Critical Exponents and Stabilizers of Infinite Words

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis concerns infinite words over finite alphabets. It contributes to two topics in this area: critical exponents and stabilizers.

Let $\mathbf{w}$ be a right-infinite word defined over a finite alphabet. The critical exponent of $\mathbf{w}$ is the supremum of the set of exponents r such that $\mathbf{w}$ contains an r-power as a subword. Most of the thesis (Chapters 3 through 7) is devoted to critical exponents.

Chapter 3 is a survey of previous research on critical exponents and repetitions in morphic words. In Chapter 4 we prove that every real number greater than 1 is the critical exponent of some right-infinite word over some finite alphabet. Our proof is constructive. In Chapter 5 we characterize critical exponents of pure morphic words generated by uniform binary morphisms. We also give an explicit formula to compute these critical exponents, based on a well-defined prefix of the infinite word. In Chapter 6 we generalize our results to pure morphic words generated by non-erasing morphisms over any finite alphabet. We prove that critical exponents of such words are algebraic, of a degree bounded by the alphabet size. Under certain conditions, our proof implies an algorithm for computing the critical exponent. We demonstrate our method by computing the critical exponent of some families of infinite words. In particular, in Chapter 7 we compute the critical exponent of the Arshon word of order n for $n \geq 3$.

The stabilizer of an infinite word $\mathbf{w}$ defined over a finite alphabet Σ is the set of morphisms $f : \Sigma^* \to \Sigma^*$ that fix $\mathbf{w}$. In Chapter 8 we study various problems related to stabilizers and their generators. We show that over a binary alphabet, there exist stabilizers with at least n generators for all n. Over a ternary alphabet, the monoid of morphisms generating a given infinite word by iteration can be infinitely generated, even when the word is generated by iterating an invertible primitive morphism. Stabilizers of strict epistandard words are cyclic when non-trivial, while stabilizers of ultimately strict epistandard words are always non-trivial. For this latter family of words, we give a characterization of stabilizer elements.

We conclude with a list of open problems, including a new problem that has not been addressed yet: the D0L repetition threshold.

Key words: Critical exponents, repetitions, morphic words, circularity, stabilizers.

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To my father

"If I've inspired anyone to read the book, I request only that the reader not ask what was the point of these centuries of mountain-climbing. You don't ask mathematicians and mountain climbers questions like that."

Wislawa Szymborska, "Nonrequired Reading"

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List of Symbols

Symbol	Meaning	Page introduced
Numbers		
$\mathbb{C}$	the set of complex numbers	7
$\mathbb{N}$	the set of nonnegative integers (0 included)	7
$\mathbb{Q}$	the set of rational numbers	7
$\mathbb{R}$	the set of real numbers	7
$\mathbb{Z}$	the set of integers	7
$\mathbb{Z}_{\geq\alpha}$	the set of integers $\geq \alpha$ (similarly: $\mathbb{Q}_{\geq\alpha}, \mathbb{R}_{\geq\alpha}$)	7
$\lceil\alpha\rceil$	the least integer greater than a real number $\alpha \in \mathbb{R}$	7
$\lfloor\alpha\rfloor$	the greatest integer smaller than a real number $\alpha \in \mathbb{R}$	7
$q(G)$	the maximal length of rank-zero subwords of D0L-system G	110
$s_k(n)$	the sum of digits in the base k expansion of n	31
τ	the golden mean	43
Sets, Monoids, Groups		
Σ, Γ	finite alphabets	7
Σ_n	the alphabet $\{0, 1, \ldots, n-1\}$	7
Σ^n	the set of words of length n over Σ	8
Σ^*	the set of finite words (free monoid) over Σ	8
Σ^+	the set of nonempty finite words (free semigroup) over Σ	8
Σ^ω	the set of right-infinite words over Σ	8
Σ^∞	$\Sigma^* \cup \Sigma^\omega$	8
Σ_0	the set of rank-zero letters in the D0L-system (Σ, f, w)	110

$\mathscr{S}$	the monoid of epistandard morphisms	160
$\mathscr{E}$	the monoid of episturmian morphisms	160
$\mathbb{F}_\Sigma$	the free group over Σ	158
F_h^*	the set of finite fixed points of a morphism h	16
$\mathcal{I}\mathcal{S}tab(\mathbf{w})$	the iterative stabilizer of a right-infinite word $\mathbf{w}$	18
$L(f)$	the language generated by a morphism f	19
$L(G)$	the language of a D0L system G	19
$\mathcal{M}(\Sigma)$	the monoid of morphisms defined over Σ	13
M_h	the set of mortal letters associated with a morphism h	15
$\mathrm{Occ}(\mathbf{w})$	the set of occurrences of an infinite word $\mathbf{w}$	9
St	the monoid of Sturmian morphisms	23
$\mathcal{S}tab(\mathbf{w})$	the stabilizer of a right-infinite word $\mathbf{w}$	17
$\mathrm{Sub}(w)$	the set of subwords of a word w	8
$\mathrm{Sub}(L)$	the set of subwords of a language $L \subseteq \Sigma^*$, $\mathrm{Sub}(L) = \bigcup_{w \in L} \mathrm{Sub}(w)$	8

Words

$\|w\|$	the length of a finite word w	7
$\|w\|_a$	the number of occurrences of a letter a in a finite word w	7
$\mathrm{alph}(w)$	the set of letters occurring in a word w	8
$\mathbf{a}_n$	the Arshon word of order n	34
$\mathbf{c}_\alpha$	the characteristic word of slope α	42
$\Delta(\mathbf{s})$	directive word of the epistandard word $\mathbf{s}$	161
ε	the empty word	8
$\mathbf{f}$	the Fibonacci word	15
$\mathrm{inv}(z)$	the ancestor of an occurrence z	138
$p_\mathbf{w}$	the subword complexity function of an infinite word $\mathbf{w}$	8
$\mathbf{r}$	the Tribonacci word	163
$\mathbf{s}_{\alpha,\rho}$	Sturmian word of slope α and intercept ρ	41
$\mathbf{t}$	the Thue-Morse word	1
$\mathbf{t}_{k,m}$	generalized Thue-Morse word	31
$\mathbf{t}_{u,m}$	$\varphi_{u,m}^\omega(0)$ (valid only when u begins with 0)	33
$u \prec w$	u is a subword of w	8

$u \prec_p w$	u is a prefix of w	8
$u \prec_s w$	u is a suffix of w	8
$w^{(+)}$	the palindromic closure of a finite word w	10
w^R	the reversal of a finite word w	10
$u^{-1}w$	v (valid only when $w = uv$)	8
wv^{-1}	u (valid only when $w = uv$)	8
$\check{z}$	the inner closure of an occurrence z	69
$\hat{z}$	the outer closure of an occurrence z	69
$z \supset z'$	the occurrence z contains the occurrence z'	9

Powers

$E(\mathbf{w})$	the critical exponent of an infinite word $\mathbf{w}$	12
$\exp(h)$	the mortality exponent of a morphism h	15
$RT(n)$	repetition threshold for n letters	13
r_n	the repetition threshold value stated in Dejean's conjecture	31
$RT_{D0L}(n)$	D0L repetition threshold for n letters	177
x^ω	$xx\cdots$ (x concatenated infinitely many times), $x \in \Sigma^*$	8
x^n	$xx\cdots x$ (x concatenated n times), $x \in \Sigma^*$	8
$x^{p/q}$	a fractional power of length p with period $\|x\|= q$	12

Morphisms

α_n	the Arshon morphism of order n, n even	35
δ	Dejean's morphism	30
$f^n(x)$	applying the morphism f to the word x iteratively n times	13
Id	the identity morphism	13
μ	the Thue-Morse morphism	1
$\mu_{k,m}$	generalized Thue-Morse morphism	32
ϕ	the Fibonacci morphism	15
$\varphi_{e,n}$	the even Arshon morphism of order n	34
$\varphi_{o,n}$	the odd Arshon morphism of order n	34
φ_n	the Arshon operator of order n	34
$\varphi_{u,m}$	symmetric morphism defined by the word $u \in \Sigma_m^*$	32

ψ_a	epistandard morphism of letter a	160
$\bar{\psi}_a$	episturmian morphism of letter a	160
$\|\psi\|$	the length of an epistandard morphism ψ	162
ψ_u	$\psi_{a_1} \cdots \psi_{a_n}$, where $u = a_1 \cdots a_n \in \Sigma^*$	162
θ_{ab}	transposition of letters a, b	160

Matrices

A^T	the transpose of a matrix A	19
A^{-1}	the inverse of a square nonsingular matrix A	19
$A(f)$	the incidence matrix associated with a morphism f	20
A_I	submatrix of A resulting from deleting rows and columns $i \in I$ from A	102
$I_0(U)$	$\{1 \leq i \leq n : u_i = 0\}$ $\quad (U = (u_1, \ldots, u_n)^T$ a column vector)	102
$\mathbf{I}_0(A, U)$	$\bigcap_{m \geq 0} I_0(A^m U)$ $\quad$ (A a square matrix, U a column vector)	102
$J_{\lambda,d}$	Jordan block of order d with eigenvalue λ	107
$M_{n \times m}(S)$	the set of $n \times m$ matrices with entries in the number set S	19
$M_n(S)$	the set of $n \times n$ matrices with entries in the number set S	19
$O_{x,y}$	square zero matrix, except for x at the top-right corner and y at the diagonal below it	107
O_x	$O_{x,0}$	107
$P[\theta]$	the permutation matrix induced by the permutation θ	103
$\mathbb{Q}[A]$	the field extension over $\mathbb{Q}$ spanned by the eigenvalues of $A \in M_n(\mathbb{Z})$	19
$r(A)$	the Perron-Frobenius eigenvalue of a nonnegative matrix A	22
$\mathrm{rad}(A)$	the spectral radius of a square matrix A	19
$r(f)$	the Perron-Frobenius eigenvalue of the incidence matrix of a morphism f	22
$\sum U$	$\sum_{i=1}^{n} u_i$ $\quad (U = (u_1, \ldots, u_n)^T$ a column vector)	102
$[u]$	the Parikh vector of a word $u \in \Sigma^*$	19
U_I	column vector resulting from deleting entries $i \in I$ from U	103
U/V	$\sum U / \sum V$ $\quad$ (U, V column vectors)	105

The π-map

π	"apply f and stretch" map	67
$\rho(z, q)$	the right stretch of occurrence z and period q	112

ρ_f	the right stretch of a uniform binary morphism f	65
$\sigma(z,q)$	the left stretch of occurrence z and period q	112
σ_f	the left stretch of a uniform binary morphism f	65
$\Lambda(z,q)$	the stretch vector of occurrence z and period q	112
λ_f	the stretch size of a uniform binary morphism f, $\lambda_f = \lvert\rho_f\rvert + \lvert\sigma_f\rvert$	65
$\varsigma(z,Q)$	the left context of occurrence z and period Q	114
$\varrho(z,Q)$	the right context of occurrence z and period Q	114

General notation

Unless stated otherwise, we typically use the following notation:

Notation	**Meaning**
a, b, c, d	letters
u, v, w, x, y, z, t	finite words
$\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{s}, \mathbf{t}$	right-infinite words
$w = w_0 w_1 \cdots w_{n-1}$	the letters of a finite word of length n
$\mathbf{w} = w_0 w_1 \cdots$	the letters of a right-infinite word
$i, j, k, l, m, n, p, q, r, s, t$	natural numbers (including indices)
α, β, γ	real numbers
A, B, C, D, F, M	square matrices
U, V, W	column vectors
f, g, h	morphisms
$\mathcal{M}, \mathcal{N}$	monoids
$f = (x_0, \ldots, x_{n-1})$	$f : \Sigma_n^* \to \Gamma^*$ is a morphism, and $f(i) = x_i \in \Gamma^*$

Chapter 1

Introduction

Combinatorics on words is an old research area in discrete and algorithmic mathematics that has recently attracted new interest. It involves the study of combinatorial, arithmetical and geometrical aspects of finite or infinite discrete sequences ("words"), composed of symbols ("letters") taken from a finite set ("alphabet"); in other words, the combinatorial properties of free monoids. For general resources on combinatorics on words, see Allouche and Shallit [6], Berstel and Karhumäki [13], Berstel and Perrin [15], Choffrut and Karhumäki [28], and Lothaire [82, 83, 84].

Research in combinatorics on words was initiated a hundred years ago by the Norwegian mathematician Axel Thue (1863–1922), with two papers that dealt with repetitions in finite and infinite words. The first, which appeared in 1906 [131], contained the construction of an infinite *square-free* word (that is, a word containing no two consecutive identical blocks) over a ternary alphabet. The second, which appeared in 1912 [132], introduced what is now called the *Thue-Morse word*, perhaps the most famous sequence in combinatorics on words (the *Fibonacci word* shares this title). The Thue-Morse word, denoted by $\mathbf{t}$, is a one-sided infinite word over a binary alphabet. Its first few terms are given by

$$\mathbf{t} = t_0 t_1 t_2 \cdots = 0110100110010110 \cdots$$

There are several different ways to define $\mathbf{t}$. Here is one:

Definition 1.1. The Thue-Morse word is the fixed point beginning with 0 of the monoid morphism μ, defined over $\{0,1\}^*$ by $\mu(0) = 01$, $\mu(1) = 10$.

Here a fixed point of a morphism f is a word w that satisfies $f(w) = w$; since $\mu(w)$ is longer than w for every finite binary word w, any fixed point of μ must be infinite. Such a fixed point is called a *pure morphic sequence* or a *D0L word*. (Note that not all infinite fixed points are pure morphic; the exact definition is given in Section 2.5.) In [132], Thue showed (among other things) that $\mathbf{t}$ contains no *overlaps*, that is, no blocks of the form $axaxa$, where a is a letter and x a word, possibly empty (see also Berstel [11] and Allouche and Shallit [4]). The way he did it was by showing that the morphism μ, now known as the *Thue-Morse morphism*, is overlap-free, that is, if $w \in \{0,1\}^*$ contains no overlap then so does $\mu(w)$. Moreover, Thue also showed that any overlap-free binary morphism is of the form μ^k or $e \cdot \mu^k$, where k is a positive integer and e is the morphism that exchanges 0 and 1.

Perhaps because Thue's papers were published in German in a Norwegian journal, they were largely ignored for a long time, and many of his results were rediscovered again and again during the twenties and thirties. In particular, the Thue-Morse sequence was rediscovered in 1921 by American mathematician Marston Morse (1892-1977), who applied it to differential geometry [93]; hence the name. We also mention Arshon [7, 8], who rediscovered in 1935 the existence of an infinite square-free ternary word and an infinite overlap-free binary word. The Arshon sequences will be discussed in Section 3.2.4.

Squares and overlaps are special cases of *repetitions*, or *fractional powers*: a square is a two-power, while an overlap is a "two-and-a-bit"-power. The existence of an infinite square-free ternary word and an infinite overlap-free binary word constitutes a positive answer to specific cases of the more general *power avoidance* question:

- *Given an integer k and a real number r, does there exist an infinite word over an alphabet of size k that contain no fractional powers of exponent r or more?*

In the case of the Thue-Morse word, the result is even stronger: over binary alphabets, 2-powers are unavoidable, as any binary word of length 4 or more must contain a square. By avoiding any power greater than 2, $\mathbf{t}$ avoids the smallest powers it is possible to avoid over two letters. Thus in the case of a binary alphabet, $r = 2$ is the answer to the more general *repetition threshold* question:

- *Given an integer k, what is the infimum of the set of exponents that can be avoided over an alphabet of size k?*

The *critical exponent* of an infinite word $\mathbf{w}$ is the supremum of the set of rational numbers $r > 1$ such that $\mathbf{w}$ contains an r-power. Thus in the case of the Thue-Morse word, $r = 2$ is the answer to the *critical exponent* question:

- *Given an infinite word* $\mathbf{w}$*, what is its critical exponent?*

Though critical exponents have been widely studied over the past years, many questions remain open. The main subject of the present thesis is a systematic study of this topic, especially in relation to pure morphic words.

The last part of the thesis deals with another aspect of infinite words: the structure of the *stabilizer* of a given infinite word, that is, the monoid of morphisms that fix a given infinite word. Again, this subject is tightly related to pure morphic words: an important type of stabilizer elements are morphisms that generate a word $\mathbf{w}$ by iteration, in which case $\mathbf{w}$ is pure morphic. In contrast to critical exponents, the subject of stabilizers of infinite words has hardly been studied, and in this thesis we mainly offer a framework for a deeper research. We solve some of the problems, but many remain open.

1.1 Thesis outline

In Chapter 2 we establish the general terminology and definitions we will use throughout the thesis, and state some background results required for our work.

Chapters 3 through 7 are devoted to the study of critical exponents. In Chapter 3 we give a survey of the work previously done on critical exponents and related topics. Most of the research done on critical exponents per se concerns either critical exponents of specific infinite words, or critical exponents of Sturmian words, but there is a large body of research done on repetitions in general, and on pure morphic words in particular.

In Chapters 4 – 7 we develop our original work. We start with critical exponents in the most general setting: in Chapter 4, we prove that every real number greater than 1 is the critical exponent of some right-infinite word over some finite alphabet. Our proof in constructive: provided that a suitable sequence of rational numbers converging to a given real number is known, we can effectively construct an infinite word having this number as a critical exponent.

Next, we concentrate on critical exponents of pure morphic words. In Chapter 5 we completely characterize critical exponents of pure morphic words generated by uniform morphisms defined over a binary alphabet. We also give an explicit formula to compute these critical exponents, based on a well-defined prefix of the infinite word. We conclude with some examples, among them a complete catalogue of critical exponents in fixed points of uniform binary morphisms of length up to 4.

In Chapter 6 we generalize our results to pure morphic words generated by non-erasing morphisms over any finite alphabet. We prove that critical exponents of such words are always algebraic, of a degree bounded by the alphabet size. Under certain conditions, our proof implies an algorithm for computing the critical exponent. We demonstrate our method by computing the critical exponents of some families of infinite words.

In Chapter 7 we apply the method we developed in Chapter 6 to the Arshon words. We prove that for all $n \geq 2$, the critical exponent of the Arshon word of order n is given by $(3n-2)/(2n-2)$.

In Chapter 8 we begin our study of stabilizers. Our ultimate goal is to answer two fundamental questions:

1. Do there exist infinitely generated stabilizers of aperiodic infinite words over finite alphabets?
2. Is there a characterization of morphisms that, when iterated, generate infinite words with cyclic stabilizers?

Though we fail to answer these questions, we succeed in shedding some light on the structure of stabilizers. We prove that over a binary alphabet, there exist stabilizers with any finite number of generators, and over general alphabets there exist infinitely generated *iterative stabilizers* (the iterative stabilizer of a given infinite word is the monoid of morphisms that generate it by iteration; it is a submonoid of the stabilizer). We also characterize the stabilizer structure for some certain classes of *epistandard words*.

In Chapter 9 we conclude the thesis, list all the open problems encountered on the way, and present a new related problem for future research: the D0L repetition threshold. Recall the repetition threshold problem: given an integer k, what is the infimum of the set of exponents that can be avoided over an alphabet of size k? Having been open for more than 30 years, this problem has been recently solved by Carpi [23, 24] in 2006 (save for a few specific values of k). However, the following variation is still open: given an integer k, what is the infimum of the set of exponents that can be avoided over an alphabet of size k *by pure morphic sequences*?

Chapter 2

Preliminaries

In this section, we give the definitions and terminology of concepts we will use in the rest of the thesis: words, powers, and morphisms. We present briefly Sturmian words, and state some fundamental results in combinatorics on words that we will need later. Most of our terminology is based on Allouche and Shallit [6] and Lothaire [82, 83].

2.1 Words

As usual, $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Q}$ denotes the set of rational numbers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. For a real number $\alpha \in \mathbb{R}$, $\lfloor \alpha \rfloor$ denotes the greatest integer smaller than α and $\lceil \alpha \rceil$ denotes the least integer greater than α. We use $\mathbb{Z}_{\geq \alpha}$ (and similarly $\mathbb{Q}_{\geq \alpha}, \mathbb{R}_{\geq \alpha}$) to denote the integers (and similarly rational or real numbers) greater than or equal to α.

An *alphabet* is a set of symbols, called *letters*. In this thesis we deal only with finite alphabets. We usually use the symbols Σ, Γ to denote alphabets. We denote by Σ_n the n-letter alphabet containing the letters $\{0, 1, \ldots, n-1\}$. To denote a generic letter, we usually use the symbols a, b, c, d.

Let Σ be a finite alphabet. A (finite) *word* over Σ is a finite sequence of letters of Σ. We usually use the symbols u, v, w, x, y, z to denote finite words, and the symbols w_i, v_i, etc., to denote the i'th letter (starting from 0) of the word. The *length* of a word w, denoted by $|w|$, is the number of letters composing w. We denote by $|w|_a$ the number of occurrences of a letter a in a word w, and by $\mathrm{alph}(w)$ the set of letters occurring in w. For example, if $w = banana$, then $|w| = 6$, $|w|_a = 3$, and $\mathrm{alph}(w) = \{a, b, n\}$. The *empty word*, denoted by ε, is the sequence containing no letters; its length is $|\varepsilon| = 0$. The set of all finite words over Σ is denoted by Σ^*. A subset of Σ^* is called a *language*. The set of all finite words of length n over Σ is denoted by Σ^n. If $u = u_0 \cdots u_{k-1}$ and $v = v_0 \cdots v_{m-1}$ are two elements of Σ^*, then the concatenation $uv = u_0 \cdots u_{k-1} v_0 \cdots v_{m-1}$ is also an element of Σ^*. The empty word is neutral with respect to concatenation: for all $w \in \Sigma^*$, $\varepsilon w = w \varepsilon = w$.

A *monoid* is a set equipped with an associative binary operation and an element neutral for this operation (the *unit element* of the monoid). Hence Σ^* has a monoid structure, with concatenation as the binary operation and ε as the unit element. Moreover, each element of Σ^* has a unique representation as a concatenation of letters of Σ. For this reason, Σ^* is called the *free monoid* over Σ. The set of nonempty words over Σ, denoted by Σ^+, is called the *free semigroup* over Σ. Concatenation is written multiplicatively, e.g., $x^n := xx \cdots x$ (n times). The infinite word $xxx \cdots$ (x concatenated to itself infinitely many times) is denoted by x^ω .

A *right-infinite word* over Σ is a mapping from $\mathbb{N}$ into Σ. Similarly, a *bi-infinite word* is a mapping from $\mathbb{Z}$ into Σ. In this thesis, the term "infinite words" refers to right-infinite words. We usually denote infinite words by bold letters, e.g., $\mathbf{w} = w_0 w_1 w_2 \ldots$, where w_i are letters for all $i \in \mathbb{N}$. The set of infinite words over Σ is denoted by Σ^ω. The set of all words (finite or infinite) is denoted by Σ^∞.

2.1.1 Subwords

A word $u \in \Sigma^*$ is a *subword* or *factor* of a word $w \in \Sigma^\infty$, denoted $u \prec w$, if $w = xuy$ for some words $x \in \Sigma^*$ and $y \in \Sigma^\infty$. If $x = \varepsilon$ (resp., $y = \varepsilon$) then u is a *prefix* (resp., a *suffix*) of w, denoted by $u \prec_p w$ (resp., $u \prec_s w$). A prefix (resp., suffix) u of w is *proper* if $u \neq w$. If $w = uy$ (resp., $w = xu$), then we denote $u^{-1}w = y$ (resp., $wu^{-1} = x$). The set of all subwords of w is denoted by $\mathrm{Sub}(w)$. If $L \subseteq \Sigma^*$ is a language, then $\mathrm{Sub}(L) = \bigcup_{w \in L} \mathrm{Sub}(w)$. A subword u of an infinite word $\mathbf{w}$ is *right-* (resp., *left-*) *special* if there exist at least two distinct letters $a \neq b \in \Sigma$, such that both ua and ub (resp., au and bu) are subwords of $\mathbf{w}$. The *subword complexity function* of an infinite word $\mathbf{w}$, denoted by $p_{\mathbf{w}}$, counts the number of distinct subwords of $\mathbf{w}$ of length n, $n \geq 0$:

$$p_{\mathbf{w}}(n) = |\mathrm{Sub}(\mathbf{w}) \cap \Sigma^n|, \quad n \in \mathbb{N}. \tag{2.1}$$

Example 2.1. Let $w = 01011 \in \Sigma_2^*$. The set of subwords of w is given by $\mathrm{Sub}(w) = \{\varepsilon, 0, 1, 01, 10, 11, 010, 101, 011, 0101, 1011, 01011\}$. The right-special subwords of w are ε, 1, and 01. The left-special subwords are ε and 1. For the prefix 010 of w, $(010)^{-1}w = 11$, where 11 is a suffix of w.

An *occurrence* of a subword z within a word $\mathbf{w} \in \Sigma^\omega$ is a triple (z, i, j), where $z \in \mathrm{Sub}(\mathbf{w})$, $0 \leq i \leq j$, and $w_i \cdots w_j = z$. In other words, z occurs in $\mathbf{w}$ at positions $i, \ldots, j$. For convenience, we usually omit the indices, and refer to an occurrence (z, i, j) as $z = w_i \cdots w_j$. The set of all occurrences of subwords within $\mathbf{w}$ is denoted by $\mathrm{Occ}(\mathbf{w})$. We say that an occurrence (z, i, j) *contains* an occurrence (z', i', j'), and denote it by $z \supset z'$, if $i \leq i'$ and $j \geq j'$.

Example 2.2. Let $\mathbf{w} = w_0 w_1 w_2 \cdots$ be an infinite ternary word, whose prefix of length 24 is given by

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
w_i	0	1	2	0	2	1	0	1	2	1	0	2	0	1	2	0	2	1	0	2	0	1	2	1

The word 202 is a subword of $\mathbf{w}$. The triple $(202, 2, 4)$ is an occurrence within $\mathbf{w}$, and so is the triple $(202, 14, 16)$. Alternately, we can write $202 = w_2 \cdots w_4$, or $202 = w_{14} \cdots w_{16}$. the occurrence $1202 = w_1 \cdots w_4$ contains the occurrence $202 = w_2 \cdots w_4$, but does not contain the occurrence $202 = w_{14} \cdots w_{16}$.

An infinite word $\mathbf{w} \in \Sigma^\omega$ is *recurrent* if every subword of $\mathbf{w}$ occurs in $\mathbf{w}$ infinitely often. It is *letter-recurrent* if every letter in Σ occurs in $\mathbf{w}$ infinitely often. It is *uniformly recurrent* if for each finite subword u of $\mathbf{w}$ there exists an integer m, such that every subword of $\mathbf{w}$ of length m contains u as a subword.

Example 2.3.

- The infinite binary word $\mathbf{w} = (01)^\omega$ is uniformly recurrent: clearly, every subword of $\mathbf{w}$ beginning with 0 occurs at every even position in $\mathbf{w}$, while every subword beginning with 1 occurs at very odd position. Therefore, every subword of $\mathbf{w}$ of length m is contained in every subword of $\mathbf{w}$ of length $m + 1$.

- The infinite binary word $\mathbf{u} = 0100011011\cdots$, obtained by concatenating all binary strings in increasing lexicographic order, is recurrent, as any binary string occurs in $\mathbf{u}$ infinitely often. It is not uniformly recurrent, since it contains the subword 1^n for all n, and so there exists no m such that every subword of $\mathbf{u}$ of length m contains the subword 0.

- The infinite binary word $\mathbf{v} = v_0 v_1 v_2 \cdots$, defined by $v_i = 1$ if and only if i is a power of 2, is letter-recurrent, since both 0 and 1 occur infinitely often in $\mathbf{v}$. It is not recurrent, since for all $n \geq 0$, the subword $10^{2^n-1}1$ occurs in $\mathbf{v}$ exactly once.

- The infinite ternary word $\mathbf{z} = 2\mathbf{v}$ (where $\mathbf{v}$ is as defined above) is not letter-recurrent, since the letter 2 occurs only once.

2.1.2 Palindromes

The *reversal* of a word $u = a_0 \cdots a_{n-1}$, $a_i \in \Sigma$, is given by $u^R = a_{n-1} \cdots a_0$. An example in English is $(\texttt{deer})^R = \texttt{reed}$. A language $L \subseteq \Sigma^*$ is *closed under reversal* if $u \in L \Leftrightarrow u^R \in L$ for all $u \in \Sigma^*$. A word $u \in \Sigma^*$ is a *palindrome* if $u = u^R$; palindromes in English (ignoring white spaces and punctuation marks) include `level`, `evil olive`, and the most famous one, `a man, a plan, a canal - panama`. The *palindromic closure* of u, denoted by $u^{(+)}$, is the unique shortest palindrome that has u as a prefix. For example, $\texttt{banana}^{(+)} = \texttt{bananab}$, and $\texttt{race}^{(+)} = \texttt{racecar}$. (As a side note, we mention that $u^{(+)}$ can be computed efficiently: $u^{(+)} = uv^{-1}u^R$, where v is the longest palindromic suffix of u. This palindromic suffix can be computed in linear time using suffix trees. See Gusfield, [56, Section 9.2].)

2.2 Periodicity

Let $w = w_0 \cdots w_{n-1} \in \Sigma^+$, $w_i \in \Sigma$. A positive integer $q \leq |w|$ is a *period* of w if $w_{i+q} = w_i$ for $i = 0, \ldots, n-1-q$. An infinite word $\mathbf{w} = w_0 w_1 \cdots \in \Sigma^\omega$ has a period $q \in \mathbb{Z}_{\geq 1}$ if $w_{i+q} = w_i$ for all $i \geq 0$; that is, $\mathbf{w} = y^\omega$, where $y = w_0 \cdots w_{q-1}$. If this is the case, we say that $\mathbf{w}$ is *purely periodic* (or just *periodic*). The *minimal period* of a purely periodic word

$\mathbf{w}$ is the unique shortest word $y \in \Sigma^+$ such that $\mathbf{w} = y^\omega$. We say that $\mathbf{w}$ is *ultimately periodic* if there exist words $x \in \Sigma^*$ and $y \in \Sigma^+$ such that $\mathbf{w} = xy^\omega$. A non-ultimately periodic word is called *aperiodic*.

The following classical theorem, due to Fine and Wilf, describes how far two periodic words have to agree in order to guarantee their equality:

Theorem 2.1 (Fine and Wilf [44]). *Let $\mathbf{u}, \mathbf{v} \in \Sigma^\omega$ be two periodic words, with periods p and q, respectively. Then $\mathbf{u} = \mathbf{v}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ agree on a prefix of length $p + q - \gcd(p, q)$.*

In the following chapters we will use another formulation of Theorem 2.1 (see Lothaire, [83, Theorem 8.1.4]):

Theorem 2.2 (Fine and Wilf [44]). *Let $w \in \Sigma^\infty$ be a word having periods p and q, and suppose that $|w| \geq p + q - \gcd(p, q)$. Then w also has period $\gcd(p, q)$.*

Theorem 2.1 can be considered as a generalization of another classical theorem: the second theorem of Lyndon and Schützenberger. The two theorems of Lyndon and Schützenberger, and some generalizations of them to systems of word equations, will be a central tool in our critical exponent analysis in Chapter 5.

Theorem 2.3 (Lyndon and Schützenberger [85]). *Let $y \in \Sigma^*$ and $x, z \in \Sigma^+$. Then $xy = yz$ if and only if there exist $u, v \in \Sigma^*$ and an integer $e \geq 0$ such that $x = uv$, $z = vu$, and $y = (uv)^e u$.*

Theorem 2.4 (Lyndon and Schützenberger [85]). *Let $x, y \in \Sigma^+$. Then the following three conditions are equivalent:*

1. *$xy = yx$;*
2. *There exist integers $i, j > 0$ such that $x^i = y^j$;*
3. *There exist $z \in \Sigma^+$ and integers $k, \ell > 0$ such that $x = z^k$ and $y = z^\ell$.*

2.3 Powers

As we have already mentioned, concatenation is written multiplicatively: for a nonempty word $x \in \Sigma^+$, the notation x^n, $n \in \mathbb{N}$, denotes concatenating n copies of x. A word of the form x^n is called an *n-power*. A 2-power is also called a *square*; a 3-power is also called a *cube*. The notion of integral powers was extended to fractional powers by Dejean [37] (Dejean's work is discussed in Chapter 3). Our formulation of fractional powers is based on Brandenburg [20].

A *fractional power* or a *repetition* is a word of the form $z = x^n y$, where $n \in \mathbb{Z}_{\geq 1}$, $x \in \Sigma^+$, and y is a proper prefix of x. Equivalently, z has a $|x|$-period and $|y| = |z| \bmod |x|$. If $|z| = p$ and $|x| = q$, we say that z is a *p/q-power*, or $z = x^{p/q}$. For example, the word `sense` is a 5/3-power, $\texttt{sense} = (\texttt{sen})^{5/3}$. In the expression $x^{p/q}$, the number p/q is the power's *exponent*, and the word x is the *power block*.

Since q stands for both the fraction's denominator and the period, we use non-reduced fractions to denote fractional powers: for example, 10101 is a 5/2-power (as well as a 5/4-power), while 1010101010 is a 10/4-power (as well as a 10/2-power). This distinction is not always made when regarding integral powers: for example, we may refer to the 8/4-power `hotshots` as a 2-power. The term "n-power", where n is an integer, refers to a whole class of fractional powers, that is, the class of nq/q-powers, where q is an arbitrary integer.

There is one type of fractional power which, like integral powers, describes a whole class: the *overlap*. As already mentioned in the introduction, an overlap is a word of the form $axaxa$, where a is a letter and x a word, possibly empty; examples in English include the words `alfalfa` and `entente`. Taking $q = |ax|$ to be the period, we get that an overlap is a $(2q+1)/q$-power, for some $q \geq 1$. Thus the term "overlap" refers to a class of fractional powers with exponents in the range $2 < p/q \leq 3$, that can get arbitrarily close to 2.

Let α be a real number. We say that a word $w \in \Sigma^\infty$ is *α-power-free*, or *avoids α-powers*, if no subword of w is an r-power for any rational $r \geq \alpha$; otherwise, w *contains an α-power*. Similarly, w is *α^+-power-free* if no subword of w is an r-power for any rational $r > \alpha$. The *critical exponent* or the *index* of an infinite word $\mathbf{w} \in \Sigma^\omega$ is defined by

$$E(\mathbf{w}) = \sup\{r \in \mathbb{Q}_{\geq 1} : \mathbf{w} \text{ contains an } r\text{-power}\} . \tag{2.2}$$

Note that while $\mathbf{w}$ contains α-powers for all $1 \leq \alpha < E(\mathbf{w})$, and no α-powers for $\alpha > E(\mathbf{w})$, it may or may not contain $E(\mathbf{w})$-powers, and there are examples of both situations. Indeed, $E(\mathbf{w})$ may be irrational, in which case $\mathbf{w}$ obviously cannot contain $E(\mathbf{w})$-powers. However, even when $E(\mathbf{w})$ is rational or integral, $\mathbf{w}$ need not attain it: in Chapter 4 we construct, for every real number $\alpha > 1$, an infinite word $\mathbf{w}_\alpha$ over a finite alphabet, such that $E(\mathbf{w}_\alpha) = \alpha$, but $\mathbf{w}_\alpha$ does not contain α-powers. When $E(\mathbf{w}) = \infty$ we say that $\mathbf{w}$ is *repetitive*.

Tightly related to the notion of critical exponent is the notion of repetition threshold. Let n be a positive integer, and let $\Sigma_n = \{0, 1, \ldots, n-1\}$. Let α be a real number. We say that α (resp., α^+) is *n-avoidable* if there exists an infinite word over Σ_n that avoids α-powers (resp., α^+-powers). The *repetition threshold for n letters* is defined by

$$RT(n) = \inf\{r \in \mathbb{R}_{>1} : r \text{ is } n\text{-avoidable}\} = \inf\{r \in \mathbb{R}_{>1} : \exists \mathbf{w} \in \Sigma_n^\omega : E(\mathbf{w}) = r\}. \quad (2.3)$$

2.4 Morphisms

A *monoid homomorphism*, or just a *morphism*, is a function f from a monoid $\mathcal{M}$ into a monoid $\mathcal{N}$ that preserves the unit elements and the operations of $\mathcal{M}$ and $\mathcal{N}$:

$$\begin{aligned} f(\mathbf{1}_{\mathcal{M}}) &= \mathbf{1}_{\mathcal{N}}\ ; \\ f(mm') &= f(m)f(m') \quad \forall m, m' \in \mathcal{M}. \end{aligned}$$

Thus a morphism $f : \Sigma^* \to \Gamma^*$, where Σ and Γ are two finite alphabets, is a mapping that preserves concatenation (which in this case implies also that $f(\varepsilon) = \varepsilon$). As such, it is enough to define it on the letters of Σ: if $w = w_0 \cdots w_{m-1} \in \Sigma^*$, then $f(w) = f(w_0) \cdots f(w_{m-1}) \in \Gamma^*$. If $\Gamma = \Sigma$, we say that f *is defined over* Σ. The set of morphisms $f : \Sigma^* \to \Sigma^*$ forms a monoid, with composition as the binary operation and the identity morphism, denoted by Id, as the identity element. We denote this monoid by $\mathcal{M}(\Sigma)$. A n-ary (resp., binary, ternary) morphism is a morphism defined over Σ_n (resp., Σ_2, Σ_3). For a morphism $f : \Sigma_n^* \to \Sigma_k^*$, we sometimes use the notation $f = (f(0), f(1), \ldots, f(n-1))$. The notation $f^n(x)$ stands for applying f to x iteratively n times.

A morphism f is *erasing* if $f(a) = \varepsilon$ for some $a \in \Sigma$; otherwise it is *nonerasing*. Nonerasing morphisms are also called *substitutions*. We say that a nonerasing morphism is *growing* if $|f^n(a)|$ is unbounded as n tends to infinity for all $a \in \Sigma$. A morphism f is called k-*uniform*, or simply *uniform*, if $|f(a)| = k$ for all $a \in \Sigma$. A 1-uniform morphism is called a *coding*. If $f = \mathrm{Id}$ we say f is *trivial*, otherwise it is *nontrivial*. A morphism f defined over Σ is *primitive* if there exists an integer $n \geq 1$ such that a occurs in $f^n(b)$ for all $a, b \in \Sigma$. By this definition, a primitive morphism is always growing. The converse is not always true: for example, the 3-uniform morphism $f \in \mathcal{M}(\Sigma_5)$ defined by $f = (012, 210, 120, 343, 434)$ is growing but not primitive, since for $a \in \{0, 1, 2\}$ and $b \in \{3, 4\}$ there exists no n such that $f^n(a)$ contains b or vice versa.

A morphism $f : \Sigma^* \to \Gamma^*$ is α-power-free if it preserves α-power-freeness, that is, whenever $w \in \Sigma^*$ is α-power-free, then $f(w) \in \Gamma^*$ is α-power-free. An α^+-power-free morphism is defined similarly.

A morphism $f : \Sigma^* \to \Gamma^*$ can be extended to a morphism $f : \Sigma^* \cup \Sigma^\omega \to \Gamma^* \cup \Gamma^\omega$ by $f(a_0 a_1 a_2 \ldots) = f(a_0) f(a_1) f(a_2) \ldots$.

2.4.1 Codes and injectivity

A *code* over Σ^* is a set $X \subseteq \Sigma^*$ such that every decomposition of a word $w \in \Sigma^*$ into elements of X is unique. A set $X \subseteq \Sigma^*$ is a *prefix set* if no element of X is a proper prefix of another element of X. *Suffix sets* are defined similarly. *Bifix sets* are sets that are both prefix and suffix. A morphism $f : \Sigma^* \to \Gamma^*$ is prefix (suffix, bifix) if the set $\{f(a) : a \in \Sigma\}$ is prefix (suffix, bifix).

Theorem 2.5 ([14, Chapter I, Proposition 1.4])**.** *Any prefix (suffix, bifix) set of words $X \neq \varepsilon$ is a code.*

Theorem 2.6 ([83, Proposition 6.1.3])**.** *A morphism $f : \Sigma^* \to \Gamma^*$ is injective if and only if it is injective on Σ, and the set $\{f(a) : a \in \Sigma\}$ is a code.*

Corollary 2.7. *Any prefix (suffix, bifix) morphism is injective.*

2.5 Fixed points and stabilizers

Let f be a morphism defined over Σ. A word $w \in \Sigma^\infty$ is a *fixed point* of f if $f(w) = w$. In this thesis we are interested only in infinite fixed points. An important class of infinite fixed points is the class of pure morphic words.

A morphism $f : \Sigma^* \to \Sigma^*$ is *prolongable* on a letter $a \in \Sigma$ if $f(a) = ax$ for some $x \in \Sigma^+$, and furthermore $f^n(x) \neq \varepsilon$ for all $n \geq 0$. If f is prolongable on a, then $f^n(a)$ is a proper prefix of $f^{n+1}(a)$ for all $n \geq 0$, and the sequence of words $a, f(a), f^2(a), \ldots$ converges in the limit to the infinite word

$$f^\omega(a) = \lim_{n\to\infty} f^n(a) = axf(x)f^2(x)f^3(x)\cdots .$$

The Thue-Morse word $\mathbf{t}$ is an example of an infinite word generated by iterating a morphism: as mentioned before, $\mathbf{t} = \mu^\omega(0)$, where μ is the Thue-Morse morphism, defined over Σ_2^* by $\mu = (01, 10)$. Iterating over the letter 0, we get:

$$\begin{aligned} \mu(0) &= 01; \\ \mu^2(0) &= \mu(0)\mu(1) = 01\mu(1) = 0110; \\ \mu^3(0) &= \mu(0)\mu(1)\mu^2(1) = 01101001; \\ &\vdots \end{aligned}$$

Another example is the *Fibonacci word*,

$$\mathbf{f} := \phi^\omega(0) = 010010100100101\cdots ,$$

where ϕ is the *Fibonacci morphism*, defined over Σ_2^* by $\phi = (01, 0)$. The Fibonacci word gets its name from its close relation to the famous Fibonacci integer sequence, defined recursively by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. In particular, the Fibonacci word can be generated by an analogous recursive process: let $\phi_{-1} = 1$, $\phi_0 = 0$, and $\phi_n = \phi_{n-1}\phi_{n-2}$ for $n \geq 3$. Then $\mathbf{f} = \lim_{n\to\infty} \phi_n$ (see Allouche and Shallit [6, Theorem 7.1.1]).

Clearly, if $\mathbf{w} = f^\omega(a)$ then $\mathbf{w}$ is a fixed point of f. Moreover, if f is growing, then $f(\mathbf{w}) = \mathbf{w}$ if and only if $\mathbf{w} = f^\omega(a)$ for some $a \in \Sigma$ on which f is prolongable. For the more general case we have the following definition and theorem:

Definition 2.1. Let Σ be a finite alphabet, and let $h : \Sigma^* \to \Sigma^*$ be a morphism. A letter $a \in \Sigma$ is said to be *mortal* under h if there exists some $t \geq 1$ such that $h^t(a) = \varepsilon$. The set of all mortal letters associated with h is denoted by M_h. A word is mortal if it belongs to M_h^*; otherwise it is *immortal*. The *mortality exponent* of h, denoted by $\exp(h)$, is the least integer $t \geq 1$ such that $h^t(a) = \varepsilon$ for all $a \in M_h$; if $M_h = \emptyset$, then $\exp(h) = 0$. We define two sets:

$$A_h = \{a \in \Sigma : \exists\, x, y \in M_h^* \text{ such that } h(a) = xay\}, \tag{2.4}$$

$$F_h = \{h^{\exp(h)}(a) : a \in A_h\}. \tag{2.5}$$

Notes:

1. Since a letter satisfying $a \prec h(a)$ cannot be mortal, there exists at most one decomposition $h(a) = xay$ with $x, y \in M_h^*$.

2. The set F_h^* is the set of finite fixed points of h (see Allouche and Shallit, [6, Theorem 7.2.3]).

3. If h is nonerasing, then $M_h = \emptyset$, $A_h = \{a \in \Sigma : h(a) = a\}$, and $F_h = A_h$.

Theorem 2.8 (Head and Lando [60])**.** *Let $h : \Sigma^* \to \Sigma^*$, and let $\mathbf{w} \in \Sigma^\omega$. Then $\mathbf{w}$ is a fixed point of h if and only if at least one of the following two conditions holds:*

1. *$\mathbf{w} \in F_h^\omega$;*

2. *$\mathbf{w} = uh^\omega(a)$, where $u \in F_h^*$ and $h(a) = xay$, with x mortal and y immortal.*

See also Allouche and Shallit, [6, Section 7.3].

Example 2.4. Let $h \in \Sigma_5$ be the morphism defined by $h = (30123, 3413, 104, 4, \varepsilon)$. Then $M_h = \{3, 4\}$, $\exp(h) = 2$, $A_h = 1$, and $F_h = \{h^2(1)\} = \{434134\}$. The morphism h has exactly one infinite fixed point that falls under classification (1) in Theorem 2.8, namely, $\mathbf{u} = (434134)^\omega$. As for classification (2), by iterating h on 0 we get an infinite fixed point:

$$\begin{aligned} h(0) &= 30123; \\ h^2(0) &= h(3)h(0)h(123) = h(3)30123h(123); \\ h^3(0) &= h^2(3)h(3)h(0)h(123)h^2(123) = h(3)h(0)h(123)h^2(123); \\ &\vdots \end{aligned}$$

Since 0 is the only letter that satisfies $h(0) = x0y$ with x mortal and y immortal, the set of infinite fixed points of h of the second type is given by $\{(434134)^n h^\omega(0) : n \in \mathbb{N}\}$.

Infinite words generated by iterating a morphism are called *pure morphic words* or *D0L words*. If $\mathbf{w} \in \Sigma^\omega$ is pure morphic and $c : \Sigma^* \to \Gamma^*$ is a coding, then $c(\mathbf{w})$ is called a *morphic word* or a *CD0L word*. Though generated by a very simple iterative process, pure morphic words can have a very complex combinatorial structure. This combination, of simple definition and complex behavior, has made them the subject of much research. In particular, the area of repetitions in morphic words was widely studied. In Chapter 3 we give a more detailed account of work done in this area. For general properties of morphic words, see Allouche and Shallit [6, Chapters 6–7].

The set of morphisms $f \in \mathcal{M}(\Sigma)$ that fix a given word $\mathbf{w} \in \Sigma^\omega$ is called the *stabilizer* of $\mathbf{w}$, and is denoted by $\mathcal{S}tab(\mathbf{w})$. Being closed under composition, this set forms a submonoid of $\mathcal{M}(\Sigma)$. We write $\mathcal{S}tab(\mathbf{w}) = \langle h_1, \cdots, h_n \rangle$ if the morphisms $h_1, \cdots, h_n$ *generate* $\mathcal{S}tab(\mathbf{w})$, that is, every element of $\mathcal{S}tab(\mathbf{w})$ can be represented as a product of elements of $\{h_1, \cdots, h_n\}$. We use a similar notation for an infinite set of generators. We say that $\mathcal{S}tab(\mathbf{w})$ is *infinitely generated* if it cannot be generated by any finite set. A word $\mathbf{w} \in \Sigma^\omega$ is called *rigid* if $\mathcal{S}tab(\mathbf{w})$ is *cyclic*, that is, $\mathcal{S}tab(\mathbf{w}) = \langle h \rangle$ for some morphism h.

By the discussion above, the stabilizer of a pure morphic word is always non-trivial; however, non-pure morphic words can have non-trivial stabilizers, as the following example demonstrates:

Example 2.5. Let $\mathbf{u} \in \Sigma_2^\omega$ be the infinite binary word constructed by concatenating all finite binary words according to the lexicographic order:

$$\mathbf{u} = 0 \cdot 1 \cdot 00 \cdot 01 \cdot 10 \cdot 11 \cdot 000 \cdot 001 \cdots$$

Define a morphism $h : \Sigma_2 \to \Sigma_3$ by $h(0) = 02$, $h(1) = 1$, and let

$$\mathbf{v} = h(\mathbf{u}) = 02 \cdot 1 \cdot 0202 \cdot 021 \cdot 102 \cdot 11 \cdot 020202 \cdot 02021 \cdots$$

Let $f : \Sigma_3 \to \Sigma_3$ be the morphism defined by $f(0) = \varepsilon$, $f(1) = 1$, $f(2) = 02$. Clearly, $f \in \mathcal{S}tab(\mathbf{v})$. But $\mathbf{v}$ is not morphic: clearly, the subword complexity of $\mathbf{v}$ is exponential, as the subword complexity of $\mathbf{u}$ is given by $p_{\mathbf{u}}(n) = 2^n$. But by a famous result due to Ehrenfeucht, Lee, and Rozenberg [41], the subword complexity of a morphic word can be at most quadratic.

In Chapter 8 we give an additional example, of an infinite word over a 4-letter alphabet, which is fixed by exactly 4 morphisms, none of which generates it by iteration (Example 8.3). In the binary case, however, a non-trivial stabilizers always implies that the word is either pure morphic or has a pure morphic suffix, as the next lemma shows:

Lemma 2.9. *Let* $\mathbf{w} \in \Sigma_2^\omega$*, and suppose that* $\mathcal{S}tab(\mathbf{w})$ *is nontrivial. Then either* $\mathbf{w}$ *is pure morphic, or there exists a letter* $a \in \Sigma_2$*, a positive integer* n*, and a pure morphic binary word* $\mathbf{w}'$*, such that* $\mathbf{w} = a^n\mathbf{w}'$*.*

Proof. If $\mathbf{w}$ is purely periodic, $\mathbf{w} = x^\omega$ for some $x \in \Sigma_2^+$, then the morphism $f = (x, x)$ generates $\mathbf{w}$ by iteration on the first letter of x (if $\mathbf{w} = a^\omega$ for some $a \in \Sigma_2$, take $x = aa$). Suppose therefore that $\mathbf{w}$ is not purely periodic. Then any morphism $f \in \mathcal{S}tab(\mathbf{w})$ must be nonerasing, for if f were erasing, say $f = (\varepsilon, u)$, we would get that $\mathbf{w} = f(\mathbf{w}) = u^\omega$, a contradiction. By Theorem 2.8, f must therefore satisfy exactly one of the following three cases:

1. $f = \mathrm{Id}$;
2. f is prolongable on some $a \in \Sigma_2$ and $\mathbf{w} = f^\omega(a)$;
3. f is prolongable on some $a \in \Sigma_2$, $f(\bar{a}) = \bar{a}$, and $\mathbf{w} = \bar{a}^n f^\omega(a)$ for some $n \geq 1$.

Here $\bar{a} = 1 - a$. Since $\mathcal{S}tab(\mathbf{w})$ is nontrivial, there must exist some $f \in \mathcal{S}tab(\mathbf{w})$ that satisfies either case 2 or case 3. $\square$

The set of morphisms that generate a word $\mathbf{w}$ by iteration (plus the identity morphism) forms a submonoid of the stabilizer of $\mathbf{w}$. We refer to this submonoid as the *iterative stabilizer* and denote it by $\mathcal{IS}tab(\mathbf{w})$. Under this terminology, an infinite word $\mathbf{w}$ is pure morphic if and only if $\mathcal{IS}tab(\mathbf{w})$ is non-trivial.

We end this section by stating a theorem that will be useful later. The theorem is due to Gottschalk [54]; a more recent reference is, e.g., Allouche and Shallit [6, Theorem 10.9.5].

Theorem 2.10 (Gottschalk)**.** *Let* $h : \Sigma^* \to \Sigma^*$ *be a primitive morphism, prolongable on* a*. Then* $h^\omega(a)$ *is uniformly recurrent.*

2.6 D0L systems

The term "D0L words" arises from the theory of *L-systems* (see e.g., Rozenberg and Salomaa [119]). L-systems (or Lindenmayer systems; the acronym "D0L" is an abbreviation for *Deterministic, Zero-sided Lindenmayer*) are string rewriting systems, introduced in 1968 by the biologist Aristid Lindenmayer as part of a mathematical theory of plant development [81]. A *D0L-system*, the simplest class of L-systems, is a triple $G = (\Sigma, f, w)$, where Σ is a finite alphabet, $f : \Sigma^* \to \Sigma^*$ is a morphism, and $w \in \Sigma^+$ is a word known as the system's *axiom*. The system's *language* is the set $L(G) = \{f^n(w) : n \geq 0\}$; thus a pure morphic word generated by f represents a D0L language for which f is prolongable on the axiom. A D0L language L is *α-power-free* if all of its elements are α-power-free; it is *repetitive* if for all $n \in \mathbb{Z}_{\geq 1}$ there exists a word $w \in \Sigma^*$ such that $w^n \in \mathrm{Sub}(L)$; it is *strongly repetitive* if there exists a word $w \in \Sigma^+$ such that $w^n \in \mathrm{Sub}(L)$ for all $n \in \mathbb{Z}_{\geq 1}$.

The *language generated by a morphism* f, denoted by $L(f)$, is the union of all f-based D0L languages that have a letter for an axiom: $L(f) = \{f^n(a) : n \geq 0, a \in \Sigma\}$.

2.7 Parikh vectors and incidence matrices

For a set of numbers S, we denote by $M_{n\times m}(S)$ the set of $n \times m$ matrices with entries in S, and by $M_n(S)$ the set of square $n \times n$ matrices with entries in S. We write $M_{n\times m}, M_n$ if the set S is of no importance. For a matrix $A \in M_{n\times m}$, the transpose of A is denoted by A^T. For a nonsingular matrix $A \in M_n$, the inverse of A is denoted by A^{-1}.

The *spectral radius* of a matrix $A \in M_n(\mathbb{C})$, denoted by $\mathrm{rad}(A)$, is the radius of the smallest origin-centered disc in the complex plane that contains all the eigenvalues of A:

$$\mathrm{rad}(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}. \tag{2.6}$$

For a matrix $A \in M_n(\mathbb{Z})$, we denote by $\mathbb{Q}[A]$ the field extension over $\mathbb{Q}$ spanned by the eigenvalues of A.

Let $u \in \Sigma_n^*$ be a finite word, and let $f : \Sigma_k^* \to \Sigma_n^*$ be a morphism. The *Parikh vector* of u, denoted by $[u]$, is a vector $[u] \in M_{n\times 1}(\mathbb{N})$, defined by

$$[u] = (|u|_0, |u|_1, \ldots, |u|_{n-1})^T \,. \tag{2.7}$$

The *incidence matrix associated with* f, denoted by $A(f)$, is a matrix $A(f) \in M_{n\times k}(\mathbb{N})$, defined by

$$A(f) = (A_{i,j})_{0\le i<n,\, 0\le j<k}\ ; \quad A_{i,j} = |f(j)|_i\ . \tag{2.8}$$

In other words, column j of $A(f)$ is the Parikh vector of $f(j)$.

Example 2.6. Over Σ_5, $[21300041] = (3,2,1,1,1)^T$. Over Σ_6, $[21300041] = (3,2,1,1,1,0)^T$. The incidence matrix of $f = (23120, 212, 111, 3) \in \mathcal{M}(\Sigma_4)$ is given by

$$A(f) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ 2 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\ .$$

Incidence matrices have been proved to be a very useful tool in studying the properties of morphic sequences. In particular, they play a central role in computing letter frequencies in morphic sequences (see Allouche and Shallit [6, Chapter 8]; Saari [120, 121]). In the present thesis, incidence matrices are one of the main tools we use to compute critical exponents in pure morphic sequences over a general alphabet. Their usefulness stems from the following two properties:

Proposition 2.11. *Let $f : \Sigma_n^* \to \Sigma_n^*$ be a morphism, and let $A = A(f)$ be the incidence matrix of f. Then for all $u \in \Sigma_n^*$, and for all $m \ge 0$, we have $[f^m(u)] = A^m[u]$.*

Proof. For all $i = 0, \cdots, n-1$ we have

$$|f(u)|_i = \sum_{k=0}^{n-1} |f(k)|_i \cdot |u|_k,$$

i.e., $[f(u)] = A[u]$. The result follows by induction on m. □

Proposition 2.12. *Let $f : \Sigma_n^* \to \Sigma_n^*$ be a morphism, and let $A = A(f)$ be the incidence matrix of f. Then for all $m \ge 0$, we have $A(f^m) = A(f)^m$.*

Proof. Let $A = (f_{ij})_{0\le i,j<n}$, where $f_{ij} = |f(j)|_i$. Denote $A(f^2) = (a_{ij})_{0\le i,j<n}$, $A(f)^2 = (b_{ij})_{0\le i,j<n}$. Then for all $0 \le i,j < n$ we have:

$$a_{ij} = |f^2(j)|_i = |f(f(j))|_i = \sum_{k=0}^{n-1} |f(k)|_i \cdot |f(j)|_k = \sum_{k=0}^{n-1} f_{ik} \cdot f_{kj} = b_{ij}.$$

The result follows by induction on m. □

As implied by the two propositions above, the subword structure of an infinite word created by iterating a morphism f largely depends on the asymptotic behavior of $A(f)^m$. Since incidence matrices are nonnegative by definition, we have very powerful tools for analyzing such behavior, namely, the theorems of Perron and Frobenius.

2.7.1 Some Perron-Frobenius Theory

Perron-Frobenius Theory is a collective name of a large body of results concerning eigenvalues and eigenvectors of square nonnegative matrices. The topic evolved from the contributions of German mathematicians Oskar Perron (1880–1975) and Ferdinand Georg Frobenius (1849–1917): the first result is the 1907 theorem of Perron [107], regarding positive matrices; in 1912, Frobenius [49] extended Perron's theorem to irreducible nonnegative matrices. Later, some results where extended to nonnegative matrices in general.

In this section we present the theorems of Perron and Frobenius and some additional results, that will be used in Chapter 6. Proofs can be found, e.g., in Minc [91, Chapters 1, 3], Horn and Johnson [61, Chapter 8], and Allouche and Shallit [6, Chapter 8].

A matrix $A = (a_{ij})_{0 \le i,j < n} \in M_n(\mathbb{C})$ is said to be *positive* (resp., *nonnegative*), denoted by $A > 0$ (resp., $A \ge 0$), if $a_{ij} > 0$ (resp., $a_{ij} \ge 0$) for all $0 \le i, j < n$.

A *permutation matrix* is a matrix $P \in M_n(\Sigma_2)$, which has exactly one entry equal to 1 in each row and each column. Thus a left multiplication of a matrix $A \in M_{n \times m}$ by a permutation matrix $P \in M_n$ is equivalent to permuting the rows of A, while a right multiplication by a permutation matrix $P \in M_m$ is equivalent to permuting the columns of A. A Permutation matrix P satisfies $P^{-1} = P^T$.

A matrix $A \in M_n$ is *reducible* if $n = 1$ and $A = 0$, or $n \ge 2$ and there exist a permutation matrix $P \in M_n(\Sigma_2)$ and an integer $1 \le s < n$, such that $A = P\left[\begin{smallmatrix} B & C \\ 0 & D \end{smallmatrix}\right]P^T$, where $B \in M_s$, $D \in M_{n-s}$, $C \in M_{s \times n-s}$, and $0 \in M_{n-s \times s}$ is the zero matrix. If A is not reducible it is said to be *irreducible*. In particular, every positive matrix is irreducible, and every matrix that has a zero row or column is reducible. An irreducible matrix A is *primitive* if there exists an integer k such that all the entries of A^k are positive. Recall that a morphism defined over Σ is primitive if there exists an integer $n \ge 1$ such that a occurs in $f^n(b)$ for all $a, b \in \Sigma$. The term stems from the matrix definition: it is easy to see that a morphism is primitive if and only if its incidence matrix is primitive.

Theorem 2.13 (Perron)**.** *Let $A \in M_n(\mathbb{C})$ be a positive matrix, and let $r = \mathrm{rad}(A)$. Then*

1. *$r > 0$;*
2. *r is an eigenvalue of A, with a real positive corresponding eigenvector;*
3. *If $\lambda \neq r$ is an eigenvalue of A, then $|\lambda| < r$;*
4. *r is a simple root of the characteristic polynomial of A.*

Theorem 2.14 (Perron-Frobenius)**.** *Let $A \in M_n(\mathbb{C})$ be a nonnegative irreducible matrix, and let $r = \mathrm{rad}(A)$. Then*

1. *$r > 0$;*
2. *r is an eigenvalue of A, with a real positive corresponding eigenvector;*
3. *Let $r = \lambda_0, \lambda_1, \ldots, \lambda_h$ be the eigenvalues of A of modulus r (counting multiplicities). Then $\lambda_0, \ldots, \lambda_{h-1}$ are the h distinct roots of r^h, $\lambda_k = e^{2ik\pi/h} r$;*
4. *If A is primitive, then*
   (a) *If $\lambda \neq r$ is an eigenvalue of A, then $|\lambda| < r$;*
   (b) *r is a simple root of the characteristic polynomial of A.*

Theorem 2.15. *Let $A \in M_n(\mathbb{C})$ be a nonnegative matrix, and let $r = \mathrm{rad}(A)$. Then*

1. *r is an eigenvalue of A, with a real nonnegative corresponding eigenvector;*
2. *there exists a positive integer n such that any eigenvalue λ of A with $|\lambda| = r$ satisfies $\lambda^n = r^n$.*

Definition 2.2. The eigenvalue $r = \mathrm{rad}(A)$ of a nonnegative square matrix A is called the *Perron-Frobenius eigenvalue* of A. We denote it by $r(A)$. If $f \in \mathcal{M}(\Sigma_n)$ is a morphism, we denote by $r(f)$ the Perron-Frobenius eigenvalue of the incidence matrix of f.

2.8 Sturmian words and morphisms

Sturmian words (named after French mathematician Jacques Charles François Sturm, 1803–55) are infinite binary words that have exactly $n+1$ distinct subwords of length n for all $n \geq 0$; that is, the subword complexity function of a Sturmian word $\mathbf{s}$ is given by $p_{\mathbf{s}}(n) = n+1$. The Fibonacci word defined in Section 2.5 is the most famous example of this family. Sturmian words have numerous combinatorial, arithmetical and geometrical properties, and have been extensively studied. In particular, there exist many results concerning critical exponents in Sturmian words. In the next chapter we survey these results in detail.

A morphism $f \in \mathcal{M}(\Sigma_2)$ is Sturmian if, whenever $\mathbf{s}$ is a Sturmian word, $f(\mathbf{s})$ is also Sturmian. A pure morphic word generated by a Sturmian morphism is itself Sturmian. The Sturmian morphisms form a submonoid of $\mathcal{M}(\Sigma_2)$, defined as follows:

Definition 2.3. *The monoid of Sturm*, denoted by St, is the submonoid of $\mathcal{M}(\Sigma_2)$ defined by $St = \langle \phi, \tilde{\phi}, e \rangle$, where $\phi, \tilde{\phi}, e \in \mathcal{M}(\Sigma_2)$ are defined by $\phi = (01, 0)$, $\tilde{\phi} = (10, 0)$, and $e = (0, 1)$. *The monoid of standard morphisms* is the submonoid of St generated by ϕ, e.

Theorem 2.16 ([83, Theorem 2.3.7])**.** *A morphism f is Sturmian if and only if $f \in St$.*

As we can see, the Fibonacci morphism is one of the generators of St. The submonoid generated by $\langle \phi, e \rangle$ is called *the monoid of standard morphisms*.

For a comprehensive background on Sturmian words, see Lothaire [83, Chapter 2].

2.9 Episturmian words and morphisms

Episturmian words, introduced by Droubay, Justin and Pirillo in [40], are a generalization of Sturmian words to alphabets of more than two letters. In Chapter 8, where we study stabilizers of episturmian words, we give a general definition of episturmian words and present many of their properties. In this section we briefly introduce the subclass of *strict epistandard words*, the critical exponent of which will be discussed in the next chapter. For a comprehensive survey on episturmian words, see Glen and Justin [53].

Let Σ be a finite alphabet, and let $\Delta = x_1x_2x_3\cdots \in \Sigma^\omega$ be any infinite sequence. Define a sequence of words $\{u_n\}_{n=0}^{\infty} \subseteq \Sigma^*$ by

$$\begin{aligned} u_0 &= \varepsilon, \\ u_n &= (u_{n-1}x_n)^{(+)},\ n \geq 1. \end{aligned}$$

Then u_i is a prefix of u_{i+1} for all $i \geq 0$, and so $\lim_{n\to\infty} u_n$ exists. The word $\mathbf{s} = \lim_{n\to\infty} u_n$ is called *the epistandard word directed by* Δ, and $\Delta = \Delta(\mathbf{s})$ is the *directive word* of the epistandard word $\mathbf{s}$. An epistandard word $\mathbf{s}$ is Σ-*strict* (or simply *strict*) if $\Delta(\mathbf{s})$ is letter-recurrent.

For a letter $a \in \Sigma$, define a morphism $\psi_a \in \mathcal{M}(\Sigma)$ by $\psi(a) = a$ and $\psi(b) = ab$ for every letter $b \neq a$. For every pair of letters $a \neq b \in \Sigma$, define a morphism $\theta_{ab} \in \mathcal{M}(\Sigma)$ by $\theta(a) = b$, $\theta(b) = a$, and $\theta(c) = c$ for every letter $c \neq a, b$. The *monoid of epistandard morphisms*, denoted by $\mathscr{S}$, is the monoid generated by the set $\{\psi_a, \theta_{ab} : a, b \in \Sigma\}$. For a word $y = a_1a_2\cdots a_n \in \Sigma^+$, the epistandard morphism $\psi_{a_1}\cdots\psi_{a_n}$ is denoted by ψ_y.

Theorem 2.17 (Justin and Pirillo [64])**.** *Let* $\mathbf{s}$ *be a* Σ*-strict epistandard word. Then* $\mathbf{s}$ *is pure morphic if and only if* $\Delta(\mathbf{s})$ *is purely periodic. More specifically, if* $\Delta(\mathbf{s}) = y^\omega$*, then* $\mathbf{s} = \psi_y^\omega(a)$*, where* a *is the first letter of* Δ*.*

Chapter 3

Critical Exponents

3.1 Introduction

Power avoidance problems — and more generally, pattern avoidance problems — are among the most studied in the area of combinatorics on words. For a comprehensive list of references, see Guy [57, §E21]. In this chapter, we survey in more detail work related to critical exponents. The results we survey can be roughly divided into four types: results concerning morphisms and morphic words, results concerning Sturmian words, results concerning episturmian words, and results concerning paperfolding words. The subject of repetition thresholds is tightly related to critical exponents in pure morphic words, and we devote the last section to it.

3.2 Repetitions in morphic words

3.2.1 Circularity

When trying to analyze repetitions in pure morphic words (to show that a given word avoids a certain power or attains a certain power), the most common approach is to use the decomposition of the word into images of the generating morphism. Let $\mathbf{w} = f^\omega(a)$ be a pure morphic word. The fact that $\mathbf{w}$ is generated by iterating f implies that every subword u of $\mathbf{w}$ with $|u| \geq \max\{|f(a)| : a \in \Sigma\}$ can be decomposed as $u = s_0 f(a_1) f(a_2) \cdots f(a_n) p_{n+1}$, where

- $n \geq 0$;
- $a_0, a_1, \ldots, a_{n+1} \in \Sigma$;
- s_0 is a suffix of $f(a_0)$;
- p_{n+1} is a prefix of $f(a_{n+1})$.

The decomposition above is called an *interpretation* of u by the D0L system (Σ, f, a). The word $u' = a_0 a_1 \cdots a_{n+1}$ is called an *ancestor* of u. Whether or not this interpretation is unique is tightly related to whether or not $\mathbf{w}$ is repetitive (recall that $\mathbf{w}$ is *repetitive* if $E(\mathbf{w}) = \infty$).

The first one to define and study the notion of circularity (though under a different name) was Mossé: in 1992 [96], she considered primitive morphisms that generate aperiodic fixed points. Let $f : \Sigma^* \to \Sigma^*$ be a primitive morphism prolongable on a letter a, and let $\mathbf{w} = f^\omega(a)$. Mossé showed that if $\mathbf{w}$ is aperiodic, then it is not repetitive, and moreover, it is recognizable (see below). A year later, Mignosi and Séébold [89] gave a stronger result: if $L(G) = L((\Sigma, f, w))$ is a non-repetitive D0L language, where $f : \Sigma^* \to \Sigma^*$ is any morphism, then $L(G)$ is circular (see definition below). Using this result, they gave an algorithm to test whether a given D0L language is k-power-free for a given k, and a simpler proof for the following result of Ehrenfeucht and Rozenberg [42]: it is decidable whether a D0L language is repetitive.

Recognizability and circularity are almost equivalent notions. Roughly speaking, a D0L language $L(G)$ is circular if every sufficiently long word $v \in \mathrm{Sub}(L(G))$ can be decomposed unambiguously into images under f, except perhaps a prefix and a suffix of bounded length. The bound on the length of these prefix and suffix is called the *synchronization delay*. Recognizability is slightly weaker, as it allows ambiguity when f is not injective. The formal definition of circularity is the following:

Definition 3.1. Let $G = (\Sigma, f, w)$ be a D0L system, and let $u \in \mathrm{Sub}(L(G))$ satisfy

$$u = s_0 f(a_1) \cdots f(a_n) p_{n+1} = y_0 f(b_1) \cdots f(b_m) x_{m+1},$$

where

- $a_i, b_j \in \Sigma$ for $0 \le i \le n+1$ and $0 \le j \le m+1$;
- s_0, y_0 are suffixes of $f(a_0)$, $f(b_0)$, respectively;
- p_{n+1}, x_{m+1} are prefixes of $f(a_{n+1})$, $f(b_{m+1})$, respectively.

Then $L(G)$ is *circular* with *synchronization delay* D if whenever $|s_0 f(a_1)\cdots f(a_{i-1})| > D$ and $|f(a_{i+1})\cdots f(a_n)p_{n+1}| > D$ for some $1 \le i \le n$, then $s_0 f(a_1)\cdots f(a_{i-1}) = y_0 f(b_1)\cdots f(b_{j-1})$ for some $1 \le j \le m$, and $a_i = b_j$ (see Fig. 3.1). If such i and j exist, we say that the two interpretations are *synchronized*. A word $u \in \mathrm{Sub}(L(G))$ is *synchronized* if any two interpretations of it are synchronized.

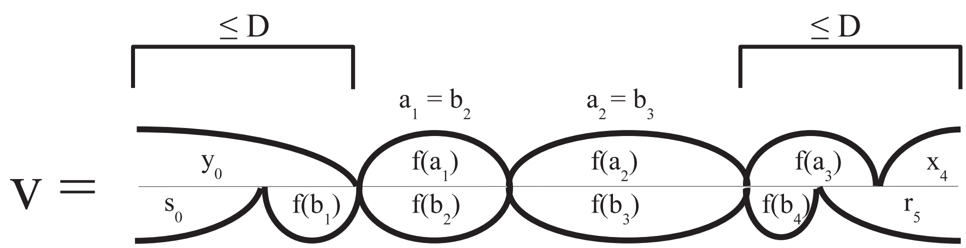

Figure 3.1: Synchronization of two interpretations.

The definition of recognizability does not require the condition $a_i = b_j$; thus, if f is not injective, two synchronized interpretations might admit different ancestors. However, for injective morphisms the definitions coincide.

Theorem 3.1 (Mignosi and Séébold [89]). *If a D0L language is k-power-free for some number k, then it is circular.*

Yet another definition of circularity was given by Cassaigne in 1994 [26], where he used circularity to design an algorithm that tests whether a given D0L language avoids a given pattern. Again, the definition coincides with the two previous ones for injective morphisms:

Definition 3.2. Let $h : \Sigma^* \to \Sigma^*$ be a morphism injective over Σ^*, and let $w \in \Sigma^*$. We say that (w_1, w_2) is a *synchronization point* of w (for h), if $w = w_1 w_2$, and for all $v_1, v_2, u \in \Sigma^*$,

$$v_1 w v_2 = h(u) \Rightarrow \exists u_1, u_2 \text{ such that } u = u_1 u_2, \text{ and } v_1 w_1 = h(u_1),\ w_2 v_2 = h(u_2).$$

A D0L system $G = (\Sigma, h, w)$ is circular with synchronization delay D if h is injective on $\mathrm{Sub}(L(G))$, and every word $u \in \mathrm{Sub}(L(G))$ with $|u| \geq D$ has at least one synchronization point.

Definitions 3.1 and 3.2 are equivalent for injective morphisms, though the synchronization delay is not necessarily the same: if f is injective over $\mathrm{Sub}(L(G))$ and $u \in \mathrm{Sub}(L(G))$ has at least two synchronization points, then any two distinct interpretations of u must synchronize. On the other hand, if $L(G)$ is circular with synchronization delay D, then every word $u \in \mathrm{Sub}(L(G))$ such that $|u| \geq 2D$ must admit a synchronization point. That is, circularity with delay D by Definition 3.1 implies circularity with delay at most $2D$ by Definition 3.2, and vice versa.

Another related result is the one of Frid, who gave in 1998 [47] necessary and sufficient conditions for a uniform *marked D0L word* to be circular. A morphism $f \in \mathcal{M}(\Sigma)$ is said to be *marked* if $f(a)$ and $f(b)$ both begin and end with different letters for all $a \neq b \in \Sigma$; an example over Σ_3 is the morphism $f = (11, 002, 210)$. A pure morphic word is marked if it is generated by a marked morphism. Frid's criterion gives another tool for checking whether a D0L language is repetitive. Her proof also implies that if a uniform marked pure morphic word is circular then it is k-power-free for some number k, and so, for uniform marked pure morphic words, Theorem 3.1 becomes an if and only if (this is not the case in general, see Section 6.5.2).

What makes circularity so useful is that we can "know where we are coming from": if a D0L language G is circular, every sufficiently long word $u \in \mathrm{Sub}(L(G))$ has a unique inverse image (modulo the edges). Therefore, if a certain property is preserved under f, or evolves under f in a predictable manner, then by observing a finite number of words $u \in \mathrm{Sub}(L(G))$ we can draw conclusions about the whole system. Power containment and avoidance are such properties. Clearly, integral powers are preserved under morphisms: if $u = x^n$ for some $n \in \mathbb{N}$, then $f(u) = (f(x))^n$. Fractional powers are not always preserved: consider, for example, the morphism $f \in \mathcal{M}(\Sigma_2)$, defined by $f(0) = 0$, $f(1) = 11$. The word $u = 01010$ is a $5/2$-power, but the word $f(u) = 0110110$ is only a $7/3$-power. However, we can compute exactly the sequence of exponents generated by successive applications of f (indeed, this is the central strategy we use in characterizing and computing critical exponents of pure morphic words, as will be shown in Chapters 5, 6). From the other direction, if a language is circular, then whenever a sufficiently long word $u \in \mathrm{Sub}(L(G))$ contains a certain power, the ancestor of u must contain a power with the same, or almost the same, exponent; this fact allows us to test only a finite number of words when trying to show that $\mathbf{w}$ avoids certain powers.

Though circularity as such was defined only fifteen years ago, the ideas sketched above were used implicitly by many authors. When trying to prove that a pure morphic word $\mathbf{w}$ is α-power-free for some real number α, there are two main strategies that have been commonly used:

1. Prove that the generating morphism is α-power-free: if $f : \Sigma^* \to \Sigma^*$ is α-power-free for some $\alpha > 1$ and $\mathbf{w} = f^\omega(a)$, then clearly $\mathbf{w}$ is α-power-free as well. Proving a morphism f to be power-free can be done by showing that if $f(u)$ contains an α-power then u must contain an α-power; this method was used by Thue to show that the Thue-Morse morphism is overlap-free. Another method is to use *finite test sets*: Let $f : \Sigma^* \to \Gamma^*$ be a morphism. A *test set* for the α-power-freeness of f is a set $X \subseteq \Sigma^*$ of α-power-free words, such that f is α-power-free if and only if $f(w)$ is α-power-free for all $w \in X$.

   Though proving a morphism to be α-power-free is a useful strategy, it is inherently limited: as we shall see, no α-power-free morphisms exist for $\alpha \leq 3/2$.

2. Prove $\mathbf{w}$ to be α-power-free using inverse image arguments: show that if an element of a certain form of $\mathrm{Sub}(\mathbf{w})$ contains an α-power, then its pre-image under f must contain an α-power. Proofs or this type make implicit use of the circularity of the given D0L-system.

3.2.2 Thue's work and Dejean's conjecture

Thue was the first to use power-freeness of morphisms. In his 1906 paper [131, 11], he constructed a square-free morphic word over a ternary alphabet, by iterating a square-free uniform morphism over Σ_4, and then applying a square-free coding $\Sigma_4^* \to \Sigma_3^*$ to the resulting pure morphic word (Thue's original construction used a slightly different terminology). In his 1912 paper [132, 11] Thue concentrated on binary alphabets. Recall that the Thue-Morse morphism μ is defined over Σ_2 by $\mu = (01, 10)$, and the Thue-Morse word $\mathbf{t}$ is given by $\mathbf{t} = \mu^\omega(0)$ (Definition 1.1). Thue proved that μ is overlap-free, thus establishing the existence of an overlap-free binary infinite word. Moreover, he showed that any overlap-free binary morphism is of the form μ^k or $e \cdot \mu^k$, where k is a positive integer and e is the morphism that exchanges 0 and 1.

Recall that an overlap is a $(2q+1)/q$-power, where $q \geq 1$ is any positive integer. Avoiding overlaps is thus equivalent to avoiding 2^+-powers (that is, avoiding any power larger than 2). On the other hand, it is easy to see that any binary word of length at least 4 must contain a square. It follows that the critical exponent of $\mathbf{t}$ is 2, and the bound is attained. Moreover, since squares are unavoidable, this result is optimal: any infinite binary word must have a critical exponent ≥ 2. That is, the repetition threshold for a binary alphabet is $RT(2) = 2$ (recall Chapter 2, Definition 2.3: the repetition threshold for an alphabet of size k is the infimum of the set of exponents that can be avoided over k letters).

The same technique was used sixty years later to compute the repetition threshold for ternary alphabets. By Thue we know that, over three letters, 2-powers can be avoided; but this bound is not optimal. In 1972, Dejean [37] constructed an infinite ternary word that avoids $(7/4)^+$-powers. She also showed that all ternary words of length 39 or more must contain a 7/4-power, and so $RT(3) = 7/4$. The infinite word she constructed was a pure morphic word generated by *Dejean's morphism*, and the technique she used was to prove the morphism to be $7/4^+$-power-free. However, while Thue's morphism is 2-uniform, Dejean's morphism is 19-uniform, and no shorter example exists (Ochem, [102]). Dejean's morphism is given below:

$$\delta : \begin{cases} 0 \rightarrow 0120212012102120210 \\ 1 \rightarrow 1201020120210201021 \\ 2 \rightarrow 2012101201021012102 \end{cases} . \tag{3.1}$$

Dejean's paper is the origin of the repetition threshold question. In the same paper she showed by exhaustive search that every word over Σ_4 of length 122 must contain a 7/5-power, thus $RT(4) \geq 7/5$. For $n \geq 5$, she observed that any word over Σ_n of length $n+2$ must contain an $n/(n-1)$-power, thus $RT(n) \geq n/(n-1)$. Dejean conjectured that these values are the actual repetition threshold values. This is the famous *Conjecture of*

Dejean:

$$RT(n) = \begin{cases} 2, & \text{if } n = 2; \\ 7/4, & \text{if } n = 3; \\ 7/5, & \text{if } n = 4; \\ n/(n-1), & \text{if } n \geq 5. \end{cases} \tag{3.2}$$

The formulation above was given in 1983 by Brandenburg, in a paper on power-free uniform morphisms [20]. Brandenburg gave necessary and sufficient conditions for a uniform morphism over an arbitrary alphabet to be square-free, and showed that for any finite alphabet Σ there exists a square-free uniform morphism $\Sigma^* \to \Sigma_3^*$ and a cube-free uniform morphism $\Sigma^* \to \Sigma_2^*$. He used these results to show that the sets of square-free ternary words and of cube-free binary words grow exponentially. He also formulated the notion of repetition thresholds. Let $\{r_n\}_{n\geq 2}$ be the sequence of values stated in Dejean's conjecture. Brandenburg showed that for $n \geq 3$, every r_n^+-power-free morphism must be uniform, and that no uniform morphism is $3/2$-power-free. Thus, for $n \geq 4$, no r_n^+-power-free morphism exists. This implies that the technique used by Thue and Dejean to show that $RT(n) = r_n$ for $n = 2, 3$ cannot be employed further. (However, it does not necessarily imply that there exist no r_n^+-power-free pure morphic words over Σ_n. This is the *D0L repetition threshold* problem that will be discussed further in Chapter 9.) In Section 3.6 we give a full account of the repetition threshold problem.

3.2.3 Generalizations of the Thue-Morse word and morphism

The Thue-Morse word $\mathbf{t}$ admits a few equivalent definitions. We have seen a definition as a pure morphic word. Another definition uses sums of digits:

Definition 3.3. Let $s_2(n)$ be the sum of digits in the binary expansion of n. Then

$$\mathbf{t} = t_0 t_1 t_2 \ldots, \text{ where } t_n = s_2(n) \bmod 2.$$

The above definition can be generalized as follows:

Definition 3.4. Let $k \geq 2, m \geq 1$ be integers, and let $s_k(n)$ be the sum of digits in the base k expansion of n. Then the *generalized Thue-Morse word* $\mathbf{t}_{k,m}$ is an infinite word over the alphabet $\Sigma_m = \{0, 1, \ldots, m-1\}$, defined by

$$\mathbf{t}_{k,m} = a_0 a_1 a_2 \ldots, \text{ where } a_n = s_k(n) \bmod m.$$

Example 3.1. Let $k = 2$. The first few terms of $(s_2(n))_{n\geq 0}$ are $0, 1, 1, 2, 1, 2, 2, 3, 1, \ldots$. Taking the sequence modulo 2, we get the Thue-Morse word, $\mathbf{t} = \mathbf{t}_{2,2} = 011010011\cdots$. Taking the sequence modulo 3 we get an infinite word over Σ_3, $\mathbf{t}_{2,3} = 011212201\cdots$.

Though Thue was the first to study explicitly the combinatorial properties of $\mathbf{t}$, the sequences $\{\mathbf{t}_{n,n} : n \geq 2\}$ already appear implicitly in a 1851 paper due to Prouhet [108]: in that paper, Prouhet used the $t_{n,n}$ sequences to realize a solution to an arithmetical problem, the so-called "Prouhet-Tarry-Escott", or "multigrades" problem (see Adler and Li [2]; Allouche and Shallit [4]; Séébold [125, 126]). The combinatorial properties of $\mathbf{t}_{k,m}$ where studied by Morton and Mourant in 1991 [95] and by Allouche and Shallit in 2000 [5]. Morton and Mourant proved, among other things, that $\mathbf{t}_{k,m}$ is ultimately periodic if and only if $m|(k-1)$. Allouche and Shallit proved that $\mathbf{t}_{k,m}$ is never square-free, and is overlap-free if and only if $m \geq k$. In other words, $E(\mathbf{t}_{k,m}) \geq 2$ for all $k \geq 2$ and $m \geq 1$, and $E(\mathbf{t}_{k,m}) = 2$ if and only if $m \geq k$. Both papers use the sum-of-digit definition of $\mathbf{t}_{k,m}$; however, in [5] Allouche and Shallit also give without proof a theorem stating that the $\mathbf{t}_{k,m}$ sequences are pure morphic.

Definition 3.5. Let $k \geq 2, m \geq 1$ be integers, and let $\Sigma_m = \{0, 1, \ldots, m-1\}$. Define the *generalized Thue-Morse morphism* $\mu_{k,m} : \Sigma_m^* \to \Sigma_m^*$ by

$$\mu_{k,m}(i) = i(i+1)(i+2)\cdots(i+k-1); \quad i = 0, 1, \ldots, m-1, \tag{3.3}$$

where all the sums are taken modulo m.

Example 3.2. For $k = m = 2$, we get that $\mu_{2,2}$ is defined over Σ_2 by $\mu_{2,2} = (01, 10)$, thus $\mu_{2,2}$ is the Thue-Morse morphism. The morphisms $\mu_{2,3}$, $\mu_{7,3}$ are defined over Σ_3 by $\mu_{2,3} = (01, 12, 20)$, $\mu_{7,3} = (0120120, 1201201, 2012012)$.

Theorem 3.2 (Allouche and Shallit [5]). *For all $k \geq 2$ and $m \geq 1$, $\mathbf{t}_{k,m} = \mu_{k,m}^{\omega}(0)$.*

The generalized Thue-Morse words and morphisms where studied from a pure-morphic view by a few authors. In 2001, Frid [48] generalized the family $\{\mu_{k,m}\}$ to the family of symmetric morphisms. Let $u = u_0 u_2 \cdots u_{k-1} \in \Sigma_m^+$. The *symmetric morphism* $\varphi_{u,m}$ is a uniform morphism defined over Σ_m, that satisfies $\varphi(i) = (i+u_0)(i+u_1)\cdots(i+u_{k-1})$ for all $i \in \Sigma_m$ (again, all the sums are taken modulo m). Thus $\mu_{k,m}$ is the symmetric morphism defined by the word $u = 012\cdots(k-1) \bmod m$. If u begins with 0, then $\varphi_{u,m}$ is prolongable on 0 (indeed, on any of the letters), and we define $\mathbf{t}_{u,m} = \varphi^{\omega}_{u,m}(0)$. Frid showed that if all the symbols occurring in $u \in \Sigma_m^*$ are distinct then $\mathbf{t}_{u,m}$ is overlap-free. However, this condition is not necessary, nor is the 2^+-avoidance optimal: observe that Dejean's morphism (3.1) is a symmetric morphism, $\delta = \varphi_{u,3}$, where $u = 0120212012102120210$. Though none of the symbols occurring in u is distinct, δ is $7/4^+$-power-free.

In 2002, Séébold [125, 126] studied the generalized Thue-Morse words and morphisms, with a special emphasis on the family $\{\mathbf{t}_{n,n} : n \geq 2\}$, also known as *Prouhet words*. Among other results, he showed that $\mu_{k,m}$ is overlap-free if and only if $k|m$. He also gave a combinatorial proof to the fact that $\mathbf{t}_{2,m}$ is overlap free for all $m \geq 2$, based on the inverse image technique.

In 2007, Blondin-Massé and Labbé [19] computed the critical exponent for all generalized Thue-Morse words. As mentioned above, $E(\mathbf{t}_{k,m}) = 2$ for all $m \geq k$; Blondin-Massé and Labbé used the inverse image technique to compute $E(\mathbf{t}_{k,m})$ in general, in terms of k and m. They also computed the positions in $\mathbf{t}_{k,m}$ where $E(\mathbf{t}_{k,m})$ is attained (as we prove in Chapter 6, finite critical exponents of fixed points of symmetric morphisms are always attained).

In 2007, Tompkins [133] suggested yet another generalization of the Thue-Morse morphism. A *Latin square* of order n is a square matrix $A \in M_n(\{0, 1, \ldots, n-1\})$, where each row and each column is a permutation of $\{0, 1, \ldots, n-1\}$. Given a Latin square L of order n, with rows $\ell_0, \ell_1 \cdots, \ell_{n-1}$, define a morphism $\lambda_L : \Sigma_n^* \to \Sigma_n^*$ by $\lambda_L(i) = \ell_i$, $i \in \Sigma_n$. Thus, when the rows of L are given by $\ell_i = i(i+1)\cdots n12\cdots(n-1)$, the morphism λ_L is exactly $\mu_{n,n}$. When the first column of L is the identity permutation $01\cdots(n-1)$, we get that λ_L is prolongable on every letter of Σ_n. Tompkins showed that if L is a Latin square of such type, then the pure morphic word $\lambda_L^{\omega}(i)$ is overlap-free for all $i \in \Sigma_n$. The proof is based both on the arithmetical properties of Latin squares and on the inverse image technique.

3.2.4 The Arshon words

In 1935, the Russian mathematician Solomon Efimovich Arshon[1] [7, 8] gave an algorithm to construct an infinite cube-free word over 2 letters, and an algorithm to construct an infinite square-free word over n letters for each $n \geq 3$. The binary word he constructed turns out to be exactly the Thue-Morse word; the square-free words are now known as the *Arshon words*, and can be considered as another generalization of the Thue-Morse word. For $n \geq 2$, we denote the Arshon word of order n by $\mathbf{a}_n = a_0 a_1 a_2 \cdots$.

For $n \geq 3$, let $e = 01 \cdots (n-1) \in \Sigma_n^*$, and let $o = e^R$ (the letters 'e' and 'o' stand for "even" and "odd", respectively). The Arshon word of order n can be generated by alternately iterating the symmetric morphisms $\varphi_{e,n}$ and $\varphi_{o,n}$: define an operator

$$\varphi_n : \Sigma^* \to \Sigma^*$$

by

$$\varphi_n(a_i) = \begin{cases} \varphi_{e,n}(a_i), & \text{if } i \text{ is even;} \\ \varphi_{o,n}(a_i), & \text{if } i \text{ is odd.} \end{cases} \tag{3.4}$$

That is, if $u = a_0 a_1 \cdots a_m \in \Sigma_n^*$, then $\varphi_n(u) = \varphi_{e,n}(a_0)\varphi_{o,n}(a_1)\varphi_{e,n}(a_2)\varphi_{o,n}(a_3)\cdots$. The Arshon word of order n is given by $\mathbf{a}_n = \lim_{k\to\infty} \varphi_n^k(0)$. Note that, though φ_n is not a morphism, $\varphi_n^k(0)$ is still a prefix of $\varphi_n^{k+1}(0)$ for all $k \geq 0$, and the limit is well defined.

Example 3.3. For $n = 3$, the even and odd Arshon morphisms are given by

$$\varphi_{e,3} : \begin{cases} 0 \to 012 \\ 1 \to 120 \\ 2 \to 201 \end{cases}, \quad \varphi_{o,3} : \begin{cases} 0 \to 210 \\ 1 \to 021 \\ 2 \to 102 \end{cases},$$

and the Arshon word of order 3 is given by

$$\mathbf{a}_3 = \lim_{k\to\infty} \varphi_3^k(0) = \underbrace{012}_{\varphi_{e,3}(0)}\ \underbrace{021}_{\varphi_{o,3}(1)}\ \underbrace{201}_{\varphi_{e,3}(2)}\ \underbrace{210}_{\varphi_{o,3}(0)} \cdots.$$

The morphisms $\varphi_{e,n}$ and $\varphi_{o,n}$ are called the *even and odd Arshon morphisms of order* n, respectively. The operator φ_n is called the *Arshon operator of order* n.

[1] Vilenkin, in his 1991 article "Formulas on cardboard" [136], says that Arshon was arrested by the Soviet regime and died in prison, most likely in the late 1930's or early 1940's.

It is not difficult to see that when n is even, the i'th letter of $\mathbf{a}_n$ is even if and only if i is an even position (for a formal proof, see Séébold [125, 126]). Therefore, when n is even, the map φ_n becomes a morphism, called the *Arshon morphism of order* n, and we denote it by α_n:

$$\alpha_n(a) = \begin{cases} \varphi_{e,n}(a), & \text{if } a \text{ is even;} \\ \varphi_{o,n}(a), & \text{if } a \text{ is odd.} \end{cases} \tag{3.5}$$

When n is odd no such partition exists, and indeed, $\mathbf{a}_n$ cannot be generated by iterating a morphism. This fact was proved for $\mathbf{a}_3$ by Berstel [10] and Kitaev [67, 68], and for any odd n by Currie [31]. However, one can still apply circularity-type arguments when trying to prove that $\mathbf{a}_n$ avoids certain powers, though the arguments would be more involved. In 2001, Klepinin and Sukhanov [70] used this approach to show that $E(\mathbf{a}_3) = 7/4$. As we saw in Section 3.2.2, this is the smallest critical exponent attainable over a ternary alphabet. In Chapter 7 we generalize this result, and compute $E(\mathbf{a}_n)$ for all $n \geq 4$.

We add two remarks:

1. The Arshon words of odd order are morphic, though not pure morphic: instead of iterating two n-uniform morphisms over n letters, we can iterate one n-uniform morphism over $2n$ letters, and get $\mathbf{a}_n$ by applying a coding to the pure morphic word thus generated. In particular, since the generating morphism is n-uniform, $\mathbf{a}_n$ is *n-automatic*: there exists a deterministic finite automaton $\mathcal{A}_n$, such that, when the input string represents the base n expansion of a number $i \in \mathbb{N}$, $\mathcal{A}_n$ outputs the i'th letter of $\mathbf{a}_n$. See Cobham, [29]; Allouche and Shallit [6, Definition 5.1.1, Theorem 6.3.2].

2. The Arshon word $\mathbf{a}_3$ was precisely the sequence used for solving the *Burnside problem for groups* (Adian [1]): let G be a finitely generated group, such that every element of G has a finite order. Must G be finite? The answer is negative, and the proof used square-free words in an essential way.

3.2.5 Repetitions in binary pure morphic words

Over binary alphabets, the question of whether a pure morphic word is repetitive or not is completely solved: In 1988, Séébold [123] characterized completely the binary morphisms that, when iterated, generate a repetitive pure morphic word. In 1997 this result was strengthened by Kobayashi, Otto and Séébold [73], to characterize all binary morphisms that generate a repetitive language (recall that the language generated by a morphism $f \in \mathcal{M}(\Sigma)$ is defined by $L(f) = \{f^n(a) : n \geq 0, a \in \Sigma\}$). Note that the class of morphisms that generate repetitive words is strictly contained in the class of morphisms whose language is repetitive, as the latter class includes morphisms that are not prolongable on any letter (e.g., $f = (0, 01)$). The characterization is the following:

Theorem 3.3 (Kobayashi, Otto and Séébold [73]). *Let $\Sigma = \Sigma_2$ and let $f \in \mathcal{M}(\Sigma)$. For $a \in \Sigma$, denote $\bar{a} = 1 - a$. Then $L(f)$ is repetitive if and only if f belongs to one of the following classes:*

1. $f(a) = \varepsilon$, $|f(\bar{a})|_{\bar{a}} \geq 2$;
2. $f(a) = a$, $f(\bar{a}) \in a\Sigma^+ \cup \Sigma^+ a$, $|f(\bar{a})|_{\bar{a}} \geq 1$;
3. $f(a) = a$, $f(\bar{a}) = \bar{a}(a^m\bar{a})^n$ *for some* $m, n \geq 1$;
4. $f(a) = a^m$ *for some* $m \geq 2$;
5. $f(0) = 0(10)^m$, $f(1) = 1(01)^n$ *for some* $m, n \geq 0$ *satisfying* $m + n \geq 1$;
6. $f(0) = 1(01)^m$, $f(1) = 0(10)^n$ *for some* $m, n \geq 0$ *satisfying* $m + n \geq 1$;
7. $f(0) = 1^m$, $f(1) = 0^n$ *for some* $m, n \geq 0$ *satisfying* $m + n \geq 3$;
8. $f(0), f(1) \in w^+$ *for some* $w \in \Sigma^+$ *satisfying* $|w| \geq 2$.

3.2.6 Avoiding powers with large power blocks

Let $\mathbf{w}$ be an infinite word. Suppose there exist some positive integers n and t, such that $\mathbf{w}$ contains n powers x^n with $|x| = t$ but avoids n-powers x^n with $|x| > t$. Then $\mathbf{w}$ also avoids m-powers x^m with $|x| > t$ for all $m > n$, and in general, $\mathbf{w}$ avoids m-powers for $m \geq n(t+1)$: if u^m is a subword of $\mathbf{w}$, and $|u| = r \leq t$, then $|u^{t+1}| = r(t+1) > t$. Since $u^m = (u^{t+1})^{m/(t+1)}$, to satisfy the condition of avoiding n-powers with power blocks $> t$, we must have $m/(t+1) < n$. Therefore, $\mathbf{w}$ satisfies the following:

1. There exists some integers m and q, $n \leq m < n(t+1)$ and $0 < q \leq t$, such that $E(\mathbf{w}) \in \{m + i/q : 0 \leq i < q\}$;

2. $E(\mathbf{w})$ is attained.

To compute $E(\mathbf{w})$ we need only to compute the set of subwords of length at most $t(m+1)$, where $m < n(t+1)$, and in many cases we need to compute even fewer subwords. Here are some examples.

As already mentioned, squares cannot be avoided over binary alphabets. However, quite a few authors have constructed binary words that avoid squares with large power blocks.

In 1974, Entringer, Jackson and Schatz [43] constructed an infinite binary word that avoids squares xx with $|x| > 2$. The construction is as follows: let $\mathbf{w}$ be any infinite ternary square-free word (not necessarily morphic), and let $h : \Sigma_3^* \to \Sigma_2^*$ be the morphism $h = (1010, 1100, 0111)$. Using inverse image arguments, Entringer et. al. showed that $h(\mathbf{w})$ contains no squares with power block of length 3 or more. This condition implies that $\mathbf{w}$ avoids 6-powers, and the only 5-powers it may contain are of the type a^5 for some $a \in \Sigma_2$. On the other hand, it can be verified that every ternary square-free word of length 14 or more must contain the subword 21. Since $h(21) = 01^5 00$, we get that $E(h(\mathbf{w})) = 5$ for every ternary square-free infinite word $\mathbf{w}$.

In 1976, Dekking [38] constructed an infinite binary word that avoids both cubes and squares xx with $|x| > 3$. His technique was similar to that of Entringer et. al. (apply a morphism to a ternary square-free word, and use inverse image arguments to attain the results). However, the morphism he used was non-uniform, and much longer: $h = (00110101100101, 001101100101101001, 001101101001011001)$. The fact that the constructed word avoids simultaneously cubes and squares with power block of length ≥ 4 implies that its critical exponent can be either 7/3, 5/2, or 8/3; it cannot be 2, since in the same paper Dekking also proved that any overlap-free binary word must contain arbitrarily large squares. Since $h(2)$ contains the 8/3-power 01101101, we get that $E(h(\mathbf{w})) = 8/3$ for every ternary square-free word $\mathbf{w}$.

In 1994, Shallit [129] refined the results of Entringer et. al. and Dekking, by constructing an infinite binary word that avoids both 3^+-powers and squares xx with $|x| > 2$, and an infinite binary word that avoids both $2\frac{1}{2}^+$-powers and squares xx with $|x| > 3$. Since the constructed words attain 3-powers and $2\frac{1}{2}$-powers, respectively, the results imply immediately the value of the critical exponents. Again, the technique consisted of applying a uniform morphism to a square-free word. For the first word, Shallit used a 10-uniform morphism $\Sigma_3^* \to \Sigma_2^*$; for the second word, he used a 1560-uniform morphism $\Sigma_8^* \to \Sigma_2^*$.

In 1995, Fraenkel and Simpson [46] constructed an infinite binary word containing only the squares 0^2, 1^2, and $(01)^2$. A word thus constructed cannot contain 4-powers, and the only 3-powers it may contain are 000, 111, and 010101. Therefore, the critical exponent can be either 3 or $3\frac{1}{2}$. However, the construction involved again applying a morphism to a square-free word (this time over Σ_5), and the morphism used never generates the word 010101. Therefore, for all words thus constructed, the critical exponent is 3.

Alternate proofs for Fraenkel and Simpson's result were given in 2005 by Rampersad, Shallit and Wang [110], and in 2006 by Harju and Nowotka [59] and by Ochem [99]. All used the technique of applying a morphism to a square-free word, and the resulting word had critical exponent 3. Rampersad et. al. also gave an alternate proof for Dekking's result. Ochem also presented a general method to construct morphisms that are (α, q)*-power-free* (see generalizations of repetition thresholds, Section 3.6).

We mention one more related result. The *Hilbert curve* is a space filling curve, that can be encoded as an infinite word over a 4-letter alphabet. Denote this word by $\mathbf{h}$. In [127] (2007), Séébold showed that $\mathbf{h}$ is morphic but not pure morphic; that $\mathbf{h}$ is 4-power-free; and that the only cubes $\mathbf{h}$ contains are of the type a^3, where a is a letter. Therefore, $E(\mathbf{h}) = 3$. See also Kitaev, Mansour, and Séébold [69].

3.2.7 Power-free morphisms and finite test sets

Power-free morphisms have been studied extensively since Thue. Perhaps the first to study power-free morphisms systematically were Bean, Ehrenfeucht, and McNulty. In 1979 they published a thorough study of pattern avoidance in words [9]. The first section of that paper was devoted to k-power-free morphisms, where $k \geq 2$ is an integer. Among other results, Bean et. al. gave sufficient conditions for a morphism $f : \Sigma_n^* \to \Sigma_k^*$ to be square-free, and to be k-power-free for all $k \geq 2$. They also established the existence of the following:

- a morphism $f : \mathbb{N}^* \to \Sigma_3^*$ which is k-power-free for all $k \geq 2$;

- a morphism $g : \mathbb{N}^* \to \Sigma_2^*$ which is k-power-free for all $k \geq 3$.

The morphisms f, g above are defined from an infinite alphabet into a finite alphabet. It is worth noting that a k-power-free morphism is not necessarily n-power-free for $n > k$. For example, the morphism $f \in \mathcal{M}(\Sigma_4)$ defined by $f = (0102101, 23012013, 230201213, 23021202)$ is square-free by the criteria given in [9], but it is not cube-free, since $f(0^2)$ contains the cube $(10)^3$ (see [9, page 271]).

As mentioned earlier, finite test sets constitute a very useful tool when trying to prove the power-freeness of morphisms. Most results that make use of this technique concern either integral powers or overlaps.

In 1984, Keränen [66] gave a finite test set for k-power-freeness of uniform morphisms $f : \Sigma^* \to \Gamma^*$, where Σ is a binary alphabet and $k \in \mathbb{Z}_{\geq 3}$.

In 1985, Leconte [80] showed that a morphism h is *power-free* (that is, k-power-free for all $k \in \mathbb{Z}_{\geq 2}$) if and only if it is square-free and $h(aa)$ is cube-free for all $a \in \Sigma$.

In 1993, Berstel and Séébold [16] proved that a morphism f defined over a binary alphabet is overlap-free if and only if $f(01101001)$ is overlap-free. Their result provides a simple proof of Thue's theorem, that the only overlap-free binary morphisms are essentially powers of the Thue-Morse morphism [132]. Their result also implies a simpler proof of a 1985 theorem of Séébold [122]: the Thue-Morse word and its complement are the only overlap-free binary pure morphic words. In 1999, Richomme and Séébold [115] extended the above result by fully characterizing all the test-sets for overlap-freeness of binary morphisms. In particular, $\{110100\}$ is a test-set, which improves the previous result by giving a shorter test word.

In 2001, Wlazinski [139] showed that a binary morphism is k-power-free ($k \geq 2$ an integer) if and only if $f(w)$ is k-power-free for every k-power-free $w \in \Sigma_2^*$ with $|w| \leq k^2$. For primitive morphisms the k^2 bound can be improved to $2k + 1$. In 2002, Richomme and Wlazinski [116] considered test-sets more generally. They characterized test-sets for cube-free binary morphisms; in particular, a binary morphism f is cube-free if and only if $f(w_0)$ is cube-free, where $w_0 = 001101011011001001010011$, and $|w_0| = 24$ is optimal. They also showed that if $k \geq 3$ and $|\Sigma| > 3$, then there exists no finite test-set for k-power-freeness of morphisms defined over Σ. However, to generate a cube-free D0L word, we do not necessarily need a cube-free morphism. In 2007, Richomme and Wlazinski [118] proved that contrary to the general case, for uniform morphisms defined over any finite alphabet there always exists a finite test-set for k-power-freeness. In 2004, Richomme and Wlazinski [117] returned to test-sets for overlap-freeness, this time for morphisms $f : \Sigma_n^* \to \Sigma_m^*$, where $m, n \geq 2$ are any integers. For each case of m and n they either characterized the finite test-sets, or proved that none exist.

There are only a few papers we are aware of that deal with α-power-free morphisms where $\alpha > 1$ is not integral, and all of them concern uniform morphisms. The first paper is the one by Brandenburg [20], already discussed in Section 3.2.2. Another is a 1985 paper due to Kobayashi [71], where he studied uniform marked morphisms. Among other results, Kobayashi gave sufficient conditions for a uniform marked morphism to be α-power-free (α^+-power-free), where $\alpha > 1$ is a real number. One of the conditions is preserving α-power-freeness on a finite test set.

Another related result is due to Ochem [99]. A morphism $f : \Sigma_n^* \to \Sigma_k^*$ is *synchronizing* if for all $a, b, c \in \Sigma_n$ and $u, v \in \Sigma_k^*$, if $f(ab) = uf(c)v$ then either $u = \varepsilon$ and $c = a$ or $v = \varepsilon$ and $c = b$. In 2006, Ochem proved that if $f : \Sigma_n^* \to \Sigma_k^*$ is uniform and synchronizing, and $\mathbf{w} \in \Sigma_n^\omega$ is an α^+-power-free word for some rational number α, then the inverse image under f of any β^+-power $z \in \mathrm{Occ}(f(\mathbf{w}))$ satisfies $|f^{-1}(z)| < 2\beta/(\beta - \alpha)$. That is, a uniform synchronizing morphism is "nearly power-free", in the sense that the fractional powers it generates have a source of bounded length.

3.2.8 Decidability results

Many results concerning repetitions in pure morphic words focus on deciding whether a given word has a bounded critical exponent. We have already mentioned Mignosi and Séébold's algorithm to decide whether a given word is k-power-free [89], and Cassaigne's algorithm to decide whether a given word avoids a given pattern [26]. Here are some other results.

In 1983, Ehrenfeucht and Rozenberg [42] showed that a D0L language is repetitive if and only if it is strongly repetitive, and that it is decidable whether a given D0L language is repetitive or not. As already mentioned, a simpler proof was given by Mignosi and Séébold in [89]. In 2000, Kobayashi and Otto [72] suggested yet another algorithm for checking repetitiveness, that can run in polynomial time if the alphabet is fixed (it is not clear whether the algorithms of Ehrenfeucht and Rozenberg and of Mignosi and Séébold can be made to run in polynomial time).

In 1983, Karhumäki [65] showed that a necessary condition for a binary pure morphic word to be cube-free is that the generating morphism h is bifix, and that this assertion does not hold for 4-power-free pure morphic words; the counterexample was the Fibonacci word, which was shown to be 4-power-free. The main result of the paper was that given a prolongable binary morphism f, it is decidable whether f generates a cube-free word, and that 10 iterations of f are enough to decide it.

In two papers that appeared side by side in *Theoretical Informatics and Applications* **20** (1986), Pansiot [106] and Harju and Linna [58] proved that it is decidable whether a given morphism h prolongable on a word u generates an ultimately periodic infinite word. Pansiot's proof relied on the notion of right-special subwords (*biprolongable* in his terminology). Harju and Linna relied on the equation $h(x) = x^n$, and showed it is decidable whether nontrivial solutions to this equation exist.

3.3 Critical exponents of Sturmian words

3.3.1 Continued fractions and standard sequences

In the previous chapter we have defined Sturmian words as infinite binary words that have exactly $n+1$ distinct subwords of length n for all $n \geq 0$. This is only one of a few equivalent definitions these words admit. The one that is most relevant to the study of critical exponents is through the *slope*:

Definition 3.6. Let α and ρ be two real numbers, with $0 \leq \alpha < 1$. Define two infinite binary words:

$$\mathbf{s}_{\alpha,\rho} = \{\lfloor \alpha(n+1)+\rho \rfloor - \lfloor \alpha n + \rho \rfloor\}_{n\geq 0},$$
$$\mathbf{s}'_{\alpha,\rho} = \{\lceil \alpha(n+1)+\rho \rceil - \lceil \alpha n + \rho \rceil\}_{n\geq 0}.$$

The words $\mathbf{s}_{\alpha,\rho}$ and $\mathbf{s}'_{\alpha,\rho}$ are called the *lower* and *upper mechanical words* with *slope* α and *intercept* ρ. An infinite binary word $\mathbf{w}$ is *mechanical irrational* if $\mathbf{w} = \mathbf{s}_{\alpha,\rho}$ or $\mathbf{w} = \mathbf{s}'_{\alpha,\rho}$ for some real intercept ρ and irrational slope α.

Theorem 3.4 (Morse and Hedlund [94]). *An infinite binary word is Sturmian if and only if it is mechanical irrational.*

When $\rho = 0$, we get that $\mathbf{s}_{\alpha,0} = 0\mathbf{c}_\alpha$ and $\mathbf{s}'_{\alpha,0} = 1\mathbf{c}_\alpha$, where

$$\mathbf{c}_\alpha = \mathbf{s}_{\alpha,\alpha} = \mathbf{s}'_{\alpha,\alpha}.$$

The word $\mathbf{c}_\alpha$ is called the *characteristic word* of slope α.

Critical exponents of Sturmian words are tightly related to the continued fraction expansion of the slope. Recall that every irrational number α can be expanded uniquely into an infinite continued fraction,

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}} = [a_0, a_1, a_2, \ldots],$$

where a_0 is a nonnegative integer and a_i is a positive integer for all $i \geq 1$. The numbers a_i, $i \geq 0$, are called the *partial quotients* of α. If the sequence of partial quotients is ultimately periodic, that is, there exists integers $m \geq 0$ and $p > 0$ such that $a_n = a_{n+p}$ for all $n > m$, we denote it by $\alpha = [a_0, a_1, \ldots, a_m, \overline{a_{m+1} \cdots a_{m+p}}\,]$. By a famous theorem of Lagrange, the sequence of partial quotients is ultimately periodic if and only if α is an algebraic real number of degree 2. The *convergents* of α are the rational numbers given by $p_k/q_k = [a_0, a_1, a_2, \ldots a_k]$ for $k \geq 0$; it is well known that $\lim_{k\to\infty} p_k/q_k = \alpha$, and that p_k and q_k satisfy the following recurrence:

$$\begin{aligned} &p_{-1} = 1,\ p_0 = a_0,\ p_k = a_k p_{k-1} + p_{k-2} \text{ for } k > 0,\\ &q_{-1} = 0,\ q_0 = 1,\ \ q_k = a_k q_{k-1} + q_{k-2} \text{ for } k > 0. \end{aligned}$$

Let $\alpha = [0, 1 + d_1, d_2, d_3, \ldots]$ be an irrational number. Define a sequence $\{s_n\}_{n\geq -1}$ of binary words by

$$s_{-1} = 1, \quad s_0 = 0, \quad s_n = s_{n-1}^{d_n} s_{n-2} \ \ (n \geq 1).$$

Note that for all $n \geq 0$, we have $|s_n| = q_n$, where $\{q_n\}_{n\geq -1}$ is the sequence of denominators of the convergents of α. The sequence $\{s_n\}_{n\geq -1}$ is called the *standard sequence directed by α*. The connection between critical exponents and continued fractions stems from the following theorem:

Theorem 3.5 (Fraenkel, Mushkin, and Tassa [45])**.** *Let $\alpha \in (0,1)$ be an irrational number, let $\mathbf{c}_\alpha$ be the characteristic word with slope α, and let $\{s_n\}_{n\geq -1}$ be the standard sequence directed by α. Then s_n is a prefix of $\mathbf{c}_\alpha$ for all $n \geq 0$, and*

$$\mathbf{c}_\alpha = \lim_{n\to\infty} s_n \ .$$

Example 3.4. Let $\mathbf{f} = 0100101001001\cdots$ be the Fibonacci word. Then $\mathbf{f} = \mathbf{c}_{1/\tau^2}$, where $\tau = (1+\sqrt{5})/2$ is the golden mean (see Lothaire, [83, Example 2.1.24]). The continued fraction expansion of $1/\tau^2$ is given by $1/\tau^2 = [0, 2, 1, 1, 1, \ldots] = [0, 1+1, 1, 1, 1, \ldots]$, and the standard sequence directed by $1/\tau^2$ is $s_{-1} = 1$, $s_0 = 0$, and $s_n = s_{n-1}s_{n-2}$ for all $n > 0$. This is exactly the sequence $\{\phi_n\}_{n\geq -1}$ we have seen in Section 2.5. The first few terms of the standard sequence are $1, 0, 01, 010, 01001, 01001010, \ldots$.

Theorem 3.5 implies immediately that if the partial quotients of α are unbounded, the critical exponent of $\mathbf{c}_\alpha$ is unbounded as well. The following theorem implies (not that immediately) that the other direction holds as well, and not only for characteristic words, but for Sturmian words in general:

Theorem 3.6 (Mignosi [87])**.** *Let $\mathbf{s}$ be a Sturmian word of slope α. Then*

1. *$Sub(\mathbf{s}) = Sub(\mathbf{c}_\alpha)$;*
2. *the set of right-special subwords of $\mathbf{s}$ is the set of reversals of prefixes of $\mathbf{c}_\alpha$.*

The formulation of the above theorem is taken from Lothaire [83, Proposition 2.1.2, Proposition 2.1.23].

3.3.2 Results

Considering the discussion above, it is not surprising that most results concerning critical exponents in Sturmian words make use of the standard sequence and of the partial quotients of the slope. In 1992, Mignosi and Pirillo [88] computed the critical exponent of the Fibonacci word $\mathbf{f}$, proving that that $E(\mathbf{f}) = 2 + \tau$. Their proof used the properties of the standard sequence, and the following result of Karhumäki [65]: $\mathbf{f}$ is 4-power-free. Mignosi and Pirillo's result was the first example of an irrational critical exponent; however, note that τ is an algebraic number of degree 2, and as we shall see, this is the general case for Sturmian words generated by morphisms.

In 1989, Mignosi [87] was the first to prove the theorem we already hinted at:

Theorem 3.7 (Mignosi [87]). *Let* $\mathbf{s}$ *be a Sturmian word of slope* α. *Then* $E(\mathbf{s}) < \infty$ *if and only if the continued fraction expansion of* α *has bounded partial quotients.*

Mignosi's proof used very involved number-theoretic arguments. In 1999, Berstel [12] gave a combinatorial proof of this result, based on Theorems 3.5 and 3.6 we stated above. He also proved the following fact:

Theorem 3.8 (Berstel [12]). *Let* $\alpha = [0, 1+d_1, d_2, d_3, \ldots]$, *and let* $\{s_n\}_{n\geq -1}$ *be the standard sequence directed by* α. *Then for all* $n \geq 0$, *the word* s_{n+4} *contains the word* $s_n^{e_n}$ *as a subword, where*

$$e_n = 2 + d_{n+1} + \frac{q_{n-1} - 2}{q_n} \ .$$

In particular, for all Sturmian words $\mathbf{s}$, $E(\mathbf{s}) > 3$.

In 2000, Vandeth [135] gave an explicit formula for $E(\mathbf{s})$, where $\mathbf{s}$ is a pure morphic Sturmian word. Let $\alpha \in (0, 1)$ be an irrational number. By Crisp et. al. [30], the characteristic word $\mathbf{c}_\alpha$ is pure morphic if and only if the continued fraction expansion of α has the form $\alpha = [0, a_0, \overline{a_1, a_2, \ldots, a_m}\ ]$, with $a_m \geq a_0 \geq 1$. Vandeth proved that if the slope α of a Sturmian word $\mathbf{s}$ has such an expansion, then

$$E(\mathbf{s}) = \max_{1\leq t\leq m} [2 + a_t, a_{t-1}, \ldots, a_1, \overline{a_m, \ldots, a_1}\ ]. \tag{3.6}$$

By Lagrange, this formula implies in particular that the critical exponent of a pure morphic Sturmian word is always algebraic quadratic. In Chapter 6, we prove that in general, binary pure morphic words can have either rational or algebraic quadratic critical exponents; the above formula shows that for Sturmian pure morphic words only the second case holds.

Alternative proofs for the results of Mignosi and Vandeth, with some generalizations, were given in 2000 by Carpi and de Luca [25], and in 2001 by Justin and Pirillo [63]. Carpi and de Luca also showed that $2 + \varphi$ is the minimal critical exponent a Sturmian word can have.

In 2002, Damanik and Lenz [36] gave a formula for critical exponents of general Sturmian words. Their result showed that Theorem 3.8 is optimal:

Theorem 3.9 (Damanik and Lenz [36]). *Let $\alpha = [0, a_1, a_2, a_3, \ldots]$, and let $\mathbf{s}$ be a Sturmian word with slope α. Then the critical exponent of $\mathbf{s}$ is given by*

$$E(\mathbf{s}) = 2 + \sup_{n \geq 0}\{a_{n+1} + \frac{q_{n-1} - 2}{q_n}\} .$$

An alternative proof for this result was given in 2003 by Cao and Wen [21].

The *initial critical exponent* of an infinite word $\mathbf{w}$ is defined by

$$\text{ice}(\mathbf{w}) = \sup\{r \in \mathbb{Q}_{\geq 1} : \mathbf{w} \text{ contains an arbitrarily long } r\text{-power as a prefix}\}.$$

In 2006, Berthé, Holton, and Zamboni [17] studied initial critical exponents of Sturmian words. Given an irrational number $\alpha \in (0, 1)$, they gave a formula for $\text{ice}(\mathbf{c}_\alpha)$ (again in terms of the partial quotients of α), and also gave necessary and sufficient conditions for the following to hold: there exists a Sturmian word $\mathbf{s}$ with slope α such that $\text{ice}(\mathbf{s}) = 2$.

3.4 Critical exponents of strict epistandard words

Unlike the case of Sturmian words, critical exponents in episturmian words have hardly been studied. In fact, we are aware of only one result in this area, due to Justin and Pirillo.

Recall from Section 2.9 that for an epistandard word $\mathbf{s}$ directed by $\Delta = x_1x_2x_3\cdots$, the sequence $\{u_n\}_{n=0}^{\infty} \subseteq \Sigma^*$ is the the sequence of words generated by successively applying palindromic closure to the letters of Δ. Recall also that $\mathbf{s}$ is pure morphic if and only if Δ is periodic.

Theorem 3.10 (Justin and Pirillo [64]). *Let $\mathbf{s} \in \Sigma^\omega$ be a pure morphic strict epistandard word, let $\Delta = x_1x_2x_3\cdots$ be its directive word, and let q be the minimal period of Δ. For $n \geq 0$, let*

$$h_n = \psi_{x_1}\cdots\psi_{x_n}(x_{n+1}).$$

Define the following numbers:

- *$\ell \in \mathbb{N}$ is the maximal integer such that Δ contains a subword of the form a^ℓ, $a \in \Sigma$;*
- *L is the set of indices $0 \leq r < q$, such that Δ contains a subword of the form a^ℓ beginning at position $r + 1$;*

- $P(n) = \sup\{p < n | x_p = x_n\}$ *(that is, if $x_n = a$, p is the position of the nearest a on the left);*

- $d(r) = r + q + 1 - P(r + q + 1)$, $0 \leq r < q$.

Then

$$E(\mathbf{s}) = \ell + 2 + \sup_{r \in L} \left\{ \lim_{i \to \infty} (|u_{r+iq-d(r)}|/|h_{r+iq}|) \right\}.$$

3.5 Critical exponents of paperfolding words

Paperfolding words are infinite binary words, so named for the following construction: take a sheet of paper, and fold it in half, left-to-right, by folding the left side either on top or under the right side. Take the folded paper and fold it again left-to-right, again with the left side either on top or under the right side (these are the so-called "folding instructions"). Continue infinitely many times. Once you've reached infinity, unfold the paper, and code the resulting sequence of "hills" and "valleys" by 0 for a hill and 1 for a valley. What you get is a paperfolding word.

If you get tired of folding somewhere on the way to infinity (it has been conjectured that no sheet of paper can be folded more than seven times), you can choose among a few equivalent, more formal definitions. Here are two:

Definition 3.7 (paperfolding words: recursive definition)**.** An infinite binary word $\mathbf{w} = w_0 w_1 w_2 \cdots$ is a paperfolding word if and only if the subsequence $\mathbf{w}_e = w_0 w_2 w_4 \cdots$ equals either $(01)^\omega$ or $(10)^\omega$, and the subsequence $\mathbf{w}_o = w_1 w_3 w_5 \cdots$ is a paperfolding word.

Definition 3.8 (paperfolding words: perturbed symmetry definition)**.** An infinite binary word is a paperfolding word if and only if it is the limit of a sequence of binary words, $\{F_n\}_{n \geq 0}$, defined in the following way: let $\mathbf{c} = c_0 c_1 c_2 \cdots \in \Sigma_2^\omega$ be an arbitrary infinite binary word, let $F_0 = c_0$, and for $n > 0$, let $F_n = F_{n-1} c_n \overline{F_{n-1}}^R$. Here $\overline{w}$ is the word resulting from exchanging 0 and 1; e.g., $\overline{01001} = 10110$. Since F_n is a prefix of F_{n+1} for all $n \geq 0$, $\lim_{n \to \infty} F_n$ is well defined.

The term "perturbed symmetry" is due to Mendès France [86, 18]. For background on paperfolding words, see Dekking, Mendès France, and van der Poorten [39].

The recursive definition of paperfolding words implies constraints on their sets of subwords. In 1994, Allouche and Bousquet-Melou [3] used these constraints to prove the following: let $\mathbf{w}$ be any paperfolding word. If xx is a subword of $\mathbf{w}$, then $|x| \in \{1, 3, 5\}$. As a result, we get that $\mathbf{w}$ avoids 4-powers, and the only cubes it contains are 000 and 111. In particular:

Theorem 3.11 (Allouche and Bousquet-Melou [3]). *Let $\mathbf{w} \in \Sigma_2^\omega$ be an arbitrary paperfolding word. Then $E(\mathbf{w}) = 3$, and the bound is attained.*

3.6 More on repetition thresholds

As we have seen in Section 3.2.2, Dejean's conjecture states that the repetition threshold values are given by

$$RT(n) = \begin{cases} 2, & \text{if } n = 2; \\ 7/4, & \text{if } n = 3; \\ 7/5, & \text{if } n = 4; \\ n/(n-1), & \text{if } n \geq 5. \end{cases}$$

Recall that $RT(2)$ and $RT(3)$ where computed (by Thue and Dejean, respectively) using power-free morphisms, and that this method was proved inadequate for larger alphabets by Brandenburg. To construct an infinite word over Σ_n avoiding r_n^+-powers, a different construction needs to be employed. Such a construction was introduced by Pansiot in 1984 [105]. Pansiot observed that in order for a word $\mathbf{w} \in \Sigma_n^\omega$ to avoid r_n^+-powers, every $n-1$ consecutive letters in $\mathbf{w}$ must be distinct. Indeed, for $n = 4$, a subword of the form aba is a 3/2-power, and $3/2 > 7/5$; and for $n \geq 5$, a subword of the form $a_1 a_2 \cdots a_{n-2} a_1$ is a $(n-1)/(n-2)$-power, and $(n-1)/(n-2) > n/(n-1)$ for all $n \geq 2$. Let $z = w_j w_{j+1} \cdots w_{j+n-2}$ be a subword of $\mathbf{w}$ of length $n-1$, occurring at position j. Then there are two possible choices for w_{j+n-1}: either use the first letter of z again (code this transition by 0), or use the single letter that did not appear in z (code this transition by 1). Therefore, the word $\mathbf{w}$ can be coded by its prefix of size $n-1$ and an infinite binary word $\mathbf{b}$. Without loss of generality, we can assume $\mathbf{w}$ begins with $01 \cdots n-2$, thus $\mathbf{b}$ codes $\mathbf{w}$ completely. Here is an example over Σ_4, that uses a code word beginning with 11010111011101:

$$\mathbf{w} = \mathbf{012}_1 3_1 0_0 2_1 1_0 0_1 3_1 2_1 1_0 3_1 0_1 2_1 1_0 0_1 3 \cdots$$

A p/q-power in $\mathbf{w}$ that satisfies $p - q \geq n - 1$ implies a $(p - n + 1)/q$-power in $\mathbf{b}$. Consider for example the 14/8-power

$$w = \mathbf{302}_1 1_0 0_1 3_1 2_1 1|_0 3_1 0_1 2_1 1_0 0_1 3.$$

This word is coded by the prefix 302 and by the binary code $b = 10111011\ 101$. The prefix of length 8 of b completes a transition from 302 to 302; what comes thereafter must repeat the previous block. Therefore, b is an 11/8-power. (The converse is not true: in the above example, the 10/4-power 0111011101 codes the word **021**0321302103, which is only a 13/8-power.) To avoid r_n^+-powers in $\mathbf{w}$, it is enough therefore to avoid a certain type of powers in $\mathbf{b}$ (these are the so-called "forbidden repetitions"). Pansiot used the pure morphic word $\mathbf{b} = f^\omega(1)$, where $f(1) = 10$ and $f(0) = 101101$, to code a word $\mathbf{w}$ over Σ_4. Using the combinatorial properties of $\mathbf{b}$, he was able to show that $\mathbf{w}$ avoided $7/5^+$-powers, thus proving that $RT(4) = 7/5$.

Pansiot's coding method was generalized by Moulin Ollagnier in 1992 [97]. Moulin Ollagnier noticed that Pansiot's coding corresponds to an action on the symmetric group S_n: if we consider the prefix of length $n - 1$ of $\mathbf{w}$ as a permutation $\sigma \in S_n$ (with the unused letter stationary), then the transitions coded by 0,1 corresponds to multiplying σ on the right by the permutations $\sigma_0, \sigma_1 \in S_n$, respectively, where σ_1 is the full cycle $1 \to 2 \to \cdots \to n \to 1$, and σ_0 is the cycle $1 \to 2 \to \cdots \to n - 1 \to 1$, that leaves n stationary. Thus, Pansiot's coding induces a monoid morphism, $f : \Sigma_2 \to S_n$, defined by $f(0) = \sigma_0$, $f(1) = \sigma_1$. Moulin Ollagnier showed that sufficiently long repetitions in the binary code word $\mathbf{b}$ are mapped to the identity permutation in S_n; for that reason, he named them *kernel repetitions*. To check whether $\mathbf{b}$ avoids kernel repetitions, we can use not only the properties of $\mathbf{b}$, but also the properties of S_n.

Moulin Ollagnier used non-repetitive binary pure morphic words as code words, generated by primitive morphisms. Using the circularity of the generating morphism, together with a property of the group of endomorphisms of S_n, he was able to show the following: in order to check whether a given binary morphism generates a code word that avoids forbidden repetitions, it is enough to check its subwords of a bounded length. He then employed his method to find binary morphisms that generate code words avoiding forbidden repetitions for $n = 5, \ldots, 11$, thus proving Dejean's conjecture for $n \leq 11$.

Mohammad-Noori and Currie continued Moulin Ollagnier's work in 2005 [92], where they used Sturmian morphic words as code words. Sturmian words have many combinatorial properties that enable one to reduce the search space of appropriate morphisms considerably. Using their algorithm, they were able to find appropriate code words for $n = 6, \ldots, 14$ (they also showed that no Sturmian code words exist for $n = 5$). Thus Dejean's conjecture was proved for $n \leq 14$.

A general proof (though incomplete) of Dejean's conjecture was found only in 2006, by Carpi [23, 24]. His proof used Pansiot's encoding and Moulin Ollagnier's algebraic reductions to show that a code word exists for all $n \geq 33$. Though the proof is only for $n \geq 33$, it is strongly believed that Dejean's conjecture holds for all n.

To finish this section, we mention some generalizations of repetition thresholds:

- In 1999, Cassaigne and Currie [27] introduced the *commutative repetition threshold*. Let $1 < q < 2$ be a rational number. A *commutative q-power* (also known as an *abelian q-power*) is a word of the form $u_1 v u_2$, where u_2 is a permutation of u_1 and $|u_1 v u_2|/|u_1 v| = q$. A word w *strongly avoids r-powers*, $1 < r < 2$ a real number, if it contains no commutative q-powers for any rational number $q \geq r$. A real number $1 < r < 2$ is *strongly n-avoidable*, or *n-avoidable in the abelian sense*, if there exists an infinite word over Σ_n that strongly avoids r-powers. The *commutative repetition threshold* (*abelian repetition threshold*) is defined by

$$CRT(n) = \inf\{r \in \mathbb{R}_{>1} : r \text{ is strongly } n\text{-avoidable}\}.$$

  Cassaigne and Currie showed that $\lim_{n\to\infty} CRT(n) = 1$, i.e., for all $1 < r < 2$ there exists an infinite word over a finite alphabet that strongly avoids r-powers. However, not much more is known, and there is no general conjecture about the actual value of $CRT(n)$. See also Currie [32] (2003).

- In 2005, Ilie, Ochem, and Shallit [62] generalized the concept of repetition thresholds to include the length of the avoided words. A *(r, q)-power* z is an r-power with period q: $z = x^r$, $|x| = q$. A word w *avoids (α, q)-powers*, $\alpha \in \mathbb{R}_{>1}$ and $q \in \mathbb{Z}_{\geq 1}$, if it contains no (r, q')-powers such that $r \geq \alpha$ and $q' \geq q$. The pair (α, q) is n-avoidable if there exists an infinite word over Σ_n that avoids (α, q)-powers. The *generalized repetition* *threshold* is defined for $n \geq 2$ and $q \geq 1$ by

$$R(n, q) = \inf\{\alpha \in \mathbb{R}_{>1} : (\alpha, q) \text{ is } n\text{-avoidable}\}.$$

In other words, instead of trying to avoid all α-powers, we try to avoid only the ones with large block sizes, and allow α-powers with a bounded period. Note that for $q = 1$, $R(n, q) = RT(n)$.

Ilie, Ochem, and Shallit showed that $R(n, q)$ always exists, and satisfies the inequality $1 + q/n^q \leq R(n, q) \leq 2$. They also computed $R(3, 2)$, $R(3, 3)$ and $R(2, 4)$, and gave two conjectures regarding the value of $R(3, q)$ and $R(4, q)$ in general. In 2006, Ochem continued the study of generalized repetition thresholds [99], and gave an algorithm to generate uniform (α, q)-power-free morphisms.

- The *frequency of the letter a* in an infinite word $\mathbf{w} = w_0 w_1 w_2 \cdots$ is defined by

$$\mathrm{frq}_{\mathbf{w}}(a) = \lim_{m \to \infty} \frac{|w_0 w_1 \cdots w_{m-1}|_a}{m},$$

if the limit exists; otherwise, the frequency of a in $\mathbf{w}$ does not exist. In 2005, Ochem [98, 101] suggested a stronger version of Dejean's conjecture that included constraints on the frequency of the letter 0:

Conjecture 3.12 (Ochem)**.**

1. *For all $n \geq 5$, there exists an infinite ${\frac{n}{n-1}}^+$-power-free word $\mathbf{w}$ over Σ_n, such that $\mathrm{frq}_{\mathbf{w}}(0) = \frac{1}{n+1}$;*
2. *For all $n \geq 6$, there exists an infinite ${\frac{n}{n-1}}^+$-power-free word $\mathbf{w}$ over Σ_n, such that $\mathrm{frq}_{\mathbf{w}}(0) = \frac{1}{n-1}$.*

In 2006, Ochem [100] proved the conjecture for $n = 5, 6$.

3.7 Conclusion

The results surveyed in this chapter are far from being a complete list of critical exponents related results. The literature regarding repetitions in infinite words is huge, and in many of the papers, though critical exponents are not mentioned explicitly, their value is implied straightforwardly. Rather than trying to cover all related literature, we have tried to demonstrate general methods and approaches of computing critical exponents. In the coming chapters, we will present in detail our contribution to this field.

Chapter 4

Every Real Number Greater Than 1 is a Critical Exponent

4.1 Introduction

In the previous chapter we have seen the first example of an irrational critical exponent: recall that $E(\mathbf{f}) = 2+\tau$, where $\mathbf{f}$ is the Fibonacci word and τ is the golden mean. However, τ is an algebraic number, and as we shall see, pure morphic words cannot have a non-algebraic critical exponent. A natural question, therefore, is whether or not there exist infinite words over finite alphabets that have transcendental critical exponents. The formula for critical exponents of Sturmian words suggests that the answer is positive: recall that if $\mathbf{s}$ is a Sturmian word of slope α, then $E(\mathbf{s}) = 2 + \sup_{n\geq 0}\{a_{n+1} + (q_{n-1} - 2)/q_n\}$, where a_n are the partial quotients of α, and q_n are denominators of the convergents. This formula may produce transcendental numbers. However, no concrete example of transcendental critical exponent was known so far. In this chapter, we construct such examples. The main result of this chapter is the following:

Theorem 4.1. *Let $\alpha > 1$ be a real number. Then there exists an infinite word $\mathbf{w}$ over some finite alphabet such that $E(\mathbf{w}) = \alpha$.*

The results in this chapter have appeared in Krieger and Shallit [79].

4.2 Proof of the result

The strategy we use to prove Theorem 4.1 is as follows. For a fixed real number $\alpha > 1$, let $\mathcal{C} = \{p_i/q_i\}_{i\geq 0}$ be a sequence of rational numbers, where p_i, q_i are positive integers, such that the following conditions are satisfied:

1. $1 < p_i/q_i < \alpha$ for all $i \geq 0$;
2. the sequence $\{p_i\}_{i\geq 0}$ is strictly increasing;
3. $\lim_{i\to\infty}(p_i/q_i) = \alpha$.

Two possible choices for $\mathcal{C}$ are the infinite sequence of decimal approximations of α, or the even-indexed convergents to the continued fraction for α.

We now construct an infinite word $\mathbf{w}$ by concatenating building blocks of size p_i, where each building block is a (p_i/q_i)-power. Clearly, the critical exponent of a word thus constructed is at least α; we need to show that we can construct such a word without creating any α^+-powers.

Before turning to prove Theorem 4.1 in detail, we state a useful lemma.

Lemma 4.2. *Let $y \in \Sigma^*$ and $x, z \in \Sigma^+$. If $xy = yz$ then xyz contains a square.*

Proof. By Lyndon and Schützenberger (Theorem 2.3), there exist $u, v \in \Sigma^*$ and an integer $e \geq 0$ such that $x = uv$, $z = vu$, and $y = (uv)^e u$. Therefore, $xyz = uv(uv)^e uvu$, containing the square $(uv)^2$. □

Proof of Theorem 4.1. For $\alpha = 2$, the Thue-Morse word $\mathbf{t}$ satisfies $E(\mathbf{t}) = 2$. We consider two cases: $\alpha > 2$ and $1 < \alpha < 2$.

Case (a): $\alpha > 2$. Let $\mathcal{C} = \{p_i/q_i\}_{i\geq 0}$ be a sequence of rational numbers that satisfy conditions 1–3 above. Let $\mathbf{t}$ be the Thue-Morse word over $\{0, 1\}$, and let $\{x_i\}_{i\geq 0}$ be the following sequence of subwords of $\mathbf{t}$:

- x_{2i} is the prefix of $\mathbf{t}$ of length $q_{2i} - 1$;
- x_{2i+1} is the subword of $\mathbf{t}$ of length $q_{2i+1} - 1$ starting at index 1.

Define an infinite word $\mathbf{w}$ over the alphabet $\{0, 1, a, b\}$ by

$$\mathbf{w} = \prod_{i \geq 0} (x_i a)^{p_i/q_i} b \ .$$

We claim that $E(\mathbf{w}) = \alpha$. Since $(x_i a)^{p_i/q_i}$ is a (p_i/q_i)-power, and $\lim_{i\to\infty}(p_i/q_i) = \alpha$, necessarily $E(\mathbf{w}) \geq \alpha$. We will show that $\mathbf{w}$ avoids α-powers.

First, let us consider a block of the form $z = (xa)^{p/q}$, where x is a subword of $\mathbf{t}$ satisfying $|x| = q - 1$. Suppose z contains an α-power. Then z contains some (r/s)-power w as a subword, where $r, s \in \mathbb{N}$ and $r/s \geq \alpha > p/q$. Since $r = |w| \leq |z| = p$, we get that $s < q$. There are three cases, which are illustrated in Fig. 4.1.

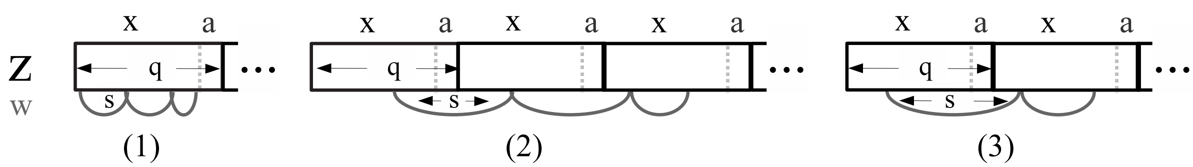

Figure 4.1: Possible alignments of z and w.

Case 1: w is contained in an xa block of z. Since a does not occur in x, this implies that x contains an (r/s)-power. This is a contradiction, since x is α-power-free, being a subword of $\mathbf{t}$.

Case 2: w contains at least one xa block. Then xa has an s-period, where $s < q = |xa|$. This is a contradiction, since a does not occur in x.

Case 3: w is a subword of xax and contains exactly one a. Then w can have exactly one full power block, i.e., $r/s < 2$. This is a contradiction, since $r/s \geq \alpha > 2$.

We conclude that the blocks $z_i := (x_i a)^{p_i/q_i}$ are α-power-free for all $i \geq 0$. It remains to show that no α-powers are created by concatenating the blocks together. Suppose $\mathbf{w}$ contains an (r/s)-power w, where $r/s \geq \alpha$. There are two cases:

Case 1: b does not occur in w. Then w is contained in a z_i block for some i. This is a contradiction, since z_i is α-power-free for all i.

Case 2: w contains a b symbol. Then every s-block of w must contain at least one b symbol, and since $\alpha > 2$, w contains at least two b's. On the other hand, the sequence of distances between two consecutive b's is given by the sequence $\{p_i\}$, which is strictly increasing, and so w cannot contain more than two b's. We get that w contains exactly two b's. Thus $s = p_i$ for some i, $2 < r/s < 3$, and $w = uuv$, where $|u| = s$, v is the prefix of u of length $r - 2s$, and u contains one b symbol. Note that the last symbol of w cannot be b, since $|v| > 0$. Therefore, the two letters following the b occurrences of w belong to w, and must equal the same symbol. But by the construction of $\mathbf{w}$, the letter following a b symbol is the first letter of an x_i block, and two consecutive x_i's begin with different symbols. Again we get a contradiction.

We conclude that for $\alpha > 2$, there exists an infinite word $\mathbf{w}$ over a 4-letter alphabet with $E(\mathbf{w}) = \alpha$. Moreover, the construction of $\mathbf{w}$ is effective, provided that a suitable sequence $\{p_i/q_i\}$ is known.

Case (b): $1 < \alpha < 2$. To use a construction similar to the one we used in the $\alpha > 2$ case, we need an infinite word over some finite alphabet that avoids α-powers. The existence of such a word is guaranteed for all $\alpha > 1$ by a construction of Carpi [22] and a probabilistic proof of Currie [33], and more recently, by Carpi's proof of Dejean's conjecture [24]. However, we cannot simply repeat the construction above with some α-power-avoiding infinite word: the fact that a word x avoids α-powers does not guarantee that $(xa)^{p/q}$, where $p/q < \alpha$, will avoid α-powers as well. Consider, for example, an irrational number α satisfying $\frac{7}{5} < \alpha < \frac{3}{2}$. Let $x = 0120$, and let $a = 3$. Then x avoids 7/5-powers (and therefore α-powers), but $(xa)^{7/5} = 0120301$, and this word contains the 3/2-power 030.

The reason for this is that for $\alpha < 2$, the fact that a word x avoids α-powers is not enough to guarantee that $(xa)^{p/q}$ avoids α-powers as well; x has to avoid α-powers *as a circular word.* A circular word (or a *necklace*) consists of a word together with all of its cyclic shifts [33]; a circular word avoids α-powers if all its shifts avoid α-powers. In the example above, the set of cyclic shifts of $xa = 01203$ is given by $\{01203, 12030, 20301, 03012, 30120\}$. The word $(xa)^{p/q}$ contains the first $p - q + 1 = 3$ cyclic shifts of xa, and two of them contain the 3/2-power 030. The problem does not exist when trying to avoid squares or higher powers, because of the uniqueness of the letter a in the period.

To ensure that the x_i blocks avoid α-powers circularly, we use the following construction. Let $\mathbf{v}$ be an infinite word over some finite alphabet $\Sigma = \{0, 1, \ldots, n-1\}$ avoiding α. Let $\mathcal{C} = \{p_i/q_i\}_{i \geq 0}$ be a sequence of rational numbers satisfying conditions 1–3. Define an alphabet $\overline{\Sigma}$ by $\overline{\Sigma} = \{\overline{a} : a \in \Sigma\}$. Let $h : \Sigma \to \overline{\Sigma}$ be the morphism defined by $h(a) = \overline{a}$ for all $a \in \Sigma$, and let $\overline{w} = h(w)$ for all $w \in \Sigma^*$. For all $i \geq 0$, let x_i be a subword of $\mathbf{v}$ of length q_i, and let $z_i = (x_i\overline{x_i})^{2p_i/2q_i}$. Then z_i is a $(2p_i/2q_i)$-power. We claim that z_i is α-power-free for all i. To show that, we consider two cases: (i) $p_i/q_i \leq \frac{3}{2}$, and (ii) $p_i/q_i > \frac{3}{2}$. For convenience, we omit the index, and refer to $z = (x\overline{x})^{2p/2q}$.

(i) $p/q \leq \frac{3}{2}$. Then $z = x\overline{x}y$, where $y \in \Sigma^*$ is the prefix of length $2p - 2q$ of x (Fig. 4.2). Suppose that z contains an α-power. Then z contains a subword that is an (r/s)-power,

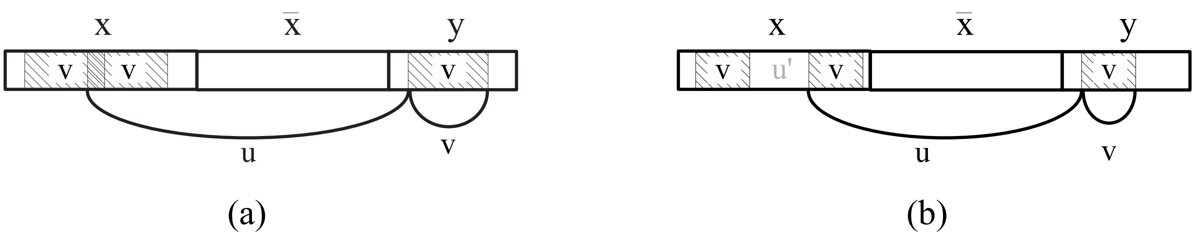

Figure 4.2: z for $p/q \leq \frac{3}{2}$. (a) overlapping occurrences of v; (b) non-overlapping occurrences of v.

where r, s are integers, and $r/s \geq \alpha > p/q$. Clearly, z is square-free; therefore $1 < r/s < 2$, and z contains a subword of the form $w = uv$, where $|u| = s$ and v is the prefix of u of length $r - s$ (Fig. 4.2). Since $2p = |z| > |w| = r$, and $r/s > p/q$, we get that $s < 2q$.

Since both x and $\overline{x}$ are α-power-free, and $\Sigma \cap \overline{\Sigma} = \emptyset$, u must begin within x and extend into y. Thus v must be contained in y. We get that x contains two occurrences of v, one derived from repeating a prefix of u, and one from repeating a prefix of x. The occurrences of v can either overlap (Fig. 4.2 (a)) or not overlap (Fig. 4.2 (b)), but they cannot coincide: if they did, we would get that $|u| = 2|x|$, a contradiction, since $s < 2q$. If the occurrences of v are overlapping, we get by Lemma 4.2 that x contains a square, a contradiction to x being α-power-free. Otherwise, x contains a word of the form $vu'v$. Let $s' = |vu'|$. Then

$vu'v$ is a $(|vu'v|/s')$-power. Now, $s' \leq |x| < |u| = s$, and so

$$\frac{|vu'v|}{s'} = \frac{s' + |v|}{s'} = 1 + \frac{|v|}{s'} > 1 + \frac{|v|}{s} = \frac{|u| + |v|}{s} = \frac{r}{s} > \alpha \, .$$

Again we get that x contains an α-power, a contradiction.

Example 4.1. $p = 7$, $q = 5$, $x = 01203$. Then $z = 01203\overline{01203}0120$, and it is easy to check that $\frac{14}{10}$ is the highest power contained in z.

(ii) $p/q > \frac{3}{2}$. Then $z = x\overline{x}x\overline{y}$, where $\overline{y}$ is the prefix of $\overline{x}$ of length $2p - 3q$ (Fig. 4.3). Suppose that z contains an α-power. Again, z is square-free, and so it contains an (r/s)-power $w = uv$, with $p/q < r/s < 2$ and $s < 2q$. We have 3 cases:

1. u begins within the first x block, and v is contained in the second x block. This case is similar to the $p/q \leq \frac{3}{2}$ case.

2. u begins within $\overline{x}$, and v is contained in $\overline{y}$. Again, this case is similar to the $p/q \leq \frac{3}{2}$ case; this time it is $\overline{x}$ that contains an α-power.

3. u begins within the first x block, and v contains both Σ and $\overline{\Sigma}$ symbols. This case is described in Fig. 4.3. Suppose u begins j positions to the left of $\overline{x}$. Since $\Sigma \cap \overline{\Sigma} = \emptyset$, v must begin j positions to the left of the $\overline{y}$; this implies that $|u| = 2q$, a contradiction, since $s < 2q$.

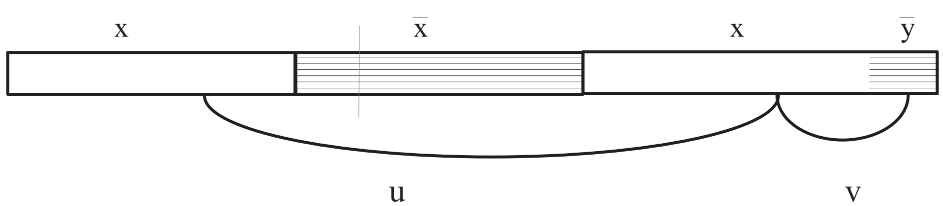

Figure 4.3: z for $p/q > \frac{3}{2}$.

Example 4.2. $p = 7$, $q = 4$, $x = 0120$. Then $z = 0120\overline{0120}0120\overline{01}$, and it is easy to check that $\frac{14}{8}$ is the highest power contained in z.

It remains to concatenate the z_i's together in a way that will not generate α-powers. We do that by padding the z_i's with "sufficiently long" α-power-free words over a third copy of Σ. Let $\Gamma = \{a_0, \cdots, a_{n-1}\}$, and let $\mathbf{u}$ be an α-power-free infinite word over Γ. Let $\{u_k\}_{k\geq 0}$ be a sequence of subwords of $\mathbf{u}$, and let $\{z_{i_k}\}_{k\geq 0}$ be a subsequence of $\{z_i\}_{i\geq 0}$, where z_{i_k} is a $(2p_{i_k}/2q_{i_k})$-power. Let

$$\mathbf{w} = \prod_{k=0}^{\infty} z_{i_k} u_k = z_{i_0} u_0 z_{i_1} u_1 z_{i_2} u_2 \cdots .$$

We choose z_{i_k} and u_k alternately, such that $|z_{i_k}| < |u_k| < |z_{i_{k+1}}|$ for all k; this can be done since $|z_i| = 2p_i$, and the sequence $\{p_i\}_{i\geq 0}$ grows to infinity. We choose successive z_{i_k}'s to begin with different symbols, and similarly for u_k's. We also impose some additional restrictions, which are given below.

Suppose $\mathbf{w}$ contains an α-power. Since the z_{i_k} and u_k blocks are all α-power free, and since they are defined over different alphabets, a power greater than α would have to include at least one whole block. Since $|z_{i_k}| < |u_k| < |z_{i_{k+1}}|$ for all k, $\mathbf{w}$ cannot contain a 2-power. Suppose $\mathbf{w}$ contains an (r/s)-power $w = xy$, where $|x| = s$, y is the prefix of x of length $r - s$, and $r/s \geq \alpha$. Assume w starts within a block z_{i_k}. Then the y part has to start within a $z_{i_{k+m}}$ block for some $m > 0$. Since the block lengths are growing, and two successive u_k's begin with different letters, either $m = 1$ and $|y| \leq |z_{i_k}|$, or $m > 1$ and $|y| \leq |z_{i_k}| + |u_k|$ (Fig. 4.4).

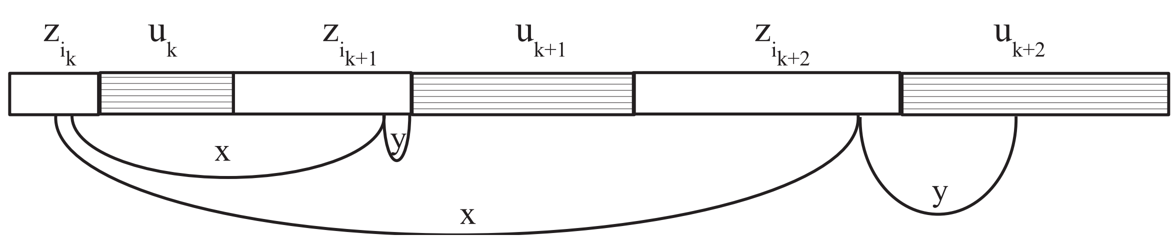

Figure 4.4: Possible powers in $\mathbf{w}$.

If $m = 1$ then x contains the block u_k and part of the block $z_{i_{k+1}}$; if $m > 1$ then x must contain at least the blocks $u_k, z_{i_{k+1}}, u_{k+1}$. Therefore, the following conditions suffice to ensure that $r/s < \alpha$:

1. $|z_{i_k}|/|u_k| < \alpha - 1$;
2. $(|z_{i_k}| + |u_k|)/|z_{i_{k+1}}| < \alpha - 1$.

Similar arguments show that if w starts within a u_k block, the following conditions imply that $r/s < \alpha$:

1. $|u_k|/|z_{i_{k+1}}| < \alpha - 1$;
2. $(|u_k| + |z_{i_{k+1}}|)/|u_{k+1}| < \alpha - 1$.

Combining these conditions, we see that in order for $\mathbf{w}$ to be α-power-free, it is enough to choose z_{i_k}, u_k as follows:

1. Let $u_{-1} = \varepsilon$, and let $z_{i_0} = z_0$.
2. Assume z_{i_k} is already chosen. Choose u_k such that $(|u_{k-1}| + |z_{i_k}|)/|u_k| < \alpha - 1$.
3. Choose $z_{i_{k+1}}$ such that $(|z_{i_k}| + |u_k|)/|z_{i_{k+1}}| < \alpha - 1$.

This completes the proof of the Theorem. □

4.3 Remarks

We conclude with some remarks.

1. In proving Theorem 4.1, we have constructed a word that contains infinitely many β-powers for all $1 < \beta < \alpha$. This shows that the theorem holds for the following stronger definition of critical exponent:

$$E'(\mathbf{w}) = \sup\{r \in \mathbb{Q}_{\geq 1} : \mathbf{w} \text{ contains an } r\text{-power infinitely often}\} .$$

2. For $\alpha > 2$, we have constructed an infinite word $\mathbf{w}$ over a 4-letter alphabet with $E(\mathbf{w}) = \alpha$. But actually it is possible to construct such a word over a 2-letter alphabet, though the construction is more complicated. This fact was proved by Shur [130] for $\alpha > 7/3$; by Currie, Rampersad, and Shallit [35] for a dense set of rational α values in the range $2 < \alpha \leq 7/3$; and by Currie and Rampersad [34] for all $\alpha > 2$.

3. For $\alpha < 2$, we need the existence of an infinite word $\mathbf{v}$ avoiding α-powers for our proof. Such a word can be constructed explicitly: in [24], Carpi constructs, for all $n \geq 33$, an infinite word over an n-letter alphabet that avoids $n/(n-1)^+$-powers. Thus we can construct a word $\mathbf{w}$ with $E(\mathbf{w}) = \alpha$ effectively (again, provided that a suitable sequence $\{p_i/q_i\}$ is known). The alphabet size is naturally not fixed, but tends to infinity as α tends to 1.

4. Given an infinite word $\mathbf{v}$ avoiding α-powers over an n-letter alphabet, we have shown in the $\alpha < 2$ case how to construct $\mathbf{w}$ over a $3n$-letter alphabet. It is an open question whether $\mathbf{w}$ can be constructed over an n-letter alphabet.

5. The construction of the z_i blocks in the $\alpha < 2$ case shows that if α is n-avoidable, then α is *circularly* $2n$*-avoidable*, that is, there are arbitrarily long circular words over a $2n$-letter alphabet that avoid α. It is an open question whether, when α is n-avoidable, it is also circularly n-avoidable [33].

Chapter 5

Critical Exponents in Uniform Binary Pure Morphic Words

5.1 Introduction

In this chapter we begin our investigation of critical exponents in pure morphic words. We focus here on a simpler special case, namely, critical exponents in pure morphic words generated by uniform binary morphisms. Our main result is a simple formula for the critical exponent that can be computed based on four iterations of the generating morphism.

Let $k \geq 2$ be an integer, let f be a binary k-uniform morphism prolongable on 0, and let $\mathbf{w} = f^{\omega}(0)$. In Section 5.2 we give some preliminary definitions, and state our main result.

In Section 5.3 we analyze the structure of powers occurring in $\mathbf{w}$. We show that when $E(\mathbf{w})$ is bounded, powers with sufficiently large power block must have a power block of length divisible by k, and must be produced by a simple iterative process, which we describe. Based on this analysis, we give a simple formula for the critical exponent, based on powers with small power block. In particular, if $E(\mathbf{w}) < \infty$, then $E(\mathbf{w})$ is rational. We also give necessary and sufficient conditions for $E(\mathbf{w})$ to be bounded, again based on powers with small power block.

In Section 5.4 we show that when $E(\mathbf{w})$ is bounded, all powers with small power block occur within the prefix $f^4(0)$ of $\mathbf{w}$. Thus, our formula can be easily computed for all uniform binary pure morphic words.

Finally, in Section 6.5, we give some applications of our results. We show that, given a rational number $0 < r < 1$, we can construct a binary k-uniform morphism f such $E(f^\omega(0)) = n + r$ for some positive integer n. We also compute the critical exponent for all prolongable k-uniform binary morphisms with $k \leq 4$.

The results presented in this chapter will be generalized in the next one to non-uniform morphisms over an arbitrary finite alphabet. Though the results of Chapter 6 are much more general, two points distinguish the results of this chapter: first, we give a simple formula for computing $E(\mathbf{w})$, based on only four iterations of f; for the general case, there are no such formula and bound. Secondly, the proof is based on very elementary tools, while the general results relies on heavy combinatorial and algebraic machinery.

Most of the results in this chapter have appeared in Krieger [74, 77].

5.2 The main result

For the rest of this chapter, $\Sigma = \Sigma_2$.

In order to state our main result, we need three definitions:

Definition 5.1. Let $z \in \Sigma^+$ be a p/q-power. We say that z is *reducible* if it contains a p'/q'-power, such that $p'/q' > p/q$, or $p'/q' = p/q$ and $q' < q$. If $p'/q' > p/q$ then z is *strictly reducible*.

Example 5.1. The $\frac{8}{4}$-power 1011 1011 is strictly reducible, since it contains the $\frac{3}{1}$-power 111. The $\frac{6}{3}$-power 101 101 is reducible since it contains the $\frac{2}{1}$-power 11. The word 1111 is strictly reducible as a $\frac{4}{2}$-power and irreducible as a $\frac{4}{1}$-power.

Definition 5.2. Let $\mathbf{w} \in \Sigma^\omega$, and let $z = w_i \cdots w_j \in \text{Occ}(\mathbf{w})$ be a p/q-power. We say that z is *left stretchable* (resp., *right stretchable*) if $w_{i-1} \cdots w_j$ (resp., $w_i \cdots w_{j+1}$) is a $(p+1)/q$-power. If z is neither left nor right stretchable we say it is an *unstretchable power*.

Example 5.2. Let $f = (01, 00)$, and let $\mathbf{w} = f^\omega(0)$ (the first few terms of $\mathbf{w}$ are given bellow). The $\frac{6}{2}$-power $w_4 \cdots w_9 = 010101$ is right-stretchable to the $\frac{7}{2}$-power $w_4 \cdots w_{10} = 0101010$, which is unstretchable.

i	0	1	2	3	4	5	6	7	8	9	10	11
w_i	0	1	0	0	0	1	0	1	0	1	0	0

Since $E(\mathbf{w})$ is an upper bound, it is enough to consider irreducible, unstretchable powers when computing it. Therefore, over a binary alphabet, we can assume that $p/q \geq 2$: since any binary word of length 4 or more contains a square, a p/q-power over $\{0,1\}^*$ with $1 < p/q < 2$ is always reducible, save for the $\frac{3}{2}$-powers 010 and 101.

Definition 5.3. Let f be a binary k-uniform morphism. The *left stretch* of f, denoted by σ_f, is the longest word $\sigma \in \Sigma^*$ satisfying $f(0) = x\sigma$ and $f(1) = y\sigma$ for some $x, y \in \Sigma^*$. Similarly, the *right stretch* of f, denoted by ρ_f, is the longest word $\rho \in \Sigma^*$ satisfying $f(0) = \rho x$, $f(1) = \rho y$ for some $x, y \in \Sigma^*$. The *stretch size* of f is the combined length $\lambda_f = |\rho_f| + |\sigma_f|$.

Example 5.3. The morphism $f = (0110, 1010)$ satisfies $\sigma_f = 10$, $\rho_f = \varepsilon$, and $\lambda_f = 2$.

We can now state our main result:

Theorem 5.1. *Let f be a binary k-uniform morphism prolongable on 0, and let $\mathbf{w} = f^\omega(0)$. Then:*

1. *$E(\mathbf{w}) = \infty$ if and only if at least one of the following holds:*
   - *(a) $f(0) = f(1)$;*
   - *(b) $f(0) = 0^k$;*
   - *(c) $f(1) = 1^k$;*
   - *(d) $k = 2m + 1$, $f(0) = (01)^m 0$, and $f(1) = (10)^m 1$.*
2. *Suppose $E(\mathbf{w}) < \infty$. Let $\mathcal{E}$ be the set of exponents $r = p/q$, such that $q < k$ and $f^4(0)$ contains an r-power. Then*

$$E(\mathbf{w}) = \max_{p/q \in \mathcal{E}} \left\{ \frac{p(k-1) + \lambda_f}{q(k-1)} \right\} .$$

   In particular, $E(\mathbf{w})$, when bounded, is always rational. The bound $E(\mathbf{w})$ is attained if and only if $\lambda_f = 0$.

Here is an example of an application of Theorem 5.1:

Example 5.4. The Thue-Morse word is overlap-free.

Proof. The Thue-Morse morphism μ satisfies $\lambda_\mu = 0$; and since the largest power in $\mu^4(0)$ is a square, we get that $E(\mu^\omega(0)) = 2$, and the bound is attained. □

5.3 Power structure

5.3.1 The basic iterative process

Lemma 5.2. *Let f be a binary k-uniform morphism prolongable on 0, and let $\mathbf{w} = f^\omega(0)$. Let $\sigma = \sigma_f$, $\rho = \rho_f$ and $\lambda = \lambda_f$. Suppose $z = w_i \cdots w_j \in Occ(\mathbf{w})$ is a p/q-power. Then*

$$E(\mathbf{w}) \geq \frac{p(k-1) + \lambda}{q(k-1)} \ .$$

Proof. If $f(0) = 0^k$ or $f(1) = 1^k$, then it is easy to see that $E(\mathbf{w}) = \infty$. Otherwise, f is primitive, thus by Theorem 2.10 $\mathbf{w}$ is recurrent, and we can assume that $i > 0$. Let $p = nq + r$, where $n, r \in \mathbb{N}$ and $r < q$, and let $z = x^n y$, where $x = a_0 \cdots a_{q-1}$ and $y = a_0 \cdots a_{r-1}$. Let $f(w_{i-1}) = u\sigma$ and $f(w_{j+1}) = \rho v$ for some $u, v \in \Sigma^*$. Applying f to $w_{i-1} \cdots w_{j+1}$, we get a subword of $\mathbf{w}$ which is a fractional power with period kq, as illustrated in Fig. 5.1.

Since σ is a common suffix of $f(0)$ and $f(1)$, it is a suffix of $f(a_{q-1})$ as well; similarly, ρ is a prefix of $f(a_r)$. Therefore, we can stretch the kq-period of $f(z)$ by σ to the left and ρ to the right. We get that $z' = \sigma f(z)\rho$ is a $(kp + \lambda)/kq$-power.

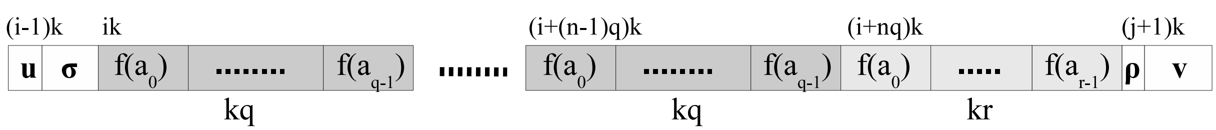

Figure 5.1: Applying f to $w_{i-1} \cdots w_{j+1}$.

The process of applying f and stretching the resulting power can be repeated infinitely. Successive applications of f give a sequence of powers, $\{p_m/q_m\}_{m \geq 0}$, which satisfy $p_0 = p$,

$q_0 = q$, and for $m > 0$, $p_m = kp_{m-1}+\lambda$, and $q_m = kq_{m-1}$. Let $\pi : \mathrm{Occ}(\mathbf{w})\times\mathbb{Q} \to \mathrm{Occ}(\mathbf{w})\times\mathbb{Q}$ be the map defined by

$$\pi\left(z, \frac{p}{q}\right) = \left(\sigma f(z)\rho, \frac{kp+\lambda}{kq}\right). \tag{5.1}$$

Let $\pi(z)$ and $\pi(p/q)$ denote the first and second component, respectively. Iterating π, we get

$$\pi^m\left(\frac{p}{q}\right) = \frac{k^m p + \lambda\sum_{i=0}^{m-1} k^i}{k^m q} = \frac{k^m p + \lambda\frac{k^m-1}{k-1}}{k^m q} \quad \xrightarrow[m\to\infty]{} \quad \frac{p(k-1)+\lambda}{q(k-1)}. \tag{5.2}$$

□

Our goal is to show that the π map defined in (5.1) is what generates $E(\mathbf{w})$, and that it is enough to apply it to powers that appear in $f^4(0)$. Though the details are a bit tedious, the proof idea is very simple:

1. Every p/q-power in $\mathbf{w}$ that satisfies $q \equiv 0 \pmod k$ is an image under the π map;
2. If $z \in \mathrm{Occ}(\mathbf{w})$ is a p/q-power that satisfies $q \not\equiv 0 \pmod k$ and $q > k$, then either z is reducible to an r/s-power that satisfies $s \le k$, or $p/q < 3$ and $\mathbf{w}$ contains a cube;
3. All the distinct p/q-powers in $\mathbf{w}$ that satisfy $q < k$ occur in $f^4(0)$.

The second item involves a subtle point. Let $\lim(p/q) = (p(k-1)+\lambda)/q(k-1)$. In general, the fact that some positive integers p, q, r, s satisfy $p/q < r/s$ does not imply that $\lim(p/q) \le \lim(r/s)$. Consider the following example:

$$f = (010011, 001111), \quad k = 6, \quad \lambda = 3, \quad \mathbf{w} = f^\omega(0) = w_0w_1w_2\cdots = 010011\mid 001111\mid\cdots$$

The word $\mathbf{w}$ contains the unstretchable 2/1-power 00 beginning at position 2, and the unstretchable 9/4-power 100110011 beginning at position 1. These powers satisfy $2 < 9/4$, but $\lim(2) = 13/5 > 12/5 = \lim(9/4)$. However, as the next lemma shows, the situation described by the above example cannot happen when a p/q-power is reducible to an r/s-power (in the above example, the 9/4-powers *contains* the 2-power, but is not *reducible* to it).

Lemma 5.3. *Let p, q, r, s be positive integers, such that*

$$\frac{p}{q} < \frac{r}{s} \quad \text{and} \quad \frac{p}{q} + \frac{\lambda}{q(k-1)} > \frac{r}{s} + \frac{\lambda}{s(k-1)} . \tag{5.3}$$

Then $q < s$, and $\lfloor p/q \rfloor = \lfloor r/s \rfloor$.

Proof. Since $\lambda \leq k-1$, Equation 5.3 implies that

$$0 < \frac{r}{s} - \frac{p}{q} < \frac{\lambda}{k-1}\left(\frac{1}{q} - \frac{1}{s}\right) < \frac{1}{q} \leq 1 .$$

In particular, $1/q - 1/s > 0$, and so $q < s$. Also, if $p/q = n + p'/q$ and $r/s = m + r'/s$, where $0 \leq p' < q$ and $0 \leq r' < s$, then $n \leq m$, and

$$m + \frac{r'}{s} + \frac{\lambda}{s(k-1)} < n + \frac{p'}{q} + \frac{\lambda}{q(k-1)} \leq n + \frac{p'+1}{q} \leq n+1.$$

Therefore, $n = m$. □

Corollary 5.4. *Let $\lim(p/q) = (p(k-1)+\lambda)/q(k-1)$, let z be a p/q-power, and let z' be an r/s-power.*

1. *If z is reducible to z', then $\lim(p/q) \leq \lim(r/s)$.*

2. *If $\lfloor p/q \rfloor < \lfloor r/s \rfloor$ then $\lim(p/q) \leq \lim(r/s)$.*

Proof. If z is reducible to z', then $p = |z| \geq |z'| = r$ and $p/q \leq r/s$. This implies that $q \geq s$, and so by Lemma 5.3 $\lim(p/q) \leq \lim(r/s)$. If $\lfloor p/q \rfloor = n < m = \lfloor r/s \rfloor$ then $\lim(p/q) \leq n+1 \leq m \leq \lim(r/s)$. □

By Corollary 5.4, it is enough to consider irreducible powers when applying π iteratively to unstretchable powers. Also, if we know that $\mathbf{w}$ contains powers of exponent n or more for some integer n, we do not need to consider powers with exponents smaller than n. This fact will be used in Section 5.3.3.

We start with a few lemmata that describe power behavior in a more general setting, namely, in an infinite word $\mathbf{v} = h(\mathbf{u})$, where h is a k-uniform binary morphism and $\mathbf{u} \in \Sigma^\omega$ is an arbitrary infinite word.

5.3.2 p/q-powers with $q \equiv 0 \pmod{k}$

Definition 5.4. Let h be a binary k-uniform morphism, and let $\mathbf{v} = h(\mathbf{u})$ for some $\mathbf{u} \in \Sigma^\omega$. We refer to the decomposition of $\mathbf{v}$ into images of h as decomposition into *k-blocks*. Let $z = v_i \cdots v_j \in \mathrm{Occ}(\mathbf{v})$. The *outer closure* and *inner closure* of z, denoted by $\hat{z}$, $\check{z}$, respectively, are defined as follows:

$$\hat{z} = v_{\hat{\imath}} \cdots v_{\hat{\jmath}}\ , \quad \hat{\imath} = \left\lfloor \frac{i}{k} \right\rfloor k\ , \quad \hat{\jmath} = \left\lceil \frac{j+1}{k} \right\rceil k - 1\ ;$$

$$\check{z} = v_{\check{\imath}} \cdots v_{\check{\jmath}}\ , \quad \check{\imath} = \left\lceil \frac{i}{k} \right\rceil k\ , \quad \check{\jmath} = \left\lfloor \frac{j+1}{k} \right\rfloor k - 1\ .$$

Thus $\hat{z} \in \mathrm{Occ}(\mathbf{v})$ consists of the minimal number of k-blocks that contain z; similarly, $\check{z} \in \mathrm{Occ}(\mathbf{v})$ consists of the maximal number of k-blocks that are contained in z. By this definition, both $\hat{z}$ and $\check{z}$ have inverse images under h, denoted by $h^{-1}(\hat{z})$ and $h^{-1}(\check{z})$, respectively. Note that $\check{z}$ may be empty.

Lemma 5.5. *Let h be an injective binary k-uniform morphism, let $\mathbf{v} = h(\mathbf{u})$ for some $\mathbf{u} \in \Sigma^\omega$, and let $z = v_i \cdots v_j \in Occ(\mathbf{v})$ be an unstretchable p/q-power. Suppose $q \equiv 0 \pmod{k}$. Then z is an image under the π map defined in (5.1).*

Proof. Let $q = mk$, $m \geq 1$. By definition, $\check{\imath} - i, j - \check{\jmath} \leq k - 1$. Therefore,

$$|\check{z}| = |z| - (\check{\imath} - i) - (j - \check{\jmath}) \geq |z| - (2k - 2).$$

Since $|z| = p$ and $p/q \geq 2$, we get that $|\check{z}| \geq 2q - (2k-2) = (2m-2)k + 2$. If $m \geq 2$, this implies that $|\check{z}| > q$; if $m = 1$, this implies that $|\check{z}| \geq 2$, and since $|\check{z}| \equiv 0 \pmod{k}$, necessarily $|\check{z}| \geq k = q$. Thus $|\check{z}| \geq q$, and since $\check{z} \prec z$, we get that $\check{z}$ has a q-period. It follows that $\check{z}$ is a p'/q-power, where $p' = nk$ for some $n \geq 1$. Let $z' = h^{-1}(\check{z})$. Since h is injective, necessarily z' is an n/m-power.

Now apply h to z'. By the proof of Lemma 5.2, the mk-period of $h(z')$ can be stretched by at least $|\sigma|$ to the left and $|\rho|$ to the right, to create a power with exponent $(kn+\lambda)/km = (p+\lambda)/q$. On the other hand, $h(z') = \check{z}$, and by inner closure definition, the q-period of $\check{z}$ can be stretched by at least $\check{\imath} - i$ to the left and at least $j - \check{\jmath}$ to the right to create the p/q-power z. But, since by assumption z is unstretchable, it cannot be stretched more than this, i.e., it can be stretched by exactly $\check{\imath} - i$ to the left and $j - \check{\jmath}$ to the right. Therefore, $\sigma = w_i \cdots w_{\check{\imath}-1}$, $\rho = w_{\check{\jmath}+1} \cdots w_j$, and

$$z = \sigma \check{z} \rho = \sigma h(z') \rho = \pi(z').$$

□

5.3.3 p/q-powers with $q > k$ and $q \not\equiv 0 \pmod{k}$

Lemma 5.6. *Let h be a binary k-uniform morphism, let $\mathbf{v} = h(\mathbf{u})$ for some $\mathbf{u} \in \Sigma^\omega$, and let $\alpha \in Occ(\mathbf{v})$ be a p/q-power with $p/q \geq 2$. Let Q be the power block. Suppose $q > 2k$ and $q \not\equiv 0 \pmod{k}$. Then at leas one of the following holds:*

1. *$h(0) = h(1)$;*
2. *$p/q < 3$ and $\mathbf{w}$ contains a cube x^3 with $|x| \leq k$;*
3. *α is reducible to a p'/q'-power, which satisfies $q' \leq k$.*

If the last case holds, then either $p/q < 5/2$, or $Q = u^c$ for some integer $c \geq 4$ and $u \in \Sigma^+$ satisfying $|u| < k$.

Proof. We start with three propositions that will be useful for the proof. All are extensions of the theorems of Lyndon and Schützenberger (Theorems 2.3, 2.4) to systems of word equations.

Proposition 5.7. *Let $x, y, z, t \in \Sigma^+$, and suppose the following equalities hold:*

1. *$xy = yx$;*
2. *$yz = zt$ (equivalently, $tz = zy$);*
3. *$|z| = |x|$.*

Then $x = z$ and $y = t$.

Proof. By Theorem 2.4, $xy = yx$ if and only if there exists $u \in \Sigma^+$ and integers $i, j > 0$ such that $x = u^i$ and $y = u^j$. Therefore, it is enough to prove the following: $u^j z = zt$ and $|z| = i|u|$ imply $z = u^i$ and $t = u^j$. We consider two cases:

$i < j$ in this case, $|z| = |x| < |y|$, and since $yz = zt$, z is a prefix (equivalently suffix) of y. Since $y = u^j$ and $|z| = i|u|$, necessarily $z = u^i = x$. This gives us $u^{j+i} = u^i t$, which implies $t = u^j = y$.

$i \geq j$ here we use induction on $|z|$. If $|z| = 1$, then either $|z| < |y|$, which implies $i < j$, or $|y| = 1$, which implies $x = y = z = t \in \Sigma$. Let $|z| > 1$. Since $|z| \geq |y|$, we get that $y = u^j$ is a prefix (suffix) of z. Let $z = u^j z'$ ($z = z'u^j$). Then $u^j z' = z't$ ($z'u^j = tz'$) and $|z'| = |u^i| - |u^j| = (i-j)|u|$. Therefore, by the induction hypothesis, $z' = u^{i-j}$, and so $z = u^i$ and $t = u^j$.

□

Figure 5.2 illustrates the proof of Proposition 5.7.

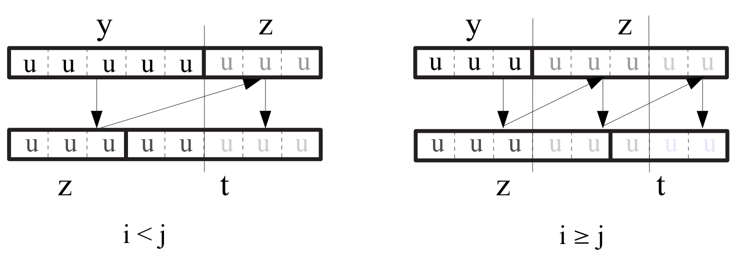

Figure 5.2: $yz = zt$, $y = u^j$, $|z| = i|u|$.

Proposition 5.8. *Let $x, y, z, t \in \Sigma^+$, and suppose the following equalities hold:*

1. $xy = yz$;
2. $yx = zt$ *(equivalently,* $tx = zy$*).*

Then $x = z$ and $y = t$.

Proof. We prove this proposition for $yx = zt$. The proof for $tx = zy$ is identical.

By Theorem 2.3, $xy = yz$ if and only if there exist $u, v \in \Sigma^*$ and an integer $e \geq 0$ such that $x = uv$, $z = vu$, and $y = (uv)^e u$. If $e > 0$, then y has uv as a prefix, and so $yx = zt$ implies $uv = vu$, i.e., $x = z$. The conditions translate to $xy = yx = xt$, thus $y = t$.

Suppose $e = 0$. Then $yx = zt$ translates to $uuv = vut$. By Theorem 2.3, there exist $\alpha, \beta \in \Sigma^*$ and a $c \geq 0$ such that $uu = \alpha\beta$, $ut = \beta\alpha$, and $v = (\alpha\beta)^c\alpha$. Therefore, $uuv = (\alpha\beta)^{c+1}\alpha = (uu)^{c+1}\alpha = u^{2c+2}\alpha$, where α is a prefix of uu. We get that uuv has a $|u|$ period, and therefore vut has a $|u|$ period. Since $|t| = |u|$ (as implied by $|uuv| = |vut|$), we get that $t = u = y$.

The equality $yx = zt$ now translates to $uuv = vuu$. By Theorem 2.4, there exists $w \in \Sigma^+$ and $i, j > 0$ such that $uu = w^i$ and $v = w^j$. If $i > 1$, then uu has both periods $|u|$ and $|w|$ and $|uu| > |u| + |w| - 1$, so by Theorem 2.2, it has a $g = \gcd(|u|, |w|)$ period; thus there exists $w' \in \Sigma^+$ such that $u = w'^{|u|/g}$, $w = w'^{|w|/g}$, and $v = w'^{j|w|/g}$. If $i = 1$, then $uu = w$, and $v = u^{2j}$. In any case, u and v are integral powers of the same word, which means $uv = vu$, and $x = z$. □

Proposition 5.9. *Let $x, y, z, t \in \Sigma^+$, and suppose the following equalities hold:*

1. $xy = yz$;
2. $zt = tx$;

Then there exists $u \in \Sigma^+$, $v \in \Sigma^*$, and integers $i \geq 1$, $j, m \geq 0$, such that $x = (uv)^i$, $z = (vu)^i$, $y = (uv)^j u$, $t = (vu)^m v$. If in addition $|y| = |t|$, then either $v = \varepsilon$ and $m = j + 1$ ($y = t = u^m$), or $|u| = |v|$ and $m = j$.

Proof. From the first equation, we get by Theorem 2.3 that $x = rs$, $z = sr$, and $y = (rs)^e r$ for some $r, s \in \Sigma^*$, and an integer $e \geq 0$. Plugging into the second equation, we get $srt = trs$, therefore $srtr = trsr$, and so by Theorem 2.4, $sr = w^i$ and $tr = w^{i'}$ for some $u \in \Sigma^+$ and $i, i' \geq 1$. This implies $s = w^a v$, $r = uw^b$, and $t = w^m v$, where $w = vu$, $0 \leq |v| < w$, $i = a + b + 1$, and $i' = m + b + 1$. Altogether we get:

$$
\begin{aligned}
x &= rs = u(vu)^b (vu)^a v = (uv)^{a+b+1} = (uv)^i; \\
z &= sr = (vu)^i; \\
y &= (rs)^e r = (uv)^{ie} u (vu)^b = (uv)^{ie+b} u; \\
t &= (vu)^m v.
\end{aligned}
$$

Thus, the proposition assertion holds for $j = ie + b$.

Now suppose that $|y| = |t|$. If $v = \varepsilon$, necessarily $u^{j+1} = y = t = u^m$. Otherwise, since $|uv| = |vu|$ and $|v| < |uv|$, the equation $|(uv)^j u| = |(vu)^m v|$ implies $j = m$ and $|u| = |v|$. $\square$

We now go back to the proof of Lemma 5.6. Let $\alpha = Q^{p/q} \in \mathrm{Occ}(\mathbf{v})$ be a p/q-power, where $p/q \geq 2$ and $q > 2k$, $q \not\equiv 0 \pmod{k}$. Since $\mathbf{v}$ is an image under a k-uniform morphism, it can be decomposed into k-blocks, which are images of either 0 or 1. Assume the decomposition of α into k-blocks starts from its first character; we will show at the end of the proof that this assumption causes no loss of generality. Since $q \not\equiv 0 \pmod{k}$, the last k-block of the first q-block extends into the second q-block; since the first and second q-blocks are identical, we get overlaps of k-blocks in the second q-block. An example is given in Fig. 5.3. The bold rectangles denote the power's q-blocks; the light grey and dark grey rectangles stand for $h(0), h(1)$, respectively; the top line of h rectangles stands for the k-decomposition of α; and the bottom line shows the repetition of the q-block.

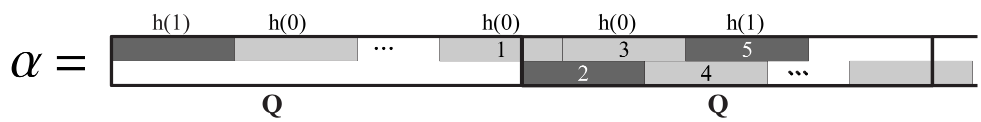

Figure 5.3: Overlaps of k-blocks. The bold rectangles represent the power blocks, the light grey ones stand for $h(0)$, the dark grey ones for $h(1)$.

The fact that $q > 2k$ implies that there are at least 5 k-blocks involved in the overlap. We shall now analyze the different overlap cases. For case notation, we order the k-blocks by their starting index (the numbers 1-5 in Fig. 5.3), and denote the case by the resulting 5-letter binary word; in the Fig. 5.3 case, it is 01001. By symmetry arguments, it is enough to consider words that start with 0, therefore we need to consider 16 overlap combinations. Fig. 5.4 shows these overlaps.

Each overlap induces a partition on the k-blocks involved, denoted by dashed lines. We mark the k-block parts by the letters x, y, z, t in the following manner: we start by marking the leftmost part by x, and then mark by x all the parts we know are identical to it. We then mark the leftmost unmarked part by y, and so on. Since the k-decomposition starts from the first letter of α, we have $|x| = q \mod k$, i.e., $|x| > 0$.

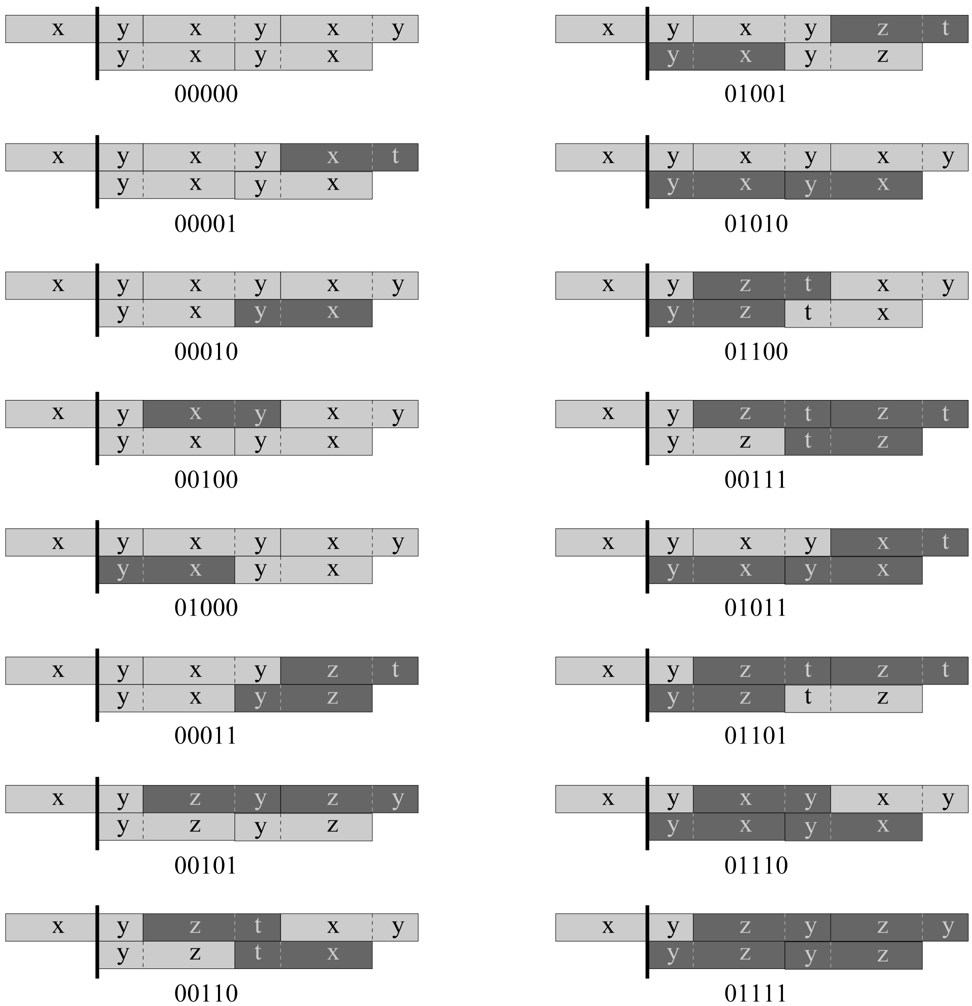

Figure 5.4: Possible overlaps of k-blocks.

We begin with combinations that imply $h(0) = h(1)$ straightforwardly.

00010, 00100, 01000, 01110: $h(0) = xy = yx = h(1)$.

00011: $h(0) = xy = yx$, $h(1) = yz = zt$. By Proposition 5.7, $x = z$ and $y = t$, i.e., $h(0) = h(1)$.

01001: $h(0) = xy = yz$, $h(1) = yx = zt$. By Proposition 5.8, $x = z$ and $y = t$.

00111: $h(0) = xy = yz$, $h(1) = zt = tz$. By Proposition 5.7 (set $x \leftrightarrow t$, $y \leftrightarrow z$), $x = z$ and $y = t$.

01101: $h(0) = xy = tz$, $h(1) = yz = zt$. By Proposition 5.8 (set $x \to t$, $y \to x$, $z \to y$, $t \to z$), $x = z$ and $y = t$.

For the rest of the combinations, we need to consider possible continuations of the q-block. As mentioned above, $q = mk + |x|$ for some $m \geq 2$. If $m \geq 3$, the q-block continues with another k-block on the bottom row; the top row continues with the k-decomposition regardless of m. Let K_t be the next k-block of the k-decomposition in the top row, and let K_b be the next k-block in the bottom row if $m \geq 3$. Note that by the first block, Q must end with an x.

00001: $h(0) = xy = yx$, $h(1) = xt$. By Theorem 2.4, $x = u^i$ and $y = u^j$ for some $u \in \Sigma^+$ and $i, j \in \mathbb{Z}_{\geq 1}$. Thus $|t| = |y| = j|u|$.

Suppose $m \geq 3$. If $K_b = h(0)$, we get $t = y$, and so $h(1) = xy = h(0)$; if $K_b = h(1)$, we get $h(1) = xt = tz$, where $|t| = |y|$, therefore by Proposition 5.7 (set $x \leftrightarrow y$ and $z \leftrightarrow t$) $y = t$ and $x = z$, and again $h(1) = xy = h(0)$. Therefore we can assume that $m = 2$. Thus $Q = yxyxx = u^{3i+2j} = u^c$, $c \geq 5$, i.e., α is reducible with $q' = |u| < k$.

00000: $h(0) = xy = yx$, therefore, by Theorem 2.4, $x = u^i$ and $y = u^j$ for some $u \in \Sigma^+$, $i, j \in \mathbb{Z}_{\geq 1}$. If $m = 2$, or Q is continued with $h(0)$ blocks all the way, then $Q = (yx)^m x = u^{m(i+j)+i} = u^c$, $c \geq 5$; we get that the p/q-power contains at least a $5 \lfloor p/q \rfloor$-power, i.e., it is reducible with $q' = |u| < k$. Otherwise, if $m > 2$ and the continuation of k-blocks is not strictly by $h(0)$ blocks, then at some point an $h(1)$ block is introduced, and the behavior is similar to one of 00010, 00011, or 00001. As was shown above, the first two imply that $h(0) = h(1)$, and the last one implies either that $h(0) = h(1)$ or that $Q = u^c$, where $u \in \Sigma^+$ satisfies $|u| < k$ and $c \geq 5$.

00110: $h(0) = xy = yz$, $h(1) = zt = tx$. By Proposition 5.9, either $x = z = u^i$ and $y = t = u^j$, or there exist $u, v \in \Sigma^+$ and integers $i \geq 1$, $j \geq 0$, such that $|u| = |v|$, $x = (uv)^i$, $z = (vu)^i$, $y = (uv)^j u$, and $t = (vu)^j v$. In the first case, $h(0) = h(1) = u^\ell$, where $\ell = i + j$; in the second, $h(0) = (uv)^\ell u$ and $h(1) = (vu)^\ell v$, thus $h(0)h(1) = (uv)^{2\ell+1}$.

Suppose $m \geq 3$. If $K_b = h(1)$, we get $y = t$, and by Proposition 5.8, $h(0) = h(1)$. If $K_b = h(0)$ and $K_t = h(0)$, we get $x = z$, and again $h(0) = h(1)$. The only way to continue the q-block without forcing $h(0) = h(1)$ is to have $h(0)h(1)h(0)h(1)\cdots$ in the bottom row and $yh(1)h(0)h(1)h(0)\cdots$ in the top row. Thus we can assume that Q has the form $(h(0)h(1))^{m/2}x$ (m even) or $(h(0)h(1))^{(m-1)/2}h(0)x$ (m odd); here $m \geq 2$.

If m is odd, then either x is a prefix of t or t is a prefix of x (Fig. 5.5); since x begins with u, t begins with v and $|u| = |v|$, we get $u = v$ and $h(0) = h(1)$. If m is even, then $Q = (h(0)h(1))^{m/2}x = ((uv)^{2\ell+1})^{m/2}(uv)^i = (uv)^c$, $c \geq 4$, thus α is reducible with $q' = |uv| = |x|/i < k$.

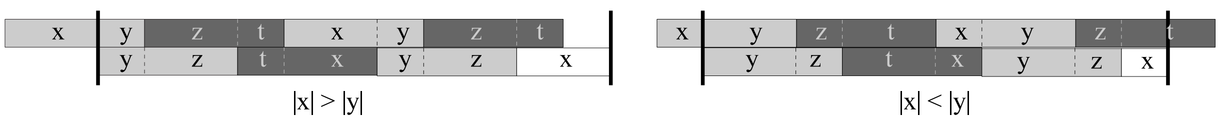

Figure 5.5: Overlap 00110, $Q = (h(0)h(1))^{(m-1)/2}h(0)x$.

01100: $h(0) = xy = tx$, $h(1) = yz = zt$. This case is similar to the 00110 case. Use Proposition 5.9 to show that either $h(0) = h(1)$ or α is reducible.

01111: $h(0) = xy$, $h(1) = yz = zy$. By Theorem 2.4 $z = u^i$, $y = u^j$, therefore $|x| = |z| = i|u|$.

Suppose $m \geq 3$. If $K_t = h(0)$, then for $K_b = h(1)$ we get $x = z$, and so $h(0) = h(1)$; for $K_b = h(0)$ we get case 00011 with 0 and 1 flipped, and again $h(0) = h(1)$. Therefore we can assume $K_t = h(1)$. If $K_b = h(0)$ we then get $h(0) = yz = h(1)$. The only continuation that does not immediately force $h(0) = h(1)$ is $K_b = K_t = h(1)$. Thus we can assume $Q = h(1)^m x = u^{m(i+j)}x$, $m \geq 2$. But then, like in the 00110 case, x is a prefix of y or a prefix of yz or yx, and since $|x| = i|u|$, we get $x = u^i = z$ and $h(0) = h(1)$.

01011: $h(0) = xy$, $h(1) = yx = xt$. If $|x| = |y|$ then $x = y$ (since both words are prefixes of $h(1)$), and so $h(0) = h(1)$. We can assume therefore that $|x| < |y|$ or $|y| < |x|$.

Suppose $m \geq 3$. The 4 possible continuations for $K_b K_t$ are 00, 01, 11, 10. Combined with the last 3 k-blocks (011), these continuations yield the combinations 01100, 01101, 01111, 01110. All were shown to imply $h(0) = h(1)$ or to be reducible. Therefore, we can assume that $m = 2$ and $Q = h(1)h(1)x$. This means that either x is a prefix of t or t is a prefix of x (Fig. 5.6).

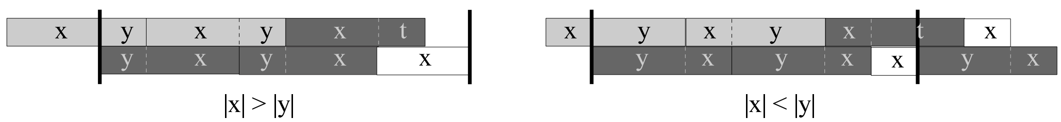

Figure 5.6: Overlap 01011, $Q = h(1)h(1)x$.

By Theorem 2.3, there exist $u, v \in \Sigma^*$ and $e \geq 0$ such that $y = uv$, $t = vu$, $x = (uv)^e u$. If $|x| > |y|$ then $e > 0$, and since t is a prefix of x we get $uv = vu = w^j$ for some $j \geq 1$, which implies $h(0) = h(1)$. If $|x| < |y|$, then $x = u$, i.e., it is a prefix of y. Since the q-block ends with xx and starts with y, α contains the cube xxx.

Suppose $p/q \geq 5/2$. Since x is a prefix of both $h(0)$, $h(1)$, the third q-block will give us the equality $tx = xy$, which together with $xt = yx$ implies (by Proposition 5.8) that $y = t$, and so $h(0) = h(1)$. Thus $h(0) \neq h(1)$ implies $p/q < 5/2$, and α is reducible with $q' = |x| < k$.

01010: $h(0) = xy$, $h(1) = yx$. If $m = 2$ then $Q = h(1)^2 x$. Suppose $m \geq 3$. Again, the possible continuations for $K_b K_t$ are 00, 01, 11, 10. The first three, when combined with the last three k-blocks (010), yield the combinations $01000, 01001, 01011$, which were shown to imply either that $h(0) = h(1)$ or that α is reducible. The 10 continuation yields the original configuration again. Therefore, we can assume that the q-block continues with $h(1)$. We conclude that $Q = h(1)^m x$ for some $m \geq 2$.

Suppose $p/q \geq 3$. The third q-block must begin with a y, to match the second one; the second one must end with an x to match the first one. Therefore, the bottom row has the form $(yx)^m xy$. But since $xy = h(0)$, we get again one of the combinations 01000, 01001, which imply $h(0) = h(1)$. We can thus assume that $\lfloor p/q \rfloor = 2$. If $m \geq 3$ this means that α is reducible, since it contains the cube $h(1)^3$. Assume therefore that $m = 2$, $Q = h(1)^2 x = yxyxx$.

Suppose $|x| < |y|$. Then x is a prefix of y, thus $Q = xwxxwxx$ for some $w \in \Sigma^*$, and $QQ = xwxxwxxxwxxwxx$, containing the cube xxx; we get that α is reducible with $q' = |x| < k$. Suppose $|x| > |y|$. Then y is a prefix of x, and since $K_t = h(0)$, x is a prefix of yx. Thus $yx = xt$ for some $t \in \Sigma^+$, and by Theorem 2.3, $y = uv$ and $x = (uv)^e u$. Since $|x| > |y|$, necessarily $e \geq 1$, and since $e \geq 2$ induces the cube $(uv)^3$, we can assume $x = uvu$. Plugging these values, we get $QQ = uvuvuuvuvuuvuuvuvuuvuvuuvu$, which contains the cube $(uvuuv)^3$. Thus α is reducible with $q' = |uvuuv| = |xy| = k$.

00101: $h(0) = xy = yz$, $h(1) = zy$. We can assume that $|x| < |y|$ or $|y| < |x|$, since $|x| = |y|$ implies by the first equation that $x = y$. Suppose $m \geq 3$, and consider the possible continuations for $K_b K_t$. If $K_b K_t = 00$ we get that $z = x$, and so $h(0) = h(1)$. If $K_b K_t = 11$ we get that $h(1) = yz = h(0)$. If $K_b K_t = 10$ we get that $h(0) = xy = yz$ and $h(1) = yx = zy$, and by Proposition 5.8 $h(0) = h(1)$. Finally, $K_b K_t = 01$ yields, when combined with the last three k-blocks (101) the combination 10101, which is symmetric to the 01010 case. We can therefore assume that $m = 2$ and $Q = yzyzx$.

By Theorem 2.3, there exist $u, v \in \Sigma^*$ and $e \geq 0$ such that $x = uv$, $z = vu$, $y = (uv)^e u$. Thus $QQ = (uv)^{e+1} u (uv)^{e+1} uuv (uv)^{e+1} u (uv)^{e+1} uuv$. For $e > 0$, QQ contains the cube $(uv)^3$, where $|uv| = |x| < k$; for $e = 0$, QQ contains the cube $(uvu)^3$, where $|uvu| = |f(0)| = k$. Thus, if $p/q < 5/2$, then α is reducible to a cube of block size at most k.

Suppose $p/q \geq 5/2$. Then the third q-block implies that either u is a prefix of v or v is a prefix of u. If they are of equal length, then $u = v$ and so $h(0) = h(1)$. Otherwise, if $u = vv'$ for some $v' \in \Sigma^+$, the third q-block begins with $h(0) = (uv)^{e+1} u = (vv'v)^{e+1} vv'$ (bottom row, to match the second q-block). However, if the top row continues with $K_t = f(1) = vu(uv)^e u$, then the third q-block also begins with $u(uv)^e u = vv'(vv'v)^e vv'$; if $K_t = f(0) = vv'v(vv'v)^e vv'$, then the third q-block also begins with $v'v(vv'v)^e vv'$. In both cases, we get that either v is a prefix of v' or vice versa. By induction, we can continue to split the k-block into shorter and shorter substrings, until finally we must get equality.

The same holds when u is a prefix of v. We conclude that for $p/q \geq 5/2$, $h(0) = h(1)$.

To finish the proof of the lemma, we need to justify our assumption that the k-decomposition starts from the first letter of α. Recall that $\alpha = v_i \cdots v_j \in \mathrm{Occ}(\mathbf{v})$. Suppose $i \not\equiv 0 \pmod{k}$. Let $\check{\alpha} = v_{\check{i}} \cdots v_{\check{j}}$ be the inner closure of α, and let $\beta = v_{\check{i}} \cdots v_j$. Let $c = \check{i} - i$. Then $\beta \in \mathrm{Occ}(\mathbf{v})$ is a $(p-c)/q$-power, $\beta \prec \alpha$, and the k-decomposition of β starts from the first letter. Also, by the definition of $\check{\alpha}$ we have $c \leq k-1$, and so $p/q - (p-c)/q \leq (k-1)/q < \frac{1}{2}$.

Let Q' be the power block of β. Suppose $p/q \geq 5/2$. If $p/q \geq 3$, then $(p-c)/q \geq 5/2$; by the analysis above, either $h(0) = h(1)$, or $Q' = u^d$ for some $u \in \Sigma^*$ and $d \geq 4$. If the second case holds, then $Q = v^d$ for some conjugate v of u and α is reducible. If $5/2 \leq p/q < 3$, then $(p-c)/q \geq 2$, and by the analysis above β contains at least a cube. In both cases, α is reducible. We can therefore assume that $p/q < 5/2$.

Let $p = 2q + r$, where $0 \leq r < q$, and let $q = mk + s$, where $0 \leq s < k$. By the theorem's conditions, $m \geq 2$ and $s \geq 1$. If $r \geq c$ then β is at least a square, and by the analysis above α is reducible to a cube. We can therefore assume that $r < c$. We can also assume that $r > 0$, since a square of length more than 4 is always reducible (recall that the alphabet is binary).

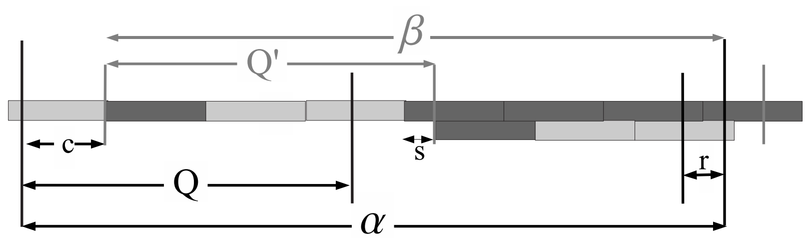

Figure 5.7: $m \geq 3$. β contains enough information for applying previous arguments.

Suppose that $m \geq 3$ (Fig. 5.7). Then

$$p - q - c = q + r - c \geq 3k + 2 - (k-1) = 2k + 3.$$

Therefore, the bottom row of k-blocks contains more than 2 blocks, and altogether we have at least 5 blocks involved in the overlap. By the analysis above, β (and therefore

α) is reducible to a cube. We can therefore assume that $m = 2$, i.e., $q = 2k + s$, and $p - q - c = 2k + s + r - c$. If $s + r \geq c$ then again we get that β is reducible to a cube, and so we can assume that $2 \leq s + r < c$. Since $c < k$ we also get that $2k + s + r - c > k + s + r$. To summarize the setting, we have:

- $2 < p/q < 5/2$;
- $q = 2k + s, p = 2q + r$;
- $1 \leq r, s;\ r + s < c \leq k - 1$;
- $k + s + r < p - q - c < 2k$.

Again we use overlap analysis. This time we have 4 combinations to consider (Fig. 5.8). We will use the following notation:

- K_1 - the white k-block in Fig. 5.8;
- K_2 - the k-block preceding β.

Since $|x| = s < c$, the $|x|$ letters to the right of β belong to α and must equal x. Therefore $K_2 = wx$ for some $w \in \Sigma^+$.

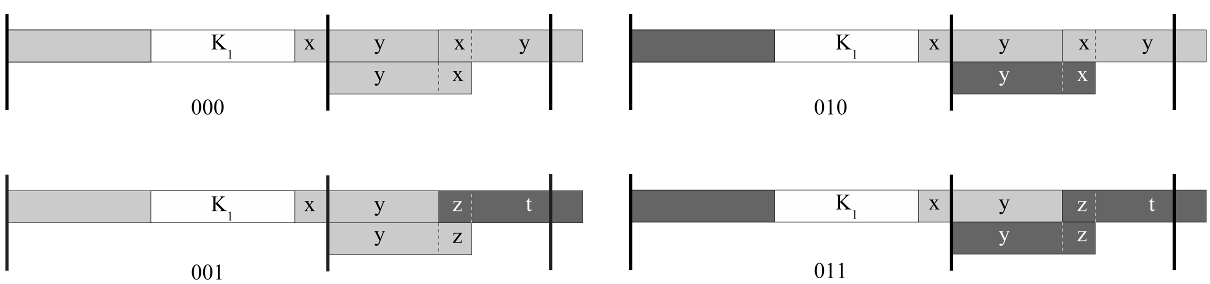

Figure 5.8: Overlaps of k-blocks, $q = 2k + s$, $p = 2q + r$.

000: $h(0) = xy = yx$. By Theorem 2.4, $xyx = u^\ell$ for some $\ell \geq 3$, and α is reducible.

001: $h(0) = xy = yz$, $h(1) = zt$. By Theorem 2.3, $x = uv$, $z = vu$, and $y = (uv)^e u$, for some $u, v \in \Sigma^*$ and $e \geq 0$. If $e > 0$, then xyz contains the cube $(uv)^3$. Suppose $e = 0$. If $K_1 = h(0)$, then α contains the cube $h(0)^3$, thus we can assume $K_1 = h(1)$. Since x is a suffix of K_2, we get that α contains the suboccurrence $xh(0)h(1) = xxyzt = uvuvuvut$, which again contains the cube $(uv)^3$.

010: $h(0) = xy$, $h(1) = yx$. Here, we do not show that α is reducible per se; rather, we show that $\mathbf{v}$ contains a cube with power block of length at most k.

Consider possible choices for K_1 and K_2. If $K_1 = h(0)$, we get the cube $h(0)^3$. Assume $K_1 = h(1)$. If $K_2 = h(1)$, we get the cube $h(1)^3$. Otherwise, if $K_2 = h(0)$, we get that $h(0) = wx = xy$, and so $w = uv$, $x = (uv)^e u$, $y = vu$. If $e > 0$, then wxy (which must occur in $\mathbf{v}$, being a prefix of $h(0)h(1)$) contains the cube $(uv)^3$; otherwise, $h(1)h(0) = yxxy = vuuuvu$, containing the cube u^3.

011: $h(0) = xy$, $h(1) = yz = zt$. By Theorem 2.3, $y = uv$, $t = vu$, and $z = (uv)^e u$, for some $u, v \in \Sigma^*$ and $e \geq 0$. We can assume both u, v are nonempty, for otherwise $h(0) = h(1)$. If $e > 0$, then yzt contains the cube $(uv)^3$. Suppose $z = u$. If $K_2 = h(1)$, then both $h(1) = yz = wx$, and since $|z| = |x|$ we get that $x = z = u$. Thus $h(0) = uuv$, $h(1) = uvu$, and $h(1)h(0)$ contains the cube uuu. If $K_2 = h(0)$, then $h(0) = wx = xy$, therefore $w = ab$, $x = (ab)^e a$, and $y = ba$ for some $a, b \in \Sigma^*$ and $e \geq 0$. Since $|x| = |u| < |uv| = |y|$, necessarily $e = 0$, and $x = a$. We get:

$$\begin{array}{lll} w = ab, & x = a, & y = ba; \\ y = uv, & z = u, & t = vu; \\ |a| = |u|, & |b| = |v|. & \end{array}$$

Assume $|b| \leq |u|$. Then $y = ba = uv$ implies that $u = bd$ and $a = dv$ for some $d \in \Sigma^*$, and so

$$h(0)h(1) = xy\ zt = aba\ uvu = dvbdv\ bdvbd,$$

containing the cube $(dvb)^3$. Now assume $|b| > |u|$. Then $b = ud$ and $v = da$ for some $d \in \Sigma^*$, and

$$h(0)h(1) = auda\ udau = (aud)^{(6|u|+2|d|)/(2|u|+|d|)}.$$

If $|d| \leq 2|u|$, then $(6|u| + 2|d|)/(2|u| + |d|) \geq 5/2$, making α reducible. Suppose $|d| > 2|u|$. Since $t = dau$, and $p - q - c > k + s + r$, either a or u must be a prefix of d, depending on whether $K_1 = h(0)$ or $K_1 = h(1)$ (recall that $|a| = |u| = s$, thus the first s letters of t are still part of β. If $K_1 = h(1)$, then u is a prefix of d, and $h(1)h(1)$ contains the cube uuu. If $K_1 = h(0)$, then a is a prefix of d, and $h(1)h(0)$ contains the subword $auaua$, which is a $5s/2s$-power. In both cases, $p/q < 5/2$ and $\mathbf{w}$ contains a cube. □

The overlap analysis we used in the previous lemma, though a bit long, was straightforward enough. This was due to the fact that when $q > 2k$, the power is sufficiently long to imply easy constraints. Things are a little more subtle when dealing with short powers, especially when $k < q < 2k$ and $\lfloor p/q \rfloor = 2$. While powers with $q > 2k$ are always strictly reducible, and in most cases to a power with a much bigger exponent, this is not necessarily the case for small powers. Consider the following example of a 10-uniform morphism:

$$\begin{aligned} h(0) &= 0110010110 \\ h(1) &= 1001011010 \\ h(101) &= 1001011010\ 0110010110\ 1001011010 \end{aligned}$$

The prefix of length 24 of $h(101)$ is a square with block size 12, which contains no overlaps; thus it is not strictly reducible, though it does contain squares of block size smaller than k. This example demonstrates that we need to be more careful when analyzing powers with $k < q < 2k$.

Definition 5.5. Let h be a binary k-uniform morphism, and let $\mathbf{v} = v_0 v_1 v_2 \cdots = h(\mathbf{u})$ for some $\mathbf{u} \in \Sigma^\omega$. Let $0 < i < k$. The *k-partition of the i-shift* of $\mathbf{v}$, denoted by $T_{i,k}(\mathbf{v})$ is the partition of $v_i v_{i+1} v_{i+2} \cdots$ into k-blocks.

Let $\alpha = v_i \cdots v_j \in \mathrm{Occ}(\mathbf{v})$ be a p/q-power, and suppose $i \not\equiv 0 \pmod{k}$. Consider α as an occurrence of $T_{i,k}(\mathbf{v})$. In general, $T_{i,k}(\mathbf{v})$ can have up to 4 different k-blocks. However, if there are only two composing α, we can do the overlap analysis of α using the blocks of $T_{i,k}(\mathbf{v})$. Clearly, if α is reducible with respect to $T_{i,k}(\mathbf{v})$ then it is also reducible with respect to $\mathbf{v}$. If the k-blocks of $T_{i,k}(\mathbf{v})$ are proved to be equal it does not necessarily imply that $h(0) = h(1)$; however, it would still imply that α is reducible.

Lemma 5.10. *Let h be a binary k-uniform morphism, let $\mathbf{v} = h(\mathbf{u})$ for some $\mathbf{u} \in \Sigma^\omega$, and let $\alpha = v_i \cdots v_j \in Occ(\mathbf{v})$ be a p/q-power with $p/q \geq 2$. Suppose $k < q < 2k$. Then at leas one of the following holds:*

1. *$h(0) = h(1)$;*
2. *$p/q < 3$ and $\mathbf{w}$ contains a cube x^3 with $|x| \leq k$;*
3. *α is reducible to a p'/q'-power, which satisfies $q' \leq k$.*

Proof. Let $q = k + s$, $1 \leq s \leq k - 1$, and let $p = nq + r = nk + ns + r$, $0 \leq r \leq q - 1$. Let $\beta = v_{\check{\imath}} \cdots v_j$, $c = \check{\imath} - i$, and let Q' be the power block of β. If $p - q - c \geq 2k$, then there are at least 5 k-blocks involved in the overlap of β, and we can use a case analysis similar to the one applied in the previous lemma. Out of the 16 possible combinations, only 8 are possible when $k < q < 2k$: to avoid violation of the period, the combination has to be of the form $0ab0c$ for some $a, b, c \in \{0, 1\}$ (see Fig. 5.9).

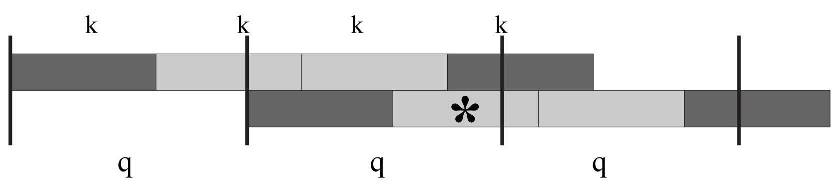

Figure 5.9: Overlaps of k-blocks, $k < q < 2k$. The k-block marked by '*' cannot be an $h(1)$ block, or the period will be violated.

Out of the 8 possible combinations, the combinations $00100, 01000, 01001, 01101$ imply straightforwardly that $h(0) = h(1)$, as in Lemma 5.6. The combinations 00000 and 00001 imply, as in Lemma 5.6, that $h(0) = xy = yx$, and so $x = u^i$ and $y = u^j$ for some $u \in \Sigma^+$ and $i, j \in \mathbb{Z}_{\geq 1}$. But since $q < 2k$, we get that $Q' = xyx = u^i$ for some $i \geq 3$ (see Fig 5.10). Since Q is a conjugate of Q', α is reducible to a power with power block $|u| < k$. The combination 01100 implies, as in Lemma 5.6, that $h(0) = xy = tx$ and $h(1) = yz = zt$. Therefore, by Proposition 5.9, there exist $u, v \in \Sigma^+$ and integers $i \geq 1$, $j, m \geq 0$, such that $y = (uv)^i$, $t = (vu)i$, $z = (uv)^j u$, and $x = (vu)^m v$. Since $Q' = yzx = (uv)^\ell$, where

$\ell = i+j+m+1 \geq 2$, we get again that α is reducible to a power with power block smaller than k. The only combination that needs a little more consideration is 00101.

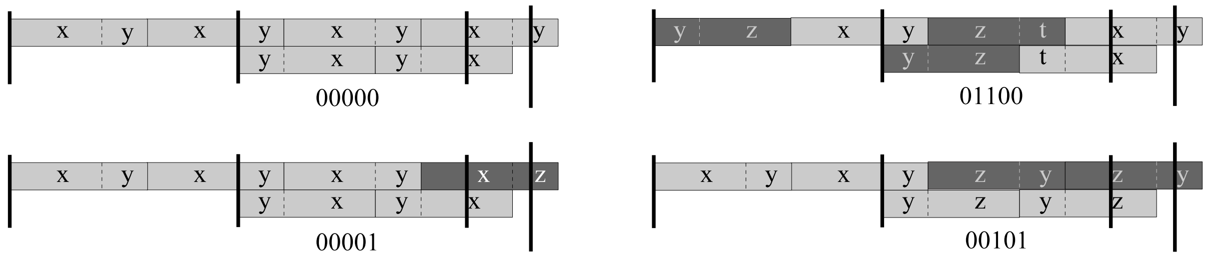

Figure 5.10: Overlaps of k-blocks, $k < q < 2k$, $p - q - c \geq 2k$.

00101: $h(0) = xy = yz$ and $h(1) = zy$, and so by Theorem 2.3, there exist $u, v, \in \Sigma^*$ and an integer $e \geq 0$ such that $x = uv$, $y = (uv)^e u$, and $z = vu$. We can assume that $|x| \neq |y|$, otherwise we get $x = y = z$ and $h(0) = h(1)$. Suppose $|x| < |y|$, and suppose $p - q - c \geq 2k + s$ (recall that $s = |x| = |z|$). Then $e \geq 1$. Also, the bottom row includes the prefix of length s of the next k-block, which by the top row must equal z. Thus, z is a prefix of y. But $z = vu$ and y begins with uv. We get that $uv = vu = w^i$ for some $w \in \Sigma^+$ and $i \geq 2$, thus $h(0) = h(1)$. Assume therefore that $p - q - c < 2k + s$. Suppose $p/q \geq 3$. Then $p - q - c \geq 2q - c$, and so $2q - c < 2k + s$. Since $q = k + s$, we get that $c > s$. Therefore, α includes the suffix of length s of the previous k-block, which must equal x to match the period. Since both $h(0)$ and $h(1)$ end with y, we get that x is a suffix of y. But $x = uv$ and y ends with vu, and again $uv = vu$ and $h(0) = h(1)$. We can assume therefore that $p/q < 3$. But then α is reducible, since $Q'Q'$ contains the cube $(uv)^3$.

Now suppose that $|x| > |y|$. Then $e = 0$, $h(0) = xy = uvu$, and $h(1) = zy = vuu$. Suppose $p - q - c \geq 2k + s$. Then $y = u$ is a prefix of $z = vu$, and as in the case 00101 in Lemma 5.6, we get that $h(0) = h(1)$. If $p - q - c < 2k + s$ and $p/q \geq 3$ then the suffix of length $|x|$ of the previous k-block is included in α, implying that u is a suffix of $x = uv$. Again we get that $h(0) = h(1)$. We can assume therefore that $p/q < 3$, but then $h(1)h(0)$ contains the cube uuu. If $h(1)h(0)$ is a subword of $\mathbf{w}$, then uuu satisfies the second condition of the lemma; otherwise, $\mathbf{w} = h(0)^n h(1)^\omega$ for some $n \geq 0$, and again the second condition is satisfied. This concludes the 00101 combination for $k < q < 2k$.

From now on, we assume that $p-q-c<2k$ and there are less than 5 overlapping k-blocks. Suppose $p-q-c=(n-1)k+(n-1)s+r-c<2k$. Then we get:

$$(n-1)k+n-1 \leq (n-1)k+(n-1)s+r<2k+c \leq 3k-1.$$

Therefore, $n \leq 4k/(k+1)<4$

Suppose $n=3$ and $p-q-c=2k+2s+r-c<2k$. Then $2s+r<c \leq k-1$, i.e., $s<k/2$. Also, $p-q-c \geq k+2s+r+1$, and so there are 3 k-blocks involved in the overlap. The cases are illustrated in Fig. 5.11. We denote by K the k-block preceding β (the white block in Fig. 5.11). Note that in all cases $|x|<|y|$, since $|x|=s<k/2$, and $|x|+|y|=k$.

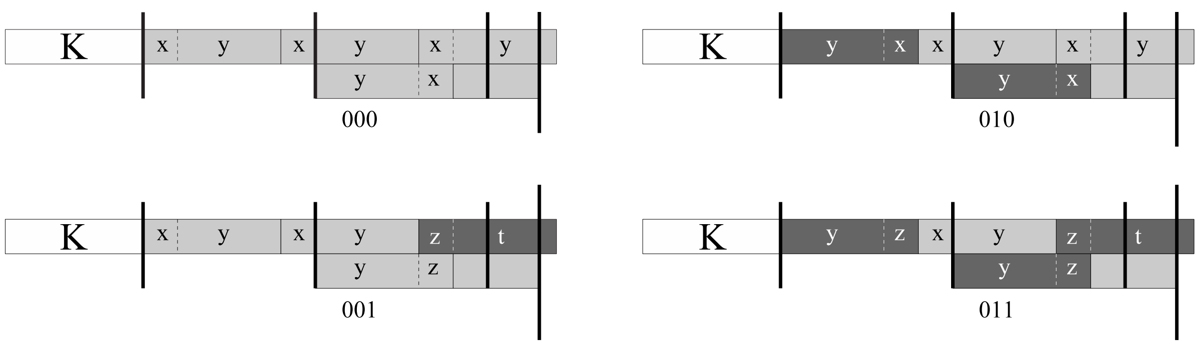

Figure 5.11: Overlaps of k-blocks, $k<q<2k$, $n=3$.

000: $h(0)=xy=yx$. By Theorem 2.4, $h(0)h(0)=u^{\ell}$ for some $\ell \geq 4$, and since $p/q<4$ α is reducible.

001: $h(0)=xy=yz$, $h(1)=zt$. Since $|x|<|y|$, we get that $x=uv$, $z=vu$, and $y=(uv)^e u$ for some $u,v \in \Sigma^*$ and $e \geq 1$. If $e>2$, then xyz contains the 4-power $(uv)^4$, making α reducible. If $e=1$, then $|t|=|y|=|uvu|=s+|u|<2s$, and so all of the $h(1)$ block is included in β, implying that $t=y$. Consider K: since $|x|=s<c$, the last $|x|$ letters of K belong to α, and must equal x. Since both $h(0)$ and $h(1)$ end with y,

and $|y| > |x|$, necessarily x is a suffix of y. Therefore, the word $yxyz$ contains the word $xxyz = (uv)^4u$. Again, α is reducible.

010: $h(0) = xy$, $h(1) = yx$. Also, the last block on the second row implies that x is a prefix of y. Let y' be the suffix of y of length s. Consider K: as in the previous case, x is a suffix of K. If $K = h(0) = xy$, then $y' = x$. If $K = h(1) = yx$, then, since $c > 2s$, the first power block of β must end with $y'x$, and again $y' = x$. We get that x is both a prefix and a suffix of y, and so $yxxy$ contains x^4, making α reducible.

011: $h(0) = xy$, $h(1) = yz = zt$. Since $|z| < |y|$, we get that $y = uv$, $z = u$, and $t = vu$ for some $u, v \in \Sigma^*$. Suppose that $K = h(1)$. Then x is a suffix of $h(1)$ therefore $x = z = u$. Also, the last block on the second row implies that x is a prefix of $t = vu$. Therefore, $zxyz = uuuvu$ contains u^4.

Now assume that $K = h(0)$. Then $\alpha \prec h(0)h(1)h(0)h(1)$, and contains at most two different k-blocks when considered as a subword of $T_{i,k}(\mathbf{v})$: the block resulting from $h(0)h(1)$, and the block resulting from $h(1)h(0)$. Using the $T_{i,k}(\mathbf{v})$ decomposition, we get 5 blocks involved in the overlap of α, making it reducible.

Finally, suppose that $n = 2$. The cases are illustrated in Fig. 5.12. We denote by K_1 the k-block that continues the k-decomposition on the first row, and by K_2 the k-block preceding it. Fig. 5.12 depicts powers that start at an index $i \equiv 0 \pmod{k}$, but we also consider the case where α begins at some index i, and $p - c - q \geq k$.

000: α contains $h(0)^3$, and must be reducible since $p/q < 3$.

001: $h(0) = xy = yz$, and so $x = uv$, $z = vu$, and $y = (uv)^e u$ for some $u, v \in \Sigma^*$ and $e \geq 0$. If $|x| = |y|$ then $h(0) = h(1)$. If $|x| < |y|$ (as illustrated in Fig. 5.11), then $e \geq 1$, and xyz contains the cube $(uv)^3$. Otherwise, $|x| > |y|$ and $h(1) = zy$ (as illustrated in Fig. 5.12). We get that $h(0) = uvu$, $h(1) = vuu$, and $h(1)h(0)$ contains the cube u^3. Since either $h(1)h(0)$ or $h(1)^3$ is a subword of $\mathbf{w}$, the second condition of the lemma is satisfied.

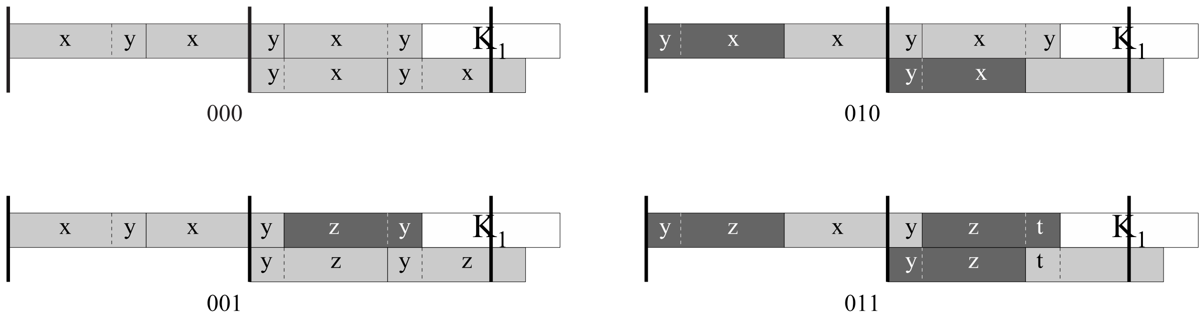

Figure 5.12: Overlaps of k-blocks, $k < q < 2k$, $n = 2$.

010: $h(0) = xy$, $h(1) = yx$. Also, the last k-block in the second row implies that either x is a prefix of y or vice versa. If $x = ya$ for some $a \in \Sigma^*$, then $h(0) = yay$, $h(1) = yya$, and $h(0)h(1)$ contains the cube y^3; if $y = xb$ for some $b \in \Sigma^*$, then $h(0) = xxb$, $h(1) = xbx$, and $h(1)h(0)$ contains the cube x^3. In any case, α is reducible.

If α starts at some index $i \not\equiv 0 \pmod{k}$ and $p - c - q \geq k$, we still get $h(0) = xy$ and $h(1) = yx$. If $K_2 = h(0)$, then y is a suffix of x or vice versa, and again we get a cube. If $K_2 = h(1)$, then either α is the square $(xyx)^2$ (in which case it must be reducible), or again either x is a prefix (suffix) of y or vice versa.

011: $h(1) = yz = zt$, therefore $y = uv$, $t = vu$, and $z = (uv)^e u$. If $|z| = |y|$, then $z = y = t$ and α contains the cube yzt. If $|z| > |y|$, then $e > 0$, and α contains the cube $uvuvuv$. Assume $|z| < |y|$, i.e., $z = u$ and $|x| = |z| = |u|$. We get the picture illustrated in Fig. 5.13(a). Here we assume $|u| < |v|$; assuming $|u| > |v|$ leads to similar results, while

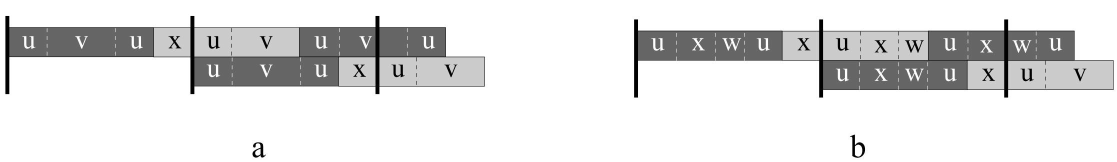

Figure 5.13: Overlaps 011, $|z| < |y|$

$|u| = |v|$ implies $QQ = (uv)^4$. Using the fact that x is a prefix of v, we get the picture illustrated in Fig. 5.13(b).

Suppose $r > 0$. Then u and w share a prefix of size $\min(r, |u|, |w|)$. If $r \geq |u|$, then either u is a prefix of w, or u is a prefix of wu. In both cases, we get that α contains the $5|u|/2|u|$-power $uxuxu$. In order for α to be irreducible, we must have $r > \frac{1}{2}q = \frac{1}{2}(4|u| + |w|)$, but then the third power block is longer than $2|u| = |u| + |x|$, and so ux is a prefix of wu or vice versa. In the first case, α contains the cube $(ux)^3$. In the second case, w is a prefix of u and wu is a prefix of ux. Therefore, $ux = wut$. If the next k-block in the top row is $h(0)$, we get that t is a prefix of x, and so $uxuxwux = wutwutwutt'$, containing the cube $(wut)^3$. If the next k-block is $h(1)$, we get that t is a prefix of u and a suffix of x. Aligning both rows, we get the equations $u = wt = tw'$ and $x = w't$. Thus there exist words a, b and an integer e such that $w = ab$, $w' = ba$, and $t = (ab)^e a$. We get that $u = ab(ab)^e a$ and $x = ba(ab)^e a$. If $e = 0$, then xu contains the cube a^3. If $e = 1$, then uxu contains the cube $(aba)^3$. Otherwise, $e \geq 2$, and u contains the cube $(ab)^3$.

Therefore, we must assume that $r < |u|$. Let $u = da$ and $w = db$, where $|d| = r$. Then $uxuxw = daxdaxdb$, containing the power $(dax)^{2+|d|/|dax|} = (dax)^{2+r/2|x|}$. Since α is a $(2 + r/q)$-power and $2|x| < k < q$, we get that α is reducible. Therefore we must have $r = 0$. But then α is a square, and must be reducible.

If α starts at some index $i \not\equiv 0 \pmod k$, consider K_2: if $K_2 = h(0)$, then $\alpha \prec h(0)h(1)h(0)h(1)$, and contains at most two different k-blocks when considered as a subword of $T_{i,k}(\mathbf{v})$. We can apply the overlap analysis of the current lemma starting from the first character of α, and get at least one of the three conditions is satisfied. If $K_2 = h(1)$, then $x = u$, and α contains the 4-power x^4. Again, α is reducible.

The only cases not yet covered are when $p - q - c < k$. Recall that $\alpha = v_i \cdots v_j$, $q = k + s$, $p = 2k + 2s + r$, and $c = \check{\imath} - i$, where $\check{\alpha} = v_{\check{\imath}} \cdots v_{\check{\jmath}}$ is the inner closure of α. By assumption, $p - q - c = k + s + r - c < k$, and so, $s + r < c$. Since $1 \leq r, s \leq k - 1$, we get that $s + r \leq k - 2$.

Suppose α spans across the k-blocks $K_1 K_2 \cdots K_n$. By assumption, α begins at the $(k-c)^{th}$ letter of K_1. Since $q = k+s$ and $s < c$, the second q-block of α begins in K_2. The remaining of α is of length $k + s + r$, and since $s + r \leq k - 2$, necessarily α ends in either K_3 or K_4. That is, α spans across either 3 or 4 k-blocks. If α spans across 3 k-blocks, then there can be at most two different k-blocks in the $T_{i,k}(\mathbf{v})$ decomposition of α, and we can apply the analysis from the first character of α. Otherwise, $j - \check{\jmath} = p - 2k - c = 2s + r - c$. Consider α^R (that is, the reverse of α), with the k-decomposition of $h(0)^R$ and $h(1)^R$. Then $c^R = j - \check{\jmath} = 2s + r - c$. Therefore, $p - q - c^R = k + (c - r) > k$, and we can show that α is reducible. This concludes the proof of the lemma. □

Corollary 5.11. *Let f be a binary k-uniform morphism prolongable on 0, and let $\mathbf{w} = f^\omega(0)$. Suppose that $E(\mathbf{w}) < \infty$, and let $\mathcal{E}'$ be the set of exponents $r = p/q$, such that $q < k$ and $\mathbf{w}$ contains an unstretchable r-power. Then*

$$E(\mathbf{w}) = \max_{p/q \in \mathcal{E}'} \left\{ \frac{p(k-1) + \lambda_f}{q(k-1)} \right\} . \tag{5.4}$$

Proof. By Lemma 5.5, if $z \in \mathrm{Occ}(\mathbf{w})$ is an unstretchable p/q-power and $q \equiv 0 \bmod k$, then z is an image under the π map; thus the exponent of every such power is an element of a sequence of the form $\{\pi^i(r)\}_{i=0}^{\infty}$, where $r \not\equiv 0 \bmod k$. By Lemma 5.2, the limit of each such sequence is given by the expression in (5.4), and each of these sequences increases towards its limit. Therefore, to compute $E(\mathbf{w})$ it is enough to apply π iteratively to p/q-powers with $q \not\equiv 0 \bmod k$.

By Lemmata 5.6, 5.10, if $z \in \mathrm{Occ}(\mathbf{w})$ is a p/q power with $q > k$ and $q \not\equiv 0 \bmod k$, then either z is reducible to a p'/q'-power with $q' \leq k$, or $p/q < 3$ and $\mathbf{w}$ contains a cube x^3 with $|x| \leq k$. By Corollary 5.4, this implies that $\lim_{m\to\infty} \pi^m(z, p/q) < E(\mathbf{w})$. Therefore, to compute $E(\mathbf{w})$ it is enough to take the limits of π-sequences generated by elements of $\mathcal{E}'$. Since $E(\mathbf{w}) < \infty$ the set $\mathcal{E}'$ is finite, therefore there are only finitely many such sequences, and the critical exponent is the maximum of those limits. □

5.3.4 p/q-powers with $q < k$

Lemma 5.12. *Let h be a binary k-uniform morphism, let $\mathbf{v} = h(\mathbf{u})$ for some $\mathbf{u} \in \Sigma^\omega$, and let $\alpha = v_i \cdots v_j \in Occ(\mathbf{v})$ be a p/q-power with $p/q \geq 2$. Suppose $q < k$. Then at least one of the following holds:*

1. *$q|k$;*
2. *$q \nmid k$ and $q|2k$;*

3. *α is reducible;*

4. *$p < 4k - 1$.*

Proof. Let $\check{\alpha}$ be the inner closure of α, and let Q be the power block of α. Suppose $p = |\alpha| \geq 4k - 1$. Then $|\check{\alpha}| \geq 2k + 1$, and since $|\check{\alpha}| \equiv 0 \bmod k$, we get that $|\check{\alpha}| \geq 3k$. Thus, α contains an occurrence of the form $h(a_1a_2a_3)$ for some $a_1, a_2, a_3 \in \Sigma$. There are two cases: either $a_1a_2a_3$ contains a square, or $a_1a_2a_3$ is a 3/2-power.

Suppose $a_1a_2a_3$ contains a square, and assume without loss of generality it is 00. Then $h(0)h(0)$ is a suboccurrence of α that has both k and q periods. Since $2k > k + q$, by Theorem 2.2 $h(0)h(0)$ has a $g = \gcd(k, q)$ period. Since $q = |Q| < |h(0)| = k$, there must be an occurrence of Q within $h(0)h(0)$, thus Q has a g period as well. We get that $Q = w^{q/g}$ for some $w \in \Sigma^*$ satisfying $|w| = g$, and $\alpha = w^{p/g}$. This implies that either $q|k$, or α is reducible: if $q \nmid k$, then $g < q$, and $p/g > p/q$.

Now suppose that $a_1a_2a_3 = 010$. Then $h(0)h(1)h(0)$ has both q and $2k$ periods, thus, as previously, either $q|2k$ or α is reducible. □

Corollary 5.13. *If $q < k$ and α is irreducible, then at least one of the following holds:*

1. *$h(0) = h(1)$;*

2. *$h^{-1}(\hat{\alpha}) = ac^\ell b$, where $a, b \in \{0, 1, \varepsilon\}$, $c \in \{0, 1\}$, and $\ell \geq 0$;*

3. *$h^{-1}(\hat{\alpha}) = ax^\ell b$, where $a, b \in \{0, 1, \varepsilon\}$, $x \in \{01, 10\}$, and $\ell \geq 0$;*

4. *$|h^{-1}(\hat{\alpha})| \leq 5$.*

Proof. By Lemma 5.12, either $q|k$, $q|2k$, or $p < 4k - 1$. Suppose $q|k$. Let $k = mq$, and let $\check{Q}$ denote the q block of $\check{\alpha}$. Let $\ell = |\check{\alpha}|/k$. Then $\check{\alpha} = \check{Q}^{m\ell} = (h(a))^\ell$ for some $a \in \Sigma$. If $h(0) \neq h(1)$ this means that $h^{-1}(\check{\alpha}) = a^\ell$ and $h^{-1}(\hat{\alpha}) = ba^\ell c$, where $b, c \in \{0, 1, \varepsilon\}$.

If $q \nmid k$ and $q|2k$, we get similarly that $\check{\alpha} = \check{Q}^{m\ell/2} = (h(x))^{\ell/2}$, where $x \in \{01, 10\}$ and $\ell \geq 0$. Suppose $q \nmid 2k$. Then $p = |\alpha| < 4k - 1$, thus $|\hat{\alpha}| \leq 5k$, and $|h^{-1}(\hat{\alpha})| \leq 5$. □

Corollary 5.14. *Let f be a k-uniform binary morphism, prolongable on 0, and let $\mathbf{w} = f^\omega(0)$. Then $E(\mathbf{w}) = \infty$ if and only if at least one of the following holds:*

1. $f(0) = f(1)$;

2. $f(0) = 0^k$;

3. $f(1) = 1^k$;

4. $f(0) = (01)^m 0$ *and* $f(1) = (10)^m 1$, *where* $k = 2m + 1$.

Proof. It is easy to see that any of the 4 conditions implies $E(\mathbf{w}) = \infty$. For the converse, suppose $f(0) \neq f(1)$, and $\mathbf{w}$ contains unbounded powers. Then by Lemmata 5.5, 5.6, 5.10, 5.12 and Corollary 5.13, $\mathbf{w}$ must contain unbounded powers of the form 0^m, 1^m, or $(01)^m$. If it contains unbounded 0^m powers, then $f(a) = 0^k$ for some $a \in \Sigma$. Suppose $f(1) = 0^k$. Then $\mathbf{w}$ must contain unbounded 1^m powers as well, and so necessarily $f(0) = 1^k$, a contradiction: f is prolongable on 0. Thus $\mathbf{w}$ contains unbounded 0^m powers if and only if $f(0) = 0^k$, and similarly it contains unbounded 1^m powers if and only if $f(1) = 1^k$. Finally, it is easy to see using similar inverse image arguments that $\mathbf{w}$ contains unbounded $(01)^m$ powers if and only if the last condition holds. □

Note: Corollary 5.14 also follows from Theorem 3.3.

5.4 Bounding the occurrence of small powers

To complete the proof of Theorem 5.1, it remains to show that in order to compute $E(\mathbf{w})$, it is enough to consider $f^4(0)$. We do this by showing that any subword of $\mathbf{w}$ of the form ab, a^ℓ, or $(a\bar{a})^\ell$, where ℓ is a positive integer, $a, b \in \Sigma$ and $\bar{a} = 1 - a$, must occur in $f^2(0)$ or $f^3(0)$. We then apply Corollary 5.13. The details are given below.

For the rest of this section $f : \Sigma^* \to \Sigma^*$ is a k-uniform morphism prolongable on 0, and $\mathbf{w} = f^\omega(0)$.

Lemma 5.15. *Let* $a, b \in \Sigma$, *and suppose* $ab \prec \mathbf{w}$. *If* $ab \in \{01, 10, 11\}$, *then* $ab \prec f^2(0)$; *if* $ab = 00$, *then either* $00 \prec f^2(0)$, *or* $00 \prec f^3(0)$ *and* $000 \not\prec \mathbf{w}$.

Proof. The assertion clearly holds for $f(0) = 0^k$, thus we can assume $1 \prec f(0)$. Suppose $ab \prec \mathbf{w}$. Then either $ab \prec f(c)$ for some $c \in \Sigma$, or $ab \prec f(a'b')$ for some $a', b' \in \Sigma$; the first case implies that $ab \prec f^2(0)$, since both $0, 1 \prec f(0)$.

For $k = 2$, it is easy to check that the assertion holds. Assume $k \geq 3$. Then $f(0)$ contains at least two distinct pairs ab. If it contains four, we are done. Assume it contains exactly two. Then necessarily $f(0) \in \{0^{k-1}1, 01^{k-1}\}$. If the first case holds, then $f^2(0) = 0^{k-1}10^{k-1}1 \cdots 0^{k-1}1f(1)$, where $k-1 \geq 2$; i.e., it contains the pairs $00, 01, 10$. Assume $ab = 11 \not\prec f^2(0)$. then necessarily $f(1) = 0y$, where $11 \not\prec y$; but then $11 \not\prec \mathbf{w}$, a contradiction. Thus the assertion holds for $f(0) = 0^{k-1}1$.

Assume now that $f(0) = 01^{k-1}$. Then $f^2(0) = 01^{k-1}f(1) \cdots f(1)$, i.e., it contains the pairs $01, 11$. Suppose $ab = 10 \not\prec f^2(0)$. Then necessarily $f(1) = 1^k$; but then $w = 01^{\omega}$, and $10 \not\prec \mathbf{w}$, a contradiction. Now suppose $ab = 00 \not\prec f^2(0)$. Then 00 is not a subword of either $f(0), f(1), f(01)$, or $f(11)$. If $00 \not\prec f(10)$, the only remaining option is $00 \prec f(00)$; but then $00 \not\prec f^3(0)$, and by induction, $00 \not\prec f^n(0)$ for all n, a contradiction. Thus $00 \prec f(10)$, and $00 \prec f^3(0)$. The assertion $000 \not\prec \mathbf{w}$ follows from the fact that $00 \not\prec f(0), f(1)$.

Finally, assume $f(0)$ contains three distinct pairs. Let $ab \not\prec f(0)$, and suppose $ab \not\prec f^2(0)$. Then ab is not a subword of either $f(0), f(1)$, or $f(a'b')$, where $a'b' \neq ab$. The only option is $ab \prec f(ab)$, but then again $ab \not\prec f^n(0)$ for all n, a contradiction. This completes the proof of the lemma. $\square$

Lemma 5.16. *Let ℓ be the maximal integer such that $0^\ell \prec \mathbf{w}$ ($\ell = \infty$ if such integer does not exist). Then*

1. $f(0) = 0^k \Rightarrow \ell = \infty$;
2. $f(0) \neq 0^k, f(1) \neq 0^k \Rightarrow \ell \leq 2k-2$ *and* $0^\ell \prec f^3(0)$;
3. $f(0) \neq 0^k, f(1) = 0^k \Rightarrow \ell \leq k^2 - k + 1$ *and* $0^\ell \prec f^3(0)$.

The bounds on ℓ and on the first occurrence of 0^ℓ are tight. If $1 < \ell < \infty$, then 0^ℓ occurs as a non-prefix subword of $f^3(0)$.

Proof. Let $z = w_i \cdots w_j = 0^\ell \prec \mathbf{w}$. Let $\hat{z}$ be the outer closure of z, and let $z' = f^{-1}(\hat{z})$. Observe that for any $m \in \mathbb{Z}_{\geq 0}$, if $\ell \geq 2k-1+mk$ then the k-decomposition of $\hat{z}$ contains at least $m+1$ consecutive 0-blocks (blocks of the form 0^k).

Clearly, $f(0) = 0^k$ implies $\ell = \infty$. Assume $f(0) \neq 0^k$, $f(1) \neq 0^k$. Suppose $\ell \geq 2k-1$. Then the k-decomposition of $\hat{z}$ must contain a 0-block, a contradiction. Thus $\ell \leq 2k-2$, and $|z'| \leq 2$. By Lemma 5.15, $z' \prec f^2(0)$, or $z' = 00$ and $z' \prec f^3(0)$. The first case implies $0^\ell \prec f^3(0)$; the second case implies that $\ell = 2$, and so again $0^\ell \prec f^3(0)$. The bounds are tight: for the Thue-Morse morphism μ, we get $\ell = 2k - 2 = 2$, and the first occurrence of 0^2 is in $f^3(0)$.

To see that 0^ℓ occurs in $f^3(0)$ as a non-prefix subword when $\ell > 1$, consider 4 possible values of z', namely $\{0, 1, 0a, 1a\}$, where $a \in \Sigma$.

- If $z' = 1$, then, since $1 \prec f(0)$ as a non-prefix, $0^\ell \prec f^2(0)$ as a non-prefix.
- If $z = 0$, then either $0 \prec f^2(0)$ as a non-prefix, or $f(0) = 01^{k-1}$ and $f(1) = 1^k$. In the first case, $0^\ell \prec f^3(0)$ as a non-prefix; in the second case, $\mathbf{w} = 01^\omega$ and $\ell = 1$.
- If $z' = 1a$ for some $a \in \Sigma$, then it must occur as a non-prefix subword of $f^2(0)$, thus $0^\ell \prec f^3(0)$ as a non-prefix.
- If $z = 0a$, and it is a prefix of $\mathbf{w}$, then $z \prec f(0)$, and again, $z \prec f^3(0)$ as a non-prefix unless $f(0) = 01^{k-1}$ and $f(1) = 1^k$.

Now assume $f(1) = 0^k$. Then $1^k \not\prec \mathbf{w}$, and for any $0 < m < k$, we get that $1^m \prec \mathbf{w}$ if and only if $1^m \prec f(0)$. Suppose $\ell \geq 2k - 1$ (at least one 0-block in the k-decomposition of $\hat{z}$). Then $z' = a1^m b$, where $a, b \in \{0, \varepsilon\}$, $1 \leq m \leq k - 1$, and $1^m \prec f(0)$. Therefore, $|f(0)|_0 \leq k - m$. Let x, y be the longest prefix and suffix of $f(0)$, respectively, that do not contain 1. Then $|x| + |y| \leq k - m$. Since $z' \prec 01^m 0$, we get that $z \prec y f(1)^m x = 0^{|x|+|y|+km}$; i.e., $\ell \leq k - m + km = k + (k-1)m \leq k + (k-1)^2 = k^2 - k + 1$. Moreover, since $1^m \prec f(0)$, we get that $z' \prec f^2(0)$, thus $0^\ell \prec f^3(0)$. The bounds are tight: let $f(0) = 01$, $f(1) = 00$. Then $\ell = k^2 - k + 1 = 3$, and the first occurrence of 0^3 is in $f^3(0)$.

To see that 0^ℓ occurs as a non-prefix subword of $f^3(0)$, observe that 0^k is not a prefix of $\mathbf{w}$ (since $1 \prec f(0)$); on the other hand, $\ell \geq k + 1$, since $f(1) = 0^k$. Therefore 0^ℓ must occur as a non-prefix subword of $f^3(0)$. $\square$

Lemma 5.17. *Let ℓ be the maximal integer such that $1^\ell \prec \mathbf{w}$ ($\ell = \infty$ if such an integer does not exist). Then*

1. $f(1) = 1^k \Rightarrow \ell = \infty$;
2. $f(1) \neq 1^k \Rightarrow \ell \leq 2k - 2$ *and* $1^\ell \prec f^3(0)$ *as a non-prefix subword.*

The bounds on ℓ and on the first occurrence of 1^ℓ are tight.

Proof. If $f(1) = 1^k$, then $1^{k^n} \prec f^n(0)$ for all n, and so $\ell = \infty$. Otherwise, if $0 \prec f(1)$, then both $f(0) \neq 1^k$ and $f(1) \neq 1^k$, and the proof is similar to the proof of Lemma 5.16. The non-prefix statement is trivial, since $\mathbf{w}$ begins with 0. For tightness of the bound on ℓ, observe that for a morphism of the form $0 \to 01^{k-1}$, $1 \to 1^{k-1}0$, we get $\ell = 2k-2$; for tightness of the bound on the first occurrence of 0^ℓ, observe that for a morphism of the form $0 \to 0^{k-1}1$, $1 \to 101^{k-2}$ we get that $\ell = k-1$, and the first occurrence of 1^ℓ is in $f^3(0)$. □

Lemma 5.18. *Let ℓ be the maximal integer such that $(a\bar{a})^\ell \prec \mathbf{w}$, where $a \in \Sigma$ and $\bar{a} = 1-a$ ($\ell = \infty$ if such an integer does not exist). Assume $f(0) \neq f(1)$. Then either $\ell = \infty$ and $\mathbf{w} = (01)^\omega$, or*

1. *k is even $\Rightarrow \ell \leq k-1+(k^2+k)/2$ and $(a\bar{a})^\ell \prec f^4(0)$;*
2. *k is odd $\Rightarrow \ell \leq k-1$ and $(a\bar{a})^\ell \prec f^3(0)$.*

The bounds on ℓ and on the first occurrence of $(a\bar{a})^\ell$ are tight. If $1 < \ell < \infty$, then $(a\bar{a})^\ell$ occurs as a non-prefix subword of $f^4(0)$.

Proof. Let $z = w_i \cdots w_j = (a\bar{a})^\ell \in \mathrm{Occ}(\mathbf{w})$. Let $\hat{z}$ be the outer closure of z, and let $z' = f^{-1}(\hat{z})$. Observe that for any $m \in \mathbb{Z}_{\geq 0}$, if $\ell > k-1+mk/2$ then the k-decomposition of z contains at least $m+1$ k-blocks. For even k, these blocks have the form $(b\bar{b})^{k/2}$, $b \in \Sigma$; for odd k, they alternate between $(b\bar{b})^{(k-1)/2}b$ and $(\bar{b}b)^{(k-1)/2}\bar{b}$. Assume $\ell > k-1$, and let m be the maximal integer satisfying $\ell > k-1+mk/2$.

Suppose k is even. Since $\ell > k-1$, we get that $f(a) = (b\bar{b})^{k/2}$ for some $a \in \Sigma$. Since $f(0) \neq f(1)$, and the $m+1$ k-blocks of z are all the same, $z' = ca^{m+1}c'$ for $a, c, c' \in \Sigma$. Suppose $m > k$. Then $a^{k+2} \prec \mathbf{w}$, and $f(a) = (b\bar{b})^{k/2}$. It is easy to see that the only way this situation is possible is if $f(\bar{a}) = a^k$. Since by assumption $f(0) = 0x$ where $1 \prec x$, this implies that $f(0) = (01)^{k/2}$ and $f(1) = 0^k$. But in this case, it is easy to check that $a^{k+2} \not\prec \mathbf{w}$, a contradiction. Thus $m \leq k$, and $\ell \leq k-1+(k+1)k/2$.

For tightness of the bound on ℓ, observe that for the f just defined, $0^{k+1} \prec \mathbf{w}$, thus $(01)^{k(k+1)/2} \prec \mathbf{w}$. The bound on the first occurrence of z follows from the first occurrence bounds given in Lemmata 5.16, 5.17. From these lemmata, we also get that z occurs as a non-prefix. For tightness of this bound, observe that for $0 \to 010101, 1 \to 000110$, we get $\ell = 12$, and the first occurrence of $(01)^{12}$ is in $f^4(0)$.

Suppose that k is odd. Then for $m \geq 1$, the k-decomposition contains both the blocks $(01)^{(k-1)/2}0$ and $(10)^{(k-1)/2}1$. This implies that $f(0) = (01)^{(k-1)/2}0$, $f(1) = (10)^{(k-1)/2}1$, and $\mathbf{w} = (01)^\omega$. Suppose $k-1 < \ell \leq k-1+k/2$. Then $|z| \geq 2k$, i.e., $z = xf(b)y = x(a\bar{a})^{(k-1)/2}ay$, where $b \in \Sigma$, $|xy| \geq k$, and x, y has the following form: $x = \bar{a}(a\bar{a})^i$, $y = (\bar{a}a)^j$, $i+j \geq k/2$; or $x = (a\bar{a})^i$, $y = (\bar{a}a)^j\bar{a}$, $i+j \geq k/2$. Since $f(b)$ begins and ends with a, this implies that x is a suffix of $f(\bar{b})$, and y is a prefix of $f(\bar{b})$; and since $|xy| \geq k$, this implies that $f(\bar{b}) = (\bar{a}a)^{k/2}\bar{a}$. Again, we get $\mathbf{w} = (01)^\omega$. Thus $\mathbf{w} \neq (01)^\omega$ implies $\ell \leq k-1$. Moreover, z' must have the form a, $a\bar{a}$, or $a\bar{a}a$. If $z' = a$ then $z' \prec f(0)$ as a non prefix, or $\mathbf{w}$ would be ultimately periodic. If $z' = a\bar{a}$, then by Lemma 5.15, $z' \prec f^2(0)$, and unless $\ell = 1$ (i.e., $\mathbf{w} = 01^\omega$), it must occur as a non-prefix; if $z' = a\bar{a}a$, it is easy to show, by similar arguments, that $z' \prec f^2(0)$ as a non-prefix. Thus $z \prec f^3(0)$ as a non-prefix. For tightness of the bound on ℓ, consider $f(0) = 010$, $f(1) = 111$. For tightness on the bound on the first occurrence of z, consider $f(0) = 01110$, $f(1) = 10101$. □

Lemma 5.19. *Let k be even, and suppose $f(0) \neq f(1)$, and there exist $n \geq 1$ and $x, y \in \Sigma^+$ such that $f(0) = (xy)^n x$ and $f(1) = (yx)^n y$. Let ℓ be the maximal integer (it it exists) such that $(a\bar{a})^\ell \prec \mathbf{w}$, where $a \in \Sigma$ and $\bar{a} = 1-a$. Then $\ell \leq k-1$ and $(a\bar{a})^\ell \prec f^3(0)$ as a non-prefix.*

Proof. Let $z = w_i \cdots w_j = (a\bar{a})^\ell \in \mathrm{Occ}(\mathbf{w})$. Let $\hat{z}$ be the outer closure of z, and let $z' = f^{-1}(\hat{z})$. Observe that the conditions imply $|x| = |y|$ and $|x|$ even. If there is at least one k-block in the k-decomposition of z', then $f(a) = (b\bar{b})^{k/2}$ for some $a, b \in \Sigma$. This implies $x = y = (b\bar{b})^t$ for some $t \geq 1$, i.e., $f(0) = f(1)$, a contradiction. Therefore $\ell \leq k-1$ and $|z'| \leq 2$. By Lemma 5.15, if $z' \in \{01, 10, 11\}$ then $z \prec f^2(0)$, and $(a\bar{a})^\ell \prec f^3(0)$ as a non-prefix. If $z' = 00$, then it might be the case that z' occurs for the first time in $f^3(0)$; however, if this is the case, then $00 \prec f(10)$, and so $f(1)$ ends with 0 and $f(0)$ begins with 0. This implies that y ends with 0 and x begins with 0, and so $00 \prec yx \prec f(0)$. Again, $(a\bar{a})^\ell \prec f^3(0)$ as a non-prefix. □

Corollary 5.20. *Let $z \prec \mathbf{w}$ be an irreducible p/q-power satisfying $q < k$. Suppose $E(\mathbf{w})$ is bounded. Then $z \prec f^4(0)$ as a non-prefix and $\lfloor p/q \rfloor \in O(k^3)$.*

Proof. Suppose $q < k$. By Corollary 5.13, either $f^{-1}(\hat{z}) = ab^\ell c$, $f^{-1}(\hat{z}) = a(b\bar{b})^\ell c$, or $|f^{-1}(\hat{z})| \leq 5$; here $\ell \geq 0$ is an integer and $a, b, c \in \{0, 1, \varepsilon\}$.

- If $f^{-1}(\hat{z}) = ab^\ell c$, then by Lemmata 5.16, 5.17, $\ell \leq k^2 - k + 1$ and $z \prec f^4(0)$ as a non-prefix.

- If $f^{-1}(\hat{z}) = a(b\bar{b})^\ell c$ and k is odd, then by Lemma 5.18, $\ell \leq k - 1$ and $z \prec f^4(0)$ as a non-prefix.

- If $f^{-1}(\hat{z}) = a(b\bar{b})^\ell c$ and k is even, then $f(b\bar{b}) = u^{2k/q}$ for some $u \in \Sigma^+$, where $|u| \nmid k$. This is possible only if there exist $n \geq 1$ and $x, y \in \Sigma^+$, such that $f(b) = (xy)^n x$ and $f(\bar{b}) = (yx)^n y$; but then by Lemma 5.19, $\ell \leq k - 1$ and $z \prec f^4(0)$ as a non-prefix.

- If neither of the above cases hold, then $|f^{-1}(\hat{z})| \leq 5$, and $q \nmid 2k$, thus $q \geq 3$. Since $q < k$, we therefore get either $q = 3$ and $k = 4$, or $k \geq 5$. In both cases, a subword $x \prec \mathbf{w}$ of length 5 satisfies $|f^{-1}(\hat{x})| \leq 2$, therefore by Lemma 5.15, $f^{-1}(\hat{z}) \prec f^3(0)$ as a non-prefix. Again we get that $z \prec f^4(0)$ as a non-prefix.

□

Corollary 5.21. *Suppose $E(\mathbf{w}) < \infty$. Let $\mathcal{E}$ be the set of exponents $r = p/q$, such that $q < k$ and $f^4(0)$ contains an r-power. Then*

$$E(\mathbf{w}) = \max_{p/q \in \mathcal{E}} \left\{ \frac{p(k-1) + \lambda_f}{q(k-1)} \right\} . \tag{5.5}$$

The bound is attained if and only if $\lambda_f = 0$.

Proof. Equation (5.5) is an immediate result of Corollaries 5.11, 5.20. The second assertion follows directly from the definition of π. □

Corollary 5.21 completes the proof of Theorem 5.1.

Example 5.5. As implied by the tightness assertions of Corollary 5.20, the prefix $f^4(0)$ is best possible. Consider the morphism $0 \to 010101$, $1 \to 000110$. In this example, $E(\mathbf{w}) = 12\frac{3}{5}$, and the first occurrence of a 12-power is in $f^4(0)$.

5.5 Applications

We conclude this chapter with some applications of Theorem 5.1. The first one is a "density theorem" for critical exponent values.

Theorem 5.22. *For any rational number $0 < r < 1$ there exist a binary uniform pure morphic word* $\mathbf{w}$*, such that $E(\mathbf{w}) = n + r$ for some $n \in \mathbb{Z}_{\geq 2}$.*

Proof. Let s, t be natural numbers satisfying $0 < s \leq t$. Let f be the following morphism:

$$f : \begin{array}{l} 0 \to 01^t \ ; \\ 1 \to 01^{s-1}0^{t-s+1} \ . \end{array}$$

Then f is an $(t+1)$-uniform morphism, satisfying $\rho_f = 01^{s-1}$, $\sigma_f = \varepsilon$, and $\lambda_f = s$. Let $\mathbf{w} = f^\omega(0)$. Then 1^t is a subword of $f^1(0)$; also, $0^{t(t+1)+1}$ is a subword of $f^3(0)$ if $s = 1$. Set $z = 1^t$ for $s > 1$ and $z = 0^{t(t+1)+1}$ for $s = 1$. It is easy to check that by applying π to z we get the maximal number in the set $\left\{\frac{p(k-1)+\lambda_f}{q(k-1)} : p/q \in \mathcal{E}\right\}$; thus

$$\begin{aligned} s > 1 &\Rightarrow E(\mathbf{w}) = \frac{t \cdot t + s}{1 \cdot t} = t + \frac{s}{t} \ ; \\ s = 1 &\Rightarrow E(\mathbf{w}) = t(t+1) + 1 + \frac{s}{t} \ . \end{aligned}$$

□

Next, we have applied Theorem 5.1 to all k-uniform binary morphisms prolongable on 0, morphisms that generate repetitive words excluded, for $k \leq 4$. The results are summarized in the following tables.

We also ran some tests to determine the shortest morphisms that generate words $\mathbf{w}$ with $2 < E(\mathbf{w}) < 3$. We tested morphisms of length up to 10. If there was more than one morphism generating a word with a given exponent, we chose the lexicographically smallest one. The results are summarized in the following table.

To get below $5/2$ we need larger k values. For example, the shortest morphism that generates a word with critical exponent $7/3$ is 19-uniform (Rampersad, [109]):

$$f = (0110100110110010110, 1001011001001101001).$$

f	$E(f^{\omega}(0))$
$(01, 00)$	4
$(01, 10)$	2
$(001, 000)$	6
$(001, 010)$	$7/2$
$(001, 011)$	3
$(001, 100)$	4
$(001, 101)$	3
$(001, 110)$	3
$(010, 000)$	6
$(010, 001)$	$7/2$
$(010, 011)$	3
$(010, 100)$	$7/2$
$(010, 110)$	3
$(011, 000)$	$15/2$
$(011, 001)$	3
$(011, 010)$	3
$(011, 100)$	3
$(011, 101)$	$7/2$
$(011, 110)$	4

Table 5.1: Critical exponents of words generated by 2- and 3-uniform binary morphisms.

f	$E(f^\omega(0))$	f	$E(f^\omega(0))$	f	$E(f^\omega(0))$
$(0001, 0000)$	8	$(0011, 0110)$	$10/3$	$(0101, 1011)$	$10/3$
$(0001, 0010)$	$14/3$	$(0011, 0111)$	4	$(0101, 1100)$	6
$(0001, 0011)$	4	$(0011, 1000)$	5	$(0101, 1101)$	4
$(0001, 0100)$	$16/3$	$(0011, 1001)$	$10/3$	$(0101, 1110)$	4
$(0001, 0101)$	4	$(0011, 1010)$	6	$(0110, 0000)$	$32/3$
$(0001, 0110)$	$13/3$	$(0011, 1011)$	4	$(0110, 0001)$	$13/3$
$(0001, 0111)$	$11/3$	$(0011, 1100)$	4	$(0110, 0010)$	4
$(0001, 1000)$	6	$(0011, 1101)$	$13/3$	$(0110, 0011)$	$10/3$
$(0001, 1001)$	4	$(0011, 1110)$	5	$(0110, 0100)$	4
$(0001, 1010)$	4	$(0100, 0000)$	8	$(0110, 0101)$	$16/3$
$(0001, 1011)$	$10/3$	$(0100, 0001)$	$16/3$	$(0110, 0111)$	4
$(0001, 1100)$	5	$(0100, 0010)$	$14/3$	$(0110, 1000)$	$13/3$
$(0001, 1101)$	$11/3$	$(0100, 0011)$	$13/3$	$(0110, 1001)$	2
$(0001, 1110)$	4	$(0100, 0101)$	4	$(0110, 1010)$	$16/3$
$(0010, 0000)$	8	$(0100, 0110)$	4	$(0110, 1011)$	$10/3$
$(0010, 0001)$	$14/3$	$(0100, 0111)$	$11/3$	$(0110, 1100)$	$10/3$
$(0010, 0011)$	4	$(0100, 1000)$	$14/3$	$(0110, 1101)$	$10/3$
$(0010, 0100)$	$14/3$	$(0100, 1001)$	$10/3$	$(0110, 1110)$	4
$(0010, 0101)$	$10/3$	$(0100, 1010)$	$10/3$	$(0111, 0000)$	$40/3$
$(0010, 0110)$	4	$(0100, 1011)$	3	$(0111, 0001)$	$11/3$
$(0010, 0111)$	$10/3$	$(0100, 1100)$	4	$(0111, 0010)$	$10/3$
$(0010, 1000)$	$16/3$	$(0100, 1101)$	3	$(0111, 0011)$	4
$(0010, 1001)$	$10/3$	$(0100, 1110)$	$10/3$	$(0111, 0100)$	$11/3$
$(0010, 1010)$	4	$(0101, 0000)$	$32/3$	$(0111, 0101)$	8
$(0010, 1011)$	3	$(0101, 0001)$	8	$(0111, 0110)$	4
$(0010, 1100)$	$13/3$	$(0101, 0010)$	$14/3$	$(0111, 1000)$	4
$(0010, 1101)$	3	$(0101, 0011)$	$16/3$	$(0111, 1001)$	$13/3$
$(0010, 1110)$	$11/3$	$(0101, 0100)$	8	$(0111, 1010)$	8
$(0011, 0000)$	$32/3$	$(0101, 0110)$	$16/3$	$(0111, 1011)$	$14/3$
$(0011, 0001)$	4	$(0101, 0111)$	4	$(0111, 1100)$	5
$(0011, 0010)$	4	$(0101, 1000)$	8	$(0111, 1101)$	$16/3$
$(0011, 0100)$	$13/3$	$(0101, 1001)$	$16/3$	$(0111, 1110)$	6
$(0011, 0101)$	$16/3$	$(0101, 1010)$	4		

Table 5.2: Critical exponents of words generated by 4-uniform binary morphisms.

f	$E(f^\omega(0))$			k
$(01001, 10110)$	$8/3$	$=$	2.6667	5
$(001011, 001101)$	$14/5$	$=$	2.8	6
$(0100110, 1011001)$	$5/2$	$=$	2.5	7
$(001010011, 011001101)$	$21/8$	$=$	2.625	9
$(001001101, 011001011)$	$11/4$	$=$	2.75	9
$(001001101, 100100101)$	$67/24$	$=$	2.7917	9
$(001001011, 010010011)$	$17/6$	$=$	2.8333	9
$(001001011, 010011011)$	$23/8$	$=$	2.875	9
$(001001101, 001100101)$	$35/12$	$=$	2.9167	9
$(001010011, 001001011)$	$47/16$	$=$	2.9375	9
$(001001011, 001010011)$	$71/24$	$=$	2.9583	9
$(0011001101, 1001011001)$	$23/9$	$=$	2.5556	10
$(0010110011, 0110011001)$	$74/27$	$=$	2.7407	10
$(0010011011, 0010110011)$	$25/9$	$=$	2.7778	10

Table 5.3: Critical exponents in the range $(2, 3)$.

Chapter 6

Critical Exponents in Words Generated by Non-Erasing Morphisms

6.1 Introduction

In this chapter we extend our results from the previous chapter to fixed points of non-erasing morphisms over any finite alphabet. Let $f \in \mathcal{M}(\Sigma_n)$ be a non-erasing morphism, prolongable on 0, and let $\mathbf{w} = f^{\omega}(0)$. We show that if $E(\mathbf{w}) < \infty$, then it is rational for a uniform f, and algebraic of degree at most n for a non-uniform f. More specifically, $E(\mathbf{w}) \in \mathbb{Q}[A(f)]$, where $A(f)$ is the incidence matrix of f, and $\mathbb{Q}[A(f)]$ is the field extension over $\mathbb{Q}$ spanned by the eigenvalues of $A(f)$.

The main tools we use to prove our result are nonnegative matrices and circular D0L languages. In Section 6.2 we prove some preliminary results concerning these tools. In Section 6.3 we state and prove our main theorem. Under certain conditions, our proof implies an algorithm for computing $E(\mathbf{w})$; in Section 6.4 we describe this algorithm. In Section 6.5 we give some examples of applying our theorem. Among other examples, we give a new proof for the result of Mignosi and Pirillo [88] regarding the critical exponent of the Fibonacci word, and prove a generalization of Theorem 5.22: every rational number greater than 2 is a critical exponent of some pure morphic word. We conclude the chapter in Section 6.6 with some open problems.

Most of the results of this chapter have appeared in Krieger [75, 76].

6.2 Preliminary results

6.2.1 The incidence matrix associated with a morphism

Some of the notation in this section was introduced in Section 2.7. In particular, $M_n(\mathbb{N})$ is the set of nonnegative square integer matrices of order n, $\mathbb{Q}[A]$ is the field extension over $\mathbb{Q}$ spanned by the eigenvalues of the matrix $A \in M_n(\mathbb{N})$, and $r(A)$ is the Perron-Frobenius eigenvalue of A.

Proposition 6.1. *Let $A \in M_n(\mathbb{N})$. Then either $r(A) = 0$ or $r(A) \geq 1$.*

Proof. Let $r(A) = r, \lambda_1 \cdots, \lambda_\ell$ be the distinct eigenvalues of A. Suppose that $r < 1$. Then $\lim_{n\to\infty} r^n = \lim_{n\to\infty} \lambda_i^n = 0$ for all $i = 1, \ldots, \ell$. Since $r^n, \lambda_1^n \cdots, \lambda_\ell^n$ are the eigenvalues of A^n, this implies that $\lim_{n\to\infty} A^n = 0$ (the zero matrix). But $A^n \in M_n(\mathbb{N})$ for all $n \in \mathbb{N}$, and the above limit can hold if and only if A is nilpotent, i.e., $r = \lambda_i = 0$ for all $i = 1, \ldots, \ell$. $\square$

Notation 6.1. Let $A \in M_n(\mathbb{R}_{\geq 0})$ be a nonnegative matrix, and let $U = (u_1, \ldots, u_n)^T$ be a nonnegative column vector. We use the following notation:

$$\sum U := \sum_{i=1}^{n} u_i \; ; \tag{6.1}$$

$$I_0(U) := \{1 \leq i \leq n : u_i = 0\} \; ; \tag{6.2}$$

$$\mathbf{I}_0(A, U) := \bigcap_{m \geq 0} I_0(A^m U) \; . \tag{6.3}$$

Let $I \subseteq \{1, 2 \ldots, n\}$, and let $\theta = [i_1, i_2, \ldots, i_n]$ be a permutation of the numbers $\{1, 2, \ldots, n\}$.

- A_I is the $(n-|I|) \times (n-|I|)$ submatrix of A resulting from deleting rows and columns $i \in I$ from A;

- U_I is the $n - |I|$ column vector resulting from deleting entries $i \in I$ from U;
- $P[\theta]$ is a permutation matrix of order n, the rows of which form the permutation θ of the rows of the identity matrix.

Lemma 6.2. *Let $A = (a_{ij})_{1 \le i,j \le n}$ be a nonnegative matrix, and let $U = (u_1, \dots, u_n)^T$ be a nonnegative column vector. Then*

1. *for all $I \subseteq \mathbf{I}_0(A,U)$, and for all $m \ge 0$, we have $\sum A^m U = \sum A_I^m U_I$;*
2. *if $U_1, \dots, U_k$ are nonnegative column vectors of size n and $K = \bigcap_{1 \le \ell \le k} \mathbf{I}_0(A, U_\ell)$, then the eigenvalues of A_K are a subset of the eigenvalues of A.*

Proof. To prove the first assertion, let $\mathbf{I} = \mathbf{I}_0(A,U)$. If $\mathbf{I} = \emptyset$ the result is trivially true. Without loss of generality, suppose that $1 \in \mathbf{I}$. For $m \ge 0$, denote

$$A^m U = U^m = (u_1^m, u_2^m, \dots, u_n^m)^T.$$

By matrix multiplication rules, for all $m \ge 0$ we have

$$A^{m+1}U = A \begin{pmatrix} 0 \\ u_2^m \\ \vdots \\ u_n^m \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} u_2^m + \begin{pmatrix} a_{13} \\ a_{23} \\ \vdots \\ a_{n3} \end{pmatrix} u_3^m + \cdots + \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix} u_n^m = \begin{pmatrix} 0 \\ u_2^{m+1} \\ \vdots \\ u_n^{m+1} \end{pmatrix}.$$

Therefore, necessarily $a_{12}u_2^m + a_{13}u_3^m + \cdots + a_{1n}u_n^m = 0$ for all $m \ge 0$. Since A and U are nonnegative, this implies that $a_{12}u_2^m = a_{13}u_3^m = \cdots = a_{1n}u_n^m = 0$ for all $m \ge 0$. Let $j \notin \mathbf{I}$. Then there exists some $m_0 \ge 0$ such that $u_j^{m_0} \ne 0$. Since $a_{1j}u_j^m = 0$ for all $m \ge 0$, necessarily $a_{1j} = 0$. We get: $a_{1j} = 0$ for all $j \notin \mathbf{I}$, and in general,

$$a_{ij} = 0 \text{ for all } i \in \mathbf{I} \text{ and } j \in \mathbf{I}^c.$$

Here $\mathbf{I}^c$ stands for the complement of $\mathbf{I}$. Therefore, A is reducible: let $[\mathbf{I}^c\,\mathbf{I}]$ be the permutation consisting of the elements of $\mathbf{I}^c$ in increasing order, followed by the elements of $\mathbf{I}$ in increasing order. Let $P = P[\mathbf{I}^c\,\mathbf{I}]$. Then

$$PAP^T = \begin{pmatrix} A_{\mathbf{I}} & C \\ 0 & D \end{pmatrix},$$

where $A_{\mathbf{I}}$ is the $(n-|\mathbf{I}|) \times (n-|\mathbf{I}|)$ matrix resulting from deleting rows and columns $i \in \mathbf{I}$ from A, D is an $|\mathbf{I}| \times |\mathbf{I}|$ matrix, C is an $|\mathbf{I}| \times (n-|\mathbf{I}|)$ matrix, and 0 is an $(n-|\mathbf{I}|) \times |\mathbf{I}|$ zero matrix. Since $u_i = 0$ for all $i \in I$, we get that $PU = (U_{\mathbf{I}}, 0)^T$, where $U_{\mathbf{I}}$ is a column vector of size $n - |\mathbf{I}|$.

Clearly, $\sum A^m U = \sum P(A^m U)$ for all $m \geq 0$. Also,

$$P(A^m U) = (PAP^T)^m (PU) = \begin{pmatrix} A_{\mathbf{I}}^m & C(m) \\ 0 & D^m \end{pmatrix} \begin{pmatrix} U_{\mathbf{I}} \\ 0 \end{pmatrix},$$

where $C(m)$ is an $|\mathbf{I}| \times (n-|\mathbf{I}|)$ matrix. Therefore, $\sum A^m U = \sum A_{\mathbf{I}}^m U_{\mathbf{I}}$ for all $m \geq 0$. Since deleting row and column $i \in \mathbf{I}$ from A is equivalent to deleting one of the last $|\mathbf{I}|$ rows and columns of PAP^T, and these rows and columns do not affect the sum, we get that for all $I \subseteq \mathbf{I}$ and for all $m \geq 0$,

$$\sum A_I^m U_I = \sum A_{\mathbf{I}}^m U_{\mathbf{I}} = \sum A^m U.$$

For the second assertion, let $\mathbf{I}_\ell = \mathbf{I}_0(A, U_\ell)$, $\ell = 1, \ldots, k$. Then $a_{ij} = 0$ for all $1 \leq \ell \leq k$ and for all $i \in \mathbf{I}_\ell$ and $j \in \mathbf{I}_\ell^c$. Let $K = \bigcap_{1 \leq \ell \leq k} \mathbf{I}_\ell$, and let $i \in K$ and $j \in K^c$. Then $i \in \mathbf{I}_\ell$ for all $1 \leq \ell \leq k$, and $j \in \mathbf{I}_\ell^c$ for at least one of $1 \leq \ell \leq k$. Therefore, $a_{ij} = 0$ for all $i \in K$ and $j \in K^c$, and for the permutation matrix $P = P[K^c\ K]$, we get that

$$PAP^T = \begin{pmatrix} A_K & E \\ 0 & F \end{pmatrix}$$

for some $|K| \times |K|$ matrix F and some $|K| \times (n-|K|)$ matrix E. By the block structure of PAP^T, the eigenvalues of A_K form a subset of the eigenvalues of A. $\square$

Example 6.1. Let $A = (a_{ij})_{1 \leq i,j \leq 5}$, let $U = (u_1, \ldots, u_5)^T$, and suppose that $\mathbf{I} = \mathbf{I}_0(A, U) = \{2, 4\}$. Then for all $m \geq 0$,

$$A \begin{pmatrix} u_1^m \\ 0 \\ u_3^m \\ 0 \\ u_5^m \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \\ a_{51} \end{pmatrix} u_1^m + \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \\ a_{43} \\ a_{53} \end{pmatrix} u_3^m + \begin{pmatrix} a_{15} \\ a_{25} \\ a_{35} \\ a_{45} \\ a_{55} \end{pmatrix} u_5^m = \begin{pmatrix} u_1^{m+1} \\ 0 \\ u_3^{m+1} \\ 0 \\ u_5^{m+1} \end{pmatrix},$$

and so necessarily $a_{21} = a_{23} = a_{25} = 0$, and $a_{41} = a_{43} = a_{45} = 0$. Therefore,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & 0 & a_{24} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & a_{42} & 0 & a_{44} & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix} , \quad PAP^T = \begin{pmatrix} a_{11} & a_{13} & a_{15} & a_{12} & a_{14} \\ a_{31} & a_{33} & a_{35} & a_{32} & a_{34} \\ a_{51} & a_{53} & a_{55} & a_{52} & a_{54} \\ 0 & 0 & 0 & a_{22} & a_{24} \\ 0 & 0 & 0 & a_{42} & a_{43} \end{pmatrix} , \quad PU = \begin{pmatrix} u_1 \\ u_3 \\ u_5 \\ 0 \\ 0 \end{pmatrix} ,$$

where $P = P[1, 3, 5, 2, 4]$. The matrix

$$A_{\mathbf{I}} = \begin{pmatrix} a_{11} & a_{13} & a_{15} \\ a_{31} & a_{33} & a_{35} \\ a_{51} & a_{53} & a_{55} \end{pmatrix}$$

is the principal submatrix derived from A by deleting rows $\{2, 4\}$ and columns $\{2, 4\}$. Its eigenvalues are a subset of the eigenvalues of A, and for all subsets $I \subseteq \mathbf{I}$ and for all $m \geq 0$, we have $\sum A_I^m U_I = \sum A_{\mathbf{I}}^m U_{\mathbf{I}} = \sum A^m U$.

Notation 6.2. Let $U = (u_1, \cdots, u_n)^T$, $V = (v_1, \cdots, v_n)^T$ be two column vectors of size n. In the next theorem, and throughout the chapter, we use the following notation:

$$\frac{U}{V} := \frac{\sum U}{\sum V} \ . \tag{6.4}$$

Theorem 6.3. *Let $A \in M_N(\mathbb{N})$ be a matrix with no zero columns, and let $r(A) = r, \lambda_1, \ldots, \lambda_\ell$ be its distinct eigenvalues. Let $U, V, W \in M_{N\times 1}(\mathbb{N})$ be column vectors with $W \neq 0$, and let*

$$\mathcal{A}(m) = \frac{A^m U + (\sum_{i=0}^{m-1} A^i)V}{A^m W} \ , \ \ m \geq 0 \ . \tag{6.5}$$

Then

1. *$\{\mathcal{A}(m)\}_{m\geq 0}$ has finitely many accumulation points.*

2. *if α is a finite accumulation point of $\mathcal{A}$, then $\alpha \in \mathbb{Q}[A] = \mathbb{Q}[r, \lambda_1, \ldots, \lambda_\ell]$. In particular, α is algebraic of degree at most N.*

3. *If A is primitive, then $\mathcal{A}$ has one accumulation point.*

Proof. First, note that $\mathcal{A}(m)$ is well defined for all m: by assumption, A has no zero columns and $W \neq 0$. Since both A and W are nonnegative and integral, this implies that $\sum A^m W \geq 1$ for all $m \geq 1$. Note also that $r \geq 1$ by Proposition 6.1.

First, we claim that we can assume without loss of generality that A satisfies the following conditions:

1. A does not have an eigenvalue λ such that $\lambda \neq r$ and $|\lambda| = r$;
2. $\mathbf{I}_0(A,U) \cap \mathbf{I}_0(A,W) \cap \mathbf{I}_0(A,V) = \emptyset$.

To see that the above assertions hold, assume that A does not satisfy condition (1). By Theorem 2.15, there exists an integer h such that $\lambda^h = r^h$ for all eigenvalues λ that satisfy $|\lambda| = r$. Let

$$\mathbf{A} = A^h, \quad \mathbf{V} = (\sum_{i=0}^{h-1} A^i)V,$$

and for $j = 0, 1, \ldots, h-1$, let

$$U_j = A^j U + (\sum_{i=0}^{j-1} A^i)V, \quad W_j = A^j W, \quad \mathcal{A}_j(m) = \frac{\mathbf{A}^m U_j + (\sum_{i=0}^{m-1} \mathbf{A}^i)\mathbf{V}}{\mathbf{A}^m W_j} .$$

Then

$$\mathcal{A}_j(m) = \frac{A^{mh}(A^j U + (\sum_{i=0}^{j-1} A^i)V) + (\sum_{k=0}^{m-1} A^{kh})(\sum_{i=0}^{h-1} A^i)V}{A^{mh}(A^j W)} = \frac{A^{mh+j}U + (\sum_{i=0}^{mh+j-1} A^i)V}{A^{mh+j}W} = \mathcal{A}(mh+j) .$$

Since the eigenvalues of $\mathbf{A}$ are $r^h, \lambda_1^h, \ldots, \lambda_\ell^h$, we get that $\mathbf{A}$ satisfies condition (1), and $\mathbb{Q}[\mathbf{A}] = \mathbb{Q}[A]$. We have thus split $\mathcal{A}$ into h subsequences, each of which has the same form of (6.5) but with a matrix that satisfies conditions (1), and we can consider each subsequence separately.

Now assume that A does not satisfy condition (2). Let

$$K = \mathbf{I}_0(A,U) \cap \mathbf{I}_0(A,W) \cap \mathbf{I}_0(A,V).$$

By Lemma 6.2, all the sums remain the same if we replace A, U, W, V by A_K, U_K, W_K, V_K. The matrix A_K and the vectors U_K, W_K, V_K are nonnegative and integral, and by Lemma 6.2

the eigenvalues of A_K form a subset of the eigenvalues of A. By induction, we can apply the theorem to A_K, U_K, W_K, V_K, and still get that the finite accumulation points belong to $\mathbb{Q}[A]$.

We will now show that if A satisfies conditions (1) and (2), then $\lim_{m\to\infty} \mathcal{A}(m)$ exists, and is a rational expression of the eigenvalues of A when finite. Note that a primitive matrix always satisfies conditions (1) and (2).

Let J be the Jordan canonical form of A, i.e., $A = SJS^{-1}$, where S is a nonsingular matrix, and J is a diagonal block matrix of Jordan blocks. The convention we use here is to arrange the blocks by the magnitude of the eigenvalues, and within the same eigenvalue, by the order of the block; thus the top-left block of J is the largest block associated with r. We call this block (which may appear more than once) the *dominating Jordan block* of A. We will use the following notation: $J_{\lambda,d}$ is a Jordan block of order d corresponding to eigenvalue λ; $O_{x,y}$ is a square matrix, where all entries are zero, except for x at the top-right corner and y at the two entries of the diagonal just below it; and $O_x = O_{x,0}$.

$$J_{\lambda,d} = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ 0 & & & \lambda \end{pmatrix}_{d\times d}, \quad O_{x,y} = \begin{pmatrix} 0 & \cdots & 0 & y & x \\ 0 & \cdots & 0 & 0 & y \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \quad O_x = \begin{pmatrix} 0 & 0 & \cdots & 0 & x \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Let $J_{r,d}$ be the dominating Jordan block of A. It is easy to verify by induction that

$$J_{r,d}^m = \begin{pmatrix} r^m & \binom{m}{1} r^{m-1} & \binom{m}{2} r^{m-2} & \cdots & \binom{m}{d-2} r^{m-d+2} & \binom{m}{d-1} r^{m-d+1} \\ 0 & r^m & \binom{m}{1} r^{m-1} & \cdots & \binom{m}{d-3} r^{m-d+3} & \binom{m}{d-2} r^{m-d+2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & r^m & \binom{m}{1} r^{m-1} \\ 0 & 0 & 0 & \cdots & 0 & r^m \end{pmatrix}.$$

Thus the first row of $J_{r,d}^m$ has the form

$$r^m \left[1 \quad \frac{m}{r} \quad \frac{m(m-1)}{2!r^2} \quad \cdots \quad \frac{m(m-1)\cdots(m-(d-2))}{(d-1)!r^{d-1}} \right],$$

and so

$$\lim_{m\to\infty} \frac{J_{r,d}^m}{m^{d-1} r^m} = O_\alpha, \quad \text{where } \alpha = \frac{1}{(d-1)!r^{d-1}} \in \mathbb{Q}[A] \;.$$

Now consider the sum $\sum_{n=0}^{m-1} J_{r,d}^n$. The c-th entry of the first row of this sum, where $0 \le c < d$, has the form

$$\sum_{n=c}^{m-1} \binom{n}{c} r^{n-c} \; = \; \frac{1}{r^c} \sum_{n=0}^{m-1} \binom{n}{c} r^n. \tag{6.6}$$

To estimate (6.6), we consider two cases: $r = 1$ and $r > 1$.

Suppose $r = 1$. Using *upper summation* [55, Identity 5.10], we get that

$$\sum_{n=0}^{m-1} \binom{n}{c} = \binom{m}{c+1} = \frac{m^{c+1}}{(c+1)!} + o(m^{c+1}),$$

and so

$$\lim_{m\to\infty} \frac{\sum_{n=0}^{m-1} J_{1,d}^n}{m^{d-1}} = O_{\infty,\beta}, \quad \text{where } \beta = \frac{1}{(d-1)!} \in \mathbb{Q}[A].$$

As for the other Jordan blocks, a block of the form $J_{1,d-1}$ (if it exists) will converge similarly to an O_β block; all other blocks will converge to blocks of zeros. Thus $\lim_{m\to\infty} \mathcal{A}(m)$ depends on the vector V: if V has zero entries at appropriate indices, the limit belongs to $\mathbb{Q}[A]$, otherwise it diverges to ∞.

Now suppose that $r > 1$. To estimate (6.6), we use the following identity:

$$\begin{aligned}\sum_{n=0}^{m-1} \binom{n}{c} r^n (r-1) &= \binom{m-1}{c} r^m - \sum_{n=0}^{m-1} \left[\binom{n}{c} r^n - \binom{n-1}{c} r^n \right] \\ &= \binom{m-1}{c} r^m - \sum_{n=0}^{m-1} \binom{n-1}{c-1} r^n \; = \; \frac{m^c r^m}{c!} + o(m^c r^m).\end{aligned}$$

Therefore,

$$\frac{1}{r^c} \sum_{n=0}^{m-1} \binom{n}{c} r^n = \frac{m^c r^m}{c! r^c (r-1)} + o(m^c r^m),$$

and so,

$$\lim_{m\to\infty} \frac{\sum_{n=0}^{m-1} J_{r,d}^n}{m^{d-1} r^m} = O_\gamma, \quad \text{where } \gamma = \frac{1}{(d-1)! r^{d-1} (r-1)} \in \mathbb{Q}[A] \; .$$

All Jordan blocks other than the dominating block converge to zero blocks. Let

$$J_1 = \lim_{m\to\infty} \frac{J^m}{m^{d-1} r^m} \; , \quad M_1 = S J_1 S^{-1} = \lim_{m\to\infty} \frac{A^m}{m^{d-1} r^m} \; ,$$

$$J_2 = \lim_{m\to\infty} \frac{\sum_{n=0}^{m-1} J^n}{m^{d-1} r^m} \ , \quad M_2 = S J_2 S^{-1} = \lim_{m\to\infty} \frac{\sum_{n=0}^{m-1} A^n}{m^{d-1} r^m} \ .$$

Note that, since S is composed of generalized eigenvectors of A, the matrices S, S^{-1} belong to $M_N(\mathbb{Q}[A])$ as well (recall that a vector v is a *generalized eigenvector* of a matrix A if $v \in \ker(A - \lambda I)^k$ for some eigenvalue λ and $k \in \mathbb{N}$). Therefore, $M_1, M_2 \in M_N(\mathbb{Q}[A])$, and

$$\lim_{m\to\infty} \frac{A^m U + (\sum_{i=0}^{m-1} A^i)V}{A^m W} = \frac{M_1 U + M_2 V}{M_1 W} \in \mathbb{Q}[A]. \tag{6.7}$$

A priori, the above limit might not be defined: since M_1, M_2 could have zero columns, it might be possible that $M_1 U = M_1 W = M_2 V = 0$. To see that this is not the case, let $M_1 = (E_1 \ E_2 \ \cdots \ E_N)$ and $M_2 = (F_1 \ F_2 \ \cdots \ F_N)$, where E_i, F_i are column vectors of size N. Let

$$I = \{1 \le i \le N : E_i \ne 0\} \ , \quad I' = \{1 \le i \le N : F_i \ne 0\} \ .$$

If $r > 1$, then $J_2 = \frac{1}{r-1} J_1$, and so $I = I'$; if $r = 1$, then $I \subseteq I'$.

By matrix multiplication rules,

$$M_1 U = E_1 u_1 + E_2 u_2 + \cdots + E_N u_N \ ,$$

and similarly for $M_1 W$ and $M_2 V$. Since M_1, M_2, U, V, W are nonnegative, $M_1 U = M_1 W = M_2 V = 0$ implies that $u_j = w_j = v_j = 0$ for all $j \in I$, where u_j, w_j, v_j are the entries of the vectors U, W, V, respectively (in addition, $v_j = 0$ for all $j \in I' \setminus I$). Moreover, the vectors $A^m U$, $A^m W$ and $(\sum_{i=0}^{m-1} A^i)V$ must have zero entries at positions $j \in I$ for all $m \ge 0$: otherwise, if for some $m_0 \ge 0$ and $j \in I$ entry j is positive in one of the three vectors, set $U' = A^{m_0} U + (\sum_{i=0}^{m_0-1} A^i)V$, $W' = A^{m_0} W$, and $V' = V$. Then

$$\{\mathcal{A}(m)\}_{m \ge m_0} = \left\{ \frac{A^m U' + (\sum_{i=0}^{m-1} A^i)V'}{A^m W'} \right\}_{m \ge 0} ,$$

and

$$\lim_{m\to\infty} \mathcal{A}(m) = \frac{M_1 U' + M_2 V'}{M_1 W'} \ .$$

Since by assumption either U' or W' must have a non-zero entry at some position $j \in I$, we get by the above that $\mathcal{A}(m)$ has a well defined limit. Therefore, the limit in (6.7) is not defined only if

$$I \subseteq K = \mathbf{I}_0(A,U) \cap \mathbf{I}_0(A,W) \cap \mathbf{I}_0(A,V).$$

But since A satisfies condition (2), $K = \emptyset$, and so $I = \emptyset$. We get that M_1 is the zero matrix, a contradiction. This completes the proof of the theorem.

□

Example 6.2. Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix} = S \begin{pmatrix} 4 & & \\ & 1 & \\ & & 0 \end{pmatrix} S^{-1} \, ; \quad S = \frac{1}{6} \begin{pmatrix} 1 & 2 & -3 \\ 1 & -4 & 3 \\ 1 & 2 & 3 \end{pmatrix} \, , \quad S^{-1} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} .$$

Since A is a primitive matrix and $r(A) = 4 > 1$, the sequence $\mathcal{A}$ converges to a finite limit for any set of vectors U, V, W with $W \neq 0$:

$$M_1 = S \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} S^{-1} = \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix} ; \quad M_2 = S \begin{pmatrix} \frac{1}{3} & & \\ & 0 & \\ & & 0 \end{pmatrix} S^{-1} = \frac{1}{3} \begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix} ;$$

$$\lim_{m\to\infty} \mathcal{A}(m) = \frac{M_1 U + M_2 V}{M_1 W} = \frac{9u_1 + 6u_2 + 3u_3 + 3v_1 + 2v_2 + v_3}{9w_1 + 6w_2 + 3w_3}.$$

6.2.2 Circular D0L languages

Recall that a D0L-system is a triple $G = (\Sigma, f, w)$, where Σ is a finite alphabet, $f \in \mathcal{M}(\Sigma)$ is a morphism, and $w \in \Sigma^+$ is a finite word. If f is non-erasing then G is called a *PD0L system*; in this chapter, when referring to a D0L system, we always mean a PD0L system.

To prove our main theorem, we will need the following fact: if $\mathbf{w} = f^\omega(0)$, and $E(\mathbf{w})$ is bounded, then subwords z of $\mathbf{w}$ satisfying $|f(z)| = |z|$ are of bounded length. To show this fact we need the following definition and lemma.

Definition 6.1. Let f be a morphism over Σ. A letter $a \in \Sigma$ has *rank zero* (with respect to f) if $L(G_a)$ is finite, where G_a is the D0L system (Σ, f, a); a word $x \in \Sigma^*$ has rank zero if it belongs to Σ_0^*, where $\Sigma_0 \subseteq \Sigma$ is the set of rank-zero letters. A D0L system $G = (\Sigma, f, w)$ is *pushy* if $\mathrm{Sub}(L(G))$ contains rank-zero words of unbounded length. If G is not pushy, we set $q(G)$ to be the maximal length of a subword composed of rank-zero letters:

$$q(G) = \max\{|v| : v \in \mathrm{Sub}(L(G)) \cap \Sigma_0^*\}. \tag{6.8}$$

Lemma 6.4 (Ehrenfeucht and Rozenberg [42]). *Let G be a D0L system. Then*

1. *it is decidable whether or not G is pushy;*
2. *if G is pushy, then $Sub(L(G))$ contains unbounded powers;*
3. *If G is not pushy, then $q(G)$ is effectively computable.*

Corollary 6.5. *Let $f : \Sigma^* \to \Sigma^*$ be a non-erasing morphism, prolongable on 0, and let $\mathbf{w} = f^\omega(0)$. Suppose $E(\mathbf{w}) < \infty$. Then there exist a non-erasing morphism $g : \Sigma^* \to \Sigma^*$, prolongable on 0, and an effectively computable number $c \in \mathbb{N}$, such that $\mathbf{w} = g^\omega(0)$, and for all $v \in Sub(\mathbf{w})$ with $|g(v)| = |v|$ we have $|v| \le c$.*

Proof. Let $\Sigma_1 = \{a \in \Sigma : |f(a)| = 1\} - \Sigma_0$. Then there exists $n \in \mathbb{N}$ such that $|f^n(a)| \ge 2$ for all $a \in \Sigma_1$, or $L(G_a)$ would be finite, a contradiction. Let $g = f^n$. Then $\mathbf{w} = g^\omega(0)$, and for all $a \in \Sigma$, if $|g(a)| = 1$ then $a \in \Sigma_0$. Thus, $|g(v)| = |v|$ implies that $v \in \Sigma_0^*$. Since $E(\mathbf{w}) < \infty$, by Lemma 6.4 there exists an effectively computable number $c = q((\Sigma, g, 0))$, such that $|v| \le c$ for all $v \in \mathrm{Sub}(\mathbf{w})$ such that $|g(v)| = |v|$. $\square$

6.3 Algebraicity of $E(\mathbf{w})$ for non-erasing morphisms

In this section we prove our main result, which is the following theorem:

Theorem 6.6. *Let $f \in \mathcal{M}(\Sigma_n)$ be a non-erasing morphism, prolongable on 0, and let $\mathbf{w} = f^\omega(0)$. Let A be the incidence matrix associated with f, and let $\mathbb{Q}[A] = \mathbb{Q}[r, \lambda_1, \cdots, \lambda_\ell]$, where $r, \lambda_1, \cdots, \lambda_\ell$ are the eigenvalues of A. Suppose $E(\mathbf{w}) < \infty$. Then $E(\mathbf{w}) \in \mathbb{Q}[A]$. In particular, $E(\mathbf{w})$ is algebraic of degree at most n.*

Though the details are a bit technical, the essential idea of the proof is rather simple. To describe it, we need a few more definitions. In what follows, $\Sigma = \Sigma_n$ is a finite alphabet; $f \in \mathcal{M}(\Sigma)$ is a non-erasing morphism, prolongable on 0; $\mathbf{w} = f^\omega(0)$; A is the incidence matrix associated with f; and $r, \lambda_1, \ldots, \lambda_\ell$ are the distinct eigenvalues of A, with r the Perron-Frobenius eigenvalue.

The following definition generalizes the left and right stretch from the previous chapter (Definition 5.3).

Definition 6.2. Let $z = w_i \cdots w_j \in \mathrm{Occ}(\mathbf{w})$ be a p/q-power. We say that (z, q) is *left-stretchable* (resp., *right-stretchable*) if the q-period of z can be stretched left (resp., right), i.e., if $w_{i-1} = w_{i+q-1}$ (resp., $w_{j+1} = w_{j-q+1}$). If (z, q) can be stretched left by $c > 0$ letters and no more, then the *left stretch* of (z, q) is defined by $\sigma(z, q) = w_{i-c} \cdots w_{i-1}$; otherwise, if (z, q) is not left-stretchable, then $\sigma(z, q) = \varepsilon$. Similarly, the *right stretch* of (z, q) is given by $\rho(z, q) = w_{j+1} \cdots w_{j+d}$ if (z, q) can be stretched right by exactly $0 < d < \infty$ letters, by $\rho(z, q) = \varepsilon$ if (z, q) is not right-stretchable, and by $\rho(z, q) = (w_m)_{m>j}$ if (z, q) can be stretched right infinitely (i.e., $(w_m)_{m\geq i}$ is periodic with period q). The *stretch vector* of (z, q), denoted by $\Lambda(z, q)$, is the Parikh vector of the left and right stretch combined:

$$\Lambda(z, q) = [\sigma(z, q)\rho(z, q)]. \tag{6.9}$$

If $\rho(z, q) \in \Sigma^\omega$, then $\Lambda(z, q)$ is not defined. Note that the order of stretching (left first or right first) does not matter.

Example 6.3. Let $\Sigma = \{0, 1, 2\}$, let $f = (012, 02, 1) \in \mathcal{M}(\Sigma)$, and let $\mathbf{w} = f^\omega(0) = w_0w_1w_2 \cdots$. Here are the first 24 terms of $\mathbf{w}$:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
w_i	0	1	2	0	2	1	0	1	2	1	0	2	0	1	2	0	2	1	0	2	0	1	2	1

1. The occurrence $z = w_2w_3w_4 = 202$ is a $3/2$-power. $(z, 2)$ is neither left nor right stretchable, since $1 = w_1 \neq w_3 = 0$, and $1 = w_5 \neq w_3 = 0$. The left and right stretch are given by $\sigma(z, 2) = \rho(z, 2) = \varepsilon$. The stretch vector is given by $\Lambda(z, 2) = [\varepsilon\ \varepsilon] = (0, 0, 0)^T$.

2. The occurrence $z' = w_5 \cdots w_9 = 10121$ is a $5/4$-power. $(z', 4)$ is left-stretchable, since $w_4 = w_8 = 2$; it is right-stretchable, since $w_{10} = w_6 = 0$. The left stretch is given by $\sigma(z', 4) = w_4$, since $(z', 4)$ can be stretched left by exactly one character. Similarly, the right stretch is given by $\rho(z', 4) = w_{10}$. The stretch vector is given by $\Lambda(z', q) = [20] = (1, 0, 1)^T$. The $7/4$-power

$$\sigma(z', 4)\ z'\ \rho(z', 4) = w_4 \cdots w_{10} = 2101210$$

is unstretchable.

Outline of proof of Theorem 6.6: Since $E(\mathbf{w})$ is an upper bound, it is enough to consider unstretchable powers when computing it. The idea of the proof is as follows:

1. Take an unstretchable power $z \in \mathrm{Occ}(\mathbf{w})$, apply f to it, and stretch the result to an unstretchable power (Equation (6.10)). Repeat the process to get an infinite sequence of unstretchable powers in $\mathrm{Occ}(\mathbf{w})$ (Equation (6.11)). We refer to such sequences as "π-sequences".

2. Show that the resulting sequence of exponents (which is a sequence of rational numbers) has its lim sup in $\mathbb{Q}[A]$ (Lemma 6.7, Corollary 6.8, Corollary 6.9).

3. Show that every sufficiently long unstretchable power in $\mathrm{Occ}(\mathbf{w})$, that has a sufficiently large exponent, belongs to one of finitely many π-sequences (Lemma 6.10).

Clearly, the three steps above suffice to prove Theorem 6.6: if $E(\mathbf{w})$ is attained by some power $z \in \mathrm{Sub}(\mathbf{w})$, then it is rational; otherwise, there exists a sequence of unstretchable powers $\mathcal{A} = \{z_i\}_{i\geq 0} \subset \mathrm{Occ}(\mathbf{w})$, such that $E(\mathbf{w}) = \lim_{i\to\infty}(z_i)$. Since every sufficiently long element of $\mathcal{A}$ with sufficiently large exponent belongs to one of finitely many π-sequences, there must be an infinite subsequence of $\mathcal{A}$ which belongs to one π-sequence, hence its limit must belong to $\mathbb{Q}[A]$.

We now turn to proving Theorem 6.6 in detail. Let $z = x^{p/q} = w_i \cdots w_j \in \mathrm{Occ}(\mathbf{w})$ be an unstretchable p/q-power. Let $P = [z]$ and $Q = [x]$. In order to keep track of the components of P and Q, we introduce the notation "z is a P/Q-power", where

$$\frac{P}{Q} \;:=\; \frac{\sum_{0\leq \ell<n} |z|_\ell}{\sum_{0\leq \ell<n} |x|_\ell} \;=\; \frac{p}{q}\,.$$

Let $i' = |f(w_0 \cdots w_{i-1})|$, $j' = |f(w_0 \cdots w_j)| - 1$, and consider the occurrence $f(z) = w_{i'} \cdots w_{j'} \in \mathrm{Occ}(\mathbf{w})$. Recall that by Proposition 2.11, $[f(z)] = AP$ and $[f(x)] = AQ$; thus under this notation, $f(z)$ is an AP/AQ-power. This power can be stretched by $\sigma(f(z), AQ)$ on the left and $\rho(f(z), AQ)$ on the right; the result (provided that ρ is finite) is an unstretchable $(AP + \Lambda)/AQ$-power, where $\Lambda = \Lambda(f(z), AQ)$ is the stretch vector of $(f(z), AQ)$. Let us define a map $\pi : \mathrm{Occ}(\mathbf{w}) \times \mathbb{Q} \;\to\; \mathrm{Occ}(\mathbf{w}) \times \mathbb{Q}$ by

$$\pi\left(z, \frac{P}{Q}\right) = \left(\sigma f(z)\rho, \frac{AP + \Lambda}{AQ}\right)\,. \tag{6.10}$$

Here $\sigma = \sigma(f(z), AQ)$, $\rho = \rho(f(z), AQ)$, and $\Lambda = \Lambda(f(z), AQ)$. Note that this definition is a generalization of the uniform binary "apply f and stretch" map, defined in (5.1). In what follows, we use $\pi(z)$ and $\pi(P/Q)$ to denote the first and second component, respectively (this is only a shorthand: when we talk of "$\pi(z)$" it should be understood that z is a P/Q-power, and similarly, when we talk of "$\pi(P/Q)$", it should be understood that P is the Parikh vector of an occurrence (z, i, j) of $\mathbf{w}$).

Iterating π on an initial unstretchable P/Q-power z, we get a sequence of unstretchable powers, $\{\pi^m(z, P/Q)\}_{m\geq 0}$. We refer to such a sequence as a *π-sequence*. A π-sequence satisfies

$$\pi^m\left(\frac{P}{Q}\right) = \frac{A^m P + \sum_{i=0}^{m-1} A^{m-1-i}\Lambda_i}{A^m Q}\ , \tag{6.11}$$

where Λ_m is the stretch vector we get at iteration m. The sequence $\{\Lambda_m\}_{m\geq 0}$ is *the stretch sequence associated with the π-sequence*. Our aim now is to show that for any π-sequence, the corresponding stretch sequence is ultimately periodic. This will enable us to reduce (6.11) to an expression of the form of (6.5), thus enabling us to apply Theorem 6.3 and show that the π-sequence has its lim sup in $\mathbb{Q}[A]$.

Definition 6.3. Let $z = w_i \cdots w_j \in \mathrm{Occ}(\mathbf{w})$ be an unstretchable P/Q-power. The *left context* of (z, Q) with respect to f, denoted by $\varsigma(z, Q)$, is the shortest occurrence $z_L \in \mathrm{Occ}(\mathbf{w})$ to the left of z such that $f(z_L)$ contains the left stretch $\sigma(f(z), FQ)$. Similarly, the *right context*, denoted by $\varrho(z, Q)$, is the shortest occurrence $z_R \in \mathrm{Occ}(\mathbf{w})$ to the right of z such that $f(z_R)$ contains the right stretch $\rho(f(z), FQ)$.

More formally, let $z_L = w_{i-c} \cdots w_{i-1}$ and $z'_L = w_{i+q-d} \cdots w_{i+q-1}$, where c, d are the minimal nonnegative integers such that $f(z_L)$ and $f(z'_L)$ are incomparable in the suffix order (i.e., neither one is a suffix of the other). If these integers do not exist (i.e., $f(w_0 \cdots w_{i-1})$ and $f(w_{i+q-d} \cdots w_{i+q-1})$ are comparable for some d), then set $c = 0$. If $i = 0$, set $z_L = z'_L = \varepsilon$. Similarly, let $z_R = w_{j+1} \cdots w_{j+r}$, $z'_R = w_{j-q+1} \cdots w_{j-q+s}$, where r, s are the minimal nonnegative integers such that $f(z_R)$ and $f(z'_R)$ are incomparable in the prefix order; if these integers do not exist, then $z_R = (w_m)_{m>j}$. Then $\varsigma(z, Q) = z_L$ and $\varrho(z, Q) = z_R$. Note that if $E(\mathbf{w}) < \infty$, then $\varrho(z, Q)$ is always finite.

Example 6.4. Continuing Example 6.3, the unstretchable 3/2-power $z = w_2 w_3 w_4 = 202$ is mapped by f to the 5/4-power $f(z) = w_5 \cdots w_9 = 10121$, which can be stretched left and right by $\sigma(f(z),4) = w_4 = 2$ and $\rho(f(z),4) = w_{10} = 0$. Now, w_4 is a suffix of $f(w_1) = w_3w_4 = 02$; w_{10} is a prefix of $f(w_5) = w_{10}w_{11} = 02$. Therefore, the left context of $(z,2)$ with respect to f is given by $\varsigma(z,2) = w_1$, and the right context of $(z,2)$ with respect to f is given by is $\varrho(z,2) = w_5$.

Lemma 6.7. *Suppose $E(\mathbf{w}) < \infty$. Then there exists a constant $C = C(\mathbf{w})$, such that for any sufficiently long unstretchable P/Q-power $z = w_i \cdots w_j \in Occ(\mathbf{w})$, we have*

1. *$C \geq |\varsigma(z,Q)|$ and $|f(w_{i-C} \cdots w_{i-1})| \geq |\sigma(f(z),FQ)| + C$;*
2. *$C \geq |\varrho(z,Q)|$ and $|f(w_{j+1} \cdots w_{j+C})| \geq |\rho(f(z),FQ)| + C$.*

In other words, $w_{i-C} \cdots w_{i-1}$ contains the left context of (z,Q), and $f(w_{i-C} \cdots w_{i-1})$ contains both the left stretch and the left context of $(f(z),FQ)$; similarly, $w_{j+1} \cdots w_{j+C}$ contains the right context of (z,Q), and $f(w_{j+1} \cdots w_{j+C})$ contains both the right stretch and the right context of $(f(z),FQ)$.

Proof. We prove the lemma for the right stretch. The proof for the left stretch is similar.

Let $\varrho = \varrho(z,Q)$, $\rho = \rho(f(z),FQ)$. Since $E(\mathbf{w}) < \infty$, ϱ is finite. Let z_R, z'_R be as in Definition 6.3. By definition, ρ is the longest common prefix of $f(z_R)$, $f(z'_R)$; this prefix is strictly shorter than both $f(z_R)$, $f(z'_R)$, since $f(z_R)$ and $f(z'_R)$ are incomparable in the prefix order.

Since $z_R \neq z'_R$, we get that $f(z_R)$ and $f(z'_R)$ constitute two different interpretations of ρ by the D0L system $(\Sigma, f, 0)$. Since $\mathbf{w}$ is circular (Theorem 3.1), these interpretations must synchronize at a distance of at most D from the edges of ρ, where D is the synchronization delay (see Definition 3.1). Let $M = \max\{|f(a)| : a \in \Sigma\}$, and suppose $|\rho| \geq 2D + M$. Then $f(z_R)$ and $f(z'_R)$ synchronize, and so z_R and z'_R have the following decomposition:

$$\begin{aligned} z_R &= xuy, \;\; x \in \Sigma^+, \;\; u \in \Sigma^+, \; y \in \Sigma^*, \\ z'_R &= x'uy', \; x' \in \Sigma^+, \; u \in \Sigma^+, \; y' \in \Sigma^*, \end{aligned}$$

which satisfies

$$x \neq x', \;\; f(x) = f(x'), \;\; \rho = f(x)f(u)v, \;\; |f(x)| < D, \;\; |v| < D.$$

Here v is the longest common prefix of $f(y)$ and $f(y')$. The picture is illustrated in Fig. 6.1.

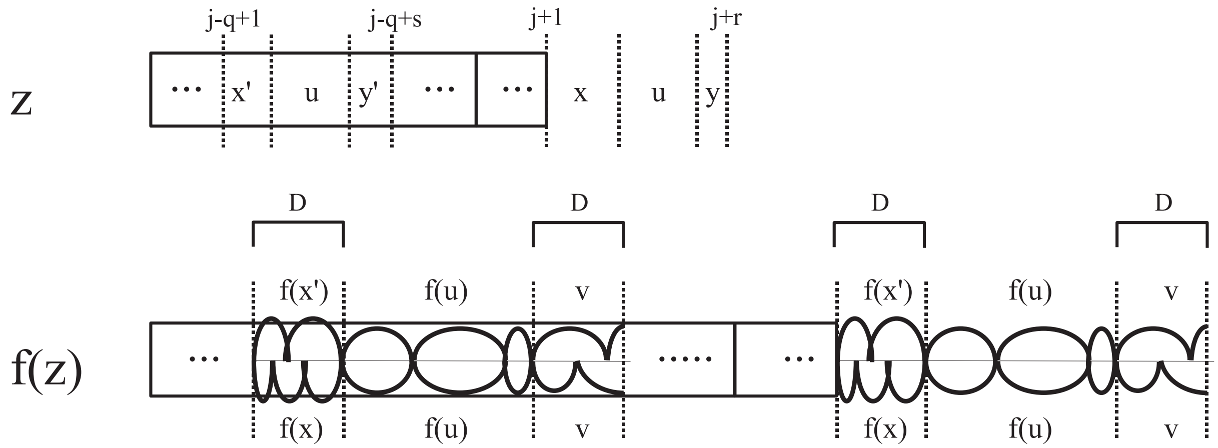

Figure 6.1: Two interpretations of $\rho(f(z), FQ)$.

We can assume that $f(z)$ contains at least D positions to the left of $f(x')$: since $E(\mathbf{w}) < \infty$, by Lemma 6.4 f is not pushy. Therefore, the power block gets bigger with every application of f. By applying π finitely many times, we get an unstretchable power with D positions to the left of $f(x')$. Therefore, for any sufficiently long power z, we can consider the FQ period of $\pi(z)$ starting D positions to the left of $f(x')$. Suppose $|f(u)v| \geq D$. Then relative to this starting point, $f(x')$ is at distance D from the edges, and should therefore synchronize with $f(x)$. But $x \neq x'$, a contradiction. Therefore, we must have $|f(u)v| < D$, and $|f(x)f(u)v| < 2D$. Since f is non-erasing, this implies that $|xuy| = |z_R| < 2D$, and the same holds for z'_R.

Now recall that by Corollary 6.5, $E(\mathbf{w}) < \infty$ implies that there exists a number $c \in \mathbb{N}$, such that every word $u \in \mathrm{Sub}(\mathbf{w})$ with $|u| > c$ must contain at least one letter a with $|f(a)| \geq 2$. Therefore, for every $k \in \mathbb{N}$ and $u \in \mathrm{Sub}(\mathbf{w})$, we have:

$$|u| \geq k(c+1) \Rightarrow |f(u)| \geq k + k(c+1) \geq k + |u| \ .$$

Let $C = (2D + M)(c + 1)$. Then $C > |\varrho(z, Q)|$, and

$$|f(w_{j+1} \cdots w_{j+C})| \geq 2D + M + C > |\rho(f(z), FQ)| + C \ .$$

$\square$

Corollary 6.8. *Suppose that $E(\mathbf{w}) < \infty$. Let $z \in Occ(\mathbf{w})$ be an unstretchable P/Q-power, let $\{\pi^m(z, P/Q)\}_{m\geq 0}$ be the π-sequence generated by z, and let $\mathcal{S} = \{\Lambda_m\}_{m\geq 0}$ be the associated stretch sequence. Then $\mathcal{S}$ is ultimately periodic.*

Proof. Let $z_m = \pi^m(z) = w_{i_m} \cdots w_{j_m} \in \mathrm{Occ}(\mathbf{w})$, and let $q_m = A^m Q$ be the size of the power block of z_m. Let $\{\rho_m\}_{m\geq 0}$, $\{\varrho_m\}_{m\geq 0}$ be the sequences of right stretches and right contexts, respectively, where $\rho_0 = \varepsilon$ and $\varrho_0 = \varrho(z, Q)$. Let C be the minimal constant which satisfies the conditions of Lemma 6.7; if the first elements of $\{\pi^m(z, P/Q)\}_{m\geq 0}$ are too short to satisfy Lemma 6.7, we simply discard them. Let $x_0 = w_{j_0+1} \cdots w_{j_0+C}$. Then by Lemma 6.7, $f(x_0)$ contains both the next stretch ρ_1 and the next context ϱ_1.

Now let $x_1 = w_{j_1+1} \cdots w_{j_1+C}$. Note that $f(x_0)$ contains x_1, since $|f(x_0)| \geq |\rho_1| + |x_1|$. Similar arguments show that $f(x_1)$ contains the next stretch ρ_2 and the next context ϱ_2. Continuing this way, we get a sequence of occurrences, $(x_m)_{m\geq 0} \subseteq \mathrm{Occ}(\mathbf{w})$, where for all m,

- $|x_m| = C$;
- $f(x_m) \supset x_{m+1}$;
- $f(x_m) \supset \rho_{m+1}\varrho_{m+1}$.

Thus the sequence $(x_m)_{m\geq 0}$ must be ultimately periodic: The word x_{m+1} depends only on the combination of x_m and the suffix of length at most C of the q_m-block of z_m. Since there is only a finite number of different words of length C, we get a period once a combination is repeated. This, in turn, implies that $\{\rho_m\}_{m\geq 0}$ is ultimately periodic. Similar arguments show that the sequence of left stretches is ultimately periodic as well. Put together, we get that $\mathcal{S}$ is ultimately periodic. □

Example 6.5. Continuing Example 6.4, the morphism $f = (012, 02, 1) \in \mathcal{M}(\Sigma_3)$ is bifix (recall that a morphism is bifix if $f(a)$ is neither a prefix nor a suffix of $f(b)$ for all $a \neq b \in \Sigma$). Therefore, if $z = w_i \cdots w_j \in \mathrm{Occ}(\mathbf{w})$ is an unstretchable P/Q-power, necessarily $|\varsigma(z, Q)| = |\varrho(z, Q)| = 1$, that is, $\varsigma(z, Q) = w_{i-1}$ and $\varrho(z, Q) = w_{j+1}$.

Consider the stretch of $(f(z), FQ)$: suppose that $w_{i-1} = 2$. Since (z, Q) is unstretchable, $w_{i+q-1} \neq 2$, therefore $f(w_{i-1})$ and $f(w_{i+q-1})$ have no common suffix, and $(f(z), FQ)$

cannot be stretched left. We get that $\varsigma(f(z), FQ) = f(w_{i-1})$. Now suppose that $w_{i-1} \neq 2$. Then $|f(w_{i-1})| \geq 1$. Since for any pair of letters $a \neq b \in \{0, 1, 2\}$ the words $f(a)$ and $f(b)$ have at most one letter as a common suffix, necessarily $|\sigma(f(z), FQ)| \leq 1$, and so $f(w_{i-1})$ contains both $\sigma(f(z), FQ)$ and $\varsigma(f(z), FQ)$. Similar reasoning shows that $f(w_{j+1})$ contains both $\rho(f(z), FQ)$ and $\varrho(f(z), FQ)$. Thus in this case, $C = 1$.

Here is a specific stretch sequence. Recall the first 24 terms of $\mathbf{w}$:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
w_i	0	1	2	0	2	1	0	1	2	1	0	2	0	1	2	0	2	1	0	2	0	1	2	1

Let $z_0 = w_2 w_3 w_4 = 202$ be the initial power of a π-sequence. As we have seen, the left and right contexts of $(z_0, 2)$ are w_1 and w_5 respectively; $\pi(z_0, 3/2) = (z_1, 7/4)$, where $z_1 = w_4 \cdots w_{10} = 2101210$; and the stretch vector of $(f(z_0), 2)$ is $\Lambda_0 = (1, 0, 1)^T$. The left and right contexts of z_1 are w_3 and w_{11}, respectively, since $f(w_3)$ and $f(w_{11})$ contain the left and right stretch of $(f(z_1), 4)$, respectively:

$$f(0\ \ 2101|210\ \ 2) = 012\ \ 10201202|102012\ \ 1 = 01\ \ 21020120|2102012\ \ 1.$$

Therefore, $\pi(z_1, 7/4) = (z_2, 15/8)$, where $z_2 = w_8 \cdots w_{22} = (21020120)^{15/8}$. The stretch vector of $(f(z_1), 8))$ is $\Lambda_1 = (0, 0, 1)^T$. The left and right contexts of $(z_2, 8)$ are w_7 and w_{23}, respectively. Since $w_7 = w_1 = 1$ and $w_{23} = w_5 = 1$, we get a periodic sequence of contexts with period 2; thus the stretch sequence is periodic with period 2:

$$\{\Lambda_m\}_{m \geq 0} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ldots$$

Corollary 6.9. *Suppose that $E(\mathbf{w}) < \infty$. Let $z = w_i \cdots w_j \in Occ(\mathbf{w})$ be an unstretchable P/Q- power. Then* $\limsup_{m \to \infty} \pi^m(P/Q) \in \mathbb{Q}[A]$.

Proof. Let $\mathcal{S} = \{\Lambda_m\}_{m \geq 0}$ be the sequence of stretch vectors associated with the π-sequence generated by z. By the previous lemma, $\mathcal{S}$ is ultimately periodic; without loss of generality, we can assume it is purely periodic. Let h be the period, and let

$$\mathbf{A} = A^h\ , \quad \mathbf{\Lambda} = \sum_{i=0}^{h-1} A^{h-1-i} \Lambda_i\ .$$

Equation (6.11) now implies that

$$\pi^{mh}\left(\frac{P}{Q}\right) = \frac{\mathbf{A}^m P + (\sum_{i=0}^{m-1}\mathbf{A}^i)\mathbf{\Lambda}}{\mathbf{A}^m Q} \ .$$

Note that since $\mathbf{A}$ is associated with a non-erasing morphism (namely, f^h) it has no zero columns, and that Q is a non-zero vector. Also, the eigenvalues of $\mathbf{A}$ are given by $r^h, \lambda_1^h, \ldots, \lambda_\ell^h \in \mathbb{Q}[A]$. Therefore, by Theorem 6.3, we get that $\{\pi^{mh}(P/Q)\}_{m\geq 0}$ has finitely many limit points, all in $\mathbb{Q}[A]$. In particular,

$$\limsup_{m\to\infty} \pi^{mh}\left(\frac{P}{Q}\right) \in \mathbb{Q}[A] \ .$$

Similar reasoning shows that for $1 \leq k \leq h-1$, the subsequence $\{\pi^{mh+k}(P/Q)\}_{m\geq 0}$ has its $\limsup$ (as well as its other limit points) in $\mathbb{Q}[A]$. We have thus partitioned $\{\pi^m(P/Q)\}_{m\geq 0}$ into h subsequences, each of which has its $\limsup$ in $\mathbb{Q}[A]$. The result follows. □

Example 6.6. Continuing Example 6.5, we have seen that the π-sequence generated by the power $z = w_2w_3w_4 = 202$ has a purely periodic stretch sequence with period $h = 2$. Therefore, the π-sequence can be partitioned into two subsequences, $\pi^{2m}(z, Q)$ and $\pi^{2m+1}(z, Q) = \pi^{2m}(\pi(z, Q))$. Let A be the incidence matrix of $f = (012, 02, 1)$. Then

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad ; \quad A^2 = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix} \ .$$

Let $\mathbf{A} = A^2$, $\mathbf{\Lambda} = A\Lambda_0 + \Lambda_1 = (1, 2, 2)^T$, $P = [202] = (1, 0, 2)^T$, $Q = [20] = (1, 0, 1)^T$. By Example 6.2,

$$\lim_{m\to\infty} \pi^{2m}\left(202, \frac{P}{Q}\right) = \lim_{m\to\infty} \frac{\mathbf{A}^m P + (\sum_{i=0}^{m-1}\mathbf{A}^i)\mathbf{\Lambda}}{\mathbf{A}^m Q} =$$

$$\frac{9\cdot 1 + 6\cdot 0 + 3\cdot 2 + 3\cdot 1 + 2\cdot 2 + 2}{9\cdot 1 + 6\cdot 0 + 3\cdot 1} = 2.$$

To compute $\lim_{m\to\infty} \pi^{2m+1}(z, P/Q)$, let

$$\left(z', \frac{P'}{Q'}\right) = \pi\left(202, \frac{P}{Q}\right) = \left(2101210, \frac{AP + \Lambda_0}{AQ}\right) = \left(2101210, \frac{(2,3,2)^T}{(1,2,1)^T}\right).$$

The stretch sequence is given by $\mathbf{\Lambda}' = A\Lambda_1 + \Lambda_0 = (1,1,1)^T$, and altogether we get:

$$\lim_{m\to\infty} \pi^{2m+1}\left(202, \frac{P}{Q}\right) = \lim_{m\to\infty} \pi^{2m}\left(2101210, \frac{AP+\Lambda_0}{AQ}\right) = \frac{9\cdot 2+6\cdot 3+3\cdot 2+3\cdot 1+2\cdot 1+1}{9\cdot 1+6\cdot 2+3\cdot 1} = 2.$$

Corollary 6.9 completes step 2 of the proof of Theorem 6.6. The next lemma proves the last step.

Lemma 6.10. *Suppose $E(\mathbf{w}) < \infty$. Then*

1. *there exists a constant $K = K(\mathbf{w}) \in \mathbb{N}$, such that every unstretchable p/q-power $z \in Occ(\mathbf{w})$ that satisfies $|z| \geq K$ and $p/q \geq 2$ is an image under the π map;*
2. *$Occ(\mathbf{w})$ contains only finitely many different sequences of the form $\{\pi^m(z, P/Q)\}_{m\geq 0}$, where $|z| < K$;*
3. *Out of the sequences $\{\pi^m(z, P/Q)\}_{m\geq 0}$, where z is a p/q-power with $p/q < 2$, only finitely many (if any) can attain the critical exponent or have it as a limit point.*

Proof. Let $e = \lfloor E(\mathbf{w})\rfloor + 1$. Then $\mathbf{w}$ is e-power-free, and hence circular. Let D be the synchronization delay, let $M = \max\{D, \{|f(a)| : a \in \Sigma\}\}$, and let $K = e(2D+M)$. Let $z = w_i \cdots w_j \in \mathrm{Occ}(\mathbf{w})$ be an unstretchable p/q power, and suppose $|z| \geq K$. Since $\mathbf{w}$ is e-power-free, we get that $e(2D+M) \leq |z| < eq$, i.e., $q > 2D+M$. Therefore, all the interpretations of a power block must synchronize at distance D from the edges, and so all power blocks have an unambiguous decomposition into images under f (Fig. 6.2).

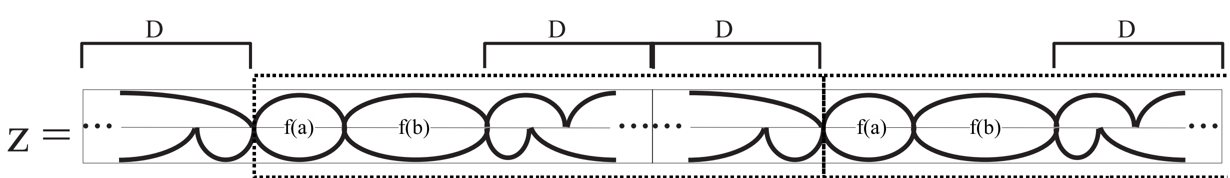

Figure 6.2: Interpretation of power blocks.

Let $n = \lfloor p/q \rfloor$, $\ell = p \mod q$. Then z is an $(n + \ell/q)$-power, that contains n full power blocks. Let $R = 1 + \frac{2D+M}{q} < 2$, and suppose that $p/q \geq R$. Then either $n \geq 2$, or $n = 1$ and $\ell \geq 2D + M$. Shifting by at most D positions to the right (the dashed line in Fig. 6.2), we get a p'/q-power, z', whose power block is an exact image under f.

Since $M \geq D$, the suffix of length D of the power block of z is at distance D from both edges of the power block of z', and must have an unambiguous decomposition as well. Therefore, the suffixes of length D of blocks $1, 2, \ldots, n-1$ of z (and also block n, if $\ell \geq D$) have the same interpretation. Similarly, by starting from the right end of z and shifting by at most D positions to the left, we get that the prefixes of length D of blocks $2, 3, \ldots, n$ has an unambiguous decomposition. Define z' to be the largest suboccurrence of z that has a unique exact decomposition under f. Then every power block of z' has an identical exact decomposition. We get that $z' = f(z'')$, where z'' is an unstretchable r/s-power, and $\lfloor r/s \rfloor = \lfloor p'/q \rfloor$. Fig. 6.3 illustrates the shift.

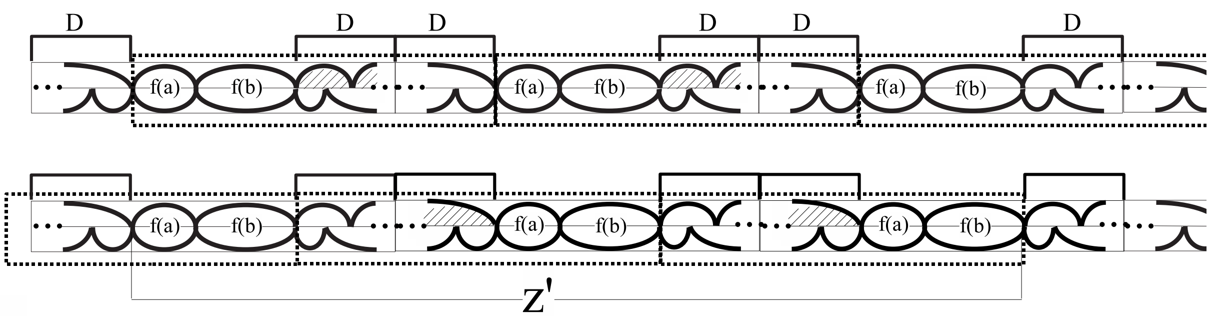

Figure 6.3: Shifting by D to the right (top row) or to the left (bottom row), shows that prefixes and suffixes of power blocks have the same interpretation.

Since $\ell \geq 2D + M$, if $n = 1$ necessarily $r/s > 1$: to illustrate, observe that in Fig. 6.3, the second power block of z' must include at least $f(a)$. Thus z'' has exponent greater than 1, and we can apply π. Now, $f(z'')$ can be stretched by π to a unique unstretchable power. On the other hand, $f(z'') = z'$, and z' can be stretched to the unstretchable power z. Therefore, $z = \pi(z'')$.

For the second assertion, recall that by Lemma 6.7, a sequence of the form $\{\pi^m(z, P/Q)\}_{m\geq 0}$ is completely determined by z and by the C letters to the left and to the right of z. Thus every sequence is generated by a word of length at most $K+2C$, and there are only finitely many such words in $\mathrm{Sub}(\mathbf{w})$.

It remains to consider powers with exponent smaller than 2, such that the second power block is too short to be synchronized (recall Definition 3.1: a word $u \in \mathrm{Sub}(\mathbf{w})$ is synchronized if any two interpretations of it are synchronized). Such powers have the form $z = xyx$, where $|xy| = q$, $|xyx| = p$, xy is synchronized, and the two occurrences of x have different ancestors. But such powers have exponent tending to 1, since y can grow arbitrarily large while $|x| < 2D+M$. Since there is only a finite number of different stretch sequences, there can be only a finite number of such powers whose π-sequences attain the critical exponent or have it as a limit point. □

Lemma 6.10 completes the proof of Theorem 6.6 as was outlined in the beginning of this section.

Definition 6.4. An *initial power* is an unstretchable power that is not an image under the π map, that is, its power blocks are not synchronized. Note that if $\mathbf{w}$ is circular then the set of initial powers occurring in $\mathbf{w}$ must be finite.

Definition 6.5. A *half-synchronized power* is a p/q-power of the form xyx, where $|xy| = q$, $|xyx| = p$, xy is synchronized, and x is unsynchronized.

Example 6.7. Continuing Example 6.6, let us check what block sizes we need to consider for initial powers of π-sequences. Recall that $f = (012, 02, 1)$. It is easy to check that $aa \notin \mathrm{Sub}(\mathbf{w})$ for all letters $a \in \{0, 1, 2\}$; thus the block size must be at least 2. We can also verify that $(ab)^2 \notin \mathrm{Sub}(\mathbf{w})$ for all letters $a, b \in \{0, 1, 2\}$, and so a power with block size 2 must be a $\frac{3}{2}$-power of the form aba, where $a \neq b \in \{0, 1, 2\}$. The only words of this form in $\mathrm{Sub}(\mathbf{w})$ are (in order of appearance) 202, 101, 121, and 020.

Now consider powers with block size 3. The set of subwords of $\mathbf{w}$ of length 3 is given by

$$\mathrm{Sub}(\mathbf{w}) \cap \{0, 1, 2\}^3 = \{012, 120, 202, 021, 210, 101, 121, 102, 020, 201\} .$$

A short case analysis shows that these words have the following synchronization points:

$$\{|012|, 12|0, 2|02|, |02|1|, 2|1|0, |1|01, 12|1|, |1|02|, |02|0, 2|01\} .$$

The only words that have less than two synchronization points are 120 and 201. However, 120 must be followed by 2, since 1201 implies the word $012|012 = f(00)$, a contradiction. Therefore, 120 cannot be a power block. The word 201 must be followed by 2, to get $2|012|$; the word 2012 can then be followed by $f(1) = 02$, and so we get the 5/3-power $2|01\ 2|0$, whose blocks are not synchronized. All half-synchronized powers have exponent smaller than 2 by definition. We get that all initial powers are of the form $z = xyx$, where $q = |xy|$ and x, y are nonempty.

Let $z = xyx \in \mathrm{Occ}(\mathbf{w})$ be an unstretchable $(q+\ell)/q$-power, where $\ell = |x|$ and $q = |xy|$. Observe for any unstretchable power $z \in \mathrm{Occ}(\mathbf{w})$, $f(z)$ can be stretched by at most one letter to the left (the common suffix of $f(0)$ and $f(1)$) and one letter to the right (the common prefix of $f(0)$ and $f(1)$). Let $\sigma = \sigma(f(z), |f(xy)|)$, $\rho = \rho(f(z), |f(xy)|)$. Then

$$\pi\left(xyx, 1 + \frac{\ell}{q}\right) = \left(x'y'x', 1 + \frac{|f(x)| + c}{|f(x)| + |f(y)|}\right) ,$$

where $c \leq 2$, $x' = \sigma f(x)\rho$, and $y' = \rho^{-1} f(y)\sigma^{-1}$. If $\rho \neq \varepsilon$, necessarily $\rho = 0$ and $f(y)$ begins with 02 or 012; if $\sigma \neq \varepsilon$, necessarily $\sigma = 2$ and $f(y)$ ends with 02 or 012. We get that $\pi((q+\ell)/q) \geq 2$ if and only if $y' = \varepsilon$, if and only if $f(y) = 02$, $\rho = 0$, and $\sigma = 2$. This implies that $y = 1$, and $z = x1x$ occurs in the context $0x1x0$. But such a word never occurs in $\mathbf{w}$: let $x = x_1 \cdots x_k$, and suppose $0x_1 \cdots x_k 1 x_1 \cdots x_k 0 \in \mathrm{Sub}(\mathbf{w})$. Since $00, 11 \notin \mathrm{Sub}(\mathbf{w})$, necessarily $x_1 = 2$. But then $x_k = 0$, since this is the only letter that can precede 12, and we get that $00 \in \mathrm{Sub}(\mathbf{w})$, a contradiction.

We conclude that a π-sequence generated by a p/q-power with $p/q < 2$ consists only of powers strictly smaller than 2. Since all the initial powers have exponents smaller than 2, and since 2 is a limit point of at least one π-sequence (as we saw in Example 6.6), we get that $E(\mathbf{w}) = 2$, and the bound is not attained.

The result that $E(\mathbf{w}) = 2$ is well known: $\mathbf{w}$ can be also characterized as the sequence of differences between two consecutive 1's in the Thue-Morse word $\mathbf{t}$ (see e.g., [6, Theorem 1.6.2]). This characterization of $\mathbf{w}$ implies that $E(\mathbf{w}) = 2$ and the bound is not attained: since $\mathbf{t}$ is overlap-free, $\mathbf{w}$ is square-free; since $\mathbf{t}$ contains arbitrarily large squares, $\mathbf{w}$ contains powers arbitrarily close to 2. We have just shown how to compute $E(\mathbf{w})$ independently of $\mathbf{t}$'s properties.

6.3.1 The uniform case

When f is a k-uniform morphism, the π-sequences have a simpler form: if z is a p/q-power, then $f(z)$ is a kp/kq-power, and the vector notation we used in the general case is unnecessary. Let $\sigma = \sigma(f(z), kq)$, $\rho = \rho(f(z), kq)$, and let $\lambda = \lambda(f(z), kq) = |\sigma| + |\rho|$ be the stretch size. The π map now has the form:

$$\pi\left(z, \frac{p}{q}\right) = \left(\sigma f(z)\rho, \frac{kp+\lambda}{kq}\right) ,$$

and

$$\pi^m\left(\frac{p}{q}\right) = \frac{k^m p + \sum_{i=0}^{m-1} k^{m-1-i}\lambda_i}{k^m q} .$$

As in the non-uniform case, when applying π successively we get an ultimately periodic sequence of stretch sizes. Without loss of generality, we can assume the stretch sequence is purely periodic; otherwise, we discard the first elements of the sequence, and start from the periodic part. Let $\lambda_0, \lambda_1, \ldots, \lambda_{h-1}$ be the period, and let $K = k^h$. For $j = 0, 1, \ldots, h-1$, define

$$\begin{aligned}
\overline{\lambda}_j &= k^{h-1}\lambda_j + k^{h-2}\lambda_{j+1} + \cdots + k^j\lambda_{h-1} + k^{j-1}\lambda_0 + \cdots + k\lambda_{j-2} + \lambda_{j-1},\\
p_j &= k^j p + k^{j-1}\lambda_0 + k^{j-2}\lambda_1 + \cdots + k\lambda_{j-2} + \lambda_{j-1},\\
q_j &= k^j q.
\end{aligned}$$

Then

$$\begin{aligned}
\pi^{mh+j}\left(\frac{p}{q}\right) &= \frac{k^{mh+j}p + \sum_{i=0}^{mh+j-1} k^{mh+j-1-i}\lambda_i}{k^{mh+j}q}\\
&= \frac{k^{mh}(k^j p + \sum_{i=0}^{j-1} k^{j-1-i}\lambda_i) + (\sum_{i=0}^{m-1} k^{ih})(k^{h-1}\lambda_j + k^{h-2}\lambda_{j+1} + \cdots + k\lambda_{j-2} + \lambda_{j-1})}{k^{mh}k^j q}\\
&= \frac{K^m p_j + \overline{\lambda}_j \sum_{i=0}^{m-1} K^i}{K^m q_j}.
\end{aligned}$$

As in the uniform binary case (see equation (5.2)), the sequence $\pi^{mh+j}(p/q)$ is increasing for all $0 \le j < h$, and converges to a rational limit as m tends to infinity:

$$\lim_{m\to\infty} \pi^{mh+j}\left(\frac{p}{q}\right) = \lim_{m\to\infty} \frac{K^m p_j + \overline{\lambda}_j \sum_{i=0}^{m-1} K^i}{K^m q_j} = \frac{(K-1)p_j + \overline{\lambda}_j}{(K-1)q_j} \in \mathbb{Q} .$$

Moreover, the limit is the same for all $j \in \{0, 1 \ldots, h-1\}$:

$$\begin{aligned}
&(K-1)p_j \quad + \overline{\lambda}_j \\
&= (K-1)k^j p + (K-1)[k^{j-1}\lambda_0 + k^{j-2}\lambda_1 + \cdots + k\lambda_{j-2} + \lambda_{j-1}] + \overline{\lambda}_j \\
&= (K-1)k^j p + [k^{h+j-1}\lambda_0 + k^{h+j-2}\lambda_1 + \cdots + k^{h+1}\lambda_{j-2} + k^h\lambda_{j-1}] \\
&\qquad\qquad - [k^{j-1}\lambda_0 + k^{j-2}\lambda_1 + \cdots + k\lambda_{j-2} + \lambda_{j-1}] \\
&\qquad\qquad + [k^{h-1}\lambda_j + k^{h-2}\lambda_{j+1} + \cdots + k^j\lambda_{h-1}] \\
&\qquad\qquad + [k^{j-1}\lambda_0 + k^{j-2}\lambda_1 + \cdots + k\lambda_{j-2} + \lambda_{j-1}] \\
&= (K-1)k^j p + k^j[k^{h-1}\lambda_0 + k^{h-2}\lambda_1 + \cdots + k\lambda_{h-2} + \lambda_{h-1}] \\
&= (K-1)k^j p + k^j\overline{\lambda}_0 \,.
\end{aligned}$$

Therefore,

$$\frac{(K-1)p_j + \overline{\lambda}_j}{(K-1)q_j} = \frac{(K-1)k^j p + k^j\overline{\lambda}_0}{(K-1)k^j q} = \frac{(K-1)p + \overline{\lambda}_0}{(K-1)q} \quad \forall\, j \in \{0, 1 \ldots, h-1\}.$$

As in the binary case, the limit is attained if and only if $\overline{\lambda}_0 = 0$. However, if f is a binary morphism, then $\overline{\lambda}_0 = 0$ if and only if $f(0)$ and $f(1)$ have no common prefix or suffix, that is, f is a marked morphism (recall from Section 3.2.1 that a morphism f is marked if $f(a)$ and $f(b)$ have no common prefix or suffix for all $a \neq b \in \Sigma$). For the general case, being marked is sufficient but not necessary: it is possible for $f(a)$ and $f(b)$ to have a non-empty common prefix (or suffix), and still never have this prefix manifest itself in a π-sequence. The even Arshon morphisms constitute such an example, as we show in the next chapter (Lemma 7.7).

In light of the above discussion, we have proved the following theorem, which generalizes Theorem 5.1:

Theorem 6.11. *Let f be a k-uniform morphism over $\Sigma = \Sigma_n$, prolongable on 0, and let $\mathbf{w} = f^\omega(0)$. For an unstretchable p/q-power $z \in Occ(\mathbf{w})$, let $\lambda_0, \lambda_1, \ldots, \lambda_{h(z)-1}$ be the period of the stretch sequence generated by z, and let $\overline{\lambda}(z) = \sum_{i=0}^{h(z)-1} k^{h(z)-1-i}\lambda_i$. Let $\mathcal{E} \subseteq Occ(\mathbf{w})$ be the set of the initial powers of the π-sequences. Suppose $E(\mathbf{w}) < \infty$. Then:*

1. *$E(\mathbf{w})$ is a rational number, given by*

$$E(\mathbf{w}) = \max\left\{ \frac{(k^{h(z)}-1)p + \overline{\lambda}(z)}{(k^{h(z)}-1)q} : z = x^{p/q} \in \mathcal{E} \right\} .$$

2. *If the stretch sequence is always the zero sequence (in particular, if f is marked), then* $E(\mathbf{w}) = \max\left\{p/q : z = x^{p/q} \in \mathcal{E}\right\}$. *In particular,* $E(\mathbf{w})$ *is attained.*

6.4 Computing critical exponents

When trying to use Theorem 6.6 for actually computing critical exponents, we encounter two main problems. The first one is that the supremum of the π-sequence is not necessarily one of its limit points, and it is not clear how to compute it; see Section 6.5.4 for an example of an oscillating π-sequence. The second problem is that we have no general bound on the size of half-synchronized powers we need to consider (although in practice, we do not know of any actual example where such a power generates the critical exponent). However, if we can prove that $E(\mathbf{w})$ is one of the limit points of the π-sequences generated by the unsynchronized powers, then we can use the following algorithm to compute $E(\mathbf{w})$:

Input: A morphism f over Σ_n, prolongable on 0.

Algorithm: Let $\mathbf{w} = f^\omega(0)$.

1. Check whether or not $E(\mathbf{w}) < \infty$. If $E(\mathbf{w}) = \infty$, return ∞.
2. Compute the number $k = 2D + M$, where D is the synchronization delay, $M = \max\{D, m\}$, and $m = \max\{|f(a)| : a \in \Sigma\}$.
3. Compute the set of powers

$$S_k(\mathbf{w}) := \{z = x^{p/q} \in \mathrm{Occ}(\mathbf{w}) : z \text{ is unstretchable and } q < k\}.$$

4. For every unstretchable power $z \in S_k(\mathbf{w})$:
   (a) compute the period h of the stretch sequence generated by iterating π on z and the starting point of this period;
   (b) compute the limit points of each of the subsequences $\{\pi^{mh+j}(z, P/Q)\}_{m\geq 0}$, where $j = 0, 1, \ldots, h-1$.

Output: The maximum of the values computed in 4(b).

Notes:

1. It is decidable whether or not $E(\mathbf{w}) < \infty$ [42, 89, 72]. Moreover, if the alphabet size is fixed, deciding whether $E(\mathbf{w})$ is bounded can be done in polynomial time, though the exact degree is not clear (Kobayashi and Otto [72, Algorithm 5.21, Theorem 5.22]). If n is part of the input the complexity becomes exponential.

2. Given that $E(\mathbf{w}) < \infty$, the synchronization delay D is computable as a function of n, m and $q(\mathbf{w})$ (Definition 6.1) [89]. By Mignosi and Séébold [89, Theorem 1, Proposition 1],
$$D \leq \max\left\{\frac{q(\mathbf{w})}{2} - m, (2m)^n - 1 - m\right\}.$$
By Kobayashi and Otto [72, Theorem 5.17], if f is non-erasing and not pushy, then $q(\mathbf{w}) \leq 2nm^n$, and so $D \leq (2m)^n - m - 1$. Again, the bound exponential in n.

3. Given a fixed integer e, it is decidable whether $\mathbf{w}$ is e-power-free [89]. Therefore, given that $E(\mathbf{w}) < \infty$, finding $e := \lfloor E(\mathbf{w}) \rfloor + 1$ is also decidable. The set $S_k(\mathbf{w})$ can thus be computed by computing the set $\{z \in \text{Sub}(\mathbf{w}) : |z| < ek + 2C\}$. The $2C$ factor is added to cover all possible contexts (see Lemma 6.7).

4. The length of the prefix of $\mathbf{w}$ which contains $S_k(\mathbf{w})$ can be exponential in n (Allouche and Shallit [6, Example 10.4.11]). Computing all maximal powers in this prefix (as initial powers for the π-sequences) can be done in time linear in the size of the prefix (Kolpakov and Koucherov [84, Section 8.4]).

5. Computing the Jordan form of an n by n matrix requires about $O(n^9 log^2 ||A||)$ bit operations, where $||A|| = \max |a_{ij}|$ (Giesbrecht and Storjohann [50]; Gil [51, 52]). In our case, $n = |\Sigma|$ and $||A|| \in O(m)$.

The complexity of this algorithm is not clear, but in light of the above discussion it can be exponential in the alphabet size. In some cases, however, computing $E(\mathbf{w})$ becomes very simple. In particular, if it is easy to show that $\mathbf{w}$ is circular with delay D, and the set $S_k(\mathbf{w})$ is easy to compute and is shown to be finite, step 1 of the algorithm becomes unnecessary, as was shown in Examples 6.3 - 6.7. In the next section we give some more examples.

6.5 Applications

6.5.1 The critical exponent of the Fibonacci word

Let ϕ be the Fibonacci morphism, $\phi(0) = 01$, $\phi(1) = 0$, and let $\mathbf{f} = \phi^{\omega}(0)$. In [88], Mignosi and Pirillo showed that $E(\mathbf{f}) = 2 + \tau$, where $\tau = (1 + \sqrt{5})/2$ is the golden mean. We give an alternative proof.

Let $z = w_i \cdots w_j \in \mathrm{Occ}(\mathbf{f})$ be an unstretchable P/Q-power, $z = x^{p/q}$. First, observe that if $q \geq 3$, then x has at least 2 synchronization points. If $q = 2$, then the only possible power block is $x = 01$, since it is easy to see that 11 and $(00)^2$ are not subwords of $\mathbf{f}$, and a power of the form $(10)^r$ will be left-stretchable. The word 01 has two synchronization points, $01 = \varepsilon|01|\varepsilon$. Since ϕ is injective (being a suffix code, see Corollary 2.7), this means that the only unsynchronized initial power in $\mathbf{f}$ is the square 00. Also, half-synchronized unstretchable powers have the form $0y0$, where $|y| \geq 1$: a power of the form $1y1$ is left-stretchable, and a power of the form xyx with $|x| > 1$ is synchronized. We get that the initial powers we have to check are p/q-powers of the form $0y0$, where $|y| \geq 0$, $p = |0y0|$, $q = |0y|$, and $\binom{p_0}{p_1} = \binom{q_0+1}{q_1}$ (here $\binom{p_0}{p_1}, \binom{q_0}{q_1}$ are the Parikh vectors of $0y0, 0y$, respectively). Note that we do not need to check separately that $E(\mathbf{f}) < \infty$.

Next, let us compute the stretch sequence of $\pi^m(z)$. Assume without loss of generality that $w_{i-1} = w_{j+1} = 1$, and $w_{i+q-1} = w_{j-q+1} = 0$. Since $\phi(0)$ and $\phi(1)$ have no common suffix, $\phi(z)$ cannot be stretched left, and $\sigma(\phi(z), FQ) = \varepsilon$. To the right, we can always stretch by the letter 0, which is the longest common prefix of $\phi(0)$ and $\phi(1)$; however, we cannot stretch by more, since we must have $w_{j+2} = 0$, or else we would get $11 \in \mathrm{Sub}(\mathbf{f})$. Thus $\phi(w_{j-q+1}) = 01$, $\phi(w_{j+1}w_{j+2}) = 001$, and $\rho(\phi(z), FQ) = 0$. We get that the stretch vector is always $\binom{1}{0}$, and the π map is given by

$$\pi^m(P/Q) = \frac{A^m P + \left(\sum_{i=0}^{m-1} A^i\right)\binom{1}{0}}{A^m Q} .$$

The incidence matrix of the Fibonacci morphism is given by $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. To compute $\lim_{m\to\infty} \pi^m(P/Q)$ we can use the Jordan decomposition of A; however, because of the special properties of the Fibonacci sequence, we can also compute it directly. Let $\{f_n\}_{n\geq 0}$ be the Fibonacci sequence, defined by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$.

It is an easy induction to show that for all $m \geq 1$,

$$A^m = \begin{pmatrix} f_{m+1} & f_m \\ f_m & f_{m-1} \end{pmatrix}.$$

Using the identity $\sum_{i=1}^n f_i = f_{n+2} - 1$ (see, e.g., [134, Chapter III, (33)]), we get that

$$\sum_{i=0}^{m-1} A^i = \begin{pmatrix} \sum_{i=1}^m f_i & \sum_{i=1}^{m-1} f_i \\ \sum_{i=1}^{m-1} f_i & \sum_{i=1}^{m-2} f_i + 1 \end{pmatrix} = \begin{pmatrix} f_{m+2} - 1 & f_{m+1} - 1 \\ f_{m+1} - 1 & f_m \end{pmatrix}.$$

Therefore, for all initial powers $0y0$,

$$\begin{aligned} \pi^m(P/Q) &= \frac{\begin{pmatrix} f_{m+1} & f_m \\ f_m & f_{m-1} \end{pmatrix}\begin{pmatrix} q_0 + 1 \\ q_1 \end{pmatrix} + \begin{pmatrix} f_{m+2} - 1 & f_{m+1} - 1 \\ f_{m+1} - 1 & f_m \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} f_{m+1} & f_m \\ f_m & f_{m-1} \end{pmatrix}\begin{pmatrix} q_0 \\ q_1 \end{pmatrix}} = \\ & \frac{(q_0+1)f_{m+2} + q_1 f_{m+1} + f_{m+2} + f_{m+1} - 2}{q_0 f_{m+2} + q_1 f_{m+1}} = 1 + \frac{2f_{m+2} + f_{m+1} - 2}{q_0 f_{m+2} + q_1 f_{m+1}}. \end{aligned}$$

For the initial power $z = 00$ we have $q_0 = 1$ and $q_1 = 0$, and so

$$\pi^m\left(00, \frac{\binom{2}{0}}{\binom{1}{0}}\right) = 1 + \frac{2f_{m+2} + f_{m+1} - 2}{f_{m+2}} = 3 + \frac{f_{m+1} - 2}{f_{m+2}} \xrightarrow{m \to \infty} 2 + \tau.$$

Also, using the identity $f_{n-1}f_{n+1} - f_n^2 = (-1)^n$ (see, e.g., [134, Chapter III, (29)]), we get that

$$\pi^m\left(\frac{\binom{2}{0}}{\binom{1}{0}}\right) - \pi^{m-1}\left(\frac{\binom{2}{0}}{\binom{1}{0}}\right) = \frac{f_{m+1}^2 - f_m f_{m+2} + 2f_m}{f_{m+2}f_{m+1}} = \frac{(-1)^m + 2f_m}{f_{m+2}f_{m+1}} > 0,$$

that is, $\{\pi^m(\binom{2}{0}/\binom{1}{0})\}_{m\geq 0}$ is an increasing sequence, and so $2+\tau$ is its supremum.

Consider now the half-synchronized powers. For $z = 010$ we have $q_0 = q_1 = 1$, and so

$$\pi^m\left(010, \frac{\binom{2}{1}}{\binom{1}{1}}\right) = 1 + \frac{2f_{m+2} + f_{m+1} - 2}{f_{m+2} + f_{m+1}} = 2 + \frac{f_{m+2} - 2}{f_{m+3}} \xrightarrow{m \to \infty} 1 + \tau.$$

Also, the π-sequence is increasing, for exactly the same argument as for the $z = 00$ case. As for powers $0y0$ with $|y| > 1$, we claim that such powers generate only exponents strictly smaller than 2:

Claim 6.12. *Suppose* $z = xyx = (xy)^{p/q} \in Occ(\mathbf{f})$ *satisfies* $[y] > \binom{0}{0}$. *Then* $\pi(z) = x'y'x' = (x'y')^{p'/q'}$, *where* $p'/q' < 2$ *and* $[y'] > \binom{0}{0}$.

Proof. Let $[x] = \binom{r_0}{r_1}$, $[xy] = \binom{q_0}{q_1}$. Then xyx is a $(1+\binom{r_0}{r_1}/\binom{q_0}{q_1})$-power. Applying π, we get the following exponent:

$$\pi\left(1+\frac{\binom{r_0}{r_1}}{\binom{q_0}{q_1}}\right) \quad = \quad 1+\frac{\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}r_0\\ r_1\end{pmatrix}+\begin{pmatrix}1\\ 0\end{pmatrix}}{\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}q_0\\ q_1\end{pmatrix}} \quad = \quad 1+\frac{\begin{pmatrix}r_0+r_1+1\\ r_0\end{pmatrix}}{\begin{pmatrix}q_0+q_1\\ q_0\end{pmatrix}} \ .$$

By assumption, $q_0 = r_0 + |y|_0 > r_0$, and $q_1 = r_1 + |y|_1 > r_1$. Therefore, $z' = \pi(z)$ is a $(1+\binom{r'_0}{r'_1}/\binom{q'_0}{q'_1})$-power, where $r'_0 = r_0 + r_1 + 1 < q_0 + q_1 = q'_0$, and $r'_1 = r_0 < q_0 = q'_1$. Obviously, $(1+\binom{r'_0}{r'_1}/\binom{q'_0}{q'_1}) < 2$. Let x' be the prefix of length $r'_0 + r'_1$ of z', and let y' be such that $z' = x'y'x'$. Then $|y'|_0 = q'_0 - r'_0 > 0$ and $|y'|_1 = q'_1 - r'_1 > 0$. □

By the above claim, when iterating π on powers of the form $z = xyx$ with $[y] > \binom{0}{0}$, we get only exponents strictly smaller than 2. Now, if $z = 0y0$ and $|y| > 1$, necessarily y contains both 0 and 1, since 11 and 000 are not subwords of $\mathbf{f}$. Therefore, we can ignore such powers when computing $E(\mathbf{f})$.

We conclude that the critical exponent is the limit of the π-sequence generated by 00:

$$E(\mathbf{f}) = \lim_{m\to\infty} \pi^m\left(00, \frac{\binom{2}{0}}{\binom{1}{0}}\right) = 2+\tau \ .$$

Note: Recall that by Vandeth (Equation (3.6)), Sturmian pure morphic words always have irrational critical exponents, and so their critical exponents are never attained. Therefore, in the Sturmian case, the critical exponent is always the maximal limit point of the π-sequences.

6.5.2 A circular word with unbounded critical exponent

As already mentioned in Chapter 3, a pure morphic word can be circular and still contain unbounded powers. If this is the case, we still get that all sufficiently long unstretchable powers belong to one of finitely many π-sequences, only some of these sequences will have a subsequence diverging to infinity. We give an example in this section.

Let $f = (0101, 1) \in \mathcal{M}(\Sigma_2)$, and let $\mathbf{w} = f^\omega(0)$. Here are the first 24 terms of $\mathbf{w}$:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
w_i	0	1	0	1	1	0	1	0	1	1	1	0	1	0	1	1	0	1	0	1	1	1	1	0

It is easy to see that $\mathbf{w} = 010x_1010x_2010x_3\cdots$, where $x_i \in \{1^m : m \geq 2\}$ for all $i \geq 1$. Therefore, any subword of length 4 or more must have at least two synchronization points. Since f is injective, being a prefix code, this implies that $\mathbf{w}$ is circular.

Consider the occurrence $w_3w_4 = 11$. Then w_3w_4 is an unstretchable $\binom{0}{2}/\binom{0}{1}$-power. Applying f, we get that $f(w_3w_4) = w_9w_{10} = 11$ can be stretched left by the letter 1, and cannot be stretched right. Therefore, $\pi(z, \binom{0}{2}/\binom{0}{1}) = (w_8w_9w_{10}, \binom{0}{3}/\binom{0}{1})$. Applying π again, we get the same stretch vector. Therefore, the π-sequence initiated by w_3w_4 is given by

$$\pi^m\left(w_3w_4, \frac{\binom{0}{2}}{\binom{0}{1}}\right) = \frac{A^m\binom{0}{2} + (\sum_{i=0}^{m-1} A^i)\binom{0}{1}}{A^m\binom{0}{1}} .$$

Let $U = \binom{0}{2}$, $V = W = \binom{0}{1}$, and let $A = \left(\begin{smallmatrix} 2 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ be the incidence matrix of f. It is an easy induction to show that for all $m \geq 1$,

$$A^m = \begin{pmatrix} 2^m & 0 \\ 2^{m+1} - 2 & 1 \end{pmatrix} .$$

Therefore, $\mathbf{I}_0(A, U) \cap \mathbf{I}_0(A, V) \cap \mathbf{I}_0(A, W) = \{1\}$ (see Equation (6.3)). By Lemma 6.2, we can replace A, U, V, W by the integers $U_{\{1\}} = 2$, $A_{\{1\}} = V_{\{1\}} = W_{\{1\}} = 1$. Therefore, the π-sequence initiated by w_3w_4 translates to

$$\pi^m\left(w_3w_4, \frac{\binom{0}{2}}{\binom{0}{1}}\right) = \frac{1^m \cdot 2 + (\sum_{i=0}^{m-1} 1^i) \cdot 1}{1^m \cdot 1} = 2 + m,$$

which results in $E(\mathbf{w}) = \infty$.

The same type of argument applies to any binary morphism of the form $f = (0x1, 1)$, where $x \in \Sigma_2^+$ contains at least one 0 (otherwise, if $f = (01^n, 1)$ for some $n \geq 1$, we get that $f^\omega(0) = 01^\omega$, which contains no unstretchable powers). In all such cases, $f^\omega(0)$ is a circular word that contains unbounded powers, and some π-sequence will diverge to infinity.

6.5.3 A "density theorem"

In the previous chapter we have proved that any rational number $0 < r < 1$ is the fractional part of $E(\mathbf{w})$ for some uniform binary pure morphic word $\mathbf{w}$ (Theorem 5.22). In this section we prove that any rational number $\alpha \geq 2$ is the critical exponent of some uniform pure morphic word defined over a finite alphabet. Compare also to Theorem 4.1 (every real number is a critical exponent): the theorem proved in the current section refines Theorem 4.1 in the sense that here we construct a pure morphic word, not just arbitrary word; on the other hand, the construction of Theorem 4.1 used only 4 letters, while in this section we use an unbounded number of letters, and it is not clear if we can do it with a bounded number.

Theorem 6.13. *For any rational number $\alpha \geq 2$ there exists a uniform pure morphic word $\mathbf{w}$ over some finite alphabet such that $E(\mathbf{w}) = \alpha$.*

Proof. If $\alpha = 2$, take the Thue-Morse word. Otherwise, let $\alpha = p/q > 2$, let $\Sigma = \Sigma_{q+1}$, and let $u = 0(12\cdots q)^{p/q}0 \in \Sigma^*$. Let $f = \varphi_{u,q+1}$ be the symmetric morphism over Σ defined by u, and let $\mathbf{w} = \mathbf{t}_{u,q+1} = f^\omega(0)$. Then f is a $(p+2)$-uniform injective morphism. Denote $k = p+2$. We prove that $E(\mathbf{w}) = p/q$.

It is easy to see that any occurrence that contains a full k-block has at least two synchronization points, and moreover, since f is marked, once an occurrence is synchronized there can be no ambiguity at the edges. Since f is a uniform marked morphism, by Frid [47] the fact that $\mathbf{w}$ is circular implies that $E(\mathbf{w}) < \infty$ (see Section 3.2.1). Therefore, by Theorem 6.11, $E(\mathbf{w})$ is the maximal exponent of the unsynchronized and half-synchronized powers. But half-synchronized powers have exponents smaller than 2, and by definition of f, $E(\mathbf{w}) > 2$. We conclude that $E(\mathbf{w})$ is the maximal exponent of the unsynchronized powers, and so it is enough to consider powers with power block of size less than $2k$.

Let $z \in \mathrm{Occ}(\mathbf{w})$ be an r/s-power with $s < 2k$. Let x be the power block. We can assume that x is either contained in a k-block or spans across two k-blocks, otherwise it would be synchronized, and we could take an inverse image. Note also that $aa \notin \mathrm{Sub}(\mathbf{w})$ for all $a \in \Sigma$, and therefore two consecutive k blocks must have distinct first and last symbols. We conclude that x is a suboccurrence of an occurrence of the form

$$a_0\ a_1a_2\cdots a_q\ \cdots a_1a_2\cdots a_q\ a_1a_2\cdots a_m\ a_0|a_0'\ a_1'a_2'\cdots a_q'\ \cdots a_1'a_2'\cdots a_q'\ a_1'a_2'\cdots a_m'\ a_0',$$

where $m = p \mod q$, $a_i \neq a_j$ for all $0 \leq i \neq j \leq q$, and there exists some $1 \leq n \leq q$ such that $a'_i = (a_i + n) \mod (q+1)$ for all $0 \leq i \leq q$. We consider three cases: $s = q$, $s < q$, and $s > q$.

1. Suppose $s = q$. If x begins at a_0, the first letter following x is $a_q \neq a_0$, which implies the exponent of z cannot be more than 1. The same holds when x begins at a'_0. If x ends at a_0, then the next x block starts with a'_0, and since a'_0 does not occur again until the end of the k block, the exponent of z can be at most 2. The same holds when x ends at a'_0. Finally, if x contains both a_0 and a'_0, then a'_0 occurs in the first x block, but cannot occur in the second x block. Therefore, the exponent of z must be smaller than 2. The only way to get an exponent greater than 2 is when x does not contain a_0 or a'_0, and in this case, it is easy to see that the maximal exponent is attained when $x = a_1 \cdots a_q$ and $z = x^{p/q}$.

2. Suppose $s < q$. If x contains a_0 or a'_0, similar arguments show that the exponent of z can be at most 2. Otherwise, the exponent must be 1, since the letters $\{a_i : 1 \leq i \leq q\}$ are all distinct.

3. Suppose $s > q$. If x does not contain a_0 or a'_0, then z is contained in the p/q-power $(a_1 \cdots a_q)^{p/q}$, and since $r = |z| \leq p$, necessarily $r/s < p/q$. Assume therefore that x contains a_0 or a'_0.

   If $x = a_0\ a_1a_2 \cdots a_q\ \ \cdots a_1a_2 \cdots a_q\ a_1a_2 \cdots a_m$ then, since a_0 does not appear in the $a_1 \cdots a_q$ period but does appear in the $a'_1 \cdots a'_q$ period, z must have exponent smaller than 2. If $s < k$ and x ends at a_0, the exponent of z can be at most 2, as was argued in the $s = q$ case. Otherwise, x contains $a_0a'_0$. Since x does not contain a whole k-block (or it would be synchronized), the second x block starts within the second k-block. Since a'_0 does not appear again until the end of the second k-block, necessarily $s \geq k - 1$.

   Consider the next k-block, $a''_0\ a''_1a''_2 \cdots a''_q\ \ \cdots a''_1a''_2 \cdots a''_q\ a''_1a''_2 \cdots a''_m\ a''_0$. Since $a''_0 \neq a'_0$, a'_0 must appear in the q period of the third k-block, and so z cannot contain two full x blocks. Again, $r/s < 2$.

□

Example 6.8. For $\alpha = 8/3$, we have $u = 0123123120 \in \Sigma_4$,

$$\varphi_{u,4} : \begin{cases} 0 \to 0123123120 \\ 1 \to 1230230231 \\ 2 \to 2301301302 \\ 3 \to 3012012013 \end{cases},$$

$$\varphi_{u,4}^{\omega}(0) = 0123123120|1230230231|2301301302|3012012013|1230230231|\cdots$$

6.5.4 An oscillation example

We have mentioned before that the π-sequences are not necessarily increasing. In this section we give an example of an oscillating sequence.

Let $f = (012212112, 3, 455454455445, 21, 50, 1221211221124) \in \mathcal{M}(\Sigma_6)$, let $A = A(f)$, and let $\mathbf{w} = f^{\omega}(0) = 01221211234554\cdots$. Then f is marked, hence the stretch vector is always the zero vector, and for every unstretchable P/Q-power $z \in \mathrm{Occ}(\mathbf{w})$ we have $\pi^m(P/Q) = A^m P / A^m Q$. Let $z = w_{75} \cdots w_{79} = 12412$. Then z is an unstretchable P/Q-power, where $P = (0,2,2,0,1,0)^T$ and $Q = (0,1,1,0,1,0)^T$. Iterating π on z, we get an oscillating sequence of powers. Fig. 6.4 shows the first 20 values of the π-sequence.

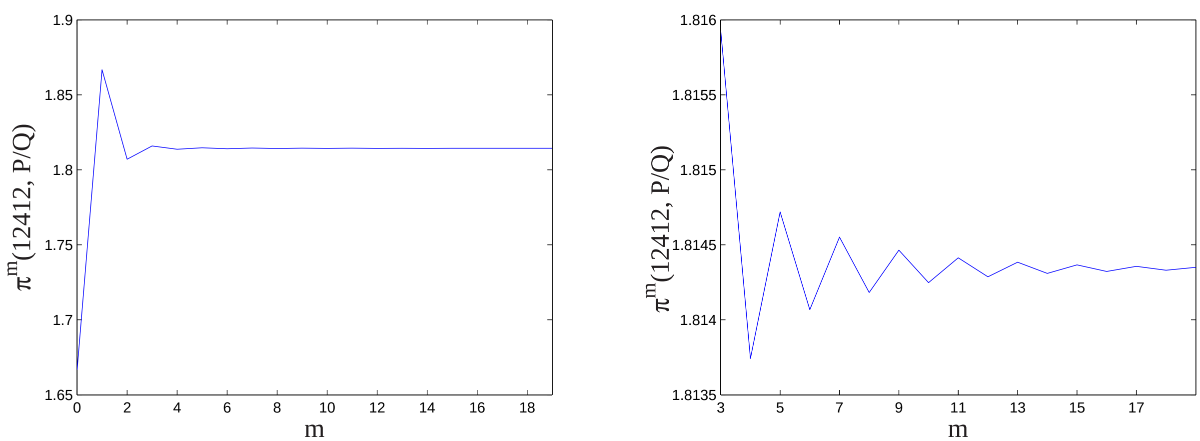

Figure 6.4: Values 0 to 19 of $\pi^m(12412, P/Q)$ (left), zoom on values 3 to 19 (right).

Here is a list of the first 20 values, rounded to 9 decimal places (read top to bottom, left to right):

1.666666666,	1.814719226,	1.814247765,	1.814366368,
1.866666666,	1.814067221,	1.814413588,	1.814322558,
1.807017543,	1.814551364,	1.814286463,	1.814356121,
1.815926892,	1.814181890,	1.814383828,	1.814330401,
1.813741875,	1.814464344,	1.814309202,	1.814350107.

6.6 Some open problems

1. We have proved that if $f \in \mathcal{M}(\Sigma_n)$ is a non-erasing morphism prolongable on 0, then $E(f^\omega(0))$ is either infinite or algebraic of degree at most n. It yet remains to prove the result for erasing morphisms. Another generalization which seems plausible is to morphic words in general. Recall from Section 2.5 that a morphic word is the image of a pure morphic word under a coding, that is, a 1-uniform morphism $h : \Sigma_n \to \Sigma_m$. Here typically $m < n$. If $\mathbf{w} = f^\omega(0)$ and $\mathbf{v} = h(\mathbf{w})$, then obviously $E(\mathbf{v}) \geq E(\mathbf{w})$. The problem is that when the inequality is strict, the relation between $E(\mathbf{v})$ and $E(\mathbf{w})$ is not clear. There are examples where $E(\mathbf{w})$ is attained and $E(\mathbf{v})$ is not, and vice versa. Proving Theorem 6.6 for morphic sequences will cover the erasing case as well, since every word generated by iterating a morphism is the image under a coding of a word generated by iterating a non-erasing morphism [6, Theorem 7.5.1].

2. Given an algebraic number α of degree d, can we construct a morphism $f : \Sigma_n \to \Sigma_n$ for some $n \geq d$ such that $E(f^\omega(0)) = \alpha$? For a rational number $\alpha \geq 2$ the answer is positive, as we have seen in Section 6.5.3. The question is much harder when trying to construct a pure morphic sequence that does not attain the critical exponent, in particular when α is irrational.

Chapter 7

The Critical Exponent of The Arshon Words

In this chapter we use the method developed in the previous chapter to compute the critical exponent of the Arshon words. Recall from Section 3.2.4 that the Arshon word of order 2 is given by $\mathbf{a}_2 = \mathbf{t}$ (the Thue-Morse word), and for $n \geq 3$, the Arshon word of order n is an infinite square-free word over Σ_n, $\mathbf{a}_n = a_0 a_1 a_2 \cdots = \lim_{k\to\infty} \varphi_n^k(0)$, where φ_n is the Arshon operator of order n,

$$\varphi_n(a_i) = \begin{cases} \varphi_{e,n}(a_i), & \text{if } i \text{ is even;} \\ \varphi_{o,n}(a_i), & \text{if } i \text{ is odd.} \end{cases}$$

Here $\varphi_{e,n}, \varphi_{o,n}$ are the symmetric morphisms defined by the words $e = 01\cdots(n-1)$ and $o = (n-1)\cdots 10$, respectively. Also, if n is even, then $\varphi_n = \alpha_n$, where

$$\alpha_n(a) = \begin{cases} \varphi_{e,n}(a), & \text{if } a \text{ is even;} \\ \varphi_{o,n}(a), & \text{if } a \text{ is odd.} \end{cases}$$

Though the odd Arshon words are not generated by iterating a morphism, the same type of arguments we used in the pure morphic case can be applied when computing their critical exponent.

So far, the critical exponent of $\mathbf{a}_n$ has been computed only for $n = 2$ and $n = 3$. For $n = 2$, $E(\mathbf{a}_2) = E(\mathbf{t}) = 2$ by Thue [132, 11]. For $n = 3$, Klepinin and Sukhanov proved in [70] that $E(\mathbf{a}_3) = 7/4$. These values are in agreement with the values stated in Dejean's conjecture: recall from Section 3.2.2 that the repetition threshold for $n = 2, 3$ is given by

$RT(2) = 2$ and $RT(3) = 7/4$. For $n \geq 4$, however, the values no longer agree, as we shall prove in this chapter. In particular, while $RT(n) < 3/2$ for all $n \geq 4$ and $RT(n)$ tends to 1 as n tends to infinity, the critical exponent of $\mathbf{a}_n$ satisfies $E(\mathbf{a}_n) > 3/2$ for all $n \geq 2$, and $E(\mathbf{a}_n)$ tends to $3/2$ as n tends to infinity. We prove the following theorem:

Theorem 7.1. *Let $n \geq 2$, and let $\mathbf{a}_n = a_0a_1a_2\cdots$ be the Arshon word of order n. Then the critical exponent of $\mathbf{a}_n$ is given by $E(\mathbf{a}_n) = (3n-2)/(2n-2)$, and $E(\mathbf{a}_n)$ is attained by a subword beginning at position 1.*

7.1 General properties of the Arshon words

Notation 7.1. For an occurrence $z \in \mathrm{Occ}(\mathbf{a}_n)$, we denote by $\mathrm{inv}(z)$ the ancestor of z under φ_n. That is, $\mathrm{inv}(z)$ is the shortest occurrence $z' \in \mathrm{Occ}(\mathbf{a}_n)$ such that $\varphi_n(z')$ contains z. Following Currie [31], we refer to the decomposition of $\mathbf{a}_n$ into images under φ_n as the *φ-decomposition*, and to the images of the letters as *φ-blocks*. We denote the borderline between two consecutive φ-blocks by '$|$'; e.g., $i|j$ means that i is the last letter of a block and j is the first letter of the following block. If $z = a_i \cdots a_j \in \mathrm{Occ}(\mathbf{a}_n)$ occurs at an even position, we write $z = a_i^{(e)}a_{i+1}^{(o)}a_{i+2}^{(e)}\cdots$, and similarly for an occurrence that occurs at an odd position.

For the rest of this chapter, all sums of letters are taken modulo n.

Lemma 7.2. *For all $n \geq 2$, $\mathbf{a}_n$ contains a $(3n-2)/(2n-2)$-power at position 1.*

Proof. For $n = 2$, $\mathbf{a}_2 = \mathbf{t} = 0110\cdots$, which contains the 2-power 11 at position 1. For $n \geq 3$, $\mathbf{a}_n$ begins with

$$\varphi_{e,n}(0)\varphi_{o,n}(1)\varphi_{e,n}(2) = 012\cdots(n-1)|0(n-1)\cdots 21|2\cdots(n-1)01 =$$
$$0\,(12\cdots(n-1)0(n-1)\cdots 2)^{(3n-2)/(2n-2)}\,1.$$

□

Example 7.1.

$$\begin{aligned}
\mathbf{a}_3 &= 012|021|201|\cdots &&= 0\,(1202)^{7/4}\,1\cdots,\\
\mathbf{a}_4 &= 0123|0321|2301|\cdots &&= 0\,(123032)^{10/6}\,1\cdots,\\
\mathbf{a}_5 &= 01234|04321|23401|\cdots &&= 0\,(12340432)^{13/8}\,1\cdots.
\end{aligned}$$

Corollary 7.3. *The critical exponent of* $\mathbf{a}_n$ *satisfies* $(3n-2)/(2n-2) \leq E(\mathbf{a}_n) \leq 2$ *for all* $n \geq 2$.

Proof. For $n = 2$, we know by Thue that $E(\mathbf{a}_n) = 2$. For $n \geq 3$, we know by Arshon [8] that $\mathbf{a}_n$ is square-free, and so $E(\mathbf{a}_n) \leq 2$. The lower bound follows from Lemma 7.2. □

Lemma 7.4. *Let* $n \geq 3$*, and let* $i, j \in \Sigma_n$*. If* $ij \in Sub(\mathbf{a}_n)$*, then* $j = i \pm 1$.

Proof. If ij occurs within a φ-block, the assertion holds by definition of φ_n. Let i be the last letter of a φ-block, let j be the first letter of the next φ-block, and let $kl = \text{inv}(ij)$. Assume $j \neq i \pm 1$, and suppose further that ij is the first pair that satisfies this inequality. Then $l = k \pm 1$, and so there are four cases:

1. $\varphi_n(kl) = \varphi_{e,n}(k)\varphi_{o,n}(k+1)$;
2. $\varphi_n(kl) = \varphi_{o,n}(k)\varphi_{e,n}(k+1)$;
3. $\varphi_n(kl) = \varphi_{e,n}(k)\varphi_{o,n}(k-1)$;
4. $\varphi_n(kl) = \varphi_{o,n}(k)\varphi_{e,n}(k-1)$.

But it is easy to check that for all the cases above, $j = i \pm 1$, a contradiction. □

Corollary 7.5. *For* $n \geq 3$*, the borderline between two consecutive* φ*-blocks has the form* $i|ji$ *or* $ij|i$*, where* $j = i \pm 1$*. Moreover, a word of the form* iji *can occur only at a borderline.*

Proof. By definition of φ_n, a φ-block is either strictly increasing or strictly decreasing, and two consecutive blocks have alternating directions. By Lemma 7.4, a change of direction can have only the form $i|ji$ or $ij|i$. □

Definition 7.1 (Currie [31]). A *mordent* is a word of the form iji, where $i, j \in \Sigma_n$ and $j = i \pm 1$. Two consecutive mordents occurring in $\mathbf{a}_n$ are either *near mordents*, *far mordents*, or *neutral mordents*, according to the position of the borderlines:

$$\begin{aligned}
i|ji\ u\ kl|k &= \text{near mordents, } |u| = n-4; \\
ij|i\ u\ k|lk &= \text{far mordents, } |u| = n-2; \\
i|ji\ u\ k|lk &= \text{neutral mordents, } |u| = n-3; \\
ij|i\ u\ kl|k &= \text{neutral mordents, } |u| = n-3.
\end{aligned}$$

Note that for $n = 3$, near mordents are overlapping: $\mathbf{a}_3 = 01\mathit{2}|\mathit{021}|\mathit{2}01|\cdots$.

Since $\mathbf{a}_n$ is square-free, a p/q-power occurring in $\mathbf{a}_n$ has the form xyx, where $q = |xy|$, $p = |xyx|$, and both x, y are nonempty.

Lemma 7.6. *Let $n \geq 3$. Let $z = xyx = (xy)^{p/q}$ be an unstretchable power in $Occ(\mathbf{a}_n)$, such that $|x| \leq n$ and x contains no mordents. Then $p/q \leq (3n-2)/(2n-2)$.*

Proof. Since $|x| \leq n$, it is enough to consider y such that $|y| \leq n-2$, for otherwise we would get $p/q < (3n-2)/(2n-2)$. Therefore, $|xy| = q \leq 2n-2$ and $|z| \leq 3n-2$. We get that xy is contained in at most 3 consecutive φ-blocks and z is contained in at most 4 consecutive φ-blocks.

Suppose z is not contained in 3 consecutive φ-blocks. Let $B_0B_1B_2B_3$ be the blocks containing z, and assume that B_0 is even (the other case is similar). Since $|x| \leq n$, necessarily xy begins in B_0 and ends in B_2. Since x contains no mordents, x has to start at the last letter of B_0: otherwise, we would get that x cannot extend beyond the first letter of B_1, and since $|y| \leq n-2$, we would get that z is contained in 3 φ-blocks. Therefore, the letters of x are decreasing. Now, since $|xy| \leq 2n-2$, the second occurrence of x begins at least 3 letters from the end of B_2. Since B_2 is an even block, we get a contradiction if $|x| > 1$. But if $|x| = 1$ then z is contained in $B_0B_1B_2$. We can assume therefore that z is contained in three consecutive φ-blocks, $B_0B_1B_2$. We assume that B_0 is even (the other case is symmetric).

If xy is contained in one block then, because B_0, B_2 are even and B_1 is odd, necessarily $|x| = 1$, and so $p/q \leq 3/2 \leq (3n-2)/(2n-2)$. If xy begins in B_0 and ends in B_2, then, since $|y| \leq n-2$, the first x occurrence has to end at the third letter of B_1 or later. Since x contains no mordents, this implies that xy begins at the last letter of B_0 and the letters of x are decreasing. Since B_2 is even, again $|x| = 1$.

Assume xy begins in B_0 and ends in B_1. Again, because B_0 is even and B_1 is odd, in order for x to contain more than one letter the second x occurrence has to start either at the last letter of B_1, or at the first letter of B_2.

Let $B_0 = \varphi_{e,n}(i)$. Then there are four cases for B_1, B_2:

1. $B_1 = \varphi_{o,n}(i+1)$, $B_2 = \varphi_{e,n}(i)$;

2. $B_1 = \varphi_{o,n}(i-1)$, $B_2 = \varphi_{e,n}(i)$;

3. $B_1 = \varphi_{o,n}(i+1)$, $B_2 = \varphi_{e,n}(i+2)$;

4. $B_1 = \varphi_{o,n}(i-1)$, $B_2 = \varphi_{e,n}(i-2)$.

We now check what the maximal possible exponent is in each of these cases. Without loss of generality, we can assume $i = 0$. We use the notation $z = xyx'$, where x' is the second occurrence of x in z.

Case 1: $B_0B_1B_2 = |01\cdots(n-1)|0(n-1)\cdots 1|01\cdots(n-1)|$.

If x' starts at the last letter of B_1 then $|x| = 1$, since 10 does not occur anywhere before. If x' starts at the first letter of B_2, the only possible power is the $3n/2n$-power $B_0B_1B_0$, which contradicts the hypothesis $|y| \leq n-2$.

Case 2: $B_0B_1B_2 = |01\cdots(n-1)|(n-2)(n-3)\cdots 0(n-1)|01\cdots(n-1)|$.

By the same argument, either $|x| = 1$ or z is a $3n/2n$-power.

Case 3: $B_0B_1B_2 = |01\cdots(n-1)|0(n-1)\cdots 1|23\cdots(n-1)01|$.

If x' starts at the last letter of B_1, we get the $(3n-2)/(2n-2)$-power described in Lemma 7.2. If x' starts at the first letter of B_2, then x has to start at the 2 in B_0. But then the power is left-stretchable, to the $(3n-2)/(2n-2)$-power described above.

Case 4: $B_0B_1B_2 = |01\cdots(n-1)|(n-2)(n-3)\cdots 0(n-1)|(n-2)(n-1)0\cdots(n-3)|$.

If x' starts at the last letter of B_1, then x has to start at the last letter of B_0. But then $|x| = 2$, since $(n-1) \neq (n-3)$. We get that z is an $(n+2)/n$-power, and $(n+2)/n < (3n-2)/(2n-2)$ for all $n \geq 3$. If x' starts at the first letter of B_2, then x has to start at the second last letter of B_0. Again, $|x| = 2$, and z is an $(n+4)/(n+2)$-power, where $(n+4)/(n+2) < (3n-2)/(2n-2)$ for all $n \geq 2$.

□

In what follows, we will show that in order to compute $E(\mathbf{a}_n)$, it is enough to consider powers xyx such that $|x| \leq n$ and x contains no mordents.

7.2 Arshon words of even order

Since even Arshon words are pure morphic, we can use the terminology and results of Chapter 6. To illustrate the power structure in Arshon words of even order, consider $\mathbf{a}_4$:

$$
\begin{array}{ccccc}
 & & \varphi_{e,4} & \varphi_{o,4} & \alpha_4 \\
0 & \to & 0123 & 3210 & 0123 \\
1 & \to & 1230 & 0321 & 0321 \\
2 & \to & 2301 & 1032 & 2301 \\
3 & \to & 3012 & 2103 & 2103
\end{array}
$$

$$\mathbf{a}_4 = 0123|0321|2301|2103|0123|2103|2301|0321|2301|2103|0123|0321|2301|0321|\cdots$$

Lemma 7.7. *Let $n \geq 4$ be even, and let $z = a_i \cdots a_j \in Occ(\mathbf{a}_n)$ be an unstretchable p/q-power. Then $\alpha_n(z)$ is an unstretchable np/nq-power.*

Proof. Since α_n is an n-uniform morphism, $\alpha_n(z)$ is an np/nq-power. To see that $\alpha_n(z)$ is unstretchable, let $a_{i-1} = k$ and let $a_{i+q-1} = l$. Since z is unstretchable, $k \neq l$. Assume $\alpha_n(z)$ is left stretchable. Then $\alpha_n(k)$ and $\alpha_n(l)$ have a common suffix, and so, by definition of α_n, $l = k \pm 1$. But then, by Lemma 7.4, $a_{i+q} = k + 1 \pm 1$, while $a_i = k \pm 1$. Since $n > 3$, this implies that $a_i \neq a_{i+q}$, a contradiction: z is a p/q-power, where $p/q > 1$. For the same reason, $\alpha_n(z)$ is not right stretchable. □

Corollary 7.8. *The critical exponent of an even Arshon word is the largest exponent of the unstretchable unsynchronized and half-synchronized powers.*

Proof. Follows diredtly from Theorem 6.11. □

A power $z = xyx$ is unsynchronized or half-synchronized if and only if x is unsynchronized: otherwise, if the two occurrences of x have the same interpretation, we could take an inverse image and get a power with the same exponent. Thus, in order to compute $E(\mathbf{a}_n)$ it is enough to consider powers of the form $z = xyx$, where x does not have a unique interpretation.

Lemma 7.9. *Let $n \geq 4$ be even. Then every subword $x \in Sub(a_n)$ with $|x| \geq n+1$ has a unique interpretation under α_n.*

Proof. In general, a mordent iji can admit two possible borderlines: $ij|i$ or $i|ji$. However, if n is even, all images under α_n begin with an even letter and end with an odd letter; images of odd letters under $\varphi_{e,n}$ and images of even letters under $\varphi_{o,n}$ are never manifested. Therefore, every mordent admits exactly one interpretation: if i is even and j is odd the interpretation has to be $ij|i$, and vice versa for odd i.

Suppose x contains no mordents. Then $|x| \leq n+2$, and the letters of x are either increasing or decreasing. Assume they are increasing. If $|x| = n+2$ then x has exactly one interpretation, $x = i|(i+1)\cdots(i-1)i|(i+1)$, or else we would get that $\mathbf{a}_n$ contains two consecutive even blocks. If $|x| = n+1$ then a priori x has two possible interpretations: $x = i|(i+1)\cdots(i-1)i|$ or $x = |i(i+1)\cdots(i-1)|i$. However, the first case is possible if and only if i is odd, since when n is even, no φ-block ends with an even letter. Similarly, the second case is possible if and only if i is even. $\square$

By the above lemma, to compute $E(\mathbf{a}_n)$ for an even n it is enough to consider powers of the form $z = xyx$, where $|x| \leq n$ and x contains no mordent. By Lemma 7.6, such powers have exponent at most $(3n-2)/(2n-2)$. This completes the proof of Theorem 7.1 for all even $n \geq 4$.

7.3 Arshon words of odd order

As already mentioned in Section 3.2.4, odd Arshon words cannot be generated by iterating a single morphism. However, we can adjust most of the terms we used in the previous chapter to the Arshon operator. In particular, the concept of synchronization applies to φ_n: a word $z \in \mathrm{Sub}(\mathbf{a}_n)$ is synchronized if any two interpretations of z by φ_n are synchronized. A word $z = z_1z_2 \in \mathrm{Sub}(\mathbf{a}_n)$ has a synchronization point at $z_1|z_2$ if, whenever $\varphi_n(u) = v_1zv_2$ for some $u, v_1, v_2 \in \mathrm{Occ}(\mathbf{a}_n)$, we have $u = u_1u_2$, $\varphi_n(u_1) = v_1z_1$, and $\varphi_n(u_2) = z_2v_2$.

To illustrate the power structure in Arshon words of odd order, consider $\mathbf{a}_5$:

$$\begin{array}{ccll} & & \varphi_{e,5} & \varphi_{o,5} \\ 0 & \rightarrow & 01234 & 43210 \\ 1 & \rightarrow & 12340 & 04321 \\ 2 & \rightarrow & 23401 & 10432 \\ 3 & \rightarrow & 34012 & 21043 \\ 4 & \rightarrow & 40123 & 32104 \end{array}$$

$$\mathbf{a}_5 = 01234|04321|23401|21043|40123|43210|40123|21043|23401|04321|23401|21043|\cdots$$

Lemma 7.10. *Let $z \in Occ(\mathbf{a}_n)$. If z has a synchronization point then z has a unique interpretation under φ_n, and inv(z) can be determined uniquely.*

Proof. Suppose z has a synchronization point, $z = u|v$. Then either $|u| > 1$ or $|v| > 1$ (or both): if $|u| = |v| = 1$, then by Lemma 7.4 $uv = i(i \pm 1)$ for some $i \in \Sigma_n$, and so uv can occur in either $\varphi_{e,n}(u)$ or $\varphi_{o,n}(u+1)$. Suppose $|u| > 1$. If the last two characters of u are increasing, we know that an even φ-block ends at u and an odd φ-block starts at v, and vice versa if the last two characters of u are decreasing. Since both $\varphi_{e,n}$ and $\varphi_{o,n}$ are uniform marked morphisms, and since we know φ-blocks alternate between even and odd, we can infer inv(z) unambiguously from $u|v$. $\square$

Lemma 7.11. *Let $n \geq 3$ be odd, and let $z = xyx \in Occ(\mathbf{a}_n)$ be an unstretchable p/q-power, such that x has a synchronization point. Then there exists an r/s-power $z' \in Occ(\mathbf{a}_n)$, such that $p = nr$, $q = ns$, and $z = \varphi_n(z')$.*

Proof. Since x has a synchronization point, it has a unique interpretation under φ_n. Suppose x does not begin at a borderline of φ-blocks. Then $x = t|w$, where t is a nonempty suffix of a φ-block, and $z = t|wyt|w$. But since the interpretation is unique, both occurrences of t must be preceded by a word s, such that st is a φ-block. Thus z can be stretched by s to the left, a contradiction. Therefore, x begins at a borderline, and so y ends at a borderline. For the same reason, x must end at a borderline, and so y must begin at a borderline. We get that both x and y have an exact decomposition under φ_n, and this decomposition is unique. In particular, both occurrences of x have the same inverse image under φ_n. Let k, l be the number of φ-blocks composing x, y, respectively. Then $p = n(2k + l)$, $q = n(k + l)$, and $\varphi_n^{-1}(z)$ is a $(2k + l)/(k + l)$-power. $\square$

Note: Lemmata 7.10, 7.11 are the equivalents of Lemma 7.7 for the odd case: since there is no ambiguity at the edges, there can be no stretch when applying φ_n to a p/q-power, and so a synchronized p/q-power is the image under φ_n of another p/q-power.

Lemma 7.12. *Let $n \geq 3$ be odd. Then every subword $z \in Sub(\mathbf{a}_n)$ with $|z| \geq 3n$ has a unique interpretation under φ_n.*

Proof. Consider a subword that contains a pair of consecutive mordents, $z = iji\ u\ klk$. If $|u| = n-4$ (that is, these are near mordents), then z contains two synchronization points, $z = i|ji\ u\ kl|k$: otherwise, we get a φ-block that contains a repeated letter, a contradiction. Similarly, if $|u| = n-2$ (a pair of far mordents), z contains the synchronization points $z = ij|i\ u\ k|lk$. To illustrate, consider $\mathbf{a}_5$: let $z = a_4 \cdots a_{10} = 404\ 3\ 212$. A borderline $40|4$ implies that 43212 is a φ-block, a contradiction; a borderline $2|12$ implies that 40432 is a φ-block, again a contradiction. Now let $z = a_7 \cdots a_{16} = 212\ 340\ 121$. A borderline $2|12$ implies that 121 is a prefix of a φ-block, while a borderline $12|1$ implies that 212 is a suffix of a φ-block. Again, we get a contradiction.

If $|u| = n-3$ (neutral mordents), then z has two possible interpretations, either $z = i|ji\ u\ k|lk$ or $z = ij|i\ u\ kl|k$. However, by Currie [31], $\mathbf{a}_n$ does not contain two consecutive pairs of neutral mordents: out of three consecutive mordents, at least one of the pairs is either near or far. (It is also easy to see that this is the case by a simple inverse image analysis: a subword of the form $ij|i\ u\ kl|k\ v\ rs|r$ or $i|ji\ u\ k|lk\ v\ r|sr$ implies that $\mathbf{a}_n$ contains a square of the form $abab$, $a, b \in \Sigma_n$, a contradiction: by Arshon, $\mathbf{a}_n$ is square-free.)

Let $z \in \mathrm{Sub}(\mathbf{a}_n)$ satisfy $|z| = 3n$. If z contains a pair of near or far mordents, then z has a unique ancestor. Otherwise, z contains a pair of neutral mordents, $iji\ u\ klk$, where $|u| = n-3$, and there are two possible interpretations: $i|ji\ u\ k|lk$ or $ij|i\ u\ kl|k$. Let $i'j'i'$ be the mordent on the left of iji, and let $k'l'k'$ be the mordent on the right of klk. Since no two consecutive neutral mordents occur, $i'j'i'$ and $k'l'k'$ must form near or far mordents with iji and klk.

If the interpretation is $i|ji\ u\ k|lk$, then $k'l'k'$ forms a near pair with klk, while $i'j'i'$ forms a far pair with iji. Therefore, $k'l'k'$ is $n-4$ letters away from klk, while $i'j'i'$ is $n-2$ letters away from iji. By assumption, z does not contain a near pair or a far pair, therefore z can contain at most $n-2$ letters to the right of klk, and at most n letters to the left of iji. Since $|z| = 3n$, this means that either $z = j'|i'\ x\ i|ji\ u\ k|lk\ v\ k'$ or $z = |i'\ x\ i|ji\ u\ k|lk\ v\ k'l'|$, where $|x| = n-2$ and $|v| = n-4$. Similarly if the interpretation is $ij|i\ u\ kl|k$, then either $z = |j'i'\ v\ ij|i\ u\ kl|k\ x\ k'|$ or $z = i'\ v\ ij|i\ u\ kl|k\ x\ k'|l'$, where $|x| = n-2$ and $|v| = n-4$. In any case, z contains enough letters to determine if the far mordent is on the left or on the right, and the interpretation is unique. $\square$

Example 7.2. For $n = 5$, the occurrence $z = a_{21} \cdots a_{34} = 01234321040123$, of length $3n - 1 = 14$, has two possible interpretations under φ_5, as illustrated in Fig. 7.1. However,

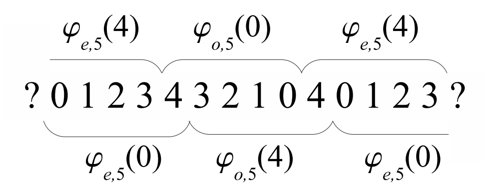

Figure 7.1: Two interpretations under φ_5.

if either of the left or right question marks is known, the ambiguity is solved: the top interpretation is valid if and only if the left question mark equals 4 (so as to complete the φ-block) and the right question mark equals 2 (so as to complete the near mordent). The bottom interpretation is valid if and only if the left question mark equals 1 (so as to complete the near mordent) and the right question mark equals 4 (so as to complete the φ-block).

Note: Lemma 7.12 is an improvement of a similar lemma of Currie [31], who proved that every occurrence of length $3n + 3$ or more has a unique interpretation.

Corollary 7.13. *The critical exponent of an odd Arshon word is the largest exponent of powers of the form $z = xyx$, such that $|x| < 3n$.*

Proof. Follows from Lemmata 7.11, 7.12. $\square$

To compute $E(\mathbf{a}_n)$ we need to consider subwords of the form xyx, with x unsynchronized. Moreover, the two occurrences of x should have different interpretations, or else we could take an inverse image under φ_n. For a fixed n, it would suffice to run a computer check on a finite number of subwords of $\mathbf{a}_n$; this is exactly the technique Klepinin and Sukhanov employed in [70]. For a general n, we need a more careful analysis.

Lemma 7.14. *Let $n \geq 3$, n odd. For all mordents in $\mathbf{a}_n$,*

1. *$inv(i(i+1)i) = (i+2)^{(e)}(i+1)^{(o)}$ or $inv(i(i+1)i) = (i+1)^{(e)}(i+2)^{(o)}$;*
2. *$inv(i(i-1)i) = (i-1)^{(o)}(i)^{(e)}$ or $inv(i(i-1)i) = (i)^{(o)}(i-1)^{(e)}$.*

Proof. A mordent iji can admit two possible borderlines: $ij|i$ or $i|ji$. Consider the mordent $i(i+1)i$. If the borderline is $i(i+1)|i$, then $i(i+1)$ is a suffix of an increasing φ-block, and so the block must be an image under $\varphi_{e,n}$. By definition of $\varphi_{e,n}$, $i(i+1)$ is the suffix of $\varphi_{e,n}(i+2)$. Since even and odd blocks alternate, the next block must be an image under $\varphi_{o,n}$, and by definition, i is the prefix of $\varphi_{o,n}(i+1)$.

If the borderline is $i|(i+1)i$, then $(i+1)i$ is the prefix of a decreasing φ-block, and by similar considerations this block is $\varphi_{o,n}(i+2)$, while the previous block is $\varphi_{e,n}(i+1)$. The assertion for $i(i-1)i$ is proved similarly. □

Lemma 7.15. *Let $n \geq 3$, n odd, and let $z \in Occ(\mathbf{a}_n)$.*

1. *If $z = i^{(e)}ui^{(o)}$ or $z = i^{(o)}ui^{(e)}$ for some $i \in \Sigma_n$, then $|u| \geq n-1$;*
2. *If $z = i^{(e)}u(i \pm 1)^{(e)}$ or $z = i^{(o)}u(i \pm 1)^{(o)}$ for some $i \in \Sigma_n$, then $|u| \geq n-2$.*

Proof. Let $z = i^{(e)}ui^{(o)}$, and suppose $i^{(o)}$ does not occur in u (otherwise, if $u = u'i^{(o)}u''$, take $z = i^{(e)}u'i^{(o)}$). If $|u| < n-1$ then z must contain a mordent in order for i to be repeated. But then the two occurrences of i have the same parity, a contradiction. The rest of the cases are proved similarly. □

Lemma 7.16. *Let $n \geq 3$ be odd, and let $z = xyx = (xy)^{p/q} \in Occ(\mathbf{a}_n)$ be an unstretchable power, such that x is unsynchronized and contains a mordent. Then $p/q < E(\mathbf{a}_n)$.*

Proof. Suppose x contains the mordent $i(i+1)i$ (the case of $i(i-1)i$ is symmetric). Then the two occurrences of the mordent have different interpretations, else we could take an inverse image under φ_n and get a power with the same exponent. By Lemma 7.14, there are two different cases, according to which interpretation comes first:

$$\overbrace{\cdots i(i+1)}^{\varphi_{e,n}(i+2)}\,|\,\overbrace{i(i-1)\cdots(i+2)(i+1)}^{\varphi_{o,n}(i+1)}\,|\,\overbrace{\cdots\cdots\cdots}^{n-1\ \ \varphi-\text{blocks}}\,|\,\overbrace{(i+1)(i+2)\cdots(i-1)i}^{\varphi_{e,n}(i+1)}\,|\,\overbrace{(i+1)i\cdots}^{\varphi_{o,n}(i+2)}$$

$$\overbrace{\cdots i}^{\varphi_{e,n}(i+1)}\;|\,\overbrace{(i+1)i\cdots(i+3)(i+2)}^{\varphi_{o,n}(i+2)}\,|\,\overbrace{\cdots\cdots\cdots}^{n-1\ \ \varphi-\text{blocks}}\,|\,\overbrace{(i+2)(i+3)\cdots i(i+1)}^{\varphi_{e,n}(i+2)}\,|\,\overbrace{i(i-1)\cdots}^{\varphi_{o,n}(i+1)}$$

By Lemma 7.15, in both cases there must be at least $n-1$ additional φ-blocks between the blocks containing the two $i(i+1)i$ occurrences. Therefore, we get that $q \geq n^2+n+1$ in the first case and $q \geq n^2+n-1$ in the second case. Now, x is unsynchronized, and so by Lemma 7.12 $|x| < 3n$. Therefore, $|x|/q \leq (3n-1)/(n^2+n-1) < n/(2n-2)$ for all $n \geq 3$, and so $p/q = (|x|+q)/q < (3n-2)/(2n-2) \leq E(\mathbf{a}_n)$. □

By Lemma 7.16, in order to compute $E(\mathbf{a}_n)$ it is enough to consider powers xyx such that x is unsynchronized and contains no mordents. The longest subword that contains no mordents is of length $n+2$, but such subword implies a far pair, and has a unique ancestor. Therefore, we can assume $|x| \leq n+1$.

Lemma 7.17. *Let $n \geq 3$ be odd, and let $z = xyx = (xy)^{p/q} \in Occ(\mathbf{a}_n)$ be an unstretchable power, such that x is unsynchronized, x contains no mordents, and $|x| = n+1$. Then $p/q < E(\mathbf{a}_n)$.*

Proof. Since $|x| = n+1$ and x contains no mordents, necessarily $x = ivi$, where $i \in \Sigma_n$ and either $v = (i+1)\cdots(n-1)01\cdots(i-2)(i-1)$, or $v = (i-1)\cdots 01(n-1)\cdots(i+2)(i+1)$. Suppose the letters of v are increasing, and assume without loss of generality that $i = 0$. Then x admits two possible interpretations: $x = 01\cdots(n-1)|0$ or $x = 0|1\cdots(n-1)0$. The ancestors of the first and second case are $\mathrm{inv}(x) = 0^{(e)}1^{(o)}$ and $\mathrm{inv}(x) = 0^{(o)}1^{(e)}$, respectively. Any other interpretation is impossible, since it implies $\mathbf{a}_n$ contains two consecutive even φ-blocks.

Since x is unsynchronized, its two occurrences in z have different interpretations. By

Lemma 7.14, there are two possible cases:

$$\overbrace{01\cdots(n-1)}^{\varphi_{e,n}(0)}\,|\,\overbrace{0\cdots}^{\varphi_{o,n}(1)}\,|\,\overbrace{\cdots\cdots\cdots}^{n-2\ \ \varphi-\text{blocks}}\,|\,\overbrace{\cdots 0}^{\varphi_{o,n}(0)}\,|\,\overbrace{1\cdots(n-1)0}^{\varphi_{e,n}(1)}\ ,$$

$$\overbrace{\cdots 0}^{\varphi_{o,n}(0)}\,|\,\overbrace{1\cdots(n-1)0}^{\varphi_{e,n}(1)}\,|\,\overbrace{\cdots\cdots\cdots}^{n-2\ \ \varphi-\text{blocks}}\,|\,\overbrace{01\cdots(n-1)}^{\varphi_{e,n}(0)}\,|\,\overbrace{0\cdots}^{\varphi_{o,n}(1)}\ .$$

By Lemma 7.15, in both cases y contains at least $n-2$ additional φ-blocks. Therefore, $q \geq n^2 - n + 1 > 2n - 2$ for all $n \in \mathbb{N}$, and so $|x|/q = (n+1)/q < n/(2n-2)$ for all $n \geq 3$. Again, $p/q < (3n-2)/(2n-2) \leq E(\mathbf{a}_n)$. □

By Lemma 7.17, to compute $E(\mathbf{a}_n)$ for an odd $n \geq 3$ it is enough to consider powers of the form $z = xyx$ such that $|x| \leq n$ and x contains no mordent. By Lemma 7.6, such powers have exponent at most $(3n-2)/(2n-2)$. This completes the proof of Theorem 7.1.

Chapter 8

Stabilizers of Infinite Words

8.1 Introduction

In this chapter we leave critical exponents, and begin our study of stabilizers. Recall from Section 2.5 that the stabilizer $\mathcal{S}tab(\mathbf{w})$ of an infinite word $\mathbf{w} \in \Sigma^\omega$ is the monoid of morphisms $f \in \mathcal{M}(\Sigma)$ that satisfy $f(\mathbf{w}) = \mathbf{w}$, and that $\mathbf{w}$ is rigid if $\mathcal{S}tab(\mathbf{w})$ is cyclic. We are interested in the structure of stabilizers of aperiodic words. In particular, we are interested in the following questions:

1. How many generators can a stabilizer of an aperiodic infinite binary word have?
2. Can we characterize morphisms that, when iterated, generate rigid words?
3. Do there exist infinitely generated stabilizers of aperiodic infinite words over finite alphabets?

We manage to give a partial answer to the first question, some negative answers to the second question, and no answers to the third one. However, through studying these questions, we manage to shed some light on the structure of stabilizers. The reason we concentrate on aperiodic words is that stabilizers of periodic ones can have any number of generators. For example, over a unary alphabet $\Sigma = \{a\}$, the only infinite word is $\mathbf{w} = a^\omega$, and the stabilizer $\mathcal{S}tab(\mathbf{w})$ satisfies

$$\mathcal{S}tab(\mathbf{w}) = \{f_m : m > 0\}, \text{ where } f_m(a) = a^m.$$

Clearly, $\mathcal{S}tab(\mathbf{w})$ is infinitely generated by the set $\{f_p : p \text{ is prime}\}$.

Unlike critical exponents, the subject of stabilizers of infinite words over free monoids has hardly been studied. The main results in this area are due to Pansiot and Séébold, and concern the rigidity of some families of infinite words. Pansiot proved the rigidity of the Thue-Morse word [103] and of the Fibonacci word [104]. Séébold proved the rigidity of all Sturmian words [124] and of all Prouhet words and all Arshon words of even order [125] (recall that the Prouhet word of order n is the generalized Thue-Morse word $\mathbf{t}_{n,n}$; see Section 3.2.3).

Other related results concern morphism monoids that are not stabilizers. The monoid of invertible morphisms (see Section 8.3.2) over a 3-letter alphabet is not finitely generated (Wen and Zhang, [138]; Richomme, [112]). Neither are the following monoids: primitive (uniform) morphisms over an alphabet of size at least 2; overlap-free (uniform) morphisms over an alphabet of size at least 3; k-power-free (uniform) morphisms over an alphabet of size at least 2, where $k \geq 3$ is an integer (Richomme, [113]. In this context, primitive morphisms are morphisms that preserve primitive words). However, these results do not imply that there exist words that have infinitely generated stabilizers.

The rest of the chapter is organized as follows: in Section 8.2 we consider stabilizers of infinite binary words. We show that for all $n \in \mathbb{N}$ there exists an aperiodic infinite binary word such that its stabilizer cannot be generated by fewer than n morphisms. Among the stabilizer elements are primitive uniform morphisms, that is, a primitive uniform morphism does not necessarily generate a rigid word when iterated.

In Section 8.3 we give an example of an aperiodic ternary word for which the monoid of morphisms generating it by iteration (the *iterative stabilizer*) is infinitely generated. Among this monoid's elements are primitive invertible morphisms. The stabilizer itself is not cyclic. Again, this shows that a primitive and invertible morphism does not necessarily generate a rigid word when iterated. In Section 8.4 we concentrate on epistandard words. We show that strict epistandard words that have a non-trivial stabilizer are always rigid, and characterize the stabilizing morphisms of ultimately strict epistandard words.

The results in this chapter are to appear in Krieger [78].

8.2 Stabilizers of binary words

In this section, we consider stabilizers of aperiodic infinite binary words. In the past it had been conjectured by Berstel [128] that all pure morphic aperiodic binary words are rigid[1]. However, the following counterexample, due to Séébold [128], proves the conjecture to be false: let

$$f = (01, 100110), \quad g = (011001, 10), \quad \mathbf{w} = f^{\omega}(0).$$

Then $f(01) = g(01)$ and $f(10) = g(10)$. Since $\mathbf{w} \in \{01, 10\}^{\omega}$, necessarily $g(\mathbf{w}) = f(\mathbf{w}) = \mathbf{w}$. On the other hand, since $|f(0)| < |g(0)|$ and $|f(1)| > |g(1)|$, f and g cannot be powers of a common morphism. Therefore, $\mathcal{S}tab(\mathbf{w})$ is generated by at least two elements.

This example can be generalized to any finite number of generators, as the following theorem shows:

Theorem 8.1. *For all $m \in \mathbb{N}$ there exists an aperiodic word $\mathbf{w} \in \Sigma_2^{\omega}$, such that $\mathcal{S}tab(\mathbf{w})$ cannot be generated by fewer than $m+1$ morphisms.*

Proof. Let $m \in \mathbb{N}$, and let $u, v \in \{01, 10\}^{+}$, such that u begins with 01 and $uv \neq vu$. Define $m+1$ morphisms, $f_0, f_1, \ldots, f_m \in \mathcal{M}(\Sigma_2)$, by

$$f_i : \begin{cases} 0 \to (uv)^i u, \\ 1 \to (vu)^{m-i} v, \end{cases} \quad 0 \leq i \leq m.$$

Let $\mathbf{w} = f_0^{\omega}(0)$. Then $\mathbf{w}$ is aperiodic: we refer the reader to Theorem 3.3, that classifies all binary morphisms whose language is repetitive. By definition, f_0 does not belong to any of the classes $1, \ldots, 7$ of Theorem 3.3, and since $uv \neq vu$, neither does f_0 belong to class 8 of Theorem 3.3. Therefore, $L(f_0)$ is not repetitive. In particular, $\mathbf{w}$ is aperiodic.

By definition, for all $0 \leq i, j \leq m$,

$$\begin{aligned} f_i(01) = f_j(01) &= (uv)^{m+1} , \\ f_i(10) = f_j(10) &= (vu)^{m+1} . \end{aligned}$$

[1] The term "rigid" is due to Berstel. Originally, he used the term to denote words that have a cyclic *iterative* stabilizer, but as Lemma 2.9 shows, in the binary case all aperiodic words that have a non-trivial stabilizer are essentially generated by iteration.

Since $\mathbf{w} \in \{01, 10\}^\omega$, this implies that $f_i(\mathbf{w}) = f_j(\mathbf{w}) = \mathbf{w}$ for all $i \neq j$, and so $f_i \in \mathcal{S}tab(\mathbf{w})$ for all $0 \leq i \leq m$.

To see that $\mathcal{S}tab(\mathbf{w})$ cannot be generated by fewer than $m+1$ morphisms, we first prove the following lemma:

Lemma 8.2. *Let $g \in \mathcal{S}tab(\mathbf{w})$. If $g \neq \mathrm{Id}$, then both $|g(0)|$ and $|g(1)|$ are even.*

Proof. Since $\mathbf{w}$ begins with 01, and since $g \neq \mathrm{Id}$, by Lemma 2.9 either $\mathbf{w} = g^\omega(0)$, or $g(0) = 0$ and $\mathbf{w} = 0g^\omega(1)$. Let $\mathbf{w} = w_0 w_1 w_2 \ldots$. Since $\mathbf{w} \in \{01, 10\}^\omega$, necessarily

$$w_{2n} \neq w_{2n+1} \text{ for } n = 0, 1, 2, \ldots \tag{8.1}$$

Also, since $\mathbf{w}$ is aperiodic, it must contain both 00 and 11 as subwords, and so it must contain both $g(0)g(0)$ and $g(1)g(1)$.

Suppose $|g(0)|$ is odd. Since $g(0)g(0)$ occurs in $\mathbf{w}$, necessarily $g(0)$ occurs both at an odd and an even position in $\mathbf{w}$. This implies that $g(0) = (01)^k 0$ for some $k \geq 0$, or else we would get a violation to (8.1). Also, since $\mathbf{w}$ begins with $g(0)g(1)$ and $g(0)$ ends with 0, $g(1)$ must begin with 1.

If $|g(1)|$ is odd, a similar argument shows that $g(1) = (10)^m 1$ for some $m \geq 0$, which implies that $\mathbf{w} = (01)^\omega$, a contradiction. Assume therefore that $|g(1)|$ is even. If $g(0) \neq 0$, it must satisfy $|g(0)| \geq 3$. In this case, $\mathbf{w}$ begins with 0101 and therefore with $g(0)g(1)g(0)g(1)$. Since $|g(1)|$ is even, the first $g(1)$ block begins at an odd position, while the second one begins at an even position. This implies that $g(1) = (10)^m$ for some $m \geq 1$. But then we get that $\mathbf{w}$ contains the occurrence 00 at an even position (the borderline between the first $g(1)$ and the second $g(0)$ blocks), a contradiction to (8.1).

If $g(0) = 0$, it is possible for $\mathbf{w}$ to begin with 0110. But if $g(0) = 0$ necessarily $g(1)$ begins and ends with 1, or else we would get that $\mathbf{w}$ is repetitive, a contradiction (class 2 of Theorem 3.3). Therefore, $\mathbf{w}$ contains the occurrence 11 at an even position (the borderline between the first and second $g(1)$ blocks), a contradiction.

We conclude that $|g(0)|$ must be even. Suppose that $|g(0)|$ is even and $|g(1)|$ is odd. A similar argument shows that both $g(0)$ and $g(1)$ occur both at odd and even positions, which implies that $\mathbf{w}$ contains a pair 00 or 11 at an even position, a contradiction. $\square$

We now continue with the proof of Theorem 8.1. Let $\mathrm{Id} \neq g \in \mathcal{S}tab(\mathbf{w})$. By Lemma 8.2, both $|g(0)|$ and $|g(1)|$ are even, and since $\mathbf{w} \in \{01, 10\}^\omega$, necessarily $|g(0)|_0 = |g(0)|_1$ and $|g(1)|_0 = |g(1)|_1$. In other words, the incidence matrix of g has the form

$$A(g) = \begin{pmatrix} a & b \\ a & b \end{pmatrix} \ ; \ a, b \in \mathbb{N}.$$

Let $\mathcal{G}$ be a set of generators for $\mathcal{S}tab(\mathbf{w})$. Then for all $h \in \mathcal{S}tab(\mathbf{w})$, we have $h = h_1 \cdots h_k$ for some $h_1, \ldots, h_k \in \mathcal{G}$. Let

$$A(h_i) = \begin{pmatrix} a_i & b_i \\ a_i & b_i \end{pmatrix} \ ; \ i = 1, 2, \ldots, k.$$

Then

$$A(h) = \begin{pmatrix} a_1 & b_1 \\ a_1 & b_1 \end{pmatrix} \cdots \begin{pmatrix} a_k & b_k \\ a_k & b_k \end{pmatrix} \ = \ (a_1 + b_1) \cdots (a_{k-1} + b_{k-1}) \begin{pmatrix} a_k & b_k \\ a_k & b_k \end{pmatrix},$$

and so

$$\begin{aligned} |h(0)| &= 2a_k(a_1 + b_1) \cdots (a_{k-1} + b_{k-1}), \\ |h(1)| &= 2b_k(a_1 + b_1) \cdots (a_{k-1} + b_{k-1}). \end{aligned}$$

We get that $|h(0)|/|h(1)| = a_k/b_k = |h_k(0)|/|h_k(1)|$. Denote this ratio by $\rho(h)$. By the above, $\rho(h)$ depends only on the last morphism in a representation of h as a product of elements of $\mathcal{G}$; if h has more than one representation, then necessarily the last morphism in each representation has the same ratio.

Now suppose that $|\mathcal{G}| < m+1$. Then there must exist i and j with $i \neq j$ and $0 \leq i, j \leq m$ such thatf_i and f_j have representations with the same last element, i.e., $\rho(f_i) = \rho(f_j)$. But then we get:

$$\rho(f_i) = \frac{(i+1)|u| + i|v|}{(m-i)|u| + (m-i+1)|v|} = \frac{(j+1)|u| + j|v|}{(m-j)|u| + (m-j+1)|v|} = \rho(f_j).$$

Simplifying, we get

$$(m+1)(i-j)(|u| + |v|)^2 = 0.$$

Since $|u|$, $|v|$ and $m+1$ are positive, necessarily $i = j$, a contradiction. Therefore, $\mathcal{G}$ must contain at least $m+1$ elements. $\square$

Example 8.1. Let $u = 01$, $v = 10$, $m = 2$. Then the morphisms $f_0 = (01, 1001100110)$, $f_1 = (011001, 100110)$, and $f_2 = (0110011001, 10)$ generate the same aperiodic word. Note that f_1 is uniform, and so the fact that a word is generated by a uniform morphism is not enough to guarantee its rigidity.

By Lemma 8.2, all the words constructed in Theorem 8.1 satisfy $\mathcal{S}tab(\mathbf{w}) = \mathcal{IS}tab(\mathbf{w})$. It remains an open question whether there exist infinitely generated iterative stabilizers over binary alphabets. We believe the answer is negative.

8.3 Stabilizers of words over ternary alphabets

Over alphabets of more than two letters, it is much easier to construct infinitely generated iterative stabilizers. Moreover, even "nice" morphisms can generate by iteration aperiodic words with infinitely generated iterative stabilizers, as we show in this section.

In this section, $\Sigma = \Sigma_3$ and $\mathcal{M} = \mathcal{M}(\Sigma)$.

8.3.1 An infinitely generated iterative stabilizer

Theorem 8.3. *There exists an aperiodic word $\mathbf{w} \in \Sigma^\omega$ such that $\mathcal{IS}tab(\mathbf{w})$ is infinitely generated.*

To prove Theorem 8.3, let $f = (02, 02, 1)$, and let $\mathbf{w} = f^\omega(0)$. Define a sequence of morphisms, $\{h_n\}_{n \geq 1}$, by $h_1 = f$, and for $n \geq 1$,

$$h_{n+1} : \begin{cases} 0 \to h_n(0), \\ 1 \to h_n(02), \\ 2 \to h_n(21). \end{cases}$$

Lemma 8.4. *Let $\phi \in \mathcal{M}(\Sigma_2)$ be the Fibonacci morphism, $\phi = (01, 0)$. Let $\mathbf{f} = \phi^\omega(0)$ be the Fibonacci word. Define a morphism $\eta : \Sigma_3^* \to \Sigma_2^*$ by $\eta = (0, 1, \varepsilon)$. Then $\eta(\mathbf{w}) = \mathbf{f}$.*

Proof. We prove by induction that $\eta f^n(0) = \phi^{n-1}(0)$ for all $n \geq 1$. The assertion clearly holds for $n = 1$ and $n = 2$. Assume $n \geq 3$. Then

$$\eta f^n(0) = \eta f^{n-1}(02) = \eta f^{n-1}(0)\eta f^{n-2}(1) = \eta f^{n-1}(0)\eta f^{n-2}(0) = \phi^{n-2}(0)\phi^{n-3}(0) = \phi^{n-1}(0).$$

□

Corollary 8.5. $\mathbf{w}$ *is aperiodic.*

Proof. The Fibonacci word $\mathbf{f}$ is aperiodic, as $E(\mathbf{f}) = 2 + \tau < \infty$ (see Section 6.5.1). □

Lemma 8.6. *For all* $n \geq 1$

$$h_n(02) = f^n(02),$$
$$h_n(1) = f^n(1).$$

Proof. The assertion clearly holds for $n = 1$. Assume it holds for n. Then we get:

$$h_{n+1}(02) = h_{n+1}(0)h_{n+1}(2) = h_n(0)h_n(21) = h_n(02)h_n(1) = f^n(02)f^n(1) = f^n(f(02)) = f^{n+1}(02),$$

$$h_{n+1}(1) = h_n(02) = f^n(02) = f^n(f(1)) = f^{n+1}(1).$$

□

Corollary 8.7. $h_n \in \mathcal{IS}tab(\mathbf{w})$ *for all* n.

Proof. By definition of f, $\mathbf{w} \in \{02, 1\}^\omega$, and therefore $h_n(\mathbf{w}) = f^n(\mathbf{w}) = \mathbf{w}$ for all n. Also, h_n is prolongable on 0 for all n, and hence belongs to $\mathcal{IS}tab(\mathbf{w})$. □

Lemma 8.8. *Let* $n \geq 1$*, and suppose* $h_n = \varphi\psi$ *for some* $\varphi, \psi \in \mathcal{IS}tab(\mathbf{w})$*. Then* $\varphi = \mathrm{Id}$ *and* $\psi = h_n$ *(or vice versa).*

Proof. If both $\varphi, \psi \neq \mathrm{Id}$ then they are both prolongable on 0, i.e., $\varphi(0) = 02x$ and $\psi(0) = 02y$ for some $x, y \in \Sigma^*$. If both φ, ψ are nonerasing, then

$$02 = h_n(0) = \varphi(\psi(0)) = \varphi(02y) = 02x\varphi(2)\varphi(y),$$

a contradiction, since $\varphi(2) \neq \varepsilon$. If ψ is erasing then so is h_n, a contradiction. The only option is that ψ is nonerasing and φ is erasing. If $\varphi(1) = \varepsilon$ then $\varphi(\mathbf{w}) = \varphi(02)^\omega$, a contradiction: $\mathbf{w}$ is aperiodic. This leaves only $\varphi(2) = \varepsilon$.

Suppose $\psi(0) = 021z$ for some $z \in \Sigma^*$. Then

$$02 = \varphi(0)\varphi(2)\varphi(1)\varphi(z).$$

Since $\varphi(0), \varphi(1) \neq \varepsilon$, the equality above holds if and only if $\varphi = (0, 2, \varepsilon)$. But then $\varphi \notin \mathcal{S}tab(\mathbf{w})$, since $\mathbf{w}$ begins with 021, but $\varphi(\mathbf{w})$ begins with 020.

Now suppose that $\psi(0) = 02$. Then $02 = \varphi(0)\varphi(2)$, and so necessarily $\varphi(0) = 02$. But then $\varphi \notin \mathcal{IS}tab(\mathbf{w})$, since $\varphi^n(0) = 02$ for all $n \geq 1$. Therefore at least one of ψ, φ must equal Id. $\square$

Corollary 8.9. *$\mathcal{IS}tab(\mathbf{w})$ is infinitely generated.*

$\mathcal{S}tab(\mathbf{w})$ itself does not seem to be infinitely generated. In particular, for $g = (0, 02, 21)$, it is a straightforward induction to show that $h_{n+1} = h_n g$ for all $n \geq 1$, that is, $h_{n+1} = fg^n$ for all $n \geq 1$ (to see that $g \in \mathcal{S}tab(\mathbf{w})$, observe that $g(02) = 021 = f(02)$ and $g(1) = 02 = f(1)$). Whether there exists an infinitely generated stabilizer over a finite alphabet is an open question. However, $\mathbf{w}$ is not rigid: clearly, f and g cannot be powers of a common morphism, and the same holds for $(02, 1, \varepsilon)$ and $(\varepsilon, 1, 02)$, which are also stabilizer elements. (Note that, since $\mathcal{IS}tab(\mathbf{w})$ is infinitely generated, $\mathbf{w}$ is not rigid in the sense of Berstel either; see comment in the beginning of Section 8.2.)

8.3.2 Invertible morphisms

Let $\Sigma = \{a_1, \cdots, a_n\}$ be a finite alphabet. *The free group over* Σ, denoted by $\mathbb{F}_\Sigma$, is the free group generated by $\Sigma \cup \bar{\Sigma}$, where $\bar{\Sigma} = \{\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_n\}$, and for all $1 \leq i \leq n$,

- $a_i\bar{a}_i = \bar{a}_i a_i = \varepsilon$;
- $\bar{\bar{a}}_i = a_i$;
- $\bar{u} = \bar{u}_k \cdots \bar{u}_2\bar{u}_1$ for all $u = u_1u_2 \cdots u_k \in (\Sigma \cup \bar{\Sigma})^*$.

The free monoid Σ^* can be naturally embedded into $\mathbb{F}_\Sigma$, and every monoid morphism $f \in \mathcal{M}(\Sigma)$ can be extended to an endomorphism of $\mathbb{F}_\Sigma$, by defining $f(\bar{a}) = \overline{f(a)}$ for all $a \in \Sigma$. A morphism $f \in \mathcal{M}(\Sigma)$ is *invertible* if when extended to a free group endomorphism it is an automorphism, that is, there exists a free group endomorphism f^{-1}, such that $ff^{-1} = f^{-1}f = \mathrm{Id}$.

Over binary alphabets, invertible morphisms are exactly the Sturmian morphisms ([90, 137, 83]), and so, by [90, 124], all invertible morphisms generate rigid words. Over general alphabets, things get much more complicated. In particular, already for three-letter alphabets, the monoid of invertible morphisms is not finitely generated [138, 112] (recall that the monoid of Sturmian morphisms is finitely generated; see Section 2.8). This fact may lead one to suspect that over alphabets of more than two letters, invertible morphisms can generate non-rigid words. The next theorem shows that this is indeed the case.

Theorem 8.10. *There exists an aperiodic word* $\mathbf{w} \in \Sigma_3^\omega$ *and a morphism* $f \in \mathcal{S}tab(\mathbf{w})$*, such that* $\mathbf{w}$ *is not rigid and* f *is invertible.*

Proof. Let $g = (0210, 021, 2)$. Extended to a group morphism, it is easy to verify that g is invertible:

$$g : \begin{cases} 0 \to 0210 \\ 1 \to 021 \\ 2 \to 2 \\ \bar{0} \to \bar{0}\bar{1}\bar{2}\bar{0} \\ \bar{1} \to \bar{1}\bar{2}\bar{0} \\ \bar{2} \to \bar{2} \end{cases} , \quad g^{-1} : \begin{cases} 0 \to \bar{1}0 \\ 1 \to \bar{2}\bar{0}11 \\ 2 \to 2 \\ \bar{0} \to \bar{0}1 \\ \bar{1} \to \bar{1}\bar{1}02 \\ \bar{2} \to \bar{2} \end{cases} .$$

Let $f = (02, 02, 1)$, and let $\mathbf{w} = f^\omega(0)$. Then $g(02) = 02102 = f^2(02)$, and $g(1) = 021 = f^2(1)$. Since $\mathbf{w} \in \{02, 1\}^\omega$, we get that $g(\mathbf{w}) = f^2(\mathbf{w}) = \mathbf{w}$, and so $g \in \mathcal{S}tab(\mathbf{w})$. As $\mathbf{w}$ is not rigid (see previous section), the result follows. $\square$

Note: Another example of an invertible element of $\mathcal{S}tab(\mathbf{w})$ is $h = (021020, 02102, 21)$: the inverse morphism is given by $h^{-1} = (\bar{1}0, \bar{1}\bar{1}02\bar{1}02, \bar{0}1\bar{2}\bar{0}11)$, and $h(\{02, 1\}) = f^3(\{02, 1\})$. The morphism h is an example of an invertible morphism which is also primitive. This shows that an invertible primitive morphism does not necessarily generate a rigid word when iterated.

It remains an open question whether there exists a characterization of morphisms that generate rigid words. The "usual suspects" – uniform, primitive, or invertible – do not form such a characterization, as we have seen in the last two sections.

8.4 Epistandard words

Episturmian words, introduced by Droubay, Justin and Pirillo in [40], are a generalization of Sturmian words to alphabets of more than two letters. As in the Sturmian case, the class of episturmian words contains a subclass of *standard* episturmian (or *epistandard*) words. In this section we consider two classes of epistandard words. We show that all strict epistandard words are rigid; however, this assertion does not hold for non-strict ones. We then characterize the stabilizers of a certain class of non-strict aperiodic epistandard words.

In this section, $\Sigma = \Sigma_n$ for some $n \geq 3$, and $\mathcal{M} = \mathcal{M}(\Sigma)$. We use the symbols $\mathbf{s}$, $\mathbf{t}$ to denote arbitrary episturmian words (not to be mixed with the Thue-Morse word, which is completely unrelated).

8.4.1 Definitions and properties of episturmian words

In this section we define episturmian and epistandard words, and give some of their properties. Some of the material presented in this section was already introduced in Section 2.9, but we repeat it here for the sake of completeness. All the definitions and properties in this section are taken from [40, 64].

Definition 8.1. An infinite word $\mathbf{s} \in \Sigma^\omega$ is *episturmian* if the set of its subwords is closed under reversal, and $\mathbf{s}$ has at most one right special subword of length n for all $n \in \mathbb{N}$. An episturmian word $\mathbf{s}$ is *standard* (or *epistandard*) if all of its left special subwords are prefixes of it.

Definition 8.2. For all $a, b \in \Sigma$, define the following morphisms:

$$\psi_a : \begin{cases} a \to a \\ b \to ab \ \forall b \neq a \end{cases}, \quad \bar{\psi}_a : \begin{cases} a \to a \\ b \to ba \ \forall b \neq a \end{cases}, \quad \theta_{ab} : \begin{cases} a \to b \\ b \to a \\ c \to c \ \forall c \neq a, b \end{cases}.$$

The monoids of *episturmian morphisms* and *epistandard morphisms*, denoted by $\mathscr{E}$, $\mathscr{S}$, respectively, are defined by

$$\begin{aligned} \mathscr{E} &= \langle \psi_a, \bar{\psi}_a, \theta_{ab} : a, b \in \Sigma \rangle, \\ \mathscr{S} &= \langle \psi_a, \theta_{ab} : a, b \in \Sigma \rangle. \end{aligned}$$

A morphism generated by the set $\{\psi_a, \bar{\psi}_a : a \in \Sigma\}$ is called *pure*; a morphism generated by the set $\{\theta_{ab} : a, b \in \Sigma\}$ is called a *permutation*. Note that the set of transpositions $\{\theta_{ab} : a, b \in \Sigma\}$ generates all permutations over Σ, and that any permutation has an inverse, which is also a permutation over Σ.

Property 8.1. Every episturmian morphism is invertible. In particular, it is injective.

Property 8.2. For every $a \in \Sigma$, and for every permutation ν over Σ, we have

$$\begin{aligned} \nu\psi_a &= \psi_{\nu(a)}\nu, \\ \nu\bar{\psi}_a &= \bar{\psi}_{\nu(a)}\nu. \end{aligned}$$

Property 8.3. For every epistandard morphism $\psi \in \mathscr{S}$ there exist *unique* letters $a_1, \ldots, a_n$ and a permutation ν, such that

$$\psi = \psi_{a_1}\psi_{a_2}\cdots\psi_{a_n}\nu.$$

Property 8.4. If $\mathbf{s}$ is an epistandard (resp., episturmian) word and $\psi \in \mathscr{S}$ (resp., $\psi \in \mathscr{E}$), then $\psi(\mathbf{s})$ is an epistandard (resp., episturmian) word.

Property 8.5. For every epistandard word $\mathbf{s} \in \Sigma^\omega$ there exists a unique infinite word $\Delta(\mathbf{s}) = x_1x_2x_3\cdots \in \Sigma^\omega$, $x_i \in \Sigma$, such that $\mathbf{s} = \lim_{n\to\infty} u_n$, where $\{u_n\}_{n=0}^{\infty} \subseteq \Sigma^*$ is defined by

$$\begin{aligned} u_0 &= \varepsilon, \\ u_n &= (u_{n-1}x_n)^{(+)},\ n \geq 1. \end{aligned}$$

Here $w^{(+)}$ denotes the palindromic closure of a word $w \in \Sigma^*$ (see Section 2.1.2).

Definition 8.3. The word $\Delta(\mathbf{s})$ defined above is called the *directive word* of the epistandard word $\mathbf{s}$. An epistandard word $\mathbf{s}$ is Σ-*strict* (or simply *strict*) if $\Delta(\mathbf{s})$ is letter-recurrent.

Property 8.6. An infinite word $\mathbf{s}$ is epistandard if and only if there exists an epistandard word $\mathbf{t}$ and a letter a such that $\mathbf{s} = \psi_a(\mathbf{t})$. Moreover, $\mathbf{t}$ and a are unique, and $\Delta(\mathbf{s}) = a\Delta(\mathbf{t})$.

Property 8.7. An epistandard word $\mathbf{s}$ is ultimately periodic if and only if $\Delta(\mathbf{s}) = ua^\omega$ for some $u \in \Sigma^*$ and $a \in \Sigma$ (if this is the case, then $\mathbf{s}$ is actually purely periodic). In particular, Σ-strict epistandard words are aperiodic when $|\Sigma| \geq 2$.

Property 8.8. If $\mathbf{s}$ and $\mathbf{t}$ are epistandard (resp., episturmian) words, with $\mathbf{s}$ aperiodic and $\mathbf{t}$ Σ-strict, and $\psi \in \mathcal{M}$ satisfies $\psi(\mathbf{t}) = \mathbf{s}$, then ψ is an epistandard (resp., episturmian) morphism and $\mathbf{s}$ is Σ-strict.

Property 8.9. Let $\mathbf{s}$ be a Σ-strict epistandard word. Then $\mathcal{S}tab(\mathbf{s})$ is non-trivial if and only if $\Delta(\mathbf{s})$ is purely periodic. More specifically, if $\Delta(\mathbf{s}) = (x_1 \cdots x_n)^\omega$, then $\psi_{x_1} \cdots \psi_{x_n} \in \mathcal{S}tab(\mathbf{s})$.

Definition 8.4. A letter $a \in \Sigma$ is *separating* for a word $w \in \Sigma^\infty$ if for any subword of length two $xy \in \mathrm{Sub}(w)$, $x = a$ or $y = a$ (or both).

Property 8.10. If $\mathbf{s}$ is an epistandard word with first letter a then a is separating for $\mathbf{s}$.

8.4.2 Stabilizers of strict epistandard words

Definition 8.5. Let $\psi = \psi_{a_1}\psi_{a_2} \cdots \psi_{a_n}\nu$ be an epistandard morphism, where ν is a permutation. We define the *length* of ψ by $\|\psi\| = n$. By Property 8.3, the length is well-defined. For a word $u = a_1 \cdots a_n \in \Sigma^*$, $a_i \in \Sigma$, we denote $\psi_u = \psi_{a_1} \cdots \psi_{a_n}$. Note that for all $u, v \in \Sigma^*$, $\psi_{uv} = \psi_u\psi_v$, and that $\|\psi_u\| = |u|$.

Theorem 8.11. *All Σ-strict epistandard words that have a non-trivial stabilizer are rigid.*

Proof. Let $\mathbf{s} \in \Sigma^\omega$ be a Σ-strict epistandard word, and suppose $\mathcal{S}tab(\mathbf{s})$ is non-trivial. By Property 8.8, every morphism $h \in \mathcal{S}tab(\mathbf{s})$ is epistandard. Let $f, h \in \mathcal{S}tab(\mathbf{s})$. Then by Property 8.3, there exist unique letters $a_1, \ldots, a_k, b_1, \ldots, b_m \in \Sigma$ and permutations ν, θ over Σ, such that $f = \psi_{a_1 \cdots a_k}\nu$ and $h = \psi_{b_1 \cdots b_m}\theta$. Therefore,

$$\mathbf{s} = \psi_{a_1}(\psi_{a_2 \cdots a_k}\nu(\mathbf{s})),$$
$$\mathbf{s} = \psi_{b_1}(\psi_{b_2 \cdots b_m}\theta(\mathbf{s})).$$

Let $\mathbf{t} = \psi_{a_2 \cdots a_k}\nu(\mathbf{s})$, $\mathbf{t}' = \psi_{b_2 \cdots b_m}\theta(\mathbf{s})$. By Property 8.4, both $\mathbf{t}$ and $\mathbf{t}'$ are epistandard. Therefore, by Property 8.6, $\mathbf{t} = \mathbf{t}'$ and $a_1 = b_1$, and similarly (assume w.l.o.g. that $k \le m$), $a_i = b_i$ for $i = 1, \ldots, k$. If $k = m$, we get that $\psi_{a_1 \cdots a_k}(\nu(\mathbf{s})) = \psi_{a_1 \cdots a_k}(\theta(\mathbf{s}))$, therefore by injectivity of episturmian morphisms $\nu = \theta$, and so $f = h$. Otherwise, $a_1 \cdots a_k$ is a proper prefix of $b_1 \cdots b_m$. We get that all the elements of $\mathcal{S}tab(\mathbf{s})$ can be strictly ordered by the prefix order. That is, there exists a sequence of words $u_0 = \varepsilon, u_1, u_2 \ldots \subseteq \Sigma^*$ and a sequence of permutations $\theta_0 = \mathrm{Id}, \theta_1, \theta_2, \ldots$ over Σ, such that u_i is a proper prefix of u_{i+1} for all $i \ge 0$, and $\mathcal{S}tab(\mathbf{s}) = \{f_i\}_{i \ge 0}$, where $f_i = \psi_{u_i}\theta_i$. We will show that f_1 generates $\mathcal{S}tab(\mathbf{s})$.

Let $\|f_1\| = k$. Clearly, $f_1^i \in \mathcal{S}tab(\mathbf{s})$ for all $i \geq 1$; also, by Property 8.2, $\|f_1^i\| = ik$. Since $\|h\| = \|g\|$ if and only if $h = g$ for all $h, g \in \mathcal{S}tab(\mathbf{s})$, any morphism $h \in \mathcal{S}tab(\mathbf{s})$ with $\|h\| \equiv 0 \pmod{k}$ must satisfy $h = f_1^m$ for some $m \geq 0$.

Suppose there exists a morphism $h \in \mathcal{S}tab(\mathbf{s})$ and some $n \geq 0$ such that $nk < \|h\| < (n+1)k$. If $n = 0$ we get that $0 < \|h\| < k$, a contradiction to the minimality of $\|f_1\|$. Assume that $n \geq 1$. Then $h = \psi_{u_1}\psi_w\nu$, for some permutation ν and some $w \in \Sigma^*$ with $|w| = \|h\| - k$. We get that

$$\begin{aligned} \mathbf{s} &= \psi_{u_1}(\psi_w\nu(\mathbf{s})), \\ \mathbf{s} &= \psi_{u_1}(\theta_1(\mathbf{s})), \end{aligned}$$

thus necessarily $\psi_w\nu(\mathbf{s}) = \theta_1(\mathbf{s})$, and $\theta^{-1}\psi_w\nu(\mathbf{s}) = \mathbf{s}$. By Property 8.2, we get that $\psi_{w'}\theta^{-1}\nu(\mathbf{s}) = \mathbf{s}$, where $|w'| = |w|$. Let $\nu' = \theta^{-1}\nu$, and let $h' = \psi_{w'}\nu'$. Then $h' \in \mathcal{S}tab(\mathbf{s})$, and $(n-1)k < \|h'\| < nk$. By induction, we must get after n steps to a morphism $g \in \mathcal{S}tab(\mathbf{s})$ that satisfies $0 < \|g\| < k$, a contradiction to the minimality of $\|f_1\|$.

We conclude that every morphism $h \in \mathcal{S}tab(\mathbf{s})$ satisfies $\|h\| \equiv 0 \pmod{k}$. Therefore, $f_i = f_1^i$ for all $i \geq 1$, and $\mathcal{S}tab(\mathbf{s}) = \langle f_1 \rangle$. □

Corollary 8.12. *All fixed points of epistandard morphisms are rigid.*

Example 8.2. The *Tribonacci* (or *Rauzy*) *word* $\mathbf{r}$, introduced by Rauzy in 1982 [111] as a generalization of the Fibonacci word, is a Σ_3-strict epistandard word whose directive word is given by $\Delta(\mathbf{r}) = (012)^\omega$. The Tribonacci word is generated by the morphism $h = (01, 02, 0)$, which has the representation $h = \psi_0\theta$, where θ is the cycle $(1, 2, 0)$. Since $\|h\| = 1$, it necessarily generates $\mathcal{S}tab(\mathbf{r})$. The order of θ is 3, therefore $h^3 = \psi_0\psi_1\psi_2$ is the first pure morphism in $\mathcal{S}tab(\mathbf{r})$. This is exactly the morphism that matches the minimal period of $\Delta(\mathbf{r})$ (see Property 8.9).

8.4.3 Stabilizers of ultimately strict epistandard words

A key point in the proof of Theorem 8.11 was the use of Property 8.8: if a morphism f fixes a Σ-strict epistandard word, then f must be an epistandard morphism. This property does not hold for non-strict words. Consider, for example, the word $\mathbf{s}$ generated by the directive word $\Delta(\mathbf{s}) = 3(012)^\omega$. By Property 8.6, $\mathbf{s} = \psi_3(\mathbf{r})$, where $\mathbf{r}$ is the Tribonacci word. But, since 3 does not occur in $\mathbf{r}$, we get that $\mathbf{r} = E_3(\mathbf{s})$, where E_3 is defined over $\{0, 1, 2, 3\}$ by $E_3(3) = \varepsilon$, $E_3(a) = a$ for all $a \neq 3$. Therefore, the non-episturmian morphism $\psi_3 E_3 = (30, 31, 32, \varepsilon)$ belongs to $\mathcal{S}tab(\mathbf{s})$.

The example given above is the general case. First, we need some definitions.

Definition 8.6. An epistandard word $\mathbf{s}$ is *ultimately strict* if there exists a decomposition $\Delta(\mathbf{s}) = x\mathbf{y}$, where $x = x_1 \cdots x_n \in \Sigma^+$ and $\mathbf{y} = y_1 y_2 y_3 \cdots \in \Sigma^\omega$, such that the following conditions hold:

1. $\mathbf{y}$ is letter-recurrent;
2. $\mathrm{alph}(x) \cap \mathrm{alph}(\mathbf{y}) = \emptyset$.

Note that if such a decomposition exists then it must be unique. The prefix x is called the *excess* of $\mathbf{s}$; the suffix $\mathbf{y}$ is called the *base* of $\mathbf{s}$. We denote $\Sigma_x = \mathrm{alph}(x)$, $\Sigma_{\mathbf{y}} = \mathrm{alph}(\mathbf{y})$, and $\hat{x} = u_n$, where u_n is as defined in Property 8.5 (the word attained by successively applying palindromic closure to the letters of x). We assume that $|\Sigma_{\mathbf{y}}| \geq 2$ (or else, by Property 8.7, $\mathbf{s}$ would be ultimately periodic). For a morphism $f \in \mathcal{M}(\Sigma)$, we denote by $f_{|\Sigma_x}$ the restriction of f to Σ_x, and similarly for $\Sigma_{\mathbf{y}}$.

Note: This definition of ultimately strict epistandard words is slightly different from the one introduced by Richomme in [114].

Lemma 8.13. *Let $\mathbf{s}$ be an ultimately strict epistandard word, with excess x and base $\mathbf{y}$. Then $\mathbf{s} = \psi_x(\mathbf{t})$, where $\mathbf{t}$ is the epistandard word given by $\Delta(\mathbf{t}) = \mathbf{y}$, and for all $a \in \Sigma_{\mathbf{y}}$, we have $\psi_x(a) = \hat{x}a$. In particular, if $\mathbf{t} = t_1 t_2 t_3 \cdots$, then $\mathbf{s} = \hat{x}t_1\hat{x}t_2\hat{x}t_3\hat{x}\cdots$.*

Proof. Follows directly from Property 8.6 and the definition of ψ_x. □

Corollary 8.14. *Every ultimately strict epistandard word has a non-trivial stabilizer.*

Proof. Let $\mathbf{s}$ be an ultimately strict epistandard word with excess x and base $\mathbf{y}$, and let $\mathbf{t}$ be the epistandard word given by $\Delta(\mathbf{t}) = \mathbf{y}$. Let $E_x \in \mathcal{M}(\Sigma)$ be the morphism defined by $E_x(a) = \varepsilon$ if $a \in \Sigma_x$, and $E_x(a) = a$ otherwise. Since $\Sigma_x \cap \Sigma_{\mathbf{y}} = \emptyset$, necessarily $E_x(\mathbf{s}) = \mathbf{t}$. Therefore, $\mathbf{s} = \psi_x(E_x(\mathbf{s}))$, and so $\psi_x E_x \in \mathcal{S}tab(\mathbf{s})$. Since $\psi_x E_x(a) = \psi_x(a) = \hat{x}a$ for all $a \in \Sigma_{\mathbf{y}}$, we get that $\psi_x E_x \neq \mathrm{Id}$, and so $\mathcal{S}tab(\mathbf{s})$ is non-trivial. □

Theorem 8.15. *Let* $\mathbf{s}$ *be an ultimately strict epistandard word with excess* x *and base* $\mathbf{y}$*, and let* $\mathbf{t}$ *be the epistandard word given by* $\Delta(\mathbf{t}) = \mathbf{y}$*. Let* Σ_x*,* $\Sigma_{\mathbf{y}}$ *and* $\hat{x}$ *be as in Definition 8.6. Then a morphism* $f \in \mathcal{M}(\Sigma)$ *belongs to* $\mathcal{S}tab(\mathbf{s})$ *if and only if there exists a prefix* z *of* $\mathbf{s}$ *and a morphism* $h \in \mathcal{S}tab(\mathbf{t})$*, such that*

1. $f(\hat{x}) = z$*;*
2. z *is a common prefix of* $\{\psi_x h(a) : a \in \Sigma_{\mathbf{y}}\}$*;*
3. $f(a) = z^{-1}\psi_x h(a)$ *for all* $a \in \Sigma_{\mathbf{y}}$*.*

Proof. Suppose $f \in \mathcal{M}(\Sigma)$ satisfies the conditions above. Let $\mathbf{t} = t_1 t_2 t_3 \cdots$. Then

$$f(\mathbf{s}) = f(\hat{x})f(t_1)f(\hat{x})f(t_2)\cdots = zf(t_1)zf(t_2)\cdots = \psi_x h(t_1)\psi_x h(t_2)\cdots = \psi_x h(\mathbf{t}) = \psi_x(\mathbf{t}) = \mathbf{s}.$$

Now suppose that $f \in \mathcal{S}tab(\mathbf{s})$. Let $h = E_x f \psi_x$. Then $h_{|\Sigma_{\mathbf{y}}} \in \mathcal{S}tab(\mathbf{t})$. We will show that for all $a \in \Sigma_{\mathbf{y}}$, $f(a) = f(\hat{x})^{-1}\psi_x h(a)$.

First, note that $f(\hat{x}a)$ must contain at least one letter of $\Sigma_{\mathbf{y}}$ for all $a \in \Sigma_{\mathbf{y}}$. For else we would get that

$$\varepsilon = E_x f(\hat{x}a) = E_x f \psi_x(a) = h(a),$$

a contradiction: $h_{|\Sigma_{\mathbf{y}}}$ is an epistandard morphism, and hence nonerasing.

Let $b = t_1$. Then $f(\hat{x}b) \prec_p \mathbf{s}$, and since $f(\hat{x}b)$ contains a letter of $\Sigma_{\mathbf{y}}$, necessarily $f(\hat{x}b) = \hat{x}u$ for some $u \in \Sigma^+$. Now, by Property 8.10, b is separating for $\mathbf{t}$, and since $\mathbf{t}$ is strict, this implies that $ab, ba \in \mathrm{Sub}(\mathbf{t})$ for all $a \in \Sigma_{\mathbf{y}}$ (including the case $a = b$). Therefore, $\hat{x}a\hat{x}b, \hat{x}b\hat{x}a \in \mathrm{Sub}(\mathbf{s})$ for all $a \in \Sigma_{\mathbf{y}}$, and so $\hat{x}uf(\hat{x}a), f(\hat{x}a)\hat{x}u \in \mathrm{Sub}(\mathbf{s})$ for all $a \in \Sigma_{\mathbf{y}}$.

Assume there exists a letter $a \in \Sigma_{\mathbf{y}}$ such that $f(\hat{x}a) = vc$ for some $v \in \Sigma^*$ and $c \in \Sigma_x$. Then $f(\hat{x}a)\hat{x}u = vc\hat{x}u$, and hence $c\hat{x} \in \mathrm{Sub}(\mathbf{s}) \cap \Sigma_x^*$, a contradiction: by Lemma 8.13, the only elements of $\mathrm{Sub}(\mathbf{s}) \cap \Sigma_x^*$ are the subwords of $\hat{x}$. Therefore, for every $a \in \Sigma_{\mathbf{y}}$, $f(\hat{x}a)$ ends with a letter of $\Sigma_{\mathbf{y}}$. In particular, $f(\hat{x}b)$ ends with a letter of $\Sigma_{\mathbf{y}}$, and since b is separating, $f(\hat{x}a)$ must begin with $\hat{x}$ for all $a \in \Sigma_{\mathbf{y}}$.

We conclude that for all $a \in \Sigma_{\mathbf{y}}$, there exist some $m \geq 1$ and letters $a_1, a_2, \ldots, a_m \in \Sigma_{\mathbf{y}}$, such that $f(\hat{x}a) = \hat{x}a_1\hat{x}a_2\cdots\hat{x}a_m$. Therefore,

$$f(\hat{x}a) = \psi_x E_x(f(\hat{x}a)) \ \ \forall a \in \Sigma_{\mathbf{y}}.$$

But $f(\hat{x}a) = f\psi_x(a)$, and so we get:

$$f(\hat{x})f(a) = f(\hat{x}a) = \psi_x E_x f \psi_x(a) = \psi_x h(a) \;\; \forall a \in \Sigma_{\mathbf{y}}. \tag{8.2}$$

$\square$

Corollary 8.16. *Let* $\mathbf{s}$ *be an ultimately strict epistandard word with an aperiodic base. Then*

1. $\mathcal{S}tab(\mathbf{s})$ *is finite. In particular, every non-trivial morphism* $f \in \mathcal{S}tab(\mathbf{s})$ *must be erasing. Moreover,* $\mathcal{S}tab(\mathbf{s})$ *depends only on the excess of* $\mathbf{s}$*.*

2. $\mathcal{IS}tab(\mathbf{s}) = \{\mathrm{Id}\}$*. In particular,* $\mathbf{s}$ *is not pure morphic.*

Proof.

1. Let $x, \mathbf{y}, \mathbf{s}, \mathbf{t}, E_x$ be as in Theorem 8.15, and let $f \in \mathcal{S}tab(\mathbf{s})$. Then there must exist a morphism $h \in \mathcal{S}tab(\mathbf{t})$ such that $f(\hat{x})f(a) = \psi_x h(a)$ for all $a \in \Sigma_{\mathbf{y}}$, and since by Property 8.9 the stabilizer of a strict epistandard with an aperiodic directive word is trivial, necessarily $h = \mathrm{Id}$. Therefore, $f(\hat{x})f(a) = \psi_x(a) = \hat{x}a$ for all $a \in \Sigma_{\mathbf{y}}$. Since $|\Sigma_{\mathbf{y}}| \geq 2$, we cannot have $f(\hat{x}) = \hat{x}b$ for some $b \in \Sigma_{\mathbf{y}}$: for a letter $c \in \Sigma_{\mathbf{y}}$ such that $c \neq b$, we would get $f(\hat{x})f(c) = \hat{x}bf(c) \neq \hat{x}c$. This implies the following:

   - $f \in \mathcal{S}tab(\mathbf{s})$ if and only if there exists a decomposition $\hat{x} = uv$, such that $f(\hat{x}) = u$, and $f(a) = va$ for all $a \in \Sigma_{\mathbf{y}}$.

   Clearly, there are only finitely many morphisms that satisfy this condition. Also, the morphism depends only on the partition of $\hat{x}$, thus any ultimately strict word over Σ with an aperiodic base over $\Sigma_{\mathbf{y}}$ and excess x will have the same stabilizer.

2. Let $\hat{x} = x_1 \cdots x_n$, and let $f \in \mathcal{S}tab(\mathbf{s})$ be a non-trivial morphism. We show that $f(x_1) = \varepsilon$. By the above, either $f(\hat{x}) = \varepsilon$, or $f(\hat{x}) = x_1 \cdots x_m$ for some $m \leq n$. Suppose the latter case holds, and suppose $f(x_1) \neq \varepsilon$. Then $f(x_1) = x_1 w$ for some $w \in \Sigma_x^*$. Recall that $\hat{x}$ is a palindrome and thus ends with x_1. If $m < n$, this implies that $|f(\hat{x})|_{x_1} < |\hat{x}|_{x_1}$, a contradiction, since x_1 occurs in $f(x_1)$. Assume that $m = n$, that is, $f(\hat{x}) = \hat{x}$ and $f(a) = a$ for all $a \in \Sigma_{\mathbf{y}}$. Since x_1 is separating for $\hat{x}$, the only

way to get $f(\hat{x}) = \hat{x}$ when $f(x_1) \neq \varepsilon$ is by having $f(b) = b$ for all $b \in \Sigma_x$. But then $f = \mathrm{Id}$, a contradiction.

We get that every non-trivial morphism $f \in \mathcal{S}tab(\mathbf{s})$ is erasing on the first letter of $\mathbf{s}$, and so $\mathbf{s}$ cannot be generated by iteration.

□

Note: Part 2 of Corollary 8.16 is true for any non-strict epistandard word that has an aperiodic directive word, as was proved in [64, Proposition 3.7].

Example 8.3. Let $\mathbf{s} \in \Sigma_5^\omega$ be an ultimately strict epistandard word, with an aperiodic base $\mathbf{y} \in \{0,1,2\}^\omega$ and excess $x = 43$. Then $\hat{x} = 434$, and $\mathcal{S}tab(s) = \{\mathrm{Id}, g_1, g_2, g_3, g_4\}$, where

$$\begin{aligned} g_1 &= (4340, 4341, 4342, \varepsilon, \varepsilon), \\ g_2 &= (340, 341, 342, 4, \varepsilon), \\ g_3 &= (40, 41, 42, 43, \varepsilon), \\ g_4 &= (0, 1, 2, 434, \varepsilon). \end{aligned}$$

By observing the multiplication table of $\mathcal{S}tab(\mathbf{s})$, we can see that $\mathcal{S}tab(\mathbf{s}) = \langle g_2, g_3, g_4 \rangle$:

	g_1	g_2	g_3	g_4
g_1	g_1	g_1	g_1	g_1
g_2	g_1	g_1	g_2	g_2
g_3	g_1	g_1	g_3	g_3
g_4	g_1	g_1	g_4	g_4

When $\mathbf{s}$ is an ultimately strict epistandard word with a periodic base, Theorems 8.11, 8.15 give an explicit way of constructing morphisms in its stabilizer. Let $x, \mathbf{y}, \mathbf{s}, \mathbf{t}, E_x$ be as in Theorem 8.15. By Theorems 8.11, there exists an epistandard morphism $h \in \mathcal{M}(\Sigma_{\mathbf{y}})$, such that $\|h\| \geq 1$ and $\mathcal{S}tab(\mathbf{t}) = \langle h \rangle$. Also, from the definition of epistandard morphisms, it is easy to see that for every prefix z of $\mathbf{s}$ there exists some $k(z) \geq 0$, such that for all $k \geq k(z)$, z is a common prefix of $\{\psi_x h^k(a) : a \in \Sigma_{\mathbf{y}}\}$. Every morphism $f \in \mathcal{S}tab(\mathbf{s})$ can be constructed in the following way:

1. Choose a prefix z of $\mathbf{s}$, such that there exists a morphism $g : \Sigma_x \to \Sigma$ satisfying $g(\hat{x}) = z$;

2. Let $k(z)$ be the minimal k such that z is a common prefix of $\{\psi_x h^k(a) : a \in \Sigma_{\mathbf{y}}\}$;

3. For all $k \geq k(z)$, and for all $g : \Sigma_x^* \to \Sigma^*$ that satisfies $g(\hat{x}) = z$ (there must be finitely many such morphisms), define the morphism $f_{z,g,k}$ by

$$f_{z,g,k}(a) = \begin{cases} z^{-1}\psi_x h^k(a), & \text{if } \ a \in \Sigma_{\mathbf{y}} \ ; \\ g(a), & \text{if } \ a \in \Sigma_x \ . \end{cases}$$

This kind of construction is always possible if we trivially choose $z = \varepsilon$. Indeed, this is exactly the morphism we constructed in Corollary 8.14 for $h = \mathrm{Id}$. More generally, if $\mathcal{S}tab(\mathbf{t}) = \langle h \rangle$, extend h to Σ by defining $h(a) = a$ for all $a \in \Sigma_x$. Then for all $k \geq 1$, $\psi_x h^k E_x \in \mathcal{S}tab(\mathbf{s})$.

From the discussion above, it follows that the elements of $\mathcal{S}tab(\mathbf{s})$ can be viewed as being generated along two orthogonal axes: one axis is indexed by the natural numbers $k \in \mathbb{N}$, while the other is indexed by prefixes z of $\mathbf{s}$ which are images of $\hat{x}$ under some morphism g. When $\mathcal{S}tab(\mathbf{t}) = \langle h \rangle$ for some non-trivial morphism $h \in \mathcal{M}(\Sigma_{\mathbf{y}})$, every such prefix z and such morphism g induce an infinite sequence of elements of $\mathcal{S}tab(\mathbf{s})$, namely $\{f_{z,g,k}\}_{k \geq k(z)}$. We now show that each of these sequences is finitely generated, that is, $\mathcal{S}tab(\mathbf{s})$ is finitely generated along the k axis.

In what follows, we use the notation $z, k(z), f_{z,g,k}$ as defined above. We assume that $\Delta(\mathbf{t})$ is periodic, thus $\mathcal{S}tab(\mathbf{t}) = \langle h \rangle$ for some epistandard morphism h with $\|h\| \geq 1$.

Lemma 8.17. *Let $f_1 = f_{z_1,g_1,k_1}, f_2 = f_{z_2,g_2,k_2}$ be two elements of $\mathcal{S}tab(\mathbf{s})$. Then*

$$f_1 f_2 = f_{f_1(z_2), f_1 f_{2|\Sigma_x}, k_1+k_2} \ .$$

Proof. Let $a \in \Sigma_{\mathbf{y}}$, and let $h^{k_2}(a) = a_1 a_2 \cdots a_n$.

$$\begin{aligned}
f_1 f_2(a) &= f_1(z_2^{-1}\psi_x h^{k_2}(a)) \\
&= f_1(z_2^{-1}\hat{x}a_1\hat{x}a_2\cdots\hat{x}a_n) \\
&= (f_1(z_2))^{-1}\cdot f_1(\hat{x}a_1\hat{x}a_2\cdots\hat{x}a_n) \\
&= (f_1(z_2))^{-1}\cdot z_1\cdot z_1^{-1}\psi_x h^{k_1}(a_1)\cdot z_1\cdot z_1^{-1}\psi_x h^{k_1}(a_2)\cdots z_1\cdot z_1^{-1}\psi_x h^{k_1}(a_n) \\
&= (f_1(z_2))^{-1}\cdot \psi_x h^{k_1}(a_1 a_2\cdots a_n) \\
&= (f_1(z_2))^{-1}\cdot \psi_x h^{k_1}(h^{k_2}(a)) \\
&= (f_1(z_2))^{-1}\cdot \psi_x h^{k_1+k_2}(a).
\end{aligned}$$

Now let $a \in \Sigma_x$, and let $g_{12} = f_1 f_{2|\Sigma_x}$. Then $g_{12}(\hat{x}) = f_1 f_2(\hat{x}) = f_1(z_2)$. By definition of $f_{z,g,k}$, we get that $f_1 f_2 = f_{f_1(z_2),g_{12},k_1+k_2}$. $\square$

Corollary 8.18. *Let z be a prefix of $\mathbf{s}$ which is the image of $\hat{x}$ under some morphism g, and let*

$$f_{\hat{x},\mathrm{Id},1}(a) = \begin{cases} \hat{x}^{-1}\psi_x h(a), & \text{if } a \in \Sigma_{\mathbf{y}}\,; \\ a, & \text{if } a \in \Sigma_x\,. \end{cases}$$

Then for all $k \geq k(z)$, $f_{z,g,k} = f_{z,g,k(z)} \cdot f_{\hat{x},\mathrm{Id},1}^{k-k(z)}$.

Proof. By Lemma 8.17 and by induction on n, we get that $f^n_{\hat{x},\mathrm{Id},1} = f_{\hat{x},\mathrm{Id},n}$ for all $n \geq 1$. Therefore,

$$f_{z,g,k(z)} \cdot f_{\hat{x},\mathrm{Id},1}^{k-k(z)} = f_{z,g,k(z)} \cdot f_{\hat{x},\mathrm{Id},k-k(z)} = f_{f_{z,g,k(z)}(\hat{x}),g,k(z)+k-k(z)} = f_{z,g,k}.$$

$\square$

By Corollary 8.18, to find a set of generators for $\mathcal{S}tab(\mathbf{s})$ it is enough to find such a set along the z axis. The following theorem demonstrates such a case.

Theorem 8.19. *Let $\mathbf{r}$ be the Tribonacci word, and let $\mathbf{s} = \psi_3(\mathbf{r})$. Then $\mathcal{S}tab(\mathbf{s}) = \langle g_\varepsilon, g_0, g_1, g_2, g_3\rangle$, where*

$$\begin{aligned}
g_\varepsilon &= f_{\varepsilon,0} &&= (\,30\,, 31\,, 32\,, \varepsilon\,), \\
g_0 &= f_{30,1} &&= (\,31\,, 32\,, \varepsilon\,, 30\,), \\
g_1 &= f_{3,1} &&= (\,031\,, 032\,, 0\,, 3\,), \\
g_2 &= f_{303,2} &&= (\,13032\,, 130\,, 1\,, 303\,), \\
g_3 &= f_{3031303,3} &&= (\,2303130\,, 23031\,, 2\,, 3031303\,).
\end{aligned}$$

Proof. As shown in Example 8.2, the Tribonacci word $\mathbf{r} = 0102010\cdots$ satisfies $\Delta(\mathbf{r}) = (012)^\omega$, and $\mathcal{S}tab(\mathbf{r}) = \langle h \rangle$, where $h = (01, 02, 0)$. Therefore, $\mathbf{y} = (012)^\omega$, $x = \hat{x} = 3$, $\Sigma_{\mathbf{y}} = \{0, 1, 2\}$, and $\Sigma_x = \{3\}$. This implies that every prefix z of $\mathbf{s}$ induces the stabilizer element $f_{z,g,k(z)}$, where $g : \Sigma_x^* \to \Sigma^*$ is uniquely defined by $g(3) = z$; since g is uniquely defined for each z, we can omit it from the subscript, and refer to $f_{z,k(z)}$. By Theorem 8.15 and Corollary 8.18, we then get:

$$\mathcal{S}tab(\mathbf{s}) = \bigcup_{z \prec_p \mathbf{s}} \{f_{z,k(z)} \cdot f_{3,1}^{k-k(z)} | k \geq k(z)\}.$$

We will show that the set $\{f_{z,k(z)} | z \prec_p \mathbf{s}\}$ is generated by the set $\mathcal{G} = \{g_\varepsilon, g_0, g_1, g_2, g_3\}$.

For a morphism $g \in \mathcal{G}$, let $z(g) = g(3)$, and let $k(g) = k(z(g))$. That is,

$$\begin{array}{ll} z(g_\varepsilon) = \varepsilon & k(g_\varepsilon) = 0, \\ z(g_0) = 30, & k(g_0) = 1, \\ z(g_1) = 3, & k(g_1) = 1, \\ z(g_2) = 303, & k(g_2) = 2, \\ z(g_3) = 3031303, & k(g_3) = 3. \end{array}$$

The proof strategy is to show that for every prefix $z \prec_p \mathbf{s}$ with $z \notin \{z(g) : g \in \mathcal{G}\}$ there exists a prefix $z' \prec_p \mathbf{s}$, such that $f_{z,k(z)} = g \cdot f_{z',k(z')}$ for some $g \in \mathcal{G}$; since $k(g) \geq 0$ for all $g \in \mathcal{G}$, after finitely many steps we must arrive at a representation of $f_{z,k(z)}$ as a product of elements of $\mathcal{G}$. By Lemma 8.17, showing that $f_{z,k(z)} = g \cdot f_{z',k(z')}$ for all $z \prec_p \mathbf{s}$ is equivalent to showing the following:

1. for every prefix $z \prec_p \mathbf{s}$ there exist a prefix $z' \prec_p \mathbf{s}$ and a morphism $g \in \mathcal{G}$, such that $z = g(z')$, and
2. $k(z) = k(g) + k(z')$.

Proof of part 1: First consider the prefixes of even length. Let $\mathbf{s} = s_1 s_2 s_3 \cdots$, and let $z_0 = \varepsilon$, and for $i \geq 1$, $z_i = s_1 \cdots s_i$. For $i = 0$, $f_{z_0,k(z_0)} = g_\varepsilon \in \mathcal{G}$. For $i > 0$, consider $g_0(z_i)$: by definition, $|g_0(z_1)| = 2$, and for $i > 1$,

$$|g_0(z_i)| = \begin{cases} |g_0(z_{i-1})| + 2 & \text{if } s_i \in \{0, 1, 3\}, \\ |g_0(z_{i-1})| & \text{if } s_i = 2. \end{cases}$$

Therefore, for every nonempty prefix $z \prec_p \mathbf{s}$ of even length there exists a nonempty prefix $z' \prec_p \mathbf{s}$ such that $|g_0(z')| = |z|$. Since $g_0(z')$ is also a prefix of $\mathbf{s}$, and two prefixes have the same length if and only if they are equal, we get that $g_0(z') = z$.

Now consider the prefixes of odd length. First, we prove an auxiliary lemma.

Lemma 8.20. *Let* $\mathbf{r} = r_1 r_2 r_3 \cdots$, *and let* $y_n = r_1 \cdots r_n$. *Then for all* $n \geq 1$ *there exists some* $m \geq 1$, *such that exactly one of the following holds:*

1. $y_n = h(y_m)$;
2. $y_n = h^2(y_m)0$;
3. $y_n = h^3(y_m)010$.

Proof. Since $\mathbf{r} = h^\omega(0)$, $\mathbf{r}$ can be decomposed over $\{h^i(a) : a = 0, 1, 2\}$ for any $i \geq 0$. We call such a decomposition an h^i*-decomposition*, and the words $\{h^i(a) : a = 0, 1, 2\}$ the h^i*-blocks*. The h-blocks are given by $\{01, 02, 0\}$; the h^2-blocks are given by $\{0102, 010, 01\}$; the h^3-blocks are given by $\{0102010, 010201, 0102\}$.

If $r_n \neq 0$, or $r_n = r_{n+1} = 0$, then y_n can be decomposed into h-blocks, and so $y_n = h(y_m)$ for some $m < n$. Suppose $r_n = 0$ and $r_{n+1} \neq 0$. If $r_{n+1} = 1$, then r_n is the first letter in an h^2-block, and so $y_n = h^2(y_m)0$ for some $m < n$. Otherwise, if $r_{n+1} = 2$, then r_n is the third letter in an h^3-block, and so $y_n = h^3(y_m)010$ for some $m < n$. $\square$

Example 8.4. Here are the first few terms of $\mathbf{r}$. Broken bars stand for h-decomposition, regular bars for h^2-decomposition, and long bars for h^3-decomposition. The first letter of an h-block is either the first letter of an h^2-block, or the third letter of an h^3-block.

$$01\text{¦}02|01\text{¦}0\Big|01\text{¦}02|01\Big|01\text{¦}02|01\text{¦}0\Big|01\text{¦}02\Big|01\text{¦}02|01\text{¦}0\Big|01\text{¦}02|01\Big|01\text{¦}02|01\text{¦}0\Big|01\text{¦}02|01\text{¦}0\Big|\cdots$$

We now continue with the proof of Theorem 8.19. Let z be a prefix of odd length. Then there exists some $n \geq 1$ such that $z = 3r_1 3r_2 \cdots 3r_n 3 = \psi_3(y_n)3$. Recall that by Equation (8.2), every stabilizer element $f_{z,k}$ satisfies

$$f_{z,k}\psi_x(u) = \psi_x h^k(u) \ \ \forall\, u \in \Sigma_{\mathbf{y}},$$

regardless of the choice of z. Applying Lemma 8.20, and letting $z' = \psi_3(z_m)$, we get three cases:

1. If $y_n = h(y_m)$, then

$$z = \psi_3(h(y_m))3 = g_1(\psi_3(y_m))3 = g_1(z')3 = g_1(z'3).$$

2. If $y_n = h^2(y_m)0$, then

$$z = \psi_3(h^2(y_m)0)3 = \psi_3(h^2(y_m))303 = g_2(\psi_3(y_m))303 = g_2(z'3).$$

3. If $y_n = h^3(y_m)010$, then

$$z = \psi_3(h^3(y_m)010)3 = \psi_3(h^3(y_m))3031303 = g_3(\psi_3(y_m))3031303 = g_3(z'3).$$

We conclude that if $\varepsilon \neq z \prec_p \mathbf{s}$ is a prefix of even length then $z = g_0(z')$ for some prefix z', and if z is of odd length then there exists exactly one $g \in \{g_1, g_2, g_3\}$ such that $z = g(z')$ for some prefix z'. This completes the proof of part 1.

Proof of part 2: we prove a more general lemma:

Lemma 8.21. *Let* $f_1 = f_{z_1,k(z_1)}, f_2 = f_{z_2,k(z_2)} \in \mathcal{S}tab(\mathbf{s})$, *where* $z_1, z_2 \prec_p \mathbf{r}$ *satisfy* $|z_1| \geq 1$ *and* $|z_2| \geq 2$. *Then* $k(f_1(z_2)) = k(z_1) + k(z_2)$.

Proof. Recall that $k(z_1)$ (and similarly $k(z_2)$) is the minimal integer n such that z_1 is a common prefix of $\{\psi_3 h^n(a) | a = 0, 1, 2\}$. Since $\psi_3 h^n(2)$ is the shortest element in this set for all n, we get that

$$|\psi_3 h^{k(z_2)-1}(2)| < |z_2| \leq |\psi_3 h^{k(z_2)}(2)|.$$

(Note that, since $|z_2| \geq 2$, z_2 begins with $30 = \psi_3 h^1(2)$, and so $k(z_2) \geq 1$.)

By Lemma 8.17, $f_1 f_2 = f_{f_1(z_2),k(z_1)+k(z_2)}$. Therefore, $f_1(z_2)$ is a common prefix of

$$\{\psi_3 h^{k(z_1)+k(z_2)}(a) | a = 0, 1, 2\}.$$

To show that $k(f_1(z_2)) = k(z_1) + k(z_2)$, we need to show that $k(z_1) + k(z_2)$ is the minimal exponent such that $f_1(z_2)$ is a common prefix, that is, we need to show that

$$|f_1(z_2)| > |\psi_3 h^{k(z_1)+k(z_2)-1}(2)|.$$

Let $z_2 = \psi_3 h^{k(z_2)-1}(2)z'$. Then

$$|f_1(z_2)| = |f_1(\psi_3 h^{k(z_2)-1}(2))f_1(z')| = |\psi_3(h^{k(z_1)}(h^{k(z_2)-1}(2)))f_1(z')| = \\ |\psi_3 h^{k(z_1)+k(z_2)-1}(2)| + |f_1(z')|.$$

Since by assumption $z' \neq \varepsilon$, necessarily $z' = 3u$ for some $u \in \Sigma^*$, thus $|f_1(z')| = |f_1(3)f_1(u)| = |z_1| + |f_1(u)| > 0$. This completes the proof of the lemma. □

Lemma 8.21 completes the proof of the theorem: if $f_{z,k(z)} \in \mathcal{S}tab(\mathbf{s})$ and $f_{z,k(z)} \notin \mathcal{G}$, then by part 1 there exists a prefix z' and a morphism $g \in \mathcal{G}$ such that $z = g(z')$, and by the above lemma, $gf_{z',k(z')} = f_{z,k(z)}$.

We have shown that all stabilizer elements of the form $f_{z,k(z)}$ are generated by $\mathcal{G}$. By Corollary 8.18, a sequence of the form $\{f_{z,k} : k \geq k(z)\}$ is generated by $\{g_1, f_{z,k(z)}\}$. Therefore, $\mathcal{G}$ generates $\mathcal{S}tab(\mathbf{s})$. □

8.5 Open problems

We conclude by stating again the problems we stated in the beginning of this chapter.

1. How many generators can a stabilizer of a binary infinite word have? We have proved that binary aperiodic stabilizers can have any finite number of generators, but can they be infinitely generated? Does there exist an aperiodic binary word with an infinitely generated iterative stabilizer? We believe the answer is negative.

2. More generally, do there exist infinitely generated stabilizers of infinite aperiodic words over finite alphabets? Again, we believe the answer is negative.

3. Is there a characterization of morphisms that generate rigid words by iteration? We have proved only negative results: uniform, primitive, or invertible morphisms can all generate non-rigid words.

4. Are strict episturmian words rigid? The uniqueness of Property 8.6 does not hold for episturmian words in general. Moreover, the decomposition of a pure episturmian morphism into $\{\psi_a, \bar{\psi}_a\}$ elements is not unique (e.g., $\psi_a\bar{\psi}_a = \bar{\psi}_a\psi_a$). However, computer tests suggest that strict episturmian words are rigid.

Chapter 9

Conclusion and Open Problems

In this thesis, we have studied various aspects of critical exponents in infinite words. After a survey of past results in Chapter 3, we began in Chapter 4 by studying critical exponents in the most general setting, namely, in arbitrary infinite words over any finite alphabet. We have proved that every real number greater than 1 is a critical exponent of some infinite word, and gave an explicit construction for such a word. Our construction raised the following questions:

Problem 9.1. *Given a real number $2 < \alpha \leq 7/3$, is it possible to construct an infinite binary word $\mathbf{w} \in \Sigma_2^\omega$ such that $E(\mathbf{w}) = \alpha$?*

Problem 9.2. *Given a real number $1 < \alpha < 2$ and an infinite word $\mathbf{v}$ avoiding α-powers over a n-letter alphabet, is it possible to construct an infinite word $\mathbf{w}$ with $E(\mathbf{w}) = \alpha$ over an n-letter alphabet?*

Problem 9.3. *Let $1 < \alpha < 2$ be an n-avoidable real number. Is α also circularly n-avoidable?* (See also Currie [33].)

In Chapter 5, we studied critical exponents in uniform binary pure morphic words. We gave necessary and sufficient conditions for the critical exponent to be bounded, and an explicit formula to compute it when it is bounded. In Chapter 6 we generalized our results to non-erasing morphisms over any finite alphabet. We showed that the critical exponent, when bounded, is algebraic of degree bounded by the alphabet size, and gave an algorithm to compute it. We then applied our algorithm to various infinite words, including the Arshon words (Chapter 7). Here are some of the open problems raised:

Problem 9.4. *Are critical exponents of words generated by iterating an erasing morphism always algebraic?*

We strongly believe the answer is positive.

Problem 9.5. *Are critical exponents of morphic words always algebraic?*

Again, we believe the answer is positive. A positive answer will also cover the erasing case.

Problem 9.6. *Given an algebraic number α of degree d, can we construct a morphism $f : \Sigma_n \to \Sigma_n$ for some $n \geq d$ such that $E(f^\omega(0)) = \alpha$?*

In Chapter 8, we began our study of stabilizers. This area has hardly been studied, and offers many interesting open problems. We find the following two the most interesting (it also seems, the most hard):

Problem 9.7. *Do there exist aperiodic infinite words over finite alphabets that have infinitely generated stabilizers?*

Problem 9.8. *Can we characterize morphisms that, when iterated, generate rigid words?*

We would like to conclude this thesis with a new open problem, which is closely related to critical exponent in pure morphic words: the problem of the *D0L repetition threshold.*

9.1 The D0L repetition threshold

Recall the repetition threshold problem: given a natural number n, what is the infimum of the set of exponents that can be avoided over an alphabet of size n? As we have seen in Section 3.6, Dejean's conjecture states that this number is given by $RT(2) = 2$, $RT(3) = 7/4$, $RT(4) = 7/5$, and $RT(n) = n/(n-1)$ for $n \geq 5$. This conjecture was recently proved by Carpi for all $n \geq 33$, and remains open only for $15 \leq n \leq 32$. However, a question which we find just as interesting is the following: what is the repetition threshold for pure morphic words?

Definition 9.1. The *D0L repetition threshold*, denoted by RT_{D0L} is the number

$$RT_{D0L}(n) = \inf\{r \in \mathbb{R}_{>1} : \exists \text{ a pure morphic word } \mathbf{w} \in \Sigma_n^\omega \text{ that avoids } r\}. \tag{9.1}$$

Recall that by Thue, $RT_{D0L}(2) = RT(2)$, and by Dejean, $RT_{D0L}(3) = RT(3)$; both results were attained using $RT(n)^+$-power-free morphisms. However, by Brandenburg, there exist no $RT(n)^+$-power-free morphisms over Σ_n for $n \geq 4$. Does this imply that there are no $RT(n)^+$-power-free pure morphic words over Σ_n? and if there are no such words, what is $RT_{D0L}(n)$?

In a paper dealing with avoiding repetitions in arithmetic progressions ([22], 1988), Carpi constructed a family of pure morphic words avoiding $(1 + 1/p)$-powers, where p is any prime number. His construction implies that $\lim_{n\to\infty} RT_{D0L}(n) = 1$, i.e., every real number $\alpha > 1$ can be avoided by some pure morphic word. The alphabet size, however, is huge: to avoid $(1 + 1/p)$-powers Carpi needed $2p^4 - 2p^2$ letters. Just to compare, to avoid $(1+1/p)^+$-powers with an arbitrary infinite sequence we need only $p+1$ letters, and $p+2$ letters would suffice to avoid $(1 + 1/p)$-powers.

The $RT_{D0L}(n)$ question can be broken into two parts:

1. Does $RT_{D0L}(n) = RT(n)$?
2. If $RT_{D0L}(n) \neq RT(n)$, what is $RT_{D0L}(n)$?

The first question seems easier. We suspect that the answer is negative. A possible approach is to use Brandenburg's proof of non-existence of $RT(n)^+$-power-free morphisms over Σ_n. Brandenburg showed that if such a morphism existed, it would have to be uniform and marked. For a uniform marked morphism f there must exist some words $x = ab$, where $a \neq b$ are letters, such that $f(x)$ contains a $3/2$-power. If we could prove that any morphism generating an $RT(n)^+$-power-free word over Σ_n must be uniform and marked, then in order to show that $RT_{D0L}(n) > RT(n)$, it would suffice to show that any sufficiently long $RT(n)^+$-power-free word over Σ_n must contain all pairs of letters $a \neq b$. This is the case for $n = 4$: computer tests show that every word $w \in \Sigma_4$ avoiding $7/5^+$-powers, of length 23 or more, must contain all pairs.

The second question seems much harder, and currently we do not know how to approach it.

Bibliography

[1] S. I. Adian. *The Burnside Problem and Identities in Groups.* Springer-Verlag, 1979.

[2] A. Adler and S.-Y. Li. Magic cubes and Prouhet sequences. *Amer. Math. Monthly* **84** (1977), 618–627.

[3] J.-P. Allouche and M. Bousquet-Mélou. Facteurs des suites de Rudin-Shapiro généralisées. *Bull. Belgian Math. Soc.* **1** (1994), 145–164.

[4] J.-P. Allouche and J. Shallit. The ubiquitous Prouhet–Thue–Morse sequence. In *Sequences and their Applications: Proceedings of SETA 98*, pp. 1–16. Springer-Verlag, 1999.

[5] J.-P. Allouche and J. Shallit. Sums of digits, overlaps, and palindromes. *Discrete Math. & Theoret. Comput. Sci.* **4** (2000), 1–10.

[6] J.-P. Allouche and J. Shallit. *Automatic Sequences: Theory, Applications, Generalizations.* Cambridge University Press, 2003.

[7] S. E. Arshon. A proof of the existence of infinite asymmetric sequences on n symbols. *Matematicheskoe Prosveshchenie (Mathematical Education)* **2** (1935), 24–33. In Russian. Available electronically at `http://ilib.mccme.ru/djvu/mp1/mp1-2.htm`.

[8] S. E. Arshon. A proof of the existence of infinite asymmetric sequences on n symbols. *Mat. Sbornik* **2** (1937), 769–779. In Russian, with French abstract.

[9] D. Bean, A. Ehrenfeucht, and G. McNulty. Avoidable patterns in strings of symbols. *Pacific J. Math.* **85** (1979), 261–294.

[10] J. Berstel. Mots sans carré et morphismes itérés. *Discrete Math.* **29** (1979), 235–244.

[11] J. Berstel. *Axel Thue's papers on repetition in words: a translation*, Vol. 20 of *Publications du Laboratoire de Combinatoire et d'Informatique Mathématique*. Université du Québec à Montréal, 1995.

[12] J. Berstel. On the index of Sturmian words. In J. Karhumäki, H. Maurer, G. Păun, and G. Rozenberg, editors, *Jewels are Forever*, pp. 287–294. Springer-Verlag, 1999.

[13] J. Berstel and J. Karhumäki. Combinatorics on words – a tutorial. *Bull. European Assoc. Theor. Comput. Sci.* **79** (2003), 178–228.

[14] J. Berstel and D. Perrin. *Theory of Codes*, Vol. 117 of *Pure and Applied Mathematics*. Academic Press, 1985.

[15] J. Berstel and D. Perrin. The origins of combinatorics on words. *European J. Combinatorics* **28** (2007), 996–1022.

[16] J. Berstel and P. Séébold. A characterization of overlap-free morphisms. *Disc. Appl. Math.* **46** (1993), 275–281.

[17] V. Berthé, C. Holton, and L. Q. Zamboni. Initial powers of sturmian sequences. *Acta Arith.* **122**(4) (2006), 315–347.

[18] A. Blanchard and M. Mendès France. Symétrie et transcendance. *Bull. Sci. Math.* **106** (1982), 325–335.

[19] A. Blondin-Massé and S. Labbé. On the critical exponent of generalized Thue–Morse sequences. In *WORDS 2007, Proc. 6th Int'l Conf. on Words*, pp. 57–62, 2007.

[20] F.-J. Brandenburg. Uniformly growing kth power-free homomorphisms. *Theoret. Comput. Sci.* **23** (1983), 69–82.

[21] W.-T. Cao and Z.-Y. Wen. Some properties of the factors of Sturmian sequences. *Theoret. Comput. Sci.* **304** (2003), 365–385.

[22] A. Carpi. Multidimensional unrepetitive configurations. *Theoret. Comput. Sci.* **56** (1988), 233–241.

[23] A. Carpi. On the repetition threshold for large alphabets. In *Proc. MFCS*, Vol. 3162 of *Lecture Notes in Computer Science*, pp. 226–237. Springer-Verlag, 2006.

[24] A. Carpi. On Dejean's conjecture over large alphabets. *Theoret. Comput. Sci.* **385** (2007), 137–151.

[25] A. Carpi and A. de Luca. Special factors, periodicity, and an application to Sturmian words. *Acta Informatica* **36** (2000), 983–1006.

[26] J. Cassaigne. An algorithm to test if a given circular HD0L-language avoids a pattern. In B. Pehrson, I. Simon, K. Brunnstein, E. Raubold, K. Duncan, and K. Krueger, editors, *Information Processing '94*, Vol. 1, pp. 459–464.

[27] J. Cassaigne and J. D. Currie. Words strongly avoiding fractional powers. *European J. Combinatorics* **20** (1999), 725–737.

[28] C. Choffrut and J. Karhumäki. Combinatorics of words. In *Handbook of Formal Languages*, Vol. 1, chapter 6, pp. 329–438. Springer-Verlag, 1997.

[29] A. Cobham. Uniform tag sequences. *Math. Systems Theory* **6** (1972), 164–192.

[30] D. Crisp, W. Moran, A. Pollington, and P. Shiue. Substitution invariant cutting sequences. *J. Théorie Nombres Bordeaux* **5** (1993), 123–137.

[31] J. D. Currie. No iterated morphism generates any Arshon sequence of odd order. *Discrete Math.* **259** (2002), 277–283.

[32] J. D. Currie. What is the abelian analogue of Dejean's conjecture? In *Grammars and Automata for String Processing: From Mathematics and Computer Science to Biology, and Back: Essays in Honour of Gheorghe Păun*, pp. 237–242. Taylor and Francis, 2003.

[33] J. D. Currie. Pattern avoidance: themes and variations. *Theoret. Comput. Sci.* **339** (2005), 7–18.

[34] J. D. Currie and N. Rampersad. For each $\alpha > 2$ there is an infinite binary word with critical exponent α. Manuscript. Available electronically at `http://arxiv.org/abs/0802.4095`.

[35] J. D. Currie, N. Rampersad, and J. Shallit. Binary words containing infinitely many overlaps. *Electronic J. Combinatorics* **13** (2006), #R82. Available electronically at `http://www.combinatorics.org/Volume_13/Abstracts/v13i1r82.html`.

[36] D. Damanik and D. Lenz. The index of Sturmian sequences. *European J. Combinatorics* **23** (2002), 23–29.

[37] F. Dejean. Sur un théorème de Thue. *J. Combin. Theory. Ser. A* **13** (1972), 90–99.

[38] F. M. Dekking. On repetitions of blocks in binary sequences. *J. Combin. Theory. Ser. A* **20** (1976), 292–299.

[39] M. Dekking, M. Mendès France, and A. van der Poorten. FOLDS! *Math. Intelligencer* **4** (1982), 130–138; 173–181; 190–195.

[40] X. Droubay, J. Justin, and G. Pirillo. Episturmian words and some constructions of de Luca and Rauzy. *Theoret. Comput. Sci.* **255** (2001), 539–553.

[41] A. Ehrenfeucht, K. P. Lee, and G. Rozenberg. Subword complexities of various classes of deterministic developmental languages without interaction. *Theoret. Comput. Sci.* **1** (1975), 59–75.

[42] A. Ehrenfeucht and G. Rozenberg. Repetitions of subwords in D0L languages. *Inform. Comput.* **59** (1983), 13–35.

[43] R. C. Entringer, D. E. Jackson, and J. A. Schatz. On nonrepetitive sequences. *J. Combin. Theory. Ser. A* **16** (1974), 159–164.

[44] N. J. Fine and H. S. Wilf. Uniqueness theorems for periodic functions. *Proc. Amer. Math. Soc.* **16** (1965), 109–114.

[45] A. S. Fraenkel, M. Mushkin, and U. Tassa. Determination of $[n\theta]$ by its sequences of differences. *Canad. Math. Bull.* **21** (1978), 441–446.

[46] A. S. Fraenkel and J. Simpson. How many squares must a binary sequence contain? *Electronic J. Combinatorics* **2** (1995), #R2. Available electronically at `http://www.combinatorics.org/Volume_2/Abstracts/v2i1r2.html`.

[47] A. E. Frid. On uniform D0L words. In *STACS 98, Proc. 15th Symp. Theoretical Aspects of Comp. Sci.*, Vol. 1373 of *Lecture Notes in Computer Science*, pp. 544–554. Springer-Verlag, 1998.

[48] A. E. Frid. Overlap-free symmetric D0L words. *Discrete Math. & Theoret. Comput. Sci.* **4** (2001), 357–362.

[49] G. Frobenius. Über Matrizen aus nicht negativen Elementen. *Sitzungsberichte Preussische Akademie der Wissenschaft, Berlin* (1912), 456–477.

[50] M. Giesbrecht and A. Storjohann. Computing rational forms of integer matrices. *J. Symbolic Comput.* **34**(3) (2002), 157–172.

[51] I. Gil. Computation of the Jordan canonical form of a square matrix (using the Axiom programming language). In *Proc. Int'l. Symp. on Symbolic and Algebraic Computation: ISSAC '92*, pp. 138–145. ACM, 1992.

[52] I. Gil. *Contribution à l'algebre linéaire formelle. Formes normales de matrices et applications*. PhD thesis, Grenoble, France, 1993.

[53] A. Glen and J. Justin. Episturmian words: a survey. Manuscript. Available electronically at `http://arxiv.org/abs/0801.1655`.

[54] W. H. Gottschalk. Substitution minimal sets. *Trans. Amer. Math. Soc.* **109** (1963), 467–491.

[55] R. L. Graham, D.E. Knuth, and O. Patashnik. *Concrete Mathematics*. Adison-Wesley, 1989.

[56] D. Gusfield. *Algorithms on Strings, Trees, and Sequences*. Cambridge University Press, 1997.

[57] R. K. Guy. *Unsolved Problems in Number Theory.* Springer-Verlag, third edition, 2004.

[58] T. Harju and M. Linna. On the periodicity of morphisms on free monoids. *RAIRO Inform. Théor. App.* **20** (1986), 47–54.

[59] T. Harju and D. Nowotka. Binary words with few squares. *Bull. European Assoc. Theor. Comput. Sci.* **89** (2006), 164–166.

[60] T. Head and B. Lando. Fixed and stationary ω-words and ω-languages. In G. Rozenberg and A. Salomaa, editors, *The Book of L*, pp. 147–156. Springer-Verlag, 1986.

[61] R. Horn and C. Johnson. *Matrix Analysis.* Cambridge University Press, 1985.

[62] L. Ilie, P. Ochem, and J. Shallit. A generalization of repetition threshold. *Theoret. Comput. Sci.* **345** (2005), 359–369.

[63] J. Justin and G. Pirillo. Fractional powers in Sturmian words. *Theoret. Comput. Sci.* **255** (2001), 363–376.

[64] J. Justin and G. Pirillo. Episturmian words and episturmian morphisms. *Theoret. Comput. Sci.* **276** (2002), 281–313.

[65] J. Karhumäki. On cube free ω-words generated by binary morphisms. *Disc. Appl. Math.* **5** (1983), 279–297.

[66] V. Keränen. On k-repetition freeness of length uniform morphisms over a binary alphabet. *Disc. Appl. Math.* **9** (1984), 301–305.

[67] S. Kitaev. *Symbolic sequences, crucial words and iterations of a morphism.* PhD thesis, Göteborg, Sweden, October 2000.

[68] S. Kitaev. There are no iterative morphisms that define the Arshon sequence and the σ-sequence. *J. Automata, Languages, and Combinatorics* **8**(1) (2003), 43–50.

[69] S. Kitaev, T. Mansour, and P. Séébold. Generating the Peano curve and counting occurrences of some patterns. *J. Automata, Languages, and Combinatorics* **9**(4) (2004), 439–455.

[70] A. V. Klepinin and E. V. Sukhanov. On combinatorial properties of the Arshon sequence. *Disc. Appl. Math.* **114** (2001), 155–169.

[71] Y. Kobayashi. Repetition-free words. *Theoret. Comput. Sci.* **44** (1986), 175–197.

[72] Y. Kobayashi and F. Otto. Repetitiveness of languages generated by morphisms. *Theoret. Comput. Sci.* **240** (2000), 337–378.

[73] Y. Kobayashi, F. Otto, and P. Séébold. A complete characterization of repetitive morphisms over the two-letter alphabet. In *Proc. 3rd Ann. Int'l. Conf. on Computing and Combinatorics*, Vol. 1276 of *Lecture Notes in Computer Science*, pp. 393–402. Springer-Verlag, 1997.

[74] D. Krieger. On critical exponents in fixed points of binary k-uniform morphisms. In *STACS 2006, Proc. 23rd Symp. Theoretical Aspects of Comp. Sci.*, Vol. 3884 of *Lecture Notes in Computer Science*, pp. 104–114. Springer-Verlag, 2006.

[75] D. Krieger. On critical exponents in fixed points of non-erasing morphisms. In *DLT 06, Proc. 10th International Conference, Developments in Language Theory 2006*, Vol. 4036 of *Lecture Notes in Computer Science*, pp. 280–291. Springer-Verlag, 2006.

[76] D. Krieger. On critical exponents in fixed points of non-erasing morphisms. *Theoret. Comput. Sci.* **376** (2007), 70–88.

[77] D. Krieger. On critical exponents in fixed points of binary k-uniform morphisms. *RAIRO Inform. Théor. App.* (2007), available electronically at `http://dx.doi.org/10.1051/ita:2007042`.

[78] D. Krieger. On stabilizers of infinite words. *Theoret. Comput. Sci.* (2008), available electronically at `http://dx.doi.org/10.1016/j.tcs.2008.03.017`.

[79] D. Krieger and J. Shallit. Every real number greater than 1 is a critical exponent. *Theoret. Comput. Sci.* **381** (2007), 177–182.

[80] M. Leconte. A characterization of power-free morphisms. *Theoret. Comput. Sci.* **38** (1985), 117–122.

[81] A. Lindenmayer. Mathematical models for cellular interaction in development i & ii. *J. Theoret. Biol.* **18** (1968), 280–315.

[82] M. Lothaire. *Combinatorics on Words*, Vol. 17 of *Encyclopedia of Mathematics*. Addison-Wesley, 1983. Reprinted in the *Cambridge Mathematical Library*, Cambridge University Press, 1997.

[83] M. Lothaire. *Algebraic Combinatorics on Words*, Vol. 90 of *Encyclopedia of Mathematics*. Cambridge University Press, 2002.

[84] M. Lothaire. *Applied Combinatorics on Words*, Vol. 105 of *Encyclopedia of Mathematics*. Cambridge University Press, 2005.

[85] R. C. Lyndon and M.-P. Schützenberger. The equation $a^m = b^n c^p$ in a free group. *Michigan Math. J.* **9** (1962), 289–298.

[86] M. Mendès France. Principe de la symétrie perturbée. *Séminaire de Théorie des Nombres de Paris* (1979–1980), 77–98.

[87] F. Mignosi. Infinite words with linear subword complexity. *Theoret. Comput. Sci.* **65** (1989), 221–242.

[88] F. Mignosi and G. Pirillo. Repetitions in the Fibonacci infinite word. *RAIRO Inform. Théor. App.* **26** (1992), 199–204.

[89] F. Mignosi and P. Séébold. If a D0L language is k-power free then it is circular. In *Proc. 20th Int'l Conf. on Automata, Languages, and Programming (ICALP)*, Vol. 700 of *Lecture Notes in Computer Science*, pp. 507–518. Springer-Verlag, 1993.

[90] F. Mignosi and P. Séébold. Morphismes sturmiens et régles de Rauzy. *J. Théorie Nombres Bordeaux* **5**(2) (1993), 221–233.

[91] H. Minc. *Nonnegative Matrices.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley and Sons Inc., New York, 1988.

[92] M. Mohammad-Noori and J. D. Currie. Dejean's conjecture and Sturmian words. *European J. Combinatorics* **28** (2007), 876–890.

[93] M. Morse. Recurrent geodesics on a surface of negative curvature. *Trans. Amer. Math. Soc.* **22** (1921), 84–100.

[94] M. Morse and G. A. Hedlund. Unending chess, symbolic dynamics and a problem in semigroups. *Duke Math. J.* **11** (1944), 1–7.

[95] P. Morton and W. J. Mourant. Digit patterns and transcendental numbers. *J. Austral. Math. Soc. Ser. A* **51** (1991), 216–236.

[96] B. Mossé. Puissances de mots et reconnaissabilité des points fixes d'une substitution. *Theoret. Comput. Sci.* **99** (1992), 327–334.

[97] J. Moulin-Ollagnier. Proof of Dejean's conjecture for alphabets with $5, 6, 7, 8, 9, 10$ and 11 letters. *Theoret. Comput. Sci.* **95** (1992), 187–205.

[98] P. Ochem. Letter frequency in infinite repetition-free words. In S. Brlek and C. Reutenauer, editors, *WORDS 2005, Proc. 5th Int'l Conf. on Words*, Vol. 36 of *Publications du LaCim*, pp. 335–340. Université du Québec à Montréal, 2005.

[99] P. Ochem. A generator of morphisms for infinite words. *RAIRO Inform. Théor. App.* **40** (2006), 427–441.

[100] P. Ochem. Dejean's conjecture and letter frequency. In *Mons Days of Theoretical Computer Science 2006.* Manuscript available at `http://www.lri.fr/~ochem/publi.htm`.

[101] P. Ochem. Letter frequency in infinite repetition-free words. *Theoret. Comput. Sci.* **380** (2007), 388–392.

[102] P. Ochem, 2007. Personal communication.

[103] J.-J. Pansiot. The Morse sequence and iterated morphisms. *Inform. Process. Lett.* **12** (1981), 68–70.

[104] J.-J. Pansiot. Mots infinis de Fibonacci et morphismes itérés. *RAIRO Inform. Théor. App.* **17** (1983), 131–135.

[105] J.-J. Pansiot. A propos d'une conjecture de F. Dejean sur les répétitions dans les mots. *Disc. Appl. Math.* **7** (1984), 297–311.

[106] J.-J. Pansiot. Decidability of periodicity for infinite words. *RAIRO Inform. Théor. App.* **20** (1986), 43–46.

[107] O. Perron. Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus. *Math. Annalen* **64** (1907), 11–76.

[108] M. E. Prouhet. Mémoire sur quelques relations entre les puissances des nombres. *C. R. Acad. Sci. Paris* **33** (1851), 225.

[109] N. Rampersad. Words avoiding 7/3-powers and the Thue–Morse morphism. *Internat. J. Found. Comp. Sci.* **16** (2005), 755–766.

[110] N. Rampersad, J. Shallit, and M.-w. Wang. Avoiding large squares in infinite binary words. *Theoret. Comput. Sci.* **339** (2005), 19–34.

[111] G. Rauzy. Nombres algébriques et substitutions. *Bull. Soc. Math. France* **110** (1982), 147–178.

[112] G. Richomme. Conjugacy and episturmian morphisms. *Theoret. Comput. Sci.* **302** (2003), 1–34.

[113] G. Richomme. Some non finitely generated monoids of repetition-free morphisms. *Inform. Process. Lett.* **85** (2003), 61–66.

[114] G. Richomme. A local balance property of episturmian words. In *DLT 07, Proc. 11th International Conference, Developments in Language Theory 2007*, Vol. 4588 of *Lecture Notes in Computer Science*, pp. 371–381. Springer-Verlag, 2007.

[115] G. Richomme and P. Séébold. Characterization of test-sets for overlap-free morphisms. *Disc. Appl. Math.* **98** (1999), 151–157.

[116] G. Richomme and F. Wlazinski. Some results on k-power-free morphisms. *Theoret. Comput. Sci.* **273** (2002), 119–142.

[117] G. Richomme and F. Wlazinski. Overlap-free morphisms and finite test-sets. *Disc. Appl. Math.* **143** (2004), 92–109.

[118] G. Richomme and F. Wlazinski. Existence of finite test-sets for k-power-freeness of uniform morphisms. *Disc. Appl. Math.* **155**(15) (2007), 2001–2016.

[119] G. Rozenberg and A. Salomaa. *The Mathematical Theory of L Systems*, Vol. 90 of *Pure and Applied Mathematics*. Academic Press, 1980.

[120] K. Saari. On the frequency of letters in pure binary morphic sequences. In *DLT 05, Proc. 9th International Conference, Developments in Language Theory 2005*, Vol. 3572 of *Lecture Notes in Computer Science*, pp. 397–408. Springer-Verlag, 2005.

[121] K. Saari. On the frequency of letters in morphic sequences. In *Proceedings of CSR 2006*, Vol. 3967 of *Lecture Notes in Computer Science*, pp. 334–345. Springer-Verlag, 2006.

[122] P. Séébold. Sequences generated by infinitely iterated morphisms. *Disc. Appl. Math.* **11** (1985), 255–264.

[123] P. Séébold. An effective solution to the D0L-periodicity problem in the binary case. *Bull. European Assoc. Theor. Comput. Sci.* **36** (1988), 137–151.

[124] P. Séébold. On the conjugation of standard morphisms. *Theoret. Comput. Sci.* **195** (1998), 91–109.

[125] P. Séébold. About some overlap-free morphisms on a n-letter alphabet. *J. Automata, Languages, and Combinatorics* **7** (2002), 579–597.

[126] P. Séébold. On some generalizations of the Thue–Morse morphism. *Theoret. Comput. Sci.* **292** (2003), 283–298.

[127] P. Séébold. Tag-systems for the Hilbert curve. *Discrete Math. & Theoret. Comput. Sci.* **9**(2) (2007), 213–226.

[128] P. Séébold, 2007. Personal communication.

[129] J. Shallit. Simultaneous avoidance of large squares and fractional powers in infinite binary words. *Internat. J. Found. Comp. Sci.* **15** (2004), 317–327.

[130] A. M. Shur. The structure of the set of cube-free $\mathbb{Z}$-words in a two-letter alphabet. *Izv. Ross. Akad. Nauk Ser. Mat.* **64** (2000), 201–224. In Russian. English translation in *Izv. Math.* **64** (2000), 847–871.

[131] A. Thue. Über unendliche Zeichenreihen. *Kra. Vidensk. Selsk. Skrifter. I. Mat. Nat. Kl.* **7** (1906), 1–22. Reprinted in *Selected Mathematical Papers of Axel Thue*, T. Nagell *et al.*, editors, Universitetsforlaget, Oslo, 1977, pp. 139–158.

[132] A. Thue. Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen. *Kra. Vidensk. Selsk. Skrifter. I. Mat. Nat. Kl.* **1** (1912), 1–67. Reprinted in *Selected Mathematical Papers of Axel Thue*, T. Nagell *et al.*, editors, Universitetsforlaget, Oslo, 1977, pp. 413–477.

[133] C. R. Tompkins. Latin square Thue–Morse sequences are overlap-free. Manuscript available at `http://front.math.ucdavis.edu/0706.0907`, 2007.

[134] S. Vajda. *Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications.* Ellis Horwood, 1989.

[135] D. Vandeth. Sturmian words and words with a critical exponent. *Theoret. Comput. Sci.* **242** (2000), 283–300.

[136] N. Ya. Vilenkin. Formulas on cardboard. *Priroda* (6) (1991), 95–104. In Russian. English summary available at `http://www.ams.org/mathscinet/index.html`, review no. MR1143732.

[137] Z.-X. Wen and Z.-Y. Wen. Local isomorphisms of invertible substitutions. *C. R. Acad. Sci. Paris* **318** (1994), 299–304.

[138] Z.-X. Wen and Y. Zhang. Some remarks on invertible substitutions on three letter alphabet. *Chinese Sci. Bull.* **44** (1999), 1755–1760.

[139] F. Wlazinski. A test-set for k-power-free binary morphisms. *RAIRO Inform. Théor. App.* **35** (2001), 437–452.