Q}$ in (2.2.8) can be written as

$$\begin{aligned} \mathcal{Q}H(x) &= \int_0^\infty H(y) dQ(x-y) = \int_{-\infty}^x H(x-y) dQ(y) \\ &= \pi \int_0^x H(x-y) dQ^+(y) + \int_0^\infty H(x+y) dQ^-(y). \end{aligned} \tag{2.5.1}$$

If the generalized Gerber-Shiu function $H(x)$ defined in (2.2.1) is bounded, then the integro-differential equation for $H(x)$ can be obtained by inserting (2.5.1) in (2.2.2),

$$cH'(x) - (\lambda+\delta)H(x) + \lambda\pi \int_0^\infty H(x+y) dQ^+(y) + \lambda(1-\pi) \int_0^x H(x-y) dQ^-(y) + l(x) = 0. \tag{2.5.2}$$

This integro-differential equation is generally difficult to solve when $Q(y)$ is an arbitrary distribution function. Instead we will look at explicit solutions for the double exponential jump case, where the jump size is given by a mixture of two exponential distributions governing insurance claims and investment returns respectively,

$$Q(y) = \pi(1 - e^{-\beta_1 y}) I(y \geq 0) + [(1-\pi) - (1-\pi)(1 - e^{\beta_2 y}) I(y < 0)]. \tag{2.5.3}$$

Thus the integro-differential equation (2.5.2) becomes

$$cV'(x) - (\lambda+\delta)V(x) + \lambda\pi\beta_1 \mathcal{T}_{\beta_1} V(x) + \lambda(1-\pi)\beta_2 \mathcal{E}_{\beta_2} V(x) + l(x) = 0. \tag{2.5.4}$$

Readers may find it interesting to read the justification given in Kennedy [32] for considering the double-sided exponential distribution for the outcome of trades.

2.5.1 Discounted Payoff at Exercise

We are now interested in a special version of the Gerber-Shiu function in the compound Poisson model with double sided jumps denoted by $\psi_\delta(x)$,

$$\psi_\delta(x) = \mathbb{E}^x[e^{-\delta\tau_0} f(|X_{\tau_0}|) I(\tau_0 < \infty)],$$

where $\delta \geq 0$ and the payoff function $f(x)$ is bounded. Since $\psi_\delta(x)$ is bounded, we can utilize Theorem 2.2.1 to find its solutions. Similarly, one can easily replace τ_0 by a general stopping time τ_d in a model where d is treated as a level of optimal exercise and the generalized Gerber-Shiu function can be used to price a contingent claim with payoff function $f(x)$.

Corollary 2.5.1. *If $Q(y)$ follows the distribution given in (2.5.3), ψ_δ admits an explicit solution given by*

$$\psi_\delta(x) = \left[(\beta_2 + \rho) \int_0^\infty f(z) e^{-\beta_2 z} dz\right] e^{\rho x}, \qquad x \geq 0, \tag{2.5.5}$$

where ρ is the unique negative root of the Lundberg fundamental equation

$$cs + \lambda \left[\pi \frac{\beta_1}{\beta_1 - s} + (1-\pi)\frac{\beta_2}{\beta_2 + s} - 1\right] = \delta. \tag{2.5.6}$$

Proof. Taking derivatives with respect to x and making a substitution in (2.5.4) gives

$$cV''(x) - (\lambda+\delta)V'(x) - c\beta_1 V'(x) + \beta_1(\lambda+\delta)V(x) - \beta_1\lambda(1-\pi)\beta_2 e^{-\beta_2 x}\int_0^x V(y)e^{\beta_2 y}dy$$
$$-\beta_1 l(x) - \lambda\pi\beta_1 V(x) + c\beta_2 V'(x) - \beta_2(\lambda+\delta)V(x) + \beta_2\lambda\pi\beta_1 e^{\beta_1 x}\int_x^\infty V(y)e^{-\beta_1 y}dy$$
$$+\beta_2 l(x) + \lambda(1-\pi)\beta_2 V(x) + l'(x) = 0.$$

Taking derivatives with respect to x again and substituting the integral terms yields

$$cV'''(x) + (c\beta_2 - c\beta_1 - \lambda - \delta)V''(x) + \left[\lambda(1-\pi)\beta_2 - \lambda\pi\beta_1 - \beta_2(\lambda+\delta) + \beta_1(\lambda+\delta) - c\beta_1\beta_2\right]V'(x)$$
$$+\delta\beta_1\beta_2 V(x) + l''(x) + (\beta_2 - \beta_1)l'(x) - \beta_1\beta_2 l(x) = 0.$$

Note that $w(x, y-x) = f(y-x)$, then

$$
\begin{aligned}
l(x) &= \lambda(1-\pi)\beta_2 \int_x^\infty f(y-x)e^{-\beta_2 y}dy \\
&= \lambda(1-\pi)\beta_2 e^{-\beta_2 x} \int_0^\infty f(z)e^{-\beta_2 z}dz.
\end{aligned} \tag{2.5.7}
$$

It is easy to verify that in this case $l''(x) + (\beta_2 - \beta_1)l'(x) - \beta_1\beta_2 l(x) = 0$.

Thus (2.5.2) reduces to

$$
\begin{aligned}
c\psi_\delta'(x) - (\lambda+\delta)\psi_\delta(x) + \lambda\pi \int_0^\infty \psi_\delta(x+y)dQ^+(y) \\
+\lambda(1-\pi)\int_0^x \psi_\delta(x-y)dQ^-(y) + \lambda(1-\pi)\beta_2 e^{-\beta_2 x}\int_0^\infty f(z)e^{-\beta_2 z}dz = 0.
\end{aligned} \tag{2.5.8}
$$

Apparently from previous analysis, it satisfies a homogenous integro-differential equation

$$
\begin{aligned}
& c\psi_\delta'''(x) + (c\beta_2 - c\beta_1 - \lambda - \delta)\psi_\delta''(x) \\
& + [\lambda(1-\pi)\beta_2 - \lambda\pi\beta_1 - \beta_2(\lambda+\delta) + \beta_1(\lambda+\delta) - c\beta_1\beta_2]\,\psi_\delta'(x) + \delta\beta_1\beta_2\psi_\delta(x) = 0.
\end{aligned}
$$

We know that the fundamental solution to $\psi_\delta(x)$ can be written as

$$
C_1 e^{\rho x} + C_2 e^{\gamma_1 x} + C_3 e^{\gamma_2 x}
$$

where $\rho \le 0, \gamma_1 \ge 0$ and $\gamma_2 > \gamma_1$ are the three real roots of the characteristic function

$$
\begin{aligned}
& cs^3 + (c\beta_2 - c\beta_1 - \lambda - \delta)s^2 \\
& + [\lambda(1-\pi)\beta_2 - \lambda\pi\beta_1 - \beta_2(\lambda+\delta) + \beta_1(\lambda+\delta) - c\beta_1\beta_2]\,s + \delta\beta_1\beta_2 = 0,
\end{aligned}
$$

which is essentially the Lundberg fundamental equation (2.5.6). We denote the left-hand side of (2.5.6) by $k(s)$. Note that $\delta > 0$. It is obvious from (2.5.6) that $k(0) = 0, k(-\beta_2-) = +\infty$, hence there must be one solution $\rho \in (-\beta_2, 0)$ for $k(s) = \delta$. We also have a solution $\gamma_1 \in (0, \beta_1)$ as $k(\beta_1-) = +\infty$, and a solution $\gamma_2 \in (\beta_1, +\infty)$ as $k(\beta_1+) = -\infty$ and $k(+\infty) = +\infty$.

Since $\lim_{u\to+\infty} \psi_\delta(u) = 0$, we must have $C_2 = C_3 = 0$, i.e.

$$\psi_\delta(x) = C_1 e^{\rho x}.$$

Substituting it into (2.5.8) gives

$$\begin{aligned} c\rho C_1 e^{\rho x} - (\lambda+\delta) C_1 e^{\rho x} + \lambda\pi\beta_1 e^{\beta_1 x} \int_x^\infty C_1 e^{-(\beta_1-\rho)y} dy \\ + \lambda\pi\beta_2 e^{-\beta_2 x} \int_0^x C_1 e^{(\beta_2+\rho)y} dy + \lambda(1-\pi) e^{-\beta_2 x} = 0. \end{aligned}$$

Rearranging terms yields,

$$\begin{aligned} \left[c\rho - (\lambda+\delta) + \frac{\lambda\pi\beta_1}{\beta_1-\rho} + \frac{\lambda(1-\pi)\beta_2}{\beta_2+\rho}\right] C_1 e^{\rho x} - \frac{\lambda(1-\pi)\beta_2 C_1}{\beta_2+\rho} e^{-\beta_2 x} \\ + \left[\lambda(1-\pi)\beta_2 \int_0^\infty f(z) e^{-\beta_2 z} dz\right] e^{-\beta_2 x} = 0. \end{aligned}$$

Note that the algebraic expression in the bracket of the first term is the lundberg equation and hence the first term vanishes. Therefore,

$$C_1 = (\beta_2+\rho) \int_0^\infty f(z) e^{-\beta_2 z} dz.$$

□

Figure 2.5 shows the three roots of Lundberg equation (2.5.6) for the compound Poisson risk model with double exponential jumps and positive drift in which $c = 20, \beta_1 = 0.1, \beta_2 = 0.2, \lambda = 1, \pi = 0.5, \theta = 0.125$. Figure 2.6 shows the three roots of Lundberg equation (2.5.6) for the compound Poisson risk model with double exponential jumps and negative drift in which $c = 1, \beta_1 = 1, \beta_2 = 0.1, \lambda = 1, \pi = 0.01, \theta = -9.89$. Denoting the left-hand side of

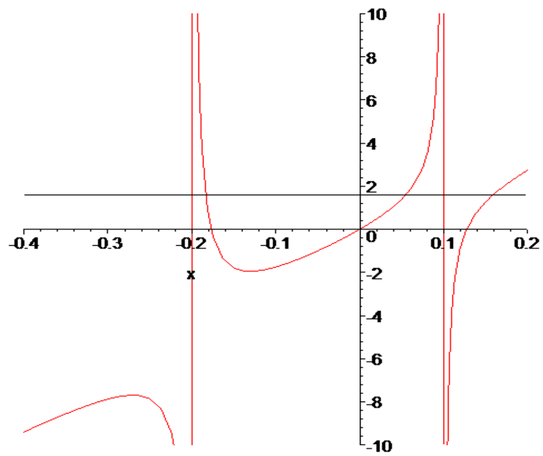

Figure 2.5: Illustration of the roots of Lundberg equation for the compound Poisson risk model with double exponential jumps and positive drift

(2.5.6) by $k(s)$, we observe that the equation $k(s) = 0$ has a negative solution if $k(s)$ crosses x-axis with positive tangent, whereas the smallest root of $k(s) = 0$ is zero if $k(s)$ has negative tangent at the origin.

We now consider the case where $\delta = 0$. Hence $\psi_\delta(x)$ simplifies to

$$\psi(x) = \mathbb{E}^x \left[f(|X_{\tau_0}|) I(\tau_0 < \infty) \right].$$

Corollary 2.5.2. *If $Q(y)$ follows the distribution given in (2.5.3), $\psi(x)$ admits an explicit solution given by*

$$\psi(x) = \begin{cases} \left[(\beta_2 + \rho) \int_0^\infty f(z) e^{-\beta_2 z} dz \right] e^{\rho x}, & \text{if } \theta > 0; \\ \beta_2 \int_0^\infty f(z) e^{-\beta_2 z} dz, & \text{if } \theta \leq 0, \end{cases}$$

where the safety loading factor $\theta = (\beta c - \lambda)/\lambda$ and ρ is the unique negative solution to the Lundberg fundamental equation

$$cs + \lambda \left[\pi \frac{\beta_1}{\beta_1 - s} + (1 - \pi) \frac{\beta_2}{\beta_2 + s} - 1 \right] = 0. \tag{2.5.9}$$

Proof. We have to determine whether there exists a negative solution for the lundberg equation (2.5.9). When the lundberg equation does not have a negative solution, we must have $\psi(x) = C_1$ with C_1 to be determined from the integro-differential equation

$$c\psi'(x) - \lambda\psi(x) + \lambda\pi \int_0^\infty \psi(x+y)dQ^+(y)$$
$$+\lambda(1-\pi)\int_0^x \psi(x-y)dQ^-(y) + \left[\lambda(1-\pi)\beta_2 \int_0^\infty f(z)e^{-\beta_2 z}dz\right] e^{-\beta_2 x} = 0.$$

By replacing $\psi(x)$ with the constant C_1, we find out that

$$C_1 = \beta_2 \int_0^\infty f(z)e^{-\beta_2 z}dz.$$

Since

$$k'(0) = c + \lambda\pi\frac{1}{\beta_1} - \lambda(1-\pi)\frac{1}{\beta_2},$$

and recall that

$$c = (1+\theta)E(Y) = (1+\theta)\left[\lambda\pi\frac{1}{\beta_1} - \lambda(1-\pi)\frac{1}{\beta_2}\right],$$

whether (2.5.9) has a negative or zero solution depends solely on θ. □

Note that when $\theta \leq 0$, the safety loading condition is violated and the surplus process has a negative drift, hence ruin is deemed to occur ultimately. If we take $f(y) = 1$ in $\psi(x)$, then $\psi(x) = \mathbb{E}^x[\tau_0 < \infty] = 1$.

2.6 Geometric Compound Poisson Model

Since the compound Poisson process converges weakly to a Brownian motion, the geometric compound Poisson model was also introduced by many authors to model the dynamics of asset prices as an approximation of the Black-Scholes model. Interested readers

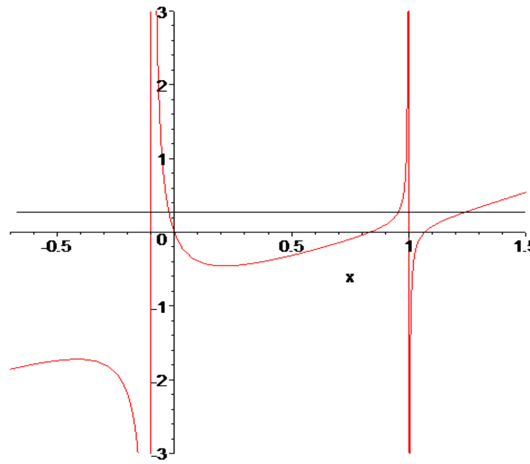

Figure 2.6: Illustration of the roots of Lundberg equation for the compound Poisson risk model with double exponential jumps and negative drift

are referred to Gerber and Shiu [25] for a detailed discussion on its implication in financial modelling.

It would also have been appropriate to put the geometric compound Poisson model in the context of jump diffusion processes in Chapter 4. However, because of the nature of geometric compound Poisson process, we shall see that the quantities of interest to us in this section can be completely solved with only references to generalized Gerber-Shiu function in compound Poisson model. Hence we treat this section as part of our discussion in the context of PDCP.

Assume that X is a shifted compound Poisson process with double-sided exponential jumps and the dynamics of asset price follows a geometric compound Poisson process $S = \{S(t), t \geq 0\}$ with

$$S(t) = e^{X(t)} = \exp\{x + ct - \sum_{i=1}^{N(t)} Y_i\}, \tag{2.6.1}$$

where the expected yield rate $c = r + \lambda[\tilde{q}(-1) - 1]$, the counting process $\{N(t), t \leq 0\}$ is Poisson process with intensity λ and the sequence of random movements $\{Y_i, i = 1, 2, \cdots\}$

are mutually independent and follow the common distribution

$$Q(y) = \pi(1 - e^{-\beta_1 y})I(y \geq 0) + [(1-\pi) - (1-\pi)(1 - e^{\beta_2 y})I(y < 0)],$$

and the mean of jumps is given by $\kappa = \pi/\beta_1 + (1-\pi)/\beta_2$. It can be shown that the geometric compound Poisson process S defined in (2.6.1) is a solution to the stochastic differential equation

$$dS(t) = rS(t)dt - S(t)dZ(t). \tag{2.6.2}$$

Readers may easily employ the approach to be discussed in Chapter 4 to recover all the results in this section.

2.6.1 Perpetual American Put Option

In Gerber and Shiu [21], it was successfully demonstrated that the Gerber-Shiu discounted penalty function can be applied to price a perpetual American put option. Following the same line of logic, we shall now derive the price of a perpetual American put option with a underlying stock price driven by the geometric compound Poisson with two-sided exponential jumps.

It has been proved in mathematical finance that the price of an American put option is the maximum of expected discounted payoff function over all possible hitting times. For notational convenience, we denote the price by

$$F(x) = \sup_a \mathbb{E}^x \left[e^{-\delta\tau_a} \Pi\big(S(\tau_a)\big) \right] = \sup_a \mathbb{E}^x \left[e^{-\delta\tau_a} \Pi\big(e^{X(\tau_a)}\big) \right], \tag{2.6.3}$$

where the payoff function

$$\Pi(s) = (K - s)_+ ,$$

with the exercise price K and

$$\tau_a = \inf\{t | S(t) < e^a\} = \inf\{t | X(t) < a\}$$

with $a < \ln K \leq x$. Now we are able to derive a result analogous to the perpetual American put option with negative jumps only, which is given in Gerber and Shiu [21].

Corollary 2.6.1. *When $\delta > 0$, the solution to $F(x)$ defined in (2.6.3) is given by*

$$F(x) = \frac{(\beta_2+\rho)K}{\beta_2(1-\rho)}\left[K\frac{\rho(\beta_2+1)}{\beta_2(\rho-1)}\right]^{-\rho} e^{\rho x},$$

where ρ is the unique negative solution to (2.5.6).

When $\delta = 0$, the solution to $F(x)$ is given by

$$F(x) = \begin{cases} \frac{(\beta_2+\rho)K}{\beta_2(1-\rho)}\left[K\frac{\rho(\beta_2+1)}{\beta_2(\rho-1)}\right]^{-\rho} e^{\rho x}, & \text{if } \theta > 0; \\ \frac{K}{\beta_2^2}, & \text{if } \theta \leq 0. \end{cases}$$

Proof. If we define a new process $Y = \{Y_t, t \geq 0\}$ such that $Y_t = X_t - a$ and its corresponding time of default $\tau_d^Y = \inf\{t|Y(t) < d\}$, then it is easy to see that $\tau_0^Y = \tau_a$. We have to keep in mind that $Y(0) = x - a$. Therefore, the discounted payoff function upon which the supremum is taken can written as a special case of $\psi_\delta(x)$,

$$\mathbb{E}^x\left[e^{-\delta\tau_a}\Pi\big(e^{X(\tau_a)}\big)\right] = \mathbb{E}^{x-a}\left[e^{-\delta\tau_0^Y}\Pi\big(e^{Y(\tau_0^Y)+a}\big)\right].$$

When $\delta > 0$, it follows immediately from Corollary 2.5.1 that

$$\mathbb{E}^x\left[e^{-\delta\tau_a}\Pi\big(e^{X(\tau_a)}\big)\right] = \left(K - \frac{\beta_2}{\beta_2+1}e^a\right)\frac{\beta_2+\rho}{\beta_2}e^{\rho(x-a)},$$

which is maximized at

$$a = \ln\left[K\frac{\rho(\beta_2+1)}{\beta_2(\rho-1)}\right].$$

Since $\rho \in (-\beta_2, 0)$, we can show that $a < \ln K$.

When $\delta = 0$, the similar result follows from Corollary 2.5.2. When $\theta \leq 0$,

$$\mathbb{E}^x \left[e^{-\delta\tau_a} \Pi\big(e^{X(\tau_a)}\big) \right] = \frac{K}{\beta_2^2} - \frac{e^a}{\beta_2(\beta_2 - 1)},$$

which is maximized at $a = -\infty$. $\square$

The last part of the corollary makes sense because the investor is better off delaying exercising the option as much as possible, as the safety loading condition $\theta > 0$ is violated and the stock price process will eventually drift towards zero.

2.6.2 Fixed-rate and Floating-rate Stochastic Annuities

Suppose there are two types of investment features to policyholders in a certain insurance product. One feature offers to credit an annuity in amount of one dollar per unit time continuously in the policyholder's account at a predetermined fixed force of interest as long as a reference equity index stays above a certain level. The second feature provides an annuity in amount of one dollar per unit time continuously in the policyholder's account with a floating interest rate according to the reference equity index until it goes below the certain level. If both features can be freely traded in the market, it would give rise to transactions where a risk-seeking party agrees to pay floating-rate annuity in return for fixed-rate annuity given up by another risk-averse party. Now we address the interesting issue of how to price such an annuity swap.

We assume that the dynamics of equity price quoted by an insurance company as a reference index is driven by a geometric compound Poisson process $\{S(t), t \geq 0\}$ with

$$S(t) = e^{X(t)} = \exp\{x + ct - \sum_{i=1}^{N(t)} Y_i\},$$

where the expected yield rate $c = r - \lambda[\tilde{q}(-1) - 1]$ and, for simplicity, the insurance claims follow an exponential distribution with mean $1/\beta$. We shall also denote $s = S(0) = e^x$.

Suppose the insurance company set up the benchmark level at $b = e^d > 0$ and promises continuous annuity payments until the reference index falls below the benchmark level. In other words, the payments are made starting from the date of issue until the stopping time

$$\tau_b \triangleq \inf\{S(t) < b\} = \inf\{X(t) < d\}.$$

Therefore the expected present value of the continuous annuity payable until the time of index default with a fixed force of interest $\delta > 0$ is given by

$$\overline{a}^{\delta}_{\overline{\tau_b}|} \triangleq \mathbb{E}^x \left[\int_0^{\tau_b} e^{-\delta t} dt \right].$$

If we define $Y = \{Y(t) = X(t) - d, 0 \leq t < \infty\}$ and $\tau_0^Y = \inf\{t | Y(t) < 0\}$, then it is easy to see that $\tau_d = \tau_0^Y$. Hence

$$\overline{a}^{\delta}_{\overline{\tau_b}|} = E^{x-d} \left[\int_0^{\tau_0^Y} e^{-\delta t} dt \right] = \frac{1}{\delta}\{1 - E^{x-d}[e^{-\delta \tau_0^Y}]\}.$$

In view of (2.3.36), we find that

$$\overline{a}^{\delta}_{\overline{\tau_b}|} = \frac{1}{\delta} - \frac{1}{\delta}\frac{\rho+\beta}{\beta} e^{\rho(x-d)} = \frac{1}{\delta} - \frac{1}{\delta}\frac{\rho+\beta}{\beta}\left(\frac{s}{b}\right)^{\rho}.$$

In the limiting case where $d \to -\infty$ or $b \to 0$, the time of default is virtually infinite and the annuity with the fixed force of interest becomes a perpetuity. Hence, in consistent with the result for perpetuity-certain,

$$\overline{a}^{\delta}_{\overline{\infty}|} = \lim_{b \to 0} \overline{a}^{\delta}_{\overline{\tau_b}|} = \frac{1}{\delta}.$$

Now we follow the notion of "stochastic life annuity" from Dufresne [17] to construct a floating rate annuity where the credited interest is linked with the equity index. Since the equity price starting from e^x at time zero accumulates to $S(t)$ at any time t, it is obvious that the amount that has to be invested initially at time zero to fund one dollar at time t is given by the discount function

$$v(t) \triangleq e^x S(t)^{-1}.$$

Hence the expected present value of the annuity with the floating rates must be

$$\begin{aligned}\overline{a}^S_{\overline{\tau_b}|} &= \mathbb{E}^x\left[\int_0^{\tau_b} v(t)dt\right] = \mathbb{E}^x\left[\int_0^{\tau_b} e^{-(X_t-x)}dt\right] \\ &= \mathbb{E}^{x-d}\left[\int_0^{\tau_0^Y} e^{-(Y_t+d-x)}dt\right] = e^{(x-d)}\mathbb{E}^{x-d}\left[\int_0^{\tau_0^Y} e^{-Y_t}dt\right]\end{aligned}$$

It follows immediately from (2.3.7) and (2.3.13) that

$$\overline{a}^S_{\overline{\tau_b}|} = \frac{1}{c-\lambda/(\beta-1)} - \frac{\lambda/(\beta-1)}{c[c-\lambda/(\beta-1)]}\left(\frac{s}{b}\right)^{-(\beta-1)[c-\lambda/(\beta-1)]/c}. \tag{2.6.4}$$

Similarly, when we set $d \to -\infty$ or $b \to 0$, the annuity with stochastic interest continues for an infinite term. It follows from (2.6.4) that the expected present value of the stochastic perpetuity converges if and only if $c > \lambda/(\beta-1)$ and $\beta > 1$,

$$\overline{a}^S_{\overline{\infty}|} \triangleq \lim_{b\to 0}\overline{a}^S_{\overline{\tau_b}|} = \frac{1}{c-\lambda/(\beta-1)}.$$

It is intuitive to interpret the condition in connection with the discount process $v(t)$. To see if $v(t)$ is a supermartingale or submartingale, we need to check that for any $t > s \geq 0$,

$$\begin{aligned}\mathbb{E}^x[v(t)|\mathcal{F}_s] &= \mathbb{E}^x[e^{-ct+\sum_{i=1}^{N(t)}Y_i}|\mathcal{F}_s] \\ &= \mathbb{E}^x[e^{-ct+\sum_{i=1}^{N(t)}Y_i} - e^{-cs+\sum_{i=1}^{N(s)}Y_i}|\mathcal{F}_s] + v(s) \\ &= e^{-cs+\sum_{i=1}^{N(s)}Y_i}\mathbb{E}^x[e^{-c(t-s)+\sum_{i=N(s)}^{N(t)}Y_i} - 1|\mathcal{F}_s] + v(s) \\ &= e^{-cs+\sum_{i=1}^{N(s)}Y_i}\{e^{-c(t-s)+\lambda(t-s)/(\beta-1)} - 1\} + v(s).\end{aligned}$$

Hence it is clear that

$$\mathbb{E}^x[v(t)|\mathcal{F}_s] < v(s) \qquad \mathbb{P}^x - a.s.$$

if and only if $c > \lambda/(\beta-1)$ and $\beta > 1$. Note that the convergence of moment generation function of exponential distribution at 1 (or $\tilde{q}(-1)$ in terms of Laplace transform) takes place only if $\beta > 1$. When the discount process is a supermartingale, the perpetuity would converge to a finite limit.

In fact, we can also see from the stochastic differential equation (2.6.2) that, in order for the equity index process to have a positive drift, we have to make sure that $r = c - \lambda[\tilde{q}(-1) - 1] > 0$, which recovers that condition that $c > \lambda/(\beta - 1)$ and $\beta > 1$.

Given that the expected present value of both fixed-rate and floating-rate annuity are obtained, a swap that exchanges a fixed-rate annuity for a floating-rate annuity until the equity index falls below b can be evaluated as

$$\begin{aligned} V_{swap} &= \overline{a}^{S}_{\overline{\tau_b}|} - \overline{a}^{\delta}_{\overline{\tau_b}|} \\ &= \frac{1}{c - \lambda/(\beta-1)} - \frac{\lambda/(\beta-1)}{c[c-\lambda/(\beta-1)]}\left(\frac{s}{b}\right)^{-(\beta-1)[c-\lambda/(\beta-1)]/c} - \frac{1}{\delta} + \frac{1}{\delta}\frac{\rho+\beta}{\beta}\left(\frac{s}{b}\right)^{\rho}. \end{aligned}$$

2.7 Compound Poisson Model with Absolute Ruin

It has been argued in the recent literature that an insurer would not go bankrupted immediately after the surplus in one line of business hits zero, rather the insurer stays in business with debts borrowed from other lines of business or investors until the premium income is no longer sufficient to cover debit interests. Gerber-Shiu functions in this model has been studied thoroughly in Cai [6]. We shall use this example to work out the integro-differential equations for the generalized Gerber-Shiu function for the compound Poisson model with absolute ruin.

On the positive side, the surplus varies much the same way as the classical model. The distinctive feature of the absolute ruin model lies in the deterministic sample path when the surplus goes below zero and debt interest rate r starts to apply. In the absence of insurance claims, the actual value at time t of the surplus process starting off from $x, x < 0$, at time 0 should be the balance of the accumulated value of premium income up to time t, $c\,\overline{s}^{(r)}_{\overline{t}|}$, less the original amount of debts at time 0 accumulated to time t, $|x|e^{rt}$. Hence,

$$\frac{d}{dt}X_t = \frac{d}{dt}(c\overline{s}^{(r)}_{\overline{t}|} - |x|e^{rt}) = \frac{d}{dt}(c\overline{s}^{(r)}_{\overline{t}|} + xe^{rt}) = rX_t + c.$$

Note that when $x \leq -c/r$, $dX_t/dt \leq 0$, which means the premium is no longer able to even cover the debit interest and the surplus process is therefore said to be absolutely ruined.

In summary, the extended generator of the deterministic path is given by

$$\mathfrak{X} = \begin{cases} c\, d/dx, & \text{if } x > 0, \\ (rx+c)d/dx, & \text{if } -c/r < x \leq 0. \end{cases}$$

Since we are now interested all ruin-related quantities up to the time of absolute ruin, the time of default in the definition of generalized Gerber-Shiu function is to be chosen as

$$\tau_{-c/r} = \inf\{t | X(t) < -c/r\}.$$

Thus, the generalized Gerber-Shiu function (2.2.1) takes the form

$$V_{abs}(x) = \mathbb{E}^x \left[\int_0^{\tau_{-c/r}} e^{-\delta t} l(X_t) dt \right].$$

Since $d = -c/r$, it follows from (2.2.9) that

$$\mathcal{Q}V_{abs}(x) = \int_0^{x+c/r} V_{abs}(x-y)dQ(y).$$

If $V_{abs}(x)$ is bounded, it follows from Theorem 2.2.1 that

$$cV'_{abs}(x) - (\lambda+\delta)V_{abs}(x) + \lambda \int_0^{x+c/r} V_{abs}(x-y)dQ(y) + l(x) = 0, \qquad x > 0,$$

$$(rx+c)V'_{abs}(x) - (\lambda+\delta)V_{abs}(x) + \lambda \int_0^{x+c/r} V_{abs}(x-y)dQ(y) + l(x) = 0, \qquad -c/r < x < 0.$$

Following the previous examples, we could derive all sorts of ruin-related quantities in the absolute ruin model. For instance, we choose the penalty function at random jumps by

$$\varpi(x,y) = \begin{cases} 0, & \text{if } x \geq -c/r, \\ w(x,-y), & \text{if } x < -c/r. \end{cases}$$

Inserting it into (2.3.31) we obtain the cost function for the Gerber-Shiu function with absolute ruin

$$l(x) = \lambda \int_{x+c/r}^{\infty} w(x, y-x)dQ(y). \tag{2.7.1}$$

Thus, the Gerber-Shiu function, defined by

$$m_{abs}(x) = \mathbb{E}^x \left[\exp\{-\delta \tau_{-c/r}\} w(X_{\tau_{-c/r}-}, |X_{\tau_{-c/r}}|) I(\tau_{-c/r} < \infty) \right],$$

where $\delta \geq 0$ and $w(x,y)$ is a bounded measurable function, is also bounded as shown in Section 2.3.5. Hence it satisfies the following equations

$$cm'_{abs}(x) - (\lambda+\delta)m_{abs}(x) + \lambda \int_0^{x+c/r} m_{abs}(x-y)dQ(y) + \lambda \int_{x+c/r}^{\infty} w(x, y-x)dQ(y) = 0, \qquad x > 0,$$

$$(rx+c)m'_{abs}(x) - (\lambda+\delta)m_{abs}(x) + \lambda \int_0^{x+c/r} m_{abs}(x-y)dQ(y) + \lambda \int_{x+c/r}^{\infty} w(x, y-x)dQ(y) = 0,$$
$$-c/r < x \leq 0,$$

which are precisely equation (2.16) and (2.15) of Cai [6] respectively.

As a generalization, it is suggested that when an insurance company is in debt, its debtor would demand debit interest that commensurate with the risk of bankruptcy. The larger the deficit, the more interest charged. We hence amend the absolute ruin model above to incorporate a varying debit interest rate $r(x), x \leq 0$. The function $r(x)$ is an increasing function in x. It is important to note that absolute ruin occurs at the new level d determined by $r(d)d + c = 0$, which means the premium income is no longer able to cover the debit interest. Therefore, the extended generator of the deterministic path is given by

$$\mathfrak{X} = \begin{cases} c\, d/dx, & \text{if } x \geq 0, \\ [r(x)x + c]\, d/dx, & \text{if } d < x < 0. \end{cases}$$

Correspondingly, the Gerber-Shiu function $m_{abs}(x)$ satisfies

$$cm'_{abs}(x) - (\lambda+\delta)m_{abs}(x) + \lambda \int_0^{x-d} m_{abs}(x-y)dQ(y) + \lambda \int_{x-d}^{\infty} w(x, y-x)dQ(y) = 0,$$
$$x \geq 0, \quad (2.7.2)$$

$$[r(x)x + c]\, m'_{abs}(x) - (\lambda+\delta)m_{abs}(x) + \lambda \int_0^{x-d} m_{abs}(x-y)dQ(y) + \lambda \int_{x-d}^{\infty} w(x, y-x)dQ(y) = 0.$$
$$d < x < 0, \quad (2.7.3)$$

For illustration, we look at an easy example of

$$\varphi_{abs}(x) = \mathbb{E}^x \left[w(X_{\tau_d-}, |X_{\tau_d}|) I(\tau_d < \infty) \right],$$

where the claim sizes are exponentially distributed with mean $1/\beta$.

Corollary 2.7.1. *If $Q(y)$ is exponentially distributed with mean $1/\beta$, $\varphi_{abs}(x)$ admits an explicit solution given by*

$$\begin{aligned}
\varphi_{abs}(x) &= C_1 + \int_0^x e^{-S(y)} \left(C_2 + \int_0^y e^{S(t)} f(t) dt \right) dy, \qquad x \geq 0, \qquad (2.7.4)\\
\varphi_{abs}(x) &= C_3 + \int_d^x e^{-G(y)} \left(C_4 + \int_d^y e^{G(t)} h(t) dt \right) dy, \qquad d < x < 0, \qquad (2.7.5)
\end{aligned}$$

where

$$\begin{aligned}
\zeta(x) &= \lambda\beta \int_{x-d}^{\infty} e^{-\beta y} w(x, y-x) dy,\\
f(x) &= -\frac{\beta\zeta(x) + \zeta'(x)}{c},\\
g(x) &= \frac{r'(x)x + r(x) + \beta r(x)x + \beta c - \lambda}{r(x)x + c},\\
h(x) &= -\frac{\beta\zeta(x) + \zeta'(x)}{r(x)x + c},\\
S(y) &= \int_0^y (\beta - \frac{\lambda}{c}) dt = (\beta - \frac{\lambda}{c}) y,\\
G(y) &= \int_d^x g(t) dt.
\end{aligned}$$

and the coefficients are determined by

$$
\begin{aligned}
C_1 &= -\int_0^\infty e^{-S(y)}\left(C_2+\int_0^y e^{S(t)}f(t)dt\right)dy,\\
C_2 &= e^{-G(0)}C_4+e^{-G(0)}\int_d^0 e^{G(t)}h(t)dt,\\
C_3 &= \zeta(d),\\
C_4 &= -\Big[\zeta(d)+\int_0^\infty e^{-S(y)}\int_0^y e^{S(t)}f(t)dtdy+\int_d^0 e^{-G(y)}\int_d^y e^{G(t)}h(t)dtdy\\
&\quad +e^{-G(0)}\int_0^d e^{-G(t)}h(t)dt\int_0^\infty e^{-S(y)}dy\Big]/\Big[\int_d^0 e^{-G(y)}dy+e^{-G(0)}\int_0^\infty e^{-S(y)}dy\Big].
\end{aligned}
$$

Proof. Multiplying $\beta+\mathcal{D}$ on both sides of (2.7.2) and (2.7.3) yields

$$
\begin{aligned}
&\varphi''_{abs}(x)+(\beta-\frac{\lambda}{c})\varphi'_{abs}(x)=f(x), \qquad x\geq 0,\\
&\varphi''_{abs}(x)+g(x)\varphi'_{abs}(x)=h(x), \qquad d<x<0.
\end{aligned}
$$

The general solution to $\varphi_{abs}(x)$ is given by (2.7.4) and (2.7.5). In order to determine those coefficients, we search for four boundary conditions, each of which gives a linear equation involving the coefficients.

Since $\varphi_{abs}(x)$ is a special case of Gerber-Shiu function, we always have

$$\lim_{x\to\infty}\varphi_{abs}(x)=0. \tag{2.7.6}$$

Letting $x\to d$ in (2.7.3) yields

$$\varphi_{abs}(d+)=\zeta(d). \tag{2.7.7}$$

By continuity of the generalized Gerber-Shiu function,

$$\varphi_{abs}(0-)=\varphi_{abs}(0+). \tag{2.7.8}$$

Letting $x = 0$ in (2.7.2) and $x \to 0$ in (2.7.3) and in view of (2.7.8) we obtain

$$\varphi'_{abs}(0-) = \varphi'_{abs}(0+). \tag{2.7.9}$$

Hence, inserting (2.7.4) and (2.7.5) into (2.7.6), (2.7.7), (2.7.8) and (2.7.9) yields the desired solutions. $\square$

2.8 Compound Poisson Model with Multiple Thresholds

As an extension to the classical compound Poisson model with a dividend threshold described by (2.3.3), Lin and Sendova [40] analyzed the Gerber-Shiu function in a compound Poisson model with n threshold levels $b_1, b_2, \cdots, b_n$, each of which specifies a different dividend payout rate $\alpha_1, \alpha_2, \cdots, \alpha_n$ respectively. In this example, we shall follow the techniques from Lin and Pavlova [39], which treats a single threshold, to find solutions to the generalized Gerber-Shiu function.

We number the threshold levels in the order from bottom to top. Let $\mathbf{b} = (b_1, b_2, \cdots, b_n)$ and $V(x; \mathbf{b})$ be the generalized Gerber-Shiu function defined by

$$V(x; \mathbf{b}) = \mathbb{E}^x \left[\int_0^{\tau_d} e^{-\delta t} l(X_t) dt \right], \tag{2.8.1}$$

where the surplus process X_t is a PDCP with local characteristics $(\mathfrak{X}, \lambda, Q)$ and the generator of the deterministic path between ith and $(i+1)$-th threshold is given by

$$\mathfrak{X} = (c - \alpha_i) \frac{d}{dx}, \qquad b_i \le x < b_{i+1},$$

with $i = 0, 1, \cdots, n, b_0 = d, \alpha_0 = 0, b_{n+1} = \infty$. The same technique that treats the compound Poisson model with multiple thresholds as a piecewise-deterministic Markov process also appears in Yin [51].

Suppose $V(x;\mathbf{b})$ considered in this section is a bounded function. Therefore, inserting the specific generator into (2.2.2), we obtain the integro-differential equation for each threshold step,

$$(c-\alpha_i)V'(x;\mathbf{b}) - (\lambda+\delta)V(x;\mathbf{b}) + \lambda\int_0^{x-d} V(x-y;\mathbf{b})\,dQ(y) + l(x) = 0,$$
$$b_i \le x < b_{i+1}, \text{ for } i = 1, 2, \cdots, n+1. \qquad (2.8.2)$$

We summarize the solution to $V(x;\mathbf{b})$ in the following corollary.

Corollary 2.8.1. *The solution to $V(x;\mathbf{b})$ defined in (2.8.1) is given by*

$$\begin{aligned} V(x;\mathbf{b}) &= \psi_i(x) + \eta_i v_i(x), \qquad b_i \le x < b_{i+1}, \qquad \textit{for } i = 0, 1, \cdots, n-1, \\ V(x;\mathbf{b}) &= \frac{1}{1-\pi_n}\int_0^{x-b_n} h_n(x-y)dv_n(y) + h_n(x), \qquad b_n \le x < \infty. \end{aligned}$$

where

$$\eta_i = \frac{\lambda\pi_{i+1}\int_0^{b_{i+1}-b_i}\psi_i(b_{i+1}-y)d\hat{Q}_{i+1}(y) + (c-\alpha_{i+1})[f_{i+1}(b_{i+1}+) - \psi_i(b_{i+1})]}{(c-\alpha_{i+1})v_i(b_{i+1}) - \lambda\pi_{i+1}\int_0^{b_{i+1}-b_i} v_i(b_{i+1}-y)d\hat{Q}_{i+1}(y)}, \qquad (2.8.3)$$

$$v_i(x) = \sum_{n=0}^{\infty}(1-\pi_i)\pi_i^n \hat{Q}_i^{\star n}(x-b_i), \qquad (2.8.4)$$

$$\psi_i(x) = \frac{1}{1-\pi_i}\int_0^{x-b_i} h_i(x-y)dv_i(y) + h_i(x). \qquad (2.8.5)$$

$$h_i(x) = \pi_i\int_{x-b_i}^{x-d} V(x-y;\mathbf{b})d\hat{Q}_i(y) + \frac{1}{c-\alpha_i}T_{\rho_i}l(x). \qquad (2.8.6)$$

$$\pi_i = \frac{\lambda}{c-\alpha_i}\int_0^{\infty} e^{\rho_i y}\int_y^{\infty} e^{-\rho_i t}dQ(t)dy \qquad (2.8.7)$$

$$\hat{Q}_i(x) = \frac{\int_0^x e^{\rho_i y}\int_y^{\infty} e^{-\rho_i t}dQ(t)dy}{\int_0^{\infty} e^{\rho_i y}\int_y^{\infty} e^{-\rho_i t}dQ(t)dy}, \qquad (2.8.8)$$

with ρ_i being the unique non-negative root of the fundamental Lundberg equation

$$(c-\alpha_i)s + \lambda\tilde{q}(s) - (\lambda+\delta) = 0.$$

Proof. As usual, (2.8.2) can be written in terms of operators,

$$(\frac{\lambda+\delta}{c-\alpha_i}\mathcal{I}-\mathcal{D})V(x;\mathbf{b}) = \frac{\lambda}{c-\alpha_i}V \star q(x) + \frac{1}{c-\alpha_i}l(x), \qquad b_i < x < b_{i+1}.$$

Since $(h\mathcal{I}-\mathcal{D})\mathcal{T}_h = \mathcal{I}$, we hence obtain

$$V(x;\mathbf{b}) = \frac{\lambda}{c-\alpha_i}T_h\{V \star q\}(x) + \frac{1}{c-\alpha_i}T_h l(x), \qquad b_i, < x < b_{i+1},$$

where $h = (\lambda+\delta)/(c-\alpha_i)$.

Repeating the usual procedure to take the Dickson-Hipp transform inside the convolution and cancel terms, it is easy to obtain

$$V(x;\mathbf{b}) = \frac{\lambda}{c-\alpha_i}V \star \mathcal{T}_{\rho_i}q(x) + \frac{1}{c-\alpha_i}\mathcal{T}_{\rho_i}l(x).$$

To make it look clear, we can rewrite it as

$$V(x;\mathbf{b}) = \frac{\lambda}{c-\alpha_i}\int_0^{x-d} V(x-y;\mathbf{b})d\mathcal{T}_{\rho_i}Q(y) + \frac{1}{c-\alpha_i}\mathcal{T}_{\rho_i}l(x), \qquad b_i < x < b_{i+1}. \quad (2.8.9)$$

Recall that $V(x)$ is absolutely continuous for all $x \in \mathbb{R}$, then we must have

$$V(b_i-;\mathbf{b}) = V(b_i+;\mathbf{b}), \qquad \text{for } i = 1, 2, \cdots, n. \quad (2.8.10)$$

We search for the solution in the form of a combination of a particular solution and fundamental solution to the corresponding homogeneous equation. For $i = 0, 1, 2, \cdots, n-1$,

$$V(x;\mathbf{b}) = \psi_i(x) + \eta_i v_i(x), \qquad b_i \le x < b_{i+1}, \qquad \text{for } i = 0, 1, \cdots, n-1, \quad (2.8.11)$$

where the particular solution $\psi_i(x)$ is obtained from the defective renewal equation

$$\psi_i(x) = \pi_i \int_0^{x-b_i} \psi_i(x-y)d\hat{Q}_i(y) + h_i(x), \qquad b_i \le x < \infty,$$

and the fundamental solution $v_i(x)$ to the corresponding homogeneous equation

$$v_i(x) = \pi_i \int_0^{x-b_i} v_i(x-y)d\hat{Q}_i(y), \qquad x \geq b_i.$$

The function $\hat{Q}_i(y), h_i(x)$ are given in (2.8.8) and (2.8.6). The constant π_i is given in (2.8.7).

Note that for each step to compute $V(x;\mathbf{b}), b_i \leq x < b_{i+1}, i = 1, 2, \cdots, n-1$, the function $h_i(x)$ is known from previous steps, depending on $V(x;\mathbf{b}), b_0 \leq x < b_i$. When $i = 0, h_0(x)$ is a function of $l(x)$. Then it is easy to prove that the solutions can be expressively written as (2.8.4) and (2.8.6).

In light of (2.8.11), we have

$$V(b_{i+1}-;\mathbf{b}) = \psi_i(b_{i+1}) + \eta_i v_i(b_{i+1}).$$

And it follows from (2.8.9) and (2.8.11) that

$$\begin{aligned} V(b_{i+1}+;\mathbf{b}) &= \frac{\lambda\pi_{i+1}}{c-\alpha_{i+1}} \int_0^{b_{i+1}-b_i} V(b_{i+1}-y;\mathbf{b})d\hat{Q}_{i+1}(y) + f_{i+1}(b_{i+1}+) \\ &= \frac{\lambda\pi_{i+1}}{c-\alpha_{i+1}} \Big[\int_0^{b_{i+1}-b_i} \psi_i(b_{i+1}-y)d\hat{Q}_{i+1}(y) + \eta_i \int_0^{b_{i+1}-b_i} v_i(b_{i+1}-y)d\hat{Q}_{i+1}(y)\Big] + f_i(b_{i+1}+), \end{aligned}$$

where

$$f_{i+1}(b_{i+1}+) = \frac{\lambda\pi_{i+1}}{c-\alpha_{i+1}} \int_{b_{i+1}-b_i}^{b_{i+1}-d} V(b_{i+1}-y;\mathbf{b})d\hat{Q}_{i+1}(y) + \frac{1}{c-\alpha_{i+1}}\mathcal{T}_{\rho_{i+1}} l(b_{i+1}+).$$

Given (2.8.10), we obtain the expressions for η_i as given in (2.8.3).

The solution to $V(x;\mathbf{b})$ for $x \geq b_n$ is rather straightforward, as it satisfies a renewal equation without upper boundary.

$$V(x;\mathbf{b}) = \pi_n \int_0^{x-b_n} V(x-y;\mathbf{b})d\hat{Q}_n(y) + h_n(x), \qquad b_n \leq x < \infty.$$

Thus we obtain the last unknown part,

$$V(x;\mathbf{b}) = \frac{1}{1-\pi_n}\int_0^{x-b_n} h_n(x-y)dv_n(y) + h_n(x), \qquad b_n \leq x < \infty.$$

$\square$

Readers are referred to Lin and Sendova [40] for alternative solutions to the Gerber-Shiu function in the same model.

Chapter 3

Sparre Andersen Risk Models

The generalization from the classical compound Poisson model to Sparre Andersen model is another milestone that shapes the modern landscape of ruin theory. Since first introduced in 1957 by E. Sparre Andersen, many new techniques have been brought in from various original backgrounds and further developed in the ruin literature.

In simple words, the Sparre Andersen model replaces the exponential inter-claim time distribution in the compound Poisson model by more general distributions while retaining the assumption on the independence between inter-claim times and insurance claims. However, with such a generosity of inter-claim time distribution, we are not always unable to obtain closed-form solutions to the probability of ruin, let alone Gerber-Shiu functions. The recent study of a particular case of Sparre Andersen model with generalized Erlang-n inter-claim time distribution has gained enormous popularity among the research community, as the model produces many elegant results analogous to those in compound Poisson models. Interested readers may refer to Gerber and Shiu [24], Li and Garrido [38] etc for a detailed account.

In this chapter, we shall begin with another case of Sparre Andersen model, which naturally leads to the construction of generalized Gerber-Shiu function in a similar manner as in previous chapter. Later on, we shall demonstrate its connection with the popular generalized Erlang-n model and reproduce some well-known results to prove the efficiency of a generalized Gerber-Shiu function and its consistency with conventional approaches.

3.1 Jacobsen Model

The risk model proposed by Jacobsen [29] assumes continuous phase-type distributed inter-claim arrival times and claim sizes governed by distribution with rational Laplace transform. Although it can be viewed as a generalization of the Sparre Andersen model previously existed, Jacobsen [29] was the first in the literature to identify and technically treat the underlying renewal process as a piecewise-deterministic Markov process. The major contribution of his work was to introduce the martingale approach under the PDMP framework to derive the Laplace transform of the time of ruin. As a result of specific assumptions on both inter-claim time distribution and claim size distribution, closed-form solutions to the Laplace transform of the time of ruin were produced, which in turn permits the calculation of probability of ultimate ruin through numerical means.

Our goal is to reconcile Jacobsen model with all other models in the thesis in the framework of a generalized Gerber-Shiu function and investigate more general ruin-related quantities.

Suppose that an insurer's surplus is driven by an indexed stochastic process $X = \{(X_t, J_t), t \geq 0\}$ where the level of surplus is given by $X_t \in \mathbb{R}$ and the index $J_t \in \{1, 2, \cdots, m\}$ is governed by the inter-claim arrival time distribution.

- Jump Arrivals

  The inter-claim arrival times $\{T_n, n = 2, 3, \cdots\}$ are independent and identically distributed with the common phase-type distribution $PH(\mathbf{a}, \mathbf{\Lambda})$, where $\mathbf{a} \triangleq (a_1, a_2, \cdots, a_m)^T$ is the initial probability vector and $\mathbf{\Lambda}$ the sub-intensity matrix of the underlying Markov chain J moving in the transient state space $E = \{1, 2, \cdots, m\}$ and an absorbing state. The absorption probability vector is given by

$$\eta \triangleq (\eta_1, \eta_2, \cdots, \eta_m)^T = -\mathbf{\Lambda}\mathbf{1}_m,$$

where vector $\mathbf{1}_m$ of dimension m consists of all elements equaling 1. Hence the intensity matrix of the underlying Markov chain can be written as

$$\begin{pmatrix} \mathbf{\Lambda} & \eta \\ \mathbf{0} & 0 \end{pmatrix}.$$

From the knowledge of phase-type distributions, tail probability of the inter-claim arrival time distribution is hence given by

$$\overline{K}(t) = \mathbb{P}\{V_n > t\} = \mathbf{a}^T e^{\mathbf{\Lambda} t}\mathbf{1}_m, \qquad n = 2, 3, \cdots,$$

and the Laplace transform of the inter-claim time distribution can be written as

$$\tilde{k}(s) = -\mathbf{a}^T(\mathbf{\Lambda} - s\mathbf{I})\eta, \tag{3.1.1}$$

where $\mathbf{I}$ denotes the identity matrix of dimension m.

In other words, we can interpret that the insurer's surplus process X jumps from index i to j with transition rates defined by the following two cases,

1. Transition from (X_t, i) to (X_t, j) with J communicating from the transient state i to j at the rate given by $\mathbf{\Lambda}_{ij}$;
2. Transition from (X_t, i) to $(X_t - y, j)$ with J first absorbed in absorbing state resulting in an insurance claim of size y and then regenerated in a transient state j at the rate given by $\eta_i a_j$.

- Jump Sizes

The jump sizes $\Delta X(T_k) = X(T_k) - X(T_k-)$ are determined by a transition measure $Q(\cdot; X(T_k-), j)$ where $j \in \{1, 2, \cdots, m\}$. Note that by definition Jacobsen model allows the dependency between claim size $\Delta X(T_k)$ and current surplus level $X(T_k-)$.

- Piecewise-deterministic Path

  In Jacobsen [29], the insurer's surplus process increases by a constant premium rate. However, we may easily extend the model to include the growth of surplus process $X_t, T_k \le t < T_{k+1}$, governed by a vector field $\mathfrak{X}$ regardless of the status of Markov chain J_t.

- Initial Value

  For practical uses, we often assume X starts at a fixed point (x, i) and all the above assumptions are made with the measure $\mathbb{P}^{(x,i)}$ under which $\mathbb{P}^{(x,i)}\{X(t) = x, J(t) = i\} = 1$. However, if the initial index i is unknown in certain situations, we may make various assumptions on i, which equivalently leads to different assumptions on V_1. It is obvious that if we let the process X starts randomly according to the entrance law $\mathbf{a}$, then V_1 has exactly the same distribution as $V_i, i = 2, 3, \cdots$, which is the case we shall study in this section. Then we shall define a new measure under which V_1 follows $PH(\mathbf{a}, \mathbf{\Lambda})$ as well as $V_i, i = 2, 3, \cdots$,

$$\mathbb{P}^x = \sum_{i=1}^{m} a_i \mathbb{P}^{(x,i)}.$$

The infinitesimal generator of the stochastic process X under the measure $\mathbb{P}^{(x,i)}$ is given by,

$$\mathfrak{A} f(x,i) = \mathfrak{X} f(x,i) + \sum_{j \in E} \mathbf{\Lambda}_{ij} f(x,j) + \eta_i \sum_{j \in E} a_j \int_{-\infty}^{\infty} f(y,j) dQ(y; x, i). \tag{3.1.2}$$

Following the notion of generalized Gerber-Shiu function defined for PDCP, we can similarly adopt a definition for the stochastic process X :

$$H(x,i) = \mathbb{E}^{(x,i)} \left[\int_0^{\tau_d} e^{-\delta t} l(X_t, J_t) dt \right], \tag{3.1.3}$$

where the expectation is taken under the measure $\mathbb{P}^{(x,i)}$, the *cost function* $l : \mathbb{R} \times E \mapsto \mathbb{R}$ is measurable and the *time of default* τ_d is defined by

$$\tau_d = \inf\{t | X(t) < d\}, \qquad d \in \mathbb{R}.$$

One may also be interested in a generalized Gerber-Shiu function under the measure $\mathbb{P}^x$,

$$H(x) = \sum_{i=1}^{m} a_i H(x,i),$$

which replies on the solution to the functions $H(x,i)$.

It can be shown that $H(x,i)$ satisfies the following integro-differential equation

$$\mathfrak{A}H(x,i) - \delta H(x,i) + l(x,i) = 0, \qquad x \geq d, \tag{3.1.4}$$

by which we shall analyze a great variety of ruin-related quantities for the rest of this chapter.

3.1.1 Dividends Paid up to Ruin

We assume that the insurer collects premiums continuously at a constant rate c before the surplus reaches a dividend threshold b. When the surplus runs above b, dividends are paid out continuously to the insurance company's shareholders at a constant rate α and accordingly the surplus increases at a reduced rate $c-\alpha$. As shown in the previous chapter, we take the infinitesimal generator of the deterministic path to be

$$\mathfrak{X} = \begin{cases} (c-\alpha)d/dx, & x \geq b, \\ cd/dx, & 0 \leq x < b. \end{cases}$$

We are interested in the expected present value of dividends paid up to the time of ruin defined by

$$V(x,i) = \mathbb{E}^{(x,i)}\left[\int_0^{\tau_d} e^{-\delta t} l(X_t, J_t)dt\right],$$

where the cost function $l(x,i)$ takes the form

$$l(x,i) = \begin{cases} \alpha, & x \geq b, \\ 0, & 0 \leq x < b. \end{cases} \tag{3.1.5}$$

The jumping mechanism $Q(y;x)$ is assumed to be independent of the current surplus position x and involves only negative jumps. Ruin occurs when the surplus hits zero, which is to say that $d = 0$. Hence, in view of (3.1.2) and (3.1.4), we obtain the following integro-differential equations for $V(x,i)$,

$$(c-\alpha)V'(x,i) - \delta V(x,i) + \sum_{j=1}^{m} \mathbf{\Lambda}_{ij} V(x,j) + \eta_i \sum_{j=1}^{m} a_j \int_0^x V(x-y,j)dQ(y) + \alpha = 0, \qquad x \geq b, \tag{3.1.6}$$

and

$$cV'(x,i) - \delta V(x,i) + \sum_{j=1}^{m} \mathbf{\Lambda}_{ij} V(x,j) + \eta_i \sum_{j=1}^{m} a_j \int_0^x V(x-y,j)dQ(y) = 0, \qquad 0 \leq x < b. \tag{3.1.7}$$

As always, we shall first start with the simplest example where the claim sizes are exponentially distributed, i.e.

$$Q(y) = 1 - e^{-\beta y}, \qquad y \geq 0.$$

Based on past experience with homogeneous equations, we try by inspection a solution to $V(x,i)$ of the form

$$V(x,i) = \begin{cases} k_i e^{\gamma x} + A, & x \geq b, \\ \sum_{n=1}^{m+1} h_{in} e^{\rho_n x}, & 0 \leq x < b, \end{cases} \tag{3.1.8}$$

where $\gamma, A, \rho_1, \rho_2, \cdots, \rho_{m+1}$ are constants to be determined later.

To find these constants, we replace $V(x)$ in (3.1.7) by the lower part of (3.1.8).

$$\sum_{n=1}^{m+1} c\rho_n h_{in} e^{\rho_n x} - \sum_{n=1}^{m+1} \delta h_{in} e^{\rho_n x} + \sum_{n=1}^{m+1} \sum_{j=1}^{m} \mathbf{\Lambda}_{ij} h_{jn} e^{\rho_n x} + \sum_{n=1}^{m+1} \eta_i \sum_{j=1}^{m} a_j \int_0^x h_{jn} e^{\rho_n (x-y)} \beta e^{-\beta y} dy = 0.$$
$$0 \leq x < b.$$

Hence we obtain

$$\sum_{n=1}^{m+1} c\rho_n h_{in} e^{\rho_n x} - \sum_{n=1}^{m+1} \delta h_{in} e^{\rho_n x} + \sum_{n=1}^{m+1} \sum_{j=1}^{m} \mathbf{\Lambda}_{ij} h_{jn} e^{\rho_n x} + \sum_{n=1}^{m+1} \eta_i \sum_{j=1}^{m} a_j h_{jn} e^{\rho_n x} \frac{\beta}{\beta + \rho_n}$$
$$+ \sum_{n=1}^{m+1} \eta_i \sum_{j=1}^{m} a_j h_{jn} e^{-\beta x} \frac{\beta}{\beta + \rho_n} = 0, \qquad 0 \leq x < b.$$

Equating all terms with $e^{\rho_1 x}, e^{\rho_2 x}, \cdots, e^{\rho_m x}, e^{\rho_{m+1} x}$ respectively with zero yields

$$c\rho_n h_{in} - \delta h_{in} + \sum_{j=1}^{m} \mathbf{\Lambda}_{ij} h_{jn} + \frac{\beta}{\beta + \rho_n} \eta_i \sum_{j=1}^{m} a_j h_{jn} = 0, \qquad \text{for } n = 1, 2, \cdots, m+1,$$

which can be written in matrix form as

$$\{\mathbf{\Lambda} - (\delta - c\rho_n)\mathbf{I}\}\mathbf{h}_{\cdot n} + \frac{\beta}{\beta + \rho_n}(\mathbf{a}^T \mathbf{h}_{\cdot n})\eta = 0,$$

where $\mathbf{h}_{\cdot n} = (h_{1n}, h_{2n}, \cdots, h_{mn})^T$.

Equating the rest of terms with $e^{-\beta x}$ with zero gives

$$\sum_{n=1}^{m} \sum_{j=1}^{m} a_j h_{jn} \frac{\beta}{\beta + \rho_n} = 0,$$

which can also be represented as

$$\sum_{n=1}^{m+1} \frac{\beta}{\beta + \rho_n} \mathbf{a}^T \mathbf{h}_{\cdot n} = 0.$$

Denote the constant $\mathbf{a}^T \mathbf{h}_{\cdot n} = D_n$, then

$$\mathbf{h}_{\cdot n} = -D_n \frac{\beta}{\beta + \rho_n} \{\mathbf{\Lambda} - (\delta - c\rho_n)\mathbf{I}\}^{-1} \eta, \qquad \text{for } n = 1, 2, \cdots, m+1, \tag{3.1.9}$$

with the constraint

$$\sum_{n=1}^{m+1} \frac{\beta}{\beta + \rho_n} D_n = 0. \tag{3.1.10}$$

Replacing the expression (3.1.9) for $\mathbf{h}_{\cdot n}$ in $\mathbf{a}^T \mathbf{h}_{\cdot n} = D_n$ gives

$$-\frac{\beta}{\beta + \rho_n} \mathbf{a}^T \Big\{ \mathbf{\Lambda} - (\delta - c\rho_n)\mathbf{I} \Big\}^{-1} \eta = 1. \tag{3.1.11}$$

Comparing (3.1.1) and (3.1.11), one soon recognizes that $\rho_1, \rho_2, \cdots, \rho_{m+1}$ satisfy the famous generalized Lundberg fundamental equation

$$\tilde{q}(s)\tilde{k}(\delta - cs) = 1.$$

By substituting the expression (3.1.8) into (3.1.6) we have

$$
\begin{aligned}
&(c-\alpha)\gamma k_i e^{\gamma x} - \delta k_i e^{\gamma x} + \sum_{j=1}^{m} \mathbf{\Lambda}_{ij} k_j e^{\gamma x} + \frac{\beta}{\beta+\gamma}\left(\sum_{j=1}^{m} a_j k_j\right)\eta_i e^{\gamma x} \\
&-\frac{\beta}{\beta+\gamma}\left(\sum_{j=1}^{m} a_j k_j\right)\eta_i e^{(\beta+\gamma)b} e^{-\beta x} + \eta_i \sum_{n=1}^{m+1} \frac{\beta}{\beta+\rho_n}\left(\sum_{j=1}^{m} a_j h_{jn}\right) e^{(\beta+\rho_n)b} e^{-\beta x} \\
&-\eta_i \sum_{n=1}^{m+1} \frac{\beta}{\beta+\rho_n}\left(\sum_{j=1}^{m} a_j h_{jn}\right) e^{-\beta x} - \eta_i \sum_{j=1}^{m} a_j e^{\beta b} A e^{-\beta x} - \delta A + \sum_{i=1}^{m} \mathbf{\Lambda}_{ij} A + \eta_i \sum_{j=1}^{m} a_j A + \alpha = 0.
\end{aligned}
$$

Recall that by definition $\sum_{j=1}^{m} a_j = 1$ and $\sum_{j=1}^{m} \mathbf{\Lambda}_{ij} + \eta_i = 0$. Thus we obtain by combining the last four constant terms that $A = \alpha/\delta$. Equating all terms with $e^{\gamma x}$ and $e^{-\beta x}$ yields

$$
(c-\alpha)\gamma k_i - \delta k_i + \sum_{j=1}^{m} \mathbf{\Lambda}_{ij} k_j + \frac{\beta}{\beta+\gamma}\left(\sum_{j=1}^{m} a_j k_j\right)\eta_i = 0
$$

and

$$
\begin{aligned}
-\frac{\beta}{\beta+\gamma}\left(\sum_{j=1}^{m} a_j k_j\right) &= -\sum_{n=1}^{m+1} \frac{\beta}{\beta+\rho_n}\left(\sum_{j=1}^{m} a_j h_{jn}\right) e^{(\rho_n-\gamma)b} \\
&+\sum_{n=1}^{m+1} \frac{\beta}{\beta+\rho_n}\left(\sum_{j=1}^{m} a_j h_{jn}\right) e^{-(\beta+\gamma)b} + \sum_{j=1}^{m} a_j A e^{-\gamma b}
\end{aligned}
$$

which can be written in matrix forms as

$$
\Big\{\mathbf{\Lambda} - [\delta - (c-\alpha)\gamma]\mathbf{I}\Big\}\mathbf{k} + \frac{\beta}{\beta+\gamma}(\mathbf{a}^T\mathbf{k})\eta = 0, \tag{3.1.12}
$$

and

$$
-\frac{\beta}{\beta+\gamma}(\mathbf{a}^T\mathbf{k}) = Ae^{-\gamma b} + \mathbf{a}^T\mathbf{h}\mathbf{d}_1 - \mathbf{a}^T\mathbf{h}\mathbf{d}_2, \tag{3.1.13}
$$

where $\mathbf{k} = (k_1, k_2, \cdots, k_m)^T$, $\mathbf{h} = (h_{in})_{m\times(m+1)}$,

$$
\mathbf{d}_1(b) = \left(\frac{\beta}{\beta+\gamma}e^{-(\beta+\gamma)b}, \frac{\beta}{\beta+\rho_2}e^{-(\beta+\gamma)b}, \cdots, \frac{\beta}{\beta+\rho_{m+1}}e^{-(\beta+\gamma)b}\right)^T
$$

and

$$
\mathbf{d}_2(b) = \left(\frac{\beta}{\beta+\gamma}e^{(\rho_1-\gamma)b}, \frac{\beta}{\beta+\rho_2}e^{(\rho_2-\gamma)b}, \cdots, \frac{\beta}{\beta+\rho_{m+1}}e^{(\rho_{m+1}-\gamma)b}\right)^T.
$$

It can be verified that $\mathbf{\Lambda} - [\delta - (c-\alpha)\rho]\mathbf{I}$ is invertible (c.f. Jacobsen [29]). Note that $\mathbf{a}^T\mathbf{k}$ is a constant. Hence in light of (3.1.12), (3.1.13) and the fact that $A = \alpha/\delta$, the solution to $\mathbf{k}$ is given by

$$\mathbf{k} = [\frac{\alpha}{\delta}e^{-\gamma b} + \mathbf{a}^T\mathbf{h}\mathbf{d}_1(b) - \mathbf{a}^T\mathbf{h}\mathbf{d}_2(b)]\Big\{\mathbf{\Lambda} - [\delta - (c-\alpha)\gamma]\mathbf{I}\Big\}^{-1}\eta. \tag{3.1.14}$$

In light of the fact that $\mathbf{a}^T\mathbf{h}_{\cdot n} = D_n$, we must have $\mathbf{a}^T\mathbf{h}\mathbf{d}_1(b) = \mathbf{D}^T\mathbf{d}_1(b)$ and $\mathbf{a}^T\mathbf{h}\mathbf{d}_2(b) = \mathbf{D}^T\mathbf{d}_2(b)$, where $\mathbf{D} = (D_1, D_2, \cdots, D_{m+1})^T$.

Inserting (3.1.14) into either (3.1.12) or (3.1.13) yields

$$-\frac{\beta}{\beta+\rho}\mathbf{a}^T\Big\{\mathbf{\Lambda} - [\delta - (c-\alpha)\gamma]\mathbf{I}\Big\}^{-1}\eta = 1. \tag{3.1.15}$$

Apparently, γ has to satisfies (3.1.15) in order to make both (3.1.12) and (3.1.13) consistent. Comparing (3.1.1) and (3.1.15), one soon recognize that γ satisfies the famous generalized Lundberg fundamental equation

$$\tilde{q}(s)\tilde{k}[\delta - (c-\alpha)s] = 1.$$

Therefore, the solution to the dividends up to ruin for $x \geq b$ can be written in matrix form,

$$\mathbf{V}(x) = \frac{\alpha}{\delta}\mathbf{1} + [\frac{\alpha}{\delta}e^{-\gamma b} + \mathbf{D}^T\mathbf{d}_1(b) - \mathbf{D}^T\mathbf{d}_2(b)]\Big\{\mathbf{\Lambda} - [\delta - (c-\alpha)\gamma]\mathbf{I}\Big\}^{-1}\eta e^{\gamma x}, \qquad x \geq b, \tag{3.1.16}$$

$$\mathbf{V}(x) = -\sum_{n=1}^{m+1} D_n \frac{\beta}{\beta+\rho_n}\{\mathbf{\Lambda} - (\delta - c\rho_n)\mathbf{I}\}^{-1}\eta e^{\rho_n x}, \qquad 0 \leq x < b. \tag{3.1.17}$$

where $\mathbf{V}(x) = \Big(V(x,1), V(x,2), \cdots, V(x,m)\Big)^T$ and $\mathbf{1} = (1, 1, \cdots, 1)^T$.

Since both (3.1.16) and (3.1.17) contain $m+1$ unknown constants, we now need $m+1$ linear equations. By continuity condition that $\mathbf{V}(b-) = \mathbf{V}(b+)$, we have

$$\frac{\alpha}{\delta}\mathbf{1} + \left[\frac{\alpha}{\delta} + \sum_{n=1}^{m+1} D_n \frac{\beta}{\beta+\rho_n}(e^{-\beta b} - e^{\rho_n b})\right]\Big\{\mathbf{\Lambda} - [\delta - (c-\alpha)\gamma]\mathbf{I}\Big\}^{-1}\eta$$
$$= -\sum_{n=1}^{m+1} D_n \frac{\beta}{\beta+\rho_n}e^{\rho_n b}\{\mathbf{\Lambda} - (\delta - c\rho_n)\mathbf{I}\}^{-1}\eta. \tag{3.1.18}$$

Now that we obtain (3.1.18) and (3.1.10), $D_1, D_2, \cdots, D_{m+1}$ can be determined.

We can recover the classical result of dividends up to ruin in the compound Poisson model by letting $m = 1$ in our model. Thus $\alpha = (1), \mathbf{\Lambda} = (-\lambda), \eta = (\lambda), \mathbf{D} = (D_1, D_2)^T$.

Together with (3.1.11) and (3.1.15), equation (3.1.18) can be simplified as

$$-\frac{\alpha\gamma}{\delta\beta} - \left(\frac{\beta+\gamma}{\beta+\rho_1}e^{-\beta b} - \frac{\beta+\gamma}{\beta+\rho_1}e^{\rho_1 b}\right) D_1 - \left(\frac{\beta+\gamma}{\beta+\rho_2}e^{-\beta b} - \frac{\beta+\gamma}{\beta+\rho_2}e^{\rho_2 b}\right) D_2 = e^{\rho_1 b} D_1 + e^{\rho_2 b} D_2.$$

And equation (3.1.10) reduces to

$$\frac{\beta}{\beta+\rho_1} D_1 + \frac{\beta}{\beta+\rho_2} D_2 = 0.$$

Solving the two linear equations yields the solution,

$$\begin{aligned} D_1 &= -\frac{\alpha\gamma(\beta+\rho_1)}{\delta\beta[(\rho_1-\gamma)e^{\rho_1 b} - (\rho_2-\gamma)e^{\rho_2 b}]}, \\ D_2 &= \frac{\alpha\gamma(\beta+\rho_2)}{\delta\beta[(\rho_1-\gamma)e^{\rho_1 b} - (\rho_2-\gamma)e^{\rho_2 b}]}. \end{aligned}$$

Inserting D_1 and D_2 into (3.1.16) and (3.1.17) gives

$$\begin{aligned} V(x) &= \frac{\alpha}{\delta} - \frac{\alpha(\beta+\gamma)}{\delta\beta}e^{\gamma(x-b)} - \frac{\alpha\gamma(\beta+\gamma)(e^{\rho_1 b} - e^{\rho_2 b})}{\delta\beta[(\rho_1-\gamma)e^{\rho_1 b} - (\rho_2-\gamma)e^{\rho_2 b}]}e^{\gamma(x-b)}, \qquad x \geq b, \\ V(x) &= \frac{-\alpha\gamma}{\delta\beta}\frac{(\beta+\rho_1)e^{\rho_1 x} - (\beta+\rho_2)e^{\rho_2 x}}{(\rho_1-\gamma)e^{\rho_1 b} - (\rho_2-\gamma)e^{\rho_2 b}}, \qquad 0 \leq x < b, \end{aligned}$$

which are exactly equation (1.2.20) and (1.2.19).

3.1.2 Total Discounted Claim Costs up to Ruin

We take as given a $\mathcal{B}(\mathbb{R})$-measurable function $\varpi[(x,i),(y,j)]$ that determines the expenses of each insurance claim according to the surplus position prior to the claim arrival (x,i) and the resulting surplus position (y,j). Hence we define the expected present value of total claim expenses up to the time of ruin by

$$A(x,i) = \mathbb{E}^{(x,i)} \left\{ \sum_{k=1}^{N} e^{-\delta T_k} \varpi[(X_{T_{k-}}, J_{T_{k-}}), (X_{T_k}, J_{T_k})] \right\},$$

where $\delta \geq 0$ and $N = \max\{k : T_k \leq \tau_0\}$.

Using similar arguments as in Theorem 2.2.1, we can show that

$$A(x,i) = \mathbb{E}^{(x,i)} \{ e^{-\delta(T_1 \wedge \tau_0)} \varpi[(X_{T_1-}, i), (X_{T_1}, J_{T_1})] + e^{-\delta(T_1 \wedge \tau_0)} A(X_{T_1}, J_{T_1}) \}.$$

The first term can be written in terms of Lebesgue-Stieljes integral. Thus,

$$A(x,i) = \mathbb{E}^{(x,i)} \left\{ \sum_{j \in E} \int_0^{\tau_0} \int_0^{\infty} e^{-\delta t} \varpi[(X_{t-}, i), (X_{t-} - y, j)] H_j(dy, dt) \right\} + \mathbb{E}^{(x,i)} \{ e^{-\delta(T_1 \wedge \tau_0)} A(X_{T_1}, J_{T_1}) \},$$

where $\{H_j, j \in E\}$ are mutually independent single jump processes defined by

$$H_j(A, t) = a_j I(t \geq T_1) Q(A).$$

Since T_1 is governed by exponential distribution with mean $1/\eta_i$, by Theorem 17 (Chapter 5, Protter [43]) we have the compensator of H

$$\tilde{H}_j(A, t) = \eta_i a_j Q(A)(t \wedge T_1).$$

Therefore,

$$\begin{aligned} A(x,i) &= \mathbb{E}^{(x,i)} \left\{ \sum_{j \in E} \int_0^{\tau_0} \int_0^{\infty} e^{-\delta t} \varpi[(X_{t-}, i), (X_{t-} - y, j)] \tilde{H}_j(dy, dt) \right\} + \mathbb{E}^{(x,i)} \{ e^{-\delta(T_1 \wedge \tau_0)} A(X_{T_1}, J_{T_1}) \} \\ &= \mathbb{E}^{(x,i)} \left\{ \int_0^{T_1 \wedge \tau_0} e^{-\delta t} \int_0^{\infty} \sum_{j \in E} \eta_i a_j \varpi[(X_{t-}, i), (X_{t-} - y, j)] Q(dy) dt \right\} + \mathbb{E}^{(x,i)} \{ e^{-\delta(T_1 \wedge \tau_0)} A(X_{T_1}, J_{T_1}) \}. \end{aligned}$$

Now we compare it with a similar equation for $V(x,i)$ obtained by strong Markov property

$$V(x,i) = \mathbb{E}^{(x,i)} \left\{ \int_0^{T_1 \wedge \tau_0} e^{-\delta t} l(X_t, i) dt \right\} + \mathbb{E}^{(x,i)} \{ e^{-\delta(T_1 \wedge \tau_0)} V(X_{T_1}, J_{T_1}) \}.$$

It becomes obvious that $A(x)$ can be obtained by taking the following cost function in (3.1.3),

$$l(x,i) = \int_0^{\infty} \sum_{j \in E} \eta_i a_j \varpi[(x,i), (x - y, j)] Q(dy). \tag{3.1.19}$$

Hence in the classical case with constant premium rate c, the integro-differential equations for $A(x,i)$ are given by

$$cA'(x,i) - \delta A(x,i) + \sum_{j=1}^{m} \mathbf{\Lambda}_{ij} A(x,j) + \eta_i \sum_{j=1}^{m} a_j \int_0^x A(x-y,j)\,dQ(y)$$
$$+\eta_i \sum_{j=1}^{m} a_j \int_0^{\infty} \varpi[(x,i),(x-y,j)]\,dQ(y) = 0, \qquad x \geq 0. \tag{3.1.20}$$

3.1.3 Gerber-Shiu Function Depending on Deficit Only

As we shown in Cai *et al.* [9], the Gerber-Shiu expected discounted penalty is a special case of the expected total discounted claim expenses. For simplicity, we choose

$$\varpi[(x,i),(y,j)] = g(-y)I(y<0).$$

Then the total claim expenses reduce to

$$m(x,i) = \mathbb{E}^{(x,i)}[e^{-\delta\tau_0} g(|X_{\tau_0}|)],$$

which is the Gerber-Shiu function depending on the deficit at ruin only.

We shall as well illustrate this example by taking the simplest assumption that the claim sizes are exponentially distributed, i.e.

$$Q(y) = 1 - e^{-\beta y}, \qquad y \geq 0.$$

By inspection, we search for the solution in the form of

$$m(x,i) = l_i e^{\gamma x}, \qquad x \geq 0, \tag{3.1.21}$$

where γ is a constant to be determined later.

Inserting (3.1.21) into (3.1.20) yields,

$$c\gamma l_i e^{\gamma x} - \delta l_i e^{\gamma x} + \sum_{j=1}^{m} \mathbf{\Lambda}_{ij} l_j e^{\gamma x} + \frac{\beta}{\beta+\gamma}\eta_i \left(\sum_{j=1}^{m} a_j l_j\right) e^{\gamma x}$$
$$-\frac{\beta}{\beta+\gamma}\eta_i \left(\sum_{j=1}^{m} a_j l_j\right) e^{-\beta x} + \eta_i \beta \left[\int_0^{\infty} g(z) e^{-\beta z} dz\right] e^{-\beta x} = 0.$$

By equating all terms with $e^{\gamma x}$ and $e^{-\beta x}$ with zero, we obtain the following equations in matrix form.

$$c\gamma\mathbf{l} - \delta\mathbf{l} + \mathbf{\Lambda}\mathbf{l} + \frac{\beta}{\beta+\gamma}(\mathbf{a}^T\mathbf{l})\eta = 0,$$

and

$$\frac{\beta}{\beta+\gamma}(\mathbf{a}^T\mathbf{l})\eta = \beta\left[\int_0^\infty g(z)e^{-\beta z}dz\right]\eta,$$

where $\mathbf{l} = (l_1, l_2, \cdots, l_m)^T$.

Hence we obtain the coefficient vector

$$\mathbf{l} = -\beta\left[\int_0^\infty g(z)e^{-\beta z}dz\right]\{\mathbf{\Lambda} - (\delta - c\gamma)\mathbf{I}\}^{-1}\eta,$$

where γ is the unique non-negative solution to the Lundberg equation

$$-\frac{\beta}{\beta+\gamma}\mathbf{a}^T\{\mathbf{\Lambda} - (\delta - c\gamma)\mathbf{I}\}^{-1}\eta = 1.$$

3.1.4 Insurer's Accumulated Utility

Another attraction of the generalized Gerber-Shiu function is the admission of an insurer's accumulated utility, which in the case of indexed compound Poisson process can be defined as

$$U(x,i) \triangleq \mathbb{E}^{(x,i)}\left[\int_0^{\tau_0} u(X_t, J_t)dt\right],$$

where $u(x,i)$ measures an insurer's utility of current surplus reserve. The most frequently quoted utility function is the exponential utility function,

$$u(x,i) = -\frac{1}{a}e^{-ax}, \qquad \text{for all i's,}$$

which implies constant absolute risk aversion. For notational brevity, we shall denote

$$W(x,i) \triangleq \mathbb{E}^{(x,i)}\left[\int_0^{\tau_0} e^{-aX_t}dt\right].$$

Once the expression $W(x,i)$ is determined, we can find $U(x,i) = -W(x,i)/a$.

Thus by (3.1.4) we have the integro-differential equation for $W(x,i)$,

$$cW'(x,i) + \sum_{j=1}^{m} \mathbf{\Lambda}_{ij} W(x,j) + \eta_i \sum_{j=1}^{m} a_j \int_0^x A(x-y,j)dQ(y) + e^{-ax} = 0, \qquad x \geq 0. \tag{3.1.22}$$

Under the exponential claim size assumption, we search for the solution to the accumulated utility of the form

$$W(x,i) = A_i e^{\rho x} + B_i e^{-ax} + D. \tag{3.1.23}$$

Substituting (3.1.23) for $W(x,i)$ in (3.1.22) gives

$$\begin{aligned} c\rho A_i e^{\rho x} - acB_i e^{-ax} + \sum_{j=1}^{m} \mathbf{\Lambda}_{ij} A_j e^{\rho x} + \sum_{j=1}^{m} \mathbf{\Lambda}_{ij} B_j e^{-ax} + D \sum_{j=1}^{m} \mathbf{\Lambda}_{ij} \\ + \eta_i \frac{\beta}{\beta+\rho} \left(\sum_{j=1}^{m} a_j A_j \right) e^{\rho x} - \eta_i \frac{\beta}{\beta+\rho} \left(\sum_{j=1}^{m} a_j A_j \right) e^{-\beta x} + \eta_i \frac{\beta}{\beta-a} \left(\sum_{j=1}^{m} a_j B_j \right) e^{-ax} \\ - \eta_i \frac{\beta}{\beta-a} \left(\sum_{j=1}^{m} a_j B_j \right) e^{-\beta x} + D\eta_i \sum_{j=1}^{m} a_j + D\eta_i \sum_{j=1}^{m} a_j e^{-\beta x} + e^{-ax} = 0. \end{aligned}$$

Letting $\mathbf{A} = (A_1, A_2, \cdots, A_m)^T, \mathbf{B} = (B_1, B_2, \cdots, B_m)^T$ and equating all terms with $e^{\rho x}, e^{-ax}$ and $e^{-\beta x}$ gives,

$$\begin{aligned} c\rho \mathbf{A} + \mathbf{\Lambda} \mathbf{A} + \frac{\beta}{\beta+\rho} \eta \mathbf{a}^T \mathbf{A} = 0, \\ \frac{\beta}{\beta+\rho} \mathbf{a}^T \mathbf{A} + \frac{\beta}{\beta-a} \mathbf{a}^T \mathbf{B} = -D, \\ -ac\mathbf{B} + \mathbf{\Lambda} \mathbf{B} + \frac{\beta}{\beta-a} \mathbf{a}^T \mathbf{B} \eta + \mathbf{1} = 0. \end{aligned}$$

Again we investigate how the compound Poisson model can be recovered from the above analysis. Let $m = 1, \mathbf{\Lambda} = (-\lambda), \mathbf{a} = (1), \eta = (\lambda)$. Thus the above system of equations become

$$\begin{aligned} c\rho A - \lambda A + \frac{\beta}{\beta+\rho} \lambda A = 0, \\ \frac{\beta}{\beta+\rho} A + \frac{\beta}{\beta-a} B = -D, \\ -acB - \lambda B + \lambda \frac{\beta}{\beta-a} B + 1 = 0. \end{aligned}$$

Hence

$$B = \frac{a-\beta}{ac(a-\beta+\lambda/c)},$$

the solution the lundberg equation is $\rho = 0$ or $\lambda/c - \beta$. When the safety loading condition is satisfied, i.e. $c > \lambda/\beta$, it can be proved that $W(\infty) = 0$, which implies $D = 0$. Then we arrive at the solution to the insurer's accumulated utility when $c > \lambda/\beta$,

$$W(x) = \frac{\lambda}{ac^2(a-\beta+\lambda/c)} e^{-(\beta-\lambda/c)x} + \frac{a-\beta}{ac(a-\beta+\lambda/c)} e^{-ax}.$$

On the other hand, if the safety loading condition is violated, since the accumulated utility function is still bounded, then A has to be zero. Therefore, when $c \leq \lambda/\beta$,

$$W(x) = \frac{\beta}{ac(a-\beta+\lambda/c)} + \frac{a-\beta}{ac(a-\beta+\lambda/c)} e^{-ax}.$$

3.2 Generalized Erlang-n Inter-claim Risk Models

The model assumes that all inter-claim time random variables are identically distributed with the generalized Erlang-n distribution, which is equivalent to say that each inter-claim time V_i is a sum of n independent exponentially distributed random variables with parameters $\lambda_1, \lambda_2, \cdots, \lambda_n$. In the context of phase-type distribution, we can treat each exponential random variable as the time the Markov chain J stays in a transient state and it must go through each state consecutively before it reaches the absorption state and regenerates thereafter. Hence, we define $\mathbf{a} = (1, 0, 0, \cdots, 0)^T$ and the corresponding sub-intensity matrix $\mathbf{\Lambda}$ can be written as

$$\mathbf{\Lambda} = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & \lambda_2 & \cdots & 0 \\ 0 & 0 & -\lambda_3 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & 0 & \cdots & -\lambda_n \end{pmatrix}.$$

Hence the absorption vector η is given by $\eta = (0, 0, 0, \cdots, \lambda_n)^T$. One can obtain from (3.1.1) that the Laplace transform of the inter-claim time distribution is given by

$$\tilde{k}(s) = \prod_{i=1}^{n} \frac{\lambda_i}{\lambda_i + s}.$$

In view of (3.1.2) and (3.1.4), we obtain the following system of integro-differential equations for $V(x, i), i = 1, 2, \cdots, n$.

$$\mathfrak{X}V(x,i) - (\lambda_i + \delta)V(x,i) + \lambda_i V(x, i+1) + l(x,i) = 0, \qquad i = 1, 2, \cdots, n-1$$
$$\mathfrak{X}V(x,n) - (\lambda_n + \delta)V(x,n) + \lambda_n \int_{-\infty}^{\infty} V(y,1)dQ(y;x) + l(x,n) = 0.$$

Rearranging the equations gives

$$V(x, i+1) = \left[\left(1 + \frac{\delta}{\lambda_i}\right)\mathcal{I} - \frac{1}{\lambda_i}\mathfrak{X}\right] V(x,i) - \frac{1}{\lambda_i} l(x,i), \qquad \text{for } i = 1, 2, \cdots, n-1, \tag{3.2.1}$$

$$\left[\left(1 + \frac{\delta}{\lambda_n}\right)\mathcal{I} - \frac{1}{\lambda_n}\mathfrak{X}\right] V(x,n) = \int_{-\infty}^{\infty} V(y,1)dQ(y,x) + \frac{1}{\lambda_i} l(x,n). \tag{3.2.2}$$

As specified by the generalized Erlang-n inter-claim time distribution, the surplus process must start with $(x, 1)$ and hence we are interested in particular the generalized Gerber-Shiu function $V(x, 1)$. Iterative substitution by (3.2.1) into (3.2.2) leads to

$$\begin{aligned} &\prod_{i=1}^{n} \left[\left(1 + \frac{\delta}{\lambda_i}\right)\mathcal{I} - \frac{1}{\lambda_i}\mathfrak{X}\right] V(x,1) \\ &= \int_{-\infty}^{\infty} V(y,1)dQ(y;x) + \sum_{i=1}^{n} \left\{ \prod_{k=i+1}^{n} \left[\left(1 + \frac{\delta}{\lambda_k}\right)\mathcal{I} - \frac{1}{\lambda_k}\mathfrak{X}\right] \frac{1}{\lambda_i} l(x,i) \right\} \end{aligned} \tag{3.2.3}$$

with the convention that $\prod_{k=n+1}^{n} \cdot = 1$.

3.2.1 Dividends Paid up to Ruin with Two-sided Jumps

Following the same notion of dividend threshold policy in Section 3.1.1, we take the infinitesimal generator of the deterministic path to be

$$\mathfrak{X} = \begin{cases} (c - \alpha)d/dx = (c - \alpha)\mathcal{D}, & x \geq b, \\ cd/dx = c\mathcal{D}, & 0 \leq x < b. \end{cases}$$

The cost function for dividends paid up to ruin is given regardless of the index of surplus process by (3.1.5). As always in traditional Sparre Andersen model, the jumping mechanism $Q(y; x)$ is assumed to be independent of the current surplus position x. The ordinary ruin level is set to be $d = 0$. To make it slightly more general than the dividends paid up to ruin covered in Wang and Dong [47] and Albrecher *et al.* [1], we assume that the surplus process jumps either upwards or downwards at random according to

$$Q(y) = \pi Q^{+}(y)I(y \geq 0) + (1-\pi)[1 - Q^{-}(-y)]I(y < 0).$$

Therefore, (3.2.3) reduces to the following system of integro-differential equations,

$$\begin{aligned}\prod_{i=1}^{n}\left[\frac{\lambda_i+\delta}{c-\alpha}\mathcal{I}-\mathcal{D}\right]V(x,1) &= \frac{\prod_{i=1}^{n}\lambda_i}{(c-\alpha)^n}\int_0^\infty V(x-y,1)dQ(y)\\ &\quad +\frac{\alpha}{(c-\alpha)^n}\sum_{i=1}^{n}\prod_{k=1}^{i-1}\lambda_k\prod_{k=i+1}^{n}(\lambda_k+\delta), \qquad x \geq b; \qquad (3.2.4)\\ \prod_{i=1}^{n}\left[\frac{\lambda_i+\delta}{c}\mathcal{I}-\mathcal{D}\right]V(x,1) &= \frac{\prod_{i=1}^{n}\lambda_i}{c^n}\int_0^\infty V(x-y,1)dQ(y),\\ &\qquad 0 \leq x < b. \qquad (3.2.5)\end{aligned}$$

When $\pi = 0$, equation (3.2.4) and (3.2.5) are precisely (2.12) and (2.13) in Wang and Dong [47] derived using traditional probabilistic arguments, and (3.2.5) is the same as (9) in the case of $m = 1$ in Albrecher *et al.* [1].

Since the Sparre Andersen model with generalized Erlang-n claim sizes is a special case of the Jacobsen model, the same technique in Section 3.1.1 would enable us to obtain general solutions for $V(x, 1)$. To avoid repetitive derivation, we shall illustrate the explicit solution to $V(x, 1)$ in an example where traditional ordinary differential equation approach applies. Assume that the random jump is governed by a mixture of two exponential distributions corresponding to insurance claims and unexpected investment returns respectively,

$$Q(y) = \pi(1 - e^{-\beta_1 y})I(y \geq 0) + [(1-\pi) - (1-\pi)(1 - e^{\beta_2 y})I(y < 0)].$$

Hence, we can write (3.2.4) and (3.2.5) in terms of operators

$$\prod_{i=1}^{n}\left[\frac{\lambda_i+\delta}{c-\alpha}\mathcal{I}-\mathcal{D}\right]V(x,1)=\frac{\prod_{i=1}^{n}\lambda_i}{(c-\alpha)^n}\Big[\pi\beta_1\mathcal{T}_{\beta_1}V(x,1)+(1-\pi)\beta_2\mathcal{E}_{\beta_2}V(x,1)\Big]$$
$$+\frac{\alpha}{(c-\alpha)^n}\sum_{i=1}^{n}\prod_{k=1}^{i-1}\lambda_k\prod_{k=i+1}^{n}(\lambda_k+\delta),\qquad x\geq b;$$

$$\prod_{i=1}^{n}\left[\frac{\lambda_i+\delta}{c}\mathcal{I}-\mathcal{D}\right]V(x,1)=\frac{\prod_{i=1}^{n}\lambda_i}{c^n}\Big[\pi\beta_1\mathcal{T}_{\beta_1}V(x,1)+(1-\pi)\beta_2\mathcal{E}_{\beta_2}V(x,1)\Big],\qquad 0\leq x\leq b,$$

where the Dickson-Hipp operator $\mathcal{T}_s$ and the exponential convolution operator $\mathcal{E}_s$ are defined in Section 1.1. Recall that $(s\mathcal{I}+\mathcal{D})\mathcal{T}_s=\mathcal{I}$ and $(s\mathcal{I}-\mathcal{D})\mathcal{E}_s=\mathcal{I}$. Thus

$$(\beta_1\mathcal{I}-\mathcal{D})(\beta_2\mathcal{I}+\mathcal{D})\prod_{i=1}^{n}\left\{\frac{\lambda_i+[\delta-(c-\alpha)\mathcal{D}]}{\lambda_i}\right\}V(x,1)=\pi\beta_1(\beta_2\mathcal{I}+\mathcal{D})V(x,1)$$
$$+(1-\pi)\beta_2(\beta_1\mathcal{I}-\mathcal{D})V(x,1)+\beta_1\beta_2\alpha\sum_{i=1}^{n}\left[\frac{1}{\lambda_i}\prod_{k=i+1}^{n}\left(1+\frac{\delta}{\lambda_k}\right)\right],\qquad x\geq b;\quad (3.2.6)$$
$$(\beta_1\mathcal{I}-\mathcal{D})(\beta_2\mathcal{I}-\mathcal{D})\prod_{i=1}^{n}\left\{\frac{\lambda_i+[\delta-(c-\alpha)\mathcal{D}]}{\lambda_i}\right\}V(x,1)=\pi\beta_1(\beta_2\mathcal{I}+\mathcal{D})V(x,1)$$
$$+(1-\pi)\beta_2(\beta_1\mathcal{I}-\mathcal{D})V(x,1),\qquad 0\leq x\leq b,\quad (3.2.7)$$

It is obvious that a constant C must be a particular solution to (3.2.6) if it satisfies

$$\prod_{i=1}^{n}\left(1+\frac{\delta}{\lambda_i}\right)C=C+\alpha\sum_{i=1}^{n}\left[\frac{1}{\lambda_i}\prod_{k=i+1}^{n}\left(1+\frac{\delta}{\lambda_k}\right)\right].$$

It can easily be proved by mathematical induction that

$$\prod_{i=1}^{n}(\lambda_i+\delta)-\prod_{i=1}^{n}\lambda_i=\delta\sum_{i=1}^{n}\left\{\prod_{k=i+1}^{n}(\lambda_k+\delta)\prod_{j=1}^{i-1}\lambda_j\right\}.\qquad (3.2.8)$$

Hence, we find that $C=\alpha/\delta$ is a particular solution to (3.2.6). The fundamental solution to the homogeneous differential equation corresponding to (3.2.6) is given by $\sum_{i=1}^{n+2}e^{s_i}$ where $s_1,s_2,\cdots,s_{n+2}$ are roots of the characteristic equation (i.e. Lundberg equation in ruin context)

$$\prod_{i=1}^{n}\left\{\frac{\lambda_i+[\delta-(c-\alpha)s]}{\lambda_i}\right\}=\pi\frac{\beta_1}{(\beta_1-s)}+(1-\pi)\frac{\beta_2}{\beta_2+s}.\qquad (3.2.9)$$

In light of the fact that

$$\lim_{x\to\infty} V(x,1) = \frac{\alpha}{\delta},$$

we must have

$$V(x,1) = ke^{-\rho x} + \frac{\alpha}{\delta}, \qquad x \geq b, \tag{3.2.10}$$

where $-\rho$ is the unique negative root to the Lundberg equation (3.2.9) and k is to be determined. Substituting (3.2.10) for $V(x,1)$ in (3.2.4) yields,

$$\begin{aligned}
&\prod_{i=1}^{n}\left[\frac{\lambda_i+\delta}{c-\alpha}+\rho\right]ke^{-\rho x} + \frac{\alpha\prod_{i=1}^{n}(\lambda_i+\delta)}{\delta(c-\alpha)^n} = \frac{\prod_{i=1}^{n}\lambda_i}{(c-\alpha)^n}\frac{\pi\beta_1}{\beta_1+\rho}ke^{-\rho x} + \pi\frac{\alpha\prod_{i=1}^{n}\lambda_i}{\delta(c-\alpha)^n} \\
&+\frac{\prod_{i=1}^{n}\lambda_i}{(c-\alpha)^n}\frac{(1-\pi)\beta_2}{\beta_2-\rho}ke^{-\rho x} - \frac{\prod_{i=1}^{n}\lambda_i}{(c-\alpha)^n}\frac{(1-\pi)\beta_2}{\beta_2-\rho}ke^{(\beta_2-\rho)b}e^{-\beta_2 x} + (1-\pi)\frac{\alpha\prod_{i=1}^{n}\lambda_i}{\delta(c-\alpha)^n} \\
&-(1-\pi)\frac{\alpha\prod_{i=1}^{n}\lambda_i}{\delta(c-\alpha)^n}e^{\beta_2 b}e^{-\beta_2 x} + \frac{\prod_{i=1}^{n}\lambda_i}{(c-\alpha)^n}(1-\pi)\sum_{i=1}^{n+2}h_i\frac{\beta_2}{\beta_2+\rho_i}e^{(\beta_2+\rho_i)b}e^{-\beta_2 x} \\
&-\frac{\prod_{i=1}^{n}\lambda_i}{(c-\alpha)^n}(1-\pi)\sum_{i=1}^{n+2}h_i\frac{\beta_2}{\beta_2+\rho_i}e^{-\beta_2 x} + \frac{\alpha}{(c-\alpha)^n}\sum_{i=1}^{n}\prod_{k=1}^{i-1}\lambda_k\prod_{k=i+1}^{n}(\lambda_k+\delta).
\end{aligned}$$

All the terms with $e^{-\rho x}$ cancel out thanks to (3.2.9) and all the constant terms collapse to zero because of (3.2.8). The only two terms left and both involving $e^{-\beta_2 x}$ gives

$$k = \frac{\rho-\beta_2}{\beta_2}[\frac{\alpha}{\delta}e^{\rho b} + \sum_{i=1}^{n+2}h_i\frac{\beta_2}{\beta_2+\rho_i}e^{(\rho-\beta_2)b} - \sum_{i=1}^{n+2}h_i\frac{\beta_2}{\beta_2+\rho_i}e^{(\rho+\rho_i)b}].$$

Since (3.2.7) is a homogeneous differential equation, the solution to $V(x,1), 0 \leq x < b$ must be in the form of

$$V(x,1) = \sum_{i=1}^{n+2}h_i e^{\rho_i x}, \qquad 0 \leq x < b, \tag{3.2.11}$$

where $\rho_i, i = 1, \cdots, n+2$ are roots of its characteristic (Lundberg) equation

$$\prod_{i=1}^{n}\left\{\frac{\lambda_i+[\delta-cs]}{\lambda_i}\right\} = \pi\frac{\beta_1}{(\beta_1-s)} + (1-\pi)\frac{\beta_2}{\beta_2+s}.$$

Inserting (3.2.11) back into (3.2.5) one finds that the coefficient of $e^{-\beta_2 x}$ must be equal to zero, which yields that

$$\frac{1}{\beta_2+\rho_1}h_1+\frac{1}{\beta_2+\rho_2}h_2+\cdots+\frac{1}{\beta_2+\rho_n}h_n+\frac{1}{\beta_2+\rho_{n+1}}h_{n+1}+\frac{1}{\beta_2+\rho_{n+2}}h_{n+2}=0. \tag{3.2.12}$$

Observe from (3.2.1) that

$$V(x,j+1)=\prod_{i=1}^{j}\left[\left(1+\frac{\delta}{\lambda_i}\right)\mathcal{I}-\frac{1}{\lambda_i}\mathfrak{X}\right]V(x,1)-\sum_{i=1}^{j}\left\{\prod_{k=i+1}^{j}\left[\left(1+\frac{\delta}{\lambda_k}\right)\mathcal{I}-\frac{1}{\lambda_k}\mathfrak{X}\right]\frac{1}{\lambda_i}l(x,i)\right\},$$

for $j=1,2,\cdots,n-1$. Because it is obvious by definition that $V(x,j+1)$ is continuous for all $x\geq 0, j=0,1,2,\cdots,n-1$, we must have

$$\prod_{i=1}^{j}\left[(\lambda_i+\delta)\mathcal{I}-(c-\alpha)\mathcal{D}\right]V(b+,1)-\alpha\sum_{i=1}^{j}\left\{\prod_{k=i+1}^{j}(\lambda_k+\delta)\prod_{k=1}^{i-1}\lambda_k\right\} \tag{3.2.13}$$

$$=\prod_{i=1}^{j}\left[(\lambda_i+\delta)\mathcal{I}-c\mathcal{D}\right]V(b-,1). \tag{3.2.14}$$

It follows from (3.2.3) that the above identity works for $j=n$ as well.

Inserting the expression (3.2.10) and (3.2.11) in (3.2.14) gives for $j=0,1,\cdots,n$

$$\prod_{i=1}^{j}[\lambda_i+\delta+(c-\alpha)\rho]ke^{-\rho b}+\prod_{i=1}^{j}(\lambda_i+\delta)\frac{\alpha}{\delta}-\alpha\sum_{i=1}^{j}\left\{\prod_{k=i+1}^{j}(\lambda_k+\delta)\prod_{k=1}^{i-1}\lambda_k\right\}$$
$$=\sum_{l=1}^{n+2}\prod_{i=1}^{j}(\lambda_i+\delta-c\rho_l)h_l e^{\rho_l b}.$$

In light of (3.2.8), we obtain for $j=0,1,\cdots,n$

$$\sum_{l=1}^{n+2}\prod_{i=1}^{j}(\lambda_i+\delta-c\rho_l)e^{\rho_l b}h_l=\prod_{i=1}^{j}[\lambda_i+\delta+(c-\alpha)\rho]e^{-\rho b}k-\frac{\alpha}{\delta}\prod_{i=1}^{j}\lambda_i. \tag{3.2.15}$$

Combing (3.2.12) and (3.2.15) in matrix form gives

$$\mathbf{Ah}=\mathbf{B},$$

where $\mathbf{h} = (h_1, h_2, \cdots, h_{n+2})^T$,

$$\mathbf{A} = \begin{pmatrix} 1/(\beta_2+\rho_1) & 1/(\beta_+\rho_2) & \cdots & 1/(\beta_+\rho_{n+2}) \\ e^{\rho_1 b} & e^{\rho_2 b} & \cdots & e^{\rho_{n+2} b} \\ (\lambda_1+\delta-c\rho_1)e^{\rho_1 b} & (\lambda_1+\delta-c\rho_2)e^{\rho_2 b} & \cdots & (\lambda_1+\delta-c\rho_{n+2})e^{\rho_{n+2} b} \\ & & \cdots & \\ \prod_{i=1}^{n+2}(\lambda_i+\delta-c\rho_1)e^{\rho_1 b} & \prod_{i=1}^{n+2}(\lambda_i+\delta-c\rho_2)e^{\rho_2 b} & \cdots & \prod_{i=1}^{n+2}(\lambda_i+\delta-c\rho_{n+2})e^{\rho_{n+2} b} \end{pmatrix},$$

$$\mathbf{B} = \left(0, e^{-\rho b}k, [\lambda_1+\delta+(c-\alpha)\rho]e^{-\rho b} - \alpha\lambda_1/\delta, \cdots, \prod_{i=1}^{n+2}[\lambda_i+\delta+(c-\alpha)\rho]e^{-\rho b} - \alpha\prod_{i=1}^{n+2}\lambda_i/\delta\right)^T.$$

Hence we finally determine the unknown coefficients h_i by

$$\mathbf{h} = \mathbf{A}^{-1}\mathbf{B}.$$

3.2.2 Total Claim Costs with Two-sided Jumps

Since insurance claims can only occur when $J(t) = n$, we suppress the indices of surplus positions and hence ϖ depends only on the surplus prior to claims x and immediately after claims y. Accordingly, the cost function for the total claim expenses can be further simplified from (3.1.19) by substituting specific transition rates.

$$l(x,i) = \begin{cases} \lambda_n \int_0^\infty \varpi(x, x-y)dQ(y), & i = n; \\ 0, & i = 1, 2, \cdots, n-1. \end{cases}$$

We have the infinitesimal generator for the classical deterministic path

$$\mathfrak{X} = c\frac{d}{dx} = c\mathcal{D}.$$

Hence, (3.2.3) reduces to

$$\prod_{i=1}^{n}\left[\left(1+\frac{\delta}{\lambda_i}\right)\mathcal{I} - \frac{c}{\lambda_i}\mathcal{D}\right]A(x,1) = \int_0^\infty A(x-y,1)dQ(y) + \int_0^\infty \varpi(x,x-y)dQ(y).$$

3.2.3 Gerber-Shiu Functions

If we let

$$\varpi[(x,n),(y,1)] = \begin{cases} 0, & \text{for } y \geq 0, \\ w(x,-y), & \text{for } y < 0, \end{cases}$$

which means

$$l(x,n) = \lambda_n \int_x^\infty w(x, y-x) dQ(y).$$

Thus (3.1.3) turns into the familiar Gerber-Shiu function

$$m(x,1) = \mathbb{E}^{(x,1)}\left[e^{-\delta\tau_0} w(X_{\tau_0 -}, |X_{\tau_0}|) I(\tau_0 < \infty)\right],$$

and the corresponding integro-differential equation (3.2.3) becomes

$$\prod_{i=1}^n \left[\left(1 + \frac{\delta}{\lambda_i}\right)\mathcal{I} - \frac{c}{\lambda_i}\mathcal{D}\right] m(x,1) = \int_0^x m(x-y,1) dQ(y) + \int_x^\infty w(x, y-x) dQ(y),$$

which is precisely the equation (5.11) in Gerber and Shiu [24].

3.3 Generalized Erlang-*2* Inter-claim Time Model with Absolute Ruin

3.3.1 Gerber-Shiu Functions

The Sparre Andersen model with Erlang-2 inter-claim time distribution was first studied in a seminal paper by Dickson and Hipp [16] and then later successfully extended to consider Erlang-n inter-claim time distribution by Li and Garrido [38]. In Gerber and Shiu [24], the model was further generalized to incorporate a broader class of inter-claim times governed by generalized Erlang-n distribution. Many inspiring new techniques and results introduced in Gerber and Shiu [24] such as operator arguments popularized the study of

Sparre Andersen model and were followed by numerous research papers such as Li and Garrido [37] which investigated dividend paid up to ruin under a constant barrier strategy, and Albrecher *et al.* [1] which derived the distribution of dividend payments in the same model.

In the classical Sparre Andersen model with generalized Erlang-2 inter-claim times, it is assumed that an insurer's surplus is driven by a stochastic process $X = \{X_t, t \geq 0\}$ with

$$X_t = x + ct - \sum_{i=1}^{N_t} Y_i, \tag{3.3.1}$$

where x is the initial surplus level, insurance premium is collected continuously at a constant rate of c, the sequence of insurance claims $\{Y_i, i = 1, 2, \cdots\}$ are i.i.d. with common distribution $Q(y)$ with density $q(y)$. The counting process $\{N(t), t \geq 0\}$ is defined by $N(t) = \min\{n | T_1 + \cdots + T_n \leq t\}$ where $\{T_i, i = 1, 2, \cdots\}$ representing the inter-claim times with a common generalized Erlang-2 distribution with Laplace transform

$$\tilde{k}(s) = \frac{\lambda_1}{\lambda_1 + s} \frac{\lambda_2}{\lambda_1 + s}.$$

Since Erlang-2 distribution can be represented as a phase-type distribution $PH(\mathbf{a}, \mathbf{\Lambda})$ where

$$\mathbf{\Lambda} = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ 0 & -\lambda_2 \end{pmatrix},$$

$\mathbf{a} = (1, 0)^T$ and $\eta = (0, \lambda_2)^T$, it is easy to see that the process $\{X_t, t \leq 0\}$ can be decomposed as a piecewise-deterministic Markov process $\{(X_t, J_t), t \geq 0, J_t = 1 \text{ or } 2\}$ with transition rates defined by the following two cases,

1. Transition from (X_t, i) to (X_t, j) with J communicating in the transient states at the rate given by $\mathbf{\Lambda}_{ij}$;

2. Transition from (X_t, i) to $(X_t - y, j)$ with J first absorbed resulting in an insurance claim of size y and then regenerated at the rate given by $\eta_i a_j$.

And the sample path of (X_t, i) in between any two consecutive claims yields the infinitesimal generator

$$\mathfrak{X} = cd/dt.$$

Hence, the classical Sparre Andersen model with generalized Erlang-2 inter-claim time distribution can be specified by the local characteristics $(\mathfrak{X}, \mathbf{\Lambda}, Q)$.

The absolute ruin probability was first introduced in the compound Poisson model by Dassios and Embrechts [13] and then analyzed through piecewise-deterministic Markov process approaches in Embrechts and Schmidli [18]. Recently, the Gerber-Shiu function was extensively studied in the context of the compound Poisson model with absolute ruin in Cai [6]. We follow the same idea to investigate the Gerber-Shiu function in the Sparre Andersen model with Erlang-2 inter-claim times with absolute ruin.

Since the insurer is allowed to borrow money from a bank at a debit force of interest r whenever in deficit, the dynamics of the surplus process is given by

$$\begin{cases} dX_t = cdt - dZ_t, & x \geq 0; \\ dX_t = (rX_t + c)dt - dZ_t, & -c/r \leq x < 0. \end{cases} \tag{3.3.2}$$

In terms of local characteristics of the piecewise deterministic Markov process, both $\mathbf{\Lambda}$ and $Q(y)$ remain the same. The infinitesimal generator for the deterministic path is now given by

$$\mathfrak{X} = \begin{cases} cd/dt, & x \geq 0; \\ (rx + c)d/dt, & -c/r \leq x < 0. \end{cases} \tag{3.3.3}$$

It is shown in Section 3.2 that the Gerber-Shiu function for such a process can be represented as

$$m(x) = \mathbb{E}^{(x,1)} \left[\int_0^{\tau} e^{-\delta t} l(X_t, J_t) dt \right],$$

where

$$l(x, i) = \begin{cases} \lambda_2 \int_{x+c/r}^{\infty} w(x, y - x) dQ(y), & i = 2; \\ 0, & i = 1. \end{cases} \tag{3.3.4}$$

The Gerber-Shiu function satisfies the following integro-differential equation

$$\begin{aligned}
&\prod_{i=1}^{2}\left[\left(1+\frac{\delta}{\lambda_i}\right)\mathcal{I}-\frac{1}{\lambda_i}\mathfrak{X}\right]m(x)\\
&=\int_{-\infty}^{\infty}m(y)dQ(y;x)+\sum_{i=1}^{2}\left\{\prod_{k=i+1}^{2}\left[\left(1+\frac{\delta}{\lambda_k}\right)\mathcal{I}-\frac{1}{\lambda_k}\mathfrak{X}\right]\frac{1}{\lambda_i}l(x,i)\right\} \qquad (3.3.5)
\end{aligned}$$

with the convention that $\prod_{k=n+1}^{n}\cdot=1$.

Due to two types of dynamics in deterministic paths of the surplus process, we introduce the notation

$$m(x)=\begin{cases} m_+(x), & x\ge 0;\\ m_-(x), & -c/r\le x<0.\end{cases} \qquad (3.3.6)$$

Inserting the expressions in (3.3.3), (3.3.4) and (3.3.6) into the equation (3.3.5), we could easily obtain the integro-differential equation satisfied by the Gerber-Shiu function.

$$\begin{aligned}
&[(\lambda_1+\delta)\mathcal{I}-c\mathcal{D}][(\lambda_2+\delta)\mathcal{I}-c\mathcal{D}]m_+(x)=\lambda_1\lambda_2\Big[\int_0^x m_+(x-y)dQ(y)\\
&+\int_x^{x+c/r}m_-(x-y)dQ(y)+\int_{x+c/r}^{\infty}w(x,y-x)dQ(y)\Big],\qquad x\ge 0, \qquad (3.3.7)\\
&[(\lambda_1+\delta)\mathcal{I}-(ru+c)\mathcal{D}]\,[(\lambda_2+\delta)\mathcal{I}-(ru+c)\mathcal{D}]\,m_-(x)=\lambda_1\lambda_2\Big[\int_0^{x+c/r}m_-(x-y)dQ(y)\\
&+\int_{x+c/r}^{\infty}w(x,y-x)dQ(y)\Big],\qquad -c/r\le x<0. \qquad (3.3.8)
\end{aligned}$$

Hence we summarize the integro-differential equations in the following theorem.

Theorem 3.3.1.

$$\begin{aligned}
m_+''(x) &= \frac{\lambda_1+\lambda_2+2\delta}{c}m_+'(x)-\frac{(\lambda_1+\delta)(\lambda_2+\delta)}{c^2}m_+(x)+\frac{\lambda_1\lambda_2}{c^2}\Big[\int_0^x m_+(x-y)dQ(y)\\
&\quad+\int_x^{x+c/r}m_-(x-y)dQ(y)+\int_{x+c/r}^{\infty}w(x,y-x)dQ(y)\Big],\qquad x\ge 0; \qquad (3.3.9)\\
m_-''(x) &= \frac{\lambda_1+\lambda_2+2\delta-r}{rx+c}m_-'(x)-\frac{(\lambda_1+\delta)(\lambda_2+\delta)}{(rx+c)^2}m_-(x)\\
&\quad+\frac{\lambda_1\lambda_2}{(rx+c)^2}\Big[\int_0^{x+c/r}m_-(x-y)dQ(y)+\int_{x+c/r}^{\infty}w(x,y-x)dQ(y)\Big],-c/r\le x<0.
\end{aligned}$$

(3.3.10)

Theorem 3.3.2. *The Gerber-Shiu function satisfies the following equations,*

$$m_+(x) = \frac{\lambda_1\lambda_2}{c^2}\int_0^x m_+(x-y)g(y)dy + h(x), \qquad x \geq 0, \tag{3.3.11}$$

$$m_-(x) = n(x) + \int_{-c/r}^x k(x,u)m_-(u)du, \qquad -c/r \leq x < 0, \tag{3.3.12}$$

where

$$\begin{aligned}
g(y) &= \int_0^\infty\int_0^\infty e^{-\rho_1 s-\rho_2 t}p(t+s+y)dtds, \\
\zeta(x) &= \int_{x+c/r}^\infty w(x,y-x)dQ(y), \\
h(x) &= \frac{\lambda_1\lambda_2}{c^2}\int_x^{x+c/r} m_-(x-y)g(y)dy + \frac{\lambda_1\lambda_2}{c^2}\int_0^\infty\int_0^\infty e^{-\rho_1 s-\rho_2 t}\zeta(t+s+x)dtds, \\
n(x) &= \frac{\lambda_1\lambda_2\int_{-c/r}^x\int_{-c/r}^u \zeta(z)dzdu}{(rx+c)^2}, \\
k(x,u) &= \frac{\lambda_1\lambda_2\int_u^x Q(y-u)dy}{(rx+c)^2} + \frac{(3r+\lambda_1+\lambda_2+2\delta)(ru+c)}{(rx+c)^2} \\
&\quad -\frac{(\lambda_1\lambda_2-\lambda_1\delta-\lambda_2\delta-\delta^2-r^2-\lambda_1 r-\lambda_2 r-2\delta r)(x-u)}{(rx+c)^2}.
\end{aligned}$$

Proof. Following the same arguments as in Gerber and Shiu [24], we can have the following Li's renewal equation

$$m_+(x) = \frac{\lambda_1\lambda_2}{c^2}\int_0^x m_+(x-y)g(y)dy + \int_0^\infty\int_0^\infty e^{-\rho_1 s-\rho_2 t}\gamma(t+s+x)dtds,$$

where

$$\gamma(x) = \int_x^{x+c/r} m_-(x-y)dQ(y) + \int_{x+c/r}^\infty w(x,y-x)dQ(y).$$

Note that

$$\int_x^{x+c/r} m_-(x-y)dQ(y) = \int_{-c/r}^0 m_-(z)q(x-z)dz.$$

Hence

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty e^{-\rho_1 s-\rho_2 t} \int_{x+s+t}^{x+s+t+c/r} m_-(x+s+t-y)dQ(y)dtds \\
= &\int_0^\infty \int_0^\infty e^{-\rho_1 s-\rho_2 t} \int_{-c/r}^{0} m_-(z)q(x+s+t-z)dzdtds \\
= &\int_{-c/r}^{0} m_-(z)g(x-z)dz = \int_x^{x+c/r} m_-(x-y)g(y)dy.
\end{aligned}
$$

Therefore we obtain the renewal equation (3.3.11).

We rewrite (3.3.10) as

$$
\begin{aligned}
&(rx+c)^2 m_-''(x) + [(r^2-\lambda_1 r-\lambda_2 r-2\delta r)x+(r-\lambda_1-\lambda_2-2\delta)c]m_-'(x) \\
&+(\lambda_1\lambda_2+\lambda_1\delta+\lambda_2\delta+\delta^2)m_-(x) = \lambda_1\lambda_2 \int_0^{x+c/r} m_-(x-y)dQ(y)+\zeta(x).
\end{aligned}
$$

Replacing x by t and integrating each term from $-c/r$ to u,

$$
\begin{aligned}
\int_{-c/r}^{u} (rt+c)^2 m_-''(t)dt &= (ru+c)^2 m_-'(u) - 2r\int_{-c/r}^{u}(rt+c)m_-'(t)dt \\
&= (ru+c)^2 m_-'(u) - 2r(ru+c)m_-(u) + 2r^2\int_{-c/r}^{u} m_-(t)dt.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_{-c/r}^{u} [(r^2-\lambda_1 r-\lambda_2 r-2\delta r)t+(r-\lambda_1-\lambda_2-2\delta)c]m_-'(t)dt \\
= &[(r^2-\lambda_1 r-\lambda_2 r-2\delta r)u+(r-\lambda_1-\lambda_2-2\delta)c]m_-(u) \\
&-(r^2-\lambda_1 r-\lambda_2 r-2\delta r)\int_{-c/r}^{u} m_-(t)dt.
\end{aligned}
$$

Conducting the same procedure,

$$\begin{aligned}\int_{-c/r}^{u} \lambda_1\lambda_2 \int_0^{t+c/r} m_-(t-y)dQ(y)dt &= \lambda_1\lambda_2 \int_{-c/r}^{u}\int_{-c/r}^{t} m_-(x)Q'(t-z)dzdt \\ &= \lambda_1\lambda_2 \int_{-c/r}^{u}\int_{z}^{u} m_-(z)Q'(x-z)dxdz \\ &= \lambda_1\lambda_2 \int_{-c/r}^{u} m_-(z)Q(u-z)dz.\end{aligned}$$

Thus we must have

$$\begin{aligned}(ru+c)^2 m_-'(u) - 2r(ru+c)m_-(u) + 2r^2\int_{-c/r}^{u} m_-(t)dt + (r^2-\lambda_1 r-\lambda_2 r-2\delta r)um_-(u) \\ +(r-\lambda_1-\lambda_2-2\delta)cm_-(u) - (r^2-\lambda_1 r-\lambda_2 r-2\delta r)\int_{-c/r}^{u} m_-(t)dt \\ +(\lambda_1\lambda_2-\lambda_1\delta-\lambda_2\delta-\delta^2)\int_{-c/r}^{u} m_-(t)dt = \lambda_1\lambda_2\int_{-c/r}^{u} m_-(z)Q(u-z)dz + \lambda_1\lambda_2\int_{-c/r}^{u}\zeta(t)dt.\end{aligned}$$

Integrating again from $-c/r$ to x yields,

$$\int_{-c/r}^{x}(ru+c)^2 m_-'(u)du = (rx+c)^2 m_-(x) - 2r\int_{-c/r}^{x}(ru+c)m_-(u)du,$$

$$\int_{-c/r}^{x}\int_{-c/r}^{u} m_-(t)dtdu = \int_{-c/r}^{x}(x-u)m_-(u)du,$$

$$\int_{-c/r}^{x}\int_{-c/r}^{u} m_-(z)Q(u-z)dzdu = \int_{-c/r}^{x}\int_{z}^{x} m_-(z)Q(u-z)dudz = \int_{-c/r}^{x}\left(\int_{z}^{x} Q(u-z)du\right)m_-(z)dz.$$

Hence, in summary we have

$$\begin{aligned}(rx+c)^2 m_-(x) - (3r+\lambda_1+\lambda_2+2\delta)\int_{-c/r}^{x}(ru+c)m_-(u)du \\ +(\lambda_1\lambda_2-\lambda_1\delta-\lambda_2\delta-\delta^2-r^2-\lambda_1 r-\lambda_2 r-2\delta r)\int_{-c/r}^{x}(x-u)m_-(u)du \\ = \lambda_1\lambda_2\int_{-c/r}^{x}\left(\int_{z}^{x} Q(u-z)du\right)m_-(z)dz + \lambda_1\lambda_2\int_{-c/r}^{x}\int_{-c/r}^{u}\zeta(t)dtdu,\end{aligned}$$

which is the Volterra equation of the second kind (3.3.12) upon rearrangement. $\square$

As a result, the general solution to the Gerber-Shiu function is given by

$$\begin{aligned} m_+(x) &= \frac{\lambda_1\lambda_2}{c^2}\int_0^x h(x-y)g(y)dy + h(x), \qquad x \geq 0, \\ m_-(x) &= n(x) + \int_{-c/r}^x K(x,u)n(u)du, \qquad -c/r \leq x < 0, \end{aligned}$$

where

$$\begin{aligned} K(x,u) &= \sum_{m=1}^{\infty} k_m(x,u), \qquad x > u \geq -c/r, \\ k_m(x,u) &= \int_u^x k(x,t)k_{m-1}(t,u)dt, \qquad m = 2,3,\cdots, x > u \geq -c/r \end{aligned}$$

with $k_1(x,u) = k(x,u)$.

For the rest of this section, we assume that all insurance claims follow the exponential distribution with mean $1/\beta$, i.e.

$$Q(y) = 1 - e^{-\beta y}, \qquad y \geq 0.$$

And the safety loading condition is also satisfied, i.e. $c(1/\lambda_1 + 1/\lambda_2) > 1/\beta$.

Theorem 3.3.3. *The Gerber-Shiu function with $w(x,y) = g(y)$ and $\delta = 0$ in the model (3.3.2) with exponential claim size distribution of mean $1/\beta$ is given by*

$$\begin{aligned} & m_+(x) = D_1 e^{s_1 x}, \qquad x \geq 0, \\ & m_-(x) = C_1 \int_{-c/r}^x e^{-\beta(t+c/r)}(t+c/r)^{\lambda_1/r-1} M\left[1+\frac{\lambda_1}{r}, 1+\frac{\lambda_1-\lambda_2}{r}; \beta(t+c/r)\right] dt \\ & C_2 \int_{-c/r}^x e^{-\beta(t+c/r)}(t+c/r)^{\lambda_1/r-1} U\left[1+\frac{\lambda_1}{r}, 1+\frac{\lambda_1-\lambda_2}{r}; \beta(t+c/r)\right] dt + C_3, \qquad -c/r \leq x < 0, \end{aligned}$$

where

$$
\begin{aligned}
s_1 &= \frac{\lambda_1+\lambda_2-\beta c-\sqrt{(\beta c-\lambda_1-\lambda_2)^2+4(\lambda_1\beta c+\lambda_2\beta c-\lambda_1\lambda_2)}}{2c}, \\
d &= [(s_1-r/c)A(0)-A'(0)][B(0)/s_1-\int_{-c/r}^{0}B(t)dt] \\
&\quad +[B'(0)-(s_1-r/c)B(0)][A(0)/s_1-\int_{-c/r}^{0}A(t)dt], \\
\zeta(-c/r) &= \beta\int_0^\infty g(y+c/r)e^{-\beta y}dy, \\
C_1 &= [(s_1-r/c)A(0)-A'(0)]\zeta(-c/r)/d, \\
C_2 &= [B'(0)-(s_1-r/c)B(0)]\zeta(-c/r)/d, \\
C_3 &= \zeta(-c/r), \\
D_1 &= [A(0)B'(0)-B(0)A'(0)]\zeta(-c/r)/s_1/d.
\end{aligned}
$$

Proof. In view of Lemma 1.1.4, we apply the operator $\beta\mathcal{I}+\mathcal{D}$ to both sides of (3.3.9).

$$(\beta\mathcal{I}+\mathcal{D})[(\lambda_1+\delta)\mathcal{I}-c\mathcal{D}][(\lambda_2+\delta)\mathcal{I}-c\mathcal{D}]m_+(x)=\beta\lambda_1\lambda_2m_+(x)+\lambda_1\lambda_2(\beta\mathcal{I}+\mathcal{D})\zeta(x),$$

which can be expanded as

$$
\begin{aligned}
&\{c^2\mathcal{D}^3+[\beta c^2-(\lambda_1+\delta)c-(\lambda_2+\delta)c]\mathcal{D}^2+[(\lambda_1+\delta)(\lambda_2+\delta)-\beta c(\lambda_1+\delta)-\beta c(\lambda_2+\delta)]\mathcal{D} \\
&\qquad +\beta(\lambda_1+\delta)(\lambda_2+\delta)\}m_+(x)=\beta\lambda_1\lambda_2m_+(x)+\lambda_1\lambda_2(\beta\mathcal{I}+\mathcal{D})\zeta(x).
\end{aligned}
$$

When $\delta=0$ and $w(x,y)=g(y)$,

$$c^2m_+'''(x)+(\beta c^2-\lambda_1c-\lambda_2c)m_+''(x)+(\lambda_1\lambda_2-\beta c\lambda_1-\beta c\lambda_2)m_+'(x)=0.$$

Hence the general solution to $m_+(x)$ is given by

$$m_+(x)=D_1e^{s_1x}+D_2e^{s_2}+D_3,$$

where

$$s_1 = \frac{\lambda_1+\lambda_2-\beta c-\sqrt{(\beta c-\lambda_1-\lambda_2)^2+4(\lambda_1\beta c+\lambda_2\beta c-\lambda_1\lambda_2)}}{2c} < 0,$$
$$s_2 = \frac{\lambda_1+\lambda_2-\beta c+\sqrt{(\beta c-\lambda_1-\lambda_2)^2+4(\lambda_1\beta c+\lambda_2\beta c-\lambda_1\lambda_2)}}{2c} > 0$$

are the roots of the characteristic equation

$$c^2s^3+(\beta c^2-\lambda_1 c-\lambda_2 c)s^2+(\lambda_1\lambda_2-\beta c\lambda_1-\beta c\lambda_2)s=0.$$

Since $\lim_{x\to\infty} m_+(x)=0$, we must have $D_2=D_3=0$. Hence

$$m_+(x)=D_1e^{s_1x}, \qquad x\geq 0. \tag{3.3.13}$$

On the other hand, applying the operator $\beta\mathcal{I}+\mathcal{D}$ to both sides of (3.3.10) gives

$$(\beta\mathcal{I}+\mathcal{D})[(\lambda_1+\delta)\mathcal{I}-(rx+c)\mathcal{D}][(\lambda_2+\delta)\mathcal{I}-(rx+c)\mathcal{D}]m_-(x)=\lambda_1\lambda_2\beta m_-(x)+\lambda_1\lambda_2(\beta\mathcal{I}+\mathcal{D})\zeta(x),$$

which means

$$\begin{aligned}\{(rx+c)^2\mathcal{D}^3&+[\beta(rx+c)^2+(rx+c)(3r-\lambda_1-\lambda_2-2\delta)]\mathcal{D}^2+[(\beta rx+c\beta)(r-\lambda_1-\lambda_2-2\delta)\\&+\lambda_1\lambda_2+\lambda_1\delta+\lambda_2\delta+\delta^2+r^2-r\lambda_1-r\lambda_2-2\delta r]\mathcal{D}+\beta(\lambda_1+\delta)(\lambda_2+\delta)\}m_-(x)\\&=\lambda_1\lambda_2\beta m_-(x)+\lambda_1\lambda_2(\beta\mathcal{I}+\mathcal{D})\zeta(x).\end{aligned}$$

In the case where $\delta=0$ and $w(x,y)=g(y)$, it can be simplified as

$$\begin{aligned}&(rx+c)^2m_-'''(x)+[\beta(rx+c)^2+(rx+c)(3r-\lambda_1-\lambda_2)]m_-''(x)\\&+[(rx+c)(\beta r-\beta\lambda_1-\beta\lambda_2)+\lambda_1\lambda_2+r^2-r\lambda_1-r\lambda_2]m_-'(x)=0.\end{aligned} \tag{3.3.14}$$

Let $z = x + c/r$ and $m'_-(x) = e^{-\beta z} f(z)$, then

$$m''_-(x) = e^{-\beta z}[f'(z) - \beta f(z)]$$
$$m'''_-(x) = e^{-\beta z}[f''(z) - 2\beta f'(z) + \beta^2 f(z)].$$

Hence (3.3.14) can be written as

$$r^2 z^2 f''(z) + [-\beta r^2 z^2 + 3r^2 z - (\lambda_1 + \lambda_2) r z] f'(z) + (-2\beta r^2 z + \lambda_1 \lambda_2 + r^2 - \lambda_1 r - \lambda_2 r) f(z) = 0.$$

Let $f(z) = z^{(\lambda_1/r)-1} k(z)$, then

$$r^2 z k''(z) + (-\beta r^2 z + r^2 + \lambda_1 r - \lambda_2 r) k'(z) + (-\beta \lambda_1 r - \beta r^2) k(z) = 0.$$

Let $k(z) = y(x)$ and $x = \beta z$, then $k'(z) = \beta y'(x)$ and $k''(z) = \beta^2 y''(x)$. Hence,

$$r^2 \beta x y''(x) + (-r^2 \beta x + \beta r^2 + \beta \lambda_1 r - \beta \lambda_2 r) y'(x) + (-\beta \lambda_1 r - \beta r^2) y(x) = 0,$$

which means

$$x y''(x) + (1 + \frac{\lambda_1 - \lambda_2}{r} - x) y'(x) - (1 + \frac{\lambda_1}{r}) y(x) = 0.$$

The Kummer's differential equation

$$x y''(x) + (b - x) y'(x) - a y(x) = 0$$

has two independent solutions denoted by $M(a, b; x)$ and $U(a, b; x)$. The Kummer function of the first kind can computed by

$$M(a, b; x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(b)_n n!},$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$. The Kummer function of the second kind is hence given by

$$U(a,b;x) = \frac{\pi}{\sin(\pi b)}\left[\frac{M(a,b;x)}{\Gamma(1+a-b)\Gamma(b)} - x^{1-b}\frac{M(1+a-b,2-b;x)}{\Gamma(a)\Gamma(2-b)}\right].$$

Inverting the variables to the originals, we have

$$\begin{aligned} m_-(x) = {} & C_1\int_{-c/r}^{x} e^{-\beta(t+c/r)}(t+c/r)^{\lambda_1/r-1}M\left[1+\frac{\lambda_1}{r}, 1+\frac{\lambda_1-\lambda_2}{r}; \beta(t+c/r)\right]dt \\ & +C_2\int_{-c/r}^{x} e^{-\beta(t+c/r)}(t+c/r)^{\lambda_1/r-1}U\left[1+\frac{\lambda_1}{r}, 1+\frac{\lambda_1-\lambda_2}{r}; \beta(t+c/r)\right]dt + C_3, \\ & -c/r \le x < 0. \end{aligned} \tag{3.3.15}$$

Now we need four linear equations to determine the unknown coefficients D_1, C_1, C_2 and C_3 in (3.3.13) and (3.3.15). First, letting $x = 0$ in (3.3.9) and $x \to 0$ in (3.3.10) gives

$$m_+(0+) = m_-(0-). \tag{3.3.16}$$

Letting $x = -c/r$ in (3.3.10) yields the second equation

$$m_-(-c/r) = \zeta(-c/r). \tag{3.3.17}$$

In view of (3.2.1) and the continuity property, it follows that

$$\begin{aligned} \left[\mathcal{I} - \frac{1}{\lambda_1}\mathfrak{X}\right] m_+(0+) &= \left[\mathcal{I} - \frac{1}{\lambda_1}\mathfrak{X}\right] m_-(0-), \\ \left[\mathcal{I} - \frac{1}{\lambda_1}\mathfrak{X}\right]\left[\mathcal{I} - \frac{1}{\lambda_2}\mathfrak{X}\right] m_+(0+) &= \left[\mathcal{I} - \frac{1}{\lambda_1}\mathfrak{X}\right]\left[\mathcal{I} - \frac{1}{\lambda_2}\mathfrak{X}\right] m_-(0-), \end{aligned}$$

Hence

$$\begin{aligned} m_+(0+) - \frac{c}{\lambda_1}m'_+(0+) &= m_-(0-) - \frac{c}{\lambda_1}m'_-(0-) \\ m_+(0+) - (\frac{c}{\lambda_1} + \frac{c}{\lambda_2})m'_+(0+) &+ \frac{c^2}{\lambda_1\lambda_2}m''_+(0+) \\ = m_-(0-) - (\frac{c}{\lambda_1} + \frac{c}{\lambda_2})m'_-(0-) &+ \frac{rc}{\lambda_1\lambda_2}m'_-(0-) + \frac{c^2}{\lambda_1\lambda_2}m''_-(0-), \end{aligned}$$

which implies that

$$m'_+(0+) \;=\; m'_-(0-), \tag{3.3.18}$$

$$rm'_-(0-) + cm''_-(0-) \;=\; cm''_+(0+). \tag{3.3.19}$$

Substituting (3.3.13) and (3.3.15) into (3.3.16), (3.3.17), (3.3.18) and (3.3.19) gives

$$\begin{aligned}
D_1 &= \int_{-c/r}^{0} A(t)dt\, C_1 + \int_{-c/r}^{0} B(t)dt\, C_2 + C_3,\\
C_3 &= \zeta(-c/r),\\
s_1 D_1 &= A(0)C_1 + B(0)C_2,\\
s_1^2 D_1 &= (r/c)s_1 D_1 + A'(0)C_1 + B'(0)C_2,
\end{aligned}$$

where

$$A(t) = e^{-\beta(t+c/r)}(t+c/r)^{\lambda_1/r-1} M\left[1+\frac{\lambda_1}{r}, 1+\frac{\lambda_1-\lambda_2}{r}; \beta(t+c/r)\right]$$
$$B(t) = e^{-\beta(t+c/r)}(t+c/r)^{\lambda_1/r-1} U\left[1+\frac{\lambda_1}{r}, 1+\frac{\lambda_1-\lambda_2}{r}; \beta(t+c/r)\right].$$

Solving the system of equations for the unknown parameters gives the desired result. □

3.3.2 Probability of Absolute Ruin and Probability of Ordinary Ruin

Now we are interested in the probability of absolute ruin defined by

$$\psi(x) = \mathbb{P}^x\{\tau < \infty\},$$

where $\tau = \inf\{X(t) < -c/r.\}$ Since it is a special case of the Gerber-Shiu function $m(x)$, we take $w(x,y) = 1$ and $\delta = 0$. Hence $\zeta(-c/r) = 1$ and we obtain explicit solutions for $\psi(x)$ by Theorem 3.3.3.

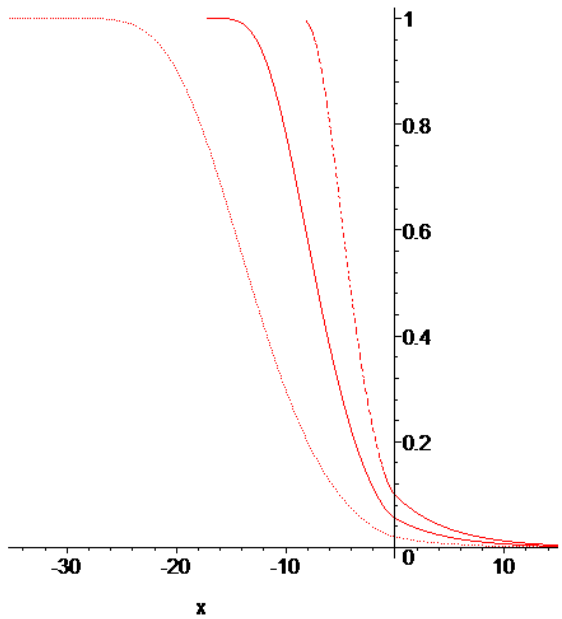

Figure 3.1: Absolute ruin probabilities

Figure 3.1 exhibits the probabilities of absolute ruin as functions of initial surplus in three scenarios with different debt interest rates. The parameters are chosen as follows. $c = 2, \beta = 0.5, \lambda_1 = 2, \lambda_2 = 1$. The three functions correspond to the debt interest rate $r_1 = 0.055$, $r_2 = 0.11$ and $r_3 = 0.22$ respectively clockwise. We observe that in all cases the probability decreases as the initial surplus increases, which indicates that the more initial surplus the less likely the insurer's surplus gets ruined. It agrees with our intuition that the probability of absolute ruin gets larger as the debt interest rate increases since the insurer has to pay more interest and it makes more difficult to break even.

For the purpose of comparison, we derive the probability of "ordinary" ruin in classical Sparre Andersen model (3.3.1) defined by

$$\varphi(x) = \mathbb{P}^x\{\tau_0 < \infty\},$$

where $\tau_0 = \inf\{X(t) < 0.\}$

We can easily find the integro-differential equation satisfied by $\varphi(x)$,

$$\varphi''(x) = \frac{\lambda_1 + \lambda_2}{c}\varphi'(x) - \frac{\lambda_1\lambda_2}{c^2}\varphi(x) + \frac{\lambda_1\lambda_2}{c^2}\Big[\int_0^x \varphi(x-y)dQ(y) + 1 - Q(x)\Big]. \quad (3.3.20)$$

By similar arguments used to derive $m_+(x)$, we find that

$$\varphi(x) = D_1^\star e^{s_1 x}, \qquad x \geq 0.$$

Substituting it into (3.3.20) and equating all terms involving $e^{-\beta x}$ with zero determines the coefficient $D_1^\star$. Hence the probability of ultimate ruin is given by

$$\varphi(x) = \frac{\beta + s_1}{\beta} e^{s_1 x}, \qquad x \geq 0.$$

We plot both the ordinary ruin probability and the three previous cases of absolute ruin probabilities in Figure 3.2. It is interesting to notice that ordinary ruin probability is always bigger than absolute ruin probability. As one would expect from the definition of absolute ruin probability, Theorem 3.3.3 shows that the absolute ruin probability approaches the ordinary probability as the debit interest r goes to infinity.

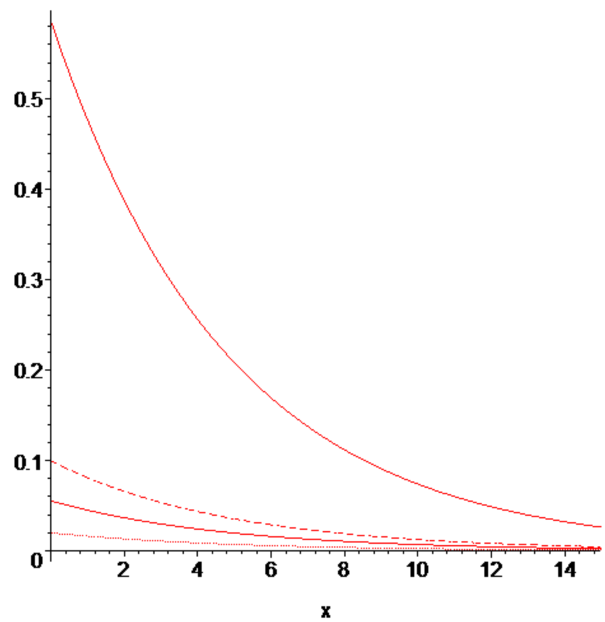

Figure 3.2: Comparison of absolute ruin and ordinary ruin probabilities

Chapter 4

Jump Diffusion Risk Models

Historically, most insurance-related problems in ruin theory are natural applications of jump processes due to the nature of discrete-time occurrences of insurance claims, whereas most classical models in financial mathematics take root in continuous processes that are believed to describe the volatility in the constantly changing financial market. Although the two disciplines of applied probability has evolved independently in the past few decades, more and more researchers came to realize the need to involve characteristics captured in each other's models. For instance, on the financial mathematics side, numerous examples such as "9.11" incidence have shown that stock price may at times increase or decrease faster than a geometric Brownian motion can. In recent years, new efforts has been made to model market prices by diffusion processes with jump components. On the ruin theory side, in addition to the traditional approach of modelling the arrival of insurance claims by jump processes, diffusion components have gained increasing popularity in the literature to allow more randomness in surplus process for the periods in between claim arrivals.

In this chapter, we aim to build up risk processes on a common ground where both traditional jump processes and diffusion processes can be accommodated in a systematic way. To this end, we shall first give a motivation for a general class of risk processes defined by a stochastic differential equation, which includes the famous Levy process, geometric Levy process and more. To make the thesis self-contained, we give a brief introduction to the Levy process and its connection with the general jump diffusion processes to be discussed.

4.1 Introduction

4.1.1 Motivation and Introduction to Levy Process

We assume throughout the chapter as given the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions and as given the filtration $(\mathcal{F}_t)_{0\leq t<\infty}$, on which all processes to be discussed are defined and adapted to the filtration.

As one would see frequently in the ruin literature, the classical compound Poisson surplus process $X = \{X(t), t \geq 0\}$ is usually presented in terms of

$$X(t) = x + ct - Z(t). \tag{4.1.1}$$

Recall from Section 1.2 that the process is interpreted as the balance of total premium income ct and aggregate claim up to time $Z(t)$. The initial deposit is given by x and the aggregate claim $Z_t = \sum_{i=1}^{N_t} Y_i$ with $\{Y_i, i = 1, 2, \cdots\}$ denoting a sequence of independent insurance claims with common distribution $Q(A) = \mathbb{P}\{Y_i \in A\}$ and mean denoted by κ. The total number of insurance claims up to time t, $\{N(t), t \geq 0\}$, is a Poisson counting process with intensity rate λ, or in other words, follows a Poisson distribution with mean λt at any time t.

An alternative approach to define the surplus process, as shown in Section 2.3, is through the stochastic differential equation given by

$$dX_t = cdt - dZ_t,$$

together with $X(0) = x$, and $\{Z(t), t \geq 0\}$ is the compound Poisson jump process defined above. The stochastic differential equation is usually set up by interpreting the instantaneous change in the surplus level at any time t as the balance of instantaneous increase due to premium income over the infinitesimal period, cdt, and instantaneous decrease as a result of changes in aggregate claim, $dZ(t)$. The great advantage of the second presentation is that for the purpose of further generalizations it is often easier to employ infinitesimal arguments or stochastic differential equations in more complicated situations with economic factors rather than expressing $X(t)$ in an explicit form as in (4.1.1).

There have been proposed in the ruin literature a series of pure diffusion risk models by stochastic differential equation approach, among which the most recent ones are Gerber and Shiu [23], Gerber and Shiu [28], Cai *et al.* [10], and more. They all can be generally put in the form of

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t,$$

with $X_0 = x$, provided some integrability conditions are satisfied to ensure pathwise uniqueness of the processes. In fact, most asset pricing models in the finance literature can be put in this category, as this type of diffusion processes naturally find their applications in a wide range of financial topics. For an introduction to the application of pure diffusion models in finance, readers are referred to Björk [3].

In recent development, actuarial researchers start to investigate classical jump surplus processes perturbed by a Brownian motion. In terms of a stochastic differential equation,

$$dX_t = cdt + \sigma dB_t - dZ_t$$

with $X_0 = x$. Interested readers are referred to Gerber and Landry [19], Tsai and Willmot [46], Li [36] for more detailed information on such models.

To find a common ground where all these models can be compared and analyzed in a unified approach, we shall seek for more general jump diffusion processes. Generally speaking, jump diffusion processes are particularly suitable for modelling in the context of insurance surplus. A drift component can be chosen to reflect the dominating trend of growth in surplus and a diffusion component demonstrates certain degree of randomness in total surplus, whereas a jump component would represent unexpected costs resulted from extreme events on a large scale. To allow for more flexibility, we would like to search for a way of measuring the actual impact of a jump caused by the jump component on the overall process level, which for practical reasons depends on both the size of a jump and the position of the process prior to the jump. In the context of risk models, the drop in surplus would most likely be different from the actual size of jumps due to insurance claims, as they are always accompanied with extra business costs. If big claims occur, its financial impact on a high surplus level might be significantly different from that on the surplus which is running low. With all of these in mind, one might want to consider the jump-diffusion process governed by the following stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t - a(X_{t-})dZ_t, \tag{4.1.2}$$

where $a(x)$ magnifies the actual impact incurred by the jump at the surplus level $X(t-)$.

However, the generalization is not quite satisfactory in some sense. For instance, if the jump size in the aggregate claim $Z(t)$ is z, then the surplus $X(t)$ will have a jump due to the impact measured by $a(X_{t-})z$. Hence we did not quite reach the ideal model where the impact is expected to depend on both $X(t-)$ and z, but not necessarily linear in z. It is not hard to imagine that the financial impact of a relatively large insurance claim might be much greater than the proportionally enlarged impact of a small claim. In order to tackle this type of non-linear dependence, we need to introduce the Levy process and the Poison random measure. Details of point processes can be found in Brémaud [4], Levy processes and Poisson random measure in Bass [2], Cont and Tankov [11], Oksendal and Sulem [42], Protter [43].

Definition 4.1.1. *An adapted process* $X = (X_t)_{t\geq 0}$ *with* $X(0) = 0$ *a.s. is a* Levy process *if*

1. *X has increments independent of the past; i.e.* $X_t - X_s$ *is independent of* $\mathcal{F}_s$ *for all* $0 \leq s < t < \infty$;

2. *X has stationary increments; i.e. $X_t - X_s$ has the same distribution as X_{t-s} for all $0 \leq s < t < \infty$;*

3. *X_t is continuous in probability; i.e. $\mathbb{P}\{\lim_{t \to s} X_t = X_s\} = 1$ for all $s \geq 0$.*

Typical examples of Levy process are compound Poisson process and Brownian motion. In fact, X always has a cadlag version and hence we shall consider it as a property of all Levy processes to be considered.

The jump of X_t is defined by

$$\Delta X_t = X_t - X_{t-}.$$

We now define a Poisson random measure $N(t, \cdot) : [0, \infty) \times \mathbb{R} \mapsto \mathbb{N}$ by given any $A \subset \mathbb{R}$ that is a Borel set whose closure does not contain 0,

$$N(t, A) = \sum_{0<s\leq t} I(\Delta X_s \in A).$$

Note that $N(t, \cdot)$ is a generalization of the Poisson counting process $N(t)$ in the compound Poisson risk model. If we let

$$\nu(A) = \mathbb{E}[N(1, A)],$$

then the set function $\nu : \Omega \mapsto \mathbb{R}^+$ is called the Levy measure of X_t. It is easy to show that for any fixed A, the process $N(t, A)$ is indeed a counting process with stationary and independent increments and hence a Poisson process with the intensity $\nu(A)$.

One would now wonder how to represent the compound Poisson process $Z(t)$ in terms of the Poisson random measure $N(t, \cdot)$ and its corresponding Levy measure $\nu(\cdot)$ in terms of the common distribution $Q(y)$. We need the following result for the Poisson random measure.

Theorem 4.1.1. *Let A be a Borel set of $\mathbb{R}$ such that the closure of A does not include 0 and f be measurable and finite on A, then*

$$\int_A f(z) N(t, dz) = \sum_{0<s\leq t} f(\Delta X_s) I(\Delta X_s \in A).$$

Hence by definition of the compound Poisson process

$$Z(t) = \sum_{i=1}^{N(t)} Y_i = \sum_{0<s\leq t} \Delta X(s) = \int_{\mathbb{R}} zN(t, dz) \tag{4.1.3}$$

with the second equality from the fact that all jumps in the surplus process are caused by the compound Poisson component. Note that in this particular case,

$$N(t, A) = \sum_{0<s\leq t} I(\Delta X_s \in A) = \sum_{i=1}^{N(t)} I(Y_i \in A). \tag{4.1.4}$$

The Levy measure that corresponds to the compound Poisson process is given by

$$\nu(0, y] = \mathbb{E}\{N(1, (0, y])\} = \mathbb{E}\left\{\sum_{i=1}^{N(1)} I(Y_i \in (0, y])\right\} = \mathbb{E}[N(1)]\mathbb{P}\{Y_i \in (0, y]\} = \lambda Q(y) \tag{4.1.5}$$

with the second last equality from the independence of $N(t)$ and Y_i's. It is obvious from the derivation that a Levy process can be represented by a compound Poisson process if and only if its Levy measure is finite.

However, there are a great number of interesting Levy processes with infinite Levy measures. For the notational brevity and practical reason, we shall only be looking at integrable Levy processes which can always be represented as follows.

Theorem 4.1.2. *If X_t is a Levy process such that*

$$\mathbb{E}[X_t] < \infty \qquad \textit{for all } t \geq 0,$$

then it has the decomposition

$$X_t = \alpha t + \beta B(t) + \int_{\mathbb{R}} z\tilde{N}(t, dz), \tag{4.1.6}$$

for some constants $\alpha, \beta \in \mathbb{R}$ and

$$\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt.$$

The compound Poisson process can be recovered from the representation by taking $\beta = 0$, (4.1.4), (4.1.5) and

$$\alpha = \int_{\mathbb{R}} z\nu(dz) = \lambda \int_{\mathbb{R}} zQ(dy) = \lambda\kappa. \tag{4.1.7}$$

Returning to our search for a general risk process, we can now represent the stochastic differential equation (4.1.2) as

$$dX_t = cdt + dB_t - \int_{\mathbb{R}} a(X_{t-})zN(dt, dz),$$

whose integral clearly indicates the linear dependency in the size of jump x by Theorem 4.1.1. Equipped with the powerful tool of Poisson random measure, we can now readily fix the problem by replacing $a(X_{t-})z$ with a more general impact function $F(X_{t-}, z)$. For mathematical convenience, the compound Poisson process term is to be replaced by a compensated compound Poisson process. One can always recover a compound Poisson process by adding a drift term to the compensated compound Poisson process. Hence we shall now investigate risk processes given by the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t + \int_{\mathbb{R}} F(X_{t-}, z)\tilde{N}(dt, dz), \tag{4.1.8}$$

with $X_0 = x$ and the Levy measure ν. Although the process is being introduced here with insurance flavors, the original model first appeared in one of the famous probabilist Skorokhod's papers on a rather theoretical background according to Bass [2].

The existence and pathwise uniqueness of the stochastic process given by (4.1.8) is proved in the following theorem due to Skorokhod.

Theorem 4.1.3. *If μ and σ are bounded and Lipschitz,*

$$\int_{\mathbb{R}} \sup_x |F(x, z)|^2 \nu(dz) < \infty \tag{4.1.9}$$

and

$$\int_{\mathbb{R}} |F(x, z) - F(y, z)|^2 \nu(dz) < c_1 |x - y|^2,$$

then there exists a solution to (4.1.8) and that solution is pathwise unique.

Remark 4.1.1. *It can be shown that the pathwise unique solution still exists, if μ and σ are bounded and piecewise Lipschitz continuous.*

As we often deal with functionals of a risk process, we now state the Ito's formula for semimartingale, of which the process (4.1.8) is an example.

Theorem 4.1.4. *Suppose X is a semimartingale and f is twice continuously differentiable. Then $f(X_t)$ is also semimartingale and satisfies*

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s + \frac{1}{2}\int_0^t f''(X_{s-})d\langle X^c\rangle_s + \sum_{0<s\leq t}[f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s].$$

It is easy to show by the Ito's formula for semimartingale that the infinitesimal generator of X given in (4.1.8) is

$$\mathfrak{A}f(x) = \hat{\mu}(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) + \int_{\mathbb{R}}\{f[x + F(x,z)] - f(x)\}\nu(dz). \tag{4.1.10}$$

where

$$\hat{\mu}(x) = \mu(x) - \int_{\mathbb{R}} F(x,z)\nu(dz).$$

It would be interesting at this point to recover the infinitesimal generator of the shifted compound Poisson process, which is also given as a special case of piecewise-deterministic Markov process in Section 2.3. In view of (4.1.7) and (4.1.3), we let $\mu(x) = c + \lambda\kappa, \sigma(x) = 0$ and $F(x,z) = z$ in (4.1.8), then the shifted compound Poisson process can be represented in the form

$$dX_t = (c + \lambda\kappa)dt + \int_0^t z\tilde{N}(dt, dz).$$

Plugging the relevant parameters into (4.1.10), we obtain the infinitesimal generator for the classical compound Poisson risk process

$$\mathfrak{A}f(x) = cf'(x) + \lambda\int_{\mathbb{R}}\{f(x+z) - f(x)\}Q(dz),$$

which is precisely (2.3.2) derived from the PDCP generator.

Theorem 4.1.5. *Let H is twice continuously differentiable. Suppose $\delta \geq 0$ and τ is the first exit time of an open bounded set such that $\mathbb{E}^x[\tau] < \infty$, then*

$$\mathbb{E}^x[e^{-\delta\tau}H(X_\tau)] = H(x) + \mathbb{E}^x\left[\int_0^\tau e^{-\delta s}\mathfrak{A}H(X_s)ds\right] - \delta\mathbb{E}^x\left[\int_0^\tau e^{-\delta s}H(X_s)ds\right].$$

Proof. Since $H(x)$ is twice differentiable function, we apply the Ito's formula for semimartingales.

$$\begin{aligned}
H(X_t) &= H(x) + \int_0^t H'(X_{s-})dX_s + \frac{1}{2}\int_0^t H''(X_s)d\langle X^c\rangle_s \\
&\quad + \sum_{s\leq t}[H(X_s) - H(X_{s-}) - H'(X_{s-})\triangle X_s] \\
&= H(x) + \int_0^t \sigma(X_s)H'(X_s)dW_s + \int_0^t \mu(X_s)H'(X_s)ds \\
&\quad + \int_0^t H'(X_{s-})\int_{\mathbb{R}} F(X_{s-}, z)\tilde{N}(ds, dz) + \frac{1}{2}\int_0^t \sigma^2(X_s)H''(X_s)ds \\
&\quad + \int_0^t \int_{\mathbb{R}} \{H[X_{s-} + F(X_{s-}, z)] - H(X_{s-}) - H'(X_{s-})F(X_{s-}, z)\}N(dt, dz).
\end{aligned}$$

By product rule and then taking expectations, we obtain

$$\begin{aligned}
&\mathbb{E}^x[e^{-\delta t}H(X_t)] = H(x) + \mathbb{E}^x\left[\int_0^t e^{-\delta s}dH(X_s) - \delta\int_0^t e^{-\delta s}H(X_s)ds\right] \\
= \; & H(x) - \mathbb{E}^x\left[\delta\int_0^t e^{-\delta s}H(X_s)ds\right] + \mathbb{E}^x\left[\int_0^t e^{-\delta s}\sigma(X_s)H'(X_s)dW_s\right] + \mathbb{E}^x\left[\int_0^t e^{-\delta s}\mu(X_s)H'(X_s)ds\right] \\
&+ \frac{1}{2}\mathbb{E}^x\left[\int_0^t e^{-\delta s}\sigma^2(X_s)H''(X_s)ds\right] + \mathbb{E}^x\left[\int_0^t e^{-\delta s}H'(X_{s-})\int_{\mathbb{R}} F(X_{s-}, z)\tilde{N}(ds, dz)\right] \\
&+ \mathbb{E}^x\left[\int_0^t e^{-\delta s}\int_{\mathbb{R}}\{H[X_{s-} + F(X_{s-}, z)] - H(X_{s-}) - H'(X_{s-})F(X_{s-}, z)\}\nu(dz)ds\right]
\end{aligned}$$

with the fact that $\tilde{N}(t, A) = N(t, A) - \nu(A)t$ is a martingale.

Using the infinitesimal generator given in (4.1.10), we have

$$\begin{aligned}\mathbb{E}^x[e^{-\delta t}H(X_\tau)] &= H(x) + \mathbb{E}^x\left[\int_0^\tau e^{-\delta s}\mathfrak{A}H(X_s)ds\right] - \mathbb{E}^x\left[\delta\int_0^\tau e^{-\delta s}H(X_s)ds\right]\\ &+\mathbb{E}^x\left[\int_0^\tau e^{-\delta s}\sigma(X_s)H'(X_s)dW_s\right] + \mathbb{E}^x\left[\int_0^\tau e^{-\delta s}H'(X_{s-})\int_{\mathbb{R}}F(X_{s-},z)\tilde{N}(ds,dz)\right].\end{aligned} \tag{4.1.11}$$

For any integer k we have

$$\begin{aligned}& \mathbb{E}^x\left[\int_0^{\tau\wedge k} e^{-\delta s}\sigma(X_s)H'(X_s)dW_s\right] + \mathbb{E}^x\left[\int_0^{\tau\wedge k} e^{-\delta s}H'(X_{s-})\int_{\mathbb{R}}F(X_{s-},z)\tilde{N}(ds,dz)\right]\\ = \;& \mathbb{E}^x\left[\int_0^{k} e^{-\delta s}I(s<\tau)\sigma(X_s)H'(X_s)dW_s\right]\\ &+\mathbb{E}^x\left[\int_0^{k} e^{-\delta s}I(s<\tau)H'(X_{s-})\int_{\mathbb{R}}F(X_{s-},z)\tilde{N}(ds,dz)\right] = 0.\end{aligned} \tag{4.1.12}$$

As there exists such a closed bounded set A that $X(s) \in A$ for all $s < \tau$, $H'(X_{s-})$ must be bounded by the continuity of the first derivative. Hence both $I(s<\tau)\sigma(X_s)H'(X_s)$ and $I(s<\tau)H'(X_{s-})\int_{\mathbb{R}}F(X_{s-},z)$ are bounded. The two integral terms in (4.1.12) are both martingales and hence their expectations are equal to zero.

Moreover,

$$\begin{aligned}& \lim_{k\to\infty}\mathbb{E}^x\left[\left(\int_0^\tau e^{-\delta s}\sigma(X_s)H'(X_s)dW_s - \int_0^{\tau\wedge k} e^{-\delta s}\sigma(X_s)H'(X_s)dW_s\right)^2\right]\\ = \;& \lim_{k\to\infty}\mathbb{E}^x\left[\int_{\tau\wedge k}^{\tau}\left\{e^{-\delta s}\sigma(X_s)H'(X_s)\right\}^2 ds\right] = 0\end{aligned}$$

from the fact that $\mathbb{E}^x[\tau] < \infty$ and dominated convergence theorem. Similarly, we use the same argument together with (4.1.9) to conclude that

$$\begin{aligned}
&\lim_{k\to\infty}\mathbb{E}^x\Big[\Big(\int_0^{\tau} e^{-\delta s}I(s<\tau)H'(X_{s-})\int F(X_{s-},z)\tilde{N}(ds,dz)\\
&-\int_0^{\tau\wedge k} e^{-\delta s}I(s<\tau)H'(X_{s-})\int F(X_{s-},z)\tilde{N}(ds,dz)\Big)^2\Big]\\
=\ &\lim_{k\to\infty}\mathbb{E}^x\left[\int_{\tau\wedge k}^{\tau}\int_{\mathbb{R}}\{e^{-\delta s}I(s<\tau)H'(X_{s-})F(X_{s-},z)\}^2\nu(dz)dt\right]=0.
\end{aligned}$$

Therefore, we must have

$$\mathbb{E}^x\left[\int_0^{\tau} e^{-\delta s}\sigma(X_s)H'(X_s)dW_s\right]+\mathbb{E}^x\left[\int_0^{\tau} e^{-\delta s}H'(X_{s-})\int_{\mathbb{R}} F(X_{s-},z)\tilde{N}(ds,dz)\right]=0.$$

Hence the desired equality follows from (4.1.11). □

4.1.2 Exponential of Levy Process and Lundberg Equation

Before proceeding to various applications of the risk process (4.1.8), we first look at some nice properties of an interesting special case and its connection with the Lundberg equation we have frequently encountered throughout the thesis.

Theorem 4.1.6. *(Levy-Khintchine Formula) Let X be a Levy process with Levy measure ν. Then*

$$\int_{\mathbb{R}}\min(1,z^2)\nu(dz)<\infty$$

and

$$\mathbb{E}[e^{iuX(t)}]=e^{t\psi(u)},$$

where the Levy exponent

$$\psi(u)=-\frac{1}{2}\beta^2u^2+i\alpha u+\int_{|z|<R}\{e^{iuz}-1-iuz\}\nu(dz)+\int_{|z|\geq R}\{e^{iuz}-1\}\nu(dz).$$

Remark 4.1.2. *If* $\mathbb{E}[X_t] < \infty$, *for all* $t \geq 0$, *then the Levy exponent*

$$\psi(u) = -\frac{1}{2}\sigma^2 u^2 + i\alpha u + \int_{\mathbb{R}} \{e^{iuz} - 1 - iuz\}\nu(dz). \tag{4.1.13}$$

The famous Levy-Khintchine formula gives an explicit expression of the characteristic function of a Levy process. However, as the title of this section alluded to, we are particularly interested in viewing the characteristic function as central moments of the exponential of Levy process

$$E(t) \triangleq e^{X(t)}, \qquad t \geq 0.$$

To provide more insight to the exponential of Levy process, we apply the Ito's formula in Theorem 4.1.4.

$$\begin{aligned} dE(t) &= E(t)dX(t) + \frac{1}{2}E(t)d\langle X^c\rangle_t + \int_{\mathbb{R}} E(t-)\{e^z - 1 - z\}\tilde{N}(dt, dz) \\ &= (\alpha - \frac{1}{2}\beta^2)E(t)dt + \beta E(t)dB(t) + \int_{\mathbb{R}} E(t-)\{e^z - 1\}\tilde{N}(dt, dz), \end{aligned}$$

which is obviously a special case of (4.1.8).

There are many ways of explaining the Lundberg equations, one of such is by a martingale approach introduced in Gerber and Shiu [22]. The major contribution of their work to reveal that the Lundberg equation is the necessary condition on which the exponential of a certain multiple of the compound Poisson risk process by discounting is a martingale. We now follow their idea to generalize the Lundberg equation for Levy process.

The goal is to find the condition on which the process

$$e^{-\delta t + uX(t)}, \qquad t \geq 0,$$

is a martingale under the measure $\mathbb{P}^x$ meaning that $\mathbb{P}^x\{X(0) = x\} = 1$. Applying the optional sampling theorem,

$$\mathbb{E}^x[e^{-\delta t + uX(t)}] = e^{ux}. \tag{4.1.14}$$

It follows from Levy-Khintchine formula that

$$\mathbb{E}^x[e^{-\delta t+uX(t)}] = \exp\left\{\left[-\delta + ux + \frac{1}{2}\beta^2u^2 - \alpha u + \int_{\mathbb{R}}\{e^{-uz} - 1 + uz\}\nu(dz)\right]t\right\}.$$

Therefore, in order for (4.1.14) to hold true for all t's, we must have the Lundberg equation

$$-\delta + \frac{1}{2}\beta^2u^2 - \alpha u + \int_{\mathbb{R}}\{e^{-uz} - 1 + uz\}\nu(dz) = 0. \tag{4.1.15}$$

We conclude the section by recovering the Lundberg equation (1.2.5) for the classical compound Poisson model.

Recall that the shifted compound Poisson process is a special case of Levy process where $\alpha = c + \lambda\kappa, \beta = 0$ and $\nu(y) = \lambda Q(y)$. Inserting the parameters, (4.1.15) reduces to

$$-\delta - (c + \lambda\kappa)u + \lambda\int_0^\infty e^{-uz}Q(dz) - \lambda + \lambda\kappa u = 0$$

which simplifies to (1.2.5) upon rearrangement.

4.2 Generalized Gerber-Shiu Functions

We are now ready to extend the notion of a generalized Gerber-Shiu function in the context of the jump-diffusion model. Hence we define

$$H(x) = \mathbb{E}^x\left[\int_0^{\tau_d} e^{-\delta t}l(X_t)dt\right], \tag{4.2.1}$$

where the cost function $l(\cdot)$ is $\mathcal{B}(\mathbb{R})$-measurable and the time of default τ_d is given by

$$\tau_d = \inf\{t : X_t < d\},$$

with the convention that $\inf\varnothing = \infty$. Intuitively speaking, a generalized Gerber-Shiu function represents the aggregation of discounted business costs up to the time at which the surplus hits the level of default d.

Theorem 4.2.1. *Suppose that $l(x)$ is continuous on (d,∞) except for a countable set of discontinuities D and*

$$\mathbb{E}^x\left[\int_0^{\tau_d}|e^{-\delta t}l(X_t)|dt\right]<\infty \qquad \textit{for all } x>d \tag{4.2.2}$$

and $H(x)$ defined in (4.2.1) has continuous first and second derivatives, then $H(x)$ is the solution to the following differential equation

$$\mathfrak{A}H(x)-\delta H(x)+l(x)=0, \qquad x>d, x\notin D. \tag{4.2.3}$$

Proof. For any $x\in(d,\infty)$ such that $x\notin D$, we let $S_n=n\wedge\inf\{t|X_t\notin(x-1/n,x+1/n)\}$,

$$Z=\int_0^{\tau_d}e^{-\delta s}l(X_s)ds \qquad \text{and} \qquad \theta_t Z=\int_0^{\tau_d}e^{-\delta s}l(X_{t+s})ds.$$

Since $(x-1/n,x+1/n)$ is an open set and X_t is cadlag, then S_n must be a stopping time with respect to $\{\mathcal{F}_{t+}\}$. Hence it is stopping time with respect to $\{\mathcal{F}_t\}$ as the filtration is right-continuous by usual conditions.

Consider a partition of the positive real line $\{t_0=0,t_1,t_2,\cdots\}$. We can approximate Z as follows.

$$\begin{aligned} Z^{(k)} &= \sum_{i=1}^{\infty}e^{-\delta t_i}l(X_{t_i})I(\tau_d\in[t_i,\infty))(t_i-t_{i-1}) \\ &= \sum_{i=1}^{\infty}e^{-\delta t_i}l(X_{t_i})I(\forall r\in[0,t_i),X_r\in[d,\infty)\&\exists s\geq t_i,X_s<d)(t_i-t_{i-1}) \end{aligned}$$

Then

$$\begin{aligned} \theta_t Z^{(k)} &= e^{\delta t}\sum_{i=1}^{\infty}e^{-\delta(t_i+t)}l(X_{t_i+t})I(\forall r\in[0,t_i),X_{t+r}\in[d,\infty)\&\exists s\geq t_i,X_{t+s}<d)(t_i-t_{i-1}) \\ &= e^{\delta t}\sum_{i=1}^{\infty}e^{-(\delta t_i+t)}l(X_{t_i+t})I(\forall r\in[t,t+t_i),X_r\in[d,\infty)\&\exists s\geq t+t_i,X_s<d)(t_i-t_{i-1}) \end{aligned}$$

Therefore, we obtain that

$$\theta_{S_n} Z = e^{\delta S_n} \int_{S_n}^{\tau_d} e^{-\delta s} l(X_s) ds, \tag{4.2.4}$$

where

$$\theta_{S_n} Z(\omega) = \theta_t Z(\omega), \qquad \text{if } S_n(\omega) = t.$$

Hence,

$$\begin{aligned}
\mathbb{E}^x[e^{-\delta S_n} H(X_{S_n}) - H(x)] &= \mathbb{E}^x\{e^{-\delta S_n} E^{X_{S_n}}[Z]\} - \mathbb{E}^x[Z] \qquad \text{(definition)} \\
&= \lim_{M\to\infty} \mathbb{E}^x\{e^{-\delta S_n} E^{X_{S_n}}[Z \wedge M]\} - \mathbb{E}^x[Z] \qquad \text{(Dom Conv Thm \& 4.2.2)} \\
&= \lim_{M\to\infty} \mathbb{E}^x\{e^{-\delta S_n} \mathbb{E}^x[\theta_{S_n} Z \wedge M | \mathcal{F}_{S_n}]\} - \mathbb{E}^x[Z] \qquad \text{(strong Markov)} \\
&= \mathbb{E}^x\{e^{-\delta S_n} \mathbb{E}^x[\theta_{S_n} Z | \mathcal{F}_{S_n}]\} - \mathbb{E}^x[Z] \qquad \text{(Dom Conv Thm \& 4.2.2)}
\end{aligned}$$

Substituting (4.2.4) yields

$$\mathbb{E}^x[e^{-\delta S_n} H(X_{S_n}) - H(x)] = \mathbb{E}^x[-\int_0^{S_n} e^{-\delta s} l(X_s) ds]. \tag{4.2.5}$$

Since H is twice continuously differentiable on the compact set $[x - 1/n, x + 1/n]$, we can re-define H on the compact support in order to apply Theorem 4.1.5,

$$\begin{aligned}
&\lim_{n\to\infty} \left| \frac{\mathbb{E}^x[e^{-\delta S_n} H(X_{S_n}) - H(x)]}{\mathbb{E}^x[S_n]} - \mathfrak{A} H(x) + \delta H(x) \right| \\
= &\lim_{n\to\infty} \left| \frac{\mathbb{E}^x\left[\int_0^{S_n} e^{-\delta s} \mathfrak{A} H(X_s) ds\right] - \delta \mathbb{E}^x\left[\int_0^{S_n} e^{-\delta s} H(X_s) ds\right]}{\mathbb{E}^x[S_n]} - \mathfrak{A} H(x) + \delta H(x) \right| \\
\leq &\lim_{n\to\infty} \sup_{y \in (x-1/n, x+1/n)} |\mathfrak{A} H(y) - \mathfrak{A} H(x) - \delta H(y) + \delta H(x)| = 0
\end{aligned}$$

since $\mathfrak{A} H(x)$ and $H(x)$ are continuous functions.

On the other hand,

$$\lim_{n\to\infty}\left|\frac{\mathbb{E}^x[\int_0^{S_n} l(X_s)ds]}{\mathbb{E}^x[S_n]} - l(x)\right| \leq \lim_{n\to\infty}\sup_{y\in(x-1/n,x+1/n)}|l(y)-l(x)| = 0,$$

as $l(x)$ is continuous when $x \notin D$.

Dividing $\mathbb{E}^x[S_n]$ on both sides of (4.2.5) and taking limit $n \to \infty$ gives (4.2.3). $\square$

4.3 Brownian Motion Risk Model

Originally used to describe the random movement of particles suspended in a liquid, Brownian motion is nowadays widely used in many other areas. One of the frequently quoted examples is the stock market fluctuation. Motivated by both its representation of randomness and mathematical convenience, Brownian motion is lately added in many ways to risk models. We shall now demonstrate the analysis of ruin-related quantities in Brownian motion models by means of generalized Gerber-Shiu function.

Taking $\mu(x) = \mu, \sigma(x) = \sigma \geq 0$ and $F(x,z) = 0$ in (4.1.8) yields the Brownian motion risk model driven by the stochastic differential equation

$$dX_t = \mu dt + \sigma dB_t.$$

We shall not allow $\mu = \sigma^2 = 0$, in which case the process becomes a trivial constant function over time.

We see from (4.1.10) that its corresponding infinitesimal generator is given by

$$\mathfrak{A}f(x) = \mu f'(x) + \frac{1}{2}\sigma^2 f''(x).$$

The graph of a sample path of shifted Brownian motion can be found in Figure 4.1 with parameters $\mu = 0.05$ and $\sigma = 0.3$.

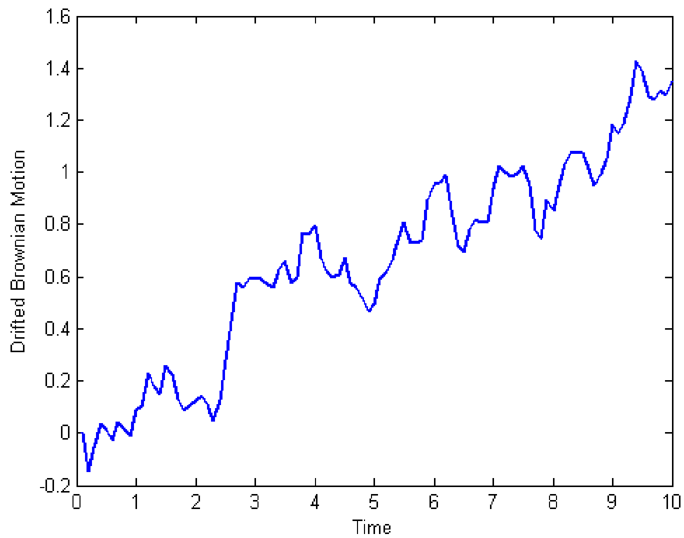

Figure 4.1: Sample path of shifted Brownian motion

4.3.1 Gerber-Shiu Function and Passage Time Distribution

Since ruin occurs exactly at the moment the continuous surplus process lands on zero, there will be no deficit below zero in contrast with the compound Poisson case. Hence it only makes sense to look at a smaller class of the Gerber-Shiu function, the expectation of the time value of ruin defined by

$$L(x) = \mathbb{E}^x[e^{-\delta\tau_0} I(\tau_0 < \infty)]. \tag{4.3.1}$$

where $\delta \geq 0$.

In order to represent it in terms of the generalized Gerber-Shiu function, we shall introduce the famous Dirac delta function (also referred to as unit impulse function) denoted by $\delta(x)$, which is defined with the following properties.

$$\delta(x) = 0 \text{ for } x \neq 0 \qquad \text{and} \int_{-\infty}^{\infty} \delta(x)dx = 1$$

For any continuous function $F(x)$,

$$\int_{-\infty}^{\infty} \delta(x)F(x)dx = F(0).$$

With the aid of Dirac delta function, the expected value of the time of ruin can be written as

$$L(x) = \mathbb{E}^x \left[\int_0^{\tau_0} e^{-\delta t} l(X_t) dt \right],$$

where $l(x) = \delta(x)$. Since $L(x)$ is bounded, by Theorem 4.2.1 we see that $L(x)$ is a solution to the differential equation

$$\frac{1}{2}\sigma^2 L''(x) + \mu L'(x) - \delta L(x) = 0, \qquad x > 0. \tag{4.3.2}$$

Corollary 4.3.1. *The solution to $L(x)$ defined in (4.3.1) is given, when $\sigma^2 > 0$, by*

$$L(x) = \exp\{\frac{-(\mu + \sqrt{\mu^2 + 2\sigma^2\delta})x}{\sigma^2}\}, \qquad x \geq 0. \tag{4.3.3}$$

When $\sigma^2 = 0$, the solution to $L(x)$ is given by

$$L(x) = \begin{cases} \exp\{\frac{\delta}{\mu}x\}, & \mu < 0, x \geq 0, \\ 0, & \mu > 0. \end{cases}$$

Proof. When $\sigma^2 > 0$, the general solution to (4.3.2) is given by

$$L(x) = C_1 e^{\gamma_1 x} + C_2 e^{\gamma_2 x},$$

where

$$\gamma_1 = \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2\delta}}{\sigma^2} \tag{4.3.4}$$

and

$$\gamma_2 = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2\delta}}{\sigma^2} \tag{4.3.5}$$

which are the two roots of the Lundberg equation

$$\frac{1}{2}\sigma^2\gamma^2 + \mu\gamma - \delta = 0.$$

In light of the fact that $\lim_{x\to\infty} L(x) = 0$ and $L(0) = 1$, the solution to the Gerber-Shiu function (4.3.1) must be (4.3.3).

When $\sigma^2 = 0$, the differential equation reduces to

$$\mu L'(x) - \delta L(x) = 0, \qquad x > 0,$$

which admits solution

$$L(x) = A + B\exp\{\frac{\delta}{\mu}x\},$$

where A, B are constants to be determined. Since $\lim_{x\to\infty} L(x) = 0$ and $L(0) = 1$, we find $A = 0, B = 1$ when $\mu < 0$, and $B = 0$ when $\mu > 0$. $\square$

Since the expectation of time value of ruin is the Laplace transform of the time of ruin, inverting (4.3.3) with respect to δ gives the density function of the time of ruin,

$$f_{\tau_0}(t) = \frac{xt^{-3/2}}{\sqrt{2\pi\sigma^2}}\exp\{-\frac{(x+\mu t)^2}{2\sigma^2 t}\}.$$

Note that the density is defective, as it does not integrate to one when $\mu > 0$. Integrating with respect to t, we would obtain the (defective) distribution function of the time of ultimate ruin

$$F_{\tau_0}(t) = \Phi(-\frac{x+\mu t}{\sqrt{\sigma^2 t}}) + \exp\{-\frac{2\mu}{\sigma^2}x\}\Phi(-\frac{x-\mu t}{\sqrt{\sigma^2 t}}),$$

where $\Phi(x)$ is the standard normal distribution function. This function is given in equation (8.29) in Klugman *et al.* [33] by taking the conventional approach of reflecting properties.

We are now ready to derive the probability of ultimate ruin by taking the limit of (4.3.3) when $\delta \to 0$. But one has to do this with caution about the sign of the drift coefficient μ. Hence we have

$$\psi(x) = \mathbb{P}^x\{\tau_0 < \infty\} = \begin{cases} \exp\{\frac{-2\mu x}{\sigma^2}\}, & \text{if } \mu > 0, \\ 1, & \text{if } \mu \le 0. \end{cases}$$

The first part of the solution was given by equation (8.33) in Klugman *et al.* [33]. Hence, the (proper) density function of the time of ultimate ruin given that it occurs is given by

$$\frac{f_{\tau_0}(t)}{\psi(x)} = \frac{xt^{-3/2}}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(x-|\mu|t)^2}{2\sigma^2 t}\},$$

which is the density of an inverse Gaussian distribution with mean $x/|\mu|$ and $x\sigma^2/|\mu|^3$ when $\mu \neq 0$ and the density of an one-sided stable law with index $1/2$ when $\mu = 0$.

In the case where $\sigma^2 = 0$ and $\mu < 0$, it is apparent that $L(x)$ is a Laplace transform of a constant

$$\tau_0 = -\frac{x}{\mu}. \tag{4.3.6}$$

When the surplus process does not have a diffusion component, it is nothing more than a linear function of t. Only if $\mu < 0$, the linear function goes from x to 0 by the time $-x/\mu$.

4.3.2 Total Dividends Paid up to Ruin by Threshold

Under the dividend threshold strategy, the sample path is generated by the stochastic differential equation (4.1.8) with $\sigma(x) = \sigma$ and

$$\mu(x) = \begin{cases} \mu - \alpha, & x \geq b, \\ \mu, & x < b. \end{cases}$$

The drift term representing the net influx of cash flow is the balance of premium income at rate of μ offset by dividend payments at rate of α, when the surplus reaches the dividend threshold b. Since we are interested in the total amount of discounted dividend payments up to the time of ruin, we define

$$V(x) \triangleq \mathbb{E}^x \left[\int_0^{\tau_0} e^{-\delta t} l(X_t) dt \right], \qquad x \geq 0,$$

where the cost function is taken to be

$$l(x) = \begin{cases} \alpha, & x \geq b, \\ 0, & x < b. \end{cases}$$

Hence (4.2.3) turns into

$$\begin{aligned}\frac{1}{2}\sigma^2 V''(x)+(\mu-\alpha)V'(x)-\delta V(x)+\alpha &= 0, \qquad x\ge b,\\ \frac{1}{2}\sigma^2 V''(x)+\mu V'(x)-\delta V(x) &= 0, \qquad 0\le x<b.\end{aligned}$$

These two equations are precisely (2.13) and (2.7) in Gerber and Shiu [28].

Absolute ruin with dividend threshold

If the insurer is allowed to borrow money with debit force of interest r, then the sample path of the deterministic part is changed with

$$\mu(x)=\begin{cases}\mu-\alpha, & x\ge b,\\ \mu, & 0\le x<b,\\ \mu+rx, & -c/r<x<0.\end{cases}$$

Therefore, the generalized Gerber-Shiu function satisfies

$$\begin{aligned}\frac{1}{2}\sigma^2 V''(x)+(\mu-\alpha)V'(x)-\delta V(x)+l(x) &= 0, \qquad x\ge b,\\ \frac{1}{2}\sigma^2 V''(x)+\mu V'(x)-\delta V(x)+l(x) &= 0, \qquad 0\le x<b,\\ \frac{1}{2}\sigma^2 V''(x)+(\mu+rx)V'(x)-\delta V(x)+l(x) &= 0, \qquad -c/r<x<0.\end{aligned}$$

4.3.3 Total Dividends Paid up to Ruin by Barrier

In the extreme case of threshold strategy, the dividend rate α can be chosen to be equal to the premium rate μ. Then the surplus process bounces back as soon as it hits from below the dividend barrier b_0 due to the nature of oscillations. In terms of the infinitesimal generator, the dynamics of surplus growth is determined by (4.1.10) with $\sigma(x)=\sigma$, $F(x,z)=0$ and

$$\mu(x)=\begin{cases}0, & x\ge b_0,\\ \mu, & 0\le x<b_0.\end{cases}$$

In order to calculate the total amount of dividends paid up to ruin, we choose the cost function in (4.2.1) to be

$$l(x) = \begin{cases} \mu, & x \geq b_0, \\ 0, & 0 \leq x < b_0. \end{cases}$$

Therefore, when $\delta > 0$, the expected present value of total dividends paid up to ruin is bounded and hence satisfies the integro-differential equation

$$\frac{1}{2}\sigma^2 V''(x) + \mu V'(x) - \delta V(x) = 0, \qquad 0 \leq x < b_0,$$

and the boundary condition stated in the following corollary.

Corollary 4.3.2. *With the dividend barrier strategy, the function $V(x)$ defined in (4.2.1) satisfies the following boundary condition*

$$\mathfrak{A}V(b_0) - \delta V(b_0) + l(b_0) = 0. \tag{4.3.7}$$

Proof. The proof mirrors that of Theorem 4.2.1.

Choose $x = b_0$ and $S_n = n \wedge \inf\{t | X_t \notin [b_0, b_0 + 1/n]\}$. Since the pure diffusion process is continuous, it is easy to see that S_n is indeed a stopping time with respect to the adapted filtration $\{\mathcal{F}_t\}$. The rest of the argument follows in exactly the same manner and the result is achieved since $V(x)$ is assumed to be continuous and $l(x)$ is right-continuous in $[b_0, b_0 + 1/n]$ for all n's. $\square$

The condition (4.3.7) is written in the format consistent with (2.3.6). Inserting all specific functions, it can be simplified as

$$\frac{1}{2}\sigma^2 V''(b_0) - \delta V(b_0) + \mu = 0. \tag{4.3.8}$$

Since $V(x)$ is twice differentiable, we must have

$$\begin{aligned} V'(b_0) = \lim_{x \to b_0} V'(x) &= \lim_{x \to b_0} \frac{\delta}{\mu} V(x) - \frac{\sigma^2}{2\mu} V''(x) \\ &= \frac{1}{\mu}[\delta V(b_0) - \frac{\sigma^2}{2} V''(b_0)], \end{aligned}$$

which, combined with (4.3.8), gives us the alternative boundary condition

$$V'(b_0) = 1.$$

Interested readers may refer to Gerber and Shiu [23] for an explicit solution for the total dividends paid up to ruin by barrier strategy in the Brownian motion risk model as a result of the boundary condition.

4.3.4 Insurer's Accumulated Discounted Utility

A measurement of an insurer's overall performance in maintaining its surplus in a particular line of business is given by the accumulated utility on the surplus from the date of inception to the date of default, which is a special case of the generalized Gerber-Shiu function,

$$U(x) = \mathbb{E}^x \left[\int_0^{\tau_d} e^{-\delta t} u(X_t) dt \right],$$

where $\delta \geq 0$ and $u(x)$ is the utility function of its surplus level. When $\delta = 0$, $U(x)$ reduces to the insurer's accumulated utility. We choose the exponential utility function $u(x) = -e^{-ax}/a$ as it is more mathematically tractable than other utility functions.

We are now interested in the function $W(x)$ in the Brownian Motion surplus model defined by

$$W(x) = \mathbb{E}^x \left[\int_0^{\tau_d} e^{-\delta t} e^{-aX_t} dt \right]. \tag{4.3.9}$$

The function $W(x)$ can be used to determine

$$U(x) = -\frac{1}{a} W(x).$$

However, one should be cautious that $W(x)$ might not converge for all a's. As the function shall be used to facilitate calculations of other quantities in other chapters, we allow a to be negative and find an as wide as possible range of a where $W(x)$ satisfies (4.2.2).

Lemma 4.3.1. *Suppose that $\sigma^2 > 0$ and*

$$a^2\sigma^2/2 - a\mu - \delta < 0. \tag{4.3.10}$$

Then the function $W(x)$ defined in (4.3.9) satisfies condition (4.2.2) and

$$W(x) \le \frac{e^{-ax}}{\delta + a\mu - a^2\sigma^2/2}. \tag{4.3.11}$$

In addition, if $a \ge 0$, $W(x)$ is bounded for all $x > d$.

Proof. Since $l(x) = e^{-ax}$ is non-negative,

$$\begin{aligned} \mathbb{E}^x\left[\int_0^{\tau_d} |e^{-\delta t} l(X_t)| dt\right] &= \mathbb{E}^x\left[\int_0^{\tau_d} e^{-\delta t} l(X_t) dt\right] \le \mathbb{E}^x\left[\int_0^{\infty} e^{-\delta t} e^{-aX_t} dt\right] \\ &= \int_0^{\infty} e^{-\delta t}\mathbb{E}^x[e^{-aX_t}] dt = e^{-ax}\int_0^{\infty} e^{-(\delta + a\mu - a^2\sigma^2/2)t} dt \\ &= \frac{1}{\delta + a\mu - a^2\sigma^2/2} e^{-ax} < \infty, \qquad \text{for all } x > d, \end{aligned}$$

with the last equality from the assumption that $a^2\sigma^2/2 - a\mu - \delta < 0$. And it follows that $W(x)$ is bounded when $a \ge 0$. □

Remark 4.3.1. *1. The condition (4.3.10) is equivalent to say, when $\sigma^2 \ne 0$,*

$$\frac{\mu - \sqrt{\mu^2 + 2\sigma^2\delta}}{\sigma^2} < a < \frac{\mu + \sqrt{\mu^2 + 2\sigma^2\delta}}{\sigma^2}. \tag{4.3.12}$$

2. When $\sigma^2 = 0$, we can similarly prove that (4.2.2) is satisfied if either of the two conditions holds:

(a) $\mu > 0$ *and* $a\mu + \delta > 0$,

(b) $\mu < 0$.

Hence if the parameters satisfy any of the conditions in Remark 4.3.1, it follows from Theorem 4.2.1 that $W(x)$ satisfies the following differential equation

$$\frac{1}{2}\sigma^2 W''(x) + \mu W'(x) - \delta W(x) + e^{-ax} = 0, \qquad x > d. \tag{4.3.13}$$

Together with boundary conditions, we obtain an explicit solution to the accumulated exponential utility.

Corollary 4.3.3. *If* $\sigma^2 > 0$ *and (4.3.10) is satisfied, then the solution to* $W(x)$ *defined in (4.3.9) is given by*

$$W(x) = \frac{2}{a^2\sigma^2 - 2a\mu - 2\delta} e^{-ad-(\mu+\sqrt{\mu^2+2\sigma^2\delta})(x-d)/\sigma^2} - \frac{2}{a^2\sigma^2 - 2a\mu - 2\delta} e^{-ax}, \qquad x > d. \tag{4.3.14}$$

If $\sigma^2 = 0, \mu > 0$ *and* $a\mu + \delta > 0$,

$$W(x) = \frac{e^{-ax}}{a\mu + \delta}, \qquad x > d.$$

If $\sigma^2 = 0$ *and* $\mu < 0$,

$$W(x) = \frac{1}{a\mu + \delta} e^{-ax} - \frac{1}{a\mu + \delta} e^{\delta x/u - (a+\delta/\mu)d}, \qquad x > d.$$

Proof. We first determine a particular solution to (4.3.13) given of the form $C_1 e^{-ax}$. Inserting into (4.3.13) gives

$$\frac{a^2\sigma^2}{2} C_1 e^{-ax} - a\mu C_1 e^{-ax} - \delta C_1 e^{-ax} + e^{-ax} = 0,$$

which yields

$$C_1 = \frac{1}{a\mu + \delta - (1/2)a^2\sigma^2}.$$

Hence when $\sigma^2 > 0$, the general solution to (4.3.13) must be in the form,

$$W(x) = C_1 e^{-ax} + C_2 e^{\gamma_1 x} + C_3 e^{\gamma_2 x},$$

where the last two terms constitute a complimentary solution to the corresponding homogeneous differential equation (4.3.2).

It follows from the condition (4.3.11) that

$$e^{ax} W(x) = C_1 + C_2 e^{(a+\gamma_1)x} + C_3 e^{(a+\gamma_2)x} \leq \text{constant}. \tag{4.3.15}$$

By (4.3.12) we know that

$$\begin{aligned} a + \gamma_1 &= a - \frac{\mu + \sqrt{\mu^2 + 2\sigma^2\delta}}{\sigma^2} < 0, \\ a + \gamma_2 &= a - \frac{\mu - \sqrt{\mu^2 + 2\sigma^2\delta}}{\sigma^2} > 0. \end{aligned}$$

Then we must have $C_3 = 0$ by taking limit $x \to \infty$ on both sides of the inequality in (4.3.15).

Based on the condition that $W(d) = 0$, we obtain the last unknown

$$C_2 = \frac{2}{a^2\sigma^2 - 2a\mu - 2\delta} e^{-ad + (\mu + \sqrt{\mu^2 + 2\sigma^2\delta})d/\sigma^2}.$$

Therefore, we obtain (4.3.14) upon substitution and rearrangement.

When $\sigma^2 = 0$, the general solution to (4.3.13) must be in the form,

$$W(x) = C_1 e^{-ax} + C_2 e^{\delta x/\mu}.$$

According to (4.3.11) we have

$$e^{ax}W(x) = C_1 + C_2 e^{(a+\delta/\mu)x} \leq \text{constant}. \tag{4.3.16}$$

If $\mu > 0$, then the condition (4.3.10) reduces to $a + \delta/\mu > 0$. By taking limit $x \to \infty$ on both sides of (4.3.16), we must have $C_2 = 0$. Hence

$$W(x) = \frac{e^{-ax}}{a\mu + \delta}.$$

If $\mu < 0$, we can not eliminate C_2 by (4.3.16). But since $W(d) = 0$, we would have

$$C_2 = -\frac{1}{a\mu + \delta} e^{-(a+\delta/\mu)d}.$$

Therefore,

$$W(x) = \frac{1}{a\mu + \delta} e^{-ax} - \frac{1}{a\mu + \delta} e^{\delta x/u - (a+\delta/\mu)d}.$$

$\square$

4.4 Geometric Brownian Motion Risk Model

Geometric Brownian motion is the most widely used stochastic process in financial modelling, much owing to its computational tractability. There has also been a growing number of papers in actuarial literature to involve geometric Brownian motion in pricing insurance and investment combined products. Understanding its significance in financial and actuarial modelling, we shall now investigate certain quantities of ruin theoretic interests arising from the geometric Brownian motion model. Applications of these quantities will be seen in credit risk modelling in the next chapter.

Suppose the insurer's equity index, denoted by $S(t)$, is represented by a geometric Brownian motion

$$S(t) = e^{X_t} = e^{x+\mu t+\sigma B_t}, \qquad t \geq 0, \tag{4.4.1}$$

where the initial value is given by $S(0) = s = e^x$. Note that both $S(t)$ and $X(t)$ are defined one-to-one correspondent on the same probability space, hence we shall use $\mathbb{P}^s$ whenever $S(t)$ appears so as to emphasize the corresponding measure's dependency on the initial value of $S(t)$.

Applying the Ito's formula, one can easily see that $S(t)$ is a solution to the stochastic differential equation

$$dS(t) = \nu S(t)dt + \sigma S(t)dB(t), \qquad t \geq 0,$$

where $S(0) = s > 0, \nu = \mu + \sigma^2/2$. Note that the geometric Brownian motion is by itself a special case of (4.1.8) in which $\mu(x) = \nu x, \sigma(x) = \sigma x$ and $F(x, z) = 0$. Hence the infinitesimal generator of the geometric Brownian motion is given by

$$\mathfrak{X} f(s) = \nu s f'(s) + \frac{1}{2}\sigma^2 s^2 f''(s).$$

To visualize the geometric Brownian motion, we now give a sample path of the geometric Brownian motion with parameters $\mu = 0.05, \sigma = 0.3$ and $s = 15$ in Figure 4.2.

We are now interested in the Laplace transform of the time of index default defined by

$$L(s) \triangleq \mathbb{E}^s[e^{-\delta\tau_b} I(\tau_b < \infty)],$$

where the first time index goes below a predetermined level of default $b > 0$ is given by

$$\tau_b \triangleq \inf\{t | S(t) < b\}$$

with the convention that $\inf \varnothing = \infty$. In the above expression, $\mathbb{E}^s$ corresponds to the probability measure $\mathbb{P}^s$. In view of (4.4.1), we must have $\tau_b = \inf\{t | X(t) < \ln b\}$. Hence we could easily obtain solutions from the relationship that

$$L(s) = \mathbb{E}^x[e^{-\delta\tau_{\ln b}} I(\tau_{\ln b} < \infty)] = \mathbb{E}^{x-\ln b}[e^{-\delta\tau_0} I(\tau_0 < \infty)].$$

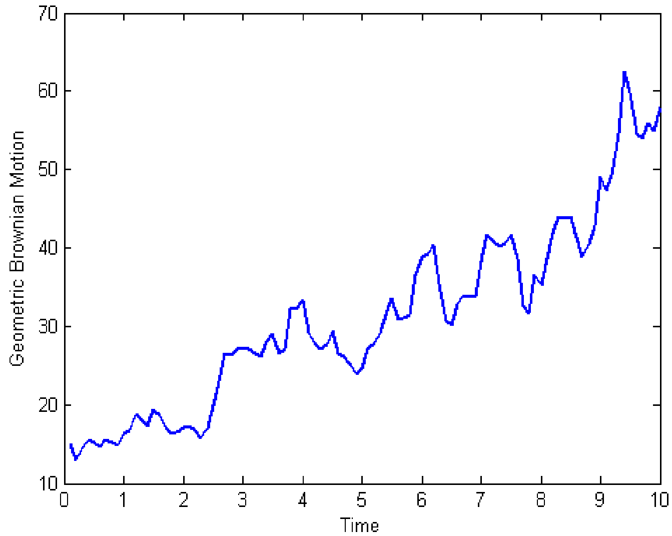

Figure 4.2: Sample path of geometric Brownian motion

However, since $S(t)$ by itself is a special case of (4.1.8), we shall use Theorem 4.2.1 to obtain solutions for the purpose of illustration. For simplicity, we skip the case where $\sigma^2 = 0$.

As shown in the previous section, an alternative way to express $L(s)$ is given by the generalized Gerber-Shiu function

$$L(s) = \mathbb{E}^s \left[\int_0^{\tau_b} e^{-\delta t} l(S_t) dt \right], \qquad s \geq b \tag{4.4.2}$$

where the cost function $l(s) = \delta(s - b)$.

It follows from Theorem 4.2.1 that $L(s)$ is a solution to the differential equation

$$\frac{1}{2}\sigma^2 s^2 L''(s) + \nu s L'(s) - \delta L(s) = 0, \qquad s > b, \tag{4.4.3}$$

which is an Euler equation.

Corollary 4.4.1. *The solution to $L(s)$ defined in (4.4.2) is given by*

$$L(s) = \left(\frac{s}{b}\right)^{\gamma_1} = \exp\left\{ -\ln\left(\frac{s}{b}\right) \frac{\mu + \sqrt{\mu^2 + 2\sigma^2\delta}}{\sigma^2} \right\}, \qquad s \geq b. \tag{4.4.4}$$

Proof. By inspection, we conjecture that the solution to $L(s)$ would be in the form of As^{γ}. Substituting into (4.4.3) yields

$$\frac{1}{2}\sigma^2 A\gamma(\gamma-1)s^{\gamma} + \nu A s^{\gamma} - \delta A s^{\gamma} = 0.$$

Hence we must have

$$\frac{1}{2}\sigma^2\gamma + (\nu - \frac{1}{2}\sigma^2)\gamma - \delta = 0,$$

which admits two roots of each sign

$$\begin{aligned} \gamma_1 &= \frac{-\nu + \sigma^2/2 - \sqrt{(\nu - \sigma^2/2)^2 + 2\sigma^2\delta}}{\sigma^2} = \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2\delta}}{\sigma^2}, \\ \gamma_2 &= \frac{-\nu + \sigma^2/2 + \sqrt{(\nu - \sigma^2/2)^2 + 2\sigma^2\delta}}{\sigma^2} = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2\delta}}{\sigma^2}. \end{aligned}$$

Therefore, the general solution to $L(s)$ must be

$$L(s) = A_1 s^{\gamma_1} + A_2 s^{\gamma_2}.$$

Recall that $\lim_{s\to\infty} L(s) = 0$, hence $A_2 = 0$. By the definition of Dirac delta function, $L(b) = 1$, which implies that $A_1 = 1/b^{\gamma_1}$. $\square$

Inverting the Laplace transform with respect to δ, we have the density function of the time of index default

$$f_{\tau_b}(t) = \frac{\ln(s/b)t^{-3/2}}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\ln(s/b) + \mu t)^2}{2\sigma^2 t}\right\}. \tag{4.4.5}$$

Hence the (defective) distribution function of the time of index default is given by

$$F_{\tau_b}(t) = \Phi(-\frac{\ln(s/b) + \mu t}{\sqrt{\sigma^2 t}}) + \left(\frac{s}{b}\right)^{-2\mu/\sigma^2} \Phi(-\frac{\ln(s/b) - \mu t}{\sqrt{\sigma^2 t}}). \tag{4.4.6}$$

4.5 Ornstein-Uhlenbeck Risk Model

Ornstein-Uhlenbeck process, also known as the mean-reverting process, is often defined by the stochastic differential equation

$$dX_t = \theta(\mu - X_t)dt + \sigma dB_t.$$

The process has a bounded variance and converges to a stationary probability distribution. One of the most prominent examples is the Vasicek model of short rate in finance literature.

It was first proposed by Cai *et al.* [10] that Ornstein-Uhlenbeck process serves as a risk process to approximate the fluctuation of insurance surplus. The version of Ornstein-Uhlenbeck process used in Cai *et al.* [10] is given by

$$dX_t = (\mu + \rho X_t)dt + \sigma dB_t, \tag{4.5.1}$$

which has the natural interpretation that the insurance surplus is continuously funded by premium income at rate of μ and investment return with a constant yield rate ρ. Since the total surplus to be invested varies from time to time, the rate of interest due at time t is proportional to the current amount of surplus and hence given by ρX_t. The source of randomness in surplus is assumed to be accurately captured in the Brownian motion component. A sample path of Ornstein-Uhlenbeck process with parameters $\mu = 0.1, \rho = 0.3, \sigma = 0.05$ and $x = 0.5$ is generated in Figure 4.3.

Having $\mu(x) = \mu + \rho x, \sigma(x) = \sigma$ and $F(x, z) = 0$ gives the Ornstein-Uhlenbeck type risk model (4.5.1). By (4.1.10) its infinitesimal generator is given by

$$\mathfrak{A} f(x) = (\mu + \rho x) f'(x) + \frac{1}{2}\sigma^2 f''(x).$$

Cai *et al.* [10] focused on the dividends paid up to ruin and the Laplace transform of the time of ruin under the dividend barrier strategy. As an application of Theorem 4.2.1, we now demonstrate by the new approach how to obtain the differential equation satisfied

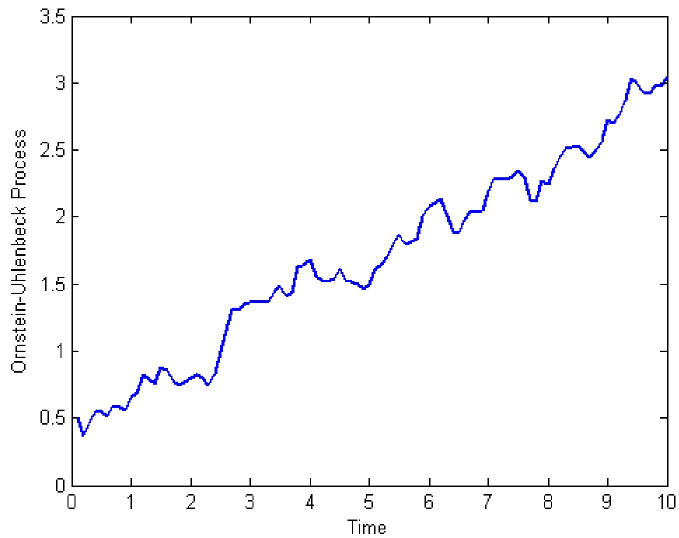

Figure 4.3: Sample path of Ornstein-Uhlenbeck process

by the dividends paid up to ruin under dividend threshold strategy

$$\begin{aligned}\frac{1}{2}\sigma^2 V''(x;b) + (\mu - \alpha + \rho x)V'(x;b) - \delta V(x;b) + \alpha &= 0, \qquad x \geq b;\\ \frac{1}{2}\sigma^2 V''(x;b) + (\mu + \rho x)V'(x;b) - \delta V(x;b) &= 0, \qquad 0 \leq x < b.\end{aligned}$$

The Gerber-Shiu function, which is simply the Laplace transform of the time of ruin, can be obtained from

$$\frac{1}{2}\sigma^2 m''(x) + (\mu + \rho x)m'(x) - \delta m(x) = 0, \qquad x > 0.$$

4.6 Kou Jump Diffusion Model

The Kou jump diffusion model was proposed out of the need to address two phenomenons observed in empirical studies, which can not be explained by the Black-Scholes model. Interested readers may read Kou [34] for its background and Kou and Wang [35], Dao and Jeanblanc [12] for applications in option pricing and credit risk modelling.

The asset price in Kou's model is driven by the exponential of a Brownian motion and a compound Poisson process with both positive and negative exponential jumps.

$$S(t) \triangleq e^{X(t)} \triangleq \exp\left\{x + \left(r - \frac{1}{2}\sigma^2 - \lambda[\tilde{q}(-1) - 1]\right)t + \sigma B(t) + \sum_{i=1}^{N(t)} Y_i\right\}, \qquad (4.6.1)$$

where r is the risk-free rate of return, σ is the volatility coefficient, $\{N(t), t \geq 0\}$ is a Poisson counting process with intensity λ and the sequence of jumps $\{Y_1, Y_2, \cdots\}$ follows the common asymmetric double exponential distribution

$$Q(y) = \pi\lambda\beta_1 e^{-\beta_1 y} I(y > 0) + (1-\pi)\lambda\beta_2 e^{\beta_2 y} I(y < 0) \qquad (4.6.2)$$

with the Laplace transform denoted by $\tilde{q}(s)$. It can be shown using Ito's formula for semi-martingale that the process $S = \{S(t), t \geq 0\}$ is a solution to the stochastic differential equation

$$dS(t) = S(t)\left\{r dt + \sigma dB(t) + \int_{\mathbb{R}} (e^z - 1)\tilde{N}(dt, dz)\right\},$$

where the Poisson random measure

$$\tilde{N}(t, A) = \sum_{i=1}^{N(t)} I(Y_i \in A) - \lambda \mathbb{P}(Y_i \in A)t, \qquad A \subset \mathbb{R}/\{0\}.$$

Note that the asset price process S in (4.6.1) is set in such a way that the discounted price process $\{e^{-rt}S(t), t \geq 0\}$ is a martingale under the measure $\mathbb{P}^x$.

We can recover Kou's model from the general process (4.1.8) by letting $\mu(s) = rs, \sigma(s) = \sigma s, F(s, z) = s(e^z - 1)$ and $Q(y)$ as defined in (4.6.2). Hence it follows from (4.1.10) that the infinitesimal generator for process S is given by

$$\mathfrak{A}f(s) = \frac{1}{2}\sigma^2 s^2 f''(s) + \hat{\mu} s f'(s) + \lambda \int_{\mathbb{R}} [f(s + e^y - 1) - f(s)] dQ(y),$$

where

$$\hat{\mu} = r - \int_{\mathbb{R}} (e^z - 1)\nu(dz) = r - \lambda[\tilde{q}(-1) - 1].$$

However, the easiest way to solve functionals of Kou's jump diffusion process is to find functionals of its exponent $X = \{X(t), t \geq 0\}$, which is a much simpler jump diffusion process, and then write the functionals of the original process S as a function of the functional of X.

The infinitesimal generator of X is easily found by replacing $\mu(x) = r - \frac{1}{2}\sigma^2 - \lambda[\tilde{q}(-1) - 1], \sigma(x) = x, F(x, z) = z$ in (4.1.10).

$$\mathfrak{A}f(x) = \frac{1}{2}\sigma^2 f''(x) + \hat{\mu}f'(x) + \lambda \int_{\mathbb{R}} [f(x+z) - f(x)]dQ(y),$$

where

$$\begin{aligned}\hat{\mu} &= r - \frac{1}{2}\sigma^2 - \lambda[\tilde{q}(-1) - 1] - \lambda\kappa \\ &= r - \frac{1}{2}\sigma^2 - \lambda\left[\pi\frac{\beta_1}{\beta_1 + 1} + (1-\pi)\frac{\beta_2}{\beta_2 - 1} - 1 - (\frac{\pi}{\beta_1} - \frac{1-\pi}{\beta_2})\right].\end{aligned}$$

4.6.1 Gerber-Shiu Function

In this subsection, we shall look at ruin-related quantities of the jump diffusion process X. Once these quantities are obtained, they could be easily used to provide solutions to ruin-related quantities of the asset price process S.

In a jump diffusion risk model, there are two types of causes for ruin. When the surplus is running low, it might be dropped to a level below zero by a large insurance claim, or gradually declines to zero by oscillation. Since the Gerber-Shiu function in either case corresponds to a different cost function, we shall treat them separately.

We define the expected discounted penalty at ruin due to jump by

$$m_J(x) = \mathbb{E}^x[e^{-\delta\tau_0}w(X(\tau_0-), |X(\tau_0)|)I(\tau < \infty, X(\tau_0) < 0)]$$

and the expected discounted penalty at ruin due to diffusion by

$$m_D(x) = \mathbb{E}^x[e^{-\delta\tau_0}w(0,0)I(\tau < \infty, X(\tau_0) = 0)].$$

By the law of total probability, the Gerber-Shiu function is a sum of the two functions

$$m(x) = m_J(x) + m_D(x).$$

Note that Brownian motion changes sign infinitely often in as small interval as one wants. Once hits zero, the surplus ruins by oscillation in an instantaneous moment. Hence by continuity one can prove that $X(\tau_0) = 0$, $\mathbb{P}^x$-$a.s.$ On the other hand, a jump does not cause ruin until it brings the surplus strictly below zero. Hence the two terms unambiguously distinguish the two situations.

Once known the cause of ruin, we can represent the expected discounted penalty at ruin in the form of the generalized Gerber-Shiu function as we did in the previous few chapters. Since jumps are governed by the embedded compound Poisson component, we can follow the same arguments as in Section 2.3.5 to prove that the expected discounted penalty due to jump

$$m_J(x) = \mathbb{E}^x \left[\int_0^{\tau_0} e^{-\delta t} l(X_t) dt \right],$$

where

$$l(x) = \lambda \int_x^{\infty} w(x, y - x) dQ(y).$$

Similarly, the expected discounted penalty due to oscillation can be written as

$$m_D(x) = \mathbb{E}^x \left[\int_0^{\tau_0} e^{-\delta t} w(0,0) \delta(X_t) dt \right],$$

where $\delta(x)$ is the Dirac delta function given in Section 4.3.1.

Applying Theorem 4.2.1, we obtain the following system of differential equations to solve for the Gerber-Shiu functions.

$$\frac{1}{2}\sigma^2 m_J''(x) + \hat{\mu} m_J'(x) - (\lambda + \delta) m_J(x) + \lambda \int m_J(x+y) dQ(y) + \lambda \int_x^{\infty} w(x, y-x) dQ(y) = 0, \quad x > 0,$$

$$\frac{1}{2}\sigma^2 m_D''(x) + \hat{\mu} m_D'(x) - (\lambda + \delta) m_D(x) + \lambda \int m_D(x+y) dQ(y) = 0, \qquad x > 0.$$

Expected Discounted Penalty at Ruin Due to Jump

For the purpose of applications in later section, it suffices to study the expected discounted penalty at ruin when the penalty function is only dependent on the surplus at ruin.

Suppose we have a bounded penalty function $f(x)$,

$$m_J(x) = \mathbb{E}^x[e^{-\delta\tau_0} f(|X(\tau_0)|) I(\tau_0 < \infty, X(\tau_0) < 0)]. \tag{4.6.3}$$

Hence it can be written in terms of the generalized Gerber-Shiu function

$$m_J(x) = \mathbb{E}^x\left[\int_0^{\tau_0} l(X_t)dt\right],$$

where

$$l(x) = \lambda\int_x^\infty f(y-x)dQ^-(y) = \lambda\beta_2 e^{-\beta_2 x}\int_0^\infty f(z)e^{-\beta_2 z}dz.$$

Substituting in the claim size distribution, we have

$$\begin{aligned}\frac{1}{2}\sigma^2 m_J''(x) + \hat{\mu}m_J'(x) - (\lambda+\delta)m_J(x) + \lambda\pi\beta_1 e^{\beta_1 x}\int_x^\infty e^{-\beta_1 y}m_J(y)dy \\ +\lambda(1-\pi)\beta_2 e^{-\beta_2 x}\int_0^x e^{\beta_2 y}m_J(y)dy + l(x) = 0, \qquad x > 0.\end{aligned} \tag{4.6.4}$$

Corollary 4.6.1. *If the claim size distribution $Q(y)$ is given by (4.6.2), the Gerber-Shiu function defined in (4.6.3) admits an explicit solution given by*

$$\begin{aligned} m_J(x) &= \left[\frac{(\beta_2+s_1)(\beta_2+s_2)}{s_1-s_2}\int_0^\infty e^{-\beta_2 z}f(z)dz\right]e^{s_1 x} \\ &\quad + \left[\frac{(\beta_2+s_1)(\beta_2+s_2)}{s_2-s_1}\int_0^\infty e^{-\beta_2 z}f(z)dz\right]e^{s_2 x}.\end{aligned}$$

Proof. Represent (4.6.4) in terms of operators,

$$\frac{1}{2}\sigma^2\mathcal{D}^2 m_J(x) + \hat{\mu}\mathcal{D}m_J(x) - (\lambda+\delta)m(x) + \lambda\pi\beta_1\mathcal{T}_{\beta_1}m_J(x) + \lambda(1-\pi)\beta_2\mathcal{E}_{\beta_2}m_J(x) + l(x) = 0.$$

Multiplying both sides by $(\beta_1\mathcal{I}-\mathcal{D})(\beta_2\mathcal{I}+\mathcal{D})$ we obtain

$$\begin{aligned}\Big\{\frac{1}{2}\sigma^2\mathcal{D}^2(\beta_1\mathcal{I}-\mathcal{D})(\beta_2\mathcal{I}+\mathcal{D}) + \hat{\mu}\mathcal{D}(\beta_1\mathcal{I}-\mathcal{D})(\beta_2\mathcal{I}+\mathcal{D}) - (\lambda+\delta)(\beta_1\mathcal{I}-\mathcal{D})(\beta_2\mathcal{I}+\mathcal{D}) \\ +\lambda\pi\beta_1(\beta_2\mathcal{I}+\mathcal{D}) + \lambda(1-\pi)\beta_2(\beta_1\mathcal{I}-\mathcal{D})\Big\}m_J(x) + (\beta_1\mathcal{I}-\mathcal{D})(\beta_2\mathcal{I}+\mathcal{D})l(x) = 0,\end{aligned}$$

which simplifies to

$$\frac{1}{2}\sigma^2 m_J^{(4)}(x)+(\hat{\mu}-\frac{1}{2}\sigma^2\beta_1+\frac{1}{2}\sigma^2\beta_2)m_J^{(3)}(x)+(\hat{\mu}\beta_2-\hat{\mu}\beta_1-\lambda-\delta-\frac{1}{2}\sigma^2\beta_1\beta_2)m_J^{(2)}(x)$$
$$+[(\lambda+\delta)\beta_1+\lambda(1-\pi)\beta_2-\lambda\pi\beta_1-(\lambda+\delta)\beta_2-\hat{\mu}\beta_1\beta_2]m_J^{(1)}(x)+\delta\beta_1\beta_2 m(x)=0.$$

Since $m_J(x)$ satisfies a homogeneous fourth order differential equation with constant coefficients, it is easy to represent $m_J(x)$ as

$$m_J(x)=C_1e^{s_1x}+C_2e^{s_2x}+C_3e^{s_3x}+C_4e^{s_4x}, \tag{4.6.5}$$

where $s_1<s_2<0<s_3<s_4$ are roots of the generalized Lundberg equation

$$\frac{1}{2}\sigma^2s^2+\hat{\mu}s-(\lambda+\delta)+\lambda\pi\frac{\beta_1}{\beta_1-s}+\lambda(1-\pi)\frac{\beta_2}{\beta_2+s}=0. \tag{4.6.6}$$

As $\lim_{x\to\infty}m(x)=0$, it is obvious that $C_3=C_4=0$. We need two more boundary conditions to determine C_1 and C_2. Letting $x\to 0$ in (4.6.4) we obtain the first condition that

$$\frac{1}{2}\sigma^2m_J''(0)+\hat{\mu}m_J'(0)-(\lambda+\delta)m_J(0)+\lambda\pi\beta_1\int_0^\infty e^{-\beta_1y}m_J(y)dy$$
$$+\lambda(1-\pi)\beta_2\int_0^\infty e^{-\beta_2z}f(z)dz=0. \tag{4.6.7}$$

Inserting (4.6.5) into (4.6.7) gives

$$\frac{C_1}{\beta_2+s_1}+\frac{C_2}{\beta_2+s_2}=\int_0^\infty e^{-\beta_2z}f(z)dz.$$

Multiplying both sides of (4.6.4) by $\beta_1\mathcal{I}-\mathcal{D}$ gives the second condition that

$$-\frac{1}{2}\sigma^2m_J^{(3)}(0)+(\frac{1}{2}\sigma^2\beta_1-\hat{\mu})m_J^{(2)}(0)+(\hat{\mu}\beta_1+\lambda+\delta)m_J^{(1)}(0)$$
$$+(\lambda\pi\beta_1-\beta_1\lambda-\beta_1\delta)m_J(0)+\beta_1l(0)-l'(0)=0. \tag{4.6.8}$$

We insert (4.6.5) into (4.6.8) and simplify the terms together with (4.6.6)

$$\frac{\beta_1 - s_1}{\beta_2 + s_1} C_1 + \frac{\beta_1 - s_2}{\beta_2 + s_2} C_2 = (\beta_1 + \beta_2) \int_0^\infty e^{-\beta_2 z} f(z) dz.$$

Hence we obtain

$$\begin{aligned} C_1 &= \frac{(\beta_2 + s_1)(\beta_2 + s_2)}{s_1 - s_2} \int_0^\infty e^{-\beta_2 z} f(z) dz \\ C_2 &= \frac{(\beta_2 + s_1)(\beta_2 + s_2)}{s_2 - s_1} \int_0^\infty e^{-\beta_2 z} f(z) dz. \end{aligned}$$

The solution is thus obtained upon substitution of coefficients. $\square$

Expected Discounted Penalty at Ruin Due to Diffusion

The expected discounted penalty at ruin is actually a Laplace transform of the time of ruin due to diffusion,

$$m_D(x) = f(0) \mathbb{E}^x [e^{-\delta \tau_0} I(\tau < \infty, X(\tau_0) = 0)]. \tag{4.6.9}$$

Hence it satisfies the integro-differential equation

$$\frac{1}{2} \sigma^2 m_D''(x) + \hat{\mu} m_D'(x) - (\lambda + \delta) m_D(x) + \lambda \pi \beta_1 e^{\beta_1 x} \int_x^\infty e^{-\beta_1 y} m_D(y) dy \tag{4.6.10}$$

$$+\lambda(1 - \pi) \beta_2 e^{-\beta_2 x} \int_0^x e^{\beta_2 y} m_D(y) dy = 0, \qquad x > 0. \tag{4.6.11}$$

Corollary 4.6.2. *If the claim size distribution $Q(y)$ is given by (4.6.2), the Gerber-Shiu function defined in (4.6.9) admits an explicit solution given by*

$$m_D(x) = \frac{\beta_2 + s_1}{s_1 - s_2} f(0) e^{s_1 x} + \frac{\beta_2 + s_2}{s_2 - s_1} f(0) e^{s_2 x}.$$

Proof. Following the same technique used previously, we would also have

$$m_D(x) = C_1 e^{s_1 x} + C_2 e^{s_2 x},$$

where s_1 and s_2 are the two negative roots of the fundamental Lundberg equation (4.6.6).

By letting $x \to 0$ and substituting in the solution with unknown coefficients, we obtain the first condition that

$$\frac{C_1}{\beta_2 + s_1} + \frac{C_2}{\beta_2 + s_2} = 0. \tag{4.6.12}$$

The second condition comes from the fact that ruins occurs immediately if the process starts at 0. Hence,

$$m_D(0) = f(0),$$

which means

$$C_1 + C_2 = f(0). \tag{4.6.13}$$

Combining (4.6.12) and (4.6.13) we obtain the solution to the expected discounted penalty at ruin due to diffusion. $\square$

4.6.2 Perpetual American Put Option

Since the American put option is priced at the expected discounted payoff at such an exercise date so that its value is maximized, we can write it as

$$F(x) = \sup_d \mathbb{E}^x[e^{-\delta\tau_d}\Pi(S(\tau_d))I(\tau_d < \infty)],$$

where

$$\Pi(s) = (K - s)_+.$$

Hence it can be expressed as

$$\begin{aligned} F(x) &= \sup_d \mathbb{E}^x[e^{-\delta\tau_d}\Pi(e^{X(\tau_d)})I(\tau_d<\infty)] \\ &= \sup_d \mathbb{E}^x[e^{-\delta\tau_0^Y}\Pi(e^{Y(\tau_0^Y)+d})I(\tau_d<\infty)] \\ &= \sup_d \mathbb{E}^{x-d}[e^{-\delta\tau_0^Y}f(|Y(\tau_0^Y)|)I(\tau_d<\infty)], \end{aligned}$$

where

$$f(y) = \Pi(e^{d-y}) = (K - e^{d-y}).$$

As shown in previous section, we need to find the solution in two parts,

$$\begin{aligned} & \mathbb{E}^x[e^{-\delta\tau_d}\Pi(e^{X(\tau_d)})I(\tau_d<\infty, X(\tau_d<d))] \\ = \ & \mathbb{E}^{x-d}[e^{-\delta\tau_0^Y}f(|Y(\tau_0^Y)|)I(\tau_d<\infty, Y(\tau_0^Y<0)] \\ = \ & \frac{(\beta_2+s_1)(\beta_2+s_2)}{s_2-s_1}\left(\frac{K}{\beta_2}-\frac{e^d}{\beta_2+1}\right)e^{s_1(x-d)} + \frac{(\beta_2+s_1)(\beta_2+s_2)}{s_1-s_2}\left(\frac{K}{\beta_2}-\frac{e^d}{\beta_2+1}\right)e^{s_2(x-d)}. \end{aligned}$$

$$\begin{aligned} & \mathbb{E}^x[e^{-\delta\tau_d}\Pi(e^{X(\tau_d)})I(\tau_d<\infty, X(\tau_d=d))] \\ = \ & \mathbb{E}^{x-d}[e^{-\delta\tau_0^Y}f(0)I(\tau_d<\infty, Y(\tau_0^Y)=0)] \\ = \ & (K-e^d)\frac{\beta_2+s_1}{s_1-s_2}e^{s_1(x-d)} + (K-e^d)\frac{\beta_2+s_2}{s_2-s_1}e^{s_2(x-d)} \end{aligned}$$

Hence,

$$\begin{aligned} & \mathbb{E}^x[e^{-\delta\tau_d}\Pi(e^{X(\tau_d)})I(\tau_d<\infty)] \\ = \ & \left\{\frac{(\beta_2+s_1)(\beta_2+s_2)}{s_2-s_1}\left(\frac{K}{\beta_2}-\frac{e^d}{\beta_2+1}\right) + (K-e^d)\frac{\beta_2+s_1}{s_1-s_2}\right\}e^{s_1(x-d)} \\ & + \left\{\frac{(\beta_2+s_1)(\beta_2+s_2)}{s_1-s_2}\left(\frac{K}{\beta_2}-\frac{e^d}{\beta_2+1}\right) + (K-e^d)\frac{\beta_2+s_2}{s_2-s_1}\right\}e^{s_2(x-d)}. \end{aligned}$$

4.7 Jang Jump Diffusion Model

Taking $\mu(x) = b + ax, \sigma(x) = \sigma\sqrt{x}$ and $F(x,z) = x$, we obtain a jump-diffusion model that was proposed by Jang [30].

$$dX_t = (b + aX_t)dt + \sigma\sqrt{X_t}dB_t + dZ_t.$$

As Jang [30] explains, the parameter a is considered as the expected market rate of return and the volatility squared is proportional to the surplus level. The last term of compound Poisson jump process is employed to keep track of unexpected substantial interest rises. We may slightly generalize the model by including both positive and negative jumps.

Hence it follows from (4.1.10) that

$$\mathfrak{A}f(x) = (b + ax)f'(x) + \frac{1}{2}\sigma^2 x f''(x) + \lambda \int [f(x+y) - f(x)]Q(dy).$$

Jang [30] used a martingale approach to find the mean and variance of the surplus process X_t when the jump size is exponentially distributed. In what follows, we shall use the approach of generalized Gerber-Shiu function to investigate the ruin-related quantities of such a surplus model. For a simple example, we search for an explicit solution to the probability of default caused by diffusion at the level d defined by

$$\psi(x) = \mathbb{P}^x(\tau_d < \infty, X_{\tau_d} = d).$$

As we have shown before, the cost function that corresponds to the probability of default caused by diffusion is given by

$$l(x) = \delta(x - d),$$

where $\delta(x)$ is the Dirac delta function. Hence, it follows from Theorem 4.2.3 that

$$\frac{1}{2}\sigma^2 x\psi''(x) + (b + ax)\psi'(x) - \lambda\psi(x) + \lambda \int_0^{x-d} \psi(x-y)dQ(y) = 0, \qquad x > d. \tag{4.7.1}$$

Corollary 4.7.1. *If $Q(y)$ follows an exponential distribution with mean $1/\beta$, the probability of default $\psi(x)$ admits an explicit solution given by*

$$\psi(x) = AZ_1(x) + BZ_2(x), \qquad x > d,$$

where the coefficients

$$\begin{aligned} A &= \frac{\lambda Z_2(d) - [(1/2)\sigma^2 dZ_2''(d) + (b + ad)Z_2''(d)]}{Z_2(d)[(1/2)\sigma^2 dZ_1''(d) + (b + ad)Z_1''(d)] - Z_1(d)[(1/2)\sigma^2 dZ_2''(d) + (b + ad)Z_2''(d)]} \\ B &= \frac{\lambda Z_1(d) - [(1/2)\sigma^2 dZ_1''(d) + (b + ad)Z_1''(d)]}{Z_1(d)[(1/2)\sigma^2 dZ_2''(d) + (b + ad)Z_2''(d)] - Z_2(d)[(1/2)\sigma^2 dZ_1''(d) + (b + ad)Z_1''(d)]} \end{aligned}$$

with the function $Z_1(x)$ and $Z_2(x)$ defined as

$$\begin{aligned} Z_1(x) &= \int_x^\infty e^{-\beta y} M\left[\frac{\sigma^2\beta+2a+2\lambda}{\sigma^2\beta-2a}, \frac{2b+\sigma^2}{\sigma^2}; \frac{\sigma^2\beta-2a}{\sigma^2}y\right] dy, \\ Z_2(x) &= \int_x^\infty e^{-\beta y} U\left[\frac{\sigma^2\beta+2a+2\lambda}{\sigma^2\beta-2a}, \frac{2b+\sigma^2}{\sigma^2}; \frac{\sigma^2\beta-2a}{\sigma^2}y\right] dy, \end{aligned}$$

where $M(a, b; y)$ and $U(a, b; y)$ are the Kummer function of the first and second kind, respectively.

Proof. Since it is given that

$$Q(y) = 1 - \beta e^{\beta y}, \qquad y < 0,$$

we take the operator $\beta\mathcal{I} + \mathcal{D}$ on both sides yields,

$$\begin{aligned} [\frac{1}{2}\sigma^2 x\mathcal{D}^3 + \frac{1}{2}\sigma^2\beta x\mathcal{D}^2 + \frac{1}{2}\sigma^2\mathcal{D}^2]\psi(x) + [(b+ax)\mathcal{D}^2 + a\mathcal{D} + (\beta b + \beta a x)\mathcal{D}]\psi(x) \\ -\lambda(\beta+\mathcal{D})\psi(x) + \lambda\beta\psi(x) = 0. \end{aligned}$$

Hence we find the integral differential equation satisfied by $\psi(x)$,

$$\frac{1}{2}\sigma^2 x\psi'''(x) + (\frac{1}{2}\sigma^2\beta x + ax + b + \frac{1}{2}\sigma^2)\psi''(x) + (\beta a x + \beta b - a - \lambda)\psi'(x) = 0, \qquad x > d.$$

Letting $\psi'(x) = e^{-\beta x} f(x)$ gives

$$\frac{1}{2}\sigma^2 x[f''(x) - 2\beta f'(x) + \beta^2 f(x)] + (\frac{1}{2}\sigma^2\beta x + ax + b + \frac{1}{2}\sigma^2)[f'(x) - \beta f(x)] + (\beta a x + \beta b - a - \lambda) f(x) = 0,$$

which simplifies to

$$\frac{1}{2}\sigma^2 x f''(x) + (ax - \frac{1}{2}\sigma^2\beta x + b + \frac{1}{2}\sigma^2) f'(x) - (\frac{1}{2}\sigma^2\beta + a + \lambda) f(x) = 0.$$

In order to further simplify the second order differential equation, we let $z = (\sigma^2\beta - 2a)x/\sigma^2$ and $g(z) = f(x)$. Hence we must have

$$g'(z)\frac{\sigma^2\beta - 2a}{\sigma^2} = f'(x),$$
$$g''(z)\left(\frac{\sigma^2\beta - 2a}{\sigma^2}\right)^2 = f''(x).$$

Hence,

$$\frac{1}{2}(\sigma^2\beta - 2a)zg''(z) + (-\frac{1}{2}\sigma^2 z + b + \frac{1}{2}\sigma^2)\frac{\sigma^2\beta - 2a}{\sigma^2}g'(z) - (\frac{1}{2}\sigma^2\beta + a + \lambda)g(z) = 0.$$

We obtain upon further rearrangement,

$$zg''(z) + \left[\frac{2b + \sigma^2}{\sigma^2} - z\right]g'(z) - \frac{\sigma^2\beta + 2a + 2\lambda}{\sigma^2\beta - 2a}g(z) = 0,$$

which is the Kummer's confluent hypergeometric equation.

Hence the probability of ruin can be written as

$$\psi(x) = AZ_1(x) + BZ_2(x),$$

in consideration of the fact that $\psi(\infty) = 0$.

Since $\psi(d) = 1$, we must have

$$AZ_1(d) + BZ_2(d) = 1. \tag{4.7.2}$$

Letting $x \to d$ in (4.7.1) gives the second boundary condition that

$$\frac{1}{2}\sigma^2 d\psi''(d) + (b + ad)\psi'(d) = \lambda,$$

which implies that

$$A[\frac{1}{2}\sigma^2 dZ_1''(d) + (b + ad)Z_1''(d)] + B[\frac{1}{2}\sigma^2 dZ_2''(d) + (b + ad)Z_2''(d)] = \lambda. \tag{4.7.3}$$

Solving the linear equation system (4.7.2) and (4.7.3) results in the desired expressions.

$\square$

Conclusion

The theme of the thesis is the development of a unifying approach to analyze ruin-related quantities.

In Chapter 1, we use traditional approaches to analyze the Gerber-Shiu function and dividends paid up to ruin respectively. Later on, a generalized Gerber-Shiu function is introduced to reconcile the two seemingly unrelated quantities. We show through heuristic arguments that the generalized Gerber-Shiu function can be derived through a general equation, which significantly reduces the amount of derivations required by traditional solution methods.

As the generalized Gerber-Shiu function is formally defined in Chapter 2, we see that not only does it recover both Gerber-Shiu function and dividends paid up to ruin which are well-studied in ruin theory, the generalized Gerber-Shiu also gives rise to many interesting new ruin-related quantities such as an insurer's accumulated utility, total claim costs up to ruin and more. We prove in Chapter 2 that the general equation holds for all piecewise-deterministic compound Poisson processes, such as the compound Poisson model with constant interest and dividend strategies.

We show in Chapter 3 that the same general equation applies to the Sparre Andersen model where inter-claim time distribution is phase-typed. Similarly we produce solutions to various ruin-related quantities in many cases of Sparre Andersen model.

To further demonstrate the generality of the unifying approach, we introduce in Chapter 4 a class of jump diffusion processes under which the general equation continues to hold. Following the same logic as in Chapter 2 and 3, we find explicit solutions to both traditional and new ruin-related quantities in different jump diffusion models, such as Brownian motion risk model and Kou's model. It is interesting to point out that the Gerber-Shiu function can also be used to find passage time distributions of all risk models.

However, the thesis by no means exhausts all quantities accommodated by the generalized Gerber-Shiu function and all risk models under which the unifying approach applies. Owing to the flexibility of cost function and infinitesimal generator associated with the function, we should be able to extend the applications in future work to even more quantities of interest in ruin theory and potentially in other financial topics.

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