# A Lagrangean Relaxation and A Heuristic for the Pooling Problem

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Hossa Almutairi

# Abstract

The pooling problem is one of the fundamental optimization problems encountered in the petroleum industry. In the pooling problem, final products are produced using two stages of blending operations. In the first stage, raw materials are mixed together to produce intermediate products. In the second stage, intermediate products and some of the raw materials are blended together according to product demand and quality requirements. Generally, the pooling problem is a nonlinear problem because the output stream qualities, which are unknown, depend on the volume, which is also unknown, and on the quality of the input streams. Specifically, nonlinearity and nonconvexity are due to the use of bilinear terms either in the quality constraints or in the objective function. Nonlinearity and nonconvexity result in several local optima, making the process of solving large-scale pooling problems to global optimality very challenging. Therefore, developing efficient heuristics for large-scale pooling problems is very desirable. Moreover, devising tight bounds on the global solutions is essential to assess the quality of the proposed heuristics.

In this thesis, we use a Lagrangean relaxation approach where feasible solutions and lower bounds are generated for the pooling problem. The procedure targets all nonlinear constraints and penalizes their violation in the objective function. The resulting Lagrangean subproblem has a nonlinear objective function and linear constraints. The Lagrangean subproblem is reformulated as a mixed integer programming problem where the nonlinearities in the objective function are eliminated at the expense of using binary variables. The obtained Lagrangean lower bounds are strengthened using valid cuts that are based on the relaxed bilinear terms. In addition, at every iteration of the Lagrangean algorithm, the subproblem solutions are used to generate feasible solutions to the pooling problem.

The procedure is applied to fifteen pooling problems collected from the literature. Some of these problems have a single quality and others have multiple qualities. Numerical results show that for eight solved cases, the obtained Lagrangean lower bounds are equal to the global optima, whereas for seven cases the obtained Lagrangean lower bound is on average 8.2% from the global optimum. Numerical results indicate the efficiency of the heuristic. For nine cases, the heuristic gives the global solution, and for the other cases the heuristic solutions are within 1.8% of the global optimum.

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# Dedication

*This thesis is dedicated to my father, who is my first teacher and greatest champion, who always believes in me, and whose support and advice are always there when I need them. This thesis is also dedicated to my mother, who taught me that hard work and patience make dreams come true. Her endless love and help are always there. I am so thankful that you are more than great parents; you are also my best friends. I love you!*

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# Chapter 1

# Introduction

Blending crude or refined petroleum is at the core of any refinery operation. Most of these blending operations involve two stages of blending. In the first stage, various raw materials are combined together, usually in pooling tanks, to produce intermediate products. In the second stage, intermediate products and some of the raw materials are mixed together to produce final products. This two-level blending process is often referred to as a pooling problem (Tawarmalani and Sahinidis, 2002). As shown in Figure 1.1, the pooling problem can be represented as a network with three sets of nodes. The first set of nodes represents raw materials with known attributes such as costs and qualities. The raw materials are distributed to the second set of nodes, the pooling tanks, to produce intermediate products or to the third set of nodes, to be used directly in blending final products. Taking into account product demand and restriction on end product qualities such as sulfur content, density, and octane number, the final products can be produced by blending raw materials and intermediate products received from the pooling tanks (Fieldhouse, 1993; Tawarmalani and Sahinidis, 2002). Given the availability of raw materials and their individual properties, as well as the demand and quality requirement of the final product, decisions are made to determine the optimal flow of raw materials to be mixed in order to minimize the difference between raw material cost and final product revenue.

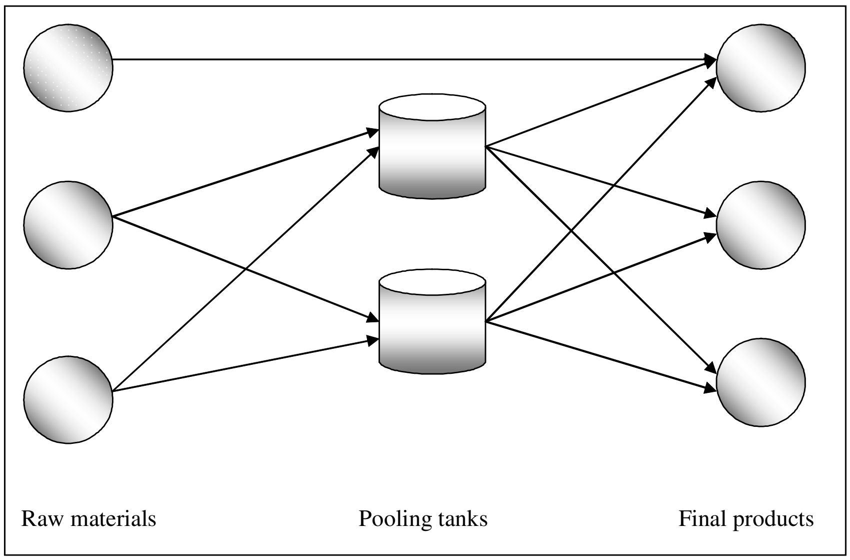


Figure 1.1: The pooling problem

As an illustration, Figure 1.2 shows a small pooling example presented by Haverly (1978). In this example, a single pool received two input streams of crude oil A and B. Each crude oil stream has different known properties, such as cost and sulfur content. The third source of crude oil, C, is used to blend directly with the pool output streams, producing two final products with some restrictions on demand and sulfur content. The formulation of this example is given in Chapter 3.

The need for pooling raw materials arises in several situations, three of which are detailed as follows. The first situation comes about when a refinery has limited

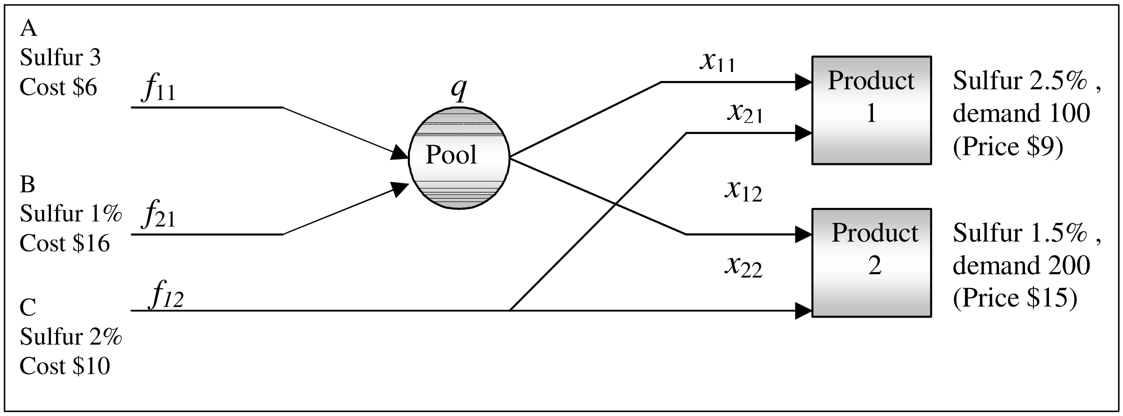


Figure 1.2: Haverly's pooling problem

storage facilities, leading to the storage of more than one product in one tank. The second situation arises when several products are transported together in a single pipeline or vessel. The third situation occurs when the requirements of the end product are not satisfied by any single feed (Floudas and Pardalos, 2004; Lasdon and Waren, 1980). In short, two products are pooled together, but the property of the resulting mixture does not match the quality requirement for the end product which makes blending a new product with the intermediate products a necessary step (Foulds et al., 1992).

Currently, with the increase of environmental regulations on refinery operations and refined product properties such as limiting sulfur and total aromatics contents in gasoline, refineries experience downward pressures on profitability (DeWitt and Lasdon, 1989; Duncan, 2000). The use of an efficient blending system plays a significant role in improving refined product quality, reducing cost and, consequently, improving profitability. Take for example the Texaco experience with its developed decision support system, OMEGA, for gasoline blending operations which is implemented in all its seven US refineries as well as its Canadian and Welsh refineries.

As stated in Dewitt and Lasdon (1989), the use of nonlinear optimization in solving pooling and blending problems results in better quality control and about 2.5 cents/gal of gasoline savings which are translated to more than 30 million dollars of annual savings.

From the modeling point of view, several approaches have been proposed for the pooling problem such as the p-formulation, the q-formulation, the pq-formulation, and the Generalized formulation. The p-formulation is the most used formulation since its introduction in Haverly (1978). Ben-Tal (1994) proposed the q-formulation as an alternative formulation for the pooling problem. The primary difference between the two formulations lies in the source of nonlinearities. In the p-formulation, explicit variables are used to represent pool qualities, and nonlinearities are due to the multiplication of quality variables by flow rate variables. In the q-formulation, proportion variables are introduced, instead of quality variables, to represent the proportion of raw materials used in each pool. As a result, nonlinearities are due to the multiplication of flow rate variables by proportion variables, and appear in the objective function and quality requirement constraints. The pq-formulation was proposed by Tawarmalani and Sahinidis (2002), as they sought to extend the q-formulation by adding new nonlinear constraints. The added constraints were derived by Quesada and Grossman (1995) using the reformulation linearization technique. Tawarmalani and Sahinidis (2002) proved that the linear programming relaxation of the pq-formulation using bilinear envelopes provides tighter lower bounds than the ones obtained through the same relaxation of the p-formulation and the q-formulation. A hybrid formulation of the p-formulation and the q-formulation called the Generalized formulation, is suggested by Audet et al. (2004). This formulation allows for interchanging intermediate products among the pools. Meyer and Floudas (2006) generalized the pooling formulation to include decisions related to pool existence and network structure. Continuous variables were used to model flow rates and stream attributes whereas binary variables were used to model the network structure.

Generally, all the proposed approaches are nonlinear formulations due to the use of bilinear terms either in the quality constraints or in the objective function. Balancing qualities around the pool introduces nonlinearity and nonconvexity to the problem because the output stream qualities, which are unknown, depend on the volume, which is also unknown, and the quality of the input streams (Fieldhouse, 1993; Tawarmalani and Sahinidis, 2002). Nonlinearity and nonconvexity result in several local optima, making the process of solving the problem to global optimality very challenging (Adhya et al., 1999; Tawarmalani and Sahinidis, 2002). Consequently, the literature is rich with suggested approaches that could be used to solve the various formulations of the pooling problem. One of the earlier developed approaches was recursion, which appears in the work of Haverly (1978). This approach is based on the idea of estimating and fixing the value of recursion variables, thus converting the nonlinear problem into a linear problem. This technique might not converge to a solution, and if it does, it leads to local optima (Haverly, 1979; Adhya et al., 1999). Successive linear-programming algorithms (SLP) have been widely used to solve the pooling problem and nonlinear blending operations which arise in the petrochemical industry. Some examples are: Baker and Lasdon (1985), Lasdon and Waren (1979), Bodington and Randall (1979) and Simon and Azma (1983). For further details, we refer the reader to the extensive survey article by Lasdon and Waren (1980). SLP solves the pooling problem through a sequence of linear programs (Baker and Lasdon, 1985; Griffith and Stewart, 1961), however, as with the recursion technique, the obtained solution is also a local optimum (Baker and Lasdon, 1985).

A decomposition strategy based on Geoffrion's generalization of Benders decomposition technique (Geoffrion, 1972) was suggested by Floudas and Aggarwal (1990) to search for the global optimum of the pooling problem. After identifying the source of nonconvexities, the variables are partitioned into complicating and noncomplicating variables. The original problem can then be decomposed into a master problem and a subproblem. Solving the master problem provides both a lower bound on the global minimum and the values for the complicating variables. In contrast, the solution of the subproblem yields an upper bound on the global minimum and the values for the noncomplicating variables. Furthermore, the solution of the subproblem provides dual information to form Benders' feasibility and optimality cuts that are added to the master problem (Sahinidis and Grossmann, 1991). Iterating between these problems continues until the specified stopping criteria are met. As is the case with previous approaches, identification of a global optimum is difficult to guarantee (Sahinidis and Grossmann, 1991; Floudas and Aggarwal, 1990 ). Visweswaran and Floudas (1990) developed a generalized approach, the Global Optimization Algorithm (GOP), to provide a global solution for the pooling problem. The GOP algorithm makes use of the duality theory to solve the pooling problem through a series of primal and relaxed dual subproblems. Visweswaran and Floudas (1993) proposed new theoretical properties of the GOP algorithm which improve its computational performance.

Audet et al. (2004) looked at the pooling problem from a slightly different angle. They investigated the application of a new branch-and-cut quadratic programming algorithm (Audet et al. 2000) to two formulations of the pooling problem: flow and proportion models. These two models are equivalent to the p-formulation and the q-formulation respectively. In addition, they developed a Multistart Alternate heuristic (MALT) and a Variable Neighborhood Search (VNS) metaheuristic to obtain starting solutions for exact algorithms and solve large instances of the pooling problem (Audet et al., 2004).

In the last two decades, most of the work on the pooling problem has focused on generating tight bounds on the global solutions using several relaxation techniques. Some examples are. Adhya et al. (1999), Foulds et al. (1992), Liberti and Pantelides (2006), Meyer and Foulds (2006), Quesada and Grossmann (1995), and Tawarmalani and Sahinidis (2002). The resulting lower bounds are integrated into global optimization techniques such as branch-and-bound algorithms in order to find global solutions. Foulds et al. (1992) used convex relaxation to underestimate and overestimate each bilinear term by its convex and concave envelopes respectively. The Reformulation-Linearization technique (RLT) has been suggested by some authors such as Liberti and Pantelides (2006), Meyer and Foulds (2006), and Quesada and Grossmann (1995) to generate tighter lower bounds on the global solutions of the pooling problem. Under this approach, additional nonlinear constraints are derived by multiplying a set of nonnegative variable factors with some of the original problem linear constraints (Sherali and Alameddine, 1992). The resulting nonlinear constraints are linearized using bilinear envelopes. Adhya et al. (1999) applied Lagrangean relaxation to the pooling problem. The basic idea behind this technique is that most difficult problems can be converted into easier ones by associating Lagrangean multipliers with complicated constraints and adding them to the objective function to penalize their violations. In the case of the pooling problem, complicated constraints are the ones that involve bilinear terms. Compared to the original problem, the Lagrangean subproblem is easier to solve and provides a lower bound on the optimal solutions for minimization problems, but as more constraints are relaxed the quality of the bound might deteriorate (Adhya et al., 1999; Ben-Tal et al., 1994).

Nowadays, global optimization techniques are quite effective in solving small-scale pooling problems from the open literature. However, real life problems tend to be large-scale problems. Solving large instances of the pooling problem to global optimality using the available global optimization algorithms is challenging. Hence, developing efficient heuristics for large-scale pooling problems is very desirable. In addition, relaxations that provide tight bounds on the global solutions are essential to assess and improve the quality of the heuristic solutions.

This thesis has two major contributions. The first contribution is the introduction of a new lower bound to the pooling problem using Lagrangean relaxation techniques. Lagrangean relaxation is applied to the p-formulation of the pooling problem to obtain lower bound on the global solution. The procedure targets all nonlinear constraints and penalizes their violation in the objective function. The resulting Lagrangean subproblem has a nonlinear objective function and linear constraints. The Lagrangean subproblem is reformulated as a mixed integer programming problem where the nonlinearities in the objective function are eliminated at the expense of using binary variables. The obtained Lagrangean lower bounds are strengthened by using valid cuts that are based on the relaxed bilinear terms.

The second contribution is the introduction of a heuristic technique, where Lagrangean subproblem solutions are used to generate feasible solutions to the pooling problem. At each iteration, the Lagrangean subproblem is solved for the values of the noncomplicating variables, flow variables. The obtained values are used to calculate the values of the complicating variables, quality variables, from the quality balance constraints. A linear programming problem, resulting from fixing the complicating variables in the original nonlinear problem, is solved at each iteration to generate feasible solutions.

The proposed Lagrangean relaxation approach and heuristic technique are applied to fifteen pooling problems collected from the literature. Some problems have a single quality, while others have multiple qualities. For eight solved cases, the obtained Lagrangean lower bounds are equal to the global optima, whereas for the other seven cases the obtained Lagrangean lower bound is on average 8.2% from the global solutions. Numerical results indicate the efficiency of the heuristic solutions. For nine cases, the heuristic gives the global solution, and for the other cases the heuristic solutions are within 1.8% of the global solution.

The thesis is organized as follows: Chapter two presents a literature review for the pooling problem in terms of modeling approaches and solution methodologies. Chapter three describes three of the pooling problem formulations and the proposed Lagrangean relaxation for the p-formulation. Chapter four reports the computational results. Finally, Chapter five concludes this thesis.

# Chapter 2

# Literature Review

The literature on the pooling problem focuses on two major complementary directions: formulations and solution methodologies. In the following two sections, we first review the problem formulations and then the solution methodologies.

## 2.1 Problem Formulations

Using the terminology presented by Ben-Tal et al. (1994), and Tawarmalani and Sahinidis (2002), pooling formulations can be classified as p-formulations, q-formulations, and pq-formulations. Meyer and Floudas (2006) proposed a generalized formulation for the pooling problem in the context of petrochemical and wastewater treatment industries to include decisions related to pool existence and network structure. Audet et al. (2004) have also proposed a general pooling formulation which is a hybrid formulation of the p-formulation and the q-formulation. Table 2.1 summarizes the work that has been done on the pooling problem in terms of formulation, number of pools, and qualities.

In this Chapter, we describe each formulation. In Chapter three, we explain the mathematical models of the p-formulation, the q-formulation, and the pq-formulation with illustrative examples.

### 2.1.1 The p-Formulation

The p-formulation was presented in Haverly (1978) and became the most common formulation for the pooling problem in the petrochemical industry. The basic idea of the p-formulation is the use of explicit variables to model the pool qualities and flow volumes. Apart from the pool quality balancing constraints and quality requirement constraints, the objective function and the remaining constraints are linear. Since the nonlinear terms are dependent on the number of pool qualities, bilinear terms increase as the number of qualities increases (Audet, 2004).

The same formulation is used by Lasdon et al. (1979), Floudas and Aggarwal (1990), and Fieldhouse (1993). Baker and Lasdon (1985) used the p-formulation to discuss the pooling problem at Exxon and developed a linearization technique using successive linear programming algorithms, a process that will be discussed later in the solution methodology section. Foulds et al. (1992) applied a generalized Benders' decomposition technique to the p-formulation to solve larger instances involving multiple pools. Amos and Gill (1997) used the p-formulation to develop a model that describes the pooling problem at the New Zealand Refining Company. In their proposed model, they added new variables to represent temperature cut points in each distillation. In addition, they introduced the use of cumulative functions to describe the distillation yields. Adhya et al. (1999) used the p-formulation

| Authors | Formulation | Pools | Qualities |
|---|---|---|---|
| Adhya et al. (1999) | p-formulation | Multiple | Multiple |
| Amos and Gill (1997) | p-formulation | Multiple | Single |
| Audet et al (2004) | Flow, Proportion, and Generalized formulation | Multiple | Multiple |
| Baker and Lasdon (1985) | p-formulation | Single | Single |
| Ben-Tal et al (1994) | q-formulation | Multiple | Multiple |
| Fieldhouse (1993) | p-formulation | Single | Single |
| Floudas and Aggarwal (1990) | p-formulation | Single | Single |
| Foulds et al. (1992) | p-formulation | Multiple | Single |
| Haverly (1978), (1979) | p-formulation | Single | Single |
| Lasdon et al. (1979) | p-formulation | Single | Single |
| Liberti and Pantelides (2006) | p-formulation | Multiple | Multiple |
| Meyer and Floudas (2006) | Generalized formulation | Multiple | Multiple |
| Quesada and Grossman (1995) | pq-formulation Generalized | Multiple | Single |
| Tawarmalani and Sahinidis (2002) | pq-formulation | Multiple | Multiple |
| Visweswaran and Floudas (1993), (1996) | p-formulation | Multiple | Multiple |

Table 2.1: Summary of the Formulation Approaches for the Pooling Problem

to develop a Lagrangian approach to the pooling problem. Audet et al. (2004) developed a flow model of the pooling problem based on the p-formulation.

### 2.1.2 The q-Formulation

The q-formulation was proposed by Ben-Tal et al. (1994). In this formulation, a new variable is introduced to represent the fraction of each stream used in each pool instead of using explicit variables for pool qualities. As a result, the nonlinearities arise in the objective function and quality requirement constraints, but with fewer nonlinear constraints since there are no quality balance constraints. Also, the number of nonlinear variables is independent of the number of pool qualities, making this formulation more practical as the number of qualities increases (Audet, 2004). Audet et al. (2004) presented a similar formulation called the proportion model which is based on the proportion of flows entering each pool.

### 2.1.3 The pq-Formulation

Tawarmalani and Sahinidis (2002) proposed the pq-formulation as an extension of the q-formulation by adding nonlinear convexification constraints. The added constraints were derived by Quesada and Grossman (1995), using the reformulation-linearization technique (RTL), to obtain a global solution of bilinear process networks. Tawarmalani and Sahinidis (2002) proved that the linear programming relaxation of the pq-formulation using bilinear envelopes dominates the linear programming relaxation and the Lagrangean relaxation of the p-formulation and the q-formulation.

### 2.1.4 The Generalized Formulation

Audet et al.(2004) proposed a hybrid formulation of the p-formulation and the q-formulation for a general pooling problem. This formulation allows for interconnecting among the pools so that each pool receives input streams of raw materials and intermediate products. The p-formulation is used to model the flows among the pools whereas the q-formulation is applied for the rest of the flows. Meyer and Floudas (2006) presented a generalized pooling problem in the context of the petrochemical and wastewater treatment industries. Binary variables were used to model decisions regarding pool existence and network configuration whereas continuous variables were used to model the flow rates and quality requirements.

## 2.2 Solution Methodologies

The main solution methodologies for the pooling problem are based on recursion, successive linear-programming algorithms, a global optimization algorithm (GOP), heuristic techniques, the reformulation-linearization technique (RLT), Benders decomposition, Lagrangean Relaxation, and Branch-and-Bound algorithms. In the following sections we describe each approach.

### 2.2.1 The Recursive Approach

The basic idea of the recursive approach is the estimation and fixing of the values of some recursion variables such as quality variables (Haverly, 1978, 1979). Haverly used two recursion variables: one as a quality variable and the other for the estimated fraction of the pool quality that was used for one of the products. Therefore,

| Authors | Solution Methodology | Optimum |
|---|---|---|
| Adhya et al (1999) | Lagrangean Relaxation, Branch-and-Bound | Global |
| Audet et al (2004) | Branch-and-cut quadratic programming, Heuristic | Global |
| Baker and Lasdon (1985) | Successive Linear Programming (SLP) | Local |
| Ben-Tal et al (1994) | Lagrangean Duality, Branch-and-Bound | Global |
| Fieldhouse (1993) | Distributed Recursion | Local |
| Floudas and Aggarwal (1990) | Generalized Benders Decomposition | Local |
| Foulds et al. (1992) | Convex Approximation, Branch-and-Bound | Global |
| Haverly (1978), (1979) | Recursion | Local |
| Lasdon et al. (1979) | Genralized reduced gradient and SLP | Local |
| Liberti and Pantelides (2006) | RRLT constraints Branch-and-Bound | Global |
| Meyer and Floudas (2006) | Reformulation-Linearization Technique (RLT) | Global |
| Quesada and Grossman (1995) | (RLT) , Branch-and-Bound | Global |
| Tawarmalani and Sahinidis (2002) | convexification techniques, Branch-and-Bound | Global |
| Visweswaran and Floudas (1993), (1996) | GOP Algorithm | Global |

Table 2.2: Summary of the Solution Methodologies for the Pooling Problem

the pooling problem is converted into a linear programming problem with only flow variables. The suggested algorithm involves two steps. First, the resulting linear program is solved, and then the quality value is calculated from the resulting flow variable values. If the obtained quality values coincide with the guessed values, the algorithm stops. Otherwise, the process is continued until the calculated quality values coincide with the estimated ones (Haverly, 1979; Adhya, 1999). Haverly shows that this approach may not converge to a solution, and when it converges, it does not always converge to a global optimum. Moreover, the obtained solutions depend on starting points. Main (1993) proved that this approach is stable for small problems and can result in computational difficulty when the number of pools and final product increase.

White and Trierwiller (1980) discussed the implementation and impact of an improved recursion technique, distributive recursion, at Socal. After adding the distributive recursion to Socal recursion program, the authors were able to model and solve pooling problems in a more realistic manner. Results reveal that the overall run time dropped 40 percent. The distributive recursion is equivalent to Successive Linear-Programming algorithms, a method that is discussed in the next subsection, as shown in Lasdon and Joffe (1990).

### 2.2.2 The Successive Linear-Programming (SLP) Approach

Successive linear-programming algorithms solve nonlinear optimization problems by using a sequence of linear programs. The key idea of this approach is the replacement of bilinear terms by the first-order Taylor series expansions in order to obtain a linearized SLP subproblem (Baker and Lasdon, 1985). Griffith and Stewart (1961) of Shell Oil described the application of SLP algorithms to nonlinear programming problems arising from petroleum refinery optimization problems. In their paper, the name they used for the proposed method is Mathematical Approximation Programming (MAP) which is later replaced with SLP algorithms.

Baker and Lasdon (1985) discussed the application of SLP algorithms at Exxon to nonlinear optimization problems involving the pooling problem. They proposed a multiplicative formulation, a method which includes nonnegative deviation variables in order to prevent infeasibility of LP subproblems, to solve linearized SLP subproblems. The multiplicative formulation has an advantage over the formulation proposed by Griffith and Stewart (1961), additive formulation, in that the obtained linearized problem is compatible with existing LP formulations (Baker and Lasdon, 1985). However, because of nonconvexities, convergence to a global optimum can not be guaranteed.

Simon and Azma (1983) reported on Exxon's experience with linear programming and SLP systems. Exxon implemented a math programming system, PLATOFORM, for planning applications. Although the PLATOFORM was implemented to solve large-scale linear programming problems, the system has evolved to handle mixed integer and nonlinear problems. The PLATOFORM employed SLP algorithms to solve nonlinear problems such as the pooling problem.

Lasdon et al. (1979) tested generalized reduced gradient and successive linear-programming algorithms on the three cases of the Haverly pooling problems. Results show that the two tested algorithms outperform the recursion approach.

SLP was used to solve nonlinear optimization problems because of its ability to utilize available LP codes and solve large-scale problems (Baker and Lasdon, 1985). However, the resulting solutions are local optima. Therefore, since 1990 most of the attempts to solve the pooling problem have focused on finding global solutions using various approaches such as decomposition, relaxation, and Branch-and-Bound algorithms. The following subsections summarize these attempts.

### 2.2.3 Benders Decomposition

Benders Decomposition has been used to solve large-scale linear and mixed-integer problems with special structures, as well as nonlinear optimization problems since its introduction in Benders (1962). The basic idea of this technique lies in fixing the complicating variables: the variables that, when fixed temporarily, result in a more tractable problem which is parameterized by the value of the complicating variables vector. The resulting relaxed problem can be decomposed into both a master problem and subproblems. The relaxed master problem is solved in order to provide a lower bound for a minimization problem as well as the value of the complicating variables. For fixed values of the complicating variables, the subproblem is solved to generate an upper bound and the value of the noncomplicating variables. Furthermore, the subproblem solutions are used to generate Benders' feasibility and optimality cuts that are added to the master problem. The process continues until the difference between lower and upper bound is sufficiently small (Sahinidis and Grossmann, 1991; Geoffrion, 1972; Benders, 1962).

Geoffrion (1972) generalized Benders' decomposition to account for broader problems where the subproblems can be nonlinear program. Nonlinear convex duality theory was used to derive the master problem. Floudas and Aggarwal (1989) presented a global optimization approach based on the generalized Benders' decomposition technique to search for the global solution of nonconvex NLP and MINLP problems. Floudas and Aggarwal (1990) extended the work that had been presented by Floudas in 1989 to determine the global optimum of the pooling problem. The proposed approach consists of four stages. The first stage is the identification of the sources of nonconvexity. In the pooling problem, bilinear terms are the source of nonconvexity. The second stage involves partitioning the variable set into complicating and noncomplicating variables. Quality and estimated pool fraction variables are selected to be the complicating variables in the pooling problem. The third stage entails the decomposition of the original problem into a subproblem and a master problem. The final stage is the iterating between the master problem and subproblem solutions based on the generalized Benders' decomposition technique until the specified stopping criteria are met. The procedure was applied to Haverly's pooling problems and one larger pooling problem. Although the global solutions of the four problems were found, there is no guarantee for a convergence to a global optimum (Sahinidis and Grossmann, 1991; Floudas and Aggarwal, 1990). Such guarantee is provided by a decomposition technique, a Global Optimization Algorithm, discussed next.

### 2.2.4 A Global Optimization Algorithm (GOP)

Visweswaran and Floudas (1990) proposed a generalized approach to determine a global optimum for several classes of nonconvex programming problems of the form:

$$
\begin{aligned}
\min_{x,y} \quad & f(x,y) \\
\\
s.t \quad & g(x,y) \leq 0 \\
& h(x,y) = 0 \\
& x \in X \\
& y \in Y
\end{aligned}
$$

Where $X$ and $Y$ are non-empty, compact, convex sets. $f(x,y), g(x,y)$, and $h(x,y)$ can be nonlinear functions leading to nonconvexities in the problem. The functions $f, h$,and $g$ should satisfy the following condition:

For fixed $y = y^k$, $f(x,y)$ and $g(x,y)$ are convex in $x$, and $h(x,y)$ is affine in $x$, and for every $x = x^k$, $f(x,y)$ and $g(x,y)$ are convex in $y$, and $h(x,y)$ is affine in $y$ (Liu and Floudas, 1995; Visweswaran and Floudas, 1990).

The GOP algorithm employs duality theory to solve nonconvex optimization problems through a series of primal and relaxed dual subproblems. The proposed algorithm was proven to have finite convergence to an $\epsilon$-global optimum (Visweswaran and Floudas, 1990). The GOP algorithm was applied to solve the three cases of the Haverly pooling problems to optimality.

Following that, Visweswaran and Floudas (1993) presented new theoretical properties of the GOP algorithm that improve the computational performance of the algorithm. Pooling problems with five products and three pools, each pool having two quality components, were solved to optimality.

Androulakis et al. (1995) identified the main computational bottlenecks of the GOP algorithm which is the requirement to solve a very large number of relaxed dual problems at a given iteration. Therefore, the authors proposed a distributed implementation of the GOP algorithm to improve the computational efficiency of the method. The proposed approach was able to solve large-scale, randomly generated, pooling problems with up to twelve components and thirty qualities.

Before discussing the third global optimization technique, we need to discuss the relaxation techniques that are used to generate lower bounds on the global solutions. These lower bounds are used within Branch-and-Bound algorithms to search for a global solution.

### 2.2.5 The Reformulation-Linearization/Convexification Technique (RLT)

The Reformulation-Linearization technique has been applied to bilinear programming problems since its introduction in Sherali and Alameddine (1992). The proposed approach consists of two fundamental steps: Reformulation and Linearization steps. In the reformulation step, additional nonlinear constraints are generated by multiplying some of the existing linear constraints with some of the original problem variables. For instance, consider the following bound constraints:

$$(x_j - x_j^L) \geq 0, (x_j^U - x_j) \geq 0 \ , \ j = 1, ...., J$$

$$\text{and} \ \ (q_i - q_i^L) \geq 0, (q_i^U - q_i) \geq 0 \ , \ i = 1, ...., I,$$

where $x_j^L \leq x_j \leq x_j^U$ and $q_i^L \leq q_i \leq q_i^U$ are the original problem variables and their range. Also, consider the linear constraints

$$\sum_i^I a_{ij} x_j - b_i = 0, \ \ i = 1, .....I$$

The RLT approach generates new constraints by multiplying bound constraints and the linear constraints. For example, $(q_i - q_i^L)\left(\sum_i^I a_{ij}x_j - b_i\right) = 0$

and $(x_j - x_j^L)(q_i - q_i^L) \geq 0$ are valid RLT constraints (Sherali and Alameddine, 1992; Liberti and Pantelides, 2004). The multiplication of bound constraints

$(x_j - x_j^L)(q_i - q_i^L) \geq 0$ and $(x_j^U - x_j)(q_i^U - q_i) \geq 0$ leads to:

$$
\begin{aligned}
x_j q_i &\geq x_j^L q_i + q_i^L x_j - x_j^L q_i^L \\
x_j q_i &\geq x_j^U q_i + q_i^U x_j - x_j^U q_i^U
\end{aligned}
$$

and the multiplication of bound constraints

$(x_j^U - x_j)(q_i - q_i^L) \geq 0$ and $(x_j - x_j^L)(q_i^U - q_i) \geq 0$ leads to:

$$
\begin{aligned}
x_j q_i &\leq x_j^U q_i + q_i^L x_j - x_j^U q_i^L \\
x_j q_i &\leq x_j^L q_i + q_i^U x_j - x_j^L q_i^U
\end{aligned}
$$

In the linearization step, the resulting nonlinear programming problem is linearized by replacing each bilinear term with a new variable. To illustrate, the above inequalities are linearized by replacing each bilinear term $x_j q_i$ with a new variable $w_{ij}$ as follows:

$$
\begin{aligned}
w_{ij} &\geq x_j^l q_i + q_i^l x_j - x_j^l q_i^l \\
w_{ij} &\geq x_j^u q_i + q_i^u x_j - x_j^u q_i^u \\
w_{ij} &\leq x_j^u q_i + q_i^l x_j - x_j^u q_i^l \\
w_{ij} &\leq x_j^l q_i + q_i^u x_j - x_j^l q_i^u
\end{aligned}
$$

Note that the obtained linearized constraints are the same as McCormick convex relaxation (McCormick, 1976), where a bilinear term $x_j q_i$ is underestimated by its convex envelope (the first two inequalities) and overestimated by its concave envelope (the last two inequalities) (Liberti and Pantelides, 2004).

Although the newly added constraints have the potential of providing tight bound in the convex relaxation, they contain redundant and inactive constraints which increase the size of the resulting relaxed problem. To reduce the size of the RLT constraints, Liberti and Pantelides (2006) proposed an algorithm, called reduced reformulation linearization technique (RRLT). The new reformulation contains fewer bilinear terms and more linear constraints. The proposed approach was applied to thirteen pooling problems. Results indicate the efficiency of the algorithm in providing tight convex relaxations.

Quesada and Grossmann (1995) used the Reformulation-Linearization technique in the context of general process networks that consist of splitters, mixers and process units that are interconnected with multicomponent streams. The technique is employed to establish a relation between two proposed formulations, composition and individual flow formulations. Moreover, the authors presented preprocessing steps to determine initial bounds on the variables involved in the nonconvex terms. Numerical results of twelve test problems imply that the proposed approach is capable of providing tight lower bounds.

Foulds et al. (1992) used McCormick convex relaxation to underestimate and overestimate bilinear terms by their convex and concave envelopes respectively. These envelopes are defined over the rectangular region derived from the bounds on the variables involved in bilinear terms. The procedure is tested on five pooling problems. For four solved problems, the resulting lower bounds are equal to the global solutions.

Tawarmalani and Sahinidis (2002) employed convexification techniques based on disjunctive programming to obtain lower bounds of the pq-formulation of the pooling problem. The authors proved that the linear programming relaxation of the pq-formulation using bilinear envelopes results in tighter lower bounds than the ones obtained using Lagrangean relaxation of the same formulation. The proposed technique is used to assess the relaxation quality of three formulations of the pooling problem. Results of solving fourteen pooling problems indicate that the linear programming relaxation of the pq-formulation using bilinear envelopes provides tighter lower bounds than the ones obtained through the same relaxation of the p-formulation and the q-formulation.

Meyer and Floudas (2006) discussed three techniques to generate a lower bound of a generalized pooling problem. In the first and second techniques, convex and concave envelopes of bilinear terms and the reformulation-linearization technique are used, respectively, to generate the lower bounds. In the third technique, piecewise linear RLT formulation, binary variables are used to model a partition of the continuous space. Then, the resulting MINLP is reformulated as a mixed-integer linear program using RLT principles. The approach is applied to industrial case study with seven sources, ten plants, single sink, and three qualities. The solution of a subnetwork with four plants was verified to be 1.2% from the optimum.

### 2.2.6 Lagrangean Relaxation

Lagrangean Relaxation is a relaxation technique that converts difficult problems into easier ones by associating Lagrangean multipliers with difficult constraints and adding them to the objective function so as to penalize their violation. On the one hand, solving the resulting Lagrangean subproblem is easier than the original problem, and it also provides a lower bound on the optimal solution of the original problem for a minimization case. On the other hand, as more constraints are relaxed, the obtained bound might not be tight enough (Held, 1971; Adhya et al., 1999). For the pooling problem case, the complicating constraints are the constraints involving bilinear terms. By relaxing these constraints, the resulting Lagrangean subproblem has a nonlinear objective function and linear constraints.

Ben-Tal et al. (1994) proposed a Lagrangean dual to obtain a lower bound to the q-formulation for the pooling problem as well as a branch-and-bound algorithm to partition the feasible set of the problem until the duality gap between the non-convex program and its Lagrangean dual is reduced. The procedure was applied to the Haverly's three problems and to other larger examples. The results show that the approach is successful in solving the examples to global optima. Adhya et al. (1999) suggested a Lagrangean relaxation approach to obtain a lower bound for the p-formulation of the pooling problem by dualizing all nonlinear and linear constraints and leaving linear bound constraints on the flow volume and pool quality. The resulting Lagrangean subproblem consists of optimizing a nonlinear objective function over a hypercube. The Lagrangean subproblem was reformulated as a mixed integer linear program. The proposed approach was applied to several previous problems and to four problems that were constructed in the course of their study. Results indicated that the proposed relaxation provided a tighter lower bound than the one obtained from the linear-programming approach based on McCormick estimators. A branch-and-bound algorithm was used to solve the proposed Lagrangean relaxation approach and obtain a global solution for the tested problem. Tawarmalani and Sahinidis (2002) applied Lagrangean relaxation for the pq-formulation and proved that this relaxation is no tighter than the linear programming relaxation obtained using bilinear envelopes for the same formulation.

Similar to Adhya et al. (1999), the Lagrangean relaxation proposed in this thesis is based on the p-formulation. However, our approach is different in that the resulting Lagrangean subproblem reserves most of the original problem structure because we relaxed only the complicating constraints which are the nonlinear constraints. Our approach is similar to Adhya et al. (1999) in that the resulting Lagrangean subproblems are reduced to mixed integer programming problems. However, the differences are in the approaches used to eliminate nonlinearities and reduce the Lagrangean subproblems to mixed integer programming problems as will be explained in Chapter three.

Having discussed the relaxation techniques used for generating a lower bound on the global solution of the pooling problem, we next describe Branch-and-Bound Algorithms.

### 2.2.7 Branch-and-Bound Algorithms

Branch and Bound is a popular technique which uses relaxation and enumeration to find global solutions of optimization problems. The algorithms partition the relaxed feasible regions into subregions and generate a tree of subproblems which have to be solved at each node. While the lower bound on the global solution of a minimization problem is given by a relaxed problem optimal solution, the upper bound on the global solution is found when the optimal solution of the subproblem is a feasible solution to the original problem. Based on the obtained bounds, some of the nodes are fathomed by optimality or feasibility arguments. The algorithms proceed until all the nodes are solved or fathomed (Horst and Tuy, 1996; Adhya et al., 1999).

Most of the reviewed papers used Branch-and-Bound algorithms to obtain a global solution of the pooling problem. Since having tight bounds is essential to reduce the search space, the proposed algorithms differ in the relaxation used to provide lower bounds on the global optima. Ben-Tal et al. (1994) provided Branch-and-Bound algorithms to partition the feasible set of the pooling problem until the duality gap between a nonconvex program and its Lagrangean dual is reduced. Numerical results of five solved pooling problems show that the optimal solutions of the five problems were found. Foulds et al. (1992) used McCormick convex relaxation to generate lower bound on the global optimum of the pooling problem. The resulting lower bounds were integrated into Branch-and-Bound algorithms to partition the hyper-rectangle region to obtain global solutions. Five pooling problems were solved to optimality using the suggested approach. However, results show that the procedure is time consuming when applied to large problems.

Quesada and Grossmann (1995) applied RLT to obtain a tight lower bound on the global optimum of the pooling problem. The obtained lower bound is used within special Branch-and-Bound algorithms. Numerical results on twelve solved pooling problems reveal that only a small number of nodes are needed in the Branch-and-Bound search to identify the global solutions. After applying convexification techniques to the pq-formulation, Tawarmalani and Sahinidis (2002) integrated the resulting lower bounds into Branch-and-Bound algorithms. Computational results demonstrate that the proposed approach leads to a significant reduction in the size of the Branch-and-Bound search tree. Adhya et al. (1999) used Lagrangean relaxation within a Branch-and-Bound framework to solve the pooling problem. Thirteen pooling problems were solved to global optimality. Audet et al. (2004) investigated how to apply a new Branch-and-Cut quadratic program, inspired by Al-Khayyal and Falk's (1983), to solve the pooling problem. Results imply that the proportion formulation, which is equivalent to the q-formulation, is preferable for these algorithms. Liberti and Pantelides (2006) used special Branch-and-Bound algorithms after deriving tight lower bounds using the reduced reformulation linearization technique (RRLT). Computational results of thirteen solved pooling problems show that having tight lower bounds speeds up the special Branch-and-Bound algorithms.

Small-scale problems from the open literature were solved to global optimality using available global optimization algorithms. However, solving large-scale problems to global optimality is challenging. Therefore, the literature often proposes heuristic techniques.

### 2.2.8 Heuristic Techniques

Audit et al. (2004) applied an Alternate heuristic and a Variable Neighborhood Search (VNS) metaheuristic to solve large instances of the pooling problem. The Alternate heuristic is based on the idea of dividing a set of variables into two subsets: complicating and noncomplicating variables and alternately solving linear programming problems, which result from fixing one of the subsets. The solution of each solved linear programming problem is used as parameters for the other one. Moreover, a multistart version of the Alternate heuristic (MALT) was used to improve the quality of the solutions.

The Variable Neighborhood Search (VNS) metaheuristic is based on the systematic change of the neighborhood within the search. We refer the readers to Hansen and Mladenović (2001) for a review of VNS and its applications to several classical optimization problems. To improve VNS solutions, MALT best solutions were used as initial solutions for VNS.

Audit et al. (2004) applied The Variable Neighborhood Search and the Multistart Alternate heuristic to the flow and the proportion models. Thirteen pooling problems were solved and a comparison of the computational properties of the two modeling approaches was given. Results show that for most cases, eleven cases, there is no significant difference between the two formulations. For the other two cases, the proportion model gives better results. Moreover, Audit et al. (2004) used the heuristic approaches to solve randomly generated pooling problems. Results reveal that VNS always gives the best results.

## 2.3 Conclusion

This chapter has reviewed the various pooling problem formulations such as the p-formulation, the q-formulation, the pq-formulation, and the generalized formulation. This chapter has also reviewed solution methodologies including recursion, successive linear-programming algorithms, Benders decomposition, a global optimization algorithm (GOP), the reformulation-linearization technique (RLT), Lagrangean Relaxation, Branch-and-Bound algorithms, and heuristic techniques.

From this review, it is clear that the challenge of solving the pooling problem is due to the appearance of bilinear terms which results in several local optima. Hence, most of the attempts to solve the pooling problem to global optimality are only achieved for small-scale problems. However, solving large-scale pooling problems to global optimality is still challenging. Practically, pooling problems involve large numbers of pools, qualities, and final products, which result in a model with a large number of bilinear terms. Therefore, using heuristic techniques to find good feasible solutions for large instances is desirable. One of the attempts in this direction is due to Audet et al (2004).

In this thesis, we focus on generating a tight lower bound using a new Lagrangean relaxation and constructing feasible solutions using a Lagrangean-based heuristic. In the following chapter, the proposed Lagrangean relaxation and heuristic techniques are presented.

# Chapter 3

# Problem Formulation and Solution Methodology

## 3.1 Problem Formulation

In the classical blending problem, end products are produced by mixing raw materials directly. Therefore, the blending problem can be formulated as a linear problem because the quality of the blend is approximated by the weighted average of the qualities of the input streams (Audet et al., 2004). In contrast, in the pooling problem, raw materials are blended together, and then the resulting blend as well as other input streams are mixed together to produce end products. Hence, the pooling problem can be formulated as a nonlinear problem because the quality of the pool, which is unknown, depends on the qualities and the volume of the input streams, which are also unknown (Audet et al., 2004 ; Fieldhouse, 1993).

As mentioned previously in Chapter two, several formulations have been proposed in the literature for the pooling problem. In this Chapter, three formulations of the pooling problem are given with illustrative examples. First, we present the p-formulation which is based on Adhya et al. (1999). This formulation is the one we use to develop the solution methodology. Then, we give the q-formulation and the pq-formulation which are based on Tawarmalani and Sahinidis (2002).

### 3.1.1 The p-Formulation

The pooling problem can be stated as follows. Given a set of $J$ pools and a set of $I$ available raw materials with known properties, as well as a set of $K$ final products and a set of $W$ pool qualities, decisions are made to determine the optimal quantity and quality of the streams. The aim is to minimize the difference between raw materials cost and final product revenue while satisfying end product demand and quality requirements (Adhya et al., 1999). Note that since not all raw materials are delivered to pool $j$, we defined $N_j$ to be the subset of raw materials $i$ that can be fed into pool $j$. The representation of the pooling problem is shown in Figure 3.1.

Before presenting the formulation, the following notation are introduced.

Indices:

$i$ available raw materials, $i = 1, ...., I$.

$j$ pools, $j = 1, ....., J$.

$k$ products, $k = 1, ...., K$.

$w$ qualities, $w = 1, ...., W$.

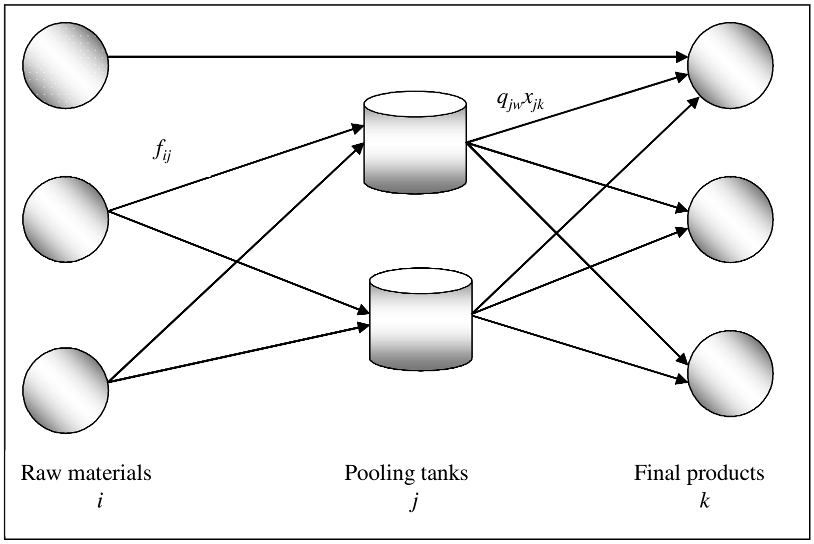


Figure 3.1: A Pooling Network

Parameters:

| | |
|---|---|
| $c_{ij}$ | unit cost of the $i$th stream into pool $j$. |
| $d_k$ | unit price of product $k$. |
| $s_k$ | demand for product $k$. |
| $z_{kw}$ | $w$th quality requirement for product $k$. |
| $t_{ijw}$ | $w$th quality specification of the $i$th stream into pool $j$. |
| $f_{ij}^l$ | lower bound on flow for $f_{ij}$ |
| $f_{ij}^u$ | upper bound on flow for $f_{ij}$ |
| $q_{jw}^l$ | lower bound on flow for $q_{jw}$ |
| $q_{jw}^u$ | upper bound on flow for $q_{jw}$ |
| $x_{jk}^l$ | lower bound on flow for $x_{jk}$ |
| $x_{jk}^u$ | upper bound on flow for $x_{jk}$ |

Variables:

| | |
|---|---|
| $f_{ij}$ | flow of $i$th input stream into pool $j$. |
| $x_{jk}$ | total flow from pool $j$ to product k. |
| $q_{jw}$ | $w$th quality of pool $j$ from pooling of streams. |

The p-formulation for the pooling problem is:

$$(PP) \quad \min \quad \sum_{j=1}^{J} \sum_{i \in N_j} c_{ij} f_{ij} - \sum_{k=1}^{K} d_k \sum_{j=1}^{J} x_{jk} \tag{1}$$

$$\begin{aligned}
s.t. \quad & \sum_{i \in N_j} f_{ij} - \sum_{k=1}^{K} x_{jk} = 0 && \forall j && (2)\\
& q_{jw} \sum_{k=1}^{K} x_{jk} - \sum_{i \in N_j} t_{ijw} f_{ij} = 0 && \forall j, w && (3)\\
& \sum_{j=1}^{J} q_{jw} x_{jk} - z_{kw} \sum_{j=1}^{J} x_{jk} \leq 0 && \forall k, w && (4)\\
& \sum_{j=1}^{J} x_{jk} \leq s_k && \forall k && (5)\\
& f_{ij}^l \leq f_{ij} \leq f_{ij}^u && \forall i, j && (6)\\
& q_{jw}^l \leq q_{jw} \leq q_{jw}^u && \forall j, w && (7)\\
& x_{jk}^l \leq x_{jk} \leq x_{jk}^u && \forall j, k && (8)
\end{aligned}$$

Note that this formulation does not allow raw materials to be mixed directly in the end products. To allow for such blending, a fake pool that receives only a single input stream and has multiple output streams can be used.

The objective function (1) minimizes the difference between raw material cost and end product revenue. Constraints (2) are mass balance constraints for each pool. Constraints (3) represent the quality mass balances around pools. Constraints (4) ensure that end product quality requirements are satisfied. Constraints (5) ensure that the total flows do not exceed demand. Constraints (3) and (4) are bilinear constraints that introduce nonconvexity to the problem. Note that constraints (4) in the "$\leq$" form are used when the end product qualities such as sulfur content are undesirable; however, when the end product qualities such as octane number are desirable, these constraints can be expressed in the "$\geq$" form. Constraints (6), (7), and (8) represent the bound on the flow of the raw materials, qualities, and the flows from pools to end products, respectively. The quality bounds are estimated from the raw material qualities (Haverly, 1978). As an illustration, from the streams fed into pool $j$, the stream with lower quality gives the lower bound and the stream with the higher quality provides the upper bound. Note also that except for those constraints which involve bilinear terms, which are constraints (3) and (4), the objective function and all other constraints are linear.

### An Illustrative Example of the p-Formulation

We use the pooling example (Haverly, 1978) presented in Chapter one to illustrate how to drive the p-formulation of the pooling problem. In this example, two input streams of crude oil, A and B are fed into a single pool. Each crude oil stream has different known properties, such as cost and sulfur content. The third source of crude oil, C, is used to blend directly with the pool output streams, producing two final products with some restrictions on demand and sulfur content as shown in Figure 3.2.

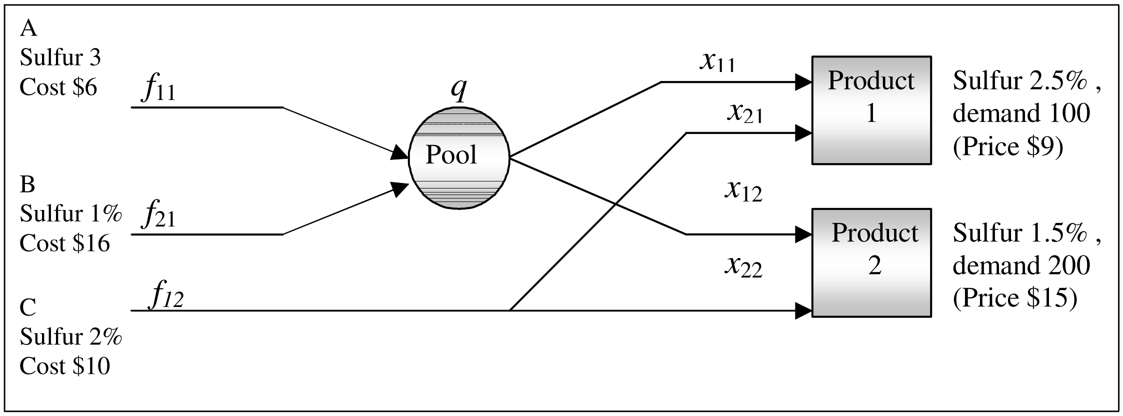


Figure 3.2: Haverly's pooling problem (p-formulation)

$$
\begin{aligned}
\min \quad & 6f_{11} + 16f_{21} + 10f_{12} - 9x_{11} - 9x_{21} - 15x_{12} - 15x_{22} \\
s.t. \quad & f_{11} + f_{21} - x_{11} - x_{12} = 0 \\
& f_{12} - x_{21} - x_{22} = 0 \\
& q(x_{11} + x_{12}) - 3f_{11} - f_{21} = 0 \\
& qx_{11} + 2x_{21} - 2.5(x_{11} + x_{21}) \leq 0 \\
& qx_{12} + 2x_{22} - 1.5(x_{12+}x_{22}) \leq 0 \\
& x_{11} + x_{21} \leq 100 \\
& x_{12} + x_{22} \leq 200 \\
& q^l \leq q \leq q^u \\
& f_{ij}, x_{jk} \geq 0; \quad \forall i, j, k
\end{aligned}
$$

### 3.1.2 The q-Formulation

Ben-Tal et al. (1994) derived the q-formulation by using new variables to represent the fraction of each input stream used in each pool instead of using explicit variables for pool qualities.

We use the same notations as in 3.1.1 and define the following additional parameters and variables.

Parameters:

| | |
|---|---|
| $I$ | number of available raw material |
| $t_{iw}$ | $w$th quality of raw material $i$ |
| $b_i$ | availability of $i$th raw material |
| $a_j$ | $j$th pool capacity |

Variables:

| | |
|---|---|
| $g_{ik}$ | direct flow of raw material $i$ to product $k$ |
| $x_{jk}$ | total flow from pool $j$ to product k |
| $p_{ij}$ | fraction of raw material $i$ used in pool $j$ |

Note that instead of using explicit variables $f_{ij}$ to represent the flow from raw material $i$ to pool $j$ and explicit variables $q_{jw}$ to represent pool qualities, only proportion variables $p_{ij}$ are used. Therefore, each $f_{ij}$ variables in the p-formulation is replaced with $p_{ij} \sum_{k}^{K} x_{jk} \quad \forall i, j$ in the q-formulation.

The resulting q-formulation for the pooling problem is:

$$\min \quad \sum_{k=1}^{K} \left( \sum_{j=1}^{J} x_{jk} \sum_{i=1}^{I} c_i p_{ij} - d_k \sum_{j=1}^{J} x_{jk} + \sum_{i=1}^{I} (c_i - d_k) g_{ik} \right) \tag{9}$$

$$s.t. \quad \sum_{j=1}^{J} \left( \sum_{i=1}^{I} t_{iw} p_{ij} - z_{kw} \right) x_{jk} + \sum_{i=1}^{I} (t_{iw} - z_{kw}) g_{ik} \leq 0 \qquad \forall k, w \tag{10}$$

$$\sum_{j=1}^{J} x_{jk} + \sum_{i=1}^{I} g_{ik} \leq s_k \qquad \forall k \tag{11}$$

$$\sum_{i=1}^{I} p_{ij} = 1 \qquad \forall j \tag{12}$$

$$p_{ij} \geq 0 \qquad \forall i, j \tag{13}$$

$$x_{jk} \geq 0 \qquad \forall j, k \tag{14}$$

$$g_{ik} \geq 0 \qquad \forall i, k \tag{15}$$

The objective function (9) minimizes the difference between raw material cost and end product revenue. Constraints (10) ensure that end product quality requirements are satisfied. Constraints (11) model end product demands. Constraints (12) represent the mass balances for each pool. Constraints (13), (14), and (15) are non-negativity constraints.

Note that raw material availability constraints, constraints (16), and pool capacity constraints, constraints (17), can be modeled as:

$$\sum_{j=1}^{J} \sum_{k=1}^{K} p_{ij} x_{jk} + \sum_{k=1}^{K} g_{ik} \leq b_i \qquad \forall i \tag{16}$$

$$\sum_{k=1}^{K} x_{jk} \leq a_j \qquad \forall j \tag{17}$$

The above constraints were presented in Tawarmalani and Sahinidis (2002).

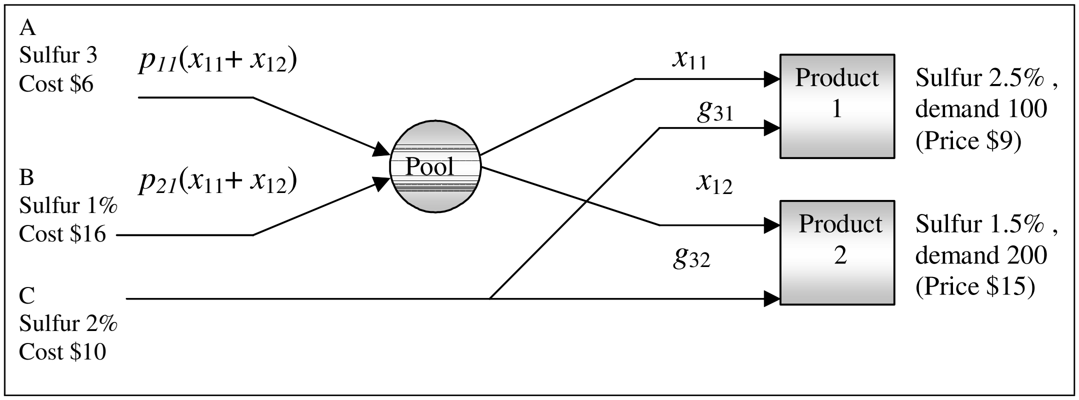


Figure 3.3: Haverly's pooling problem (q-formulation)

### An illustrative example for the q-Formulation

Haverly's pooling example can be written using the q-formulation as follows:

$$
\begin{aligned}
\min \quad & 6(p_{11}x_{11} + p_{11}x_{12}) + 16(p_{21}x_{11} + p_{21}x_{12}) + 10(g_{31} + g_{32}) \\
& -9(x_{11} + g_{31}) - 15(x_{12} + g_{32}) \\
s.t. \quad & x_{11} + g_{31} \leq 100 \\
& x_{12} + g_{32} \leq 200 \\
& (3p_{11} + p_{21} - 2.5)x_{11} - 0.5g_{31} \leq 0 \\
& (3p_{11} + p_{21} - 1.5)x_{12} + 0.5g_{32} \leq 0 \\
& p_{11} + p_{21} = 1 \\
& p_{ij},\, x_{jk},\, g_{ik} \geq 0; \quad \forall i, j, k
\end{aligned}
$$

### 3.1.3 The pq-Formulation

Starting from the q-formulation, Tawarmalani and Sahinidis (2002) derived the pq-formulation by adding the following convexification constraints:

$$\sum_{i=1}^{I} p_{ij} x_{jk} = x_{jk} \quad \forall j, k \quad (18)$$

The convexification constraints were derived using the idea of reformulation linearization technique (RLT) which appeared in the work of Sherali and Adams (1990), as well as the work of Quesada and Grossman (1995). Constraints (18) are obtained by multiplying $\sum_{i=1}^{I} p_{ij} = 1$ constraints with $x_{jk}$. In a similar way, Tawarmalani and Sahinidis (2002) derived constraints (19) by multiplying $\sum_{k=1}^{K} x_{jk} \leq a_j$ constraints with $p_{ij}$ in order to strengthen the obtained lower bound if the pq-formulation is relaxed using the convex relaxation technique. The idea of deriving constraints (19) was inspired by the work of Sherali et al. (1999).

$$\sum_{k=1}^{K} p_{ij} x_{jk} \leq a_j p_{ij} \quad \forall i, j \quad (19)$$

The resulting pq-formulation for the pooling problem is:

$$
\min \quad \sum_{k=1}^{K}\left(\sum_{j=1}^{J} x_{jk}\sum_{i=1}^{I} c_i p_{ij} - d_k \sum_{j=1}^{J} + \sum_{i=1}^{I}(c_i - d_k) g_{ik}\right) \tag{20}
$$

$$
\begin{aligned}
s.t.\quad & \sum_{j=1}^{J}\left(\sum_{i=1}^{I} t_{iw} p_{ij} - z_{kw}\right) x_{jk} + \sum_{i=1}^{I}(t_{iw} - z_{kw}) g_{ik} \le 0 && \forall k, w && (21)\\
& \sum_{j=1}^{J} x_{jk} + \sum_{i=1}^{I} g_{ik} \le s_k && \forall k && (22)\\
& \sum_{i=1}^{I} p_{ij} = 1 && \forall j && (23)\\
& p_{ij} \ge 0 && \forall i, j && (24)\\
& x_{jk} \ge 0 && \forall j, k && (25)\\
& g_{ik} \ge 0 && \forall i, k && (26)\\
& \sum_{j=1}^{J}\sum_{k=1}^{K} p_{ij} x_{jk} + \sum_{k=1}^{K} g_{ik} \le b_i && \forall i && (27)\\
& \sum_{k=1}^{K} x_{jk} \le a_j && \forall j && (28)\\
& \sum_{i=1}^{I} p_{ij} x_{jk} = x_{jk} && \forall i, j && (29)\\
& \sum_{k=1}^{K} p_{ij} x_{jk} \le a_j p_{ij} && \forall i, j && (30)
\end{aligned}
$$

The objective function (20) minimizes the difference between raw material cost and end product revenue. Constraints (21) ensure that end product quality requirements are satisfied. Constraints (22) model end product demands. Constraints (23) represent the mass balances for each pool. Constraints (24), (25), and (26) are non-negativity constraints. Constraints (27) and (28) model raw materials availabilities and pool capacities, respectively. Constraints (29) and (30) are derived using the reformulation-linearization technique.

**An illustrative example for the pq-Formulation**

Haverly's pooling example can be written using the pq-formulation as follows:

$$
\begin{aligned}
\min \quad & 6(p_{11}x_{11} + p_{11}x_{12}) + 16(p_{21}x_{11} + p_{21}x_{12}) + 10(g_{31} + g_{32}) \\
& -9(x_{11} + g_{31}) - 15(x_{12} + g_{32}) \\
s.t. \quad & x_{11} + g_{31} \leq 100 \\
& x_{12} + g_{32} \leq 200 \\
& (3p_{11} + p_{21} - 2.5)x_{11} - 0.5g_{31} \leq 0 \\
& (3p_{11} + p_{21} - 1.5)x_{12} + 0.5g_{32} \leq 0 \\
& p_{11} + p_{21} = 1 \\
& p_{11}x_{11} + p_{21}x_{11} = x_{11} \\
& p_{11}x_{12} + p_{21}x_{12} = x_{12} \\
& p_{ij},\, x_{jk},\, g_{ik} \geq 0; \qquad \forall i, j, k
\end{aligned}
$$

## 3.2 Proposed Lagrangean Relaxation

As mentioned previously in Chapter two, Lagrangean relaxation converts difficult problems into easier ones by dualizing complicating constraints. However, as more constraints are relaxed the resulting Lagrangean bound might not be tight enough. Adhya et al. (1999) constructed a Lagrangean relaxation for the p-formulation of the pooling problem by relaxing all constraints except the bound constraints on the flow and quality variables. Note also that Tawarmalani and Sahinidis (2002) applied the same Lagrangean relaxation presented in Adhya et al. (1999) to the pq-formulation. Ben-Tal et al. (1994) proposed a Lagrangean dual based on relaxing the entire constraint set except pool mass balance constraints.

In this Section, we present a new Lagrangean relaxation for the pooling problem based on the p-formulation. The proposed relaxation differs from the relaxation of Adhya et al. (1999) in that it targets only the nonlinear constraints. As a result, the resulting Lagrangean subproblem has most of the original problem structure. Moreover, the way we eliminate nonlinearities from the Lagrangean subproblem objective function is different from the one used in Adhya et al. (1999).

### 3.2.1 Lagrangean Relaxation for the General p-Formulation

To apply Lagrangean relaxation, we associate unrestricted Lagrangean multipliers $\alpha_{jw}$ with constraints (33) and positive Lagrangean multipliers $\beta_{kw}$ with constraints (34) in $(PP)$.

$$(PP) \quad \min \quad \sum_{j=1}^{J} \sum_{i \in N_j} c_{ij} f_{ij} - \sum_{k=1}^{K} d_k \sum_{j=1}^{J} x_{jk} \tag{31}$$

$$
\begin{array}{llll}
s.t. & \sum_{i \in N_j} f_{ij} - \sum_{k=1}^{K} x_{jk} = 0 & \forall j & (32) \\
& q_{jw} \sum_{k=1}^{K} x_{jk} - \sum_{i \in N_j} t_{ijw} f_{ij} = 0 & \forall j, w \quad \leftarrow \alpha_{jw} & (33) \\
& \sum_{j=1}^{J} q_{jw} x_{jk} - z_{kw} \sum_{j=1}^{J} x_{jk} \leq 0 & \forall k, w \quad \leftarrow \beta_{kw} \geq 0 & (34) \\
& \sum_{j=1}^{J} x_{jk} \leq s_k & \forall k & (35) \\
& f_{ij}^l \leq f_{ij} \leq f_{ij}^u & \forall i, j & (36) \\
& q_{jw}^l \leq q_{jw} \leq q_{jw}^u & \forall j, w & (37) \\
& x_{jk}^l \leq x_{jk} \leq x_{jk}^u & \forall j, k & (38)
\end{array}
$$

The resulting Lagrangean subproblem is:

$$
\begin{aligned}
(SPP) \quad \min \quad & \sum_{i \in N_j} \sum_{j=1}^{J} f_{ij}(c_{ij} - \sum_{w=1}^{W} t_{ijw}\alpha_{jw}) + \sum_{j=1}^{J} \sum_{k=1}^{K} x_{jk}(-d_k + \sum_{w=1}^{W} q_{jw}(\alpha_{jw} + \beta_{kw}) \\
& - \sum_{w=1}^{W} z_{kw}\beta_{kw})
\end{aligned}
$$

$$
\begin{aligned}
s.t. \quad & \sum_{i \in N_j} f_{ij} - \sum_{k=1}^{K} x_{jk} = 0 & \forall j \\
& \sum_{j=1}^{J} x_{jk} \leq s_k & \forall k \\
& f_{ij}^l \leq f_{ij} \leq f_{ij}^u & \forall i, j \\
& q_{jw}^l \leq q_{jw} \leq q_{jw}^u & \forall j, w \\
& x_{jk}^l \leq x_{jk} \leq x_{jk}^u & \forall j, k
\end{aligned}
$$

For a given value of Lagrangean multipliers ($\alpha_{jw}$, $\beta_{kw}$), the optimal solution of (*SPP*) provides a lower bound on the global solution of the original nonlinear problem (*PP*).

Rearranging terms in (*SPP*), we get

$$
\begin{aligned}
\min \quad & \sum_{i \in N_j} \sum_{j=1}^{J} f_{ij}(c_{ij} - \sum_{w=1}^{W} t_{ijw}\alpha_{jw}) + \sum_{j=1}^{J} \sum_{k=1}^{K} x_{jk}(-d_k - \sum_{w=1}^{W} z_{kw}\beta_{kw}) \\
& + \sum_{j=1}^{J} \sum_{k=1}^{K} x_{jk}(\sum_{w=1}^{W} q_{jw}(\alpha_{jw} + \beta_{kw}))
\end{aligned}
$$

$$
\begin{aligned}
s.t. \quad & \sum_{i \in N_j} f_{ij} - \sum_{k=1}^{K} x_{jk} = 0 && \forall j \\
& \sum_{j=1}^{J} x_{jk} \leq s_k && \forall k \\
& f_{ij}^{l} \leq f_{ij} \leq f_{ij}^{u} && \forall i, j \\
& q_{jw}^{l} \leq q_{jw} \leq q_{jw}^{u} && \forall j, w \\
& x_{jk}^{l} \leq x_{jk} \leq x_{jk}^{u} && \forall j, k
\end{aligned}
$$

To solve $(SPP)$ to global optimality, we are interested in eliminating the nonlinearity from the Lagrangean subproblem objective function. To do that, we reformulate $(SPP)$ into a mixed integer program as follows. First, if we define a new continuous variable $u_{jw}$ to satisfy the following relationship

$$
u_{jw} = q_{jw} \left( \sum_{k=1}^{K} x_{jk}(\alpha_{jw} + \beta_{kw}) \right), \qquad \forall j, w
$$

then the nonlinearity can be eliminated by using the linear bound constraints $q_{jw}^{l} \leq q_{jw} \leq q_{jw}^{u}$ and the fact that $q_{jw}$ does not appear in the rest of the constraints. Depending on the sign of $\sum_{k=1}^{K} x_{jk}(\alpha_{jw}+\beta_{kw})$ two cases should be considered:

$$
\begin{cases}
(\sum_{k=1}^{K}(\alpha_{jw}+\beta_{kw})x_{jk})q_{jw}^{l} \leq u_{jw} \leq (\sum_{k=1}^{K}(\alpha_{jw}+\beta_{kw})x_{jk})q_{jw}^{u}, \\
\text{if } (\sum_{k=1}^{K} x_{jk}(\alpha_{jw}+\beta_{kw})) \geq 0, \\
(\sum_{k=1}^{K}(\alpha_{jw}+\beta_{kw})x_{jk})q_{jw}^{u} \leq u_{jw} \leq (\sum_{k=1}^{K}(\alpha_{jw}+\beta_{kw})x_{jk})q_{jw}^{l}, \\
\text{if } (\sum_{k=1}^{K} x_{jk}(\alpha_{jw}+\beta_{kw})) \leq 0,
\end{cases}
$$

Hence, Replacing each $q_{jw}\left(\sum_{k=1}^{K} x_{jk}(\alpha_{jw}+\beta_{kw})\right)$ with $u_{jw}$ in $(SPP)$ reduces the Lagrangean subproblem to:

$$
\begin{aligned}
(SPPU) \quad \min \quad & \sum_{i\in N_j}\sum_{j=1}^{J} f_{ij}(c_{ij}-\sum_{w=1}^{W} t_{ijw}\alpha_{jw}) + \sum_{j=1}^{J}\sum_{k=1}^{K} x_{jk}(-d_k-\sum_{w=1}^{W} z_{kw}\beta_{kw}) \\
& + \sum_{j=1}^{J}\sum_{w=1}^{W} u_{jw}
\end{aligned}
$$

$$
\begin{aligned}
s.t. \quad & \sum_{i\in N_j} f_{ij} - \sum_{k=1}^{K} x_{jk} = 0 && \forall j \\
& \sum_{j=1}^{J} x_{jk} \leq s_k && \forall k \\
& f_{ij}^{l} \leq f_{ij} \leq f_{ij}^{u} && \forall i,j \\
& x_{jk}^{l} \leq x_{jk} \leq x_{jk}^{u} && \forall j,k \\
& (\sum_{k=1}^{K}(\alpha_{jw}+\beta_{kw})x_{jk})q_{jw}^{l} \leq u_{jw} \leq (\sum_{k=1}^{K}(\alpha_{jw}+\beta_{kw})x_{jk})q_{jw}^{u}, && \forall j,w \\
& \text{if } (\sum_{k=1}^{K} x_{jk}(\alpha_{jw}+\beta_{kw})) \geq 0 \\
& (\sum_{k=1}^{K}(\alpha_{jw}+\beta_{kw})x_{jk})q_{jw}^{u} \leq u_{jw} \leq (\sum_{k=1}^{K}(\alpha_{jw}+\beta_{kw})x_{jk})q_{jw}^{l}, && \forall j,w \\
& \text{if } (\sum_{k=1}^{K} x_{jk}(\alpha_{jw}+\beta_{kw})) \leq 0
\end{aligned}
$$

Second, to model the if-then constraints in $(SPPU)$, we define a binary variable

$$y_{jw} = \begin{cases} 1 \text{ if } (\sum_{k=1}^{K} x_{jk}(\alpha_{jw} + \beta_{kw})) \geq 0 \\ 0 \text{ otherwise} \end{cases}$$

and introduce big-M constraints as follows:

$$\begin{aligned}
&(\sum_{k=1}^{K}(\alpha_{jw} + \beta_{kw})x_{jk})q^{l}_{jw} \leq u_{jw} + M(1 - y_{jw}) && \forall j, w \\
&u_{jw} \leq (\sum_{k=1}^{K}(\alpha_{jw} + \beta_{kw})x_{jk})q^{u}_{jw} + M(1 - y_{jw}) && \forall j, w \\
&(\sum_{k=1}^{K}(\alpha_{jw} + \beta_{kw})x_{jk})q^{u}_{jw} \leq u_{jw} + My_{jw} && \forall j, w \\
&u_{jw} \leq (\sum_{k=1}^{K}(\alpha_{jw} + \beta_{kw})x_{jk})q^{l}_{jw} + My_{jw} && \forall j, w \\
&(\sum_{k=1}^{K}(\alpha_{jw} + \beta_{kw})x_{jk}) \geq -M(1 - y_{jw}) && \forall j, w \\
&(\sum_{k=1}^{K}(\alpha_{jw} + \beta_{kw})x_{jk}) \leq My_{jw} && \forall j, w
\end{aligned}$$

The above constraints can be written as:

$$\begin{aligned}
&(\sum_{k=1}^{K}(\alpha_{jw} + \beta_{kw})x_{jk})q^{l}_{jw} - u_{jw} + My_{jw} \leq M && \forall j, w \\
&-(\sum_{k=1}^{K}(\alpha_{jw} + \beta_{kw})x_{jk})q^{u}_{jw} + u_{jw} + My_{jw} \leq M && \forall j, w \\
&(\sum_{k=1}^{K}(\alpha_{jw} + \beta_{kw})x_{jk})q^{u}_{jw} - u_{jw} - My_{jw} \leq 0 && \forall j, w \\
&-(\sum_{k=1}^{K}(\alpha_{jw} + \beta_{kw})x_{jk})q^{l}_{jw} + u_{jw} - My_{jw} \leq 0 && \forall j, w \\
&(\sum_{k=1}^{K} -(\alpha_{jw} + \beta_{kw})x_{jk}) + My_{jw} \leq M && \forall j, w \\
&(\sum_{k=1}^{K}(\alpha_{jw} + \beta_{kw})x_{jk}) - My_{jw} \leq 0 && \forall j, w
\end{aligned}$$

Therefore, $(SPPU)$ is equivalent to the following mixed integer program:

$$
\begin{aligned}
(SMIP) \quad \min \quad & \sum_{i \in N_j} \sum_{j=1}^{J} f_{ij}(c_{ij} - \sum_{w=1}^{W} t_{ijw}\alpha_{jw}) + \sum_{j=1}^{J} \sum_{k=1}^{K} x_{jk}(-d_k - \sum_{w=1}^{W} z_{kw}\beta_{kw}) \\
& + \sum_{j=1}^{J} \sum_{w=1}^{W} u_{jw} \\
s.t. \quad & \sum_{i \in N_j} f_{ij} - \sum_{k=1}^{K} x_{jk} = 0 && \forall j \\
& \sum_{j=1}^{J} x_{jk} \leq s_k && \forall k \\
& (\sum_{k=1}^{K} (\alpha_{jw} + \beta_{kw}) x_{jk}) q_{jw}^{l} - u_{jw} + M y_{jw} \leq M && \forall j, w \\
& -(\sum_{k=1}^{K} (\alpha_{jw} + \beta_{kw}) x_{jk}) q_{jw}^{u} + u_{jw} + M y_{jw} \leq M && \forall j, w \\
& (\sum_{k=1}^{K} (\alpha_{jw} + \beta_{kw}) x_{jk}) q_{jw}^{u} - u_{jw} - M y_{jw} \leq 0 && \forall j, w \\
& -(\sum_{k=1}^{K} (\alpha_{jw} + \beta_{kw}) x_{jk}) q_{jw}^{l} + u_{jw} - M y_{jw} \leq 0 && \forall j, w \\
& (\sum_{k=1}^{K} -(\alpha_{jw} + \beta_{kw}) x_{jk}) + M y_{jw} \leq M && \forall j, w \\
& (\sum_{k=1}^{K} (\alpha_{jw} + \beta_{kw}) x_{jk}) - M y_{jw} \leq 0 && \forall j, w \\
& f_{ij}^{l} \leq f_{ij} \leq f_{ij}^{u} && \forall i, j \\
& x_{jk}^{l} \leq x_{jk} \leq x_{jk}^{u} && \forall j, k \\
& y_{jw} \in \{0, 1\} && \forall j, w
\end{aligned}
$$

The best Lagrangean lower bound is given by the optimal solution of the Lagrangean dual problem:

$$\max_{\beta\geq 0,\alpha}\left\{\begin{array}{lll} \min & \sum_{i\in N_j}\sum_{j=1}^{J} f_{ij}(c_{ij}-\sum_{w=1}^{W} t_{ijw}\alpha_{jw})+\sum_{j=1}^{J}\sum_{k=1}^{K} x_{jk}(-d_k+\sum_{w=1}^{W} q_{jw}(\alpha_{jw}+\beta_{kw}) & \\ & -\sum_{w=1}^{W} z_{kw}\beta_{kw}) & \\ & & \\ s.t. & \sum_{i\in N_j} f_{ij}-\sum_{k=1}^{K} x_{jk}=0 & \forall j \\ & \sum_{j=1}^{J} x_{jk}\leq s_k & \forall k \\ & f_{ij}^{l}\leq f_{ij}\leq f_{ij}^{u} & \forall i,j \\ & q_{jw}^{l}\leq q_{jw}\leq q_{jw}^{u} & \forall j,w \\ & x_{jk}^{l}\leq x_{jk}\leq x_{jk}^{u} & \forall j,k \end{array}\right\}$$

Which is equivalent to:

$$\max_{\beta\geq 0,\alpha}\left\{\begin{array}{ll} \min_{h\in H} & \sum_{i\in N_j}\sum_{j=1}^{J} f_{ij}^{h}(c_{ij}-\sum_{w=1}^{W} t_{ijw}\alpha_{jw})+\sum_{j=1}^{J}\sum_{k=1}^{K} x_{jk}^{h}(-d_k+\sum_{w=1}^{W} q_{jw}^{h}(\alpha_{jw}+\beta_{kw}) \\ & -\sum_{w=1}^{W} z_{kw}\beta_{kw}) \end{array}\right\}$$

where $H$ is the index set of extreme points to the set:

$$
H: \left\{
\begin{array}{lll}
(f^h, x^h, u^h, y^h): & \sum_{i \in N_j} f_{ij}^h - \sum_{k=1}^{K} x_{jk}^h = 0 & \forall j \\
& \sum_{j=1}^{J} x_{jk}^h \leq s_k & \forall k \\
& (\sum_{k=1}^{K} (\alpha_{jw} + \beta_{kw}) x_{jk}^h) q_{jw}^l - u_{jw}^h + M y_{jw}^h \leq M & \forall j, w \\
& -(\sum_{k=1}^{K} (\alpha_{jw} + \beta_{kw}) x_{jk}^h) q_{jw}^u + u_{jw}^h + M y_{jw}^h \leq M & \forall j, w \\
& (\sum_{k=1}^{K} (\alpha_{jw} + \beta_{kw}) x_{jk}^h) q_{jw}^u - u_{jw}^h - M y_{jw}^h \leq 0 & \forall j, w \\
& -(\sum_{k=1}^{K} (\alpha_{jw} + \beta_{kw}) x_{jk}^h) q_{jw}^l + u_{jw}^h - M y_{jw}^h \leq 0 & \forall j, w \\
& (\sum_{k=1}^{K} -(\alpha_{jw} + \beta_{kw}) x_{jk}^h) + M y_{jw}^h \leq M & \forall j, w \\
& (\sum_{k=1}^{K} (\alpha_{jw} + \beta_{kw}) x_{jk}^h) - M y_{jw}^h \leq 0 & \forall j, w \\
& f_{ij}^l \leq f_{ij}^h \leq f_{ij}^u & \forall i, j \\
& x_{jk}^l \leq x_{jk}^h \leq x_{jk}^u & \forall j, k \\
& y_{jw}^h \in \{0, 1\} & \forall j, w
\end{array}
\right\}
$$

which are also feasible to

$$
H: \left\{
\begin{array}{lll}
(f^h, x^h, q^h): & \sum_{i \in N_j} f_{ij}^h - \sum_{k=1}^{K} x_{jk}^h = 0 & \forall j \\
& \sum_{j=1}^{J} x_{jk}^h \leq s_k & \forall k \\
& f_{ij}^l \leq f_{ij}^h \leq f_{ij}^u & \forall i, j \\
& q_{jw}^l \leq q_{jw}^h \leq q_{jw}^u & \forall j, w \\
& x_{jk}^l \leq x_{jk}^h \leq x_{jk}^u & \forall j, k
\end{array}
\right\}
$$

If we define

$$\theta = \min_{h \in H} \left\{ \begin{array}{l} \sum_{i \in N_j} \sum_{j=1}^{J} f_{ij}^h (c_{ij} - \sum_{w=1}^{W} t_{ijw} \alpha_{jw}) + \sum_{j=1}^{J} \sum_{k=1}^{K} x_{jk}^h (-d_k + \sum_{w=1}^{W} q_{jw}^h (\alpha_{jw} + \beta_{kw}) \\ - \sum_{w=1}^{W} z_{kw} \beta_{kw}) \end{array} \right\}$$

The Lagrangean dual problem can be written as the following linear program, which we refer to as the master problem $(MPP)$:

$$\begin{array}{lll} (MPP) & \max_{\alpha,\beta,\theta} & \theta \\ & s.t. & \theta + \sum_{j=1}^{J} \sum_{w=1}^{W} \alpha_{jw} (\sum_{i \in N_j} t_{ijw} f_{ij}^h - q_{jw}^h \sum_{k=1}^{K} x_{jk}^h) + \sum_{k=1}^{K} \sum_{w=1}^{W} \beta_{kw} \\ & & (\sum_{j=1}^{J} x_{jk}^h (z_{kw} - q_{jw}^h)) \leq \sum_{i \in N_j} \sum_{j=1}^{J} c_{ij} f_{ij}^h - \sum_{k=1}^{K} d_k \sum_{j=1}^{J} x_{jk}^h; \quad \forall h \in H \\ & & \beta_{kw} \geq 0 \end{array}$$

## 3.3 An Illustrative Example

In this Section, we apply the proposed Lagrangean relaxation to the pooling example presented in Section 3.1.

### 3.3.1 Lagrangean Relaxation for The Pooling Example

We construct the Lagrangean relaxation by associating the unrestricted Lagrangean multiplier $\lambda$ with constraint (42) and positive Lagrangean multipliers $\alpha$ and $\beta$ with constraints (43) and (44) respectively.

$$
\begin{aligned}
(PP) \quad \min \quad & 6f_{11} + 16f_{21} + 10f_{12} - 9x_{11} - 9x_{21} - 15x_{12} - 15x_{22} && (39)\\
s.t. \quad & f_{11} + f_{21} - x_{11} - x_{12} = 0 && (40)\\
& f_{12} - x_{21} - x_{22} = 0 && (41)\\
& -3f_{11} - f_{21} + qx_{11} + qx_{12} = 0 \qquad \longleftarrow \lambda && (42)\\
& (q - 2.5)x_{11} - .5x_{21} \leq 0 \qquad \longleftarrow \alpha \geq 0 && (43)\\
& (q - 1.5)x_{12} + .5x_{22} \leq 0 \qquad \longleftarrow \beta \geq 0 && (44)\\
& x_{11} + x_{21} \leq S_1 = 100 && (45)\\
& x_{12} + x_{22} \leq S_2 = 200 && (46)\\
& q^l \leq q \leq q^u && (47)\\
& f_{ij}, x_{jk} \geq 0; \quad \forall i, j, k && (48)
\end{aligned}
$$

The resulting Lagrangean subproblem is:

$$
\begin{aligned}
(SPP) \quad \min \quad & (6 - 3\lambda)f_{11} + (16 - \lambda)f_{21} + 10f_{12} + (-9 + q\lambda + q\alpha - 2.5\alpha)x_{11}\\
& +(-9 - .5\alpha)x_{21} + (-15 + q\lambda + q\beta - 1.5\beta)x_{12} + (-15 + .5\beta)x_{22}\\
s.t. \quad & f_{11} + f_{21} - x_{11} - x_{12} = 0\\
& f_{12} - x_{21} - x_{22} = 0\\
& x_{11} + x_{21} \leq S_1 = 100\\
& x_{12} + x_{22} \leq S_2 = 200\\
& q^l \leq q \leq q^u\\
& f_{ij}, x_{jk} \geq 0; \quad \forall i, j, k
\end{aligned}
$$

For a given value of Lagrangean multipliers ($\lambda$, $\alpha$, $\beta$), the optimal solution of ($SPP$) provides a lower bound on the global solution of the original nonlinear problem ($PP$).

After rearranging the terms in $(SPP)$, we get:

$$\begin{aligned}
\min \quad & (6-3\lambda)f_{11} + (16-\lambda)f_{21} + 10f_{12} + (-9-2.5\alpha)x_{11} + (-9-.5\alpha)x_{21} \\
& +(-15-1.5\beta)x_{12} + (-15+.5\beta)x_{22} + q(\lambda+\alpha)x_{11} + q(\lambda+\beta)x_{12} \\
s.t. \quad & f_{11} + f_{21} - x_{11} - x_{12} = 0 \\
& f_{12} - x_{21} - x_{22} = 0 \\
& x_{11} + x_{21} \leq 100 \\
& x_{12} + x_{22} \leq 200 \\
& q^l \leq q \leq q^u \\
& f_{ij}, x_{jk} \geq 0; \quad \forall i, j, k
\end{aligned}$$

.

To solve $(SPP)$ to global optimality, we are interested in eliminating the nonlinearity from the Lagrangean subproblem objective function. To do that, we reformulate $(SPP)$ into a mixed integer program as follows. First, if we define a new variable $u$ to satisfy the following relationship

$$u = q\left((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}\right),$$

then the nonlinearity can be eliminated by using the linear bound constraints $q^l \leq q \leq q^u$ and the fact that $q$ does not appear in the other constraints. Two cases should be considered depending on the sign of $(\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}$ :

$$\begin{cases}
\left((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}\right) q^l \leq u \leq \left((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}\right) q^u, \\
\text{if } \left((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}\right) \geq 0, \\
\left((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}\right) q^u \leq u \leq \left((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}\right) q^l, \\
\text{if } \left((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}\right) \leq 0
\end{cases}$$

where $q^l = 1$, and $q^u = 3$.

Hence, the subproblem is reduced to:

$$
\begin{aligned}
(SPPU) \quad \min \quad & (6-3\lambda)f_{11} + (16-\lambda)f_{21} + 10f_{12} + (-9-2.5\alpha)x_{11} + (-9-.5\alpha)x_{21} \\
& +(-15-1.5\beta)x_{12} + (-15+.5\beta)x_{22} + u \\
s.t. \quad & f_{11} + f_{21} - x_{11} - x_{12} = 0 \\
& f_{12} - x_{21} - x_{22} = 0 \\
& x_{11} + x_{21} \leq 100 \\
& x_{12} + x_{22} \leq 200 \\
& ((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}) \leq u \leq 3\,((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12})\,, \\
& \text{if } ((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}) \geq 0, \\
& 3\,((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}) \leq u \leq ((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12})\,, \\
& \text{if } ((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}) \leq 0 \\
& f_{ij}, x_{jk} \geq 0; \quad \forall i, j, k
\end{aligned}
$$

Second, to model the if-then constraints in $(SPPU)$, we define a binary variable

$$
y = \begin{cases} 1 \text{ if } ((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}) \geq 0 \\ 0 \text{ otherwise} \end{cases}
$$

and introduce big-M constraints as follows:

$$
\begin{aligned}
& ((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}) \leq u + M(1-y) \\
& u \leq 3\,((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}) + M(1-y) \\
& 3\,((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}) \leq u + My \\
& u \leq ((\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12}) + My. \\
& (\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12} \geq -M(1-y) \\
& (\lambda+\alpha)x_{11} + (\lambda+\beta)x_{12} \leq My
\end{aligned}
$$

The above constraints can be written as follows

$$
\begin{aligned}
&(\lambda+\alpha)x_{11}+(\lambda+\beta)x_{12}-u+My \leq M \\
&u-((\lambda+\alpha)3x_{11}-(\lambda+\beta)3x_{12})+My \leq M \\
&(\lambda+\alpha)3x_{11}+(\lambda+\beta)3x_{12}-u-My \leq 0 \\
&-(\lambda+\alpha)x_{11}-(\lambda+\beta)x_{12}+u-My \leq 0 \\
&-(\lambda+\alpha)x_{11}-(\lambda+\beta)x_{12}+My \leq M \\
&((\lambda+\alpha)x_{11}+(\lambda+\beta)x_{12})-My \leq 0
\end{aligned}
$$

Therefore, $(SPPU)$ is equivalent to the following mixed integer program:

$$
\begin{aligned}
(SMIP) \quad \min \quad & (6-3\lambda)f_{11}+(16-\lambda)f_{21}+10f_{12}+(-9-2.5\alpha)x_{11}+(-9-.5\alpha)x_{21} \\
& +(-15-1.5\beta)x_{12}+(-15+.5\beta)x_{22}+v \\
s.t. \quad & f_{11}+f_{21}-x_{11}-x_{12}=0 \\
& f_{12}-x_{21}-x_{22}=0 \\
& x_{11}+x_{21} \leq 100 \\
& x_{12}+x_{22} \leq 200 \\
& (\lambda+\alpha)x_{11}+(\lambda+\beta)x_{12}-u+My \leq M \\
& u-((\lambda+\alpha)3x_{11}-(\lambda+\beta)3x_{12})+My \leq M \\
& (\lambda+\alpha)3x_{11}+(\lambda+\beta)3x_{12}-u-My \leq 0 \\
& -(\lambda+\alpha)x_{11}-(\lambda+\beta)x_{12}+u-My \leq 0 \\
& -(\lambda+\alpha)x_{11}-(\lambda+\beta)x_{12}+My \leq M \\
& ((\lambda+\alpha)x_{11}+(\lambda+\beta)x_{12})-My \leq 0 \\
& f_{ij}, x_{jk} \geq 0; \quad \forall i,j,k; \quad y \in \{0,1\}.
\end{aligned}
$$

The optimal solution of the above mixed integer Lagrangean subproblem provides a lower bound on the global solution of the original nonlinear problem However, the best Lagrangean lower bound is given by the optimal solution of the Lagrangean dual problem.

$$\max_{\alpha \geq 0, \beta \geq 0} \left\{ \begin{array}{ll} \min & (6-3\lambda)f_{11} + (16-\lambda)f_{21} + 10f_{12} + (-9 + q\lambda + q\alpha - 2.5\alpha)x_{11} \\ & +(-9-.5\alpha)x_{21} + (-15 + q\lambda + q\beta - 1.5\beta)x_{12} + (-15+.5\beta)x_{22} \\ \text{s.t.} & f_{11} + f_{21} - x_{11} - x_{12} = 0 \\ & f_{12} - x_{21} - x_{22} = 0 \\ & x_{11} + x_{21} \leq 100 \\ & x_{12} + x_{22} \leq 200 \\ & q^l \leq q \leq q^u \\ & f_{ij}, x_{jk} \geq 0; \quad \forall i, j, k \end{array} \right\}$$

Which is equivalent to:

$$\max_{\alpha \geq 0, \beta \geq 0} \left\{ \begin{array}{ll} \min\limits_{h \in H} & (6-3\lambda)f_{11}^h + (16-\lambda)f_{21}^h + 10f_{12}^h + (-9 + q^h\lambda + q^h\alpha - 2.5\alpha)x_{11}^h \\ & +(-9-.5\alpha)x_{21}^h + (-15 + q^h\lambda + q^h\beta - 1.5\beta)x_{12}^h + (-15+.5\beta)x_{22}^h \end{array} \right\}$$

where $H$ is the index set of extreme points to the set:

$$\left\{\begin{array}{ll} (f^h, x^h, u^h, y^h): & f^h_{11} + f^h_{21} - x^h_{11} - x^h_{12} = 0 \\ & f^h_{12} - x^h_{21} - x^h_{22} = 0 \\ & x^h_{11} + x^h_{21} \le 100 \\ & x^h_{12} + x^h_{22} \le 200 \\ & (\lambda+\alpha)x^h_{11} + (\lambda+\beta)x^h_{12} - u^h + My^h \le M \\ & u^h - \big((\lambda+\alpha)3x^h_{11} - (\lambda+\beta)3x^h_{12}\big) + My^h \le M \\ & (\lambda+\alpha)3x^h_{11} + (\lambda+\beta)3x^h_{12} - u^h - My^h \le 0 \\ & -(\lambda+\alpha)x^h_{11} - (\lambda+\beta)x^h_{12} + u^h - My^h \le 0 \\ & -(\lambda+\alpha)x^h_{11} - (\lambda+\beta)x^h_{12} + My^h \le M \\ & \big((\lambda+\alpha)x^h_{11} + (\lambda+\beta)x^h_{12}\big) - My^h \le 0 \\ & f^h_{ij}, x^h_{jk} \ge 0; \quad \forall i,j,k; \quad y^h \in \{0,1\}. \end{array}\right\}$$

which are also feasible to

$$H: \left\{\begin{array}{ll} (f^h, x^h, q^h): & f^h_{11} + f^h_{21} - x^h_{11} - x^h_{12} = 0 \\ & f^h_{12} - x^h_{21} - x^h_{22} = 0 \\ & x^h_{11} + x^h_{21} \le 100 \\ & x^h_{12} + x^h_{22} \le 200 \\ & q^l \le q^h \le q^u, \\ & f^h_{ij}, x^h_{jk} \ge 0; \quad \forall i,j,k \end{array}\right\}$$

If we define

$$\theta = \min_{h\in H}\left\{\begin{array}{l} (6-3\lambda)f^h_{11} + (16-\lambda)f^h_{21} + 10f^h_{12} + (-9 + q^h\lambda + q^h\alpha - 2.5\alpha)x^h_{11} \\ +(-9-.5\alpha)x^h_{21} + (-15 + q^h\lambda + q^h\beta - 1.5\beta)x^h_{12} + (-15+.5\beta)x^h_{22} \end{array}\right\}$$

The Lagrangean dual problem can be written as the following linear program, which we refer to as the master problem.

$$
\begin{aligned}
(MPP) \quad & \max_{\alpha,\beta,\theta} \quad \theta \\
& s.t. \quad \theta + (3f_{11}^h + f_{21}^h + q^h(-x_{11}^h - x_{12}^h))\lambda + (2.5x_{11}^h + .5x_{21}^h - q^h x_{11}^h)\alpha \\
& \qquad +(1.5x_{12}^h - .5x_{22}^h - qx_{12}^h)\beta \leq 6f_{11}^h + 16f_{21}^h + 10f_{12}^h - 9x_{11}^h - 9x_{21}^h \\
& \qquad -15x_{12}^h - 15x_{22}^h;\ \forall h \in H \\
& \qquad \alpha \geq 0, \beta \geq 0
\end{aligned}
$$

## 3.4 Strengthening the Lagrangean Lower Bounds

Generally, for global optimization techniques such as Branch-and-Bound algorithms, tight bounds reduce the search space and, consequently, improve the performance of the algorithms. For heuristic techniques, tight bounds can also be useful in improving the quality of the heuristic solutions.

In this Section, we seek to improve the quality of the heuristic solutions by improving the Lagrangean lower bounds. In order to do that, we generate valid cuts using the idea of replacing each bilinear term with a new linear variable and adding linear constraints to the Lagrangean subproblem to bound the value of the linear variables. For instance, if we define a new nonnegative variable $v_{jkw}$ to replace each bilinear term $q_{jw}x_{jk}$ we have the following relationship

$$v_{jkw} = q_{jw}x_{jk} \quad \forall j, k, w$$

Using the linear bound constraints $q_{jw}^l \leq q_{jw} \leq q_{jw}^u$ , we can eliminate the nonlinearity and bound the value of the introduced variable $v_{jkw}$ as follows:

$$q_{jw}^l x_{jk} \leq v_{jkw} \leq q_{jw}^u x_{jk} \quad \forall j, k, w \quad (49)$$

Thus, the quality constraints

$$\begin{aligned} q_{jw} \sum_{k=1}^{K} x_{jk} - \sum_{i \in N_j} t_{ijw} f_{ij} = 0 \qquad \forall j, w \\ \sum_{j=1}^{J} q_{jw} x_{jk} - z_{kw} \sum_{j=1}^{J} x_{jk} \leq 0 \qquad \forall k, w \end{aligned}$$

can be written as:

$$\sum_{k=1}^{K} v_{jkw} - \sum_{i \in N_j} t_{ijw} f_{ij} = 0 \qquad \forall j, w \quad (50)$$

$$\sum_{j=1}^{J} v_{jkw} - z_{kw} \sum_{j=1}^{J} x_{jk} \leq 0 \qquad \forall k, w \quad (51)$$

Recall that in Section 3.2, we defined another new variable $u_{jw}$ to satisfy the following relationship

$$u_{jw} = q_{jw} \left( \sum_{k=1}^{K} x_{jk} (\alpha_{jw} + \beta_{kw}) \right) \quad \forall j, w$$

The above constraints can also be written as:

$$u_{jw} = \sum_{k=1}^{K} v_{jkw} (\alpha_{jw} + \beta_{kw}) \quad \forall j, w \quad (52)$$

Constraints $(49), (50), (51)$,and $(52)$ are added to the Lagrangean subproblem to strengthen the Lagrangean lower bound.

## 3.5 A Heuristic Approach and Overall Algorithm

In Section 3.2, we generated a lower bound on the global solution of the pooling problem using Lagrangean relaxation. However, the solutions $(\bar{f}, \bar{x})$ we get from solving the Lagrangean subproblem, are mostly likely to be infeasible for the original nonlinear problem. Hence, we use a Lagrangean heuristic to construct feasible solutions and to provide an upper bound on the original nonlinear problem optimal solution. Lagrangean heuristics typically work on the Lagrangean subproblem solution by modifying it to be feasible to the original nonlinear problem. In our approach, the Lagrangean heuristics works as follows:

1. Start with initial values for Lagrangean multipliers ($\alpha$, $\beta$).

2. Solve the Lagrangean subproblem ($SMIP$) and find the optimal values ($\bar{f}$, $\bar{x}$) of the flow variables.

3. Keep the best Lagrangean lower bound $\bar{Z}_{Lag}$ found so far.

4. Using the Lagrangean subproblem optimal solution ($\bar{f}$, $\bar{x}$), calculate the quality values from the quality mass balance constraints

$$q_{jw} \sum_{k=1}^{K} x_{jk} - \sum_{i \in N_j} t_{ijw} f_{ij} = 0 \qquad \forall j, w$$

   as follows:

$$\bar{q}_{jw} = \left( \frac{\sum_{i \in N_j} t_{ijw} \bar{f}_{ij}}{\sum_{k=1}^{K} \bar{x}_{jk}} \right) \qquad \forall j, w$$

5. Using the obtained $\bar{q}_{jw}$ values from 4, fix the quality values in the quality balance and requirement constraints in ($PP$) and solve the resulting linear program:

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{J} \sum_{i \in N_j} c_{ij} f_{ij} - \sum_{k=1}^{K} d_k \sum_{j=1}^{J} x_{jk} \\
\\
s.t. \quad & \sum_{i \in N_j} f_{ij} - \sum_{k=1}^{K} x_{jk} = 0 && \forall j \\
& \bar{q}_{jw} \sum_{k=1}^{K} x_{jk} - \sum_{i \in N_j} t_{ijw} f_{ij} = 0 && \forall j, w \\
& \sum_{j=1}^{J} \bar{q}_{jw} x_{jk} - z_{kw} \sum_{j=1}^{J} x_{jk} \leq 0 && \forall k, w \\
& \sum_{j=1}^{J} x_{jk} \leq s_k && \forall k \\
& f_{ij}^{l} \leq f_{ij} \leq f_{ij}^{u} && \forall i, j \\
& x_{jk}^{l} \leq x_{jk} \leq x_{jk}^{u} && \forall j, k
\end{aligned}
$$

6. Keep the best found feasible solution $\bar{Z}_{Heur}$.

7. Solve the master problem ($MPP$) to get new values for Lagrangean multipliers ($\alpha$, $\beta$) and get the value of its objective function $\bar{Z}_{Mas}$.

8. If the stopping condition $Z_{Mas} - Z_{Lag} > \varepsilon$ is not met, update Lagrangean multipliers in step 1 and repeat steps 2-8.

## 3.6 Conclusion

In this chapter, we explained with illustrative examples three mathematical models of the pooling problem. These models are the p-formulation, the q-formulation, and the pq-formulation. This chapter also presented a new Lagrangean relaxation for the pooling problem based on the p-formulation. We relaxed only the complicating constraints which are the quality balance and the quality requirement constraints. We showed how the Lagrangean subproblem was reduced to a mixed integer programming problem using binary variables. We improved the resulting Lagrangean lower bounds by adding valid cuts to the Lagrangean subproblem. The added constraints are derived using the idea of replacing each bilinear term with a new continuous variable and adding linear constraints to the Lagrangean subproblem to bound the value of the new variable. To construct feasible solutions, we used a Lagrangean heuristic that modified the Lagrangean subproblem solutions to be feasible to the relaxed constraints. The following chapter reports on numerical results of fifteen pooling problems from the literature.

# Chapter 4

# Computational Results

The proposed approach is coded in Matlab 7. The master problem and the subproblem are solved using GLPK. Fifteen pooling problems collected from the literature were solved using the proposed approach. Table 4.1 shows test problem characteristics in terms of number of pools, qualities, raw materials, and end products.

Table 4.2 shows a comparison between the proposed Lagrangean lower bound and those lower bounds proposed in the literature. The second column displays the linear-programming relaxation, LP, for the p-formulation using McCormick overestimators and underestimators (Foulds et al., 1992; Adhya et al, 1999). The third column shows the obtained bounds from linear-programming relaxation , $\text{LP}_{pq}$, of the pq-formulation using standard bilinear envelopes as presented in Tawarmalani and Sahinidis (2002). The fourth column, $\text{LP}_{RRLT}$, shows the lower bounds presented in Liberti and Pantelides (2006). These lower bounds result from using RRLT constraints. The fifth column shows the obtained bounds from the Lagrangean relaxation of Adhya et al. (1999) for the p-formulation ($\text{LR}_{\text{AST}}$). The

| Problem | Number of | | | |
|---|---|---|---|---|
| | Raw Materials | Pools | Qualities for Each Pool | End Products |
| Haverly1 | 3 | 1 | 1 | 2 |
| Haverly2 | 3 | 1 | 1 | 2 |
| Haverly3 | 3 | 1 | 1 | 2 |
| Foulds2 | 6 | 2 | 1 | 4 |
| Foulds3 | 11 | 8 | 1 | 16 |
| Foulds4 | 11 | 8 | 1 | 16 |
| Foulds5 | 11 | 4 | 1 | 16 |
| Ben-Tal4 | 4 | 1 | 1 | 2 |
| Ben-Tal5 | 5 | 3 | 2 | 5 |
| Adhya1 | 5 | 2 | 4 | 4 |
| Adhya2 | 5 | 2 | 6 | 4 |
| Adhya3 | 8 | 3 | 6 | 4 |
| Adhya4 | 8 | 2 | 4 | 5 |
| RT1 | 3 | 2 | 4 | 3 |
| RT2 | 3 | 2 | 4 | 3 |

Table 4.1: Test Problem Characteristics

previous lower bounds values are taken from Adhya et al. (1999), Tawarmalani and Sahinidis (2002), and Liberti and Pantelides (2006). The sixth column shows the proposed Lagrangean lower bounds. The seventh column shows the evaluation of the quality of the Lagrangean lower bounds with respect to the global optima computed as $\left(\frac{\text{Global optimum -Lagrangean lower bound}}{\text{Global optimum}}\right) \times 100$. Finally, the last column, GO, shows the global optimum values. Results reveal that for eight solved cases the obtained Lagrangean lower bounds are equal to the global optima, whereas for seven cases the obtained Lagrangean lower bounds are within 8.2% of the global solutions on average. Numerical results also indicate that for Haverly2, Haverly3, and RT2 the proposed Lagrangean relaxation gives tighter lower bounds than the ones obtained in the literature. The resulting lower bounds for Haverly1, Haverly2, Foulds2, Foulds3, Foulds4, Foulds5, Ben-Tal4, and Ben-Tal5 are equal to the global optima. Figure 4.1 summarizes this comparison.

Table 4.3 shows a comparison between the proposed Lagrangean lower bounds and the heuristic solutions. The second column gives Lagrangean lower bounds (Lag. LB). The third column shows the Lagrangean heuristic results obtained from solving a linear program with fixed quality values calculated from the quality mass balance constraints using the Lagrangean subproblem optimal solution at each iteration. The fourth and fifth columns show the evaluation of the heuristic quality with respect to the Lagrangean lower bound (LB) and global optimum (GO) computed as $\left(\frac{\text{Heuristic solution -Lagrangean lower bound}}{\text{Lagrangean lower bound}}\right) \times 100$ , $\left(\frac{\text{Heuristic solution -Global optimum}}{\text{Global optimum}}\right) \times 100$ respectively. The sixth column shows the global solutions. Numerical results indicate the efficiency of the Lagrangean heuristic technique. For nine cases the heuristic gives the global optima, and for the other cases the heuristic solutions are within 1.8% of the global solution on average.

| Problem | Lower Bounds | | | | | | GO. |
| --- | --- | --- | --- | --- | --- | --- | --- |
| | LP | $LP_{pq}$ | $LP_{RRLT}$ | $LR_{AST}$. | Lag. LB | Quality of LB w.r.t GO | |
| Haverly1 | -500 | -500 | -400 | -500 | -400 | 0 | -400 |
| Haverly2 | -1000 | -1000 | -1000 | -1000 | -600 | 0 | -600 |
| Haverly3 | -800 | -800 | -800 | -800 | -781.67 | 4.2% | -750 |
| Foulds2 | -1100 | -1100 | -1133.3 | -1100 | -1100 | 0 | -1100 |
| Foulds3 | -8.00 | -8.00 | -8.00 | -8.00 | -8.00 | 0 | -8.00 |
| Foulds4 | -8.00 | -8.00 | -8.00 | -8.00 | -8.00 | 0 | -8.00 |
| Foulds5 | -8.00 | -8.00 | -8.00 | -8.00 | -8.00 | 0 | -8.00 |
| Ben-Tal4 | -550 | -550 | -450 | -550 | -450 | 0 | -450 |
| Ben-Tal5 | -3500 | -3500 | -3500 | -3500 | -3500 | 0 | -3500 |
| Adhya1 | -999.31 | -840.27 | -572.4 | -939.29 | -775.07 | 40.9% | -549.80 |
| Adhya2 | -854.10 | -574.78 | -572.4 | -825.59 | -642.55 | 16.9% | -549.80 |
| Adhya3 | -882.84 | -574.78 | -571.1 | -864.81 | -687.19 | 22.5% | -561.05 |
| Adhya4 | -1012.50 | -961.93 | -1029 | -988.50 | -969.27 | 10.4% | -877.65 |
| RT1 | - | - | - | - | -4287.98 | 3.6% | -4136.21 |
| RT2 | -6331.73 | -6034.87 | - | - | -5485.38 | 24.8% | -4391.83 |

Table 4.2: A Comparison of Lower Bounds and Global Optima (GO)

| | | | Quality of Heuristic% | | |
|---|---|---|---|---|---|
| Problem | Lag. LB | Lag Heuristic | with respect to LB | with respect to GO | GO. |
| Haverly1 | $-400$ | $-400$ | 0 | optimal | $-400$ |
| Haverly2 | $-600$ | $-600$ | 0 | optimal | $-600$ |
| Haverly3 | $-781.67$ | $-750$ | 4.1% | optimal | $-750$ |
| Foulds2 | $-1100$ | $-1000$ | 0 | 9% | $-1100$ |
| Foulds3 | $-8.00$ | $-8.00$ | 0 | optimal | $-8.00$ |
| Foulds4 | $-8.00$ | $-8.00$ | 0 | optimal | $-8.00$ |
| Foulds5 | $-8.00$ | $-8.00$ | 0 | optimal | $-8.00$ |
| Ben-Tal4 | $-450$ | $-450$ | 0 | optimal | $-450$ |
| Ben-Tal5 | $-3500$ | $-3500$ | 0 | optimal | $-3500$ |
| Adhya1 | $-775.07$ | $-539.17$ | 30.4% | 1.9% | $-549.80$ |
| Adhya2 | $-642.55$ | $-549.42$ | 14.5% | 0.07% | $-549.80$ |
| Adhya3 | $-687.19$ | $-548.29$ | 20.2% | 2.3% | $-561.05$ |
| Adhya4 | $-969.27$ | $-865.23$ | 10.7% | 1.4% | $-877.65$ |
| RT1 | $-4287.98$ | $-4136.21$ | 3.5% | optimal | $-4136.21$ |
| RT2 | $-5485.38$ | $-3785.53$ | 30.9% | 13.8% | $-4391.83$ |

Table 4.3: Lagrangean Bounds and Heuristic Results

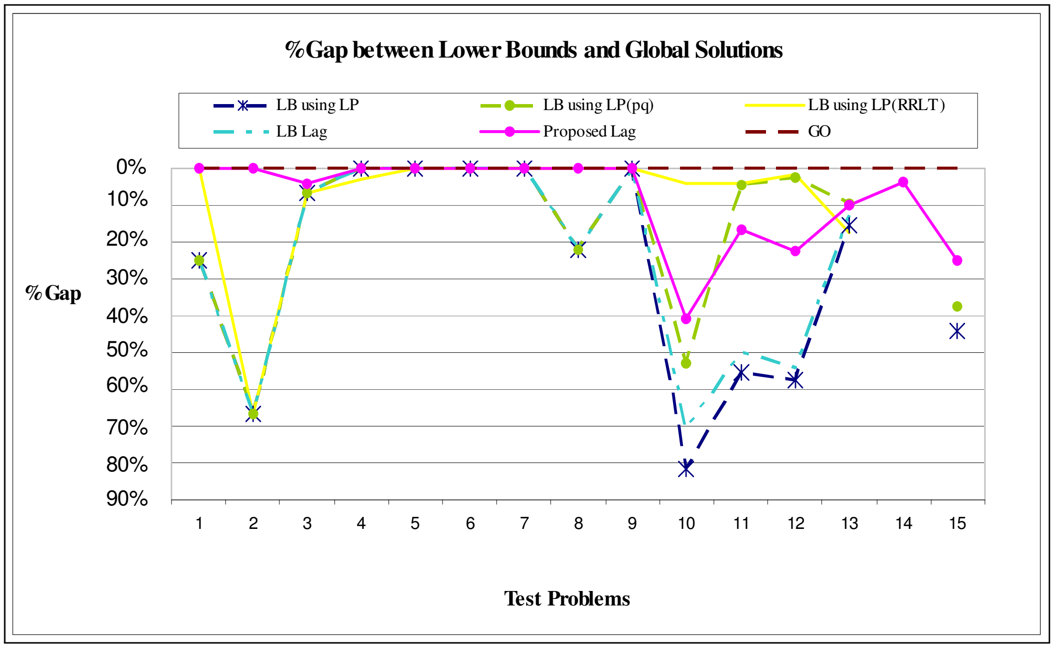


Figure 4.1: A Comparison of Lower Bounds and Optimal Values

Table 4.4 compares the proposed Lagrangean heuristic to two heuristics from the literature. Columns two and three display results for the Variable Neighborhood Search (VNS) and the Multistart Alternate heuristic (MALT) respectively. These results were presented in Audet et al. (2004) and were obtained from solving the flow model of the poling problem. Column four gives the Lagrangean heuristic results, and Column five provides the global solutions. Figure 4.2 summarizes these results. VNS solutions are on average 4.08% from the global solutions, whereas MALT solutions are within 6.76% of the global solutions. The Lagrangean heuristic solutions are on average 1.8% from the global solutions. Numerical results indicate that the proposed Lagrangean heuristic outperforms VNS and MALT in Adhya3 which has the largest number of qualities and pools. Although numerical results of RT2 indicate that there is a significant difference between the VNS and MALT

| Problem | VNS | MALT | Lag Heuristic | Global Optimum |
|---|---|---|---|---|
| Haverly1 | −400 | −400 | −400 | −400 |
| Haverly2 | −600 | −600 | −600 | −600 |
| Haverly3 | −750 | −700 | −750 | −750 |
| Foulds2 | −1100 | −1070.86 | −1000 | −1100 |
| Foulds3 | – | – | −8.00 | −8.00 |
| Foulds4 | – | – | −8.00 | −8.00 |
| Foulds5 | – | – | −8.00 | −8.00 |
| Ben-Tal4 | −450 | −450 | −450 | −450 |
| Ben-Tal5 | −3500 | −3240 | −3500 | −3500 |
| Adhya1 | −545.27 | −532.9 | −539.17 | −549.80 |
| Adhya2 | −543.909 | −535.6 | −549.42 | −549.80 |
| Adhya3 | −412.14 | −397.4 | −548.29 | −561.05 |
| Adhya4 | −876.2 | −876.2 | −865.23 | −877.65 |
| RT1 | −4136.21 | −4136.21 | −4136.21 | −4136.21 |
| RT2 | −4391.83 | −4330.78 | −3785.53 | −4391.83 |

Table 4.4: A Comparison of Heuristic Solutions and Global Optima

results and Lagrangean heuristic results, finding a feasible solution for this problem is difficult as stated in Audet et al. (2004). In the process of generating feasible solution to RT2, Audet et al. (2004) generated 10,000 sets of proportion values at random and there is no one feasible solution. Therefore, to improve VNS solutions Audet et al. (2004) used MALT best solutions as initial solutions for VNS and tried some tricks to find feasible solutions. Nevertheless, when we solve RT2, Lagrangean heuristic gives three feasible solutions without using any tricks.

Table 4.5 shows a comparison between the obtained Lagrangean lower bounds

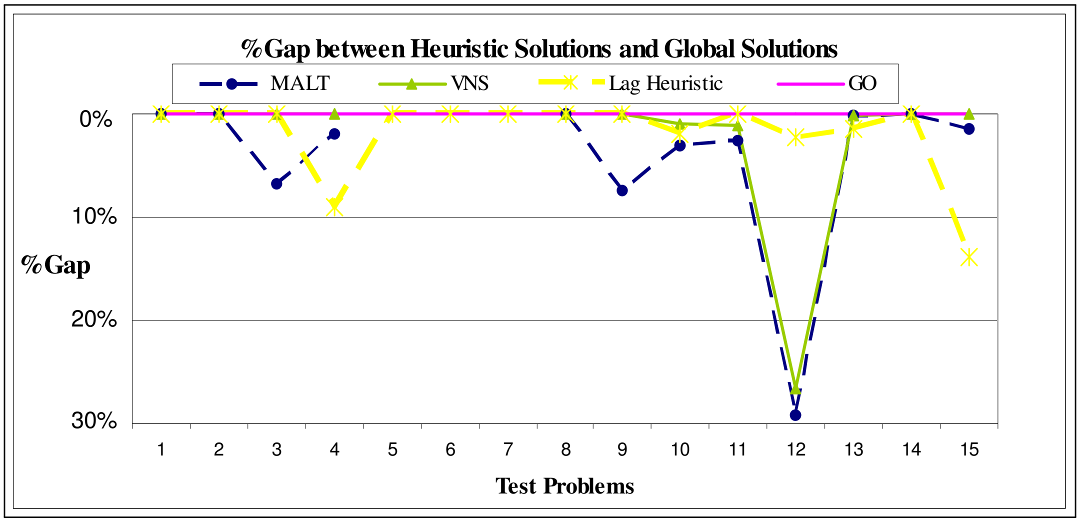


Figure 4.2: A Comparison of Heuristic Solutions and Global Solutions

without cuts and the ones obtained by adding the cuts proposed in Section 3.4. A comparison between heuristic solutions before and after adding the cuts are also given in the same table. "—" means that we do not add cuts to the problem because the Lagrangean lower bound is equal to the optimal solution. Results show that the added cuts are effective in providing tighter lower bounds and, consequently, improving the heuristic solutions. These comparisons are summarized in Figures 4.3 and 4.4

Tables 4.6 reports on the computational time partition and the number of iterations before and after adding the cuts. Columns two and three show the number of iterations before and after adding the cuts. Column seven shows the total CPU time in seconds before adding the cuts whereas Columns 4, 5, and 6 show the computational time of the subproblem (SP), master problem (MP), and heuristic (Lag Heur.) respectively as a percentage of the total computational time before adding the cuts. Column eleven shows the total CPU time in seconds after adding the

| Problem | Lag. Bound | Lag. Bound with Cuts | Lag Heur without Cuts | Lag Heur with Cuts | GO. |
|---|---|---|---|---|---|
| Haverly1 | $-500$ | $-400$ | $-400$ | $-400$ | $-400$ |
| Haverly2 | $-1000$ | $-600$ | $-600$ | $-600$ | $-600$ |
| Haverly3 | $-800$ | $-781.67$ | $-700$ | $-750$ | $-750$ |
| Foulds2 | $-1100$ | $-1100$ | $-1000$ | $-1000$ | $-1100$ |
| Foulds3 | $-8.00$ | – | $-8.00$ | – | $-8.00$ |
| Foulds4 | $-8.00$ | – | $-8.00$ | – | $-8.00$ |
| Foulds5 | $-8.00$ | – | $-8.00$ | – | $-8.00$ |
| Ben-Tal4 | $-550$ | $-450$ | $-450$ | $-450$ | $-450$ |
| Ben-Tal5 | $-3500$ | – | $-3500$ | – | $-3500$ |
| Adhya1 | $-937.59$ | $-775.07$ | $-462.5$ | $-539.17$ | $-549.80$ |
| Adhya2 | $-820.08$ | $-642.55$ | $-462.5$ | $-549.42$ | $-549.80$ |
| Adhya3 | $-864.55$ | $-687.19$ | $-525$ | $-548.29$ | $-561.05$ |
| Adhya4 | $-986.89$ | $-969.27$ | $-470.83$ | $-865.23$ | $-877.65$ |
| RT1 | $-4827.59$ | $-4287.98$ | $-4136.21$ | $-4136.21$ | $-4136.21$ |
| RT2 | $-6134.04$ | $-5485.38$ | $-3749.88$ | $-3785.53$ | $-4391.83$ |

Table 4.5: A Comparison of Lower Bounds and Heuristic Solutions with and without cuts

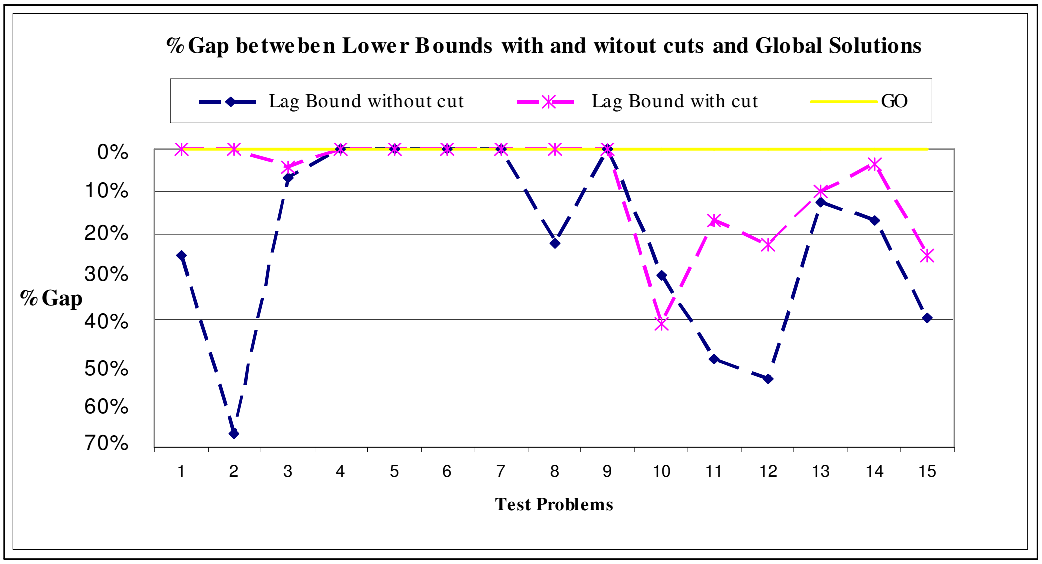


Figure 4.3: A Comparison of Lower bounds with and without cuts

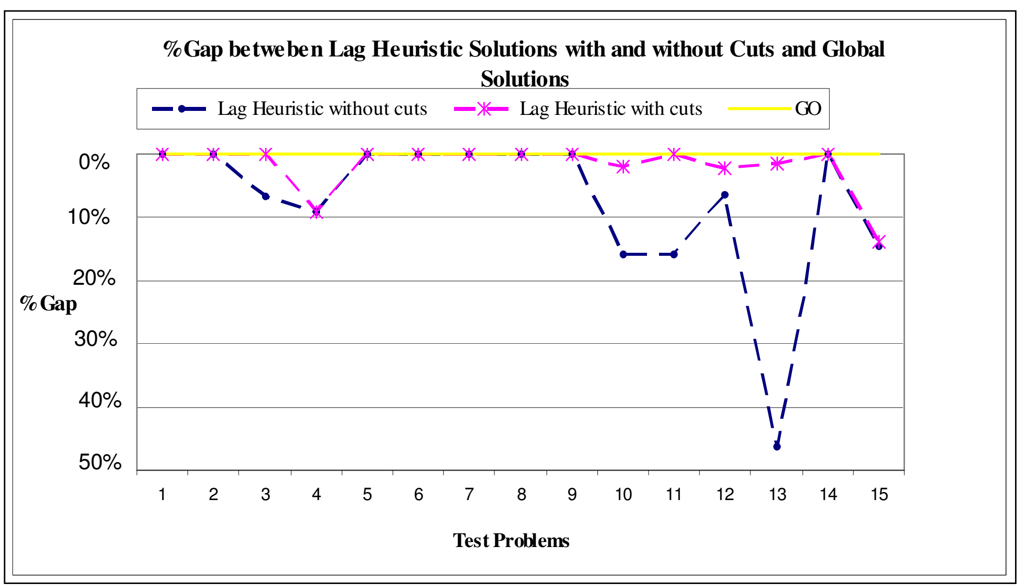


Figure 4.4: A Comparison of Heuristic Solutions with and without cuts

cuts whereas Columns 8, 9, and 10 show the computational time of the subproblem (SP), master problem (MP), and heuristic (Lag Heur.) respectively as a percentage of the total computational time after adding the cuts. "—" means that we do not add cuts to the problem because the Lagrangean lower bound is equal to the optimal solution. Clearly, subproblem solutions always account for most of the computational time. On average, subproblem solutions use 91% of the total CPU time before adding the cuts and 87.8% after adding the cuts. Heuristics solutions, on average, account for 3.8% of the total CPU time before adding the cuts and 5% after adding the cuts. For most of the cases there is an improvement in the total CPU time; however, the significant improvement is in Adhaya2 and Adhaya3 problems. For Adhaya2, the total CPU time dropped from 4.1 minutes to 27.03 seconds and for Adhaya3 the total CPU time dropped from 52.2 hours to 2.4 minutes.

## 4.1 Conclusion

This chapter reported on the computational results of solving fifteen pooling problems from the open literature. Some of the solved problems have a single quality while others have multiple qualities. As the number of pools, qualities, and final products increases, the number of bilinear terms also increases which increases the computational complexity. However, the proposed Lagrangean relaxation and heuristic performed well when the number of bilinear terms increased. The Lagrangean heuristic outperformed VNS and MALT in Adhya3 when the number of bilinear terms is the highest. Except for Foulds2, Lagrangean heuristic found the global solutions for all the single quality instances.

From the comparison between lower bounds obtained using the new Lagrangean

| Problem | Number of iter | | Computational Time without cuts | | | | Computational Time with cuts | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | No cuts | with cuts | SP% | MP% | LH% | Total CPU (sec.) | SP% | MP% | LH% | Total CPU (sec.) |
| Haverly1 | 9 | 6 | 50% | 50% | 0% | 0.02 | 100% | 0% | 0% | 0.015 |
| Haverly2 | 9 | 6 | 66.7% | 33.3% | 0% | 0.03 | 100% | 0% | 0% | 0.01 |
| Haverly3 | 7 | 18 | 100% | 0% | 0% | 0.02 | 100% | 0% | 0% | 0.015 |
| Foulds2 | 20 | 8 | 87.5% | 0% | 12.5% | 0.08 | 62.5% | 12.5% | 25% | 0.08 |
| Foulds3 | 44 | - | 98.77% | 0.54% | 0.69% | 18.7 | - | - | - | - |
| Foulds4 | 35 | - | 98.77% | 0.68% | 0.55% | 16.2 | - | - | - | - |
| Foulds5 | 35 | - | 98.37% | 1.06% | 0.57% | 14.09 | - | - | - | - |
| Ben-Tal4 | 14 | 5 | 66.7% | 0% | 33.3% | 0.06 | 75% | 0% | 25% | 0.04 |
| Ben-Tal5 | 35 | - | 86.95% | 6.1% | 6.95% | 1.15 | - | - | - | - |
| Adhya1 | 81 | 109 | 94.94% | 4.28% | 0.78% | 10.28 | 88.5% | 11.1% | 0.4% | 7.48 |
| Adhya2 | 107 | 153 | 99.4% | 0.57% | 0.03% | 246.19 | 90.42% | 9.14% | 0.44% | 27.03 |
| Adhya3 | 192 | 194 | 99.99% | 0.003% | 0.0001% | 187940 | 96.54% | 3.3% | 0.16% | 144.62 |
| Adhya4 | 89 | 193 | 92.15% | 6.48% | 1.37% | 8.03 | 82.3% | 17% | 0.7% | 19.92 |
| RT1 | 33 | 84 | 98% | 1.6% | 0.4% | 2.3 | 74.5% | 21.5% | 4% | 1.49 |
| RT2 | 66 | 73 | 95.25% | 4.4% | 0.35% | 8.5 | 96.4% | 3.37% | 0.23% | 12.8 |

Table 4.6: Computational Time Partition without cuts

relaxation and the ones proposed in the literature using other relaxations such as LP relaxation using convex and concave envelopes, RRLT, LP relaxation of the pq-formulation, and Lagrangean relaxation, the new Lagrangean relaxation provided tighter lower bounds for Haverly2, Haverly3, and RT2. Moreover, the new Lagrangean relaxation provided tighter lower bounds than the ones obtained using Lagrangean relaxation proposed in Adhya et al. (1999) for all the multiple quality instances and for Haverly2, Haverly3, and Ben-Tal4. For the other test problems the two relaxations provided lower bounds equal to the global optima. On the other hand, there are might be numerical difficulties with the approach. First, the subproblem was reduced to a mixed integer programming problem where the number of binary variables increases as the number of qualities and pools increases. This is expected to increase the computational time of the subproblem. Second, when the number of bilinear terms increases, the number of linear constraints and continuous variables used to strengthen Lagrangean lower bounds increases.

# Chapter 5

# Conclusion

This thesis proposed a heuristic approach and a Lagrangean relaxation technique to the p-formulation of the pooling problem. The Lagrangean relaxation converts the pooling problem into an easier one by associating Lagrangean multipliers with nonlinear constraints and adding them to the objective function to penalize their violations. The resulting Lagrangean subproblem is a nonlinear problem since the nonlinearity is transformed from the constraints to the objective function. Nonlinearity is eliminated by using linear bound constraints and binary variables. Hence, the Lagrangean subproblem is reduced to a mixed integer programming problem. Lagrangean lower bounds are strengthened by adding valid cuts to the Lagrangean subproblem. The valid cuts are generated using the idea of replacing each bilinear term in the nonlinear constraints with a nonnegative variable and adding some linear constraints to the Lagrangean subproblem to bound the value of the introduced variable.

The Lagrangean heuristic works on the Lagrangean subproblem solutions by modifying it to be feasible to the original nonlinear problem. At each iteration, Lagrangean subproblem solutions are used to calculate the quality values from the quality mass balance constraints. Then, a linear programming problem, resulting from fixing the quality variable value, is solved at each iteration. The approach is applied to fifteen pooling problems collected from the literature. Numerical results indicate the efficiency of the procedure. For most cases, Lagrangean lower bounds and heuristic solutions are equal to the proven global solutions.

Several extensions of this work are possible. First of all, despite the fact that heuristic results indicate the efficiency of the technique in providing a good feasible solution, future work should focus on further computational testing to demonstrate the potential of the procedure in solving large instances of the pooling problem in the chemical and wastewater treatment industries. Moreover, the obtained lower bounds can be used within a Branch-and-Bound framework to find a global solution for the pooling problem. Another possible extension of this work is the implementation of this approach to generalized formulations of the pooling problem to account for strategic decisions related to pool existence and network configuration. Furthermore, this approach can be applied to solve a general petroleum supply chain involving pooling problems. Finally, future work may investigate how to apply the proposed approach to other optimization problems involving bilinear terms. Such problems arise in economics, game theory, production and nonlinear multi-commodity network flows.

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# Appendix A

# Some Illustrative Examples

In this appendix, we derive the Lagrangean subproblem and Lagrangean dual for two pooling examples. The first example has multiple pools each with single quality, and the second example has multiple pools each with multiple qualities.

## A.1 Lagrangean Relaxation for Foulds2

This example has six raw materials, two pools each with single quality, and four end products. See Table 4.1.

### A.1.1 Problem Formulation

$$
\begin{aligned}
\min \quad & 6f_{11} + 16f_{21} + 10f_{12} + 3f_{13} + 13f_{23} + 7f_{14} - 9x_{11} - 9x_{21} - 9x_{31} - 9x_{41} \\
& \text{-}15x_{12} - 15x_{22} - 15x_{32} - 15x_{42} - 6x_{13} - 6x_{23} - 6x_{33} - 6x_{43} - 12x_{14} \\
& \text{-}12x_{24} - 12x_{34} - 12x_{44} \\
\text{s.t.} \quad & f_{11} + f_{21} - x_{11} - x_{12} - x_{13} - x_{14} = 0 \\
& f_{12} - x_{21} - x_{22} - x_{23} - x_{24} = 0 \\
& f_{13} + f_{23} - x_{31} - x_{32} - x_{33} - x_{34} = 0 \\
& f_{14} - x_{41} - x_{42} - x_{43} - x_{44} = 0 \\
& x_{11} + x_{21} + x_{31} + x_{41} \leq 100 \\
& x_{12} + x_{22} + x_{32} + x_{42} \leq 200 \\
& x_{13} + x_{23} + x_{33} + x_{43} \leq 100 \\
& x_{14} + x_{24} + x_{34} + x_{44} \leq 200 \\
& -3f_{11} - f_{21} + q_1x_{11} + q_1x_{12} + q_1x_{13} + q_1x_{14} = 0 \\
& -3.5f_{13} - 1.5f_{23} + q_3x_{31} + q_3x_{32} + q_3x_{33} + q_4x_{34} = 0 \\
& (q_1 - 2.5)x_{11} - 0.5x_{21} + (q_3 - 2.5)x_{31} \leq 0 \\
& (q_1 - 1.5)x_{12} + 0.5x_{22} + (q_3 - 1.5)x_{32} + x_{42} \leq 0 \\
& (q_1 - 3)x_{13} - x_{23} + (q_3 - 3)x_{33} - 0.5x_{43} \leq 0 \\
& (q_1 - 2)x_{14} + (q_3 - 2)x_{34} + 0.5x_{44} \leq 0 \\
& q_1^l \leq q_1 \leq q_1^u \\
& q_3^l \leq q_3 \leq q_3^u
\end{aligned}
$$

### A.1.2 Lagrangean Relaxation

$$
\begin{array}{lll}
\min & 6f_{11} + 16f_{21} + 10f_{12} + 3f_{13} + 13f_{23} + 7f_{14} - 9x_{11} - 9x_{21} - 9x_{31} & \\
& -9x_{41} - 15x_{12} - 15x_{22} - 15x_{32} - 15x_{42} - 6x_{13} - 6x_{23} - 6x_{33} & \\
& -6x_{43} - 12x_{14} - 12x_{24} - 12x_{34} - 12x_{44} & \\
s.t. & f_{11} + f_{21} - x_{11} - x_{12} - x_{13} - x_{14} = 0 & \\
& f_{12} - x_{21} - x_{22} - x_{23} - x_{24} = 0 & \\
& f_{13} + f_{23} - x_{31} - x_{32} - x_{33} - x_{34} = 0 & \\
& f_{14} - x_{41} - x_{42} - x_{43} - x_{44} = 0 & \\
& x_{11} + x_{21} + x_{31} + x_{41} \leq 100 & \\
& x_{12} + x_{22} + x_{32} + x_{42} \leq 200 & \\
& x_{13} + x_{23} + x_{33} + x_{43} \leq 100 & \\
& x_{14} + x_{24} + x_{34} + x_{44} \leq 200 & \\
& -3f_{11} - f_{21} + q_1x_{11} + q_1x_{12} + q_1x_{13} + q_1x_{14} = 0 & \leftarrow \lambda_1 \\
& -3.5f_{13} - 1.5f_{23} + q_3x_{31} + q_3x_{32} + q_3x_{33} + q_4x_{34} = 0 & \leftarrow \lambda_2 \\
& (q_1 - 2.5)x_{11} - 0.5x_{21} + (q_3 - 2.5)x_{31} \leq 0 & \leftarrow \alpha \\
& (q_1 - 1.5)x_{12} + 0.5x_{22} + (q_3 - 1.5)x_{32} + x_{42} \leq 0 & \leftarrow \beta \\
& (q_1 - 3)x_{13} - x_{23} + (q_3 - 3)x_{33} - 0.5x_{43} \leq 0 & \leftarrow \gamma \\
& (q_1 - 2)x_{14} + (q_3 - 2)x_{34} + 0.5x_{44} \leq 0 & \leftarrow \delta \\
& q_1^l \leq q_1 \leq q_1^u & \\
& q_3^l \leq q_3 \leq q_3^u & \\
& f, x \geq 0 &
\end{array}
$$

The resulting subproblem is:

$$\begin{aligned}
\min \quad & (6-3\lambda_1)f_{11}+(16-\lambda_1)f_{21} \;+10f_{12}+(3-3.5\lambda_2)f_{13}+(13-1.5\lambda_2)f_{23} \\
& +7f_{14}+(-9+q_1\lambda_1+q_1\alpha-2.5\alpha)x_{11}+(-9-0.5\alpha)x_{21}+(-9+q_3\lambda_2 \\
& +q_3\alpha-2.5\alpha)x_{31}-9x_{41}+(-15+q_1\lambda_1+q_1\beta-1.5\beta)x_{12}+(-15+0.5\beta)x_{22} \\
& +(-15+q_3\lambda_2+q_3\beta-1.5\beta)x_{32}+(-15+\beta)x_{42}+\;(-6+q_1\lambda_1+q_1\gamma-3\gamma)x_{13} \\
& +(-6-\gamma)x_{23}+\;(-6+q_3\lambda_2+q_3\gamma-3\gamma)x_{33}+(-6-0.5\gamma)x_{43}+\;(-12+q_1\lambda_1 \\
& +q_1\delta-2\delta)x_{14}-12x_{24}+(-12+q_3\lambda_2+q_3\delta-2\delta)x_{34}+(-12+0.5\delta)x_{44} \\
\text{s.t.} \quad & f_{11}+f_{21}-x_{11}-x_{12}-x_{13}-x_{14}=0 \\
& f_{12}-x_{21}-x_{22}-x_{23}-x_{24}=0 \\
& f_{13}+f_{23}-x_{31}-x_{32}-x_{33}-x_{34}=0 \\
& f_{14}-x_{41}-x_{42}-x_{43}-x_{44}=0 \\
& x_{11}+x_{21}+x_{31}+x_{41}\le 100 \\
& x_{12}+x_{22}+x_{32}+x_{42}\le 200 \\
& x_{13}+x_{23}+x_{33}+x_{43}\le 100 \\
& x_{14}+x_{24}+x_{34}+x_{44}\le 200 \\
& q_1^l\le q_1\le q_1^u \\
& q_3^l\le q_3\le q_3^u \\
& f,x\ge 0
\end{aligned}$$

which can be written as:

$$
\begin{aligned}
\min \quad & (6-3\lambda_1)f_{11}+(16-\lambda_1)f_{21}+10f_{12}+(3-3.5\lambda_2)f_{13}+(13-1.5\lambda_2)f_{23} \\
& +7f_{14}+(-9-2.5\alpha)x_{11}+(-9-.5\alpha)x_{21}+(-9-2.5\alpha)x_{31}-9x_{41}+(-15 \\
& -1.5\beta)x_{12}+(-15+0.5\beta)x_{22}+(-15-1.5\beta)x_{32}+(-15+\beta)x_{42}+(-6 \\
& -3\gamma)x_{13}+(-6-\gamma)x_{23}+(-12+0.5\delta)x_{44}+(-6-3\gamma)x_{33}+(-6-0.5\gamma)x_{43} \\
& +(-12-2\delta)x_{14}-12x_{24}+(-12-2\delta)x_{34}+(\lambda_1+\alpha)q_1x_{11}+(\lambda_1+\beta)q_1x_{12} \\
& +(\lambda_1+\gamma)q_1x_{13}+(\lambda_1+\delta)q_1x_{14}+(\lambda_2+\alpha)q_3x_{31}+(\lambda_2+\beta)q_3x_{32} \\
& +(\lambda_2+\gamma)q_3x_{33}+(\lambda_2+\delta)q_3x_{34} \\
\text{s.t.} \quad & f_{11}+f_{21}-x_{11}-x_{12}-x_{13}-x_{14}=0 \\
& f_{12}-x_{21}-x_{22}-x_{23}-x_{24}=0 \\
& f_{13}+f_{23}-x_{31}-x_{32}-x_{33}-x_{34}=0 \\
& f_{14}-x_{41}-x_{42}-x_{43}-x_{44}=0 \\
& x_{11}+x_{21}+x_{31}+x_{41}\leq 100 \\
& x_{12}+x_{22}+x_{32}+x_{42}\leq 200 \\
& x_{13}+x_{23}+x_{33}+x_{43}\leq 100 \\
& x_{14}+x_{24}+x_{34}+x_{44}\leq 200 \\
& q_1^l\leq q_1\leq q_1^u \\
& q_3^l\leq q_3\leq q_3^u \\
& f,x\geq 0.
\end{aligned}
$$

if we define

$$
\begin{aligned}
u_1 &= q_1\left((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14}\right), \\
u_2 &= q_3\left((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34}\right),
\end{aligned}
$$

then the nonlinearity can be eliminated by using the linear bound constraints $q_1^l\leq q_1\leq q_1^u$ , $q_3^l\leq q_3\leq q_3^u$ and the fact that $q_1$ and $q_3$ do not appear in the constraints. Four cases should be considered depending on the sign of $(\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14}$ and $\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34}$ :

$$\begin{cases} ((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14})\, q_1^l \leq u_1 \\ u_1 \leq ((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14})\, q_1^u, \\ \text{if } ((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14}) \geq 0, \\ ((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14})\, q_1^u \leq u_1 \\ u_1 \leq ((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14})\, q_1^l, \\ \text{if } ((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14}) \leq 0 \\ ((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34})\, q_3^l \leq u_2 \\ u_2 \leq q_3^u\, ((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34}) \\ \text{if } ((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34}) \geq 0, \\ ((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34})\, q_3^u \leq u_2 \\ u_2 \leq q_3^l\, ((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34}) \\ \text{if } ((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34}) \leq 0 \end{cases}$$

Hence, the subproblem reduces to:

$$\begin{aligned}
\min_{x,f} \quad & (6-3\lambda_1)f_{11}+(16-\lambda_1)f_{21}+10f_{12}+(3-3.5\lambda_2)f_{13} \\
& +(13-1.5\lambda_2)f_{23}+7f_{14}+(-9-2.5\alpha)x_{11}+(-9-.5\alpha)x_{21} \\
& +(-9-2.5\alpha)x_{31}-9x_{41}+(-15-1.5\beta)x_{12}+(-15+0.5\beta)x_{22} \\
& +(-15-1.5\beta)x_{32}+(-15+\beta)x_{42}+(-6-3\gamma)x_{13}+(-6-\gamma)x_{23} \\
& +(-6-3\gamma)x_{33}+(-6-0.5\gamma)x_{43}+(-12-2\delta)x_{14}-12x_{24} \\
& +(-12-2\delta)x_{34}+(-12+0.5\delta)x_{44}+v_1+v_2 \\
\text{s.t.} \quad & f_{11}+f_{21}-x_{11}-x_{12}-x_{13}-x_{14}=0 \\
& f_{12}-x_{21}-x_{22}-x_{23}-x_{24}=0 \\
& f_{13}+f_{23}-x_{31}-x_{32}-x_{33}-x_{34}=0 \\
& f_{14}-x_{41}-x_{42}-x_{43}-x_{44}=0 \\
& x_{11}+x_{21}+x_{31}+x_{41}\le 100 \\
& x_{12}+x_{22}+x_{32}+x_{42}\le 200 \\
& x_{13}+x_{23}+x_{33}+x_{43}\le 100 \\
& x_{14}+x_{24}+x_{34}+x_{44}\le 200
\end{aligned}$$

$$
\begin{aligned}
&((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14})\, q_1^l \leq u_1\\
&u_1 \leq ((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14})\, q_1^u,\\
&\text{if } ((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14}) \geq 0,\\
&((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14})\, q_1^u \leq u_1\\
&u_1 \leq ((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14})\, q_1^l,\\
&\text{if } ((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14}) \leq 0\\
&((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34})\, q_3^l \leq u_2\\
&u_2 \leq q_3^u\, ((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34})\\
&\text{if } \; ((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34}) \geq 0,\\
&((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34})\, q_3^u \leq u_2\\
&u_2 \leq q_3^l\, ((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34})\\
&\text{if } \; ((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34}) \leq 0\\
&f, x \geq 0.
\end{aligned}
$$

upon defining a binary variable $y_1$ that takes value 1 if $((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14}) \geq 0$,and 0 otherwise, and a binary variable $y_2$ that takes value 1 if

if $((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34}) \geq 0,$ the if constraints can be modelled as

$$((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14})\,q_1^l \leq u_1+M(1-y_1)$$
$$u_1 \leq ((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14})\,q_1^u+M(1-y_1)$$
$$((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14})\,q_1^u \leq u_1+My_1$$
$$u_1 \leq ((\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14})\,q_1^l+My_1.$$
$$(\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14} \geq -M(1-y_1)$$
$$(\lambda_1+\alpha)x_{11}+(\lambda_1+\beta)x_{12}+(\lambda_1+\gamma)x_{13}+(\lambda_1+\delta)x_{14} \leq My_1$$

$$((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34})\,q_3^l \leq u_2+M(1-y_2)$$
$$u_2 \leq q_3^u\,((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34})+M(1-y_2)$$
$$((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34})\,q_3^u \leq u_2+My_2$$
$$u_2 \leq q_3^l\,((\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34})+My_2$$
$$(\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34} \geq -M(1-y_2)$$
$$(\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34} \leq My_2$$

The resulting linear subproblem is:

$$
\begin{aligned}
\min_{x,f} \quad & (6-3\lambda_1)f_{11} + (16-\lambda_1)f_{21} + 10f_{12} + (3-3.5\lambda_2)f_{13} \\
& +(13-1.5\lambda_2)f_{23} + 7f_{14} + (-9-2.5\alpha)x_{11} + (-9-.5\alpha)x_{21} \\
& +(-9-2.5\alpha)x_{31} - 9x_{41} + (-15-1.5\beta)x_{12} + (-15+0.5\beta)x_{22} \\
& +(-15-1.5\beta)x_{32} + (-15+\beta)x_{42} + (-6-3\gamma)x_{13} + (-6-\gamma)x_{23} \\
& +(-6-3\gamma)x_{33} + (-6-0.5\gamma)x_{43} + (-12-2\delta)x_{14} - 12x_{24} \\
& +(-12-2\delta)x_{34} + (-12+0.5\delta)x_{44} + v_1 + v_2 \\
\text{s.t.} \quad & f_{11} + f_{21} - x_{11} - x_{12} - x_{13} - x_{14} = 0 \\
& f_{12} - x_{21} - x_{22} - x_{23} - x_{24} = 0 \\
& f_{13} + f_{23} - x_{31} - x_{32} - x_{33} - x_{34} = 0 \\
& f_{14} - x_{41} - x_{42} - x_{43} - x_{44} = 0 \\
& x_{11} + x_{21} + x_{31} + x_{41} \leq 100 \\
& x_{12} + x_{22} + x_{32} + x_{42} \leq 200 \\
& x_{13} + x_{23} + x_{33} + x_{43} \leq 100 \\
& x_{14} + x_{24} + x_{34} + x_{44} \leq 200
\end{aligned}
$$

$$
\begin{aligned}
& (\lambda_1+\alpha)x_{11}q_1^l + (\lambda_1+\beta)x_{12}q_1^l + (\lambda_1+\gamma)x_{13}q_1^l + (\lambda_1+\delta)x_{14}q_1^l - u_1 + My_1 \leq M \\
& u_1 - (\lambda_1+\alpha)x_{11}q_1^u - (\lambda_1+\beta)x_{12}q_1^u - (\lambda_1+\gamma)x_{13}q_1^u - (\lambda_1+\delta)x_{14}q_1^u + My_1 \leq M \\
& (\lambda_1+\alpha)x_{11}q_1^u + (\lambda_1+\beta)x_{12}q_1^u + (\lambda_1+\gamma)x_{13}q_1^u + (\lambda_1+\delta)x_{14}q_1^u - u_1 - My_1 \leq 0 \\
& u_1 - (\lambda_1+\alpha)x_{11}q_1^l - (\lambda_1+\beta)x_{12}q_1^l - (\lambda_1+\gamma)x_{13} - (\lambda_1+\delta)x_{14}q_1^l - My_1 \leq 0 \\
& -(\lambda_1+\alpha)x_{11} - (\lambda_1+\beta)x_{12} - (\lambda_1+\gamma)x_{13} - (\lambda_1+\delta)x_{14} + My_1 \leq M \\
& (\lambda_1+\alpha)x_{11} + (\lambda_1+\beta)x_{12} + (\lambda_1+\gamma)x_{13} + (\lambda_1+\delta)x_{14} - My_1 \leq 0
\end{aligned}
$$

$$(\lambda_2+\alpha)x_{31}q_3^l+(\lambda_2+\beta)x_{32}q_3^l+(\lambda_2+\gamma)x_{33}q_3^l+(\lambda_2+\delta)x_{34}q_3^l-u_2+My_2\le M$$

$$u_2-(\lambda_2+\alpha)x_{31}q_3^u-(\lambda_2+\beta)x_{32}q_3^u-(\lambda_2+\gamma)x_{33}q_3^u-(\lambda_2+\delta)x_{34}q_3^u+My_2\le M$$

$$(\lambda_2+\alpha)x_{31}q_3^u+(\lambda_2+\beta)x_{32}q_3^u+(\lambda_2+\gamma)x_{33}q_3^u+(\lambda_2+\delta)x_{34}q_3^u-u_2-My_2\le 0$$

$$u_2-(\lambda_2+\alpha)x_{31}q_3^l-(\lambda_2+\beta)x_{32}q_3^l-(\lambda_2+\gamma)x_{33}q_3^l-(\lambda_2+\delta)x_{34}q_3^l-My_2\le 0$$

$$-(\lambda_2+\alpha)x_{31}-(\lambda_2+\beta)x_{32}-(\lambda_2+\gamma)x_{33}-(\lambda_2+\delta)x_{34}+My_2\le M$$

$$(\lambda_2+\alpha)x_{31}+(\lambda_2+\beta)x_{32}+(\lambda_2+\gamma)x_{33}+(\lambda_2+\delta)x_{34}-My_2\le 0$$

$$f,x\ge 0.,y\in\{0,1\}$$

The best Lagrangean lower bound is given by the optimal solution of the Lagrangean dual problem.

$$\max_{\alpha,\beta,\gamma,\delta\geq 0}\left\{\begin{array}{ll}\min & (6-3\lambda_1)f_{11}+(16-\lambda_1)f_{21}\ +10f_{12}+(3-3.5\lambda_2)f_{13}\\ & +(13-1.5\lambda_2)f_{23}+7f_{14}+(-9+q_1\lambda_1+q_1\alpha-2.5\alpha)x_{11}\\ & +(-9-0.5\alpha)x_{21}+(-9+q_3\lambda_2+q_3\alpha-2.5\alpha)x_{31}-9x_{41}\\ & +(-15+q_1\lambda_1+q_1\beta-1.5\beta)x_{12}+(-15+0.5\beta)x_{22}\\ & +(-15+q_3\lambda_2+q_3\beta-1.5\beta)x_{32}+(-15+\beta)x_{42}\\ & +\ (-6+q_1\lambda_1+q_1\gamma-3\gamma)x_{13}\ +(-6-\gamma)x_{23}\\ & +\ (-6+q_3\lambda_2+q_3\gamma-3\gamma)x_{33}+(-6-0.5\gamma)x_{43}\\ & +\ (-12+q_1\lambda_1+q_1\delta-2\delta)x_{14}-12x_{24}\\ & +(-12+q_3\lambda_2+q_3\delta-2\delta)x_{34}+(-12+0.5\delta)x_{44}\end{array}\right\}$$

$$\begin{array}{ll} s.t & f_{11}+f_{21}-x_{11}-x_{12}-x_{13}-x_{14}=0\\ & f_{13}+f_{23}-x_{31}-x_{32}-x_{33}-x_{34}=0\\ & f_{13}+f_{23}-x_{31}-x_{32}-x_{33}-x_{34}=0\\ & f_{14}-x_{41}-x_{42}-x_{43}-x_{44}=0\\ & x_{11}+x_{21}+x_{31}+x_{41}\leq 100\\ & x_{12}+x_{22}+x_{32}+x_{42}\leq 200\\ & x_{13}+x_{23}+x_{33}+x_{43}\leq 100\\ & x_{14}+x_{24}+x_{34}+x_{44}\leq 200\\ & q_1^l\leq q_1\leq q_1^u\\ & q_3^l\leq q_3\leq q_3^u\\ & f,x\geq 0\end{array}$$

Which is equivalent to:

$$
\max_{\alpha\geq 0,\beta\geq 0}\left\{\begin{array}{ll}
\min & (6-3\lambda_1)f_{11}^h+(16-\lambda_1)f_{21}^h\ +10f_{12}^h+(3-3.5\lambda_2)f_{13}^h \\
h\in H & +(13-1.5\lambda_2)f_{23}^h+7f_{14}^h+(-9+q_1^h\lambda_1+q_1^h\alpha-2.5\alpha)x_{11}^h \\
 & +(-9-0.5\alpha)x_{21}^h+(-9+q_3^h\lambda_2+q_3^h\alpha-2.5\alpha)x_{31}^h-9x_{41}^h \\
 & +(-15+q_1^h\lambda_1+q_1^h\beta-1.5\beta)x_{12}^h+(-15+0.5\beta)x_{22}^h \\
 & +(-15+q_3^h\lambda_2+q_3^h\beta-1.5\beta)x_{32}^h+(-15+\beta)x_{42}^h \\
 & +\ (-6+q_1^h\lambda_1+q_1^h\gamma-3\gamma)x_{13}^h\ +(-6-\gamma)x_{23}^h \\
 & +\ (-6+q_3^h\lambda_2+q_3^h\gamma-3\gamma)x_{33}^h+(-6-0.5\gamma)x_{43}^h \\
 & +\ (-12+q_1^h\lambda_1+q_1^h\delta-2\delta)x_{14}^h-12x_{24}^h \\
 & +(-12+q_3^h\lambda_2+q_3^h\delta-2\delta)x_{34}^h+(-12+0.5\delta)x_{44}^h
\end{array}\right\}
$$

where

$$
H:\left\{\begin{array}{ll}
(f^h,x^h,q^h): & f_{11}+f_{21}-x_{11}-x_{12}-x_{13}-x_{14}=0 \\
 & f_{13}+f_{23}-x_{31}-x_{32}-x_{33}-x_{34}=0 \\
 & f_{13}+f_{23}-x_{31}-x_{32}-x_{33}-x_{34}=0 \\
 & f_{14}-x_{41}-x_{42}-x_{43}-x_{44}=0 \\
 & x_{11}+x_{21}+x_{31}+x_{41}\leq 100 \\
 & x_{12}+x_{22}+x_{32}+x_{42}\leq 200 \\
 & x_{13}+x_{23}+x_{33}+x_{43}\leq 100 \\
 & x_{14}+x_{24}+x_{34}+x_{44}\leq 200 \\
 & q_1^l\leq q_1\leq q_1^u \\
 & q_3^l\leq q_3\leq q_3^u \\
 & f,x\geq 0,
\end{array}\right\}
$$

The master problem can be written as the following linear program.

$$
\begin{aligned}
\max_{\alpha,\beta,\gamma,\delta,\theta} \quad & \theta \\
\text{s.t.} \quad & \theta + (3f_{11}^h + f_{21}^h - q_1^h(x_{11}^h + x_{12}^h + x_{13}^h + x_{14}^h))\lambda_1 + (3.5f_{13}^h + 1.5f_{23}^h \\
& -q_3^h(x_{31}^h + x_{32}^h + x_{33}^h + x_{34}^h))\lambda_2 + ((2.5 - q_1^h)x_{11}^h + .5x_{21}^h + (2.5 - q_3^h)x_{31}^h)\alpha \\
& +((1.5 - q_1^h)x_{12}^h - .5x_{22}^h + (1.5 - q_3^h)x_{32}^h - x_{42}^h)\beta + ((3 - q_1^h)x_{13}^h + x_{23}^h \\
& +(3 - q_3^h)x_{33}^h + 0.5x_{43}^h)\gamma + ((2 - q_1^h)x_{14}^h + (2 - q_3^h)x_{34}^h - 0.5x_{44}^h)\delta \leq \\
& 6f_{11}^h + 16f_{21}^h + 10f_{12}^h + 3f_{13}^h + 13f_{23}^h + 7f_{14}^h - 9x_{11}^h - 9x_{21}^h - 9x_{31}^h - 9x_{41}^h \\
& -15x_{12}^h - 15x_{22}^h - 15x_{32}^h - 15x_{42}^h - 6x_{13}^h - 6x_{23}^h - 6x_{33}^h - 6x_{43}^h - 12x_{14}^h \\
& -12x_{24}^h - 12x_{34}^h - 12x_{44}^h; \quad \forall h \in H \\
& \alpha \geq 0, \beta \geq 0
\end{aligned}
$$

## A.2 Lagrangean Relaxation for Adhya2

This example has five raw materials, two pools each with six qualities, and four end products. See Table 4.1.

### A.2.1 Problem formulation

This example is proposed in Adhya et al. (1999). Figure A.1

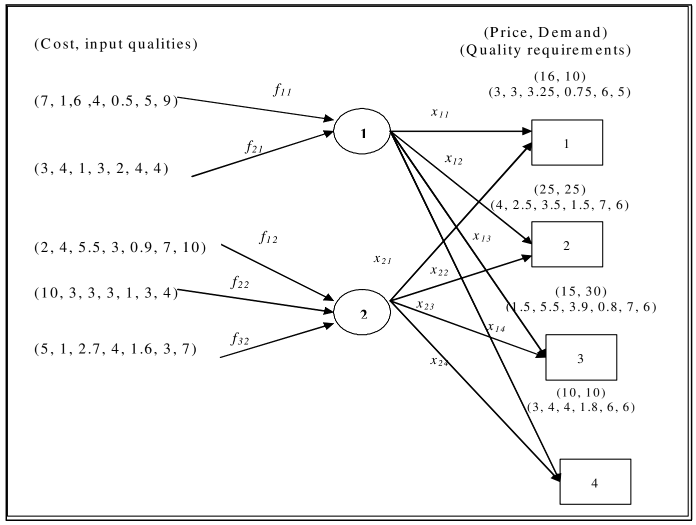


Figure A.1: Flowchart for Example 2.

$$
\begin{aligned}
\min \quad & 7f_{11} + 3f_{21} + 2f_{12} + 10f_{22} + 5f_{32} - 16(x_{11} + x_{21}) - 25(x_{12} + x_{22}) \\
& -30(x_{13} + x_{23}) - 10(x_{14} + x_{24}) \\
s.t \quad & f_{11} + f_{21} - x_{11} - x_{12} - x_{13} - x_{14} = 0 \\
& f_{12} + f_{22} + f_{32} - x_{21} - x_{22} - x_{23} - x_{24} = 0 \\
& x_{11} + x_{21} \leq 10 \\
& x_{12} + x_{22} \leq 25 \\
& x_{13} + x_{23} \leq 30 \\
& x_{14} + x_{24} \leq 10
\end{aligned}
$$

$$q_{11}(x_{11}+x_{12}+x_{13}+x_{14})-f_{11}-4f_{21}=0$$
$$q_{12}(x_{11}+x_{12}+x_{13}+x_{14})-6f_{11}-f_{21}=0$$
$$q_{13}(x_{11}+x_{12}+x_{13}+x_{14})-4f_{11}-3f_{21}=0$$
$$q_{14}(x_{11}+x_{12}+x_{13}+x_{14})-0.5f_{11}-2f_{21}=0$$
$$q_{15}(x_{11}+x_{12}+x_{13}+x_{14})-5f_{11}-4f_{21}=0$$
$$q_{16}(x_{11}+x_{12}+x_{13}+x_{14})-9f_{11}-4f_{21}=0$$
$$q_{21}(x_{21}+x_{22}+x_{23}+x_{24})-4f_{12}-3f_{22}-f_{32}=0$$
$$q_{22}(x_{21}+x_{22}+x_{23}+x_{24})-5.5f_{12}-3f_{22}-2.7f_{32}=0$$
$$q_{23}(x_{21}+x_{22}+x_{23}+x_{24})-3f_{12}-3f_{22}-4f_{32}=0$$
$$q_{24}(x_{21}+x_{22}+x_{23}+x_{24})-0.9f_{12}-f_{22}-1.6f_{32}=0$$
$$q_{25}(x_{21}+x_{22}+x_{23}+x_{24})-7f_{12}-3f_{22}-3f_{32}=0$$
$$q_{26}(x_{21}+x_{22}+x_{23}+x_{24})-10f_{12}-4f_{22}-7f_{32}=0$$

$$\left\{\begin{array}{l} q_{11}x_{11}+q_{21}x_{21}-3(x_{11}+x_{21})\leq 0 \\ q_{12}x_{11}+q_{22}x_{21}-3(x_{11}+x_{21})\leq 0 \\ q_{13}x_{11}+q_{23}x_{21}-3.25(x_{11}+x_{21})\leq 0 \\ q_{14}x_{11}+q_{24}x_{21}-0.75(x_{11}+x_{21})\leq 0 \\ q_{15}x_{11}+q_{25}x_{21}-6(x_{11}+x_{21})\leq 0 \\ q_{16}x_{11}+q_{26}x_{21}-5(x_{11}+x_{21})\leq 0 \end{array}\right\} \quad \text{Meeting product one requirements}$$

$$\left.\begin{cases} q_{11}x_{12} + q_{21}x_{22} - 4(x_{12} + x_{22}) \leq 0 \\ q_{12}x_{12} + q_{22}x_{22} - 2.5(x_{12} + x_{22}) \leq 0 \\ q_{13}x_{12} + q_{23}x_{22} - 3.5(x_{12} + x_{22}) \leq 0 \\ q_{14}x_{12} + q_{24}x_{22} - 1.5(x_{12} + x_{22}) \leq 0 \\ q_{15}x_{12} + q_{25}x_{22} - 7(x_{12} + x_{22}) \leq 0 \\ q_{16}x_{12} + q_{26}x_{22} - 6(x_{12} + x_{22}) \leq 0 \end{cases}\right\} \text{Meeting product two requirements}$$

$$\left.\begin{cases} q_{11}x_{13} + q_{21}x_{23} - 1.5(x_{13} + x_{23}) \leq 0 \\ q_{12}x_{13} + q_{22}x_{23} - 5.5(x_{13} + x_{23}) \leq 0 \\ q_{13}x_{13} + q_{23}x_{23} - 3.9(x_{13} + x_{23}) \leq 0 \\ q_{14}x_{13} + q_{24}x_{23} - 0.8(x_{13} + x_{23}) \leq 0 \\ q_{15}x_{13} + q_{25}x_{23} - 7(x_{13} + x_{23}) \leq 0 \\ q_{16}x_{13} + q_{26}x_{23} - 6(x_{13} + x_{23}) \leq 0 \end{cases}\right\} \text{Meeting product three requirements}$$

$$\left.\begin{cases} q_{11}x_{14} + q_{21}x_{24} - 3(x_{14} + x_{24}) \leq 0 \\ q_{12}x_{14} + q_{22}x_{24} - 4(x_{14} + x_{24}) \leq 0 \\ q_{13}x_{14} + q_{23}x_{24} - 4(x_{14} + x_{24}) \leq 0 \\ q_{14}x_{14} + q_{24}x_{24} - 1.8(x_{14} + x_{24}) \leq 0 \\ q_{15}x_{14} + q_{25}x_{24} - 6(x_{14} + x_{24}) \leq 0 \\ q_{16}x_{14} + q_{26}x_{24} - 6(x_{14} + x_{24}) \leq 0 \end{cases}\right\} \text{Meeting product four requirements}$$

$1 \leq q_{11} \leq 4,\ 1 \leq q_{12} \leq 6,\ 3 \leq q_{13} \leq 4,\ 0.5 \leq q_{14} \leq 2,$

$4 \leq q_{15} \leq 5,\ 4 \leq q_{16} \leq 91 \leq q_{21} \leq 4,\ 2.7 \leq q_{22} \leq 5.5,$

$3 \leq q_{23} \leq 4,\ 0.9 \leq q_{24} \leq 1.6,\ 3 \leq q_{25} \leq 7,\ 4 \leq q_{26} \leq 10$

$x_{jk,}f_{ij} \geq 0$

### A.2.2 Lagrangean Relaxation

$$
\begin{array}{lll}
\min & 7f_{11}+3f_{21}+2f_{12}+10f_{22}+5f_{32}-16(x_{11}+x_{21})-25(x_{12}+x_{22}) & \\
 & -30(x_{13}+x_{23})-10(x_{14}+x_{24}) & \\
s.t & f_{11}+f_{21}-x_{11}-x_{12}-x_{13}-x_{14}=0 & \\
 & f_{12}+f_{22}+f_{32}-x_{21}-x_{22}-x_{23}-x_{24}=0 & \\
 & x_{11}+x_{21}\leq 10 & \\
 & x_{12}+x_{22}\leq 25 & \\
 & x_{13}+x_{23}\leq 30 & \\
 & x_{14}+x_{24}\leq 10 & \\
 & q_{11}(x_{11}+x_{12}+x_{13}+x_{14})-f_{11}-4f_{21}=0 & \leftarrow \alpha_{11} \\
 & q_{12}(x_{11}+x_{12}+x_{13}+x_{14})-6f_{11}-f_{21}=0 & \leftarrow \alpha_{12} \\
 & q_{13}(x_{11}+x_{12}+x_{13}+x_{14})-4f_{11}-3f_{21}=0 & \leftarrow \alpha_{13} \\
 & q_{14}(x_{11}+x_{12}+x_{13}+x_{14})-0.5f_{11}-2f_{21}=0 & \leftarrow \alpha_{14} \\
 & q_{15}(x_{11}+x_{12}+x_{13}+x_{14})-5f_{11}-4f_{21}=0 & \leftarrow \alpha_{15} \\
 & q_{16}(x_{11}+x_{12}+x_{13}+x_{14})-9f_{11}-4f_{21}=0 & \leftarrow \alpha_{16} \\
 & q_{21}(x_{21}+x_{22}+x_{23}+x_{24})-4f_{12}-3f_{22}-f_{32}=0 & \leftarrow \alpha_{21} \\
 & q_{22}(x_{21}+x_{22}+x_{23}+x_{24})-5.5f_{12}-3f_{22}-2.7f_{32}=0 & \leftarrow \alpha_{22} \\
 & q_{23}(x_{21}+x_{22}+x_{23}+x_{24})-3f_{12}-3f_{22}-4f_{32}=0 & \leftarrow \alpha_{23} \\
 & q_{24}(x_{21}+x_{22}+x_{23}+x_{24})-0.9f_{12}-f_{22}-1.6f_{32}=0 & \leftarrow \alpha_{24} \\
 & q_{25}(x_{21}+x_{22}+x_{23}+x_{24})-7f_{12}-3f_{22}-3f_{32}=0 & \leftarrow \alpha_{25} \\
 & q_{26}(x_{21}+x_{22}+x_{23}+x_{24})-10f_{12}-4f_{22}-7f_{32}=0 & \leftarrow \alpha_{26}
\end{array}
$$

$$\left\{\begin{array}{ll} q_{11}x_{11} + q_{21}x_{21} - 3(x_{11} + x_{21}) \leq 0 & \leftarrow \beta_{11} \\ q_{12}x_{11} + q_{22}x_{21} - 3(x_{11} + x_{21}) \leq 0 & \leftarrow \beta_{12} \\ q_{13}x_{11} + q_{23}x_{21} - 3.25(x_{11} + x_{21}) \leq 0 & \leftarrow \beta_{13} \\ q_{14}x_{11} + q_{24}x_{21} - 0.75(x_{11} + x_{21}) \leq 0 & \leftarrow \beta_{14} \\ q_{15}x_{11} + q_{25}x_{21} - 6(x_{11} + x_{21}) \leq 0 & \leftarrow \beta_{15} \\ q_{16}x_{11} + q_{26}x_{21} - 5(x_{11} + x_{21}) \leq 0 & \leftarrow \beta_{16} \end{array}\right\}$$

$$\left\{\begin{array}{cc} q_{11}x_{12} + q_{21}x_{22} - 4(x_{12} + x_{22}) \leq 0 & \leftarrow \beta_{21} \\ q_{12}x_{12} + q_{22}x_{22} - 2.5(x_{12} + x_{22}) \leq 0 & \leftarrow \beta_{22} \\ q_{13}x_{12} + q_{23}x_{22} - 3.5(x_{12} + x_{22}) \leq 0 & \leftarrow \beta_{23} \\ q_{14}x_{12} + q_{24}x_{22} - 1.5(x_{12} + x_{22}) \leq 0 & \leftarrow \beta_{24} \\ q_{15}x_{12} + q_{25}x_{22} - 7(x_{12} + x_{22}) \leq 0 & \leftarrow \beta_{25} \\ q_{16}x_{12} + q_{26}x_{22} - 6(x_{12} + x_{22}) \leq 0 & \leftarrow \beta_{26} \end{array}\right\}$$

$$\left\{\begin{array}{cc} q_{11}x_{13} + q_{21}x_{23} - 1.5(x_{13} + x_{23}) \leq 0 & \leftarrow \beta_{31} \\ q_{12}x_{13} + q_{22}x_{23} - 5.5(x_{13} + x_{23}) \leq 0 & \leftarrow \beta_{32} \\ q_{13}x_{13} + q_{23}x_{23} - 3.9(x_{13} + x_{23}) \leq 0 & \leftarrow \beta_{33} \\ q_{14}x_{13} + q_{24}x_{23} - 0.8(x_{13} + x_{23}) \leq 0 & \leftarrow \beta_{34} \\ q_{15}x_{13} + q_{25}x_{23} - 7(x_{13} + x_{23}) \leq 0 & \leftarrow \beta_{35} \\ q_{16}x_{13} + q_{26}x_{23} - 6(x_{13} + x_{23}) \leq 0 & \leftarrow \beta_{36} \end{array}\right\}$$

$$\left\{\begin{array}{cc} q_{11}x_{14} + q_{21}x_{24} - 3(x_{14} + x_{24}) \leq 0 & \leftarrow \beta_{41} \\ q_{12}x_{14} + q_{22}x_{24} - 4(x_{14} + x_{24}) \leq 0 & \leftarrow \beta_{42} \\ q_{13}x_{14} + q_{23}x_{24} - 4(x_{14} + x_{24}) \leq 0 & \leftarrow \beta_{43} \\ q_{14}x_{14} + q_{24}x_{24} - 1.8(x_{14} + x_{24}) \leq 0 & \leftarrow \beta_{44} \\ q_{15}x_{14} + q_{25}x_{24} - 6(x_{14} + x_{24}) \leq 0 & \leftarrow \beta_{45} \\ q_{16}x_{14} + q_{26}x_{24} - 6(x_{14} + x_{24}) \leq 0 & \leftarrow \beta_{46} \end{array}\right\}$$

$$1 \le q_{11} \le 4,\ 1 \le q_{12} \le 6,\ 3 \le q_{13} \le 4,\ 0.5 \le q_{14} \le 2,$$
$$4 \le q_{15} \le 5,\ 4 \le q_{16} \le 9, 1 \le q_{21} \le 4,\ 2.7 \le q_{22} \le 5.5,$$
$$,3 \le q_{23} \le 4, 0.9 \le q_{24} \le 1.6,\ 3 \le q_{25} \le 7,\ 4 \le q_{26} \le 10$$
$$x_{jk,} f_{ij} \ge 0$$

**The resulting subproblem is:**

$$\begin{aligned}
\min \quad & (7 - \alpha_{11} - 6\alpha_{12} - 4\alpha_{13} - 0.5\alpha_{14} - 5\alpha_{15} - 9\alpha_{16}) f_{11} \\
& + (3 - 4\alpha_{11} - \alpha_{12} - 3\alpha_{13} - 2\alpha_{14} - 4\alpha_{15} - 4\alpha_{16}) f_{21} \\
& + (2 - 4\alpha_{21} - 5.5\alpha_{22} - 3\alpha_{23} - 0.9\alpha_{24} - 7\alpha_{25} - 10\alpha_{26}) f_{12} \\
& + (10 - 3\alpha_{21} - 3\alpha_{22} - 3\alpha_{23} - \alpha_{24} - 3\alpha_{25} - 4\alpha_{26}) f_{22} \\
& + (5 - \alpha_{21} - 2.7\alpha_{22} - 4\alpha_{23} - 1.6\alpha_{24} - 3\alpha_{25} - 7\alpha_{26}) f_{32}
\end{aligned}$$

$$\begin{aligned}
& (-16 + q_{11}(\alpha_{11} + \beta_{11}) + q_{12}(\alpha_{12} + \beta_{12}) + q_{13}(\alpha_{13} + \beta_{13}) \\
& + q_{14}(\alpha_{14} + \beta_{14}) + q_{15}(\alpha_{15} + \beta_{15}) + q_{16}(\alpha_{16} + \beta_{16}) - 3\beta_{11} \\
& - 3\beta_{12} - 3.25\beta_{13} - 0.75\beta_{14} - 6\beta_{15} - 5\beta_{16}) x_{11}
\end{aligned}$$

$$\begin{aligned}
& + (-16 + q_{21}(\alpha_{21} + \beta_{11}) + q_{22}(\alpha_{22} + \beta_{12}) + q_{23}(\alpha_{23} + \beta_{13}) \\
& + q_{24}(\alpha_{24} + \beta_{14}) + q_{25}(\alpha_{25} + \beta_{15}) + q_{26}(\alpha_{26} + \beta_{16}) - 3\beta_{11} \\
& - 3\beta_{12} - 3.25\beta_{13} - 0.75\beta_{14} - 6\beta_{15} - 5\beta_{16}) x_{21}
\end{aligned}$$

$$\begin{aligned}
& + (-25 + q_{11}(\alpha_{11} + \beta_{21}) + q_{12}(\alpha_{12} + \beta_{22}) + q_{13}(\alpha_{13} + \beta_{23}) \\
& + q_{14}(\alpha_{14} + \beta_{24}) + q_{15}(\alpha_{15} + \beta_{25}) + q_{16}(\alpha_{16} + \beta_{26}) - 4\beta_{21} \\
& - 2.5\beta_{22} - 3.5\beta_{23} - 1.5\beta_{24} - 7\beta_{25} - 6\beta_{26}) x_{12}
\end{aligned}$$

$$\begin{aligned}
& + (-25 + q_{21}(\alpha_{21} + \beta_{21}) + q_{22}(\alpha_{22} + \beta_{22}) + q_{23}(\alpha_{23} + \beta_{23}) \\
& + q_{24}(\alpha_{24} + \beta_{24}) + q_{25}(\alpha_{25} + \beta_{25}) + q_{26}(\alpha_{26} + \beta_{26}) - 4\beta_{21} \\
& - 2.5\beta_{22} - 3.5\beta_{23} - 1.5\beta_{24} - 7\beta_{25} - 6\beta_{26}) x_{22}
\end{aligned}$$

$$+(-15+q_{11}(\alpha_{11}+\beta_{31})+q_{12}(\alpha_{12}+\beta_{32})+q_{13}(\alpha_{13}+\beta_{33})$$
$$+q_{14}(\alpha_{14}+\beta_{34})+q_{15}(\alpha_{15}+\beta_{35})+q_{16}(\alpha_{16}+\beta_{36})-1.5\beta_{31}$$
$$-5.5\beta_{32}-3.9\beta_{33}-0.8\beta_{34}-7\beta_{35}-6\beta_{36})x_{13}$$

$$+(-15+q_{21}(\alpha_{21}+\beta_{31})+q_{22}(\alpha_{22}+\beta_{32})+q_{23}(\alpha_{23}+\beta_{33})$$
$$+q_{24}(\alpha_{24}+\beta_{34})+q_{25}(\alpha_{25}+\beta_{35})+q_{26}(\alpha_{26}+\beta_{36})-1.5\beta_{31}$$
$$-5.5\beta_{32}-3.9\beta_{33}-0.8\beta_{34}-7\beta_{35}-6\beta_{36})x_{23}$$

$$+(-10+q_{11}(\alpha_{11}+\beta_{41})+q_{12}(\alpha_{12}+\beta_{42})+q_{13}(\alpha_{13}+\beta_{43})$$
$$+q_{14}(\alpha_{14}+\beta_{44})+q_{15}(\alpha_{15}+\beta_{45})+q_{16}(\alpha_{16}+\beta_{46})-3\beta_{41}$$
$$-4\beta_{42}-4\beta_{43}-1.8\beta_{44}-6\beta_{45}-6\beta_{46})x_{14}$$

$$+(-10+q_{21}(\alpha_{21}+\beta_{41})+q_{22}(\alpha_{22}+\beta_{42})+q_{23}(\alpha_{23}+\beta_{43})$$
$$+q_{24}(\alpha_{24}+\beta_{44})+q_{25}(\alpha_{25}+\beta_{45})+q_{26}(\alpha_{26}+\beta_{46})-3\beta_{41}$$
$$-4\beta_{42}-4\beta_{43}-1.8\beta_{44}-6\beta_{45}-6\beta_{46})x_{24}$$

$$
\begin{aligned}
s.t\quad & f_{11}+f_{21}-x_{11}-x_{12}-x_{13}-x_{14}=0\\
& f_{12}+f_{22}+f_{32}-x_{21}-x_{22}-x_{23}-x_{24}=0\\
& x_{11}+x_{21}\le 10\\
& x_{12}+x_{22}\le 25\\
& x_{13}+x_{23}\le 30\\
& x_{14}+x_{24}\le 10\\
& 1\le q_{11}\le 4,\ 1\le q_{12}\le 6,\ 3\le q_{13}\le 4, 0.5\le q_{14}\le 2,\\
& \ 4\le q_{15}\le 5,\ 4\le q_{16}\le 9, 1\le q_{21}\le 4,\ 2.7\le q_{22}\le 5.5,\\
& \ 3\le q_{23}\le 4,\ 0.9\le q_{24}\le 1.6,\ 3\le q_{25}\le 7,\ 4\le q_{26}\le 10\\
& x_{jk}, f_{ij}\ge 0
\end{aligned}
$$

If we define

$$u_{11} = q_{11}((\alpha_{11} + \beta_{11})x_{11} + (\alpha_{11} + \beta_{21})x_{12} + (\alpha_{11} + \beta_{31})x_{13} + (\alpha_{11} + \beta_{41})x_{14})$$
$$u_{12} = q_{12}((\alpha_{12} + \beta_{12})x_{11} + (\alpha_{12} + \beta_{22})x_{12} + (\alpha_{12} + \beta_{32})x_{13} + (\alpha_{12} + \beta_{42})x_{14})$$
$$u_{13} = q_{13}((\alpha_{13} + \beta_{13})x_{11} + (\alpha_{13} + \beta_{23})x_{12} + (\alpha_{13} + \beta_{33})x_{13} + (\alpha_{13} + \beta_{43})x_{14})$$
$$u_{14} = q_{14}((\alpha_{14} + \beta_{14})x_{11} + (\alpha_{14} + \beta_{24})x_{12} + (\alpha_{14} + \beta_{34})x_{13} + (\alpha_{14} + \beta_{44})x_{14})$$
$$u_{15} = q_{15}((\alpha_{15} + \beta_{15})x_{11} + (\alpha_{15} + \beta_{25})x_{12} + (\alpha_{15} + \beta_{35})x_{13} + (\alpha_{15} + \beta_{45})x_{14})$$
$$u_{16} = q_{16}((\alpha_{16} + \beta_{16})x_{11} + (\alpha_{16} + \beta_{26})x_{12} + (\alpha_{16} + \beta_{36})x_{13} + (\alpha_{16} + \beta_{46})x_{14})$$

$$u_{21} = q_{21}((\alpha_{21} + \beta_{11})x_{21} + (\alpha_{21} + \beta_{21})x_{22} + (\alpha_{21} + \beta_{31})x_{23} + (\alpha_{21} + \beta_{41})x_{24})$$
$$u_{22} = q_{22}((\alpha_{22} + \beta_{12})x_{21} + (\alpha_{22} + \beta_{22})x_{22} + (\alpha_{22} + \beta_{32})x_{23} + (\alpha_{22} + \beta_{42})x_{24})$$
$$u_{23} = q_{23}((\alpha_{23} + \beta_{13})x_{21} + (\alpha_{23} + \beta_{23})x_{22} + (\alpha_{23} + \beta_{33})x_{23} + (\alpha_{23} + \beta_{43})x_{24})$$
$$u_{24} = q_{24}((\alpha_{24} + \beta_{14})x_{21} + (\alpha_{24} + \beta_{24})x_{22} + (\alpha_{24} + \beta_{34})x_{23} + (\alpha_{24} + \beta_{44})x_{24})$$
$$u_{25} = q_{25}((\alpha_{25} + \beta_{15})x_{21} + (\alpha_{25} + \beta_{25})x_{22} + (\alpha_{25} + \beta_{35})x_{23} + (\alpha_{25} + \beta_{45})x_{24})$$
$$u_{26} = q_{26}((\alpha_{26} + \beta_{16})x_{21} + (\alpha_{26} + \beta_{26})x_{22} + (\alpha_{26} + \beta_{36})x_{23} + (\alpha_{26} + \beta_{46})x_{24})$$

then the nonlinearity can be eliminated by using the linear bound constraints $q^l_{jw} \leq q_{jw} \leq q^u_{jw}$ , and the fact that $q_{jw}$ does not appear in the constraints.

The resulting linear subproblem is:

$$
\begin{aligned}
\min \quad & (7 - \alpha_{11} - 6\alpha_{12} - 4\alpha_{13} - 0.5\alpha_{14} - 5\alpha_{15} - 9\alpha_{16})f_{11} \\
& +(3 - 4\alpha_{11} - \alpha_{12} - 3\alpha_{13} - 2\alpha_{14} - 4\alpha_{15} - 4\alpha_{16})f_{21} \\
& +(2 - 4\alpha_{21} - 5.5\alpha_{22} - 3\alpha_{23} - 0.9\alpha_{24} - 7\alpha_{25} - 10\alpha_{26})f_{12} \\
& +(10 - 3\alpha_{21} - 3\alpha_{22} - 3\alpha_{23} - \alpha_{24} - 3\alpha_{25} - 4\alpha_{26})f_{22} \\
& +(5 - \alpha_{21} - 2.7\alpha_{22} - 4\alpha_{23} - 1.6\alpha_{24} - 3\alpha_{25} - 7\alpha_{26})f_{32} \\
& (-16 - 3\beta_{11} - 3\beta_{12} - 3.25\beta_{13} - 0.75\beta_{14} - 6\beta_{15} - 5\beta_{16})x_{11} \\
& (-16 - 3\beta_{11} - 3\beta_{12} - 3.25\beta_{13} - 0.75\beta_{14} - 6\beta_{15} - 5\beta_{16})x_{21} \\
& (-25 - 4\beta_{21} - 2.5\beta_{22} - 3.5\beta_{23} - 1.5\beta_{24} - 7\beta_{25} - 6\beta_{26})x_{12} \\
& (-25 - 4\beta_{21} - 2.5\beta_{22} - 3.5\beta_{23} - 1.5\beta_{24} - 7\beta_{25} - 6\beta_{26})x_{22} \\
& (-15 - 1.5\beta_{31} - 5.5\beta_{32} - 3.9\beta_{33} - 0.8\beta_{34} - 7\beta_{35} - 6\beta_{36})x_{13} \\
& (-15 - 1.5\beta_{31} - 5.5\beta_{32} - 3.9\beta_{33} - 0.8\beta_{34} - 7\beta_{35} - 6\beta_{36})x_{23} \\
& (-10 - 3\beta_{41} - 4\beta_{42} - 4\beta_{43} - 1.8\beta_{44} - 6\beta_{45} - 6\beta_{46})x_{14} \\
& (-10 - 3\beta_{41} - 4\beta_{42} - 4\beta_{43} - 1.8\beta_{44} - 6\beta_{45} - 6\beta_{46})x_{24} \\
& u_{11} + u_{12} + u_{13} + u_{14} + u_{15} + u_{16} + u_{21} + u_{22} + u_{23} + u_{24} + u_{25} + u_{26}
\end{aligned}
$$

$$
\begin{aligned}
s.t \quad & f_{11} + f_{21} - x_{11} - x_{12} - x_{13} - x_{14} = 0 \\
& f_{12} + f_{22} + f_{32} - x_{21} - x_{22} - x_{23} - x_{24} = 0 \\
& x_{11} + x_{21} \leq 10 \\
& x_{12} + x_{22} \leq 25 \\
& x_{13} + x_{23} \leq 30 \\
& x_{14} + x_{24} \leq 10
\end{aligned}
$$

$$(\alpha_{11}+\beta_{11})x_{11}+(\alpha_{11}+\beta_{21})x_{12}+(\alpha_{11}+\beta_{31})x_{13}+(\alpha_{11}+\beta_{41})x_{14}-u_{11}$$
$$+My_{11}\leq M$$
$$-(\alpha_{11}+\beta_{11})4x_{11}-(\alpha_{11}+\beta_{21})4x_{12}-(\alpha_{11}+\beta_{31})4x_{13}-(\alpha_{11}+\beta_{41})4x_{14}+u_{11}$$
$$+My_{11}\leq M$$
$$(\alpha_{11}+\beta_{11})4x_{11}+(\alpha_{11}+\beta_{21})4x_{12}+(\alpha_{11}+\beta_{31})4x_{13}+(\alpha_{11}+\beta_{41})4x_{14}-u_{11}$$
$$-My_{11}\leq 0$$
$$-(\alpha_{11}+\beta_{11})x_{11}-(\alpha_{11}+\beta_{21})x_{12}-(\alpha_{11}+\beta_{31})x_{13}-(\alpha_{11}+\beta_{41})x_{14}+u_{11}$$
$$+My_{11}\leq M$$
$$-(\alpha_{11}+\beta_{11})x_{11}-(\alpha_{11}+\beta_{21})x_{12}-(\alpha_{11}+\beta_{31})x_{13}-(\alpha_{11}+\beta_{41})x_{14}$$
$$+My_{11}\leq M$$
$$(\alpha_{11}+\beta_{11})x_{11}+(\alpha_{11}+\beta_{21})x_{12}+(\alpha_{11}+\beta_{31})x_{13}+(\alpha_{11}+\beta_{41})x_{14}$$
$$-My_{11}\leq 0$$

$$(\alpha_{12}+\beta_{12})x_{11}+(\alpha_{12}+\beta_{22})x_{12}+(\alpha_{12}+\beta_{32})x_{13}+(\alpha_{12}+\beta_{42})x_{14}$$
$$-u_{12}+My_{12}\leq M$$
$$-(\alpha_{12}+\beta_{12})6x_{11}-(\alpha_{12}+\beta_{22})6x_{12}-(\alpha_{12}+\beta_{32})6x_{13}-(\alpha_{12}+\beta_{42})6x_{14}$$
$$+u_{12}+My_{12}\leq M$$
$$(\alpha_{12}+\beta_{12})6x_{11}+(\alpha_{12}+\beta_{22})6x_{12}+(\alpha_{12}+\beta_{32})6x_{13}+(\alpha_{12}+\beta_{42})6x_{14}$$
$$-u_{12}-My_{12}\leq 0$$
$$-(\alpha_{12}+\beta_{12})x_{11}-(\alpha_{12}+\beta_{22})x_{12}-(\alpha_{12}+\beta_{32})x_{13}-(\alpha_{12}+\beta_{42})x_{14}$$
$$+u_{12}-My_{12}\leq 0$$
$$-(\alpha_{12}+\beta_{12})x_{11}-(\alpha_{12}+\beta_{22})x_{12}-(\alpha_{12}+\beta_{32})x_{13}-(\alpha_{12}+\beta_{42})x_{14}$$
$$+My_{12}\leq M$$
$$(\alpha_{12}+\beta_{12})x_{11}+(\alpha_{12}+\beta_{22})x_{12}+(\alpha_{12}+\beta_{32})x_{13}+(\alpha_{12}+\beta_{42})x_{14}$$
$$-My_{12}\leq 0$$

$$(\alpha_{13}+\beta_{13})3x_{11}+(\alpha_{13}+\beta_{23})3x_{12}+(\alpha_{13}+\beta_{33})3x_{13}+(\alpha_{13}+\beta_{43})3x_{14}$$
$$-u_{13}+My_{13}\leq M$$
$$-(\alpha_{13}+\beta_{13})4x_{11}-(\alpha_{13}+\beta_{23})4x_{12}-(\alpha_{13}+\beta_{33})4x_{13}-(\alpha_{13}+\beta_{43})4x_{14}$$
$$+u_{13}+My_{13}\leq M$$
$$(\alpha_{13}+\beta_{13})4x_{11}+(\alpha_{13}+\beta_{23})4x_{12}+(\alpha_{13}+\beta_{33})4x_{13}+(\alpha_{13}+\beta_{43})4x_{14}$$
$$-u_{13}-My_{13}\leq 0$$
$$-(\alpha_{13}+\beta_{13})3x_{11}-(\alpha_{13}+\beta_{23})3x_{12}-(\alpha_{13}+\beta_{33})3x_{13}-(\alpha_{13}+\beta_{43})3x_{14}$$
$$+u_{13}-My_{13}\leq 0$$
$$-(\alpha_{13}+\beta_{13})x_{11}-(\alpha_{13}+\beta_{23})x_{12}-(\alpha_{13}+\beta_{33})x_{13}-(\alpha_{13}+\beta_{43})x_{14}$$
$$+My_{13}\leq M$$
$$(\alpha_{13}+\beta_{13})x_{11}+(\alpha_{13}+\beta_{23})x_{12}+(\alpha_{13}+\beta_{33})x_{13}+(\alpha_{13}+\beta_{43})x_{14}$$
$$-My_{13}\leq 0$$

$$(\alpha_{14}+\beta_{14})0.5x_{11}+(\alpha_{14}+\beta_{24})0.5x_{12}+(\alpha_{14}+\beta_{34})0.5x_{13}+(\alpha_{14}+\beta_{44})0.5x_{14}$$
$$-u_{14}+My_{14}\leq M$$
$$-(\alpha_{14}+\beta_{14})2x_{11}-(\alpha_{14}+\beta_{24})2x_{12}-(\alpha_{14}+\beta_{34})2x_{13}-(\alpha_{14}+\beta_{44})2x_{14}$$
$$+u_{14}+My_{14}\leq M$$
$$(\alpha_{14}+\beta_{14})2x_{11}+(\alpha_{14}+\beta_{24})2x_{12}+(\alpha_{14}+\beta_{34})2x_{13}+(\alpha_{14}+\beta_{44})2x_{14}$$
$$-u_{14}-My_{14}\leq 0$$
$$-(\alpha_{14}+\beta_{14})0.5x_{11}-(\alpha_{14}+\beta_{24})0.5x_{12}-(\alpha_{14}+\beta_{34})0.5x_{13}-(\alpha_{14}+\beta_{44})0.5x_{14}$$
$$+u_{14}-My_{14}\leq 0$$
$$-(\alpha_{14}+\beta_{14})x_{11}-(\alpha_{14}+\beta_{24})x_{12}-(\alpha_{14}+\beta_{34})x_{13}-(\alpha_{14}+\beta_{44})x_{14}$$
$$+My_{14}\leq M$$
$$(\alpha_{14}+\beta_{14})x_{11}+(\alpha_{14}+\beta_{24})x_{12}+(\alpha_{14}+\beta_{34})x_{13}+(\alpha_{14}+\beta_{44})x_{14}$$
$$-My_{14}\leq 0$$

$$(\alpha_{15}+\beta_{15})4x_{11}+(\alpha_{15}+\beta_{25})4x_{12}+(\alpha_{15}+\beta_{35})4x_{13}+(\alpha_{15}+\beta_{45})4x_{14}$$
$$-u_{15}+My_{15}\leq M$$
$$-(\alpha_{15}+\beta_{15})5x_{11}-(\alpha_{15}+\beta_{25})5x_{12}-(\alpha_{15}+\beta_{35})5x_{13}-(\alpha_{15}+\beta_{45})5x_{14}$$
$$+u_{15}+My_{15}\leq M$$
$$(\alpha_{15}+\beta_{15})5x_{11}+(\alpha_{15}+\beta_{25})5x_{12}+(\alpha_{15}+\beta_{35})5x_{13}+(\alpha_{15}+\beta_{45})5x_{14}$$
$$-u_{15}-My_{15}\leq 0$$
$$-(\alpha_{15}+\beta_{15})4x_{11}-(\alpha_{15}+\beta_{25})4x_{12}-(\alpha_{15}+\beta_{35})4x_{13}-(\alpha_{15}+\beta_{45})4x_{14}$$
$$+u_{15}-My_{15}\leq 0$$
$$-(\alpha_{15}+\beta_{15})x_{11}-(\alpha_{15}+\beta_{25})x_{12}-(\alpha_{15}+\beta_{35})x_{13}-(\alpha_{15}+\beta_{45})x_{14}$$
$$+My_{15}\leq M$$
$$(\alpha_{15}+\beta_{15})x_{11}+(\alpha_{15}+\beta_{25})x_{12}+(\alpha_{15}+\beta_{35})x_{13}+(\alpha_{15}+\beta_{45})x_{14}$$
$$-My_{15}\leq 0$$

$$(\alpha_{16}+\beta_{16})4x_{11}+(\alpha_{16}+\beta_{26})4x_{12}+(\alpha_{16}+\beta_{36})4x_{13}+(\alpha_{16}+\beta_{46})4x_{14}$$
$$-u_{16}+My_{16}\leq M$$
$$-(\alpha_{16}+\beta_{16})9x_{11}-(\alpha_{16}+\beta_{26})9x_{12}-(\alpha_{16}+\beta_{36})9x_{13}-(\alpha_{16}+\beta_{46})9x_{14}$$
$$+u_{16}+My_{16}\leq M$$
$$(\alpha_{16}+\beta_{16})9x_{11}+(\alpha_{16}+\beta_{26})9x_{12}+(\alpha_{16}+\beta_{36})9x_{13}+(\alpha_{16}+\beta_{46})9x_{14}$$
$$-u_{16}-My_{16}\leq 0$$
$$-(\alpha_{16}+\beta_{16})4x_{11}-(\alpha_{16}+\beta_{26})4x_{12}-(\alpha_{16}+\beta_{36})4x_{13}-(\alpha_{16}+\beta_{46})4x_{14}$$
$$+u_{16}-My_{16}\leq 0$$
$$-(\alpha_{16}+\beta_{16})x_{11}-(\alpha_{16}+\beta_{26})x_{12}-(\alpha_{16}+\beta_{36})x_{13}-(\alpha_{16}+\beta_{46})x_{14}$$
$$+My_{16}\leq M$$
$$(\alpha_{16}+\beta_{16})x_{11}+(\alpha_{16}+\beta_{26})x_{12}+(\alpha_{16}+\beta_{36})x_{13}+(\alpha_{16}+\beta_{46})x_{14}$$
$$-My_{16}\leq 0$$

$$(\alpha_{21}+\beta_{11})x_{21}+(\alpha_{21}+\beta_{21})x_{22}+(\alpha_{21}+\beta_{31})x_{23}+(\alpha_{21}+\beta_{41})x_{24}$$
$$-u_{21}+My_{21}\leq M$$
$$-(\alpha_{21}+\beta_{11})4x_{21}-(\alpha_{21}+\beta_{21})4x_{22}-(\alpha_{21}+\beta_{31})4x_{23}-(\alpha_{21}+\beta_{41})4x_{24}$$
$$+u_{21}+My_{21}\leq M$$
$$(\alpha_{21}+\beta_{11})4x_{21}+(\alpha_{21}+\beta_{21})4x_{22}+(\alpha_{21}+\beta_{31})4x_{23}+(\alpha_{21}+\beta_{41})4x_{24}$$
$$-u_{21}-My_{21}\leq 0$$
$$-(\alpha_{21}+\beta_{11})x_{21}-(\alpha_{21}+\beta_{21})x_{22}-(\alpha_{21}+\beta_{31})x_{23}-(\alpha_{21}+\beta_{41})x_{24}$$
$$+u_{21}-My_{21}\leq 0$$
$$-(\alpha_{21}+\beta_{11})x_{21}-(\alpha_{21}+\beta_{21})x_{22}-(\alpha_{21}+\beta_{31})x_{23}-(\alpha_{21}+\beta_{41})x_{24}$$
$$+My_{21}\leq M$$
$$(\alpha_{21}+\beta_{11})x_{21}+(\alpha_{21}+\beta_{21})x_{22}+(\alpha_{21}+\beta_{31})x_{23}+(\alpha_{21}+\beta_{41})x_{24}$$
$$-My_{21}\leq 0$$

$$(\alpha_{22}+\beta_{12})2.7x_{21}+(\alpha_{22}+\beta_{22})2.7x_{22}+(\alpha_{22}+\beta_{32})2.7x_{23}+(\alpha_{22}+\beta_{42})2.7x_{24}$$
$$-u_{22}+My_{22}\leq M$$
$$-(\alpha_{22}+\beta_{12})5.5x_{21}-(\alpha_{22}+\beta_{22})5.5x_{22}-(\alpha_{22}+\beta_{32})5.5x_{23}-(\alpha_{22}+\beta_{42})5.5x_{24}$$
$$+u_{22}+My_{22}\leq M$$
$$(\alpha_{22}+\beta_{12})5.5x_{21}+(\alpha_{22}+\beta_{22})5.5x_{22}+(\alpha_{22}+\beta_{32})5.5x_{23}+(\alpha_{22}+\beta_{42})5.5x_{24}$$
$$-u_{22}-My_{22}\leq 0$$
$$-(\alpha_{22}+\beta_{12})2.7x_{21}-(\alpha_{22}+\beta_{22})2.7x_{22}-(\alpha_{22}+\beta_{32})2.7x_{23}-(\alpha_{22}+\beta_{42})2.7x_{24}$$
$$+u_{22}-My_{22}\leq 0$$
$$-(\alpha_{22}+\beta_{12})x_{21}-(\alpha_{22}+\beta_{22})x_{22}-(\alpha_{22}+\beta_{32})x_{23}-(\alpha_{22}+\beta_{42})x_{24}$$
$$+My_{22}\leq M$$
$$(\alpha_{22}+\beta_{12})x_{21}+(\alpha_{22}+\beta_{22})x_{22}+(\alpha_{22}+\beta_{32})x_{23}+(\alpha_{22}+\beta_{42})x_{24}$$
$$-My_{22}\leq 0$$

$$(\alpha_{23}+\beta_{13})3x_{21}+(\alpha_{23}+\beta_{23})3x_{22}+(\alpha_{23}+\beta_{33})3x_{23}+(\alpha_{23}+\beta_{43})3x_{24}$$
$$-u_{23}+My_{23}\leq M$$
$$-(\alpha_{23}+\beta_{13})4x_{21}-(\alpha_{23}+\beta_{23})4x_{22}-(\alpha_{23}+\beta_{33})4x_{23}-(\alpha_{23}+\beta_{43})4x_{24}$$
$$+u_{23}+My_{23}\leq M$$
$$(\alpha_{23}+\beta_{13})4x_{21}+(\alpha_{23}+\beta_{23})4x_{22}+(\alpha_{23}+\beta_{33})4x_{23}+(\alpha_{23}+\beta_{43})4x_{24}$$
$$-u_{23}-My_{23}\leq 0$$
$$-(\alpha_{23}+\beta_{13})3x_{21}-(\alpha_{23}+\beta_{23})3x_{22}-(\alpha_{23}+\beta_{33})3x_{23}-(\alpha_{23}+\beta_{43})3x_{24}$$
$$+u_{23}-My_{23}\leq 0$$
$$-(\alpha_{23}+\beta_{13})x_{21}-(\alpha_{23}+\beta_{23})x_{22}-(\alpha_{23}+\beta_{33})x_{23}-(\alpha_{23}+\beta_{43})x_{24}$$
$$+My_{23}\leq M$$
$$(\alpha_{23}+\beta_{13})x_{21}+(\alpha_{23}+\beta_{23})x_{22}+(\alpha_{23}+\beta_{33})x_{23}+(\alpha_{23}+\beta_{43})x_{24}$$
$$-My_{23}\leq 0$$

$$(\alpha_{24}+\beta_{14})0.9x_{21}+(\alpha_{24}+\beta_{24})0.9x_{22}+(\alpha_{24}+\beta_{34})0.9x_{23}+(\alpha_{24}+\beta_{44})0.9x_{24}$$
$$-u_{24}+My_{24}\leq M$$
$$-(\alpha_{24}+\beta_{14})1.6x_{21}-(\alpha_{24}+\beta_{24})1.6x_{22}-(\alpha_{24}+\beta_{34})1.6x_{23}-(\alpha_{24}+\beta_{44})1.6x_{24}$$
$$+u_{24}+My_{24}\leq M$$
$$(\alpha_{24}+\beta_{14})1.6x_{21}+(\alpha_{24}+\beta_{24})1.6x_{22}+(\alpha_{24}+\beta_{34})1.6x_{23}+(\alpha_{24}+\beta_{44})1.6x_{24}$$
$$-u_{24}-My_{24}\leq 0$$
$$-(\alpha_{24}+\beta_{14})0.9x_{21}-(\alpha_{24}+\beta_{24})0.9x_{22}-(\alpha_{24}+\beta_{34})0.9x_{23}-(\alpha_{24}+\beta_{44})0.9x_{24}$$
$$+u_{24}-My_{24}\leq 0$$
$$-(\alpha_{24}+\beta_{14})x_{21}-(\alpha_{24}+\beta_{24})x_{22}-(\alpha_{24}+\beta_{34})x_{23}-(\alpha_{24}+\beta_{44})x_{24}$$
$$+My_{24}\leq M$$
$$(\alpha_{24}+\beta_{14})x_{21}+(\alpha_{24}+\beta_{24})x_{22}+(\alpha_{24}+\beta_{34})x_{23}+(\alpha_{24}+\beta_{44})x_{24}$$
$$-My_{24}\leq 0$$

$$(\alpha_{25} + \beta_{15})3x_{21} + (\alpha_{25} + \beta_{25})3x_{22} + (\alpha_{25} + \beta_{35})3x_{23} + (\alpha_{25} + \beta_{45})3x_{24}$$
$$-u_{25} + My_{25} \leq M$$
$$-(\alpha_{25} + \beta_{15})4x_{21} - (\alpha_{25} + \beta_{25})4x_{22} - (\alpha_{25} + \beta_{35})4x_{23} - (\alpha_{25} + \beta_{45})4x_{24}$$
$$+u_{25} + My_{25} \leq M$$
$$(\alpha_{25} + \beta_{15})4x_{21} + (\alpha_{25} + \beta_{25})4x_{22} + (\alpha_{25} + \beta_{35})4x_{23} + (\alpha_{25} + \beta_{45})4x_{24}$$
$$-u_{25} - My_{25} \leq 0$$
$$-(\alpha_{25} + \beta_{15})3x_{21} - (\alpha_{25} + \beta_{25})3x_{22} - (\alpha_{25} + \beta_{35})3x_{23} - (\alpha_{25} + \beta_{45})3x_{24}$$
$$+u_{25} - My_{25} \leq 0$$
$$-(\alpha_{25} + \beta_{15})x_{21} - (\alpha_{25} + \beta_{25})x_{22} - (\alpha_{25} + \beta_{35})x_{23} - (\alpha_{25} + \beta_{45})x_{24}$$
$$+My_{25} \leq M$$
$$(\alpha_{25} + \beta_{15})x_{21} + (\alpha_{25} + \beta_{25})x_{22} + (\alpha_{25} + \beta_{35})x_{23} + (\alpha_{25} + \beta_{45})x_{24}$$
$$-My_{25} \leq 0$$

$$(\alpha_{26} + \beta_{16})4x_{21} + (\alpha_{26} + \beta_{26})4x_{22} + (\alpha_{26} + \beta_{36})4x_{23} + (\alpha_{26} + \beta_{46})4x_{24}$$
$$-u_{26} + My_{26} \leq M$$
$$-(\alpha_{26} + \beta_{16})10x_{21} - (\alpha_{26} + \beta_{26})10x_{22} - (\alpha_{26} + \beta_{36})10x_{23} - (\alpha_{26} + \beta_{46})10x_{24}$$
$$+u_{26} + My_{26} \leq M$$
$$(\alpha_{26} + \beta_{16})10x_{21} + (\alpha_{26} + \beta_{26})10x_{22} + (\alpha_{26} + \beta_{36})10x_{23} + (\alpha_{26} + \beta_{46})10x_{24}$$
$$-u_{26} - My_{26} \leq 0$$
$$-(\alpha_{26} + \beta_{16})4x_{21} - (\alpha_{26} + \beta_{26})4x_{22} - (\alpha_{26} + \beta_{36})4x_{23} - (\alpha_{26} + \beta_{46})4x_{24}$$
$$+u_{26} - My_{26} \leq 0$$
$$-(\alpha_{26} + \beta_{16})x_{21} - (\alpha_{26} + \beta_{26})x_{22} - (\alpha_{26} + \beta_{36})x_{23} - (\alpha_{26} + \beta_{46})x_{24}$$
$$+My_{26} \leq M$$
$$(\alpha_{26} + \beta_{16})x_{21} + (\alpha_{26} + \beta_{26})x_{22} + (\alpha_{26} + \beta_{36})x_{23} + (\alpha_{26} + \beta_{46})x_{24}$$
$$-My_{26} \leq 0$$

**The master problem is:**

$$
\begin{aligned}
\max \quad & \theta \\
s.t \quad & \theta + (f^h_{11} + 4f^h_{21} - q^h_{11}(x^h_{11} + x^h_{12} + x^h_{13} + x^h_{14}))\alpha_{11} \\
& +(6f^h_{11} + f^h_{21} - q^h_{12}(x^h_{11} + x^h_{12} + x^h_{13} + x^h_{14}))\alpha_{12} \\
& +(4f^h_{11} + 3f^h_{21} - q^h_{13}(x^h_{11} + x^h_{12} + x^h_{13} + x^h_{14}))\alpha_{13} \\
& +(0.5f^h_{11} + 2f^h_{21} - q^h_{14}(x^h_{11} + x^h_{12} + x^h_{13} + x^h_{14}))\alpha_{14} \\
& +(5f^h_{11} + 4f^h_{21} - q^h_{15}(x^h_{11} + x^h_{12} + x^h_{13} + x^h_{14}))\alpha_{15} \\
& +(9f^h_{11} + 4f^h_{21} - q^h_{14}(x^h_{11} + x^h_{12} + x^h_{13} + x^h_{14}))\alpha_{16} \\
& +(4f^h_{12} + 3f^h_{22} + f^h_{32} - q^h_{21}(x^h_{21} + x^h_{22} + x^h_{23} + x^h_{24}))\alpha_{21} \\
& +(5.5f^h_{12} + 3f^h_{22} + 2.7f^h_{32} - q^h_{22}(x^h_{21} + x^h_{22} + x^h_{23} + x^h_{24}))\alpha_{22} \\
& +(3f^h_{12} + 3f^h_{22} + 4f^h_{32} - q^h_{23}(x^h_{21} + x^h_{22} + x^h_{23} + x^h_{24}))\alpha_{23} \\
& +(0.9f^h_{12} + f^h_{22} + 1.6f^h_{32} - q^h_{24}(x^h_{21} + x^h_{22} + x^h_{23} + x^h_{24}))\alpha_{24} \\
& +(7f^h_{12} + 3f^h_{22} + 3f^h_{32} - q^h_{25}(x^h_{21} + x^h_{22} + x^h_{23} + x^h_{24}))\alpha_{25} \\
& +(10f^h_{12} + 4f^h_{22} + 7f^h_{32} - q^h_{26}(x^h_{21} + x^h_{22} + x^h_{23} + x^h_{24}))\alpha_{26} \\
& +((3 - q_{11})x_{11} + (3 - q_{21})x_{21})\beta_{11} + ((3 - q_{12})x_{11} + (3 - q_{22})x_{21})\beta_{12} \\
& +((3.25 - q_{13})x_{11} + (3.25 - q_{23})x_{21})\beta_{13} + ((0.75 - q_{14})x_{11} + (0.75 - q_{24})x_{21})\beta_{14} \\
& +((6 - q_{15})x_{11} + (6 - q_{25})x_{21})\beta_{15} + ((5 - q_{16})x_{11} + (5 - q_{26})x_{21})\beta_{16} \\
& +((4 - q_{11})x_{12} + (4 - q_{21})x_{22})\beta_{21} + ((2.5 - q_{12})x_{12} + (2.5 - q_{22})x_{22})\beta_{22} \\
& +((3.5 - q_{13})x_{12} + (3.5 - q_{23})x_{22})\beta_{23} + ((1.5 - q_{14})x_{12} + (1.5 - q_{24})x_{22})\beta_{24} \\
& +((7 - q_{15})x_{12} + (7 - q_{25})x_{22})\beta_{25} + ((6 - q_{16})x_{12} + (6 - q_{26})x_{22})\beta_{26} \\
& +((1.5 - q_{11})x_{13} + (1.5 - q_{21})x_{23})\beta_{31} + ((5.5 - q_{12})x_{13} + (5.5 - q_{22})x_{23})\beta_{32}
\end{aligned}
$$

$$
\begin{aligned}
&+((3.9-q_{13})x_{13}+(3.9-q_{23})x_{23})\beta_{33}+((0.8-q_{14})x_{13}+(10.8-q_{24})x_{23})\beta_{34}\\
&+((7-q_{15})x_{13}+(7-q_{25})x_{23})\beta_{35}+((6-q_{16})x_{13}+(6-q_{26})x_{23})\beta_{36}\\
&+((3-q_{11})x_{14}+(3-q_{21})x_{24})\beta_{41}+((4-q_{12})x_{14}+(4-q_{22})x_{24})\beta_{42}\\
&+((4-q_{13})x_{14}+(4-q_{23})x_{24})\beta_{43}+((1.8-q_{14})x_{14}+(1.8-q_{24})x_{24})\beta_{44}\\
&+((6-q_{15})x_{14}+(6-q_{25})x_{24})\beta_{45}+((6-q_{16})x_{14}+(6-q_{26})x_{24})\beta_{46}\leq\\
&7f_{11}^h+3f_{21}^h+2f_{12}^h+10f_{22}^h+5f_{32}^h-16x_{11}-16x_{21}-25x_{12}-25x_{22}-15x_{13}\\
&-15x_{23}-10x_{14}-10x_{24};\qquad \forall h\in H
\end{aligned}
$$