Dominating sets in Kneser graphs

by

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Author's declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis investigates dominating sets in Kneser graphs as well as a selection of other topics related to graph domination. Dominating sets in Kneser graphs, especially those of minimum size, often correspond to interesting combinatorial incidence structures.

We begin with background on the dominating set problem and a review of known bounds, focusing on algebraic bounds. We then consider this problem in the Kneser graphs, discussing basic results and previous work. We compute the domination number for a few of the Kneser graphs with the aid of some original results. We also investigate the connections between Kneser graph domination and the theory of combinatorial designs, and introduce a new type of design that generalizes the properties of dominating sets in Kneser graphs. We then consider dominating sets in the vector space analogue of Kneser graphs. We end by highlighting connections between the dominating set problem and other areas of combinatorics. Conjectures and open problems abound.

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Contents

Introduction

A *dominating set* in a graph is a vertex subset S such that every vertex not in S has a neighbor in S, and the *domination number* of a graph is the size of its smallest dominating set. The *dominating set problem* asks to determine the domination number of a given graph. Formal study of the dominating set problem began in the 1960s, the term itself first appearing in the 1967 book on graph theory [45] by Ore. However, the problem has historical roots in the dominating queens problem, which occupied European chess enthusiasts in the mid- to late 19^{th} century; this problem asks for the minimum number of queens that can be placed on a chessboard such that every square not containing a queen is under attack by one.

Besides being of theoretical interest, the dominating set problem also finds a natural application in numerous facility location problems. In these problems, the vertices of a graph correspond to locations, adjacency represents some notion of accessability, and the purpose is to find a subset of locations accessible from all other locations at which to install fire stations, bus stops, post offices, or other such facilities. Dominating sets have also been applied in the analysis of social networks (Kelleher and Cozzens [35]). See the monograph by Haynes et al. [30] for more information on the history and applications of the dominating set problem.

This thesis investigates dominating sets in Kneser graphs as well as a selection of other topics related to graph domination. The *Kneser graph* $K_{n:k}$ is the graph whose vertices are the k-subsets of an n-element set, where two vertices are adjacent if the corresponding sets are disjoint. Dominating sets in Kneser graphs, especially those of minimum size, often correspond to interesting combinatorial incidence structures. Throughout the thesis we pay particular attention to lower bounds on the domination number.

The thesis is divided into three parts. Part I contains background on the dominating set problem. In Chapter 1 we formally introduce dominating sets as well as two important variants: total dominating sets, which are dominating sets that induce subgraphs with no isolated vertices, and independent dominating sets, whose name is self-explanatory. Chapter 2 reviews known bounds on the domination and total domination number, and includes a detailed discussion of algebraic upper bounds. Chapter 3 reviews the fundamentals of the theory of combinatorial designs, which will be integral to our discussion of dominating sets in Kneser graphs.

Part II deals with the dominating set problem in Kneser graphs and related problems. All of the author's original results are contained in Part II.

Chapter 4 is devoted to the dominating set problem in Kneser graphs. We begin with a review of previous work on this problem, providing proofs for many basic results. We then discuss work by Hartman and West in which they determine $\gamma(K_{n:k})$ for a certain range of n. Next, we compute $\gamma(K_{n:k})$ for a few small values of n and k with the aid of some original results, in particular computing $\gamma(K_{n:3})$ for all n.

In Chapter 5 we consider dominating sets in the odd graphs, which are the Kneser graphs of the form $K_{2k+1:k}$. Small dominating sets in odd graphs correspond to certain Steiner systems, and determining their existence is a long-standing open problem. We review important results and prove that an odd graph that contains such a small dominating set admits a nontrivial equitable partition.

In Chapter 6 we introduce *semi-covering designs*, which are combinatorial designs, similar to covering designs, that generalize certain properties of dominating sets in Kneser graphs.

In Chapter 7 we consider the dominating set problem in q-Kneser graphs, which are vector space analogues of Kneser graphs. We prove some basic results then highlight previous work by Clark and Shekhtman, using it to reflect on the differences between graph domination of Kneser and q-Kneser graphs. We compute the domination number of a small q-Kneser graph, improving on a proof by Clark and Shekhtman.

In Part III we relate the dominating set problem to two disparate areas of combinatorics. In Chapter 8 we consider maximal intersecting families of sets, a topic in extremal combinatorics. We review existing results, focusing on the role played by the Kneser graph domination problem. In Chapter 9 we review some fundamental problems in coding theory as viewed through the lens of the dominating set problem in the hypercube.

Conjectures and open problems abound, and many chapters conclude with several. In addition, we collect in Chapter 10 some miscellaneous open problems in graph domination, most of which concern the existence of small dominating sets in particular graph families.

Part I

Background

Chapter 1

Graph domination and its variants

In this chapter we formally introduce the dominating set problem, some of its variants, and the notion of a perfect 1-code in a graph.

We assume familiarity with basic graph theory terminology and notation, which can be found in Appendix A. All logarithms are to base e.

1.1 Dominating sets

A *dominating set* in a graph G is a vertex subset $S \subseteq V$ such that every vertex in $V \setminus S$ is adjacent to some vertex in S. The *domination number* of G, written $\gamma(G)$, is the minimum size of a dominating set in G.

Fix an ordering of V and let $|V| = v$. The *characteristic vector* of a vertex subset $S \subseteq V$, written $\mathbf{z}_S$, is a 0-1 vector in $\mathbb{R}^v$ whose xth entry is 1 if and only if $x \in S$. If A is the adjacency matrix of G then S is a dominating set if and only if

$$(A + I)\,\mathbf{z}_S \geq \mathbf{1}$$

where $\mathbf{1}$ is the all-1 vector and the inequality is componentwise. Thus the dominating set problem can be phrased as an integer program:

$$\begin{gathered} \min\ \mathbf{z}^T\mathbf{1} \\ (A + I)\,\mathbf{z} \geq \mathbf{1} \\ \mathbf{z} \in \{0,1\}^v \end{gathered}$$

The problem of finding a minimum size dominating set is in general NP-hard (Garey and Johnson [19]) but is approximable within $1 + \log(v)$ (Johnson [33]). This problem is unlikely to be approximable within $(1 - \epsilon)\log(v)$ for any $\epsilon > 0$ (Feige [17]). However, there are polynomial time algorithms for finding a minimum size dominating set in trees and series parallel graphs (details in Haynes et al. [30]), and there also exists a polynomial time approximation scheme for planar graphs (Baker [2]).

Given $S \subseteq V$ and some $x \in S$, a vertex $y \in V \setminus S$ is a *private neighbor* of x if x is the only neighbor of y in S. The set of private neighbors of $x \in S$ in $V \setminus S$ will be written $\mathrm{pn}_S(x)$, or simply $\mathrm{pn}(x)$ when S is clear from context.

A dominating set S is *minimal* if, as usual, no proper subset of S is a dominating set. Minimum size dominating sets are obviously minimal, but the converse is not always true. Ore gave in [45] a useful characterization of minimal dominating sets using the notion of private neighbors.

1.1.1 Theorem. *A dominating set S is minimal if and only if every $v \in S$ that has a neighbor in S has a private neighbor in $V \setminus S$.*

Proof. Assume first that S is minimal and consider a vertex $x \in S$ that is adjacent to some $y \in S$. If x has no private neighbors in $V \setminus S$ then $S \setminus \{x\}$ is a dominating set since x as well as every vertex in $\Gamma(x)$ is adjacent to a vertex in $S \setminus \{x\}$. This contradicts the minimality of S, so x must have a private neighbor.

Conversely, let S be a dominating set for which the condition holds; we will show that S is minimal. If it is not, then there exists $x \in S$ such that $S \setminus \{x\}$ is a dominating set. This implies that x is adjacent to some $y \in S$, so by assumption it has a private neighbor in $V \setminus S$, call it w. Then w is not adjacent to any vertex in $S \setminus \{x\}$, contradicting the assumption that this is a dominating set. □

The following basic bounds for regular graphs follow immediately from the definitions.

1.1.2 Proposition. *If G is a connected d-regular graph on v vertices then*

$$\frac{1}{d+1} \leq \frac{\gamma(G)}{v} \leq \frac{1}{2}.$$

Proof. Let S be a minimum size dominating set in G. To show the upper bound, note that every $x \in S$ is adjacent to some $y \in V \setminus S$, for otherwise $S \setminus \{x\}$ would be a dominating set, contradicting the minimality of S. This implies that the complement of a minimal dominating set is also a dominating set, so in particular there exists a dominating set of size at most $v/2$.

For the lower bound, note that since G is d-regular, each vertex in S has at most d neighbors outside of S. Thus $|V \setminus S|$ is at most $d|S|$ which implies $v \leq (d+1)|S|$ and the bound. □

The upper bound above first appeared in Ore [45]. There are only finitely many connected graphs for which $\gamma(G) = \lfloor v/2 \rfloor$; see Haynes et al. [30] for the very short list. There is a much better general upper bound for $\gamma(G)$ based on the greedy algorithm; we discuss it in Chapter 2

The lower bound in Proposition 1.1.2 is called the *sphere-covering bound*. If S is a dominating set in a d-regular graph G that meets the sphere-covering bound we call S a *perfect 1-code*; this terminology will be justified in Chapter

9, which introduces coding theory. It is easy to see that S is a perfect 1-code if and only if $(A+I)\,\mathbf{z}_S = \mathbf{1}$.

Perfect 1-codes in graphs were introduced by Biggs in [3]. Deciding whether a graph has a perfect 1-code is NP-complete, even for bipartite graphs and planar graphs of maximum degree 3 (see Haynes et al. [30] for more).

Our discussions will often turn to the existence of perfect 1-codes in particular graph families, and so before moving on we make note of the following useful observation. Given $S \subseteq V$ we define the *measure* of S to be the ratio $\mu(S) = |S|/v$, and then define

$$\mathbf{w}_S = \mathbf{z}_S - \mu(S)\mathbf{1}$$

where $\mathbf{1}$ is the all-1 vector. Note that $\mathbf{w}_S$ is the projection of $\mathbf{z}_S$ to the subspace $\mathbf{1}^\perp$. Biggs observed in [3] that if a d-regular graph G has a perfect 1-code S then $\mathbf{w}_S$ is an eigenvector of A with eigenvalue -1; this follows easily from the facts $(A+I)\,\mathbf{z}_S = \mathbf{1}$ and $A\mathbf{1} = d\mathbf{1}$, where the latter is implied by the fact that G is regular.

1.2 Total dominating sets

Total domination is a variant of graph domination introduced by Cockayne et al. in [10]. A vertex subset $S \subseteq V$ is a *total dominating set* if every vertex in V is adjacent to some vertex in S. In other words, a dominating set is total if the subgraph it induces has no isolated vertices. The minimum size of a total dominating set in G is the *total domination number* of G, written $\gamma_t(G)$.

In analogy to dominating sets, if A is the adjacency matrix of G and $\mathbf{z}_S$ is the characteristic vector of a set S then S is a total dominating set if and only if $A\,\mathbf{z}_S \geq \mathbf{1}$. Thus the dominating set problem can also be phrased as an integer program:

$$\begin{gathered} \min\ \mathbf{z}^T\mathbf{1} \\ A\,\mathbf{z} \geq \mathbf{1} \\ \mathbf{z} \in \{0,1\}^v \end{gathered}$$

In analogy to Proposition 1.1.2, if G is a d-regular graph on v vertices then

$$\gamma_t(G) \geq \frac{v}{d}.$$

This bound follows from the fact that every vertex in a minimum size total dominating set has at most $d-1$ neighbors outside the set. An upper bound is far less trivial; see Section 2.2 for more information. There is also a general upper bound on $\gamma_t(G)$ based on the greedy algorithm, as was the case with $\gamma(G)$. We will discuss it in Chapter 2

Taking further the analogy with standard domination, we define a *totally perfect 1-code* to be a total dominating set whose size meets the lower bound

v/d. In other words, the neighborhoods of the vertices of a totally perfect 1-code partition the vertex set.

Finding a minimum size total dominating set is NP-hard, and deciding whether a graph has a totally perfect 1-code is NP-complete. However, just like in the case of standard domination, there exists a polynomial time algorithm for finding minimum size total dominating sets in trees and series parallel graphs; details on this and other aspects of total domination can be found in the monograph [30] by Haynes et al.

We note that since every total dominating set is a dominating set we have $\gamma(G) \leq \gamma_t(G)$. Moreover, it is easy to see that $\gamma_t(G) \leq 2\gamma(G)$. Thus domination and total domination are asymptotically not very different, and the numerous analogies between the two are not surprising. However, we next meet a variant of domination that, as we will see later, can exhibit quite different behavior.

1.3 Independent dominating sets

A vertex subset $S \subseteq V$ is an *independent dominating set* if it is both a dominating set and independent. It is easy to see that a vertex subset is an independent dominating set if and only if it is a maximal independent set. The minimum size of a total dominating set in G is the *independent domination number* of G and will be written $\gamma_i(G)$.

Let M be the *incidence matrix* of G, the $e \times v$ matrix with rows and columns indexed by edges and vertices, respectively, in which $M_{ij} = 1$ if and only if edge i is incident to vertex j. Then if A is the adjacency matrix of G, the minimum size of a dominating set in G is the solution to the following integer program:

$$\begin{gathered} \min \mathbf{z}^T \mathbf{1} \\ (A + I)\,\mathbf{z} \geq \mathbf{1} \\ M\,\mathbf{z} \leq \mathbf{1} \\ \mathbf{z} \in \{0,1\}^v \end{gathered}$$

Unlike the case with total domination, bounds on $\gamma(G)$ do not tend to generalize to bounds on $\gamma_i(G)$. The sphere-covering bound of Proposition 1.1.2 still clearly applies, but there is no general upper bound like the greedy bounds on $\gamma(G)$ and $\gamma_t(G)$ that we mentioned above and will encounter in Chapter 2.

Nonetheless, independent domination has been studied extensively and many interesting results are known. For instance, $\gamma(G) = \gamma_i(G)$ when G is claw-free.[1] For this and more on independent domination see Haynes et al. [30].

As expected, finding a minimum size independent dominating set is NP-hard (Garey and Johnson [19]), and remains NP-hard for bipartite graphs and line graphs. There is, however, a polynomial time algorithm for minimum size independent dominating sets in trees; as usual, see Haynes et al. [30] for more.

[1] A graph is claw-free if no four vertices induce the complete bipartite graph $K_{1,3}$.

Chapter 2

Known bounds

In this chapter we review known bounds on the domination number of regular graphs, focusing on algebraic bounds. Almost all bounds considered are upper bounds, as nontrivial lower bounds are practically nonexistent.

We begin with greedy bounds on $\gamma(G)$ and $\gamma_t(G)$. We then present a detailed discussion of the only eigenvalue upper bound in the literature, found in Lu et al. [38]. We also make the observation that an eigenvalue lower bound proved by Lu et al. in the same paper is useless. We end with a proof of a new algebraic upper bound on $\gamma(G)$ due to Godsil.

2.1 General bounds

Perhaps the best known upper bound on $\gamma(G)$ is due to Alon:

2.1.1 Theorem. *If G is a d-regular graph on v vertices then*

$$\frac{\gamma(G)}{v} \leq \frac{1+\log(d+1)}{d+1}. \qquad \square$$

The bound also holds for non-regular graphs, with d replaced by the minimum degree. A proof can be found in the book [1] by Alon and Spencer on probabilistic methods. Not surprisingly, the proof is probabilistic, but the authors also give a second, algorithmic proof. The latter proof constructs a dominating set S greedily; vertices are added to S one by one, where at each step the vertex that covers the maximum number of yet uncovered vertices is picked (a vertex is covered if it is adjacent to a vertex in S). It is possible to show that after $\frac{v}{d+1}\log(d+1)$ steps there are at most $\frac{v}{d+1}$ uncovered vertices, and adding these to S gives a dominating set of the desired size.

The following upper bound for regular graphs is proven in a sequence of three papers by Caro and Roditty [7], Liu and Sun [36], and Xing et al. [49]:

2.1.2 Theorem. *If G is a d-regular graph on v vertices then*

$$\frac{\gamma(G)}{v} \leq \frac{d}{3d-1}. \qquad \square$$

The following lower bound appears in Haynes et al. [30]. We will not make any use of it, nor is it especially useful to begin with, but we nonetheless note it because it is one of the very few nontrivial lower bounds on $\gamma(G)$ in the literature.

2.1.3 Theorem. *If G is a connected graph with diameter D then*

$$\gamma(G) \geq \frac{D+1}{3}.$$

Proof. Let S be a dominating set of minimum size. Choose two vertices x, y such that the shortest length of a path between x and y is D, and let P be an $x - y$ path of this length.

The closed neighborhood of any given vertex in S contributes at most two edges to P. Also, P contains at most $|S| - 1$ edges joining neighborhoods of vertices in S. Thus

$$D \leq 2|S| + |S| - 1 = 3|S| - 1$$

and the bound follows. $\square$

We observe that this lower bound is tight for the complete bipartite graphs $K_{1,n}$. It is unlikely that it is tight for any interesting graph families.

2.2 Total domination

As we have seen, v/d is a trivial lower bound on $\gamma_t(G)$ when G is d-regular. The most general upper bound on $\gamma_t(G)$, which holds for any graph G with $v \geq 3$ vertices, is $2v/3$. For a proof see Cockayne et al. [10] or the monograph [30] by Haynes et al.

A more interesting upper bound follows from the Johnson-Stein-Lovász theorem on 0-1 matrices (a proof can be found in Cohen et al. [11]).

2.2.1 Theorem. *Let A be a 0-1 matrix with n rows and m columns. Assume that each row of A contains at least r ones and each column contains at most c. Then A has an $n \times k$ submatrix B with*

$$k \leq \frac{n}{c} + \frac{m}{r} \log c$$

such that no row of B is an all-zero row. $\square$

Applying the theorem to the adjacency matrix of a d-regular graph yields a submatrix whose columns correspond to vertices in a total dominating set. This gives an upper bound on $\gamma_t(G)$ that to the author's knowledge is unpublished.

2.2.2 Corollary. *If G is a d-regular graph on v vertices then*

$$\frac{\gamma_t(G)}{v} \leq \frac{1+\log d}{d}. \qquad \square$$

Corollary 2.2.2 is total domination's answer to Theorem 2.1.1. The analogy between the two bounds is in fact quite profound because the proof of Theorem 2.2.1 uses a greedy algorithm to construct B, just like the algorithmic proof of Theorem 2.1.1. The details are somewhat different because multiple columns, not just one, can be added to B at each step of the algorithm; see Cohen et al. [11] for more.

Observe that Theorem 2.2.1 can also be used to derive the greedy upper bound on $\gamma(G)$ in Theorem 2.1.1.

2.3 An eigenvalue bound

Let G be a connected d-regular graph on v vertices and let λ_2 be the second-largest eigenvalue of G (since G is d-regular, d is the largest eigenvalue). The value of λ_2 has many significant implications for the structure of G. For example, $\lambda_2 = d$ if and only if G is disconnected (see Godsil and Royle [22]). More profoundly, λ_2 can be used to bound the expansion of a regular graph, which becomes very useful in numerous computer science applications (see Hoory et al. [31] for an excellent introduction to the theory and applications of graph expansion).

In this section we prove that

$$\frac{\gamma(G)}{v} \leq \frac{1+\delta}{v+\delta} \tag{2.3.1}$$

where $\delta = v - (d - \lambda_2)$, a result due to Lu et al. [38]. This is the only eigenvalue bound on $\gamma(G)$ in the literature.

The proof requires some preliminary results. Unless stated otherwise, these are all from [38] by Lu et al.

If G is a graph with vertex set V and $S, S' \subseteq V$ we define $e(S, S')$ to be the set of edges with one endvertex in S and the other in S'. If $S' = V \setminus S$ then we will write $e(S, S')$ as ∂S; this is called the *cut* defined by S. We allow $S' = S$ in this definition, in which case $e(S, S')$ is the set of edges in the subgraph induced by S.

Recall that if S is a vertex subset then $\mu(S) = |S|/v$, $\mathbf{z}_S$ is its characteristic vector, and $\mathbf{w}_S = \mathbf{z}_S - \mu(S)\mathbf{1}$, so that in particular $\mathbf{w}_S \perp \mathbf{1}$.

The proof of the bound (2.3.1) makes use of Rayleigh's inequality for the second-largest eigenvalue of a symmetric matrix: if A is a symmetric matrix with second-largest eigenvalue λ_2, and $\mathbf{x} \perp \mathbf{1}$, then

$$\mathbf{x}^T A \mathbf{x} \leq \lambda_2 \mathbf{x}^T \mathbf{x}. \tag{2.3.2}$$

For details and a proof see Section 9.5 of Godsil and Royle [22]. Rayleigh's inequality is used to prove the following eigenvalue bound on the size of a cut in a regular graph, due to Mohar and Poljak (see [44]).

2.3.1 Theorem. *Let G be a connected d-regular graph on v vertices with second-largest eigenvalue λ_2. Then for any $S \subseteq V$ we have*

$$\frac{|\partial S|}{|S|} \geq (d - \lambda_2)\left(1 - \frac{|S|}{v}\right).$$

Proof. Let A be the adjacency matrix of G. By Rayleigh's inequality (2.3.2),

$$\mathbf{w}_S^T\, A\, \mathbf{w}_S \leq \lambda_2\, \mathbf{w}_S^T\, \mathbf{w}_S. \tag{2.3.3}$$

Now, $\mathbf{w}_S^T\, \mathbf{w}_S = |S|(1 - \mu(S))$ so

$$\mathbf{w}_S^T\, A\, \mathbf{w}_S \leq \lambda_2 |S|(1 - \mu(S)). \tag{2.3.4}$$

As for the left side of the inequality,

$$\mathbf{w}_S^T\, A\, \mathbf{w}_S = \mathbf{z}_S^T\, A\, \mathbf{z}_S - k|S|\mu(S).$$

Now, it is easy to see that $\mathbf{z}_S^T\, A\, \mathbf{z}_S = 2|e(S,S)|$. Since G is d-regular we know that $d|S| = 2|e(S,S)| + |\partial S|$ so that

$$\begin{aligned}\mathbf{w}_S^T\, A\, \mathbf{w}_S &= 2|e(S,S)| - k|S|\mu(S)\\ &= k|S| - |\partial S| - k|S|\mu(S)\\ &= k|S|\,(1 - \mu(S)) - |\partial S|.\end{aligned}$$

Substituting this into inequality (2.3.4) gives

$$k|S|\,(1 - \mu(S)) - |\partial S| \leq \lambda_2 |S|\,(1 - \mu(S))$$

and the claim follows after some rearranging. □

We are almost ready to prove the bound 2.3.1. First we must prove the existence of a particular kind of minimum size dominating set. Recall that given $S \subseteq V$ and some $x \in S$, $\mathrm{pn}(x)$ is the set of private neighbors of x in $V \setminus S$. The following lemma appears in Lu et al. [38] as well as the monograph [30] by Haynes et al.

2.3.2 Lemma. *Let G be a connected graph. Then there exists a minimum size dominating set S such that $|\,\mathrm{pn}(x)| \geq 1$ for every $x \in S$.*

Proof. Let S be a dominating set of minimum size that maximizes $|e(S,S)|$; we will show that S satisfies the claim. Assume the contrary, so that $|\,\mathrm{pn}(x)| = 0$ for some $x \in S$. Then, since S must be a minimal dominating set, we know by Theorem 1.1.1 that x has no neighbors in S.

Let y be an arbitrary neighbor of x; it is adjacent to some vertex in S besides x. The set $S' = (S \setminus \{x\}) \cup \{y\}$ is a dominating set of minimum size, and

$$|e(S', S')| = |e(S,S)| + 1 > |e(S,S)|.$$

But this contradicts our choice of S, so we conclude that there can be no $x \in S$ with $|\,\mathrm{pn}(x)| = 0$. □

2.3.3 Theorem. *Let G be a connected graph on $v \geq 2$ vertices. Then there exists a minimum size dominating set S in G such that*

$$\frac{|\partial S|}{|S|} \leq v - 2\gamma(G) + 1.$$

Proof. By Lemma 2.3.2 there exists a minimum size dominating set S in G such that $|\operatorname{pn}(x)| \geq 1$ for every $x \in S$. It follows that

$$\begin{aligned} |\partial S| &= |\{xy \in E \mid x \in S, y \in V \setminus S\}| \\ &\leq \sum_{x \in S} |\operatorname{pn}(x)| + |S| \left(v - |S| - \sum_{x \in S} |\operatorname{pn}(x)| \right) \\ &= |S|(v - |S|) - (|S| - 1) \sum_{x \in S} |\operatorname{pn}(x)| \\ &\leq |S|(v - |S|) - (|S| - 1)|S| \\ &= (v - 2|S| + 1)|S| \end{aligned}$$

which gives the desired inequality. □

We are now ready to prove the promised eigenvalue bound on $\gamma(G)$.

2.3.4 Theorem. *Let G be a connected d-regular graph with second-largest eigenvalue λ_2. Then*

$$\frac{\gamma(G)}{v} \leq \frac{1 + \delta}{v + \delta}$$

where $\delta = v - (d - \lambda_2)$.

Proof. By Lemma 2.3.3, there exists a dominating set S in G with $|S| = \gamma(G)$ and

$$\frac{|\partial S|}{|S|} \leq v - 2\gamma(G) + 1.$$

Combining the above with the inequality in Theorem 2.3.1 gives

$$(k - \lambda_2) \left(1 - \frac{|S|}{v} \right) \leq v - 2\gamma(G) + 1.$$

which, after some rearranging, yields the desired bound on $\gamma(G)$. □

It should also be noted that Lu et al. prove a version of Theorem 2.3.4 that applies to arbitrary, not necessarily regular, connected graphs by working with the Laplacian matrix of a graph rather than its adjacency matrix.

2.3.1 A lower bound, too

Before moving on, we note that Lu et al. also prove in [38] an eigenvalue lower bound on $\gamma(G)$:

$$\frac{\gamma(G)}{v} \geq \frac{1}{d - \lambda_v} \tag{2.3.5}$$

where λ_v is the smallest eigenvalue of G. This lower bound, however, is useless. Before discussing why, we briefly sketch the proof for completeness.

The proof of this lower bound is almost identical to the proof of the upper bound. The first component of the proof is the inequality

$$\frac{|\partial S|}{|S|} \leq (d - \lambda_v)\left(1 - \frac{|S|}{v}\right), \tag{2.3.6}$$

an analogue of the inequality in Theorem 2.3.1. Not surprisingly, the proof of former is very similar to the proof of the latter, save that it uses Rayleigh's inequality for the *least*, rather than second-largest, eigenvalue of a symmetric matrix A: for any vector $\mathbf{x}$,

$$\mathbf{x}^T A \mathbf{x} \geq \lambda_v \mathbf{x}^T \mathbf{x}$$

(again, see Section 9.5 of Godsil and Royle [22] for details). To get inequality 2.3.6 we substitute $\mathbf{x} = \mathbf{w}_S$ and simplify.

Now, let S be a dominating set. Then $|\partial S| \geq n - |S|$ because every vertex in $V \setminus S$ has a neighbor in S, hence is an endvertex of at least one edge whose other endvertex is in S. Writing this as

$$\frac{|\partial S|}{|S|} \geq \frac{n}{|S|} - 1$$

and combining with inequality (2.3.6) yields the bound (2.3.5).

This lower bound, however, is no better than the sphere-covering bound, and is usually worse. To see why, observe that this bound beats the sphere-covering bound if

$$\frac{1}{d+1} \leq \frac{1}{d - \lambda_v}$$

which happens if and only if $\lambda_v \geq -1$. It is possible to show, however, that $\lambda_v \geq -1$ if and only if G is a union of complete graphs (in which case equality holds). One way to prove this claim is using the "interlacing" of eigenvalues, which is discussed in Chapter 9 of Godsil and Royle [22] (see also Theorem 5.3 of Godsil [20]).

2.4 Domination and the minimum rank problem

Again, let G be a graph on v vertices. A real symmetric $v \times v$ matrix M is *consistent* with G if for every distinct i and j, $M_{ij} \neq 0$ if and only ij is an edge in G.

The set of real symmetric matrices consistent with G will be written $\mathcal{M}(G)$. We define the *minimum rank* of a graph G to be

$$\operatorname{mr}(G) = \min\{\operatorname{rank}(M) \,:\, M \in \mathcal{M}(G)\}.$$

The *minimum rank problem* for a graph G asks to determine $\mathrm{mr}(G)$. There is an extensive literature on this problem treating various families of graphs; at present, the best reference is the survey [16] by Fallat and Hobgen.

In this section we prove that a graph's minimum rank is an upper bound on its domination number. This unpublished result, due to Godsil, is based on an argument by Rowlinson.

The *maximum multiplicity* of a graph G, written $\mathrm{mm}(G)$, is the maximum multiplicity taken over all eigenvalues of matrices in $\mathcal{M}(G)$. Note that the maximum multiplicity of G is equal to the maximum nullity of a matrix in $\mathcal{M}(G)$. This is because we can translate the eigenvalues of any matrix in $M(G)$ by adding a multiple of the identity matrix, yielding another matrix consistent with G. This observation implies the following useful lemma.

2.4.1 Lemma. *For any graph G on v vertices,* $\mathrm{mr}(G) + \mathrm{mm}(G) = v$. □

We are now ready to prove the main result.

2.4.2 Theorem. $\gamma(G) \leq \mathrm{mr}(G)$.

Proof. Let M be a matrix consistent with G that has an eigenvalue whose multiplicity equals $\mathrm{mm}(G)$. By the discussion above, we may assume that this eigenvalue is 0 so that $\mathrm{null}(M) = \mathrm{mm}(G)$.

Let U be a matrix whose columns form an orthonormal basis for $\ker(M)$, and let U_i denote the ith row of U. Let $B \subseteq V$ be such that $\{U_i\}_{i \in B}$ is a basis for the row space of U. Note that $|B| = \mathrm{null}(M) = \mathrm{mm}(G)$.

We claim that $V \setminus B$ is a dominating set in G. To see why this is true, assume that there exists a vertex $\ell \in B$ whose neighborhood $\Gamma(\ell)$ is contained in B. Let M_ℓ denote the ℓth row of M. Then

$$M_\ell U = \sum_{j=1}^{v} M_{\ell j} U_j = M_{\ell\ell} U_\ell + \sum_{j \sim \ell} M_{\ell j} U_j$$

where the second equality follows from the fact that if $j \neq \ell$ then $M_{\ell j} = 0$ if and only if $j \sim \ell$. But $M_\ell U = \mathbf{0}$ by definition of U, so that the set of vectors

$$\{U_\ell\} \cup \{U_j\}_{j \in \Gamma(\ell)}$$

is a linearly dependent subset of B, a contradiction.

Thus $V \setminus B$ is a dominating set and, by Lemma 2.4.1,

$$\mathrm{mr}(G) = v - \mathrm{mm}(G) = |V \setminus B| \geq \gamma(G)$$

as desired. □

Observe that $\mathrm{mr}(K_n) = 1$ since the all-ones matrix is consistent with K_n (in fact, if G is connected then $\mathrm{mr}(G) = 1$ if and only if G is a complete graph; see Fallat and Hobgen [16]). It follows that the bound in Theorem 2.4.2 is tight for the complete graphs.

Chapter 3

Some design theory

In this chapter we introduce some topics from the theory of combinatorial designs that will be essential to later discussions. We will use the notation $[n] = \{1, 2, \ldots, n\}$.

3.1 Steiner systems

A λ-(v, k, t) *design* is set of k-subsets of $[v]$, called *blocks*, such that every t-subset of $[v]$ is contained in exactly λ blocks. A 1-(v, k, t) design is called a (v, k, t) *Steiner system*. Designs and Steiner systems have been studied extensively; good introductions can be found in the books by MacWilliams and Sloane [39], Cameron and van Lint [6], and van Lint and Wilson [47].

We shall require the following well-known property of designs.

3.1.1 Lemma. *If $\mathcal{S}$ is a λ-(v, k, t) design and $s \le t$, then $\mathcal{S}$ is also a λ_s-(v, k, t) design, where*

$$\lambda_s = \lambda \frac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}.$$

Proof. Given a fixed s-subset of $[v]$, count in two ways the number of pairs (T, B) where T is a t-set that contains it and B is a block that contains T. $\square$

If $\mathcal{S}$ is a (v, k, t) Steiner system then it follows from the definitions that

$$|\mathcal{S}| = \binom{v}{t} \Big/ \binom{k}{t}. \tag{3.1.1}$$

Given a (v, k, t) Steiner system $\mathcal{S}$, let $\mathcal{S}[i]$ be the set of blocks containing a fixed $i \in [v]$. We can define a $(v-1, k-1, t-1)$ Steiner system $\mathcal{S}'$ by taking all of the blocks in $\mathcal{S}[i]$ for some i and removing i from them; it is easy to verify that this is indeed a Steiner system with the desired parameters. We say that $\mathcal{S}'$ is *derived* from $\mathcal{S}$. Conversely, if a (v, k, t) Steiner system $\mathcal{S}'$ can be derived from a $(v+1, k+1, t+1)$ Steiner system $\mathcal{S}$ we say that $\mathcal{S}$ is an *extension* of $\mathcal{S}'$.

3.2 Covering designs

Our investigation of dominating sets in Kneser graphs will intersect the much-studied topic of covering designs, which we now introduce. As usual, we keep our treatment brief; for details, additional information, and an extensive bibliography the reader is referred to the survey by Mills and Mullin [43].

An (n, r, k) *covering design* is a family of r-subsets of $[n]$, called *blocks*, such that every k-subset of $[n]$ is contained in at least one block. We define $C(n, r, k)$ to be the size of the smallest (n, r, k) covering design (use of the notation $C(n, r, k)$ will always imply $n \geq r \geq k$).

3.2.1 Lower bounds

Each block of an (n, r, k) covering design covers exactly $\binom{r}{k}$ k-sets and each k-set is covered by at least one block, so we have the trivial lower bound

$$C(n, r, k) \geq \binom{n}{k} \bigg/ \binom{r}{k} . \tag{3.2.1}$$

It follows from the definitions that $C(n, r, k)$ meets this lower bound if and only if there exists an (n, r, k) Steiner system.

A less trivial lower bound follows from the observation that all blocks of an (n, r, k) covering design that contain some fixed $i \in [n]$ cover those k-sets that contain i. Consequently, if $\mathcal{S}$ is an (n, r, k) covering design and we let, as before, $\mathcal{S}[i]$ be the set of blocks containing a fixed $i \in [n]$, then

$$|\mathcal{S}[i]| \geq C(n-1, r-1, k-1). \tag{3.2.2}$$

We can use this observation to prove a useful inequality.

3.2.1 Lemma.
$$C(n, r, k) \geq \left\lceil \frac{n}{r} C(n-1, r-1, k-1) \right\rceil .$$

Proof. Let $\mathcal{S}$ be a minimum size (n, r, k) covering design. From inequality (3.2.2) we know that

$$\sum_i |\mathcal{S}[i]| \geq n\, C(n-1, r-1, k-1),$$

but we also have

$$\sum_i |\mathcal{S}[i]| = r|\mathcal{S}| = r\, C(n, r, k)$$

which yields the desired inequality. □

Iterating the inequality in Lemma 3.2.1 and using the observation that $C(n, r, 1) = \lceil n/r \rceil$ yields what is known as the Schönheim bound, proved by Schönheim in [46]:

3.2.2 Theorem.

$$C(n,r,k) \geq \left\lceil \frac{n}{r} \left\lceil \frac{n-1}{r-1} \left\lceil \cdots \left\lceil \frac{n-k+1}{r-k+1} \right\rceil \cdots \right\rceil \right\rceil \right\rceil . \qquad \square$$

In light of our earlier observations, we conclude that if there exists an (n, r, k) Steiner system then $C(n, r, k)$ meets the Schönheim bound. The same hypothesis can be shown to imply that $C(n+1, r, k)$ meets the Schönheim bound (see Mills and Mullin [43]). In fact, $C(n, r, k)$ often meets the Schönheim bound, independently of the existence of particular Steiner systems. Proving that $C(n, r, k)$ is strictly greater than the Schönheim bound for a particular set of parameters tends to be difficult.

The value of $C(n, r, k)$ has been established for certain small values of r and k. For instance, it is known that $C(n, 3, 2)$ meets the Schönheim bound. It is also known that $C(n, 4, 2)$ meets the Schönheim bound unless $n \in \{7, 9, 10\}$ or $n = 19$, in which case it exceeds it by 1 and 2, respectively. $C(n, 5, 2)$ has been determined for many values of n, but many more remain open. $C(n, r, k)$ has also been determined in many individual cases in which n is small. See Mills and Mullin [43] for details on these results.

Turán proved that $C(n, n-2, k)$ meets the Schönheim bound, essentially a graph-theoretic result. He also made an interesting conjecture about the $(n, n-3, n-4)$ covering designs of minimum size; we defer to Mills and Mullin [43] for the details. The conjecture has been verified for $n \leq 13$, but the methods used do not appear to generalize, and the case $n = 13$ required a computer proof.

3.2.2 Upper bounds

Upper bounds on $C(n, r, k)$ (that is, constructions of small covering designs) are an active avenue of research and will be quite useful to us in later chapters.

In a celebrated use of the probabilistic method, Rödl proved the existence of covering designs whose size is asymptotically, as $n \to \infty$, equal to the trivial lower bound (3.2.1), thereby showing that this bound is asymptotically optimal:

$$C(n,r,k) = (1 + o(1)) \binom{n}{k} \bigg/ \binom{r}{k} .$$

See the book [1] by Alon and Spencer for more.

We will be interested in upper bounds on $C(n, r, k)$ for particular parameter sets, rather than asymptotic bounds. For these we will often cite the La Jolla Covering Repository [23], an online database of known values of $C(n, r, k)$ maintained by D. M. Gordon. It should be noted that most of the best upper bounds on $C(n, r, k)$ for particular parameter sets are implied by constructions due to Gordon et al. in [24].

Part II

Dominating Kneser graphs and related topics

Chapter 4

Dominating Kneser graphs

Given positive integers k and n define the *Kneser graph* $K_{n:k}$ to be the graph whose vertices are the k-subsets of $[n]$ and where two vertices are adjacent if the corresponding k-subsets are disjoint.[1]

If $n < 2k$ then $K_{n:k}$ has no edges, so we will always assume that $n \geq 2k$. In this case it is easy to see that $K_{n:k}$ is an $\binom{n-k}{k}$-regular graph on $\binom{n}{k}$ vertices. The reader is invited to verify the fact that $K_{5:2}$ is the Petersen graph. Kneser graphs were introduced by Lovász in [37] in order to prove Kneser's long standing conjecture, which is equivalent to the claim that the chromatic number of $K_{n:k}$ is $n-2k+2$. More information on this and other aspects of Kneser graphs can be found in Chapter 7 of Godsil and Royle [22].

In this chapter we consider the domination number of Kneser graphs, motivated by the fact that dominating sets in Kneser graphs, especially those of minimum size, often correspond to interesting combinatorial structures such as covering designs, projective planes, and Steiner systems. We will use the shorthand $\gamma(n : k)$ for $\gamma(K_{n:k})$, the minimum size of a dominating set in $K_{n:k}$. Similarly, $\gamma_t(n : k)$ will be shorthand for $\gamma_t(K_{n:k})$.

We begin with a brief discussion of the sphere-covering bound and then present a few basic results on $\gamma(n : k)$. In particular, we present a complete proof of the fact that $\gamma(n : k)$ is nonincreasing in n (the only proof in the literature encountered by the author is incomplete). Next we discuss the results of Hartman and West, who determined $\gamma(n : k)$ and $\gamma_t(n : k)$ for certain values of n and explored the connection between total dominating sets in Kneser graphs and covering designs. Finally, we compute $\gamma(n : k)$ for a few small values of n and k, determining in particular the value of $\gamma(n : 3)$ for all n. To facilitate these computations we prove two useful results. The first, generalizing the work of Hartman and West, shows that when n is not too small, a small dominating set is total. The second is a recursive lower bound on $\gamma(n : k)$ when n is a multiple of k.

[1] The vertices of $K_{n:k}$ will sometimes be referred to as elements of the set $V(K_{n:k})$ and sometimes as k-subsets of $[n]$, depending on context.

4.1 The sphere-covering bound

The sphere-covering bound on $\gamma(G)$ from Proposition 1.1.2 implies

$$\gamma(n:k) \geq \frac{\binom{n}{k}}{\binom{n-k}{k}+1}.$$

This bound is hard to apply in its present form; however, we can deduce from it a weaker but more elucidative statement using estimates for binomial coefficients. This was first done by Füredi in [18].

It is well known that

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

The lower bound is trivial, and the upper bound follows from the binomial theorem. It is also possible to prove a stronger estimate for $\binom{n}{k}$ (exercise I.15 of the book [34] by Jukna): if $k \leq k+x < n$ then

$$\left(\frac{n-k-x}{n-x}\right)^x \leq \binom{n-x}{k}\binom{n}{k}^{-1} \leq \left(\frac{n-k}{n}\right)^x \leq e^{-kx/n}. \tag{4.1.1}$$

The following lemma is from Füredi [18]. Our proof follows Jukna [34] (see Theorem 8.4), whose proof is not very different from the original one but slightly clearer. See Chapter 8 for more on Füredi's work.

4.1.1 Lemma. *If* $2k+1 \leq n \leq \frac{k^2}{c\log k+1}$ *then* $\gamma(n:k) > k^c$.

Proof. Let $\mathcal{S}$ be a minimum size dominating set in $K_{n:k}$. We begin with the sphere-covering bound

$$|\mathcal{S}| \geq \frac{\binom{n}{k}}{\binom{n-k}{k}+1}$$

and observe that

$$\frac{\binom{n}{k}}{\binom{n-k}{k}+1} \geq \frac{1}{2}\frac{\binom{n}{k}}{\binom{n-k}{k}}.$$

Combining this inequality with the estimate 4.1.1 gives

$$|\mathcal{S}| \geq \frac{1}{2}\binom{n}{k}\binom{n-k}{k}^{-1} \geq \frac{1}{2}e^{k^2/n} > e^{k^2/n-1}$$

so that our assumption on n implies $|\mathcal{S}| > e^{c\log k} = k^c$, as desired. □

4.2 Basic Results

In this section we prove a few basic results about dominating sets in $K_{n:k}$. We begin by observing that $\gamma(2k : k)$ is quite easy to compute. This is because $K_{2k:k}$ has $\binom{2k}{k}$ vertices and is regular of degree $\binom{k}{k} = 1$, hence is a matching of $\frac{1}{2}\binom{2k}{k}$ edges. It follows that $\gamma(2k : k) = \frac{1}{2}\binom{2k}{k}$. It is also easy to compute $\gamma(n : k)$ when n is sufficiently large.

4.2.1 Theorem. *If $n \geq k(k+1)$ then $\gamma(n : k) = k + 1$.*

Proof. First we show that $\gamma(n : k) \leq k + 1$. Since $n \geq k(k+1)$ it is possible to choose a set $\mathcal{S}$ of $k+1$ pairwise disjoint k-subsets of $[n]$. Any k-subset must be disjoint from at least one member of $\mathcal{S}$, since any subset of $[n]$ that intersects them all must contain at least $k+1$ elements. Thus $\mathcal{S}$ is a dominating set and $\gamma(n : k) \leq |\mathcal{S}| = k + 1$.

Next we prove $\gamma(n : k) \geq k + 1$. Let $\mathcal{S}$ be a vertex subset of $K_{n:k}$ of size k; we will show that $\mathcal{S}$ cannot be a dominating set. If all k-sets in $\mathcal{S}$ are pairwise disjoint then we can create a transversal of $\mathcal{S}$ of size k by choosing an arbitrary element from each member of $\mathcal{S}$. This transversal cannot be in $\mathcal{S}$ because it cannot equal any of the elements of $\mathcal{S}$, which are pairwise disjoint. Thus there exists a k-subset not in $\mathcal{S}$ and not disjoint from any member of $\mathcal{S}$, meaning $\mathcal{S}$ is not a dominating set.

If on the other hand some pair of members of $\mathcal{S}$ has nontrivial intersection then there exists a set U of size at most $k-1$ that intersects every member of $\mathcal{S}$. Since there are at least $n - k|\mathcal{S}| \geq k$ elements of $[n]$ that are not contained in any member of $\mathcal{S}$, we can use these elements to complete U to a k-subset that is not disjoint from any member of S but cannot itself be in S. Again, this implies that $\mathcal{S}$ is not a dominating set. □

Theorem 4.2.1 allows us to determine $\gamma(n, 2)$ for all allowed values of n. This was originally done by Ivančo and Zelinka in [32].

4.2.2 Corollary. *If $n \geq 4$ then $\gamma(n, 2) = 3$.*

Proof. If $n \geq 6$, the claim follows from Theorem 4.2.1, so it remains to verify it for $n = 4, 5$. This is easy to do, as $K_{4:2}$ is a matching on six vertices and $K_{5:2}$ is the Petersen graph. □

It is quite easy, then, to determine $\gamma(n : k)$ for nearly all values of n. Indeed, if $n \geq k(k+1)$ we know exactly what the minimum size dominating sets of $K_{n:k}$ are, since it follows from the proof of Theorem 4.2.1 that all k-subsets in a minimum size dominating sets must be pairwise disjoint. But what can we say about $\gamma(n : k)$ when $2k < n < k(k+1)$? This question will occupy us for the remainder of this chapter.

One fundamental fact is that $\gamma(n : k)$ is nonincreasing with n. Before proving this, we define the *upward shadow* of a $(k-1)$-subset of $[n]$ to be the set of k-subsets of $[n]$ that contain it. Observe that the upward shadow contains $n-k+1$ k-subsets.

4.2.3 Lemma. *If $n \geq 2k+1$, then there exists a minimum size dominating set in $K_{n:k}$ that does not contain the upward shadow of any $(k-1)$-subset of $[n]$.*

Proof. Let $\mathcal{S}$ be a minimum size dominating set in $K_{n:k}$ in which every vertex has at least one private neighbor in $V(K_{n:k}) \setminus \mathcal{S}$ (such dominating sets exist by Lemma 2.3.2), and let U be an arbitrary $(k-1)$-subset. If $\mathcal{S}$ does not contain any k-sets in the upward shadow of U then we are done. Otherwise let S be such a k-set in $\mathcal{S}$, so that $S = U \cup \{i\}$ for some $i \in [n]$. Let N be the private neighbor of S in $V(K_{n:k}) \setminus \mathcal{S}$.

By definition N is disjoint from S, but since $|S| = |N| = k$ and $n \geq 2k+1$ there must exist some $j \in [n]$ that is not in $S \cup N$. Then the set $S' = U \cup \{j\}$ is disjoint from N and is in the upward shadow of U. We must have $S' \notin S$ for otherwise N would not be a private neighbor of S, from which it follows that $\mathcal{S}$ does contain the upward shadow of U or any other $(k-1)$-subset. □

4.2.4 Proposition. *If $n \geq 2k+1$ then $\gamma(n:k) \geq \gamma(n+1:k)$.*

Proof. Let $\mathcal{S}$ be a minimum size dominating set in $K_{n:k}$ that does not contain the upward shadow of any $(k-1)$-subset of $[n]$, as in Lemma 4.2.3. It suffices to show that $\mathcal{S}$ is a dominating set in $K_{n+1:k}$.

Fix $S \in V(K_{n+1:k}) \setminus \mathcal{S}$; if $n+1 \notin S$ then S is adjacent to a vertex in $\mathcal{S}$ by assumption. If $n+1 \in S$ then define $S^- = S \setminus \{n+1\}$. Since $\mathcal{S}$ does not contain the upward shadow of S^- there exists $i \in [n]$ such that $S^- \cup \{i\} \notin \mathcal{S}$. Since $S^- \cup \{i\}$ is a vertex in $K_{n:k}$ it is adjacent to some vertex in $\mathcal{S}$; but then S must be adjacent to the same vertex in $K_{n+1:k}$. Thus $\mathcal{S}$ dominates $K_{n+1:k}$, as desired. □

Our proof of Proposition 4.2.4 is based on the one given by Hartman and West in [27]. Hartman and West, however, are not explicit about their use of Lemma 2.3.2; they assume without proof the existence of a minimum size dominating set in which every vertex has a private neighbor, which they use to prove the claim in Lemma 4.2.3.

Theorem 4.2.1 and Proposition 4.2.4 both appear in Hartman and West [27], and are also independently attributed to Jurkiewicz and Wager by Chris Godsil in [21]. Both results are likely much older than either of these references suggest, though an original reference is unknown.

4.2.1 Monotonicity via graph homomorphisms

It is possible to prove additional monotonicity results for $\gamma(n:k)$ using the notion of a graph homomorphism. Given graphs X and Y, a *homomorphism* from X to Y is a function $f : V(X) \to V(Y)$ such that if $x \sim x'$ in X then $f(x) \sim f(x')$ in Y. By convention, homomorphisms never map adjacent vertices to the same vertex. It is easy to see that the image of a dominating set under a surjective homomorphism is again a dominating set. This implies the following.

4.2.5 Lemma. *If there exists a surjective homomorphism from a graph X to a graph Y then $\gamma(X) \geq \gamma(Y)$.* □

It follows from the lemma that if there exists a surjective homomorphism from $K_{n:k}$ to $K_{n':k'}$ then $\gamma(n:k) \geq \gamma(n':k')$.

It is clear that removing an arbitrary element from every vertex of $K_{n:k}$ is a surjective homomorphism from that graph to $K_{n:k-1}$. This implies that

$$\gamma(n:k) \geq \gamma(n:k-1).$$

We can improve this result by noting that removing the *largest* element of every vertex of $K_{n:k}$ is a surjective homomorphism to $K_{n-1:k-1}$, so that

$$\gamma(n:k) \geq \gamma(n-1:k-1).$$

This homomorphism can be extended to a surjective homomorphism from $K_{n:k}$ to $K_{n-2:k-1}$, a result due to Stahl; see Theorem 7.9.2 of Godsil and Royle [22] for a proof. This implies

$$\gamma(n:k) \geq \gamma(n-2:k-1).$$

It is unlikely that any of these inequalities is ever tight. However, homomorphisms might be helpful in exploring the structure of minimum size dominating sets in Kneser graphs. See Section 7.9 of Godsil and Royle [22] for more on homomorphisms between Kneser graphs.

4.2.2 Projective planes

We make note of the following result, first proved by Meyer in [42] (though Godsil in [21] also independently attributes it to Jurkiewicz and Wager).

4.2.6 Theorem. *If $n \geq k^2 - k + 1$ then the projective plane of order $k-1$ is a dominating set in $K_{n:k}$.* □

The projective plane of order $k-1$ has size $k^2 - k + 1$ so Theorem 4.2.6 implies that $\gamma(n:k) \leq k^2 - k + 1$ when $n \geq k^2 - k + 1$ and a projective plane of order $k-1$ exists. However, computational evidence suggests that this bound is not close to tight; we will see in Section 4.6 that even though the bound is tight for $k = 3$ and $n = k^2 - k + 1 = 7$, we have

$$\gamma(13,4) \leq 10 < 13 = k^2 - k + 1$$

for $k = 4$ and

$$\gamma(21,5) \leq 12 < 21 = k^2 - k + 1$$

for $k = 5$, which does not bode well. Moreover, the theorem can only be applied to a particular k when a projective plane of order $k-1$ exists, which is only known to be happen when $k-1$ is a power of a prime.

Thus the projective plane is most likely useless in the search for minimum size dominating sets. Observe, however, that by definition it is an *independent* dominating set. This will have great significance in Chapter 8, where we discuss independent dominating sets in Kneser graphs and their relevance to a problem in extremal combinatorics.

4.3 Total domination and covering designs

Recall that a total dominating set is a vertex subset $S \subseteq V$ such that every $x \in V$ has a neighbor in S. Compare this to the definition of a dominating set, where only vertices not in S are required to have neighbors in S. It follows that every total dominating set is a dominating set, but the gap between the size of the smallest dominating and smallest total dominating sets can be, in general, large.

In this section we make some basic observations about total dominating sets in Kneser graphs as a preface to our discussion, in the next section, of the work of Hartman and West. Recall that $\gamma_t(n:k)$ is the total domination number of $K_{n:k}$, the size of the smallest total dominating set. If $\mathcal{T}$ is a total dominating set in $K_{n:k}$, then every k-subset of $[n]$ is disjoint from some k-subset in $\mathcal{T}$.

As Hartman and West observed, this means that $\mathcal{T}$ is a total dominating set if and only if the complements of the k-subsets in $\mathcal{T}$ form a $(n, n-k, k)$ covering design, which implies that

$$\gamma_t(n:k) = C(n, n-k, k).$$

Now the lower bound

$$\gamma_t(n:k) \geq \binom{n}{k} \Big/ \binom{n-k}{k}$$

is suggested both by equation (3.2.1) and by the total domination analogue of the sphere-covering bound that we saw in Section 1.2.

Recall that a totally perfect 1-code in a regular graph is a total dominating set whose neighborhoods partition the vertex set. Thus $\mathcal{T}$ is a totally perfect 1-code in $K_{n:k}$ if and only if every k-subset is disjoint from exactly one member of $\mathcal{T}$, i.e. if and only if the complements of the k-subsets in $\mathcal{T}$ form an $(n, n-k, k)$ Steiner system. This suggests that totally perfect 1-codes in Kneser graphs are rare, but to prove it would be immensely difficult. We will see more connections between Kneser graphs and Steiner systems when we discuss the odd graphs in Chapter 5.

Hartman and West observe in [27] that if $n \geq r(k+1)$ then

$$C(n, n-r, k) = k+1.$$

The proof is identical to the proof of Theorem 4.2.1. To wit, one first observes that the set of complements of $k+1$ disjoint r-subsets is an $(n, n-r, k)$ covering design. One then shows that given any collection $\mathcal{T}$ of k r-subsets it is possible to find a subset of size at most k that intersects every member of $\mathcal{T}$.

Taking $r = k$ above, we conclude that when $n \geq k(k+1)$ we have

$$C(n, n-k, k) = \gamma_t(n:k) = \gamma(n:k) = k+1.$$

Thus when n is sufficiently large, $\gamma_t(n:k)$ and $\gamma(n:k)$ are equal (in fact, it is easy to conclude this directly from the proof of Theorem 4.2.1). At the other extreme, when n is the minimum value of $2k$, we have $\gamma(n:k) = \frac{1}{2}\binom{2k}{k}$ and $\gamma_t(n:k) = \binom{2k}{k}$, so that the total domination number is significantly larger than the domination number.

The obvious question is, How do the domination number and the total domination number compare when $2k < n < k(k+1)$? Hartman and West tackled this question in [27], showing that the two values are equal when $n \geq \frac{3}{4}k^2 + k$. The next section deals with their work.

4.4 Determining $\gamma(n:k)$ for large n

In [27], Hartman and West determine $\gamma(n:k)$ when $n \geq \frac{3}{4}k^2+k$, and also show that $\gamma(n:k) = \gamma_t(n:k)$ for these values of n. In this section we highlight their results and discuss some of the proofs.

Before we do so, we make the following observation. When dealing with Kneser graphs, domination and total domination can be phrased as follows. If $\mathcal{S}$ is a collection of subsets of $[n]$, a *transversal* of $\mathcal{S}$ is a subset of $[n]$ that intersects every subset in $\mathcal{S}$. If we let $\mathcal{S}$ be a collection of k-subsets, we see that $\mathcal{S}$ dominates $K_{n:k}$ if and only if every transversal of $\mathcal{S}$ of size at most k is contained in some member of $\mathcal{S}$. Similarly, $\mathcal{S}$ is a total dominating set if and only if all transversals of $\mathcal{S}$ are of size at least $k+1$.

The main theorem in [27] is the following:

4.4.1 Theorem. *If $k(k+1) - \ell\lfloor k/2\rfloor \leq n < k(k+1) - (\ell-1)\lfloor k/2\rfloor$, where $0 \leq \ell \leq \lfloor k/2\rfloor$, then $\gamma_t(n:k) = \gamma(n,k) = k+1+\ell$.* □

To prove the theorem, Hartman and West first prove an upper bound on $\gamma_t(n:k)$ with an explicit construction of a total dominating set.

4.4.2 Theorem. *If $n \geq k(k+1)-\ell\lfloor k/2\rfloor$ and $\ell \leq \lceil k/2\rceil$ then $\gamma_t(n:k) \leq k+1+\ell$.*

Proof. By monotonicity we may assume that $n = k(k+1) - \ell\lfloor k/2\rfloor$.

Define a *triangle configuration* to be the following set of three k-subsets of $\{1,2,\ldots,\lceil 3k/2\rceil\}$. If k is even, the three sets have pairwise intersections of size $k/2$ and no common elements. If k is odd, two sets intersects at $\frac{k+1}{2}$ elements and the third set is formed by taking the remaining $\frac{k-1}{2}$ elements from each of the previous two sets and adding one final element. Note that in either case, the smallest transversal of a triangle configuration has size 2.

Now define $\mathcal{S}$, a set of k-subsets of $[n]$, to consist of $k+1-2\ell$ disjoint k-sets and ℓ triangle configurations formed from the remaining elements of $[n]$. Note that this is possible because

$$k(k+1-2\ell) + \ell\left\lceil\frac{3k}{2}\right\rceil = k(k+1) - \ell\lfloor k/2\rfloor = n.$$

Since each triangle configuration consists of three subsets, we have

$$|\mathcal{S}| = (k+1-2\ell) + 3t = k+1+\ell.$$

Finally, it is easy to see that by construction, the smallest transversal of $\mathcal{S}$ has size $k+1$, so that $\mathcal{S}$ is a total dominating set. $\square$

Next, Hartman and West prove the following technical lemma, which will imply a lower bound on $\gamma(n : k)$ that matches the upper bound above.

4.4.3 Lemma. *Fix positive integers n, k, r, ℓ such that $n < r(k+1)-(\ell-1)\lfloor r/2 \rfloor$. If $\mathcal{S}$ is a collection of r-subsets of $[n]$ of size at least $k+\ell$ then there exists $S \subseteq [n]$ that intersects at least $|S| + \ell$ members of $\mathcal{S}$.* $\square$

Given any collection of k-subsets $\mathcal{S}$ of size $k+\ell$, the lemma can be used to construct a transversal of $\mathcal{S}$ of size k that is not in $\mathcal{S}$, proving the following lower bound on $\gamma(n : k)$:

4.4.4 Theorem. *If $k \geq 3$ and $n < k(k+1)-(\ell-1)\lfloor k/2 \rfloor$, where $0 \leq \ell \leq \lfloor k/2 \rfloor$, then $\gamma(n, k) \geq k+1+\ell$.* $\square$

See [27] for details. The condition on k only excludes the case $K_{5:2}$, the Petersen graph, which has domination number 3.

Since $\gamma_t(n : k) \geq \gamma(n : k)$, Theorems 4.4.2 and 4.4.4 combine to give Theorem 4.4.1. It is convenient to rewrite this theorem in a more user-friendly manner by setting n equal to its lower bound and using the bound on ℓ to eliminate that variable; we do so in a corollary.

4.4.5 Corollary. *If $k(k+1) \geq n \geq \frac{3}{4}k^2 + k$ then*

$$\gamma(n : k) = \gamma_t(n : k) = k + 1 + \left\lceil \frac{k(k+1)-n}{\lfloor k/2 \rfloor} \right\rceil . \qquad \square$$

It is straightforward to generalize the above arguments, as Hartman and West do, to show that

$$C(n, n-r, k) = k + 1 + \left\lceil \frac{r(k+1)-n}{\lfloor r/2 \rfloor} \right\rceil . \tag{4.4.1}$$

when $\frac{3}{4}rk + r \leq n \leq r(k+1)$.

4.5 Two useful results

In this section we prove two results that will be useful when we attempt to determine $\gamma(n : k)$ for some small values of n and k. The first result is derived to a large extent from the work of Hartman and West, and states that when n is not too small, a small dominating set must be total. The second result is a recursive lower bound on $\gamma(n : k)$ when n is a multiple of k.

4.5.1 Small dominating sets are total

A slightly weaker version of the following theorem is implicit in the work of Hartman and West (in particular, the proof of their Theorem 7 from [27]).

4.5.1 Theorem. *Let $\mathcal{S}$ be a dominating set in $K_{n:k}, k \geq 4$. If $n \geq k + |\mathcal{S}|$ and $|\mathcal{S}| \leq 2k$ then $\mathcal{S}$ is a total dominating set.*

Proof. It suffices to show that $\mathcal{S}$ has no transversal of fewer than $k+1$ elements. It is easy to see that $\mathcal{S}$ cannot have transversal of $k-1$ or fewer elements, since such a transversals would have at least $n-k+1 > |\mathcal{S}|$ extensions to a k-set, so it is impossible for $\mathcal{S}$ to contain all of them. It remains to show that $\mathcal{S}$ has no transversal of size k.

For the sake of contradiction assume that $\mathcal{S}$ has a transversal S of size k, which it must then contain. We have already observed that no $(k-1)$-subset of S can be a transversal of $\mathcal{S}$, so for every $i \in S$ there exists at least one member of $\mathcal{S}$ that intersects S precisely at i. Since $|\mathcal{S} \setminus \{S\}| < 2k$ there exists $i \in S$ such that only one such subset exists. In other words, there exists $i \in S$ and $S_i \in \mathcal{S}$ such that $S \setminus \{i\}$ is a transversal of $\mathcal{S} \setminus \{S_i\}$.

Define $S^- = S \setminus \{i\}$. Since S^- is a transversal of $\mathcal{S} \setminus \{S_i\}$, adding any $j \in S_i$ to S^- yields a transversal of $\mathcal{S}$, hence every such k-set is in $\mathcal{S}$; denote this subset of $\mathcal{S}$ by $\mathcal{T}$.

Define $\mathcal{T}' = \mathcal{S} \setminus (\mathcal{T} \cup \{S_i\})$. Since $|\mathcal{T}| = k$ it follows that

$$|\mathcal{T}'| = |\mathcal{S}| - k - 1 \leq k - 1.$$

But by our observations above, for every $i \in S^-$ there exists at least one k-set in $\mathcal{S}$ that intersects S^- precisely at i, from which it follows that $\mathcal{T}'$ is exactly the set of these k-sets, and each of these k-sets intersects S^- at a unique, distinct element.

Pick a transversal T of $\mathcal{T}'$ of size at most $k-1$ by choosing one element not in S^- from every set in $\mathcal{T}'$. It is possible to extend T to a transversal of $\mathcal{S}$ in $k-1$ ways by the addition of an element in S^-. Since any such extension intersects S^- in exactly one element, the set of these intersections is exactly $\mathcal{T}'$, meaning that every set in $\mathcal{T}'$ is an extension of T (we note that since $|\mathcal{T}'| = |\mathcal{S}| - k - 1$, this forces $|\mathcal{S}| = 2k$).

It follows that every member of $\mathcal{S}$, except S_i, contains either S^- or T. This implies that there exists a transversal of $\mathcal{S}$ of size at most $3 \leq k-1$, a possibility that we had ruled out at the very beginning of this proof. We have, then, our desired contradiction, and we conclude that $\mathcal{S}$ cannot have a transversal of size k and thus is a total dominating set. □

It follows from the theorem that If $k \geq 4$, $n \geq k + \gamma(n:k)$ and $\gamma(n:k) \leq 2k$ then $\gamma(n:k) = \gamma_t(n:k)$.

4.5.2 A recursive lower bound

We begin with a definition. The *incidence graph* of $K_{n:k}$, written $\mathcal{I}_{n:k}$, is a bipartite graph with one part of size n corresponding to $[n]$ and another part of size $\binom{n}{k}$ corresponding to the vertices of $K_{n:k}$. There is an edge between $i \in [n]$ and $S \in V(K_{n:k})$ if and only if $i \in S$.

Given a vertex subset $\mathcal{S} \subseteq V(K_{n:k})$, let $\mathcal{I}_{n:k}$ denote the subgraph of $\mathcal{I}_{n:k}^{\mathcal{S}}$ induced by $\mathcal{S}$ and $[n]$, called the *incidence subgraph.*

Now, let $\mathcal{S}$ be a vertex subset of the Kneser graph $K_{n:k}$, $n \geq 2k+1$. Given $i \in [n]$, recall that $\mathcal{S}[i]$ is defined to consist of the members of S that contain i. An elementary averaging argument proves the following useful lemma.

4.5.2 Lemma. *Given any $\mathcal{S} \subseteq V(K_{n:k})$, there exists $i \in [n]$ such that*

$$\big|\mathcal{S}[i]\big| \geq k|\mathcal{S}|/n.$$

Proof. Consider the incidence graph $\mathcal{I}_{n:k}^{\mathcal{S}}$; recall that the two parts correspond to $[n]$ and $\mathcal{S}$. Every vertex in the $\mathcal{S}$ part has degree k, so the total number of edges is $k|\mathcal{S}|$. Thus the average degree of a vertex in the $[n]$ part is $k|\mathcal{S}|/n$, and there exists $i \in n$ whose degree is at least this value. □

We now prove the promised recursive lower bound.

4.5.3 Lemma. *Assume that $n = \alpha k$, where $\alpha, k \geq 2$. Then*

$$\gamma(n:k) \geq \frac{\alpha}{\alpha-1}\big(\gamma(n+k:k)-1\big).$$

Proof. Let $\mathcal{S}$ be a minimum size dominating set in $K_{n:k}$. Choose $i \in [n]$ such that $\big|\mathcal{S}[i]\big| \geq k\gamma(n:k)/n$; such an i exists by Lemma 4.5.2. Since $n = \alpha k$, we have

$$\big|\mathcal{S}[i]\big| \geq \frac{k\gamma(n:k)}{n} = \frac{\gamma(n:k)}{\alpha}.$$

Now regard the members of $\mathcal{S}$ as vertices in $K_{n+k:k}$, and define

$$\mathcal{S}^+ = \big(\mathcal{S} \setminus \mathcal{S}[i]\big) \bigcup \{U\}$$

where $U = \{n+1, \ldots, n+k\} \in V(K_{n+k:k})$.

We claim that $\mathcal{S}^+$ is a dominating set in $K_{n+k:k}$. To see this, let W be a vertex in $K_{n+k:k} \setminus \mathcal{S}^+$. If $W \cap U = \emptyset$ then $W \sim U$ in $K_{n+k:k}$, so assume that $|W \cap U| > 0$. Then W contains at most $k-1$ elements in $[n]$. Define $W' \subset [n]$ by adding i to $W \cap [n]$, and then adding arbitrary elements of $[n]$ until $|W'| = k$. Since $W' \in V(K_{n:k})$ and $\mathcal{S}$ is a dominating set in that graph, there exists $S \in \mathcal{S}$ such that $W' \cap S = \emptyset$. Since $i \in W'$ we know that $S \in \mathcal{S} \setminus \mathcal{S}[i]$, so $S \in \mathcal{S}^+$. But $W \cap [n] \subseteq W'$ so $W' \cap S = \emptyset$ implies $W \cap S = \emptyset$, as desired. Thus $\mathcal{S}^+$ is a dominating set in $K_{n+k:k}$.

It follows that $|\mathcal{S}^+| \geq \gamma(n+k:k)$. But by our choice of i,

$$|\mathcal{S}^+| = \big|(\mathcal{S} \setminus \mathcal{S}[i]) \cup \{U\}\big| = |\mathcal{S}| - \big|\mathcal{S}[i]\big| + 1 \leq \gamma(n:k) - \frac{\gamma(n:k)}{\alpha} + 1$$

so that

$$\gamma(n:k) - \frac{\gamma(n:k)}{\alpha} + 1 \geq \gamma(n+k:k)$$
$$\gamma(n:k) \geq \frac{\alpha}{\alpha-1}\big(\gamma(n+k:k) - 1\big)$$

as desired. □

It is possible to use Lemma 4.5.3 to prove a lower bound on $\gamma(n : k)$ by induction, starting with the fact that $\gamma(n : k) = k + 1$ when $n = k(k + 1)$.

4.5.4 Proposition. *Assume that $n = \alpha k$, where $2 \leq \alpha \leq k + 1$. Then*

$$\gamma(n:k) \geq \frac{1}{2}\left(\frac{k(k+1)}{\alpha-1} + \alpha\right). \qquad \square$$

This bound, however, is not very good since it is only tight for $\alpha = k+1$; the bound gets worse and worse with every inductive application of the inequality in Lemma 4.5.3. It is even worse than the sphere-packing bound when α is small.

It is often much more helpful to use Lemma 4.5.3 directly to generate a lower bound on $\gamma(\alpha k : k)$ after $\gamma(\alpha k + k : k)$ has been determined. For instance, applying Proposition 4.5.4 to $K_{12:4}$ gives $\gamma(12 : 4) \geq 7$. However, Corollary 4.4.5 tells us that $\gamma(16 : 4) = 7$, and combining this with Lemma 4.5.3 gives the better bound $\gamma(12 : 4) \geq 9$.

4.6 Known values of $\gamma(n : k)$

In this section we compute $\gamma(n : k)$ for some small values of n, k. These values are recorded in Table 4.1. The companion Table 4.2 records known values of $\gamma_t(n : k)$ (all data in this table are taken from the La Jolla Covering Repository [23]). Recall that we have already shown in Corollary 4.2.2 that $\gamma(n : 2) = 3$ for all $n \geq 4$. Below we discuss the case $k = 3$, which is completely resolved, and the cases $k = 4, 5$, which are not.

Both Tables 4.1 and 4.2 obey the following conventions. Rows in the table correspond to values of n while columns correspond to values of k. An entry is in boldface when $n = k(k + 1)$, and it follows that all entries below a boldface entry are equal to $k + 1$. When an entry is unknown the table lists the best known upper and lower bounds.

There is an additional convention in Table 4.1: an entry is italicized when $n < k(k + 1)$ and $\gamma(n : k) = \gamma_t(n : k)$.

4.6.1 $k = 3$

If $n \geq 12$ then $\gamma(n : 3) = 4$ by Theorem 4.2.1. Also, $\gamma(6 : 3) = \frac{1}{2}\binom{6}{3} = 10$.

n/k	2	3	4	5
4	3	-	-	-
5	3	-	-	-
6	**3**	10	-	-
7	3	7	-	-
8	3	7	35	-
9	3	*7*	$22 \leq \cdots \leq 27$	-
10	3	*6*	$14 \leq \cdots \leq 20$	126
11	3	*5*	$10 \leq \cdots \leq 17$	66
12	3	**4**	$10 \leq \cdots \leq 12$	$36 \leq \cdots \leq 59$
13	3	4	*10*	$23 \leq \cdots \leq 42$
14	3	4	*9*	$16 \leq \cdots \leq 32$
15	3	4	*8*	$14 \leq \cdots \leq 27$
16	3	4	*7*	$11 \leq \cdots \leq 22$
17	3	4	*7*	$11 \leq \cdots \leq 17$
18	3	4	*6*	$11 \leq \cdots \leq 15$
19	3	4	*6*	$11 \leq \cdots \leq 14$
20	3	4	**5**	$11 \leq \cdots \leq 12$
21	3	4	5	$11 \leq \cdots \leq 12$

Table 4.1: Some values of $\gamma(n:k)$.

n/k	2	3	4	5
4	6	-	-	-
5	4	-	-	-
6	**3**	20	-	-
7	3	12	-	-
8	3	8	70	-
9	3	7	30	-
10	3	6	20	252
11	3	5	17	$96 \leq \cdots \leq 100$
12	3	**4**	12	$55 \leq \cdots \leq 59$
13	3	4	10	$33 \leq \cdots \leq 42$
14	3	4	9	$28 \leq \cdots \leq 32$
15	3	4	8	$24 \leq \cdots \leq 27$
16	3	4	7	$18 \leq \cdots \leq 22$
17	3	4	7	17
18	3	4	6	15
19	3	4	6	14
20	3	4	**5**	12
21	3	4	5	12

Table 4.2: Some values of $\gamma_t(n:k)$ (see La Jolla Covering Repository [23]).

When $n = 10, 11$ we can use Corollary 4.4.5 to determine $\gamma(n : 3)$:

$$\gamma(10 : 3) = 4 + \left\lceil \frac{12 - 10}{\lfloor 3/2 \rfloor} \right\rceil = 6$$
$$\gamma(11 : 3) = 4 + \left\lceil \frac{12 - 11}{\lfloor 3/2 \rfloor} \right\rceil = 5$$

When $n = 7$, Theorem 4.2.6 tells us that the projective plane of order 2 is a dominating set in $K_{7:3}$. However, we know from the sphere-covering bound that

$$\gamma(7 : 3) \geq \frac{\binom{7}{3}}{\binom{4}{3} + 1} = 7 \tag{4.6.1}$$

so that $\gamma(7 : 3) = 7$. In particular, the projective plane of order 2 is a perfect 1-code in $K_{7:3}$.

We are left with $n = 8, 9$. By monotonicity (Proposition 4.2.4) we know that $\gamma(8 : 3)$ and $\gamma(9 : 3)$ are at most 7, so it suffices to show that neither $K_{8:3}$ nor $K_{9:3}$ has a dominating set of size 6.

4.6.1 Proposition. $\gamma(8 : 3) \geq 7$.

Proof. Assume for the sake of contradiction that $\mathcal{S}$ is a dominating set of size 6 in $K_{8:3}$. Recall that $\mathcal{S}[i]$ is the set of members of $\mathcal{S}$ that contain some fixed i. Choose $i \in [n]$ that maximizes $|\mathcal{S}[i]|$; by Lemma 4.5.2, there exists $j \in [8]$ such that $|\mathcal{S}[j]| \geq 3$, so we know that $|\mathcal{S}[i]| \geq 3$ for our chosen i.

We cannot have $|\mathcal{S}[i]| = 6$ since then $\mathcal{S}$ would have to contain all 3-subsets of $[8]$ that contain i, of which there are $\binom{7}{2} = 21$. Thus we can assume $|\mathcal{S}[i]| \leq 5$.

If there exists $j \in [n]$ such that every subset in $\mathcal{S} \setminus \mathcal{S}[i]$ contains j, then $\mathcal{S}$ must contain every 3-subset that contains both i and j, of which there are 6, implying the contradiction $|\mathcal{S}[i]| \geq 6$. This implies in particular that there must be more than one set in $\mathcal{S} \setminus \mathcal{S}[i]$, so that $|\mathcal{S}[i]| \leq 4$.

If $|\mathcal{S}[i]| = 4$ then $|\mathcal{S} \setminus \mathcal{S}[i]| = 2$ and by the above the two 3-subsets in $\mathcal{S} \setminus \mathcal{S}[i]$, call them S and T, must be disjoint. Then there are nine 3-subsets that contain i and intersect both S and T that must thus be in $\mathcal{S}$, a contradiction.

If $|\mathcal{S}[i]| = 3$ then there are three 3-subsets not containing i, call them S, T, R. Clearly not all three can be disjoint, so assume that S and T contain some j in their intersection. By the above, j cannot be in R, so $\mathcal{S}$ must contain the three 3-subsets that contain i, j, and an element of R; indeed, $\mathcal{S}[i]$ must be the set of these three 3-subsets. But then $|\mathcal{S}[j]| \geq 2 + |\mathcal{S}[i]| = 5$, which we already proved cannot be.

Since $|\mathcal{S}[i]| \geq 3$, all cases have been considered and the claim is true. □

The case $n = 9$ is similar to $n = 8$, but is significantly more tedious.

4.6.2 Proposition. $\gamma(9 : 3) \geq 7$.

Proof. Assume for the sake of contradiction that $\mathcal{S}$ is a dominating set of size 6 in $K_{9:3}$. As before, we choose $i \in [9]$ that maximizes $|\mathcal{S}[i]|$.

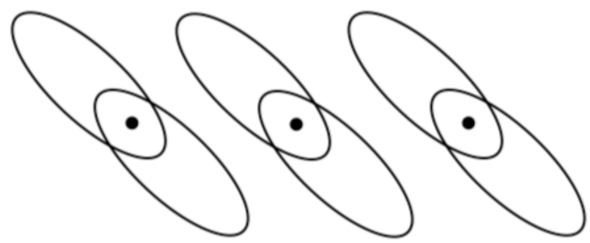

Figure 4.1: A bad partition of a dominating set of size 6 in $K_{9:3}$.

Assume first that $|\mathcal{S}[i]| \geq 3$. Then the proof of Proposition 4.6.1, the case $n = 8$, extends to the case $n = 9$ in its entirety. We conclude that we must have $|\mathcal{S}[i]| \leq 2$. However, unlike in the $n = 8$ case, Lemma 4.5.2 cannot now be used to end the proof because it no longer guarantees the existence of $i \in [9]$ such that $|\mathcal{S}[i]| \geq 3$. All it implies is that $|\mathcal{S}[j]| \geq 2$ for some $j \in [9]$.

It follows that $|\mathcal{S}[i]| = 2$, but we can say even more: $|\mathcal{S}[j]| = 2$ for every $j \in [9]$. To see why, consider the incidence subgraph $\mathcal{I}_{9:3}^{\mathcal{S}}$ and note that the degree of every $j \in [9]$ equals $|\mathcal{S}[j]|$. Now, $\mathcal{I}_{9:3}^{\mathcal{S}}$ has 18 edges since $|\mathcal{S}| = 6$, meaning the degree of every vertex corresponding to an element of [9] cannot be less than 2, yet must be at most 2 because $|\mathcal{S}[i]| \leq 2$ for the i that maximizes $|\mathcal{S}[i]|$.

Before we go on, we make an important observation. Define a *bad partition* of $\mathcal{S}$ to be a partition of its six elements into three pairs $\{S_1, S_1'\}, \{S_2, S_2'\}, \{S_3, S_3'\}$, such that $S_\ell \cap S_\ell' \neq \emptyset$ for each ℓ. This is illustrated in Figure 4.1. If a bad partition of $\mathcal{S}$ exists then, under the assumption that $|\mathcal{S}[j]| = 2$ for every $j \in [9]$, $\mathcal{S}$ cannot be a dominating set. This is because if it were, and we defined i_ℓ to be an element in $S_\ell \cap S_\ell'$, then $\mathcal{S}$ would have to contain $\{i_1, i_2, i_3\}$, which implies that $|\mathcal{S}[i_\ell]| \geq 3$ for one of the ℓ, a contradiction. We will often make use of this observation in order to produce contradictions.

To proceed, we first claim that $\mathcal{S}$ contains three pairwise disjoint subsets. To see this, observe first that it must contain two subsets that are disjoint, for otherwise any two of its members have a nonempty intersection so an arbitrary partition of $\mathcal{S}$ into pairs would be a bad partition. We can assume that the two disjoint subsets are $S = \{1, 2, 3\}$ and $R = \{4, 5, 6\}$. Let the other four subsets in $\mathcal{S}$ be T, U, V, W. We will show that one of these must equal $\{7, 8, 9\}$.

Assume otherwise. Since $\mathcal{S}[7] = \mathcal{S}[8] = \mathcal{S}[9] = 2$ and we assume that $\{7, 8, 9\} \notin \mathcal{S}$, there must exist at least 2 subsets in $\{T, U, V, W\}$ such that each intersects $\{7, 8, 9\}$ at 2 elements (this is clear from the incidence subgraph induced by the blocks $\{T, U, V, W\}$, illustrated in Figure 4.2). Say T, U are these two subsets. They then must intersect at some element in $\{7, 8, 9\}$, say 7. What about V and W? Well, since we must satisfy $\mathcal{S}[7] = \mathcal{S}[8] = \mathcal{S}[9] = 2$, neither contains 7 and there two possibilities; either one of V, W contains both 8 and 9, or one contains 8 and the other 9. If the former is true, say $\{8, 9\} \subset V$, then V intersects either S or R; assume S. Then $\{T, U\}, \{V, S\}, \{W, R\}$ is a bad partition. If the latter is true then it is similarly possible to create a bad partition by pairing up T and U, S and the subset in $\{V, W\}$ that intersects it, and R and the other subset in $\{V, W\}$. Thus assuming $\{7, 8, 9\} \notin \mathcal{S}$ always yields a bad partition, so we must have $\{7, 8, 9\} \in \mathcal{S}$, hence a set of three pairwise disjoint subsets in $\mathcal{S}$.

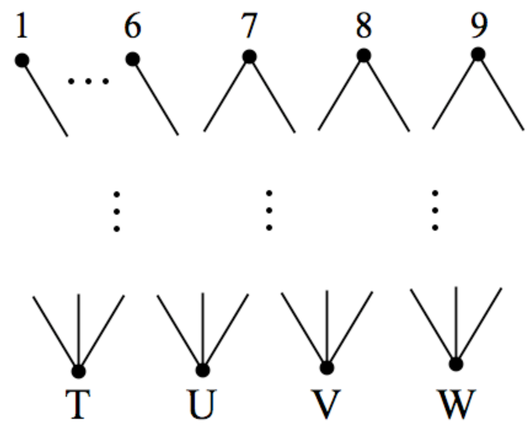

Figure 4.2: The incidence subgraph induced by the blocks $\{T, U, V, W\}$ in the proof of Proposition 4.6.2.

Now that we have established the claim, it is easy to produce the final contradiction. Let $\mathcal{T}$ be the set of the three pairwise disjoint subsets of $\mathcal{S}$, and let $\mathcal{T}' = \mathcal{S} \setminus \mathcal{T}$. Recall that we are also still assuming that $|\mathcal{S}[i]| = 2$ for every $i \in [9]$.

We create a bad partition of $\mathcal{S}$ as follows. If there exist $T \in \mathcal{T}$ and $T' \in \mathcal{T}'$ such that $|T \cap T'| = 2$ then pair up T and T'. After doing so, take the remaining subsets in $\mathcal{T}$ and $\mathcal{T}'$ and pair them up arbitrarily. Since the three sets in $\mathcal{T}$ partition [9] and $|\mathcal{S}[i]| = 2$ for every $i \in [9]$, this procedure is guaranteed to yield a bad partition. We can finally conclude that there cannot exist a dominating set $\mathcal{S}$ of size 6 in $K_{9:3}$. $\square$

4.6.2 $k = 4$

When $16 \leq n \leq 19$, Corollary 4.4.5 tells us that

$$\gamma(16:4) = 5 + \left\lceil \frac{20 - 16}{\lfloor 4/2 \rfloor} \right\rceil = 7 \tag{4.6.2}$$

and similarly, that $\gamma(17:4) = 7$ and $\gamma(18:4) = \gamma(19:4) = 6$.

When $n = 9$ the graph in question cannot have a perfect 1-code (we will see why in Chapter 5); thus

$$\gamma(9:4) > \frac{\binom{9}{4}}{\binom{5}{4}+1} = 21.$$

An upper bound of 27 on $\gamma(9:4)$ follows from a simple construction. Define $U_1 = \{1,2,3\}, U_2 = \{4,5,6\}, U_3 = \{7,8,9\}$. Now let $\mathcal{S}$ be the set containing every 4-subset S such that $|S \cap U_i| = |S \cap U_j| = 2$ for some $i \neq j$. It follows that $|\mathcal{S}| = \binom{3}{2}^3 = 27$. Now, if R is a 4-subset of [9] not in $\mathcal{S}$ then either R intersects all three U_i or R contains some U_i. In either case it is easy to see that R is disjoint from a member of $\mathcal{S}$.

Recall that $\gamma(n:k) \leq \gamma_t(n:k) = C(n, n-k, k)$, so the number of blocks in an $(n, n-k, k)$ covering design provides an upper bound on $\gamma(n:k)$. We can thus deduce upper bounds for $10 \leq n \leq 15$ by consulting the list of known covering designs at the La Jolla Covering Repository [23].

All of these covering designs are in fact optimal, so that $\gamma(n:4)$ will be determined for some n in this range if we can show that

$$\gamma(n:4) = \gamma_t(n:4).$$

When $n = 15$ this follows from Theorem 4.5.1, so that $\gamma(15:4) = 8$. Monotonicity then implies that

$$8 \leq \gamma(14:4) \leq \gamma_t(14:4) = 9.$$

If we assume that $\gamma(14:4) = 8$ then we can again apply Theorem 4.5.1 and conclude that $\gamma_t(14:4) = 8$, a contradiction. Thus it is also implied by this theorem that $\gamma(14:4) = 9$. Monotoncity now implies that $\gamma(12:4) \geq \gamma(13:4) \geq 9$. Recall from our discussion in Section 4.5.2 that this lower bound on $\gamma(12:4)$ can be deduced using Lemma 4.5.3.

Let us now consider $\gamma(13:4)$. Monotonicity implies that $\gamma(13:4) \geq 9$, and the La Jolla Covering Repository, which lists an optimal $(13, 9, 4)$ covering design with 10 blocks, tells us that $\gamma(13:4) \leq 10$. To show that $\gamma(13:4) = 10$ it suffices to prove, in the style of Theorem 4.5.1, that if there is a dominating set of size 9 in $K_{13:4}$ then it must be a total dominating set. We cannot apply Theorem 4.5.1 directly since $9 > 2k$ when $k = 4$, but we can deduce enough information from its proof to establish the claim.

4.6.3 Lemma. *Let $\mathcal{S}$ be a dominating set of size $2k+1$ in $K_{n:k}$, $k \geq 4$. If $3k+1 \leq n < k^2 - 1$ and S is a transversal of $\mathcal{S}$ then*

(i) for every $i \in S$ there are two blocks whose intersection with S is $\{i\}$;

(ii) every block that is not equal to S must intersect S at precisely one element;

(iii) if S_i and T_i are the two blocks that intersect S at i for some fixed i then $S_i \cap T_i = \{i\}$.

Proof. We proceed as we did in the proof of Theorem 4.5.1. Observe that once again $\mathcal{S}$ cannot have a transversal of size $k-1$ because $n \geq k + |\mathcal{S}|$. As before this implies that for every $i \in S$ there exists at least one block that intersects S precisely at i.

If we assume that there exists $i \in S$ such that precisely one block intersects S at i, then it is possible to reach a contradiction by making some simple modifications to the proof of Theorem 4.5.1. We omit the details, mentioning only that because the assumption $|\mathcal{S}| \leq 2k$ has been relaxed to $|\mathcal{S}| = 2k+1$, the set $\mathcal{T}'$ can have k elements and not, as before, at most $k-1$. However, the additional hypothesis $n < k^2 - 1$ guarantees that $\mathcal{T}'$ still has a transversal of size at most $k-1$, which is all that is required for the proof.

It follows that for every $i \in S$ there are at least two blocks in $\mathcal{S}$ whose intersection with S is $\{i\}$. But $|\mathcal{S} \setminus \{S\}| = 2k$, so for every $i \in S$ there must be *exactly* two blocks in $\mathcal{S}$ with this property. This proves the first two claims.

It remains to prove the last claim. Now, if for some $i \in S$ there is an element in $S_i \cap T_i$ not equal to i, say j, then replacing i with j in S yields a k-subset that is a transversal of $\mathcal{S}$ hence is a block, and that intersects S at $k-1$ elements, which is impossible by the second claim. Thus $S_i \setminus \{i\}$ and $T_i \setminus \{i\}$ are disjoint for every $i \in S$. □

Let us apply this result in the case $k = 4$, $n = 13$. We want to show that a dominating set of size 9 in $K_{13:4}$ is total, so let $\mathcal{S}$ be such a dominating set and assume for the sake of contradiction that it has a transversal S.

Given $i \in S$, let S_i and T_i be the two blocks of $\mathcal{S}$ that intersect S at i. The lemma tells us that $(S_i \cap T_i) \setminus \{i\}$ is a subset of $[13] \setminus S$ of size 6 for every i. Since $|[13] \setminus S| = 9$, it is easy to show that for any fixed $i \in S$ there must exist $j \in S$ such that $S_i \setminus \{i\}$ intersects one of $S_j \setminus \{j\}$ or $T_j \setminus \{j\}$, say at a, and $T_i \setminus \{i\}$ intersects the other, say at b (the details are straightforward and tedious). This means that replacing i and j in S with a and b yields a transversal of $\mathcal{S}$, which must be a block, that intersects S at exactly two elements; this is impossible by the lemma.

Thus the assumption that $\mathcal{S}$ has a transversal of size k produces a contradiction, meaning that $\mathcal{S}$ is a total dominating set of size 9. But $\gamma_t(13 : 4) = 10$, so this cannot be, meaning that we cannot have a dominating set of size 9 in $K_{13:4}$ to begin with, i.e. $\gamma(13 : 4) = 10$.

We note that monotonicity now implies that $\gamma(12 : 4) \geq 10$; when $n = 11$ this already follows from the sphere-covering bound.

4.6.3 $k = 5$

We can say very little about $k = 5$. The most important observation is that $K_{11:5}$ has a perfect 1-code, which is not known to happen often, so $\gamma(11 : 5)$

meets the sphere-covering bound of 66. This fact is implied by the existence of a (unique) $(11, 5, 4)$-Steiner system, originally discovered by Witt (see Cameron and van Lint [6]). There is in fact a general equivalence between perfect 1-codes in Kneser graphs of the form $K_{2k+1:k}$ and certain Steiner systems; we have already seen an example with $K_{7:3}$. We defer an in-depth discussion of this interesting topic to the next chapter.

All of the upper bounds for $n \geq 12$ are again those implied by the existence of certain covering designs, as listed in the La Jolla Covering Repository [23].

As for lower bounds, Corollary 4.4.5 only applies for $n \geq 24$, so we cannot apply it directly. However, we can use it once again in conjunction with Lemma 4.5.3. To wit, the corollary tells us that $\gamma(25 : 5) = 8$, and two applications of the lemma tell us that $\gamma(20 : 5) \geq 10$ and $\gamma(15 : 5) \geq 14$. Monotonicity then implies that $\gamma(n : 5) \geq 10$ for $n \leq 19$.

We observe that since $\gamma_t(n : 5) \geq 12$ for the values of n in the table, Theorem 4.5.1 can be used to increase a lower bound of 10 to 11 for $n \geq 15$. If $n \leq 15$, the sphere-covering bound is already greater than 11.

4.7 Open problems

The data in Table 4.1 suggest that even though $\gamma(n : k) = \gamma_t(n : k)$ when n is large, as formalized by Hartman and West, the two quantities gradually diverge until $\gamma(n : k)$ is much smaller than $\gamma_t(n : k)$ when n is near $2k$. One way to formalize this phenomenon would be to prove the following conjecture.

4.7.1 Conjecture. $\gamma_t(n + 1 : k) - \gamma(n + 1 : k) \leq \gamma_t(n : k) - \gamma(n : k)$.

Observe that this conjecture implies that if $\gamma_t(n : k) - \gamma(n : k) = 0$ for some n then $\gamma_t(n' : k) - \gamma(n' : k) = 0$ for all $n' > n$. Perhaps it is possible to estimate the value of n at which the difference $\gamma_t(n : k) - \gamma(n : k)$ first becomes nonzero, at least asymptotically; that seems like a neat though very difficult problem.

Another interesting (and difficult) open problem is the classification of minimum size dominating sets in $K_{n:k}$ when $2k + 1 \leq n < \frac{3}{4}k^2 + k$. This might be possible when n is quadratic in k, as suggested by the work of Hartman and West, but is likely to be very difficult when n is linear in k. There are, however, less ambitious but interesting questions. We could ask, for instance, whether $\gamma(n+1 : k) = \gamma(n : k)$ implies that the minimum size dominating sets in $K_{n+1:k}$ are the minimum size dominating sets in $K_{n:k}$.

Chapter 5

Perfect 1-codes in odd graphs

We have mentioned before that the *odd graphs* are the Kneser graphs of the form $K_{2k+1:k}$. Let us fix the notation $O(k)$ for the odd graph $K_{2k+1:k}$. Since odd graphs are Kneser graphs, it follows that $O(k)$ is a regular graph on $\binom{2k+1}{k}$ vertices of degree $k+1$.

In this chapter we consider the existence of perfect 1-codes in odd graphs. We prove that if a perfect 1-code exists in $O(k)$ then $k+2$ is prime, a corollary of the well-known result that perfect 1-codes in $O(k)$ are equivalent to certain Steiner systems. We end with a new result showing that an odd graph with a perfect 1-code admits a nontrivial equitable partition. This result is interesting but is unlikely to be useful.

5.1 Introduction

When does $O(k)$ have a perfect 1-code? This question is motivated by the fact that perfect 1-codes in $O(k)$ tend to have a very special structure. We will formalize this fact in the next section; for now, let us whet our palate by recalling some examples.

We saw, for instance, in Chapter 4 that the projective plane of order 2 is a perfect 1-code in $O(3)$. We saw in the same chapter that there exists a (unique) Steiner system with parameters $(11, 5, 4)$ that is a dominating set in $O(5)$. But a Steiner system with these parameters must contain 66 blocks, and 66 is precisely the sphere-covering bound for $O(5)$, meaning this Steiner system must be a perfect 1-code in $O(5)$.

There are no known perfect 1-codes in $O(k)$ for any other values of k, and it is believed that no more exist.

In certain cases we know that $O(k)$ cannot have a perfect 1-code. Clearly this is the case if the sphere-covering bound on $\gamma(2k+1 : k)$ is not an integer.

Recall that this lower bound is

$$\frac{1}{k+2}\binom{2k+1}{k}$$

and it is straightforward to show that this quantity is not an integer if and only if $k+2$ is a power of 2.

More generally, we can show that $O(k)$ cannot have a perfect 1-code for any even k. This is because, as we saw in Chapter 1, a graph with a perfect 1-code must have -1 as an eigenvalue. Now, the eigenvalues of $O(k)$ are

$$(-1)^i(k+1-i) \tag{5.1.1}$$

for $i = 0, \ldots, k$ (see Theorem 9.4.3 of Godsil and Royle [22]), so -1 is an eigenvalue if and only if k is odd.

We can strengthen this condition even further once we arm ourselves with the promised result on Steiner systems.

5.2 Odd graphs and Steiner systems

The following theorem is due to Hammond and Smith (see [25]), and a proof also appears in Godsil [20] (Lemma 11.8.3).

5.2.1 Theorem. *A set of k-subsets of $[2k+1]$ is a perfect 1-code in $O(k)$ if and only if it is a $(2k+1, k, k-1)$ Steiner system.* □

We omit the proof, which is straightforward but involved, and instead focus on the following corollary.

5.2.2 Corollary. *If a perfect 1-code exists in $O(k)$ then $k+2$ is prime.*

Proof. Our proof follows Godsil [20]. It is easy to verify the claim for $k < 3$ so assume $k \geq 3$. Let $\mathcal{S}$ be a perfect 1-code in $O(k)$. By Theorem 5.2.1 $\mathcal{S}$ is a $(2k+1, k, k-1)$ Steiner system, so that by Lemma 3.1.1 we know that the numbers

$$\lambda_s = \frac{\binom{2k+1-s}{k-1-s}}{\binom{k-s}{k-1-s}} = \frac{1}{k+2}\binom{2k+1-s}{k+1}$$

must be integers for $0 \leq s \leq k-1$.

Let p be a prime less than k. Then

$$\lambda_{k-p} = \frac{1}{k+2}\binom{k+1+p}{p} = \frac{(k+p+1)(k+p)\ldots(k+3)}{p!}$$

so that there exists some j, $3 \leq j \leq p+1$, such that p divides $k+j$. It follows that p cannot divide $k+2$, so that no prime less than k divides $k+2$. Since $k \geq 3$ neither k nor $k-1$ can divide $k+2$, so $k+2$ must be prime. □

It follows that $O(7)$ cannot have a perfect 1-code since 9 is not a prime. What about $O(9)$? Since it is conjectured that perfect 1-codes do not exist in $O(k)$ for $k > 5$ we would not expect $O(9)$ to have one, but we can no longer use Corollary 5.2.2 to prove it. This claim is true, however, because of the following.

It was shown by Mendelsohn in [40] that any $(2k+1, k, k-1)$ Steiner system can be uniquely extended to a $(2k+2, k+1, k)$ Steiner system. Thus the existence of a perfect 1-code in $O(9)$ would imply the existence of a $(19, 9, 8)$ and hence a $(20, 10, 9)$ Steiner system. The latter, however, cannot exist, as was shown by Mendelsohn and Rung in [41] using extensive computer calculations.

5.3 Partitioning odd graphs

Let G be a graph with vertex set V. A *partition* of V is, as usual, a collection of nonempty pairwise disjoint subsets of V whose union equals V. A partition $\pi = \{C_1, \ldots, C_t\}$ is *equitable* if there exist nonnegative integers b_{ij}, $1 \leq i, j \leq t$, such that any vertex in C_i has precisely b_{ij} neighbors in C_j. In such a case we associate with π a $t \times t$ matrix $B(\pi)$ whose i, j entry is b_{ij}.

We make note of the following lemma (for a proof see Lemma 5.2.2 in Godsil [20] or Theorem 9.3.3 in Godsil and Royle [22]).

5.3.1 Lemma. *If π is an equitable partition of a graph G with adjacency matrix A then the characteristic polynomial of $B(\pi)$ divides the characteristic polynomial of A.* □

If a d-regular graph G has a perfect 1-code S then $\pi = \{S, V \setminus S\}$ constitutes a trivial equitable partition: every vertex in S has d neighbors in $V \setminus S$ and every vertex in $V \setminus S$ has one neighbor in S. It follows that $B(\pi)$ is

$$B(\pi) = \begin{pmatrix} 0 & d \\ 1 & d-1 \end{pmatrix}.$$

The characteristic polynomial of $B(\pi)$ is $(x - d)(x + 1)$, so Lemma 5.3.1 now provides an alternate proof that -1 is an eigenvalue of G when G has a perfect 1-code.

If G is one of the odd graphs then the existence of a perfect 1-code implies the existence of a more interesting equitable partition.

5.3.2 Theorem. *If $k \geq 2$ and the odd graph $O(k)$ contains a perfect 1-code then there exists an equitable partition π of $O(k)$ with*

$$B(\pi) = \begin{pmatrix} 0 & 0 & k & 1 & 0 \\ 0 & 0 & 0 & 0 & k+1 \\ 1 & 0 & 0 & 1 & k-1 \\ 1 & 0 & k & 0 & 0 \\ 0 & 1 & k-1 & 0 & 1 \end{pmatrix}.$$

Proof. Recall that $O(k)$ is a regular graph of degree $k+1$. Let $\mathcal{T}$ be the set of vertices of $O(k)$ that contain a fixed element of $[2k+1]$, say 1, and define $\mathcal{R} = V \setminus \mathcal{T}$, the rest of the vertices. It follows that the vertices of $\mathcal{R}$ induce a matching, since every $S \in \mathcal{R}$ has precisely one neighbor not containing 1.

Let $\mathcal{S}$ be a perfect 1-code in $O(k)$ and consider $\mathcal{S} \cap \mathcal{T}$. Because $\mathcal{S}$ is a Steiner system by Theorem 5.2.1, it follows that $\mathcal{S} \cap \mathcal{T}$ is neither empty nor equal to $\mathcal{T}$, so that neither is $\mathcal{S} \setminus \mathcal{T}$ empty.

Now, $\mathcal{S} \setminus \mathcal{T}$ is contained in $\mathcal{R}$, which induces a matching. Since $\mathcal{S}$ is an independent set, every pair of neighbors in $\mathcal{R}$ can contain at most one vertex in $\mathcal{S} \setminus \mathcal{T}$. Define $\mathcal{R}_1$ to be the set of vertices in $\mathcal{R}$ whose neighbor in $\mathcal{R}$ is in $\mathcal{S} \setminus \mathcal{T}$, and finally define

$$\mathcal{R}_2 = \mathcal{R} \setminus \left(\mathcal{R}_1 \bigcup (\mathcal{S} \setminus \mathcal{T}) \right).$$

Clearly,

$$\pi = \{\mathcal{S} \setminus \mathcal{T}, \mathcal{S} \cap \mathcal{T}, \mathcal{T} \setminus \mathcal{S}, \mathcal{R}_1, \mathcal{R}_2\}$$

is a partition of the vertex set V. Observe that $\mathcal{R}_2$ induces a matching while the other four cells are independent sets.

We claim that this partition is in fact equitable, and the number of neighbors that a vertex from a given cell has in some other cell is given by the appropriate entry in Table 5.1. We now prove this claim by considering each cell of the partition in turn.

	$\mathcal{S} \setminus \mathcal{T}$	$\mathcal{S} \cap \mathcal{T}$	$\mathcal{T} \setminus \mathcal{S}$	$\mathcal{R}_1$	$\mathcal{R}_2$
$\mathcal{S} \setminus \mathcal{T}$	0	0	k	1	0
$\mathcal{S} \cap \mathcal{T}$	0	0	0	0	$k+1$
$\mathcal{T} \setminus \mathcal{S}$	1	0	0	1	$k-1$
$\mathcal{R}_1$	1	0	k	0	0
$\mathcal{R}_2$	0	1	$k-1$	0	1

Table 5.1: The partition in Theorem 5.3.2.

We begin with $\mathcal{S} \setminus \mathcal{T}$. Every $S \in \mathcal{S} \setminus \mathcal{T}$ has one neighbor in $\mathcal{R}_1$ and none in $\mathcal{R}_2$, by definition of the latter two sets. Since $\mathcal{S}$ is an independent set, S has no neighbors in $\mathcal{S} \setminus \mathcal{T}$ or $\mathcal{S} \cap \mathcal{T}$, so its k remaining neighbors must be in $\mathcal{T} \setminus \mathcal{S}$.

Moving on to $\mathcal{S} \cap \mathcal{T}$, we observe that $S \in \mathcal{S} \cap \mathcal{T}$ has no neighbors in $\mathcal{S} \cap \mathcal{T}$ or $\mathcal{S} \setminus \mathcal{T}$ since $\mathcal{S}$ is independent, nor in $\mathcal{T} \setminus \mathcal{S}$ since $\mathcal{T}$ is also independent. If S had a neighbor $U \in \mathcal{R}_1$ it would share it with the neighbor of U in $\mathcal{R}$, which by definition is in $\mathcal{S} \setminus \mathcal{T}$; this cannot be since $\mathcal{S}$ is a perfect 1-code, so that every vertex in $V \setminus \mathcal{S}$ has a unique neighbor in $\mathcal{S}$. Thus $\Gamma(S) \subseteq \mathcal{R}_2$.

Now consider $S \in \mathcal{T} \setminus \mathcal{S}$. It has no neighbors in $\mathcal{T}$, but since $\mathcal{S}$ is a perfect 1-code it has a single neighbor in $\mathcal{S} \setminus \mathcal{T}$. Now, if $U \in \mathcal{R}_1$ and we let N_U denote the unique neighbor of U in $\mathcal{S} \setminus \mathcal{T}$, then $S \sim U$ if and only if $S \setminus \{1\} \subset N_U$. But $|S \setminus \{1\}| = k-1$ and $\mathcal{S}$ is a $(2k+1, k, k-1)$ Steiner system by Theorem

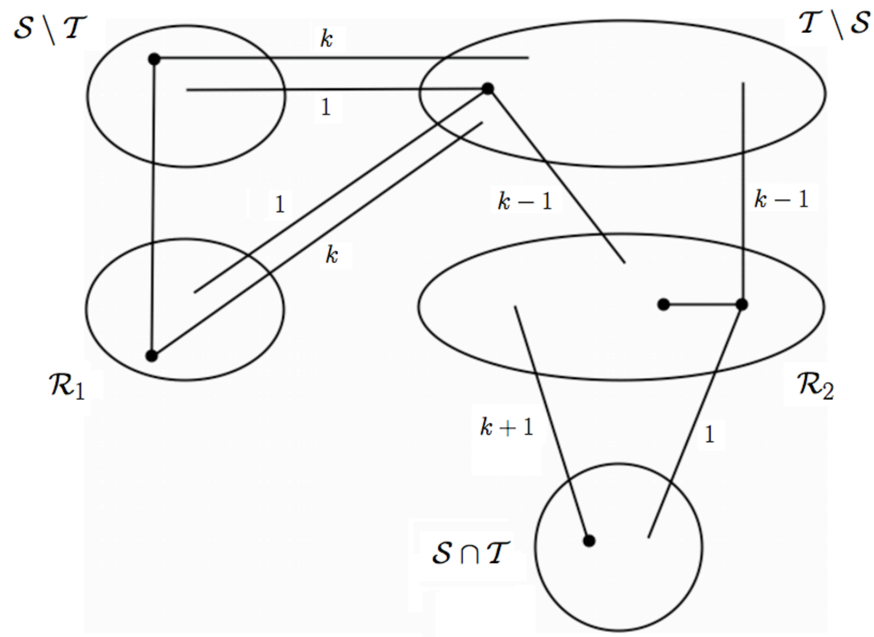

Figure 5.1: The partition of $O(k)$ in Theorem 5.3.2. The label on an edge between two cells is the number of neighbors that a vertex in one of the cells (the one that contains the solid circle attached to the line) has in the other.

5.2.1, so there is exactly one vertex in $\mathcal{S}$ that contains $S \setminus \{1\}$. This implies that S has exactly one neighbor in $\mathcal{R}_1$, and it follows that its remaining $k-1$ neighbors are in $\mathcal{R}_2$.

Next is $S \in \mathcal{R}_1$; it has a unique neighbor in $\mathcal{S} \setminus \mathcal{T}$ and so has none in $\mathcal{S} \cap \mathcal{T}$. By construction it has no neighbors in $\mathcal{R}_1$ or $\mathcal{R}_2$, so its remaining k neighbors are in $\mathcal{T} \setminus \mathcal{S}$.

Finally, if $S \in \mathcal{R}_2$ then, since $\mathcal{R}_2$ induces a submatching of the matching induced by $\mathcal{R}$, S has one neighbor in $\mathcal{R}_2$ and none in $\mathcal{R}_1 \cup (\mathcal{S} \setminus \mathcal{T})$. Also, S must have one neighbor in $\mathcal{S}$, so this neighbor must be in $\mathcal{S} \cap \mathcal{T}$, meaning the remaining $k-1$ neighbors of S are in $\mathcal{T} \setminus \mathcal{S}$.

This completes the proof of the claim and the theorem. □

The partition in Theorem 5.3.2 is illustrated in Figure 5.1. It follows from the theorem that $(\mathcal{S} \cap \mathcal{T}) \cup \mathcal{R}_1$ is also a perfect 1-code in $O(k)$.

One would hope that Theorem 5.3.2 could be used with Lemma 5.3.1 to provide additional information about k when $O(k)$ has a perfect 1-code. This, unfortunately, is not the case since the characteristic polynomial of the matrix

$B(\pi)$ in Theorem 5.3.2 is

$$(x-2)(x+1)^2(x+k)(x-k-1)$$

so every eigenvalue of $B(\pi)$ is already guaranteed to be an eigenvalue of A by the mere fact that k is odd (see equation 5.1.1).

Finally, for completeness we record the sizes of the cells in the partition of Theorem 5.3.2. Recall that $\mu(\mathcal{S})$ is the measure of a vertex subset $\mathcal{S}$. Since $\mathcal{T}$ is the set of k-subsets that contain 1,

$$\mu(\mathcal{T}) = \frac{\binom{2k}{k-1}}{\binom{2k+1}{k}} = \frac{k}{2k+1}$$

Substituting $s = 1$ in Lemma 3.1.1 tells us that

$$\mu(\mathcal{S} \cap \mathcal{T}) = \frac{k}{(2k+1)(k+2)}$$

which then implies

$$\mu(\mathcal{S} \setminus \mathcal{T}) = \frac{k+1}{(2k+1)(k+2)}$$
$$\mu(\mathcal{T} \setminus \mathcal{S}) = \frac{k(k+1)}{(2k+1)(k+2)}.$$

Finally, the data in Table 5.1 imply that that $|\mathcal{R}_1| = |\mathcal{S} \setminus \mathcal{T}|$ and $|\mathcal{R}_2| = |\mathcal{T} \setminus \mathcal{S}|$.

Chapter 6

Semi-covering designs

Motivated by the search for lower bounds on $\gamma(n:k)$ as well as the relationship between total dominating sets in Kneser graphs and covering designs, we make the following definition.

An (n,r,k) *semi-covering design* is a family of r-subsets of $[n]$, again called *blocks*, such that for every k-subset S there exists at least one block that either contains S or whose complement contains S.

The minimum number of blocks in an (n,r,k) semi-covering design will be written $SC(n,r,k)$. Clearly, $SC(n,r,k) = SC(n,n-r,k)$. We also have

$$SC(n,r,k) \leq \min\{C(n,r,k),\, C(n,n-r,k)\}$$

since every covering design is a semi-covering design.

Observe that an (n,k,k) semi-covering design $\mathcal{S}$ consists of a family of k-subsets of $[n]$ such that every k-subset is either a block of $\mathcal{S}$ or is in the complement of a block. In other words $\mathcal{S}$ is a dominating set in $K_{n:k}$, which implies that

$$\gamma(n:k) = SC(n,k,k) = SC(n,n-k,k).$$

Thus semi-covering designs can be regarded as combinatorial generalizations of dominating sets in Kneser graphs.

Our aim in this chapter is to prove a recursive lower bound on $SC(n,r,k)$ analogous to the Schönheim bound (Theorem 3.2.2) for covering designs. We begin with some basic results on semi-covering designs, including a proof that $SC(n,r,k) = k+1$ when n is large. Then we prove the promised lower bound, and finally close with some open questions.

In this chapter we will always assume that $k \leq \min\{r, n-r\}$, for otherwise an (n,r,k) semi-covering design is forced to be either an (n,r,k) or $(n,n-r,k)$ covering design. We will also assume $r \leq n$, an obvious necessity.

6.1 Some basic results

Let $\mathcal{S}$ be a collection of r-subsets of $[n]$ and let $\mathcal{S}^c$ denote the set of complements of members of $\mathcal{S}$. If $\mathcal{S}$ is an (n, r, k) semi-covering design then every k-subset that is not contained in a member of $\mathcal{S}$ is contained in a member of $\mathcal{S}^c$. Rephrasing slightly, $\mathcal{S}$ is an (n, r, k) semi-covering design if and only if every transversal of $\mathcal{S}$ of size k is contained in a member of $\mathcal{S}$ and every transversal of $\mathcal{S}^c$ of size k is contained in a member of $\mathcal{S}^c$. We will use this observation below.

We begin with a trivial lower bound. Every block in an (n, r, k) semi-covering design covers $\binom{r}{k}$ k-subsets and its complement covers another $\binom{n-r}{k}$, which implies a trivial lower bound analogous to inequality (3.2.1):

$$SC(n, r, k) \geq \frac{\binom{n}{k}}{\binom{r}{k} + \binom{n-r}{k}}. \tag{6.1.1}$$

Observe that when $r = k$ this lower bound reduces to

$$SC(n, k, k) \geq \frac{\binom{n}{k}}{\binom{n-k}{k} + 1},$$

the sphere-covering bound on $\gamma(n : k)$, no surprise since $SC(n, k, k) = \gamma(n : k)$. In the next section we will prove a less trivial lower bound.

Pursuing the analogy with covering designs, one might ask whether the lower bound (6.1.1) is asymptotically optimal, as was the case with covering designs. The answer, however, is no, as the next result shows.

6.1.1 Theorem. *If $n \geq r(k+1)$ then $SC(n, r, k) = k + 1$.*

Proof. We saw in Section 4.3 that $C(n, n-r, k) = k + 1$ when $n \geq r(k+1)$, which implies that

$$SC(n, r, k) = SC(n, n-r, k) \leq C(n, n-r, k) = k + 1$$

for such n. It remains to show that $SC(n, r, k) \geq k + 1$; our proof will be quite similar to the proof of Theorem 4.2.1, where we proved that $\gamma(n : k) = k + 1$ when $n \geq k(k+1)$, which is in fact the special case $r = k$ of the present claim.

Let $\mathcal{S}$ be a collection of k r-subsets of $[n]$; we will show that $\mathcal{S}$ cannot be a semi-covering design. If all r-sets in $\mathcal{S}$ are pairwise disjoint then we can create a transversal of $\mathcal{S}$ of size k by choosing an arbitrary element from each member of $\mathcal{S}$. This transversal cannot be contained in any member of $\mathcal{S}$ because they are pairwise disjoint, meaning $\mathcal{S}$ is not a semi-covering design.

If on the other hand some pair of members of $\mathcal{S}$ has nontrivial intersection then there exists a set U of size at most $k-1$ that intersects every member of $\mathcal{S}$. Since there are at least $n - r|\mathcal{S}| \geq r \geq k$ elements of $[n]$ that are not contained in any member of $\mathcal{S}$, we can use these elements to complete U to a k-subset that cannot be contained in any member of $\mathcal{S}$. But this set is a transversal of $\mathcal{S}$ so again it follows that $\mathcal{S}$ is not a dominating set. □

Theorem 6.1.1 reveals a fundamental difference between covering designs and semi-covering designs; the latter, unlike the former, become trivial once n is large enough with respect to r and k.

6.2 A nontrivial lower bound

In this section we prove a recursive lower bound on $SC(n, r, k)$ in the style of the Schönheim bound (Theorem 3.2.2). We begin with a definition: if $\mathcal{S}$ is an (n, r, k) semi-covering design and i, j are two distinct elements of $[n]$ then we define

$$\mathcal{S}[ij] = \mathcal{S}[i] \cap \mathcal{S}[j]$$

the set of blocks containing both i and j, and

$$\mathcal{S}\left[\overline{ij}\right] = \overline{\mathcal{S}[i]} \cap \overline{\mathcal{S}[j]}$$

the set of blocks that contains neither i nor j.

6.2.1 Lemma. *If $\mathcal{S}$ is an (n, r, k) semi-covering design and i, j are two distinct elements of $[n]$ then*

$$|\mathcal{S}[ij]| + \left|\mathcal{S}\left[\overline{ij}\right]\right| \geq SC(n-2, r-2, k-2).$$

Proof. It suffices to show that we can modify the blocks in $\mathcal{S}[ij] \cup \mathcal{S}\left[\overline{ij}\right]$ to yield an $(n-2, r-2, k-2)$ semi-covering design. The modification is as follows: if $S \in \mathcal{S}[ij]$ then remove i and j from S, while if $S \in \mathcal{S}\left[\overline{ij}\right]$ then remove from S two arbitrary elements. Let $\mathcal{T}$ be the set of these modified blocks; observe that

$$|\mathcal{T}| \leq |\mathcal{S}[ij]| + \left|\mathcal{S}\left[\overline{ij}\right]\right|.$$

Now, we claim that $\mathcal{T}$ is an $(n-2, r-2, k-2)$ semi-covering design, where we identify $[n-2]$ with $[n] \setminus \{i, j\}$. To prove this claim, let T be a $(k-2)$-subset of $[n] \setminus \{i, j\}$, and define $T^+ = T \cup \{i, j\}$. By assumption, there exists $S \in \mathcal{S}$ such that T^+ is either contained in S or in its complement. If the former is the case then $\{i, j\} \subset S$ so $S \in \mathcal{S}[ij]$, and it follows that $S \setminus \{i, j\}$ is in $\mathcal{T}$ and contains T. If the latter is true then S contains neither i nor j, so $S \in \mathcal{S}\left[\overline{ij}\right]$, and it follows that a $(k-2)$-subset of S is in $\mathcal{T}$ and its complement contains T.

We conclude that $\mathcal{T}$ is an $(n-2, r-2, k-2)$ semi-covering design, so that

$$|\mathcal{T}| \geq SC(n-2, r-2, k-2)$$

and the proof is complete. □

We can now prove a semi-covering analogue of Lemma 3.2.1.

6.2.2 Lemma. *If $n \geq 3$ then*

$$SC(n, r, k) \geq \left\lceil \left(1 - 2\frac{r(n-r)}{n(n-1)}\right)^{-1} SC(n-2, r-2, k-2) \right\rceil.$$

Proof. Let $\mathcal{S}$ be a minimum size (n, r, k) semi-covering design. We know from Lemma 6.2.1 that

$$\sum_{i<j} |\mathcal{S}[ij]| + \left|\mathcal{S}\left[\overline{ij}\right]\right| \geq \binom{n}{2} SC(n-2, r-2, k-2).$$

Now, each block of $\mathcal{S}$ contains $\binom{r}{2}$ pairs of elements and its complement contains another $\binom{n-r}{2}$ pairs, so that

$$\sum_{i<j} |\mathcal{S}[ij]| + \left|\mathcal{S}\left[\overline{ij}\right]\right| = \left(\binom{r}{2} + \binom{n-r}{2}\right) |\mathcal{S}| = \left(\binom{r}{2} + \binom{n-r}{2}\right) SC(n, r, k).$$

Combining this with the above inequality yields

$$\begin{aligned} SC(n, r, k) &\geq \frac{\binom{n}{2}}{\binom{r}{2} + \binom{n-r}{2}} SC(n-2, r-2, k-2) \\ &= \frac{n(n-1)}{r(r-1) + (n-r)(n-r-1)} SC(n-2, r-2, k-2) \\ &= \frac{n(n-1)}{n(n-1) - 2r(n-r)} SC(n-2, r-2, k-2) \\ &= \left(1 - 2\frac{r(n-r)}{n(n-1)}\right)^{-1} SC(n-2, r-2, k-2) \end{aligned}$$

as desired. $\square$

Lemma 6.2.2 is an improvement due to Jim Geelen of an earlier result of the author's.

We can repeatedly apply the inequality in Lemma 6.2.2 to derive a Schönheim-esque lower bound on $SC(n, r, k)$. The situation here is slightly more complicated than it was for covering designs, however, because every application of the recursive inequality in Lemma 6.2.2 reduces the value of k by 2, so that if k is odd we can apply the inequality until we have reduced k to 1, while if k is even we must stop when we have reached 2.

In the former case we then use the trivial observation $SC(n, r, 1) = 1$ for all n and r. In the latter case we take our cue from the trivial lower bound (6.1.1), which tells us that

$$SC(n, r, 2) \geq \frac{\binom{n}{2}}{\binom{r}{2} + \binom{n-r}{2}} = \left(1 - 2\frac{r(n-r)}{n(n-1)}\right)^{-1}.$$

We summarize this in a hideous theorem.

6.2.3 Theorem. *If k is odd, then*

$$SC(n,r,k) \geq$$
$$\left\lceil \left(1-2\frac{r(n-r)}{n(n-1)}\right)^{-1} \left\lceil \left(1-2\frac{(r-2)(n-r)}{(n-2)(n-3)}\right)^{-1} \left\lceil \left(1-2\frac{(r-4)(n-r)}{(n-4)(n-5)}\right)^{-1} \cdots \right.\right.\right.$$
$$\left.\left.\left. \left\lceil \left(1-2\frac{(r-k+3)(n-r)}{(n-k+3)(n-k+2)}\right)^{-1} \right\rceil \cdots \right\rceil \right\rceil \right\rceil.$$

If k is even, then

$$SC(n,r,k) \geq$$
$$\left\lceil \left(1-2\frac{r(n-r)}{n(n-1)}\right)^{-1} \left\lceil \left(1-2\frac{(r-2)(n-r)}{(n-2)(n-3)}\right)^{-1} \left\lceil \left(1-2\frac{(r-4)(n-r)}{(n-4)(n-5)}\right)^{-1} \cdots \right.\right.\right.$$
$$\left.\left.\left. \left\lceil \left(1-2\frac{(r-k+4)(n-r)}{(n-k+4)(n-k+3)}\right)^{-1} \left\lceil \left(1-2\frac{(r-k+2)(n-r)}{(n-k+2)(n-k+1)}\right)^{-1} \right\rceil \right\rceil \cdots \right\rceil \right\rceil \right\rceil.$$

6.3 Open questions

An obvious question presents itself: for what parameter sets does $SC(n,r,k)$ equal $C(n,r,k)$, or $C(n,n-r,k)$? As always we assume $k \leq \min\{r, n-r\}$, otherwise the question is vacuous. It follows from Theorem 6.1.1, and our discussion in Section 4.3, that

$$SC(n,r,k) = SC(n,n-r,k) = C(n,n-r,k) = k+1$$

when $n \geq r(k+1)$. Furthermore, since $SC(n,k,k) = \gamma(n:k)$, the work of Hartman and West implies that $SC(n,k,k) = C(n,n-k,k)$ when $n \geq \frac{3}{4}k^2+k$. One promising approach would be to generalize this statement by proving that $SC(n,r,k) = C(n,n-r,k)$ when $n \geq \frac{3}{4}rk+r$. Recall that Hartman and West determined $C(n,n-r,k)$ in this range (see equation (4.4.1)).

Parameter sets for which $SC(n,r,k) = C(n,r,k)$ appear harder to come by. Perhaps it is possible to prove a lower bound on $C(n,r,k) - SC(n,r,k)$ that is often positive.

Finally, since the definition of semi-covering designs was motivated by the dominating set problem in Kneser graphs, one obvious question is whether or not Theorem 6.2.3 actually implies good lower bounds on $\gamma(n:k)$.

Chapter 7

Dominating q-Kneser graphs

In this chapter we consider dominating sets in q-Kneser graphs, which are vector-space generalizations of Kneser graphs. In analogy to the Kneser graphs, the q-Kneser graphs are the setting for a number of extremal vector space problems. Moreover, studying q-Kneser graphs sometimes provides new insight into the Kneser graphs.

We begin by introducing q-Kneser graphs and proving q-analogues of some of the basic results on the domination number of Kneser graphs that we saw in Chapter 4. We then discuss previous work on the domination number of q-Kneser graphs by Clark and Shekhtman.

7.1 Introduction

Consider positive integers n, k, q such that q is a prime power and $n \geq 2k$, let $\mathbb{F}_q$ be the finite field of order q, and let $\mathbb{F}_q^n$ be the n-dimensional vector space over $\mathbb{F}_q$.

The *q-Kneser graph* $qK_{n:k}$ is the graph whose vertices are the k-dimensional subspaces of $\mathbb{F}_q^n$, where two vertices are adjacent if the intersection of the corresponding subspaces is trivial. For the sake of brevity we will refer to k-dimensional subspaces of $\mathbb{F}_q^n$ as k-subspaces, suppressing the field.

In order to compute the number of vertices in $qK_{n:k}$ as well as its degree (since it is vertex-transitive and hence regular) we must introduce the q-binomial coefficients, which are polynomial generalizations of binomial coefficients that are used to enumerate subspaces, just as binomial coefficients are used to enumerate subsets.

7.1.1 q-binomial coefficients

Assume for the moment that q is a variable rather than a fixed prime power. Given an integer $n \geq 0$, define $(q)_n$ to be the product

$$(q)_n = (q^n - 1)(q^{n-1} - 1) \cdots (q - 1)$$

where $(q)_0 = 1$.

Given $n \geq k \geq 0$, we define the *q-binomial coefficient* $\begin{bmatrix} n \\ k \end{bmatrix}$ to be

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_{n-k}(q)_k} = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.$$

The q-binomial coefficient is sometimes written $\begin{bmatrix} n \\ k \end{bmatrix}_q$ to emphasize the dependence on q, but we shall omit this subscript.

It can be shown that

$$\lim_{q \to 1^+} \begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k}$$

so that the q-binomial coefficients can be regarded as generalizations of binomial coefficients. In fact, it is possible to define q-binomial coefficients in a way that makes this relationship more explicit, but these types of details will not concern us.

There are q-analogues for many familiar binomial identities, including the identity $\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}$ and the Pascal identities:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

where $1 \leq k \leq n-1$. As expected, these all reduce to the familiar binomial identities in the limit $q \to 1^+$.

The q-Pascal identities and the fact that $\begin{bmatrix} n \\ 0 \end{bmatrix} = 1$ can be used to inductively prove that $\begin{bmatrix} n \\ k \end{bmatrix}$ is a polynomial in q with integer coefficients, which is not obvious from the definition. This justifies the fact that q-binomial coefficients, which were first studied by Gauss, are often called *Gaussian polynomials*. Moreover, the coefficients of $\begin{bmatrix} n \\ k \end{bmatrix}$ have a very special combinatorial interpretation; however, this phenomenon plays no role in our work so we shall say no more and direct the interested reader to van Lint and Wilson [47] for more.

7.1.2 The parameters of $qK_{n:k}$

Now assume that q is a prime power and let $\mathbb{F}_q$ be the field of order q. Let $v(n:k)$ be the number of vertices in $qK_{n:k}$ and let $d(n:k)$ be this graph's degree. We first compute $v(n:k)$.

There are $(q^n-1)(q^n-q) \cdots (q^n-q^{k-1})$ ways to choose a set of k independent nonzero vectors in $\mathbb{F}_q^n$. Not all of these, however, generate distinct k-subspaces. There are $(q^k-1)(q^k-q) \cdots (q^k-q^{k-1})$ ways to choose a set of k independent nonzero vectors in any given k-subspace, so the number of distinct k-subspaces of $\mathbb{F}_q^n$ is

$$v(n:k) = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}$$
$$= \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} = \begin{bmatrix} n \\ k \end{bmatrix}$$

and so $qK_{n:k}$ has $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ vertices.

It remains to determine $d(n:k)$. If $n < 2k$ then no two vertices of $qK_{n:k}$ are adjacent and $d(n:k) = 0$, so assume $n \geq 2k$, and let U be a k-subspace.

There are $(q^n - q^k)(q^n - q^{k+1}) \cdots (q^n - q^{2k-1})$ ways to choose a set of k independent nonzero vectors in $\mathbb{F}_q^n \setminus U$, each of which yields a k-subspace that intersects U trivially. Again, however, not all of these vector sets generate distinct k-subspaces, so to find the number of distinct k-subspaces that have trivial intersection with U we must again divide by the number of ways of choosing a set of k independent nonzero vectors in a given k-subspace:

$$d(n:k) = \frac{(q^n-q^k)(q^n-q^{k+1})\cdots(q^n-q^{2k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}$$
$$= \frac{(q^n-q^k)(q^{n-1}-q^k)\cdots(q^{n-k+1}-q^k)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} = q^{k^2}\begin{bmatrix} n-k \\ k \end{bmatrix}.$$

Thus the degree of U, hence of the graph $qK_{n:k}$, is $q^{k^2}\left[\begin{smallmatrix} n-k \\ k \end{smallmatrix}\right]$. We summarize the above in a proposition.

7.1.1 Proposition. *If $n \geq 2k$ then $qK_{n:k}$ has $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ vertices and is regular of degree $q^{k^2}\left[\begin{smallmatrix} n-k \\ k \end{smallmatrix}\right]$.*

Observe that in the limit $q \to 1^+$ the parameters of $qK_{n:k}$ reduce to the parameters of $K_{n:k}$.

In light of the analogous role played by the binomial and q-binomial coefficients, one might have been tempted to guess that the degree of $qK_{n:k}$ is $\left[\begin{smallmatrix} n-k \\ k \end{smallmatrix}\right]$. This guess underestimates the degree by the necessary factor of q^{k^2} because it relies on the false assumption that any k-subspace that trivially intersects a k-subspace S must be in $S^\perp$. On the contrary, if $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a basis for a k-subspace of $S^\perp$ then adding vectors in S to the $\mathbf{x}_i$ yields a basis for another, distinct k-subspace that intersects S trivially but is not in $S^\perp$.

We are now ready to engage this chapter's main topic, the domination number of q-Kneser graphs. We begin with a few q-analogues of basic results from Chapter 4.

7.2 Basic results

We will write $\gamma(qK_{n:k})$ as $q\gamma(n:k)$ and $\gamma_t(qK_{n:k})$ as $q\gamma_t(n:k)$. In analogy to set systems, a *transversal* of a set $\mathcal{S}$ of subspaces is a subspace that intersects each subspace in $\mathcal{S}$ nontrivially.

We begin by showing that $q\gamma(n:k)$ is monotone in n. As before, define the *upward shadow* of a $(k-1)$-subspace to be the set of k-subspaces that contain it as a subspace.

7.2.1 Lemma. *There exists a minimum size dominating set in $qK_{n:k}$ that does not contain the upward shadow of any $(k-1)$-subspace.*

Proof. Let $\mathcal{S}$ be a minimum size dominating set in $qK_{n:k}$ in which every vertex has at least one private neighbor in $V(qK_{n:k}) \setminus \mathcal{S}$ (such dominating sets exist by Lemma 2.3.2), and let U be an arbitrary $(k-1)$-subspace.

If $\mathcal{S}$ contains no k-subspaces in the upward shadow of U we are done. Otherwise, let S be such a k-subspace in $\mathcal{S}$, so that $S = \langle U, \mathbf{x}\rangle$ for some $\mathbf{x} \notin U$. Let N be the private neighbor of S in $V(qK_{n:k}) \setminus \mathcal{S}$.

Now, $S \cap N = \mathbf{0}$ so if $\mathbf{y}$ is a nonzero vector in N then $S' = \langle U, \mathbf{x}+\mathbf{y}\rangle$ is in the upward shadow of U and $S \neq S'$. But $\langle S', N\rangle = \langle S, N\rangle$ so that $\dim\langle S', N\rangle = 2k$, hence $S' \cap N = \mathbf{0}$. It follows that $S' \notin \mathcal{S}$ for otherwise N would not be a private neighbor of S. Thus $\mathcal{S}$ does not contain the upward shadow of U or any other $(k-1)$-subspace. □

7.2.2 Proposition. *If $n \geq 2k$ then $q\gamma(n:k) \geq q\gamma(n+1:k)$.*

Proof. We omit the proof, which is nearly identical to the proof of Proposition 4.2.4 and consists of taking a minimum size dominating set in $qK_{n:k}$ that does not contain the upward shadow of any $(k-1)$-subspace, as in Lemma 7.2.1, and showing that it dominates $qK_{n+1:k}$. □

Finally, recall that $\gamma(n:k) = \gamma_t(n:k) = k+1$ when $n \geq k(k+1)$; this result has a q-analogue, proved by Clark and Shekhtman in [9].

7.2.3 Theorem. *If $n \geq k(k+1)$ then $q\gamma(n:k) = q\gamma_t(n:k) = k+1$.* □

The proof is a direct linear-algebraic analogue of the corresponding result for Kneser graphs; in this case, apply the argument not to $[n]$ and its k-subsets, but rather to the set of labels of the standard basis for $\mathbb{F}_q^n$ and the k-subspaces that correspond to k-subsets of basis vectors.

7.3 Previous work

7.3.1 Clark and Shekhtman

In this section we present previous work on the domination number of $qK_{n:k}$. All of the results in this section are taken from [9] and [8] by Clark and Shekhtman, which are, to the author's knowledge, the only papers in the literature on this topic.

As mentioned above, Clark and Shekhtman proved in [9] that if $n \geq k(k+1)$ then
$$q\gamma(n:k) = q\gamma_t(n:k) = k+1.$$
They extended the argument to prove the following.

7.3.1 Proposition. *If $k \geq 2$ and $n = k(k+1)-1$ then*
$$q\gamma(n:k) = q\gamma_t(n:k) = k+2. \qquad \square$$

Clark and Shekhtman proved, also in [9], the following upper bound.

7.3.2 Theorem. *If $k \geq 2, n \geq 2k$, and q is sufficiently large with respect to n and k then*
$$q\gamma_t(n:k) \leq k+1+\left\lfloor \frac{k(k-1)}{n-2k+1} \right\rfloor. \qquad \square$$

The proof of this upper bound is not linear-algebraic, but rather relies on the special combinatorial interpretation of the q-binomial coefficients that we mentioned before.

We should note that this upper bound does not follow from a construction of a dominating set, but rather from a general upper bound on the domination number of an arbitrary regular graph. One would not expect such a general approach to yield a very useful bound on $q\gamma(n:k)$, and yet, as Clark and Shekhtman show, it does when q is large.

7.3.3 Corollary. *If $k \geq 2$ and $\frac{1}{2}k^2 + \frac{3}{2}k \leq n \leq k(k+1)-1$, and q is sufficiently large with respect to n and k then*
$$q\gamma(n:k) = q\gamma_t(n:k) = k+2.$$

Proof. The lower bound follows from Proposition 7.3.1 and monotonicity (Proposition 7.2.2), while the upper bound follows from Theorem 7.3.2 and the fact that $\frac{k(k-1)}{n-2k+1} < 2$ if and only if $n \geq \frac{1}{2}k^2 + \frac{3}{2}k$. $\square$

We will elaborate on the significance of Corollary 7.3.3 after first revisiting the results of Hartman and West on dominating Kneser graphs to see how well they extend to q-Kneser graphs.

7.3.2 Hartman and West

Recall that the major result in Hartman and West [27] is Corollary 4.4.5: If $k(k+1) \geq n \geq k\left(\frac{3}{4}k+1\right)$ then
$$\gamma(n:k) = \gamma_t(n:k) = k+1+\left\lceil \frac{k(k+1)-n}{\lfloor k/2 \rfloor} \right\rceil.$$

To this end, Hartman and West first prove an upper bound using an easy construction, and then prove a matching lower bound. The upper bound easily generalizes to q-Kneser graphs, but the lower bound does not. This is because the lower bound follows from the technical Lemma 4.4.3, which has no obvious q-analogue.

In fact, it follows from the results in the previous section that Hartman and West's lower bound cannot be extended to q-Kneser graphs. In other words, the q-analogue of Hartman and West's upper bound is not tight (at least not for all q). Before elaborating, let us first prove this q-analogue.

7.3.4 Theorem. *If $n \geq k(k+1) - \ell\lfloor k/2 \rfloor$ and $\ell \leq \lceil k/2 \rceil$ then*

$$q\gamma_t(n:k) \leq k+1+\ell.$$

Proof. By monotonicity we may assume that $n = k(k+1) - \ell\lfloor k/2 \rfloor$.

Recall the definition of a triangle configuration from the proof of Theorem 4.4.2. We define a *q-triangle configuration* to be the following set of three k-subspaces of a $\lceil 3k/2 \rceil$-subspace. Fix a basis of the $\lceil 3k/2 \rceil$-subspace, create a triangle configuration T from the set of basis vectors, and let the q-triangle configuration be the set of three k-subspaces generated by the three k-sets of vectors in T. It is easy to see that the smallest transversal of a q-triangle configuration has size 2.

Returning to $qK_{n:k}$, fix a basis of $\mathbb{F}_q^n$. Let $\mathcal{S}$ be the set of k-subspaces that consists of $k+1-2\ell$ disjoint k-subspaces created using the first $k+1-2\ell$ basis vectors, as well as ℓ q-triangle configurations formed from the remaining basis vectors. Note that this is possible because

$$k(k+1-2\ell) + \ell\left\lceil \frac{3k}{2} \right\rceil = k(k+1) - \ell\lfloor k/2 \rfloor = n.$$

Since each q-triangle configuration consists of three subspaces, we have

$$|\mathcal{S}| = (k+1-2\ell) + 3t = k+1+\ell.$$

Finally, it is easy to see that by construction, the smallest transversal of $\mathcal{S}$ has size $k+1$, so that $\mathcal{S}$ is a total dominating set. □

Theorem 7.3.4 implies that

$$q\gamma(n:k) \leq q\gamma_t(n:k) \leq k+1+\left\lceil \frac{k(k+1)-n}{\lfloor k/2 \rfloor} \right\rceil$$

when $n \geq \frac{3}{4}k^2 + k$, as was the case for $\gamma(n:k)$. Though this bound was tight for Kneser graphs, Corollary 7.3.3 implies that it is not at all tight for q-Kneser graphs when q is large. This seems to suggest a fundamental difference between the domination numbers of Kneser and q-Kneser graphs, as well as a dependence of the latter on the size of the underlying field.

We conclude that results on $\gamma(n:k)$ will not simply translate to results on $q\gamma(n:k)$ by applying the same proof to k-subsets of a set of basis vectors of $\mathbb{F}_q^n$ instead of k-subsets of $[n]$; rather, new linear-algebraic techniques will have to devised.

7.4 $q\gamma(2k:k)$

The structure of $qK_{2k:k}$ is of particular interest because while the degree of $K_{2k:k}$ is $\binom{k}{k} = 1$, making $K_{2k:k}$ a trivial graph, the degree of $qK_{2k:k}$ is q^{k^2} so that it is not trivial for any field.

In this section we prove that $q\gamma(4:2) = 4$, which was shown by Clark and Shekhtman in [8] and [9]. Our proof is shorter and differs from theirs in an important way, but we will present the proof first and elaborate on these issues afterwards. Recall that we denote the standard basis of $\mathbb{F}_q^n$ by $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

7.4.1 Proposition. $q\gamma(4:2) = 4$.

Proof. The lower bound follows from Proposition 7.3.1 and monotonicity (Proposition 7.2.2):

$$q\gamma(4:2) \geq q\gamma(5:2) = 4.$$

The upper bound follows from exhibiting a dominating set of size 4, which we now do. Let $\mathcal{S}^2$ be the set of the following four 2-subspaces:

$$U_1^2 = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle \tag{7.4.1}$$
$$U_2^2 = \langle \mathbf{e}_3, \mathbf{e}_4 \rangle \tag{7.4.2}$$
$$U_3^2 = \langle \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_4 \rangle \tag{7.4.3}$$
$$U_4^2 = \langle \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4 \rangle \tag{7.4.4}$$

To show that $\mathcal{S}^2$ dominates $qK_{4:2}$ it suffices to prove that the only transversal of $\mathcal{S}^3$ is U_4^2.

So let U be a transversal of $\mathcal{S}$. Since U has nontrivial intersection with U_1^2 and U_2^2, and these two 2-subspaces intersect trivially, we can assume that $U = \langle \mathbf{x}, \mathbf{y} \rangle$ where $\mathbf{x} \in U_1^2$, $\mathbf{y} \in U_2^2$. Now form a 2×4 matrix whose rows are $\mathbf{x}$ and $\mathbf{y}$ and put it in reduced-row echelon form. We will associate U with this matrix, whose row space equals U. By the above we know that

$$U = \begin{pmatrix} * & * & 0 & 0 \\ 0 & 0 & * & * \end{pmatrix}.$$

Now, since $U \cap U_4^2 \neq \mathbf{0}$ then in at least one of the rows both of the unknown entries are equal and nonzero, and so must equal 1 (since U is in reduced-row echelon form). We can assume without loss of generality that this is true for the second row, since otherwise we can apply the permutation (13)(24) to the labels of the basis vectors and attain this outcome (note that this does not change the members of $\mathcal{S}^2$). Thus

$$U = \begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

for some $a, b \in \mathbb{F}_q$. But the fact that $U \cap U_3^2 \neq \mathbf{0}$ implies that $a = b$, so that in particular $a = b = 1$ and $U = \langle \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4 \rangle = U_3^2$ as desired. □

Now a note on the proof. As mentioned above, Clark and Shekhtman proved Proposition 7.4.1 in [8] and [9]. Our proof differs from theirs in that while we prove the claim directly, Clark and Shekhtman first proved $q\gamma_t(4:2) = 4$ in [8] and then $q\gamma(4:2) \geq 4$ in [9].

The result from [8] is in fact even stronger; Clark and Shekhtman show that any total dominating set of size 4 must be isomorphic to a total dominating set $\mathcal{S}'$ consisting of the four 2-subspaces

$$\begin{aligned}
U_1' &= \langle \mathbf{e}_1, \mathbf{e}_2 \rangle \\
U_2' &= \langle \mathbf{e}_3, \mathbf{e}_4 \rangle \\
U_3' &= \langle \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_4 \rangle \\
U_4' &= \langle \mathbf{e}_1 + a\mathbf{e}_3 + b\mathbf{e}_4, \mathbf{e}_2 + \mathbf{e}_3 \rangle
\end{aligned}$$

where $a, b \in \mathbb{F}_q$ are such that the polynomial $x^2 + ax + b$ is irreducible over $\mathbb{F}_q$ (such a pair always exists for all prime powers q).

Though this is a strong result, it is of limited use in computing $q\gamma(2k : k)$ because it does not appear to generalize easily. Our proof uses a dominating set that, though not total, has a purely combinatorial definition, eliminating the need for introducing irreducible polynomials and allowing for the use of a symmetry argument. We believe that our proof, and our dominating set, will be much easier to generalize to larger values of k.

Finally, we note that our proof method differs slightly from the one used by Clark and Shekhtman in that the latter involves directly checking that every 2-subspace has trivial intersection with some element of $\mathcal{S}'$ by computing a large number of determinants, while in our proof we restrict our attention to transversals, thereby shortening the proof and eliminating the need for computing any determinants.

Part III

Related topics

Chapter 8

Maximal intersecting families

Recall that an independent dominating set is a dominating set in which no two vertices are adjacent, and the independent domination number of a graph is the size of its smallest independent dominating set. In this chapter we relate the independent domination number of Kneser graphs to a well-studied problem in extremal combinatorics. We will denote the independent domination number of $K_{n:k}$ by $\gamma_i(n:k)$.

8.1 Background

A *maximal intersecting family of k-sets* is a family $\mathcal{S}$ of k-sets with nonempty pairwise intersection such that every k-subset not in $\mathcal{S}$ is disjoint from at least one in $\mathcal{S}$. In other words, it is an independent dominating set in $K_{n:k}$ for some n. Define $m(k)$ to be the size of the smallest maximal intersecting family of k-sets; in the language of dominating sets,

$$m(k) = \min_n \gamma_i(n:k).$$

Many attempts to bound the quantity $m(k)$ were made by extremal combinatorists in the 1980s; see the papers by Füredi [18], Dow et al. [13], Blokhuis [4], and Boros et al. [5].

Recall that if $n \geq k(k+1)$, the values of $\gamma(n:k)$ and $\gamma_t(n:k)$ are easy to determine and the minimum size dominating and total dominating sets are easy to characterize. The minimum size independent dominating sets, however, present a much greater challenge; there is no clear relationship between them and minimum size dominating sets, and they have not been characterized for any range of n. Indeed, the lower bounds on $m(k)$ discussed in the next section imply that minimum size independent dominating sets in Kneser graphs tend to be much larger than minimum size dominating sets, even for very large n. This is why unlike $\gamma(n:k)$ and $\gamma_t(n:k)$, the behavior of $\gamma_i(n:k)$ (and by extension $m(k)$) is naturally an extremal question (and a much more difficult one).

Let us now, however, abandon this pessimism and review the bounds on $m(k)$ in the literature.

8.2 Lower bounds

A review of existing lower bounds on $m(k)$, of which there are very few, reveals that even though none of the papers that tackle this problem use the language of dominating sets in Kneser graphs, the notion of domination is often implicit in the proofs therein.

The very first lower bound on $m(k)$ is due to Erdős and Lovász, who showed that $m(k) \geq 8k/3-3$. Next came a lower bound proven by Füredi in [18], which states (in the language of independent domination) that if $m(k) = \gamma_i(n:k)$ with $n \leq \frac{k^2}{2\log k}$ then $m(k) > k^2$. This is the case $c = 2$ of Lemma 4.1.1 from Chapter 4, which is why we credited Füredi with that result. Indeed, Füredi's original proof is almost identical to the one we gave in Chapter 4 (due to Jukna); it begins with a derivation of the sphere-covering bound (though it is not identified as such) and uses estimates for binomial coefficients to establish the claim.

In [13], Dow et al. improved the Erdős and Lovász bound to $m(k) \geq 3k$. The proof is quite similar to our proof of Theorem 4.5.1, which in turn is based on the work of Hartman and West. This remains the best known bound, though it is very unlikely to be tight. Indeed, Boros et al. conjecture in [5] that $m(k)/k \to \infty$ as $k \to \infty$.

8.3 Upper bounds

Several authors have proven upper bounds for $m(k)$ by constructing small maximal intersecting families of k-sets. Nearly all of these constructions, however, are based on projective planes of order $k-1$ (or k), hence are only possible when $k-1$ (or k) is a prime power.

The most fundamental of these is Meyer's result that the projective plane of order $k-1$ is a maximal intersecting family; we saw this in Theorem 4.2.6. Füredi also showed in [18] that if a projective plane of order k exists then

$$m(2k) \leq 3k^2.$$

Drake and Sane proved in [14] that under the same hypothesis,

$$m(k^r + k^{r-1}) \leq k^{2r} + k^{2r-1} + k^{2r-2}$$

for $r \geq 2$, the $r = 2$ case having been previously proven by Füredi in [18].

Blokhuis proved in [4] that if there exists a projective plane of order $k-1$ and $k \geq 8$ then

$$m(k) \leq \frac{3}{4}k^2 + \frac{3}{2} - 1.$$

In that same paper, Blokhuis used a deep result on the difference between consecutive primes to prove that

$$m(k) < k^5 \tag{8.3.1}$$

for all k, the only construction that has achieved this generality. Most recently, Boros et al. proved in [5] that if there exists a projective plane of order $k-1$ then

$$m(k) \leq \frac{1}{2}k^2 + 5k + o(k).$$

8.4 Open questions

The most obvious open question asks for better bounds on $m(k)$, both above and below. In particular, it would be nice to prove an upper bound on $m(k)$ that is valid for all k (or at least all large enough k), and that improves on inequality (8.3.1), Blokhuis' bound. Ideally this bound would involve a more direct construction that does not depend on deep number-theoretic results.

Improved lower bounds are also desirable. It might be possible to prove a separation between $\gamma(n : k)$ and $\gamma_i(n : k)$ that, combined with the fact that $\gamma(n : k)$ is known for large n, would establish a lower bound on $m(k)$ that is superlinear in k.

Less ambitiously, there are many open questions about minimum size independent dominating sets in Kneser graphs. For instance, is $\gamma_i(n : k)$ monotone in n? The proof of Proposition 4.2.4, in which we proved that $\gamma(n : k)$ is monotone in n, can only be used to show that $\gamma_i(n : k)$ is monotone when it is known that there exists a minimum size independent dominating set that does not contain the upward shadow of any $(k-1)$-subset.

On a related note, it is seems likely that while minimum size dominating sets are total for $n \geq k(k+1)$, they become more independent (that is, decreasingly connected) as n decreases. It would be nice to prove an analogue of Theorem 4.5.1 stating that if n is sufficiently small then $\gamma(n : k) = \gamma_i(n : k)$. So far, however, the author has been unable to even prove the conjecture that minimum size dominating sets in the odd graphs $K_{2k+1:k}$ are independent.

Chapter 9

Graph domination and coding theory

Graph domination is closely related to some of the central questions of coding theory, which deals with efficient means of transmitting information over noisy channels. In a more concrete sense, coding theory studies sets of constant length binary strings that possess certain desirable properties.

In this chapter we briefly highlight some topics in coding theory that are immediately relevant to graph domination. We discuss well-known results on perfect 1-codes and totally perfect 1-codes in the hypercube, as well as a non-trivial lower bound on the domination number of the hypercube due to van Wee. We close by considering an interesting open problem on independent domination of the hypercube.

We make no attempt to do this field justice in such a brief treatment. For a real introduction to the subject the reader is referred to the books by MacWilliams and Sloane [39] and Cameron and van Lint [6]. In addition, the monograph by Cohen et al. [11] is very much in the spirit of this chapter, and contains copious introductory material.

9.1 Codes and the hypercube

Recall that $\mathbb{F}_2$ is the field of order 2. The setting for much of coding theory is the *n-dimensional hypercube* Q_n, the graph whose vertex set is $\mathbb{F}_2^n$ with two vertices adjacent if they differ in exactly one coordinate. It follows from the definitions that Q_n is an n-regular graph on 2^n vertices.

The definition of the hypercube (and indeed many other constructions in coding theory) can be extended to arbitrary finite fields. To simplify matters, however, this chapter will focus on the binary case.

A *code of length n* is simply a vertex subset of Q_n; elements of a code are called *codewords*. Given a code $\mathbf{C}$ its parameters of greatest interest are the minimum distance d between codewords and the smallest r, called the *covering radius* of the code, such that every vertex not in $\mathbf{C}$ is at distance at most r from a codeword.

The purpose of a code is to serve as an alphabet for transmissions over a noisy channel, where the presence of noise is formally interpreted as a probability that an arbitrary coordinate of a transmitted word will change its value in the course of transmission. If a code has large minimum distance and large covering radius, then even if a transmission results in the corruption of a message it will likely be possible to reconstruct the original message from the corrupted one.

Graph domination emerges naturally in coding theory as dominating sets in the hypercube are precisely codes of covering radius 1. Of particular interest are perfect 1-codes in the hypercube (and this is the promised explanation for the origin of this term), as these would be able to correct single-bit errors with no ambiguity since every non-codeword is at distance 1 from a unique codeword. Perfect 1-codes in the hypercube are well-understood; we consider them in the next section.

9.2 Perfect 1-codes in Q_n

In this section we determine the values of n for which Q_n has a perfect 1-code. Again, we will treat this topic very briefly; details and additional information can be found in the references given at the beginnning of this chapter.

Recall that a perfect 1-code in a graph is a dominating set that meets the sphere-covering bound. The sphere-covering bound for the hypercube is

$$\gamma(Q_n) \geq \frac{2^n}{n+1}. \tag{9.2.1}$$

A trivial necessary condition for the existence of a perfect 1-code is that the sphere-covering bound for Q_n take an integer value. It is clear that this can only happen when $n+1$ is a power of 2. It follows that the existence of a perfect 1-code in Q_n implies that $n = 2^t - 1$ for some t.

This trivial necessary condition is in fact sufficient. One of the earliest results of coding theory was the discovery of a family of perfect 1-codes in Q_n when $n = 2^t - 1$ for some $t \geq 2$. These are the *Hamming codes*, which we now define.

Given some $t \geq 2$ define the matrix H_t to be the $t \times (2^t - 1)$ matrix whose columns consist of all nonzero vectors in $\mathbb{F}_2^t$. Set $n = 2^t - 1$. The *Hamming code* $\mathbf{H}_t$ is defined to be the kernel of H_t:

$$\mathbf{H}_t = \{\mathbf{x} \in Q_n \,:\, H_t\,\mathbf{x} = \mathbf{0}\}.$$

Thus $\mathbf{H}_t$ is a subspace of $\mathbb{F}_2^n$. Naturally we ask, What is the dimension of $\mathbf{H}_t$? It is easy to see that the rows of H_t are linearly independent because the set of columns of H_t contains the standard basis for $\mathbb{F}_2^t$. Thus H_t has rank t and so

$\mathbf{H}_t$, its kernel, has dimension $n - t = 2^t - 1 - t$. The fact that $\mathbf{H}_t$ is a perfect 1-code follows almost immediately from this observation.

9.2.1 Lemma. *If $t \geq 2$ and $n = 2^t - 1$ then $\mathbf{H}_t$ is a perfect 1-code in Q_n.*

Proof. Observe first that the minimum distance between any two codewords in $\mathbf{H}_t$ is 3. This is because the minimum distance of $\mathbf{H}_t$ is equal to the smallest d such that there exists a set of d linearly dependent columns of H_t. Since no two distinct columns can be linearly dependent, $d \geq 3$ for $\mathbf{H}_t$ (it is in fact easy to show that $d = 3$, but this fact makes no difference in this proof).

It follows that the closed neighborhoods of the codewords in $\mathbf{H}_t$ are disjoint. But then these closed neighborhoods must partition the vertex set of Q_n, because there are 2^{n-t} codewords and each corresponds to a closed neighborhood of size $n + 1 = 2^t$, so that the union of these closed neighborhoods contains 2^n vertices and hence all of $V(Q_n)$. This means that $\mathbf{H}_t$ is a perfect 1-code in Q_n as desired. □

Lemma 9.2.1 is enough to imply the following.

9.2.2 Theorem. *Q_n has a perfect 1-code if and only if $n = 2^t - 1$, $t \geq 2$.* □

Not all perfect 1-codes in hypercubes are Hamming codes. There exist families of perfect 1-codes in Q_n, $n = 2^t - 1$, that are *nonlinear*, meaning that unlike the Hamming codes they do not form a subspace of $\mathbb{F}_2^n$. The classification of all perfect 1-codes in Q_n is still open.

Before moving on, we note that coding theory is also concerned with perfect e-codes in the hypercube, which are generalizations of perfect 1-codes in which every two codewords are at distance $2e+1$ and every non-codeword is at distance e from exactly one codeword. One of the remarkable results of coding theory, proven by Tietäväinen and van Lint, is that $n = 23$, $e = 3$ is the only pair of values n, e with $1 < e < n$ such that Q_n has a perfect e-code. There is in fact only one such code, known as the binary Golay code; see the references for more.

9.2.1 Totally perfect 1-codes in Q_n

We have seen the Hamming codes used to determine the values of n for which Q_n has a perfect 1-code. They can also be used to determine the n for which Q_n has a totally perfect 1-code.

Recall that for a d-regular graph G on v vertices, the trivial lower bound on $\gamma_t(G)$ is v/d, and a totally perfect 1-code is a total dominating set that meets this bound. When G is the hypercube, we have

$$\gamma_t(Q_n) \geq \frac{2^n}{n}.$$

As before, Q_n can only have a totally perfect 1-code when this bound is an integer, which only happens when n divides 2^n. Thus a necessary condition for the existence of a totally perfect 1-code in Q_n is $n = 2^t$ for some t. It is easy to see that this trivial condition is once again sufficient, for if $n = 2^t$ then the direct sum of $\{0, 1\}$ and the Hamming code $\mathbf{H}_t$ is a total dominating set in Q_n of size 2^{n-t} (in fact it induces a matching).

We summarize this in an analogy of Theorem 9.2.2.

9.2.3 Theorem. *Q_n has a totally perfect 1-code if and only if $n = 2^t$, $t \geq 2$.* □

The direct sum construction above was first used by van Wee in [48]. The next section is dedicated to the main result of that paper, which is an improvement of the sphere-covering bound on $\gamma(Q_n)$ when n is even.

9.3 Improving the sphere-covering bound

We saw in the previous section that Q_n cannot have a perfect 1-code when n is even. In this situation we can use a simple parity argument to slightly improve the sphere-covering bound (9.2.1). This result is due to van Wee [48], and our proof will follow Cohen et al. [11].

Recall that $\Gamma[\mathbf{x}]$ is the closed neighborhood of the vertex $\mathbf{x}$. Given a dominating set $\mathbf{C}$ of Q_n and a vertex $\mathbf{x}$, define the *excess of* $\mathbf{x}$ *with respect to* $\mathbf{C}$ to be $E_{\mathbf{C}}(\mathbf{x}) = |\Gamma[\mathbf{x}] \cap \mathbf{C}| - 1$. Given $i \geq 0$ define $\mathbf{C}_i \subseteq V(Q_n)$ to be the set of vertices with excess equal to i; clearly every vertex is in some $\mathbf{C}_i$. We note that $\mathbf{C}$ is a perfect 1-code if and only if $\mathbf{C} = \mathbf{C}_0$.

If $\mathbf{S}$ is a vertex subset of Q_n define

$$E_{\mathbf{C}}(\mathbf{S}) = \sum_{\mathbf{x} \in \mathbf{S}} E_{\mathbf{C}}(\mathbf{x}).$$

It follows from these definitions that

$$E_{\mathbf{C}}(V(Q_n)) = \sum_{i \geq 0} i|\mathbf{C}_i| = |\mathbf{C}|(n+1) - 2^n. \tag{9.3.1}$$

Now, consider some $\mathbf{x} \in V(Q_n) \setminus \mathbf{C}$ and some codeword $\mathbf{c} \in \mathbf{C}$. If the distance between $\mathbf{x}$ and $\mathbf{c}$ is greater than 2 then $\Gamma[\mathbf{x}] \cap \Gamma[\mathbf{c}] = \emptyset$. If, however, the distance between them is 1 or 2 then it follows from the definition of adjacency in the hypercube that $|\Gamma[\mathbf{x}] \cap \Gamma[\mathbf{c}]| = 2$. Therefore, every codeword $\mathbf{c}$ that covers any of the vertices in $\Gamma[\mathbf{x}]$ covers exactly two of them.

If n is even then $|\Gamma[\mathbf{x}]| = n + 1$ is odd, and since $\mathbf{C}$ is a dominating set at least one vertex in $\Gamma[\mathbf{x}]$ must be covered by more than one codeword. This implies that $E_{\mathbf{C}}(\Gamma[\mathbf{x}]) \geq 1$ and consequently

$$E_{\mathbf{C}}\big(V(Q_n) \setminus \mathbf{C}\big) \geq 2^n - |\mathbf{C}|. \tag{9.3.2}$$

Van Wee used these facts in [48] to prove the following so-called excess bound.

9.3.1 Theorem. *If n is even then $\gamma(Q_n) \geq 2^n/n$.*

Proof. Let $\mathbf{C}$ be a dominating set in Q_n and define $\overline{\mathbf{C}} = V(Q_n) \setminus \mathbf{C}$. It follows from equations (9.3.2) and (9.3.1) that

$$\begin{aligned} 2^n - |\mathbf{C}| &\leq E_{\mathbf{C}}\big(\overline{\mathbf{C}}\big) \\ &= \sum_{\mathbf{x}\in\overline{\mathbf{C}}} \sum_{i>0} i|\mathbf{C}_i \cap \Gamma[\mathbf{x}]| \\ &= \sum_{i>0} i \sum_{\mathbf{x}\in\overline{\mathbf{C}}} |\mathbf{C}_i \cap \Gamma[\mathbf{x}]| \\ &= \sum_{i>0} i \sum_{\mathbf{y}\in\mathbf{C}_i} |\overline{\mathbf{C}} \cap \Gamma[\mathbf{y}]| \\ &\leq (n-1) \sum_{i>0} i|\mathbf{C}_i| = (n-1)\big(|\mathbf{C}|(n+1) - 2^n\big) \end{aligned}$$

where the last inequality used the fact that if $\mathbf{x}$ is a vertex with excess greater than 0 then $|\overline{\mathbf{C}} \cap \Gamma[\mathbf{x}]| \leq n-1$. The desired inequality follows after some rearranging. □

Observe that the lower bound in Theorem 9.3.1 is precisely the trivial lower bound on $\gamma_t(Q_n)$ (see the previous section), so that we can interpret this theorem as saying that when n is even, the trivial lower bound on $\gamma_t(Q_n)$ is a lower bound on $\gamma(Q_n)$.

Moreover, it follows from Theorem 9.2.3 that when $n = 2^t$ there exists a total dominating set that meets this bound. We conclude that if $n = 2^t$ for some $t \geq 2$ then

$$\gamma(Q_n) = \gamma_t(Q_n) = \frac{2^n}{n} = 2^{n-t}.$$

The observation $\gamma(Q_n) = 2^n/n$ when $n = 2^t$ was made by van Wee in [48].

Before moving on we note that a general excess bound has been proved by Honkala (see Cohen et al. [11] for details).

9.3.2 Theorem. *If $s+1$ is an odd prime that divides $n+1$ then*

$$\gamma(Q_n) \geq \frac{(V(n,s)+s)2^n}{V(n,s)(n+1)}$$

where

$$V(n,s) = \sum_{i=0}^{s} \binom{n}{s},$$

the number of vertices at distance at most s from a given vertex in Q_n. □

9.4 Independent domination of the hypercube

We conclude this chapter with a discussion of an open problem on hypercube domination posed by Harary and Livingston in [26].

Recall that the independent domination number of a graph G, written $\gamma_i(G)$, is the minimum size of an independent dominating set in G. It follows from Theorem 9.2.2 that $\gamma_i(Q_n) = \gamma(Q_n)$ when $n = 2^t - 1$ for $t \geq 2$, because these two quantities are equal for any graph that has a perfect 1-code.

Harary and Livingston observed in [26] that $\gamma_i(Q_n) = \gamma(Q_n)$ also holds when $n \in \{1, 2, 3, 4, 6\}$, but for $n = 5$ we have $\gamma(Q_5) = 7$ and $\gamma_i(Q_5) = 8$. They ask whether $\gamma(Q_n) \neq \gamma_i(Q_n)$ for any $n \neq 5$. Oddly, in their discussion of this question they interpret the results of van Wee to imply that $\gamma(Q_n) = \gamma_i(Q_n)$ when $n = 2^t$, even though what van Wee's results imply, as we have already observed, is that $\gamma(Q_n) = \gamma_t(Q_n)$ for such n. In fact, van Wee's results tempted the author to conjecture that $\gamma(Q_n) \neq \gamma_i(Q_n)$ when $n = 2^t$ because it seemed unlikely that $\gamma_t(Q_n) = \gamma_i(Q_n)$.

And yet this is the case when $t = 2$, as noted by Harary and Livingston, because

$$\gamma(Q_4) = \gamma_i(Q_4) = \gamma_t(Q_4) = 4.$$

So even though the author believes that Harary and Livingston misunderstood van Wee's results, their interpretation is valid for $t = 2$. Is it really the case that

$$\gamma(Q_n) = \gamma_i(Q_n) = \gamma_t(Q_n)$$

when $n = 2^t$, as Harary and Livingston imply? This still appears unlikely to the author, who is willing to make the following conjecture.

9.4.1 Conjecture. *If $n = 2^t$ and $t \geq 3$ then every minimum size dominating set in Q_n is a totally perfect 1-code.*

If proven true, this conjecture would imply that $\gamma(Q_n) \neq \gamma_i(Q_n)$ when $n = 2^t$ and $t \geq 3$.

Chapter 10

Open problems

We collect some miscellaneous open problems on dominating sets, most of an algebraic nature.

10.1 Perfect 1-codes

The most frequently asked question about dominating sets in graphs is whether or not graphs in some particular family contain perfect 1-codes. We give some examples below. Recall that $\square$ is the Cartesian product.

Let $\mathbb{Z}^n$ be the infinite graph whose vertices are strings of length n with entries in $\mathbb{Z}$, where two vertices are adjacent if they differ in exactly one entry. In [12], Dorbec and Mollard consider perfect 1-codes in graphs of the form $\mathbf{Z}^n \,\square\, Q_k$, where Q_k is, as usual, the hypercube of order k. One could consider this Cartesian product as a graph whose vertices are strings of length $n + k$ with the first n entries in $\mathbb{Z}$ and the last k in $\{0, 1\}$, and where two vertices are adjacent if they differ in one entry.

Dorbec and Mollard construct a perfect 1-code in $\mathbf{Z}^n \,\square\, Q_k$ whenever there exist positive integers a, b such that $k = 2^a - 1$ and $n = b2^{a-1}$. They also prove that if $k \geq 2n$ then a perfect 1-code exists if and only if there exists an integer c such that $2n + k = 2^c - 1$. There are many cases in which the existence of a perfect 1-code in $\mathbf{Z}^n \,\square\, Q_k$ is open.

Let S_n be the symmetric group on n elements. Edelman and White define in [15] the Cayley graph G_n whose vertex set is S_n and where two permutations σ_1, σ_2 are adjacent if $\sigma_1\sigma_2^{-1}$ is a transposition of the form $(i, i + 1)$ for some $1 \leq i \leq n - 1$. The authors show that a perfect 1-code exists in G_3 and conjecture that there do not exist any in G_n when $n > 3$, proving this for $n < 12$ and all prime n. Their proofs use the representation theory of S_n.

Let p be a prime, define $n = \frac{p-1}{2}$, and let C_p be the cycle of length p. In

[28], Hatami and Hatami exhibit a family of perfect 1-codes in C_p^n, the n-fold Cartesian product of C_p (this can also be interpreted as a Cayley graph on $\mathbb{Z}_p^n$). They conjecture that all perfect 1-codes in C_p^n belong to this family.

10.2 Vizing's conjecture

In 1968, V.G. Vizing conjectured that

$$\gamma(G \,\square\, H) \geq \gamma(G)\gamma(H)$$

for any two graphs G and H.

Significant progress has been made on this conjecture, though it has not yet been proven in full; see the monograph by Haynes et al. [29] for more information and an extensive bibliography.

Appendix A

Appendix: Basic definitions and notation

This appendix defines notation used throughout the thesis, most of which is standard. For additional information consult a textbook such as Godsil and Royle [22] or van Lint and Wilson [47].

A.1 Graphs

A *graph* G consists of a set V of *vertices* and a set E of *edges*, where the elements of E are unordered pairs of vertices. We will refer to an edge $\{x, y\}$ using the shorthand xy. We will write v for $|V|$, which will always be finite, and e for $|E|$.

If xy is an edge then x is *adjacent* to y and vice verse. We will sometimes denote this relation by $x \sim y$. Adjacent vertices may also be called *neighbors*. An edge is *incident* to a vertex x if it contains x; in such a case we say that x is an *endvertex* of the edge. The *neighborhood* of $x \in V$ is

$$\Gamma(x) = \{\, y \in V \,:\, xy \in E \,\}$$

and its *closed neighborhood* is $\Gamma[x] = \Gamma(x) \cup \{x\}$. The *degree* of x, written $\deg(x)$, is $|\Gamma(x)|$. A graph is *regular of degree d*, or *d-regular*, if every vertex has degree d.

A vertex subset is *independent* if no two vertices in it are adjacent. The *distance* between two vertices x and y is the length of the shortest path between them.

The *adjacency matrix* of a graph G on v vertices is a 0-1 matrix A with v rows and columns, both indexed by the vertices, such that $A_{xy} = 1$ if and only if $x \sim y$. The adjacency matrix of G will be written $A(G)$, or simply A when G is clear from context. We say that a real number is an *eigenvalue of G* if it is an eigenvalue of $A(G)$; similarly, a vector $\mathbf{v} \in \mathbb{R}^v$ is an *eigenvector of G* if it is an eigenvector of $A(G)$.

If G is d-regular then d is an eigenvalue of G, and indeed is the largest eigenvalue. Its multiplicity is equal to the number of connected components in G (see Godsil and Royle [22] for more on the spectrum of regular graphs).

The *Cartesian product* of two graphs G and H with vertex sets $V(G)$ and $V(H)$, respectively, is the graph whose vertex set is

$$\{(x, y) : x \in V(G),\ y \in V(H)\}$$

and in which two vertices (x, y) and (x', y') are adjacent when either $x = x'$ and $y \sim y'$ or $x \sim x'$ and $y = y'$. The Cartesian product of G and H will be written $G \,\square\, H$.

The Cartesian product of G with itself t times will be written G^t. It follows from the above definition that the vertices of G^t are of the form $(x_1, \ldots, x_t)$ with $x_i \in V(G)$, and that two vertices $(x_1, \ldots, x_t)$ and $(y_1, \ldots, y_t)$ are adjacent when there exists $i \in \{1, \ldots, t\}$ such that $x_i \sim y_i$ and $x_j = y_j$ for $j \neq i$.

A.2 Vector spaces

Let q be a prime power and let $\mathbb{F}_q$ be the finite field of order q. The n-dimensional vector space over $\mathbb{F}_q$ will be written $\mathbb{F}_q^n$. The standard basis for $\mathbb{F}_q^n$ will be written $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ where the vector $\mathbf{e}_i$ has 1 in the ith coordinate and 0 in all others.

All vector spaces discussed in this thesis will be finite-dimensional. If U is a subspace of a vector space then $U^\perp$ is its orthogonal complement. The subspace generated by a set of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_\ell\}$ will be written $\langle \mathbf{x}_1, \ldots, \mathbf{x}_\ell \rangle$. If U is a subspace then $\langle U, \mathbf{x}_1, \ldots, \mathbf{x}_\ell \rangle$ is the subspace generated by the vectors in U and the vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_\ell\}$. The dimension of U will be denoted by $\dim U$. If two subspaces U, U' intersect trivially, meaning $U \cap U' = \{\mathbf{0}\}$, we will abuse notation somewhat and write $U \cap U' = \mathbf{0}$.

A.3 Projective planes

A *projective plane* of order q is a set P of $q^2 + q + 1$ points and a set L of $q^2 + q + 1$ lines, where a line is a set of $q + 1$ points, such that any one point is in exactly $q + 1$ lines, any pair of points is contained in exactly one line, and any pair of lines intersects at exactly one point.

A projective plane of order q is known to exist when q is a power of a prime; in this case a projective plane can be formed by defining P to be the set of one-dimensional subspaces of $\mathbb{F}_q^3$ and L to be the set of two-dimensional subspaces. It is conjectured that no projective planes exist for any other values of q.

Bibliography

[1] N. Alon and J. H. Spencer, *The Probabilistic Method*, Wiley-Interscience, New York, 2000.

[2] B. S. Baker, *Approximation algorithms for NP-complete problems on planar graphs*, J. Assoc. Comput. Mach. **41** (1994), no. 1, 153–180.

[3] N. Biggs, *Perfect codes in graphs*, J. Comb. Theor. Ser. B **15** (1973), 289–296.

[4] A. Blokhuis, *More on maximal intersecting families of finite sets*, J. Combin. Theory Ser. A **44** (1987), no. 2, 299–303.

[5] E. Boros, Z. Füredi, and J. Kahn, *Maximal intersecting families and affine regular polygons in* $\mathrm{PG}(2,q)$, J. Combin. Theory Ser. A **52** (1989), no. 1, 1–9.

[6] P. J. Cameron and J. H. van Lint, *Designs, Graphs, Codes and their Links*, Cambridge University Press, Cambridge, 1991.

[7] Y. Caro and Y. Roditty, *A note on the k-domination number of a graph*, Internat. J. Math. Math. Sci. **13** (1990), no. 1, 205–206.

[8] W. E. Clark and B. Shekhtman, *Covering by complements of subspaces*, Linear Multilinear Algebra **40** (1995), no. 1, 1–13.

[9] ———, *Domination numbers of q-analogues of Kneser graphs*, Bull. Inst. Combin. Appl. **19** (1997), 83–92.

[10] E. J. Cockayne, R. M. Dawes, and S. T. Hedetniemi, *Total domination in graphs*, Networks **10** (1980), no. 3, 211–219.

[11] G. Cohen, I. Honkala, S. Litsyn, and A. Lobstein, *Covering Codes*, North-Holland Publishing Co., Amsterdam, 1997.

[12] P. Dorbec and M. Mollard, *Perfect codes in Cartesian products of 2-paths and infinite paths*, Electron. J. Combin. **12** (2005), Research Paper 65, 9 pp. (electronic).

[13] S. J. Dow, D. A. Drake, Z. Füredi, and J. A. Larson, *A lower bound for the cardinality of a maximal family of mutually intersecting sets of equal size*, Congr. Numer. **48** (1985), 47–48.

[14] D. A. Drake and S. S. Sane, *Maximal intersecting families of finite sets and n-uniform Hjelmslev planes*, Proc. Amer. Math. Soc. **86** (1982), no. 2, 358–362.

[15] P. H. Edelman and D. White, *Codes, transforms and the spectrum of the symmetric group*, Pacific J. Math. **143** (1990), no. 1, 47–67.

[16] S. C. Fallat and L. Hogben, *The minimum rank of symmetric matrices described by a graph: a survey*, to appear, Linear Algebra Appl., 2007.

[17] U. Feige, *A threshold of* $\ln n$ *for approximating set cover*, J. ACM **45** (1998), no. 4, 634–652.

[18] Z. Füredi, *On maximal intersecting families of finite sets*, J. Combin. Theory Ser. A **28** (1980), no. 3, 282–289.

[19] M. R. Garey and D. S. Johnson, *Computers and Intractability: a Guide to the Theory of NP-Completeness*, W. H. Freeman and Co., San Francisco, Calif., 1979.

[20] C. D. Godsil, *Algebraic Combinatorics*, Chapman & Hall, New York, 1993.

[21] ———, *Interesting graphs and their colourings*, `quoll.uwaterloo.ca/pstuff/colours.pdf`, 2006.

[22] C. D. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001.

[23] D. M. Gordon, *La Jolla Covering Repository*, `http://www.ccrwest.org/cover.html`.

[24] D. M. Gordon, O. Patashnik, and G. Kuperberg, *New constructions for covering designs*, J. Combin. Des. **3** (1995), no. 4, 269–284.

[25] P. Hammond and D. H. Smith, *Perfect codes in the graphs* O_k, J. Combinatorial Theory Ser. B **19** (1975), no. 3, 239–255.

[26] F. Harary and M. Livingston, *Independent domination in hypercubes*, Appl. Math. Lett. **6** (1993), no. 3, 27–28.

[27] C. Hartman and D. B. West, *Covering designs and domination in kneser graphs*, manuscript, 2003.

[28] H. Hatami and P. Hatami, *Perfect dominating sets in the Cartesian products of prime cycles*, Electron. J. Combin. **14** (2007), no. 1, Note 8, 7 pp. (electronic).

[29] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker Inc., New York, 1998.

[30] ______, *Fundamentals of Domination in Graphs*, Marcel Dekker Inc., New York, 1998.

[31] S. Hoory, N. Linial, and A. Wigderson, *Expander graphs and their applications*, Bulletin of the AMS **43** (2006), no. 4, 439–561.

[32] J. Ivančo and B. Zelinka, *Domination in kneser graphs*, Math. Bohem. **118** (1993), 147–152.

[33] D. S. Johnson, *Approximation algorithms for combinatorial problems*, J. Comput. System Sci. **9** (1974), 256–278.

[34] S. Jukna, *Extremal Combinatorics: with Applications in Computer Science*, Springer-Verlag, Berlin, 2001.

[35] L. L. Kelleher and M. B. Cozzens, *Dominating sets in social network graphs*, Math. Social Sci. **16** (1988), no. 3, 267–279.

[36] H. Liu and L. Sun, *On domination number of 4-regular graphs*, Czechoslovak Math. J. **54** (2004), 889–898.

[37] L. Lovász, *Kneser's conjecture, chromatic number, and homotopy*, J. Combin. Theory Ser. A **25** (1978), no. 3, 319–324.

[38] M. Lu, H. Liu, and F. Tian, *Bounds of Laplacian spectrum of graphs based on the domination number*, Linear Algebra Appl. **402** (2005), 390–396.

[39] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland Publishing Co., Amsterdam, 1998.

[40] N. S. Mendelsohn, *A theorem on Steiner systems*, Canad. J. Math. **22** (1970), 1010–1015.

[41] N. S. Mendelsohn and S. H. K. Rung, *On the Steiner systems $S(3,4,14)$ and $S(4,5,15)$*, Utilitas math. **1** (1972), 5–95.

[42] J. C. Meyer, *Quelques problèmes concernant les cliques des hypergraphes h-complets et q-parti h-complets*, Lecture Notes in Math., Vol. 411, Springer, Berlin, 1974, pp. 127–139.

[43] W. H. Mills and R. C. Mullin, *Coverings and packings*, Contemporary Design Theory, Wiley, New York, 1992, pp. 371–399.

[44] B. Mohar and S. Poljak, *Eigenvalues and max-cut problem*, Czechoslovak Math. J. **40(115)** (1990), no. 2, 343–352.

[45] O. Ore, *Theory of Graphs*, American Mathematical Society, Providence, R.I., 1967.

[46] J. Schönheim, *On coverings*, Pacific J. Math. **14** (1964), 1405–1411.

[47] J. H. van Lint and R. M. Wilson, *A course in combinatorics*, Cambridge University Press, Cambridge, 2001.

[48] G. J. M. van Wee, *Improved sphere bounds on the covering radius of codes*, IEEE Trans. Inform. Theory **34** (1988), no. 2, 237–245.

[49] H.-M. Xing, L. Sun, and X.-G. Chen, *An upper bound for domination number of 5-regular graphs*, Czechoslovak Math. J. **56** (2006), 1049–1061.

List of Notation

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