Amenability for the Fourier Algebra

by

Aaron P. Tikuišis

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Pure Mathematics

Waterloo, Ontario, Canada, 2007

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The Fourier algebra $\mathrm{A}(G)$ can be viewed as a dual object for the group G and, in turn, for the group algebra $\mathrm{L}^1(G)$. It is a commutative Banach algebra constructed using the representation theory of the group, and from which the group G may be recovered as its spectrum. When G is abelian, $\mathrm{A}(G)$ coincides with $\mathrm{L}^1(\hat{G})$; for non-abelian groups, it is viewed as a generalization of this object. B. Johnson has shown that G is amenable as a group if and only if $\mathrm{L}^1(G)$ is amenable as a Banach algebra. Hence, it is natural to expect that the cohomology of $\mathrm{A}(G)$ will reflect the amenability of G. The initial hypothesis to this effect is that G is amenable if and only if $\mathrm{A}(G)$ is amenable as a Banach algebra. Interestingly, it turns out that $\mathrm{A}(G)$ is amenable only when G has an abelian group of finite index, leaving a large class of amenable groups with non-amenable Fourier algebras.

The dual of $\mathrm{A}(G)$ is a von Neumann algebra (denoted $\mathrm{VN}(G)$); as such, $\mathrm{A}(G)$ inherits a natural operator space structure. With this operator space structure, $\mathrm{A}(G)$ is a completely contractive Banach algebra, which is the natural operator space analogue of a Banach algebra. By taking this additional structure into account, one recovers the intuition behind the first conjecture: Z.-J. Ruan showed that G is amenable if and only if $\mathrm{A}(G)$ is operator amenable.

This thesis concerns both the non-amenability of the Fourier algebra in the category of Banach spaces and why Ruan's Theorem is actually the proper analogue of Johnson's Theorem for $\mathrm{A}(G)$. We will see that the operator space projective tensor product behaves well with respect to the Fourier algebra, while the Banach space projective tensor product generally does not. This is crucial to explaining why operator amenability is the right sort of amenability in this context, and more generally, why $\mathrm{A}(G)$ should be viewed as a completely contractive Banach algebra and not merely a Banach algebra.

Acknowledgements

The author would like to acknowledge the financial support provided by the National Sciences and Engineering Research Council. The author would also like to thank Cameron Zwarich for proofreading this document. Finally, the author would like to thank his wife, Tara, for all her support.

Contents

Chapter 1

Introduction

The Fourier algebra $\mathrm{A}(G)$ is an important tool of non-commutative harmonic analysis. First studied in depth by Eymard in [9], it is a commutative algebra that contains certain representation theoretic information about the underlying group. For abelian groups, it coincides with $\mathrm{L}^1(\hat{G})$, and more generally, it plays many of the same important roles that $\mathrm{L}^1(\hat{G})$ plays in commutative harmonic analysis. For instance, the group G is recovered from $\mathrm{A}(G)$ as its Gelfand spectrum; this result is presented in Chapter 2.4. Much effort in non-commutative harmonic analysis concerns how various properties of the group G may be seen as properties of its Fourier algebra. The picture of the Fourier algebra as a sort of non-commutative $\mathrm{L}^1(\hat{G})$ provides intuition about how properties of the group and properties of the Fourier algebra relate.

One particularly rich property for groups to have is amenability, which is the ability for a left invariant mean to be put on $\mathrm{L}^\infty(G)$. There are a large number of interesting equivalent conditions to G being amenable; one condition with a particularly nice ring to it is that $\mathrm{L}^1(G)$ is amenable as a Banach algebra, as shown by Johnson in [21]. Banach algebra amenability essentially concerns the cohomology of a Banach algebra; the name "amenable" is inspired by this theorem of Johnson's.

The amenability of G has long been known to appear as a property of the Fourier algebra, as Leptin's Theorem states that G is amenable if and only if $\mathrm{A}(G)$ has a bounded approximate identity [23]. However, under the view that $\mathrm{A}(G)$ is the non-commutative analogue of $\mathrm{L}^1(\hat{G})$, and in light of Johnson's Theorem, it seems natural to expect that

A(G) is amenable precisely when G is amenable. Unfortunately, the intuition appears to fail here: Johnson first showed that certain compact (and therefore amenable) groups had non-amenable Fourier algebras [20]. A complete characterization of when A(G) is amenable was later carried out by Forrest and Runde in [13], and is presented in Chapter 3. It turns out that A(G) is amenable only in the simplest cases, when G has an abelian subgroup of finite index. As is evident in the proof of this fact, it is quite simple to show that for such groups, A(G) is amenable; the converse direction is the difficult one.

By considering the operator space structure on A(G), one finds that there is some truth to the intuition that A(G) should be amenable when G is. Operator amenability, a property for algebras with certain operator space structures, is a generalization of Johnson's amenability for Banach algebras. Ruan's Theorem, as presented in Chapter 4, states that G is amenable if and only if A(G) is operator amenable [28]. Moreover, it is appropriate to expect operator amenability, since using the natural operator space structure on $\mathrm{L}^1(G)$, amenability and operator amenability coincide, and therefore Johnson's Theorem may be restated: G is amenable if and only if $\mathrm{L}^1(G)$ is operator amenable.

Finally, one asks why it is that operator amenability reveals itself to be the right sort of amenability to look for on A(G). The key, it seems, is in how the operator space projective tensor product $\hat{\otimes}$ works on A(G). With the Banach space projective tensor product $\otimes^\gamma$, recall that there is a natural identification of $\mathrm{L}^1(G) \otimes^\gamma \mathrm{L}^1(G)$ with $\mathrm{L}^1(G \times G)$. When applied to the Fourier algebra, the map $(u \otimes v)(x, y) = u(x)v(y)$ extends to a map $\mathrm{A}(G) \otimes^\gamma \mathrm{A}(G) \mapsto \mathrm{A}(G \times G)$. However, it turns out that the extended map is only an isomorphism in the same limited situations that A(G) is amenable [24]. A special case of this is proven as a corollary to the main results in this thesis, in Corollary 4.0.9. On the other hand, $\mathrm{A}(G)\hat{\otimes}\mathrm{A}(G) = \mathrm{A}(G \times G)$, which is the natural thing to expect, and permits Ruan's result. The different projective tensor products are intimately connected to the respective amenability conditions for algebras, and for this reason, it is unsurprising that the right sort of amenability is tied to the right sort of projective tensor product on A(G). From Ruan's Theorem, and the very fact that $\mathrm{A}(G)\hat{\otimes}\mathrm{A}(G) = \mathrm{A}(G \times G)$, one finds that it is natural to view A(G) not merely as a Banach algebra but as a completely contractive Banach algebra, in which the operator space structure is always taken into account. This perspective has spurred many advances in the understanding of the Fourier algebra.

Chapter 2

Preliminaries

In this chapter, the necessary background from harmonic analysis and operator space theory is presented. Unreferenced basic facts about harmonic analysis may be found in [10].

Notation:

For a vector space $\mathcal{V}$, a vector $v \in \mathcal{V}$, and a functional $\phi : \mathcal{V} \mapsto \mathbb{C}$, evaluation of ϕ at v is denoted by the dual pairing:

$$\langle v, \phi \rangle_{\mathcal{V}} .$$

(Often, the subscript $\mathcal{V}$ will be omitted). For a Hilbert space $\mathcal{H}$, the inner product of vectors ξ, η is denoted

$$\langle \xi \,|\, \eta \rangle .$$

Whereas a dual pairing $\langle \cdot, \cdot \rangle$ is bilinear, the inner product $\langle \cdot \,|\, \cdot \rangle$ is sesquilinear (linear in the first coordinate, conjugate-linear in the second). For groups G, H, $H \leq G$ says that H is a subgroup of G. Finally, $\amalg$ denotes disjoint union.

2.1 Locally Compact Groups and their Representations

Let G be a locally compact group. Denote by m the Haar measure on G and by Δ the modular function on G. $\mathrm{M}^1(G)$ is the space of finite Borel measures on G; it is a $*$-algebra under the convolution product $(\mu,\nu)\mapsto\mu*\nu$ and involution, $\mu\mapsto\mu^*$, where for $f\in\mathcal{C}_c(G)$

$$\int f(x)d(\mu*\nu)(x) = \iint f(xy)\,d\mu(x)\,d\nu(y)$$
$$\int f(x)d(\mu^*)(x) = \overline{\int \overline{f(x^{-1})}\,d\mu(x)}.$$

$\mathrm{L}^1(G)$, identified with the subspace of measures that are absolutely continuous with respect to m, is a closed ideal; for $f,g\in\mathrm{L}^1(G)$,

$$(f*g)(x) = \int f(y)g(y^{-1}x)\,dy, \text{ and} \tag{2.1.1}$$
$$f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1}).$$

For $x\in G$, let $\delta_x\in\mathrm{M}^1(G)$ be the point-mass measure at x; that is, for $f\in\mathcal{C}_b(G)$,

$$\int f\,d\delta_x = f(x).$$

It may be noted that for $f\in\mathrm{L}^1(G)$, δ_x*f is the left translate of f by x^{-1}, that is:

$$(\delta_x*f)(y) = f(x^{-1}y),$$

and likewise, $f*\delta_x$ is the right translate of f by x^{-1}, appropriately scaled by the modular function:

$$(f*\delta_x)(y) = f(yx^{-1})\Delta(x^{-1}).$$

In what follows, for any function f on G, δ_x*f will be used to denote left translation of f by x^{-1}. Also, for a function f on G, let $\check{f}$ denote the function on G which is given for $x\in G$ by

$$\check{f}(x) = f(x^{-1}). \tag{2.1.2}$$

A **representation** of G is a homomorphism π from G to $\mathcal{U}(\mathcal{H}_\pi)$, the group of unitary operators on the Hilbert space $\mathcal{H}_\pi$, which is continuous in the induced strong operator topology on $\mathcal{U}(\mathcal{H}_\pi)$. A representation on G lifts to a $*$-representation on $\mathrm{L}^1(G)$, that is, a continuous $*$-algebra homomorphism, also denoted by π, from $\mathrm{L}^1(G)$ to $\mathcal{B}(\mathcal{H}_\pi)$, this time equipped with the norm topology (by [4], Proposition 13.3.1). To be precise, for $f \in \mathrm{L}^1(G)$, $\pi(f)$ is the operator such that for $\xi, \eta \in \mathcal{H}_\pi$,

$$\langle \pi(f)\xi \mid \eta \rangle = \int \langle \pi(x)\xi \mid \eta \rangle \, f(x) \, dx.$$

A representation π is **irreducible** if there is no closed subspace of $\mathcal{H}_\pi$ that is invariant under $\pi(x)$ for all $x \in G$, or equivalently, if there is no closed subspace of $\mathcal{H}_\pi$ that is invariant under $\pi(f)$ for all $f \in \mathrm{L}^1(G)$ ([4], 13.3.5).

The **left** and **right regular representation** of G are representations λ and ρ on the Hilbert space $\mathrm{L}^2(G)$, defined as follows: for $\xi \in \mathrm{L}^2(G), x, y \in G$,

$$\begin{aligned} (\lambda(x)\xi)(y) &= (\delta_x * \xi)(y) \\ &= \xi(x^{-1}y) \\ (\rho(x)\xi)(y) &= \xi(yx)\Delta(x)^{\frac{1}{2}}. \end{aligned}$$

This gives, for $f \in \mathrm{L}^1(G)$,

$$\begin{aligned} \lambda(f)\xi &= f * \xi \\ \rho(f)\xi &= \xi * (\Delta^{-\frac{1}{2}}\check{f}), \end{aligned}$$

where the convolution of L^1 functions by L^2 functions is defined exactly as in (2.1.1).

These left and right regular representations are unitarily equivalent. To be precise, the map $V : \mathrm{L}^2(G) \mapsto \mathrm{L}^2(G)$ given for $\xi \in \mathrm{L}^2(G)$ by

$$(V\xi)(x) = \xi(x^{-1})\Delta(x)^{-1/2} \tag{2.1.3}$$

is self-adjoint and self-inverse, and thus unitary. For $x \in G$,

$$\lambda(x) = V^*\rho(x)V = V\rho(x)V.$$

The **universal representation** ω of G is the direct sum of all non-equivalent cyclic representations of G [1].

[1] The standard definition of the universal representation is the direct sum over cyclic representations associated with positive forms of G. The definition given here is only quasi-equivalent to the standard definition, but this is enough for it to provide all the the properties used in this thesis. The standard definition is not used in order to avoid introducing the theory of positive forms.

2.2 The Fourier and Fourier-Stieltjes Algebras

Given a representation π of G, $\mathrm{L}^1(G)$ inherits a C*-seminorm $\|\cdot\|_\pi$ given by

$$\|f\|_\pi = \|\pi(f)\|.$$

As they are C*-seminorms, they are dominated by $\|\cdot\|_1$ (by [4], Proposition 1.3.7). Since the left regular representation and the universal representation are both faithful, $\|\cdot\|_\lambda$ and $\|\cdot\|_\omega$ are both non-degenerate, and thus are norms. The **reduced group C*-algebra** $\mathrm{C}^*_r(G)$ and the **group C*-algebra** $\mathrm{C}^*(G)$ of G are the completions of $\mathrm{L}^1(G)$ with respect to the norms $\|\cdot\|_\lambda$ and $\|\cdot\|_\omega$ respectively (both of these representations are faithful on $\mathrm{L}^1(G)$, by [4], 13.3.6). $\mathrm{C}^*(G)$ has the important property that every representation of G extends continuously from a $*$-representation on $\mathrm{L}^1(G)$ to a $*$-representation on all of $\mathrm{C}^*(G)$; denote this extension by $\pi|_{\mathrm{C}^*}$ ([4], Proposition 2.7.4). Moreover, every non-degenerate $*$-representation of $\mathrm{C}^*(G)$ induces a representation on the group G in an inverse manner. These C*-algebras embed naturally into $\mathcal{B}(\mathrm{L}^2(G))$ and $\mathcal{B}(\oplus_\pi \mathcal{H}_\pi)$ (where the direct sum is taken over all cyclic representations up to equivalence) respectively.

The von Neumann algebras generated by these C*-algebras (with these concrete realizations) are also of importance; respectively, they are the **group von Neumann algebra** $\mathrm{VN}(G)$ and the **universal enveloping von Neumann algebra** $\mathrm{VN}_\omega(G)$ of G. $\mathrm{VN}_\omega(G)$ may be identified with the second dual of $\mathrm{C}^*(G)$ ([9], (1.1)). The von Neumann subalgebra of $\mathcal{B}(\mathrm{L}^2(G))$ generated by the right regular representation ρ will also come up in this thesis, and will be denoted $\mathrm{VN}_\rho(G)$. Since the left and right regular representations are unitarily equivalent by the operator V, $\mathrm{VN}(G)$ and $\mathrm{VN}_\rho(G)$ are normal spatially $*$-isomorphic, via conjugation by V.

The **Fourier-Stieltjes algebra** $\mathrm{B}(G)$ of G is the dual of $\mathrm{C}^*(G)$. Every element of $\mathrm{B}(G)$ is given as a coefficient function, that is, as $\xi *_\pi \eta = \langle \pi(\cdot)\xi \,|\, \eta\rangle$, where π is a representation, $\xi, \eta \in \mathcal{H}_\pi$ ([9], Proposition (2.1)). $\mathrm{B}(G)$ is a Banach algebra under pointwise multiplication, with

$$(\xi_1 *_{\pi_1} \eta_1)(\xi_2 *_{\pi_2} \eta_2) = (\xi_1 \otimes \xi_2) *_{\pi_1 \otimes \pi_2} (\eta_1 \otimes \eta_2).$$

The **Fourier algebra** $\mathrm{A}(G)$ of G is the closed subspace of $\mathrm{B}(G)$ spanned by elements $\xi *_\lambda \eta$ where $\xi, \eta \in \mathrm{L}^2(G)$. It may be realized as a quotient of the Banach space projective tensor product of $\mathrm{L}^2(G)$ with its conjugate ([3], proof of Théorème (2.2)). That is,

$$\mathrm{A}(G) \cong \left(\mathrm{L}^2(G) \otimes^\gamma \overline{\mathrm{L}^2(G)}\right) /N$$

isometrically, where N is the kernel of the map given by extending $\xi \otimes \overline{\eta} \mapsto \xi *_\lambda \eta$ linearly and continuously, and $\overline{\mathrm{L}^2(G)}$ denotes the conjugate Hilbert space of $\mathrm{L}^2(G)$. Every element of $\mathrm{A}(G)$ has the form

$$\sum_{i=1}^{\infty} \xi *_\lambda \eta,$$

where $\xi_i, \eta_i \in \mathrm{L}^2(G)$ for all i and the sum converges absolutely ([3], Théorème (2.2)). Also, $\mathrm{A}(G) \subset \mathcal{C}_0(G)$ ([9], Proposition (3.7), 1°). The dual of $\mathrm{A}(G)$ is the annihilator of N in $\mathcal{B}(\mathrm{L}^2(G))$, which is precisely $\mathrm{VN}(G)$. Duality is given by the following formula, where $T \in \mathrm{VN}(G), \xi_i, \eta_i \in \mathrm{L}^2(G)$ with $\sum_{i=1}^{\infty} \|\xi_i\|\|\eta_i\| < \infty$:

$$\left\langle \sum_{i=1}^{\infty} \xi_i *_\lambda \eta_i, T \right\rangle = \sum_{i=1}^{\infty} \langle T\xi_i \mid \eta_i \rangle .$$

When $T = \lambda(f), f \in \mathrm{L}^1(G)$ and $u \in \mathrm{A}(G)$,

$$\langle u, T\rangle = \int uf\, dm.$$

$\mathrm{A}(G)$ is an ideal in $\mathrm{B}(G)$ ([9], following the proof of (3.4)). $\mathrm{A}(G) \cap \mathcal{C}_c(G)$ is dense in $\mathrm{A}(G)$ (by [9], Proposition (3.4)).

Among the nice functorial properties of the Fourier algebra is the following. A nice alternate proof of this fact may be found in [3], Proposition (3.23).

Theorem 2.2.1. *(Herz' Restriction Theorem,[14])*
If $H \leq G$ is a closed subgroup then the restriction map $u \mapsto u|_H$ is a surjective contraction from $\mathrm{A}(G)$ to $\mathrm{A}(H)$.

The duality relations

$$\mathrm{A}(G)^* = \mathrm{VN}(G) \text{ and } (\mathrm{C}^*(G)^*)^* = \mathrm{B}(G)^* = \mathrm{VN}_\omega(G)$$

are important in revealing the properties of the Fourier and Fourier-Stieltjes algebras. As will be seen shortly, these relations allow these algebras to realize natural operator space structures, which are crucial later in this thesis.

When G is abelian, there are nice concrete realizations for many of these objects, all given by applying the Fourier transform:

$$\begin{aligned} \mathrm{A}(G) &\cong \mathrm{L}^1(\hat{G}), & \mathrm{B}(G) &\cong \mathrm{M}^1(\hat{G}) \\ \mathrm{VN}(G) &\cong \mathrm{L}^\infty(\hat{G}), & \mathrm{C}^*(G) &\cong \mathcal{C}_0(\hat{G}). \end{aligned}$$

2.3 Operator Spaces

To begin this section, some notational comments are in order. Matrices are used heavily in the theory of operator spaces. For a vector space $\mathcal{V}$, denote by $\mathcal{M}_{n,m}(\mathcal{V})$ the space of $n \times m$ matrices with entries in $\mathcal{V}$, and $\mathcal{M}_n(\mathcal{V}) := \mathcal{M}_{n,n}(\mathcal{V})$. Let $\mathcal{M}_{n,m} := \mathcal{M}_{n,m}(\mathbb{C})$ and likewise, $\mathcal{M}_n := \mathcal{M}_n(\mathbb{C})$. $\mathcal{M}_{n_1\times n_2, m_1\times m_2}(\mathcal{V})$ denotes matrices whose rows and columns are doubly-indexed; essentially, it is the same as $\mathcal{M}_{n_1 n_2, m_1 m_2}(\mathcal{V})$. When $\mathcal{V}$ is a Banach space, for $a = [a_{i,j}] \in \mathcal{M}_{n,m}(\mathcal{V})$ and $f = [f_{k,l}] \in \mathcal{M}_{p,q}(\mathcal{V}^*)$, let $\langle\langle a, f\rangle\rangle \in \mathcal{M}_{n\times p, m\times q}$ given by

$$\langle\langle a, f\rangle\rangle = [\langle a_i j, f_{k,l}\rangle] \,.$$

An **operator space** is a vector space $\mathcal{V}$ along with a sequence of norms $(\|\cdot\|_n : \mathcal{M}_n(\mathcal{V}) \mapsto [0,\infty))_{n\in\mathbb{N}}$, satisfying two axioms:

(OS1): For $v \in \mathcal{M}_n(\mathcal{V}), w \in \mathcal{M}_n(\mathcal{W})$,

$$v \oplus w := \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$$

has norm $\|v \oplus w\|_{n+m} = \max\{\|v\|_n, \|w\|_m\}$.

(OS2): For $v \in \mathcal{M}_n(\mathcal{V})$ and $\alpha \in \mathcal{M}_{m,n}, \beta \in \mathcal{M}_{n,m}$,

$$\alpha v \beta := \left[\sum_{k=1}^{n}\sum_{\ell=1}^{n} \alpha_{i,k}\beta_{\ell,j} v_{k,\ell}\right]$$

has norm $\|\alpha v\beta\|_m \leq \|\alpha\|\|v\|_n\|\beta\|$. Notice that the left and right scalar matrix multiplication here is defined in exactly the same way as usual matrix multiplication; the matrix norms $\|\alpha\|$ and $\|\beta\|$ are the usual ones, that is, the operator norms given by the identification $\mathcal{M}_{n,m} = \mathcal{B}(\ell^2(m), \ell^2(n))$.

Effros and Ruan's book [8] is an excellent source for operator space theory; all unproven, uncited results in this section may be found there.

A motivating and important example of an operator space is the space of bounded operators on a Hilbert space, $\mathcal{B}(\mathcal{H})$. For each $n \in \mathbb{N}$, there is an obvious identification $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))) \cong \mathcal{B}(\mathcal{H}^n)$. By making this identification isometric, norms are defined on each matrix space $\mathcal{M}_n(\mathcal{B}(\mathcal{H})))$, satisfying the operator space axioms. It is clear that a subspace of an operator space is itself one, so that any subspace of $\mathcal{B}(\mathcal{H})$ is an operator space. In particular, via the Gelfand-Naimark-Segal construction, any C*-algebra may be realized as a subspace of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, and thus as operator space; in fact, the operator space structure is independent of the realization. For a C*-algebra, this operator space structure is the canonical one.

For operator spaces $\mathcal{V}$ and $\mathcal{W}$, a linear map $T : \mathcal{V} \mapsto \mathcal{W}$ induces natural maps $T^{(n)} : \mathcal{M}_n(\mathcal{V}) \mapsto \mathcal{M}_n(\mathcal{W})$ given by

$$T^{(n)}[v_{i,j}] = [T(v_{i,j})] \,.$$

Taking interesting analytical properties of linear maps and seeking their uniform expression over all $T^{(n)}$ brings about interesting operator space theoretic properties for T:

- T is **completely bounded** if there is a uniform bound on the operator norms of the maps $T^{(n)}$; denote
$$\|T\|_{cb} = \sup\left\{\|T^{(n)}\| : n \in \mathbb{N}\right\},$$
- T is a **complete contraction** if $\|T^{(n)}\| \leq 1$ for all n (i.e. $\|T\|_{cb} \leq 1$),
- T is a **complete isometry** if each $T^{(n)}$ is an isometry.

Let $\mathcal{CB}(\mathcal{V}, \mathcal{W}) = \{T : \mathcal{V} \mapsto \mathcal{W} : T \text{ is linear and completely bounded}\}$, a normed vector space under $\|\cdot\|_{cb}$. It is clear that the composition of two completely bounded maps gives a completely bounded map.

An interesting, and important, example of a map that is not completely bounded is the transpose map $T \mapsto T^t$ on $\mathcal{B}(\mathcal{H})$, when $\mathcal{H}$ is infinite dimensional, as shown in the following:

Proposition 2.3.1.

(1) *The transpose map on $\mathcal{M}_n$ has completely bounded norm at least n.*

(2) *If $\mathcal{H}$ is infinite dimensional then the transpose map on $\mathcal{B}(\mathcal{H})$ is not completely bounded.*

Proof.

(1) For $i_0, j_0 = 1, \dots n$, let
$$A_{i_0,j_0} = (\delta_{ii_0}\delta_{jj_0}) \in \mathcal{M}_n.$$
That is, A_{i_0,j_0} has a 1 in the (i_0, j_0) position, and 0's elsewhere; clearly, $A^t_{i_0,j_0} = A_{j_0,i_0}$.

Let $B = (A_{j,i}) \in \mathcal{M}_n(\mathcal{M}_n)$, so that the nth amplification of the transpose map applied to B gives $A = (A_{i,j})$. To give a lower bound on the completely bounded norm of the transpose map, the n-norms of A and B will be computed. These norms coincide with the $\mathcal{M}_{n^2}$ matrix norms.

B, viewed as an element of $\mathcal{M}_{n^2}$, contains exactly one 1 in each row and column, and zeroes elsewhere; that is, B is a permutation matrix, and is thus unitary. So $\|B\| = 1$.

On the other hand, noting that $A_{i,j}A_{j,k} = A_{i,k}$, it is easy to compute $AA^* = A^2 = nA$. So
$$\begin{aligned}\|A\|^2 &= \|AA^*\|\\ &= n\|A\|\end{aligned}$$
and, since $\|A\| \neq 0$, $\|A\| = n$.

It follows that the transpose has a completely bounded norm of at least n.

(2) For each n, $\mathcal{M}_n$ embeds $*$-homomorphically into $\mathcal{B}(\mathcal{H})$. Since the transose map is preserved by this embedding, its completely bounded norm on $\mathcal{B}(\mathcal{H})$ is at least n, by (1). Thus, the transpose map is not completely bounded.

□

When $\mathcal{V}$ is an operator space, $\mathcal{V}^*$ is itself also an operator space, by identifying $\mathcal{M}_n(\mathcal{V}^*)$ with $\mathcal{B}(\mathcal{V}, \mathcal{M}_n)$. Every bounded functional on $\mathcal{V}$ is automatically completely bounded (with the completely bounded norm equal to the usual norm), and thus, every bounded map from $\mathcal{V}$ to $\mathcal{M}_n$ is also completely bounded (though in this case, the completely bounded norm may be larger). The completely bounded norm on $\mathcal{B}(\mathcal{V}, \mathcal{M}_n)$ produces an operator space structure on $\mathcal{V}^*$. The usual injection of $\mathcal{V}$ into $\mathcal{V}^{**}$ is a complete isometry (see [7] for all these facts).

Much operator space theory is motivated by, and analogous to, Banach space theory. Analogous to the Hahn-Banach Theorem is the following:

Theorem 2.3.2. *(Wittstock's Extension Theorem, [35])*
Let $\mathcal{W} \subset \mathcal{V}$ be operator spaces, and let $T \in \mathcal{CB}(\mathcal{W}, \mathcal{B}(\mathcal{H}))$. Then there exists $\tilde{T} \in \mathcal{CB}(\mathcal{V}, \mathcal{B}(\mathcal{H}))$ such that $\|\tilde{T}\|_{cb} = \|T\|_{cb}$ and $\tilde{T}|_{\mathcal{W}} = T$.

Recall that for Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, the projective tensor product norm $\|\cdot\|_\gamma : \mathcal{X} \otimes \mathcal{Y} \mapsto [0, \infty)$ is defined for $u \in \mathcal{X} \otimes \mathcal{Y}$ by

$$\|u\| = \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i \right\},$$

and the Banach space projective tensor product of $\mathcal{X}$ and $\mathcal{Y}$ is the completion of $\mathcal{X} \otimes \mathcal{Y}$ with respect to $\|\cdot\|_\gamma$; it is denoted $\mathcal{X} \otimes^\gamma \mathcal{Y}$.

Analogous to the Banach space projective tensor product is the operator space projective tensor product $\hat{\otimes}$. For operator spaces $\mathcal{V}$ and $\mathcal{W}$, the sequence of operator space projective norms, $\|\cdot\|_{\wedge,n} : \mathcal{M}_n(\mathcal{V}) \otimes \mathcal{M}_n(\mathcal{W}) \mapsto [0, \infty)$ is defined by

$$\begin{aligned}\|z\|_{\wedge,n} = \inf \big\{ \|\alpha\| \|v\| \|w\| \|\beta\| : \alpha \in \mathcal{M}_{n,p\times q}, v \in \mathcal{M}_p(\mathcal{V}), w \in \mathcal{M}_q(\mathcal{W}),\\ \beta \in \mathcal{M}_{p\times q,n}, z = \alpha(v \otimes w)\beta \big\},\end{aligned}$$

where for $v = [v_{i,j}] \in \mathcal{M}_p(\mathcal{V}), w = [w_{k,l}] \in \mathcal{M}_q(\mathcal{W})$,

$$v \otimes w = [v_{i,j} \otimes w_{k,l}] \in \mathcal{M}_{p\times q}(\mathcal{V} \otimes \mathcal{W}).$$

The **operator space projective tensor product** of $\mathcal{V}$ and $\mathcal{W}$ is the completion of $\mathcal{V} \otimes \mathcal{W}$ under $\|\cdot\|_{\wedge,1}$, denoted $\mathcal{V} \hat{\otimes} \mathcal{W}$. The analogy of $\otimes^\gamma$ and $\hat{\otimes}$ is not completely transparent from the definition (although with some thought, this definition of $\hat{\otimes}$ can be motivated from the definition of $\otimes^\gamma$); the universal properties of $\hat{\otimes}$ confirm the parallelism. $(\|\cdot\|_{\wedge,n})_{n\in\mathbb{N}}$ is the maximum sequence of operator space norms on $\mathcal{M}_n(\mathcal{V}\otimes\mathcal{W})$ such that for $v\in\mathcal{M}_p(\mathcal{V})$, $w\in\mathcal{M}_q(\mathcal{W})$,

$$\|v\otimes w\|_{\wedge,pq}\leq\|v\|_p\|w\|_q.$$

One nice property of $\hat{\otimes}$ is the following:

Proposition 2.3.3. *Let $\mathcal{V},\mathcal{W}$ be operator spaces. Then $\mathcal{CB}(\mathcal{V},\mathcal{W}^*)\cong(\mathcal{V}\hat{\otimes}\mathcal{W})^*$ completely isometrically.*

In general, the dual of $\mathcal{V}\hat{\otimes}\mathcal{W}$ is somewhat complicated, but there is an important special case. Suppose M and N are von Neumann subalgebras of $\mathcal{B}(\mathcal{H}),\mathcal{B}(\mathcal{K})$ respectively. The **(von Neumann) spatial tensor product** of M and N, denoted $M\overline{\otimes}N$, is the weak* closure of $M\otimes N$ in $\mathcal{B}(\mathcal{H}\otimes_2\mathcal{K})$. The preduals M_*,N_* of M,N become operator spaces by taking their respective inclusions in M^*,N^* to be completely isometric; this operator space structure is such that $(M_*)^*\cong M$ completely isometrically and likewise for N (by [8], Proposition 4.2.2). Moreover,

$$(M_*\hat{\otimes}N_*)^*\cong M\overline{\otimes}N$$

completely isometrically [6].

A **Hopf von Neumann algebra** is a von Neumann algebra M along with a "co-multiplication", an injective normal unital $*$-homomorphism $\nabla: M\mapsto M\overline{\otimes}M$ that is co-associative, i.e. such that

$$(\nabla\otimes id)\circ\nabla=(id\otimes\nabla)\circ\nabla. \tag{2.3.1}$$

Since ∇ is a $*$-homomorphism, it is automatically a complete contraction.

If M is a Hopf von Neumann algebra, then ∇ produces a multiplication on M_*; for $a,b\in M_*$, ab is the element of M_* such that for $T\in M$,

$$\langle ab,T\rangle=\langle(a\otimes b),\nabla(T)\rangle.$$

The co-associative property (2.3.1) of ∇ is equivalent to the associativity of this multiplication operation. The multiplication extends to a completely contractive map

$\Delta : M_*\hat{\otimes}M_* \mapsto M_*$, given for $u \in M_*, T \in M$ by

$$\langle \Delta(u), T\rangle = \langle u, \nabla(T)\rangle .$$

This uses the identification of $(M_*\hat{\otimes}M_*)^*$ with $M\overline{\otimes}M$. That is, Δ is the pre-adjoint of ∇. M_* is thus an example of an operator space with an algebra structure such that the multiplication extends to a complete contraction on the operator space projective tensor product; such spaces are called **completely contractive Banach algebras**.

VN(G) is an example of a Hopf von Neumann algebra. First, note that

$$\mathrm{VN}(G)\overline{\otimes}\mathrm{VN}(G) \cong \mathrm{VN}(G \times G)$$

by extending the map $\lambda_G(x) \otimes \lambda_G(y) \mapsto \lambda_{G\times G}(x, y)$, so that

$$\mathrm{A}(G)\hat{\otimes}\mathrm{A}(G) \cong \mathrm{A}(G \times G), \tag{2.3.2}$$

by extending the map $u \otimes v \mapsto u \times v$, where $(u \times v)(x, y) = u(x)v(y)$.

By Herz' Restriction Theorem (Theorem 2.2.1), the map $\Delta_{\mathrm{A}(G)} : \mathrm{A}(G \times G) \mapsto \mathrm{A}(G)$ may be defined for $f \in \mathrm{A}(G \times G), x \in G$ by

$$\Delta_{\mathrm{A}(G)}(f)(x) = f(x, x).$$

and this map is a contractive surjection, so $\nabla_{\mathrm{VN}(G)} = \Delta^*_{\mathrm{A}(G)} : \mathrm{VN}(G) \mapsto \mathrm{VN}(G)\overline{\otimes}\mathrm{VN}(G)$ is normal and injective. For $x \in G$, $\nabla_{\mathrm{VN}(G)}$ is given by

$$\nabla_{\mathrm{VN}(G)}(\lambda(x)) = \lambda(x) \otimes \lambda(x).$$

Thus, $\nabla_{\mathrm{VN}(G)}$ is a unital $*$-homomorphism on $\mathrm{span}\,(\lambda(G))$, which is weak*-dense in VN(G), so $\nabla_{\mathrm{VN}(G)}$ is in fact a unital $*$-homomorphism on all of VN(G). This shows that the Fourier algebra is a completely contractive Banach algebra.

It can also be shown that $\mathrm{VN}_\omega(G)$ is a Hopf von Neumann algebra, using the universal property of the universal representation ω; this fact will not, however, be used here.

Given an arbitrary Banach space $\mathcal{X}$, there are two interesting operator space structures that can automatically be put on $\mathcal{X}$: the MIN and MAX structures. For $x \in \mathcal{M}_n(\mathcal{X})$, define

$$\|x\|_{\mathrm{MIN},n} = \sup\left\{\|[\langle x_{i,j}, f\rangle]\|_{\mathcal{M}_n} : f \in b_1(\mathcal{X}^*)\right\}$$
$$\|x\|_{\mathrm{MAX},n} = \sup\left\{\|[Tx_{i,j}]\|_{\mathcal{M}_{m\times n}} : T \in b_1\left(\mathcal{B}(\mathcal{X}, \mathcal{M}_m)\right), m \in \mathbb{N}\right\}.$$

Then $\mathrm{MIN}(\mathcal{X}) := \big(\mathcal{X}, (\|\cdot\|_{\mathrm{MIN},n})_{n\in\mathbb{N}}\big)$ and $\mathrm{MAX}(\mathcal{X}) := \big(\mathcal{X}, (\|\cdot\|_{\mathrm{MAX},n})_{n\in\mathbb{N}}\big)$ are operator spaces. To see this, note that they embed completely isometrically into the C*-algebras

$$\begin{aligned}&\ell^\infty\left(b_1(\mathcal{X}^*)\right) \text{ and}\\ &\ell^\infty\text{-}\oplus\left\{\mathcal{M}_m^T : T \in b_1\left(\mathcal{B}(\mathcal{X},\mathcal{M}_m)\right)\right\}\end{aligned}$$

respectively via the maps

$$\begin{aligned}&x \mapsto \left(\langle x, f\rangle\right)_{f\in b_1(\mathcal{X}^*)}, \text{ and}\\ &x \mapsto (Tx)_{T\in b_1(\mathcal{B}(\mathcal{X},\mathcal{M}_m))}.\end{aligned}$$

These operator space structures have some special properties.

Proposition 2.3.4. *Let $\mathcal{X}$ be a Banach space. If $(\|\cdot\|_n)_{n\in\mathbb{N}}$ is an operator space structure then for all $x \in \mathcal{M}_n(\mathcal{X})$,*

$$\|x\|_{\mathrm{MIN},n} \le \|x\|_n \le \|x\|_{\mathrm{MAX},n}.$$

Proof. For $n \in \mathbb{N}$,

$$\left\{f^{(n)} : f \in b_1(\mathcal{X}^*)\right\} \subset \bigcup_{m\in\mathbb{N}} b_1\left(\mathcal{CB}(\mathcal{X},\mathcal{M}_m)\right) \subset \bigcup_{m\in\mathbb{N}} b_1\left(\mathcal{B}\left(\mathcal{X},\mathcal{M}_m\right)\right).$$

For $x \in \mathcal{M}_n(\mathcal{X})$, $\|x\|_{\mathrm{MIN},n}$, $\|x\|_n$, and $\|x\|_{\mathrm{MAX},n}$ are given by taking the supremem of the norms of maps in the respective sets above applied to x (for $\|\cdot\|_n$, this is because the injection $\mathcal{X} \mapsto \mathcal{X}^{**}$ is a complete isometry). Thus,

$$\|x\|_{\mathrm{MIN},n} \le \|x\|_n \le \|x\|_{\mathrm{MAX},n}.$$

□

Proposition 2.3.5. *Let $\mathcal{V}$ be an operator space and $\mathcal{X}$ a Banach space. Then:*

(1) $\mathcal{B}(\mathcal{V},\mathcal{X}) = \mathcal{CB}\left(\mathcal{V},\mathrm{MIN}(\mathcal{X})\right)$ *isometrically, and*

(2) $\mathcal{B}(\mathcal{X},\mathcal{V}) = \mathcal{CB}\left(\mathrm{MAX}(\mathcal{X}),\mathcal{V}\right)$ *isometrically*

Proof.

(1) Let $T \in \mathcal{B}(\mathcal{V}, \mathcal{X})$, and $v = [v_{i,j}] \in \mathcal{M}_n(\mathcal{V})$. Then

$$\begin{aligned}
\|T^{(n)}v\|_{\mathrm{MIN},n} &= \sup\left\{\left\|f^{(n)}T^{(n)}v\right\| : f \in b_1(\mathcal{X}^*)\right\} \\
&= \sup\left\{\left\|(f \circ T)^{(n)}v\right\| : f \in b_1(\mathcal{X}^*)\right\} \\
&\leq \sup\left\{\left\|(f \circ T)^{(n)}\right\| \|v\| : f \in b_1(\mathcal{X}^*)\right\} \\
&= \sup\left\{\|f \circ T\| \|v\| : f \in b_1(\mathcal{X}^*)\right\}, \\
&\quad \text{since } \|f \circ T\| = \|f \circ T\|_{cb} \text{ because } f \circ T \in \mathcal{V}^* \\
&= \|T^*\|\|v\| \\
&= \|T\|\|v\|.
\end{aligned}$$

Thus, $T \in \mathcal{CB}\left(\mathcal{V}, \mathrm{MIN}(\mathcal{X})\right)$, and $\|T\|_{cb} \leq \|T\|$. Conversely, if $T \in \mathcal{CB}\left(\mathcal{V}, \mathrm{MIN}(\mathcal{X})\right)$ then $\|T\| = \|T\|_1 \leq \|T\|_{cb}$.

(2) Let $T \in \mathcal{B}(\mathcal{X}, \mathcal{V})$, WLOG with norm 1, and let $x = [x_{i,j}] \in \mathcal{M}_n(\mathcal{X})$. Then

$$\begin{aligned}
\|T^{(n)}x\| &= \sup\left\{\left\|S^{(n)}T^{(n)}x\right\| : S \in b_1\left(\mathcal{CB}(\mathcal{V}, \mathcal{M}_m)\right), m \in \mathbb{N}\right\} \\
&\leq \sup\left\{\left\|(ST)^{(n)}x\right\| : S \in b_1\left(\mathcal{B}(\mathcal{V}, \mathcal{M}_m)\right), m \in \mathbb{N}\right\} \\
&\leq \sup\left\{\left\|R^{(n)}x\right\| : R \in b_1\left(\mathcal{B}(\mathcal{X}, \mathcal{M}_m)\right), m \in \mathbb{N}\right\} \\
&= \|x\|_{\mathrm{MAX},n}.
\end{aligned}$$

The norm on the right side of the first line is the given by the complete isometry $\mathcal{V} \mapsto \mathcal{V}^{**}$. As in (1), the converse is trivial.

□

Proposition 2.3.6. *Let $\mathcal{V}$ be an operator space. Then:*

(1) $\mathcal{V} = \mathrm{MIN}(\mathcal{V})$ completely isometrically if and only if $\mathcal{V}^* = \mathrm{MAX}(\mathcal{V}^*)$ completely isometrically.

(2) $\mathcal{V} = \mathrm{MAX}(\mathcal{V})$ completely isometrically if and only if $\mathcal{V}^* = \mathrm{MIN}(\mathcal{V}^*)$ completely isometrically.

Proof. For each $n \geq 1$, consider the map from $\mathcal{M}_n(\mathrm{MIN}(\mathcal{V}))$ to $\mathcal{B}((\mathcal{V}^*, \mathcal{M}_n)$ that takes $[x_{ij}] \in \mathcal{M}_n(\mathrm{MIN}(\mathcal{V}))$ to $[\hat{x}_{ij}]$, where for $x \in \mathcal{X}, f \in \mathcal{V}^*$, $\hat{x}(f) = \langle x, f \rangle$. The range of this map is the space of weak*-continuous maps, denoted $\mathcal{B}^\sigma(\mathcal{V}^*, \mathcal{M}_n)$. Indeed, to see it is surjective, note that $[F_{ij}] \in \mathcal{B}(\mathcal{V}^*, \mathcal{M}_n)$ is weak*-continuous if and only if each F_{ij} is weak*-continuous. Moreover,

$$\begin{aligned} \|[x_{ij}]\|_{\mathrm{MIN}} &= \sup \{\|[\langle x_{ij}, f \rangle]\| : f \in b_1(\mathcal{V}^*)\} \\ &= \sup \{\|[\hat{x}_{ij}(f)]\| : f \in b_1(\mathcal{V}^*)\} \\ &= \|[\hat{x}_{ij}]\| . \end{aligned}$$

Hence $\mathcal{M}_n(\mathrm{MIN}(\mathcal{V})) \cong \mathcal{B}^\sigma(\mathcal{V}^*, \mathcal{M}_n)$ isometrically.

The map $T \mapsto T^{**}$ from $\mathcal{B}(\mathcal{V}, \mathcal{M}_n)$ to $\mathcal{B}^\sigma(\mathcal{V}^{**}, \mathcal{M}_n)$ is a surjective isometry, so

$$\begin{aligned} \mathcal{M}_n((\mathrm{MAX}(\mathcal{V}))^*) &\cong \mathcal{CB}(\mathrm{MAX}(\mathcal{V}), \mathcal{M}_n) \\ &= \mathcal{B}(\mathcal{V}, \mathcal{M}_n) \\ &\cong \mathcal{B}^\sigma(\mathcal{V}^{**}, \mathcal{M}_n) \\ &\cong \mathcal{M}_n(\mathrm{MIN}(\mathcal{V}^*)) , \end{aligned}$$

and thus, if $\mathcal{V} = \mathrm{MAX}(\mathcal{V})$ then $\mathcal{V}^* = \mathrm{MIN}(\mathcal{V}^*)$.

So $(\mathrm{MAX}(\mathcal{V}^*))^* \cong \mathrm{MIN}(\mathcal{V}^{**})$ and by the Hahn-Banach theorem, $\mathrm{MIN}(\mathcal{V})$ injects completely isometrically into $\mathrm{MIN}(\mathcal{V}^{**})$. Thus by the definition of dual norms,
$\mathrm{MAX}(\mathcal{V}^*) \mapsto (\mathrm{MIN}(\mathcal{V}))^*$ is completely contractive.

On the other hand, since $(\mathrm{MIN}(\mathcal{V}))^*$ is an operator space structure on $\mathcal{V}^*$, the (identity) map

$$\mathrm{MAX}(\mathcal{V}^*) \mapsto (\mathrm{MIN}(\mathcal{V}))^*$$

is completely contractive. Hence, if $\mathcal{V} = \mathrm{MIN}(\mathcal{V})$ then $\mathcal{V}^* = \mathrm{MAX}(\mathcal{V}^*)$.

Now, if $\mathcal{V}^* = \mathrm{MAX}(\mathcal{V}^*)$ then $\mathcal{V}^{**} = \mathrm{MIN}(\mathcal{V}^{**})$, and by Hahn-Banach, $\mathcal{V} = \mathrm{MIN}(\mathcal{V})$. Likewise, using Wittstock's Extension Theorem (Theorem 2.3.2) in place of the Hahn-Banach Theorem, it can be shown that if $\mathcal{V}^* = \mathrm{MIN}(\mathcal{V}^*)$ then $\mathcal{V} = \mathrm{MAX}(\mathcal{V})$. □

The following result can be found in [8], Section 8.2. The proof of second part uses some further theory about the MAX operator space structure; for this reason, only the first part will be proven here.

Proposition 2.3.7. *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and $\mathcal{V}$ an operator space. Then*

(1) $\mathrm{MAX}(\mathcal{X})\hat{\otimes}\mathcal{V} \cong \mathcal{X} \otimes^\gamma \mathcal{V}$ *isometrically, by extending the identity on* $\mathcal{X} \otimes \mathcal{V}$*.*

(2) $\mathrm{MAX}(\mathcal{X})\hat{\otimes}\mathrm{MAX}(\mathcal{Y}) \cong \mathrm{MAX}(\mathcal{X} \otimes^\gamma \mathcal{Y})$ *completely isometrically, in the same way.*

Proof (of (1)): This will be shown by considering the duals.

$$\begin{aligned}(\mathrm{MAX}(\mathcal{X}) \otimes \mathcal{V}, \|\cdot\|_\wedge)^* &\cong \mathcal{CB}(\mathrm{MAX}(\mathcal{X}), \mathcal{V}^*), \text{ by Proposition 2.3.3}\\ &\cong \mathrm{B}(\mathcal{X}, \mathcal{V}^*), \text{ by Proposition 2.3.5, (2)}\\ &\cong (\mathcal{X} \otimes \mathcal{V}, \|\cdot\|_\gamma)^*\end{aligned}$$

The last line is a well-known fact about $\otimes^\gamma$. Thus $\|\cdot\|_\wedge = \|\cdot\|_\gamma$ on $\mathrm{MAX}(\mathcal{X}) \otimes \mathcal{V}$, and so the completed spaces, $\mathrm{MAX}(\mathcal{X})\hat{\otimes}\mathcal{V}$ and $\mathcal{X} \otimes^\gamma \mathcal{V}$, are equal. □

It is not too difficult to show that for a locally compact Hausdorff space X, the operator space structure on $\mathcal{C}(X)$ as a C*-algebra is the MIN structure. Any commutative C*-algebra $\mathcal{A}$ can be identified completely isometrically with $\mathcal{C}_0(\sigma(\mathcal{A}))$, where $\sigma(\mathcal{A})$ denotes the Gelfand spectrum of $\mathcal{A}$. Hence, for a commutative C*-algebra $\mathcal{A}$, the canonical operator space structure on $\mathcal{A}$ coincides with the MIN structure. It follows that the canonical operator space structure on $\mathrm{L}^1(G)$, as the pre-dual to $\mathrm{L}^\infty(G)$, is the MAX structure. That is:

$$\mathrm{L}^1(G) = \mathrm{MAX}\left(\mathrm{L}^1(G)\right). \tag{2.3.3}$$

2.4 The Spectrum of $\mathrm{A}(G)$

In this section, it will be shown that the Gelfand spectrum of $\mathrm{A}(G)$ – that is, the space of multiplicative linear functionals, with the induced weak* topology – can be identified naturally with G. This result shows in one respect why $\mathrm{A}(G)$ is seen as a dual object to the group G. In truth, to recover the group structure of G (and thus to view $\mathrm{A}(G)$ as a true dual object), one must use the $*$-algebraic structure of $\mathrm{VN}(G)$, and not merely its Banach space structure. Likewise, it will be seen that $\mathrm{A}(G)$ by itself fails as a dual object when considering amenability – the proper form of amenability on $\mathrm{A}(G)$ requires some account of the richer structure of $\mathrm{VN}(G)$, which is conveyed via the operator space structure on $\mathrm{A}(G)$.

The proof here is a synthesis of ideas from Takesaki [34], VII §3, and Saitô [31]. The original proof, done by Eymard in [9], is difficult to fill in; the proof presented here uses modern techniques, similar to those used in the rest of this thesis, and avoids the pitfalls of Eymard's proof.

Define the map $W : \mathrm{L}^2(G \times G) \mapsto \mathrm{L}^2(G \times G)$ by

$$(W\psi)(x, y) = \psi(x, xy). \tag{2.4.1}$$

To see that W is well-defined (that is, that it takes almost-everywhere equal functions to almost-everywhere equal functions), note that W can be defined on $\mathcal{C}_c(G \times G)$, and continuously extended to all of $\mathrm{L}^2(G \times G)$. It is easy to check that W is unitary. The map from $\mathrm{VN}(G)$ to $\mathrm{VN}(G \times G)$ given by $T \mapsto W^*\,(T \otimes I)\,W$ is weak*-weak* continuous, and one can readily see that for $x \in G$,

$$W^*\,(\lambda(x) \otimes I)\,W = \lambda(x) \otimes \lambda(x) = \nabla(\lambda(x)).$$

It follows by the weak* density of $\mathrm{span}\lambda(G)$ in $\mathrm{VN}(G)$ that for all $T \in \mathrm{VN}(G)$,

$$W^*(T \otimes I)W = \nabla T.$$

Denote the spectrum of $\mathrm{A}(G)$ by $\sigma\left(\mathrm{A}(G)\right)$. To begin, this spectrum will be characterized:

Proposition 2.4.1. *For $T \in \mathrm{VN}(G) \setminus \{0\}$, TFAE:*

(1) $T \in \sigma\left(\mathrm{A}(G)\right)$.

(2) $\nabla T = T \otimes T$.

(3) $W^*(T \otimes I)W = T \otimes T$.

Proof. For $u, v \in \mathrm{A}(G)$,

$$\begin{aligned} \langle u \otimes v, \nabla T \rangle &= \langle \Delta(u \otimes v), T \rangle \\ &= \langle uv, T \rangle , \end{aligned}$$

and

$$\langle u \otimes v, T \otimes T \rangle = \langle u, T \rangle \langle v, T \rangle .$$

Therefore,

$$\begin{aligned} T \in \sigma(\mathrm{A}(G)) &\iff \forall u, v \in \mathrm{A}(G), \langle uv, T \rangle = \langle u, T \rangle \langle v, T \rangle \\ &\iff \forall u, v \in \mathrm{A}(G), \langle u \otimes v, \nabla T \rangle = \langle u \otimes v, T \otimes T \rangle \\ &\iff \nabla T = T \otimes T. \end{aligned}$$

Also, (2) $\iff$ (3) is apparent since $\nabla T = W^*(T \otimes I)W$. ☐

Corollary 2.4.2. $G \cong \lambda(G) \subset \sigma(\mathrm{A}(G))$.

Proof. It is clear that λ is a homeomorphism onto its range $\lambda(G)$. Moreover, $T = \lambda(G)$ satisfies condition (2), and thus, is in $\sigma(\mathrm{A}(G))$. ☐

Define $C : \mathrm{L}^2(G) \mapsto \mathrm{L}^2(G)$ by, for $\xi \in \mathrm{L}^2(G)$,

$$(C\xi)(x) = \overline{\xi(x)}.$$

For $T \in \mathrm{VN}(G)$, let $\tilde{T} = CTC$. Note that $\tilde{T}^* = \widetilde{T^*}$. Using the characterization result and the simple observation that $C \otimes C$ commutes with W, the following is apparent:

Corollary 2.4.3. $\sigma(\mathrm{A}(G)) \cup \{0\} \subset \mathrm{VN}(G)$ *is closed under multiplication and the maps* $T \mapsto T^*$, $T \mapsto \tilde{T}$.

Lemma 2.4.4. *For* $T \in \sigma(\mathrm{A}(G))$, $a \in \mathrm{L}^2(G) \cap \mathrm{L}^\infty(G)$, *letting* m_a *denote the operator on* $\mathrm{L}^2(G)$ *given by pointwise multiplication by* a,

$$m_a T = T\tilde{T}^* m_a T.$$

Proof. First note that for $a, b, c, d \in \mathrm{L}^2(G)$,

$$\begin{aligned}\langle W(a \otimes b) \,|\, c \otimes d\rangle &= \iint a(x)b(xy)\overline{c(x)d(y)}\,dy\,dx \\ &= \int b(y) \int a(x)\overline{c(x)d(x^{-1}y)}\,dx\,dy \\ &= \langle b \,|\, (\bar{a}c) * d\rangle\,. \qquad (2.4.2)\end{aligned}$$

So, for $a, b \in \mathrm{L}^2(G) \cap \mathrm{L}^\infty(G), \xi, \eta \in \mathcal{C}_c(G) \subset \mathrm{L}^2(G)$,

$$\begin{aligned}\langle \eta \,|\, (m_a Tb) * \xi\rangle &= \langle W(\bar{a} \otimes \eta) \,|\, (Tb) \otimes \xi\rangle\,, \text{ by } (2.4.2) \\ &= \langle \bar{a} \otimes \eta \,|\, W^*(T \otimes I)(b \otimes \xi)\rangle \\ &= \langle \bar{a} \otimes \eta \,|\, (T \otimes T)W^*(b \otimes \xi)\rangle\,, \text{ since } T \in \sigma\left(\mathrm{A}(G)\right) \\ &= \langle W((T^*\bar{a}) \otimes (T^*\eta) \,|\, b \otimes \xi\rangle \\ &= \left\langle T^*\eta \,|\, (m_b\overline{T^*\bar{a}}) * \xi\right\rangle\,, \text{ by } (2.4.2) \\ &= \left\langle \eta \,|\, (Tm_b\tilde{T}^*a) * \xi\right\rangle\,.\end{aligned}$$

The last step above needs some justification; letting $c = m_b\tilde{T}^*a \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$,

$$\begin{aligned}\langle T^*\eta \,|\, c * \xi\rangle &= \iint (T^*\eta)(x)\overline{c(xy)\xi(y^{-1})}\,dy\,dx \\ &= \int \xi(y^{-1})\Delta(y^{-1})\left\langle T^*\eta \,|\, c * \delta_{y^{-1}}\right\rangle\,dy, \\ &\qquad \text{by Fubini's theorem, which is valid since } \xi \in \mathcal{C}_c(G) \\ &= \int \xi(y^{-1})\Delta(y^{-1})\left\langle \eta \,|\, T(c * \delta_{y^{-1}})\right\rangle\,dy \\ &= \int \xi(y^{-1})\Delta(y^{-1})\left\langle \eta \,|\, (Tc) * \delta_{y^{-1}}\right\rangle\,dy, \\ &\qquad \text{since } T \in \mathrm{VN}(G) \text{ commutes with right translations} \\ &= \langle \eta \,|\, (Tc) * \xi\rangle\,.\end{aligned}$$

Since $\mathcal{C}_c(G)$ is dense in $\mathrm{L}^2(G)$, this shows that $m_aTb = Tm_b\tilde{T}^*a$. As $\tilde{T}^* \in \sigma\left(\mathrm{A}(G)\right)$, this may be repeated with the roles of a and b reversed. Using the fact that $\tilde{T}^* = \widetilde{T^*}$, this gives $m_aTb = T\tilde{T}^*m_aTb$. Since $b \in \mathrm{L}^2(G) \cap \mathrm{L}^\infty(G)$ is arbitrary and $\mathrm{L}^2(G) \cap \mathrm{L}^\infty(G)$ is dense in $\mathrm{L}^2(G)$, $m_aT = T\tilde{T}^*m_aT$. $\square$

Proposition 2.4.5. *$\sigma\left(\mathrm{A}(G)\right)$ is a subgroup of $\mathcal{B}\left(\mathrm{L}^2(G)\right)^{-1}$.*

Proof. This proof basically entails showing that for $T \in \sigma\left(\mathrm{A}(G)\right)$, T is invertible and $T^{-1} \in \sigma\left(\mathrm{A}(G)\right)$.

Let $T = U|T|$ be the polar decomposition of T. Since ∇ is a $*$-homomorphism, $\nabla(|T|) \geq 0$. For the same reason, and noting that an operator S is a partial isometry if and only if SS^* is a projection, $\nabla(U)$ is a partial isometry.

Moreover, since W is unitary,

$$\begin{aligned} \ker \nabla(U) &= W^* \ker(U \otimes I) \\ &\supset W^* \ker(T^* \otimes I), \text{ since } \ker U \supset \ker T^* \\ &= \ker \nabla(T^*) \\ &= \ker \nabla(T)^*. \end{aligned}$$

Thus, $\nabla(T) = \nabla(U)\nabla(|T|)$ is the polar decomposition of $\nabla(T)$.

But $\nabla(T) = T \otimes T = (U \otimes U)\left(|T| \otimes |T|\right)$ is the polar decomposition, so

$$\begin{aligned} \nabla(U) &= U \otimes U, \text{ and} \\ \nabla(|T|) &= |T| \otimes |T|. \end{aligned}$$

It follows that $U, |T| \in \sigma\left(\mathrm{A}(G)\right)$

Now, letting E be either UU^* or U^*U, E is a projection contained in $\sigma\left(\mathrm{A}(G)\right)$; using this fact, it will be shown that $E = I$.

Let $a \in \mathrm{L}^2(G) \cap \mathrm{L}^\infty(G)$. Then

$$\begin{aligned} m_a E &= E\tilde{E}^* m_a E \\ &= E(E\tilde{E}^* m_a E), \text{by Lemma 2.4.4} \\ &= E m_a E. \end{aligned}$$

Next, using this, but replacing a with $\bar{a}$, gives:

$$\begin{aligned} E m_a &= (m_{\bar{a}} E)^* \\ &= (E m_{\bar{a}} E)^* \\ &= E m_a E \\ &= m_a E. \end{aligned}$$

As multiplication operators, $\mathrm{L}^2(G) \cap \mathrm{L}^\infty(G)$ is WOT-dense in $\mathrm{L}^\infty(G)$. So, again viewing $\mathrm{L}^\infty(G) \subset \mathcal{B}(\mathrm{L}^2(G))$, this shows that

$$E \in \mathrm{L}^\infty(G)' = \mathrm{L}^\infty(G).$$

Now, it is easy to see that W commutes with $\mathrm{L}^\infty(G) \otimes I$, so in particular,

$$\begin{aligned} E \otimes I &= W^*(E \otimes I)W \\ &= E \otimes E. \end{aligned}$$

Since $E \neq 0$, it follows that $E = I$.

So, $U^*U = UU^* = I$; that is, U is unitary, so T has dense range. Now, for $a \in \mathrm{L}^2(G) \cap \mathrm{L}^\infty(G)$, $m_a T = T\tilde{T}^* m_a T$. Since T has dense range, $\{m_a T\xi : \xi \in \mathrm{L}^2(G)\}$ is dense in $\mathrm{L}^2(G)$, whence

$$T\tilde{T}^* = I.$$

$\tilde{T}^*$ is also in $\sigma(\mathrm{A}(G))$, so this shows also that $\tilde{T}^*T = I$. Thus, $T^{-1} = \tilde{T}^* \in \sigma(\mathrm{A}(G))$.

Since $\sigma(\mathrm{A}(G))$ is already known to be closed under multiplication, it follows that it is a group. □

Lemma 2.4.6. *Let G be a Hausdorff topological group and $H < G$ be locally compact (in the relative topology). Then H is closed in G.*

In general, a locally compact subset of a topological space may not be closed; in fact, any locally compact, non-compact, space is not closed in its one-point compactification. This result is analogous to the fact that a complete subspace of a metric space is closed, and indeed, it implicitly uses the fact that the uniform structure on a locally compact group is complete.

Proof. Let $(x_\alpha) \subset H$ be a net converging to $x \in G$; it will be shown that $x \in H$. Let U be a compact neighbourhood in H of the identity; since it contains an open set, let $U' \subset G$ be open such that $U \supset U' \cap H$. Let $V \subset G$ be a symmetric neighbourhood of the identity such that $VV \subset U'$.

Let α_0 be such that for all $\alpha \geq \alpha_0$, $x_\alpha \in xV$. In particular, $x_{\alpha_0} \in xV$, which can be rewritten $x \in x_{\alpha_0}V$ (using the symmetry of V). So for $\alpha \geq \alpha_0$,

$$\begin{aligned} x_\alpha &\in xV \\ &\subset x_{\alpha_0}VV \\ &\subset x_{\alpha_0}U' \end{aligned}$$

Moreover, $x_\alpha \in H$, so that $x_\alpha \in x_{\alpha_0}U$.

By compactness, $(x_\alpha)_{\alpha\geq\alpha_0}$ has a cluster point in $x_{\alpha_0}U \subset H$. Since G is Hausdorff, this cluster point is x, so that $x \in H$. □

Proposition 2.4.7. *$\lambda(G)$ is a closed subgroup of $\sigma(\mathrm{A}(G))$*

Proof. It is readily seen that λ is a homeomorphism from G onto $\lambda(G)$, when $\lambda(G)$ is equipped with the induced weak* topology (that is, the topology of pointwise convergence on $\mathrm{A}(G)$). Thus, $\lambda(G)$ is a locally compact subgroup of $\sigma(\mathrm{A}(G))$. So by the last lemma, it is closed. □

For ease of notation, G will be identified with its image in $\sigma(\mathrm{A}(G))$ under λ. $\mathrm{VN}(G)$ may be viewed as a module over itself. As the pre-dual, $\mathrm{A}(G)$ may be viewed as a $\mathrm{VN}(G)$-module; for $S, T \in \mathrm{VN}(G), u \in \mathrm{A}(G)$, the actions are defined by:

$$\begin{aligned} \langle u.T, S\rangle &= \langle u, TS\rangle\,, \text{ and} \\ \langle T.u, S\rangle &= \langle u, ST\rangle \;\; \text{([32], 1.8.1).} \end{aligned}$$

Lemma 2.4.8. *Let $u \in \mathrm{L}^2(G) \cap \mathrm{A}(G)$, $T \in \mathrm{VN}(G)$. Then $u.T = \tilde{T}^* u$.*

Proof. Since $\lambda(\mathrm{L}^1(G))$ is WOT-dense in $\mathrm{VN}(G)$, it suffices to show the result for T in this subalgebra. Let $T = \lambda(f)$ where $f \in \mathrm{L}^1(G)$. Then for $g \in \mathrm{L}^1(G) \cap \mathrm{L}^2(G)$,

$$\begin{aligned}\left\langle \tilde{T}^* u \,|\, g \right\rangle &= \left\langle u \,|\, \tilde{T} g \right\rangle \\ &= \int u(T\bar{g})\, dm \\ &= \int u(f * \bar{g})\, dm \\ &= \langle u, \lambda(f * \bar{g}) \rangle \\ &= \langle u, T\lambda(\bar{g}) \rangle \\ &= \langle u.T, \lambda(\bar{g}) \rangle \\ &= \langle u.T \,|\, g \rangle .\end{aligned}$$

As $\mathrm{L}^1(G) \cap \mathrm{L}^2(G)$ is dense in $\mathrm{L}^2(G)$, $\tilde{T}^* = u.T$ as required. □

Define the Borel measure $\tilde{m}$ on $\sigma(\mathrm{A}(G))$ by, for $A \subset G$ a Borel set,

$$\tilde{m}(A) = m(A \cap G).$$

By this definition, when the 2-norm of a function on $\sigma(\mathrm{A}(G))$ is used, it makes no difference whether this is regarded as the norm in $\mathrm{L}^2(\sigma(\mathrm{A}(G)), \tilde{m})$ or, by restricting the range of the function, in $\mathrm{L}^2(G)$.

In the following, note that $\mathrm{A}(G) \subset \mathcal{C}_0(\sigma(\mathrm{A}(G)))$ via the Gelfand transform, so it makes sense to speak of $\delta_T * u$ where $u \in \mathrm{A}(G)$ and $T \in \sigma(\mathrm{A}(G))$.

Lemma 2.4.9. *For $f \in \mathcal{C}_c(\sigma(G)), T \in \sigma(\mathrm{A}(G)), \epsilon > 0$, there exists $u \in \mathrm{A}(G) \cap \mathcal{C}_c(G)$ such that $\|f - u\|_2 < \epsilon$ and $\|\delta_T * f - \delta_T * u\|_2 < \epsilon$.*

Proof. Since supp$f$ is compact, and left translation by T is a continuous function, Tsupp$f$ is also compact, so both have finite $\tilde{m}$-measure. Let $K_1 = \max\{\tilde{m}(\mathrm{supp} f), \tilde{m}(T\mathrm{supp} f)\}^{\frac{1}{2}} < \infty$. Let $\delta > 0$ such that

$$2\delta + \delta^2 < \frac{\epsilon}{3(\|f\|_\infty + 1)K_1}.$$

By Urysohn's lemma, let $e \in \mathcal{C}_c(\sigma(\mathrm{A}(G)))$ such that $e|_{\mathrm{supp} f} = 1$ and the range of e is $[0, 1]$. Since $\mathrm{A}(G)$ is uniformly dense in $\mathcal{C}_0(\sigma(\mathrm{A}(G)))$ and $\mathrm{A}(G) \cap \mathcal{C}_c(G)$ is dense in $\mathrm{A}(G)$, let $u_{11}, u_{12} \in \mathrm{A}(G) \cap \mathcal{C}_c(G)$ such that $\|u_{11} - e\|_\infty < \delta$ and $\|u_{12} - \delta_T * e\|_\infty < \delta$, and define

$u_1 = u_{11}(u_{12}.T) \in \mathrm{A}(G) \cap \mathcal{C}_c(G)$. Note that $u_{12}.T = \delta_{T^{-1}} * u_{12}$. Hence u_1 is such that $u_1.T^{-1} = (u_{11}.T^{-1})u_{12} \in \mathrm{A}(G) \cap \mathcal{C}_c(G)$, and for $S \in \mathrm{supp} f$,

$$\begin{aligned}
|u_1(S) - 1| &\leq |u_{11}(S)|\,|\delta_{T^{-1}} * u_{12}(S) - 1| + |u_{11}(S) - 1| \\
&< (\|e\|_\infty + \delta)\delta + \delta \\
&= 2\delta + \delta^2 \\
&< \frac{\epsilon}{3(\|f\|_\infty + 1)K_1}.
\end{aligned}$$

Now, let $K_2 = \max\left\{m(\mathrm{supp} u_1), m\left(\mathrm{supp}(u_1.T^{-1})\right)\right\}^{\frac{1}{2}} < \infty$. Let $u_2 \in \mathrm{A}(G)$ such that $\|u_2 - f\|_\infty < \min\left\{\frac{\epsilon}{3K_1}, \frac{\epsilon}{2\|u_1\|_\infty K_2}, 1\right\}$, and define $u = u_1 u_2$. For $S \in \mathrm{supp} f$,

$$\begin{aligned}
|u(S) - f(S)| &\leq |u_1(S) - 1|\;|u_2(S)| + |u_2(S) - f(S)| \\
&< \frac{\epsilon}{3\left(\|f\|_\infty + 1\right)K_1}\left(\|f\|_\infty + 1\right) + \frac{\epsilon}{3K_1} \\
&= \frac{2\epsilon}{3K_1},
\end{aligned}$$

and for $S \notin \mathrm{supp} f$,

$$\begin{aligned}
|u(S)| &= |u_1(S)|\;|u_2(S)| \\
&< \|u_1\|_\infty \frac{\epsilon}{2\|u_1\|_\infty K_2} \\
&= \frac{\epsilon}{2K_2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|u - f\|_2^2 &\leq \int_{\mathrm{supp} f} |u(S) - f(S)|^2\, d\tilde{m}(S) + \int_{\mathrm{supp} u_1} |u(x)|^2\, dm(x) \\
&\leq \tilde{m}(\mathrm{supp} f)\left(\frac{2\epsilon}{3K_1}\right)^2 + m(\mathrm{supp} u_1)\left(\frac{\epsilon}{2K_2}\right)^2 \\
&\leq \frac{4\epsilon^2}{9} + \frac{\epsilon^2}{2} \\
&< \epsilon^2,
\end{aligned}$$

and likewise,

$$\begin{aligned}\|\delta_T * u - \delta_T * f\|_2^2 &\leq \int_{T\mathrm{supp} f} |u(T^{-1}S) - f(T^{-1}S)|^2 \, d\tilde{m}(S) + \int_{\mathrm{supp}(u_1 . T^{-1})} |u(T^{-1}x)| \, dm(x) \\ &\leq \tilde{m}(T\mathrm{supp} f) \left(\frac{2\epsilon}{3K_1}\right)^2 + m(T\mathrm{supp} u_1) \left(\frac{\epsilon}{2K_2}\right)^2 \\ &\leq \frac{4\epsilon^2}{9} + \frac{\epsilon^2}{2} \\ &< \epsilon^2,\end{aligned}$$

as required. ☐

Proposition 2.4.10. *$\tilde{m}$ is a Haar measure on $\sigma(\mathrm{A}(G))$.*

Proof. Since G is closed in $\sigma(\mathrm{A}(G))$ and m is a non-zero Radon measure, it follows that $\tilde{m}$ is also a non-zero Radon measure. For $u, v \in \mathrm{A}(G) \cap \mathcal{C}_c(G)$, $T \in \sigma(\mathrm{A}(G))$,

$$\begin{aligned}\int u(Tx)\overline{v(x)} \, dm(x) &= \int (u.T)\bar{v} \, dm \\ &= \langle u.T \mid v \rangle \\ &= \left\langle \tilde{T}^* u \mid v \right\rangle, \text{ by Lemma 2.4.8} \\ &= \left\langle u \mid \tilde{T} v \right\rangle \\ &= \int u(x)\overline{v(T^*x)} \, dm(x), \text{ again by Lemma 2.4.8.}\end{aligned}$$

Now, fix $f, g \in \mathcal{C}_c(\sigma(\mathrm{A}(G)))$, and let $\epsilon > 0$. By the last lemma, let $u, v \in \mathrm{A}(G) \cap \mathcal{C}_c(G)$ such that

$$\begin{gathered}\|u - f\|_2 < \epsilon, \\ \|v - g\|_2 < \epsilon, \\ \|\delta_{T^{-1}} * u - \delta_{T^{-1}} * f\|_2 < \epsilon \text{ and} \\ \|\delta_{T^{*-1}} * v - \delta_{T^{*-1}} * g\|_2 < \epsilon.\end{gathered}$$

Then,

$$\begin{aligned}
\left|\int f(TS)g(S)\,d\tilde{m}(S) - \int f(S)g(T^*S)\,d\tilde{m}(S)\right| &\leq \left|\int f(TS)(g(S)-v(S))\,d\tilde{m}(S)\right| \\
&\quad + \left|\int (f(TS)-u(TS))v(S)\,d\tilde{m}(S)\right| \\
&\quad + \left|\int u(S)(g(T^*S)-v(T^*S))\,d\tilde{m}(S)\right| \\
&\quad + \left|\int (f(S)-u(S))g(T^*S)\,d\tilde{m}(S)\right| \\
&\leq \|\delta_{T^{-1}} * f\|_2 \|g-v\|_2 \\
&\quad + \|\delta_{T^{-1}} * f - \delta_{T^{-1}} * u\|_2 \|v\|_2 \\
&\quad + \|u\|_2 \|\delta_{T^{*-1}} * g - \delta_{T^{*-1}} * v\|_2 \\
&\quad + \|f-u\|_2 \|\delta_{T^{*-1}} * v\|_2 \\
&\leq \big(\|\delta_{T^{-1}} * f\|_2 + (\|g\|_2+\epsilon) \\
&\quad + (\|f\|_2+\epsilon) + \|\delta_{T^{-1}} * g\|_2\big)\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary,

$$\int f(TS)g(S)\,d\tilde{m}(S) = \int f(S)g(T^*S)\,d\tilde{m}(S).$$

Now, fixing just $f \in \mathcal{C}_c\left(\sigma\left(\mathrm{A}(G)\right)\right)$, $(T^{-1}\mathrm{supp} f) \cup (T^*\mathrm{supp} f)$ is compact, so let $g \in \mathcal{C}_c\left(\sigma\left(\mathrm{A}(G)\right)\right)$ be such that $g = 1$ on this set. Then

$$\begin{aligned}
\int f(TS)\,d\tilde{m}(S) &= \int f(TS)\overline{g(S)}\,d\tilde{m}(S) \\
&= \int f(S)\overline{g(T^*S)}\,d\tilde{m}(S) \\
&= \int f(S)\,d\tilde{m}(S).
\end{aligned}$$

Thus, $\tilde{m}$ is left-invariant, as required. □

Finally, the conclusion of this section occurs as a simple corollary:

Theorem 2.4.11. $\sigma\left(\mathrm{A}(G)\right) = G$

Proof. Suppose for a contradiction that $S \in \sigma(\mathrm{A}(G)) \setminus G$. Then $SG \cap G = \emptyset$, so

$$0 = \tilde{m}(SG) = \tilde{m}(G) \neq 0,$$

and this is a contradiction. □

2.5 Amenability

A locally compact group G is **amenable** if there is a left translation-invariant mean on $\mathrm{L}^\infty(G)$, that is, a functional $m \in \mathrm{L}^\infty(G)^*$ such that

$$\langle 1, m \rangle = \|m\| = 1,$$

and for all $x \in G$, $f \in \mathrm{L}^\infty(G)$,

$$\langle \delta_x * f, m \rangle = \langle f, m \rangle .$$

Paterson's book [27] is a good reference for basic facts about amenability; all uncited results about group amenability may be found there. The class of groups that are most easily seen to be amenable are the compact groups, since if G is compact then $\mathrm{L}^\infty(G) \subset \mathrm{L}^1(G)$, so that normalized Haar measure induces a translation-invariant mean. All abelian groups are also amenable; this follows from Leptin's Theorem (Theorem 2.5.2, (4), below), since for amenable groups, $\mathrm{A}(G) \cong \mathrm{L}^1(\hat{G})$ has a bounded approximate identity (this is by no means the most direct proof of this fact). For $n \geq 2$, the free group $\mathbb{F}_n$ on n generators is not amenable. Amenability of groups has nice functorial properties, among them:

Proposition 2.5.1. *Let G be an amenable group.*

(1) If H is also amenable then so is $G \times H$.

(2) If $H < G$ is closed then H is amenable.

(3) If $N \lhd G$ is closed then G/N is amenable.

A few relevant, well-known characterizations of amenability are as follows:

Theorem 2.5.2. *Let G be a locally compact group. TFAE:*

(1) G is amenable.

(2) (Følner Condition [11])
For every $\epsilon > 0$ and every compact set $K \subset G$, there is a Borel set $E \subset G$ such that $0 < m(E) < \infty$ and for all $x \in K$,

$$m(E \triangle xE) < m(E)\epsilon.$$

(3) (Hulanicki's Theorem [18])
$\mathrm{C}^*(G) = \mathrm{C}^*_r(G)$, *via* $\lambda|_{\mathrm{C}^*(G)}$.

(4) (Leptin's Theorem [23])
$\mathrm{A}(G)$ *has a bounded approximate identity.*

Amenability for Banach algebras and a more refined notion of amenability for completely contractive Banach algebras will be introduced next. Banach algebra amenability was introduced in [21], as a special cohomological property. Operator amenability was introduced substantially later, in [28]; the preliminary concepts, basic results, and even their proofs parallel those of classical amenability. Indeed, Proposition 2.5.4 demonstrates that Banach algebra amenability may be viewed as a special case of operator amenability, and so some classical amenability results will be proven here as special cases of operator amenability results. The two concepts will be introduced simultaneously.

In this section, when $\mathcal{A}$ is a Banach algebra, let $\Delta : \mathcal{A} \otimes^\gamma \mathcal{A} \mapsto \mathcal{A}$ be the continuous linear extension of the multiplication map on $\mathcal{A} \times \mathcal{A}$. Similarly, when $\mathcal{A}$ is a completely contractive Banach algebra, as defined in Section 2.3, let $\Delta : \mathcal{A}\hat{\otimes}\mathcal{A} \mapsto \mathcal{A}$ be the continuous linear extension of the multiplication map.

For a Banach algebra $\mathcal{A}$, a **Banach $\mathcal{A}$-bimodule** is a Banach space $\mathcal{X}$ that is an $\mathcal{A}$-bimodule such that the module action maps $(a, x) \mapsto a.x$, $(x, a) \mapsto x.a$ are each jointly bounded, so that they extend to bounded maps $\mathcal{A} \otimes^\gamma \mathcal{X} \mapsto \mathcal{X}$ and $\mathcal{X} \otimes^\gamma \mathcal{A} \mapsto \mathcal{X}$ respectively. When $\mathcal{A}$ is a completely contractive Banach algebra, $\mathcal{X}$ is an **operator $\mathcal{A}$-bimodule** if these left and right module action maps extend to completely bounded maps $\mathcal{A}\hat{\otimes}\mathcal{X} \mapsto \mathcal{X}$ and $\mathcal{X}\hat{\otimes}\mathcal{A} \mapsto \mathcal{X}$ respectively.

If $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule, then so is $\mathcal{X}^*$, where for $f \in \mathcal{X}^*, a \in \mathcal{A}$, $a.f, f.a \in \mathcal{X}^*$ are defined by, for $x \in \mathcal{X}$,

$$\langle x, a.f \rangle = \langle x.a, f \rangle \text{ and } \langle x, f.a \rangle = \langle a.x, f \rangle .$$

Likewise, if $\mathcal{X}$ is an operator $\mathcal{A}$-bimodule, then this module structure makes $\mathcal{X}^*$ an operator $\mathcal{A}$-bimodule, using the dual operator space structure (the proof of this, while straightforward, is more involved than the Banach space case).

For a Banach $\mathcal{A}$-bimodule $\mathcal{X}$, a map $\delta : \mathcal{A} \mapsto \mathcal{X}$ is a **derivation** if for $a, b \in \mathcal{A}$,

$$\delta(ab) = a.\delta(b) + \delta(a).b.$$

The derivation δ is **inner** if there is some $x \in \mathcal{X}$ such that $\delta = \delta_x$, where δ_x is given for $a \in \mathcal{A}$ by

$$\delta_x(a) = a.x - x.a.$$

Note that inner derivations are automatically bounded and, in fact, that when $\mathcal{X}$ is an operator $\mathcal{A}$-bimodule they are completely bounded. $\mathcal{A}$ is **amenable** if for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$, every bounded derivation of $\mathcal{A}$ into $\mathcal{X}^*$ is inner. If $\mathcal{A}$ is a completely contractive Banach algebra, $\mathcal{A}$ is **operator amenable** if for every operator $\mathcal{A}$-bimodule $\mathcal{X}$, every completely bounded derivation of $\mathcal{A}$ into $\mathcal{X}^*$ is inner.

Amenability of Banach algebras may seem, at first glance, to be a completely unrelated concept to group amenability. Johnson's Theorem demonstrates a connection between the two notions of amenability given thus far:

Theorem 2.5.3. *(Johnson's Theorem, [21])*
A locally compact group G is amenable if and only if the group algebra $\mathrm{L}^1(G)$ is amenable.

The connection between these two notions of amenability and operator amenability, at this point, is simply that the definition of operator amenability is given by adding appropriate operator space overtones to the definition of Banach algebra amenability.

Suppose that a completely contractive Banach algebra $\mathcal{A}$ is amenable, as a Banach algebra. Every operator $\mathcal{A}$-bimodule is automatically a Banach $\mathcal{A}$-bimodule, and every completely bounded derivation is automatically bounded. Thus, it is clear that $\mathcal{A}$ is operator amenable. So operator amenability is a weaker condition than plain Banach algebra amenability. To realize classical amenability in terms of operator amenability, the MAX operator space structure is needed:

Proposition 2.5.4. *Let $\mathcal{A}$ be a Banach algebra. Then:*

(1) $\mathrm{MAX}(\mathcal{A})$ *is a completely contractive Banach algebra.*

(2) $\mathcal{A}$ *is amenable if and only if* $\mathrm{MAX}(\mathcal{A})$ *is operator amenable.*

Proof.

(1)

$$\begin{aligned}\Delta &\in b_1\left(\mathcal{B}(\mathcal{A}\otimes^{\gamma}\mathcal{A},\mathcal{A})\right)\\ &= b_1\left(\mathcal{CB}\left(\mathrm{MAX}(\mathcal{A}\otimes^{\gamma}\mathcal{A}),\mathrm{MAX}(\mathcal{A})\right)\right), \text{ by Proposition 2.3.5}\\ &= b_1\left(\mathcal{CB}\left(\mathrm{MAX}(\mathcal{A})\hat{\otimes}\mathrm{MAX}(\mathcal{A}),\mathrm{MAX}(\mathcal{A})\right)\right), \text{ by Proposition 2.3.7.}\end{aligned}$$

Thus, $\mathrm{MAX}(\mathcal{A})$ is a completely contractive Banach algebra.

(2) By the comments above, if $\mathcal{A}$ is amenable then $\mathrm{MAX}(\mathcal{A})$ is operator amenable. Conversely, suppose that $\mathrm{MAX}(\mathcal{A})$ is operator amenable. Let $\mathcal{X}$ be a $\mathcal{A}$-bimodule and $\delta : \mathcal{A} \mapsto \mathcal{X}^*$ a bounded derivation. Then, by a similar computation as in (1), $\mathrm{MAX}(\mathcal{X})$ is an operator $\mathrm{MAX}(\mathcal{A})$-bimodule. Moreover,

$$\begin{aligned}\delta &\in \mathcal{B}(\mathcal{A},\mathcal{X}^*)\\ &= \mathcal{CB}\left(\mathrm{MAX}(\mathcal{A}),\mathrm{MIN}(\mathcal{X}^*)\right), \text{ by Proposition 2.3.5}\\ &= \mathcal{CB}\left(\mathrm{MAX}(\mathcal{A}),\mathrm{MAX}(\mathcal{X})^*\right), \text{ by Proposition 2.3.6 (1).}\end{aligned}$$

So δ is a completely bounded derivation, and by the operator amenability of $\mathrm{MAX}(\mathcal{A})$, it must be inner.

□

Since the natural operator space structure on $\mathrm{L}^1(G)$ is the MAX structure, by (2.3.3), $\mathrm{L}^1(G)$ is operator amenable if and only if it is amenable. So, Johnson's Theorem (Theorem 2.5.3) may be restated:

Corollary 2.5.5. *G is amenable if and only if $\mathrm{L}^1(G)$ is operator amenable.*

Proposition 2.5.6. *If $\mathcal{A}$ and $\mathcal{B}$ are amenable Banach algebras or operator amenable completely contractive Banach algebras, then so respectively is $\mathcal{A} \oplus \mathcal{B}$.*

Proof. In light of the last theorem, it suffices to show only that if $\mathcal{A}$ and $\mathcal{B}$ are operator amenable then so is $\mathcal{A} \oplus \mathcal{B}$. Let $\mathcal{X}$ be an operator $(\mathcal{A} \oplus \mathcal{B})$-bimodule and $\delta : \mathcal{A} \oplus \mathcal{B} \mapsto \mathcal{X}^*$ a derivation. Then $\mathcal{X}$ is both an operator $\mathcal{A}$- and a $\mathcal{B}$-bimodule, and $\delta|_{\mathcal{A}}, \delta|_{\mathcal{B}}$ are derivations. Since $\mathcal{A}$ and $\mathcal{B}$ are operator amenable, let $x, y \in \mathcal{X}$ be such that $\delta|_{\mathcal{A}} = \delta_x$ and $\delta|_{\mathcal{B}} = \delta_y$. Then $\delta = \delta_{(x,y)}$ is inner, as required. □

The remaining proofs in this section are adapted from the arguments of Runde in [29], Section 2.2. The proofs found there concern only Banach algebra amenability. Some modifications and additional checks are needed to make them work for operator amenability, particularly in the proof of Proposition 2.5.8. In [28], Ruan does note that these results from Banach algebra amenability may be lifted to operator amenability, although he does not provide the adapted arguments.

Theorem 2.5.7. *Let $\mathcal{A}$ be a Banach algebra or completely contractive Banach algebra that is amenable or operator amenable respectively. Then $\mathcal{A}$ has a bounded approximate identity.*

Proof. Once again, it suffices to show the statement only when $\mathcal{A}$ is operator amenable. View $\mathcal{A}$ as a bimodule over itself, with usual left multiplication and trivial right multiplication. It is clear that this makes $\mathcal{A}$ an operator bimodule.

Let $\iota : \mathcal{A} \mapsto \mathcal{A}^{**}$ be the injection map, a complete isometry. For $a, b \in \mathcal{A}$, it is clear that the second dual module action of $\mathcal{A}$ on $\mathcal{A}^{**}$ produces $a.\iota(b) = \iota(ab)$. Since $\iota(a).b = 0$, ι is a derivation.

So by operator amenability, let $E \in \mathcal{A}^{**}$ be such that $\iota = \delta_E$. By Goldstine's Theorem, let $(e_\alpha) \subset \mathcal{A}$ be a net such that $\iota(e_\alpha) \to E$ weak*. So, for $a \in \mathcal{A}, f \in \mathcal{A}^*$,

$$\begin{aligned}
\langle ae_\alpha, f\rangle_{\mathcal{A}} &= \langle e_\alpha, f.a\rangle_{\mathcal{A}} \\
&= \langle f.a, \iota(e_\alpha)\rangle_{\mathcal{A}^*} \\
&\to \langle f.a, E\rangle_{\mathcal{A}^*} \\
&= \langle f, a.E - E.a\rangle_{\mathcal{A}^*}, \text{ since } \mathcal{A} \text{ acts trivially on the right} \\
&= \langle f, \iota(a)\rangle_{\mathcal{A}^*} \\
&= \langle a, f\rangle_{\mathcal{A}}.
\end{aligned}$$

That is, ae_α converges weakly to a.

Therefore, for any $a_1, \ldots, a_n$,

$$(a_1e_\alpha, \ldots, a_ne_\alpha) \to (a_1, \ldots, a_n).$$

weakly in ℓ^∞-$\oplus_{i=1}^n\mathcal{A}$. It is a well-known result of functional analysis that the weak closure of a convex set coincides with its norm closure, so that for every $\epsilon > 0$, there exists $f_{\{a_1,\ldots a_n\},\epsilon} = f \in \operatorname{conv}\{e_\alpha\}$ such that for $i = 1, \ldots, n$,

$$\|a_if - a_i\| < \epsilon.$$

To make this a net, $I = \{(S,\epsilon) : S \subset \mathcal{A} \text{ is finite}, \epsilon > 0\}$ becomes a directed set by $(S,\epsilon) \leq (S',\epsilon') \iff S \subset S'$ and $\epsilon \geq \epsilon'$ This shows that there is a net $(f_\beta)_{\beta\in I} \subset \operatorname{conv}\{e_\alpha\}$ that is a right bounded approximate identity.

By symmetry, $\mathcal{A}$ has a left bounded approximate identity (g_γ), so in fact $(f_\beta + g_\gamma - f_\beta g_\gamma)_{(\beta,\gamma)}$ is a two-sided bounded approximate identity. □

An $\mathcal{A}$-bimodule $\mathcal{X}$ is **neo-unital** if $\mathcal{A}.\mathcal{X}.\mathcal{A} = \mathcal{X}$.

Proposition 2.5.8. *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity and let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. Define*

$$\mathcal{X}_1 = \{a.x.b : a, b \in \mathcal{A}, x \in \mathcal{X}\}.$$

Then $\mathcal{X}_1$ is a neo-unital submodule of $\mathcal{X}$ and every bounded derivation of $\mathcal{X}^*$ is inner if and only if every bounded derivation of $\mathcal{X}_1^*$ is inner.

If $\mathcal{A}$ is a completely contractive Banach algebra and $\mathcal{X}$ is an operator bimodule, then every completely bounded derivation of $\mathcal{X}^*$ is inner if and only if every completely bounded derivation of $\mathcal{X}_1^*$ is inner.

Proof. Again, we need only consider the operator space version. Let (e_α) be a bounded approximate identity. Let $\theta : \mathcal{A} \mapsto \mathcal{CB}(\mathcal{X}^*, \mathcal{X}^*)$ be given for $a \in \mathcal{A}, x \in \mathcal{X}, f \in \mathcal{X}^*$ by

$$\langle x, \theta(a)(f)\rangle = \langle x, f.a\rangle .$$

θ is bounded and $\mathcal{CB}(\mathcal{X}^*, \mathcal{X}^*) \cong (\mathcal{X}\hat{\otimes}\mathcal{X}^*)^*$, so by Banach-Alaoglu's Theorem, WLOG, let $(\theta(e_\alpha))$ converge weak* to $E \in \mathcal{CB}(\mathcal{X}^*, \mathcal{X}^*)$. Thus, for $x \in \mathcal{X}, f \in \mathcal{X}^*$,

$$\langle x, Ef\rangle = \lim \langle x, f.e_\alpha\rangle .$$

Let

$$\begin{aligned} \mathcal{X}_2 &= \{a.x : a \in \mathcal{A}, x \in \mathcal{X}\} \\ &= \{x \in \mathcal{X} : e_\alpha.x \to x\}, \text{ by Cohen's Factorization Theorem.} \end{aligned}$$

By the second line, $\mathcal{X}_2$ is a closed submodule of $\mathcal{X}$. For $f \in \mathcal{X}^*, x \in \mathcal{X}_2$

$$\begin{aligned} \langle x, f\rangle &= \lim \langle e_\alpha.x, f\rangle \\ &= \langle x, Ef\rangle , \end{aligned}$$

so $I - E$ has range in $\mathcal{X}_2^\perp$. Moreover, for $f \in \mathcal{X}_2^\perp, x \in \mathcal{X}$,

$$\begin{aligned} \langle x, f\rangle &= \lim \langle (1 - e_\alpha).x, f\rangle , \text{ since } f \in \mathcal{X}_2^\perp \\ &= \langle x, (I - E)f\rangle . \end{aligned}$$

Thus, $I - E$ is the projection of $\mathcal{X}^*$ onto $\mathcal{X}_2^\perp$. So $\mathcal{X}^*$ can be decomposed

$$\mathcal{X}^* = \mathcal{X}_2^\perp \oplus \mathcal{Y},$$

where $\mathcal{Y} = E\mathcal{X}^*$. Note that if $f, g \in \mathcal{X}^*$ are such that $f|_{\mathcal{X}_2} = g|_{\mathcal{X}_2}$ then $Ef = Eg$. Thus, $\mathcal{Y} \cong \mathcal{X}_2^*$ via the restriction map

$$j : \mathcal{X}^* \mapsto \mathcal{X}_2^* : f \mapsto f|_{\mathcal{X}_2}.$$

The inverse map, denoted by ψ, takes $f \in \mathcal{X}_2^*$ to $E\tilde{f}$, where $\tilde{f}$ is a Hahn-Banach extension of f – that is, $\tilde{f}|_{\mathcal{X}_2} = f$.

For $f \in \mathcal{M}_n(\mathcal{X}^*)$,

$$\begin{aligned}\|j^{(n)}f\|_n &= \sup\{\|\langle\langle x, f\rangle\rangle\|_{nm} : x \in b_1(\mathcal{M}_m(\mathcal{X}_2))\}\\ &\leq \sup\{\|\langle\langle x, f\rangle\rangle\|_{nm} : x \in b_1(\mathcal{M}_m(\mathcal{X}))\}\\ &= \|f\|_n.\end{aligned}$$

This shows that j is a complete contraction from $\mathcal{Y}$ to $\mathcal{X}_2^*$. ψ is also completely bounded. To see this, note that for $f \in \mathcal{M}_n(\mathcal{X}_2^*) \cong \mathcal{CB}(\mathcal{X}_2, \mathcal{M}_n)$, by Wittstock's Extension Theorem (Theorem 2.3.2), there exists $\tilde{f} \in \mathcal{CB}(\mathcal{X}, \mathcal{M}_n)$ such that $\|\tilde{f}\| = \|f\|$ and $\tilde{f}|_{\mathcal{X}_2} = f$. So $\psi^{(n)}(f) = E^{(n)}\tilde{f}$, and

$$\|\psi^{(n)}(f)\| \leq \|E\|_{cb}\|\tilde{f}\| = \|E\|_{cb}\|f\|.$$

It is routine to verify from its definition that E is a module map. ψ is also a module map, since for $a \in \mathcal{A}, f \in \mathcal{X}_2^*, x \in \mathcal{X}$,

$$\begin{aligned}\langle x, \psi(a.f)\rangle &= \lim\langle e_\alpha.x, a.f\rangle\text{ , since each } e_\alpha.x \in \mathcal{X}_2\\ &= \lim\langle e_\alpha.x.a, f\rangle\\ &= \langle x.a, \psi(f)\rangle\\ &= \lim\langle e_\alpha.x, a.\psi(f)\rangle\text{ ,}\end{aligned}$$

and

$$\begin{aligned}\langle x, \psi(f.a)\rangle &= \lim\langle e_\alpha.x, f.a\rangle\text{ , since each } e_\alpha.x \in \mathcal{X}_2\\ &= \lim\langle ae_\alpha.x, f\rangle\\ &= \langle a.x, f\rangle\\ &= \lim\langle e_\alpha a.x, f\rangle\\ &= \langle a.x, \psi(f)\rangle\\ &= \langle x, \psi(f).a\rangle\text{ .}\end{aligned}$$

Now, suppose first that every completely bounded derivation into $\mathcal{X}_2^*$ is inner, and let $\delta : \mathcal{A} \mapsto \mathcal{X}^*$ be a completely bounded derivation. Since j is a completely bounded module map, $j \circ \delta$ is a completely bounded derivation of $\mathcal{A}$ into $\mathcal{X}_2^*$, so by the supposition, let $f \in \mathcal{X}_2^*$ be such that $j \circ \delta = \delta_f$. Consider $\delta' = \delta - \delta_{\psi(f)}$, which is the difference of derivations and thus is itself one. For $a \in \mathcal{A}, x \in \mathcal{X}_2$,

$$\begin{aligned} \langle x, \delta'(a) \rangle &= \langle x, \delta(a) - a.\psi(f) + \psi(f).a \rangle \\ &= \langle x, j \circ \delta(a) - \delta_f(a) \rangle \text{, since } x \in \mathcal{X}_2 \\ &= 0. \end{aligned}$$

Thus, the range of δ' is in $\mathcal{X}_2^\perp$, so for $a \in \mathcal{A}$,

$$\begin{aligned} \delta'(a) &= \lim_\alpha \delta'(ae_\alpha) \\ &= \lim_\alpha a.\delta'(e_\alpha) + \delta'(a).e_\alpha \\ &= \lim_\alpha a.\delta'(e_\alpha) \text{, since } \delta'(a) \in (1-E)\mathcal{X}^*. \end{aligned}$$

Let g be a weak* cluster point of $\delta'(e_\alpha)$ in $\mathcal{X}^*$. Since $\mathcal{X}_2^\perp$ is clearly weak*-closed, in fact $g \in \mathcal{X}_2^\perp$. For $x \in \mathcal{X}^*$,

$$\begin{aligned} \langle x, \delta'(a) \rangle &= \lim_\alpha \langle x, a.\delta'(e_\alpha) \rangle \\ &= \lim_\alpha \langle x.a, \delta'(e_\alpha) \rangle \\ &= \langle x.a, g \rangle \\ &= \langle x, a.g - g.a \rangle \text{, since } g.a = 0 \text{, because } g \in \mathcal{X}_2^\perp \\ &= \langle x, \delta_g(a) \rangle . \end{aligned}$$

So $\delta' = \delta_g$, and thus, $\delta = \delta_{\psi(f)} + \delta_g = \delta_{\psi(f)+g}$ is inner.

Conversely, suppose that every completely bounded derivation into $\mathcal{X}^*$ is inner, and let $\delta : \mathcal{A} \mapsto \mathcal{X}_2^*$ be a derivation. Since ψ is a completely bounded module map,
$\psi \circ \delta : \mathcal{A} \mapsto \mathcal{Y} \subset \mathcal{X}^*$ is a completely bounded derivation, so let $f \in \mathcal{X}^*$ be such that $\psi \circ \delta = \delta_f$. Then

$$\delta = j \circ \psi \circ \delta = j \circ \delta_f = \delta_{j(f)}$$

is inner.

Thus, every completely bounded derivation into $\mathcal{X}^*$ is inner if and only if every completely bounded derivation into $\mathcal{X}_2^*$ is inner. The same argument, but defining the operator

E using the left module action instead of the right one, shows that $\mathcal{X}_1$ is a submodule of $\mathcal{X}_2$ and that every completely bounded derivation into $\mathcal{X}_2^*$ is inner if and only if every completely bounded derivation into $\mathcal{X}_1^*$ is inner. Finally, it is clear from its definition that $\mathcal{X}_1$ is neo-unital. $\square$

There is a natural way to make $\mathcal{A} \otimes^\gamma \mathcal{A}$ and $\mathcal{A}\hat{\otimes}\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule and an operator $\mathcal{A}$-bimodule respectively. It is given in both cases by extending linearly and continuously the maps given for $a, b, c \in \mathcal{A}$ by

$$a.(b \otimes c) = (ab) \otimes c \text{ and } (a \otimes b).c = a \otimes (bc).$$

When $\mathcal{A}$ is a Banach algebra, a **bounded approximate diagonal** in $\mathcal{A} \otimes^\gamma \mathcal{A}$ is a bounded net $(d_\alpha) \in \mathcal{A} \otimes^\gamma \mathcal{A}$ such that:

(AD1): $\|a.x_\alpha - x_\alpha.a\| \to 0$ for all $a \in \mathcal{A}$, and

(AD2): $\Delta(x_\alpha)a \to a$ for all $a \in \mathcal{A}$
(that is, $(\Delta(x_\alpha))$ is a left approximate identity in $\mathcal{A}$).

For a completely contractive Banach algebra $\mathcal{A}$, a **bounded approximate diagonal** in $\mathcal{A}\hat{\otimes}\mathcal{A}$ is a bounded net $(d_\alpha) \in \mathcal{A}\hat{\otimes}\mathcal{A}$ satisfying (AD1) and (AD2).

Similarly, a **virtual diagonal** in $(\mathcal{A} \otimes^\gamma \mathcal{A})^{**}$ is an element $D \in (\mathcal{A} \otimes^\gamma \mathcal{A})^{**}$ such that

(VD1): $a.D - D.a = 0$ for all $a \in \mathcal{A}$, and

(VD2): $\Delta^{**}(D)a = a$ for all $a \in \mathcal{A}$.

For a completely contractive Banach algebra $\mathcal{A}$, an **virtual diagonal** in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ is an element $D \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ satisfying (VD1) and (VD2).

The following important theorem relates amenability and operator amenability to the existence of a bounded approximate diagonal and of a virtual diagonal.

Theorem 2.5.9.

- *For a Banach algebra $\mathcal{A}$, TFAE:*

  (1) $\mathcal{A}$ is amenable

(2) $\mathcal{A} \otimes^{\gamma} \mathcal{A}$ *has a bounded approximate diagonal*

(3) $(\mathcal{A} \otimes^{\gamma} \mathcal{A})^{**}$ *has a virtual diagonal*

- *For a completely contractive Banach algebra* $\mathcal{A}$*, TFAE:*

  (1) $\mathcal{A}$ *is operator amenable*

  (2) $\mathcal{A}\hat{\otimes}\mathcal{A}$ *has a bounded approximate diagonal*

  (3) $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ *has a virtual diagonal*

Proof. Since $\mathrm{MAX}(\mathcal{A})\hat{\otimes}\mathrm{MAX}(\mathcal{A}) = \mathcal{A} \otimes^{\gamma} \mathcal{A}$ and by Proposition 2.5.4, it suffices to show only the operator space version.

(1) ⇒ (3): Let $\mathcal{A}$ be operator amenable. Then let $(e_\alpha) \subset \mathcal{A}$ be a bounded approximate identity and by Banach-Alaoglu, WLOG by taking a subnet if necessary, let I be the weak* limit of $(e_\alpha \otimes e_\alpha)$ in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$. For $a \in \mathcal{A}$

$$\Delta^{**}(\delta_I(a)) = \lim_\alpha ae_\alpha^2 - e_\alpha^2 a = 0.$$

Thus, δ_I is a derivation from $\mathcal{A}$ into $\ker \Delta^{**}$.

Let

$$\mathcal{X} = (\mathcal{A}\hat{\otimes}\mathcal{A})^*/\overline{\mathrm{Im}\Delta^*},$$

so $\mathcal{X}^* \cong \ker \Delta^{**}$ completely isometrically and such that the module action of $\mathcal{A}$ on $\ker \Delta^{**}$ agrees with the dual module action on $\mathcal{X}^*$, given by taking the quotient of the dual module action on $(\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Hence, $\ker \Delta^{**}$ is a dual operator $\mathcal{A}$-bimodule. By the operator amenability of $\mathcal{A}$, let $J \in \ker \Delta^{**}$ be such that $\delta_I = \delta_J$. Let $D = I - J \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$. Then for $a \in \mathcal{A}$,

$$\begin{aligned} D.a - a.D &= \delta_I(a) - \delta_J(a) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \Delta^{**}(D)a &= \Delta^{**}(I)a, \text{ since } J \in \ker \Delta^{**} \\ &= \lim_\alpha e_\alpha^2 a \\ &= a, \end{aligned}$$

whence D is a virtual diagonal.

(3) $\Rightarrow$ (2): Let $D \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ be a virtual diagonal, and let $(d_\alpha) \in \mathcal{A}\hat{\otimes}\mathcal{A}$ be a bounded net converging weak* in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ to D. Then for $a \in \mathcal{A}$,

$$\begin{aligned} a.d_\alpha &\to a.D \\ d_\alpha.a &\to D.a, \end{aligned}$$

and

$$\Delta(d_\alpha)a \to \Delta^{**}(D)a,$$

where all of these converge weakly, uniformly on finite subsets of $\mathcal{A}$. As in the proof of Theorem 2.5.7, using the fact that the weak closure of a convex set is the same as its norm closure, it follows that $\operatorname{conv}\{d_\alpha\}$ contains an approximate diagonal.

(2) $\Rightarrow$ (1): Suppose that $(d_\alpha) \subset \mathcal{A}\hat{\otimes}\mathcal{A}$ is an approximate diagonal. WLOG, let $D \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ be the weak* limit of (d_α).

$\Delta(d_\alpha)$ is a bounded left approximate identity; moreover, for $a \in \mathcal{A}$,

$$\begin{aligned} \|a\Delta(d_\alpha) - a\| &\leq \|a\Delta(d_\alpha) - \Delta(d_\alpha)a\| + \|\Delta(d_\alpha)a - a\| \\ &= \|\Delta(a.d_\alpha - d_\alpha.a)\| + \|\Delta(d_\alpha)a - a\| \\ &\to 0. \end{aligned}$$

Therefore, it is also a bounded right approximate identity. Thus, by Proposition 2.5.8, it suffices to show that for all neo-unital operator $\mathcal{A}$-bimodules $\mathcal{X}$, all completely bounded derivations $\delta : \mathcal{A} \mapsto \mathcal{X}^*$ are inner.

Let $\mathcal{X}$ be such a module and $\delta : \mathcal{A} \mapsto \mathcal{X}^*$ a completely bounded derivation. For $x \in \mathcal{X}$, define $A_x : \mathcal{A} \mapsto \mathcal{A}^*$ by, for $a, b \in \mathcal{A}$,

$$\langle a, A_x(b)\rangle = \langle x, a.\delta(b)\rangle .$$

By the complete boundedness of δ and of the module action $\mathcal{A}\hat{\otimes}\mathcal{X}^* \mapsto \mathcal{X}^*$, $A_x \in \mathcal{CB}(\mathcal{A}, \mathcal{A}^*) \cong (\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Let $f \in \mathcal{X}^*$ by

$$\begin{aligned} \langle x, f\rangle &= \langle A_x, D\rangle \\ &= \lim_\alpha \langle d_\alpha, A_x\rangle , \text{ by the definition of } D. \end{aligned}$$

It will be shown that $\delta = \delta_f$, and thus that δ is inner.

For $a, b, c \in \mathcal{A}, x \in \mathcal{X}$, via the identification $\mathcal{CB}(\mathcal{A}, \mathcal{A}^*) \cong (\mathcal{A}\hat{\otimes}\mathcal{A})^*$,

$$\begin{aligned}
\langle a \otimes b, A_{x.c-c.x} \rangle_{\mathcal{A}\hat{\otimes}\mathcal{A}} &= \langle a, A_{x.c-c.x}(b) \rangle_{\mathcal{A}} \\
&= \langle x.c - c.x, a.\delta(b) \rangle_{\mathcal{A}} \\
&= \langle x, ca.\delta(b) - a.\delta(b).c \rangle_{\mathcal{A}} \\
&= \langle x, ca.\delta(b) - a.\delta(bc) + ab.\delta(c) \rangle_{\mathcal{A}}, \text{since } \delta \text{ is a derivation} \\
&= \langle (ca) \otimes b - a \otimes (bc), A_x \rangle_{\mathcal{A}\hat{\otimes}\mathcal{A}} + \langle x, ab.\delta(c) \rangle_{\mathcal{A}}.
\end{aligned}$$

So by continuity, for $u \in \mathcal{A}\hat{\otimes}\mathcal{A}$,

$$\langle u, A_{x.c-c.x} \rangle_{\mathcal{A}\hat{\otimes}\mathcal{A}} = \langle c.u - u.c, A_x \rangle_{\mathcal{A}\hat{\otimes}\mathcal{A}} + \langle x, \Delta(u).\delta(c) \rangle_{\mathcal{A}}.$$

Hence for $x \in \mathcal{X}, a \in \mathcal{A}$,

$$\begin{aligned}
\langle x, \delta_f(a) \rangle &= \langle x, a.f - f.a \rangle \\
&= \langle x.a - a.x, f \rangle \\
&= \lim_{\alpha} \langle d_\alpha, A_{x.a-a.x} \rangle \\
&= \lim_{\alpha} \langle a.d_\alpha - d_\alpha.a, A_x \rangle + \langle x, \Delta(d_\alpha).\delta(a) \rangle \\
&= \lim_{\alpha} \langle x.\Delta(d_\alpha), \delta(a) \rangle, \text{ by (AD1)} \\
&= \langle x, \delta(a) \rangle.
\end{aligned}$$

This last line holds since $\mathcal{X}$ is neo-unital and $(\Delta(d_\alpha))$ is an approximate identity. Thus, $\delta = \delta_f$, as required.

□

Chapter 3

Amenability and Non-Amenability of $A(G)$

When G is abelian, $\mathrm{A}(G)$ is isometrically isomorphic to $\mathrm{L}^1(\hat{G})$ via the Fourier transform, where $\hat{G}$ is the dual group to G. For non-abelian G, $\mathrm{A}(G)$ is intuitively thought of as an L^1 space on some dual object $\hat{G}$, although in many cases, it is incoherent to regard $\hat{G}$ as an actual object.

This intuition may be made concrete when G is compact: $\hat{G}$ can be taken to be the space of all irreducible representations of G, each of which is finite-dimensional. Then $\mathrm{L}^2(G)$ is an ℓ^2-direct sum of finite dimensional spaces $(\mathbb{C}^{d_\pi})^{d_\pi}$, taken over all representations $\pi \in \hat{G}$, where d_π is the dimension of the representation π. Corresponding to this decomposition, $\mathrm{VN}(G)$ is an ℓ^∞-direct sum of matrix algebras $\mathcal{M}_{d_\pi}$, taken over all representations $\pi \in \hat{G}$; and $\mathcal{M}_{d_\pi}$ acts identically on each of the d_π copies of $\mathbb{C}_{d_\pi}$. The pre-dual $\mathrm{A}(G)$ is then an ℓ^1-direct sum of matrix spaces $\mathcal{M}_{d_\pi}$, but with a trace class norm on each space. In this sense, $\mathrm{A}(G)$ is a viewed as non-commutative ℓ^1 space on $\hat{G}$.

Johnson's Theorem states that the amenability of $\mathrm{L}^1(G)$ corresponds to the amenability of G. Although we cannot grasp the actual object $\hat{G}$ in general, we would speculate that it would have the property of being amenable if and only if G is; and thus, it is expected that $\mathrm{A}(G)$ is amenable if and only if G is.

Unfortunately, it turns out that this isn't the case. In [20], Johnson demonstrated for the first time that certain compact (and thus amenable) groups have non-amenable

Fourier algebras. Johnson's precise result is that for infinite compact groups G, if for each n, only finitely many irreducible representations of G have dimension n, then $\mathrm{A}(G)$ is not amenable; SO(3) is an example of such a group. On the other hand, in the same paper he showed that for compact groups G, if there is a bound on the dimension of the irreducible representations of G, then $\mathrm{A}(G)$ is amenable.

It turns out that this last condition is key to the amenability of $\mathrm{A}(G)$ in general. A group G is thus called **almost abelian** if the dimensions of the irreducible representations of G are uniformly bounded. Such groups were studied extensively by Moore in [25]. Among things, Moore demonstrated the following characterization of these groups, which motivates the name "almost abelian":

Theorem 3.0.1. *Moore's Theorem*
For a locally compact group G, G is almost abelian if and only if G has an abelian subgroup of finite index.

First, make the easy observation that the Fourier algebra of these groups is indeed amenable.

Proposition 3.0.2. *If G is almost abelian then* $\mathrm{A}(G)$ *is amenable [22].*

Proof. Let $H \leq G$ be an abelian subgroup such that $[G : H] = n < \infty$. Then $\hat{H}$ is an abelian, and thus amenable, group, so $\mathrm{A}(H) \cong \mathrm{L}^1(\hat{H})$ is amenable. Since H is open in G, $\mathrm{A}(H)$ embeds isometrically into $\mathrm{A}(G)$ by taking $u \in \mathrm{A}(H)$ to the function $\theta(u) : G \mapsto H$, given for $x \in G$ by

$$\theta(u)(x) = \begin{cases} u(x), & \text{if } x \in H \\ 0, & \text{if } h \notin H \end{cases}.$$

Abusing notation, use $\mathrm{A}(H)$ to denote its image in $\mathrm{A}(G)$.

Let $x_1H, \ldots, x_nH$ be all the cosets of H. For each $i = 1, \ldots, n$, $\chi_{x_iH} \in \mathrm{B}(G)$, so

$$\begin{aligned} \mathrm{A}(x_iH) &:= \{u \in \mathrm{A}(G) : \mathrm{supp} u \subset x_iH\} \\ &= \chi_{x_iH}\mathrm{A}(G) \\ &\subset \mathrm{A}(G). \end{aligned}$$

In fact, it is clear that $\mathrm{A}(x_iH)$ is a closed ideal in $\mathrm{A}(G)$.

Moreover, since $1 = \chi_{x_1H} + \cdots + \chi_{x_nH}$,

$$\mathrm{A}(G) = \oplus_{i=1}^{n} \mathrm{A}(x_iH).$$

However, left translation by x_i^{-1} is an isometry of $\mathrm{A}(G)$ that takes $\mathrm{A}(H)$ to $\mathrm{A}(x_iH)$ Thus, $\mathrm{A}(x_iH)$ is amenable for each i. It follows that $\mathrm{A}(G)$ is the direct sum of finitely many amenable algebras, and thus, by Proposition 2.5.6, it is itself amenable. □

The converse of this result will be proven in steps. Let

$$\Gamma = \left\{(x, x^{-1}) : x \in G\right\} \subset G \times G, \tag{3.0.1}$$

and for the locally compact group G, let G_d denote G with the discrete topology. The following is taken from [29], Lemma 3.1.

Proposition 3.0.3. *If* $\mathrm{A}(G)$ *is amenable then* $\chi_\Gamma \in \mathrm{B}(G_d \times G_d)$.

Proof. Let $(d_\alpha) \subset \mathrm{A}(G) \otimes^\gamma \mathrm{A}(G)$ be a bounded approximate diagonal. By implicitly mapping $\mathrm{A}(G) \otimes^\gamma \mathrm{A}(G)$ to $\mathrm{A}(G \times G)$ by the map $u \otimes v \mapsto u \times v$, (d_α) converges pointwise to χ_D, where $D = \{(x, x) : x \in G\} \subset G \times G$. Surely, for $x, y \in G$, if $x \neq y$ then let $u \in A(G)$ be such that $u(x) = 0, u(y) = 1$. Since $u.d_\alpha - d_\alpha.u \to 0$ in $\mathrm{A}(G) \otimes^\gamma \mathrm{A}(G)$, it converges pointwise. So

$$\begin{aligned} 0 &= \lim_\alpha (u.d_\alpha - d_\alpha.u)(x, y) \\ &= \lim_\alpha (u(x) - u(y))d_\alpha(x, y) \\ &= \lim_\alpha d_\alpha(x, y). \end{aligned}$$

Likewise, since $(\Delta(d_\alpha))$ is a bounded approximate identity, for all $x \in G$,

$$\begin{aligned} 1 &= \lim_\alpha \Delta(d_\alpha)(x) \\ &= \lim_\alpha d_\alpha(x, x) \end{aligned}$$

Now, let $\vee : \mathrm{A}(G) \mapsto \mathrm{A}(G)$ take u to $\check{u}$ as defined in (2.1.2); $\vee$ is an isometry and for $\xi, \eta \in \mathrm{L}^2(G)$, it takes $\xi *_\lambda \eta$ to $\bar{\eta} *_\lambda \bar{\xi}$. So

$$\mathrm{id} \otimes \vee : \mathrm{A}(G) \otimes^\gamma \mathrm{A}(G) \mapsto \mathrm{A}(G) \otimes^\gamma \mathrm{A}(G)$$

is a contraction. To be somewhat explicit, for $u \in \mathrm{A}(G) \otimes^\gamma \mathrm{A}(G)$, (and again considering this as a function on $G \times G$)

$$(\mathrm{id} \otimes \vee)u(x, y) = u(x, y^{-1}).$$

Since $\mathrm{id} \otimes \vee$ is bounded, $((\mathrm{id} \otimes \vee)d_\alpha)$ is a bounded net, and for $x, y \in G$,

$$\begin{aligned} ((\mathrm{id} \otimes \vee)d_\alpha)(x, y) &= d_\alpha(x, y^{-1}) \\ &\to \chi_D(x, y^{-1}) \\ &= \chi_\Gamma(x, y) \end{aligned}$$

[9], Corollaire (2.25) states that: if a net in $b_1(\mathrm{B}(H))$ converges pointwise to a continuous function u, then $u \in b_1(\mathrm{B}(H))$. In particular, when H is discrete then $\mathrm{B}(H)$ is closed under pointwise limits of bounded nets. Thus, $\chi_\Gamma \in \mathrm{B}(G_d \times G_d)$. □

3.1 The Coset Ring and Piecewise Affine Maps

The theory of piecewise affine maps ties into the last result through Host's Idempotent Theorem (Theorem 3.1.3). All uncited results in this section are taken from [19]. This section begins with a very easy characterization of cosets. The proof is so simple that it is left to the reader.

Lemma 3.1.1. *Let $C \subset G$. Then C is a coset of some subgroup of G if and only if for every $x, y, z \in C$, $xy^{-1}z \in C$.*

Motivated by this characterization is the following definition of an affine map. Let G, H be groups, $C \subset G$ a coset, and $\alpha : C \mapsto H$. α is **affine** if for all $x, y, z \in C$,

$$\alpha(xy^{-1}z) = \alpha(x)\alpha(y)^{-1}\alpha(z).$$

Affine maps may also be characterized easily, as in the following lemma, whose proof is also left to the reader.

Lemma 3.1.2. *Let G, H be groups, $G_0 \leq G$, and $x_0 \in G$ so that $C = x_0G_0$ is a coset in G. Let $\alpha : C \mapsto H$. Then α is affine if and only if there is a homomorphism $\phi : G_0 \mapsto H$ and an element $y_0 \in H$ such that for all $y \in C$,*

$$\alpha(y) = y_0\phi(x_0^{-1}y).$$

A **ring of subsets** of a set X is a collection of subsets of X that is closed under union and set difference. For a group G, the **coset ring** of G, denoted $\Omega(G)$, is the smallest ring of subsets of G which contains all cosets of subsets of G. The following result gives significance to this object:

Theorem 3.1.3. *Host's Idempotent Theorem [17]*
For a discrete group G, the idempotents in $\mathrm{B}(G)$ are precisely the functions χ_A, where $A \in \Omega(G)$.

(In fact, more generally, for non-discrete groups G, [17] shows that idempotents are the characteristic functions of sets in the ring generated by open cosets).

Each set $Y \in \Omega(G)$ can be written in the form

$$Y = \coprod_{i=1}^{n} \left(L_i \setminus \coprod_{j=1}^{m} M_{i,j} \right), \tag{3.1.1}$$

where each L_i and $M_{i,j}$ is a coset or $\emptyset$, $M_{i,j} \subset L_i$ for all i, j. To see this, using the fact that the intersection of cosets is itself a coset, simply note that the collection of sets of the form (3.1.1) does, in fact, form a ring.

The following group-theoretic lemma will be needed shortly:

Lemma 3.1.4. *[26] Let G be a group and $H_1, \ldots, H_n \leq G$, $x_1, \ldots, x_n \in G$ such that*

$$G = x_1H_1 \cup \cdots \cup x_nH_n.$$

Then for some i, $[G : H_i] < \infty$.

Proof. The statement will be proven in the following form, by induction on m: Let $K_1, \ldots, K_m$ be subgroups of G, and for $i = 1, \ldots, m, j = 1, \ldots, \ell$, let $y_{i,j} \in G$. If

$$G = \bigcup_{i=1}^{m} \bigcup_{j=1}^{\ell} y_{i,j}K_i$$

then $[G : K_i] < \infty$ for some i.

Clearly this statement is true when $m = 1$. Now, suppose that it holds for $m-1$ and that

$$G = \bigcup_{i=1}^{m} \bigcup_{j=1}^{\ell} y_{i,j} K_i.$$

Then either

$$G = \bigcup_{j=1}^{\ell} y_{m,j} K_m,$$

in which case $[G : K_m] < \infty$, or else for some $z \in G$,

$$\begin{aligned} zK_m &\subset G \setminus \bigcup_{j=1}^{\ell} y_{m,j} K_m \\ &\subset \bigcup_{i=1}^{m-1} \bigcup_{j=1}^{\ell} y_{i,j} K_i. \end{aligned}$$

In the second case, this gives:

$$\begin{aligned} G &= \left(\bigcup_{i=1}^{m-1} \bigcup_{j=1}^{\ell} y_{i,j} K_i \right) \cup \bigcup_{k=1}^{\ell} y_{m,k} K_m \\ &= \left(\bigcup_{i=1}^{m-1} \bigcup_{j=1}^{\ell} y_{i,j} K_i \right) \cup \bigcup_{k=1}^{\ell} y_{m,k} z^{-1} z K_m \\ &\subset \left(\bigcup_{i=1}^{m-1} \bigcup_{j=1}^{\ell} y_{i,j} K_i \right) \cup \bigcup_{k=1}^{\ell} \bigcup_{i=1}^{m-1} \bigcup_{j=1}^{\ell} y_{m,k} z^{-1} y_{i,j} K_i \\ &= \bigcup_{i=1}^{m-1} \bigcup_{j=1}^{\ell} \left(y_{i,j} K_i \cup \bigcup_{k=1}^{\ell} y_{m,k} z^{-1} y_{i,j} K_i \right). \end{aligned}$$

Therefore by induction, $[G : K_i] < \infty$ for some $i \le m-1$. $\square$

Let $Y \in \Omega(G)$ and $\alpha : Y \mapsto H$. α is **piecewise affine** if:

(1) There are sets $Y_1, \ldots, Y_n \in \Omega(G)$ such that

$$Y = \coprod_{i=1}^{n} Y_i, \text{ and}$$

(2) For each $i = 1, \ldots, n$, there is a coset $L_i \supset Y_i$ and an affine map $\alpha_i : L_i \mapsto H$ such that

$$\alpha|_{Y_i} = \alpha_i|_{Y_i}.$$

Piecewise affine maps may be characterized purely in terms of a coset ring, as follows:

Proposition 3.1.5. *Let $Y \subset G, \alpha : Y \mapsto H$. Then α is a piecewise affine map if and only if its graph,*

$$\Gamma_\alpha := \{(x, \alpha(x)) : x \in Y\}$$

is in $\Omega(G \times H)$

Proof. If α is piecewise affine, then by using the definition and (3.1.1), let

$$Y = \coprod_{i=1}^{n} Y_i,$$

where $Y_i = L_i \setminus \cup_{j=1}^{m} M_{i,j}$, with each L_i a coset and $M_{i,j}$ a coset or $\emptyset$, $M_{i,j} \subset L_i$ for all i, j, and

$$\alpha_i : L_i \mapsto H$$

an affine map such that $\alpha_i|_{Y_i} = \alpha|_{Y_i}$.

Then for each i, $\{(x, \alpha_i(x)) : x \in L_i\}$ is a coset of $G \times H$, as is each non-empty subset $\{(y, \alpha_i(y)) : y \in M_{i,j}\}$. So

$$\Gamma_\alpha = \coprod_{i=1}^{n} \left(\{(x, \alpha_i(x)) : x \in L_i\} \setminus \bigcup_{j=1}^{m} \{(y, \alpha_i(y)) : y \in M_{i,j}\} \right),$$

and this is in $\Omega(G \times H)$.

Conversely, suppose that $\Gamma_\alpha \in \Omega(G \times H)$. Fix a finite set Σ of subgroups of $G \times H$ such that Γ_α is in the ring generated by cosets of sets in Σ, and such that for any two subgroups $A, B \in \Sigma$, if $A < B$ then $[B : A] = \infty$. Then, let

$$\Gamma_\alpha = \coprod_{i=1}^{n} \left(L_i \setminus \bigcup_{j=1}^{m} M_{i,j} \right),$$

where each L_i is a coset of a set in Σ, each $M_{i,j}$ is $\emptyset$ or a coset of a set in Σ, and $M_{i,j}$ is a proper subset of L_i for all i, j. For each i, $E_i = L_i \setminus \cup_{j=1}^{m} M_{i,j}$ is a subset of Γ_α, and is thus a graph.

It will be shown that L_i is in fact a graph. Supposing otherwise, let $(s, t_1), (s, t_2) \in L_i$, where $t_1 \neq t_2$. Then $(e, t) = (e, t_1^{-1} t_2) \in L_i^{-1} L_i$. For each $(s, \alpha(s)) \in E_i \subset L_i$,

$$(s, \alpha(s)t) \in L_i L_i^{-1} L_i = L_i,$$

and since Γ_α is a graph, $(s, \alpha(s)t) \in M_{i,j}$ for some j. That is, $(s, \alpha(s)) \in M_{i,j}(e, t)^{-1}$.

Thus,

$$E_i \subset \bigcup_{j=1}^{m} M_{i,j}(e, t)^{-1},$$

and so

$$L_i \subset \bigcup_{j=1}^{m} \left(M_{i,j} \cup M_{i,j}(e, t)^{-1} \right),$$

So if $L_i = Kx$ where $K \in \Sigma$ then

$$K \subset \bigcup_{j=1}^{m} \left(M_{i,j} x^{-1} \cup M_{i,j}(e, t)^{-1} x^{-1} \right).$$

That is, K can be covered by finitely many cosets, each of which has infinite index in K. By the last lemma, this is impossible.

L_i is thus a graph of a function. Let

$$L_i = \{(x, \alpha_i(x)) : x \in L'_i\},$$

for some set L'_i. Since L_i is a coset, it is easy to see that L'_i is a coset in G and α_i is an affine map. For each i, j, $M'_{i,j} \subset L'_i$ can be taken such that

$$M_{i,j} = \left\{(x, \alpha_i(x)) : x \in M'_{i,j}\right\}.$$

Thus,

$$Y = \coprod_{i=1}^{n} \left(L'_i \setminus \bigcup_{j=1}^{m} M'_{i,j} \right),$$

and for each i, $\alpha|_{L'_i} = \alpha_i|_{L'_i}$, where α_i is affine. $\square$

Piecewise affine maps are significant in the context of the Fourier algebra, due to the following result:

Proposition 3.1.6. *Let G and H be discrete groups, and let $\alpha : Y \subset G \mapsto H$ be piecewise affine. Define the map $\theta_\alpha : \mathrm{A}(H) \mapsto \mathcal{C}(G)$ by: for $u \in \mathrm{A}(H), x \in G$,*

$$(\theta_\alpha(u))\,(x) = \begin{cases} u(\alpha(x)), & \text{if } x \in Y \\ 0, & \text{if } x \notin Y \end{cases}.$$

Then θ_α is a completely bounded map from $\mathrm{A}(H)$ into $\mathrm{B}(G)$.

Proof. This will be shown by considering progressively more complex cases. In the simplest case, $Y = G$ and α is a homomorphism. Then for $\xi, \eta \in \mathrm{L}^2(H)$,

$$\theta_\alpha(\xi *_{\lambda_H} \eta) = \xi *_{\lambda_H \circ \alpha} \eta \in \mathrm{B}(G),$$

since $\lambda_H \circ \alpha$ is a representation of G. As it is clear that θ_α is continuous, it follows that the range of θ_α is contained in $\mathrm{B}(G)$. Consider the adjoint $\theta_\alpha^* : \mathrm{VN}_\omega(G) \mapsto \mathrm{VN}(H)$; for $x \in G, u \in \mathrm{A}(H)$,

$$\begin{aligned} \langle u, \theta_\alpha^*(\omega(x))\rangle &= \langle \theta_\alpha(u), \omega(x)\rangle \\ &= (\theta_\alpha(u))\,(x) \\ &= u(\alpha(x)) \\ &= \langle u, \lambda_H\,(\alpha(x))\rangle\,. \end{aligned} \tag{3.1.2}$$

Thus, $\theta_\alpha^*\,(\omega(x)) = \lambda\,(\alpha(x))$, and θ_α^* is a $*$-homomorphism on span $(\omega(G))$. Since this $*$-subalgebra is weak*-dense in $\mathrm{VN}_\omega(G)$, θ_α^* is in fact a $*$-homomorphism on all of $\mathrm{VN}_\omega(G)$, and is thus completely bounded.

Next, allow for Y to be a subgroup of G, but keep α a homomorphism. In this case, θ_α will be viewed as a composition of maps. $\mathrm{B}(Y)$ embeds into $\mathrm{B}(G)$ via the map ι, where for $u \in \mathrm{B}(Y), x \in G$,

$$(\iota(u))\,(x) = \begin{cases} u(x), & \text{if } x \in Y \\ 0, & \text{if } x \notin Y \end{cases}.$$

Let $m_Y : \mathrm{B}(G) \mapsto \mathrm{B}(G)$ be multiplication by χ_Y, so that the range of ι is precisely the range of m_Y.

Viewing $m_Y\mathrm{B}(G)$ as an operator space via its embedding into $\mathrm{B}(G)$ makes m_Y a complete isometry from $m_Y\mathrm{B}(G)$ into $\mathrm{B}(G)$. So $m_Y^* : \mathrm{VN}_\omega(G) \mapsto (m_Y\mathrm{B}(G))^*$ is a complete quotient map, and $(m_Y\mathrm{B}(G))^* = m_Y^*\mathrm{VN}_\omega(G)$.

Now $m_Y^*\mathrm{VN}_\omega(G)$ is, in fact, the weak*-closed span of $\{\omega_G(y) : y \in Y\}$. To see this, note that if $T \in \mathrm{VN}_\omega(G)$ is such that $m_Y^*T \notin \overline{\mathrm{span}\,\{\omega_G(y) : y \in Y\}}^{w*}$ then by the Hahn-Banach Theorem, let $u \in \mathrm{B}(G)$ be such that

$$u|_Y = 0 \quad \text{and} \quad \langle u, m_Y^*T\rangle \neq 0.$$

The second condition says that $0 \neq \langle m_Y(u), T\rangle = \langle \chi_Y u, T\rangle$, whereas $\chi_Y u = 0$, a contradiction.

By the same sort of calculation as done in (3.1.2), it can be seen that for $y \in Y$,

$$\iota^*\left(\omega_G(y)\right) = \omega_Y(y).$$

and thus, that $\iota^* : \overline{\mathrm{span}\,\{\omega_G(y) : y \in Y\}}^{w*} \mapsto \mathrm{VN}_\omega(Y)$ is a $*$-homomorphism and hence is completely bounded. So $\theta_\alpha = \iota \circ \theta_{\alpha|_Y}$ is completely bounded.

The next case to consider is that $Y \subset G$ is a coset and α is affine. Let $Y = y_0X$, where $X \leq G$, and let $\alpha(y) = x_0\phi(y_0^{-1}y)$ where $\phi : X \mapsto H$ is a homomorphism and $x_0 \in H$. Then for $u \in \mathrm{A}(H)$,

$$\theta_\alpha(u) = \delta_{x_0^{-1}} * \theta_\phi(\delta_{y_0} * u).$$

The adjoint of the map $u \mapsto \delta_{y_0} * u$ on $\mathrm{A}(H)$ is left multiplication in $\mathrm{VN}(H)$ by the unitary $\lambda(y_0^{-1})$, and it is easy to see that this multiplication map is completely bounded. Likewise, the adjoint of the map $u \mapsto \delta_{x_0^{-1}} * u$ on $\mathrm{B}(G)$ is left multiplication in $\mathrm{VN}_\omega(G)$ by the unitary $\omega(x_0)$. These adjoints are each completely bounded, so that θ_α is the composition of completely bounded maps, and is thus completely bounded.

Finally, consider the general case: let

$$Y = \coprod_{i=1}^{n} Y_i,$$

where

$$Y_i = L_i \setminus \coprod_{j=1}^{m_i} M_{i,j},$$

such that each L_i and $M_{i,j}$ is a coset, and $M_{i,j} \subset L_i$ for all i, j, and for each i, let

$$\alpha_i : L_i \mapsto H$$

be affine such that $\alpha|_{Y_i} = \alpha_i|_{Y_i}$. Then

$$\begin{aligned}\theta_\alpha &= \sum_{i=1}^{n} \theta_{\alpha|_{Y_i}} \\ &= \sum_{i=1}^{n} \left(\theta_{\alpha_i|_{L_i}} - \sum_{j=1}^{m} \theta_{\alpha_i|_{M_{i,j}}} \right).\end{aligned}$$

Thus, θ_α is in the linear span of $\{\theta_\beta : \beta \text{ is affine}\}$. Since the space of completely bounded maps form a vector space, θ_α is completely bounded, as required. □

Using this theory of piecewise affine maps, the following result is obtained:

Proposition 3.1.7. *If* $\mathrm{A}(G)$ *is amenable then the map* $\vee : \mathrm{A}(G_d) \mapsto \mathrm{A}(G_d) : u \mapsto \check{u}$ *is completely bounded.*

Proof. Let $\mathrm{A}(G)$ be amenable.

Let $\alpha : G \mapsto G$ be given by $\alpha(x) = x^{-1}$ for $x \in G$. The graph of α is Γ, as defined in (3.0.1), and since $\mathrm{A}(G)$ is amenable, $\chi_\Gamma \in \mathrm{B}(G_d \times G_d)$. Thus, by Host's Idempotent Theorem (Theorem 3.1.3), $\Gamma \in \Omega(G \times G)$.

Hence, α is piecewise affine, so $\theta_\alpha : \mathrm{A}(G_d) \mapsto \mathrm{B}(G_d)$ is completely bounded. But θ_α is precisely the map $u \mapsto \check{u}$, and its range is $\mathrm{A}(G_d)$. Since $\mathrm{A}(G_d)$ is completely isometrically contained in $\mathrm{B}(G_d)$, $\vee = \theta_\alpha$ is completely bounded as a map from $\mathrm{A}(G_d)$ to itself. □

3.2 Irreducible Representations of $\ell^1(G)$

The last steps in the proof that $\mathrm{A}(G)$ is amenable only if G is almost abelian involve drawing conclusions about the dimensions of the irreducible representations of $\ell^1(G)$. The argument here is from [13]. First, a strong statement can be made about irreducible representations of $\mathrm{VN}(G_d)$:

Proposition 3.2.1. *If* $\mathrm{A}(G)$ *is amenable then there is a finite bound* n_0 *on the dimension of the irreducible representations of* $\mathrm{VN}(G_d)$*. That is,* $\mathrm{VN}(G_d)$ *contains no copy of* $\mathcal{M}_n$ *for* $n > n_0$*.*

Proof. Since $\mathrm{VN}(G_d)$ is a von Neumann algebra, one may appeal to the theory of subhomogeneous von Neumann algebras to see that the two statements of this proposition are equivalent (see [33], V §1).

Let $\mathrm{A}(G)$ be amenable. Then by the last proposition, $\vee$ is completely bounded. Recall that for $\xi, \eta \in \mathrm{L}^2(G_d)$, $\vee(\xi *_\lambda \eta) = \bar{\eta} *_\lambda \bar{\xi}$.

Now, consider $\vee^*$. For $T \in \mathrm{VN}(G_d), \xi, \eta \in \mathrm{L}^2(G_d)$,

$$\begin{aligned} \langle \vee^*(T)\xi \,|\, \eta \rangle &= \langle \xi *_\lambda \eta, \vee^*(T) \rangle \\ &= \langle \vee(\xi *_\lambda \eta), T \rangle \\ &= \left\langle \bar{\eta} *_\lambda \bar{\xi}, T \right\rangle \\ &= \left\langle T\bar{\eta} \,|\, \bar{\xi} \right\rangle \\ &= \left\langle T^t \xi \,|\, \eta \right\rangle . \end{aligned}$$

Thus, $\vee^*$ is the transpose map.

Suppose that $\mathrm{VN}(G_d)$ contains a copy of $\mathcal{M}_n$ (that is, $\mathcal{M}_n$ injects into $\mathrm{VN}(G_d)$ $*$-homomorphically). On this copy of $\mathcal{M}_n$, $\vee^*$ restricts to the transpose map on $\mathcal{M}_n$, which has completely bounded norm of at least n, by Proposition 2.3.1. Thus, $\| \vee^* \|_{cb} \geq n$.

Since $\vee$ is completely bounded, so is $\vee^*$; thus, there is a bound on the degree n of the irreducible representations of $\mathrm{VN}(G_d)$. □

To get from the preceding result about irreducible representations of $\mathrm{VN}(G_d)$ to an analogous result for irreducible representations of $\ell^1(G)$, which embeds as a subalgebra of $\mathrm{VN}(G_d)$, some algebraic theory about polynomial identities is needed.

Let p be a polynomial in n non-commuting variables. An algebra $\mathcal{A}$ **satisfies the polynomial identity** $p = 0$ if, for all $a_1, \ldots, a_n \in \mathcal{A}$,

$$p(a_1, \ldots, a_n) = 0.$$

For a positive integer n, let S_n be the polynomial in n non-commuting variables given by

$$S_n(x_1, \ldots, x_n) = \sum_{\sigma} \operatorname{sgn}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)},$$

where the sum is over all the permutations σ of $\{1, \ldots, n\}$.

The significance of polynomial S_n is the following result, found in [2], Theorems 1 and 2:

Proposition 3.2.2. *$\mathcal{M}_n$ satisfies the polynomial identity $S_{2m} = 0$ if and only if $m \geq n$.*

Equipped with this result, the main result will be all but proven.

Theorem 3.2.3. *If* $\mathrm{A}(G)$ *is amenable then there is a finite bound on the degree of the irreducible representations of $\ell^1(G)$.*

Proof. If $\mathrm{A}(G)$ is amenable then for some positive integer n_0, $\mathrm{VN}(G_d)$ contains no copy of $\mathcal{M}_n$ for $n > n_0$. Considering that $\mathrm{VN}(G_d)$ is a von Neumann algebra, it follows from the last Proposition that $\mathrm{VN}(G_d)$ satisfies $S_{2n_0} = 0$. As $\ell^1(G)$ embeds as a subalgebra of $\mathrm{VN}(G_d)$, it also satisfies $S_{2n_0} = 0$.

Now suppose for a contradiction that $\ell^1(G)$ has an irreducible representation π of degree strictly greater than n_0. So let $\mathcal{K} \leq \mathcal{H}_\pi$ be a subspace of dimension $n > n_0$, so that $\mathcal{B}(\mathcal{K}) \cong \mathcal{M}_n$. Then $\pi\left(\ell^1(G)\right)' = \mathbb{C}I$, by Schur's Lemma, so that Jacobson's Density Theorem gives that for $b \in \mathcal{M}_n$, there exists $f \in \ell^1(G)$ such that $\pi(f)|_{\mathcal{K}} = b$. Using this, for $b_1, \ldots, b_{2n_0} \in \mathcal{M}_n$, let $f_1, \ldots, f_{2n_0} \in \ell^1(G)$ be such that $\pi(f_i)|_{\mathcal{K}} = b_i$. Then

$$\begin{aligned} S_{2n_0}(b_1, \ldots, b_{2n_0}) &= S_{2n_0}\left(\pi(f_1)|_{\mathcal{K}}, \ldots, \pi(f_{2n_0})|_{\mathcal{K}}\right) \\ &= \pi\left(S_{2n_0}(f_1, \ldots, f_{2n_0})\right)|_{\mathcal{K}} \\ &= 0, \text{ since } \ell^1(G) \text{ satisfies } S_{2n_0} = 0. \end{aligned}$$

Hence $\mathcal{M}_n$ satisfies $S_{2n_0} = 0$, which is a contradiction since $n > n_0$. It follows that $\ell^1(G)$ only has irreducible representations of degree n_0 or less. □

Theorem 3.2.4. $\mathrm{A}(G)$ *is amenable if and only if G is almost abelian.*

Proof. $\mathrm{A}(G)$ is amenable if G is almost abelian by Proposition 3.0.2.

Conversely, if $\mathrm{A}(G)$ is amenable then by the last proposition, there is a finite bound on the degree of the irreducible representations of $\ell^1(G)$, which correspond to the irreducible representations of G. So G_d is almost abelian. But by Moore's Theorem (Theorem 3.0.1), G being almost abelian is independent of the topological structure on G. So G is almost abelian. $\square$

Chapter 4

Operator Amenability of $\mathrm{A}(G)$

The non-amenability of $\mathrm{A}(G)$ for the amenable groups which do not have an abelian subgroup of finite index begs the question: what goes wrong? Considerations about the projective tensor product provide some insight into the answer.

For an algebra $\mathcal{A}$, the object $\mathcal{A} \otimes^{\gamma} \mathcal{A}$ is of fundamental importance to the amenability (or non-amenability) of $\mathrm{A}(G)$, due to its role in the approximate diagonal condition. For example, consider the group algebra $\mathrm{L}^1(G)$; recall that Johnson's Theorem states that the amenability of this algebra coincides with the amenability of the underlying group G. In this case, there is the nice relation

$$\mathrm{L}^1(G) \otimes^{\gamma} \mathrm{L}^1(G) \cong \mathrm{L}^1(G \times G)$$

isometrically, via the natural map

$$f \otimes g \mapsto f \times g,$$

where $(f \times g)(x, y) = f(x)g(y)$. (In fact, for any measure spaces X and Y, $\mathrm{L}^1(X) \otimes^{\gamma} \mathrm{L}^1(Y) \cong \mathrm{L}^1(X \times Y)$). As a result, the projective tensor product space is a manageable object, and the existence of an approximate diagonal has clear implications for the group itself.

In the case of the Fourier algebra $\mathrm{A}(G)$, the map $u \otimes v \mapsto u \times v$ does extend to all of $\mathrm{A}(G) \otimes^{\gamma} \mathrm{A}(G)$, with an image lying inside of $\mathrm{A}(G \times G)$. However, it is not always the case that the image is all of $\mathrm{A}(G\times G)$, or that the extended map is an isometry. In fact, Losert showed in [24] that

$$\mathrm{A}(G)\otimes^{\gamma}\mathrm{A}(G)\cong\mathrm{A}(G\times G)$$

isomorphically via the given map precisely when G is almost abelian, and that G must be abelian for it to be an isometry. Note that when G is abelian, $\mathrm{A}(G)\cong\mathrm{L}^1(\hat{G})$ isometrically via the Fourier transform and $\hat{G}\times\hat{G}\cong\widehat{G\times G}$, so that the map $\mathrm{A}(G)\otimes^{\gamma}\mathrm{A}(G)\mapsto\mathrm{A}(G\times G)$ can be identified with the isometric isometry $\mathrm{L}^1(\hat{G})\otimes^{\gamma}\mathrm{L}^1(\hat{G})\mapsto\mathrm{L}^1(\hat{G}\times\hat{G})$. Historically, this result by Losert precedes even Johnson's result about the non-amenability of $\mathrm{A}(G)$ for certain compact groups.

It will be shown here that $\mathrm{A}(G\times G)$ is exactly the space in which it makes sense to look for an approximate diagonal, since a bounded approximate diagonal exists in $\mathrm{A}(G\times G)$ exactly when G is amenable. This was proven by Ruan in [28], and his argument is used here. To put this into context with regards to amenability conditions, recall from (2.3.2) that the space $\mathrm{A}(G\times G)$ occurs naturally with the operator projective tensor product. That is, the same natural map $u\otimes v\mapsto u\times v$ extends to an isometric isomorphism

$$\mathrm{A}(G)\hat{\otimes}\mathrm{A}(G)\cong\mathrm{A}(G\times G),$$

and thus, operator amenability is the right sort of amenability condition to look for on the Fourier algebra.

To begin, the following lemma improves on Theorem 2.5.9 for $\mathrm{A}(G)$. In this result, $\mathrm{B}(G\times G)$ is an $\mathrm{A}(G)$-bimodule; the actions are given for $u\in\mathrm{A}(G)$, $w\in\mathrm{B}(G\times G)$ by

$$(u.w)=(u\times\mathbb{1})w,\text{ and}$$
$$(w.u)=(\mathbb{1}\times u)w,$$

where $\mathbb{1}$ is the function that is constantly one (which is in $\mathrm{B}(G)$). This is a natural action, since it extends the module action of $\mathrm{A}(G)$ on $\mathrm{A}(G)\hat{\otimes}\mathrm{A}(G)=\mathrm{A}(G\times G)$. Also, the map $\Delta_{\mathrm{B}(G)}:\mathrm{B}(G\times G)\mapsto\mathrm{B}(G)$ given by $\left(\Delta_{\mathrm{B}(G)}(w)\right)(x)=w(x,x)$ is used.

Lemma 4.0.1. *Let G be amenable. Then $\mathrm{A}(G)$ is operator amenable if and only if there is a bounded net $(w_\alpha)\subset\mathrm{B}(G\times G)$ such that for all $u\in\mathrm{A}(G)$,*

$$\|u.w_\alpha-w_\alpha.u\|_{\mathrm{B}(G\times G)}\to 0,\textit{ and}$$
$$\left\|\Delta_{\mathrm{B}(G)}(w_\alpha)u-u\right\|_{\mathrm{B}(G)}\to 0.$$

Such a net may be called an approximate diagonal in B$(G \times G)$ *for* A(G)*.*

Proof. Since $\mathrm{A}(G \times G) \subset \mathrm{B}(G \times G)$, one direction is trivial, using a bounded approximate diagonal in $\mathrm{A}(G)\hat{\otimes}\mathrm{A}(G) = \mathrm{A}(G \times G)$.

Conversely, let $(w_\alpha) \subset \mathrm{B}(G \times G)$ satisfy the given conditions. Since G is amenable, by Leptin's Theorem (Theorem 2.5.2, (4)), let $(u_\beta) \subset \mathrm{A}(G)$ be a bounded approximate identity, so that $(u_\beta \times u_\beta)$ is a bounded approximate identity for $\mathrm{A}(G \times G)$. Since $\mathrm{A}(G \times G)$ is an ideal in $\mathrm{B}(G \times G)$, $(u_\beta \times u_\beta)w_\alpha \in \mathrm{A}(G \times G)$ for all α, β, and it is easy to see that the given conditions ensure that $((u_\beta \times u_\beta)w_\alpha)_{(\alpha,\beta)}$ is an approximate diagonal. □

For $\xi \in b_1(\mathrm{L}^2(G))$, define the map $M_\xi : G \times G \mapsto \mathbb{C}$ by

$$M_\xi(x, y) = \langle \lambda(x)\rho(y)\xi \mid \xi \rangle \,.$$

Proposition 4.0.2. *If G is amenable, then $M_\xi \in \mathrm{B}(G \times G)$ with norm one.*

Proof. Define the map $\tau : G \times G \mapsto \mathcal{U}(\mathrm{L}^2(G))$ by, for $x, y \in G$,

$$\tau(x, y) = \lambda(x)\rho(y).$$

Since $\lambda(G)$ commutes with $\rho(G)$, τ is a group homomorphism. Moreover, λ and ρ are each strongly continuous, and multiplication of operators is strongly continuous, whence τ is strongly continuous. This means that τ is a representation of $G \times G$, and thus,

$$M_\xi = \xi *_\tau \xi \in \mathrm{B}(G \times G).$$

Also, $\|M_\xi\| \le \|\xi\| \, \|\xi\| = 1$. □

For C*-algebras $\mathcal{A}$ and $\mathcal{B}$, note that whenever π and σ are $*$-representations of $\mathcal{A}$ and $\mathcal{B}$ respectively, $\pi \otimes \sigma$ is a $*$-representation of $\mathcal{A} \otimes \mathcal{B}$. So for $u \in \mathcal{A} \otimes \mathcal{B}$,

$$\|u\|_\vee := \sup \{\|(\pi \otimes \sigma)(u)\| : \pi, \sigma \text{ are *-reps. of } \mathcal{A}, \mathcal{B}\}$$

defines a C*-norm on $\mathcal{A} \otimes \mathcal{B}$. The **injective tensor product** of $\mathcal{A}$ and $\mathcal{B}$, denoted $\mathcal{A}\check{\otimes}\mathcal{B}$, is the completion of $\mathcal{A} \otimes \mathcal{B}$ under $\| \cdot \|_\vee$. This coincides with the operator injective tensor product on C*-algebras, but this fact will not be shown nor used here. An important feature of this tensor product is the following; for a proof, see [33], IV, Proposition 4.9 (iii).

Proposition 4.0.3. *If $\mathcal{A},\mathcal{B}$ are C^*-algebras with faithful representations π,σ respectively, then $\pi\otimes\sigma$ is a faithful representation of $\mathcal{A}\check{\otimes}\mathcal{B}$.*

This last proposition implies that, if π,σ are faithful representations of $\mathcal{A},\mathcal{B}$ respectively, then for $u\in\mathcal{A}\otimes\mathcal{B}$,

$$\|u\|_\vee = \|(\pi\otimes\sigma)(u)\|.$$

Proposition 4.0.4. $\mathrm{C}^*_r(G\times G)\cong \mathrm{C}^*_r(G)\check{\otimes}\mathrm{C}^*_r(G)$ *as C^*-algebras, and this space is contained in* $\mathrm{VN}(G)\check{\otimes}\mathrm{VN}(G)$.

Proof. $\mathrm{L}^1(G)\otimes\mathrm{L}^1(G)$ is a dense subspace of both $\mathrm{C}^*_r(G)\check{\otimes}\mathrm{C}^*_r(G)$ and of $\mathrm{C}^*_r(G\times G)$ (because it is dense in $\mathrm{L}^1(G\times G)$). So it suffices to show that on this space, the inherited norms from $\mathrm{C}^*_r(G\times G)$ and from $\mathrm{C}^*_r(G)\check{\otimes}\mathrm{C}^*_r(G)$ agree.

Note that $\lambda_{G\times G}=\lambda_G\otimes\lambda_G$, so for $a\in\mathrm{L}^1(G)\otimes\mathrm{L}^1(G)$,

$$\begin{aligned}\|a\|_{\mathrm{C}^*_r(G)\check{\otimes}\mathrm{C}^*_r(G)} &= \|(\lambda_G\otimes\lambda_G)(a)\|\\ &= \|\lambda_{G\times G}(a)\|\\ &= \|a\|_{\mathrm{C}^*_r(G\times G)}\end{aligned}$$

The fact that $\mathrm{C}^*_r(G)\check{\otimes}\mathrm{C}^*_r(G)\subset\mathrm{VN}(G)\check{\otimes}\mathrm{VN}(G)$ is due to the last proposition. □

Proposition 4.0.5. *If G is amenable then the multiplication map* $\mathrm{VN}(G)\times\mathrm{VN}_\rho(G)\mapsto\mathcal{B}(\mathrm{L}^2(G))$ *extends to a contraction $\bar{m}$ on* $\mathrm{VN}(G)\check{\otimes}\mathrm{VN}_\rho(G)$.

Proof. This is due to the fact that if G is amenable then $\mathrm{VN}(G)$ and $\mathrm{VN}_\rho(G)$ are hyperfinite von Neumann algebras, and $\mathrm{VN}(G)'=\mathrm{VN}_\rho(G)$. For details, see [5] □

In the next result, recall the operator $W\in\mathcal{U}(\mathrm{L}^2(G\times G))$ defined in (2.4.1).

Lemma 4.0.6. *Let G be amenable and suppose that $(\xi_\alpha)\subset\mathrm{L}^2(G)$ is a net consisting of unit vectors, such that*

(1) For all $\eta\in\mathrm{L}^2(G)$, $\|W(\xi_\alpha\otimes\eta)-(\xi_\alpha\otimes\eta)\|\to 0$, and

(2) $\|\lambda(x)\rho(x)\xi_\alpha-\xi_\alpha\|\to 0$ uniformly on compact subsets of G.

Then $(M_{\xi_\alpha}) \subset \mathrm{B}(G \times G)$ *is a contractive approximate diagonal for* $\mathrm{A}(G)$*.*

Proof. The elements of this net are already known to be contractive, so only the approximate diagonal conditions need to be met. Moreover, by polarization,

$$\mathrm{A}(G) = \overline{\operatorname{span}\{\eta *_\lambda \eta : \eta \in \mathrm{L}^2(G)\}},$$

so it suffices to show that (M_{ξ_α}) satisfies (AD1) and (AD2) for all $a = \eta *_\lambda \eta$.

First (AD1) will be shown. Fix $\eta \in \mathrm{L}^2(G)$, and let $a = \eta *_\lambda \eta$. For $x, y \in G$,

$$\begin{aligned}
(a.M_{\xi_\alpha})(x,y) &= a(x) M_{\xi_\alpha}(x,y) \\
&= \langle \lambda(x)\eta \,|\, \eta \rangle \, \langle \lambda(x)\rho(y)\xi_\alpha \,|\, \xi_\alpha \rangle \\
&= \langle (\lambda(x)\rho(y) \otimes \lambda(x))\, (\xi_\alpha \otimes \eta) \,|\, (\xi_\alpha \otimes \eta) \rangle \,,
\end{aligned}$$

and likewise,

$$\begin{aligned}
(M_{\xi_\alpha}.a)(x,y) &= \langle (\lambda(x)\rho(y) \otimes \lambda(y))\, (\xi_\alpha \otimes \eta) \,|\, (\xi_\alpha \otimes \eta) \rangle \\
&= \langle (\lambda(x) \otimes I)\, (V\lambda(y)V \otimes \lambda(y))\, (\xi_\alpha \otimes \eta) \,|\, (\xi_\alpha \otimes \eta) \rangle \,, \\
&\qquad \text{since } \rho(y) = V\lambda(y)V \\
&= \langle (\lambda(x) \otimes I)\, (V \otimes I)W^* \, (\lambda(y) \otimes I)\, W(V \otimes I)(\xi_\alpha \otimes \eta) \,|\, (\xi_\alpha \otimes \eta) \rangle \,, \\
&\qquad \text{since } \lambda(x) \otimes \lambda(x) = W^*(\lambda(x) \otimes I)W \\
&= \langle W^* \, (\lambda(x) \otimes I)\, (V \otimes I)W^* \, (\lambda(y) \otimes I)\, W(V \otimes I)(\xi_\alpha \otimes \eta) \,|\, W^*(\xi_\alpha \otimes \eta) \rangle \,, \\
&\qquad \text{since } W \text{ is unitary} \\
&= \langle W^* \, (\lambda(x) \otimes I)\, W(V \otimes I)\, (\lambda(y) \otimes I)\, (V \otimes I)W^*(\xi_\alpha \otimes \eta) \,|\, W^*(\xi_\alpha \otimes \eta) \rangle \,, \\
&\qquad \text{since } W(V \otimes I) = (V \otimes I)W^* \\
&= \langle (\lambda(x) \otimes \lambda(x))\, (\rho(y) \otimes I)\, W^*(\xi_\alpha \otimes \eta) \,|\, W^*(\xi_\alpha \otimes \eta) \rangle \\
&= \langle (\lambda(x)\rho(y) \otimes \lambda(x))\, W^*(\xi_\alpha \otimes \eta) \,|\, W^*(\xi_\alpha \otimes \eta) \rangle \,.
\end{aligned}$$

Each justification above is easy to check.

Combining these,

$$\begin{aligned}(a.M_{\xi_\alpha} - M_{\xi_\alpha}.a)(x,y) &= \langle (\lambda(x)\rho(y)\otimes\lambda(x))\,(\xi_\alpha\otimes\eta)\,|\,(\xi_\alpha\otimes\eta)\rangle \\ &\quad - \langle (\lambda(x)\rho(y)\otimes\lambda(x))\,W^*(\xi_\alpha\otimes\eta)\,|\,W^*(\xi_\alpha\otimes\eta)\rangle \\ &= \langle (\lambda(x)\rho(y)\otimes\lambda(x))\,(1-W^*)(\xi_\alpha\otimes\eta)\,|\,(\xi_\alpha\otimes\eta)\rangle \\ &\quad + \langle (\lambda(x)\rho(y)\otimes\lambda(x))\,W^*(\xi_\alpha\otimes\eta)\,|\,(1-W^*)(\xi_\alpha\otimes\eta)\rangle\,.\end{aligned}$$

Since $G\times G$ is amenable, use Proposition 4.0.5, to define the contractive multiplication map

$$\tilde{m} : \mathrm{VN}(G\times G)\check{\otimes}\mathrm{VN}_\rho(G\times G) \mapsto \mathcal{B}\left(\mathrm{L}^2(G\times G)\right).$$

Once again, letting $\Phi_V : \mathrm{VN}(G) \mapsto \mathrm{VN}_\rho(G)$ be conjugation by V, use

$$\Phi_V(\cdot)\otimes 1 : \mathrm{VN}(G) \mapsto \mathrm{VN}_\rho(G)\otimes\mathrm{VN}_\rho(G) \subset \mathrm{VN}_\rho(G\times G)$$

to denote an amplification of Φ_V; that is, for $T\in\mathrm{VN}(G)$,

$$(\Phi_V(\cdot)\otimes 1)(T) = (VTV^*)\otimes 1.$$

Then

$$\Phi = \tilde{m}\left(\nabla\otimes(\Phi_V(\cdot)\otimes 1)\right) : \mathrm{VN}(G)\check{\otimes}\mathrm{VN}(G) \mapsto \mathcal{B}\left(\mathrm{L}^2(G\times G)\right)$$

is a contraction. Explicitly, for $x,y\in G$,

$$\begin{aligned}\Phi\left(\lambda(x)\otimes\lambda(y)\right) &= \tilde{m}\big(\,(\lambda(x)\otimes\lambda(x))\otimes(\rho(y)\otimes 1)\,\big) \\ &= \lambda(x)\rho(y)\otimes\lambda(x),\end{aligned}$$

so

$$\begin{aligned}(a.M_{\xi_\alpha} - M_{\xi_\alpha}.a)(x,y) &= \langle \Phi\left(\lambda(x)\otimes\lambda(y)\right)(1-W^*)(\xi_\alpha\otimes\eta)\,|\,(\xi_\alpha\otimes\eta)\rangle \\ &\quad + \langle \Phi\left(\lambda(x)\otimes\lambda(y)\right)W^*(\xi_\alpha\otimes\eta)\,|\,(1-W^*)(\xi_\alpha\otimes\eta)\rangle\,.\end{aligned}$$

By Proposition 4.0.4 and Hulanicki's Theorem (Theorem 2.5.2, (3)), Φ restricts to a contraction on $\mathrm{C}^*(G\times G)$. For $f\in\mathrm{L}^1(G\times G)$,

$$\begin{aligned}\Phi(f) &= \int f(x,y)\Phi\left(\lambda(x)\otimes\lambda(y)\right)\,dx\,dy \\ &= \int f(x,y)\lambda(x)\rho(y)\otimes\lambda(x)\,dx\,dy,\end{aligned}$$

as WOT-converging integrals.

So, as a functional on $\mathrm{C}^*(G \times G)$,

$$\begin{aligned} a.M_{\xi_\alpha} - M_{\xi_\alpha}.a = &\langle \Phi(\cdot)(1-W^*)(\xi_\alpha \otimes \eta) \,|\, \xi_\alpha \otimes \eta \rangle \\ &+ \langle \Phi(\cdot) W^*(\xi_\alpha \otimes \eta) \,|\, (1-W^*)\xi_\alpha \otimes \eta \rangle \,, \end{aligned}$$

and since the $\mathrm{B}(G \times G)$-norm is given by duality with $\mathrm{C}^*(G \times G)$,

$$\begin{aligned} \|a.M_{\xi_\alpha} - M_{\xi_\alpha}.a\| &\le 2\|\eta\| \|W^*\| \, \|(1-W^*)(\xi_\alpha \otimes \eta)\| \,, \text{ since } \|\xi_\alpha \otimes \eta\| = 1 \\ &= 2\|\eta\| \, \|(W-1)(\xi_\alpha \otimes \eta)\| \,, \text{ since } W \text{ is unitary} \\ &\to 0, \end{aligned}$$

as required.

Next, since $\mathcal{C}_c(G)$ is dense in $\mathrm{L}^2(G)$, it suffices to show (AD2) merely for elements $a = \eta *_\lambda \eta$ where $\eta \in \mathcal{C}_c(G)$. Fix $\eta \in \mathcal{C}_c(G)$ and let $a = \eta *_\lambda \eta$. For $x \in G$,

$$\begin{aligned} (\Delta(M_{\xi_\alpha})a)(x) &= M_{\xi_\alpha}(x,x)a(x) \\ &= \langle \lambda(x)\rho(x)\xi_\alpha \,|\, \xi_\alpha \rangle \, \langle \lambda(x)\eta \,|\, \eta \rangle \\ &= \langle (\lambda(x) \otimes \lambda(x)) \, (I \otimes \rho(x)) \, (\eta \otimes \xi_\alpha) \,|\, (\eta \otimes \xi_\alpha) \rangle \\ &= \langle W^* \, (\lambda(x) \otimes I) \, W \, (I \otimes \rho(x)) \, (\eta \otimes \xi_\alpha) \,|\, (\eta \otimes \xi_\alpha) \rangle \\ &= \langle W^* \, (\lambda(x) \otimes I) \, (I \otimes \rho(x)) \, W(\eta \otimes \xi_\alpha) \,|\, (\eta \otimes \xi_\alpha) \rangle \,, \\ &\qquad \text{since } W \text{ and } I \otimes \rho(x) \text{ commute} \\ &= \langle W^*(I \otimes V^*) \, (\lambda(x) \otimes \lambda(x)) \, (I \otimes V) W(\eta \otimes \xi_\alpha) \,|\, (\eta \otimes \xi_\alpha) \rangle \,, \\ &= \langle W^*(I \otimes V^*) W^* \, (\lambda(x) \otimes I) \, W(I \otimes V) W(\eta \otimes \xi_\alpha) \,|\, (\eta \otimes \xi_\alpha) \rangle \\ &= \langle (\lambda(x) \otimes I) \, W(I \otimes V) W(\eta \otimes \xi_\alpha) \,|\, W(I \otimes V) W(\eta \otimes \xi_\alpha) \rangle \\ &= \langle (I \otimes V) \, (\lambda(x) \otimes I) \, W(I \otimes V) W(\eta \otimes \xi_\alpha) \,|\, (I \otimes V) W(I \otimes V) W(\eta \otimes \xi_\alpha) \rangle \,, \\ &\qquad \text{since } I \otimes V \text{ is unitary.} \end{aligned}$$

So

$$\begin{aligned}(\Delta(M_{\xi_\alpha})a - a)(x) &= \langle(\lambda(x)\otimes I)\,(I\otimes V)W(I\otimes V)W(\eta\otimes\xi_\alpha)\,|\,(I\otimes V)W(I\otimes V)W(\eta\otimes\xi_\alpha)\rangle \\ &\quad - \langle(\lambda(x)\otimes I)\,(\eta\otimes\xi_\alpha)\,|\,(\eta\otimes\xi_\alpha)\rangle \\ &= \Big\langle\,(\lambda(x)\otimes I)\,((I\otimes V)W(I\otimes V)W - 1)\,(\eta\otimes\xi_\alpha) \\ &\qquad |\,(I\otimes V)W(I\otimes V)W(\eta\otimes\xi_\alpha)\Big\rangle \\ &\quad + \langle(\lambda(x)\otimes I)\,(\eta\otimes\xi_\alpha)\,|\,\,((I\otimes V)W(I\otimes V)W - 1)\,(\eta\otimes\xi_\alpha)\rangle\,.\end{aligned}$$

Since $\lambda(\cdot)\otimes I$ is a contraction on $\mathrm{C}^*(G)$,

$$\|\Delta(M_{\xi_\alpha})a - a\|_{\mathrm{B}(G)} \le 2\|\eta\|\,\|((I\otimes V)W(I\otimes V)W - 1)\,(\eta\otimes\xi_\alpha)\|\,.$$

For $x, y\in G$, $f\in \mathrm{L}^2(G\times G)$,

$$\begin{aligned}((V\otimes I)Wf)\,(x,y) &= (Wf)(x,y^{-1})\Delta(y)^{-\frac{1}{2}} \\ &= f(x,xy^{-1})\Delta(y)^{-\frac{1}{2}},\end{aligned}$$

so

$$\begin{aligned}((V\otimes I)W(V\otimes I)Wf)\,(x,y) &= ((V\otimes I)Wf)\,(x,xy^{-1})\Delta(y)^{-\frac{1}{2}} \\ &= f(x,xyx^{-1})\Delta(xy^{-1})^{-\frac{1}{2}}\Delta(y)^{-\frac{1}{2}} \\ &= f(x,xyx^{-1})\Delta(x)^{-\frac{1}{2}},\end{aligned}$$

and, using this,

$$
\begin{aligned}
&\left\|(I\otimes V)W(I\otimes V)W(\eta\otimes\xi_\alpha)-(\eta\otimes\xi_\alpha)\right\|^2\\
&\qquad = \iint |(I\otimes V)W(I\otimes V)W(\eta\otimes\xi_\alpha)(x,y)-(\eta\otimes\xi_\alpha)(x,y)|^2\,dy\,dx\\
&\qquad = \iint \left|\eta(x)\xi_\alpha(xyx^{-1})\Delta(x)^{-\frac{1}{2}}-\eta(x)\xi_\alpha(y)\right|^2\,dy\,dx\\
&\qquad = \int |\eta(x)|^2 \int \left|\xi_\alpha(xyx^{-1})\Delta(x)^{-\frac{1}{2}}-\xi_\alpha(y)\right|^2\,dy\,dx\\
&\qquad = \int |\eta(x)|^2 \int \left|\xi_\alpha(y)-\xi_\alpha(x^{-1}yx)\Delta(x)^{\frac{1}{2}}\right|^2\,dy\,dx, \text{ by changing } y \text{ to } x^{-1}yx\\
&\qquad \le \int |\eta(x)|^2 \left\|\xi_\alpha-\lambda(x)\rho(x)\xi_\alpha\right\|^2\,ds\\
&\qquad \le \|\eta\|^2 \sup\left\{\|\xi_\alpha-\lambda(x)\rho(x)\xi_\alpha\|^2 : x\in \operatorname{supp}\eta\right\}\\
&\qquad \to 0, \text{ since } \operatorname{supp}\eta \text{ is compact.}
\end{aligned}
$$

Thus, $\|\Delta(M_{\xi_\alpha})a-a\|_{\mathrm{B}(G)}\to 0$, as required. ☐

Next, such a net in $\mathrm{L}^2(G)$ is constructed:

Proposition 4.0.7. *Let G be amenable. Then there exists a net $(\xi_\alpha)\subset \mathrm{L}^2(G)$ satisfying the conditions of the last lemma, that is, such that:*

(1) $\|\xi_\alpha\|=1$ *for all* α,

(2) For all $\eta\in\mathrm{L}^2(G)$, $\|W(\eta\otimes\xi_\alpha)-(\eta\otimes\xi_\alpha)\|\to 0$, *and*

(3) $\|\lambda(x)\rho(x)\xi_\alpha-\xi_\alpha\|\to 0$ *uniformly in x on compact sets.*

Proof. Using the Følner Condition (Theorem 2.5.2, (2)), for $\epsilon>0$ and $K\subset G$ compact, there exists a Borel set $E\subset G$ with positive finite measure such that $m(E\triangle xE)<m(E)\epsilon$ for all $x\in K$. By inner regularity of m, it may be assumed that E is in fact compact. Letting $g_{K,\epsilon}=m(E)^{-1}\chi_E$, this means that

$$
\begin{gathered}
g_{K,\epsilon}\ge 0,\\
\|g_{K,\epsilon}\|_1=1, \text{ and}\\
\|g_{K,\epsilon}-\delta_x * g_{K,\epsilon}\|_1<\epsilon \text{ for } x\in K.
\end{gathered}
$$

For a pre-compact, open neighbourhood U of e, let $f_U = m(U)^{-1}\chi_U$. Then

$$f_U \geq 0, \text{ and}$$
$$\|f_U\|_1 = 1,$$

and it is well-known that (f_U) is a bounded approximate identity for $\mathrm{L}^1(G)$, as U decreases to $\{e\}$.

Then for each pre-compact open neighbourhood U of e, each $K \subset G$ compact, and each $\epsilon > 0$, define $f_{U,K,\epsilon}$ by

$$\begin{aligned} f_{U,K,\epsilon}(x) &= \int g_{K,\epsilon}(y)(\delta_y * f_U * \delta_{y^{-1}})(x)\,dx \\ &= \int g_{K,\epsilon}(y) f_U(y^{-1}xy)\Delta(y)\,dx. \end{aligned}$$

Then

$$\begin{aligned} f_{U,K,\epsilon} &\geq 0 \\ \|f_{U,K,\epsilon}\|_1 &= \iint g_{K,\epsilon}(x) f_U(x^{-1}yx)\Delta(x)\,dx\,dy \\ &= \int g_{K,\epsilon}(x) \int f_U(x^{-1}yx)\Delta(x)\,dy\,dx, \text{ by Fubini's theorem} \\ &= \int g_{K,\epsilon}(x) \int f_U(y)\,dy\,dx \\ &= 1. \end{aligned}$$

For each U, K, ϵ, let $\xi_{U,K,\epsilon} = f_{U,K,\epsilon}^{\frac{1}{2}} \in \mathrm{L}^2(G)$. Using the above properties of $f_{U,K,\epsilon}$, it is plain that $\xi_{U,K,\epsilon} \geq 0$ and $\|\xi_{U,K,\epsilon}\|_2 = 1$.

For each $x \in K$,

$$\begin{aligned}
\|\lambda(x)\rho(x)\xi_{U,K,\epsilon} - \xi_{U,K,\epsilon}\|_2^2 &= \int \left|\xi_{U,K,\epsilon}(x^{-1}yx)\Delta(y)^{\frac{1}{2}} - \xi_{U,K,\epsilon}(y)\right|^2 \, dy \\
&\leq \int \left|(\xi_{U,K,\epsilon}(x^{-1}yx)\Delta(x)^{\frac{1}{2}})^2 - \xi_{K,U,\epsilon}(y)^2\right| \, dy, \\
&\qquad \text{since for } a, b \geq 0, |a-b|^2 \leq |a^2 - b^2| \\
&= \int \left| f_{U,K,\epsilon}(x^{-1}yx)\Delta(x) - f_{U,K,\epsilon}(y)\right| \, dy \\
&= \int \Big| \int g_{K,\epsilon}(z) f_U(z^{-1}x^{-1}yxz)\Delta(xz))\, dz \\
&\quad - \int g_{K,\epsilon}(z) f_U(z^{-1}yz)\Delta(z)\, dz \Big| \, dy \\
&= \int \Big| \int g_{K,\epsilon}(x^{-1}z)(\delta_z * f_U * \delta_{z^{-1}})(y)\, dz \\
&\quad - \int g_{K,\epsilon}(z)(\delta_z * f_U * \delta_{z^{-1}})(y)\, dz \Big| \, dy, \\
&\qquad \text{by changing } z \text{ to } x^{-1}z \text{ in the first subintegral} \\
&= \int \Big| \int (g_{K,\epsilon}(x^{-1}z) - g_{K,\epsilon}(z))(\delta_z * f_U * \delta_{z^{-1}})(y)\, dz \Big| \, dy \\
&\leq \int \int |\delta_z * f_U * \delta_{z^{-1}}(y)\, dy \left| g_{K,\epsilon}(x^{-1}z) - g_{K,\epsilon}(z)\right| \, dz \qquad < \epsilon.
\end{aligned}$$

So, for each $x \in G$, as K increases and ϵ decreases, $\|\lambda(x)\rho(x)\xi_{U,K,\epsilon} - \xi_{U,K,\epsilon}\|_2 \to 0$.

Now, for $\eta_1, \ldots \eta_n \in \mathrm{L}^2(G)$, since λ is strongly continuous, let U_1 be an open, pre-compact neighbourhood of e such that for $x \in U_1$ and $i = 1, \ldots, n$,

$$\int |\eta_i(xy) - \eta_i(y)|^2 \, dy = \left\|\lambda(x^{-1})\eta_i - \eta_i\right\|_2^2 < \epsilon.$$

Then, for $K \subset G$ compact, let U_2 be an open neighbourhood of e such that for all

$x \in K$, $xU_2x^{-1} \subset U_1$. So, for any $U \subset U_2$ and $i = 1, \dots, n$,

$$\begin{aligned}
\|W(\xi_{U,K,\epsilon} \otimes \eta_i) - \xi_{U,K,\epsilon} \otimes \eta_i\|^2 &= \iint \left|\xi_{U,K,\epsilon}(x)\eta_i(xy) - \xi_{U,K,\epsilon}(x)\eta_i(y)\right|^2 \, dy\, dx \\
&= \iint |\xi_{U,K,\epsilon}(x)|^2 \left|\eta_i(xy) - \eta_i(y)\right|^2 \, dy\, dx \\
&= \iiint g_{K,\epsilon}(z) f_U(z^{-1}xz)\Delta(z) \left|\eta_i(xy) - \eta_i(y)\right|^2 \, dz\, dy\, dx \\
&= \int g_{K,\epsilon}(z) \int f_U(z^{-1}xz)\Delta(z) \int \left|\eta_i(xy) - \eta_i(y)\right|^2 \, dy\, dx\, dz \\
&= \int_K g_{K,\epsilon}(z) \int_U f_U(x) \int |\eta_i(zxz^{-1}y) - \eta_i(y)|^2 \, dy\, dx\, dz \\
&\leq \|g_{K,\epsilon}\|_1 \|f_U\|_1 \epsilon, \text{ since } zxz^{-1} \in U_1 \text{ for } z \in K, x \in U \subset U_2 \\
&= \epsilon.
\end{aligned}$$

So, in order to satisfy (2), we need to refine the indexing on the net $(\xi_{U,K,\epsilon})$.

Let

$$\begin{aligned}
I = \big\{(E, K, \epsilon, U) : \; & E \subset \mathrm{L}^2(G) \text{ finite}, \\
& K \subset G \text{ compact}, \\
& \epsilon > 0, \text{ and} \\
& U \text{ is an open neighbourhood of } e \text{ as given above}\big\}.
\end{aligned}$$

I is a directed set under the ordering

$$\begin{gathered}
(E_1, K_1, \epsilon_1, U_1) \leq (E_2, K_2, \epsilon_2, U_2) \\
\iff E_1 \subset E_2, K_1 \subset K_2, \epsilon_1 \geq \epsilon_2, \text{ and } U_1 \supset U_2.
\end{gathered}$$

Finally, for $\alpha = (E, K, \epsilon, U) \in I$, let $\xi_\alpha = \xi_{U,K,\epsilon}$; this defines the net $(\xi_\alpha)_{\alpha \in I}$ satisfying (2) and (3). $\square$

Thus, the following is proven:

Theorem 4.0.8. *(Ruan's Theorem, [28])*
G is amenable if and only if $\mathrm{A}(G)$ is operator amenable.

Proof. If G is amenable, then by the last two lemmas, there exists a bounded approximate diagonal for $\mathrm{A}(G)$ in $\mathrm{B}(G \times G)$. Hence, $\mathrm{A}(G)$ is operator amenable.

Conversely, if $\mathrm{A}(G)$ is operator amenable then it has a bounded approximate identity, so by Leptin's Theorem (Theorem 2.5.2, (4)), G is amenable. □

A special case of Losert's result occurs as a simple corollary of the main results of this thesis:

Corollary 4.0.9. *Let G be amenable. Then the map $u \otimes v \mapsto u \times v$ extends to an isomorphism $\mathrm{A}(G) \otimes^\gamma \mathrm{A}(G) \cong \mathrm{A}(G \times G)$ if and only if G is almost abelian.*

Proof. Suppose that $H \leq G$ is abelian with $[G : H] < \infty$. Then, as in the proof of Proposition 3.0.2,

$$\begin{aligned} A(G) &= \oplus_{i=1}^n \mathrm{A}(x_i H) \\ &= \oplus_{i=1}^n \delta_{x_i} * \mathrm{A}(H), \end{aligned}$$

and likewise,

$$\mathrm{A}(G \times G) = \oplus_{i,j} \delta_{(x_i, x_j)} * \mathrm{A}(H \times H).$$

Since H is abelian,

$$\begin{aligned} \mathrm{A}(H) \otimes^\gamma \mathrm{A}(H) &\cong \mathrm{L}^1(\hat{H}) \otimes^\gamma \mathrm{L}^1(\hat{H}) \\ &\cong \mathrm{L}^1(\hat{H} \times \hat{H}) \\ &\cong \mathrm{L}^1(\widehat{H \times H}) \\ &\cong \mathrm{A}(H \times H), \end{aligned}$$

preserving the map $u \otimes v \mapsto u \times v$. So

$$\begin{aligned} \mathrm{A}(G) \otimes^\gamma \mathrm{A}(G) &= (\oplus_{i=1}^n \delta_{x_i} * \mathrm{A}(H)) \otimes^\gamma \left(\oplus_{j=1}^n \delta_{x_j} * \mathrm{A}(H)\right) \\ &\cong \oplus_{i,j} \left(\delta_{x_i} * \mathrm{A}(H)\right) \otimes^\gamma \left(\delta_{x_j} * \mathrm{A}(H)\right) \\ &= \oplus_{i,j} (\delta_{x_i} \otimes \delta_{x_j}) * (\mathrm{A}(H) \otimes^\gamma \mathrm{A}(H)) \\ &\cong \oplus_{i,j} \delta_{(x_i, x_j)} * \mathrm{A}(H \times H) \\ &= \mathrm{A}(G \times G). \end{aligned}$$

Conversely, supposing that the given map is indeed bijective, then by the Open Mapping Theorem, the inverse map $\Phi : \mathrm{A}(G \times G) \mapsto \mathrm{A}(G) \otimes^{\gamma} \mathrm{A}(G)$ is bounded. Since G is amenable, let $(u_\alpha) \subset \mathrm{A}(G \times G)$ be a bounded approximate diagonal for $\mathrm{A}(G)$. Then $(\Phi(u_\alpha)) \subset \mathrm{A}(G) \otimes^{\gamma} \mathrm{A}(G)$ is a bounded approximate diagonal, whence $\mathrm{A}(G)$ is amenable. Consequently, G must be almost abelian. □

In essence, what's going on here is that G must be almost abelian in order for the operator space structure on $\mathrm{A}(G)$ to be sufficiently close to the MAX operator space structure, which is what is required to obtain $\mathrm{A}(G) \otimes^{\gamma} \mathrm{A}(G) \cong \mathrm{A}(G) \hat{\otimes} \mathrm{A}(G)$, in light of Proposition 2.3.7.

Historically, Losert proved this result before it was known that $\mathrm{A}(G)$ is not amenable for some amenable groups G, and before Ruan's published his operator amenability paper; indeed, Losert's paper prompted many in the field to conjecture the non-amenability result. However, the fact that this result occurs as a corollary to the non-amenability result demonstrates exactly how fundamental it is to the non-amenability of $\mathrm{A}(G)$.

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Index