Root-Locus Theory for Infinite-Dimensional Systems

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis, the root-locus theory for a class of diffusion systems is studied. The input and output boundary operators are co-located in the sense that their highest order derivatives occur at the same endpoint. It is shown that infinitely many root-locus branches lie on the negative real axis and the remaining finitely many root-locus branches lie inside a fixed closed contour. It is also shown that all closed-loop poles vary continuously as the feedback gain varies from zero to infinity.

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"The hardest arithmetic to master is that which enables us to count our blessings," Eric Hoffer (1898-1983). I include amongst my blessings having had Dr. Kirsten Morris as my supervisor during the course of my master's degree. Her dedication and knowledge was a source of invaluable influence in my academic achievements and career.

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To my parents

Notation

$\hat{f}(s)$	Laplace transform of $f(t)$
$G(s)$	Open-loop Transfer function
$G_k(s)$	Closed-loop Transfer function
k	Feedback gain
$T(t)$	C_0-semigroup
ω_0	Growth bound of a semigroup
$D(A)$	Domain of the operator A
$R(\lambda, A)$	Resolvent operator
$\rho(A)$	Resolvent set
(A, B, C)	Finite-dimensional state-space realization
$(\mathcal{A}, \mathcal{B}, \mathcal{C})$	Boundary control system
$\mathcal{L}(X, Y)$	Space of bounded linear operators from X to Y
$\mathbb{O}(f)$	Order of an entire function
$\hat{\tau}$	Instantaneous gain
s_{ig}	Sign of $\hat{\tau}$
$\overline{\lambda}$	Complex conjugate of λ

Contents

Chapter 1

Introduction

We begin this chapter with a perspective on control systems in section 1. We classify control systems into two main classes: open-loop and closed-loop systems. Open-loop and closed-loop controllers are then compared and some examples are provided. Then we move to introducing some general concepts in control systems. Based on the equations of the system, any control system is divided into two types: finite and infinite-dimensional systems. We then focus on infinite-dimensional systems and introduce systems governed by partial differential equations, in particular, boundary control systems, as examples of infinite-dimensional systems. In section 2, we narrow our introduction over systems governed by partial differential equations and outline recent research done in this area. The last section of this chapter is devoted to giving a summary of this thesis.

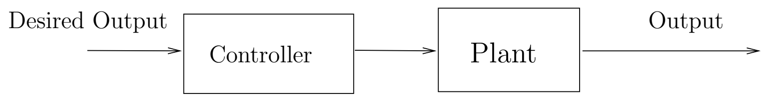

Figure 1.1: Block diagram of an open-loop control system

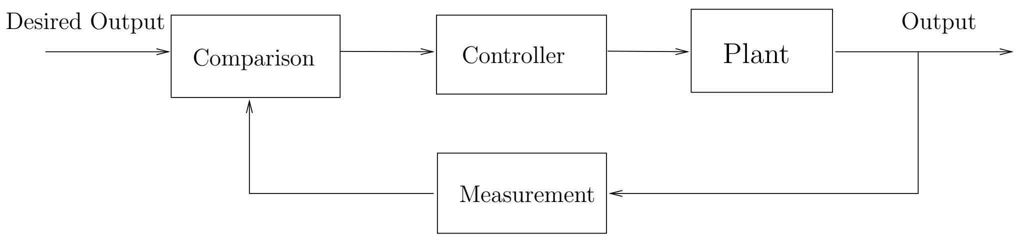

Figure 1.2: Block diagram of a closed-loop control system

1.1 A Perspective on Control Systems

Controlling a system means forcing the system to provide a desired system response. The process that is being controlled is called the plant and all actuating devices, sensors and other components that are employed to control the process are called the controller. Control systems can be divided into two classes: open-loop and closed-loop systems. In an open-loop control system, the controller can be an actuating device that controls the process directly and the output has no effect on the controller. In contrast, in a closed-loop control system the output is measured, compared with the desired output, and then this error, or a function of it, is fed to the controller so as to reduce the discrepancy between the actual and the desired output. A closed-loop system is also called a feedback system. An open-loop and a closed-loop system are shown in Figure 1.1 and Figure 1.2, respectively. An example of an open-loop control system is a washing machine. All operations of a washing machine are on a time basis and the output of the system, that is, the cleanliness of the clothes, is not measured and has no effect on the soaking, washing, or rinsing time. Another example of an open-loop system is traffic control by means of traffic lights operated on a time basis. In general, any control system that operates only on a time basis is open-loop [21].

A simple example of a closed-loop control system is the speed control of a car with eyes. The driver measures the speed by looking at the speedometer and compares the actual speed with the speed limit. If the error, that is, the difference between the actual and the desired speed, is nonzero, the driver decreases or increases the speed so as to lower the error. Room temperature control systems and cruise control systems are examples of closed-loop control systems.

Open-loop controllers are more convenient when the output is hard to measure or measuring the output is economically not feasible. The construction and maintenance of open-loop systems are simpler and less expensive than closed-loop systems. However, in most actual systems, unwanted parameters like noise and internal or external disturbances are involved and hence an exact mathematical model of the system is not available. In these cases, feeding back the output to correct the errors caused by these unwanted parameters is indispensable.

Among many types of feedback controllers used in industry, proportional-integral-derivative(PID) controllers are widely used. A PID controller involves three parts: the proportional, the integral, and the derivative values of the error. The summation of these three values, as illustrated in Figure 1.3, determines the reaction of the plant to the error. Roughly speaking,

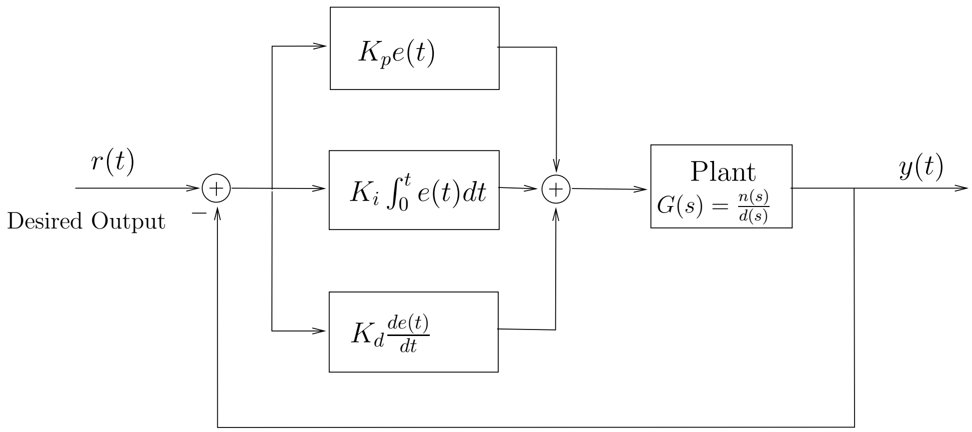

Figure 1.3: Block diagram of a PID controller

a pure proportional control($K_i = K_d = 0$) moves the output towards the desired output but usually with a steady-state error. The contribution of the integral term, when the proportional term is set to an appropriate value, accelerates the movement of the output towards the desired value by eliminating the residual steady-state error. Finally, the contribution of the derivative term, when the proportional and integral terms are set to suitable values, slows the rate of change of the output. Note that PI, PD, and proportional controllers are special cases of PID controllers with $K_d = 0$, $K_i = 0$ and $K_d = K_i = 0$, respectively.

For any linear time-invariant system a transfer function can be defined. For single-input single-output (SISO) systems, $G(s)$ is the transfer function if for any input $u(t)$ and its corresponding output $y(t)$ the relation $\hat{y}(s) = G(s)\hat{u}(s)$ holds, where $\hat{u}(s), \hat{y}(s)$ represent the Laplace transform of $u(t), y(t)$, respectively.

The stability of control systems is a very important concern in designing control systems. A system is L_2-stable if any input $u(t) \in L_2(0,\infty;U)$ results in an output $y(t) \in L_2(0,\infty;Y)$, where U, Y are the input and output spaces, respectively. Stability has a very close relation to the location of the poles of the system transfer function. (A point $p \in \mathbb{C}$ is a pole of $G(s)$ if $\lim_{s\to p} G(s) = \infty$, a point $z \in \mathbb{C}$ is a zero of $G(s)$ if $\lim_{s\to z} G(s) = 0$).

If the system has a variable gain, then the location of the closed-loop poles depend on the value of the loop gain chosen. For example, in the system shown in Figure 1.3, if K_d and K_i are set to zero and the only variable parameter is $k = K_p$, then the equations of the closed-loop system are $\hat{e}(s) = \hat{r}(s) - k\hat{y}(s)$, and $\hat{y}(s) = G(s)\hat{e}(s)$. Thus, the closed-loop transfer function (assuming the open-loop transfer function being $G(s)$) is

$$G_k(s) = \frac{\hat{y}(s)}{\hat{r}(s)} = \frac{kG(s)}{1 + kG(s)}. \tag{1.1}$$

The closed-loop poles are the roots of the characteristic equation $1 + kG(s) = 0$.

A simple method for finding the roots of the characteristic equation has been developed by W.R. Evans for finite-dimensional systems and used extensively in control systems. In this method, called the root-locus method, the roots of the characteristic equation are plotted for all values of a system parameter. The roots corresponding to a particular value of this parameter can then be located on the resulting graph. For example, for the characteristic equation $1 + kG(s) = 0$, as k approaches zero, the roots of the characteristic equation approach the poles of $G(s)$ and as k tends to infinity, the roots approach the zeros of $G(s)$.

Any system modeled by $x^{(n)}(t)+c_1x^{(n-1)}(t)+\cdots+c_{n-1}x'(t)+c_nx(t) = f(t)$ can be represented as $\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0$, where $z(.) \in R^n$

is called the system state, $u(\cdot) \in R^m$ is called the input, and $y \in R^p$ is the output. The matrix $A \in R^{n\times n}$ is the state matrix, and $B \in R^{m\times n}$ is the input operator. Now if the output is a linear mapping of the state and input, we can represent the system as

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t), \quad z(0) = z_0, \\ y(t) &= Cz(t) + Du(t), \end{aligned} \tag{1.2}$$

where $C \in R^{p\times n}$ and $D \in R^{p\times m}$. This representation is called the state-space realization of the system and is denoted by (A, B, C, D). In many cases the input term does not appear in the output measurement, so we also assume that $D = 0$ and denote the system by (A, B, C). For example, the motion of a mass m under a force $f(t)$ can be modeled by $m\frac{d^2x}{dt^2} + c_1\frac{dx}{dt} = f(t)$, where $x(\cdot)$ is the position of the mass. Taking the position and velocity as the state variables, we obtain a state-space realization for the system with R^2 as its state-space.

The states of the system (1.2) lie in a finite-dimensional vector space and hence it is called a finite-dimensional system. Now consider the system governed by the partial differential equation

$$\begin{cases} \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + u(x,t), \\ z_x(0,t) = 0 = z_x(1,t) \\ z(x,0) = z_0(x), \quad x \in [0,1], \end{cases} \tag{1.3}$$

where $u(x, \cdot)$ is a continuous function of x. At each time t, the solution $z(\cdot, t)$ lies in $\{z \in H^2(0,1); z'(0) = z'(1) = 0\}$ which is an infinite-dimensional vector space. Note that $H^2(0,1)$ is the space of all functions whose derivatives up to the second order are square integrable on $(0,1)$. In general, any state-space realization whose state-space is an infinite-dimensional vector space is called an infinite-dimensional system. Systems governed by partial differential equations and delay equations are examples of infinite-dimensional systems. The function $u(x,t)$ in equation (1.3) is the source term, or the input, and is applied through the whole region $x \in [0,1]$.

There are systems governed by partial differential equations for which the input is applied only through the boundary and the output is also taken from the boundary. These systems are called boundary control systems. For example, consider the heat equation system

$$\begin{cases} \dot{z}(t) = \frac{\partial^2 z}{\partial x^2}, \\ z_x(0,t) = u(t), \\ z_x(1,t) = 0 \\ z(x,0) = z_0(x), \quad x \in [0,1], \\ z(0,t) = y(t). \end{cases} \tag{1.4}$$

Root-locus method, stability analysis, and locating poles and zeros of finite-dimensional linear time-invariant systems are classical subjects in control theory and can be found in many books [11, 18, 21]. However, studying these subjects for infinite-dimensional systems is an open research area with many unsolved problems. One problem with infinite dimensional systems is that their transfer function may not be bounded at infinity and hence many standard arguments, required to determine the asymptotic behavior of the roots, are not generally valid [4]. Some background knowledge on infinite-dimensional systems can be found in [10, 17]. In the following section, we outline some recent advances on the zeros of infinite-dimensional systems.

1.2 Recent Advances on the Zeros of Infinite Dimensional Systems

The zeros of transfer functions for systems governed by partial differential equations are studied in e.g. [4, 5, 7, 12, 23, 30].

In [4], Byrnes et al. developed a root-locus analysis for a special class of boundary control systems with first order time derivative and linear n'th-order spatial derivative in a bounded one-dimensional space. In their work, they assumed that n is even and the derivative of order n-1 does not appear in the spatial operator. Further, the input and output operators are co-located in the sense that their highest order derivatives occur at the same point. The state operator in their work was originally studied by Birkhoff in [1], where he obtained the asymptotic eigenvalues and eigenfunctions of this class of operators with suitable homogeneous boundary conditions. Byrnes et al. showed that infinitely many root-locus branches of the considered system lie on the negative real axis and the remaining finitely many branches can be embedded within a finite closed contour. Elaborating the results in [4] for second-order systems is the main part of this thesis.

In [23], Pohjolainen studied a state-space system (A, B, C) with a linear self-adjoint state operator on a Hilbert space with bounded linear input and output operators. He proved a necessary and sufficient condition for $\lambda \in \mathbb{C}$ to be a finite eigenvalue of $A + kBC$ when k tends to infinity.

A finite-dimensional system is minimum-phase if all zeros of the system lie in the left-half s-plane. A bounded analytic function $g \in \mathcal{H}^\infty(\mathbb{C}_0; \mathbb{C}^m)$ on the closed right-half plane is minimum phase if for all functions $f \in$

$\mathcal{H}^2(\mathbb{C}_0; \mathbb{C}^m)$, the operator $\Lambda f = gf$ defined on $\mathcal{H}^2(\mathbb{C}_0; \mathbb{C}^m)$ has a dense range in $\mathcal{H}^2(\mathbb{C}_0; \mathbb{C}^m)$. Note that

$$\mathcal{H}^\infty(\mathbb{C}_0; \mathbb{C}^m) = \{g : \mathbb{C}_0 \to \mathbb{C}^m, g \text{ is analytic and } \sup_{Re(s)>0} \|g(s)\| < \infty\},$$

and,

$$\mathcal{H}^2(\mathbb{C}_0; \mathbb{C}^m) = \{f : \mathbb{C}_0 \to \mathbb{C}^m, f \text{ is analytic and } \|f\|_2 < \infty\}.$$

For infinite-dimensional systems, besides right-half plane poles, some aspects of the zero dynamics may lead to a nonminimum-phase behavior. (The zero dynamics of a system is the system obtained by setting the output to zero). For example, the transfer function of a pure delay system has no zeros but is not minimum phase [12]. One disadvantage of a non-minimum phase systems is that they are slow in responding and so in designing a system, if fast speed of response is of primary importance, we should not use non-minimum phase components [21]. Furthermore, the sensitivity of a minimum phase system can be reduced to an arbitrarily small level, resulting in good output disturbance rejection [18]. So it is desired to design minimum phase control systems.

In [30], the location of zeros of an infinite-dimensional state-space system with bounded input and output operators is discussed and sufficient conditions for the zeros to be real and negative is given. It is proved that if the state operator is self-adjoint and the output operator is the transpose conjugate of the input operator, i.e. the system is co-located, then the transmission zeros are real. They gave a sufficient condition for the transfer function to have interlacing zeros and poles on the negative real axis.

The invariant zeros of the system (A, B, C) are the set of all $\lambda \in \mathbb{C}$ such that the system of equations $(\lambda I - A)x + Bu = 0$, $Cx = 0$ has a solution for $u \in U$ and non-zero $x \in D(A)$. A subspace $Z \subseteq H$ is $A-invariant$ if $A(Z \cap D(A)) \subset Z$. A subspace $Z \subseteq H$ is feedback invariant if Z is closed and there exists an A-bounded feedback K such that Z is $A+BK$-invariant. A subspace $Z \subseteq H$ is open-loop invariant if for every $x_0 \in Z$ there exists an input $u \in C([0,\infty); U)$ such that the solution of $\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0 \in H,\ u \in U$ remains in Z. In [29], Zwart discussed the concept of open-loop and closed-loop invariance for infinite-dimensional systems and proved that they are equivalent for closed linear subspaces. In [19], SISO systems on a Hilbert space H with bounded input and output operators are studied. It is shown that for any feedback gain matrix K, the set of eigenvalues of $A + BK$ is identical to the set of invariant zeros of the system.

In [5], Byrnes et. al. studied the problem of output regulation for a co-located boundary control system governed by a PDE with first order time derivative and second order self-adjoint elliptic spatial operator on a bounded region in R^n. It is assumed that the zero-dynamics of the system is asymptotically stable, that is, the system is minimum phase. Under this assumption, the input signal guaranteeing asymptotically perfect tracking of a reference output is obtained.

In [24], conditions for continuity of the spectrum are derived for a class of well-posed boundary control systems. Rebarber and Townley in this paper studied the robustness of the spectrum of the closed-loop system operator and hence the closed-loop poles for this class of systems. Under some conditions, they derived a stability radius for robustness of the closed-loop poles. They also showed that under some conditions, the closed-loop behavior of the system approaches the system zero-dynamics as the feedback gain tends to infinity.

In [12], Jacob et al. studied a class of systems governed by a second order PDE. They derived a sufficient condition for the system that guarantees the minimum-phase property of the transfer function with either position or velocity measurements.

For some classes of systems, determining the poles and zeros of infinite-dimensional systems is closely related to locating the zeros of analytic functions. In [14], an algorithm for approximating the zeros of an analytic function is given. This algorithm also gives the multiplicity of the zeros. The problem of finding zeros of analytic functions is also studied in [2]. However, transfer functions of infinite-dimensional systems are difficult to compute in general.

1.3 Summary and Organization

The organization of this thesis is as follows. In chapter 2, first we present preliminary notions for infinite-dimensional systems. We introduce semigroup theory as the starting point of studying infinite-dimensional systems. Then, we introduce boundary control systems. In the next section, we formulate the problem of our interest [4], where we analyze the open-loop system and derive some key results about the open-loop transfer function. In the last section of chapter 2, we introduce the concept of stability, and we show that for all finite-dimensional state-space representations and some classes of infinite-dimensional systems the stability is associated with the poles of the transfer function. The material in this chapter are generally based on [6, 10, 17, 18].

We begin chapter 3 with an introduction to the root-locus theory for finite-dimensional systems. This finite-dimensional root-locus analysis is based on [11, 18]. In this section, we derive some results on the closed-loop stability of a feedback system when the feedback gain is sufficiently large. The main part of chapter 3 discusses the generalization of root-locus method to the class of infinite-dimensional systems defined in chapter 2. The stability of the closed-loop system is also discussed and a necessary and sufficient condition is given for the closed-loop system to be stable. The infinite-dimensional root-locus theory in this thesis is based on [4]. Finally, the thesis is concluded in chapter 4.

Chapter 2

Infinite-Dimensional Systems

Many problems arising in control systems are in infinite dimensional spaces. For example systems governed by partial differential equations and delay systems are infinite-dimensional systems. In this thesis, we deal with systems of partial differential equations and so this chapter is devoted to providing a background on infinite dimensional systems.

This chapter is organized as follows. In section 2.1, we introduce semigroups of operators, and present some important theorems in this context. The material in this section is generally based on [10]. In section 2.2, boundary control systems(BCS's) are described and well-posed BCS's are defined. The transfer function of BCS's is also characterized in this section. The material in this section is mainly based on [6, 7, 8]. Section 2.3 of this chapter is dedicated to the special class of boundary control systems considered in [4]. After defining the system, some of its properties are analyzed. Finally, in section 2.4 stability of finite and infinite-dimensional systems is discussed. This section is based on [18, 17]

2.1 Semigroup Theory

Consider the linear time-invariant state-space realization(SSR)

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0, \tag{2.1}$$

$$y(t) = Cz(t), \tag{2.2}$$

where $z(\cdot) \in R^n$ is the system state, $u(\cdot) \in R^m$ is the input, and $y \in R^p$ is the output. The matrix $A \in R^{n\times n}$ is the state matrix, and $B \in R^{m\times n}$ is the input operator. Also, $C \in R^{p\times n}$. From the theory of ordinary differential equations, the state of this system can be represented as

$$z(t) = e^{At}z_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \quad t > 0, \tag{2.3}$$

where

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}. \tag{2.4}$$

But there are many cases where the state-space is an infinite-dimensional space, as in the following example.

Example 2.1. *Consider the 1-D heat equation*

$$\begin{cases} \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + u(x,t), \\ z_x(0,t) = 0 = z_x(1,t) \\ z(x,0) = z_0(x), \quad x \in [0,1] \end{cases} \tag{2.5}$$

The solution of (2.5) is

$$z(x,t) = \int_0^1 g(t,x,y)z_0(y)dy + \int_0^t \int_0^1 g(t-s,x,y)u(y,s)dyds, \tag{2.6}$$

where $g(t,x,y)$ is the Green's function

$$g(t,x,y) = 1 + \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t}\cos n\pi x \cos n\pi y. \tag{2.7}$$

Now, if we define the operator $T(t)z(x) = \int_0^1 g(t,x,y)z(y)dy$*, then we can rewrite (2.6) as*

$$z(x,t) = T(t)z_0(x) + \int_0^t T(t-\tau)u(\tau)d\tau. \tag{2.8}$$

Clearly, this equation is very similar to equation (2.3) with e^{At} *replaced by* $T(t)$*. If we define* $B = I$*, and* $A = \partial^2/\partial x^2$ *with* $D(A) = \{z \in H^2(0,1), \frac{dz}{dx}(0) = 0 = \frac{dz}{dx}(1)\}$*, we observe that the system (2.5) is formulated as an abstract differential equation (2.1) on the infinite-dimensional state-space* $L_2(0,1)$ *and the solution is given by (2.8).*

This example shows that we need to generalize the notion of matrix exponential e^{At} to deal with problems in more general spaces where A is unbounded. Semigroup theory gives an abstract framework to derive solutions for equations of the form (2.1) when A is an unbounded operator on an infinite-dimensional vector space.

Assume the system dynamics from the initial state z_0 to $z(t)$ are linear, time-invariant and autonomous. For each time $t \geq 0$, a linear operator $T(t)$ on a Hilbert space H can be defined as follows

$$\begin{aligned} T(t) : H \to H, \quad T(0) = I \\ z(t) = T(t)z_0, \quad \forall z_0 \in H \end{aligned} \tag{2.9}$$

We assume that the state of the system is continuous at time $t = 0$. We also assume that for all $z_0 \in Z$ the state of the system after time $t + s$ with the initial condition z_0 is the same as that after time t with the initial condition $z(s) = T(s)z_0$. This leads to the following important property of the map $T(t)$:

$$T(t+s) = T(t)T(s) \quad \forall t, s \geq 0 \tag{2.10}$$

Such a family of dynamical systems can be described using the concept of strongly continuous semigroups:

Definition 2.2. *Consider an operator-valued function* $T(t) : R^+ \rightarrow \mathcal{L}(X, X)$ *where* X *is a Banach space and* $\mathcal{L}(X, X)$ *is the space of all bounded linear operators on* X*. T(t) is called a semigroup on* X *if it satisfies the following properties:*

(a) $T(t + s) = T(t)T(s) \quad \forall t, s \geq 0,$

(b) $T(0) = I.$

A semigroup $T(t)$ *is uniformly continuous if it satisfies*

$$\lim_{t\rightarrow 0^+} \| T(t) - I \| = 0, \tag{2.11}$$

and is strongly continuous if

$$\lim_{t\rightarrow 0^+} \| T(t)z_0 - z_0 \| = 0, \quad \forall z_0 \in X. \tag{2.12}$$

The notation C_0-semigroup is used for a strongly continuous semigroup.

Remark 2.3. *A* C_0*-semigroup is a special case of a semigroup. In general, a set* G *equipped with a binary operation* $\circ : G \times G \mapsto G$ *is called a semigroup if it is associative and possesses the identity element. It can be verified that the set* $G = \{T(t); t \geq 0\}$ *equipped with the binary operator* $T(t) \circ T(s) = T(t+s)$ *forms a semigroup.*

Example 2.4. *Let* A *be a bounded linear operator on a Banach space* X*. First we prove that the series* $\sum_{n=0}^{\infty} \frac{(At)^n}{n!}$ *is convergent in* $\mathcal{L}(X, X)$ *and then* *we show that it forms a semigroup on X. For $M > N$, we have that*

$$\begin{aligned} \|\sum_{n=0}^{M}\frac{(At)^n}{n!} - \sum_{n=0}^{N}\frac{(At)^n}{n!}\| &= \|\sum_{n=N+1}^{M}\frac{(At)^n}{n!}\| \\ &\le \sum_{n=N+1}^{M}\|\frac{(At)^n}{n!}\| \\ &\le \sum_{n=N+1}^{M}\frac{\|A\|^n t^n}{n!}. \end{aligned}$$

The series $\sum_{n=0}^{\infty}\frac{\|A\|^n t^n}{n!}$ is convergent because it is the Taylor series for $e^{\|A\|t}$. Therefore, the sequence $\{\sum_{i=0}^{n}\frac{\|A\|^i t^i}{i!}, n \ge 1\}$ is Cauchy. Thus, for any $\epsilon > 0$, there is some integer $N_1 > 0$ such that for all $M > N > N_1$,

$$\|\sum_{n=0}^{M}\frac{\|A\|^n t^n}{n!} - \sum_{n=0}^{N}\frac{\|A\|^n t^n}{n!}\| < \epsilon, \tag{2.13}$$

which implies

$$\|\sum_{n=0}^{M}\frac{(At)^n}{n!} - \sum_{n=0}^{N}\frac{(At)^n}{n!}\| < \epsilon. \tag{2.14}$$

Thus, the sequence $\{\sum_{i=0}^{n}\frac{(At)^i}{i!}, n \ge 1\}$ is Cauchy and by completeness of $\mathcal{L}(X, X)$ converges to some point in $\mathcal{L}(X, X)$. The series $\sum_{i=0}^{\infty}\frac{(At)^i}{i!}$ can be considered as a generalization of the matrix exponential. Hence, we define

$$e^{At} = \sum_{n=0}^{\infty}\frac{(At)^n}{n!} \tag{2.15}$$

Now we prove that e^{At} *is a semigroup on X. It's clear that* $e^{A(0)} = I$. *Consider*

$$\begin{aligned}
\|e^{At}z_0 - z_0\| &= \|\sum_{n=1}^{\infty} \frac{(At)^n}{n!} z_0\| \\
&\leq \sum_{n=1}^{\infty} \|\frac{(At)^n}{n!} z_0\| \\
&\leq \sum_{n=1}^{\infty} \frac{\|A\|^n t^n}{n!} \|z_0\| \\
&= (e^{\|A\|t} - 1)\|z_0\|.
\end{aligned}$$

The last term tends to zero as t approaches zero and hence strong continuity at zero is satisfied. Now take any $t, s \geq 0$. *We have*

$$\begin{aligned}
e^{A(t+s)} &= \sum_{n=0}^{\infty} \frac{(A(t+s))^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{k=0}^{n} C_k^n t^k s^{n-k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{A^k t^k A^{n-k} t^{n-k}}{k!(n-k)!} \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{A^k t^k A^{n-k} t^{n-k}}{k!(n-k)!} \\
&= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \sum_{n=k}^{\infty} \frac{A^{n-k} t^{n-k}}{(n-k)!} \\
&= e^{At} e^{As}.
\end{aligned}$$

Hence, by definition, the operator-valued function e^{At} *forms a* C_0*-semigroup on X.*

Before we begin another example, which will be applicable in our work, we recall the Generalized Fourier Series theorem.

Theorem 2.5. ***Generalized Fourier Series*** *[15]*

Let H be a separable Hilbert space with an orthonormal sequence $\{\phi_n\}$. The following statements are equivalent:

(1) $\{\phi_n\}$ is maximal, that is, $\langle x, \phi_n \rangle = 0 \ \forall n$ implies $x = 0$.

(2) For all $x \in H$, $x = \sum_{n=1}^{\infty} \langle x, \phi_n \rangle \phi_n$.

(3) For all $x \in H$, $\|x\|^2 = \sum_{n=1}^{\infty} \langle x, \phi_n \rangle^2$.

A maximal orthonormal sequence in a separable Hilbert space is called an orthonormal basis.

In the following example, we describe a class of semigroups on an infinite-dimensional Hilbert space.

Example 2.6. *Let H be a separable Hilbert space with an orthonormal basis $\{\phi_n, n \geq 1\}$, and let $\{\lambda_n, n \geq 1\}$ be a sequence in $\mathbb{C}$ with*

$$\sup_{n \geq 1} Re(\lambda_n) = \tilde{\lambda} < \infty. \tag{2.16}$$

Define the operator

$$T(t)z = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle z, \phi_n \rangle \phi_n. \tag{2.17}$$

First we show that $T(t)$ is a bounded linear operator on H. Linearity can be *easily verified. In order to prove boundedness, take any $z \in H$. We have that*

$$\begin{aligned}
\|T(t)z\|^2 &= \|\sum_{n=1}^{\infty} e^{\lambda_n t}\langle z, \phi_n\rangle\phi_n\|^2 \\
&= \sum_{n=1}^{\infty} |e^{\lambda_n t}|^2 |\langle z, \phi_n\rangle|^2 \\
&= \sum_{n=1}^{\infty} e^{2Re(\lambda_n)t} |\langle z, \phi_n\rangle|^2 \\
&\leq \sum_{n=1}^{\infty} e^{2\tilde{\lambda}t} |\langle z, \phi_n\rangle|^2 \\
&= e^{2\tilde{\lambda}t}\|z\|^2,
\end{aligned}$$

Thus,

$$\|T(t)\| \leq e^{\tilde{\lambda}t}, \tag{2.18}$$

which implies that $T(t)$ is a bounded operator on H. Now, we can prove that $T(t)$ is a semigroup on H: For any $z \in H$, $T(0)z = z$ and hence $T(0) = I$. Furthermore, for any $t, s \geq 0$ and $z \in H$,

$$\begin{aligned}
T(t)T(s)z &= \sum_{n=1}^{\infty} e^{\lambda_n t}\langle T(s)z, \phi_n\rangle\phi_n \\
&= \sum_{n=1}^{\infty} e^{\lambda_n t}\langle \left(\sum_{j=1}^{\infty} e^{\lambda_j s}\langle z, \phi_j\rangle\phi_j\right), \phi_n\rangle\phi_n \\
&= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} e^{\lambda_n t} e^{\lambda_j s}\langle z, \phi_j\rangle\langle\phi_j, \phi_n\rangle\phi_n \\
&= \sum_{n=1}^{\infty} e^{\lambda_n (t+s)}\langle z, \phi_n\rangle\phi_n \\
&= T(t+s)z,
\end{aligned}$$

which implies that

$$T(t)T(s) = T(t+s), \quad \forall t, s \geq 0. \tag{2.19}$$

Now we prove strong continuity at $t = 0$. For any $z \in H$, we have

$$\begin{aligned} \|T(t)z - z\|^2 &= \|\sum_{n=1}^{\infty} \left(e^{\lambda_n t} - 1\right) \langle z, \phi_n \rangle \phi_n\|^2 \\ &= \sum_{n=1}^{\infty} \left|e^{\lambda_n t} - 1\right|^2 |\langle z, \phi_n \rangle|^2. \end{aligned} \tag{2.20}$$

Since $\sum_{n=1}^{\infty} |\langle z, \phi_n \rangle|^2$ is a convergent sequence, for any $\epsilon > 0$ there exists $N_1 > 0$ such that $\sum_{n=N+1}^{\infty} |\langle z, \phi_n > |^2 < \epsilon$. The sequence $\{|e^{\lambda_n t} - 1|^2\}$ is also a bounded sequence on $0 \leq t \leq 1$ and hence there exists $N > 0$ such that $\sum_{n=N+1}^{\infty} \left|e^{\lambda_n t} - 1\right|^2 |\langle z, \phi_n \rangle|^2 < \epsilon$. On the other hand,

$$\lim_{t \downarrow 0} \sum_{n=1}^{N} \left|e^{\lambda_n t} - 1\right|^2 |\langle z, \phi_n \rangle|^2 = 0.$$

Therefore,

$$\lim_{t \downarrow 0} \sum_{n=1}^{\infty} \left|e^{\lambda_n t} - 1\right|^2 |\langle z, \phi_n \rangle|^2 < \epsilon \tag{2.21}$$

that verifies strong continuity at $t = 0$. Thus, $T(t)$ forms a C_0-semigroup on H.

Some important properties of a C_0-semigroup are gathered in the following theorem.

Theorem 2.7. *([10] Theorem 2.1.6) For a C_0-semigroup the following properties hold:*

1. *$T(t)$ is bounded on every finite interval $[0, a]$ for all $a \in R^+$.*

2. *$T(t)$ is strongly continuous for all $t \geq 0$,i.e.,*

$$\lim_{s \to 0^+} \|T(t+s)z_0 - T(t)z_0\| = 0, \quad \forall z_0 \in H.$$

3. $\lim_{t \to 0^+} \|\frac{1}{t}\int_0^t T(s)zds - z\| = 0, \quad \forall z \in H.$

4. *There exists an $\omega_0 \in R$ such that $\omega_0 = \inf_{t>0} \frac{1}{t} \ln \|T(t)\|$, Furthermore, $\omega_0 = \lim_{t \to \infty} \frac{1}{t} \ln \|T(t)\|$. The constant ω_0 is called the growth bound of the semigroup.*

5. *$\forall \omega > \omega_0$, there is a constant $M_\omega > 0$ s.t. for all $t > 0$, $\|T(t)\| \leq M_\omega e^{\omega t}$.*

Definition 2.8. *Let $T(t)$ be a C_0-semigroup on a Hilbert space H. If the operator A satisfies*

$$Az = \lim_{t \to 0^+} \frac{1}{t}(T(t) - I)z, \tag{2.22}$$

then A is called the infinitesimal generator of the semigroup. The domain of A, denoted by $D(A)$, contains all elements $z \in H$ for which this limit exist.

Example 2.9. *Consider the semigroup $e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$ defined in Example 2.4. For any $z \in H$,*

$$\begin{aligned} \frac{1}{t}\left(e^{At}z - z\right) &= \frac{1}{t}\sum_{n=1}^{\infty} \frac{(At)^n}{n!} z, \\ &= \sum_{n=1}^{\infty} \frac{A^n t^{n-1}}{n!} z. \end{aligned}$$

As t approaches zero only the first term of the series remains nonzero. Thus,

$$\lim_{t \to 0} \frac{1}{t}\left(e^{At}z - z\right) = Az, \quad \forall z \in H. \tag{2.23}$$

Therefore, A is the infinitesimal generator of the C_0-semigroup e^{At} with $D(A) = H$.

Example 2.10. *Consider the semigroup $T(t)$ defined in Example 2.6. For any $z \in H$,*

$$\begin{aligned}
\lim_{t\to 0} \frac{1}{t}(T(t)z - z) &= \lim_{t\to 0} \sum_{n=1}^{\infty} \frac{e^{\lambda_n t} - 1}{t} \langle z, \phi_n\rangle \phi_n \\
&= \sum_{n=1}^{\infty} \lim_{t\to 0} \frac{e^{\lambda_n t} - 1}{t} \langle z, \phi_n\rangle \phi_n \\
&= \sum_{n=1}^{\infty} \lambda_n \langle z, \phi_n\rangle \phi_n.
\end{aligned}$$

Thus the operator

$$Az = \sum_{n=1}^{\infty} \lambda_n \langle z, \phi_n\rangle \phi_n \tag{2.24}$$

with domain

$$D(A) = \{z \in H; \sum_{n=1}^{\infty} |\lambda_n \langle z, \phi_n\rangle|^2 < \infty\} \tag{2.25}$$

is the infinitesimal generator of the C_0-semigroup $T(t)$.

As we mentioned earlier, semigroup theory has been developed to solve problems of the form $\dot{z}(t) = Az$ where A is an unbounded operator. The following theorem relates any semigroup T(t) with an infinitesimal generator A and the solution of the problem $\dot{z}(t) = Az$.

Theorem 2.11. *([10] Theorem 2.1.10) Let $T(t)$ be a C_0-semigroup on a Hilbert space H with infinitesimal generator A. The following properties hold:*

1. *$z_0 \in D(A)$ implies $T(t)z_0 \in D(A)$, $\forall t \geq 0$.*
2. *$\frac{d}{dt}(T(t)z_0) = AT(t)z_0 = T(t)Az_0$, for all $z_0 \in D(A)$, $t > 0$.*

3. The operator A is densely defined, that is, $D(A)$ is dense in H.

4. The operator A is a closed linear operator, where closedness means that the set $\{(z, Az); z \in D(A)\}$ is closed in $H \times H$.

This theorem guarantees that $z(t) = T(t)z_0$ solves $\dot{z} = Az$ with $z(0) = z_0$ if A is the infinitesimal generator of the C_0-semigroup T(t). The question is whether or not we can make sure that a given operator A is the infinitesimal generator of a C_0-semigroup. The Hille-Yosida theorem below gives us the answer. First we define the resolvent set and spectrum of an operator, used frequently in our work.

Definition 2.12. *The resolvent set for an operator A, denoted by $\rho(A)$, is the set of all complex numbers λ for which the $\lambda I - A$ has a bounded inverse. The inverse operator $(\lambda I - A)^{-1}$ is called the resolvent operator and denoted by $R(\lambda, A)$. The spectrum of A, denoted by $\sigma(A)$, is the set of all complex numbers that do not belong to $\rho(A)$.*

The Hille-Yosida theorem gives necessary and sufficient conditions for an operator A to be the infinitesimal generator of a C_0-semigroup.

Theorem 2.13. ***Hille-Yosida Theorem****.([10] Theorem 2.1.12)*

Let A be a closed, densely defined, linear operator on a Banach space X. A is the infinitesimal generator of a C_0-semigroup if and only if there exist real numbers M, ω such that, for all $\alpha \in \mathbb{C}$ satisfying $Re(\alpha) > \omega$, we have $\alpha \in \rho(A)$, and

$$\|R(\alpha, A)^r\| \leq \frac{M}{(Re(\alpha) - \omega)^r}, \quad \forall r \geq 1. \tag{2.26}$$

In this case,

$$\|T(t)\| \leq Me^{\omega t}. \tag{2.27}$$

In the following example we use this theorem to show that each operator A in the class defined in example 2.10 is the infinitesimal generator of a C_0-semigroup.

Example 2.14. *Let H be a separable Hilbert space with an orthonormal basis $\{\phi_n, n \geq 1\}$, and assume that $\{\lambda_n, n \geq 1\}$ is a sequence in $\mathbb{C}$ satisfying*

$$\sup_{n\geq 1} Re(\lambda_n) = \tilde{\lambda} < \infty. \tag{2.28}$$

We will prove that the operator

$$Az = \sum_{n=1}^{\infty} \lambda_n \langle z, \phi_n \rangle \phi_n \tag{2.29}$$

with domain

$$D(A) = \{z \in H; \sum_{n=1}^{\infty} |\lambda_n \langle z, \phi_n \rangle|^2 < \infty\} \tag{2.30}$$

is the infinitesimal generator of a C_0-semigroup.

First we verify that A is a closed, densely defined linear operator on H. Linearity is obvious. In order to prove that A is densely defined, take any $z \in H$ and represent it as

$$z = \sum_{j=1}^{\infty} \langle z, \phi_j \rangle \phi_j. \tag{2.31}$$

Construct the sequence $\{z_n, n \geq 1\}$ such that

$$z_n = \sum_{j=1}^{n} \langle z, \phi_j \rangle \phi_j. \tag{2.32}$$

The sequence $\{z_n\}$ belongs to $D(A)$ and is convergent to z. Thus for any $z \in H$ there is a sequence in $D(A)$ converging to z, meaning that $D(A)$ is dense in H.

Now we must prove that A is a closed operator, that is, if the sequence $\{(x_n, Ax_n)\}$ with $x_n \in D(A)$ converges to $(x, y) \in H \times H$, then $x \in D(A)$ and $y = Ax$. Since $\{x_n\}$ converges to x, $\|x_n\|$ converges to $\|x\|$ in R^+. It can be shown that any convergent sequence in R is bounded. Therefore $\|Ax_n\|$ is bounded which means

$$\sum_{j=1}^{\infty}(\lambda_j\langle x_n, \phi_j\rangle)^2 < \infty,\ n \geq 1. \tag{2.33}$$

Thus

$$\sum_{j=1}^{\infty}(\lambda_j\langle x, \phi_j\rangle)^2 < \infty,\ n \geq 1. \tag{2.34}$$

Hence $x \in D(A)$ and $y = Ax$. Consequently, the operator A is a closed, densely defined linear operator on H.

Now we shall prove that all $\lambda \in \mathbb{C}$ with $Re(\lambda) > \tilde{\lambda}$ belong to the resolvent set of A. Take any $z \in D(A)$ and $\lambda \in \mathbb{C}$ such that $\inf_{n\geq 1}|\lambda - \lambda_n| > 0$. We have

$$(\lambda I - A)z = \sum_{n=1}^{\infty}(\lambda - \lambda_n)\langle z, \phi_n\rangle\phi_n. \tag{2.35}$$

Define the following operator:

$$A_\lambda x = \sum_{n=1}^{\infty}\frac{1}{(\lambda - \lambda_n)}\langle x, \phi_n\rangle\phi_n. \tag{2.36}$$

We claim that the operator A_λ is bounded on $D(A_\lambda)$: First recall that

$$\sup_{n\geq 1}\left|\frac{1}{\lambda - \lambda_n}\right| = \frac{1}{\inf_{n\geq 1}|\lambda - \lambda_n|} \tag{2.37}$$

which is bounded by assumption. Thus we have

$$\begin{aligned}\|A_\lambda x\|^2 &= \sum_{n=1}^{\infty}\left|\frac{1}{\lambda-\lambda_n}\right|^2\langle x,\phi_n\rangle^2\\ &\leq \sum_{n=1}^{\infty}\left|\sup_{n\geq 1}\frac{1}{\lambda-\lambda_n}\right|^2\langle x,\phi_n\rangle^2\\ &= \left|\sup_{n\geq 1}\frac{1}{\lambda-\lambda_n}\right|^2\|x\|^2,\end{aligned}$$

which implies that the operator A_λ is bounded. Then, for all $z\in D(A)$,

$$A_\lambda(\lambda I-A)z=(\lambda I-A)A_\lambda z=z. \tag{2.38}$$

Thus A_λ is the inverse of $(\lambda I-A)$ and hence it is the resolvent operator $R(\lambda,A)$. Therefore, for all $Re(\lambda)>\tilde{\lambda}$, λ belongs to the resolvent set of A. By simple calculations it can be shown that

$$R(\lambda,A)^r x=\sum_{n=1}^{\infty}\frac{1}{(\lambda-\lambda_n)^r}\langle x,\phi_n\rangle\phi_n, \forall x\in H \tag{2.39}$$

and hence

$$\begin{aligned}\|R(\lambda,A)^r\| &\leq \sup_{n\geq 1}\frac{1}{|\lambda-\lambda_n|^r}\\ &\leq \sup_{n\geq 1}\frac{1}{|Re(\lambda)-\lambda_n|^r}\\ &= \left(\sup_{n\geq 1}\frac{1}{|Re(\lambda)-\lambda_n|}\right)^r.\end{aligned}$$

For any real $\omega\geq\tilde{\lambda}$, any λ with $Re(\lambda)>\omega$, and any $r\geq 1$ we have

$$\left(\sup_{n\geq 1}\frac{1}{|Re(\lambda)-\lambda_n|}\right)^r\leq\frac{1}{(Re(\lambda)-\omega)^r}. \tag{2.40}$$

Therefore,

$$\|R(\lambda,A)^r\|\leq\frac{1}{(Re(\lambda)-\omega)^r}\ \forall r\geq 1, Re(\lambda)>\omega. \tag{2.41}$$

Thus, by the Hille-Yosida theorem A is the infinitesimal generator of a C_0-semigroup satisfying $\|T(t)\| \leq e^{\omega t}$. Now we wish to calculate this semigroup. From Theorem 2.11 we have that

$$\begin{aligned} \frac{d}{dt}T(t)\phi_n &= T(t)A\phi_n, \\ &= \lambda_n T(t)\phi_n. \end{aligned}$$

From this and the fact that $T(0) = I$, we obtain $T(t)\phi_n = e^{\lambda_n t}\phi_n$. Since $T(t)$ is linear and bounded and $\{\phi_n\}$ forms an orthonormal basis in H, for any $z \in H$ we have

$$T(t)z = \sum_{n=1}^{\infty} e^{\lambda_n t}\langle z, \phi_n\rangle \phi_n, \tag{2.42}$$

previously considered in Example 2.6.

The one-dimensional heat equation considered in Example 2.1 is a special case of the above example with $\lambda_n = -n^2\pi^2$ and $\phi_n(x) = \sqrt{2}\cos(n\pi x)$ for $n \geq 1$, and $\lambda_0 = 0$ with $\phi_0(x) = 1$. Also, $\langle \cdot, \cdot \rangle$ is the inner product on $L_2(0,1)$. Since the set $\{1, \sqrt{2}\cos(n\pi x), n \geq 1\}$ is an orthonormal basis for $L_2(0,1)$, the operator $T(t)$ defined in Example 2.1 is a C_0-semigroup on $L_2(0,1)$.

While the Hille-Yosida theorem is a strong theorem, it's not always easy to verify (2.26) for suitable M, ω. On the other hand, in most applications, we usually wish to know whether a given operator is the infinitesimal generator of a C_0-semigroup on a Hilbert space. Thus, only establishing a sufficient condition for the operator to be the infinitesimal generator of a C_0-semigroup on a Hilbert space would suffice. The Lumer-Phillips theorem gives a useful sufficient condition.

Theorem 2.15. *Lumer-Phillips Theorem ([10] Corollary 2.2.3)*

Let A be a closed, densely defined operator on a Banach space X. The operator A is the infinitesimal generator of a C_0-semigroup if there exists a real number ω such that

$$Re\langle A\phi, \phi\rangle \leq \omega\|\phi\|^2, \quad \forall\phi \in D(A), \tag{2.43}$$

$$Re\langle A^*\psi, \psi\rangle \leq \omega\|\psi\|^2, \quad \forall\psi \in D(A^*), \tag{2.44}$$

where A^* denotes the adjoint of A. In this case, the corresponding semigroup $T(t)$ satisfies

$$\|T(t)\| \leq e^{\omega t}. \tag{2.45}$$

Example 2.16. *Consider the operator A defined in Example 2.14. It can be verified that for $\omega = \sup_{n\geq 1} Re(\lambda_n)$ the inequalities (2.43)-(2.44) are satisfied, implying that A is the infinitesimal generator of a C_0-semigroup that satisfies the inequality (2.45).*

2.2 Boundary Control Systems

A boundary control system(BCS) is a system in which the input signal and the output measurement occur on the boundary. An abstract representation of such a system is

$$\dot{z}(t) = \mathcal{A}z, \; z(0) = z_0 \tag{2.46}$$

$$\mathcal{B}z(t) = u(t), \tag{2.47}$$

$$\mathcal{C}z(t) = y(t), \tag{2.48}$$

where $z(x,t)$ represents the state of the system, $u(t)$ and $y(t)$ are the input and output, respectively. The operator $\mathcal{A} \in \mathcal{L}(Z, H)$ is the state operator, where $\mathcal{L}(Z, H)$ represents the space of bounded linear operators from Z into H. The operator $\mathcal{B} \in \mathcal{L}(Z, U)$ is the input operator, and $\mathcal{C} \in \mathcal{L}(Z, Y)$ is the output operator. The spaces Z, H, U, Y are all Hilbert spaces and Z is a dense subspace of H. We denote the boundary control system (2.46)-(2.48) by $(\mathcal{A}, \mathcal{B}, \mathcal{C})$.

Example 2.17. *Consider temperature control along a bar of length 1 with Neumann boundary conditions. The mathematical representation of this system is*

$$\begin{cases} \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}, \quad x \in (0,1) \\ z(x,0) = 0, \quad x \in [0,1] \\ \frac{\partial z}{\partial x}(0,t) = 0, \\ \frac{\partial z}{\partial x}(1,t) = u(t), \\ y(t) = z(x_1, t), \quad 0 < x_1 \leq 1. \end{cases} \tag{2.49}$$

By defining $\mathcal{A}z = \frac{\partial^2 z}{\partial x^2}$ *with* $\mathcal{D}(\mathcal{A}) = \{z \in H^2[0,1]; z_x(0) = 0 = z_x(1)\}$, $\mathcal{B}z(t) = \frac{\partial}{\partial x} z(1,t)$, *and* $\mathcal{C}z(t) = z(x_1, t)$, *we obtain the standard form of a boundary control system defined by equations (2.46)-(2.48).*

2.2.1 Well-posedness

A system is well-posed if the state and the output of the system continuously depend on the initial state and input. Salamon was the first to discuss the well-posedness of boundary control systems [26]. He also derived conditions under which a BCS can be represented as a state-space realization. But in general if a BCS is transformed into a state-space realization, the input and output operators in the state-space form are unbounded. For a BCS, the definition of well-posedness can be phrased as follows.

Definition 2.18. *[26] The boundary control system (2.46)-(2.48) is well-posed if the following conditions hold.*

1. *For every initial condition $z_0 \in Z$, if $\mathcal{B}z_0 = 0$, there exists a unique $z(t) \in C^1([0,T], Z)$ which solves (2.46)-(2.48) with $u = 0$ and depends continuously on z_0.*

2. *For every $u \in H^1([0,T], U)$, if $z_0 = 0$ and $\mathcal{B}z_0 = 0$, there exists a unique $z(t) \in C^1([0,T], Z)$ which solves (2.46)-(2.48) and depends continuously on u.*

3. *There exists a constant $c > 0$ such that for all $z_0 \in Z$ with $\mathcal{B}z_0 = 0$*
$$\int_0^T \|\mathcal{C}z(t; z_0, 0)\|_Y^2 dt \leq c\|z_0\|^2. \tag{2.50}$$

4. *There exists a constant $c > 0$ such that for all $u \in H^2([0,T], U)$,*
$$\int_0^T \|y(t; 0, u)\|_Y^2 dt \leq c \int_0^T \|u(t)\|_U^2 dt. \tag{2.51}$$

The first condition in the definition of well-posedness is satisfied if $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup, say $T(t)$. In this case, the solution to (2.46)-(2.48) if $u = 0$ is $z(t) = T(t)z_0$. Condition (2) implies that the map from input to state is bounded. Condition (3) is equivalent to saying that the map from initial state to output is bounded. Similarly, condition (4) is equivalent to saying that the map from input to output is bounded.

In [27], Salamon proved that every bounded time-invariant, causal, linear input/output operator has a well-posed state-space realization. Thus, in what follows, we will only focus on well-posedness of the input/output map. In the following example, it is shown that for a finite-dimensional state-space realization the input/output map is always bounded.

Example 2.19. *Consider the finite-dimensional state-space realization* (A, B, C)*, where* $A \in R^{n\times n}, B \in R^{n\times m}$*, and* $C \in R^{n\times p}$*. The zero-state solution of this system satisfies*

$$\begin{aligned}\|y(t;0,u)\|_{R^p} &= \|\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau\| \\ &\leq \|C\|\|B\|\sqrt{\int_0^t e^{A(t-\tau)}d\tau}\|u(.)\|_{L_2([0,t];R^m)}.\end{aligned}$$

Thus, for $u \in L_2([0,t];R^m)$*, the output norm is bounded by the input norm up to a constant coefficient. Therefore, the input/output map is bounded for any finite-dimensional state-space realization* (A, B, C)*.*

In general, for an infinite-dimensional control system, if the input and output operators are not chosen properly, the input/output map may not be bounded. In the following subsection, after introducing a necessary and sufficient condition for the boundedness of the input/output map, an example of an ill-posed boundary control system is presented.

2.2.2 Transfer Functions

For a single-input single-output(SISO) linear time-invariant system the transfer function is known as the Laplace transform of the output divided by that of the input. For multiple-input multiple-output(MIMO) systems a transfer matrix can be defined whose entry (i,j) is the Laplace transform of the j'th output divided by that of the i'th input. In general, we define the system transfer function as follows.

Definition 2.20. *Let* $\hat{y}(s)$ *and* $\hat{u}(s)$ *denote the Laplace transform of the output and the input of a system. The input and the output belong to the vector* *space U and Y, respectively. The system transfer function is the operator $G(s)$ that satisfies*

$$\hat{y}(s) = G(s)\hat{u}(s), \tag{2.52}$$

for all complex s with $Re(s) > \sigma$, for some real σ. The transfer function $G(s)$ is proper if for some $\sigma \in R$

$$\sup_{Re(s)>\sigma} \|G(s)\|_{\mathcal{L}(U,Y)} < \infty. \tag{2.53}$$

The transfer function $G(s)$ is strictly proper if it is proper and

$$\lim_{Re(s)\to\infty} \|G(s)\|_{\mathcal{L}(U,Y)} = 0. \tag{2.54}$$

Definition 2.21. *A point $p \in \mathbb{C}$ is called a pole of the transfer function $G(s)$, if $G(s)$ tends to infinity as s tends to p. A point $z \in \mathbb{C}$ is called a zero of the transfer function $G(s)$, if $G(s)$ tends to zero as s tends to z.*

Example 2.22. *Consider the finite-dimensional state-space realization (A, B, C) defined in (2.1)-(2.2). By taking the Laplace transform of the equations we obtain*

$$\begin{aligned} \hat{y}(s) &= C\hat{z}(s) \\ &= C(sI - A)^{-1}B\hat{u}(s) \\ &= \frac{CAdj(sI - A)B}{det(sI - A)}\hat{u}(s). \end{aligned}$$

Thus the transfer matrix is

$$G(s) = \frac{CAdj(sI - A)B}{det(sI - A)}. \tag{2.55}$$

The function $Adj(sI-A)$ is a polynomial of order at most n-1, while $det(sI-A)$ is a polynomial of order n. We can observe that each entry in the transfer *matrix is a rational function of s with the order of its denominator greater than that of its numerator, implying that each entry approaches zero as s approaches infinity. Thus the transfer matrix $G(s)$ satisfies (2.54). Therefore, for any finite-dimensional state-space representation (A, B, C), the transfer function is strictly proper.*

Example 2.23. *Consider the system defined in Example 2.17. By taking the Laplace transform of the system we obtain the system transfer function in the form*

$$G(s) = \frac{e^{\sqrt{s}x_1} + e^{-\sqrt{s}x_1}}{\sqrt{s}(e^{\sqrt{s}} - e^{-\sqrt{s}})}, \quad s \in \mathbb{C},\ Re(s) > 0. \tag{2.56}$$

This system is strictly proper, because

$$\begin{aligned} \lim_{Re(s)\to\infty} |G(s)| &= \lim_{Re(s)\to\infty} \frac{e^{\sqrt{s}x_1}}{\sqrt{s}e^{\sqrt{s}}}, \\ &= \lim_{Re(s)\to\infty} \frac{1}{\sqrt{s}e^{\sqrt{s}(1-x_1)}}, \\ &= 0. \end{aligned}$$

The following theorem relates the transfer function and the input/output map of a system.

Theorem 2.24. *[9] Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ denote a boundary control system. The input/output map of the system is bounded, in the sense of Definition 2.18(4), if and only if the system transfer function is proper.*

An inappropriate choice of the input and boundary operators may lead to an improper transfer function and hence an unbounded input/output map, as in the following example.

Example 2.25. *Consider the 1-D heat equation with Dirichlet boundary conditions:*

$$\begin{cases} \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}, \quad x \in (0,1) \\ z(x,0) = 0, \quad x \in [0,1] \\ \frac{\partial z}{\partial x}(0,t) = 0, \\ z(1,t) = u(t), \\ y(t) = \frac{\partial z}{\partial x}(1,t). \end{cases} \tag{2.57}$$

Taking the Laplace transform of the system yields the system transfer function, which is

$$G(s) = \frac{\sqrt{s}(e^{\sqrt{s}} - e^{-\sqrt{s}})}{e^{\sqrt{s}} + e^{-\sqrt{s}}}. \tag{2.58}$$

The norm of $G(s)$ tends to infinity as $Re(s)$ approaches infinity, implying that the system transfer function is improper. Thus, the input/output map is unbounded.

2.3 A Special Class of Boundary Control Systems

The following class of systems represents a large class of BCS's which we encounter in diffusion control systems.

Consider the system

$$\begin{cases} \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + a(x)z, \quad t > 0, x \in (0,1), \; a(x) \in C^\infty(0,1) \\ \beta \frac{\partial z}{\partial x}(0,t) + \beta_1 z(0,t) + \beta_2 z(1,t) = u(t), \\ \gamma \frac{\partial z}{\partial x}(1,t) + \gamma_1 z(0,t) + \gamma_2 z(1,t) = 0, \\ \alpha z(0,t) = y(t), \\ z(x,0) = f(x), \end{cases} \tag{2.59}$$

where $f \in L_2[0,1]$, $u(t)$ and $y(t)$ are the input and output, respectively. All coefficients are real-valued with $\alpha, \beta, \gamma \neq 0$.

The system (2.59) is a special case of the problem analyzed in [4]. Byrnes and Gilliam in [4] characterized a linear partial differential system with first order time derivative and linear n'th-order spatial derivative in a bounded one-dimensional space. In their work, they assumed that n is even and the derivative of order n-1 does not appear in the spatial operator. Further, the input and output operators are co-located in the sense that their highest order derivatives occur at the same point. Differential operators of this form are first studied by Birkhoff in [1] and he derived the asymptotic eigenvalues and eigenfunctions of this class of linear differential operators with suitable homogeneous boundary conditions.

We can rewrite the system (2.59) as

$$\dot{z}(t) = \mathcal{A}z, z(.,0) = f(.) \tag{2.60}$$

$$\mathcal{B}z(t) = u(t), \tag{2.61}$$

$$\mathcal{C}z(t) = y(t), \tag{2.62}$$

where the state, input and output operators are

$$\mathcal{A}z = \frac{\partial^2 z}{\partial x^2} + a(x)z, \quad x \in (0,1),\ a(x) \in C^\infty(0,1) \tag{2.63}$$

$$\mathcal{B}z = \beta\frac{\partial z}{\partial x}(0) + \beta_1 z(0) + \beta_2 z(1),\ \beta \neq 0 \tag{2.64}$$

$$\mathcal{C}z = \alpha z(0),\ \alpha \neq 0. \tag{2.65}$$

Define the homogeneous boundary operator $\mathcal{W}$ as

$$\mathcal{W}z = \gamma\frac{\partial z}{\partial x}(1) + \gamma_1 z(0) + \gamma_2 z(1),\ \gamma \neq 0. \tag{2.66}$$

The domain $D(\mathcal{A})$ of the operator $\mathcal{A}$ becomes

$$D(\mathcal{A}) = \left\{ z \in H^2(0,1); \mathcal{W}z = 0 \right\}, \tag{2.67}$$

where the Sobolev space $H^2(0,1)$ is the space of functions f whose derivatives up to the second order are in $L_2(0,1)$. As we can see in (2.64) and (2.65), it is assumed that $\beta \neq 0$ and so the order of the input operator $\mathcal{B}$ is greater than that of the output operator $\mathcal{C}$. This assumption guarantees the strictly properness of the system transfer function, as we will see in Proposition 2.33.

2.3.1 Transfer function

In this part some properties of the transfer function for the system (2.59) are stated. Before we assert the essential proposition about the transfer function, we recall a lemma from differential equations. The definition of the order of an entire function is also needed.

Lemma 2.26. *([20],Chapter IV) Let $a(x)$ be a real-valued continuous function. The equation*

$$\frac{\partial^2 g}{\partial x^2} + (a(x) - s)g = 0, \tag{2.68}$$

possesses two unique independent solutions $g_1(x,s), g_2(x,s)$ that are entire functions of s and are real functions of s in the sense that $\overline{g_i(x,s)} = g_i(x,\overline{s}), i = 1,2,$ and satisfy

$$\begin{aligned} g_1(0,s) = 1 \quad &, \quad g_{1_x}(0,s) = 0, \\ g_2(0,s) = 0 \quad &, \quad g_{2_x}(0,s) = 1. \end{aligned} \tag{2.69}$$

It's not easy to find the independent solutions of the equation (2.68) when $a(x)$ is a non-constant function. However, for the case where $a(x)$ is constant, the two unique independent solutions given in Lemma 2.26 can be found. In the following example the solutions satisfying Lemma 2.26 are obtained when $a(x)$ is identically zero. A slight manipulation of the obtained result yields the result when $a(x)$ is a non-zero constant.

Example 2.27. *Consider the equation*

$$\frac{\partial^2 g}{\partial x^2} - sg = 0. \tag{2.70}$$

Define

$$g_1(x,s) = \cosh(\sqrt{s}x), \;\; g_2(x,s) = \frac{\sinh(\sqrt{s}x)}{\sqrt{s}}. \tag{2.71}$$

The Taylor series expansion for $g_1(x,s)$ and $g_2(x,s)$ are

$$g_1(x,s) = \sum_{n=0}^{\infty} \frac{s^n x^{2n}}{(2n)!} \tag{2.72}$$

and

$$g_2(x,s) = \sum_{n=0}^{\infty} \frac{s^n x^{2n+1}}{(2n+1)!} \tag{2.73}$$

that are clearly independent solutions of (2.70) and are entire functions of s. From this series expansion it is also evident that $g_i(x,s)$ are real functions of s and $\overline{g_i(x,s)} = g_i(x,\overline{s})$, $i = 1,2$. Moreover,

$$\begin{aligned} g_1(0,s) = 1 \quad &, \quad g_{1_x}(0,s) = 0, \\ g_2(0,s) = 0 \quad &, \quad g_{2_x}(0,s) = 1, \end{aligned} \tag{2.74}$$

as claimed in Lemma 2.26. We can also show that the solutions $g_1(x,s), g_2(x,s)$ obtained in (2.71) is the unique set of solutions that satisfies (2.69). To this end, assume $\hat{g}_1(x,s), \hat{g}_2(x,s)$ is any other two solutions for (2.70). Then

$$\begin{aligned} \hat{g}_1(x,s) &= a_1(s)g_1(x,s) + a_2(s)g_2(x,s), \\ \hat{g}_2(x,s) &= a_3(s)g_1(x,s) + a_4(s)g_2(x,s), \end{aligned} \tag{2.75}$$

for some functions $a_i(s), i = 1, .., 4$. If $\hat{g}_1(x,s), \hat{g}_2(x,s)$ satisfy (2.69), since $g_1(x,s), g_2(x,s)$ satisfy (2.74), we must have $a_1(s) = 1, a_2(s) = 0, a_3(s) = 0$, and $a_4(s) = 1$. Thus,

$$\hat{g}_1(x,s) = g_1(x,s), \ \hat{g}_2(x,s) = g_2(x,s) \tag{2.76}$$

and the uniqueness result follows.

The order of an entire function is defined in the following example. We observe that it can be interpreted as a growth bound on the absolute value of the function.

Definition 2.28. *([16], Lecture 1) An entire function $f(s), s \in \mathbb{C}$, is said to be of finite order if there exist $\alpha, r > 0$ such that*

$$|f(s)| \leq e^{|s|^\alpha}, \quad \forall |s| > r \tag{2.77}$$

The infimum of all such α, is called the order of $f(s)$, denoted by $\mathbb{O}(f)$.

Example 2.29. *We show that the function $f(s) = se^s$ has order one. To this end, we must verify that for any $\epsilon > 0$,*

(i) there exists $r > 0$ such that for all $s \in \mathbb{C}$ with $|s| > r$, the inequality $|se^s| \leq e^{|s|^{1+\epsilon}}$ holds, and

(ii) for any $r > 0$, there exists an $s \in \mathbb{C}$ with $|s| > r$ such that $|se^s| > e^{|s|^{1-\epsilon}}$.

Write $s = \rho e^{j\phi}$. Then,

$$\begin{aligned} |se^s| &= \rho e^{\rho \cos\phi}, \\ &\leq \rho e^{\rho}, \end{aligned}$$

and,

$$e^{|s|^{1+\epsilon}} = e^{\rho^{1+\epsilon}}.$$

In order to prove that $|se^s| \le e^{|s|^{1+\epsilon}}$ *for large* s *we need to show that* $\rho e^\rho \le e^{\rho^{1+\epsilon}}$ *for large* ρ*. Equivalently, we need to prove that* $f(\rho) = \rho + \ln\rho - \rho^{1+\epsilon} \le 0$ *for large* ρ*. However,* $f'(\rho) = 1 + \frac{1}{\rho} - (1+\epsilon)\rho^\epsilon$*. For any* $\epsilon > 0$ *there exists an* $R > 0$ *such that* $f'(\rho) < 0$ *for all* $\rho > R$*. Hence, for any* $\epsilon > 0$ *there is* $r > 0$ *such that* $f(\rho) < 0$ *for* $\rho > r$*, and* $|se^s| \le e^{|s|^{1+\epsilon}}$*.*

For the proof of (ii), it suffices to show that the function $f(\rho, \phi) = \rho e^{\rho\cos\phi} - e^{\rho^{1-\epsilon}}$ *is positive for sufficiently large* ρ *and* $\phi = 0$*. Since* $\rho > \rho^{1-\epsilon}$ *for* $\rho > 1$*, then* $\ln\rho + \rho > \rho^{1-\epsilon}$ *for* $\rho > 1$*. Thus,* $e^{\ln\rho+\rho} > e^{\rho^{1-\epsilon}}$ *for* $\rho > 1$ *and the result follows.*

Now we can state the properties of the system transfer function for (2.59).

Proposition 2.30. *[4] For the system (2.59) with zero-initial condition, the transfer function has the form*

$$G(s) = \frac{\mathcal{N}(s)}{\mathcal{D}(s)}, \tag{2.78}$$

where $\mathcal{D}$ *and* $\mathcal{N}$ *are entire functions of* s *and* $G(s)$ *is a real function of* s*, i.e.,*

$$G(\overline{s}) = \overline{G(s)}, \tag{2.79}$$

and hence the complex poles and zeros occur in conjugate pairs. Furthermore, the asymptotic form for $\mathcal{N}(s)$ *is*

$$\mathcal{N}(s) = (1 + \mathcal{O}(\frac{1}{\sqrt{s}}))\alpha\gamma\cosh(\sqrt{s}), \tag{2.80}$$

and the asymptotic form for $\mathcal{D}(s)$ *is*

$$\mathcal{D}(s) = -(1 + \mathcal{O}(\frac{1}{\sqrt{s}}))\beta\gamma\sqrt{s}\sinh(\sqrt{s}). \tag{2.81}$$

Thus, $\mathcal{D}$ and $\mathcal{N}$ have order $\frac{1}{2}$ with infinitely many zeros diverging to infinity.

Proof. Take the Laplace transform of the equations in the system (2.59) to obtain

$$\tfrac{\partial^2 \hat{z}}{\partial x^2} + (a(x) - s)\hat{z} = 0,\ \ a(x) \in C^\infty(0,1), \tag{2.82}$$

$$\mathcal{B}\hat{z}(s) = \beta\tfrac{\partial \hat{z}}{\partial x}(0,s) + \beta_1\hat{z}(0,s) + \beta_2\hat{z}(1,s) = \hat{u}(s), \tag{2.83}$$

$$\mathcal{W}\hat{z}(s) = \gamma\tfrac{\partial \hat{z}}{\partial x}(1,s) + \gamma_1\hat{z}(0,s) + \gamma_2\hat{z}(1,s) = 0, \tag{2.84}$$

$$\mathcal{C}\hat{z}(s) = \alpha\hat{z}(0,s) = \hat{y}(s) \tag{2.85}$$

By Lemma 2.26, there is a unique set of independent solutions $z_1(x,s)$ and $z_2(x,s)$ of equation (2.82) that are entire functions of s and are real in the sense that $z_j(x,\overline{s}) = \overline{z_j(x,s)},\ j = 1,2.$ Any solution $\hat{z}(x,s)$ of equation (2.82) has the form

$$\hat{z}(x,s) = \sum_{j=1}^{2} a_j(s) z_j(x,s) \tag{2.86}$$

Since the input operator and homogeneous boundary operator are linear, substituting (2.86) into (2.83)-(2.84) yields the system of equations

$$\begin{aligned} a_1(s)\mathcal{B}z_1(s) + a_2(s)\mathcal{B}z_2(s) &= \hat{u}(s), \\ a_1(s)\mathcal{W}z_1(s) + a_2(s)\mathcal{W}z_2(s) &= 0. \end{aligned} \tag{2.87}$$

On the other hand, the transfer function is

$$\begin{aligned} G(s) &= \frac{\hat{y}(s)}{\hat{u}(s)} \\ &= \frac{\sum a_j(s)\mathcal{C}z_j(s)}{\sum a_j(s)\mathcal{B}z_j(s)} \\ &= \alpha\frac{a_1(s)z_1(0,s) + a_2(s)z_2(0,s)}{\hat{u}(s)}. \end{aligned} \tag{2.88}$$

Now if we solve the system of equations (2.87) for $a_1(s)$ and $a_2(s)$ and plug the result in (2.88), we obtain the transfer function as

$$G(s) = \alpha \frac{z_1(0,s)\mathcal{W}z_2(s) - z_2(0,s)\mathcal{W}z_1(s)}{\mathcal{B}z_1(s)\mathcal{W}z_2(s) - \mathcal{B}z_2(s)\mathcal{W}z_1(s)} \tag{2.89}$$

Define

$$\mathcal{N}(s) = \alpha(z_1(0,s)\mathcal{W}z_2(s) - z_2(0,s)\mathcal{W}z_1(s)), \tag{2.90}$$

$$\mathcal{D}(s) = \mathcal{B}z_1(s)\mathcal{W}z_2(s) - \mathcal{B}z_2(s)\mathcal{W}z_1(s). \tag{2.91}$$

Since the operator $\mathcal{W}$ is linear and z_1 and z_2 are entire functions of s, the numerator and denominator of the obtained transfer function are entire functions of s.

The proof of $G(s)$ being real is straightforward since z_1 and z_2 are real functions of s.

Now we must find an asymptotic form for $\mathcal{N}(s)$ and $\mathcal{D}(s)$. To this end, we need to find the asymptotic form of $z_1(x,s)$ and $z_2(x,s)$. In equation (2.82), $a(x)$ is a continuous function of x on a closed and bounded interval which implies that $a(x)$ attains its extremum values on $[0,1]$. Therefore, a positive real r can be so chosen that

$$\max_{x\in[0,1]} |a(x)| << r. \tag{2.92}$$

Birkhoff in [1] proved that the asymptotic behavior of the solutions of (2.82) can be obtained by neglecting the term $a(x)y$ compared to the other two terms for $|s| > r$ and solving

$$\frac{\partial^2 \hat{z}}{\partial x^2} - s\hat{z} = 0. \tag{2.93}$$

In Example 2.27 we obtained the unique basis of solutions for the above equation satisfying Lemma 2.26. Birkhoff proved that the asymptotic independent solutions of (2.82) are

$$\begin{aligned} z_1(x,s) &= (1+\mathcal{O}(\tfrac{1}{\sqrt{s}}))\cosh(\sqrt{s}x),\\ z_2(x,s) &= (1+\mathcal{O}(\tfrac{1}{\sqrt{s}}))\tfrac{\sinh(\sqrt{s}x)}{\sqrt{s}}. \end{aligned} \tag{2.94}$$

This result on asymptotic solutions of differential equations is also discussed in [20], chapter VII. By substituting z_1 and z_2 in (2.90) we obtain

$$\begin{aligned} \mathcal{N}(s) &= \alpha\mathcal{W}z_2(s)\\ &= (1+\mathcal{O}(\frac{1}{\sqrt{s}}))\Big(\alpha\gamma\cosh(\sqrt{s})+\alpha\gamma_2\frac{\sinh(\sqrt{s})}{\sqrt{s}}\Big). \end{aligned}$$

Thus, for sufficiently large $|s|$, $\mathcal{N}(s)$ can be written as

$$\mathcal{N}(s) = \alpha\gamma(1+\mathcal{O}(\frac{1}{\sqrt{s}}))\cosh(\sqrt{s}). \tag{2.95}$$

Clearly, $\mathcal{N}(s)$ has order $\frac{1}{2}$ with infinitely many asymptotic zeros $s = -(n\pi + \frac{\pi}{2})^2$, for integer numbers $n \geq 0$, that are diverging to infinity. Similarly, we substitute (2.94) in (2.91) and simplify to obtain

$$\begin{aligned} \mathcal{D}(s) &= (1+\mathcal{O}(\frac{1}{\sqrt{s}}))\Big(\beta_2\gamma-\beta\gamma_1+(\beta_1\gamma-\beta\gamma_2)\cosh(\sqrt{s})\\ &\quad -\beta\gamma\sqrt{s}\sinh(\sqrt{s})+(\beta_1\gamma_2-\beta_2\gamma_1)\frac{\sinh(\sqrt{s})}{\sqrt{s}}\Big). \end{aligned}$$

As $|s|$ approaches infinity, $\mathcal{D}(s)$ can be written as

$$\mathcal{D}(s) = -\beta\gamma(1+\mathcal{O}(\frac{1}{\sqrt{s}}))\sqrt{s}\sinh(\sqrt{s}). \tag{2.96}$$

It is clear that this function has infinitely many asymptotic zeros $s = -n^2\pi^2$, for integer numbers $n \geq 1$, that are diverging to infinity. Similar to Example 2.29, we can show that the obtained $\mathcal{D}(s)$ has order $\frac{1}{2}$. □

Example 2.31. *Find the transfer function of the following system*

$$\begin{cases} \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}, \\ -\frac{\partial z}{\partial x}(0,t) = u(t), \\ z(1,t) = 0, \\ z(0,t) = y(t), \\ z(x,0) = f(x) \in L_2(0,1) \end{cases} \tag{2.97}$$

The Laplace transform of the system (3.12), assuming zero initial state, satisfies

$$s\hat{z} = \frac{\partial^2 \hat{z}}{\partial x^2}, \tag{2.98}$$

$$-\frac{\partial \hat{z}}{\partial x}(0,s) = \hat{u}(s), \tag{2.99}$$

$$\hat{z}(1,s) = 0, \tag{2.100}$$

$$\hat{z}(0,s) = \hat{y}(s). \tag{2.101}$$

Solving (2.98) with the homogeneous boundary condition (2.100) yields

$$\hat{z}(x,s) = a(s)\sinh\left((x-1)\sqrt{s}\right), \tag{2.102}$$

where $a(s)$ is an arbitrary function of s. Therefore, the transfer function is

$$G(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{\hat{z}(0,s)}{-\frac{\partial \hat{z}}{\partial x}(0,s)} = \frac{\sinh\sqrt{s}}{\sqrt{s}\cosh\sqrt{s}}\ . \tag{2.103}$$

In this example, the functions $\mathcal{N}(s)$ and $\mathcal{D}(s)$ defined in (2.90)-(2.91) are

$$\mathcal{N}(s) = \frac{\sinh(\sqrt{s})}{\sqrt{s}},\ \mathcal{D}(s) = \cosh(\sqrt{s}). \tag{2.104}$$

Definition 2.32. *A function $g(s)$ belongs to $H_\infty(\mathbb{C}_+^a)$ if there exists $M > 0$ such that $g(s)$ is analytic in $\mathbb{C}_+^a$ and $\sup_{s\in\mathbb{C}_+^a}|g(s)| < M$, where $\mathbb{C}_+^a = \{z \in \mathbb{C}, Re(z) \geq a\}$. The space $H_\infty(\mathbb{C}_+^0)$ is usually indicated by H_∞.*

Proposition 2.33. *For the system (2.59), the transfer function $G(s)$ is in $H^\infty(C_+^a)$, for some $a \in R$ and satisfies*

$$\lim_{|s|\to+\infty} G(s) = 0, \quad s \in \mathbb{C}_+^a. \tag{2.105}$$

Hence, the system (2.59) is strictly proper.

Proof. In the proof of Proposition 2.30, we obtained $\mathcal{N}(s)$ and $\mathcal{D}(s)$ for sufficiently large $|s|$. From (2.80) and (2.81), we obtain the asymptotic transfer function as

$$G(s) = \frac{\mathcal{N}(s)}{\mathcal{D}(s)} = \left(1 + \mathcal{O}(\frac{1}{\sqrt{s}})\right) \frac{\alpha\gamma\cosh(\sqrt{s})}{-\beta\gamma\sqrt{s}\sinh(\sqrt{s})}. \tag{2.106}$$

This function approaches zero as $|s|$ tends to infinity, as was to be shown. □

From Proposition 2.33 and Theorem 2.24, the following result follows.

Corollary 2.34. *For the system (2.59) the input/output map is bounded.*

Example 2.35 (Example 2.31 continued)**.** *The transfer function obtained in (2.103) is analytic on the open right-half plane and hence lies in $H^\infty(C_+^a)$, $a = 0$. Also, it is easy to verify that the system transfer function satisfies*

$$\lim_{|s|\to+\infty} \frac{\sinh\sqrt{s}}{\sqrt{s}\cosh\sqrt{s}} = 0. \tag{2.107}$$

Proposition 2.36. *For the system (2.59), the transfer function $G(s)$ satisfies*

$$\lim_{|s|\to\infty} \sqrt{s}G(s) = \widehat{\tau}, \ \ s \in C_+^a, \tag{2.108}$$

for some nonzero real number $\widehat{\tau}$ where the limit is taken on the positive real axis. We refer to the number $\widehat{\tau}$ as the instantaneous gain. Denoting the sign of $\widehat{\tau}$ by s_{ig},

$$s_{ig} = (-1)^r, \tag{2.109}$$

where

$$r = \frac{1}{\pi} arg\left(\frac{\alpha}{\beta}\right) + 1, \tag{2.110}$$

and the constant real numbers α and β are the coefficients of the highest order terms of the output and input operators, respectively.

Proof. From the asymptotic transfer function in (2.106) follows

$$\lim_{s\to\infty} \sqrt{s}G(s) = -\frac{\alpha}{\beta}, \tag{2.111}$$

which is a nonzero real number, as required. Thus, $\hat{\tau} = -\frac{\alpha}{\beta}$ is the instantaneous gain of the system. Verifying (2.109) is now straightforward. □

The name *instantaneous gain* for $\hat{\tau}$ is introduced in [4]. In the classical case, the instantaneous gain can be computed as the value of the inverse Laplace transform of the transfer function at time zero. For this class of systems, this value does not exist, but $\hat{\tau}$ can be considered as an analogous to this value.

Example 2.37 (example 2.31 continued)**.** *The transfer function of the system (2.97) satisfies*

$$\lim_{|s|\to+\infty} \sqrt{s}G(s) = \lim_{|s|\to+\infty} \frac{\sinh\sqrt{s}}{\cosh\sqrt{s}} = 1. \tag{2.112}$$

Therefore, the instantaneous gain of the system (2.97) is one.

2.3.2 Zero-input form and zero dynamics

Definition 2.38. *The uncontrolled or zero-input form of the system (2.59) is the system obtained by setting the input to be zero, that is,*

$$\begin{aligned} \tfrac{\partial z}{\partial t} &= \mathcal{A}_0 z, \quad t > 0, x \in (0,1) \\ \mathcal{C}z(t) &= y(t) \\ z(x,0) &= f(x) \in L^2[0,1] \end{aligned} \tag{2.113}$$

where the operator $\mathcal{A}_0$ *is*

$$\mathcal{A}_0 z = \frac{\partial^2 z}{\partial x^2} + a(x)z, \tag{2.114}$$

$$D(\mathcal{A}_0) = \{z \in H^2(0,1); \mathcal{W}z = \mathcal{B}z = 0\}. \tag{2.115}$$

Another definition is the definition of the zero dynamics of the system.

Definition 2.39. *The system obtained by constraining the output to zero is known as the zero dynamics of the system (2.59).*

$$\begin{aligned} \tfrac{\partial z}{\partial t} &= \mathcal{A}_\infty z, \\ z(x,0) &= f(x) \in L_2(0,1) \end{aligned} \tag{2.116}$$

where

$$\mathcal{A}_\infty z = \frac{\partial^2 z}{\partial x^2} + a(x)z, \tag{2.117}$$

$$D(\mathcal{A}_\infty) = \{z \in H^2(0,1); \mathcal{W}z = \mathcal{C}z = 0\} \tag{2.118}$$

We have the following relationship between the system poles and the spectrum of the operator $\mathcal{A}_0$.

Proposition 2.40. *The poles of the system (2.59) form a subset of the spectrum of the operator* $\mathcal{A}_0$ *with domain* $D(\mathcal{A}_0)$ *defined in (2.114)-(2.115).*

Proof. The Laplace transform of the system (2.59) is given in equations (2.82)-(2.85). The transfer function of this system is

$$G(s) = \frac{\hat{y}(s)}{\hat{u}(s)} \tag{2.119}$$

$$= \frac{\alpha \hat{z}(0,s)}{\beta \frac{\partial \hat{z}}{\partial x}(0,s) + \beta_1 \hat{z}(0,s) + \beta_2 \hat{z}(1,s)}, \tag{2.120}$$

where $\hat{z}(x,s)$ is the Laplace transform of $z(x,t)$. A complex number s is a pole of the transfer function if $\hat{z}(x,s)$ satisfies (2.82) and (2.84) and s is a zero of the the denominator of the transfer function, that is, $\mathcal{B}\hat{z}(s) = \beta \frac{\partial \hat{z}}{\partial x}(0,s) + \beta_1 \hat{z}(0,s) + \beta_2 \hat{z}(1,s) = 0$. On the other hand, a complex number s lies in the spectrum of $\mathcal{A}_0$ if it satisfies $\mathcal{A}_0 \hat{z} = s\hat{z}$, for some nonzero $\hat{z} \in D(\mathcal{A}_0)$, that is equivalent to

$$\frac{\partial^2 \hat{z}}{\partial x^2} + (a(x) - s)\hat{z} = 0, \tag{2.121}$$

$$\mathcal{B}\hat{z}(s) = \beta \frac{\partial \hat{z}}{\partial x}(0,s) + \beta_1 \hat{z}(0,s) + \beta_2 \hat{z}(1,s) = 0, \tag{2.122}$$

$$\mathcal{W}\hat{z}(s) = \gamma \frac{\partial \hat{z}}{\partial x}(1,s) + \gamma_1 \hat{z}(0,s) + \gamma_2 \hat{z}(1,s) = 0. \tag{2.123}$$

Clearly, the system of equations that yields the poles of the system transfer function is the same as the system of equations that yields the spectrum of $\mathcal{A}_0$ with domain $D(\mathcal{A}_0)$. Considering the possibility of a pole/zero cancelation in the system transfer function, we conclude that the poles of the system (2.59) form a subset of the spectrum of the operator $\mathcal{A}_0$ with domain $D(\mathcal{A}_0)$. □

The following relationship holds between the system zeros and the spectrum of the operator $\mathcal{A}_\infty$.

Proposition 2.41. *The zeros of the system (2.59) form a subset of the spectrum of the operator $\mathcal{A}_\infty$ with domain $D(\mathcal{A}_\infty)$.*

The proof of this proposition is similar to that of Proposition 2.40.

Example 2.42. *[Example 2.31 continued] In (2.103) we obtained the system transfer function as*

$$G(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{\hat{z}(0,s)}{-\frac{\partial \hat{z}}{\partial x}(0,s)} = \frac{\sinh\sqrt{s}}{\sqrt{s}\cosh\sqrt{s}}, \tag{2.124}$$

and the functions $\mathcal{N}(s)$ and $\mathcal{D}(s)$ introduced in (2.90)-(2.91) are

$$\mathcal{N}(s) = \frac{\sinh(\sqrt{s})}{\sqrt{s}},\ \mathcal{D}(s) = \cosh(\sqrt{s}). \tag{2.125}$$

The poles of this system are $s = -(n\pi + \pi/2)^2,\ n = 0, 1, 2, \ldots.$ On the other hand, the spectrum of $\mathcal{A}_0$ can be obtained by solving $\frac{\partial \hat{z}}{\partial x}(0,s) = 0$ with (2.102), which results in solving $\cosh\sqrt{s} = 0$. The roots of this equation are $s = -(n\pi + \pi/2)^2,\ n = 0, 1, 2, \ldots,$ corresponding to eigenfunctions $\cos(n\pi + \pi/2)x$. Thus, the spectrum of $\mathcal{A}_0$ is $s = -(n\pi + \pi/2)^2,\ n = 0, 1, 2, \ldots.$ Hence, the spectrum of $\mathcal{A}_0$ with domain $D(\mathcal{A}_0)$ are exactly the open-loop poles. Furthermore, the open-loop zeros are $s = -n^2\pi^2,\ n = 1, 2, ..,$ while the spectrum of the zero-dynamics can be obtained by solving $\hat{z}(0,s) = 0$ with (2.102), which results in solving $\frac{1}{\sqrt{s}}\sinh\sqrt{s} = 0$. Thus, the spectrum of $\mathcal{A}_\infty$ is $s = -n^2\pi^2,\ n = 1, 2, ..,$ corresponding to eigenfunctions $\sin n\pi x$. Consequently, the spectrum of $\mathcal{A}_\infty$ with domain $D(\mathcal{A}_\infty)$ are exactly the zeros of the transfer function.

2.4 Stability

One of the most important design objectives in control systems is the stability of the system. In actual systems we like that a bounded input always result in a bounded output. We refer to such systems as L_2-stable systems. The mathematical expression of stability for a finite-dimensional system is as follows.

Definition 2.43. *([18], Definition 3.1) A system is L_2-stable or externally stable if for any input $u \in L_2(0,\infty;U)$, the output y lies in $L_2(0,\infty;Y)$, where U, Y are the vector spaces of the input and output, respectively.*

Theorem 2.44. *([18], Theorem 3.6) A finite-dimensional system is L_2-stable if and only if the system transfer function $G(s)$ lies in H_∞(Definition 2.32).*

This result is also true for general well-posed systems [7]. We will prove this equivalence for the class of boundary control systems introduced in (2.59). First we recall a theorem from complex analysis.

Theorem 2.45. *([25],Theorem 10.18) Let f be an analytic function on a domain C in the complex plane. If f is not identically zero, then the zeros of f on C have the following properties.*

(a) If s_0 is a zero of f, then the order of s_0 is finite, that is, there exists an m such that $f(s) = (s - s_0)^m g(s)$ and $g(s_0) \neq 0$.

(b) The zeros of f are isolated, that is, if $f(s_0) = 0$, there exists $\delta > 0$ such that $f(s) \neq 0$ whenever $0 < |s - s_0| < \delta$.

(c) The zeros of f do not have any accumulation points in C. As a result, in any compact subset of C, the function f possesses only a finite number of zeros.

We also need Parseval's Theorem.

Theorem 2.46. *(Parseval's Theorem [22]) For $f(t), g(t) \in L_2(-\infty, \infty; \mathbb{R})$ with Fourier transform $\hat{f}(j\omega), \hat{g}(j\omega)$, $\omega \in \mathbb{R}$, the following equality holds.*

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}dt = \int_{-\infty}^{\infty} \hat{f}(j\omega)\overline{\hat{g}(j\omega)}ds. \tag{2.126}$$

The following theorem gives a necessary and sufficient condition for L_2-stability of the system (2.59).

Theorem 2.47.

Proof. In Proposition 2.30 we proved that the transfer function of the system (2.59) is

$$G(s) = \frac{\mathcal{N}(s)}{\mathcal{D}(s)}, \tag{2.127}$$

where $\mathcal{N}$ and $\mathcal{D}$ are entire functions of s. We also proved in Proposition 2.33 that $G(s)$ is strictly proper and hence it satisfies the properness condition $\sup_{Re(s)>\sigma} \|G(s)\| < \infty$, for some $\sigma \in \mathbb{R}$. If $G(s) \in H_\infty$, then $\mathcal{D}(s)$ has no roots in the closed right-half plane, and by Theorem 2.45, these roots are isolated. There exists some $M > 0$ such that the transfer function satisfies

$$|G(s)| \leq M, \tag{2.128}$$

for all $s \in \mathbb{C}_0^+ = \{s \in \mathbb{C}, Re(s) \geq 0\}$. Take any $u \in L_2(0, \infty; U)$ with Fourier transform $\hat{u}(j\omega)$. by Parseval's theorem,

$$\int_0^{\infty} |u(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{u}(j\omega)|^2 d\omega. \tag{2.129}$$

Thus, the Fourier transform of the output satisfies

$$\begin{aligned}
\int_{-\infty}^{\infty} |\hat{y}(j\omega)|^2 d\omega &= \int_{-\infty}^{\infty} |G(j\omega)\hat{u}(j\omega)|^2 d\omega \\
&\leq \|G(j\omega)\|^2 \int_{-\infty}^{\infty} |\hat{u}(j\omega)|^2 d\omega \\
&\leq M\|u\|_2^2,
\end{aligned} \tag{2.130}$$

where the right-hand side is bounded by assumption. Thus, by Parseval's Theorem

$$\int_0^\infty |y(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{y}(j\omega)|^2 d\omega < \infty. \tag{2.131}$$

Therefore, the system is L_2-stable.

Now assume $G(s)$ is not in H_∞. Since G is proper and $\mathcal{N}(s)$ and $\mathcal{D}(s)$ are entire functions, G has at least one pole in $\mathbb{C}_0^+$. Choose a pole p with $Re(p) \geq 0$. Choose an input $u \in L_2(0, \infty; U)$ such that $\hat{u}(p) \neq 0$. The Laplace transform of the corresponding output is

$$\hat{y}(s) = G(s)\hat{u}(s) = G_1(s) + \frac{A}{s-p}, \tag{2.132}$$

where $G_1(s)$ is analytic in a neighborhood of p and A is the residue of $G(s)\hat{u}(s)$ at $s = p$. If $Re(p) > 0$, the inverse Laplace transform of $\hat{y}(s)$ has a term that is growing exponentially and hence y is not in $L_2(0, \infty; Y)$. If p is imaginary, then the inverse Laplace transform has an oscillating term which implies that y is not in $L_2(0, \infty; Y)$. In both cases, the system is not L_2-stable. The argument is similar when p is a pole of multiplicity more than one. This completes the proof. □

The above theorems concern the external stability of a system. Another concept that has a significant importance is whether the states eventually become zero for all initial states. In [17], asymptotically stable and exponentially stable semigroups are introduced.

Definition 2.48. *(Definition 3.1 [17]) A C_0-semigroup $T(t)$ on a Banach space X is asymptotically stable if for any $x \in X$, $\|T(t)x\| \to 0$ as t approaches infinity.*

Definition 2.49. *(Definition 3.1 [17]) A C_0-semigroup $T(t)$ on a Banach space X is exponentially stable if*

$$\|T(t)\| \leq Me^{-\alpha t}, \tag{2.133}$$

for some constants $M \geq 1, \alpha > 0$ and all $t \geq 0$.

The constant α is the *decay rate of the semigroup.*

In both asymptotic and exponential stability of the semigroup $T(t)$ with infinitesimal generator A, the solution to $\dot{z}(t) = Az(t),\ z(0) = z_0$ tends to zero as t approaches infinity. It is easy to show that in finite-dimensional systems, asymptotic and exponential stability are equivalent. However, exponential stability is stronger than asymptotic stability for infinite-dimensional systems. In the following example, we show an asymptotically stable semigroup that is not exponentially stable.

Example 2.50. *Let $X = l^2$, the space of sequences $\{x_n, n \geq 1\}$ with $\sum_{n=1}^{\infty} x_n^2 < \infty$. Define the semigroup*

$$T(t)x = \{e^{-t/n}x_n,\ n \geq 1\},\ t \geq 0, \tag{2.134}$$

for all $x = \{x_n, n \geq 1\} \in l^2$. We have $\|T(t)x\|^2 = \sum_{n=1}^{\infty} e^{-2t/n}x_n^2$ which is convergent to zero as t tends to infinity. Thus, $T(t)$ is asymptotically stable. However, for any $t \geq 0$ and any $x \in l^2$,

$$\begin{aligned} \|T(t)x\|^2 &= \sum_{n=1}^{\infty} e^{-2t/n}x_n^2 \\ &\leq \sum_{n=1}^{\infty} x_n^2 \\ &= \|x\|^2. \end{aligned}$$

Thus, $\|T(t)\| \leq 1$*. Consider the sequence* $x = \{x_n, n \geq 1\}$ *such that* $x_i = 1$ *for some* $i \geq 1$ *and* $x_n = 0$ *for* $n \neq i$*. For this sequence,* $\|T(t)x\| = e^{-t/i}|x_i|$ *and* $\|x\| = |x_i|$*. Therefore, with a suitable choice of* i*,* $\|T(t)x\|$ *can be arbitrarily close to* $\|x\|$*. Thus,* $\|T(t)\| = 1$ *for all* $t \geq 0$*. Hence,* $T(t)$ *is not exponentially stable.*

Note that the infinitesimal generator of $T(t)$ *is*

$$Ax = \{\frac{-x_n}{n}, n \geq 1\}, \tag{2.135}$$

and the spectrum of A *is* $\sigma(A) = \{\frac{-1}{n}, n \geq 1\}$*.*

Now we define the internal stability of a control system.

Definition 2.51. *A well-posed control system with state operator* A *and associated semigroup* $T(t)$ *is internally stable if* $T(t)$ *is exponentially stable.*

Corollary 2.52. *([18], Definition 3.14) A finite-dimensional system with state-space realization* (A, B, C) *is internally stable if* $\max_{1\leq i\leq n} Re(\lambda_i(A)) < 0$*, where* $\lambda_i(A), 1 \leq i \leq n$ *are the eigenvalues of* A*.*

The following theorem gives necessary and sufficient conditions for a semigroup to be exponentially stable.

Theorem 2.53. *([17], Theorem 3.35) Let* $T(t)$ *be a semigroup on a Hilbert space* H *with generator* A*. Then* $T(t)$ *is exponentially stable if and only if* $\{\lambda \in \mathbb{C}, Re\lambda \geq 0\}$ *belongs to the resolvent set of* A *and the resolvent operator satisfies*

$$\|R(\lambda, A)\| \leq M, \tag{2.136}$$

for some positive constant M *and all* λ *with* $Re\lambda \geq 0$*.*

In [18](Theorem 3.16), it is shown that an internally stable finite dimensional system is always L_2-stable. This property also holds for general well-posed systems. We show below that for the system (2.59) exponential stability of the semigroup generated by the state operator is a sufficient condition for L_2-stability of the system.

Theorem 2.54. *The system (2.59) is L_2-stable if the semigroup generated by the operator $\mathcal{A}_0$ is exponentially stable.*

Proof. If $\mathcal{A}_0$ is exponentially stable then by Theorem 2.53 all of its eigenvalues are in the open left-half plane. From Proposition 2.40, we also know that the poles of the system transfer function form a subset of the eigenvalues of $\mathcal{A}_0$. Thus, all poles of the transfer function lie in the open left-half plane and hence by Theorem 2.47 the system is L_2-stable. □

In this chapter we provided background knowledge on infinite-dimensional linear systems and basic concepts in this context. Through examples in finite and infinite-dimensional systems we tried to clarify the material. But, so far, we have not mentioned what is meant by controlling a system. In the following chapter we will introduce feedback control systems and stability analysis.

Chapter 3

Root-Locus Theory

Controlling a system generally means forcing the system to operate in the way we wish and to produce our desired output. If an exact mathematical model of a system was available and there was no disturbance on the system, one would be able to apply the suitable input so that the desired output is produced. But in all actual systems unwanted disturbances are involved. So a feedback mechanism is required to measure a quantity and correct errors if any discrepancies from the desired value of that quantity occur. There are many examples of feedback control systems. In this chapter we consider the

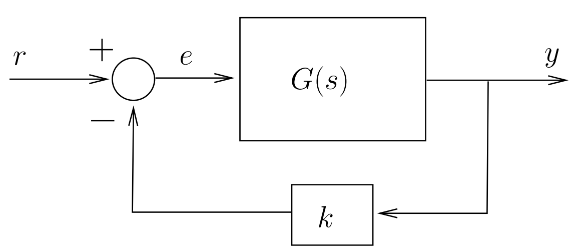

Figure 3.1: the closed-loop system

feedback system shown in Figure 3.1. The system that is being controlled is called the plant and is shown by transfer function $G(s)$. The controller, a proportional controller, is a constant but adjustable gain k. The open-loop system is the system obtained by setting the feedback gain to zero. The transfer function $G(s)$ is called the open-loop transfer function. In contrast, the system with feedback is called the closed-loop system. The equations of the closed-loop system are

$$\begin{aligned} \hat{e}(s) &= \hat{r}(s) - k\hat{y}(s), \\ \hat{y}(s) &= G(s)\hat{e}(s). \end{aligned} \tag{3.1}$$

Therefore, the closed-loop transfer function is

$$G_k(s) = \frac{\hat{y}(s)}{\hat{r}(s)} = \frac{G(s)}{1 + kG(s)}. \tag{3.2}$$

The poles of $G_k(s)$ are the closed-loop poles of the system. We can see that the closed-loop poles move in the s-plane as k varies. The graph of all trajectories of the closed-loop poles as k varies from zero to infinity, is known as the root-locus graph of the system.

The root-locus graph shows us for what values of the variable parameter k the closed-loop poles lie in the open left-half plane, making the closed-loop system stable. It is usually possible to set the parameter k such that an unstable open-loop system becomes a stable closed-loop one.

The root-locus method for finite-dimensional systems is totally established and known. But for infinite-dimensional systems the root-locus theory is still an open research area. This chapter is organized as follows. In section 1, we discuss the root-locus theory for finite-dimensional systems. The material in this section is generally based on [11], [18]. In section 2, we deal with the feedback control of the system described in (2.59) and the root-locus method is applied to this class of infinite-dimensional systems. The stability problem is then discussed for this system. This section is mainly based on [4].

3.1 Finite-dimensional systems

In this section, we describe the root-locus method to locate the closed-loop poles of a finite-dimensional state-space realization as the feedback gain parameter k varies from zero to infinity. Particularly, we deal with the stability of the closed-loop system for sufficiently large k. We begin with a motivating example.

Example 3.1. *Consider the system shown in Fig.3.1 with open-loop transfer function* $G(s) = \frac{s}{(s+1)(s+2)}$*. The closed-loop transfer function is*

$$G_k(s) = \frac{(s+1)(s+2)}{(s+1)(s+2)+ks}. \tag{3.3}$$

The closed-loop poles of the system are the roots of

$$(s+1)(s+2)+ks = 0, \tag{3.4}$$

that is,

$$s_1 = \frac{1}{2}\left(-(3+k)+\sqrt{(3+k)^2-8}\right), \;\; s_2 = \frac{1}{2}\left(-(3+k)-\sqrt{(3+k)^2-8}\right). \tag{3.5}$$

For $k \geq 0$*,* s_1, s_2 *are real-valued. As* k *varies from zero to plus infinity,* s_1 *moves from* -1 *to the origin on the real axis in the s-plane, and* s_2 *moves from -2 to* $-\infty$ *on the real axis. Furthermore,* s_1 *and* s_2 *are continuous*

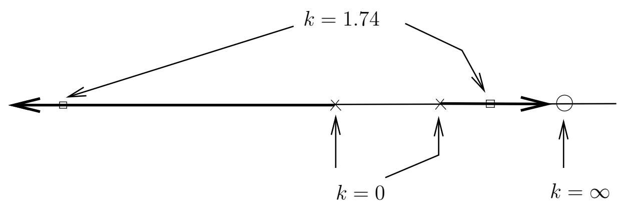

Figure 3.2: the root-locus graph for example 3.1

monotone functions of k *for* $k \geq 0$. *The root-locus graph of this system for* $0 \leq k < \infty$ *is shown in Figure 3.2. From the root-locus graph we realize that for* $0 < k < \infty$ *the closed-loop poles always are in the open left-half plane implying that the closed-loop system is stable for all* $0 < k < \infty$.

In Figure 3.3, an example of a root-locus graph produced by MATLAB for the system shown in Figure 3.1 is illustrated. The open-loop transfer function of this system is $G(s) = \frac{s}{(s^2+2s+2)(s-1)(s+1)(s+2)}$.

Consider the finite-dimensional state-space realization

$$\begin{aligned} \dot{x}(t) &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{3.6}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, and $C \in R^{p\times n}$. We obtained the transfer function of this system in Example 2.22 as

$$G(s) = \frac{CAdj(sI - A)B}{det(sI - A)}. \tag{3.7}$$

Thus, for any state-space realization (A, B, C) the transfer function is a rational function and the numerator and denominator have finite orders. Let $G(s) = n(s)/d(s)$ be the open-loop transfer function of this system, with $n(s)$ and $d(s)$ being polynomials of order n_z and n_p, respectively. We form

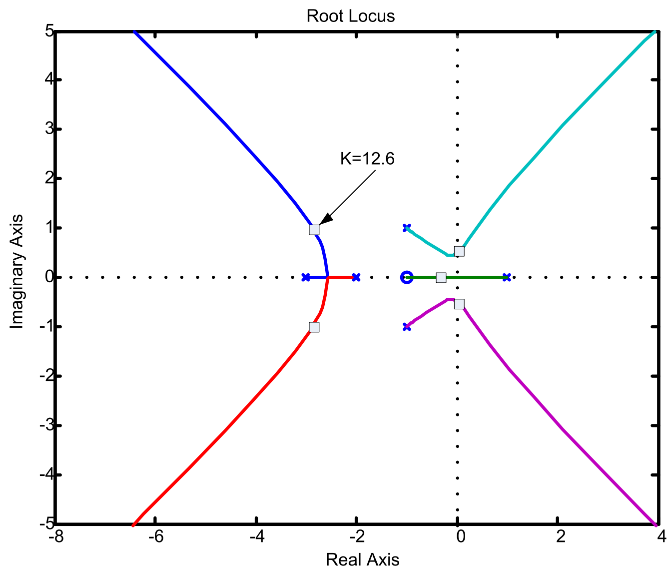

Figure 3.3: the root-locus graph for $G(s) = \frac{s}{(s^2+2s+2)(s-1)(s+1)(s+2)}$

the closed-loop system by a proportional feedback of the form

$$u = -ky + r, \tag{3.8}$$

as shown in Figure 3.1. The closed loop transfer function is

$$G_k(s) = \frac{G(s)}{1 + kG(s)} = \frac{n(s)}{d(s) + kn(s)} \tag{3.9}$$

The stability of the closed-loop system depends on the closed-loop poles of the system, that is, the zeros of

$$d(s) + kn(s) = 0. \tag{3.10}$$

which is called the characteristic equation of the closed-loop system. For $k = 0$, the closed-loop poles are exactly the open-loop poles of the system, which may not be in the open left-half plane. Therefore, we wish to choose k

so that all closed-loop poles lie in the open left-half plane. We can tackle this problem using the root-locus method. Let us rewrite the open-loop transfer function as

$$G(s) = \frac{\prod_{j=1}^{n_z}(s+z_j)}{\prod_{j=1}^{n_p}(s+p_j)}, \tag{3.11}$$

where $\{-z_j\}_{j=1}^{n_z}$ are the open-loop zeros and $\{-p_j\}_{j=1}^{n_p}$ are the open-loop poles. The characteristic equation can now be written as

$$k.\frac{\prod_{j=1}^{n_z}(s+z_j)}{\prod_{j=1}^{n_p}(s+p_j)} = -1. \tag{3.12}$$

From this equation follows

$$\left| k\frac{\prod_{j=1}^{n_z}(s+z_j)}{\prod_{j=1}^{n_p}(s+p_j)} \right| = 1, \tag{3.13}$$

$$arg\left(k\frac{\prod_{j=1}^{n_z}(s+z_j)}{\prod_{j=1}^{n_p}(s+p_j)} \right) = \pi + 2m\pi, \quad m = 0, \pm 1, \pm 2, ... \tag{3.14}$$

These equations can be simplified in the following form.

$$|k|.\frac{\prod_{j=1}^{n_z}|s+z_j|}{\prod_{j=1}^{n_p}|s+p_j|} = 1, \tag{3.15}$$

and

$$arg(k) + \sum_{j=1}^{n_z} arg(s+z_j) - \sum_{j=1}^{n_p} arg(s+p_j) = \pi + 2m\pi, \quad m \in Z. \tag{3.16}$$

Thus, a point $s \in \mathbb{C}$ lies on the root-locus if it satisfies the equations (3.15)-(3.16) for some $k \in R^+$.

All finite-dimensional systems are well-posed systems and so the map from input to output is bounded. For these systems, as we discussed in chapter 2, the transfer function is proper which means that the order of the denominator is not less than that of the numerator. Thus, in this section, we assume that the system described by (3.11) satisfies the condition $n_p \geq n_z$.

The first step to sketch the root-locus for a system with characteristic equation (3.12) is to locate the closed-loop poles for $k = 0$, that is the open-loop poles. In the second step, the closed-loop poles when k approaches infinity must be located, that is the open-loop zeros.

In what follows, some properties of the root-locus graph for a system shown in Figure 3.1 with open-loop transfer function given in (3.11) are stated.

Proposition 3.2. *If all coefficients of the open-loop transfer function $G(s)$ are real, then the root-locus graph is symmetric with respect to the real axis. In other words, for each $k \geq 0$, the closed loop poles are either real or occur in conjugate pairs.*

Proof. The closed-loop poles are obtained by solving equation (3.10). Since all coefficients are real, we have that $\overline{d(s) + kn(s)} = d(\overline{s}) + kn(\overline{s})$, for all $s \in \mathbb{C}$. Thus, for each $k \geq 0$, if $d(s_0) + kn(s_0) = 0,\ s_0 \in \mathbb{C}$, then

$$\overline{d(s_0) + kn(s_0)} = 0,$$

which implies

$$d(\overline{s_0}) + kn(\overline{s_0}) = 0$$

Thus, if s_0 is a closed-loop pole, then is so $\overline{s_0}$. □

In particular, the open-loop poles and zeros occur in conjugate pairs. In general, for any closed-loop system with characteristic equation $f(s, k) = 0$, Proposition 3.2 is satisfied if the $f(s, k)$ is a real function of s, that is

$\overline{f(s,k)} = f(\overline{s},k)$. Thus, this proposition can be applied to a variety of systems including many infinite-dimensional systems.

The following proposition indicates which intervals on the real axis are sections of the root-locus.

Proposition 3.3. *For $k > 0$, $x \in R$ belongs to the root-locus if and only if x lies in a section of the real axis to the left of an odd number of poles and zeros.*

Proof. First we prove that if for some $k \in [0, \infty)$, the point $x \in R$ satisfies the magnitude and phase conditions (3.15)-(3.16), then x lies to the left of an odd number of poles and zeros. For any $x \in R$, we have that $\prod_{j=1}^{n_z} |x + z_j| / \prod_{j=1}^{n_p} |x + p_j|$ is a positive real number, say, α. Thus, for $k = 1/\alpha$ the magnitude condition is satisfied. It only remains to prove that x satisfies the phase condition (3.16), if and only if x lies in a section of the real axis to the left of an odd number of poles and zeros. Let r be the number of real open-loop zeros to the right of x, $\hat{r}$ be the total number of complex conjugate open-loop zeros' pairs, s be the number of real open-loop poles to the right of x, and $\hat{s}$ be the total number of complex conjugate open-loop pole pairs. As illustrated in Figure 3.4, we have

$$\sum_{j=1}^{n_z} arg(x + z_j) = 2\pi\hat{r} + \pi r, \tag{3.17}$$

$$\sum_{j=1}^{n_p} arg(x + p_j) = 2\pi\hat{s} + \pi s. \tag{3.18}$$

Since $arg(k) = 0$, the phase condition (3.16) is

$$arg(k) + \textstyle\sum_{j=1}^{n_z} arg(x + z_j) - \sum_{j=1}^{n_p} arg(x + p_j)$$
$$= 2\pi(\hat{r} - \hat{s}) + \pi(r - s)$$

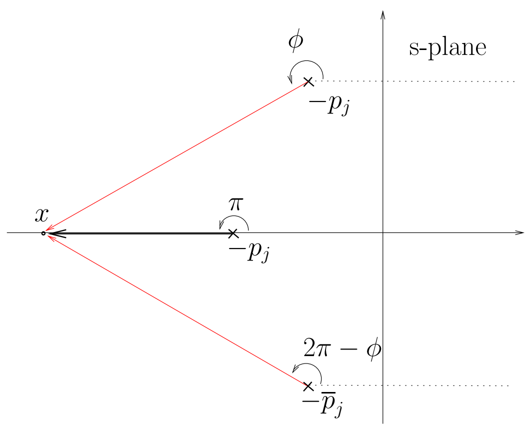

Figure 3.4: Any real pole $-p_i$ to the right of x satisfies $arg(x+p_i) = \pi$. Any pair of complex poles $-p_j, -\bar{p}_j$ satisfy $arg(x+p_j) + arg(x+\bar{p}_j) = 2\pi$

where $\hat{r} - \hat{s}$ is an integer number. For every real x on the root-locus, the equation (3.19) is equal to an odd multiplicity of π which implies $r - s$ is an odd number and hence $r + s$ is an odd number, that is, the number of real poles and zeros to the right of x is an odd number. This completes the proof. □

Definition 3.4. *A rational transfer function $G(s)$ possesses a zero at infinity of multiplicity $m_z > 0$ if*

$$\lim_{|s|\to\infty} s^{m_z} G(s) = c, \tag{3.19}$$

for some nonzero constant $c \in R$.

A pole at infinity can also be defined in a similar way, but since finite-dimensional systems have proper transfer functions, they never have poles at infinity.

The following proposition can be easily verified.

Proposition 3.5. *A finite-dimensional system with transfer function $G(s)$ given in (3.11), possesses a zero of multiplicity $n_p - n_z$ at infinity.*

Proof. Write

$$G(s) = \frac{s^{n_z} + a_1 s^{n_z-1} + ... + a_{n_z}}{s^{n_p} + b_1 s^{n_p-1} + ... + b_{n_p}}, \tag{3.20}$$

where n_z indicates the number of zeros of G and n_p the number of poles. Then,

$$\lim_{|s|\to\infty} s^{n_p-n_z} G(s) = 1 \tag{3.21}$$

Thus, $G(s)$ has $n_p - n_z$ zeros at infinity. □

The following proposition follows from a theorem in algebra that proves the roots of any polynomial continuously depend on its coefficients.

Proposition 3.6. *[28] For a system shown in Figure 3.1 with a well-posed open-loop transfer function $G(s)$ given in (3.11), all closed-loop poles vary continuously as k varies from zero to infinity.*

From Proposition 3.6 the following corollary follows.

Corollary 3.7. *For a system shown in Figure 3.1 with a rational open-loop transfer function $G(s)$ given in (3.11), the number of separate loci is equal to n_p.*

Proof. Consider the characteristic equation $d(s) + kn(s) = 0$, with the transfer function $G(s)$ given in (3.11). For each $k \geq 0$ there are n_p roots for this equation that vary continuously with k. As k approaches infinity, the roots of the characteristic equation tend to the zeros of $G(s)$. We showed that $G(s)$ possesses $n_p - n_z$ zeros at infinity in addition to n_z finite zeros. Thus,

n_z branches end at n_z finite open-loop zeros and $n_p - n_z$ branches tend to $n_p - n_z$ zeros at infinity. Thus, the number of separate loci is n_p. □

The following important property describes the behavior of the infinite branches.

Proposition 3.8. *For a system shown in Figure 3.1 with a well-posed open-loop transfer function $G(s)$ given in (3.11), $n_p - n_z$ sections of loci must end at zeros at infinity. These sections of loci proceed to the zeros at infinity along asymptotes as k approaches infinity. These linear asymptotes are centered at a point on the real axis given by*

$$\sigma = \frac{\sum(\textit{poles of } G(s)) - \sum(\textit{zeros of } G(s))}{n_p - n_z} \tag{3.22}$$

The angles of the asymptotes w.r.t the real axis are,

$$\phi = \frac{(2m-1)\pi}{n_p - n_z}, \quad m = 1, 2, ..., n_p - n_z \tag{3.23}$$

Proof. Write the open-loop transfer function in the form (3.20). For sufficiently large $|s|$, $G(s)$ can be approximated as

$$\begin{aligned} G(s) &= \frac{s^{n_z} + a_1 s^{n_z - 1} + \mathcal{O}(s^{n_z - 2})}{s^{n_p} + b_1 s^{n_p - 1} + \mathcal{O}(s^{n_p - 2})}, \\ &= \frac{1}{s^{n_p - n_z}} \frac{1 + \frac{a_1}{s} + \mathcal{O}(\frac{1}{s^2})}{1 + \frac{b_1}{s} + \mathcal{O}(\frac{1}{s^2})}. \end{aligned}$$

Dividing the numerator and denominator of the above equation by $1 + \frac{a_1}{s}$ yields

$$\begin{aligned} G(s) &= \frac{1}{s^{n_p - n_z}} \frac{1 + \mathcal{O}(\frac{1}{s^2})}{1 + \frac{b_1 - a_1}{s} + \mathcal{O}(\frac{1}{s^2})}, \\ &= \frac{1 + \mathcal{O}(\frac{1}{s^2})}{s^{n_p - n_z} + (b_1 - a_1) s^{n_p - n_z - 1} + \mathcal{O}(s^{n_p - n_z - 2})}. \end{aligned} \tag{3.24}$$

On the other hand, for sufficiently large $|s|$, we can approximate the binomial expansion

$$\left(s+\frac{b_1-a_1}{n_p-n_z}\right)^{n_p-n_z} = s^{n_p-n_z}+(b_1-a_1)s^{n_p-n_z-1}+\mathcal{O}(s^{n_p-n_z-2}). \quad (3.25)$$

Substitute (3.25) into (3.24) to obtain

$$G(s)=\frac{1+\mathcal{O}(\frac{1}{s^2})}{\left(s+\frac{b_1-a_1}{n_p-n_z}\right)^{n_p-n_z}+\mathcal{O}(s^{n_p-n_z-2})}. \quad (3.26)$$

Hence the characteristic equation $1+kG(s)=0$ becomes

$$1+k\frac{1+\mathcal{O}(\frac{1}{s^2})}{\left(s+\frac{b_1-a_1}{n_p-n_z}\right)^{n_p-n_z}+\mathcal{O}(s^{n_p-n_z-2})}=0. \quad (3.27)$$

From this equation it follows that

$$\left(s+\frac{b_1-a_1}{n_p-n_z}\right)^{n_p-n_z}+\mathcal{O}(s^{n_p-n_z-2})=ke^{j\pi(2m-1)}(1+\mathcal{O}(\frac{1}{s^2})),\ m\in Z. \quad (3.28)$$

Therefore, the asymptotic approximation of the obtained equation is

$$s+\frac{b_1-a_1}{n_p-n_z}=k^{\frac{1}{n_p-n_z}}e^{j\pi\frac{2m-1}{n_p-n_z}},\ m=1,2,...,n_p-n_z. \quad (3.29)$$

Thus, the asymptotes are centered at

$$\sigma=-\frac{b_1-a_1}{n_p-n_z}, \quad (3.30)$$

with angles $\phi=\frac{(2m-1)\pi}{n_p-n_z}$, $m=1,...,n_p-n_z$.

Recall that the summation of the roots of a polynomial in s of order n is equal to the negative of the coefficient of the term s^{n-1}. Hence, in the transfer function $G(s)$, $b_1=-\sum$(poles of $G(s)$) and $a_1=-\sum$(zeros of $G(s)$). The proof is now complete. □

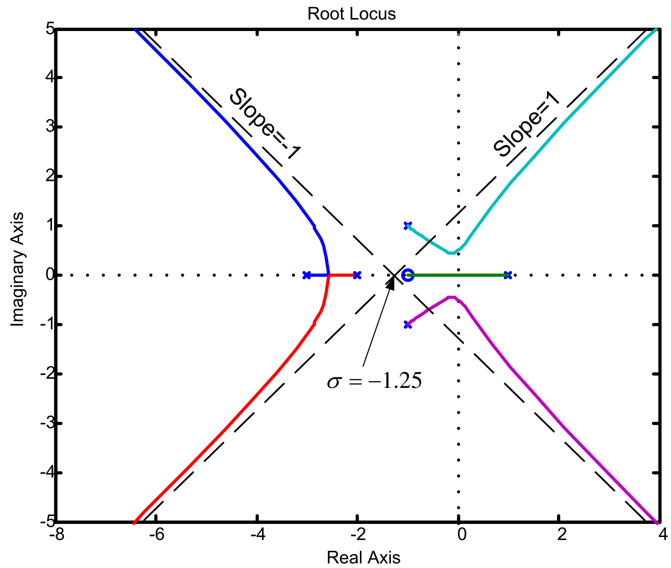

Figure 3.5: the root-locus graph for $G(s) = \frac{s}{(s^2+2s+2)(s-1)(s+1)(s+2)}$, asymptotes are centered at $\sigma = -1.25$

The behavior of the asymptotes for an example is illustrated in 3.5.

In our finite-dimensional root-locus analysis we always assumed that the feedback gain k is positive. Let us see what happens to the system shown in Figure 3.1 with open-loop transfer function (3.11), if k is negative. By simple manipulations of the proof of Proposition 3.3, we can show that a real point $x \in \mathbb{R}$ belongs to the root-locus if and only if x lies in a section of the real axis to the left on an even number of real poles and zeros. Furthermore, similar to the proof of Proposition 3.8 we can show that if $n_p - n_z > 0$, the infinite branches have asymptotes with angles $\phi = 2m\pi/(n_p - n_z)$, $m = 0, 1, .., n_p - n_z - 1$. Thus, if the system is strictly proper, that is, $n_p - n_z \geq 1$, one of the root-locus branches is the interval $[a, \infty)$, where $a \in \mathbb{R}$ is the right-most real open-loop pole or zero. The closed-loop pole that approaches infinity on this branch, makes the system unstable for sufficiently large $|k|$. Hence, we observe that the assumption $k > 0$ is plausible.

From Proposition 3.8, we have the following results on the stability of the system shown in Figure 3.1 with open-loop transfer function (3.11).

1. If $n_p - n_z > 2$, then there is an asymptote with phase $-\pi/2 < \theta < \pi/2$, which means for k sufficiently large, there is a closed-loop pole lying in the right-half plane, thus, the closed-loop system is unstable for sufficiently large k.

2. If $n_p - n_z = 2$ and $\sigma \geq 0$, σ defined in (3.22), the two asymptotic branches lie in the closed right-half plane, which means the system is unstable for sufficiently large k.

3. If $n_p - n_z = 2$ and $\sigma < 0$, then two asymptotes lie in the open left-half plane, which means that as k tends to infinity, the two closed-loop poles diverging to the two open-loop zeros at infinity, are on the open left-half plane. In this case, for the closed-loop system to be eventually stable(for sufficiently large k), it suffices that all open-loop zeros lie in the open left-half plane.

4. If $n_p - n_z = 1$, the only asymptote is the negative real axis, which means the closed-loop pole diverging to the zero at infinity, is stable for sufficiently large k and the only criterion for the closed-loop system to be stable is that all open-loop zeros lie in the open left-half plane.

5. If $n_p - n_z = 0$, there is no infinite branches in the root-locus and all closed-loop poles are finite. In this case, the only criterion for the closed-loop system to be stable is that all open-loop zeros lie in the open left-half plane.

3.2 Root-Locus Theory for a Class of Infinite-Dimensional Systems

In this section, we will evaluate the feedback control of the open-loop system

$$\begin{cases} \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + a(x)z, \quad t > 0, x \in (0,1),\ a(x) \in C^\infty(0,1) \\ \mathcal{B}z(t) = \beta \frac{\partial z}{\partial x}(0,t) + \beta_1 z(0,t) + \beta_2 z(1,t) = u(t), \\ \mathcal{W}z(t) = \gamma \frac{\partial z}{\partial x}(1,t) + \gamma_1 z(0,t) + \gamma_2 z(1,t) = 0, \\ \mathcal{C}z(t) = \alpha z(0,t) = y(t), \\ z(x,0) = f(x), \end{cases} \tag{3.31}$$

where $f \in L_2[0,1]$, $u(t)$ and $y(t)$ are the input and output, respectively. All coefficients are real-valued with $\alpha, \beta, \gamma \neq 0$. The state operator of the system is

$$\mathcal{A}z = \frac{\partial^2 z}{\partial x^2} + a(x)z, \tag{3.32}$$

$$D(\mathcal{A}) = \{z \in H^2(0,1); \mathcal{W}z = \mathcal{B}z = 0\}. \tag{3.33}$$

This system was introduced in section 2.3 and some of its properties derived. In this section, we want to evaluate the closed-loop system when a feedback shown in Figure 3.1 is applied to the system. The output control is obtained by the feedback

$$u = -ky + r, \tag{3.34}$$

which changes the system as follows

$$\begin{aligned} \frac{\partial z}{\partial t} &= \mathcal{A}_k z, \\ \mathcal{B}z(t) + k\mathcal{C}z(t) &= r(t), \\ \mathcal{C}z(t) &= y(t), \\ z(x,0) &= f(x), \end{aligned} \tag{3.35}$$

where

$$\begin{aligned} \mathcal{A}_k z &= \mathcal{A}z, \\ D(\mathcal{A}_k) &= \{z \in H^2(0,1); \mathcal{W}z = 0,\ (\mathcal{B} + k\mathcal{C})z = 0\}. \end{aligned} \tag{3.36}$$

The stability analysis of the closed-loop system mostly consists of determining if the closed-loop poles lie in the open left-half plane. This in general takes a lot of effort and requires background in complex and asymptotic analysis. However, for the specific case where the operator $\mathcal{A}_k$ with domain $D(\mathcal{A}_k)$ is self-adjoint, this analysis is much easier due to the fact that the eigenvalues of a self-adjoint operator are real, which we will prove in the following theorem.

Theorem 3.9. *([20], Chapter IV) If the operator A with domain $D(A)$ is self-adjoint, all eigenvalues of A are real.*

Proof. Let λ be an eigenvalue of A and $\bar{\lambda}$ be its complex conjugate. Take $\phi, \psi \in D(A)$. We have

$$\langle \phi, A\psi \rangle = \langle \phi, \lambda\psi \rangle = \bar{\lambda}\langle \phi, \psi \rangle,$$
$$\langle A\phi, \psi \rangle = \langle \lambda\phi, \psi \rangle = \lambda\langle \phi, \psi \rangle.$$

Self-adjointness implies

$$(\bar{\lambda} - \lambda)\langle \phi, \psi \rangle = 0. \tag{3.37}$$

Since ϕ, ψ are chosen arbitrarily, we must have $\lambda = \bar{\lambda}$ meaning that the eigenvalues are all real. □

But the operator $\mathcal{A}_k$ defined in (3.36) is not necessarily self-adjoint. In the following proposition a sufficient condition for $\mathcal{A}_k$ to be self-adjoint is given.

Proposition 3.10. *Consider the operator $\mathcal{A}_k$ with domain $D(\mathcal{A}_k)$ defined in (3.36). A sufficient condition for the operator $\mathcal{A}_k$ to be self-adjoint is*

$$\frac{\gamma_1}{\gamma} + \frac{\beta_2}{\beta} = 0. \tag{3.38}$$

Proof. For $v, w \in D(\mathcal{A}_k)$ we have

$$\begin{aligned} \langle v, \mathcal{A}_k w\rangle - \langle \mathcal{A}_k v, w\rangle &= \int_0^1 v(x)\left(\frac{d^2w}{dx^2} + a(x)w\right)dx \\ &\quad - \int_0^1 w(x)(\frac{d^2v}{dx^2} + a(x)v)dx \\ &= \int_0^1 v(x)\frac{d^2w}{dx^2}dx - \int_0^1 w(x)\frac{d^2v}{dx^2}dx. \end{aligned}$$

Integrating by parts gives

$$\langle v, A_k w\rangle - \langle A_k v, w\rangle = \left[v\frac{dw}{dx} - w\frac{dv}{dx}\right]_0^1. \tag{3.39}$$

For all $v, w \in D(\mathcal{A}_k)$ the following equations hold:

$$\mathcal{W}v = \gamma\tfrac{\partial v}{\partial x}(1) + \gamma_1 v(0) + \gamma_2 v(1) = 0, \tag{3.40}$$

$$\mathcal{W}w = \gamma\tfrac{\partial w}{\partial x}(1) + \gamma_1 w(0) + \gamma_2 w(1) = 0, \tag{3.41}$$

$$(\mathcal{B} + k\mathcal{C})v = \beta\tfrac{\partial v}{\partial x}(0) + (\beta_1 + k\alpha)v(0) + \beta_2 v(1) = 0, \tag{3.42}$$

$$(\mathcal{B} + k\mathcal{C})w = \beta\tfrac{\partial w}{\partial x}(0) + (\beta_1 + k\alpha)w(0) + \beta_2 w(1) = 0. \tag{3.43}$$

Multiplying both sides of equation (3.40) by $w(1)$, (3.41) by $v(1)$, (3.42) by $w(0)$, and (3.43) by $v(0)$, and doing simple calculations, we obtain

$$v(1)\frac{\partial w}{\partial x}(1) - w(1)\frac{\partial v}{\partial x}(1) = \frac{\gamma_1}{\gamma}(v(0)w(1) - v(1)w(0)), \tag{3.44}$$

$$w(0)\frac{\partial v}{\partial x}(0) - v(0)\frac{\partial w}{\partial x}(0) = \frac{\beta_2}{\beta}(v(0)w(1) - v(1)w(0)). \tag{3.45}$$

The summation of the left-hand sides of equations (3.44) and (3.45) is equal to the right-hand side of (3.39). Thus,

$$\langle v, \mathcal{A}_k w\rangle - \langle \mathcal{A}_k v, w\rangle = (\frac{\gamma_1}{\gamma} + \frac{\beta_2}{\beta})(v(0)w(1) - v(1)w(0)). \tag{3.46}$$

The operator $\mathcal{A}_k$ with domain $D(\mathcal{A}_k)$ is self-adjoint if

$$(\frac{\gamma_1}{\gamma} + \frac{\beta_2}{\beta})(v(0)w(1) - v(1)w(0)) = 0, \tag{3.47}$$

Thus, a sufficient condition for $\mathcal{A}_k$ to be self-adjoint is

$$\frac{\gamma_1}{\gamma} + \frac{\beta_2}{\beta} = 0. \tag{3.48}$$

□

From Proposition 3.10, we observe that in particular if $\gamma_1 = \beta_2 = 0$, the feedback operator $\mathcal{A}_k$ is self-adjoint. This is the case when the input and output operators are purely applied to one end and the homogeneous boundary operator is purely applied to the other end. These systems are referred to as co-located systems.

Consider the closed-loop system shown in Figure 3.1, with output y and input r. We obtained the closed-loop transfer function in (3.2) as

$$G_k(s) = \frac{G(s)}{1 + kG(s)}. \tag{3.49}$$

On the other hand, in Proposition 2.30, we showed that the open-loop transfer function for the system (2.59) can be written as

$$G(s) = \frac{\mathcal{N}(s)}{\mathcal{D}(s)}, \tag{3.50}$$

where $\mathcal{N}(s)$ and $\mathcal{D}(s)$ are entire functions of s. Thus, the closed-loop transfer function is

$$G_k(s) = \frac{\mathcal{N}(s)}{\mathcal{D}(s) + k\mathcal{N}(s)}. \tag{3.51}$$

The equation

$$\mathcal{D}(s) + k\mathcal{N}(s) = 0 \tag{3.52}$$

is hereafter called the characteristic equation.

Proposition 3.11. *For any k, the poles of the transfer function $G_k(s)$ given in (3.51) form a subset of the spectrum of the closed-loop operator $\mathcal{A}_k$.*

Proof. Obtaining the spectrum of the operator $\mathcal{A}_k$ requires solving the system of equations

$$\tfrac{\partial^2 \hat{z}}{\partial x^2} + (a(x) - s)\hat{z} = 0, \tag{3.53}$$

$$\mathcal{W}\hat{z} = 0, \tag{3.54}$$

$$(\mathcal{B} + k\mathcal{C})\hat{z} = 0. \tag{3.55}$$

By Proposition 2.30, there are two unique independent solutions $z_1(x, s)$ and $z_2(x, s)$ that are entire functions of s and satisfy the properties given in the mentioned proposition. Any solution $\hat{z}(x, s)$ of equation (3.53) has the form

$$\hat{z}(x, s) = \sum_{j=1}^{2} a_j(s) z_j(x, s), \tag{3.56}$$

where $a_1(s)$ and $a_2(s)$ are arbitrary functions of s. Since the input, output and homogeneous boundary operators are linear, we have

$$a_1(s)\mathcal{W}z_1 + a_2(s)\mathcal{W}z_2 = 0, \tag{3.57}$$

$$a_1(s)(\mathcal{B} + k\mathcal{C})z_1 + a_2(s)(\mathcal{B} + k\mathcal{C})z_2 = 0. \tag{3.58}$$

In order to have non-zero solutions a_1 and a_2 for this system, we must have

$$\begin{vmatrix} \mathcal{W}z_1 & \mathcal{W}z_2 \\ (\mathcal{B}+k\mathcal{C})z_1 & (\mathcal{B}+k\mathcal{C})z_2 \end{vmatrix} = 0 \tag{3.59}$$

On the other hand, the functions $\mathcal{N}(s)$ and $\mathcal{D}(s)$ for the system (3.31) are obtained in (2.90)-(2.91) and plugging them into (3.51) yields the closed-loop transfer function

$$G_k(s) = \frac{\alpha(z_1(0,s)\mathcal{W}z_2(s) - z_2(0,s)\mathcal{W}z_1(s))}{\mathcal{B}z_1(s)\mathcal{W}z_2(s) - \mathcal{B}z_2(s)\mathcal{W}z_1(s) + k\alpha(z_1(0,s)\mathcal{W}z_2(s) - z_2(0,s)\mathcal{W}z_1(s))}.$$

The roots of equation (3.59) are the roots of the function in the denominator of the above closed-loop transfer function, that is, $\mathcal{D}(s) + k\mathcal{N}(s) = 0$. Note that the functions $\mathcal{N}(s)$ and $\mathcal{D}(s)$ obtained in Proposition 2.30 are not necessarily co-prime. Thus, in the above closed-loop transfer function some pole/zero cancellations may occur. Therefore, closed-loop poles form a subset of the spectrum of the closed-loop operator $\mathcal{A}_k$. □

Example 3.12. *Consider the following system*

$$\begin{cases} \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}, \\ -\frac{\partial z}{\partial x}(0,t) = u(t), \\ z(1,t) = 0, \\ z(0,t) = y(t), \\ z(x,0) = f(x) \in L_2(0,1), \end{cases} \tag{3.60}$$

under the feedback law $u = -ky + r$. We derived the properties of the open-loop system in chapter 2. Now we want to evaluate the closed-loop system. Denote by $\mathcal{A}_0, \mathcal{A}_k$, and $\mathcal{A}_\infty$ the zero-input, feedback, and zero-dynamics operators of the system, respectively.

1. *Find the closed-loop transfer function.*

2. *Find the closed-loop poles and zeros.*

3. *Verify that the closed-loop poles are exactly the spectrum of $\mathcal{A}_k$ with domain $D(\mathcal{A}_k)$.*

4. *Verify that the closed-loop and open-loop zeros are exactly the spectrum of $\mathcal{A}_\infty$ with domain $D(\mathcal{A}_\infty)$.*

For this system the operators $\mathcal{A}_0, \mathcal{A}_k, \mathcal{A}_\infty$ are

$$\begin{aligned}
&\mathcal{A}_0 z = \tfrac{\partial^2 z}{\partial x^2}, \quad D(\mathcal{A}_0) = \{z \in H^2(0,1); z(1,t) = 0, \tfrac{\partial z}{\partial x}(0,t) = 0\},\\
&\mathcal{A}_k z = \tfrac{\partial^2 z}{\partial x^2}, \quad D(\mathcal{A}_k) = \{z \in H^2(0,1); z(1,t) = 0, -\tfrac{\partial z}{\partial x}(0,t) + kz(0,t) = 0\},\\
&\mathcal{A}_\infty z = \tfrac{\partial^2 z}{\partial x^2}, \quad D(\mathcal{A}_\infty) = \{z \in H^2(0,1); z(1,t) = 0, z(0,t) = 0\}.
\end{aligned}$$

Denote by $\hat{z}(x,s), \hat{u}(s)$, and $\hat{y}(s)$ the Laplace transform of $z(x,t), u(t)$, and $y(t)$, respectively. The Laplace transform of the system (3.60), assuming zero initial state, satisfies

$$s\hat{z} = \tfrac{\partial^2 \hat{z}}{\partial x^2}, \tag{3.61}$$

$$-\tfrac{\partial \hat{z}}{\partial x}(0,s) = \hat{u}(s), \tag{3.62}$$

$$\hat{z}(1,s) = 0, \tag{3.63}$$

$$\hat{z}(0,s) = \hat{y}(s). \tag{3.64}$$

Solving (3.61) with the homogeneous boundary condition (3.63) yields

$$\hat{z}(x,s) = a(s)\sinh\left((x-1)\sqrt{s}\right), \tag{3.65}$$

where $a(s)$ is an arbitrary function of s. Therefore, the open-loop transfer function is

$$G(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{\hat{z}(0,s)}{-\frac{\partial \hat{z}}{\partial x}(0,s)} = \frac{\sinh\sqrt{s}}{\sqrt{s}\cosh\sqrt{s}}\ . \tag{3.66}$$

The closed-loop transfer function is

$$G_k(s) = \frac{\hat{y}(s)}{\hat{u}(s) + k\hat{y}(s)} = \frac{\hat{z}(0,s)}{-\frac{\partial \hat{z}}{\partial x}(0,s) + k\hat{z}(0,s)} = \frac{\sinh\sqrt{s}}{\sqrt{s}\cosh\sqrt{s} + k\sinh\sqrt{s}} .$$

It can be seen that the functions $\mathcal{N}(s)$ and $\mathcal{D}(s)$ obtained in Proposition 2.30 are $\mathcal{N}(s) = \sinh(\sqrt{s})/\sqrt{s}$ and $D(s) = \cosh(\sqrt{s})$. These functions are co-prime, that is, they have no common roots. The closed-loop poles are the zeros of $\cosh\sqrt{s} + k\sinh\sqrt{s}/\sqrt{s} = 0$. Also, the spectrum of $\mathcal{A}_k$ must be obtained by solving $-\frac{\partial \hat{z}}{\partial x}(0,s) + k\hat{z}(0,s) = 0$ with (3.65), that is, the equation $\cosh\sqrt{s} + k\sinh\sqrt{s}/\sqrt{s} = 0$. Thus, the closed-loop poles are exactly the spectrum of $\mathcal{A}_k$ with domain $D(\mathcal{A}_k)$.

The closed-loop zeros of the system are exactly the open-loop zeros of the system and in Example 2.42 we showed that the open-loop zeros are exactly the spectrum of the operator A_∞ with domain $D(A_\infty)$.

In the rest of this section, we present a general root-locus analysis for the closed-loop system (3.35), where the operator $\mathcal{A}_k$ with domain $D(\mathcal{A}_k)$ is not necessarily self-adjoint. Recall that s_{ig} is the sign of the instantaneous gain discussed in Proposition 2.36 and proved to be

$$s_{ig} = (-1)^r,\ r = \frac{1}{\pi} arg(\frac{\alpha}{\beta}) + 1, \tag{3.67}$$

with α and β being the coefficients of highest order derivatives of output and input operators, respectively.

In Proposition 3.9, it was shown that when the closed-loop operator $\mathcal{A}_k$ is self-adjoint, the closed-loop poles of the system, that is, the zeros of the characteristic equation are all real. We show that in this case for all $k \in R$ with $k \cdot s_{ig} > 0$, The closed-loop transfer function has no poles on the right of some $a \in R$. If the state operator $\mathcal{A}_k$ is not self-adjoint, the closed-loop poles of the system may be either real or complex and the root-locus branches may move to the right or left. An asymptotic analysis of the characteristic equation shows that if $k \cdot s_{ig} > 0$, then for large $|s|$, the infinitely many real closed-loop poles vary continuously from real open-loop poles to real open-loop zeros that interlace on the negative real axis. Moreover, the number of complex closed-loop poles are finite and all branches containing complex points, can be embedded in a closed contour Ω.

3.2.1 Open-loop Poles and Zeros

As for the finite-dimensional systems, locating the open-loop poles and zeros of an infinite-dimensional system is necessary for a root-locus analysis of the system.

Let us first define the map $z = i\sqrt{s}$ and divide the complex plane into 4 regions

$$S_j = \left\{ z |\ j\frac{\pi}{2} \leq \arg(z) < (j+1)\frac{\pi}{2} \right\},\ j = 0, 1, 2, 3. \tag{3.68}$$

The map $z = i\sqrt{s}$ is a bijection between P_0 and S_0, and between P_1 and S_3, as shown in Figure 3.6. Thus, we can substitute $s = -z^2$ in the characteristic equation and solve the problem of finding the roots of $\mathcal{D}(-z^2) + k\mathcal{N}(-z^2) = 0$ in the regions $S_0 \cup S_3$ for z. Then we can find the roots of $\mathcal{D}(s) + k\mathcal{N}(s) = 0$ in the s-plane.

The following theorem concerns the open-loop poles of large modulus.

Theorem 3.13. *For the system (3.35), all but a finite number of open-loop poles and zeros are real and interlace on the negative real axis. All poles and*

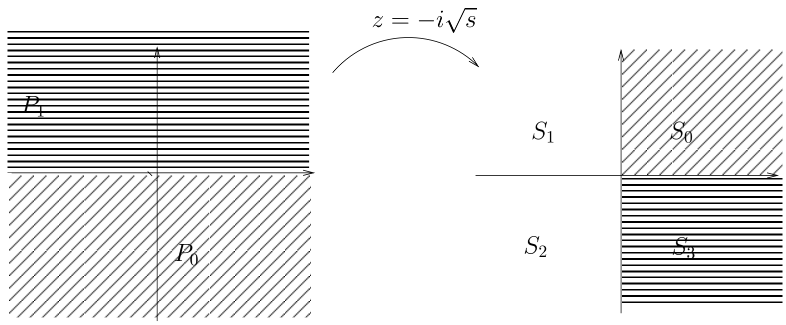

Figure 3.6: The map $z = -i\sqrt{s}$ is a bijection that transforms P_0 onto S_0 and P_1 onto S_3.

zeros diverge to minus infinity.

Proof. In the proof of Proposition 2.30 it was shown that the asymptotic form for $\mathcal{N}(s)$ is

$$\mathcal{N}(s) = (1 + \mathcal{O}(\frac{1}{\sqrt{s}}))\alpha\gamma\cosh(\sqrt{s}), \tag{3.69}$$

and the asymptotic form for $\mathcal{D}(s)$ is

$$\mathcal{D}(s) = -(1 + \mathcal{O}(\frac{1}{\sqrt{s}}))\beta\gamma\sqrt{s}\sinh(\sqrt{s}). \tag{3.70}$$

Denote $\mathcal{D}(-z^2)$ and $\mathcal{N}(-z^2)$ by $\mathcal{D}_z(z)$ and $\mathcal{N}_z(z)$, respectively. In terms of the variable $z = i\sqrt{s}$, the asymptotic form of $\mathcal{D}_z$ and $\mathcal{N}_z$ are

$$\mathcal{D}_z(z) = [\mathbf{1}]\beta\gamma z\sin(z), \tag{3.71}$$

$$\mathcal{N}_z(z) = [\mathbf{1}]\alpha\gamma\cos(z), \tag{3.72}$$

where $[\mathbf{1}]$ is defined to be $[\mathbf{1}] = 1 + \mathcal{O}(\frac{1}{z})$, [1].

For sufficiently large $|z|$, the zeros of $\mathcal{N}_z$ and $\mathcal{D}_z$ are the zeros of $\sin z$ and $\cos z$ and since $z \in S_0 \cup S_3$, the zeros of $\mathcal{D}_z$ and $\mathcal{N}_z$ interlace on the positive real axis. Therefore, the zeros of $\mathcal{D}(s)$ and $\mathcal{N}(s)$, for sufficiently large $|s|$ interlace on the negative real axis and diverge to minus infinity. $\square$

3.2.2 Closed-loop Poles

The theorems in this section are the main results about the closed-loop behavior of the system (3.35). All analysis is based on the assumption that k is chosen so that $k \cdot s_{ig} > 0$ where s_{ig} is the sign of the instantaneous gain discussed in Proposition 2.36 and proved to be $s_{ig} = (-1)^r$, where $r = \frac{1}{\pi} arg(\frac{\alpha}{\beta}) + 1$. The first theorem gives the properties of the asymptotic form of the closed-loop transfer function. The second theorem deals with the remaining finitely many closed-loop poles. Finally, the third theorem deals with the real closed-loop poles.

Theorem 3.14. *For the system (3.35), if k is so chosen that $k \cdot s_{ig} > 0$, the following properties hold.*

(***P1***) *The root-locus branches corresponding to the infinitely many real open-loop poles and zeros are real, simple, countable and diverging to minus infinity.*

(***P2***) *The root-locus branches corresponding to the infinitely many real open-loop poles and zeros move continuously to the left from an open-loop pole to an open-loop zero.*

The following theorem guarantees that the remaining finitely many root-locus branches are bounded and move continuously from an open-loop pole to an open-loop zero .

Theorem 3.15. *For the system (3.35), if k is so chosen that $k \cdot s_{ig} > 0$, then the following properties hold.*

(***Q1***) *There exists a fixed simple closed contour Ω that contains finitely many closed-loop branches such that all branches outside Ω satisfy Theorem*

3.14.

(***Q2***) *All of the closed-loop poles inside* Ω *vary continuously from the open-loop poles to the open-loop zeros.*

As for the finite-dimensional case, the real branches of the root-locus for the system of our interest are branches that lie to the left of an odd number of real open-loop poles and zeros.

Theorem 3.16. *For the feedback system (3.35), if* $k \cdot s_{ig} > 0$*, then a real point on the root-locus always lies to the left of an odd number of real poles and zeros.*

The proof of these theorems are quite long and make use of the results of several subsidiary theorems. Thus, we devote a section to the proof of each theorem. The proofs are based on [4] except the proof of **Q2** where we use a different approach. In this new approach, we deal with the zeros of analytic functions.

3.2.3 Proof of Theorem 3.14

Substituting $\mathcal{D}_z$ and $\mathcal{N}_z$ from (3.71)-(3.72) in the characteristic equation $\mathcal{D}_z + k\mathcal{N}_z = 0$, and simplifying, we obtain

$$-iz\beta[\mathbf{1}](1 - e^{-2iz}) + k\alpha[\mathbf{1}](1 + e^{-2iz}) = 0, \tag{3.73}$$

which implies

$$e^{-2iz} = [\mathbf{1}]\frac{z + ik\alpha/\beta}{z - ik\alpha/\beta}. \tag{3.74}$$

In the proof of Theorem 3.13, we discussed that the map $z = i\sqrt{s}$ is a one-to-one map from the whole complex plane onto the right-half z-plane. Thus, the

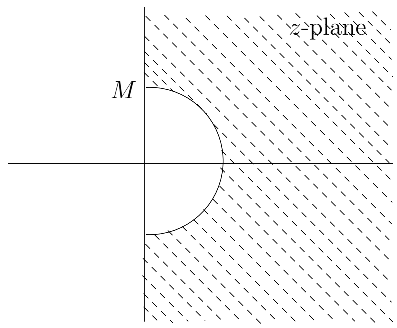

Figure 3.7: The right-hand side of the asymptotic characteristic equation (3.74) is bounded above in the shaded area.

roots of equation (3.74) in the region $S_0 \cup S_3$ defined in (3.68) are associated with the system closed-loop poles via the relation $s = -z^2$. Thus, Theorem 3.14 will follow if it is shown that the roots of (3.74) for sufficiently large $|z|$ are real, simple and move to the right from a root of (3.74) with $k = 0$ to a root of (3.74) when k approaches infinity. In this section, a proof of this statement is developed. To clarify the steps of the proof, we first give an outline of the proof.

(1) We show, through a lemma, that the absolute value of the right-hand side of equation (3.74) is bounded above in the region $(S_0 \cup S_3) \cap \{z; |z| > M\}$, for some $M > 0$ (Figure 3.7). Then, using this lemma, we prove that there exists $y_0 > 0$ such that equation (3.74) has no roots in the region $(S_0 \cup S_3) \cap \{z;\ |z| > M, |Im(z)| > y_0\}$ (Figure 3.8). Therefore, all roots of large modulus lie inside a strip $S = \{z;\ |Im(z)| \leq y_0\} \cap (S_0 \cup S_3)$.

(2) Divide the strip $S = \{z;\ Im(z) \leq y_0\} \cap (S_0 \cup S_3)$ into rectangular regions of width π, starting from a region V. Now we know that the solution belongs to the regions $V_p = V + p\pi, p = 1, 2, ...$ Each $z \in S$ can be written

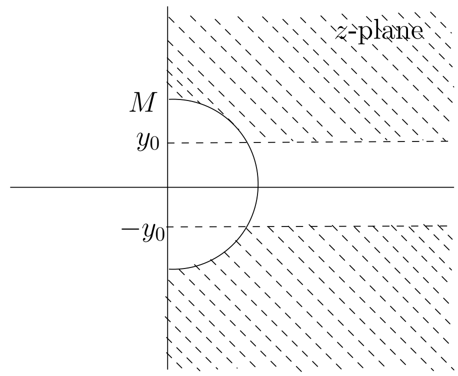

Figure 3.8: The asymptotic characteristic equation (3.74) has no roots in the shaded area

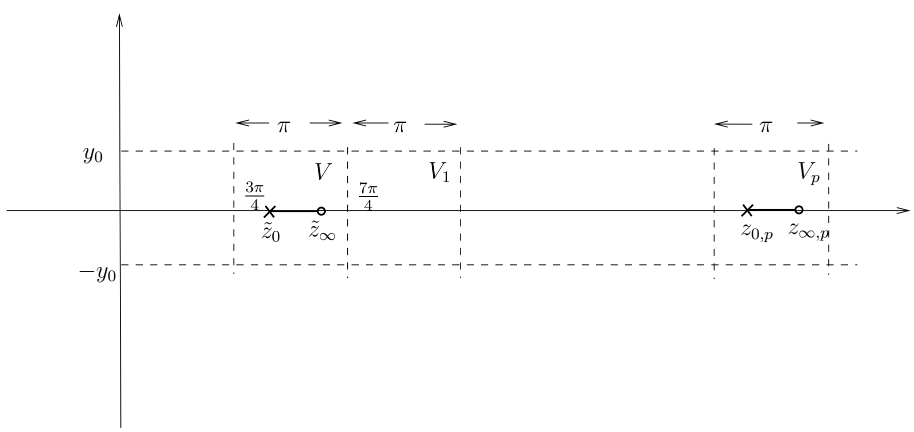

Figure 3.9: The roots of the asymptotic characteristic equation (3.74) lie inside the strip S. Any $z \in V_p$ can be written as $z = \tilde{z} + p\pi$, for some $\tilde{z} \in V$, $p \in \mathbb{N}$

as $z = \tilde{z} + p\pi$ for some $p \in \mathbb{N}$ and $\tilde{z} \in V$. Thus, solving equation (3.74) for $z \in S$ is equivalent to substituting $z = \tilde{z} + p\pi$ in (3.74) and solving for $\tilde{z} \in V$ and $p \in \mathbb{N}$. The region V is chosen so that for all $p > P$, some $P \in \mathbb{N}$, and all $|k| \in [0, \infty)$, the equation (3.74) has no roots on the boundary of $V + p\pi$. This step is illustrated in Figure 3.9.

(3) By using Rouche's theorem, we prove that for each sufficiently large p, the asymptotic characteristic equation has only one root $\tilde{z}$ in V and thus only one root in any region V_p, for all $p \geq P$, for some $P \in \mathbb{N}$. This root has to be real since the roots occur in conjugate pairs.

(4) In the final step of the proof, we show that the roots of equation (3.74) are continuous, monotone increasing functions of k and hence the corresponding roots of the characteristic equation $\mathcal{D}(s) + k\mathcal{N}(s) = 0$ are continuous and move to the left from an open-loop pole to an open-loop zero.

In what follows, we go into the details of all the steps outlined above.

Lemma 3.17. *There exists an $M > 0$ such that*

$$\left|[\mathbf{1}]\frac{z + ik\alpha/\beta}{z - ik\alpha/\beta}\right| \leq 3/2, \tag{3.75}$$

for all k and z with $k \cdot s_{ig} > 0$ and $z \in S_0 \cap \{z : |z| > M\}$.

Proof. The gain k is chosen so that $k \cdot s_{ig} > 0$, or

$$k(-1)^{\frac{1}{\pi}\arg(\frac{\alpha}{\beta})+1} > 0, \tag{3.76}$$

where α and β are coefficients of the highest order derivatives in the output and input operators of the system (3.31), respectively. This implies that if

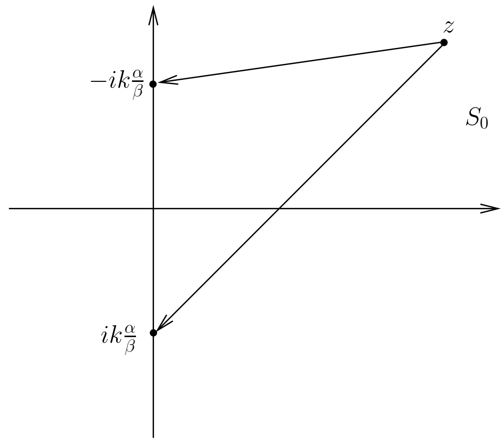

Figure 3.10: For $z \in S_0$, $\left|\frac{z+ik\alpha/\beta}{z-ik\alpha/\beta}\right| \leq 1$ and for $z \in S_3$, $\left|\frac{z+ik\alpha/\beta}{z-ik\alpha/\beta}\right| \geq 1$.

$\frac{\alpha}{\beta} > 0$ then $k < 0$ and if $\frac{\alpha}{\beta} < 0$ then $k > 0$. Therefore, the condition $k \cdot s_{ig} > 0$ implies $k\frac{\alpha}{\beta} < 0$ and we have

$$\arg(-ik\frac{\alpha}{\beta}) = \pi/2. \tag{3.77}$$

Choose $M > 0$ and $C > 2$ such that

$$|\mathcal{O}(\frac{1}{z})| \leq \frac{1}{C}, \ |z| > M. \tag{3.78}$$

From (3.77) and Figure 3.10, for any $z \in S_0$

$$\left|\frac{z + ik\alpha/\beta}{z - ik\alpha/\beta}\right| \leq 1. \tag{3.79}$$

Thus,

$$\left|(1 + \mathcal{O}(\frac{1}{z}))\frac{z + ik\alpha/\beta}{z - ik\alpha/\beta}\right| \leq (1 + 1/C) \leq 3/2, \tag{3.80}$$

for all $z \in S_0 \cap \{z; |z| > M\}$. This completes the proof. $\square$

Note that in the proof of Lemma 3.17 if we choose smaller upper bounds for $|\mathcal{O}(\frac{1}{z})|$, we will end up with a tighter upper bound for the right-hand side of (3.74). To avoid calculation complexities, we simply use this bound.

Lemma 3.18. *Suppose k is so chosen that $k \cdot s_{ig} > 0$. Then, there exist positive constants M, y_0, independent of k, such that the equation*

$$e^{-2iz} = [\mathbf{1}]\frac{z + ik\alpha/\beta}{z - ik\alpha/\beta}, \tag{3.81}$$

has no roots for $z \in (S_0 \cup S_3) \cap \{z : |z| > M, |Im(z)| > y_0\}$.

Proof. By Lemma 3.17, there exist $M > 0$ and $C > 2$ such that the modulus of the right-hand side of equation (3.81) is not greater than $3/2$ for $z \in S_0 \cap \{z; |z| > M\}$. Write $z = x + iy_0$ for real $x > 0$ and $y_0 \geq 0$. The left-hand side of (3.81) satisfies

$$|e^{-2iz}| = |e^{-2ix+2y_0}| = e^{2y_0}. \tag{3.82}$$

We can find $y_0 > 0$ such that $e^{2y_0} = 2 > 3/2$. For this value of y_0, the equation (3.74) has no roots in $z \in S_0 \cap \{z; |z| > M\}$.

On the other hand, the closed-loop poles occur in complex conjugate pairs, because in Proposition 2.30 we proved that the functions $\mathcal{N}(s)$ and $\mathcal{D}(s)$ are real functions of s in the sense that $\mathcal{N}(\bar{s}) = \overline{\mathcal{N}(s)}$ and $\mathcal{D}(\bar{s}) = \overline{\mathcal{D}(s)}$. Therefore, with the obtained value of y_0, equation (3.74) has no roots in $z \in (S_0 \cup S_3) \cap \{z : |z| > M, |Im(z)| > y_0\}$. Now the result follows. ☐

We need the following two lemmas to show that for sufficiently large $|z|$ the poles are real and simple.

Lemma 3.19. *The roots of the equation*

$$e^{-2iz} = \frac{z + ik\alpha/\beta}{z - ik\alpha/\beta} \tag{3.83}$$

are all real and simple for $z \in (S_0 \cup S_3) \cap \{z; |z| > c\}$, for some positive constant c and all k such that $k \cdot s_{ig} > 0$.

Proof. Write $z = x + iy$. In the proof of Lemma 3.17 we showed that if $z \in S_0$, then $|\frac{z+ik\alpha/\beta}{z-ik\alpha/\beta}| \leq 1$. Thus, from equation (3.83),

$$|\frac{z+ik\alpha/\beta}{z-ik\alpha/\beta}| = |e^{-2iz}| = e^{2y} \leq 1, \ z \in S_0. \tag{3.84}$$

Thus we must have $y = 0$. Similarly, if $z \in S_3$, from (3.77) and Figure 3.10, we have $|\frac{z+ik\alpha/\beta}{z-ik\alpha/\beta}| \geq 1$ and hence

$$|\frac{z+ik\alpha/\beta}{z-ik\alpha/\beta}| = |e^{-2iz}| = e^{2y} \geq 1, \ z \in S_3. \tag{3.85}$$

Thus, y has to be zero. Hence, if $z = x + iy$ satisfies (3.83), then $y = 0$, which implies that the roots are all real.

To prove that the roots are simple, we take the derivative of equation (3.81) with respect to z,

$$\frac{\partial}{\partial z}(-e^{-2iz} + \frac{z+ik\alpha/\beta}{z-ik\alpha/\beta}) = 2ie^{-2iz} - \frac{2ik\alpha/\beta}{(z-ik\alpha/\beta)^2}. \tag{3.86}$$

A point $z \in \mathbb{C}$ is a multiple root of equation (3.83) if z is a zero of (3.83) and a zero of (3.86). We proved that the roots of equation (3.83) are all real, so we need to find real roots of (3.86). Simple calculations show that the function given in (3.86) is never zero for $x \in \mathbb{R}$, $x > 1/2$. Therefore, all roots are real and simple for $\{z; |z| > 1/2\} \cap (S_0 \cup S_3)$. □

Lemma 3.20. *Suppose k is chosen so that $k \cdot s_{ig} > 0$. Define the functions*

$$F_p(\tilde{z}, k) = e^{-2i\tilde{z}} - [\mathbf{1}]\frac{\tilde{z}+p\pi+ik\alpha/\beta}{\tilde{z}+p\pi-ik\alpha/\beta}, \tag{3.87}$$

$$f_p(\tilde{z}, k) = e^{-2i\tilde{z}} - \frac{\tilde{z}+p\pi+ik\alpha/\beta}{\tilde{z}+p\pi-ik\alpha/\beta}, \tag{3.88}$$

$$h_p(\tilde{z}, k) = e^{-2i\tilde{z}} - \frac{p\pi+ik\alpha/\beta}{p\pi-ik\alpha/\beta}, \tag{3.89}$$

where p is a positive integer and $\tilde{z}$ belongs to

$$V = \{\tilde{z}; |Im(\tilde{z})| \leq y_0,\ 3\pi/4 \leq Re(\tilde{z}) \leq 7\pi/4\}, \tag{3.90}$$

and $y_0 > 0$ is obtained in Lemma 3.18. The following statements hold.

(1) $|f_p(\tilde{z}, k) - h_p(\tilde{z}, k)|$ tends to zero uniformly in k for $\tilde{z} \in V$ as $p \to \infty$.

(2) $h_p(\tilde{z}, k)$ has exactly one root in the interior of V.

(3) There exists a constant C such that $|h_p(\tilde{z}, k)| > C,\ \tilde{z} \in \partial V$, for all k, where ∂V is the boundary of the region V.

(4) There exists a positive integer P_1 such that for $p > P_1$,

$$|f_p(\tilde{z}, k)| > C/2,\ \ for\ \tilde{z} \in \partial V, \forall k. \tag{3.91}$$

(5) $|F_p(\tilde{z}, k) - f_p(\tilde{z}, k)|$ tends to zero uniformly in k for $\tilde{z} \in V$ as $p \to \infty$.

(6) There exists a positive integer $P_2 > P_1$ such that for $p > P_2$,

$$|F_p(\tilde{z}, k)| > C/3, for\ \tilde{z} \in \partial V, \forall k. \tag{3.92}$$

Proof. As we have shown in Lemma 3.18, the asymptotic characteristic equation has no roots in the region $(S_0 \cup S_3) \cap \{z; |Im(z)| > y_0, |z| > M\}$. Thus, all roots of large modulus lie inside the strip $S = (S_0 \cup S_3) \cap \{z; |Im(z)| \leq y_0\}$. We divide this strip into sections of width π starting from the region V defined in (3.90). For any $z \in S$ (with $Re(z) > 7\pi/4$) we have $z = \tilde{z} + p\pi$, for some$p \in \mathbb{N}$ and some $\tilde{z} \in V$. If $z \in S$ is a root of the characteristic equation (3.74), we have $z = \tilde{z} + p\pi$, for some $\tilde{z} \in V$. Since we know that the roots of large modulus belong to rectangles $V + p\pi, p \in \mathbb{N}$, we substitute $z = \tilde{z} + p\pi$ in equation (3.74) to obtain

$$e^{-2i\tilde{z}} = [1]\frac{\tilde{z} + p\pi + ik\alpha/\beta}{\tilde{z} + p\pi - ik\alpha/\beta}. \tag{3.93}$$

Thus, finding the roots of the characteristic equation (3.74) is equivalent to finding the roots of $F_p(\tilde{z},k)$ for $\tilde{z}\in V$ and $p\in\mathbb{N}$. This is illustrated in Figure 3.9. Thus, to say that the equation (3.74) has real and simple roots for large $|z|$, is to say that for all sufficiently large p, the roots of $F_p(\tilde{z},g)$ in the region $\overline{V}$ are real and simple. To this end, we make use of the two defined auxiliary functions, i.e., $f_p(\tilde{z},k)$ and $h_p(\tilde{z},k)$.

We have that

$$\begin{aligned} |f_p(\tilde{z},k)-h_p(\tilde{z},k)| &= \left|\frac{\tilde{z}+p\pi+ik\alpha/\beta}{\tilde{z}+p\pi-ik\alpha/\beta}-\frac{p\pi+ik\alpha/\beta}{p\pi-ik\alpha/\beta}\right| \\ &= \left|\frac{2kp\pi\alpha/\beta}{(\tilde{z}+p\pi-ik\alpha/\beta)(p\pi-ik\alpha/\beta)}\right|\cdot\left|\frac{\tilde{z}}{p\pi}\right|. \end{aligned} \tag{3.94}$$

We want to prove that $w(p,k)=\left|\frac{2kp\pi\alpha/\beta}{(\tilde{z}+p\pi-ik\alpha/\beta)(p\pi-ik\alpha/\beta)}\right|$ is a bounded function of p,k. To be able to use Extreme Value Theorem, for the moment, we assume that p can take any positive real number. The term in the denominator of $w(p,k)$ is nonzero for all $\tilde{z}\in V$, all k, and all p and hence, $w(p,k)$ is a continuous function of p and k. It is easy to show that $\lim_{p\to\infty}w=0$, for any k. Also, $\lim_{|k|\to\infty}w=0$, for any p. Thus, $w(p,k)$ is arbitrarily small outside a compact set. By the Extreme Value Theorem, $w(p,k)$ attains its extreme values inside that compact set. Thus, $w(p,k)$ is a bounded function of p,k and from (3.94)we conclude that

$$|f_p(\tilde{z},k)-h_p(\tilde{z},k)|=\mathcal{O}(1/p), \tag{3.95}$$

and hence it goes to zero uniformly in k as $p\to\infty$. This completes the proof of (1).

Proof of (2): Lemma 3.19 implies that the roots of $h_p(\tilde{z},k)$ in V are real and simple for all k and $p \in \mathbb{N}$. The real roots of $h_p(\tilde{z}, k)$ are

$$\tilde{z}_{k,p} = \frac{1}{2}\left(arg(p\pi + ik\alpha/\beta) - arg(p\pi - ik\alpha/\beta) + \pi + 2n\pi\right), \ n \in \mathbb{Z}. \quad (3.96)$$

Hence,

$$\tilde{z}_{k,p} = \pi - arg(p\pi - ik\alpha/\beta) + \pi/2 + n\pi, \ n \in \mathbb{Z}. \quad (3.97)$$

Recall that if $k \cdot s_{ig} > 0$, then $k\alpha/\beta < 0$. Thus, $0 \leq arg(p\pi - ik\alpha/\beta) \leq \pi/2$ and hence each root satisfies

$$\pi + n\pi \leq \tilde{z}_{k,p} \leq 3\pi/2 + n\pi, \ n \in \mathbb{Z}. \quad (3.98)$$

Among the roots of $h_p(\tilde{z}, k)$ the only root that lies in V is the one that satisfies

$$\pi \leq \tilde{z}_{k,p} \leq 3\pi/2. \quad (3.99)$$

The proof of (2) is now complete.

Proof of (3): We showed that the roots of $h_p(\tilde{z}, k)$ in V are always real and belong to the interval $[\pi, 3\pi/2]$ and hence h_p is not zero on ∂V. Moreover, it is easy to show that h_p is a continuous function of $\tilde{z}, k$ on V. The continuity of $h_p(\tilde{z}, k)$ on V requires the existence of a non-zero constant C so that $|h| > C > 0$ on the boundaries.

Proof of (4): Let P_1 be a number such that for $p > P_1$ and $\tilde{z} \in V$,

$$|f_p(\tilde{z}, k) - h_p(\tilde{z}, k)| < C/2, \quad (3.100)$$

and by the previous part,

$$|h_p(\tilde{z}, k)| > C, \ \tilde{z} \in \partial V \quad (3.101)$$

Thus, by using the triangle inequality,

$$|f_p(\tilde{z}, k)| > C/2, \ for \ \tilde{z} \in \partial V, \forall k, p. \quad (3.102)$$

This completes the proof of (4).

The proofs of (5) and (6) are similar to those of (1) and (4), respectively.

□

Now we use Rouche's theorem to conclude that $F_p(\tilde{z}, k)$ has only one root in V, which must be real since the roots appear in conjugate pairs.

Theorem 3.21. ***Rouche's theorem*** *([25], Theorem 10.43) If f_1 and f_2 are two analytic functions on a domain D that contains a simple, closed contour Γ, and if $|f_1(z)| > |f_2(z)|$, $z \in \Gamma$, then f_1 and $f_1 + f_2$ have the same number of zeros in the interior of the region enclosed by Γ.*

Lemma 3.22. *Consider the function $F_p(\tilde{z}, k)$ defined in (3.87), where $\tilde{z}$ belongs to the set V defined in (3.90).*

The following statements hold,

(1) There exists an integer number $P > 0$ such that the function $F_p(\tilde{z}, k)$ has only one root in V, for each $p > P$.

(2) For $p > P$, the only zero of $F_p(\tilde{z}, k)$ in V is a continuous, monotone increasing function of k for $|k| \to \infty$, with $k \cdot s_{ig} > 0$.

Proof. Proof of (1): in Lemma 3.20(4)-(5), we proved that there exists a number $\hat{P}$ such that for $p > P$,

$$|F_p(\tilde{z}, k) - f_p(\tilde{z}, k)| < C/2 < |f_p(\tilde{z}, k)|, \ \forall \tilde{z} \in \partial V. \tag{3.103}$$

Thus, by Rouche's Theorem, $f_p(\tilde{z}, k)$ and $F_p(\tilde{z}, k)$ have the same number of zeros in V for $p > \hat{P}$. Similarly, from Lemma 3.20(1)-(3), there exists a number $P > \hat{P}$ such that for $p > P$,

$$|h_p(\tilde{z}, k) - f_p(\tilde{z}, k)| < C/2 < |h(\tilde{z}, k)|, \ \forall \tilde{z} \in \partial V. \tag{3.104}$$

Thus, $f_p(\tilde{z},k)$ and $h_p(\tilde{z},k)$ have the same number of zeros in V for $p > P$. Since, from Lemma 3.20(2), $h_p(\tilde{z},k)$ has only one root in V, $F_p(\tilde{z},k)$ has also only one root in V which has to be real since all roots appear in conjugate pairs. Proof of (1) is now complete.

Before we prove (2), we need to recall the Implicit Function Theorem.

Theorem 3.23. ***Implicit Function Theorem*** *([3], page 194) Let $\phi(z,w)$ be a complex-valued function having continuous partial derivatives. Suppose $\phi(z_0,w_0) = 0$ and $\phi_y(z_0,w_0) \neq 0$. Then in some interval around z_0, there is a unique continuously differentiable function $w = \psi(z)$ such that $w_0 = \psi(z_0)$ and $\phi(z,\psi(z)) = 0$. If ϕ has continuous partial derivatives of $n \geq 1$ order, then ψ has n continuous derivatives.*

Proof of (2): Denote by $\tilde{z}_{k,p}$ the only root of $F_p(\tilde{z},k)$ in V. First we show that $\tilde{z}_{k,p}$ is continuous in k. For any fixed $p > P$, P obtained in (1), the roots of $F_p(\tilde{z},k)$ are simple and hence $\frac{\partial F_p(\tilde{z},k)}{\partial \tilde{z}} \neq 0$ for all $\tilde{z} \in V$ and all k. Therefore, by the Implicit Function Theorem, if $F_p(\tilde{z}_{k_0,p},k_0) = 0$, then $\tilde{z}_{k,p}$ can be written as a continuous function of k in some neighborhood of k_0. thus, $\tilde{z}_{k,p}$ is a continuous function of k for all k.

Now we show that $\tilde{z}_{k,p}$ is monotone in k. Suppose $\tilde{z}_{k_1,p} = \tilde{z}_{k_2,p} = \tilde{z}$. Since $F_p(\tilde{z},k_1) = F_p(\tilde{z},k_2) = 0$, we have that

$$\frac{\tilde{z} + p\pi + ik_1\alpha/\beta}{\tilde{z} + p\pi - ik_1\alpha/\beta} = \frac{\tilde{z} + p\pi + ik_2\alpha/\beta}{\tilde{z} + p\pi - ik_2\alpha/\beta} \tag{3.105}$$

which implies $k_1 = k_2$. From this and the continuity of $\tilde{z}_{k,p}$ with respect to k, we conclude that $\tilde{z}_{k,p}$ is a monotone function of k. To prove that $\tilde{z}_{k,p}$ is increasing with k, it suffices to compare $\tilde{z}_{0,p}$ and $\lim_{|k|\to\infty} \tilde{z}_{k,p}$. As k tends to zero, the root in V of $F_p(\tilde{z}, k)$ satisfies

$$e^{-2i\tilde{z}_{0,p}} = 1, \tag{3.106}$$

which implies that $\tilde{z}_{0,p} = \pi$. Similarly, as k approaches infinity, the root in V of $F_p(\tilde{z}, k)$ satisfies

$$e^{-2i\tilde{z}_{\infty,p}} = -1, \tag{3.107}$$

which implies $\tilde{z}_{\infty,p} = 3\pi/2$. Thus, $\pi = \tilde{z}_{0,p} < \tilde{z}_{g,p} < \tilde{z}_{\infty,p} = 3\pi/2$ and hence $\tilde{z}_{k,p}$ is a monotone increasing real-valued function of k. The proof of (2) in now complete. □

Note that for all $p > P$, P obtained in the above lemma, $z_0 = \tilde{z}_{0,p} + p\pi$ is a zero of the characteristic equation for $k = 0$, which implies that $z_{0,p}$ is the only open-loop pole in $V + p\pi$.

In the proof of Lemma 3.20 we showed $z \in S$ is a root of the asymptotic characteristic equation (3.74) if and only if $\tilde{z} = z - p\pi$ is a root of $F_p(\tilde{z}, k)$ in V, for some $p \in \mathbb{N}$. So we have the following result about the roots of the asymptotic characteristic equation.

Lemma 3.24. *Define*

$$F(z, k) = e^{-2iz} - [1]\frac{z + ik\alpha/\beta}{z - ik\alpha/\beta}, \tag{3.108}$$

where $z \in S_0 \cup S_3$ with S_0, S_3 defined in (3.68). For sufficiently large $|z|$ and for any given k satisfying $k \cdot s_{ig} > 0$, the roots of $F(z, k)$ are real, simple, countable, and divergent to plus infinity. Furthermore, as $|k|$ migrates from zero to infinity, these roots move to the right monotonically and continuously from a root of $F(z, 0)$ to a root of $F(z, \infty)$.

Proof. In Lemma 3.22 we proved that there exists $P \in \mathbb{N}$ such that for any $p > P$ the function $F_p(\tilde{z}, k)$ has only one root $\tilde{z}_{k,p}$ in V. Thus, for any $p > P$, $F(z, k)$ has only one root $z_k = \tilde{z}_{k,p} + p\pi$ in $V + p\pi$, which implies that for $|z| > P\pi + 7\pi/4$ all roots of $F(z, k)$ are real, simple, countable and divergent to plus infinity. Furthermore, since for any $p > P$ the only root of $F_p(\tilde{z}, k)$ moves monotonically and continuously to the right from $\tilde{z}_{0,p}$ to $\tilde{z}_{\infty,p}$ as k moves from zero to infinity, we can say that in each region $V + p\pi$, for $p > P$, the only root of $F(z, k)$, $z_k = \tilde{z}_{k,p} + p\pi$, moves to the right from the only root of $F(z, 0)$ to the only root of $F(z, \infty)$ in $V + p\pi$. This completes the proof. □

The proof of Theorem 3.14 is now straightforward due to the mapping $s = -z^2$:

Verification of **P1**: In Lemma 3.24, we proved that for sufficiently large $|z|$ the roots of $F(z, k)$, or the roots of asymptotic characteristic equation (3.74), are positive real, simple, countable and diverge to plus infinity. Thus, so are the roots of $\mathcal{D}_z + k\mathcal{N}_z = 0$ for sufficiently large $|z|$. Thus, the roots of the characteristic equation $\mathcal{D}(s) + k\mathcal{N}(s) = 0$ for sufficiently large $|s|$ are negative real, simple, countable and diverge to minus infinity, via the relation $s = -z^2$.

Verification of **P2**: The root-locus branches corresponding to the infinitely many real open-loop poles and zeros move continuously to the left from an open-loop pole, that is, a zero of the characteristic equation for $k = 0$ to an open-loop zero, that is, a zero of the characteristic equation when k approaches infinity.

The proof of Theorem 3.14 is now complete. This theorem ensures that outside some fixed contour Ω, all closed loop poles are real and negative. Thus, as far as stability is concerned, we only need to assess if the finitely many closed-loop poles inside Ω lie in the open left-half plane or for which values of the feedback gain k the closed-loop poles are in the left-half plane.

3.2.4 Proof of Theorem 3.15

Let us first restate Theorem 3.15. We want to prove that for the system (3.35), if k is so chosen that $k \cdot s_{ig} > 0$, then the following properties hold.

(**Q1**) There exists a fixed simple closed contour Ω that contains finitely many closed-loop branches such that all branches outside Ω satisfy Theorem 3.14.

(**Q2**) All of the closed-loop poles inside Ω vary continuously from the open-loop poles to the open-loop zeros.

Proof of Q1: We must prove that the remaining finitely many root-locus branches that are not considered in Theorem 3.14, lie inside a fixed simple closed contour Ω. The closed-loop poles of the system (3.35) are the roots of the characteristic equation $\mathcal{D}(s) + k\mathcal{N}(s) = 0$. In Proposition 2.30, we proved that $\mathcal{N}(s)$ and $\mathcal{D}(s)$ are entire functions of s. Therefore, by Theorem 2.45, the zeros of the characteristic equation are isolated with no accumulation points in the complex plane. Thus, for each k there exists a closed contour Ω_k that separates the roots of the the characteristic equation into a finite part embedded inside Ω_k and an infinite part outside Ω_k. Furthermore from Theorem 3.14(**P1**), we know that for sufficiently large $|s|$ equation (3.74) has no roots outside some fixed closed contour with the exception of those infinitely many closed-loop poles that we dealt with in Theorem 3.13.

Thus, the family of contours Ω_k is uniformly bounded. Thus, a separating closed contour Ω can be found that is independent from k. This closed contour embeds the remaining finite number of root-locus branches that are not considered in Theorem 3.14. This completes the proof of **Q1**.

To prove **Q2**, we need the following lemma.

Lemma 3.25. *Let $g(s)$ be an analytic function in a domain containing $s = 0$ with $g(0) \neq 0$. Consider the equation*

$$s^n g(s) = ty(s), \tag{3.109}$$

where $n \geq 1$, $t \in \mathbb{R}$ and y is an analytic function in a domain containing $s = 0$. Then, as t goes to zero, exactly n roots of equation (3.109) approach zero. In other words, for all sufficiently small values of $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|t| < \delta$, equation (3.109) possesses exactly n roots in $B_\epsilon(0) = \{s \in \mathbb{C}; |s| < \epsilon\}$.

Proof. Since $g(0) \neq 0$ and from analyticity of g, there exist $r_1 > 0$ and $0 < l < L_1$ such that $l < |g(s)| < L_1$, for all s in the closed ball $\overline{B_{r_1}(0)}$. Also, by analyticity of y, there exist $r_2 > 0$ and $l < L_2$ such that $|y(s)| < L_2$, for all s in the closed ball $\overline{B_{r_2}(0)}$. Let $r = \min\{r_1, r_2\}$ and $L = \max\{L_1, L_2\}$ We use Rouche's theorem (Theorem 3.21) to prove this lemma.

Take any ϵ such that $0 < \epsilon < r$. Pick a positive real number $\alpha < \frac{l}{2}$. Define functions

$$g_1(s) = s^n g(s) - \alpha s^n, \tag{3.110}$$

$$g_2(s) = \alpha s^n - ty(s). \tag{3.111}$$

For $s \in \partial B_\epsilon(0)$ we can write $s = \epsilon e^{j\phi}$ and hence

$$\begin{aligned} |g_1(s)| &= |s^n||g(s) - \alpha| \\ &= \epsilon^n |g(s) - \alpha| \\ &\geq \epsilon^n \Big| |g(s)| - \alpha \Big| \\ &\geq \epsilon^n (l - \alpha), \end{aligned} \tag{3.112}$$

for all $s \in \partial B_\epsilon(0)$. Moreover, for $s \in \partial B_\epsilon(0)$,

$$\begin{aligned} |g_2(s)| &= |\alpha \epsilon^n e^{jn\phi} - ty(s)| \\ &= \epsilon^n |\alpha e^{jn\phi} - \frac{ty(s)}{\epsilon^n}| \\ &\leq \epsilon^n (|\alpha e^{jn\phi}| + |\frac{ty(s)}{\epsilon^n}|) \\ &\leq \epsilon^n (\alpha + \frac{tL}{\epsilon^n}). \end{aligned} \tag{3.113}$$

Set $\delta = \epsilon^n (l - 2\alpha)/L$. For $|t| < \delta$ we have

$$\alpha + \frac{tL}{\epsilon^n} < l - \alpha \tag{3.114}$$

and therefore, $|g_2| < |g_1|$ on $\partial B_\epsilon(0)$ for $|t| < \delta$. Thus, by Rouche's Theorem, the number of the roots of g_1 and $g_1 + g_2$ inside $B_\epsilon(0)$ are equal when $|t| < \delta$. But g_1 has exactly n roots in $B_\epsilon(0)$. These n roots are $s = 0$ with multiplicity n (Note that $g(s) - \alpha \neq 0$ in $B_\epsilon(0)$, because we know that in this ball $|g| > l$ and we picked $\alpha < l/2$ when we defined g_1 and g_2 in (3.110)-(3.111)). Therefore, $g_1 + g_2 = s^n g(s) - ty(s)$ has exactly n roots in $B_\epsilon(0)$ when $|t| < \delta$. This completes the proof. □

Proof of $Q2$**:** In the previous section, we proved that for the system (3.35), if k is so chosen that $k \cdot s_{ig} > 0$, then for sufficiently large $|s|$, the closed-loop poles vary continuously from the open-loop poles to the open-loop zeros. Now we give a general proof. The closed-loop poles of the system (3.35) are the roots of the characteristic equation $f(s,k) = \mathcal{D}(s) + k\mathcal{N}(s)$, where $\mathcal{D}(s)$ and $\mathcal{N}(s)$ are entire functions of s. Thus, $f(s,k)$ has continuous partial derivatives of all orders. Thus, by the Implicit Function Theorem, if (s_0, k_0) satisfies $f(s_0, k_0) = 0$ and $f_s(s_0, k_0) \neq 0$, then s can be represented as a differentiable function $s = v(k)$ in some neighborhood of k_0 such that $v(k_0) = s_0$ and $f(v(k), k) = 0$.

It remains to prove **Q2** for the points (s_0, k_0) where $f(s_0, k_0) = f_s(s_0, k_0) = 0$. These points correspond to multiple roots of f and need to be treated differently. Since we proved that outside the closed contour Ω all roots of the characteristic equation are simple roots, we know that all multiple roots lie inside Ω. Since f is an entire function of s, by Theorem 2.45, any zero of f has a finite order. Let $s = s_0$ be a zero of order n for $f(s, k_0) = 0$. Then

$$\frac{\partial^i f}{\partial s^i}(s_0, k_0) = 0, \ 0 \leq i \leq n-1, \tag{3.115}$$

$$\frac{\partial^n f}{\partial s^n}(s_0, k_0) \neq 0. \tag{3.116}$$

Furthermore,

$$\begin{aligned} f(s, k_0) &= \mathcal{D}(s) + k_0\mathcal{N}(s) \\ &= (s - s_0)^n g(s, k_0), \end{aligned} \tag{3.117}$$

where $g(s_0, k_0) \neq 0$. Moreover, we can show that $g(s, k_0)$ is an analytic function in some neighborhood of s_0. Using equations (3.115)-(3.116), we have

$$\begin{aligned}(s-s_0)^n g(s,k_0) &= f(s,k_0) \\ &= \sum_{i=1}^{\infty} \frac{f^{(i)}(s_0,k_0)}{i!}(s-s_0)^i \\ &= (s-s_0)^n \sum_{i=n}^{\infty} \frac{f^{(i)}(s_0,k_0)}{i!}(s-s_0)^{i-n}, \qquad (3.118)\end{aligned}$$

which implies $g(s,k_0)$ has a Taylor expansion in a neighborhood of s_0 and hence is analytic in this neighborhood.

Now let $k = k_0 + \epsilon$. The function $f(s,k)$ can be written as

$$\begin{aligned}f(s,k) &= \mathcal{D}(s) + k_0\mathcal{N}(s) + \epsilon\mathcal{N}(s) \\ &= (s-s_0)^n g(s,k_0) + \epsilon\mathcal{N}(s). \qquad (3.119)\end{aligned}$$

We need to prove that as ϵ approaches zero, exactly n roots of $f(s,k)$ approach s_0. Since $\mathcal{N}(s)$ is an entire function, it is bounded on the region surrounded by Ω which contains $s = s_0$. Also, $g(s,k_0)$ is analytic in some neighborhood of $s = s_0$ and $g(s_0,k_0) \neq 0$. Now Lemma 3.25 can be applied to conclude that exactly n roots of $f(s,k)$ approach s_0 when ϵ approaches 0, that is,k approaches zero. This completes the proof of **Q2**.

In [4], the statement of **Q2** is proved using a different argument. In this paper, the proof is separated into two parts; the infinite branches of poles are treated similar to our proof of **P2**, and the finite number of poles that are embedded in the closed curve Ω, are treated based on [13] (chapter III, §6.4, theorem 6.17 and chapter VII, §1.3, theorem 1.7). Kato proved that if $\mathcal{A}_k$ is a family of operators dependent on k analytically, and the spectrum $\Sigma(\mathcal{A}_k)$ of $\mathcal{A}_k$ has two separate parts; a finite part Σ' and an infinite part Σ", then the space $D(\mathcal{A}_k)$ can be decomposed into M' and M", and the operator $\mathcal{A}_k$ can be decomposed to $\mathcal{A}_k(M')$ and $\mathcal{A}_k(M")$ in such a way that the spectrum of $\mathcal{A}_k(M')$ and $\mathcal{A}_k(M")$ lie in M' and M", respectively. Furthermore, $\mathcal{A}_k(M')$ and $\mathcal{A}_k(M")$ depend on k analytically.

3.2.5 Proof of Theorem 3.16

Let us first restate Theorem 3.16. We want to prove that for the feedback system (3.35), if $k \cdot s_{ig} > 0$, then a real point on the root-locus always lies to the left of an odd number of real poles and zeros.

Theorem 3.26. ***Hadamard factorization theorem*** *([16], Lecture 4).*

An entire function f of order $\mathbb{O}(f)$ (Definition 2.28) may be represented in the form

$$f(z) = z^m e^{P(z)} \prod_{n=1}^{\infty} Q(\frac{z}{a_n}, p), \tag{3.120}$$

where $a_1, a_2, ...$ are nonzero roots of $f(z)$, $p \leq \mathbb{O}(f)$, m is the multiplicity of the root at the origin, $P(z)$ is a polynomial of degree $q \leq \mathbb{O}(f)$, and

$$Q(w,p) = \begin{cases} 1-w & p = 0, \\ (1-w)e^{w+w^2/2+...+w^p/p} & p > 0. \end{cases} \tag{3.121}$$

The number p is the smallest integer number for which $\sum_{n=1}^{\infty} |a_n|^{-p-1} < \infty$, called the rank of $f(z)$.

Proposition 3.27. *For the system (2.59) with transfer function $G(s)$ given by Proposition 2.30, the functions $\mathcal{N}(s)$ and $\mathcal{D}(s)$ have rank zero.*

Proof. By Theorem 3.13, all but a finite number of the roots of $\mathcal{N}(s) = 0$ are negative real numbers asymptotic to $\{s_n = -n^2\pi^2\}_{n=N}^{\infty}$, where N is some positive integer number. Therefore, $\sum_{n=N}^{\infty} |s_n|^{-1} < \infty$ from which it can be deduced that $\mathcal{N}(s)$ has rank zero. The argument for $\mathcal{D}(s)$ is similar. □

By Proposition 2.30, $\mathcal{N}(s)$ and $\mathcal{D}(s)$ are entire functions of s. Thus, from Hadamard Factorization Theorem and Proposition 3.27 the following important property follows.

Proposition 3.28. *For the system (2.59) with transfer function $G(s)$ given by Proposition 2.30, the functions $\mathcal{N}(s)$ and $\mathcal{D}(s)$ can be represented as*

$$\begin{aligned} \mathcal{N}(s) &= C_1 s^p \prod_{n=p+1}^{\infty} \left(1 - \frac{s}{z_n}\right), \\ \mathcal{D}(s) &= C_2 s^q \prod_{n=q+1}^{\infty} \left(1 - \frac{s}{p_n}\right), \end{aligned} \tag{3.122}$$

where $\{z_n\}_{n=p+1}^{\infty}$ and $\{p_n\}_{n=q+1}^{\infty}$ are non-zero zeros and poles of the transfer function $G(s)$, respectively, and $C_1, C_2 \in \mathbb{R}$.

To prove Theorem 3.16, we need the following lemma.

Lemma 3.29. *For the system (2.59), let s_{ig} be the sign of the instantaneous gain given in Proposition 2.36 and the system transfer function be $G(s) = \mathcal{N}(s)/\mathcal{D}(s)$ with the representation of $\mathcal{N}(s)$ and $\mathcal{D}(s)$ given by (3.122). We have*

$$\arg(k \cdot s_{ig}) = \arg\left(k(-1)^r \frac{C_1}{C_2}\right), \tag{3.123}$$

where r is the number of positive real poles and zeros.

Proof. Since all parameters $k,\ s_{ig},\ C_1,\ C_2,\ r$ are real numbers, the problem reduces to showing that the sign of s_{ig} is equal to that of $(-1)^r \frac{C_1}{C_2}$. Using the representation (3.122) and the definition of the instantaneous gain given in

Proposition 2.36, we have

$$\begin{aligned}\arg(s_{ig}) &= \arg\left(\lim_{s\to+\infty}\sqrt{s}G(s)\right)\\ &= \arg\left(\frac{C_1}{C_2}\lim_{s\to+\infty} s^{p-q+\frac{1}{2}}\frac{\prod_{n=p+1}^{\infty}(1-\frac{s}{z_n})}{\prod_{n=q+1}^{\infty}(1-\frac{s}{p_n})}\right).\end{aligned}$$

Separate the positive real poles and zeros and rewrite the equation as

$$\arg(s_{ig}) = \arg\left(\frac{C_1}{C_2}\lim_{s\to+\infty} s^{p-q+\frac{1}{2}}\frac{\prod_{j=p+1}^{p+r_1}(1-\frac{s}{z_j})\prod_{n=p+r_1+1}^{\infty}(1-\frac{s}{z_n})}{\prod_{j=q+1}^{q+r_2}(1-\frac{s}{p_j})\prod_{n=q+1+r_2}^{\infty}(1-\frac{s}{p_n})}\right), \tag{3.124}$$

where $\{z_j, p+1 \le j \le p+r_1\}$ are positive real zeros, $\{p_j, q+1 \le j \le q+r_2\}$ are positive real poles, $\{z_j, 1+p+r_1 \le j\}$ are complex zeros, and $\{p_j, 1+q+r_2 \le j\}$ are complex poles. Note that negative real poles and zeros do not have any contribution to the argument, because $\arg(1-\frac{s}{\alpha}) = 0,\ \alpha \in \mathbb{R},\ \alpha < 0$. The complex poles and zeros have no contribution either, because, given $\alpha \in \mathbb{C},\ \alpha \neq 0$, we have

$$\arg\left((1-\frac{s}{\alpha})(1-\frac{s}{\overline{\alpha}})\right) = \arg\left(1 - s\frac{2Re(\alpha)}{|\alpha|^2} + \frac{s^2}{|\alpha|^2}\right).$$

Thus, as s tends to $+\infty$, this term tends to zero and does not affect the argument in (3.124). An illustrative example is shown in Figure 3.4. Consequently, (3.124) can be simplified as

$$\begin{aligned}\arg(s_{ig}) &= \arg\left(\frac{C_1}{C_2}\lim_{s\to+\infty}\frac{\prod_{j=1+p}^{r_1+p}(1-\frac{s}{z_j})}{\prod_{j=1+q}^{r_2+q}(1-\frac{s}{p_j})}\right)\\ &= \arg\left(\frac{C_1}{C_2}(-1)^{r_1-r_2}\right)\\ &= \arg\left(\frac{C_1}{C_2}(-1)^{r_1+r_2}\right)\\ &= \arg\left(\frac{C_1}{C_2}(-1)^{r}\right).\end{aligned}$$

This completes the proof. □

Now we can prove Theorem 3.16.

Proof of Theorem 3.16: Consider the open-loop transfer function $G(s) = \frac{\mathcal{N}(s)}{\mathcal{D}(s)}$. Use the representation (3.122) for $\mathcal{N}(s)$ and $\mathcal{D}(s)$.

Let us take an arbitrary $a \in \mathbb{R}$. The point a belongs to the root-locus if it satisfies the characteristic equation $k\frac{\mathcal{N}(s)}{\mathcal{D}(s)} + 1 = 0$, i.e.,

$$k\frac{\mathcal{N}(a)}{\mathcal{D}(a)} = -1, \tag{3.125}$$

The phase condition then becomes

$$\arg\left(k\frac{\mathcal{N}(a)}{\mathcal{D}(a)}\right) = \pi, \tag{3.126}$$

On the other hand, from (3.122),

$$\begin{aligned}
\arg\left(k\frac{\mathcal{N}(a)}{\mathcal{D}(a)}\right) &= \arg\left(k\frac{C_1}{C_2}a^{p-q}\right) + \arg\left(\prod_{n=p+1}^{\infty}(1-\frac{a}{z_n})\right) \\
&\quad - \arg\left(\prod_{n=q+1}^{\infty}(1-\frac{a}{p_n})\right) \\
&= \arg\left(k\frac{C_1}{C_2}a^{p-q}\right) + \sum_{n=p+1}^{\infty}\arg(a-z_n) - \sum_{n=q+1}^{\infty}\arg(a-p_n) \\
&\quad - \sum_{n=p+1}^{\infty}\arg(-z_n) + \sum_{n=q+1}^{\infty}\arg(-p_n)
\end{aligned}$$

By Theorem 2.30, $\mathcal{N}(s)$ and $\mathcal{D}(s)$ are entire and real functions of s. Hence, if $\alpha \in \mathbb{C}$ is a complex pole(zero) of $G(s)$ (Figure 3.4), then $\overline{\alpha} \in \mathbb{C}$ is also a pole(zero) and hence for any $a \in \mathbb{R}$,

$$\arg(a-\alpha) + \arg(a-\overline{\alpha}) = 0.$$

Thus, we conclude the following results:

(1) The contribution of each conjugate pair of zeros in $\sum_{n=p+1}^{\infty} \arg(a-z_n)$ is 0, that is $\arg(a-z_i)+arg(a-\overline{z_i})=0$, where z_i is a complex zero.

(2) The contribution of each real zero to the right of a is π, that is, $\arg(a-z_j)=\pi$, where z_j is a real zero to the right of a.

(3) Similarly, the contribution of each conjugate pair of poles in the sum $\sum_{n=q+1}^{\infty} \arg(a-p_n)$ is 0.

(4) The contribution of each real pole to the right of a is π.

(5) The contribution of each real pole or zero to the left of a is 0.

(6) For any zero z_i on the positive real axis $arg(-z_i)=\pi$ and for any pole p_i on the positive real axis $arg(-p_i)=\pi$. Both quantities are 0 if z_i or p_i are negative.

We also use the fact that $\arg(\alpha),\ \alpha\in\mathbb{C}$ can be replaced by $\arg(\alpha)+2m\pi,\ m\in\mathbb{Z}$. Let r_1 and r_2 be the number of zeros and poles on the positive real axis, respectively, and s_1 and s_2 be the number of real non-zero zeros and poles to the right of a. Therefore, we have

$$\arg\left(k\frac{\mathcal{N}(a)}{\mathcal{D}(a)}\right)=\arg\left(k\frac{C_1}{C_2}a^{p-q}\right)+s_1\pi-s_2\pi-r_1\pi+r_2\pi.$$

The total number of zeros and poles on the positive real axis is r_1+r_2, and the total number of real nonzero zeros and poles to the right of a is s_1+s_2. Let $\hat{r}=r_1+r_2$ and $\hat{r}=r_1+r_2$. We add $2\pi s_2+2\pi r_1$ to the argument of the above equation to obtain

$$\begin{aligned}
\arg\left(k\frac{\mathcal{N}(a)}{\mathcal{D}(a)}\right) &= \arg\left(k\frac{C_1}{C_2}a^{p-q}\right) + \hat{s}\pi + \hat{r}\pi \\
&= \arg\left(k(-1)^{\hat{r}}\frac{C_1}{C_2}a^{p-q}\right) + \hat{s}\pi \\
&= \arg\left(k(-1)^{\hat{r}}\frac{C_1}{C_2}\right) + \arg\left(a^{p-q}\right) + \hat{s}\pi, \\
&= \arg(k\cdot s_{ig}) + \arg\left(a^{p-q}\right) + \hat{s}\pi.
\end{aligned}$$

By Lemma 3.29 and from the hypothesis of Theorem 3.16, we have $\arg(k \cdot s_{ig}) = 0$ and therefore,

$$\arg\left(k\frac{\mathcal{N}(a)}{\mathcal{D}(a)}\right) = \arg\left(a^{p-q}\right) + \hat{s}\pi, \tag{3.127}$$

Thus, for a real number a to be on the root-locus we must have

$$\arg\left(a^{p-q}\right) + \hat{s}\pi = \pi + 2j\pi,\ j \in \mathbb{Z}, \tag{3.128}$$

The number of real poles and zeros to the right of a is $\hat{s}$ if $a > 0$, and is $\hat{s} + p + q$ if $a < 0$ (Note that $\hat{s}$ is the total number of real nonzero zeros and poles to the right of a). Now we are set to make the final conclusion of the theorem. By equation (3.128),

1. If $a > 0$, then a lies on the root-locus if and only if $\hat{s}$ is odd, that is, the number of real poles and zeros to the right of a is odd.

2. If $a < 0$ and $p + q$ is even, then a lies on the root-locus if and only if s is odd, that is, the number of real poles and zeros to the right of a, i.e., $\hat{s} + p + q$, is odd.

3. If $a < 0$ and $p + q$ is odd, then a lies on the root-locus if and only if $\hat{s}$ is even, thus the number of real poles and zeros to the right of a, i.e., $\hat{s} + p + q$, is odd.

This completes the proof of Theorem 3.16.

3.2.6 Stability of the Closed-loop System

So far we have shown that the infinitely many closed-loop poles are real and negative, and that there is a fixed contour Ω which contains all of the finitely many closed-loop poles that may be complex, multiple, or on the right-half plane. In chapter 2, we presented some theorems about the stability of the open-loop system. Here we want to investigate the closed-loop system (3.35) and derive some results regarding the stability of this system as the feedback gain k becomes sufficiently large.

Proposition 3.30. *Let $\mathbb{C}_+^a = \{z \in C; Re(z) > a\}$. Choose k so that $k \cdot s_{ig} > 0$. For the system (3.35), the transfer function $G_k(s)$ is in $H^\infty(C_+^a)$, for some $a \in R$ and satisfies*

$$\lim_{|s|\to+\infty} G_k(s) = 0, \quad s \in \mathbb{C}_+^a. \tag{3.129}$$

Hence, the system (2.59) is strictly proper.

Proof. In (2.95) and (2.96) we obtained the asymptotic forms of $\mathcal{N}(s)$ and $\mathcal{D}(s)$. Therefore, the asymptotic form of the closed-loop transfer function for the system (3.35) is

$$\begin{aligned} G_k(s) &= \frac{\mathcal{N}(s)}{\mathcal{D}(s) + k\mathcal{N}(s)} \\ &= \left(1 + O(\frac{1}{\sqrt{s}})\right) \frac{\alpha\gamma \cosh(\sqrt{s})}{-\beta\gamma\sqrt{s}\sinh(\sqrt{s}) + k\alpha\gamma\cosh(\sqrt{s})}. \end{aligned} \tag{3.130}$$

It is easy to show that $\lim_{|s|\to+\infty} G_k(s) = 0$ and therefore, $|G_k(s)|$ is bounded on $\mathbb{C}_+^a$ for some $a \in \mathbb{R}$. Hence, the closed-loop system is strictly proper. □

From Theorem 2.24 and the above proposition, the following corollary follows.

Corollary 3.31. *Choose k so that $k \cdot s_{ig} > 0$. For the system (3.35), the input/output map is bounded in sense of Definition 2.18-(4).*

In Theorem 2.47, we proved that the open-loop system (2.59) is L_2-stable if and only if the system transfer function lies in H_∞. This also holds for the feedback system (3.35). In other words, For any k with $k \cdot s_{ig} > 0$, the system (3.35) is L_2-stable if and only if the system closed-loop transfer function lie in H_∞.

Theorem 3.32. *For any k with $k \cdot s_{ig} > 0$, the system (3.35) is L_2-stable if all roots of the characteristic equation $\mathcal{D}(s) + k\mathcal{N}(s) = 0$ lie in the open left-half plane.*

Proof. In Proposition 3.11, we proved that the closed-loop poles are a subset of the roots of the characteristic equation. If all roots of the characteristic equation lie in the open left-half plane, then are so all closed-loop poles. Thus, the closed-loop transfer function $G_k(s)$ is analytic on the closed right-half plane. Moreover, the closed-loop poles are isolated since the characteristic equation is an analytic function of s (Theorem 2.45). Therefore, $|G_k(s)|$ is bounded on the closed right-half plane. Thus, $G_k(s)$ lies in H_∞ and hence the closed-loop system is L_2-stable. □

The following conclusion can be made.

Proposition 3.33. *If all open-loop zeros of the system (3.35) are in the open left-half plane, then there exists k_0 with $k_0 \cdot s_{ig} > 0$ such that for all k with $k \cdot s_{ig} > k_0 \cdot s_{ig}$ the system (3.35) is L_2-stable.*

Proof. The proof of this theorem is straightforward. □

3.2.7 Example

Consider the open-loop system

$$\begin{cases} \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2}, \quad t > 0, x \in (0,1), \\ -\frac{\partial z}{\partial x}(0,t) + z(1,t) = u(t), \\ \frac{\partial z}{\partial x}(1,t) = 0, \\ z(0,t) = y(t), \\ z(x,0) = f(x), \end{cases} \tag{3.131}$$

where $f \in L_2[0,1]$, $u(t)$ and $y(t)$ are the input and output, respectively. The output control is obtained by the feedback

$$u = -ky, \tag{3.132}$$

We want to locate the closed-loop poles of this system for all values of $k > 0$ and derive some results about the stability of this system. The Laplace transform of (3.131) is

$$\begin{cases} s\hat{z} = \frac{\partial^2 \hat{z}}{\partial x^2}, \quad t > 0, x \in (0,1), \\ -\frac{\partial \hat{z}}{\partial x}(0,s) + \hat{z}(1,s) = \hat{u}(s), \\ \frac{\partial \hat{z}}{\partial x}(1,s) = 0, \\ \hat{z}(0,s) = \hat{y}(s), \end{cases} \tag{3.133}$$

The open-loop transfer function is

$$G(s) = \frac{\cosh\sqrt{s}}{1 + \sqrt{s}\sinh\sqrt{s}}. \tag{3.134}$$

The open-loop poles are the roots of

$$1 + \sqrt{s}\sinh\sqrt{s} = 0. \tag{3.135}$$

As $|s|$ approaches infinity, the open-loop poles tend to the zeros of $\sinh\sqrt{s}$. Thus, for sufficiently large $|s|$, the open-loop poles are $-n^2\pi^2, n \geq N$, for some $N \in \mathbb{N}$, and hence the open-loop transfer function is analytic on some right-half plane. For the closed-loop poles with bounded modulus, we make the change of variable $z = -i\sqrt{s}$ in (3.135) to obtain $1 - z\sin z = 0$ and solve it for $z \in \{z; Rez \geq 0, Imz \geq 0\}$. Write $z = x + iy$. The equation $z \sin z = 1$ can be rewritten as

$$x\sin x\cosh y - y\cos x\sinh y = 1, \qquad (3.136)$$

$$y\sin x\cosh y + x\cos x\sinh y = 0. \qquad (3.137)$$

We consider three cases:

(1) For $x = 0$, the system of equations (3.136)-(3.137) has no solutions.

(2) For $y = 0$, we need to solve $x\sin x = 1, x \in \mathbb{R}^+$. It can be geometrically observed that this equation has infinitely many roots that form a countable set. Thus, the system of equations (3.136)-(3.137) possesses a countable set of real solutions $z_n = x_n, n \geq 1$, where x_n's are the roots of $x\sin x = 1$.

(3) For any $y > 0$, we need to solve $(x^2+y^2)\sinh y\cos x = -y$ (Substitute $\sin x\cosh y = \frac{-x}{y}\cos x\sinh y$ into (3.136)). Geometrically, the left-hand side of this equation is an oscillating function of x with an increasing magnitude, while the right-hand side is constant for any y. Thus, the system of equations (3.136)-(3.137) has infinitely many solutions $z_n = \hat{x}_n + iy, n \geq 1$.

From (1)-(2) we conclude that all real open-loop poles are negative due to $s = -z^2$ which means that all real branches of the root-locus lie on the negative real axis for $k > 0$. However, (3) implies that there are infinitely many open-loop poles in the complex plane and hence the open-loop system is not L_2-stable.

The closed-loop transfer function is

$$G_k(s) = \frac{\cosh\sqrt{s}}{1 + \sqrt{s}\sinh\sqrt{s} + k\cosh\sqrt{s}}. \tag{3.138}$$

The open-loop zeros are the roots of $\cosh(\sqrt{s}) = 0$, that are $s = -(\frac{2n+1}{2}\pi)^2$, $n \geq 0$. Since the open-loop zeros are in the open left-half plane, the closed-loop system is L_2-stable for sufficiently large k.

Chapter 4

Conclusions and Future Research

We started this thesis with an introduction to control systems in chapter 1. We continued this chapter with a literature review on the zeros of infinite-dimensional systems.

In chapter 2, we introduced the concept of C_0-semigroups as an essential component of analyzing infinite-dimensional systems. We presented the Hille-Yosida Theorem as a strong theorem that gives a necessary and sufficient condition for an operator on a Banach space to be the infinitesimal generator of a C_0-semigroup. Then we introduced boundary control systems and their transfer functions. After that, we formulated an open-loop second order boundary control system with co-located input and output in the sense that the highest order derivatives of the input and output operators occur at the same endpoint of a one-dimensional spatial domain. We proved some propositions about the open-loop transfer function of this system. Also, we proved that the open-loop system is L_2-stable if and only if the open-loop transfer function lies in H_∞.

In chapter 3, we introduced the root-locus method for locating the closed-loop poles of a system and applied this method to finite-dimensional systems. We derived some results about the closed-loop stability of this class of systems. Denoting the difference between the number of poles and zeros by $n_p - n_z$, if $n_p - n_z > 2$ the closed-loop system is unstable for sufficiently large feedback gain. If $n_p - n_z = 0$ or $n_p - n_z = 1$ and all open-loop zeros are in the open left-half plane, the closed-loop system is stable for sufficiently large k. If $n_p - n_z = 2$ and the asymptotes intersect on the positive real axis, then the two asymptotic branches lie in the closed right-half plane, which means the system is unstable for sufficiently large k. If $n_p - n_z = 2$ and the asymptotes intersect on the negative real axis, then two asymptotes lie in the open left-half plane, which means that as k tends to infinity, the two closed-loop poles diverging to the two open-loop zeros at infinity, are on the open left-half plane. In this case, for the closed-loop system to be eventually stable(for sufficiently large k), it suffices that all open-loop zeros lie in the open left-half plane, that is, the system is minimum-phase.

We continued chapter 3 with presenting a complete root-locus analysis for a second order diffusion problem with control and observation on the boundary. We showed that outside a fixed contour Ω, the closed-loop poles, for any feedback gain k, are negative real, simple and form a divergent sequence. In another word, all root-locus branches lie on the negative real line. We also showed that all closed-loop poles vary continuously, from open-loop poles to open-loop zeros, as k varies from zero to infinity. Particularly, for the real branches outside Ω, we proved that the closed-loop poles move continuously to the left.

We further showed that poles with negative real values do not lead to instability in the system response. Hence, the infinitely many closed-loop poles on the negative real axis are stable poles. However, the finitely many poles inside Ω may be complex, multiple, or on the right-half plane and hence the root-locus branches may lie on the right-half plane and lead to an unstable system response. If the open-loop system has no closed right-half plane zeros, the closed loop system is stable for sufficiently large values of the feedback gain. On contrast, if there is at least one open-loop zero in the closed right-half plane, then for sufficiently large k, the closed-loop system has at least one unstable pole and so the system is unstable.

The root-locus theory for infinite-dimensional systems has many open-problems. For example, we can extend our study to the root-locus analysis of PDE systems with second order time derivatives, e.g., vibrating systems. As another example, we can study boundary control systems with non-co-located input and output operators.

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