On Excluded Minors
for Even Cut Matroids

by
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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

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Chapter 1

Introduction

In this thesis we will present two main theorems that can be used to study minor minimal non even cut matroids.

Given any signed graph we can associate an even cut matroid. However, given an even cut matroid, there are in general several signed graphs which represent that matroid. This is in contrast to, for instance, graphic (or cographic) matroids (see [9]), where all graphs corresponding to a particular graphic matroid are essentially equivalent. To tackle the multiple non-equivalent representations of even cut matroids we use the concept of Stabilizer first introduced by Whittle (see [10]). Namely, we show the following: given a "substantial" signed graph, which represents a matroid N that is a minor of a matroid M, if the signed graph extends to a signed graph which represents M then it does so uniquely. Thus the representations of the small matroid determine the representations of the larger matroid containing it. This allows us to consider each representation of an even cut matroid essentially independently.

Consider a small even cut matroid N that is a minor of a matroid M that is not an even cut matroid. We would like to prove that there exists a matroid N' which contains N and is contained in M such that the size of N' is small and such that N' is not an even cut matroid (this would imply in particular that there are only finitely many minimally non even cut matroids containing N). Clearly, none of the representations of N extends to M. We will show that (under certain technical conditions) starting from a fixed representation of N, there exists a matroid N' which contains N and is contained in M such that the size of N' is small and such that the representation of N does not extend to N'.

We will now need some definitions and observations before explaining these concepts more in details. We will refer to Oxley [5] and West [8] for the definitions that are not given here. If not otherwise specified, all the matroids are binary, and all the matrices are represented over $GF(2)$.

Given a matroid M, we will denote with EM its ground set, with $r_M(F)$ the rank of the set $F \subseteq EM$ and with $r(M)$ the rank of EM. Moreover $\lambda_M(E_1, E_2)$, where (E_1, E_2) is a partition of EM, denotes the connectivity of the partition, that is $\lambda_M(E_1, E_2) = r_M(E_1) + r_M(E_2) - r(M) + 1$. If M and N are two binary matroids, the notation $M \geq N$ means that N is a minor of M.

Given a graph G we indicate with VG the set of its vertices and with EG the set of its edges. Moreover, if $U \subseteq VG$ (respectively $F \subseteq EG$) we denote with $\overline{U}$ (respectively $\overline{F}$) the complement of U in VG (respectively the complement of F in EG). If P is a path in G and $u, v \in VP$, we denote with P_{uv} the part of P between u and v.

A *cut* $\delta_G(U)$ of G, where $U \subseteq VG$, is the set of edges having one end in U and the other in $VG \setminus U$. Moreover U and $VG \setminus U$ are the *shores* of the cut. A *bond* is a minimal nonempty cut. A *circuit* of G is a connected subgraph of G in which every vertex has degree two. A *cycle* is a subgraph of G in which every vertex has even degree. We will often refer to cuts and cycles as edge sets.

A *signed graph* is a pair (G, Σ) consisting of an undirected graph G and a collection Σ of its edges. In this case Σ is a *signature* of the graph. A set $F \subseteq EG$ is called *odd* in (G, Σ) if $|F \cap \Sigma|$ is odd, otherwise F is called *even*. In particular, we will refer to odd and even edges, paths, circuits and cuts. If Σ is a signature on G and H is a subgraph of G, then Σ_H indicates the signature induced on H by Σ, that is $\Sigma_H = \Sigma \cap EH$.

1.1 Even cut matroids

Given a signed graph (G, Σ), let A be the matrix obtained by adding the row incidence vector of Σ to a full row-rank matrix whose rows span the circuit space of G. Then the *even cut matroid* of (G, Σ), written $\text{ecut}(G, \Sigma)$, is the matroid represented by A. If $M = \text{ecut}(G, \Sigma)$, we will say that (G, Σ) *represents* M. As $C \subseteq EG$ is a cut if and only if it intersects every cycle in an even number of edges, the cycles of $\text{ecut}(G, \Sigma)$ are the even cut of (G, Σ). Therefore the circuits of $\text{ecut}(G, \Sigma)$ are the even bonds of (G, Σ) and the edge disjoint unions of two odd bonds of (G, Σ).

The *cut matroid* of G is $\text{cut}(G) = \text{ecut}(G, \emptyset)$.

The problem that motivates this work is describing the minor minimal 3-connected non even cut matroids.

Given a signed graph (G, Σ) we denote with $T(G, \Sigma)$ the set of vertices of G that have odd degree in $G[\Sigma]$, where $G[\Sigma]$ is the subgraph of G induced by the edges in Σ.

Remark 1.1. *Let G be a graph and $\Sigma, \Gamma \subseteq EG$. Then $T(G, \Sigma \triangle \Gamma) = T(G, \Sigma) \triangle T(G, \Gamma)$.*

Proof. The proof follows immediately from the fact that for every $v \in VG$ we clearly have $\delta(v) \cap (\Sigma \triangle \Gamma) = (\delta(v) \cap \Sigma) \triangle (\delta(v) \cap \Gamma)$.

□

Remark 1.2. *Let (G, Σ) be a signed graph and $U \subseteq VG$. Then $\delta(U)$ is an even cut if and only if $|U \cap T(G, \Sigma)|$ is even.*

Proof. To prove the Remark it is sufficient to proceed by induction on $|EG|$ and apply Remark 1.1. □

A pair (G, T), where G is a graph and $T \subseteq VG$, is called a *graft*. It follows from the previous Remark that we can represent even cut matroids by either signed graphs or by grafts. We will use both representations, depending on which is more convenient.

The main difficulty in dealing with even cut matroids is that in general they can have many completely different representations. It was shown by Whitney (see [9]) that this does not happen with cut matroids.

Let G be a graph and G_1 and G_2 two subgraphs of G. If (EG_1, EG_2) is a partition of EG, $VG_1 \cup VG_2 = VG$ and $VG_1 \cap VG_2 = \{v_1, ..., v_k\}$, with all the v_i distinct, $k \geq 1$ and $VG_i \setminus \{v_1, ..., v_k\} \neq \emptyset$ for $i = 1, 2$, we say that G has a *k-separation*. Moreover $\{v_1, ..., v_k\}$ form a *k-vertex cutset* and G_1, G_2 are the *sides* of the separation. A graph G is *k-connected* if it has no l-separation with $l < k$. A vertex v is a *cutvertex* if $\{v\}$ is a 1-vertex cutset. A subgraph $H \subseteq G$ is *non-separating* if $G \setminus VH$ is either empty or connected.

Given two vertices v_1, v_2 in two different components of a graph G, the operation of *vertex identification* consists in identifying v_1 and v_2 in a single vertex v. If v is a cutvertex of a graph G, the operation of *vertex cleaving* on v is the reverse operation of vertex identification.

Let G be a graph with a 2-vertex cutset $\{u, v\}$ and sides G_1, G_2. Rename u and v as u_i and v_i respectively in G_i for $i = 1, 2$. Then a *switch* of G on $\{u, v\}$ consists in identifying u_1 with v_2 and u_2 with v_1.

We say that two graphs G and G' are *equivalent* if G' can be obtained from G with a series of vertex identifications, vertex cleavings and switches. We denote this equivalence with $G \sim G'$.

Clearly vertex identifications, vertex cleavings and switches do not change the cycles, and hence the cuts, of the graph, so for all $G \sim G'$ we have $\text{cut}(G) = \text{cut}(G')$. We indicate with $\text{cycle}(G)$ the matroid whose circuits are the circuits of G. Hence we also have $\text{cycle}(G) = \text{cycle}(G')$ for all $G \sim G'$. The other direction was proved by Whitney (see [9]).

Theorem (Whitney 1933). *Let G and G' be graphs with no isolated vertices. Then cycle(G) is isomorphic to cycle(G') if and only if $G \sim G'$.*

From now on, we will deal only with 2-connected graphs, so we will not consider the operations of vertex identification and vertex cleaving.

Given a signed graph (G, Σ), a *resigning* of Σ on a set $F \subseteq EG$ is $\Sigma' = \Sigma \triangle F$. In particular, we will consider resigning on cycles and resigning on cuts.

Note that if C is a cycle of G, then the parity of the cuts in (G, Σ) and in $(G, \Sigma \triangle C)$ is the same, hence $\text{ecut}(G, \Sigma) = \text{ecut}(G, \Sigma \triangle C)$.

Two representations (G, Σ) and (G', Σ') are *equivalent* if $G \sim G'$ and Σ' is a resigning of Σ on a cycle of G. We denote this with $(G, \Sigma) \sim (G', \Sigma')$. Note that the operations of resigning on a cycle and switches can be done in an arbitrary order, because a switch on G does not change the set of cycles of G. Moreover if $(G, \Sigma) \sim (G', \Sigma')$ then $\text{ecut}(G, \Sigma) = \text{ecut}(G', \Sigma')$.

Unfortunately, in general an even cut matroid can have different non equivalent representations. An example is given in Figure 1.1.

We say that (H, Γ) is a *cut-minor* of (G, Σ) if (H, Γ) can be obtained from (G, Σ) by replacing it with an equivalent signed graph, and a sequence of contractions and deletions, which are defined next. Let e be an edge of (G, Σ). The *contraction* of e is $(G, \Sigma)/e = (G/e, \Sigma \setminus \{e\})$. If e is not an odd bridge of (G, Σ), we may assume (after possibly resigning on a cycle) that e is even. Then the *deletion* of e is $(G, \Sigma) \setminus e = (G \setminus e, \Sigma)$. If e is an odd bridge of (G, Σ), then $(G, \Sigma) \setminus e = (G/e, \emptyset)$. We write $(G, \Sigma) \geq_{cut} (H, \Gamma)$ to indicate that

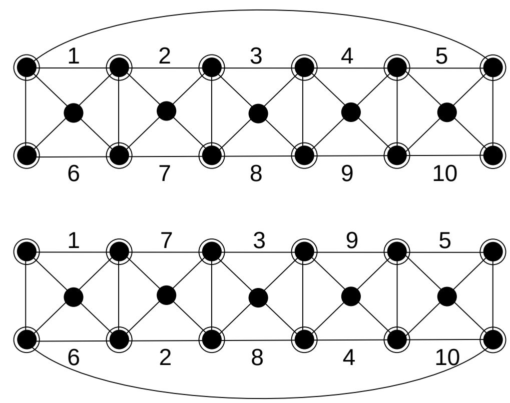

Figure 1.1: *Two non equivalent representations (as grafts) of the same even cut matroid. Circled vertices are the vertices of odd degree in the subgraph induced by the signature.*

(H, Γ) is a cut-minor of (G, Σ).

There is a one-to-one correspondence between cut-minor operations and matroid minor operations on even cut matroids. More precisely

Remark 1.3. *$ecut((G, \Sigma)/I \setminus J) = ecut(G, \Sigma) \setminus I/J$.*

This motivates the following definition. Given a matroid M and a signed graph (H, Γ) we say that the representation (H, Γ) *extends* to M if there exists a signed graph (G, Σ) such that $M = \text{ecut}(G, \Sigma)$ and (H, Γ) is a cut-minor of (G, Σ).

To deal with the problem of even cut matroids having many non equivalent representations, we will use the following Stabilizer Theorem, that is proved in the next chapter.

Theorem (Stabilizer Theorem for Even Cut Matroids). *Let (H, Γ) be a substantial signed graph, $N = ecut(H, \Gamma)$ and $M \geq N$ a binary matroid. If the representation (H, Γ) extends to M then it extends uniquely (up to equivalence).*

The definition of substantial signed graph will be given later. An important property of substantial signed graphs is the following.

Proposition 1.4. *If (H, Γ) is substantial and $(G, \Sigma) \geq_{cut} (H, \Gamma)$, then (G, Σ) is substantial.*

The Stabilizer Theorem for Even Cut Matroids and Proposition 1.4 imply that if M is a matroid such that $M \geq N$, where N is an even cut matroid with k representations all substantial, then M has at most k representations.

Now let us investigate the effect of switches on grafts. Let H be a graph with a 2-separation $\{u, v\}$, with sides H_1, H_2 and let H' be obtained from H by a switch on $\{u, v\}$. Given $T \subseteq VH$ we define T' as $T \triangle \{u, v\}$ if $|T \cap (VH_1 \setminus \{u, v\})|$ is odd and $T' = T$ otherwise. We write $\Psi_{\{uv\}}(H, T) = (H', T')$. Given a sequence of switches $\{u_1v_1, ..., u_kv_k\}$ where $k \geq 2$ we define recursively $\Psi_{\{u_1v_1,...,u_kv_k\}}(H, T) = \Psi_{\{u_1v_1\}}(\Psi_{\{u_2v_2,...,u_kv_k\}}(H, T))$.

Consider a signed graph (H, Γ) and $T = T(H, \Gamma)$. We write $(H', T') \sim (H, T)$ if for some sequence S of switches $(H', T') = \Psi_S(H, T)$. Note that if $(H', T') \sim (H, T)$ and $\Sigma \subseteq EH$ is such that $T = T(H, \Sigma)$, then $T' = T(H', \Sigma)$. In particular the order and choice of the sequence of switches is irrelevant.

We say that a graft (H, T) is *substantial* if for every pair $\{u, v\} \subseteq VH$ and any $(H', T') \sim (H, T \triangle \{u, v\})$ we have $|T'| \geq 4$. So being substantial somehow assures that $\text{ecut}(H, \Gamma)$ is not too close to being cographic, i.e. to being a cut matroid. In fact if $|T(H', \Gamma)| = 2$ for some $H' \sim H$ then $\text{ecut}(H, \Gamma)$ is cographic.

A *bipath* in (G, Σ) is an induced subgraph of (G, Σ) formed by an odd and an even edge incident with a vertex of degree two. (G, Σ) is *nearly* 3*-connected* if it is simple, 2-connected and for every 2-vertex cutset one of the sides is a bipath.

In Chapter 3 we will prove the following.

Theorem (**Escape Theorem for Even Cut Matroids**). *Let (H_0, Γ_0) be a nearly 3-connected and substantial signed graph. Let M be a 3-connected binary matroid, $M \geq$ ecut(H_0, Γ_0). If the representation (H_0, Γ_0) does not extend to M, then there exists a 3-connected matroid N such that*

- *$M \geq N$,*
- *N contains as minor a matroid isomorphic to ecut(H_0, Γ_0),*
- *$|EN| = O(|VH_0|^2)$, and*
- *(H_0, Γ_0) does not extend to N.*

We will show later that if (G_0, Σ_0) is nearly 3-connected and substantial then the matroid $\text{ecut}(G_0, \Sigma_0)$ is 3-connected. However, $\text{ecut}(G_0, \Sigma_0)$ may be 3-connected even if G_0 has a 2-separation.

Suppose that in the Escape Theorem we can replace the hypothesis of (G_0, Σ_0) being nearly 3-connected with the weaker hypothesis of $\text{ecut}(G_0, \Sigma_0)$ being 3-connected. Then we could show the following.

Consider any even cut matroid N_0 such that all its representations as graft are substantial. Then there exists a finite number of matroids $M \geq N_0$, where M is 3-connected and is a minor minimal, non even cut matroid.

Proof. Suppose N_0 is an even cut matroid having k representations $(H_1, T_1), ..., (H_k, T_k)$ as graft, and suppose all these representations are substantial. It follows from the Escape Theorem that there exists a matroid $N_1 \leq M$ that contains a matroid isomorphic to $\text{ecut}(H_1, T_1)$ but such that (H_1, T_1) does not extend to N_1. By the Stabilizer Theorem the representations $(H_2, T_2), ..., (H_k, T_k)$ extend each to at most one representation $(H_2', T_2'), ..., (H_k', T_k')$. By Proposition 1.4, each of (H_i', T_i') for $i = 2, ..., k$ is substantial. Hence to complete the proof it is sufficient to repeat the argument at most $k - 1$ times.

□

The proof of the Escape Theorem is constructive. We are currently developing an algorithm to find all these matroids.

1.2 Even cycle matroids

Given a signed graph (G, Σ), the *even cycle matroid* of (G, Σ), written $\text{ecycle}(G, \Sigma)$, is the matroid represented by the matrix obtained by adding the row incidence vector of Σ to a full row-rank matrix whose rows span the cut space of G. If $M = \text{ecycle}(G, \Sigma)$, we will say that (G, Σ) *represents* M.

Note that the cycles of $\text{ecycle}(G, \Sigma)$ are the even cycles of (G, Σ). Therefore the circuits of $\text{ecycle}(G, \Sigma)$ are the even circuits of (G, Σ) and the edge disjoint unions of two odd circuits of (G, Σ).

If $\delta(U)$ is a cut of G, then the parity of the cycles in (G, Σ) and in $(G, \Sigma \triangle \delta(U))$ is the same, hence $\text{ecycle}(G, \Sigma) = \text{ecycle}(G, \Sigma \triangle \delta(U))$.

Two representations (G, Σ) and (G', Σ') are *equivalent* if $G \sim G'$ and Σ' is a resigning of Σ on a cut of G. We denote this with $(G, \Sigma) \sim (G', \Sigma')$. Note that the operations of resigning on a cut and switches can be done in an arbitrary order. Moreover if $(G, \Sigma) \sim (G', \Sigma')$ then $\text{ecycle}(G, \Sigma) = \text{ecycle}(G', \Sigma')$.

As for even cut matroids, the difficulty in dealing with even cycle matroids is that they can have different non equivalent representations.

We say that (H, Γ) is a *cycle-minor* of (G, Σ) if (H, Γ) can be obtained from (G, Σ) by replacing it with an equivalent signed graph, and a sequence of contractions and deletions, which are defined next. Let e be an edge of (G, Σ). The *deletion* of e is $(G, \Sigma) \setminus e = (G \setminus e, \Sigma \setminus \{e\})$. If e is not an odd loop of (G, Σ), we may assume (after possibly resigning on a cut) that e is even. Then the *contraction* of e is $(G, \Sigma)/e = (G/e, \Sigma)$. If e is an odd loop of (G, Σ), then $(G, \Sigma)/e = (G \setminus e, \emptyset)$.

We write $(G, \Sigma) \geq_{cycle} (H, \Gamma)$ to indicate that (H, Γ) is a cycle-minor of (G, Σ).

There is a one-to-one correspondence between cycle-minor operations and matroid minor operations on even cycle matroids.

Remark 1.5. *ecycle*$((G, \Sigma) \setminus I/J) =$ *ecycle*$(G, \Sigma) \setminus I/J$.

Analogously to what we did for even cut matroids, given a matroid M and a signed graph (H, Γ) we say that the representation (H, Γ) *extends* to M if there exists a signed graph (G, Σ) such that $M = \text{ecycle}(G, \Sigma)$ and (H, Γ) is a cycle-minor of (G, Σ).

For even cycle matroids we proved similar results as for even cut matroids. Here we give the statement of the two main Theorems.

Theorem (Stabilizer Theorem for Even Cycle Matroids). *Let (H, Γ) be a substantial signed graph, $N =$ ecycle(H, Γ) and $M \geq N$ a binary matroid. If the representation (H, Γ) extends to M then it extends uniquely (up to equivalence).*

In the Stabilizer Theorem for Even Cycle matroids we can consider two different definitions of substantial representation. Both those definitions work in the proof of the Theorem and will be given later.

When referring to substantial representation of even cycle matroids, the following property holds.

Proposition 1.6. *If (H, Γ) is substantial and $(G, \Sigma) \geq_{cycle} (H, \Gamma)$, then (G, Σ) is substantial.*

Similarly to before, by the Stabilizer Theorem for Even Cycle Matroids and Proposition 1.6, if M is a matroid such that $M \geq N$, where N is an even cycle matroid with k representations all substantial, then M has at most k representations as even cycle matroid.

A signed graph (G, Σ) is *almost simple* if G has no loops and no series edges, and for every pair of parallel edges $e_1, e_2 \in EG$, $e_1 \in \Sigma$ and $e_2 \notin \Sigma$ (or vice versa). It can be checked that if (G, Σ) is 3-connected and almost simple, then ecycle(G, Σ) is 3-connected.

Theorem (Escape Theorem for Even Cycle Matroids). *Let (H_0, Γ_0) be a substantial signed graph, with H_0 3-connected and almost simple. Let M be a 3-connected binary matroid, $M \geq$ ecycle(H_0, Γ_0). If the representation (H_0, Γ_0) does not extend to M, then there exists a 3-connected matroid N such that*

- *$M \geq N$,*
- *N contains as minor a matroid isomorphic to ecut(H_0, Γ_0),*
- *$|EN| = O(|VH_0|^2)$, and*
- *(H_0, Γ_0) does not extend to N.*

To explain what being substantial means for the representation of an even cycle matroid, we need the following definitions. A signed graph (G, Σ) is *Σ-bipartite* if it has no odd circuits. A vertex $v \in VG$ is a *blocking vertex* in (G, Σ) if $(G, \Sigma) \setminus \{v\} = (G \setminus \{v\}, \Sigma \setminus \delta(v))$ is Σ-bipartite. Two vertices $\{u, v\} \subseteq VG$ form a *blocking pair* if $(G, \Sigma) \setminus \{u, v\}$ is Σ-bipartite. Finally, a *blocking triple* for (G, Σ) is a set of vertices $\{u, v, w\} \subseteq VG$ such that $(G, \Sigma) \setminus \{u, v, w\}$ is Σ-bipartite.

The binary matroid $AG(3, 2)$ is the even cycle matroid represented by the signed graph in Figure 1.2.

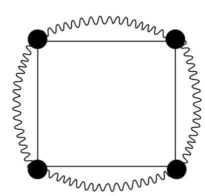

Figure 1.2: *Representation of $AG(3, 2)$ as even cycle matroid. Straight edges are even, wavy edges are odd.*

In the Stabilizer Theorem and Escape Theorem for Even Cycle Matroids we can define a representation (G, Σ) to be *substantial* in one of the following ways:

- (G, Σ) has no blocking pair and no $AG(3, 2)$ minor.
- (G, Σ) has no blocking triple.

Note that, for each of the definitions, being substantial assures that ecycle(H, Γ) is not too close to being graphic, i.e. to being a cycle matroid. In fact, if (G, Σ) has a blocking vertex then ecycle(H, Γ) is graphic.

Similarly as before, suppose that in the Escape Theorem for even cycle matroids we can replace the hypothesis of (G_0, Σ_0) being 3-connected and almost simple with the weaker hypothesis of ecut(G_0, Σ_0) being 3-connected. Then we could show

Consider any even cycle matroid N such that all its representations as signed graphs are substantial. Then there exists a finite number of matroids $M \geq N$, where M is 3-connected and is a minor minimal non even cycle matroid.

Our original motivation was to look at the class of matroids which is the union of the following classes:

1. even cycle matroids
2. even cut matroids
3. duals of even cycle matroids
4. duals of even cut matroids.

Let $\mathcal{C}$ be such class.

Under the technical condition that N is a matroid such that:

1. for all (G, T) such that $N = \text{ecut}(G, T)$, (G, T) is substantial, and
2. for all (G, T) such that $N^* = \text{ecut}(G, T)$, (G, T) is substantial, and
3. for all (G, Σ) such that $N = \text{ecycle}(G, T)$, (G, Σ) is substantial, and
4. for all (G, Σ) such that $N^* = \text{ecycle}(G, T)$, (G, Σ) is substantial,

there exists a finite number of matroids $M \geq N$ with M 3-connected and M minor minimal not in $\mathcal{C}$.

Finally note that we can apply the same kind of argument to any of the following classes $\mathcal{C}'$, where $\mathcal{C}'$ is either

- the union of even cut and dual of even cut matroids
- the union of even cycle and dual of even cycle matroids
- the union of even cut and even cycle matroids
- the union of even cut and dual of even cycle matroids
- the union of even cycle, dual of even cycle and dual of even cut matroids
- the union of even cut, dual of even cycle and dual of even cut matroids.

Note that together with $\mathcal{C}$ and the classes of even cycle and even cut matroids, up to duality these are all the classes of matroids obtained by taking the union of even cycle, even cut, dual of even cycle and dual of even cut matroids.

More in general, if we know the excluded minors for a class of matroids $\mathcal{M}$, we can use this method to find the excluded minors for the union of $\mathcal{M}$ and the class of even cut (or even cycle) matroids, starting from the excluded minors for $\mathcal{M}$.

1.3 Applications

Given a graph G, two vertices $s, t \in VG$ and a vector of weight $w \in \mathbb{R}_+^{EG}$, consider the following problems:

$$\begin{array}{lll} \min & w^T x & \\ \text{s.t.} & x(P) \geq 1 \quad \forall\ st\text{-path } P & \text{(IP)} \\ & x \in \{0,1\}^{EG} & \end{array}$$

$$\begin{array}{lll} \max & e^T y & \\ \text{s.t.} & \sum(y_P : e \in EP, P st\text{-path}) \leq w_e \quad \forall e \in EG & \text{(D)} \\ & y \geq 0 & \end{array}$$

Note that (D) is the dual of the LP relaxation of (IP). A solution to (IP) can be interpreted as a minimum *st*-cut, while a solution to (D) gives a fractional maximal *st*-flow.

By the Max-Flow Min-Cut Theorem by Ford and Fulkerson (see [3]), for all $w \in \mathbb{R}_+^{EG}$ the optimal value of (IP) is equal to the optimal value of (D).

We can generalize the concept of minimum cut and maximum flow to binary matroids. Given a matroid M and $f \in EM$, a set of the form $C \setminus \{f\}$, where C is a circuit of M using f, is called an *f-path*. We can define the analogue of (IP) and (D) in terms of f-paths.

Let M be a matroid, $f \in EM$ and $w \in \mathbb{R}_+^{EM\setminus\{f\}}$. Consider

$$\begin{array}{lll} \min & w^T x & \\ \text{s.t.} & x(P) \geq 1 \quad \forall\ f\text{-path } P & \text{(IP')} \\ & x \in \{0,1\}^{EM\setminus\{f\}} & \end{array}$$

$$\begin{array}{llll} \max & e^T y & & \\ \text{s.t.} & \sum(y_P : e \in EP, P f\text{-path}) \leq w_e & \forall e \in EM \setminus \{f\} & \text{(D')} \\ & y \geq 0 & & \end{array}$$

We say that M is *f-flowing* if for all $w \in \mathbb{R}_+^{EM\setminus\{f\}}$, the optimal values of (IP') and (D') are the same. M is 1-*flowing* if it is f-flowing for all $f \in EM$.

We will now introduce a conjecture by Seymour about 1-flowing matroids.

The matroid $AG(3,2)$ is represented by the matrix

$$\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & f \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

We also consider two other particular binary matroids, T_{11} and its dual T_{11}^*. The representation of T_{11}^* is

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & f & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

A representation of T_{11} can easily be deduced from the representation of T_{11}^*. Now choose f to be any element of $AG(3,2)$, and the element labeled as f in the above representation of T_{11}^*, and in the correspondent representation of T_{11}. Then the following holds.

Proposition 1.7. *$AG(3,2)$, T_{11} and T_{11}^* are not f-flowing for the choice of f above.*

Proof. We will prove the statement for $AG(3,2)$.

It is easy to check that for any element $f \in EAG(3,2)$, the f-paths are represented by the Fano lines (see Figure 1.3). For example, the element 1 is in an f-path together with 2 and 3, because $\{1,2,3,f\}$ is a circuit of $AG(3,2)$.

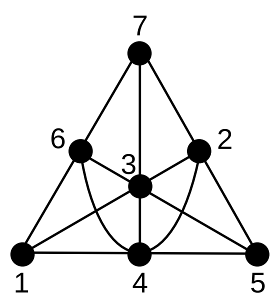

Figure 1.3: *Representation of the f-paths for some element f of $AG(3,2)$.*

Take $w = \mathbf{1}$. Now the vectors $x_e^* = 1/3 \,\forall\, e \in EAG(3,2)\backslash\{f\}$ and $y_P^* = 1/3 \,\forall\, f$-paths P are feasible solutions for (LP') and (D') respectively, where (LP') is the LP-relaxation of

(IP'). Moreover $\sum_{e\in EAG(3,2)\setminus\{f\}} x^*_e = 7/3 = \sum_{f\text{-paths } P} y^*_P$, so x^* and y^* are optimal solutions for (LP') and (D'). Hence (D') has an optimal value that is fractional, even if w is integral, and $AG(3,2)$ is not f-flowing.

An analogous proof holds for T_{11} and T^*_{11}. The f-paths for T^*_{11} are the odd circuits of K_5, while the f-paths for T_{11} are the complements of cuts of K_5.

□

Proposition 1.8. *The property of being* 1*-flowing is closed under minors.*

Proof. We will show that, given $f \in EM$, with M f-flowing, every minor of M in which f is not contracted or deleted is f-flowing. By induction it is sufficient to show that the statement holds for a single deletion or contraction of an element $e \in EM \setminus \{f\}$.

Let $e \in EM \setminus \{f\}$, $N = M \setminus e$ and $w \in \mathbb{R}_+^{EM\setminus\{e,f\}}$.

Consider (IP') and (D') for N, f, w. Define a new weight vector $\hat{w} \in \mathbb{R}_+^{EM\setminus\{f\}}$ by setting $\hat{w}_g = w_g$ if $g \in EM \setminus \{e, f\}$ and $\hat{w}_e = 0$.

Let x^* and y^* be optimal solutions for the minimum cut and the maximum (fractional) flow for $M, f, \hat{w}$. As M is f-flowing, $\hat{w}^T x^* = e^T y^*$.

Note that as $w_e = 0$, we may assume $y^*_P = 0$ for all f-path P using e. So x^* and y^* restricted to $EM \setminus \{e, f\}$ are feasible solutions for (IP') and (D').

Hence the optimal value of (IP') is not greater than

$$\sum_{g\in EM\setminus\{e,f\}} \hat{w}_g^T x^*_g = \sum_{g\in EM\setminus\{e,f\}} \hat{w}_g^T x^*_g + \hat{w}_e^T x^*_e = \hat{w}^T x^* = e^T y^* =$$

$$= \sum_{P f\text{-}path, e\notin P} y^*_P + \sum_{P f\text{-}path, e\in P} y^*_P = \sum_{P f\text{-}path, e\notin P} y^*_P \le (D').$$

So by weak duality it follows that N is f-flowing.

For the case N/e the proof is similar, setting $w_e = \infty$.

□

Remark 1.9. *It is easy to check that* $U_{2,4}$ *is not* 1*-flowing. Hence, by the above Proposition it follows that all non binary matroids are not* 1*-flowing.*

The results presented so far imply that any binary matroid containing $AG(3,2)$, T_{11} or T_{11}^* as minor is not 1-flowing. Seymour conjectured that the converse is also true (see [6]), that is

Conjecture (Seymour 1977). *A binary matroid M is 1-flowing if and only if it contains no $AG(3,2)$, T_{11} or T_{11}^* minor.*

Guenin showed that Seymour's conjecture holds for even cycle and even cut matroids (see [4]). In fact he proved the following two theorems.

Theorem 1.10. *Let M be an even cycle matroid. Then M is 1-flowing if and only if it has no $AG(3,2)$ and no T_{11}^* minor.*

Theorem 1.11. *Let M be an even cut matroid. Then M is 1-flowing if and only if it has no $AG(3,2)$ and no T_{11} minor.*

Note that T_{11} is not an even cycle matroid, and T_{11}^* is not an even cut matroid.

It was shown by Seymour (see [7]) that the following holds.

Proposition 1.12. *The property of being 1-flowing is closed under duality.*

Hence Seymour's conjecture is proved also for duals of even cycle and duals of even cut matroids.

A further result proved by Cornuéjols and Guenin (see [1]) is the following.

Theorem 1.13. *If M is a minor minimal non 1-flowing matroid, then it is internally 4-connected (i.e. it is 3-connected and all its 3-separations have one side of size ≤ 3).*

The tools developed in this work may be used to attempt to prove Seymour's Conjecture. Let $\mathcal{C}$ be the class of matroids which is the union of even cycle, even cut, dual of even cycle and dual of even cut matroids. To prove Seymour's Conjecture we first want to characterize the minor minimal matroids not in $\mathcal{C}$ which are 3-connected and which do not contain $AG(3,2)$, T_{11} or T_{11}^* as minor.

Then, if any matroids are left, we will try to bridge any 3-separation using blocking sequences.

Chapter 2

Stabilizer Theorem for Even Cut Matroids

The aim of this chapter is to prove the following.

Theorem 2.1 (**Stabilizer Theorem for Even Cut Matroids**). *Let (H, Γ) be a substantial signed graph, $N = ecut(H, \Gamma)$ and $M \geq N$ a binary matroid. If the representation (H, Γ) extends to M then it extends uniquely (up to representations equivalence).*

The condition that (H, Γ) is substantial cannot be removed. For example the two grafts in Figure 1.1 are obtained respectively from the grafts $(H, T), (H', T')$ in Figure 2.1 by adding the edge uv or $u'v'$. Note that H' is obtained from H with four switches. The representation (H, T) is not substantial, since $(H, T \triangle \{u, v\}) \sim (H', \{u', v'\})$.

Note that the planar dual of (H, Γ) is a signed graph with a blocking vertex, hence the matroid $\mathrm{ecut}(H, T)$ is in fact cographic.

Proposition 2.2. *If a graft contains a substantial graft as a minor, then it is substantial.*

Before proceeding with the proof of this result we shall require some notation and remarks. Let (G, Σ) be a signed graph and $T = T(G, \Sigma)$. Given $uv \in EG$ we indicate with $(G, T) \backslash uv$ (respectively $(G, T)/uv$) the graft corresponding to $(G, \Sigma) \backslash uv$ (respectively $(G, \Sigma)/uv$), where the operations of deletion and contraction are defined as in cut-minor operation. Then it is easy to check that $(G, T) \setminus uv = (G \setminus uv, T)$ and $(G, T)/uv = (G/uv, T')$, where T' is defined as follows. Let w be the vertex obtained by contracting

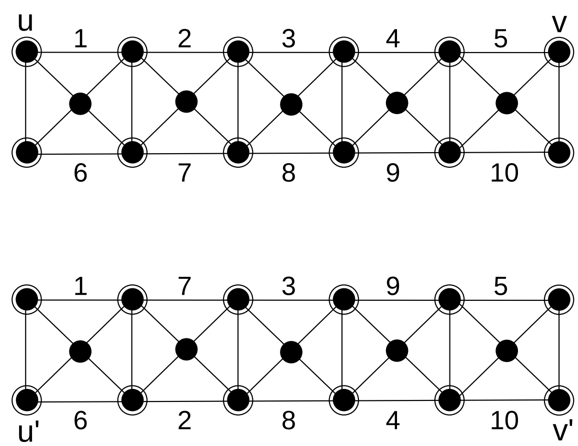

Figure 2.1: *Circled vertices are the vertices of odd degree in the subgraph induced by the signature.*

uv. Then $w \in T'$ if and only if $|\{u, v\} \cap T| = 1$ and for all $x \neq w$, $x \in T'$ if and only if $x \in T$.

Let G be a graph and let $T_1, T_2 \subseteq VG$. We denote the graft $(G, T_1 \triangle T_2)$ by $(G, T_1) \triangle (G, T_2)$. We leave the following observation as an exercise.

Remark 2.3. *Let G be a graph, $L_1, L_2 \subseteq VG$, $I, J \subseteq EG$ where $I \cap J = \emptyset$, and let S be a sequence of switches that can be applied to G.*

1. *If $(H, T) = (G, L_1) \setminus I/J$ then $|T| \leq |L_1|$*

2. *$[\Psi_S(G, L_1)] \setminus I/J = \Psi_S[(G, L_1) \setminus I/J]$*

3. *$(G, L_1 \triangle L_2) \setminus I/J = [(G, L_1) \setminus I/J] \triangle [(G, L_2) \setminus I/J]$.*

Proof of Proposition 2.2. Suppose we have two grafts (G, L_1) and $(H, T_1) = (G, L_1) \setminus I/J$ and suppose that (G, L_1) is not substantial, i.e. there exists $L_2 \subseteq VG$ with $|L_2| = 2$ and there is a sequence S of flips such that $\Psi_S(G, L_1 \triangle L_2) = (\hat{G}, \hat{L})$ where $|\hat{L}| = 2$. It follows from Remark 2.3 (1) that $(\hat{G}, \hat{L}) \setminus I/J = (\hat{H}, \hat{T})$ for some $\hat{T}$ where $|\hat{T}| \leq 2$. By the same Remark it also follows that $(G, L_2) \setminus I/J = (H, T_2)$ for some T_2 where $|T_2| \leq 2$.

Thus

$$\begin{aligned}(\hat{H}, \hat{T}) &= \Psi_S(G, L_1 \triangle L_2) \setminus I/J = \\ &= \Psi_S[(G, L_1 \triangle L_2) \setminus I/J] = \\ &= \Psi_S[((G, L_1) \setminus I/J) \triangle ((G, L_2) \setminus I/J)] = \\ &= \Psi_S[(H, T_1) \triangle (H, T_2)] = \\ &= \Psi_S(H, T_1 \triangle T_2)\end{aligned}$$

where the second equality follows from Remark 2.3 (2) and the third from Remark 2.3 (3).

Since $|T_2|, |\hat{T}| \leq 2$, (H, T_1) is not substantial. Note that some of the switches S of G may correspond to switches of H where one of the sides is empty.

□

Before we can proceed with the proof of Theorem 2.1, we will require a number of preliminary definitions and lemmas. Consider a graph G and let R be the vertex edge incidence matrix of G (i.e. the rows of R correspond to the vertices of G, its columns to the edges of G and column uv of R has exactly two non-zero entries, namely 1's in positions u and v). It is easy to check that R is a representation of cycle(G) (the graphic matroid of G). Let M be a (binary) extension of cycle(G), i.e. there is an element Ω of M such that $M \setminus \Omega =$ cycle(G). Then there exists a matrix R' representing M where R' is obtained from R by adding an extra column b. Define T to be the set of vertices of G corresponding to the entries of b with a 1 entry. Then it can be easily checked that the cycles of M avoiding Ω are the cycles of G, and the cycles C of M using Ω are exactly the sets for which $C \setminus \{\Omega\}$ is a T-join of G, i.e. a set of edges F such that the vertices of odd degree in $G[F]$ are exactly the vertices in T.

We call M the *graft matroid* of (G, T). Note that if $|T| = 2$ then the T-join consists of a set of cycles and a path with ends in T. This implies the following remark.

Remark 2.4. *Let M be the graft matroid of (G, T). If $T = \{u, v\}$ then $M =$ cycle$(G+uv)$.*

We will make repeated use of Whitney's Theorem, which for convenience we report here again.

Theorem 2.5 (Whitney). *Let G, G' be graphs. Then cycle$(G) =$ cycle(G') if and only if $G \sim G'$.*

Lemma 2.6. *Consider two graphs G_1, G_2 where $G_1 \sim G_2$ and let $B \subseteq EG_1$. Suppose $T(G_i, B) = \{u_i, v_i\}$ for $i = 1, 2$. Then $G_1 + u_1v_1 \sim G_2 + u_2v_2$.*

Proof. For $i = 1, 2$ let $T_i := T(G_i, B)$ and let M_i be the graft matroid of (G_i, T_i).

Claim 1. *$M_1 = M_2$.*

Proof (claim). It suffices to show that M_1 and M_2 have the same cycles. Since $G_1 \sim G_2$, $\text{cycle}(G_1) = \text{cycle}(G_2)$. Note that for $i = 1, 2$ every T_i-join of G_i is of the form $B \triangle C$, where C is a cycle of G_i. It follows that the T_1-joins of G_1 and T_2-joins of G_2 are the same, hence M_1, M_2 have the same cycles.

◇

Remark 2.4 implies that for $i = 1, 2$, $M_i = \text{cycle}(G_i + u_iv_i)$. Thus $\text{cycle}(G_1 + u_1v_1) = \text{cycle}(G_2 + u_2v_2)$. It follows from Theorem 2.5 that $G_1 + u_1v_1 \sim G_2 + u_2v_2$.

□

Remark 2.7. *Let (H, Γ) be a signed graph and let F be a spanning tree of H. Then we can resign on cycles so that $\Gamma \subseteq EF$.*

Proof. For every edge $g \in \Gamma \setminus EF$ let Q be the unique path of F linking the ends of g. Then resigning on $Q + g$ we make g become even and we do not change the parity of the other edges in $\Gamma \setminus EF$.

□

In the following next two Lemmas (H, Γ) will denote a non-eulerian signed graph, i.e. Γ is not a cycle. Moreover N will denote the matroid $\text{ecut}(H, \Gamma)$ and I a basis of N. For every element $f \in \overline{I}$, $I \cup \{f\}$ contains a unique circuit of N, i.e. an even cut $\delta(U_f)$ of (H, Γ) which we call the *fundamental even cut* for f.

Lemma 2.8. *There exist disjoint trees F_1, F_2 of H and $f_0 \in EH$ such that $F := F_1 \cup F_2 \cup \{f_0\}$ is a spanning tree of H and $\overline{I} = EF_1 \cup EF_2$. Moreover we may assume, possibly after resigning on cycles, that $\Gamma \subseteq EF$ and that $f_0 \in \Gamma$.*

Proof. Let $f \in \overline{I}$. Then $I \cup \{f\}$ contains an even cut $\delta(U_f)$. Suppose $\overline{I}$ contains a cycle C and let $f \in EC$, then $I \cup \{f\}$ and I contain the same cuts, a contradiction. Thus $\overline{I}$ is acyclic. Suppose $\overline{I}$ is a spanning tree of H. By Remark 2.7 we may assume $\Gamma \subseteq \overline{I}$. Since (H, Γ) is not eulerian, there exists $f_0 \in \overline{I} \cap \Gamma$. But then $I \cup \{f_0\}$ contains a unique cut which is odd, contradiction. Suppose $\overline{I}$ is a forest with at least three components, then there exist distinct cuts $\delta(U_1), \delta(U_2)$ included in I. Since I is a basis, $\delta(U_1), \delta(U_2)$ must be odd, but then $\delta(U_1) \triangle \delta(U_2) = \delta(U_1 \triangle U_2)$ is an even cut included in I, a contradiction. Thus $\overline{I} = F_1 \cup F_2$, where $F_1 \cup F_2$ spans VH and F_1, F_2 are connected. Since G is connected, there exists $f_0 \in EH$ such that $F := F_1 \cup F_2 \cup \{f_0\}$ is a spanning tree. By Remark 2.7 we may assume $\Gamma \subseteq F$. Finally, since $\delta(U_{f_0})$ is an odd cut, $f_0 \in \Gamma$.

□

Lemma 2.9. *Suppose $F := F_1 \cup F_2 \cup \{f_0\}$ is as in Lemma 2.8. Let $uv \in I$ and let B be the set of all $f \in EF_1 \cup EF_2$ such that the fundamental even cut $\delta(U_f)$ contains uv. Let Q be the unique uv-path in F.*

1. *If $u, v \in VF_i$ for some $i \in \{1, 2\}$, then $B = EQ$.*
2. *If $u \in VF_i$, $v \in VF_{3-i}$ for some $i \in \{1, 2\}$, then $B = EQ \triangle \Gamma$.*

Proof. For (1) and (2) we show $B \subseteq EQ$ (respectively $B \subseteq EQ \triangle \Gamma$), leaving the other inclusions as an easy exercise. Consider $f \in B$ and $\delta(U_f)$ as in the statement. Then $\delta(U_f)$ intersects the circuit $Q + uv$ exactly twice. Then for (1) it implies $f \in EQ$. Now consider case (2). If $f \in EQ$, then as $f_0 \notin \delta(U_f)$ we must have $f \notin \Gamma$. If $f \notin EQ$, then $f_0 \in \delta(U_f)$, hence $f \in \Gamma$.

□

We are now ready for the proof of the main result of this chapter.

Proof of Stabilizer Theorem for Even Cut Matroids. We have $N = \text{ecut}(H, \Gamma)$, where Γ is a T-join of H. We may assume that either $N = M \setminus e$ or $N = M/e$.

Case 1: $N = M/e$.

Since M is an even cut matroid, $M = \text{ecut}(G, \Sigma)$ for some signed graph (G, Σ). Remark 1.3 implies that either $(H, \Gamma) = (G, \Sigma) \setminus e$, and we may assume $\Sigma = \Gamma$, or $(H, \Gamma) = (G, \Sigma)/e$ where e is an odd bridge and $\Gamma = \emptyset$. The latter case cannot occur, for otherwise $T = \emptyset$ and (H, Γ) is not substantial. Thus $(H, \Gamma) = (G, \Gamma) \setminus e$. Suppose M has distinct representations (G, Γ) and (G', Γ) where (H, Γ) is obtained from (G, Γ) by deleting an edge $e = uv$, and (H', Γ) is obtained from (G', Γ) by deleting an edge $e = u'v'$. Let I be a basis of M containing e. Define $B \subseteq \overline{I}$ to be the set of all $f \in \overline{I}$ such that the fundamental circuit

$\delta(U_f)$ in $I \cup \{f\}$ uses element e. Applying Lemma 2.9 to (G, Γ) it follows that there exists a uv-path Q of H and either

(a) $B = EQ$, or

(b) $B = EQ \triangle \Gamma$.

Similarly, applying Lemma 2.9 to (G', Γ), there exists a $u'v'$-path Q' of H' and either

(a) $B = EQ'$, or

(b) $B = EQ' \triangle \Gamma$.

Suppose (a) occurs for both G and G', then B with Lemma 2.6 implies that $G \sim G'$, as required. Suppose (b) occurs for both G and G'. Then $B \triangle \Gamma$ with Lemma 2.6 implies that $G \sim G'$. Hence, we may assume that (b) occurs for G, i.e. $B = EQ \triangle \Gamma$ and (a) occurs for G', i.e. $B = EQ'$. Then $EQ \triangle \Gamma = EQ'$. Thus $(H, \Gamma \triangle \{u, v\}) \sim (H', \{u', v'\})$, which implies that (H, Γ) is not substantial.

Case 2: $N = M \setminus e$.

Since M is an even cut matroid, $M = \text{ecut}(G, \Sigma)$. Remark 1.3 implies that $(H, \Gamma) = (G, \Sigma)/e$. Suppose M has distinct representations $(G, \Sigma), (G', \Sigma')$, where $(G, \Sigma)/e = (G', \Sigma')/e = (H, \Gamma)$.

Claim 1. (G, Σ) *and* (G', Σ') *have the same odd cuts.*

Proof (claim). Let B_1 be an odd cut of (G, Σ). It suffices to show that B_1 is an odd cut of (G', Σ'), since then we can prove similarly that every odd cut of (G', Σ') is an odd cut of (G, Σ). Since $(G, \Sigma)/e \sim (G', \Sigma')/e$, we may assume that B_1 does not use the edge e. Since (H, T) is substantial there exists an odd cut B_2 of (H, Γ) avoiding e. Then $B_1 \triangle B_2$ is an even cut of (G, Σ). Since $\text{ecut}(G, \Sigma) = \text{ecut}(G', \Sigma')$, $B_1 \triangle B_2$ is an even cut of (G', Σ'). Since B_2 does not use e, B_2 is an odd cut of (G', Σ'). It follows that $(B_1 \triangle B_2) \triangle B_2 = B_1$ is an odd cut of (G', Σ'), as required.

◇

By the above Claim and the fact that $\text{ecut}(G, \Sigma) = \text{ecut}(G', \Sigma')$, $\text{cut}(G) = \text{cut}(G')$, hence $\text{cycle}(G) = \text{cycle}(G')$. Then Theorem 2.5 implies that $G \sim G'$. Because of the Claim, $|(\Sigma \triangle \Sigma') \cap \delta(U)|$ is even for every cut $\delta(U)$. It follows that $(\Sigma \triangle \Sigma')$ is eulerian, i.e. Σ' is obtained from Σ by resigning on a cycle. Hence, the representations (G, Σ) and (G', Σ') are equivalent.

$\square$

Chapter 3

Main Theorem

In this chapter we will prove the other main result, that is the following.

Theorem 3.1 (**Escape Theorem for Even Cut Matroids**). *Let (H_0, Γ_0) be a nearly 3-connected and substantial signed graph. Let M be a 3-connected binary matroid, $M \geq$ ecut(H_0, Γ_0). If the representation (H_0, Γ_0) does not extend to M, then there exists a 3-connected matroid N such that*

- $M \geq N$,
- *N contains as minor a matroid isomorphic to ecut(H_0, Γ_0),*
- $|EN| = O(|VH_0|^2)$, *and*
- (H_0, Γ_0) *does not extend to N.*

As the proof of Theorem 3.1 is quite long and requires some intermediate results, we will first give a general idea of how it is constructed. We first need some definitions that will be used later.

Given a graph G with two signatures Σ and τ, we define ecut(G, Σ, τ) as follows. Let A' be the representation of (G, Σ), that is the rows $1, ..., k-1$ of A' are the incidence vectors of the cycles of G which span the cycle space of G, and row k of A' is the incidence vector of Σ. Let A be the matrix with k rows and columns indexed by $EG \cup \{\Omega\}$ defined as follows: the matrix obtained from A by removing $col_\Omega(A)$ (i.e. the column indexed by Ω) is A', and the entry k of $col_\Omega(A)$ is 1, while for every $i = 1, ..., k-1$ entry i of $col_\Omega(A)$ is

1 if the circuit corresponding to row i of A' is τ-odd, it is 0 otherwise. Then $\mathrm{ecut}(G,\Sigma,\tau)$ is the (binary) matroid represented over $GF(2)$ by A.

Remark 3.2. *Let G be a graph with two signatures Σ and τ.*

1. *If Ω is defined as above, $ecut(G,\Sigma,\tau)\setminus\Omega = ecut(G,\Sigma)$*
2. *Let M be a matroid and $\Omega\in EM$ such that $M\setminus\Omega = ecut(G,\Sigma)$ and Ω is not a loop of M. Then for some resigning $\Sigma' = \Sigma\triangle C$ on a cycle C and some signature τ we have $M = ecut(G,\Sigma',\tau)$.*

Proof. Part (1) is straightforward. Now suppose that $M\setminus\Omega = \mathrm{ecut}(G,\Sigma)$ and Ω is not a loop of M. Let A be the matrix representing M obtained by adding the column corresponding to Ω to the matrix A' representing (G,Σ) (A' defined as in the definition above). Then if $col_\Omega(A)$ has entry k equal to 1, pick $\Sigma'=\Sigma$, otherwise there exists an index $i\neq k$ such that $col_\Omega(A)$ has entry i equal to 1. Let C_i be the cycle corresponding to row i of A', and $\Sigma'=\Sigma\triangle C_i$. Now the statement is satisfied taking as τ the signature on G such that the cycles corresponding to indices i for which $col_\Omega(A)$ has a 1 entry are odd, while those for which $col_\Omega(A)$ has a 0 entry are even.

□

Remark 3.3. *A representation (G,Σ) extends to a representation of $ecut(G,\Sigma,\tau)$ if and only if (G',τ) has a blocking vertex for some $G'\sim G$.*

Proof. Suppose (G',τ) has a blocking vertex v for some $G'\sim G$. We may assume we resigned τ on a cut so that $\tau\subseteq\delta_{G'}(v)$. Let $N_\tau(v)=\{u\in VG' : uv\in\tau\}$ and $N_{\overline{\tau}}(v)=\{u\in VG' : uv\in EG'\setminus\tau\}$. Then we construct a new signed graph $(\hat{G},\hat{\Sigma})$ from (G,Σ) by splitting v into two new vertices v_1,v_2, and dividing the edges of $\delta(v)$ into $\{uv_1 : u\in N_\tau(v)\}$ and $\{uv_2 : u\in N_{\overline{\tau}}(v)\}$. Finally we add an edge v_1v_2 and define $\hat{\Sigma}=\Sigma\cup\{v_1v_2\}$. It is easy to check that $(\hat{G},\hat{\Sigma})$ is a representation of $\mathrm{ecut}(G,\Sigma,\tau)$ as even cut matroid.

Now suppose that the representation (G,Σ) extends to a representation $(\hat{G},\hat{\Sigma})$ of $\mathrm{ecut}(G,\Sigma,\tau)$. Let Ω be such that $\mathrm{ecut}(G,\Sigma)=\mathrm{ecut}(G,\Sigma,\tau)\setminus\Omega$. Then Ω corresponds to an edge v_1v_2 in $(\hat{G},\hat{\Sigma})$ such that $(G,\Sigma)=(\hat{G},\hat{\Sigma})/v_1v_2$. By definition, a circuit of $\mathrm{ecut}(G,\Sigma,\tau)$ contains Ω if and only if it is τ-odd. Then all τ-odd circuits use v_1v_2 in $\hat{G}$, hence the vertex in G obtained by contracting v_1v_2 is a τ-blocking vertex.

□

Given two matroids M, N, we say that M is a *row extension* of N if it is obtained from N by uncontracting an element, that is $N = M/e$ for some $e \in EM$. We say that M is a *column extension* of N if it is obtained from N by undeleting an element, that is $N = M \setminus e$ for some $e \in EM$.

From now on, whenever we will refer to the *base graph*, we will mean the signed graph (H_0, Γ_0) defined as in Theorem 3.1.

We say that a triple (G, Σ, τ) is a *certificate* for the base graph (H_0, Γ_0) and the matroid M if

- (G, Σ) contains a cut-minor isomorphic to (H_0, Γ_0),
- $\mathrm{ecut}(G, \Sigma, \tau) \leq M$,
- $\mathrm{ecut}(G, \Sigma, \tau)$ is 3-connected, and
- the representation (G, Σ) does not extend to $\mathrm{ecut}(G, \Sigma, \tau)$.

Remark 3.4. *To prove the Escape Theorem for Even Cut Matroids it suffices to show that there exists a certificate* (G, Σ, τ) *such that* $|EG| = O(|VH_0|^2)$.

Proof. Choose $N = \mathrm{ecut}(G, \Sigma, \tau)$. Then (G, Σ) does not extend to N. Since the Stabilizer Theorem implies that (G, Σ) is the only representation of $\mathrm{ecut}(G, \Sigma)$ which extends (H_0, Γ_0), it follows that (H_0, Γ_0) does not extend to N.

□

Lemma 3.5. *Let the base graph* (H_0, Γ_0) *and the matroid* M *be defined as in Theorem 3.1. Then we may assume that there exists a certificate for* $(H_0, \Gamma_0), M$.

Proof. Let N be a minor minimal 3-connected matroid such that $M \geq N$, N contains as minor a matroid isomorphic to $\mathrm{ecut}(H_0, \Gamma_0)$ and the representation (H_0, Γ_0) does not extend to N.

The matroid $\mathrm{ecut}(H_0, \Gamma_0)$ is neither a whirl, as it is binary, nor a wheel, as (H_0, Γ_0) is substantial. Hence by Seymour's Splitter Theorem, N can be obtained from $\mathrm{ecut}(H_0, \Gamma_0)$ by a series of row extensions and column extensions so that every intermediate matroid is 3-connected. We may assume that there is at least one column extension, otherwise the statement of Theorem 3.1 holds trivially. Let Ω be the element corresponding to the last column extension and let $N' = N \setminus \Omega$. We may assume that the representation (H_0, Γ_0)

extends to N', so there exists (G, Σ) such that $(G, \Sigma) \geq_{cut} (H_0, \Gamma_0)$ and $N' = \text{ecut}(G, \Sigma)$. Hence, by Remark 3.2, $N = \text{ecut}(G, \Sigma, \tau)$ for some signature τ.

□

The outline of the proof of the Escape Theorem is the following. Given a base graph (H_0, Γ_0) and a matroid M we will:

1. Give a construction for a triple (G_1, Σ_1, τ_1).
2. Show that (G_1, Σ_1, τ_1) is a certificate for (H_0, Γ_0) and M and that $|EG_1| = O(|VH_0|^2)$.
3. Prove that for any base graph and any matroid M we may assume that we can always find a construction as in point (1).

3.1 Construction of a certificate

Subdividing an edge uv of (G, Σ) is replacing it by a uv-path P that is internally disjoint from G and replacing Σ by $(\Sigma \setminus \{uv\}) \cup \Sigma_P$ where Σ_P is a signature of P with at least one even edge if uv is even and at least one odd edge if uv is odd. A *subdivision* of (G, Σ) is the result of a series of subdivisions of edges in (G, Σ). Given a subdivision (H, Γ), the *root vertices* of (H, Γ) are the vertices having at least three neighbours in H. A *leg* of the subdivision is a path such that its ends are root vertices and none of its internal vertices is a root vertex.

A *bridge* B of a graph G is either an edge $uv \notin EG$ with $u, v \in VG$, or a maximal connected graph having at least two vertices in common with G and such that $B \setminus VG$ is connected and there is no $uv \in EB$ with $u, v \in VG$. A bridge on a signed graph is defined similarly.

Let (G, Σ) be the signed graph resulting by adding a bridge B with signature Γ' to a graph (H, Γ). B is *removable* if (H, Γ) can be obtained as cut-minor of (G, Σ). Hence B is removable if and only if there is a resigning of Γ' on a cycle of B such that there is no path with only odd edges and both ends on H.

Remark 3.6. *Let (G, Σ, τ) be a certificate for the base graph (H_0, Γ_0) and the matroid M. Then we may assume that (G, Σ) is obtained from (H_0, Γ_0) by*

- *taking a subdivision (H_1, Γ_1) of (H_0, Γ_0), where $\Gamma_0 \subseteq \Gamma_1$, and*

- *adding removable bridges to* (H_1, Γ_1).

Proof. Since deletion and contraction commute, we can assume that G is obtained from H_0 by first uncontracting edges and then adding edges. The number of uncontractions that keep the degree of each new vertex at least three is bounded for every vertex v by $|\delta(v)| - 3$. So there are certainly less than $2|EH_0|$ of these uncontractions in total and in proving Theorem 3.1 we may assume that there is none. The uncontractions that produce a vertex of degree two give the subdivision of H_0. Finally, the uncontractions that produce vertices of degree one and the addition of edges give the bridges.

□

3.1.1 Dongles

A *dongle* is a triple (D, Σ^D, τ^D), where D is a graph, Σ^D, τ^D are signatures on D and D has special distinct vertices s, t where $st \in ED$. The vertices s, t are the *ends* of the dongle.

A dongle (D, Σ^D, τ^D) is a *complete dongle* if for all $D' \sim D$, (D', τ^D) has no blocking vertex. A dongle with ends s, t is a *partial dongle* if for all $D' \sim D$ neither s nor t is a blocking vertex of (D', τ^D), but there exists $D' \sim D$ for which (D', τ^D) has a blocking vertex.

A dongle (D, Σ^D, τ^D) is *solid* if for every 2-vertex cutset $\{u, v\}$ of D with sides (D_1, D_2), such that the ends $s, t \in VD_2$, the signed graph obtained from (D_1, Σ_{D_1}) by identifying u, v has an odd cut.

Consider a triple (G, Σ, τ) and a dongle (D, Σ^D, τ^D). We say that (G', Σ', τ') is obtained by *gluing* a dongle onto a pair of vertices $u, v \in VG$ if:

- $u \neq v$,
- G' is obtained by identifying the ends s, t of $D \setminus st$ with u, v, and
- $\Sigma' = \Sigma \cup \Sigma^D$, $\tau' = \tau \cup \tau^D$.

We say that (G', Σ', τ') is obtained by *gluing* a dongle onto a leg L of G if (G', Σ', τ') is obtained by first gluing the dongle onto the root vertices of L and then removing the leg L.

3.1.2 Widgets

Consider the graph H in Figure 3.1. Let W be the graph obtained from H by replacing each edge u_iu_j for $i, j = 1, 2, 3$, $i < j$, by a non empty path P_{ij}, and by replacing each edge u_iv_i for $i = 1, 2, 3$ by a (possibly empty) path P_i. A *widget* with ends v_1, v_2, v_3 is a triple (W, Σ^W, τ^W), where W is constructed as above, P_1, P_2, P_3 are τ^W-even and P_{12}, P_{23}, P_{13} are τ^W-odd (see Figure 3.2).

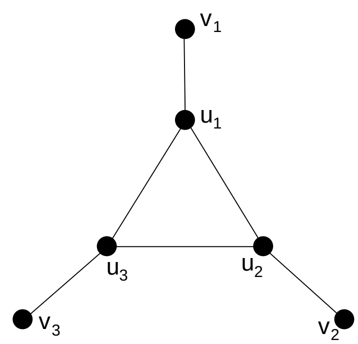

Figure 3.1:

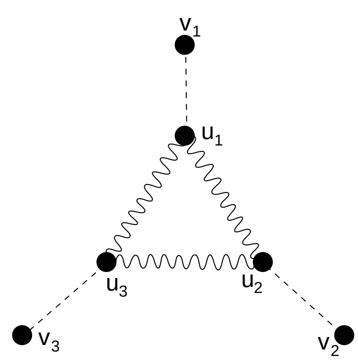

Figure 3.2: *Representation of a widget. Wavy paths are τ^W-odd, dotted paths are τ^W-even and may be empty.*

We say that (G', Σ', τ') is obtained from (G, Σ, τ) by *gluing* a widget (W, Σ^W, τ^W) with ends v_1, v_2, v_3 if $\Sigma' = \Sigma \cup \Sigma^W$, $\tau' = \tau \cup \tau^W$ and either

- v_1, v_2, v_3 are identified with three distinct vertices of G which are not all in the same leg of G, or
- $EP_2 = EP_3 = \emptyset$, $EP_{23} \subseteq EL$ for some leg L of G and v_1 is identified with a vertex of G that is not in the leg L.

3.1.3 Gadgets

Consider the graph H in Figure 3.3. Let J be the graph obtained from H by replacing each edge u_iu_j for $i, j = 1, 2, 3$, $i < j$, by a non empty path P_{ij}, by replacing the edges $u_1'v_1, u_2v_2, u_3v_3$ respectively by (possibly empty) paths P_1, P_2, P_3 and by replacing the two edges u_1u_1' by two non empty paths Q_1, Q_2. A *gadget* with ends v_1, v_2, v_3 is a triple (J, Σ^J, τ^J), where J is constructed as above, P_1, P_2, P_3, P_{12} are τ^J-even, P_{23}, P_{13} are τ^J-odd, Q_1 is τ^J-odd and Q_2 is τ^J-even (see Figure 3.4).

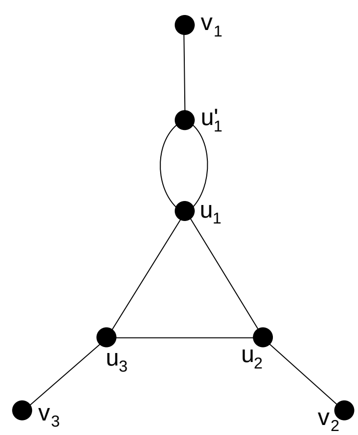

Figure 3.3:

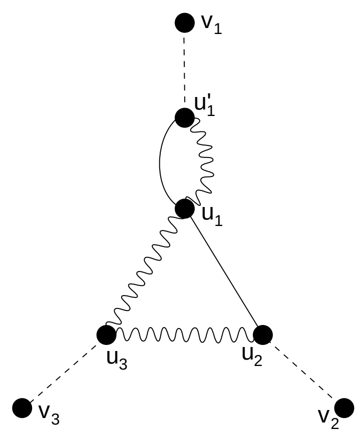

Figure 3.4: *Representation of a gadget. Wavy paths are τ^J-odd, straight paths are τ^J-even, dotted paths are τ^J-even and may be empty.*

We say that (G', Σ', τ') is obtained from (G, Σ, τ) by *gluing* a gadget (J, Σ^J, τ^J) with ends v_1, v_2, v_3 if $\Sigma' = \Sigma \cup \Sigma^J$, $\tau' = \tau \cup \tau^J$ and either

- v_1, v_2, v_3 are identified with three distinct vertices of G which are not all in the same leg of G, or
- $EP_2 = EP_3 = \emptyset$, $EP_{23} \subseteq EL$ for some leg L of G and v_1 is identified with a vertex of G that is not in the leg L.

3.1.4 Odd-K_4 configurations

Consider the graph H in Figure 3.5. Let K be the graph obtained from H by replacing K_4 by a subdivision of K_4 and replacing each edge u_iv_i for $i = 1, 2, 3$ by a (possibly empty) path P_i. An *odd-K_4 configuration* with ends v_1, v_2, v_3 is a triple (K, Σ^K, τ^K), where K is constructed as above, P_1, P_2, P_3 are τ^K-even and all circuits of K using exactly three of v_1, v_2, v_3, v_4 are τ^K-odd.

We say that (G', Σ', τ') is obtained from (G, Σ, τ) by *gluing* an odd-K_4 configuration (K, Σ^K, τ^K) with ends v_1, v_2, v_3 if $\Sigma' = \Sigma \cup \Sigma^K$, $\tau' = \tau \cup \tau^K$ and v_1, v_2, v_3 are identified with three distinct vertices of G which are not all in the same leg of G.

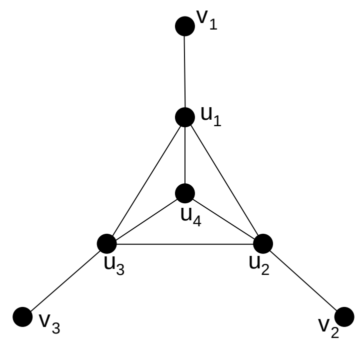

Figure 3.5:

3.1.5 Adding τ-odd paths

Let P be a path with signatures Σ^P, τ^P. We say that (G', Σ', τ') is obtained from (G, Σ, τ) by *adding a τ-odd path* if G' is obtained from G by identifying the ends of P to distinct vertices of G and $\Sigma' = \Sigma \cup \Sigma^P$, $\tau' = \tau \cup \tau^P$.

By *adding a τ-odd edge* we mean adding a τ-odd path which consists of a single edge.

3.1.6 Construction of (G_1, Σ_1, τ_1)

We now indicate the type of constructions for (G_1, Σ_1, τ_1) from the base graph (H_0, Γ_0).

I Choose a subdivision (H_1, Γ_1) of (H_0, Γ_0).

II Construct a triple (G_0, Σ_0, τ_0) by applying one of the following rules:

1. $G_0 = H_0$, $\Sigma_0 = \Gamma_0$ and $\tau \subseteq EG$ is such that (G, τ) has no blocking vertex.
2. (G_0, Σ_0, τ_0) is obtained from $(H_1, \Gamma_1, \emptyset)$ by gluing a solid complete dongle on a pair of vertices or a leg.
3. (G_0, Σ_0, τ_0) is obtained by gluing two solid partial dongles, each of them either on a pair of vertices or on a leg of $(H_1, \Gamma_1, \emptyset)$.
4. Choose τ such that (H_1, τ) has a blocking vertex but is not bipartite.

Then either:

- glue a solid partial dongle onto a pair of vertices of (H_1, Γ_1, τ), or
- glue a solid partial dongle onto a leg L of (H_1, Γ_1, τ) such that not all the τ-odd circuits of (H_1, τ) use the leg L.

5. Glue a solid partial dongle on a pair of vertices or a leg of $(H_1, \Gamma_1, \emptyset)$ and then add a τ-odd path P (with ends of P contained in VH_1).
6. Choose τ such that (H_1, τ) has a blocking vertex but is not bipartite. Then either:
   - choose u, v such that neither u nor v are blocking vertices of (H', τ) for all $H' \sim H_1$, then add a τ-odd uv-path P, or
   - add a τ-odd path P and a path Q such that for all $G \sim H_1 + Q + P$ the ends of P are not blocking vertices.
7. Add disjoint τ-odd paths P_1, P_2 to $(H_1, \Gamma_1, \emptyset)$ so that for all graphs $G \sim H_1 + P_1 + P_2$ the ends of P_1 and P_2 are distinct.
8. Add disjoint τ-odd paths P_1, P_2 to $(H_1, \Gamma_1, \emptyset)$, then add a path Q to the resulting graph G' so that for all $\hat{G} \sim G' + Q$ the ends of P_1 and P_2 are distinct.
9. Glue a widget onto $(H_1, \Gamma_1, \emptyset)$.
10. Glue a gadget onto $(H_1, \Gamma_1, \emptyset)$.
11. Glue an odd-K_4 configuration onto $(H_1, \Gamma_1, \emptyset)$.

III Let F be the graph formed by the edges of EG_0 that are not edges of EH_1 or of any dongle added to H_1. Then (G_1, Σ_1, τ_1) is obtained from (G_0, Σ_0, τ_0) by adding Σ-odd edges with at least one endpoint in $VF \setminus VH_1$. We will call these edges *dangling edges*.

The construction will assure in addition that:

(P1) The number of dangling edges is $O(|EH_0|)$.

(P2) For every path Q in $G_1 \setminus EH_1$ having both ends in H_1, $EQ \not\subseteq \Sigma$.

(P3) If e_1, e_2 are edges in series in G_1, then $e_1 \in \Sigma, e_2 \notin \Sigma$ (or vice versa). If e_1, e_2 are parallel edges in G_1, then $e_1 \in \tau, e_2 \notin \tau$ (or vice versa).

(P4) The dongles added to obtain (G_1, Σ_1, τ_1) have size bounded by a constant.

The fact that we can indeed attain property (P1) will not be proved in this thesis, while property (P4) will be partially proved.

The dangling edges are added to make sure that the graphs we added to (H_1, Γ_1) in the construction (2) are removable, that is they can be removed without loosing the basic construction of the minor (H_0, Γ_0). An example of a dangling edge that is necessary to keep is given in Figure 3.6. Here the basic construction is of type (6): we added a τ-odd path P, and u_2 is a blocking vertex for (H_1, τ_{H_1}). In the example we cannot contract the edge u_1u_2, otherwise we loose the path P. Moreover we cannot resign on a cycle to make u_1u_2 even, so that we could delete it, because that resigning would change the signature on (H_1, Γ_1).

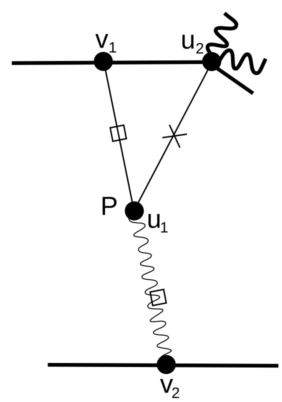

Figure 3.6: *The bold lines are edges in H_1. Wavy edges are τ-odd, straight edges are τ-even, the edge with a cross is Σ-odd, the edges with a square are Σ-even.*

We want to show that the triple (G_1, Σ_1, τ_1) obtained with the construction above is indeed a certificate. This will be proved in the following Lemmas.

Lemma 3.7. *$(H_0, \Gamma_0) \leq_{cut} (G_1, \Sigma_1)$.*

Proof. This holds by property (P2) and Lemma 3.19, that will be proved in one of the following sections. The Lemma says that if (D, Σ^D, τ^D) is a solid dongle, then we can obtain a bipath as a cut-minor of (D, Σ^D). $\square$

Lemma 3.8. *The representation* (G_1, Σ_1) *does not extend to ecut*(G_1, Σ_1, τ_1).

Proof. By Remark 3.3 it is sufficient to show that for all the constructions (1)-(11) the signed graph (G_0, τ_0) has no blocking vertex, even up to switches. This is trivially true for (1) and (2).

Now suppose by contradiction that (G_0, τ_0) has a blocking vertex x.

First suppose that to obtain (G_0, Σ_0, τ_0) we added a partial dongle (D, Σ^D, τ^D) with ends $\{s, t\}$. By definition of partial dongle, $x \in VD \setminus \{s, t\}$. In case (3) suppose $(D', \Sigma^{D'}, \tau^{D'})$ is the other partial dongle added to (H_1, Γ_1), with ends s', t'. Then $x \in VD' \setminus \{s', t'\}$, a contradiction, as $(VD \cap VD') \setminus \{s, s', t, t'\} = \emptyset$. In case (4), by the construction there is a τ-odd circuit in $G_0 \setminus D$, so x cannot be a blocking vertex of (G_0, τ_0). The same holds in case (5).

For case (6), as x needs to be a blocking vertex for (G_0, τ_0) and (H_1, τ) is not bipartite, $x \in VH_1$ but $x \neq u, v$. Then, as H_1 is 2-connected, there exists an uv-path Q in $H_1 \setminus x$. Then $P \cup Q$ forms a τ-odd circuit not using v, contradiction (note that this holds also for equivalent graphs).

Cases (7) and (8) are solved by the following.

Claim 1. *If* (G, τ) *is obtained from a 2-connected graph* H *with* $\tau_H = \emptyset$ *by gluing two disjoint* τ*-odd paths* P_1, P_2*, then* (G, τ) *has no blocking vertex.*

Proof (claim). Let u_i, v_i be the ends of P_i for $i = 1, 2$. Suppose by contradiction that (G, τ) has a blocking vertex x. If $x \notin VP_1$, then by connectivity $(G \setminus x) \cup P_1$ contains an odd circuit. The same holds for P_2. But then $x \in VP_1 \cap VP_2 = \emptyset$, contradiction.

◇

Finally widgets, gadgets and odd-K_4 configurations clearly do not have a blocking vertex, so the Lemma is proved.

□

To prove that $\mathrm{ecut}(G_1, \Sigma_1, \tau_1)$ is 3-connected we need the following results.

Lemma 3.9. *Given an even cut matroid $M = ecut(G, \Sigma)$, where (G, Σ) is not eulerian and G is simple, a 2-separation in the matroid corresponds to either*

- *a 1-separation in G, or*
- *a 2-separation G_1, G_2 in G, where, if $\{u, v\} = VG_1 \cap VG_2$, for $i = 1$ or $i = 2$ the signed graph obtained from G_i by identifying u, v is eulerian, or*
- *a 3-separation G_1, G_2 in G, where, if $\{u, v, w\} = VG_1 \cap VG_2$, both the signed graphs obtained from G_1 and G_2 by identifying u, v, w are eulerian.*

Proof. Let G_1, G_2 be a k-separation in (G, Σ), and $\hat{G}_i$ be the graph obtained by identifying the vertices of the vertex cutset in G_i. Let $V = VG$, $E = EG$ and $V_i = V\hat{G}_i$, $E_i = E\hat{G}_i$ for $i = 1, 2$.

For $i = 1, 2$, let $\delta_i = 1$ if G_i is not eulerian, $\delta_i = 0$ otherwise.

Then by Lemma 2.8

$$r_M(E_i) = |E_i| - |V_i| + 1 + \delta_i \qquad \text{and} \qquad |V| - |V_1| - |V_2| = k - 2.$$

Therefore

$$\lambda_M(E_1, E_2) = r_M(E_1) + r_M(E_2) - r(M) + 1 =$$

$$= |E_1| - |V_1| + 1 + \delta_1 + |E_2| - |V_2| + 1 + \delta_2 - |EG| + |VG| - 2 + 1 = k - 1 + \delta_1 + \delta_2$$

So $\lambda_M(E_1, E_2) \leq 2$ if and only if $k + \delta_1 + \delta_2 \leq 3$.

Then either $k = 1$, and we have the first case, or $k = 2$ and $\delta_i = 0$ for some $i = 1, 2$, or $k = 3$ and $\delta_i = 0$ for both $i = 1, 2$.

□

Note that if (G, Σ) is substantial, then the third case in the above Lemma cannot occur. This is because if G_1, G_2 is a 3-separation in G, where $\{u, v, w\} = VG_1 \cap VG_2$, and both the signed graphs obtained from G_1 and G_2 by identifying u, v, w are eulerian, then $T(G, \Sigma) \subseteq \{u, v, w\}$.

Lemma 3.10. *Let G be a graph with signatures Σ, τ. Let $N = ecut(G, \Sigma)$ and $M = ecut(G, \Sigma, \tau)$, where Ω denotes the element in $EM \backslash EN$. Then for every partition (E_1, E_2) of EN, $\lambda_M(E_1, E_2 \cup \{\Omega\}) = \lambda_N(E_1, E_2) + \delta$ where $\delta = 1$ if there exists a τ-odd circuit $C \subseteq E_1$ and $\delta = 0$ otherwise.*

Proof. Let I_2 be a basis of E_2.

Claim 1. *$I_2 \cup \{\Omega\}$ is a basis of $E_2 \cup \{\Omega\}$ if and only if there exists a τ-odd circuit $C \subseteq E_1$.*

Proof (claim). Consider the case where we have a τ-odd circuit $C \subseteq E_1$. Suppose $I_2 \cup \{\Omega\}$ is not a basis of $E_2 \cup \{\Omega\}$. Then it must contain a circuit B, with $\Omega \in B$. By definition of $\mathrm{ecut}(G, \Sigma, \tau)$, the circuits of M using Ω are of the form $\tau' \cup \Omega$, where $\tau' = \tau \triangle \delta_G(U)$, for some $U \subseteq VG$. But then $C \subseteq E_1$ implies that $B \cap E_1 \neq \emptyset$, contradiction.

Consider the case where there is no τ-odd circuit $C \subseteq E_1$. Let $G_2 = G/E_1$. We know (by Lemma 2.8) that $I_2 = E_2 \setminus ET$, where T is some forest in G_2. Since all τ-odd circuits of G are included in G_2, we have $\tau' \subseteq E_2$ for some $\tau' = \tau \triangle \delta_G(U)$. Moreover, since $I_2 = E_2 \setminus ET$, we may assume (after resigning on cuts) that $\tau' \subseteq I_2$. But then $\tau' \cup \{\Omega\}$ is a circuit of M, a contradiction. ◇

Now by definition,

$$\lambda_N(E_1, E_2) = r_N(E_1) + r_N(E_2) - r_N(E_1 \cup E_2) + 1$$

and

$$\lambda_M(E_1, E_2 \cup \{\Omega\}) = r_M(E_1) + r_M(E_2 \cup \{\Omega\}) - r_M(E_1 \cup E_2 \cup \Omega) + 1$$

Since $N = M \setminus \Omega$, $r_N(E_1 \cup E_2) = r_M(E_1 \cup E_2 \cup \Omega)$. So we have $r_N(E_2) = |I_2|$ and the result follows from the Claim.

□

Lemma 3.11. *$\mathrm{ecut}(G_1, \Sigma_1, \tau_1)$ is 3-connected.*

Proof. Lemma 3.9 and the definition of solid dongle imply that $\mathrm{ecut}(G_1, \Sigma_1, \tau_1)$ is 3-connected except for possibly parallel edges f, g, where $f, g \in \overline{\Sigma}_1$ and $f \in \tau_1, g \in \overline{\tau}_1$. Define $N = \mathrm{ecut}(G_1, \Sigma_1)$ and $M = \mathrm{ecut}(G_1, \Sigma_1, \tau_1)$. Let (E_1, E_2) be a partition of $EG_1 = EN$. We need to show that $\lambda_M(E_1, E_2 \cup \{\Omega\}) \geq 3$ for all (E_1, E_2) such that $|E_1|, |E_2 \cup \{\Omega\}| \geq 2$. We may assume that $\lambda_N(E_1, E_2) \leq 2$, for otherwise the result follows by Lemma 3.10. Since N is 3-connected except for possibly parallel edges, $\lambda_N(E_1, E_2) = 2$. Because of Lemma 3.10, to prove $\lambda_M(E_1, E_2 \cup \{\Omega\}) \geq 3$ it suffices to show that there exists a τ_1-odd circuit $C \subseteq E_1$. If $|E_2| \leq 2$ this follows from the fact that (G_1, τ_1) has no blocking vertex (by Lemma 3.8). Thus we may assume that $|E_2| > 2$. It follows from the hypothesis that

$E_1 = \{f, g\}$ where f, g are as described above. But then $C = \{f, g\}$ is the required τ_1-odd circuit.

□

The proof that (G_1, Σ_1, τ_1) is a certificate is completed by the following.

Lemma 3.12. $|EG_1| = O(|VH_0|^2)$.

Proof. The proof derives immediately from (P1), (P3) and (P4). □

3.2 Structure of the graph G in the certificate (G, Σ, τ)

Let (H_1, Γ_1) be a subdivision of the base graph (H_0, Γ_0) such that (G, Σ) is obtained from (H_1, Γ_1) by adding removable bridges ((H_1, Γ_1) does exist by Remark 3.6). Note that every 2-separation of (G, Σ) has one side containing at most a path of (H_1, Γ_1). Then *simplifying* a 2-separation with vertex cutset $\{u, v\}$ in (G, Σ) means replacing the side containing at most a path of (H_1, Γ_1) with a bipath with ends u, v. The *core* of (G, Σ) (written $core(G, \Sigma)$) is the graph obtained from (G, Σ) by recursively simplifying all its 2-separations.

Remark 3.13. *$core(G, \Sigma)$ is nearly 3-connected.*

The main result of this section is the following.

Lemma 3.14. *Let (G, Σ, τ) be a certificate for the base graph (H_0, Γ_0) and the matroid M. Then*

1. $(G, \Sigma) \geq_{cut} core(G, \Sigma) \geq_{cut} (H_0, \Gamma_0)$.
2. *We may assume that $core(G, \Sigma)$ is obtained from a subdivision (H_1, Γ_1) of (H_0, Γ_0) by adding removable good bridges and bipaths.*

The definition of good bridge is given next. If B is a bridge on a subdivision H, the *attachments* of B on a leg L of H are the vertices of B in L. The set of all attachments of a bridge B is denoted by $att(B)$. An attachment $u \in L$ is *extreme* if it is a root vertex or if one component of $L \setminus u$ does not contain any other attachment of B. The vertices of B

that are not attachments are the *internal vertices* of B. The set of all internal vertices of B is denoted by $int(B)$. B is *good* if its attachments are not all contained in one leg of H, it is *bad* otherwise.

A path P of (G, Σ) is *totally even* (respectively *totally odd*) if all its edges are even (respectively odd). It is *mixed* if it has both even and odd edges.

Given two subdivisions G_1 and G_2 of the same graph, (G_2, Σ_2) *dominates* (G_1, Σ_1) if for every leg L of (G_1, Σ_1), the corresponding leg of (G_2, Σ_2) is mixed if L is mixed, and totally even (respectively totally odd) or mixed if L is totally even (respectively totally odd).

The proof of Lemma 3.14 will be given at the end of the section and it is based on the following.

Theorem 3.15. *Let (H, Γ) be a 3-connected graph and let (H_1, Γ_1) be a subdivision of (H, Γ). Let (G, Σ) be obtained from (H_1, Γ_1) by adding disjoint bridges $B_1, ..., B_k$, where none of B_i is a single edge with both ends on root vertices of H_1. Suppose (G, Σ) is nearly 3-connected and $B_1, ..., B_k$ are removable. Then there exist a subdivision (H_2, Γ_2) of (H, Γ) and disjoint good removable bridges $B'_1, ..., B'_r$ of (H_2, Γ_2) such that*

- *(H_2, Γ_2) dominates (H_1, Γ_1), and*
- *(G, Σ) is obtained from (H_2, Γ_2) by adding $B'_1, ..., B'_r$.*

To prove both Lemma 3.14 and Theorem 3.15 we need some intermediate results that will be given in the next sections.

3.2.1 Rerouting

We will use the following Theorem by Tutte (1963) to prove the first Lemma of this section. For the proof of the Theorem see Diestel [2].

Theorem 3.16 (Tutte)**.** *If H is a 3-connected graph, the cycle space of H is generated by its non-separating chordless circuits.*

Lemma 3.17. *Let s, t, z be three distinct vertices of G. If $G \cup \{st\}$ is 3-connected, then there exists a non-separating chordless st-path P avoiding z.*

Proof. If $st \in EG$ the Lemma is proved. Then suppose $st \notin EG$. Let $\hat{G}$ be the graph obtained by gluing two copies G_1 and G_2 of G on s, t, z. As $G \cup \{st\}$ is 3-connected, $\hat{G}$ is 3-connected. Hence there is a circuit in $\hat{G} \cup \{st\}$ using st. By Theorem 3.16 it is generated by non-separating chordless circuits. Hence there is a non-separating chordless circuit using st. So $\hat{G}$ has a non-separating chordless st-path P. If P does not use z, it must be all contained in G_i for some i, so it is a non-separating chordless st-path in $G \setminus z$. Now suppose that P uses z. As P is non-separating, it must span one of G_i, and as it is chordless G_i (and hence G) is just a path. But then G is not 2-connected, contradiction. □

Let B be a bad bridge with extreme attachments s, t. Suppose v is a cutvertex of $B \setminus att(B)$ and one of the sides B' of the separation induced by v has no attachments in $\{s, t\}$. We call *satellite* such subgraph B'. Then the operation of *cleaning* B' consists in identifying with v all the internal vertices of B'. Note that cleaning does not change $att(B)$.

In the remaining part of this section G will be a 3-connected graph formed by the subdivision H of a 3-connected graph and removable bridges $B_1, ..., B_k$, where none of B_i is an edge parallel to an edge of H. Note that changing the order in which the bridges are added to H does not change the resulting graph. Moreover, if B is a bad bridge on a leg L that is not a single edge, it must have at least three attachments on L.

Given a bad bridge B on a leg L, where B is not a single edge and has extreme attachments s, t, the operation of *rerouting* consists of the following steps:

1. Clean all the satellites of B.
2. Identify the part of L containing the non extreme attachments of B to a single vertex z. Construct L'_{st} by contracting L_{sz} and L_{zt} to single edges.
3. Let G' be the new bridge plus L'_{st}. In G' find a non-separating chordless st-path Q avoiding z.
4. In G replace the leg L by $L \setminus L_{st} \cup Q$.

Note that $G' \cup \{st\}$ is 3-connected. To see this suppose $G' \cup \{st\}$ has a 2-vertex cutset S. As $B \setminus att(B)$ is connected, $S \not\subseteq \{s, t, z\}$. Moreover $S \cap \{s, t, z\} \neq \emptyset$, because G is

3-connected. So there is a vertex $u \in int(B)$ that together with $x \in \{s, t, z\}$ forms a 2-vertex cutset. If $x = s, t$ then the original graph was not 3-connected. Hence $x = z$, but then u is separating a satellite of which we disposed before. So Lemma 3.17 can be applied to G' and the operation in point (3) is possible.

Note that a non-separating chordless st-path Q avoiding z in G' corresponds to a chordless st-path in the original bridge, as all the attachments of a satellite are in z.

If B is a bridge that is an edge with ends s, t on a leg L, where s and t are not both root vertices of L, *rerouting* on B consists of replacing the leg L by $L \setminus L_{st} \cup B$.

Remark 3.18. *The new subdivision obtained by rerouting a bad bridge B has fewer bridges than the original one.*

Proof. First assume that B is not a single edge. Q does not separate $B \cup L_{st}$, as it is non-separating for the cleaned bridge plus L_{st} and every satellite of B must have an attachment that is not extreme. Moreover, we do not create new bridges formed by a single edge, as Q is chordless. So rerouting does not split existing bridges. Moreover, as B is a bad bridge and $\{s, t\}$ is not a 2-vertex cutset of G, there exists another bridge B' that has at least one attachment on $L_{st} \setminus \{s, t\}$ and one not in L_{st}. Hence rerouting we merge B with B'. The same reasoning holds if B is a single edge. It follows that the new subdivision has fewer bridges than the original one.

□

Note that all the results of this section still hold if we consider nearly 3-connected graphs instead of 3-connected graphs. To see this, consider the graph G' obtained from G by substituting the bipaths by single edges. Note that we do not create parallel edges in G', because if we have an edge between the two ends of a bipath we have a 2-separation in G with one side formed by a bipath plus an edge. Hence we can apply the result to G' and then convert it to a result for G.

3.2.2 Rank and minor-mixed paths

A path P of (G, Σ) is *minor-mixed* if P appears mixed as some cut-minor of (G, Σ). It is *minor-totally even* (respectively *minor-totally odd*) if it appears totally even (respectively totally odd) as some cut-minor of (G, Σ).

An st-path P of (G, Σ) is *nice* if $|EP| \geq 2$ and $\{s, t\}$ does not separate P from $G \setminus P$.

Lemma 3.19. *Let (G, Σ) be a signed graph with G 2-connected and with no loops, and $st \in EG$. Suppose G contains a nice st-path. The following are equivalent:*

1. $r(ecut(G/st, \Sigma)) = r(cut(G/st)) + 1$.
2. *There exists an odd bond of (G, Σ) with s and t on the same shore.*
3. *There exist an odd and an even st-bond in (G, Σ).*
4. *Every nice st-path is minor-mixed.*

Proof. (1) ⇔ (2) First suppose $r(\mathrm{ecut}(G/st, \Sigma)) = r(\mathrm{cut}(G/st)) + 1$. A basis of $\mathrm{cut}(G/st)$ is a maximal set of edges not containing a cut, hence it is the complement of a spanning tree of G. If $(G, \Sigma)/st$ does not contain an odd bond, it is eulerian and a basis of $\mathrm{ecut}(G/st, \Sigma)$ is again the complement of a spanning tree. Hence there is an odd bond of $(G, \Sigma)/st$, that corresponds to an odd bond of (G, Σ) with s, t on the same shore.

Now suppose that there exists an odd bond $\delta_G(U)$ of (G, Σ) with $s, t \notin U$. Then by Lemma 2.8 a basis of $\mathrm{ecut}(G/st, \Sigma)$ is the complement of a spanning tree plus an edge of such tree. Hence $r(\mathrm{ecut}(G/st, \Sigma)) = r(\mathrm{cut}(G/st)) + 1$.

(2) ⇔ (3) First suppose there exists an odd bond of (G, Σ) with s and t on the same shore. As G is 2-connected and has no loops, $\mathrm{ecut}(G/st, \Sigma)$ is connected. We may assume, up resigning on a cycle, that st is even. Let $M_{(G,\Sigma)}$ be the matrix obtained by adding the row incidence vector of Σ to a full row-rank matrix whose rows span the circuit space of G. Let $\hat{M}$ be the matrix obtained from $M_{(G,\Sigma)}$ by adding a column having a one in correspondence of the Σ row, and zero everywhere else. Let Ω be the element of $M_{GF(2)}(\hat{M})$ corresponding to the column added. As $M_{GF(2)}(\hat{M})$ is obtained from $\mathrm{ecut}(G, \Sigma)$ by adding an element that is not a loop or a coloop, it is connected. For any connected matroid M and $e, f \in EM$, there exists a circuit C of M with $e, f \in C$ (see Oxley [5]). So there exists a circuit of $M_{GF(2)}(\hat{M})$ containing both st and Ω. This corresponds to an odd st-bond in (G, Σ). To find an even st-bond in (G, Σ) it is sufficient to repeat the same reasoning starting with st signed odd.

Now assume that there exist an odd st-bond $\delta_G(U_1)$ and an even st-bond $\delta_G(U_2)$ in (G, Σ). Then $\delta_G(U_1) \triangle \delta_G(U_2) = \delta_G(U_1 \triangle U_2)$ is an odd cut with s and t on the same shore. As a cut is a disjoint union of bonds and $st \in (EG \setminus \delta_G(U_1 \triangle U_2))$, (G, Σ) has an odd bond with s and t on the same shore.

(3) $\Leftrightarrow$ (4) First suppose there exist an odd and an even st-bond in (G,Σ). Let P be a nice st-path. We may assume that we resigned Σ on a cycle so that we can delete as many edges as possible keeping P as a cut-minor. We may also assume that P is totally even or odd. As we can obtain P as minor, (G,Σ) has no totally odd circuits and no totally odd paths with both ends on P. Moreover, if D is the set of edges that we can delete to obtain P, $(G,\Sigma)\setminus D$ still has an even and an odd st-bond. So there is an odd edge $f \notin EP$. Since $\{s,t\}$ does not separate P from G, there are two independent paths Q_1, Q_2 in G (possibly empty) from the ends of f to P such that the ends of $Q_1 \cup Q_2$ do not contain both s and t. Let $L = Q_1 \cup Q_2 \cup \{f\}$. Choose f, Q_1, Q_2 such that L has as few even edges as possible. Let C be the circuit closed by L with part of P.

Claim 1. *Resigning on C, deleting f and contracting Q_1, Q_2 gives P as a mixed-minor.*

Proof (claim). Let u_i be the end of Q_i on P, and v_i the end of Q_i on f.

If there is a totally odd path Q from Q_i to P, let w be the end of Q on Q_i. As (G,Σ) has no totally odd paths with both ends on P, L_{u_1w} and L_{wu_2} are not totally odd. Then $L \setminus L_{u_1w} \cup Q$ or $L \setminus L_{wu_2} \cup Q$ contradicts the choice of f, Q_1, Q_2.

If there is a totally odd path Q from Q_1 to Q_2, let w_i be the end of Q on Q_i. As the circuit closed by Q with $L_{w_1w_2}$ is not totally odd, $L \setminus L_{w_1w_2} \cup Q$ contradicts the choice of f, Q_1, Q_2. Hence all the internal vertices of L have degree two and thus we can resign on C. So the Claim is proved.

◇

Now assume that every nice st-path is minor-mixed. Let P be a nice st-path. As P is obtain mixed as cut-minor, (G,Σ) has a cut-minor that has an odd and an even st-bond. Minor operations do not create new cuts. Hence (G,Σ) contains an odd and an even st-bond.

□

Corollary 3.20. *Suppose $M = ecut(G,\Sigma)$, with G having a 2-vertex cutset $\{s,t\}$ with sides G_1, G_2. Let $E_i = EG_i$ for $i = 1,2$. Then $\lambda_M(E_1,E_2) = 3$ if and only if for $i = 1,2$ (G,Σ) contains as a minor the graph obtained from (G,Σ) by replacing G_i with a bipath.*

Corollary 3.21. *Let s,t be distinct vertices of a 2-connected signed graph (G,Σ). If there exists a minor-mixed st-path of (G,Σ), then every nice st-path of G is minor-mixed.*

Corollary 3.22. *Let s, t be distinct vertices of a 2-connected signed graph (G, Σ). If there exists a minor-totally even (respectively odd) st-path of (G, Σ), then every nice st-path of G is minor-totally even (respectively odd) or minor-mixed.*

3.2.3 Final proofs

Proof of Theorem 3.15. Let $(\hat{H}, \hat{\Gamma})$ be a subdivision of (H, Γ) dominating (H_1, Γ_1), and $\hat{B}_1, ..., \hat{B}_h$ removable bridges of $\hat{H}$ such that none of $\hat{B}_i$ is a single edge with both ends on root vertices of $\hat{H}$ and (G, Σ) is obtained from $(\hat{H}, \hat{\Gamma})$ by adding $\hat{B}_1, ..., \hat{B}_h$. Note that $(\hat{H}, \hat{\Gamma})$ exists, as (H_1, Γ_1) is such a subdivision. Choose it so that the number of bridges is minimized. We want to show that $(\hat{H}, \hat{\Gamma})$ is the desired subdivision, that is none of the $\hat{B}_i$ is bad.

Since $\hat{B}_1, ..., \hat{B}_h$ are removable, we can resign (G, Σ) so that $(\hat{H}, \hat{\Gamma}) = (G, \Sigma) \setminus I / J$ with $I \cap \Sigma = \emptyset$. We may assume that $J \subseteq \Sigma$, because if there is an even edge in J we can delete it instead of contracting it.

Now suppose by contradiction that one of the bridges $\hat{B}_i$ is bad. Let $\hat{B}$ be such bridge and $\hat{L}$ the leg containing all its attachments. Let s, t be the extreme attachments of B.

For every bridge B with some attachments in L define $pod(B, L)$ to be either B if B is bad, or the graph induced by all vertices of B reachable from L by a totally odd path, if B is good.

Note that $pod(B, L)$ will not contain vertices of any leg different from L, except possibly root vertices that are in L, as we resigned so that every bridge has no totally odd path with ends in two distinct legs.

Let $\mathcal{L}$ be the union of all $pod(\hat{B}_i, \hat{L})$ for all bridges $\hat{B}_i$ with attachments on $\hat{L}$.

Now rerout $(\hat{H}, \hat{\Gamma})$ on the bridge $\hat{B}$. Let $(\hat{H}', \hat{\Gamma}')$ be the new subdivision and $\hat{L}'$ the new leg obtained by rerouting. Let $\hat{B}'_1, ..., \hat{B}'_r$ be the bridges of $(\hat{H}', \hat{\Gamma}')$. By Remark 3.18, $r < h$.

Note that $\hat{L}'$ is nice as every $\hat{B}_i$ is not a single edge with both ends on root vertices of H_1.

By the definitions of $\mathcal{L}$, $(\hat{H}, \hat{\Gamma}) \cup \mathcal{L}$ is clearly a cut-minor of (G, Σ). Moreover, $(\hat{H}, \hat{\Gamma})$ is also a cut-minor of (G, Σ). So by Corollary 3.22 if $\hat{L}$ was minor-mixed (respectively totally even or odd), $\hat{L}'$ is minor-mixed (respectively totally even or odd, or mixed). So $(\hat{H}', \hat{\Gamma}')$ dominates $(\hat{H}, \hat{\Gamma})$, and $\hat{B}'_1, ..., \hat{B}'_r$ are removable, contradiction.

□

Proof of Lemma 3.14. The fact that $(G_0, \Sigma_0) \geq_{cut} core(G_0, \Sigma_0)$ follows immediately from Corollary 3.20. Now note that resigning on a circuit containing a bipath, keeps the bipath with an odd and an even edge. Moreover every bipath is removable by contracting the odd edge and deleting the even edge. Hence, the bridges remain removable in $core(G_0, \Sigma_0)$. So $core(G_0, \Sigma_0) \geq_{cut} (H_0, \Gamma_0)$, and part (1) is proved.

The proof of part (2) follows immediately from Theorem 3.15 and from the fact that we can add at most $2|EH_0|$ edges having both ends on root vertices of (H_1, Γ_1).

□

3.3 Gadgets and widgets

In this and the next section we will study the property of signed graphs having no blocking vertex or being not bipartite. For the completion of the proof of the Escape Theorem only the main results of these sections are needed.

We will extensively use the following

Lemma 3.23. *Let (G, τ) be a signed graph with no τ-blocking vertex. Then (G, τ) contains one of the graph in Figure 3.7 as a cycle-minor.*

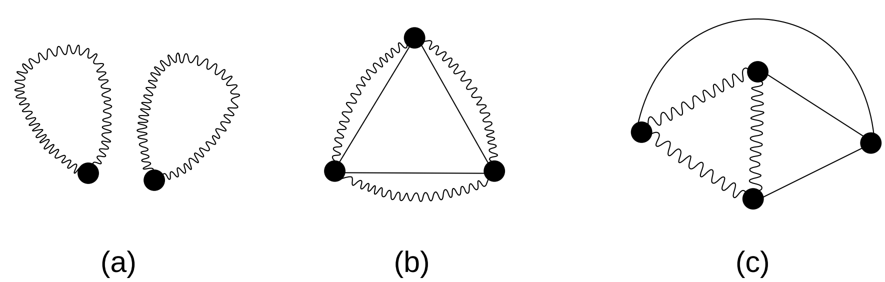

Figure 3.7: *Wavy edges are odd, straight edges are even.*

We call the graph in Figure 3.7 (b) *double triangle* and we denote the graph in Figure 3.7 (c) or any of its resigning on a cut by $\widetilde{K}_4$. Note that the graph in Figure 3.7 (a) is formed by two disjoint odd loops.

Proof. Through this proof, when referring to a path P we assume that P may be formed by a single vertex. Let (G,τ) be a minor minimal signed graph with no blocking vertex. Then for every $uv \in EG$ that is not a loop, $\{u,v\}$ is a blocking pair. So every edge of G intersects every odd circuit. We may assume that (G,τ) does not contain two disjoint odd circuits. Then (G,τ) does not contain any odd loop, otherwise if L is an odd loop incident in a vertex v, every other odd circuit would intersect L in v, so v would be a blocking vertex.

Claim 1. *There exist two odd circuits C_1 and C_2 whose intersection is a path.*

Proof (claim). Suppose not. Let C_1, C_2 be two odd circuits and let $C_1 \cap C_2 = P_1 \cup ... \cup P_k$, where P_i are disjoint paths. Choose (C_1, C_2) such that k is as small as possible. As any two odd circuits intersect and C_1 and C_2 don't intersect in a single path, $k \geq 2$. Let u_i, v_i be the ends of path P_i. So there exist u_j, v_h with $j \neq h$ such that for $i = 1,2$ one part of C_i between u_j and v_h does not contain any other vertex in $\{u_1, ..., u_k, v_1, ..., v_k\}$. Let $\hat{P}_i$ for $i = 1,2$ be such part and let $C = \hat{P}_1 \cup \hat{P}_2$. By the choice of u_j, v_h, C is a circuit. Now if C is odd, (C, C_1) contradicts the choice of (C_1, C_2). Otherwise $(C_1 - \hat{P}_1 \cup \hat{P}_2, C_2)$ contradicts the choice of (C_1, C_2).

◇

Now let C_1, C_2 be two odd circuits that intersect in a path P. Choose them such that P is as short as possible.

First suppose that $EP = \emptyset$. Then P is formed by a single vertex v. As every edge of G intersects every odd circuit and G does not contain any loop, C_1 and C_2 have exactly two edges. Let u_i be the vertex of C_i distinct from v. As v is not a blocking vertex, there must be an odd circuit C not using v. Moreover $\{v, u_i\}$ is a blocking pair for $i = 1,2$. So $u_1, u_2 \in VC$. But then (G,τ) contains a double triangle as a cycle-minor.

So $EP \neq \emptyset$. Let u, v be the ends of P. We can resign so that the only odd edge in $C_1 \cup C_2$ is incident with u (and contained in P).

Claim 2. *For $i = 1,2$ there is no path from $C_i - P$ to $P - \{u,v\}$.*

Proof (claim). Suppose there is a path Q from $C_i - P$ to $P - \{u,v\}$. Let Q_1 and Q_2 be the two part of C_i between the ends of Q. Then one of $Q \cup Q_1$ and $Q \cup Q_2$ is an odd circuit intersecting C_{3-i} in a path shorter than P, contradicting the choice of C_1, C_2.

◇

Note that as before $|EC_i - EC_{3-i}| \leq 2$. If $|EC_1 - EC_2| = 1$, then $\{u, v\}$ is a blocking pair. But as u is not a blocking vertex, there is an odd circuit C not using u (and hence using v). But then there are two odd circuits intersecting in a path shorter than P, contradiction.

So, by symmetry, $|EC_1 - EC_2|, |EC_2 - EC_1| = 2$. Let w_i be the vertex of $C_i \setminus C_{3-i}$ for $i = 1, 2$. Now $\{u, w_i\}$ is a blocking pair for $i = 1, 2$ but u is not a blocking vertex, so there exists an odd circuit C such that $u \notin VC$ and $w_1, w_2 \in VC$. But then either there is a path contradicting Claim 2, or two odd circuits contradicting the choice of C_1, C_2, or there is an odd path Q between w_1 and w_2 internally disjoint from $C_1 \cup C_2$. In the last case (G, τ) contains $\widetilde{K}_4$ as a cycle-minor.

□

An $\widetilde{K}_4$-*subdivision* is a graph obtained by substituting each edge of $\widetilde{K}_4$ with a nonempty path of the same parity.

Remark 3.24. *If (G, τ) contains two disjoint odd loops as a cycle-minor, it contains two induced disjoint odd circuits.*

If (G, τ) contains a double triangle as a cycle-minor, then either: it contains as subgraph a subdivision of a double triangle; it contains two disjoint odd circuits; or it contains an $\widetilde{K}_4$-subdivision.

Proof. The first part of the Remark is straightforward. For the second part, assume that (G, τ) does not contain a subdivision of a double triangle. Then it contains one of the graphs in Figure 3.8 as a cycle-minor (Note that these are all the possible uncontractions on vertices of degree > 3 that do not give a subdivision of a double triangle). In case (a) we have two disjoint odd circuits, in cases (b) and (c) an $\widetilde{K}_4$.

□

3.3.1 Local minors and local resigning

In all this section (G, τ) is a signed graph and $S \subseteq VG$.

A *local resigning* of (G, τ, S) is a resigning of τ on a cut $\delta(U)$ with $S \cap U = \emptyset$.

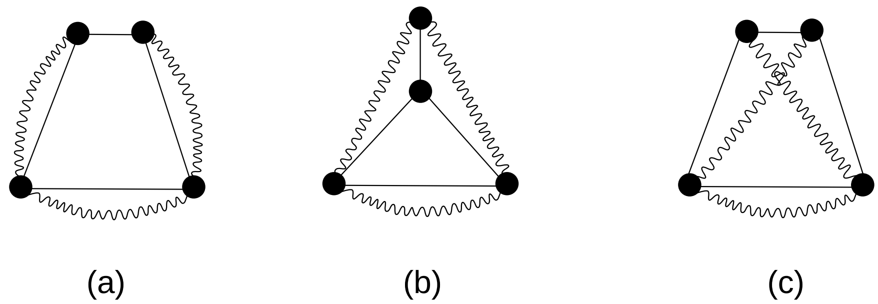

Figure 3.8: *Wavy edges are odd, straight edges are even.*

(G', τ') is a *local minor* of (G, τ) with respect to S if (G', τ') is obtained as a cycle-minor of (G, τ) where all the resignings are local and no two vertices in S are identified.

A path P in (G, τ) is an S-*path* if it is odd, its ends are in S and its internal vertices are not in S.

Remark 3.25. *Local resigning does not change the S-paths.*

We denote with $G + K_S$ the graph obtained from G by gluing a $K_{|S|}$ on the vertices of S.

Lemma 3.26. *Let (G, τ) be a signed graph and $S \subseteq VG$, $|S| \geq 3$, such that $G + K_S$ is 3-connected and (G, τ) has no odd loops. If no vertex of G intersects all the S-paths, then (G, τ) contains one of the graphs in Figure 3.9 as a local cycle-minor.*

Proof. Let (G, τ) be a minor minimal counterexample. Then (G, τ) does not contain two disjoint S-paths.

Claim 1. *G has no vertex intersecting all the S-paths if and only if $(G + K_S, \tau)$ has no blocking vertex.*

Proof (claim). First assume that $(G+K_S, \tau)$ has a blocking vertex v. Let P be an S-path in (G, τ) with ends s, t. Then $C = P \cup \{st\}$ is an odd circuit in $(G+K_S, \tau)$, hence $v \in VC$. So v intersect every S-path. Now suppose that $v \in VG$ intersects all the S-paths. Let C be an odd circuit in $(G + K_S, \tau)$. Suppose by contradiction that $v \notin VC$. Then, as (G, τ)

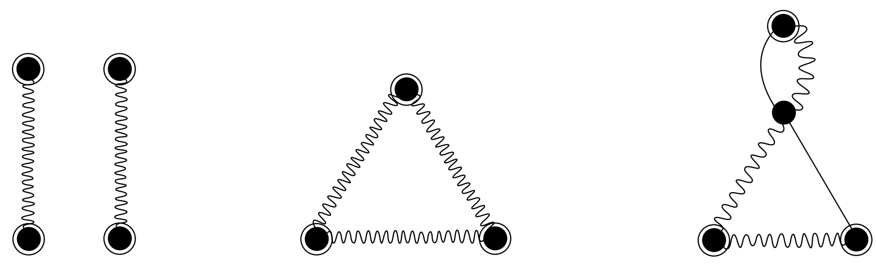

Figure 3.9: *Wavy edges are odd, straight edges are even. Circled vertices are in S.*

has no loops and $G+K_S$ is 3-connected, there exist two disjoint paths P_1, P_2 in $G+K_S \setminus v$ from VC to S (P_1, P_2 may be just a single vertex). Then $P_1 \cup P_2 \cup C$ contains an S-path not using v, contradiction.

$\diamond$

By Claim 1 and Lemma 3.23, $(G+K_S, \tau)$ contains one of the graphs in Figure 3.7 as a cycle-minor.

Case 1: $(G+K_S, \tau)$ contains two disjoint odd circuits.

Then it contains two induced disjoint odd circuits C_1 and C_2. As $G+K_S$ is 3-connected, there exist three disjoint paths from S to $C_1 \cup C_2$. We may assume that at least two of these paths end on C_1. Hence $(G+K_S, \tau)$ contains an odd circuit C and an S-path P that are disjoint. Moreover, $|VC \cap S| \leq 1$, as (G, τ) has no two disjoint S-paths.

Claim 2. *Let C be an odd circuit of (G, τ). Let Q_1, Q_2, Q_3 be three internally disjoint totally even paths from C to three distinct vertices of S, with Q_1 disjoint from Q_2, Q_3. If there is an odd path P internally disjoint from $C \cup Q_1 \cup Q_2 \cup Q_3$, that has one end on $Q_2 \setminus C$ and the other on $Q_3 \setminus C$, then (G, Σ) contains one of the graphs in Figure 3.9 (b) or (c) as local minor.*

Proof (claim). Let u_i be the end of Q_i on C. If $u_2 = u_3$, we can resign on u_2, contract Q_1 and the even part of C, and the remaining graph clearly contains the graph in Figure

3.9 (b) as local minor. Hence we may assume that $u_2 \neq u_3$ and the part of C between u_2 and u_3 not containing u_1 is odd (otherwise we contract it and reduce to the case before). Now resigning in u_2 and contracting the part of C between u_2 and u_3 not containing u_1, we obtain a graph that clearly contains the graph in Figure 3.9 (c) as local minor.

◇

Claim 3. *Let C be an odd circuit of (G, τ), $VC = \{u, v\}$. Let Q_1, Q_2, Q_3 be three internally disjoint totally even paths from C to three distinct vertices of S, with $u \in Q_1$, $v \in Q_2, Q_3$ and both Q_2 and Q_3 with at least one edge. If there is an odd path P internally disjoint from $C \cup Q_1 \cup Q_2 \cup Q_3$, that has one end on $Q_1 \setminus C$ and the other on $Q_2 \setminus C$, then (G, Σ) contains one of the graphs in Figure 3.9 (b) or (c) as local minor.*

Proof (claim). Let w_i be the end of P on Q_i. As $\{v, w_1\}$ is not a 2-vertex cutset and (G, τ) does not contain two disjoint S-paths, there exists a path P' from $(Q_1)_{u,w_1} \setminus w_1$ to $Q_2 \cup Q_3 \cup P \setminus v$. First suppose P' ends on $Q_2 \cup P \setminus v$. We may assume, eventually after contracting and resigning, that the ends of P' are u, w_2. Now, depending if P' is even or odd, (G, Σ) contains the graph in Figure 3.9 (b) or (c) as local minor.

So we may assume P' has an end on u and the other on $Q_3 \setminus v$. Now if P' is odd, we can resign in u, contract P' and $(Q_2)_{vw_2}$, and the graph we obtain clearly contains the graph in Figure 3.9 (b) as local minor. If P' is even, we again resign on u and contract the even edge in C, and obtain the graph in Figure 3.9 (c).

◇

If $|VC| \geq 3$, there exist three disjoint paths $P_i, i = 1, 2, 3$ from C to three distinct vertices in S. We can resign locally so that P_i is even for every i. As (G, τ) has no two disjoint S-paths, either it contains the graph in Figure 3.9 (b), or P contains an odd path internally disjoint from $C \cup P_1 \cup P_2 \cup P_3$ and with one end on P_i and the other on P_j, $i \neq j$. So we have the situation as in Claim 2.

So we may assume $VC = \{u, v\}$, with $v \notin S$. Then there exist three internally disjoint paths $P_i, i = 1, 2, 3$ from C to three distinct vertices in S, with $u \in P_1$, $v \in P_2, P_3$. As $v \notin S$, P_2 and P_3 have at least one edge. Similarly to before, we may assume that P contains an odd path internally disjoint from $C \cup P_1 \cup P_2 \cup P_3$ and with one end on $P_1 \setminus C$ and the other on $(P_2 \cup P_3) \setminus C$, or one end on $P_2 \setminus C$ and the other on $P_3 \setminus C$. So we have one of the situations as in Claim 2 or 3.

Case 2: $(G + K_S, \tau)$ contains an $\widetilde{K}_4$ as cycle minor.

Then, as $\widetilde{K}_4$ has maximum degree three, $(G + K_S, \tau)$ contains a subdivision (H, Γ) of $\widetilde{K}_4$ as subgraph.

Claim 4. *If (H, Γ) is a subdivision of $\widetilde{K}_4$ and P is a path internally disjoint from H with ends u, v on different legs of H, then $H \cup P$ contains a subdivision of $\widetilde{K}_4$ distinct from (H, Γ).*

Proof (claim). By symmetry, we may assume that we have one of the situations in Figure 3.10. In the first case, if P is even we remove A, if P is odd we resign on v and obtain the same situation. In the second case, if P is even we remove A, if P is odd we remove B. In all the cases we obtain a subdivision of $\widetilde{K}_4$ distinct from (H, Γ).

◇

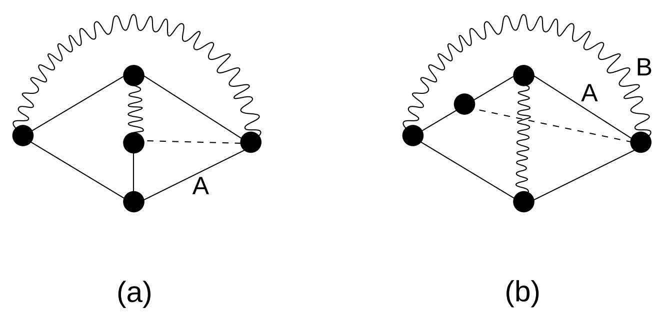

Figure 3.10: *Wavy paths are odd, straight paths are even. The dotted path is P.*

We may assume H has at most two vertices of S on the same leg L, because otherwise either we can rewrite L using an edge of the added $K_{|S|}$ or the graph contains two disjoint S-paths.

So, by connectivity and Claim 4, there are three disjoint paths $P_i, i = 1, 2, 3$ from H to distinct vertices of S, with not all these paths ending on the same leg of H. We can locally resign so that P_i is totally even for every i, and then contract each of them. So we can obtain a local minor (G', τ') of (G, τ) that is a resigning on a cut of $\widetilde{K}_4$ containing at least three vertices of S. If four vertices of S are in (G', τ'), (G, τ) either contains two disjoint S-paths, or the graph in Figure 3.9 (b) as a local minor. If exactly three vertices of S are in (G', τ'), it is easy to check that (G, τ) contains the graph in Figure 3.9 (b) as a local minor.

Case 3: $(G + K_S, \tau)$ contains an double triangle as cycle minor.

As we are not in Case 1 or 2, by Remark 3.24 we may assume (G, τ) contains as subgraph a subdivision (H, Γ) of a double triangle. We may assume H has at most two vertices of S on the same leg L, because otherwise either we can rewrite L using an edge of the added $K_{|S|}$ or the graph contains two disjoint odd circuits, and we reduce to Case 1. Moreover, there is no path between two non parallel legs of H, otherwise we reduce to Case 1 or Case 2. Hence by connectivity there exist three disjoint paths $P_i, i = 1, 2, 3$ from H to distinct vertices of S, with not all these paths ending on the same leg of H. We can locally resign so that P_i is totally even for every i, and then contract them. So (G, τ) contains as local minor a double triangle with vertices in S, hence it contains the graph in Figure 3.9 (b) as a local minor.

This concludes the proof of Lemma 3.26.

□

We will need one more result on signed graphs having more than one blocking vertex.

Lemma 3.27. *Let (H, τ) be a subdivision of a 3-connected signed graph, with (H, τ) not bipartite and H simple. Suppose (H, τ) has two distinct blocking vertices u and v. Then u and v are on the same leg L of H. Moreover all the vertices of L are blocking vertices for (H, τ).*

Proof. For the first part of the statement, we may assume $uv \notin EH$, otherwise clearly u and v are on the same leg. We can also assume that $\tau \subseteq \delta(u)$. Then there exists $U \subseteq VH$ such that $\tau \triangle \delta(U) \subseteq \delta(v)$. Hence $\delta(U) = (\delta(U) \cap \tau) \cup (\delta(U) \setminus \tau) \subseteq \delta(u) \cup \delta(v)$.

Note that $U \neq VH, \emptyset$, otherwise $\tau \subseteq \delta(u) \cap \delta(v) = \emptyset$. So if $VH \setminus (U \cup \{u, v\}) \neq \emptyset$, then $\{u, v\}$ is a 2-vertex cutset of H, hence u and v are on the same leg. Moreover (H, τ) is not bipartite, so $\tau, \tau \triangle \delta(u) \neq \emptyset$, and this implies $u \in U$. If $U = VH \setminus v$ then $\delta(U) = \delta(v)$ and $uv \in EH$, contradiction. So u, v are on the same leg.

For the second part of the statement, we may assume that $\tau \subseteq \delta(u) \setminus \delta(v)$. Let $w \in VL \setminus \{u, v\}$. Then $\tau \triangle \delta(VL_{uw}) \subseteq \delta(w)$, so w is a blocking vertex.

□

3.4 Dongles

In this section we will give an idea of how we can construct minimal partial and complete dongles.

We will first prove some results. We will do extensive use of the following

Lemma (Bixby). *Given a* 3*-connected graph* G*, for every edge* $e \in EG$*, either* G/e *is* 3*-connected, or* $G \setminus e$ *is* 3*-connected up to series edges.*

Lemma 3.28. *Let* (G, τ) *be a signed graph containing two disjoint odd circuits, with* G 3*-connected and with no loops, and let* s *and* t *be two distinct vertices of* G*. Then* (G, τ) *contains one of the graphs in Figure 3.11 as a cycle-minor, with* s *and* t *not identified.*

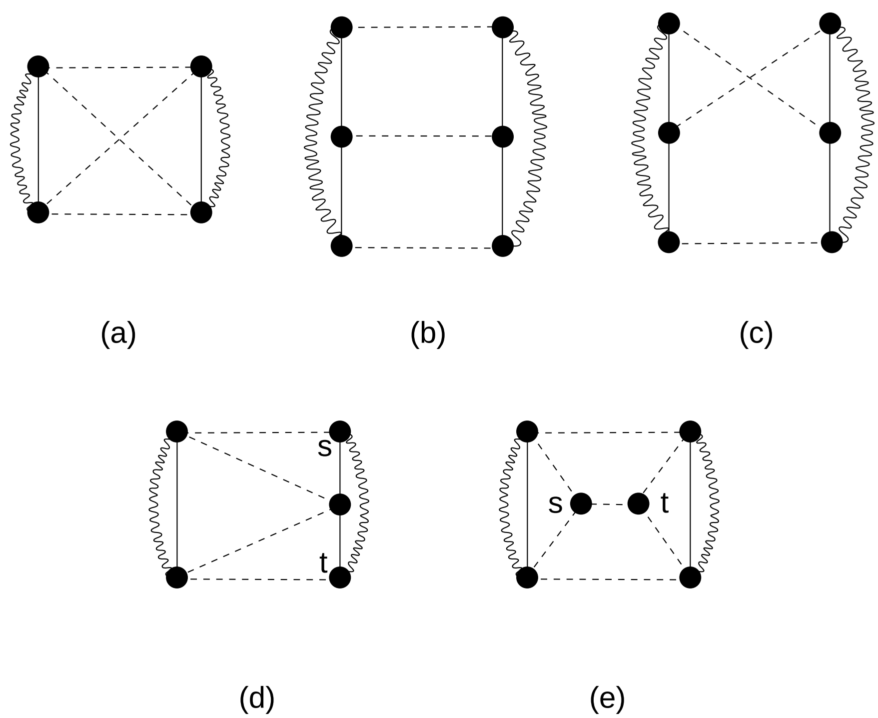

Figure 3.11: *Wavy edges are odd, straight edges are even, dotted edges can be even or odd.*

Proof. Let (G, τ) be a minor minimal counterexample, with $s, t \in VG$. Let C_1 and C_2 be two disjoint odd circuits. Choose them to be chordless. By Bixby's Lemma each edge of G is either contractible or deletable maintaining the graph 3-connected, up to series edges.

Claim 1. *For all $uv \in EG$ such that $|\{u, v\} \cap (VC_1 \cup VC_2)| < 2$ and $\{u, v\} \neq \{s, t\}$, G/uv is not 3-connected.*

Proof (claim). Otherwise we would contract it and obtain a smaller counterexample.

$\diamond$

Claim 2. *Let $uv \in EG$ such that $|\{u, v\} \cap (VC_1 \cup VC_2)| < 2$ and $\{u, v\} \neq \{s, t\}$. Then one of u, v has exactly three neighbours, is contained in C_i for $i = 1$ or $i = 2$ and C_i has exactly two edges. Moreover, if u is such vertex and w is the neighbour distinct from v that is not in C_i, then either $w \in C_{3-i}$ or $\{u, w\} = \{s, t\}$.*

Proof (claim). By Claim 1 and Bixby's Lemma, $G \setminus uv$ is 3-connected, up to series edges. As $G \setminus uv$ is not a counterexample but still contains C_1 and C_2, $G \setminus uv$ has two edges in series. Hence one of u, v has exactly three neighbours. Assume it is u, and let w_1, w_2 be it's neighbours distinct from v. As G/uw_j is not a counterexample for every $j = 1, 2$, contracting uw_j either we merge C_1 and C_2 together or we contract an edge of C_i making it become a loop, or we identify s and t. Each case cannot occur for both uw_1 and uw_2, as C_1 and C_2 are disjoint and u has exactly three neighbours.

If u is not in a circuit of length two, then we may assume $\{u, w_1\} = \{s, t\}$ and contracting uw_2 we merge C_1, C_2. But then $u \in C_i$, with $|VC_i| \geq 3$ and $w_2 \notin VC_i$, violating the condition on the degree of u.

Then we may assume $VC_1 = \{u, w_1\}$.

Now either $\{u, w_2\} = \{s, t\}$ or contracting uw_2 we merge C_1 and C_2, hence $w_2 \in VC_2$.

$\diamond$

Now let $S = EG \setminus (EC_1 \cup EC_2)$.

We have two different cases.

Case 1: for all $uv \in S$, $u \in C_i, v \in C_{3-i}$.

We may assume that there is no matching having three edges with one end in C_1 and the other in C_2, otherwise we have one of the graphs in Figure 3.11 (b) or (c) as a cycle-minor. Then by König's Matching Theorem (see West [8]) there exists a cover of S of size at most two. Let $\{u, v\}$ be such cover.

If $u \in VC_i$ and $v \in VC_{3-i}$, then $\{u, v\}$ is a 2-vertex cutset of G, contradiction. So we may assume $u, v \in VC_1$. Then, as G is 3-connected, C_1 has size two and there are at least two edges from each of u, v to C_2. So, eventually contracting part of C_2, we obtain one of the graphs in Figure 3.11 (a) or (b) as a cycle-minor. Note that we can do that without identifying s and t.

Case 2: there exists $v \notin (VC_1 \cup VC_2)$.

First assume $v \notin \{s, t\}$. Note that by Claim 2 all the neighbours of v are in $C_1 \cup C_2$. Moreover v has a neighbour $u_1 \in VC_i$ (say $i = 1$) and C_1 has exactly two edges. Then, as by connectivity v has at least three neighbours, there is an edge vu_2 such that $u_2 \notin VC_1$. Again by Claim 2, $u_2 \in VC_2$ and C_2 has exactly two edges. Let w_i be the vertex in VC_i distinct from u_i. As v has at least three neighbours, we may assume $vw_1 \in EG$.

Now if $u_2 \notin \{s, t\}$ then we may assume $u_1u_2 \in EG$. G is 3-connected, hence there exist two internally disjoint paths P_1 and P_2 in $G \setminus u_2$ from w_2 to two distinct vertices in $\{v, u_1, w_1\}$. If P_1 ends in u_1 we can contract vw_1 and obtain the graph in Figure 3.11 (a) as a minor with s, t not identified. Otherwise we may assume that P_1 ends on v and P_2 on w_1. By symmetry we may assume $\{u_1, w_1\} \neq \{s, t\}$. Then we can resign on a cut so that vu_1 is even and vw_1 is odd. Now contracting the even u_1w_1 edge and deleting the odd u_1w_1 edge we obtain the graph in Figure 3.11 (a).

So we may assume $u_2 = s$. If $u_1s \in EG$ or $w_1s \in EG$ we are in the case before. Otherwise $u_1w_2, w_1w_2 \in EG$. So contracting vs (eventually after resigning) we obtain the graph in Figure 3.11 (a) as cycle-minor.

Hence the case $v \notin \{s, t\}$ is proved and we may assume $v = s$ and $VG = VC_1 \cup VC_2 \cup \{s, t\}$.

By Claim 2 and connectivity, s has a neighbour u_1 distinct from t in C_i (say $i = 1$). Hence C_1 has exactly two edges, and u_1 has exactly three neighbours, one of which is in C_2. Let u_2 be such neighbour. Now if $t \in VC_1 \cup VC_2$ or $st \notin EG$ we reduce to the case before. So we may assume $t \notin VC_1 \cup VC_2$ and $st \in EG$. Not all the neighbours of s, t are in C_1, otherwise VC_1 would be a 2-vertex cutset. So one of s, t has a neighbour in C_2, and by Claim 2 C_2 has exactly two edges. Hence G is a 3-connected graph on six vertices. Now, using the fact that whenever a vertex $x \notin \{s, t\}$ is adjacent to one of s, t, x has exactly three neighbours, it is easy to check that we have one of the graphs in Figure 3.11

as cycle-minor. The only case that is not straightforward is when $w_2 s \notin EG$. In that case $w_1 w_2, tw_2 \in EG$ and hence $u_1 s, u_2 s \in EG$. So we can resign so that st is odd and su_2, tw_2 are even. Then contracting the even $w_1 w_2$ edge and deleting the odd one we obtain the graph in Figure 3.11 (d).

□

Corollary 3.29. *Let (G, τ) be a complete dongle, with G 3-connected and with no loops, and let s and t be two distinct vertices of G. Then (G, τ) contains either a double triangle, or $\widetilde{K}_4$ or one of the graphs in Figure 3.11 as a cycle-minor, with s and t not identified.*

Proof. By Lemma 3.23 (G, τ) contains either a double triangle, $\widetilde{K}_4$ or two disjoint odd circuits as cycle-minors. If it contains two disjoint odd circuits, it is sufficient to apply Lemma 3.28. Suppose not. If (G, τ) contains $\widetilde{K}_4$ as a cycle-minor, it contains a subdivision of $\widetilde{K}_4$ as a subgraph, because in $\widetilde{K}_4$ every vertex has degree three. By connectivity there exist two disjoint paths, possibly empty, from $\{s, t\}$ to such subdivision. Hence we can contract s, t to two different vertices of $\widetilde{K}_4$. Finally assume that (G, τ) contains a double triangle but no $\widetilde{K}_4$ and no two disjoint odd circuits. Then by Remark 3.24 (G, Σ) contains a subdivision of a double triangleas a subgraph. Again we can find two disjoint paths, possibly empty, from $\{s, t\}$ to such subdivision. Hence we can contract s, t to two different vertices of the double triangle.

□

Lemma 3.30. *If (G, τ) is a partial dongle with special vertices s, t, with $G \cup \{st\}$ 3-connected and with no loops, then (G, τ) contains one of the graphs in Figure 3.12 as a cycle-minor.*

Proof. First suppose that (G, τ) has no odd circuit disjoint from s, t. Then there exist two odd circuits C_s and C_t such that $s \in VC_s \setminus VC_t$ and $t \in VC_t \setminus VC_s$. With a similar reasoning as in Claim 1, we may assume that C_s and C_t intersect in at most one single path. If they don't intersect, by connectivity there is a path from $C_s \setminus s$ to $C_t \setminus t$. Then it is easy to check that (G, τ) contains the graph in Figure 3.12 (b) as a cycle-minor.

So we may assume that (G, τ) contains an odd circuit C_1 disjoint from s, t. Now let $(\hat{G}, \hat{\tau})$ be the graph obtained from (G, τ) by adding an even and an odd st-edge. These two dummy edges form an odd circuit C_2 of length two in $(\hat{G}, \hat{\tau})$. Then we can apply Lemma

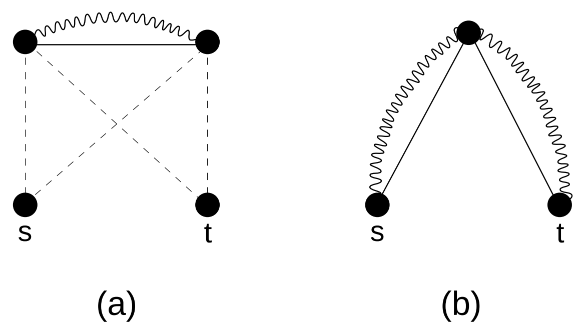

Figure 3.12: *Wavy edges are odd, straight edges are even, dotted edges can be even or odd.*

3.28 to $(\hat{G}, \hat{\tau})$. We can choose one of the two odd circuits in the proof of the Lemma to be C_2. Hence the only possible outcomings are the graphs in Figure 3.11 (a), (d) or (e) where one of the two odd circuits is C_2. Hence (G, τ) contains the graph in Figure 3.12 (a) as a cycle-minor.

□

Let (G, τ) be a signed graph, where G is 2-connected and with no loops and contains a subdivision (H, τ_H) of a 3-connected graph. Given a 2-separation (G_1, G_2) of G, we say that G_i is the *outer* side of the separation if G_i contains at most one path of H. If G_i is the outer side, then G_{3-i} is the *inner* side of the separation. An outer side G_i is *extremal* if there is no 2-separation G'_1, G'_2 in G with $G'_1 \subseteq G_i$ or $G'_2 \subseteq G_i$.

Remark 3.31. *If there is a 2-separation in G, than there is an extremal outer side.*

Now let S be an outer side of G, with vertex cutset $\{s, t\}$. Suppose that (S, τ_S) has no blocking vertex, that is (S, Σ_S, τ_S) is a complete dongle if Σ_S is a signature on S.

We give now an idea of an iterative procedure that constructs a minimal complete dongles, but we will not give all the details and will not consider what happens to the Σ-edges.

Cleaning Procedure

Let S' be an extremal outer side of G contained in S, with ends s', t'. By connectivity we may assume that there exist two (possibly empty) disjoint paths P_s, P_t, where P_s is an ss'-path and P_t is a tt'-path. We can resign τ_S on a cut $\delta(U)$, with $s, t \notin U$ so that both P_s, P_t are even. Now we add an even $s't'$-edge.

Case 1: $S' \cup \{s't'\}$ is τ-bipartite.

Then all the uv-paths in S' have the same parity. We replace S' with an edge of the same parity, and we proceed iteratively on an extreme outer side of the new graph.

Case 2: $S' \cup \{s't'\}$ is not τ-bipartite, and either s' or t' is a blocking vertex.

In this case we replace S' by an even and an odd edge, and again we repeat the procedure on an extreme outer side of the new graph.

Case 3: $S' \cup \{s't'\}$ is not τ-bipartite, it has a blocking vertex, but neither s' nor t' is a blocking vertex.

Claim 1. *(S, τ_S) contains as minor one of the graphs in Figure 3.13, where D is one of the graphs in Figure 3.12.*

Proof (claim). We will not give the details of the proof.

As we are in Case 3, by Lemma 3.30 $S' \cup \{s't'\}$ contains as cycle-minor one of the graphs in Figure 3.12 with attachments s' and t'. Moreover, as (S, τ_S) does not contain a blocking vertex, there must be an odd cycle C not completely contained in S'.

$\diamond$

Case 4: $(S' \cup \{s't'\}, \tau_{S'})$ has no blocking vertex.

By Corollary 3.29, $(S' \cup \{s't'\}, \tau_{S'})$ contains either a subdivision $(H', \tau_{H'})$ of a double triangle or of $\widetilde{K}_4$, or one of the graphs in Figure 3.11 as cycle-minor, with s' and t' not identified. In the first case, by 2-connectivity we can find two (possibly empty) disjoint paths from s and t to H', so we can obtain a double triangle or an $\widetilde{K}_4$ as cycle-minor with s and t not identified. In the second case, we can contract the paths P_s and P_t and obtain one of the graphs in Figure 3.11 as cycle-minor, with s and t not identified.

Note that these minors may use the dummy edge $s't'$, but we will use this result in the case in which there is an even st-path in $H \setminus ES$.

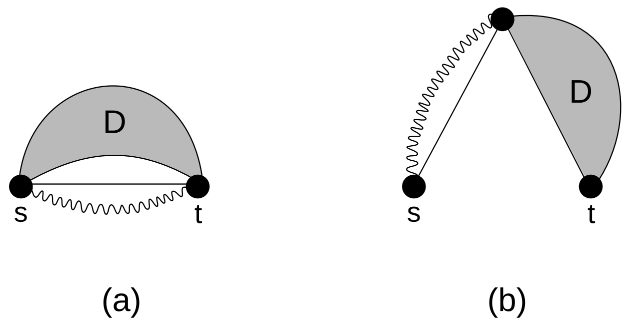

Figure 3.13: *Wavy edges are odd, straight edges are even, D is one of the graphs in Figure 3.12.*

3.5 Finale: proof of Escape Theorem for Even Cut Matroids

In this section we will show that we can indeed find a certificate as in Lemma 3.14 by one of the constructions (1)-(11).

Let the matroid M and the base graph (H_0, Γ_0) be defined as in the Escape Theorem. Let N be a minor minimal 3-connected binary matroid such that $M \geq N \geq \text{ecut}(H_0, \Gamma_0)$ and the representation (H_0, Γ_0) does not extend to N. We wish to show that $|EN| = O(|VH_0|^2)$.

Let (G, Σ, τ) be a certificate defined as in Lemma 3.14. Note that a certificate exists by Lemma 3.5.

We will consider different cases depending on the signature τ in G.

If $G' \sim G$, we will indicate with H' the subdivision in G' corresponding to H_1.

Case 1: (H', τ) does not have a blocking vertex for all $G' \sim G$. Hence (H_0, τ) has no blocking vertex, so we find (G_1, Σ_1, τ_1) as in construction (1).

Case 2: (H', τ) has a blocking vertex for some $G' \sim G$, but (H_1, τ) is not bipartite.

Let v be a blocking vertex for (H', τ) for some $G' \sim G$. We can resign τ on a cut so that all the odd edges in H' are incident with v. As v is not a blocking vertex for (G, τ), there exists an odd circuit C in (G, τ) not using v. Then there exists either a partial or a complete dongle with one end on v, or an odd path P in $G \setminus EH_1$ with ends on H_1, P not ending on v. In the first case we have the construction (4), otherwise, as $H' \setminus v$ is connected, $H' \cup P \setminus v$ contains an odd circuit. Hence if v is the unique blocking vertex in (H', τ), we have construction (6).

Now suppose v is not the unique blocking vertex in H'. By Lemma 3.27 all the vertices of a leg L of H' are blocking vertices, and we can resign τ on a cut so that all the odd edges are incident in a root vertex u and none of the odd edges in contained in EL.

If $G \setminus EH_1$ contains an odd path P with ends in legs of H_1 different from L, then again we have construction (6).

Otherwise we may assume that all the odd paths contained in $G \setminus EH_1$ have one end on L.

Claim 2. *We may assume that there are no two path P_1, P_2 having distinct ends in L and other end respectively $x_1, x_2 \notin L$, such that they intersect in a vertex y and $(P_1)_{yx_1}$ is odd and $(P_2)_{yx_2}$ is even and nonempty (see Figure 3.14).*

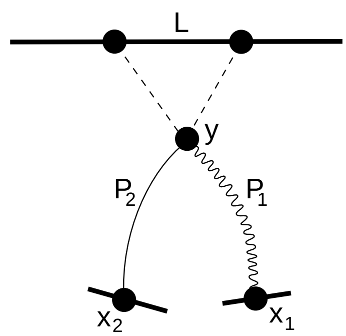

Figure 3.14: *Bold paths are in H', the straight path is even, the wavy path is odd and the dotted paths can be even or odd.*

Proof (claim). If such situation happens, we have either construction (9) or (10), depending on the parity of P_1, P_2. ◇

Claim 3. *There exist two paths P_1, P_2 which are respectively even and odd, and such that, if w_i is the end of P_i on L, then $w_1 \neq w_2$ and $w_1 \in P_{uw_2}$ (see Figure 3.15).*

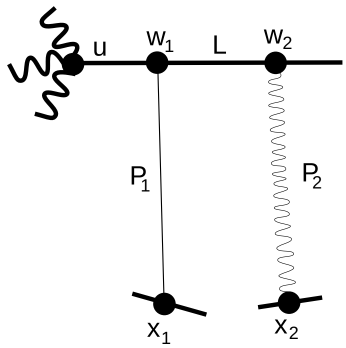

Figure 3.15: *Bold paths are in H', the straight path is even, the wavy path is odd.*

Proof (claim). Suppose not. As (G, τ) has no blocking vertex and all the odd paths contained in $G \setminus EH_1$ with ends in H_1 have one end on L, there must be at least two paths with one end in L and the other in a leg different from L. Let w be the other root vertex of L and $P_1, ..., P_l$ the paths ending on L, with ends $x_1, ..., x_l \in VL$. Suppose the P_i's are ordered so that $x_i \in L_{ux_{i+1}}$ for all $i = 1, ..., l-1$. Let e_i be the edge in P_i with end x_i. By Claim 2, we can resign on a cut without changing the signature on H' so that there exists $h \in \{1, ..., l-1\}$ such that $\tau_{P_i} = \{e_i\}$ for $i = 1, ..., h$ and $\tau_{P_i} = \emptyset$ for $i = h+1, ..., l$. But then we can resign on $\delta(VL_{ux_h})$ and all the odd edges become incident in x_h, contradicting the fact that (G, τ) has no blocking vertex. ◇

Case 3: (H_1, τ) is bipartite.

We can resign τ on a cut so that all the edges in H are even.

Case 3a: $(H \cup B_1 \cup ... \cup B_k, \tau_{H \cup B_1 \cup ... \cup B_k})$ has no blocking vertex, even up to graph equivalence.

We can apply Lemma 3.26, where S is the set of attachments of the bridges $B_1, ..., B_k$. So $\cup_{i=1}^{k} B_i$ contains one of the graphs in Figure 3.9 as a local minor. This means that we can obtain one of these graphs as a cycle-minor of (G, τ) without changing the signature τ on H.

First suppose that we have two disjoint S-paths P_1, P_2 as local minors, such that $H_1 \cup P_1 \cup P_2$ has no blocking vertex.

Then one of the situations in Figure 3.16 occurs.

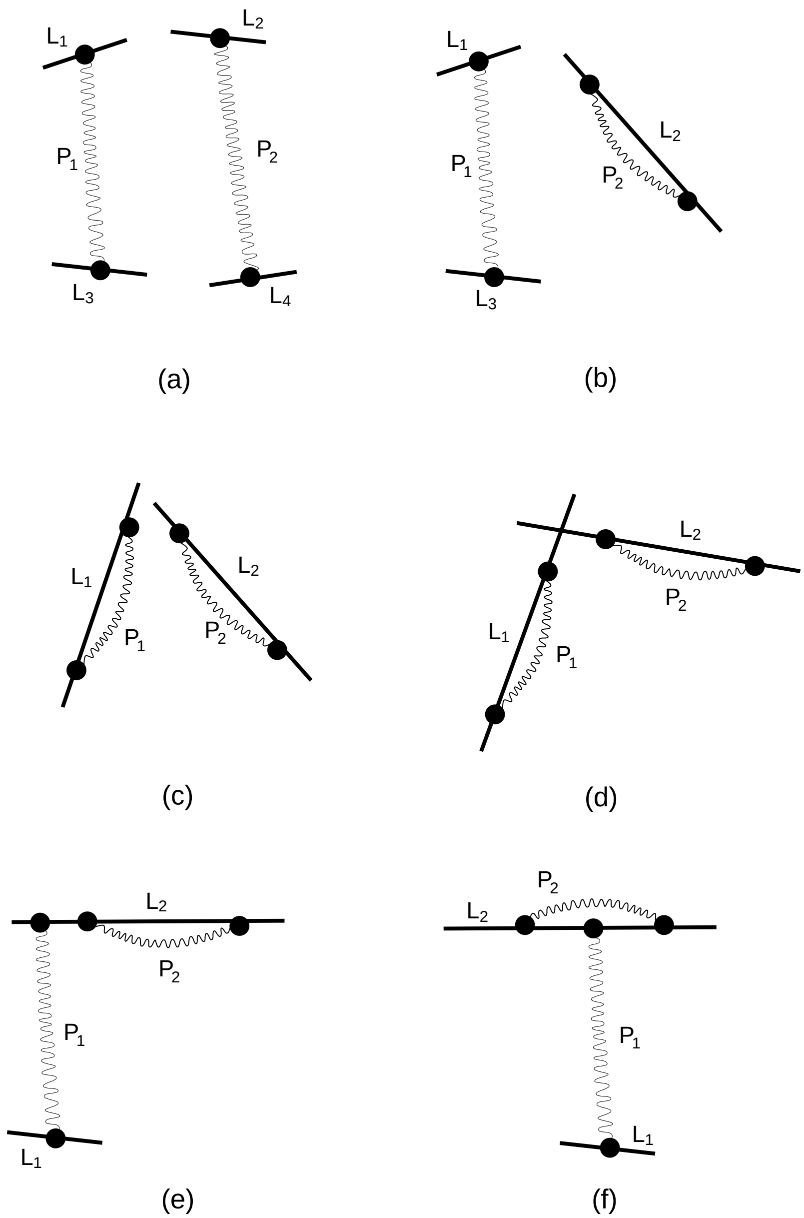

Figure 3.16: *Bold paths are in H.*

In case (a) $L_1 \neq L_3$ and $L_2 \neq L_4$, so we have the construction (7). In case (b) $L_2 \neq L_1, L_3$, and again we have construction (7). The same holds in case (c), where $L_1 \cap L_2 = \emptyset$.

In case (d), as all the bridges are good, we can find a path from P_1 to a leg L_3 different from L_1. If L_3 is different from L_2, we reduce to case (b), otherwise there must be a path Q from P_1 to P_2 that blocks the switches that would make P_1 and P_2 intersect. So we have the construction (8).

In case (e), as all the bridges are good, we can find a path from P_2 to a leg L_3 different from L_2. So either we reduce to case (a), or there is a path Q from P_2 to $L_1 \cap P_1$. If this happens again we we have construction (8).

Finally, in case (f) we have again construction (7).

Now suppose $\cup_{i=1}^{k} B_i$ contains the graph in Figure 3.9 (b) or (c) as local minor. Let F be such graph. If all the S-vertices of such graph are in the same leg, as all bridges are good we can reduce to the previous case. Otherwise we have either a widget or a gadget.

Case 3b: (H, τ) is bipartite, and there is a graph $\hat{G} \sim (H \cup B_1 \cup ... \cup B_k)$ such that $(\hat{G}, \tau_{H \cup B_1 \cup ... \cup B_k})$ has a blocking vertex.

If (G, Σ) contains a complete dongle, or two partial dongles, we have case (2) or (3) (note that if it contains two partial dongles that are contained in the same side of some 2-separation, we can reduce to the case of having a complete dongle).

Suppose not. Then (G, Σ) contains a partial dongle (D, Σ^D, τ^D) and $(H \cup B_1 \cup ... \cup B_k, \tau_{H \cup B_1 \cup ... \cup B_k})$ is not bipartite, so there exists a τ-odd path P in one of the bridges. Then we have construction (5).

This completes the proof of Theorem 3.1.

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