

# Scarf's Theorem and Applications in Combinatorics

by

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## Abstract

A theorem due to Scarf in 1967 is examined in detail. Several versions of this theorem exist, some which appear at first unrelated. Two versions can be shown to be equivalent to a result due to Sperner in 1928: for a proper labelling of the vertices in a simplicial subdivision of an  $n$ -simplex, there exists at least one elementary simplex which carries all labels  $\{0, 1, \dots, n\}$ . A third version is more akin to Dantzig's simplex method and is also examined.

In recent years many new applications in combinatorics have been found, and we present several of them. Two applications are in the area of fair division: cake cutting and rent partitioning. Two others are graph theoretic: showing the existence of a fractional stable matching in a hypergraph and the existence of a fractional kernel in a directed graph. For these last two, we also show the second implies the first.

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# Chapter 1

## Introduction

In 1928, Sperner ([30]) proved a lemma which a year later led to a short proof in [19] of Brouwer's Fixed Point Theorem ([6]). This theorem states that any continuous function of a space onto itself must have a point which is fixed by the mapping.

Although the existence of a fixed point was known, no procedure short of exhaustive search could produce such a point. Just under 40 years later, Scarf published two papers ([25, 26]) which presented a procedure for finding a fixed point and which were later shown to be an abstraction of Sperner's Lemma ([20]). In the same year, and independently of Scarf, Cohen ([9]) provided a constructive proof of Sperner's Lemma which also led to such a procedure. See [11, 12, 20, 25] for more details on calculating fixed points in a continuous mapping using Scarf's procedure.

There has been some confusion in recent years on whether Scarf's result should be known as Scarf's Theorem or Scarf's Lemma. Adding to this confusion is the fact that Scarf's result and procedure have been stated in many different forms, some of which appear unrelated. For example, it is stated in [1] that Scarf himself used the result to prove his famous theorem on the core of an  $n$ -person game, and so it acquired the title of lemma. However, in Scarf's original paper ([26]) the main theorem was a corollary to the result we are discussing. Given that it was originally stated as a theorem and that it is referred to as such in other literature ([2, 20, 34]), we will use the term Scarf's Theorem throughout this thesis.

Being an economist and game theorist, Scarf used this result to draw many conclusions in these fields ([28]). As mentioned before, this included finding the core of an  $n$ -person

game ([26]), as well as finding the equilibrium price in a market economy ([27]). More recently, Scarf's Theorem has been applied to combinatorics and graph theory, and it is these applications which this thesis will examine fully.

In order to see the similarities and differences between Scarf's Theorem and Sperner's Lemma, we present the latter here, restricted and simplified to 2 dimensions. Let  $T$  be a triangle whose vertices have labels 0, 1, 2. Subdivide  $T$  into smaller triangles ensuring that the intersection of two small triangles is either empty, a vertex, or a full face of both triangles. The vertices that lie on a face of  $T$  must take a label from one of the face's endpoints. The vertices in the interior of  $T$  can be labelled arbitrarily from 0, 1, 2.

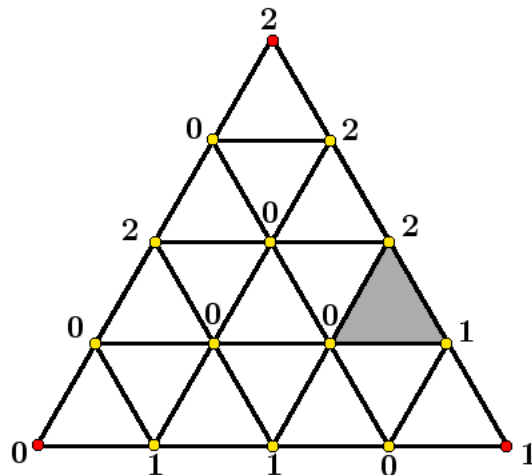


Figure 1.1: Sperner's Lemma for triangles

**Sperner's Lemma for triangles.** *For such a subdivision and labelling of  $T$ , there exists at least one small triangle which carries all labels 0, 1, 2.*

In Sperner's Lemma, there are two important concepts: the triangulation and the labelling. Each of these concepts has been abstracted by Scarf. As mentioned before, Scarf's Theorem has been stated in many forms. What ties them together is their proofs, which all follow the same pattern.

In this thesis we will examine three statements of Scarf's Theorem. In the first, called Scarf's Theorem on primitive sets, the concept of a triangulation is replaced by a set of "primitive" sets, which are sets of points in  $\mathbb{R}^n$  with special properties. The labelling however is very similar to the Sperner labelling described above. The second, Scarf's

Theorem on subdivisions, also involves triangulations and can be viewed as a simplification of Sperner’s Lemma. Instead of having many triangles touch the three faces of  $T$ , the triangulation is restricted so that only one triangle is allowed to touch each face. This leads to a simpler proof of Sperner’s Lemma that will not require induction. Again, the labelling used in this version is similar to Sperner’s labelling. Both these versions can be shown to be equivalent to Sperner’s Lemma, and vary only in formulation and proof technique. They are also both topological in nature. The third, called Scarf’s Extension Theorem, abstracts both the idea of labelling and the triangulation given in Sperner’s Lemma. Instead of integers, the labels become columns of a matrix  $B$ . The triangulation is also replaced by a matrix  $C$ , in which certain sets of columns can be viewed as primitive sets. Further, if we associate a system of equations  $B\vec{x} = \vec{b}$  with the matrix  $B$ , Scarf’s Extension Theorem dictates a basic feasible solution to this system that also has a meaningful interpretation in the matrix  $C$ . Here the formulation more closely resembles a linear program than a topological statement.

It is worth mentioning that similar work has been done by Lovász in [21], where the labels of Sperner’s Lemma are abstracted to the ground set of a matroid. We will however focus just on the work of Scarf in this thesis.

As mentioned earlier we will also examine combinatorial applications of Scarf’s Theorem. We will apply Scarf’s Theorem on primitive sets and Scarf’s Theorem on subdivisions to problems in fair division. Fair division is an area of mathematics which involves the partition of goods between parties in a way which is “fair”, based on different definitions of fair. Here we focus on envy-free division: a partitioning in which not only does every party believe they have received a fair portion of the whole with respect to their preference set, but further do not envy any one else’s share. The first application will be cake-cutting using Sperner’s Lemma, and the second will be rent partitioning and room assignment using Scarf’s Theorem. See [22] for more on fair division.

Scarf’s Extension Theorem will also be applied to graph theory. Two problems will be examined, the first being finding fractional kernels in directed graphs, work done by Aharoni and Fleiner in [1]. The second application will be finding fractional stable matchings in hypergraphs, an extension of the stable marriage problem. This is one application out of many presented by Aharoni and Holzman in [2]. It is interesting to note that although Scarf’s Theorem was used initially to provide an explicit procedure for finding solutions, in the applications we will see it will be used only to show the existence of such solutions. However, given Scarf’s procedure it would be possible to find an explicit solution, and we do so for fractional stable matchings in the appendix. Finally, we show that the theorem on fractional stable matchings can be implied by the theorem on fractional kernels in directed

graphs.

## 1.1 Outline of the thesis

Given the topological and linear programming versions of Scarf's Theorem, this thesis will be logically separated into two halves.

In Chapter 2 we will introduce the necessary topological background for the understanding of the theorem as stated in Chapters 3 and 4. Chapter 4 will further focus on the similarities between Scarf's Theorem and Sperner's Lemma. Applications of Scarf's Theorem in this form will be examined in Chapter 5.

Similarly in Chapter 6, the basics of linear programming and the simplex method are examined, followed by a generalized statement of Scarf's Theorem in Chapter 7. Finally, Chapter 8 will present two applications of Scarf's Extension Theorem as seen in Chapter 7.

The graph theory terminology used throughout is standard. Refer to [10] for basic terminology and results.

# Chapter 2

## Simplicial Complexes

### 2.1 Geometric Complexes

This section serves to provide basic definitions about simplicial complexes when viewed as geometric objects in Euclidean space. The following set of basic definitions are taken from [18, 23, 24].

**Definition 2.1.** *The set  $\mathbb{R}^m$  (Euclidean  $m$ -space) is the totality of ordered  $m$ -tuples of real numbers. An element  $X$  of  $\mathbb{R}^m$  is written*

$$X = (x_0, x_1, \dots, x_{m-1})$$

where each  $x_i$  is a real number called a **coordinate** of the element  $X$ . An element of  $\mathbb{R}^m$  is called a **point**.

**Definition 2.2.** *Let  $P = \{X^0, X^1, \dots, X^n\}$  be a finite subset of  $\mathbb{R}^m$ . Then the **hyperplane spanned by  $P$** , denoted  $\pi(X^0, X^1, \dots, X^n)$  or  $\pi(P)$ , is the set of all points  $X$  of  $\mathbb{R}^m$  that can be written*

$$X = \sum_{i=0}^n \lambda_i X^i \text{ where } \sum_{i=0}^n \lambda_i = 1$$

and each  $\lambda_i$  is a real number.

For example, consider  $X^0 = (0, 1)$ ,  $X^1 = (1, 0)$  and  $X^2 = (2, -1)$  (see Figure 2.1). Then  $\pi(X^0, X^1)$  is the line going through  $X^0$  and  $X^1$ . Also note that  $\pi(X^0, X^1, X^2) = \pi(X^0, X^1)$  since  $X^2$  lies on the same line as  $X^0$  and  $X^1$  ( $X^2 = (-1)X^0 + 2X^1$ ).

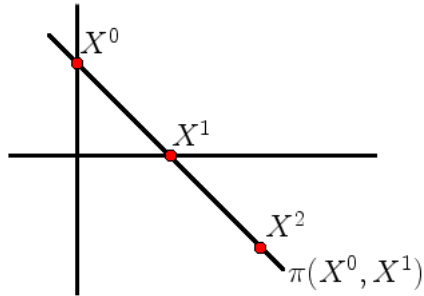


Figure 2.1: Hyperplane spanned by points

These three points do not create a plane as might be expected because they are not geometrically independent, as defined here.

**Definition 2.3.** A finite subset  $P = \{X^0, X^1, \dots, X^n\}$  of  $\mathbb{R}^m$  is said to be **geometrically independent** if  $P$  is not contained in  $\pi(Q)$  for any proper subset  $Q$  of  $P$ .

**Definition 2.4.** Let  $P = \{X^0, X^1, \dots, X^n\}$  be a geometrically independent subset of  $\mathbb{R}^m$ . The hyperplane  $\pi(P)$  is called an **n-dimensional hyperplane** or **n-hyperplane**.

For example, a 1-hyperplane is a line, a 2-hyperplane is a plane, and so on.

Now we are ready to introduce the concepts of a simplex, a simplicial complex and a pseudomanifold.

**Definition 2.5.** Let  $P = \{X^0, X^1, \dots, X^n\}$  be a geometrically independent subset of  $\mathbb{R}^m$ . Then the **simplex spanned by P**, denoted  $\sigma(P)$  or  $\sigma(X^0, X^1, \dots, X^n)$ , is the set of all points  $X$  of  $\mathbb{R}^m$  that can be written

$$X = \sum_{i=0}^n \lambda_i X^i, \quad \sum_{i=0}^n \lambda_i = 1, \quad \lambda_i \geq 0 \text{ for } i = 0, 1, \dots, n.$$

The set  $\sigma(P)$  is also called the **convex hull** of  $P$ . We say  $\sigma(P)$  is an **n-dimensional simplex** or simply an **n-simplex**.

For example, a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex a triangle and so on, as pictured in Figure 2.2.

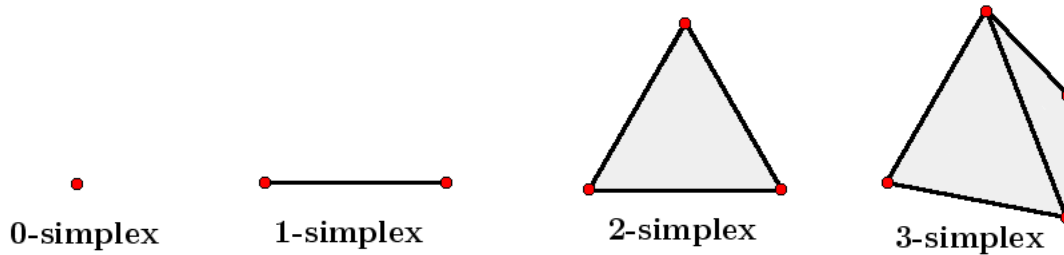


Figure 2.2: The first four  $n$ -simplices

Each point  $X^i$  is a **vertex** or equivalently an **extreme point** of  $\sigma(X^0, X^1, \dots, X^n)$ .

**Definition 2.6.** *The  $k$ -simplex spanned by a subset  $\{X^{j_0}, X^{j_1}, \dots, X^{j_k}\}$  of  $P = \{X^0, X^1, \dots, X^n\}$  is called a  **$k$ -face**, or simply **face** of  $\sigma(P)$ . In particular,  $\sigma(P)$  is its own face.*

Note that 0-faces are vertices of  $\sigma(P)$ . We call  $(n - 1)$ -faces **facets**. Further, each  $n$ -simplex has a single  $(-1)$ -face: the empty set. All faces of  $\sigma(P)$  apart from its  $(-1)$ -face and  $\sigma(P)$  itself are called **proper faces** of  $\sigma(P)$ . The union of all proper faces is called the **boundary** of  $\sigma(P)$ .

Simplices can be viewed as the building blocks for simplicial complexes.

**Definition 2.7.** *A (**geometric**) **simplicial complex**  $K$  is a set of simplices of varying dimensions satisfying the following conditions:*

1. *if  $\sigma \in K$ , then every proper face of  $\sigma$  is also in  $K$ .*
2. *if  $\sigma^1, \sigma^2 \in K$ , then  $\sigma^1 \cap \sigma^2$  is a face of both of them.*

*The highest dimension of the simplices of  $K$  is the **dimension** of  $K$ . A 0-simplex in  $K$  is called a **vertex** of  $K$ .*

Note that  $\sigma^1, \sigma^2$  can be disjoint in  $K$  when they solely share their  $(-1)$ -face, the empty set.

Figure 2.3 presents a simplicial complex  $K$  of dimension 2. Since  $\sigma(b, c, f) \in K$ , we have  $\sigma(b, c), \sigma(c, f), \sigma(b, f) \in K$  and  $\sigma(b), \sigma(c), \sigma(f) \in K$ . Note that not all maximal simplices need to be of dimension 2. For example  $\sigma(c, d) \in K$ , but it is not a face of any 2-simplex of  $K$ .



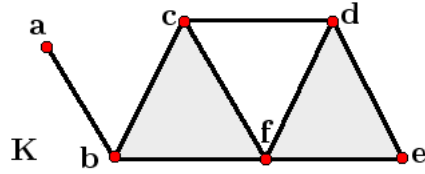


Figure 2.3: Simplicial complex of dimension 2

**Definition 2.8.** *If  $L$  is a subcollection of a simplicial complex  $K$  that contains all faces of its elements, then  $L$  is a simplicial complex in its own right; it is called a **subcomplex** of  $K$ .*

One subcomplex of  $K$  is the collection of all simplices of  $K$  of dimension at most  $p$ , denoted  $K^{(p)}$ . In particular,  $K^{(0)}$  are the vertices of  $K$ .

Finally, a special type of simplicial complex, the pseudomanifold, is introduced.

**Definition 2.9.** *An  **$n$ -dimensional pseudomanifold**, or simply an  **$n$ -pseudomanifold** is a simplicial complex  $K$  such that*

1. *Every simplex of  $K$  is a face of some  $n$ -simplex of  $K$ .*
2. *Every  $(n - 1)$ -simplex of  $K$  is the face of exactly two  $n$ -simplices of  $K$ .*
3. *If  $\sigma$  and  $\sigma'$  are  $n$ -simplices of  $K$ , there is a finite sequence  $\sigma = \sigma^1, \sigma^2, \dots, \sigma^m = \sigma'$  of  $n$ -simplices of  $K$  such that  $\sigma^i$  and  $\sigma^{i+1}$  have an  $(n - 1)$ -face in common for  $1 \leq i \leq m - 1$ .*

Figure 2.4 presents two examples of pseudomanifolds. The first is of dimension 1. Note that each 0-simplex (vertex) is the face of exactly two 1-simplices. The second is a 2-pseudomanifold, realized on the surface of a sphere. The area “outside” the main triangle is also a 2-simplex of the pseudomanifold. The dotted path outlines a sequence of 2-simplices that satisfies condition 3 for two particular simplices in the complex. Several admissible sequences between these two simplices exist.

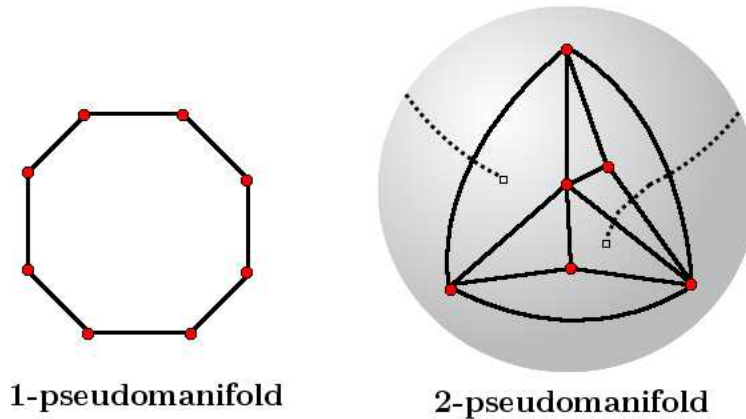


Figure 2.4: Examples of pseudomanifolds

## 2.2 Abstract Complexes

The geometric objects of Section 2.1 can be abstracted to sets with special properties. We will show at the end of the section that these two views are equivalent. We first abstract each of the definitions introduced in the previous section, using [18, 20, 29] as sources.

**Definition 2.10.** An (*abstract*) **simplicial complex**  $K$  is a finite set of elements  $\{X^0, X^1, \dots, X^n\}$  called **vertices** together with a collection of subsets called **simplices** satisfying the following conditions:

1. Every set consisting of a single vertex is a simplex.
2. Any non-empty subset of a simplex is a simplex.

A simplex containing  $n+1$  vertices is called an **n-simplex** or **n-dimensional simplex**. The highest dimension of the simplices of  $K$  is the **dimension** of  $K$ . If  $\sigma' \subseteq \sigma$  then  $\sigma'$  is called a **face** of  $\sigma$ , (a **proper face** if  $\emptyset \neq \sigma' \neq \sigma$ ). A subcollection of  $K$  that is itself a simplicial complex is called a **subcomplex** of  $K$ . One subcomplex of  $K$  is the collection of all simplices of  $K$  of dimension at most  $p$ , denoted  $K^{(p)}$ . As before,  $K^{(0)}$  denotes the vertices of  $K$ .

Note that a graph on vertex set  $V$  and edge set  $E$  can be viewed as a simplicial complex of dimension 1. We simply need to associate each  $v \in V$  with a 0-simplex (satisfies condition 1) and each  $e \in E$  with a 1-simplex. Then condition 2 is satisfied, as any non-empty subset of a 1-simplex (one of its end points) is also a simplex.

Because the collection of simplices is a set, we cannot allow multiple edges, nor can we allow loops, as each 1-simplex must be a subset of the 0-simplices.

In Figure 2.4, we can view the left pseudomanifold as a graph, in which case it is simply a cycle.

**Definition 2.11.** A *pseudomanifold* on the point set  $P = \{X^0, X^1, \dots, X^N\}$ , with  $N \geq n$ , is a family  $D$  of sets of  $n + 1$  points from  $P$ , called  $n$ -simplices, satisfying:

1. if a set of  $n$  points is a subset of a set of  $D$ , then it is a subset of exactly two sets of  $D$ .
2. if  $F, F' \in D$  then there exists a sequence of subsets of  $D$ ,  $F = F_1, F_2, \dots, F_k = F'$ , such that  $|F_i \cap F_{i+1}| = n$  for  $i = 1, \dots, k - 1$ .

**Definition 2.12.** Two simplicial complexes (abstract or geometric)  $K_1, K_2$  are said to be *isomorphic* provided there exists a bijective function  $f : K_1^{(0)} \rightarrow K_2^{(0)}$  from the vertices of  $K_1$  to the vertices of  $K_2$  having the property that a subset  $\{X^{j_0}, X^{j_1}, \dots, X^{j_k}\}$  of  $K_1^{(0)}$  is the set of vertices of a simplex in  $K_1$  if and only if  $\{f(X^{j_0}), f(X^{j_1}), \dots, f(X^{j_k})\}$  is the set of vertices of a simplex in  $K_2$ .

If an abstract simplicial complex  $K_1$  is isomorphic to a geometric simplicial complex  $K_2$ , then  $K_2$  is said to be a **realization** of  $K_1$ .

Abstract simplicial complexes can always be realized in some Euclidean space. Below are two proofs of this fact: one which lets the dimension of  $\mathbb{R}^m$  grow quite freely, and another which provides a tight bound on the dimension of  $\mathbb{R}^m$ .

**Theorem 2.13.** ([18, Chapter 1, Theorem 18]) Every abstract simplicial complex  $K_1$  has a realization  $K_2$  in some Euclidean space  $\mathbb{R}^m$ .

*Proof.* Let  $\{X^0, X^1, \dots, X^m\}$  be the vertices of the abstract simplicial complex  $K_1$ . In Euclidean space  $\mathbb{R}^m$  consider an  $m$ -simplex  $s = \sigma(Y^0, Y^1, \dots, Y^m)$ . The simplicial complex  $K_2$  is taken to be the subcomplex of  $s$  given by the rule:  $\sigma(Y^{j_0}, Y^{j_1}, \dots, Y^{j_k})$  is in  $K_2$  if and only if  $\sigma(X^{j_0}, X^{j_1}, \dots, X^{j_k})$  is a simplex in  $K_1$ .

It is now clear that the function  $f : K_1^{(0)} \rightarrow K_2^{(0)}$  given by  $f(X^i) = Y^i$  is an isomorphism.  $\square$

**Theorem 2.14.** (*[31, Section 0.2.3], [15, Theorem 3.19]*) *Any abstract simplicial complex of dimension  $n$  has a realization in  $\mathbb{R}^{2n+1}$ .*

First, examine an example for the case of  $n = 1$ . We want to realize a 1-dimensional simplicial complex, in other words a graph, in  $\mathbb{R}^3$ . If we allow the edges to bend or curve, then the realization is easy.

However, we can strive for an embedding where all edges are straight. For this, we need to put the vertices of the graph on a curve that bends enough not to admit 4 coplanar points. If we find such a curve, then any two edges cannot intersect other than at common end-points, as that would imply the four end points of those two edges are coplanar.

We claim the following curve, defined by a parameter  $t$ , admits no 4 coplanar points.

$$x = t \quad y = t^2 \quad z = t^3.$$

Suppose for a contradiction that  $t_1, t_2, t_3, t_4$  define four distinct points on the plane  $ax + by + cz = d$ . Then the equation

$$at + bt^2 + ct^3 = d$$

has 4 distinct roots, which is impossible.

*Proof (of Theorem 2.14).* We can apply the argument used in the above example to a simplicial complex of arbitrary dimension. Let  $K$  be an abstract simplicial complex of dimension  $n$  on the point set  $P = \{X^0, \dots, X^N\}$ ,  $N > n$ . We want to find a geometric simplicial complex  $K'$  in  $\mathbb{R}^{2n+1}$  and a bijection  $f$  from the vertices of  $K$  to the vertices of  $K'$  that preserves the simplices of  $K$  in  $K'$ .

Let

$$x_1 = t, \quad x_2 = t^2, \quad \dots, \quad x_{2n+1} = t^{2n+1},$$

be the parametric equation of a curve in  $\mathbb{R}^{2n+1}$ . Further, let  $f$  be the bijection  $f(X^i) = (i, i^2, \dots, i^{2n+1})$  mapping the points of  $P$  to points on the curve.

We claim that all sets of  $2n + 2$  points  $\{f(X^i)\}$  are geometrically independent. If this were not true, then the  $2n + 2$  points would lie in some  $2n$ -hyperplane, say  $a_1x_1 + a_2x_2 + \dots + a_{2n+1}x_{2n+1} = d$ . This cannot be, as it implies the curve  $a_1t + a_2t^2 + \dots + a_{2n+1}t^{2n+1} = d$  admits  $2n + 2$  distinct roots.

Recall that if a set of points  $Q$  is geometrically independent, then no point  $X$  lies on a hyperplane spanned by a set  $P \subset Q$  unless  $X \in P$ .

Now we want to show that if two simplices  $\sigma_1, \sigma_2$  of  $K$  have non-empty intersection, then under  $f$ , the intersection  $f(\sigma_1) \cap f(\sigma_2)$  is a face of both  $f(\sigma_1)$  and  $f(\sigma_2)$ .

Let  $\sigma_1 = \{X^{j_0}, \dots, X^{j_s}, \dots, X^{j_t}\}$  and  $\sigma_2 = \{X^{j_s}, \dots, X^{j_t}, \dots, X^{j_u}\}$ . Note that since  $\sigma_1$  and  $\sigma_2$  intersect in at least one point and the dimension of  $K$  is  $n$ , we have  $u < 2n + 2$ . Further,  $f(\sigma_1)$  and  $f(\sigma_2)$  share at least the simplex  $f(\sigma(X^{j_s}, \dots, X^{j_t}))$  of dimension  $t - s$ . We only need to show they intersect in no more than this common face.

Suppose to the contrary that  $f(\sigma_1) \cap f(\sigma_2) \supset f(\sigma(X^{j_s}, \dots, X^{j_t}))$ , and let  $X \in (f(\sigma_1) \cap f(\sigma_2)) \setminus f(\sigma(X^{j_s}, \dots, X^{j_t}))$ . For ease of notation, let  $Y^i = f(X^i)$ .

Since  $X$  lies in both simplices  $f(\sigma_1)$  and  $f(\sigma_2)$ , we have

$$\begin{aligned} X &= \lambda_0 Y^{j_0} + \dots + \lambda_s Y^{j_s} + \dots + \lambda_t Y^{j_t}, \text{ where } \lambda_0 + \dots + \lambda_t = 1, \lambda_0, \dots, \lambda_t \geq 0 \text{ and} \\ X &= \delta_s Y^{j_s} + \dots + \delta_t Y^{j_t} + \dots + \delta_u Y^{j_u}, \text{ where } \delta_s + \dots + \delta_u = 1, \delta_s, \dots, \delta_u \geq 0. \end{aligned}$$

Now given that  $X \notin f(\sigma(X^{j_s}, \dots, X^{j_t}))$ , there must be a  $\lambda_i \neq 0$  with  $i < s$ . Say it is  $\lambda_k$ . Then we have

$$\begin{aligned} \lambda_k Y^{j_k} &= X - (\lambda_0 Y^{j_0} + \dots + \lambda_{k-1} Y^{j_{k-1}} + \lambda_{k+1} Y^{j_{k+1}} + \dots + \lambda_s Y^{j_s} + \dots + \lambda_t Y^{j_t}) \\ &= (\delta_s Y^{j_s} + \dots + \delta_u Y^{j_u}) - (\lambda_0 Y^{j_0} + \dots + \lambda_{k-1} Y^{j_{k-1}} + \lambda_{k+1} Y^{j_{k+1}} + \dots + \lambda_t Y^{j_t}) \end{aligned}$$

and so

$$Y^{j_k} = \frac{(\delta_s Y^{j_s} + \dots + \delta_u Y^{j_u}) - (\lambda_0 Y^{j_0} + \dots + \lambda_{k-1} Y^{j_{k-1}} + \lambda_{k+1} Y^{j_{k+1}} + \dots + \lambda_t Y^{j_t})}{\lambda_k}$$

Now we have  $(\delta_s + \dots + \delta_u) - (\lambda_0 + \dots + \lambda_{k-1} + \lambda_{k+1} + \dots + \lambda_t) = 1 - (1 - \lambda_k) = \lambda_k$  hence the sum of all scalars on the right hand side is 1. This means that  $f(X^{j_k})$  lies on the hyperplane  $\pi(f(X^{j_0}), \dots, f(X^{j_{k-1}}), f(X^{j_{k+1}}), \dots, f(X^{j_u}))$ . Contradiction, the points  $\{f(X^{j_0}), \dots, f(X^{j_u})\}$  are not geometrically independent.

We can similarly prove that if  $\sigma_1$  and  $\sigma_2$  are disjoint, then  $f(\sigma_1) \cap f(\sigma_2)$  must also be disjoint.

Therefore our bijection  $f$  is indeed a realization of  $K$ . □

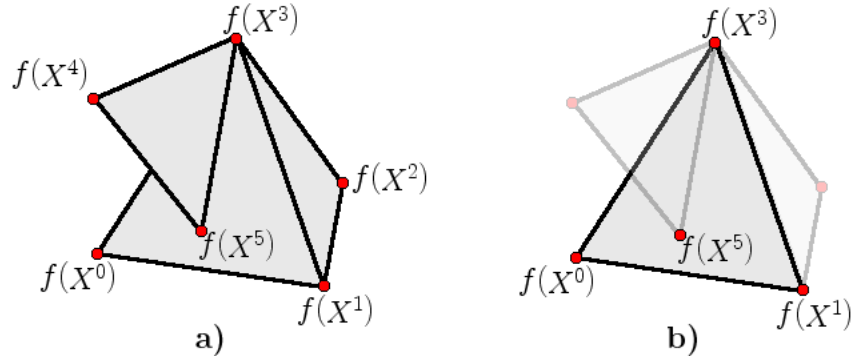


Figure 2.5: Incorrect realization of an abstract simplex

Figure 2.5 shows an example of the above argument for  $\sigma_1 = \{X^0, X^1, X^2, X^3\}$ ,  $\sigma_2 = \{X^3, X^4, X^5\}$ . Here we have the point  $f(X^5)$  which lies on  $\pi(f(X^0), f(X^1), f(X^3))$ .

The  $2n + 1$  bound is tight. For  $n = 1$ , this is Kuratowski's Theorem: not all graphs (simplicial complexes of dimension 1) can be embedded in the plane without edge crossings. For higher dimensions, this was proved by Van Kampen in 1932 ([35]).

## 2.3 Subdivisions

We examine simplicial complexes further by allowing simplices to be subdivided into smaller simplices. This procedure is also called a **triangulation** because of the 2-dimensional case. Definitions come from [17, 27].

**Definition 2.15.** *Let  $\sigma$  be an  $n$ -simplex. A collection of  $n$ -simplices  $\sigma^1, \sigma^2, \dots, \sigma^k$  is called a **simplicial subdivision of  $\sigma$**  if*

1.  $\sigma$  is contained in the union of the simplices  $\sigma^1, \sigma^2, \dots, \sigma^k$ ,
2.  $\sigma^1, \sigma^2, \dots, \sigma^k$  are each contained in  $\sigma$ , and
3. The intersection of any two simplices is a face of both of them.

Vertices of  $\sigma^1, \sigma^2, \dots, \sigma^k$  that are not vertices of  $\sigma$  are called **vertices of the subdivision**.

The simplices  $\sigma^1, \sigma^2, \dots, \sigma^k$  are called **simplices of the subdivision** or simply **elementary simplices**.

Condition 3 forbids the simplices of the subdivision from interacting in non intuitive ways, some of which are drawn in Figure 2.6 for 2-simplices.

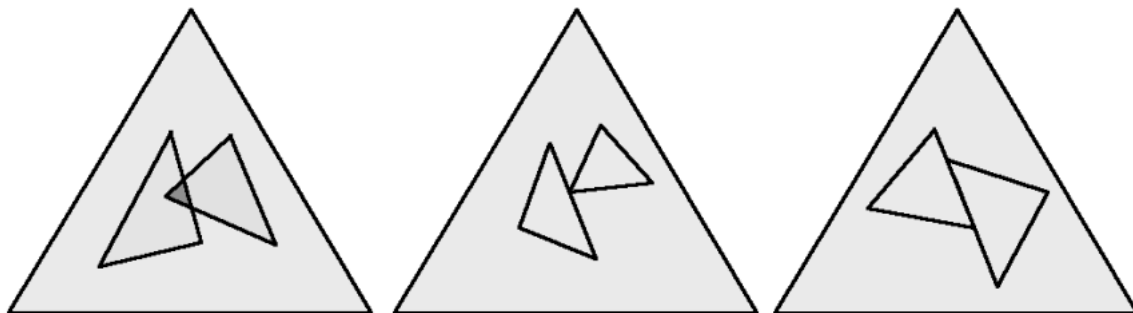


Figure 2.6: Forbidden intersections of two 2-simplices of a subdivision

Note that the collection of all faces of  $\sigma^1, \sigma^2, \dots, \sigma^k$  forms a simplicial complex.

**Definition 2.16** (Restricted Subdivision). *A simplicial subdivision of  $\sigma$  is said to be **restricted** if no vertices of the subdivision lie on the boundary of  $\sigma$ .*

Figure 2.7 gives three subdivisions of a 2-simplex. The first is the trivial subdivision, while the third (Figure 2.7 c)) is restricted.

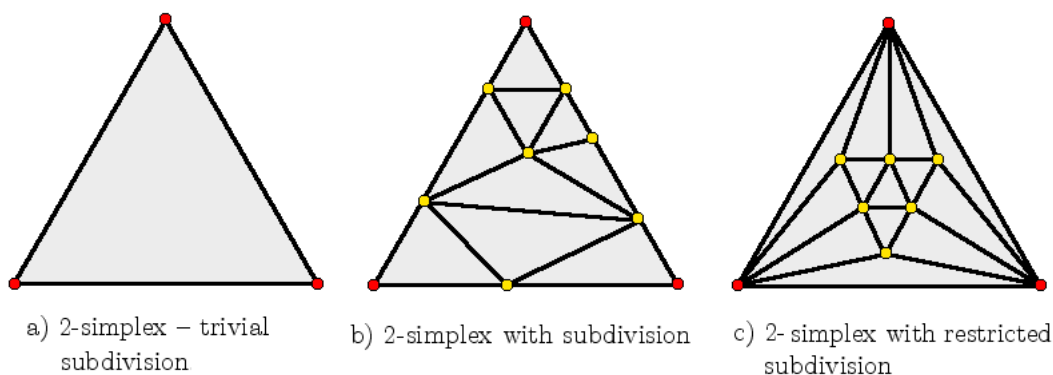


Figure 2.7: Example of simplex subdivisions

It can be shown that the simplices of a restricted subdivision of a simplex  $\sigma$  along with the original simplex  $\sigma$  form a pseudomanifold. For example, the restricted subdivision of Figure 2.7 c) is realized in Figure 2.8 by adding the original simplex as the outside face on the sphere.

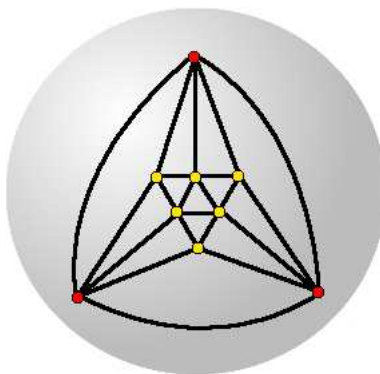


Figure 2.8: Restricted subdivision realized as a pseudomanifold

**Lemma 2.17.** *Let  $\sigma^1, \sigma^2, \dots, \sigma^k$  be a collection of simplices that form a restricted subdivision of a simplex  $\sigma$ . Then the simplices  $\sigma, \sigma^1, \sigma^2, \dots, \sigma^k$  along with all their proper faces form a pseudomanifold.*

One proof of this result can be found in [7, Section 7-5, Theorem 10] but the technique used is beyond the scope of this thesis. To motivate the lemma, we examine the case where  $\sigma$  is a 2-simplex.

*Proof (for  $\sigma$  a 2-simplex).* Relabel  $\sigma$  as  $\sigma^0$  to help with notation. Let  $K$  be a simplicial complex consisting of  $\sigma^0, \sigma^1, \sigma^2, \dots, \sigma^k$  and all their proper faces. To show  $K$  is a pseudomanifold, we must show 3 properties:

1. **Every simplex of  $K$  is a face of one of  $\sigma^0, \sigma^1, \dots, \sigma^k$** , which is true by construction of  $K$ .
2. **Every 1-simplex of  $K$  is a face of exactly 2 2-simplices from  $\sigma^0, \sigma^1, \dots, \sigma^k$ .** Suppose first that a 1-simplex  $\tau$  is a face of 3 2-simplices, say  $\sigma^h, \sigma^i$  and  $\sigma^j$ . Then 3 triangles intersect in that line segment. By Definition 2.15, each pair of triangles must intersect in a full face. Note that if one of the triangles is  $\sigma^0$ , this is ensured by



the fact that the subdivision is restricted. Further, each triangle must lie within  $\sigma^0$ . This is impossible: three triangles cannot both lie in the same plane and intersect only in a common line (see Figure 2.9). Suppose second that a 1-simplex  $\tau$  is a face of

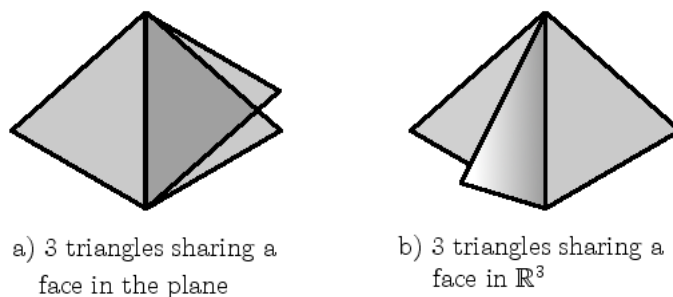


Figure 2.9: Possible intersections of 3 triangles in  $\mathbb{R}^3$

only 1 2-simplex. Then  $\tau$  must lie on the boundary  $\sigma^0$  as otherwise it would separate at least 2 triangles. Now since the subdivision is restricted, there are no boundary points, hence  $\tau$  must be one of the faces of  $\sigma^0$ . Moreover  $\tau$  must be the face of at least one simplex of the subdivision, the one that covers the interior that borders on  $\tau$  (given that the subdivision must by definition cover all of  $\sigma^0$ ). Hence  $\tau$  is the face of exactly 2 triangles.

3. **Every 2-simplex in the subdivision is connected by an alternating path of 1-simplices and 2-simplices.** This easily follows from part 2 and the fact that the subdivision is of a single connected simplex.  $\square$

Lastly, we introduce another special subdivision, called the barycentric subdivision.

**Definition 2.18.** The **barycentre** (centre of mass) of a simplex  $\sigma = (X^0, \dots, X^n)$  is the point  $Y = \sum_{i=0}^n \lambda_i X^i$  where the coefficients  $\lambda_i$  are all equal, namely  $\lambda_i = 1/(n+1)$  for each  $i$ .

The barycentre of a 0-simplex is itself, the barycentre of a 1-simplex is its midpoint, and so forth.

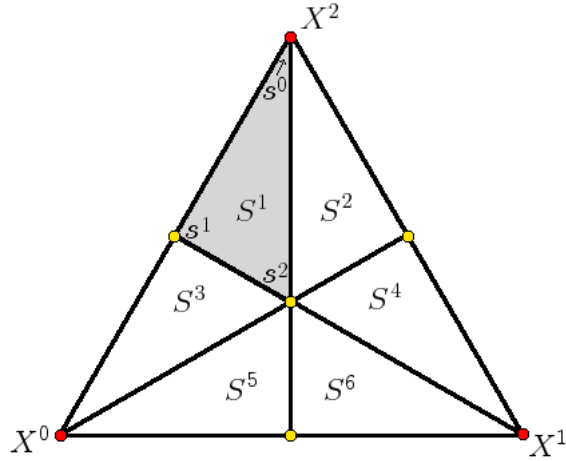


Figure 2.10: Barycentric subdivision of a 2-simplex

**Definition 2.19.** The **barycentric subdivision** of an  $n$ -dimensional simplex  $\sigma = \sigma(X^0, X^1, \dots, X^n)$  consists of  $(n + 1)!$  simplices. Each one, say  $\sigma(t^0, t^1, \dots, t^n)$ , can be associated with a permutation  $\{s^0, s^1, \dots, s^n\}$  of  $\{X^0, X^1, \dots, X^n\}$ , in such a way that each vertex  $t^i$  is the barycentre of the simplex  $\sigma(s^0, s^1, \dots, s^i)$ .

Figure 2.10 shows the barycentric subdivision of a 2-simplex. For example, the simplex  $S^1 = \sigma(s^0, s^1, s^2)$  is associated with the permutation  $\{X^2, X^0, X^1\}$ , because  $s^0$  is the barycentre of  $\sigma(X^2)$ ,  $s^1$  is the barycentre of  $\sigma(X^2, X^0)$  and  $s^2$  is the barycentre of  $\sigma(X^2, X^0, X^1)$ . Similarly  $S^2, S^3, \dots, S^6$  have associated permutations.

# Chapter 3

## Scarf's Theorem on Primitive Sets

In this chapter we will examine the first of many versions of Scarf's Theorem. We will need the concepts of primitive sets, covering simplices and labellings.

### 3.1 Primitive Sets and Covering Simplices

We will first restrict our attention to a specific simplex, called the standard simplex, and then study a set  $P$  of points taken from this simplex and its hyperplane. Definitions in this section come from [20, 25].

Denote by  $S$  the  $n$ -simplex  $\sigma(e_0, e_1, \dots, e_n)$ , where  $\{e_0, e_1, \dots, e_n\}$  are the points  $\{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)\}$ . The simplex  $S$  is called the **standard simplex** and can also be written as

$$S = \{X = (x_0, x_1, \dots, x_n) \mid \text{all } x_k \geq 0, \sum x_k = 1\}.$$

Refer to Figure 3.1 for representations of the standard 1- and 2-simplices.

Now consider a set  $P$  of points in  $\mathbb{R}^{n+1}$  of size  $N + 1$ ,  $N \geq n$ . The first  $n + 1$  points in

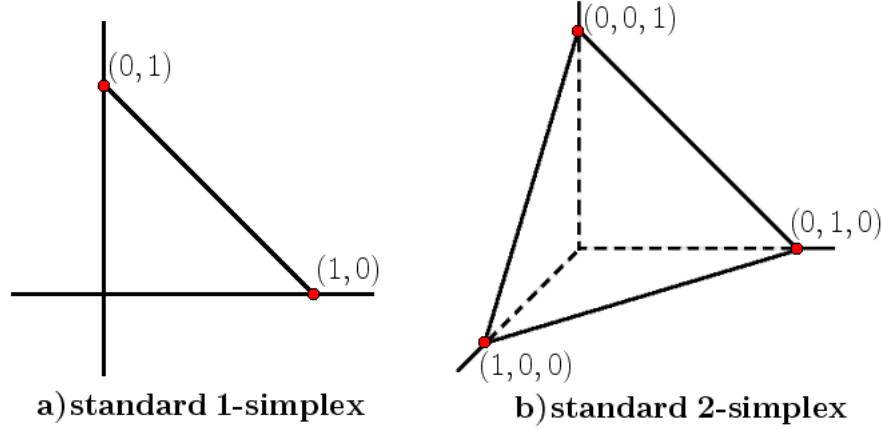


Figure 3.1: Example of standard simplices

the collection do not lie in  $S$  and have the form:

$$\begin{aligned}
 X^0 &= (1 - n, 1, 1, \dots, 1) \\
 X^1 &= (1, 1 - n, 1, \dots, 1) \\
 &\vdots \\
 X^n &= (1, 1, 1, \dots, 1 - n)
 \end{aligned}$$

Note that the points  $X^0, X^1, \dots, X^n$  lie on the same hyperplane as the points  $e_0, e_1, \dots, e_n$ , that is,  $\pi(X^0, X^1, \dots, X^n) = \pi(e_0, e_1, \dots, e_n)$ . One way to see this is to note that the sum of the coordinates of all points is 1.

Next, the points  $X^{n+1}, X^{n+2}, \dots, X^N$  of  $P$  are points taken arbitrarily within the interior of  $S$ .

From this ground set  $P$  we will be interested in subsets of size  $n + 1$ . Such a subset will be called primitive if it satisfies the following condition.

**Definition 3.1.** A set of  $n+1$  points  $W = \{X^{j_0}, X^{j_1}, \dots, X^{j_n}\}$  chosen from  $P = \{X^0, \dots, X^N\}$  is called **primitive** if there is no point  $X^j \in P$  such that  $x_k^j > \min\{x_k^{j_0}, x_k^{j_1}, \dots, x_k^{j_n}\}$  for all  $k$ . We call a point  $X^j \in W$  with  $x_k^j = \min\{x_k^{j_0}, x_k^{j_1}, \dots, x_k^{j_n}\}$  a **minimizer in  $k$** .

For example, for  $S$  a 2-simplex and

$$P = \{X^0 = (-1, 1, 1), X^1 = (1, -1, 1), X^2 = (1, 1, -1), X^3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), X^4 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})\},$$

the set  $\{X^0, X^1, X^4\}$  is not primitive because every coordinate  $x_i^3$  of  $X^3$  is greater than the  $i$ th coordinate of the minimizer in  $i$ . Indeed,

$$\begin{aligned} x_0^3 = \frac{1}{3} &> \min\{x_0^0, x_0^1, x_0^4\} = -1, \\ x_1^3 = \frac{1}{3} &> \min\{x_1^0, x_1^1, x_1^4\} = -1, \\ x_2^3 = \frac{1}{3} &> \min\{x_2^0, x_2^1, x_2^4\} = \frac{1}{4}. \end{aligned}$$

A more intuitive way of understanding primitive sets will be presented later in this section. It requires the notion of a covering simplex.

**Definition 3.2.** The **covering simplex** of a set of  $n+1$  points  $W = \{X^{j_0}, X^{j_1}, \dots, X^{j_n}\}$  from  $P$  is defined as  $\{X | x_k \geq \min\{x_k^{j_0}, x_k^{j_1}, \dots, x_k^{j_n}\} \text{ for all } k\} \cap S$ .

The proof that a covering simplex is indeed a simplex is deferred to the end of this section. Geometrically, the covering simplex of a set  $W$  is the minimal simplex whose faces are parallel to the faces of  $S$  and that contains all the points of  $W$  which are also in  $S$ .

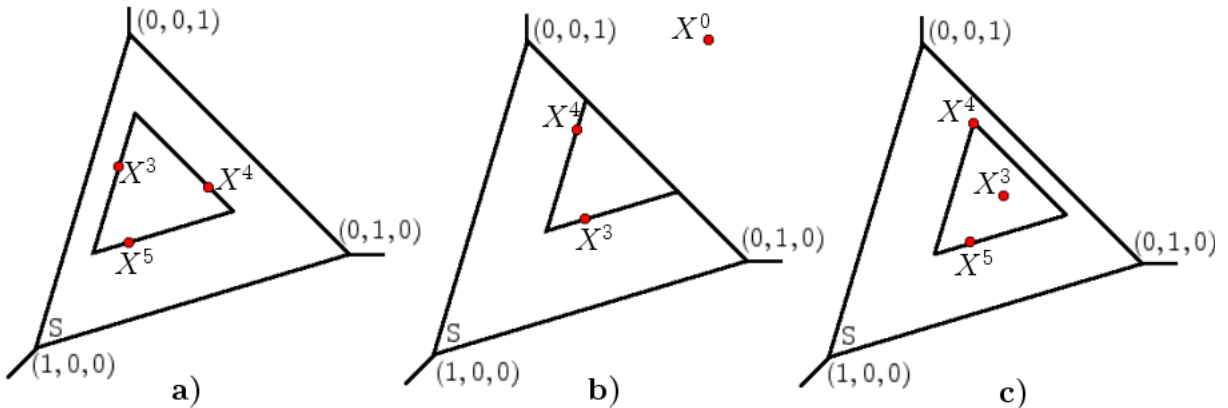


Figure 3.2: Examples of covering simplices

Refer to Figure 3.2 for some examples of covering simplices. Figure 3.2 a) exhibits the covering simplex of the set  $W = \{X^3, X^4, X^5\}$ . If  $W$  includes points from  $\{X^0, \dots, X^n\}$ , which lie outside of  $S$ , then the covering simplex of  $W$  will not contain them, as shown in Figure 3.2 b). Also note that not every point of  $W$  need be contained in a face of its covering simplex, as in Figure 3.2 c).

Primitive sets can also be interpreted geometrically. A set  $W$  of  $n + 1$  points from  $P$  is primitive if no point of  $P$  lies in the interior of its covering simplex. Indeed, if there existed a point  $X$  of  $P$  in the interior of the covering simplex, then  $x_i > \min\{x_i^{j_0}, x_i^{j_1}, \dots, x_i^{j_n}\}$  (note the strictly greater than), for all  $i$ , which contradicts the definition of a primitive set.

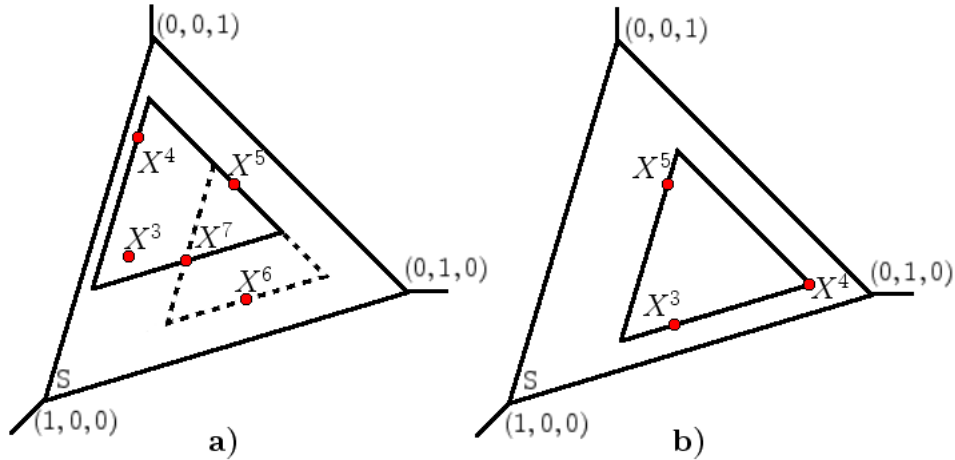


Figure 3.3: Primitive sets

In Figure 3.3 a), for  $P = \{X^0, \dots, X^7\}$ , there are two sets  $W^1 = \{X^4, X^5, X^7\}$  and  $W^2 = \{X^5, X^6, X^7\}$  of  $n + 1$  points. The first is not primitive because  $X^3$  lies in the interior of its covering simplex. The set  $W^2$  however is primitive because no point of  $P$  lies in the interior of its covering simplex (in dotted lines). It is also possible for a point in  $W$  itself to lie in the interior of its covering simplex, as seen in Figure 3.2 c). That set is thus not primitive.

The set  $W = \{X^3, X^4, X^5\}$  in Figure 3.3 b) is primitive, yet one of its points,  $X^3$ , is redundant. Primitive sets of this form are called **degenerate** and are not desirable. To avoid them, the following assumption is made.

**Definition 3.3** (Non-Degeneracy Assumption). *For each  $i = 0, \dots, n$ , the value of*

$\min\{x_i^{j_0}, x_i^{j_1}, \dots, x_i^{j_n}\}$  is unique for every set of  $n + 1$  points  $\{X^{j_0}, X^{j_1}, \dots, X^{j_n}\}$  of  $P$ .

**Lemma 3.4.** *The Non-Degeneracy Assumption ensures that every point in a primitive set of  $n + 1$  points  $W = \{X^{j_0}, X^{j_1}, \dots, X^{j_n}\}$  from  $P$  is the minimizer in exactly one coordinate. That is, there is a one-to-one correspondence between a point of  $W$  and a minimum-valued coordinate.*

*Proof.* Let  $W = \{X^{j_0}, X^{j_1}, \dots, X^{j_n}\}$  be a primitive set of  $n + 1$  members of  $P$ . Define a function  $f_W : \{0, 1, \dots, n\} \rightarrow W$ . The function takes a coordinate  $k \in \{0, 1, \dots, n\}$  to a point  $X$  in  $W$  which achieves the minimum in coordinate  $k$ .

By the Non-Degeneracy Assumption, that minimum is unique for every  $W$ , thus  $f_W$  is well defined.

The codomain of  $f_W$  is finite and has the same size as the domain. Hence to show  $f_W$  is one-to-one and onto, only one of these properties needs to be demonstrated.

Suppose  $f_W$  is not onto, and there exists a point  $X^{j_i} \in W$  which achieves the minimum in no coordinate. Then it must be that for all  $k$ ,  $x_k^{j_i} > \min\{x_k^{j_0}, x_k^{j_1}, \dots, x_k^{j_n}\}$ . This implies that  $W$  is not primitive. Contradiction.

Hence  $f_W$  is one-to-one and onto, and there exists a unique correspondence between points in a primitive set and a minimum-valued coordinate.  $\square$

As promised earlier, the proof that a covering simplex is indeed a simplex is presented here. It is not difficult but it is lengthy.

**Lemma 3.5.** *The covering simplex of a set of  $n + 1$  points  $\{X^{j_0}, X^{j_1}, \dots, X^{j_n}\}$  from  $P$  is indeed a simplex.*

*Proof.* Let  $W = \{X^{j_0}, X^{j_1}, \dots, X^{j_n}\}$  be any set of  $n + 1$  members of  $P$ , and let  $C$  be the covering simplex of  $W$ , where  $C = \{X | x_k \geq \min\{x_k^{j_0}, x_k^{j_1}, \dots, x_k^{j_n}\} \text{ for all } k\} \cap S$ .

To show  $C$  is a simplex, we display  $n + 1$  geometrically independent points whose convex hull is  $C$ .

First, let  $m_i = \max\{\min\{x_i^{j_0}, x_i^{j_1}, \dots, x_i^{j_n}\}, 0\}$ . Notice that  $m_k \geq 0$ .

Then rewrite  $C = \{X | x_k \geq m_k \text{ for all } k\} \cap S$ . The extra condition that  $m_k \geq 0$  is redundant because all points in  $S$  have coordinates which are greater than or equal to 0.

**Claim 3.5.1.**  $m_0 + m_1 + \dots + m_n < 1$ .

*Proof.* First, consider the case where  $W \cap S = \emptyset$ . There is only one such set of  $n+1$  points in  $P$ , namely  $W = \{X^0, \dots, X^n\}$ . Given that  $X^i = (x_0^i, x_1^i, \dots, x_n^i)$  where

$$x_k^i = \begin{cases} 1 - n & k = i \\ 1 & \text{otherwise} \end{cases} ,$$

then we have  $m_0 = m_1 = \dots = m_n = 0$ . Hence  $m_0 + m_1 + \dots + m_n = 0 < 1$ .

Now assume  $W \cap S \neq \emptyset$  and begin by showing  $m_0 + m_1 + \dots + m_n \leq 1$ .

Take a point  $X \in W \cap S$ . Then  $X \in S$  and so  $x_0 + x_1 + \dots + x_n = 1$  and  $x_k \geq 0$  for all  $k$ . Further  $X \in W$ , and since  $m_k = \max\{\min\{x_i^{j_0}, x_i^{j_1}, \dots, x_i^{j_n}\}, 0\}$ ,  $x_k \geq m_k$  for all  $k$ . Therefore,

$$\begin{aligned} 1 &= x_0 + x_1 + \dots + x_n \\ &\geq m_0 + m_1 + \dots + m_n \end{aligned} \tag{3.1}$$

The equality in equation (3.1) only holds if  $x_i = m_i$  for all  $i$ . We consider two cases:

1. Say there exists any other point  $X' \in W \cap S$ . Then  $x'_k > m_k$ , for some  $k$  (note the strictly greater than, given that  $X'$  cannot achieve the minimum in every coordinate, otherwise  $X' = X$ ).

Then  $1 = \sum_{k=0}^n x'_k > \sum_{k=0}^n m_k = 1$ . Contradiction.

2. If no such second point  $X' \in W \cap S$  exists, then all other points in  $W$  come from  $\{X^0, \dots, X^n\}$ . Say they are the points  $\{X^{j_1}, \dots, X^{j_n}\}$ . But then  $m_{j_k} = 0$ , for all  $k = 1, \dots, n$ .

Hence  $1 = \sum_{k=0}^n x_k = \sum_{k=0}^n m_k = m_{j_0} + m_{j_1} + \dots + m_{j_n} = m_{j_0} + 0 + \dots + 0 = m_{j_0}$ .

Then  $X$  is one of the points  $e_i$ , an extreme point of  $S$ . Contradiction,  $X$  must be taken from the interior of  $S$ .

Therefore we have  $m_0 + m_1 + \dots + m_n < 1$  □



Now consider the following points  $Y^0, \dots, Y^n$ :

$$\begin{aligned} Y^0 &= (1 - (m_1 + m_2 + \dots + m_n), m_1, m_2, \dots, m_n) \\ Y^1 &= (m_0, 1 - (m_0 + m_2 + \dots + m_n), m_2, \dots, m_n) \\ &\vdots \\ Y^n &= (m_0, m_1, m_2, \dots, 1 - (m_0 + m_1 + m_2 + \dots + m_{n-1})) \end{aligned}$$

We claim that these points are extremal in  $C$ . First we show  $Y^0, Y^1, \dots, Y^n$  are geometrically independent.

**Claim 3.5.2.** *The points  $Y^0, Y^1, \dots, Y^n$  are geometrically independent.*

*Proof.* Suppose for a contradiction that  $Y^i = \lambda_0 Y^0 + \dots + \lambda_{i-1} Y^{i-1} + \lambda_{i+1} Y^{i+1} + \dots + \lambda_n Y^n$ , with  $\lambda_0 + \lambda_1 + \dots + \lambda_{i-1} + \lambda_{i+1} + \dots + \lambda_n = 1$ . Rewriting we have  $\lambda_0 Y^0 + \dots + \lambda_{i-1} Y^{i-1} + \lambda_i Y^i + \lambda_{i+1} Y^{i+1} + \dots + \lambda_n Y^n = 0$  with  $\lambda_i = -1$ . Hence equivalently, we show that  $\lambda_0 Y^0 + \dots + \lambda_n Y^n = 0$  and  $\lambda_0 + \lambda_1 + \dots + \lambda_n = 0$  implies  $\lambda_0 = \lambda_1 = \dots = \lambda_n = 0$ . Consider the following augmented system  $[Y^0 Y^1 \dots Y^n | 0]$ .

$$\left[ \begin{array}{cccc|c} 1 - m_0 - m_1 - \dots - m_n & m_0 & \dots & m_0 & 0 \\ m_1 & 1 - m_0 - m_2 - \dots - m_n & \dots & m_1 & 0 \\ m_2 & m_2 & \dots & m_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_n & m_n & \dots & 1 - m_0 - m_1 - \dots - m_{n-1} & 0 \end{array} \right].$$

If we can row reduce the first  $n + 1$  columns to the identity matrix, the final column will yield the unique zero solution for  $\lambda$ . First, add rows  $1, \dots, n$  and add to row 0 to get:

$$\left[ \begin{array}{cccc|c} 1 & 1 & \dots & 1 & 0 \\ m_1 & 1 - m_0 - m_2 - \dots - m_n & \dots & m_1 & 0 \\ m_2 & m_2 & \dots & m_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_n & m_n & \dots & 1 - m_0 - m_1 - \dots - m_{n-1} & 0 \end{array} \right].$$

Next, eliminate the entries below the 1 in the 0th column. First multiply the 0th row

by  $m_1$  and subtract it from the first row. Follow similarly on rows  $2, \dots, n$  to get:

$$\left[ \begin{array}{cccc|c} 1 & 1 & \dots & 1 & 0 \\ 0 & 1 - m_0 - m_1 - \dots - m_n & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 - m_0 - m_1 - \dots - m_n & 0 \end{array} \right].$$

We can divide all rows except the 0th by  $1 - m_0 - m_1 - \dots - m_n$ , which by Claim 3.5.1 is greater than 0. Finally, by subtracting the sum of rows  $1, \dots, n$  from the 0th row, we get the identity matrix.

$$\left[ \begin{array}{cccc|c} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{array} \right].$$

Therefore the only solution to  $\lambda_0 Y^0 + \dots + \lambda_n Y^n = 0$  and  $\lambda_0 + \lambda_1 + \dots + \lambda_n = 0$  is  $\lambda_0 = \lambda_1 = \dots = \lambda_n = 0$ . Hence the points  $Y^0, Y^1, \dots, Y^n$  are geometrically independent.  $\square$

It remains to be shown that  $\sigma(Y^0, Y^1, \dots, Y^n) = C$ . The following two claims conclude the proof.

**Claim 3.5.3.**  $\sigma(Y^0, Y^1, \dots, Y^n) \subseteq C$ .

**Claim 3.5.4.**  $C \subseteq \sigma(Y^0, Y^1, \dots, Y^n)$ .

*Proof of Claim 3.5.3.* To show the convex hull of  $Y^0, Y^1, \dots, Y^n$  is contained in  $C$ , we ensure all linear combinations of  $Y^0, Y^1, \dots, Y^n$  where the sum of the scalars is 1 belong to  $C$ .

Let  $\lambda_0, \lambda_1, \dots, \lambda_n$  be non-negative scalars such that  $\sum_{k=0}^n \lambda_k = 1$ . Examine

$$Z = \lambda_0 Y^0 + \lambda_1 Y^1 + \dots + \lambda_n Y^n = \begin{pmatrix} \lambda_0(1 - (m_1 + m_2 + \dots + m_n)) + \lambda_1 m_0 + \dots + \lambda_n m_0 \\ \lambda_0 m_1 + \lambda_1(1 - (m_0 + m_2 + \dots + m_n)) + \dots + \lambda_n m_1 \\ \vdots \\ \lambda_0 m_n + \lambda_1 m_n + \dots + \lambda_n(1 - (m_0 + m_1 + \dots + m_{n-1})) \end{pmatrix}.$$

We have  $Z \in C$  if and only if  $\sum_{k=0}^n z_k = 1$ ,  $z_k \geq 0$  for all  $k$ , and  $z_k \geq m_k$  for all  $k$ . Given that  $m_k \geq 0$  for all  $k$ , only the first and last are shown.

First,  $\sum_{k=0}^n z_k = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ , and  $\lambda_0 + \lambda_1 + \cdots + \lambda_n = 1$  by definition.

Now to show  $z_k \geq m_k$  for all  $k$ , look at the first component; the others are worked similarly.

$$\begin{aligned}
z_0 &= \lambda_0(1 - (m_1 + m_2 + \cdots + m_n)) + \lambda_1 m_0 + \lambda_2 m_0 + \cdots + \lambda_n m_0 \\
&= \lambda_0 - \lambda_0(m_1 + m_2 + \cdots + m_n) + m_0(\lambda_1 + \lambda_2 + \cdots + \lambda_n) \\
&= \lambda_0 - \lambda_0(m_1 + m_2 + \cdots + m_n) + m_0(1 - \lambda_0), \quad \text{since } \lambda_0 + \lambda_1 + \cdots + \lambda_n = 1 \\
&= \lambda_0 - \lambda_0(m_1 + m_2 + \cdots + m_n) + m_0 - \lambda_0 m_0 \\
&= \lambda_0 - \lambda_0(m_0 + m_1 + m_2 + \cdots + m_n) + m_0 \\
&= \lambda_0(1 - (m_0 + m_1 + m_2 + \cdots + m_n)) + m_0 \\
&\geq m_0, \quad \text{since } m_0 + m_1 + \cdots + m_n < 1 \text{ by Claim 3.5.1 and } \lambda_0 \geq 0.
\end{aligned}$$

Hence the convex hull of  $Y^0, Y^1, \dots, Y^n$  is contained in  $C$ . □

*Proof of Claim 3.5.4.* To show  $C \subseteq \sigma(Y^0, Y^1, \dots, Y^n)$ , consider any point  $X = (x_0, \dots, x_n) \in C$ . It is necessary to find non-negative scalars  $\lambda_0, \lambda_1, \dots, \lambda_n$  with  $\sum_{k=0}^n \lambda_k = 1$  where  $\sum_{k=0}^n \lambda_k Y^k = X$ . Then  $X$  will belong to the convex hull of  $\{Y^0, Y^1, \dots, Y^n\}$ .

Let

$$\begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} \frac{x_0 - m_0}{1 - m_0 - m_1 - \cdots - m_n} \\ \frac{x_1 - m_1}{1 - m_0 - m_1 - \cdots - m_n} \\ \frac{x_2 - m_2}{1 - m_0 - m_1 - \cdots - m_n} \\ \vdots \\ \frac{x_n - m_n}{1 - m_0 - m_1 - \cdots - m_n} \end{bmatrix}$$

It remains to be shown that these scalars satisfy all conditions.

**Claim 3.5.5.**  $\lambda_k \geq 0$  for all  $k \in \{0, 1, \dots, n\}$  and  $\sum_{k=0}^n \lambda_k = 1$ .

*Proof.* Because  $X \in C$ , by definition  $x_k \geq m_k$  for all  $k$ , and by Claim 3.5.1,  $m_0 + m_1 + \dots + m_n < 1$  so the numerator is non-negative and the denominator is positive. Hence  $\lambda_k \geq 0$ .

Next examine

$$\begin{aligned}
\sum_{k=0}^n \lambda_k &= \frac{x_0 - m_0}{1 - m_0 - m_1 - \dots - m_n} + \frac{x_1 - m_1}{1 - m_0 - m_1 - \dots - m_n} + \dots + \frac{x_n - m_n}{1 - m_0 - m_1 - \dots - m_n} \\
&= \frac{x_0 + x_1 + \dots + x_n - m_0 - m_1 - \dots - m_n}{1 - m_0 - m_1 - \dots - m_n} \\
&= \frac{1 - m_0 - m_1 - \dots - m_n}{1 - m_0 - m_1 - \dots - m_n}, \quad \text{since } x_0 + x_1 + \dots + x_n = 1 \\
&= 1.
\end{aligned}$$

□

Finally we show that  $\sum_{k=0}^n \lambda_k Y^k = X$ .

$$\sum_{k=0}^n \lambda_k Y^k = \begin{pmatrix} \lambda_0(1 - (m_1 + m_2 + \dots + m_n)) + \lambda_1 m_0 + \dots + \lambda_n m_0 \\ \lambda_0 m_1 + \lambda_1(1 - (m_0 + m_2 + \dots + m_n)) + \dots + \lambda_n m_1 \\ \vdots \\ \lambda_0 m_n + \lambda_1 m_n + \dots + \lambda_n(1 - (m_0 + m_1 + \dots + m_{n-1})) \end{pmatrix}.$$

We show the first component is  $x_0$ , the others are worked similarly.

$$\begin{aligned}
&\lambda_0(1 - (m_1 + m_2 + \dots + m_n)) + \lambda_1 m_0 + \lambda_2 m_0 + \dots + \lambda_n m_0 \\
&= \lambda_0(1 - (m_0 + m_1 + m_2 + \dots + m_n)) + m_0, \quad \text{as shown in Claim 3.5.3} \\
&= \frac{x_0 - m_0}{1 - m_0 - m_1 - \dots - m_n}(1 - (m_0 + m_1 + m_2 + \dots + m_n)) + m_0 \\
&= x_0.
\end{aligned}$$

Therefore we conclude that  $C \subseteq \sigma(Y^0, Y^1, \dots, Y^n)$ .

□

□

## 3.2 Labellings

Recall that  $S$  is the standard  $n$ -simplex and that  $P$  is a set of points from  $\mathbb{R}^{n+1}$  whose first  $n + 1$  points have the form:

$$\begin{aligned} X^0 &= (1 - n, 1, 1, \dots, 1) \\ X^1 &= (1, 1 - n, 1, \dots, 1) \\ &\vdots \\ X^n &= (1, 1, 1, \dots, 1 - n) \end{aligned}$$

and whose remaining points  $X^{n+1}, X^{n+2}, \dots, X^N$  are taken arbitrarily within the interior of  $S$ .

**Definition 3.6.** *Points in  $P$  are said to be **properly labelled** by  $\ell$  if  $\ell(X) \in \{0, 1, \dots, n\}$  and if  $\ell(X^j) = j$  for  $j = 0, \dots, n$ .*

Additionally, a primitive set whose points carry a complete set of labels  $\{0, 1, \dots, n\}$  is said to be **fully labelled** by  $\ell$ .

## 3.3 Scarf's Theorem on Primitive Sets

We will examine many forms of Scarf's Theorem; this one is a version slightly modified by Kuhn ([20]) and involves proper labellings and primitive sets.

**Theorem 3.7** (Scarf's Theorem on primitive sets, [20]). *Let subdivision points  $X^{n+1}, \dots, X^N$  be chosen from the interior of  $S$  so as to satisfy the Non-Degeneracy Assumption. Let  $\ell$  be a proper labelling of  $P$ . Then there exists an odd number of fully labelled primitive sets.*

In order to prove Theorem 3.7, we will require the following important lemma, due to Scarf in [25].

**Lemma 3.8** ([25, Lemma 1]). *Let  $W = \{X^{j_0}, X^{j_1}, \dots, X^{j_n}\}$  be a primitive set, and let  $X^{j_t}$  be a given point in that list. Then there exists exactly one other point  $X^{j_h}$  in  $P$  with  $j_h \neq j_t$  such that the set  $\{X^{j_0}, X^{j_1}, \dots, X^{j_{t-1}}, X^{j_h}, X^{j_{t+1}}, \dots, X^{j_n}\}$  is either a primitive set or (exceptionally) the set  $\{X^0, X^1, \dots, X^n\}$ .*

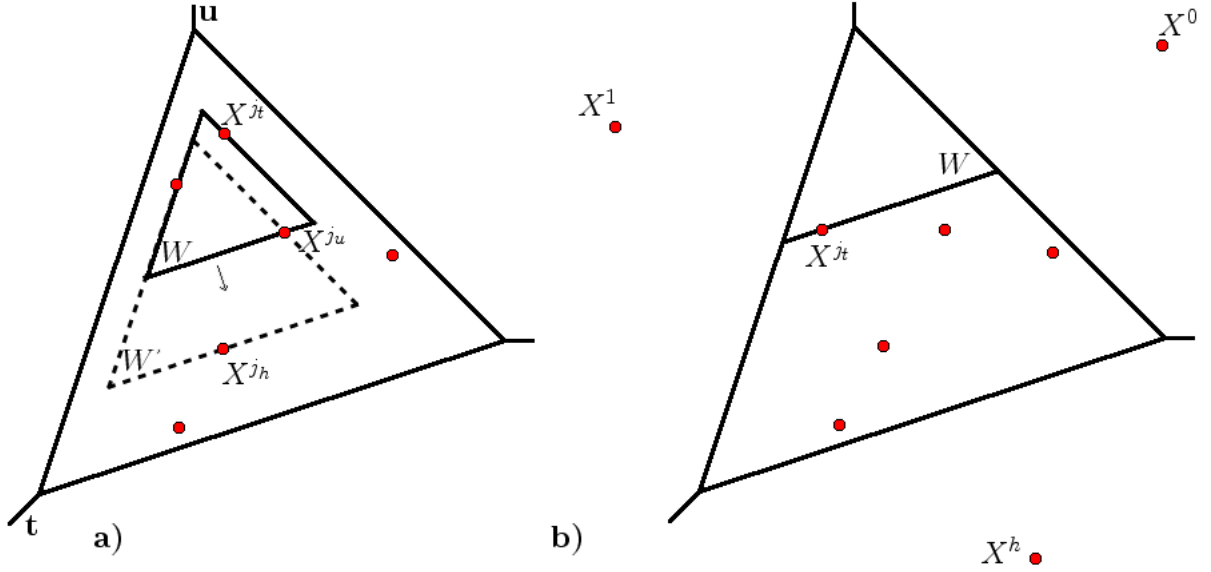


Figure 3.4: Change of primitive set on 2-simplex

*Proof.* First consider the non-exceptional case. Refer to Figure 3.4 a) to help with the proof.

Let  $W = \{X^{j_0}, X^{j_1}, \dots, X^{j_n}\}$  be a primitive set and  $X^{j_t}$  a point in that set. Label the points in  $W$  such that the minimizer in  $i$  is  $X^{j_i}$ , that is, such that  $x_i^{j_i} = \min\{x_i^{j_0}, x_i^{j_1}, \dots, x_i^{j_n}\}$ . We are guaranteed by Lemma 3.4 to find such a unique minimum.

To find the substitution point  $X^{j_h}$  we will appeal to the geometric interpretation. When  $X^{j_t}$  is removed from  $W$ , we lose the minimizer in  $t$ . Let  $X^{j_u}$  be the point in  $W$  with the second lowest value in the  $t$ th coordinate (This point is also unique by Lemma 3.4).

Note that  $X^{j_u}$  is not one of the first  $n+1$  points of  $P$ , namely  $j_u > n$ . If on the contrary  $X^i, i \neq t$  had the second minimum (with a  $t$ th coordinate of value 1), then no other point in  $W$  would lie in the interior of  $S$ , which is the exceptional case. Further,  $X^t$  cannot have the second minimum, as its  $t$ th coordinate is  $1-n$ , which is minimum over all of  $P$ . Hence  $j_u > n$  and  $X^{j_u}$  is the new minimizer in  $t$ .

The goal is to find a new point  $X^{j_h}$  to replace  $X^{j_u}$  as the minimizer in  $u$ . To do this we gradually move the face (of the covering simplex) containing  $X^{j_u}$ , parallel to itself, by lowering the  $u$ th coordinate and taking  $X^{j_h}$  to be the first point to touch this face and

satisfying:

$$x_i^{j_h} > x_i^{j_i}, i \neq t, u$$

and

$$x_t^{j_h} > x_t^{j_u}.$$

These conditions ensure  $X^{j_h}$  does not become a minimizer in any coordinate but  $u$ . Note that the first point satisfying those conditions could be  $X^{j_h} = X^u$ . This is the case when no point in  $S$  satisfies the above conditions, hence we must look outside of  $S$ . First, we claim that  $X^u$  cannot already lie in  $W$ . This follows from the fact that  $X^{j_u}$  is the minimizer in  $u$ , yet  $X^u$  has the minimum  $u$ th coordinate over all of  $P$  (with  $u$ th coordinate  $1 - n$ ). Hence if  $X^u$  were in  $W$ , it would be  $X^{j_u}$ . Yet we concluded above that  $j_u > n$ , therefore  $X^u$  is not in  $W$ .

It follows that some replacement point  $X^{j_h}$  can be found.

By taking  $X^{j_u}$  as the point with the second minimum in the  $t$ th position and taking  $X^{j_h}$  as the first point encountered, we guarantee that  $W' = \{X^{j_0}, \dots, X^{j_{t-1}}, X^{j_h}, X^{j_{t+1}}, \dots, X^{j_n}\}$  is a primitive set.

It remains to show that  $X^{j_h}$  is unique in  $P$ . We do so with the following three observations.

1. If  $W' = \{X^{j_0}, \dots, X^{j_{t-1}}, X', X^{j_{t+1}}, \dots, X^{j_n}\}$  forms a primitive set, then for  $i \neq t, u$ ,  $X^{j_i}$  must be the minimizer in  $i$ . Suppose not, and that some  $X^{j_i}$  becomes the minimizer in a coordinate other than  $i$ .

The only point to leave  $W$  is  $X^{j_t}$ . Hence  $X^{j_i}, i \neq t, u$  cannot become the minimizer in any coordinate but  $t$ , otherwise it would replace a minimizer which was originally in  $W$  and is still in  $W'$ . Now  $X^{j_i}$  cannot be the minimizer in  $t$ , unless  $i = u$ , since  $X^{j_u}$  has the second minimum in the  $t$ th coordinate by construction, and is in  $W'$ .

Thus  $x_i^{j_i} = \min\{x_i^{j_0}, \dots, x_i^{j_{t-1}}, x_i', x_i^{j_{t+1}}, \dots, x_i^{j_n}\}, i \neq t, u$ .

There are two remaining possibilities: either  $X'$  is the minimizer in  $t$  and  $X^{j_u}$  the minimizer in  $u$ , or vice versa.

2. If  $W' = \{X^{j_0}, \dots, X^{j_{t-1}}, X', X^{j_{t+1}}, \dots, X^{j_n}\}$  forms a primitive set, and  $X' \neq X^{j_t}$ , then  $X'$  must be the minimizer in  $u$  and  $X^{j_u}$  the minimizer in  $t$ . Otherwise we would have  $X^{j_i}$  as the minimizer in  $i$  for all  $i \neq t$ , and  $X'$  as the minimizer in  $t$ . But

this would either leave  $X'$  in the interior of the covering simplex of  $W$  (if  $x'_t > x_t^{j_t}$ ) or  $X^{j_t}$  in the interior of the covering simplex of  $W'$  (if  $x'_t < x_t^{j_t}$ ), both of which are impossible. We conclude that  $X' = X^{j_t}$ , and we are back where we started. Therefore we must have  $X'$  as the minimizer in  $u$  and  $X^{j_u}$  as the minimizer in  $t$ .

3. If  $\{X^{j_0}, \dots, X^{j_{t-1}}, X', X^{j_{t+1}}, \dots, X^{j_n}\}$  forms a primitive set, and  $X' \neq X^{j_t}$ , then  $X'$  must be the point  $X^{j_h}$  described above. This follows from the fact that we picked  $X^{j_h}$  to be the first to touch the face containing  $X^{j_u}$  as we lowered its  $u$ th coordinate.

Now we need only handle the exceptional case. Refer to Figure 3.4 b). Let  $W = \{X^{j_0}, X^{j_1}, \dots, X^{j_n}\}$  be a primitive set and  $X^{j_t}$  be the only point of  $W$  in the interior of  $S$ . Say the only point from  $\{X^0, X^1, \dots, X^n\}$  not in  $W$  is  $X^h$ .

Then it must be that  $X^{j_t}$  is the minimizer in  $h$ , since  $X^i$  is clearly the minimizer in  $i$ ,  $i \neq h$  (with  $i$ th coordinate  $1 - n$ ). If this is the case, then there is no unique point with a second minimum  $h$ th coordinate, as all other points in  $W$  have  $h$ th coordinate 1. Furthermore, no interior point can be introduced to replace  $X^{j_t}$  as that would make  $X^{j_t}$  lie in the interior of the new covering simplex, and cause the new set not to be primitive.

In this case we replace  $X^{j_t}$  by  $X^h$ , and the new set is  $\{X^0, X^1, \dots, X^n\}$ . □

We can now return to Theorem 3.7, whose proof is adapted from [27].

*Proof.* (of Theorem 3.7)

We first find one primitive set with a complete set of labels.

Consider the set  $W = \{X, X^1, X^2, \dots, X^n\}$ , where  $X$  is the point in the interior of  $S$  with the largest 0th coordinate. Note that  $W$  is primitive, otherwise a point in  $P$  has 0th coordinate larger than  $\min\{x_0, x_0^1, x_0^2, \dots, x_0^n\} = \min\{x_0, 1, 1, \dots, 1\} = x_0$ , which is impossible by construction.

If  $\ell(X) = 0$ , we are done, as  $\ell(X^1) = 1, \ell(X^2) = 2, \dots, \ell(X^n) = n$  by the rules of a proper labelling. Otherwise  $\ell(X)$  is one of  $\{1, 2, \dots, n\}$  and collides with one of  $\ell(X^i), i \neq 0$ , say  $\ell(X^k)$ . If we remove  $X^k$  from  $W$ , by Lemma 3.8, we can find a unique point  $X'$  such that  $W' = \{X, X^1, \dots, X^{k-1}, X', X^{k+1}, \dots, X^n\}$  is primitive.

Now, if  $\ell(X') = 0$ , we are done. At each step we have a primitive set of  $n + 1$  points,  $n$  of which have labels  $\{1, 2, \dots, n\}$  and one newly introduced point  $X$ . If  $\ell(X) = 0$ , we



are done, otherwise by removing the point  $X^k$  with  $\ell(X^k) = \ell(X)$ , we can find exactly one replacement point.

Because the number of primitive sets is finite, if we show there is no revisiting of primitive sets, this process must terminate. Either it terminates with a fully labelled primitive set, or with the set  $\{X^0, X^1, X^2, \dots, X^n\}$ , which is fully labelled but not primitive. In this latter case,  $X^0$  must be the last point to have been introduced, because had it been introduced earlier, our procedure would have terminated. ( $\ell(X^0) = 0$ ). Hence the state before this one was a primitive set  $\{X, X^1, X^2, \dots, X^n\}$ , with  $X$  an interior point. In order for this set to be primitive  $X$  must be the point in the interior of  $S$  with the largest 0th coordinate, and is unique by the Non-Degeneracy Assumption. This state is therefore our starting state. Hence we only need to show that no state, including our starting state, is ever revisited.

Assume for a contradiction that  $W$  is the first primitive set to be revisited. If it is not the initial set then it can be reached by one of two subsets of  $W$  with  $n$  distinct labels. But both of these subsets would have been encountered in our first pass through  $W$ , therefore  $W$  is not the first set to be revisited. Similarly, if  $W$  is the initial set ( $\{X^1, \dots, X^n\}$  plus one other interior point) then it can only be reached through one subset of  $n$  points of  $W$  having distinct labels, since the set  $\{X^0, X^1, \dots, X^n\}$  is not primitive. But this set we would have seen in our first pass. Therefore no set is ever revisited.

Now suppose there exists another fully labelled set  $W$ , one which was not found by the above procedure. Then by a similar argument, we must be able to remove the point having label 0 in  $W$  and find another primitive set. Proceeding as above, we can travel from primitive set to primitive set, without revisiting any set and reach another fully labelled primitive set. That is, every additional fully labelled sets must come in pairs. Along with the original one we found, there exists an odd number of fully labelled sets.  $\square$

# Chapter 4

## Scarf's Theorem and Sperner's Lemma

Scarf's Theorem of Chapter 3 can be reformulated slightly and shown to be equivalent to a well known result in Topology: Sperner's Lemma. We defer to the end of the chapter the details of the lemma as well as the proof of equivalence between the two results. First we examine the reformulation of Scarf's Theorem, which we call Scarf's Theorem on subdivisions. Though it is significantly different from the version seen in Chapter 3, what links these two versions together is their proofs: they are the same but written in different language.

### 4.1 Scarf's Theorem on Subdivisions

This version of the theorem comes from Scarf in [27]. Let  $\sigma = \sigma(X^0, X^1, \dots, X^n)$  be a simplex. Suppose  $\sigma$  has a restricted subdivision and that  $\{X^{n+1}, \dots, X^N\}$  are the vertices of the subdivision.

Note that the definition of  $X^0, X^1, \dots, X^n$  here is different from that of Chapter 3. Here they represent the extreme points of  $\sigma$ , and not the special points lying on the hyperplane of  $\sigma$ .

We introduce a very similar labelling rule to the one presented in Definition 3.6, this time acting on vertices of the subdivision.

**Definition 4.1.** Let  $\sigma^1, \dots, \sigma^k$  form a restricted subdivision of a simplex  $\sigma = \sigma(X^0, X^1, \dots, X^n)$ , and let  $X^{n+1}, \dots, X^N$  be the vertices of the subdivision. We say the restricted subdivision is **properly labelled** by  $\ell$  if for the points  $\{X^0, X^1, \dots, X^N\}$ ,  $\ell(X^j) \in \{0, 1, \dots, n\}$  for all  $j$  and  $\ell(X^j) = j$ , for  $j = 0, \dots, n$ .

Further, a simplex whose vertices have all labels  $\{0, 1, \dots, n\}$  is said to be **fully labelled** by  $\ell$ .

See Figure 4.1 for an example of a properly labelled 2-simplex.

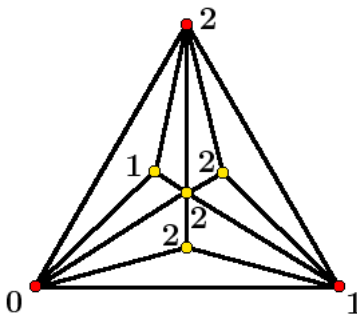


Figure 4.1: Proper labelling of a 2-simplex

**Theorem 4.2** (Sarf's Theorem on Subdivisions, [27, Lemma 3.3]). Let  $\sigma^1, \dots, \sigma^k$  be a collection of simplices that form a restricted subdivision of the simplex  $\sigma$ , and let  $\ell$  be a proper labelling. Then there exists an odd number of fully labelled elementary simplices.

*Proof.* Relabel  $\sigma$  to  $\sigma^0$  to help with notation.

First construct a graph  $G = (V, E)$  from the simplex and its subdivision. Associate a vertex  $v^i$  to each simplex  $\sigma^i$ ,  $i = 0, \dots, k$ .

If a facet of the subdivision has vertices with labels  $\{1, 2, \dots, n\}$ , we call it a **fully labelled facet**. Two vertices of  $V$  are joined by an edge if their associated simplices share a fully labelled facet.

Note that because the subdivision is restricted,  $\sigma^0$  shares its facets with elementary simplices.

We make the following claims about  $G$ :

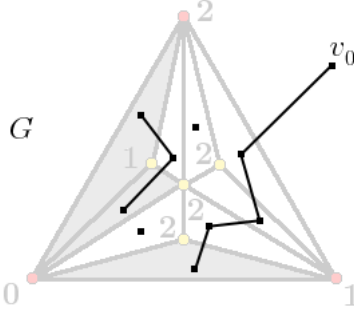


Figure 4.2: Constructing a graph from a simplicial subdivision

**Claim 4.2.1.** *All vertices have degree at most 2. Further, vertices of degree 1 correspond to fully labelled simplices.*

*Proof.* A vertex  $v^i$  is connected to  $v^j$  if and only if  $\sigma^i$  and  $\sigma^j$  share a fully labelled facet. If no facet of  $\sigma^i$  is fully labelled, then  $\deg(v^i) = 0$ . Now suppose  $\sigma^i$  has a facet with labels  $\{1, 2, \dots, n\}$ . Then  $\sigma^i$  only has one more extreme point  $X$  which is not contained in that facet.

If  $\ell(X) = 0$ , then  $\sigma^i$  is fully labelled and cannot contain any other fully labelled facets. In this case  $\deg(v^i) = 1$ .

Otherwise  $\ell(X) \in \{1, 2, \dots, n\}$ , and collides with the label of a unique point in the fully labelled facet, say  $X'$ . There exists exactly one other fully labelled facet of  $\sigma^i$ , namely the one obtained by interchanging  $X$  and  $X'$ . Hence  $\deg(v^i) = 2$ .  $\square$

**Claim 4.2.2.** *The degree of  $v^0$  is one.*

*Proof.* Here  $v^0$  corresponds to the standard simplex  $\sigma^0$ , which is fully labelled by construction, hence by Claim 4.2.1,  $v^0$  has degree 1.  $\square$

**Claim 4.2.3.** *The number of vertices of degree 1 in  $G$  is even.*

*Proof.*  $G$  is a collection of paths and cycles, hence by a standard Graph Theory result, the number of vertices of degree 1 is even (adding up the pairs of path end points).  $\square$

One of the fully labelled simplices is  $\sigma^0$ , which is not an elementary simplex, hence the number of fully labelled elementary simplices is odd.  $\square$

## 4.2 Sperner's Lemma

As noted in the beginning of the chapter, Scarf's Theorem on subdivisions is equivalent to Sperner's Lemma ([30]).

First, we examine Sperner's Lemma in its most common form, and prove it using a combinatorial method, taken from [4]. It will involve the standard simplex  $S = \sigma(e_0, e_1, \dots, e_n)$ , where the points  $e_0, e_1, \dots, e_n$  are  $\{(1, 0, \dots, 0), \dots, (0, 0, \dots, 1)\}$ .

The labelling rule we will use is called a Sperner labelling. It is very similar to the labelling rule introduced in Section 4.1.

**Definition 4.3.** *Let  $S^1, \dots, S^k$  form a simplicial subdivision of the standard simplex  $S$ , and let  $X^{n+1}, \dots, X^N$  be the vertices of the subdivision. Let  $\ell$  be a labelling rule such that  $\ell(X)$  is one of the indices  $i$  for which  $x_i > 0$ . Then we say  $\ell$  is a **Sperner labelling**.*

This labelling is similar to the labelling in Definition 4.1 because if we considered a restricted subdivision of the standard simplex, the points  $X^0, X^1, \dots, X^n$  would receive labels  $\ell(X^i) = i$ , given that  $x_k^i = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}$ . Further, the interior points of  $S$  could receive any label from  $\{0, 1, \dots, n\}$  since none of their coordinates have value 0. Refer back to Figure 4.1.

Note however that the simplicial subdivision need not be restricted. As Figure 4.3 demonstrates, vertices on the boundary of  $S$  have a restricted choice for labels. These vertices lie on one of the hyperplanes  $x_i = 0$ , and therefore cannot be labelled by  $i$ . For example, the face  $\sigma((1, 0, 0), (0, 1, 0))$  is on the plane  $x_2 = 0$  and so vertices on that face can have label 0 or 1 but not 2.

Note also that the labelling rule can be abstracted to any simplex, not only the standard simplex. The rule is then more cumbersome to state, but note that the proof below will not assume the simplex is standard.

**Theorem 4.4** (Sperner's Lemma, [4, p. 421]). *Let  $S^1, \dots, S^k$  form a simplicial subdivision of the standard  $n$ -simplex  $S$ , and  $\ell$  be a Sperner labelling on that subdivision. Then there exists an odd number of elementary simplices  $S^i$  with vertices carrying a complete set of labels. In particular, there is at least one.*

*Proof.* The proof is by induction on  $n$ , the dimension of the simplex, with  $n \geq 1$ .

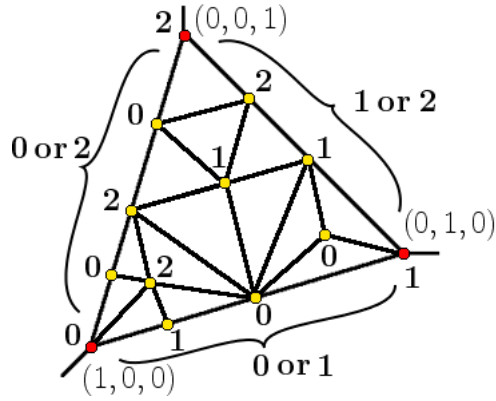


Figure 4.3: Sperner labelling for a standard 2-simplex

A simplicial subdivision of a 1-simplex  $S$  is simply a line segment which has been subdivided into smaller elementary segments. The vertices of  $S$  are labelled 0 and 1, while the vertices of the subdivision are labelled arbitrarily from  $\{0, 1\}$ . We must show that the number of elementary segments carrying both labels  $\{0, 1\}$  is odd.

We first count the occurrences of the label 0 by considering the elementary segments with one end point labelled 0. They fall in two categories, either they have the other end point labelled 0, or the other end point labelled 1. Let  $a$  be the number of elementary segments labelled  $\{0, 0\}$ , and  $b$  the number of elementary segments labelled  $\{0, 1\}$ . The first contributes  $2a$  occurrences of the label 0, while the second only accounts for  $b$  occurrences of the label 0. In total, there are  $2a + b$  occurrences of the 0 label.

Now we examine  $S$  as a whole. Only one of its end points is labelled 0, and this endpoint is adjacent to only one elementary segment. Therefore all other occurrences of the label 0 occur in the interior of the segment, and are twice accounted for. Hence  $2a + b$  must be odd, which implies  $b$  is odd, which is what we wanted to show.

Assume now that a properly labelled  $k$ -simplex contains an odd number of fully labelled elementary simplices and examine the case where  $n = k + 1$ .

We will count the number of facets in the subdivision that carry all labels  $\{1, 2, \dots, k\}$  (but not 0). Let  $b$  be the number of fully labelled elementary simplices in  $S$ . Each of those contributes one fully labelled facet to our count. Let  $a$  be the number of elementary simplices with a fully labelled facets but are missing the label 0. These simplices only have one more vertex not in the facet, whose label must collide with a label of a point in the

facet. Interchanging these two points yields exactly one other fully labelled facet. Hence we account for  $2a$  fully labelled facets, and the total is  $2a + b$ .

Some of these facets lie in the interior of  $S$ , others on the boundary. If they occur on the boundary they must occur on the unique facet of  $S$  admitting labels  $\{1, 2, \dots, k\}$ , because all other facets of  $S$  admit the label 0. This facet is of dimension  $k$ , hence by the induction hypothesis the number of fully labelled facets contributed from the boundary is odd. If the fully labelled facets occur on the interior of  $S$ , they are accounted for twice, once for each simplex on either side of the facet. Hence the total number of fully labelled facets accounted for is odd, implying that  $2a + b$  and so  $b$  is also odd.

In a Sperner labelled  $n$ -simplex, the number of fully labelled elementary simplices is odd. □

### 4.3 Scarf's Theorem and Sperner's Lemma

Now we are ready to show that Sperner's Lemma and Scarf's Theorem are equivalent. To show Scarf's Theorem as a corollary to Sperner's Lemma, we will use the fact that restricted subdivisions are a special class of subdivisions.

**Theorem 4.4** (Sperner's Lemma). *Let  $S^1, \dots, S^k$  form a simplicial subdivision of the standard  $n$ -simplex  $S$ , and  $\ell$  be a Sperner labelling on that subdivision. Then there exists an odd number of elementary simplices  $S^i$  with vertices carrying a complete set of labels. In particular, there is at least one.*

**Corollary 4.5** (Scarf's Theorem). *Let  $\sigma^1, \dots, \sigma^k$  be a collection of simplices that form a restricted subdivision of a simplex  $\sigma$ , and let  $\ell$  be a proper labelling. Then there exists an odd number of fully labelled elementary simplices.*

*Proof.* As mentioned in the previous section, Sperner's Lemma applies even if  $\sigma$  is not the standard simplex, as long as the labelling rule mimics the Sperner labelling. Indeed, it was also discussed previously how a proper labelling rule on a restricted subdivision is the same as a Sperner labelling.

Now given that a restricted subdivision is a special case of a simplicial subdivision, Theorem 4.4 applies and yields an odd number of fully labelled elementary simplices. □

In reformulating the theorem on subdivisions, Scarf has contributed to the proof of Sperner's Lemma in two ways.

First, the proof of Theorem 4.2 is constructive, unlike the counting argument used in the proof of Theorem 4.4. It makes it possible to find a fully labelled simplex without the use of exhaustive search: by following the unique path in  $G$  which started at  $v^0$ . We will argue at the end of Section 7.2 why this is better than exhaustive search.

Secondly, the proof of Theorem 4.2 does not require induction. This is because we examine restricted subdivisions only, and so the boundary of the simplex contains only one fully labelled facet, as opposed to an odd number of them.

Here is how we can show Sperner's Lemma using Scarf's Theorem on subdivisions.

**Theorem 4.2** (Scarf's Theorem). *Let  $\sigma^1, \dots, \sigma^k$  be a collection of simplices that form a restricted subdivision of a simplex  $\sigma$ , and let  $\ell$  be a proper labelling. Then there exists an odd number of fully labelled elementary simplices.*

**Corollary 4.6** (Sperner's Lemma). *Let  $S^1, \dots, S^k$  form a simplicial subdivision of the standard  $n$ -simplex  $S$ , and  $\ell$  a Sperner labelling on that subdivision. Then there exists an odd number of elementary simplices  $S^i$  with vertices carrying a complete set of labels.*

*Proof.* To prove Sperner's Lemma using Scarf's Theorem, we first embed the standard simplex  $S$  in a larger simplex  $\sigma$  with vertices  $t^0, t^1, \dots, t^n$ . We then extend the simplicial subdivision of  $S$  to a restricted simplicial subdivision of  $\sigma$  by introducing the following simplices.

Take an arbitrary subset  $U$  of  $\{0, 1, \dots, n\}$  with  $u < n + 1$  members. Consider a collection of  $n + 1 - u$  vertices of the face of  $S$  defined by  $x_i = 0$  for  $i \in U$ , and which lie in a single simplex of the subdivision of  $S$ . These  $n + 1 - u$  vertices in  $S$  are augmented by the  $u$  vertices  $t^i$  for  $i \in U$  in order to define a simplex in the larger subdivision. See Figure 4.4.

Apply Theorem 4.2 with  $l(t^i) = i + 1 \pmod{n + 1}$ .

It remains to be shown that any completely labelled simplex must lie entirely in  $S$ . A completely labelled simplex may contain the vertices  $t^i$  for  $i \in U$ , and  $n + 1 - u$  other vertices on the face  $x_i = 0$  for  $i \in U$ . Since these  $n + 1 - u$  vertices (by Sperner labelling) will be different from the members of  $U$ , the vertices  $t^i$ 's must bear all  $U$  labels. But the  $t^i$ 's have labels that differ from  $U$  by construction, since  $l(t^i) = i + 1 \pmod{n + 1}$ . Hence the fully labelled simplex must lie entirely in  $S$ .  $\square$



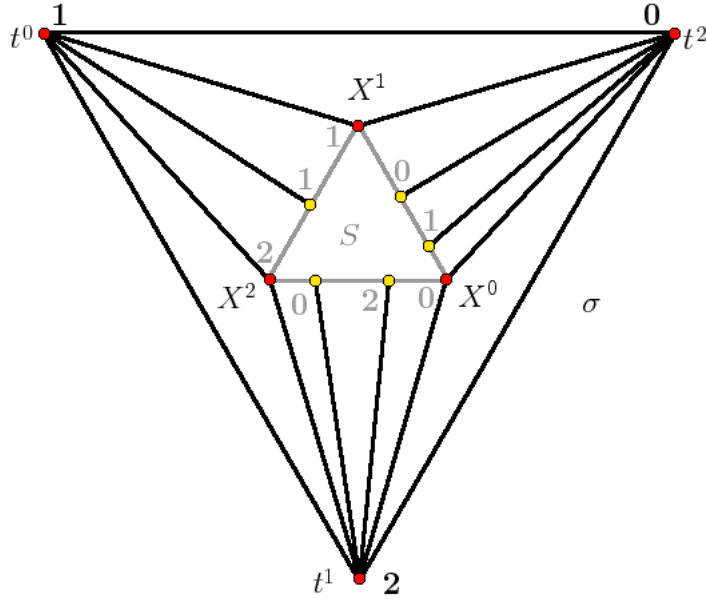


Figure 4.4: Forming a restricted subdivision from  $S$  in  $\sigma$

Kuhn ([20]) also showed Scarf's Theorem on primitive sets can be implied by Sperner's Lemma. To make the equivalence we will use the fact that the family of primitive sets of  $P$  defines a pseudomanifold. We will also need a modified version of Sperner's Lemma, one that is abstracted to pseudomanifolds.

**Theorem 4.7** (Sperner's Lemma on pseudomanifolds, [20]). *Let  $D$  be an abstract  $(n + 1)$ -pseudomanifold on the point set  $P = \{X^0, X^1, \dots, X^n, X^{n+1}, \dots, X^N\}$ , and let  $P$  be properly labelled according to Definition 3.6. If we remove the simplex  $\{X^0, X^1, \dots, X^n\}$ , then there exists an odd number of sets in  $D$ , each with a complete set of labels  $\{0, 1, \dots, n\}$ .*

The pseudomanifold embodies the properties of a subdivision that are required for the proof to hold. Also recall that a restricted subdivision defines a pseudomanifold (Lemma 2.17), which is yet another way to relate Sperner's Lemma and Scarf's Theorem on subdivisions.

*Proof.* We will use a counting argument as in the proof of Theorem 4.4. However, we will consider  $D$  as a whole first, and later remove the set  $\{X^0, X^1, \dots, X^n\}$ .

Let  $b$  be the number of fully labelled simplices in  $D$ .

First we count the number of  $n$ -subsets of the simplices of  $D$  that carry all labels  $\{1, 2, \dots, n\}$  (but not 0). Call these fully labelled  $n$ -subsets.

Each fully labelled set of  $D$  contributes exactly one fully labelled  $n$ -subset to our count. There are  $b$  of these.

Let  $a$  be the number of sets in  $D$  with a fully labelled  $n$ -subset but missing the label 0. These sets only have one more element not in the fully labelled  $n$ -subset, whose label must collide with a label of a point in the subset. Interchanging these two points yields exactly one other fully labelled  $n$ -subset. Hence we account for  $2a$  fully labelled  $n$ -subsets, and the total is  $2a + b$ .

Because  $D$  is a pseudomanifold, every  $n$ -subset is a subset of exactly 2 sets of  $D$ . Hence  $2a + b$  must be even, as we have counted each fully labelled  $n$ -subset twice. Therefore  $b$ , the number of fully labelled sets of  $D$ , is even.

However, because the fully labelled set  $\{X^0, X^1, \dots, X^n\}$  has been removed from  $D$ , we have one fewer. Therefore there exists an odd number of sets in  $D$ , each with a complete set of labels  $\{0, 1, \dots, n\}$ .  $\square$

Note that no induction was needed here, because a pseudomanifold has no boundary.

Now Scarf's Theorem on primitive sets states that within the family of properly labelled primitive sets, an odd number of them are fully labelled. Hence if we can establish that the family of primitive sets, along with the extra set  $\{X^0, \dots, X^n\}$  forms a pseudomanifold, we can use the above form of Sperner's Lemma to conclude that an odd number of primitive sets are fully labelled.

Therefore we establish that Scarf's Theorem on primitive sets is implied by Sperner's Lemma as stated above. Here is the required lemma.

**Lemma 4.8** ([20]). *Let subdivision points  $X^{n+1}, X^{n+2}, \dots, X^N$  be chosen arbitrarily within the interior of a standard simplex  $S$  so as to satisfy the Non-Degeneracy Assumption. Then the family of primitive sets from  $P$  together with  $\{X^0, X^1, \dots, X^n\}$  defines a pseudomanifold.*

*Proof.* It was established in the proof of Theorem 3.7 that there exists at least one primitive set, namely the set  $\{X^1, \dots, X^n\}$  along with the subdivision point  $X$  in the interior of  $S$  with the largest 0th coordinate.

Now we want to show that if a set of  $n$  points is a subset of a primitive set, then it

is a subset of exactly two primitive sets, or (exceptionally) of a primitive set and the set  $\{X^0, X^1, \dots, X^n\}$ .

In other words, if given a primitive set  $\{X^{j_0}, X^{j_1}, \dots, X^{j_n}\}$  and a specific point  $X^{j_t}$  in that set, we can find exactly one other point  $X^{j_h}$  in  $P$  with  $j_h \neq j_t$  such that the set  $\{X^{j_0}, X^{j_1}, \dots, X^{j_{t-1}}, X^{j_h}, X^{j_{t+1}}, \dots, X^{j_n}\}$  is either primitive or (exceptionally) the set  $\{X^0, X^1, \dots, X^n\}$ , we are done.

But this is exactly Lemma 3.8.

Lastly, we need to show that this pseudomanifold is connected, namely that we can find a sequence between any two primitive sets. This follows from the fact that a single simplex is considered.

Therefore by associating an abstract simplex with each primitive set and taking those simplices along with the set  $\{X^0, X^1, \dots, X^n\}$ , we define a pseudomanifold.  $\square$

In order to show that Sperner's Lemma is implied by Scarf's Theorem as in Theorem 3.7, we just need to observe that the only property of the primitive sets of Theorem 3.7 that is required for the proof is that they form a pseudomanifold. Therefore it is possible to show that Scarf's Theorem on primitive sets is also equivalent to Sperner's Lemma.

# Chapter 5

## Applications of Scarf's Theorem

In order to motivate the combinatorial value of this topology-based theorem, we present here two applications of Scarf's Theorem as presented in Chapter 4. The first application is more theoretical: using Scarf's Theorem we prove a result which is similar but uses a different labelling rule.

Second, we will examine an application to an area called fair division, which deals with the partition of goods among parties in an equitable way.

### 5.1 Dual labelling

As a first application we show how Scarf's Theorem can be used to prove a similar result using a slightly different labelling rule. This was first presented in [27].

**Definition 5.1.** *Let  $S^1, S^2, \dots, S^k$  form a subdivision of the standard  $n$ -simplex  $S$ . We say the subdivision is **sufficiently fine** if no simplex of the subdivision has non-empty intersection with every hyperplane  $x_i = 0$ ,  $i \in \{0, \dots, n\}$ .*

The trivial subdivision is an obvious subdivision which is not sufficiently fine. However, it is not the only one. Figure 5.1 shows a subdivision of the standard 2-simplex which is not sufficiently fine because  $S_3$  intersects with the planes  $x_0 = 0$ ,  $x_1 = 0$ , and  $x_2 = 0$ .

**Definition 5.2.** *Let  $S = \sigma(X^0, X^1, \dots, X^n)$  be the standard simplex and  $S^1, S^2, \dots, S^k$  form a subdivision which is sufficiently fine. Further, let  $\{X^{n+1}, X^{n+2}, \dots, X^N\}$  be the*

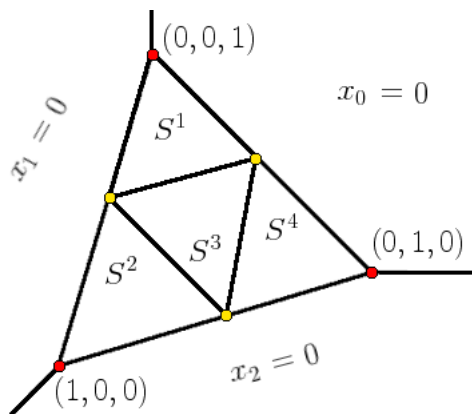


Figure 5.1: A subdivision which is not sufficiently fine

vertices of the subdivision. Let  $\ell$  be a labelling rule on  $P = \{X^0, X^1, \dots, X^N\}$  such that  $\ell(X) \in \{0, 1, \dots, n\}$  and for points  $X$  on the boundary of  $S$ ,  $\ell(X)$  is an index for which  $x_i = 0$ . Then we say that  $P$  is **Scarf labelled**.

See Figure 5.2 for a Scarf labelled 2-simplex. This rule can be thought of as the dual to the Sperner labelling of Definition 4.3.

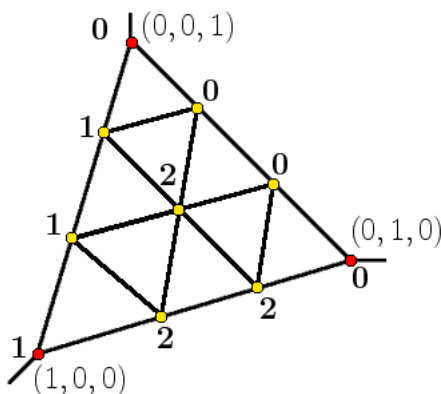


Figure 5.2: Scarf labelled 2-simplex

**Lemma 5.3** ([27, Lemma 3.5]). *Let  $S = \sigma(X^0, X^1, \dots, X^n)$  be the standard simplex and  $S^1, S^2, \dots, S^k$  form a subdivision which is sufficiently fine. Further, let  $\{X^{n+1}, X^{n+2}, \dots, X^N\}$  be the vertices of the subdivision, and let  $\ell$  be a Scarf labelling on  $P = \{X^0, X^1, \dots, X^N\}$ .*

Then there exists an odd number of fully labelled simplices of the subdivision.

*Proof.* We embed  $S$  in a larger simplex  $\sigma = \sigma(t^0, t^1, \dots, t^n)$ . Then we extend the subdivision of  $S$  to a restricted subdivision of  $\sigma$ .

Consider a subset  $U$  of  $\{0, 1, \dots, n\}$  with  $u < n + 1$  members. Take  $n + 1 - u$  vertices from the boundary of  $S$  that lie on the intersection of  $x_i = 0, i \in U$  and which lie in a single simplex of the subdivision of  $S$ . Extend these  $n + 1 - u$  vertices with  $u$  vertices from  $\sigma$ , namely  $t^i, i \in U$ . See Figure 5.3.

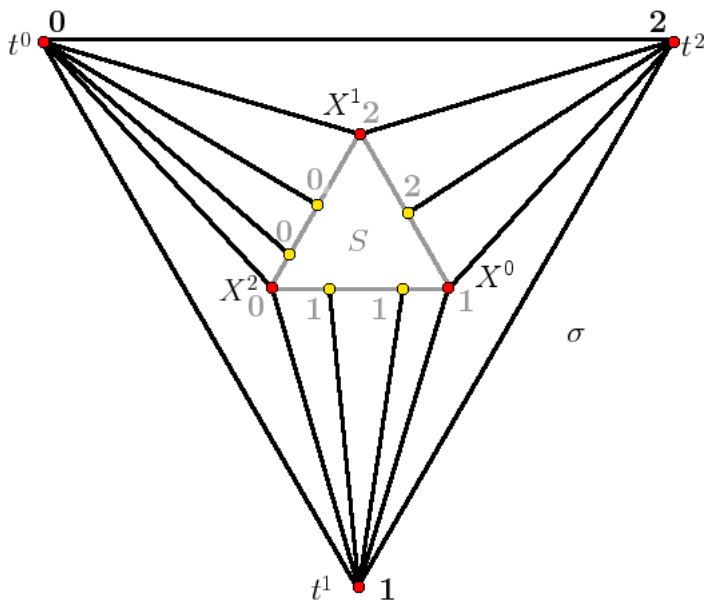


Figure 5.3: Extending a subdivision of  $S$  to a restricted subdivision of  $\sigma$

We label the vertices  $t^0, t^1, \dots, t^n$  by  $l(t^i) = i$ . This new construction satisfies the conditions of Theorem 4.2, namely we have a restricted subdivision with a proper labelling. Hence the new subdivision must contain an odd number of fully labelled simplices. We must show these simplices lie entirely within  $S$ .

Suppose for a contradiction that a fully labelled simplex contains all vertices  $t^i, i \in U$ , for some fixed  $U$ , and  $n + 1 - u$  vertices from the intersection of the faces  $x_i = 0, i \in U$ .

Vertices  $t^i$  have label  $i$ , which implies that the  $n + 1 - u$  vertices from the boundary of  $S$  must have all labels from  $\{0, 1, \dots, n\} \setminus U$ . Being on the boundary of  $S$ , these vertices

have a particular label only if the corresponding coordinate is zero. Hence for each label  $i$  in  $\{0, 1, \dots, n\} \setminus U$ , there exists a vertex on the face  $x_i = 0$ .

Also, being on the intersection of all faces  $x_i = 0$ ,  $i \in U$  it follows that these  $n + 1 - u$  vertices have their  $i$ th coordinate 0,  $i \in U$ .

So there are vertices from this simplex that lie on every face of  $S$ , that is,  $x_i = 0$ ,  $i \in \{0, 1, \dots, n\}$ , which contradicts our assumption that the subdivision is sufficiently fine.

It follows that any fully labelled simplex must lie entirely within  $S$ . □

## 5.2 Fair Division

The problem of fair division aims at partitioning a resource between multiple parties in a way that is “fair”. There are many possible definitions of fair, the easiest of which is “equal”. For example, to divide a chocolate cake between 3 people we could say it is fair to give each person  $1/3$  of the cake’s volume.

Yet it is possible to be much more precise, and ask that each participant, with respect to their own preference set, feel they received a fair share of the whole.

In **envy-free division**, we ensure that further, no participant prefers any other participant’s share.

We will apply Scarf’s Theorem to the problem of partitioning rent between house-mates in a lodging house. To introduce some of the techniques, we will first start by examining an application of Sperner’s Lemma, whose formulation was given by Su in [32].

### 5.2.1 Cake-Cutting using Sperner’s Lemma

Suppose a cake is to be divided between multiple parties. We will show that under minimal assumptions, there exists a partition of the cake which is envy-free, according to each participant’s preference set.

**Theorem 5.4** ([32, Section 3]). *Assume that the participants are hungry; meaning a participant prefers any piece of cake over nothing. Then there exists an envy-free division of the cake.*

*Proof.* Suppose we have  $n + 1$  participants,  $\{p^0, \dots, p^n\}$ .

Consider the standard  $n$ -simplex  $S = \sigma(X^0, X^1, \dots, X^n)$ . Suppose we can find a subdivision and a labelling of  $S$  which, when using labels  $\{p^0, p^1, \dots, p^n\}$ , has every simplex of the subdivision fully labelled. We call such a labelling a **labelling by ownership** and if a vertex  $X$  is labelled by  $p^i$ , we say  $p^i$  is the **owner** of  $X$ .

We claim such a subdivision and labelling pair always exists. Recall that the barycentric subdivision of an  $n$ -dimensional simplex  $S$  consists of  $(n + 1)!$  simplices. Each one, say with vertices  $t^0, t^1, \dots, t^n$ , can be associated with a permutation  $\{s^0, s^1, \dots, s^n\}$  of the vertices of  $S$ , in such a way that each vertex  $t^i$  is the barycentre of  $\sigma(s^0, s^1, \dots, s^i)$ .

Label the vertices as follows: the barycentre of each 0-simplex of  $S$  is labelled  $p^0$ , the barycentre of each 1-simplex of  $S$  is labelled  $p^1$ , and so on, until we label the barycentre of the  $n$ -simplex  $S$  by  $p^n$ . This way the triangulation will indeed be labelled by ownership. To see this, note that each simplex in the subdivision has as vertices various barycentres. In fact, exactly one vertex is the barycentre of a 0-simplex, exactly one is the barycentre of a 1-simplex, and so on. See Figure 5.4 a).

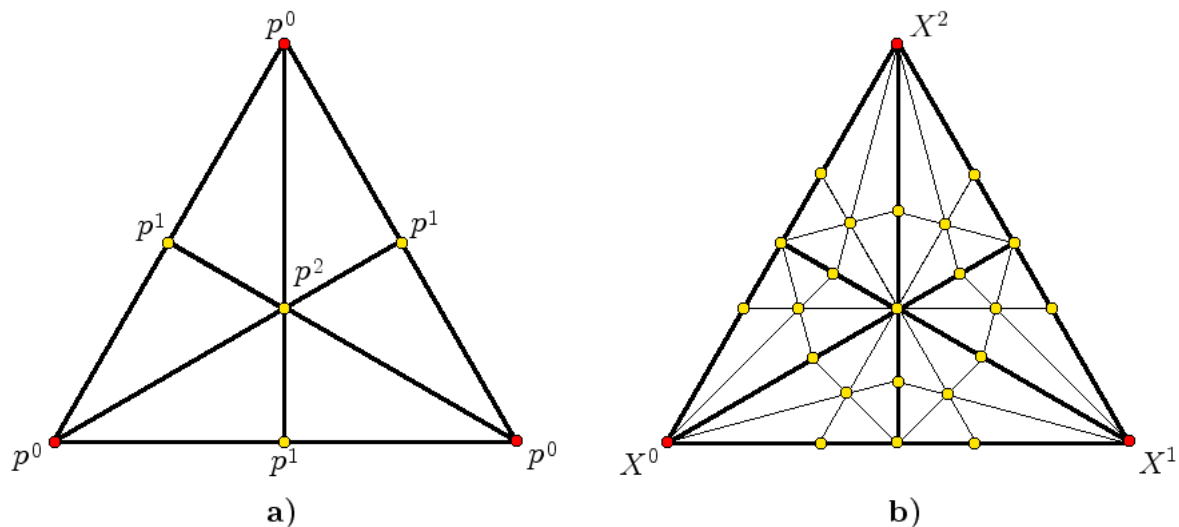


Figure 5.4: Barycentric subdivisions and labelling by ownership

For a finer subdivision we can iterate this process, subdividing each simplex of the subdivision using barycentric subdivision again. To label such a mesh, say the process was carried out  $m$  times. If we let all the vertices from the  $(m - 1)$ th iteration be labelled  $p^0$ ,



then we can proceed as above for each elementary simplex being subdivided in the  $m$ th iteration. Again, we ensure the labelling is by ownership.

Figure 5.4 b) displays the process iterated once more ( $m = 2$ ) on a 2-simplex.

Now we want to associate a particular partition of the cake with each vertex. One natural way to do this is to use the coordinates of each vertex as percentages of cake. Recall that on the standard simplex, the sum of the coordinates of any point is 1.

Participants are allowed to have different preference sets. In particular, one person could prefer a piece at the end of the cake over one in the middle. Further, the cake may be heterogeneous, half vanilla, half chocolate for example, which may also influence one's preference.

Therefore it is not sufficient to state what amount of cake each participant receives, instead we must decide exactly which piece is assigned to each participant. To represent these ideas, consider a rectangular cake of length 1 with  $n$  knives floating over it. All knives are parallel to each other and positioned perpendicular to the long side of the cake. Each knife's movement is restricted to the space between its two neighbours, or one neighbour and the end of the cake.

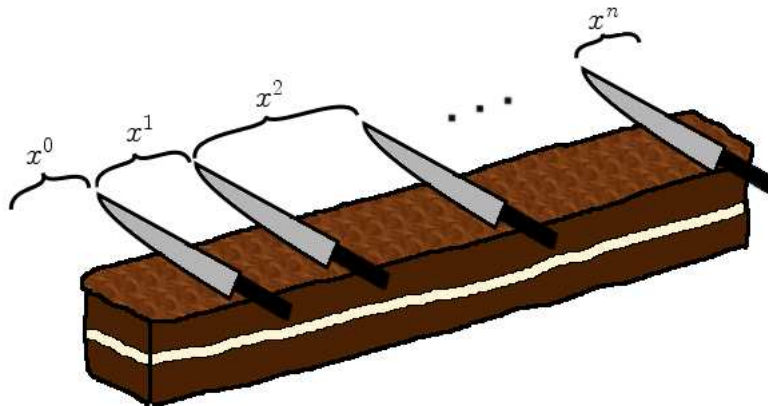


Figure 5.5: Partitioning a cake into  $n + 1$  pieces

Now measure the size of a piece by the distance between two knives, or between a knife and the end of the cake. Then the sum of the sizes of all  $n + 1$  pieces is 1, and the size of each piece is non-negative.

We associate with each vertex  $X = (x_0, x_1, \dots, x_n)$  of the subdivision a cake partition.

For example the subdivision point  $X^0 = (1, 0, \dots, 0)$  represents the partition where the first piece is the whole cake, while all other pieces have size 0.

In order to apply Sperner's Lemma, we require a Sperner labelling of the vertices. We form one by asking the owner of vertex  $X$  which piece (from  $\{0, 1, \dots, n\}$ ) they would prefer if the subdivision given by  $X$  was carried out. Because the players are hungry, the vertices on the boundary of  $S$  will necessarily receive a label corresponding to a non-zero coordinate. This new labelling is therefore a Sperner labelling.

Applying Sperner's Lemma to this simplex, we are guaranteed to find a fully labelled simplex of the subdivision, that is one with labels  $\{0, \dots, n\}$ . This simplex was labelled by ownership, hence we have found a simplex where each participant picked a different piece.

If the subdivision is not very fine, the  $n + 1$  vertices of the simplex represent very different partitions of the cake. Luckily we can take a subdivision that is very fine, say one where the distance between any two vertices of the same elementary simplex is less than  $\epsilon$ . Then by picking an acceptable small amount  $\epsilon$ , say at the level of crumbs, we can find a fully labelled simplex where each participant picked a different piece and all vertices represent the same partition of the cake up to  $\epsilon$ .  $\square$

### 5.2.2 Rent Partitioning using Scarf's Theorem

Suppose a few people decide to rent a house together. We are interested in finding a partition of the rent and an assignment of rooms which is envy-free.

What makes rent partitioning more difficult than cake-cutting is that each participant is trying to simultaneously maximize his room assignment (according to his preference set) while minimizing the cost. Therefore we are handling both goods and burdens in the same problem. Further, this problem involves both a discrete component (room assignments) and a continuous component (price of the rooms), whereas the cake-cutting problem only involved a continuous component.

We will show that under a small set of assumptions, it is always possible to find such an envy-free partition. This formulation is also due to Su in [32].

**Theorem 5.5** ([32, Section 6]). *Suppose  $n + 1$  house-mates in an  $(n + 1)$ -bedroom house must find a room assignment and partition the rent. Also, suppose that the following conditions hold:*

1. **Good House.** *In any partition of the rent, each person finds some room acceptable.*

2. **Miserly Tenants.** Each person prefers a free room (one that costs no rent) to a non-free room.

Then there exists a partition of the rent so that each person prefers a different room.

*Proof.* Suppose there are  $n+1$  house-mates, and  $n+1$  rooms to assign, numbered  $0, 1, \dots, n$ . Let  $x_i$  denote the price of the  $i$ th room, and suppose that the total rent is 1. Then  $x_0 + x_1 + \dots + x_n = 1$  and  $x_i \geq 0$ . We see that the set of all pricing schemes  $S$  forms a standard  $n$ -simplex.

We choose an acceptable  $\epsilon$ , a difference in rent which is inconsequential to the house-mates, say a penny. As done previously, we triangulate  $S$  using a barycentric subdivision, and ensure that no two vertices of an elementary simplex lie more than  $\epsilon$  apart. Then, label the subdivision by ownership.

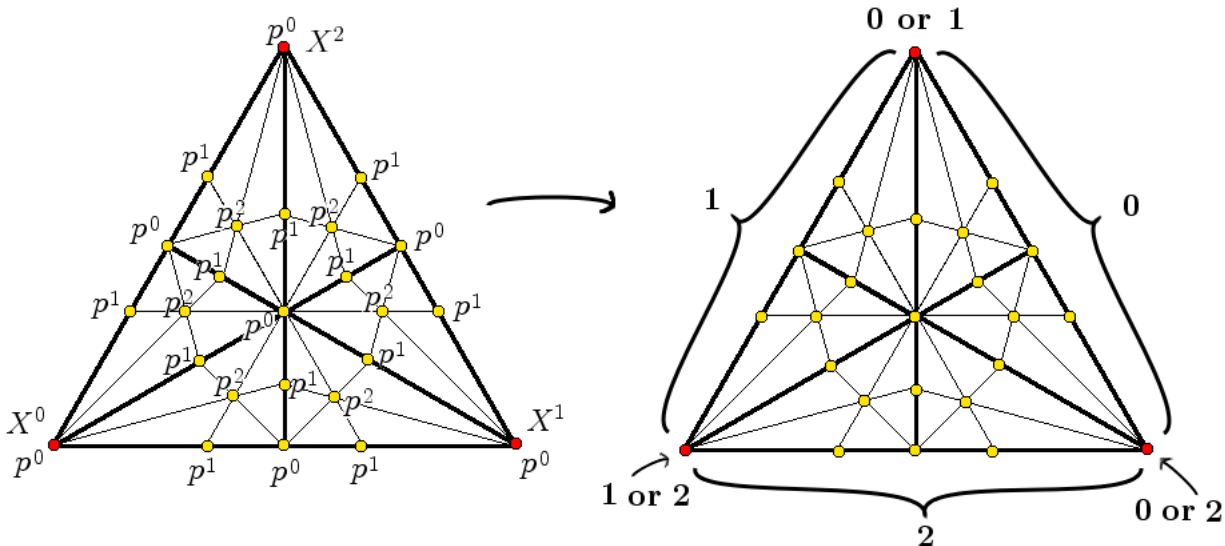


Figure 5.6: From a labelling by ownership to a Scarf labelling

Construct a new labelling from the old by asking the owner at each vertex  $X$  in the triangulation which room they would prefer if the rent was partitioned according to  $X$ . By Condition 1, each tenant will find some room acceptable.

We claim that this new labelling is a Scarf labelling as in Definition 5.2. To see this,

note that along the boundary of  $S$ , some rooms are free ( $x_i = 0$ ), and so by Condition 2, a tenant will always prefer one of those rooms.

By Lemma 5.3, we are guaranteed to find a fully labelled simplex in the subdivision. Given that the triangulation was also labelled by ownership, we have found a simplex where each tenant picked a different room. Since the subdivision was very fine, the vertices of this simplex represent the same pricing scheme, up to  $\epsilon$ .  $\square$

# Chapter 6

## Linear Programming and the Simplex Method

Although the two formulations of Scarf's Theorem presented earlier are very topological in nature, Scarf's original formulation in [26] was as a series of constraints, not unlike a linear program.

Despite several differences between Scarf's procedure and classic linear programming, it will be useful to study Dantzig's simplex algorithm for guidance. We begin with some linear algebra background and then introduce linear programs and the simplex method.

In this section we will prefer the term **vector** over **point** because of its historical use in this field. They are to be treated equivalently, though we will often overline symbols for vectors with an arrow ( $\vec{\phantom{x}}$ ) for emphasis.

### 6.1 Linear Algebra for Linear Programming

The mathematical background for this section is adapted from [3, 16].

**Definition 6.1.** A vector  $X$  from  $\mathbb{R}^m$  is said to be a **linear combination** of the vectors  $X^0, X^1, \dots, X^k$  from  $\mathbb{R}^m$  if  $X$  can be written

$$X = \lambda_0 X^0 + \lambda_1 X^1 + \dots + \lambda_k X^k \tag{6.1}$$

for some real set of scalars  $\lambda_i$ .

**Definition 6.2.** A set of vectors  $X^0, X^1, \dots, X^n$  from  $\mathbb{R}^m$  is said to be **linearly dependent** if there exists scalars  $\lambda_i$  not all zero such that

$$\lambda_0 X^0 + \lambda_1 X^1 + \dots + \lambda_n X^n = 0. \quad (6.2)$$

If the only set of  $\lambda_i$  for which (6.2) holds is  $\lambda_0 = \lambda_1 = \dots = \lambda_n = 0$ , then the set of vectors is said to be **linearly independent**.

**Definition 6.3.** A set of vectors  $X^0, X^1, \dots, X^n$  from  $\mathbb{R}^m$  is said to **span**  $\mathbb{R}^m$  if every vector in  $\mathbb{R}^m$  can be written as a linear combination of  $X^0, X^1, \dots, X^n$ .

Note that the unit vectors  $e_0, e_1, \dots, e_{m-1}$  span  $\mathbb{R}^m$ .

Now the concept of a basis of a set of vectors can be introduced.

**Definition 6.4.** A **basis** for a set of vectors  $S$  is a linearly independent spanning subset of  $S$ .

For example, for  $S = \mathbb{R}^m$ , the subset consisting of the unit vectors  $e_0, e_1, \dots, e_{m-1}$  is linearly independent and spans  $\mathbb{R}^m$ , therefore it is a basis for  $\mathbb{R}^m$ .

Let  $\{X^0, X^1, \dots, X^N\}$  be a finite set of candidate vectors for the basis of some subset  $S$  of  $\mathbb{R}^{n+1}$ . Suppose we place the vectors  $X^i$  as columns of a matrix  $A$  and want to find a subset of the columns of  $A$  that spans  $S$  and is linearly independent. This is equivalent to finding a square  $(n+1) \times (n+1)$  submatrix  $B$  of  $A$  which is non-singular. To express any vector  $X$  as a linear combination of vectors in the basis, we can solve  $B\vec{\lambda} = X$  as  $\vec{\lambda} = B^{-1}X$ .

Recall that if  $B$  is non-singular then there exists a series of elementary row operations that can be performed on  $B$  to obtain the identity matrix  $I_{n+1}$ . See [3, Section 2.2] for more on elementary row operations and matrix inverses.

## 6.2 Linear Programs in Standard Form

Linear Programming is the task of optimizing (maximizing or minimizing) some function with respect to a set of linear constraints. Definitions in this section are taken from [3, 13].

Consider the general system of  $n + 1$  linear equations in  $N + 1$  unknowns

$$\begin{aligned}
 a_{00}x_0 + a_{01}x_1 + \cdots + a_{0N}x_N &= b_0 \\
 a_{10}x_0 + a_{11}x_1 + \cdots + a_{1N}x_N &= b_1 \\
 \vdots & \\
 a_{n0}x_0 + a_{n1}x_1 + \cdots + a_{nN}x_N &= b_n
 \end{aligned} \tag{6.3}$$

where  $x_0, x_1, \dots, x_n$  are the unknowns and the other quantities are given constants.

Supposing that these equations are consistent, it may still be impossible to determine a solution uniquely. This is the case if  $(n + 1) < (N + 1)$ , or if  $(n + 1) \geq (N + 1)$  but the system is linearly dependent.

In that case the following constraints can be added:

$$x_j \geq 0 \quad j = 0, 1, \dots, N \tag{6.4}$$

and

$$\text{maximize } c_0x_0 + c_1x_1 + \cdots + c_Nx_N \tag{6.5}$$

where  $c_0, c_1, \dots, c_N$  are given constants. The task is then to maximize (6.5) subject to (6.4) and the indeterminate system (6.3).

**Definition 6.5.** *A system of equality constraints as in (6.3), together with a **non-negativity constraint** on the variables as in (6.4) and an **objective function** to maximize as in (6.5) is called a **linear program in standard equality form**.*

It is also possible to study linear programs in other forms, the most notable being the **standard inequality form** where all equality signs ( $=$ ) in (6.3) are replaced by less than or equal to signs ( $\leq$ ).

Such a linear program can be written as

$$\begin{aligned}
 &\text{maximize} && c_0x_0 + c_1x_1 + \cdots + c_Nx_N \\
 &\text{subject to} && a_{00}x_0 + a_{01}x_1 + \cdots + a_{0N}x_N \leq b_0 \\
 & && a_{10}x_0 + a_{11}x_1 + \cdots + a_{1N}x_N \leq b_1 \\
 & && \vdots \\
 & && a_{n0}x_0 + a_{n1}x_1 + \cdots + a_{nN}x_N \leq b_n \\
 & && x_0, x_1, \dots, x_N \geq 0
 \end{aligned}$$





First, the geometric interpretation of this system is pictured in Figure 6.1. Each inequality represents a half-plane in  $\mathbb{R}^2$ , hence their intersection together with the non-negativity constraints defines a region of possible solutions. We call this region the **feasible region**. Further, the “corners” of the feasible region  $F$  are called the **extreme points** of  $F$ , as we can think of this region as the convex hull of these points.

On the right, the dotted lines represent the objective functions at different evaluations of  $x_0, x_1$ .

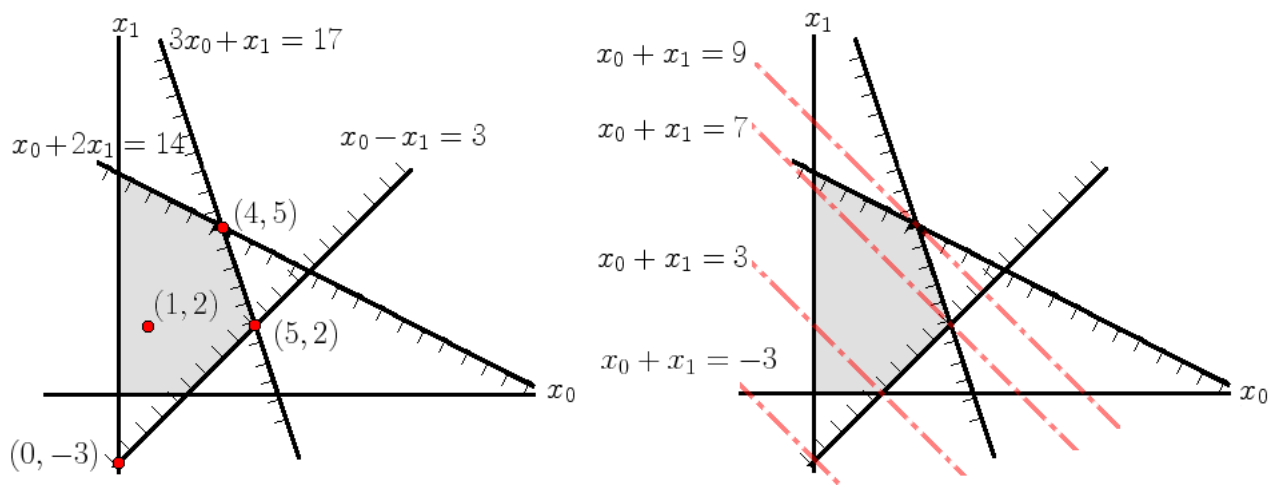


Figure 6.1: Geometric interpretation of a linear program

We examine several solutions that satisfy the inequality constraints.

$$(x_0, x_1) = (1, 2) \tag{6.7}$$

$$(x_0, x_1) = (0, -3) \tag{6.8}$$

$$(x_0, x_1) = (5, 2) \tag{6.9}$$

$$(x_0, x_1) = (4, 5) \tag{6.10}$$

Rewriting the problem in standard equality form:

$$\begin{array}{rcll}
\text{maximize } z & = & x_0 + x_1 & \\
\text{subject to} & & x_0 + 2x_1 + s_0 & = 14 \\
& & 3x_0 + x_1 + s_1 & = 17 \\
& & x_0 - x_1 + s_2 & = 3 \\
& & x_0, x_1, s_0, s_1, s_2 & \geq 0
\end{array}$$

The above solutions now correspond to

$$(x_0, x_1, s_0, s_1, s_2) = (1, 2, 9, 12, 4) \quad (6.11)$$

$$(x_0, x_1, s_0, s_1, s_2) = (0, -3, 20, 20, 0) \quad (6.12)$$

$$(x_0, x_1, s_0, s_1, s_2) = (5, 2, 5, 0, 0) \quad (6.13)$$

$$(x_0, x_1, s_0, s_1, s_2) = (4, 5, 0, 0, 4) \quad (6.14)$$

Note that the vector that maximizes  $x_0 + x_1$ , namely the vector  $(4, 5)$ , is one of the extreme points of the feasible region. Further, note that all extreme points have 2 of their coordinates zero. This is either because two of the original inequalities are met with equality, hence the “slack” drops to zero, or because the vector lies on one of the axes  $x_i = 0$ . Solutions of this form will be important in the simplex method. Lastly, the vector  $(0, -3)$  satisfies all inequalities but does not lie in the feasible region because  $x_1 < 0$ .

An equivalent but simpler way of writing a linear program in standard equality form is

$$\begin{array}{ll}
\max & \vec{c}^T \vec{x} \\
\text{s.t.} & A\vec{x} = \vec{b} \\
& \vec{x} \geq 0
\end{array}$$

where  $\vec{c}$  is a vector in  $\mathbb{R}^{N+1}$ ,  $\vec{b}$  is a vector in  $\mathbb{R}^{n+1}$ ,  $\vec{x}$  is a variable vector in  $\mathbb{R}^{N+1}$  and  $A = (a_{ij})$  is an  $(n+1) \times (N+1)$  real matrix.

In our example,

$$\vec{c} = (1, 1, 0, 0, 0)^T, \quad A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and } \vec{b} = (14, 17, 3)^T.$$

Suppose  $A$  contains  $n+1$  linearly independent columns. Then it is possible to rearrange  $A$  as  $A = [B|M]$ , where  $B$  is a non-singular matrix. We call the matrix  $B$  the **basic matrix**

since the columns of  $B$  form a basis for  $\mathbb{R}^{n+1}$ . Accordingly, the matrix  $M$  is called the **non-basic matrix**.

Let us decompose  $\vec{x}$  into components of  $B$ ,  $\vec{x}_B = (x_0, x_1, \dots, x_n)$  and components of  $M$ ,  $\vec{x}_M = (x_{n+1}, x_{n+2}, \dots, x_N)$ . Now  $A\vec{x} = \vec{b}$  can be rewritten  $[B|M] \begin{bmatrix} \vec{x}_B \\ \vec{x}_M \end{bmatrix} = \vec{b}$  or equivalently  $B\vec{x}_B + M\vec{x}_M = \vec{b}$ . Since  $B$  is non-singular, we can solve for  $\vec{x}_B = B^{-1}\vec{b} - B^{-1}M\vec{x}_M$ . If we want a unique solution to the system, we can assign arbitrary values to the components of  $\vec{x}_M$ . Assigning  $\vec{x}_M = \vec{0}$  yields what is called a **basic solution** defined by  $B$ .

Further, if the basic solution lies within the feasible region, we say it is a **basic feasible solution**. Note that a basic feasible solution is simply a basic solution which is also non-negative. The basis associated with a basic feasible solution is called a **feasible basis**.

The solutions (6.13) and (6.14) in the previous example were basic feasible solutions. Solution (6.12) was basic but not feasible.

If a basic solution has some *basic* variables with value 0, namely if the basic solution contains more than  $|M|$  0-valued coordinates, then we say that the basis associated with that solution is **degenerate**. If a system contains any basis that is degenerate, then we say the system is degenerate.

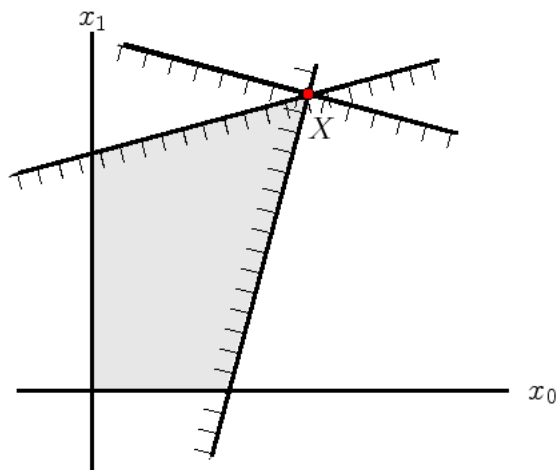


Figure 6.2: Degenerate system of linear equations

See an example of a degenerate system in Figure 6.2. The vector  $X$  is a degenerate

basic feasible solution because it lies on three constraints at once. Degenerate solutions are not desirable and we will see in the next section how to avoid them.

## 6.3 The Simplex Method

It can be shown that  $\vec{x}$  is a basic feasible solution to the system  $\{A\vec{x} = \vec{b}, \vec{x} \geq 0\}$  if and only if  $\vec{x}$  is an extreme point of the feasible region (see [3, Section 3.2, Theorem 1]). Further, it can be shown that if an optimal solution exists, then there exists an optimal solution which is a basic feasible solution (see [3, Section 3.2, Theorem 3]). This is the basis of Dantzig's simplex method: instead of searching the entire feasible region  $F$ , restrict the search to the extreme points of  $F$ . Further, we travel from one extreme point to the next only if it increases our objective function.

Suppose we have a linear program with  $N + 1$  variables and  $n + 1$  constraints. It can be shown that in practice the simplex algorithm is quite fast: the total number of iterations grows linearly in the number of constraints  $n + 1$ . Given that there are  $\binom{N+1}{n+1}$  basic feasible solutions in total, this is much faster than looking at all basic feasible solutions. However, there exist examples where the simplex method does not perform well, and where the number of iterations grows exponentially in the number of variables  $N + 1$ . These examples are hard to construct and rarely happen in practice, hence the simplex method is still a practical and efficient way to solve linear programs. For more details on the efficiency of the simplex method, see [8, Chapter 4].

To show how the simplex method works, we will consider the example (6.6) from above.

For ease of notation, we will let an index set  $J \subset \{0, 1, \dots, N + 1\}$  denote a basis for a particular system. For example the set  $J = \{0, 2, 5\}$  represents the basis consisting of columns  $\{0, 2, 5\}$  of the matrix  $A$ .

From our original problem in standard equality form we have:

$$\begin{array}{rcll} \text{maximize} & z & = & x_0 + x_1 \\ \text{subject to} & & & x_0 + 2x_1 + x_2 & = & 14 \\ & & & 3x_0 + x_1 & + & x_3 & = & 17 . \\ & & & x_0 - x_1 & & + & x_4 & = & 3 \\ & & & x_0 & , & x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0 \end{array}$$

Note that the basis associated with the problem is  $J^0 = \{2, 3, 4\}$ , since these columns

form an identity matrix. We rewrite it in **tableau** form, a form that ignores the non-negativity constraints and includes the objective function as the first equality. We get

$$\begin{array}{rcccccccl}
 z & - & x_0 & - & x_1 & & & & = & 0 \\
 & & x_0 & + & 2x_1 & + & x_2 & & = & 14 \\
 & & 3x_0 & + & x_1 & & & + & x_3 & = & 17 \\
 & & x_0 & - & x_1 & & & & + & x_4 & = & 3
 \end{array}$$

From the basis  $J^0 = \{2, 3, 4\}$  we can determine the basic feasible solution associated with the tableau. Recall that non-basic variables have value 0, so here  $x_0$  and  $x_1$  are 0. Therefore the basic feasible solution can be read easily from the tableau:  $\vec{x} = (0, 0, 14, 17, 3)$ . Further, the objective function has value  $z = 0$ , since  $x_0$  and  $x_1$  have value 0.

In order to increase the value of the objective function, consider all variables in the first row that have *negative* coefficient. If we choose one of these variables and let its value augment from 0 to a non-zero value  $t$ , keeping all other non-basic variables at 0, then the value of the objective function will increase, as all variables are non-negative.

However, we must make sure in doing so that the basic solution remains feasible. For example, say we let the value of  $x_0$  increase, with value  $t$ ,  $t > 0$ , and keep the value of all other non-basic variables at 0.

This induces the following solution  $x(t)$ :

$$\begin{aligned}
 z(t) &= 0 + t \\
 x_0(t) &= t \\
 x_1(t) &= 0 \\
 x_2(t) &= 14 - t \\
 x_3(t) &= 17 - 3t \\
 x_4(t) &= 3 - t
 \end{aligned}$$

Yet all variables  $x_i$  must remain non-negative, hence  $14 - t \geq 0$ ,  $17 - 3t \geq 0$  and  $t - 3 \geq 0$ . This yields the following conditions on  $t$ :  $t \leq 14$ ,  $t \leq \frac{17}{3}$  and  $t \leq 3$ . Therefore if we pick  $t = 3$ , all variables will remain feasible, and exactly one variable will become 0, namely  $x_4$ . Hence we say one variable  $x_0$  **entered** the basis and one variable  $x_4$  **left** the basis. Our new basis is  $J^1 = \{0, 2, 3\}$ .

We want to rearrange the tableau to reflect this new basis, namely we want the columns  $\{0, 2, 3\}$  to form an identity matrix. Columns 2 and 3 are already in the right form, but column 0 needs to be reduced to  $(0, 0, 1)^T$ . To do so we eliminate the two unwanted entries in that column by performing elementary row operations on the tableau. We also eliminate the variable  $x_0$  from the objective function row. This change of basis operation is called a **pivot step**. Here is the new tableau.

$$\begin{array}{rcccccc}
 z & & - & 2x_1 & & & + & x_4 & = & 3 \\
 & & & 3x_1 & + & x_2 & & - & x_4 & = & 11 \\
 & & & 4x_1 & & & + & x_3 & - & 3x_4 & = & 8 \\
 x_0 & - & & x_1 & & & & & + & x_4 & = & 3
 \end{array}$$

From this tableau the basic feasible solution can be read right away, since the value for non-basic variables is 0:  $\vec{x} = \{3, 0, 11, 8, 0\}$ . Further, the new objective value is the top number in the last column: 3, which is bigger than our old value 0. Let us perform a few more pivot steps.

This time, say we pick  $x_1$ , with coefficient  $-2$  to enter the basis, with value  $t > 0$ . Then the following solution  $x(t)$  can be derived:

$$\begin{aligned}
 z(t) &= 3 + 2t \\
 x_0(t) &= 3 + t \\
 x_1(t) &= t \\
 x_2(t) &= 11 - 3t \\
 x_3(t) &= 8 - 4t \\
 x_4(t) &= 0
 \end{aligned}$$

This leads to the following conditions on  $t$ :

$$\begin{aligned}
 3 + t \geq 0 &\implies t \geq -3, \\
 11 - 3t \geq 0 &\implies t \leq \frac{11}{3}, \text{ and} \\
 8 - 4t \geq 0 &\implies t \leq 2.
 \end{aligned}$$

Note that because the coefficient of  $t$  for the first condition is positive, it does not help determine a value of  $t$  for which exactly one variable will become 0, as every non-negative value of  $t$  will keep this variable non-negative. If every condition has positive  $t$  coefficient, we conclude the problem is **unbounded**, and no maximum objective value can be found.



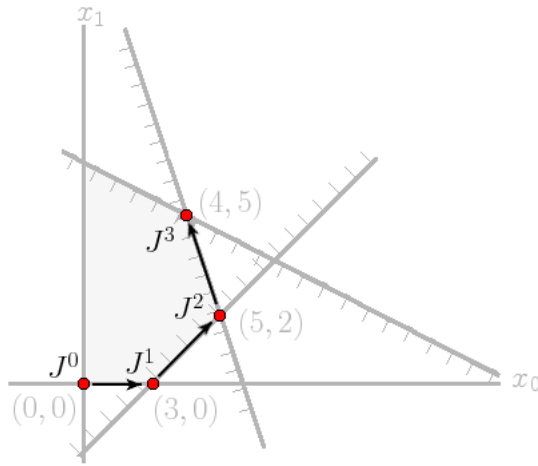


Figure 6.3: Path followed by the simplex method

2. As mentioned before, if the set of solutions  $\{\vec{x} \geq 0 : A\vec{x} = \vec{b}\}$  is bounded, then we ensure that at every step some leaving variable can be found.
3. In order for our system to be non-degenerate, we must ensure that at every pivot, only one variable leaves the basis. In other words, if picking a particular value of  $t$  causes two or more variables to become 0 simultaneously, then our new solution will be degenerate.

Examples of cycling exist, but are rare and difficult to construct. Still, we would like a way to avoid it. Luckily, there are many rules that prevent cycling. One method is due to Bland ([5]) and is called the **smallest subscript rule**: of all possible entering variables choose the one with the smallest subscript, and of all leaving variables, choose one with the smallest subscript. This guarantees no cycling. Given that Scarf's Extension Theorem uses the simplex method in a slightly different way, a proof that cycling does not occur is deferred to Chapter 7, in the proof of Theorem 7.4.

To address Observation 2, we present the following theorem which will be sufficient for our application of the simplex method in Sections 8.1 and 8.2.

**Theorem 6.6.** *Let  $A$  be an  $(n+1) \times (N+1)$  matrix where  $a_{ij} \in \mathbb{R}_+$ , and where  $A$  contains no column of zeros. Let  $\vec{b} \in \mathbb{R}_+^{n+1}$  and  $\vec{x} \in \mathbb{R}^{N+1}$ . Then the set  $\{\vec{x} \geq 0 : A\vec{x} = \vec{b}\}$  is bounded.*



*Proof.* Rewrite  $A\vec{x} = \vec{b}$  as

$$\begin{bmatrix} a_{00}x_0 + a_{01}x_1 + a_{02}x_2 + \dots + a_{0N}x_N \\ a_{10}x_0 + a_{11}x_1 + a_{12}x_2 + \dots + a_{1N}x_N \\ a_{20}x_0 + a_{21}x_1 + a_{22}x_2 + \dots + a_{2N}x_N \\ \vdots \\ a_{n0}x_0 + a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nN}x_N \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then because  $A$  contains no column of zeros, for all  $i \in \{0, \dots, N\}$ ,  $x_i$  has a non-zero coefficient on some row of  $A\vec{x}$ . Further, since all values  $a_{ij}$ , and all values  $b_i$  are non-negative, there must be a bound on each  $x_i$ . Therefore the set  $\{\vec{x} \geq 0 : A\vec{x} = \vec{b}\}$  is bounded.  $\square$

Combining Observation 3 and Observation 2, we have the following theorem.

**Theorem 6.7.** *Let the system  $A\vec{x} = \vec{b}$  be non-degenerate and the set of solutions  $\{\vec{x} \geq 0 : A\vec{x} = \vec{b}\}$  be bounded. For any non-basic variable which enters the basis at some step of the simplex method, there is a unique leaving variable.*

Of course introducing a variable with non-negative coefficient in the objective function may not be useful, but a pivot step can still be performed.

Finally, to address Observation 3, the idea will be to perturb the values of  $\vec{b}$  slightly. This is shown for two dimensions in Figure 6.4. If we perturb each value  $b_i$  by a different small amount  $\epsilon_i$ , with  $\epsilon_0 > \epsilon_1 > \dots > \epsilon_n$ , then it can be shown that with a special rule to choose the leaving variable, this system is equivalent to the original and yet non-degenerate.

In general this is called the **perturbed problem**. See [8, Theorem 3.2] for full details, including a proof of correctness. We will examine a special case of the perturbation method, called the Lexicographical Method.

## 6.4 The Lexicographical Simplex Method

To guarantee that the system  $\{A\vec{x} = \vec{b}, \vec{x} \geq 0\}$  is non-degenerate, we perturb the elements of  $\vec{b}$  by small amounts  $\epsilon_0, \epsilon_1, \dots, \epsilon_n$  such that  $\epsilon_0 > \epsilon_1 > \dots > \epsilon_n$ . Instead of picking values for  $\epsilon_0, \epsilon_1, \dots, \epsilon_n$ , we note that if we choose one small value  $\epsilon$  and set  $\epsilon_i = \epsilon^{i+1}$  then we

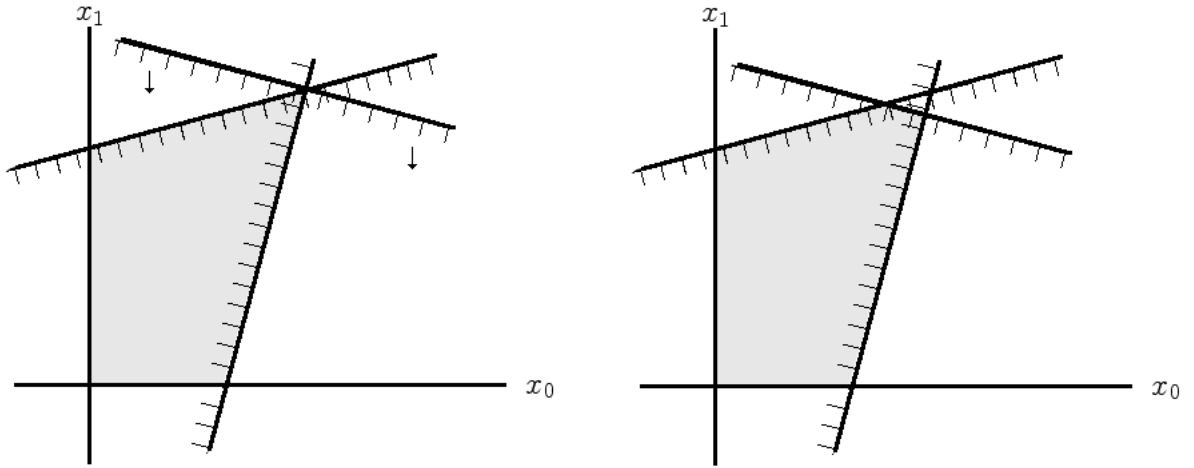


Figure 6.4: The Perturbed Problem

indeed have  $\epsilon_0 > \epsilon_1 > \dots > \epsilon_n$ . Further, if we can pick  $\epsilon$  such that  $\epsilon_i \gg \epsilon_{i+1}$ , then the  $\epsilon$ 's do not influence each other when we take small linear combinations of them. We will see later how we can pick such an  $\epsilon$ .

Because of this, in practise we do not choose an explicit value for  $\epsilon$ , but instead keep track of the coefficients of  $\epsilon^i$  as they are manipulated by the elementary row operations on  $A$ .

More precisely, at any point in the simplex method, the right hand side of the system will be of the form

$$\bar{b}_i + \beta_{i0}\epsilon^1 + \dots + \beta_{in}\epsilon^{n+1}.$$

Therefore in our tableau form, we can add one column for each  $\epsilon^i$ , and perform the same elementary row operations on these added columns as we do on the original system. This will record the values  $\beta_{ij}$ .

Examine the example (6.6), now with the right hand side perturbed.

$$\begin{array}{rcccccc}
z & - & x_0 & - & x_1 & & = & 0 \\
& & x_0 & + & 2x_1 & + & x_2 & = & 14 & + & \epsilon \\
& & 3x_0 & + & x_1 & & + & x_3 & = & 17 & & + & \epsilon^2 \\
& & x_0 & - & x_1 & & & + & x_4 & = & 3 & & + & \epsilon^3
\end{array}$$

We examine a pivot step for the tableau above. According to the smallest subscript rule, we pick  $x_0$  to enter the basis and let it have value  $t$ ,  $t > 0$ . Then we have the following solution  $\vec{x}(t)$ :

$$\begin{aligned}
z(t) &= 0 + t \\
x_0(t) &= t \\
x_1(t) &= 0 \\
x_2(t) &= 14 + \epsilon - t \\
x_3(t) &= 17 + \epsilon^2 - 3t \\
x_4(t) &= 3 + \epsilon^3 - t
\end{aligned}$$

Now similarly to before, we want to pick the first variable which becomes 0 as  $t$  increases. Here we easily detect  $x_4$  to be such a variable, but if two of the  $b_i$  entries were equal, we would need to look at the  $\epsilon$ 's to break the tie. We will express the conditions on  $t$  in the following way:

$$\begin{aligned}
t &\leq \frac{(14, 1, 0, 0)}{1} \\
t &\leq \frac{(17, 0, 1, 0)}{1} \\
t &\leq \frac{(3, 0, 0, 1)}{1}
\end{aligned}$$

where the first coordinate is  $b_i$  and the others represent the  $\epsilon$ 's. By

$$\frac{(\bar{b}_j, \beta_{j0}, \beta_{j1}, \dots, \beta_{jn})}{a_{jk}},$$

we mean

$$\left( \frac{\bar{b}_j}{a_{jk}}, \frac{\beta_{j0}}{a_{jk}}, \frac{\beta_{j1}}{a_{jk}}, \dots, \frac{\beta_{jn}}{a_{jk}} \right).$$

If we pick the **lexicographical minimum** of those three conditions, we will pick the first condition to reach 0 as  $t$  is increased.

Recall that  $(a_0, a_1, \dots, a_n)$  is lexicographically smaller than  $(b_0, b_1, \dots, b_n)$  if  $(a_0 < b_0)$  or  $(a_0 = b_0$  and  $a_1 < b_1)$  or  $(a_0 = b_0, a_1 = b_1$  and  $a_2 < b_2)$ , and so forth until  $(a_i = b_i, i < n$  and  $a_n < b_n)$ .

The general procedure is as follows. Choosing an entering variable  $x_k$  is done as before, by choosing a variable whose coefficient in the first row is negative.

Recall that we are trying to ensure a unique leaving variable at every step, and guarantee the associated basis is non-degenerate. To choose the leaving variable  $x_\ell$  in such a tableau, consider all rows  $j$  such that the coefficient of  $x_k$ ,  $a_{jk}$  is positive: this ensures the coefficient of  $t$  is negative, as before. If the problem is bounded we are guaranteed to find at least one such row. Of those, pick the one where

$$\frac{(\overline{b}_j, \beta_{j0}, \beta_{j1}, \dots, \beta_{jn})}{a_{jk}}$$

is the lexicographical minimum. Then the unique basis vector in that row is the leaving variable.

This will always yield a unique choice, otherwise we have  $\frac{\beta_{j\ell}}{a_{jk}} = \frac{\beta_{i\ell}}{a_{ik}}$  for all  $\ell$  and  $i \neq j$ . Yet we started with the columns for  $\epsilon_0, \epsilon_1, \dots, \epsilon_n$  as an identity matrix and only performed elementary row operations on it. Hence we cannot have two rows be multiples of one another, as that would imply we have a singular matrix.

It may seem like adding these extra  $n + 1$  columns will create more work and require double the storage, but in our first tableau, the basis columns also form an identity matrix. Therefore we can forego the recording of the  $\epsilon_i$  altogether, and instead keep track of the coefficients  $\beta_{ij}$  by examining the corresponding coefficients of the original basis columns.

Two pivots steps later, the above example would become:

$$\begin{array}{rccccccc} z & & + & \frac{4}{5}x_2 & + & \frac{2}{5}x_3 & & = & 9 & + & \frac{4}{5}\epsilon & + & \frac{2}{5}\epsilon^2 \\ & & & \frac{4}{5}x_2 & - & \frac{3}{5}x_3 & + & x_4 & = & 4 & + & \frac{4}{5}\epsilon & - & \frac{3}{5}\epsilon^2 & + & \epsilon^3 \\ & x_1 & + & \frac{3}{5}x_2 & - & \frac{1}{5}x_3 & & & = & 5 & + & \frac{3}{5}\epsilon & - & \frac{1}{5}\epsilon^2 & & \\ x_0 & & - & \frac{1}{5}x_2 & + & \frac{2}{5}x_3 & & & = & 4 & - & \frac{1}{5}\epsilon & + & \frac{2}{5}\epsilon^2 & & \end{array} .$$

Note that the original basis columns  $\{2, 3, 4\}$  are the same as the columns for  $\{\epsilon, \epsilon^2, \epsilon^3\}$ .

We need to ensure we can pick an  $\epsilon$  value which is small enough to make this possible. The number of iterations is bounded by the number of possible bases, given that we do not cycle. There are  $\binom{N+1}{n+1}$  possible bases. Next, when we perform elementary row operations, we will multiply a row by at most the largest entry in the matrix  $A$ . Suppose the largest entry has value  $m$ . A coefficient in our matrix will then be at most  $m^{\binom{N+1}{n+1}}$ . Hence picking

$$\epsilon = \left(m^{\binom{N+1}{n+1}}\right)^{-1}$$

ensures the  $\epsilon^i$ 's do not interfere with each other, and maintain the property  $\epsilon^i > \epsilon^{i+1}$ .

An example using the lexicographical simplex method is presented in Appendix A.

# Chapter 7

## An Extension Theorem of Scarf

The first two versions of Scarf's Theorem were topological in nature, and were shown to be equivalent to Sperner's Lemma. The final version we examine, called Scarf's Extension Theorem, abstracts the concepts introduced earlier and gives a result which is more general than Sperner's Lemma.

At first this version may seem unrelated to the other two versions, since it involves matrices and solutions to linear programs in standard equality form. Therefore we will start with an intermediary formulation which will first express Scarf's Theorem on primitive sets in matrix form. Once this transformation is understood, it is a small step to understand the full Extension Theorem.

### 7.1 Scarf's Theorem in Matrix form

We start with Scarf's Theorem on primitive sets (Theorem 3.7). Recall that in this version of the Theorem, we have a set  $P = \{X^0, \dots, X^N\}$  of  $N + 1$  points, where the first  $n + 1$ ,  $\{X^0, X^1, \dots, X^n\}$ , are the points

$$\begin{aligned} X^0 &= (1 - n, 1, 1, \dots, 1) \\ X^1 &= (1, 1 - n, 1, \dots, 1) \\ &\vdots \\ X^n &= (1, 1, 1, \dots, 1 - n) \end{aligned}$$

and the remaining points  $\{X^{n+1}, \dots, X^N\}$  are chosen arbitrarily within the interior of the standard  $n$ -simplex  $S = \sigma(e_0, e_1, \dots, e_n)$ , so as to satisfy the Non-Degeneracy Assumption (Definition 3.3).

Place each point  $X^j$  as the  $j$ th column of an  $(n+1) \times (N+1)$  matrix  $C$ .

The Non-Degeneracy Assumption ensures that within all sets of  $n+1$  columns of  $C$ , each row has a unique minimum.

Recall that for a subset of  $n+1$  points  $W = \{X^{j_0}, X^{j_1}, \dots, X^{j_n}\}$  of  $P$ , a minimizer in  $k$  is a point in  $W$  that achieves the minimum value in coordinate  $k$ . We extend the terminology to this matrix form and say that for an index set  $J = \{j_0, \dots, j_n\}$  of columns of  $C$ , a column which achieves the minimum in row  $k$  is called a **minimizer in  $k$** .

Now we can also extend the idea of a primitive set. A set  $W = \{X^{j_0}, X^{j_1}, \dots, X^{j_n}\}$  of  $n+1$  points of  $P$  is primitive if no point  $X$  of  $P$  has all coordinates  $x_k > \min\{x_k^{j_0}, x_k^{j_1}, \dots, x_k^{j_n}\}$ .

**Definition 7.1.** Let  $J = \{j_0, j_1, \dots, j_n\}$  be an index set representing  $n+1$  columns of  $C$ . If no column  $q$  of  $C$  has  $c_{kq} > \min\{c_{kj_0}, c_{kj_1}, \dots, c_{kj_n}\}$  for all rows  $k$ , we say that  $J$  forms an **ordinal basis** for  $C$ .

The term “ordinal basis” is used here instead of “primitive set” to imply a parallel with linear programming of Chapter 6, and its use will become clear later on.

Let  $q, s$  be two columns of  $C$ . If for a particular row  $k$ ,  $c_{kq} \leq c_{ks}$ , then we say that column  $s$  **dominates** column  $q$  at (row)  $k$ . Note that if  $c_{kq} = c_{ks}$ , then  $q$  also dominates  $s$ . In particular, every column dominates itself at every row.

For a set of columns  $J$ , if a minimizer in  $k$  dominates a column  $q$  at  $k$ , then all columns in  $J$  dominate  $q$  at  $k$ , as the minimizer has the minimum  $k$ th entry. Hence alternatively,  $J$  forms an ordinal basis for  $C$  if each column of  $C$  is dominated by all columns of  $J$  at some row.

For example, consider the following matrix  $C$ , corresponding to Figure 3.3 a):

$$C = \left[ \begin{array}{ccc|ccc} X^0 & X^1 & X^2 & X^3 & X^4 & [X^5] & [X^6] & [X^7] \\ \hline -1 & 1 & 1 & 0.45 & 0.3 & [0.20] & 0.35 & 0.40 \\ 1 & -1 & 1 & 0.15 & 0.1 & 0.35 & 0.40 & [0.25] \\ 1 & 1 & -1 & 0.40 & 0.6 & 0.45 & [0.25] & 0.35 \end{array} \right]$$

and the set of columns  $J = \{5, 6, 7\}$ . The row minima with respect to  $J$  are marked within square brackets ( $[ \ ]$ ) for each row. Then  $J$  forms an ordinal basis because columns  $\{0, 5\}$

are dominated by all columns of  $J$  at 0, columns  $\{1, 3, 4, 7\}$  are dominated by all columns of  $J$  at 1 and columns  $\{2, 6\}$  are dominated by all columns of  $J$  at 2.

We can also easily verify that the set  $J' = \{4, 5, 7\}$  is not an ordinal basis because column 3 is not dominated at any row by all columns of  $J'$ .

$$C = \left[ \begin{array}{ccc|cccc} X^0 & X^1 & X^2 & X^3 & [X^4] & [X^5] & X^6 & [X^7] \\ \hline -1 & 1 & 1 & 0.45 & 0.3 & [0.20] & 0.35 & 0.40 \\ 1 & -1 & 1 & 0.15 & [0.1] & 0.35 & 0.40 & 0.25 \\ 1 & 1 & -1 & 0.40 & 0.6 & 0.45 & 0.25 & [0.35] \end{array} \right]$$

Now that we have abstracted the primitive sets in a matrix  $C$ , we form a second  $(n+1) \times (N+1)$  matrix  $B$  to represent the labelling. The  $ij$ th entry of  $B$  is 1 if the point  $X^j$  has label  $i$ . Because the first  $n+1$  points of  $P$ ,  $X^0, \dots, X^n$ , are labelled  $0, 1, \dots, n$  respectively, the first  $n+1$  columns of  $B$  form an identity matrix. Note that each column contains exactly one 1, since each vertex is given exactly one label. Below is an example of a matrix  $B$ . Note that the set  $J = \{5, 6, 7\}$  is fully labelled.

$$\left[ \begin{array}{ccc|ccccc} X^0 & X^1 & X^2 & X^3 & X^4 & X^5 & X^6 & X^7 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

If  $B$  and  $C$  are matrices as above, we say they are in **standard form**.

In order to simplify the test for a fully labelled simplex, we introduce the concept of a characteristic vector.

**Definition 7.2.** The **characteristic vector**  $\vec{\alpha}$  of a subset  $J$  of  $\{0, \dots, N\}$  is given by

$$\alpha_i = \begin{cases} 1 & i \in J \\ 0 & \text{otherwise} \end{cases} .$$

Now a fully labelled set of  $n+1$  points corresponds to a set  $J$  whose characteristic vector  $\vec{\alpha}$  satisfies  $B\vec{\alpha} = \vec{1}$ . To see this, notice that multiplying  $B$  by  $\vec{\alpha}$  counts (within the columns  $J$ ) the 1's on each row of  $B$ . Hence if  $B\vec{\alpha} = \vec{1}$ , there is exactly one 1 in each row of the columns of  $J$ , namely a vector with each label.

**Definition 7.3.** Let  $J$  be an index set representing  $n+1$  columns of  $C$ , and let  $\vec{\alpha}$  be its characteristic vector. Further, let  $\vec{b} = \vec{1}$ . If  $B\vec{\alpha} = \vec{b}$ , then we say  $J$  is a **feasible basis** for  $B$ .



To draw a parallel with the ideas of Chapter 6, the system  $\{B\vec{\alpha} = \vec{b}, \vec{\alpha} \geq 0\}$  can be thought of as a linear program in standard equality form. In fact, the first  $n + 1$  columns of  $B$  always form an identity matrix and so act as slack variables for the system. One element is missing to make it complete: an objective function. Here instead of an objective function to maximize, we will use the whole of  $C$  to guide the choice of entering variables. We defer the full details until later in the chapter. Note further that  $\vec{\alpha}$  is the basic feasible solution to  $B\vec{x} = \vec{b}$  with respect to  $J$ .

Now to conclude the abstraction, we note that a fully labelled primitive set corresponds to a set  $J$  of  $n + 1$  columns which is both a feasible basis for  $B$  and an ordinal basis for  $C$ .

We are ready to formulate Scarf's Theorem in matrix form.

**Theorem 7.4.** *Let  $B$  and  $C$  be two  $(n+1) \times (N+1)$  matrices in standard form. Then there exists an odd number of bases  $J$  which are feasible for  $B$  and ordinal for  $C$ . In particular, there is at least one.*

This theorem is a reformulation of Theorem 3.7, hence we already know it to be true. However, it will be useful to prove it once more using the tools of linear programming, which will help for the Extension Theorem we will see later on. The procedure we will use is very similar to the simplex method studied in Chapter 6. Given that we have two matrices and no objective function, we will first need to define two different pivoting rules.

**Lemma 7.5.** *Let  $J = \{j_0, \dots, j_n\}$  represent the columns of a feasible basis for  $B$ , and let  $h$  be an arbitrary column not in this collection. Then there exists a unique feasible basis  $J'$  consisting of column  $h$  and  $n$  columns of the original basis. We call this change of basis a **feasible pivot step**.*

*Proof.* If  $J$  is feasible for  $B$ , then it must be that columns  $\{j_0, j_1, \dots, j_n\}$  form (up to a reordering of the columns) an identity matrix. This is equivalent to having one column with each label. When we consider  $J \cup \{h\}$ , there must then be 2 columns with the same label,  $h$  and say  $j_k$ . Then  $J' = J \cup \{h\} \setminus \{j_k\}$  is the required basis. It is clearly unique.  $\square$

**Lemma 7.6** ([26, Lemma 2]). *Let  $J = \{j_0, \dots, j_n\}$  represent the columns of an ordinal basis for  $C$  and  $j_t$  an arbitrary column in  $J$ . Assume  $j_0, j_1, \dots, j_{t-1}, j_{t+1}, \dots, j_n$  are not all selected from the first  $n + 1$  columns of  $C$ . Then there is a unique column  $j_h \neq j_t$  for which  $J' = \{j_0, j_1, \dots, j_{t-1}, j_h, j_{t+1}, \dots, j_n\}$  is an ordinal basis. We call this change of basis an **ordinal pivot step**.*

*Proof.* Because  $J$  is an ordinal basis, each column is a minimizer in some row (by the Non-Degeneracy Assumption). Relabel the columns of  $J$  so that the column  $j_i$  is the minimizer in row  $i$ .

By removing  $j_t$ , one column (say column  $j_u$ ) will become the minimizer in two rows, one of which is new (row  $t$ ) and the other which is from the original basis (row  $u$ ). We want column  $j_u$  to become the minimizer in row  $t$ , so we need to find a new column  $j_h$  to act as the minimizer in row  $u$ .

Let  $U$  be the set of columns  $U = \{j : c_{ij} > c_{ij_i}, i \neq u, t \text{ and } c_{tj} > c_{tj_u}\}$ . Columns in  $U$  cannot become the minimizer in any row but  $u$ , as  $c_{ij_i}$  is the minimum value in row  $i$ ,  $i \neq u, t$  and  $c_{tj_u}$  is the new minimum value in row  $t$ . Note in particular that no column of  $J$  can belong to  $U$ .

First we show  $U \neq \emptyset$ . We claim that column  $u$  of  $C$  belongs to  $U$ . Recall that

$$c_{iu} = \begin{cases} 1 - n, & i = u \\ 1 & \text{otherwise} \end{cases} .$$

If  $u$  were already in  $J$ , then  $u$  would be the minimizer in  $u$ , as  $c_{uu}$  is minimal over all of  $C$ , hence  $u = j_u$ . Yet  $j_u$  was picked to be the new minimizer in row  $t$ , and column  $u$  has a  $t$ th entry which is maximal over the row. This implies that no other column  $j$ ,  $j > n$  lies in  $J$ , contradicting our assumption. Hence  $u \notin J$ . Further,  $1 = c_{iu} > c_{ij_i}, i \neq u, t$  and  $1 = c_{tu} > c_{tj_u}$  and so  $u \in U$ .

Of all columns  $j$  in  $U$ , pick the column  $j_h$  that maximizes  $c_{uj}$ . This will ensure the new basis is ordinal. Let  $J' = J \cup \{j_h\} \setminus \{j_t\}$ .

Column  $j_i, i \neq u, t$  is still the minimizer in row  $i$  for  $J'$ . Column  $j_u$  is now the minimizer in  $t$ , and  $j_h$  is the new minimizer in  $u$ .

Now we examine every column in  $C$  to verify that  $J'$  is indeed an ordinal basis.

- If a column  $j_k$  was previously dominated by all columns of  $J$  at  $i = 0, 1, \dots, n$ ,  $i \neq t, u$ , then it is still dominated at  $i$  by all columns of  $J'$ , since those minimizers did not change.
- If  $j_k$  was dominated by all columns of  $J$  at row  $t$ , then it is still dominated at  $t$  by all columns of  $J'$ , as the new minimizer in  $t$ ,  $j_u$ , has  $t$ th entry larger than the old minimizer's  $t$ th entry,  $j_t$ , by choice of  $j_u$ .

- Finally, if  $j_k$  was dominated by all columns of  $J$  solely at row  $u$ ,  $j_k \in U$ , by construction of  $U$ . Since  $j_h$  was taken as the column in  $U$  with the maximal  $u$ th coordinate,  $j_k$  is now dominated at  $u$  by all of  $J'$ .

Each column in  $C$  is dominated by all columns of  $J'$  at some row, hence this new set is an ordinal basis for  $C$ .

To show  $j_h$  is the only possible column leading to an ordinal basis, suppose to the contrary that  $J^* = \{j_0, j_1, \dots, j_{t-1}, j^*, j_{t+1}, \dots, j_n\}$  is an ordinal basis. We make the following observations.

1. When  $j^*$  is added to replace  $j_t$ ,  $j_i, i \neq u, t$  must be the minimizer in  $i$  for  $J^*$ . If it became the minimizer in  $i \neq u, t$ , then column  $j_i$ , which remains in  $J^*$  would now not be dominated by all columns of  $J^*$  at any row. Therefore all  $j_i, i \neq u, t$  stay as minimizers in  $i$ .
2. For rows  $t$  and  $u$  we have two possibilities. Either column  $j^*$  is the new minimizer in  $t$  and  $j_u$  remains the minimizer in  $u$  or vice-versa. We claim that the former case is the original basis  $J$ . To see this, we will show that in  $J$ ,  $j_t$  dominated  $j^*$  at  $t$ , and that in  $J^*$ ,  $j^*$  dominates  $j_t$  at  $t$ . This can only mean  $c_{tj^*} = c_{tj_t}$  and by the Non-Degeneracy Assumption,  $j^* = j_t$ , implying  $J^* = J$ .

The basis  $J$  was ordinal, hence every column of  $J$  dominated  $j^*$  at some row. To find which row, note that in the new basis  $J^*$ ,  $j^*$  is dominated at  $t$  and no other row. Since columns  $j_i, i \neq t$  are the minimizers in  $i$  for both  $J$  and  $J^*$ ,  $j^*$  must be dominated at  $t$  with respect to  $J$  also. Therefore every column of  $J$  dominates  $j^*$  at row  $t$ . In particular,  $j_t$  dominates  $j^*$  at  $t$ .

Now since the new basis  $J^*$  is ordinal, every column of  $J^*$  dominates  $j_t$  at some row. To find which row, we note that in  $J$ , column  $j_t$  was dominated at  $t$  and no other row. Given that columns  $j_i, i \neq t$  are the minimizers in  $J$  and  $J^*$  in row  $i$ , every column of  $J^*$  dominates  $j_t$  at row  $t$ . In particular,  $j^*$  dominates  $j_t$  at  $t$ .

3. Hence the only possibility is to have column  $j^*$  be the new minimizer in  $u$  and  $j_u$  is the new minimizer in  $t$ . Then  $j^* = j_h$ , as from all eligible columns we picked one which maximized the  $u$ th coordinate.

□

To prove Theorem 7.4, the procedure will be very similar to the simplex method on  $B$ . However, instead of an objective function from which to choose entering variables, we use a pivot on  $C$  to indicate the variables to enter the basis.

Now we are ready to prove Scarf's Theorem, restated here for convenience.

**Theorem 7.4** *Let  $B$  and  $C$  be two  $(n+1) \times (N+1)$  matrices in standard form. Then there exists an odd number of bases  $J$  which are feasible for  $B$  and ordinal for  $C$ . In particular, there is at least one.*

*Proof.* First we exhibit a set which is an ordinal basis for  $C$  and another very similar set which is a feasible basis for  $B$ . The basis  $J_B = \{0, 1, \dots, n\}$  is clearly feasible for  $B$ , as it represents an identity matrix.

Now, consider  $J_C = \{j, 1, \dots, n\}$  where  $j$  is taken from all columns  $k > n$  so as to maximize  $c_{0k}$ . Then  $J_C$  is ordinal for  $C$ . This follows because all columns  $c$  of  $C$  are either in  $J$  (and then dominated by themselves) or are the columns  $\{0, n+1, n+2, \dots, N+1\}$ , in which case they are dominated by all columns of  $J$  at 0.

We will say the two bases are in **proper form** if they have the following relationship:

$$J_B = \{0, j_1, \dots, j_n\} \tag{7.1}$$

$$J_C = \{j_0, j_1, \dots, j_n\} \tag{7.2}$$

with  $j_0 \neq 0$ . Clearly the initial bases stated above are in proper form.

The cleverness of the algorithm is to maintain this relationship at each step. Few operations on the two bases will enable us to maintain proper form.

The only operation that can be performed on (7.1) is to add  $j_0$  to it (feasible pivot step). By Lemma 7.5, some column must then leave. If column 0 leaves, meaning that  $J'_B = J_C = \{j_0, j_1, \dots, j_n\}$ , we are done, as we have found a basis which is both feasible for  $B$  and ordinal for  $C$ . Otherwise some other column  $j_i$  leaves. We can rearrange  $J'_B, J_C$  to get

$$\begin{aligned} J'_B &= \{0, j_0, j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_n\} \\ J_C &= \{j_i, j_0, j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_n\}. \end{aligned}$$

Then  $J'_B, J_C$  are in proper form.

As for (7.2), the only possibility is to pivot by removing  $j_0$ . Then some column must enter the basis, by Lemma 7.6. It is possible that column 0 enters the basis, at which point

$J'_C = J_B = \{0, j_1, \dots, j_n\}$  and we are done. Otherwise some column  $h \neq 0$  is introduced, leaving

$$\begin{aligned} J_B &= \{0, j_1, \dots, j_n\} \\ J'_C &= \{h, j_1, \dots, j_n\}. \end{aligned}$$

There are however restrictions of ordinal pivot steps, namely we cannot pivot in the case where  $j_1, \dots, j_n$  are all selected from the first  $n + 1$  columns of  $C$ . This only happens when

$$\begin{aligned} J_B &= \{0, 1, \dots, n\} \text{ and} \\ J_C &= \{j, 1, \dots, n\} \end{aligned}$$

which is precisely our initial state. Hence from this initial state only one pivot step can be taken, the feasible pivot step of introducing  $j$  to  $J_B$ . From all other positions where the bases have the correct relationship, we have two options: to take a feasible pivot step or an ordinal pivot step.

The complete algorithm is as follows. Starting from the above stated initial pair, introduce  $j_0$  to  $J_B$  to obtain  $J'_B = J_B \cup \{j_0\} \setminus \{j_i\}$ . From here, we have two pivots possible, but one will take us back to the initial position (to introduce  $j_i$  to  $J_B$ ). Hence the only option is to do an ordinal pivot step on  $J_C$  by removing  $j_i$  to get  $J'_C = \{h, j_0, \dots, j_{i-1}, j_{i+1}, \dots, j_n\}$ .

At each step we have exactly one pivot step to take, and we only stop when we arrive to a state where  $J_B = J_C$ . There are a finite number of steps, and if cycling is impossible, then we are guaranteed to reach a winning configuration.

To see that cycling is impossible, say the initial pair of bases is the first state to be repeated. Then this means there are two ways to reach the initial position, the one we used to leave the state the first time through, and the one used to return to it. This is a contradiction, by Lemma 7.6.

Now say that some other state which is not the initial state is the first to be repeated. In this case we have three ways to reach this state, the two used to enter and leave the state the first time through, and the third used to revisit the state. Again by Lemmas 7.6 and 7.5, only 2 ways exist.

Hence no state is ever repeated and there exists a set  $J$  which is both feasible for  $B$  and ordinal for  $C$ .

Now suppose there exists any other basis  $J'$  with this property. We have either

$$\begin{aligned} J'_B &= \{0, j_1, \dots, j_n\} \\ J'_C &= \{0, j_1, \dots, j_n\} \\ &\text{or} \\ J'_B &= \{j_0, j_1, \dots, j_n\} \\ J'_C &= \{j_0, j_1, \dots, j_n\}. \end{aligned}$$

In the first case we can pivot on  $J'_C$  by removing 0 from it and proceeding as above until we find another state  $J''$  which is feasible for  $B$  and ordinal for  $C$ .

In the second case our only option is to pivot on  $J'_B$  by adding 0 to it. Again we can then pivot from state to state until we arrive at another basis  $J''$  which is feasible for  $B$  and ordinal for  $C$ .

This means that all other additional bases  $J'$  come in pairs. Hence the total number of bases which are feasible for  $B$  and ordinal for  $C$  is odd.  $\square$

## 7.2 Scarf's Extension Theorem

The key to understanding Scarf's Extension Theorem is to note that working with primitive sets introduces structure that is not necessary to the proof of the theorem. By isolating the properties of primitive sets and labellings that are required for the functioning of the procedure, we can abstract the theorem to a much more general case.

In fact, it was not relevant that  $B$  represented a labelling rule. Any system of linear constraints with associated non-negativity constraints will do, as long as the first  $n + 1$  columns form an identity matrix.

As seen in Chapter 6, Theorem 6.7, if the problem is non-degenerate and the set  $\{\vec{\alpha} \in \mathbb{R}_+^{n+1} : B\vec{\alpha} = \vec{b}\}$  is bounded, any column can be introduced to a feasible basis and exactly one column will be eliminated.

Now for the matrix  $C$ , if we can guarantee that given an ordinal basis  $J$  and one column to remove, we can find a unique replacement that preserves  $J$  as ordinal, we can apply the procedure outlined in the previous section.

Here are the properties of  $C$  that are required:

1. The Non-Degeneracy Assumption must hold, namely for every set of  $n + 1$  columns of  $C$ , the minimum in each row must be unique.
2. Let  $C = [C_1|C_2]$  where  $C_1$  is the square submatrix defined by the first  $n + 1$  columns of  $C$ . Then the elements on the diagonal of  $C_1$  must be minimal in their row over all of  $C$ .
3. The elements off the diagonal of  $C_1$ , say in row  $i$ , must be greater than the elements of  $C_2$  in row  $i$ .

Note that it is not important for the points to lie in any simplex.

Finally, we also need initial bases for  $B$ ,  $C$  that are in proper form. If the first  $n + 1$  columns of  $B$  form an identity matrix then the bases  $J_B = \{0, 1, \dots, n\}$  and  $J_C = \{j, 1, 2, \dots, n\}$  where  $j$  is the column in  $C$  with the largest 0th component, are in proper form.

Therefore we can abstract the theorem in the following way.

**Theorem 7.7** (Scarf's Extension Theorem, [26, Theorem 2]). *Let  $n < N$  and let  $B$  be an  $(n + 1) \times (N + 1)$  matrix such that the first  $n + 1$  columns of  $B$  form an identity matrix. Let  $\vec{b}$  be a non-negative vector in  $\mathbb{R}_+^{n+1}$ , such that the set  $\{\vec{x} \in \mathbb{R}_+^{n+1} : B\vec{x} = \vec{b}\}$  is bounded.*

*Let  $C$  be an  $(n + 1) \times (N + 1)$  matrix for which the Non-Degeneracy Assumption holds and such that  $c_{ii} < c_{ik} < c_{ij}$  for  $i, j \leq n, i \neq j$  and  $k > n$ .*

*Then there exists an odd number of sets  $J$  that form a feasible basis for  $B$  and an ordinal basis for  $C$ . In particular, there is at least one.*

Note that once we have a feasible basis for  $B$ , we can find a basic feasible solution  $\vec{\alpha}$  to  $B\vec{x} = \vec{b}$ , as described in Chapter 6.

To see that this is indeed an abstraction of the previous theorem, see for example the matrices  $B$  and  $C$  below. They are in the form of Theorem 7.7 but cannot be realized on a simplex with labelling. Note that if all the entries on a row of  $C$  are distinct, then the Non-Degeneracy Assumption is trivially met.

$$B = \left[ \begin{array}{ccc|cccc} 1 & 0 & 0 & 2 & 5 & 3 & 7 & 3 \\ 0 & 1 & 0 & 1 & 4 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 & 3 & 1 & 1 \end{array} \right] \quad C = \left[ \begin{array}{ccc|cccc} 0 & 11 & 12 & 3 & 4 & 5 & 6 & 7 \\ 10 & 0 & 12 & 4 & 5 & 6 & 7 & 8 \\ 10 & 11 & 0 & 5 & 6 & 7 & 8 & 9 \end{array} \right]$$

To prove this version of the theorem not much needs to be done. We require two unique replacement lemmas as in the previous section.

**Lemma 7.8** ([26, Lemma 1]). *Let  $\vec{b}$  be a non-negative vector such that the set  $\{\vec{x} \geq 0 : B\vec{x} = \vec{b}\}$  is bounded. Let  $J$  represent the columns of a feasible basis for  $B$ , and let  $h$  be an arbitrary column not in  $J$ . Then there exists a procedure to determine a feasible basis  $J'$  consisting of column  $h$  and  $n$  columns of the original basis. Moreover, the basis found by this procedure is unique.*

*Proof.* If the system  $\{B\vec{x} = \vec{b}\}$  is non-degenerate, then we can perform a feasible pivot step and from Theorem 6.7, we have that the new basis is unique. If the system  $\{B\vec{x} = \vec{b}\}$  is degenerate, we can solve instead the perturbed problem  $\{B\vec{x} = \vec{b}'\}$  as discussed in Section 6.4. Now since the set  $\{\vec{x} \geq 0 : B\vec{x} = \vec{b}\}$  is bounded, so is  $\{\vec{x} \geq 0 : B\vec{x} = \vec{b}'\}$ . Then we can apply Theorem 6.7 to find a unique leaving variable, and hence a unique feasible basis.  $\square$

Recall that this change of basis is called a feasible pivot step.

**Lemma 7.9** ([26, Lemma 2]). *Let  $J = \{j_0, \dots, j_n\}$  represent the columns of an ordinal basis for  $C$  and  $j_t$  an arbitrary column in  $J$ . Assume  $j_0, j_1, \dots, j_{t-1}, j_{t+1}, \dots, j_n$  are not all selected from the first  $n+1$  columns of  $C$ . Then there is a unique column  $h \neq j_t$  for which  $J' = \{j_0, j_1, \dots, j_{t-1}, j_h, j_{t+1}, \dots, j_n\}$  is an ordinal basis. We call this change of basis an **ordinal pivot step**.*

Given that we chose the properties of  $C$  to ensure the proof of Lemma 7.6 is still valid, the proof is the same as before.

Further, the proof of Theorem 7.7 is the same as the proof of Theorem 7.4, replacing the two unique replacement lemmas by the ones above.

Finally, a note on efficiency. It was mentioned in the introduction that Scarf's procedure is an efficient way to find fixed points. Because this procedure follows the simplex method closely, it has similar running time. As mentioned in Section 6.3, even though in theory the number of iterations of the simplex method can be exponential in the number of columns of  $B$  and  $C$ , in practice it is linear in the number of rows of  $B$  and  $C$ . Because by exhaustive search we would have to examine  $\binom{N+1}{n+1}$  possible bases, Scarf's procedure is more efficient.



# Chapter 8

## Applications of Scarf's Extension Theorem

This abstraction of Scarf's Theorem has many direct applications to combinatorics and graph theory. We will examine two of these applications, the first is work done by Aharoni and Holzman in [2]. The second is one of many applications published by Aharoni and Fleiner in [1].

### 8.1 Fractional Kernels in Directed Graphs

We will be interested to show the existence of fractional kernels in directed graphs under minimal assumptions. We first introduce some definitions about directed graphs and kernels.

#### 8.1.1 Background

Recall that a **directed graph** or **digraph**  $D$  is a pair  $(V, A)$  where  $V$  is the set of **vertices** and  $A$  is a subset of  $V \times V$  called **arcs**.

If the arc  $(u, v) \in A$  but  $(v, u) \notin A$ , we say  $(u, v)$  is **irreversible**. A **directed cycle**, of a digraph is a cycle in which all arcs have the same direction. A directed cycle is called **proper** if all of its arcs are irreversible.

A set of vertices  $K$  is called a **clique** if every pair of vertices in  $K$  is joined by at least one arc.

**Definition 8.1.** The *in-neighbourhood* of  $v \in V$ , denoted  $I(v)$ , is  $v$  together with all the vertices sending an arc to  $v$ .

**Definition 8.2.** A subset  $U$  of the vertex set  $V$  is said to be **dominating** if for all vertices  $v$ ,  $U \cap I(v) \neq \emptyset$ . The subset  $U$  is said to be **independent** if no two vertices of  $U$  are joined by an arc. Further, if  $U$  is both independent and dominating, it is a **kernel** of  $D$ .

For example in Figure 8.1 a) the set of vertices  $\{d, e\}$  is dominating but not independent. However, the set  $\{c, d\}$  is both dominating and independent, therefore it forms a kernel for  $D$ . Also note that all arcs apart from  $(d, e)$  and  $(e, d)$  are irreversible.

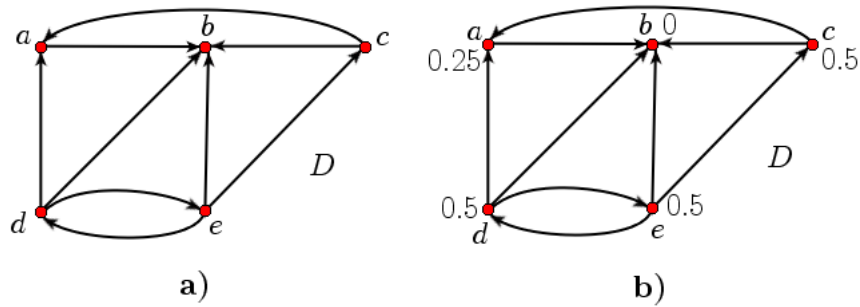


Figure 8.1: Kernel and fractional kernel of a digraph  $D$

We now define the fractional counterparts to the above definitions.

**Definition 8.3.** A non-negative function  $f$  on  $V$  is called **fractionally dominating** if for every vertex  $v$ ,  $\sum_{u \in I(v)} f(u) \geq 1$ . This can be strengthened to  $\sum_{u \in K} f(u) \geq 1$  for some clique  $K$  in  $I(v)$ . If this holds for every vertex  $v$  in  $V$ , we call  $f$  **strongly dominating**.

**Definition 8.4.** A non-negative function  $f$  on  $V$  is called **fractionally independent** if  $\sum_{u \in K} f(u) \leq 1$  for every clique  $K$  in  $D$ .

**Definition 8.5.** A **(strong) fractional kernel** is a function on  $V$  which is both fractionally independent and fractionally (strongly) dominating.

In Figure 8.1 b) the function  $f$  on  $V$  is a fractional kernel. One may easily verify that  $f$  is fractionally dominating. To see that  $f$  is fractionally independent, first recall that each arc in  $A$  is also a clique. Now each clique  $K \in A \cup \{\{a, b, d\}, \{a, b, c\}, \{b, c, e\}, \{b, d, e\}\}$  satisfies  $\sum_{u \in K} f(u) \leq 1$ , for example for  $K = \{a, b, d\}$ ,  $\sum_{u \in K} f(u) = 0.25 + 0 + 0.5 \leq 1$ .

However  $f$  is not strongly dominating because none of the cliques  $K \in \{\{a, c\}, \{a, d\}\}$  in  $I(a)$ , have  $\sum_{u \in K} f(u) \geq 1$ . Changing  $f(a)$  to 0.5 would however yield a strong fractional kernel.

### 8.1.2 Fractional Kernels in Digraphs

Let  $D$  be a graph consisting of a single clique that contains a proper directed cycle going through all vertices (Hamiltonian). Such a graph cannot contain a fractional kernel. To see this, recall that in order for the fractional kernel to be independent, we need  $\sum_{u \in K} f(u) \leq 1$

for every clique  $K$  in  $D$ , therefore  $\sum_{v \in V} f(v) \leq 1$ . If the clique contains a proper directed

cycle going through all vertices, then no vertex has in-degree greater than  $|V| - 2$ , therefore the size of any in-neighbourhood is at most  $|V| - 1$ . Therefore there exist vertices  $v$  and  $x$  such that  $f(v) > 0$  and  $v \notin I(x)$ , and so  $\sum_{u \in I(x)} f(u) < 1$ . In order to be dominating we

need  $\sum_{u \in I(v)} f(u) \geq 1$  for all vertices  $v$ . Contradiction.

In fact we show that directed cycles in cliques are the only obstacle in finding a fractional kernel. We will examine **clique-acyclic digraphs**, that is digraphs where no clique contains a proper directed cycle.

Note however that the complete 2-clique, namely vertices  $u, v$  with arcs  $(u, v)$  and  $(v, u)$  does not constitute a proper directed cycle since neither of its arcs are irreversible. We can now state the following theorem.

**Theorem 8.6** ([2, Theorem 1.1]). *Every clique-acyclic digraph has a strong fractional kernel.*

*Proof.* Let  $D = (V, A)$  be a clique-acyclic digraph. Let  $K^0, \dots, K^n$  be all maximal cliques in  $D$ . We form a digraph  $D'$  from  $D$  as follows:

1. Add  $n + 1$  vertices, say  $z^0, z^1, \dots, z^n$  to  $V$  to get  $V'$ .
2. For all  $v$  in each maximal clique  $K^i$ , add the arcs  $(v, z^i)$  to  $A$ .

Now define  $K^{i'} = K^i \cup \{z^i\}$  to be the maximal cliques in  $D'$ . See Figure 8.2.

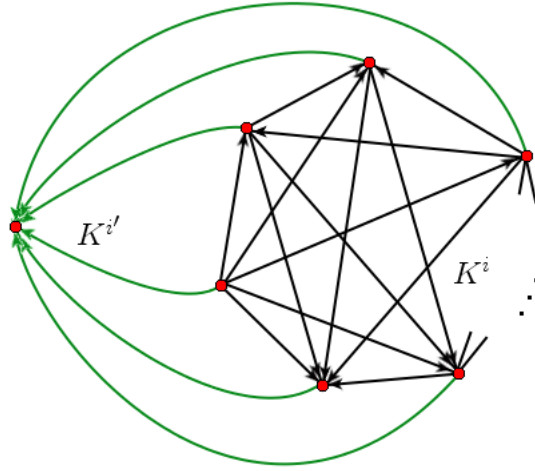


Figure 8.2: Constructing  $K^{i'}$  from  $K^i$

If  $D$  is clique-acyclic, so is  $D'$ , since all arcs added to  $K^i$  end at  $z^i$ . Hence there exists a linear order in the vertices of  $K^{i'}$ ,  $>_i$ , where if  $(u, v)$  is an irreversible arc of  $K^{i'}$ , then  $u >_i v$ . Note that some vertices may lie in many cliques and so may be in many linear orders. Further,  $z^i$  is the minimum in  $>_i$ , as all arcs point to it. We say that the **height** of  $z^i$  in  $>_i$  is 0.

Let  $w^0 = z^0, w^1 = z^1, \dots, w^n = z^n, w^{n+1}, w^{n+2}, \dots, w^N$  be an enumeration of all the vertices in  $D'$ .

Define an  $(n + 1) \times (N + 1)$  matrix  $C$  as follows:

1. If  $w^j \notin K^{i'}$  for some  $i, j$  then  $c_{ij} = M - j$ , where  $M$  is some number bigger than  $|V| + N$ .
2. If  $w^j \in K^{i'}$  then let  $c_{ij}$  be the height of  $w^j$  in the linear order  $>_i$  of  $K^{i'}$ .

Note that  $j \leq N$  and  $M > |V| + N$ . Therefore  $M - j > |V| + N - j \geq |V|$  and so is strictly greater than the labels  $c_{ij}$  which are the height of  $w^j$  in the linear order  $>_i$  of  $K^{i'}$ .

Let  $B$  be the incidence matrix of the cliques  $K^{i'}$ , namely  $b_{ij} = \begin{cases} 1 & w^j \in K^{i'} \\ 0 & \text{otherwise} \end{cases}$ . Then  $B$  is also an  $(n + 1) \times (N + 1)$  matrix. You can see an example construction for matrices  $B, C$  in Figure 8.3 and Table 8.1.

Finally, let  $\vec{b} = \vec{1}$  be the column vector of all 1's. Now we can check that the matrices  $B, C$  satisfy the conditions of Scarf's Theorem.

Since the vertices  $z^i$  belong solely to the clique  $K^{i'}$ , the first  $n + 1$  columns of  $B$  indeed form an identity matrix. Further, the set  $\{\vec{x} \in \mathbb{R}_+^{n+1} : B\vec{x} = \vec{b}\}$  is bounded by Theorem 6.6, since all entries of  $B$  are positive,  $\vec{b}$  is positive, and  $B$  contains no column of zeros, as every vertex lies in some clique.

As for the matrix  $C$ , we have  $c_{ii} = 0$  as the vertices  $z^i$  have height 0 in the linear order  $>_i$ . Also, since  $z^i$  belongs solely to  $K^{i'}$ ,  $c_{ij} = M - j$ , for  $j \leq n, j \neq i$ . Hence for  $j \leq n, k > n$ , we have  $c_{ii} < c_{ik} < c_{ij}$  because either  $0 < M - k < M - j$  or  $0 < \text{height of } w^k < M - j$ . Further, the Non-Degeneracy Assumption is met as all elements of a row are distinct. Therefore the conditions for Theorem 7.7 are met.

Apply Theorem 7.7 to obtain at least one  $J$ , a subset of size  $n + 1$  of  $\{0, 1, \dots, N\}$ , and an associated basic feasible solution  $\vec{\alpha}$  to  $B\vec{x} = \vec{1}$ .

For the example in Figure 8.3, one solution is  $J = \{1, 5, 8\}$  and  $\vec{\alpha} = (0, 0, 0, 0, 0, 1, 0, 0, 1)^T$ . Here the solution is integer, not fractional, but in general we would expect a fractional solution.

Define a function  $f$  on  $V$  by  $f(w^j) = \alpha_j, j = n + 1, \dots, N$ . We claim that  $f$  is a fractional kernel of  $D$ .

To see that  $f$  is fractionally independent in  $D$ , note first that  $f(w^j)$  is non-negative. Also, multiplying the clique incidence matrix  $B$  by  $\vec{\alpha}$  yields exactly the vector of all 1's. This means that  $\sum_{w^j \in K^{i'}} \alpha_j = 1$ , and so, restricting to  $K^i$  yields  $\sum_{w^j \in K^i} \alpha_j = \sum_{w^j \in K^i} f(w^j) \leq \sum_{w^j \in K^{i'}} \alpha_j = 1$  for each clique  $K^i$ , as required.

We now show  $f$  is strongly dominating. We know  $J$  forms an ordinal basis for  $C$ . That is, each column of  $C$  is dominated by all columns in  $J$  at some row  $i$ . Let  $w^k \in V$ , so

$k \geq n + 1$ , and say column  $k$  is dominated at row  $t$ . Restrict the clique  $K^{t'}$  to elements of  $J$  by  $K_J^t = \{w^j \in K^{t'} : j \in J\}$ .

**Claim 8.6.1.**  $K_J^t$  is a subset of  $I(w^k)$ .

*Proof.* First note that the in-neighbourhood of  $w^k$  is the same in  $D$  as in  $D'$ .

Assume there exists a vertex  $w^j$  in  $K_J^t$  but not in  $I(w^k)$ . Then since  $w^j \in K^{t'}$ ,  $c_{tj}$  is the height of  $w^j$  in  $>_t$ , and so  $c_{tj} \leq |K^{t'}| - 1$ .

However, column  $j$  is in the basis  $J$ , and column  $k$  is dominated at row  $t$ , hence  $j$  dominates  $k$  at  $t$ , namely

$$c_{tk} \leq c_{tj}. \quad (8.1)$$

Therefore,  $c_{tk} \leq c_{tj} \leq |K^{t'}| - 1$ , hence  $w^k$  must also belong to  $K^{t'}$ , otherwise  $c_{tk}$  would be  $M - k > |K^{t'}| - 1$ .

So we have both  $w^k$  and  $w^j$  in the clique  $K^{t'}$ , meaning the arc  $(w^k, w^j)$  or  $(w^j, w^k)$  exists. By assumption,  $w^j$  is not in the in-neighbourhood of  $w^k$ , hence only the arc  $(w^k, w^j)$  exists, and is irreversible. This means  $w^k$  is higher in the linear order  $>_t$  than  $w^j$ , and hence  $c_{tk} > c_{tj}$ . This contradicts (8.1).  $\square$

Now  $K_J^t \subseteq I(w^k)$  so

$$\sum_{w^j \in K^{i'}} \alpha_j = \sum_{w^j \in K_J^i} \alpha_j = 1 \geq 1. \quad (8.2)$$

We have equality between  $\sum_{w^j \in K^{i'}} \alpha_j$  and  $\sum_{w^j \in K_J^i} \alpha_j$  because the vertices in  $K_J^i$  are the only ones with non-zero  $\alpha_j$  values, given that  $J$  is a basis. Hence  $f$  is strongly dominating, and a fractional kernel of  $D$ .  $\square$

## 8.2 Fractional Stable Matchings

The last application we examine is an abstraction of stable matchings to hypergraphs. First we review some definitions, which come from [1]. See [10] for a full graph theory introduction.

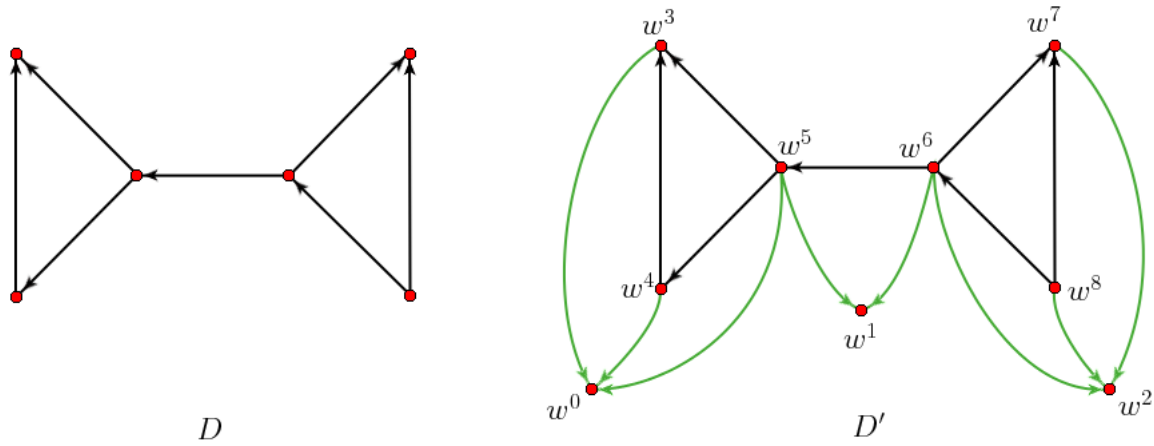


Figure 8.3: Example of  $D'$  construction

Table 8.1: Appropriate  $B$  and  $C$  matrices for Figure 8.3

$$B = \left[ \begin{array}{ccc|cccccc} w^0 & w^1 & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 & w^8 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right]$$

$$C = \left[ \begin{array}{ccc|cccccc} w^0 & [w^1] & w^2 & w^3 & w^4 & [w^5] & w^6 & w^7 & [w^8] \\ 0 & 14 & 13 & 1 & 2 & [3] & 9 & 8 & 7 \\ 15 & [0] & 13 & 12 & 11 & 1 & 2 & 8 & 7 \\ 15 & 14 & 0 & 12 & 11 & 10 & 2 & 1 & [3] \end{array} \right]$$

Here are example matrices  $B$ ,  $C$  for the graph  $D'$  in Figure 8.3. We have  $|V| = 6$ ,  $N = 8$ , so pick  $M = 6 + 8 + 1 = 15$ .

## 8.2.1 Background

Recall that in a graph  $G = (V, E)$ , a **matching**  $M$  is a subset of  $E$  such that each vertex in  $V$  is incident to no more than one edge in  $M$ .

Let  $<_v$  be a linear order on the edges incident with vertex  $v$ . A matching is said to be **stable** if for every edge  $e$  not in the matching, there exists an edge  $m$  in the matching which meets  $e$  at  $v$  and where  $e <_v m$ . The origins of stable matchings are in the stable marriage problem: given a set of men, a set of women and pairs of male-female acquaintances, can we marry acquainted men and women so that of all pairs that are acquainted but not married, at least one person in the pair prefers their spouse.

Gale and Shapley ([14]) proved that for a bipartite graph, such a stable matching always exists.

Stable matchings cannot always be found in general graphs, for example a 3-cycle  $\{a, b, c\}$  with  $\{a, c\} <_a \{a, b\}$ ,  $\{a, b\} <_b \{b, c\}$  and  $\{b, c\} <_c \{a, c\}$  admits no stable matching. If we allow the matching to be fractional however, we can find a fractional stable matching, as proved in [33].

**Definition 8.7.** A **fractional matching** is a non-negative function  $f$  on the edges of a graph  $G$  such that for all  $v \in V$ , 
$$\sum_{v \text{ incident to } e} f(e) \leq 1.$$

A fractional matching is said to be **stable** if each edge  $e$  has an endpoint  $v$  for which 
$$\sum_{v \text{ incident to } e', e \leq_v e'} f(e') = 1.$$
 Here  $e \leq_v e'$  means  $e <_v e'$  or  $e = e'$ .

For the example above, a function  $f$  which assigns 1/2 to every edge is a fractional stable matching. We can extend these definitions to hypergraphs.

**Definition 8.8.** A **hypergraph**  $H$  is a pair  $(V, E)$  where  $V$  is a set of **vertices** and  $E$  is a set of subsets of  $V$  called **hyperedges**.

Note that in a hypergraph, edges link a family of vertices, as opposed to the traditional 2 vertices.

**Definition 8.9.** A **hypergraphic preference system** is a pair  $(H, \mathcal{O})$ , where  $H = (V, E)$  is a hypergraph, and  $\mathcal{O} = \{<_v: v \in V\}$  is a family of linear orders  $<_v$ . Each linear order  $<_v$  acts on the set of edges containing the vertex  $v$ .



**Definition 8.10.** A non-negative function  $f$  on the hyperedges of  $H$  is said to be a **fractional matching** if  $\sum_{v \in h} f(h) \leq 1$  for all vertices  $v$ . Further, we say a fractional matching is **stable** if every hyperedge  $e$  contains a vertex  $v$  such that  $\sum_{v \in h, e \leq_v h} f(h) = 1$ .

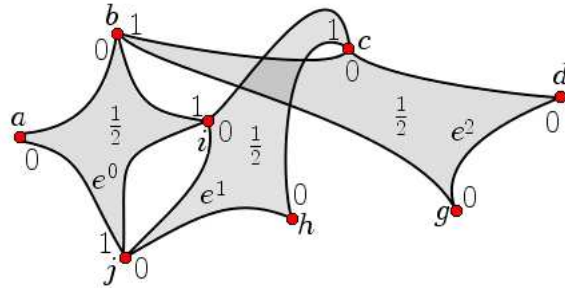


Figure 8.4: Fractional stable matching on a hypergraph

For example, Figure 8.4 shows a hypergraph on 8 vertices and 3 hyperedges. Each vertex has an ordering of the edges to which it is incident. The function  $f$  associated with each edge, namely  $f(e^0) = f(e^1) = f(e^2) = 1/2$ , is a fractional stable matching. Since no vertex lies in more than 2 edges, it is easy to see  $f$  is a fractional matching. To see it is stable, note that for  $e^0$ , there exists a vertex,  $b$ , for which  $\sum_{b \in h, e^0 \leq_b h} f(h) = f(e^0) + f(e^1) = 1$ . Similarly, there exists such a vertex,  $i$  or  $j$ , for  $e^1$  and one for  $e^2$ , namely  $c$ .

## 8.2.2 Fractional Stable Matchings in Hypergraphs

We will show, using Scarf's Extension Theorem, that there exists a fractional stable matching for every hypergraphic preference system.

**Theorem 8.11** ([1, Theorem 2.1]). *In any hypergraphic preference system, there exists a fractional stable matching.*

*Proof.* Let  $(H, \mathcal{O})$  be a hypergraphic preference system, and say that the hypergraph  $H$  is on  $n + 1$  vertices and  $N - n$  hyperedges.

Let  $B'$  be the incidence matrix of  $H$ , namely the  $(n + 1) \times (N - n)$  matrix where the rows correspond to vertices of  $H$  and the columns to hyperedges. If a vertex  $v$  belongs to a hyperedge  $e$ , then  $b'_{ve} = 1$ , and 0 otherwise. Now let  $B$  be  $B'$  with an identity matrix adjoined to its left.

Let  $C'$  be the  $(n + 1) \times (N - n)$  matrix defined as follows:

1. If the vertex  $v$  lies in hyperedge  $e$ , then let  $c'_{ve}$  be the height of  $e$  in the linear order  $<_v$ .
2. If the vertex  $v$  does not lie in some hyperedge  $e$ , then let  $c'_{ve} = M - j$ , where  $M \geq |V| + 2|E|$  and  $j$  is the index of the column of  $e$  plus  $|V|$ .

Construct  $C$  by adjoining a square matrix to the left of  $C'$ , ensuring the conditions of Theorem 7.7 are met. For instance let the diagonal entries be  $-1$  and the off diagonal entries be integers  $M - j$ , where  $j$  is the index of the column.

Finally, let  $\vec{b} = \vec{1}$ , where  $\vec{1}$  is the vector of all ones.

For the example in Figure 8.4, the appropriate matrices  $B$ ,  $C$  are outlined in Table 8.2.

Now check that  $B$ ,  $C$  satisfy the conditions of Theorem 7.7. Both  $B$  and  $C$  are  $(n + 1) \times (N + 1)$  matrices.

By construction, the first  $n+1$  columns of  $B$  form an identity matrix. Further, all entries of  $B$  are from  $\{0, 1\}$ , and there are no columns of zeros, as every hyperedge (column) contains at least one vertex. Since  $\vec{b} = \vec{1}$ , the set  $\{\vec{x} \in \mathbb{R}_+^{N+1} | B\vec{x} = \vec{b}\}$  is bounded by Theorem 6.6.

The matrix  $C$  has  $c_{ii} < c_{ik} < c_{ij}$  for  $0 \leq i, j \leq n, i \neq j$  and  $k > n$  by construction. Further,  $M \geq |V| + 2|E|$  and  $j \leq |V| + |E|$ , we have  $M - j \geq |E|$  and so the entries from vertices not in a particular hyperedge do not interfere with the entries which are the height of a vertex in a linear order. Therefore the Non-Degeneracy Assumption is met because all entries in a row are distinct.

We can apply Theorem 7.7 and find at least one subset  $J$  of  $\{0, 1, \dots, N\}$  with  $|J| = n+1$  which is feasible for  $B$  and ordinal for  $C$ . Associated with  $J$  is a basic feasible solution  $\vec{\alpha} \in \mathbb{R}_+^{N+1}$  to  $B\vec{x} = \vec{b}$ .

First we restrict  $\vec{\alpha}$  to the hyperedges  $E$ , and denote it  $\vec{\alpha}_{|E}$ . Let  $k$  be a column of  $B'$ , and  $e^k$  be the hyperedge represented by  $k$ . Define  $f(e^k) = \alpha_k$ .

Table 8.2: Appropriate matrices  $B$  and  $C$  for Figure 8.4

$$B = \left[ \begin{array}{c|cccccccc|ccc} a & 1 & & & & & & & & 1 & & & \\ b & & 1 & & & & & & & 1 & 1 & & \\ c & & & 1 & & & & & & & 1 & 1 & \\ d & & & & 1 & & & & & & & 1 & \\ g & & & & & 1 & & & & & & & 1 \\ h & & & & & & 1 & & & & 1 & & \\ i & & & & & & & 1 & & 1 & 1 & & \\ j & & & & & & & & 1 & 1 & 1 & & \end{array} \right]$$

$$C = \left[ \begin{array}{c|cccccccc|ccc} & [a] & b & c & [d] & [g] & [h] & i & j & [e^0] & [e^1] & [e^2] \\ \hline a & -1 & 14 & 13 & 12 & 11 & 10 & 9 & 8 & 0 & 6 & 5 \\ b & 15 & -1 & 13 & 12 & 11 & 10 & 9 & 8 & 0 & 6 & 1 \\ c & 15 & 14 & -1 & 12 & 11 & 10 & 9 & 8 & 7 & 1 & 0 \\ d & 15 & 14 & 13 & -1 & 11 & 10 & 9 & 8 & 7 & 6 & 0 \\ g & 15 & 14 & 13 & 12 & -1 & 10 & 9 & 8 & 7 & 6 & 0 \\ h & 15 & 14 & 13 & 12 & 11 & -1 & 9 & 8 & 7 & 0 & 5 \\ i & 15 & 14 & 13 & 12 & 11 & 10 & -1 & 8 & 1 & 0 & 5 \\ j & 15 & 14 & 13 & 12 & 11 & 10 & 9 & -1 & 1 & 0 & 5 \end{array} \right]$$

We claim that  $f$  is a fractional stable matching.

To show  $f$  is a fractional matching, first note that  $f$  is non-negative. Secondly, we require that for all  $v \in V$ ,  $\sum_{v \text{ incident to } e} f(e) \leq 1$ . That is equivalent to multiplying the row representing  $v$  of  $B'$  by  $\vec{\alpha}_{|E}$ . Yet we have  $B\vec{\alpha} = \vec{1}$ , and so restricting to  $E$ , we have  $B'\vec{\alpha}_{|E} \leq B\vec{\alpha} = \vec{1}$ .

To show  $f$  is stable, we must show that for every hyperedge  $e$ , there exists a vertex  $v$  such that  $\sum_{v \in h, e \leq_v h} f(h) = 1$ . Because  $J$  is an ordinal basis for  $C$ ,  $e$  must be dominated at some row (i.e., vertex)  $v$  by every column in  $J$ .

**Claim 8.11.1.** *The vertex  $v$  is a vertex in  $e$  for which  $\sum_{v \in h, e \leq_v h} f(h) = 1$ .*

*Proof.* First we show  $v \in e$ .

Note that the column  $v$  cannot be in  $J$ . This is because  $c_{vv} = -1$  is minimal over the entire row, and therefore cannot dominate  $e$  at  $v$ . However, if  $v \notin J$ , in the equation  $B\vec{\alpha} = \vec{1}$  we get no contribution from  $b_{vv} = 1$ . Hence there must be some edge in  $J$  that contains  $v$ , say edge  $h$ .

Since  $h \in J$ ,  $h$  must dominate  $e$  at  $v$  ( $c_{ve} \leq c_{vh}$ ) and so by Condition 2 above,  $v \in e$ , otherwise we would have  $c_{ve} > c_{vh}$ .

Therefore any column  $h$  that contributes to  $B\vec{\alpha} = \vec{1}$  is an edge of  $H$  and dominates  $e$  at  $v$ , and so we have  $\sum_{v \in h, e \leq_v h} f(h) = 1$ . □

Thus  $f$  is a fractional stable matching. □

### 8.3 A Second Proof for Fractional Stable Matchings

It is in fact possible to abstract Theorem 8.6 of Section 8.1 and from it derive Theorem 8.11 from Section 8.2.

From the hypergraphic preference system  $(H, \mathcal{O})$ , construct a digraph  $D = (V, A)$  as follows:

- For every hyperedge  $e$  of  $H$ , add a vertex  $v_e$  to  $V$ .
- For any two vertices  $v_e, v_f$  of  $D$ , send an arc from  $v_e$  to  $v_f$  if there exists a vertex  $w$  in  $H$  common to  $e$  and  $f$  and  $f <_w e$ .

See Figure 8.5 for an example construction of  $D$ .

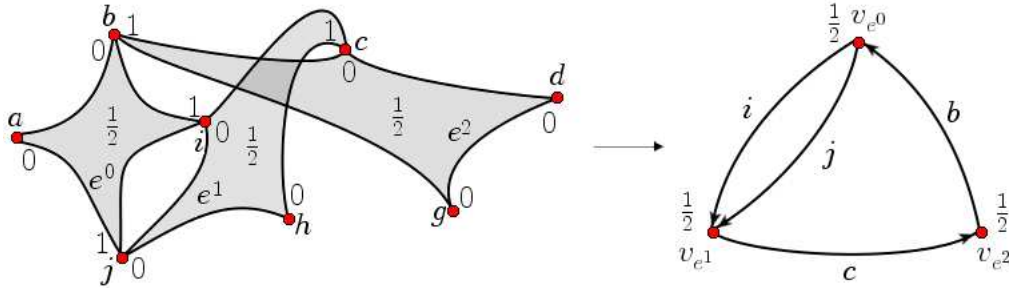


Figure 8.5: Constructing a directed graph from a hypergraphic preference system

We make the following claim.

**Claim 8.12.** *Each vertex of  $H$  induces an acyclic clique in  $D$ .*

*Proof.* Let  $v$  be any vertex of  $H$ , and say it is incident to  $k$  hyperedges,  $e^1, e^2, \dots, e^k$ . If  $k = 1$ , then  $v$  will induce no directed edges in  $D$ . Otherwise, there exists a linear order  $<_v$  on all the  $e^i$ , say  $e^k <_v e^{k-1} <_v \dots <_v e^1$ . In  $D$ , each  $e^i$  corresponds to a vertex  $v_{e^i}$ , and since the hyperedges  $e^i$  share a vertex  $v$  in  $H$ , there are directed edges between them in  $D$ , corresponding to the linear order  $<_v$ .

Namely, there is a directed edge from  $v_{e^k}$  to each of  $\{v_{e^{k-1}}, \dots, v_{e^1}\}$ , a directed edge from  $v_{e^{k-1}}$  to each of  $\{v_{e^{k-2}}, \dots, v_{e^1}\}$ , and so on until we have a directed edge between  $v_{e^2}$  and  $v_{e^1}$ .

Clearly the vertices  $v_{e^1}, v_{e^2}, \dots, v_{e^k}$ , form a clique, and further it is acyclic because the edges are always directed from  $v_{e^i}$  to  $v_{e^j}$ , where  $i > j$ .  $\square$

There may be “incidental” cliques formed in  $D$ , namely cliques that do not correspond to a vertex of  $H$ . For example in Figure 8.5 there are 2 3-cliques which are not acyclic, but neither corresponds to a vertex of  $H$ . Hence we can modify Theorem 8.6 to focus only on the cliques in which we are interested. We state this abstraction here.

**Theorem 8.13.** *Let  $D = (V, A)$  be a digraph that is a union of acyclic cliques  $K^1, K^2, \dots, K^t$ . Then  $D$  has a strong fractional kernel with respect to these specified cliques.*

Here by a strong fractional kernel with respect to the cliques  $K^1, K^2, \dots, K^t$ , we mean a non-negative function  $f$  on  $V$  which satisfies the following two conditions:

1. **The function  $f$  is strongly dominating:** for every vertex  $v$  in  $V$ , there exists  $j$  such that  $\sum_{u \in K} f(u) \geq 1$  for some clique  $K \in I(v) \cap K^j$ .
2. **The function  $f$  is fractionally independent:** for every clique  $K^j$ ,  $1 \leq j \leq t$ ,  $\sum_{u \in K^j} f(u) \leq 1$ .

The proof is the same as before, except that we consider the maximal cliques within this list as opposed to all maximal cliques in the directed graph.

Now we have this final claim.

**Claim 8.14.** *Let  $D$  be the directed graph constructed as above from a hypergraphic preference system  $(H = (V, E), \mathcal{O})$ , and let  $\{K_v : v \in V\}$  be the cliques of  $D$  corresponding to vertices in  $H$ . Then a strong fractional kernel with respect to the cliques  $\{K_v : v \in V(H)\}$  is a fractional stable matching for  $(H, \mathcal{O})$ .*

*Proof.* Let  $v_{e^1}, \dots, v_{e^k}$  be an enumeration of the vertices of  $D$ , and let  $v^1, \dots, v^t$  be an enumeration of the vertices of  $H$ . Then by Theorem 8.13, we can find a strong fractional kernel with respect to the cliques  $K_{v^1}, \dots, K_{v^t}$ . Namely, we have a non-negative function  $f$  on the vertices of  $D$  so that:

1. For every vertex  $v_{e^i}$ ,  $i = 1, \dots, k$ , there exists  $j$  such that  $\sum_{u \in K} f(u) \geq 1$  for some clique  $K$  in  $I(v_{e^i}) \cap K_{v^j}$ .
2. For every clique  $K_{v^j}$ ,  $j = 1, \dots, t$ ,  $\sum_{u \in K_{v^j}} f(u) \leq 1$ .

Recall that every vertex  $v_{e^i}$  of  $D$  corresponds to a hyperedge  $e^i$  of  $H$ , and that each clique  $K_{v^j}$  of  $D$  corresponds a vertex  $v^j$  of  $H$ . The in-neighbourhood of a vertex  $v_{e^i}$  of

$D$  is  $v_{e^i}$  along with all vertices  $v_e$  sending an arc to  $v_{e^i}$ . Therefore in  $H$  it represents the hyperedge  $e^i$  along with all hyperedges  $e$  which are higher than  $e^i$  in the linear order of some vertex  $v$  in  $e^i$ . When we restrict the in-neighbourhood to a particular clique  $K_{v^j}$  in  $D$ , we are in  $H$  restricting to all hyperedges  $e$  which are higher than  $e^i$  in the linear order of  $v^j$ . Therefore in  $H$ , we have a non-negative function  $f$  on the hyperedges of  $H$  such that

1. For every hyperedge  $e^i$ ,  $i = 1, \dots, k$ ,  $\sum_{e^i \leq_{v^j} e} f(e) \geq 1$  for some vertex  $v^j$  in  $e^i$ .
2. For every vertex  $v^j$ ,  $j = 1, \dots, t$ ,  $\sum_{e \text{ incident to } v^j} f(e) \leq 1$ .

This is almost a fractional stable matching for  $H$ . Recall that we want a non-negative function  $f$  for which:

1. Every hyperedge  $e$  contains a vertex  $v$  such that  $\sum_{v \in h, e \leq_v h} f(h) = 1$ , and
2. For every vertex  $v$ ,  $\sum_{v \in h} f(h) \leq 1$ .

But equality holds in the first condition, as was shown in equation (8.2). □

Hence we have shown the existence of a fractional stable matching in a hypergraph is implied by the existence of a strong fractional kernel in an associated directed graph.

# Appendix A

## Example of Stable Fractional Matchings in Hypergraphs

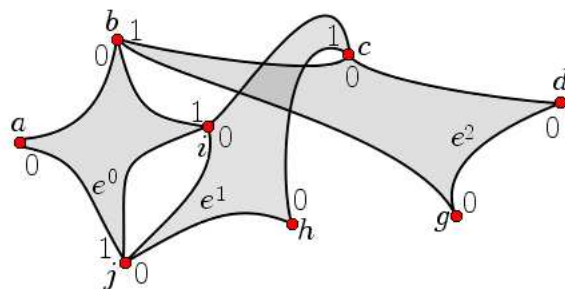


Figure A.1: Example hypergraph with preference sets

We will find a fractional matching for the hypergraphic preference system  $(H, \mathcal{O})$  in Figure A.1, using Scarf's Extension Theorem. The purpose of this section is to show how Scarf's Extension Theorem works, and so this example was chosen because it is solved in a small number of steps while still giving a matching which is not integral.

We could solve it directly using the method of Section 8.2, but instead we will transform the system into a directed graph and find a strong fractional kernel using the method of Section 8.3.

Recall that to form the digraph  $D$  from a hypergraphic preference system  $(H, \mathcal{O})$ , we do the following:



- For every hyperedge  $e$  of  $H$ , add a vertex  $v_e$  to  $D$ .
- For any two vertices  $v_e, v_f$  of  $D$ , send an arc from  $v_e$  to  $v_f$  if there exists a vertex  $w$  in  $H$  common to  $e$  and  $f$  and  $f <_w e$ .

Applying this to the hypergraph of Figure A.1, we get Figure A.2.

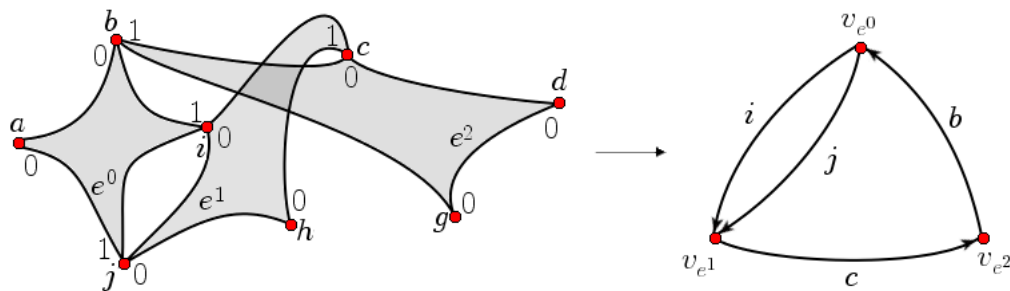


Figure A.2: Transforming a hypergraphic preference system into a directed graph

The cliques in which we are interested are the ones corresponding to vertices of  $H$ , namely  $i, j, b$  and  $c$ . Given that  $i$  and  $j$  represent the same clique, we will consider only one of them, say  $i$ . Note further that  $D$  contains a 3-clique which is not acyclic. Since it does not correspond to a vertex of  $H$ , we can still apply Theorem 8.13. According to the proof of this theorem, we add 3 vertices to  $D$ , one for each clique, and send arcs from all vertices in each clique to these new vertices. See Figure A.3. We also give a linear order on the vertices in each clique.

There are  $N + 1 = 6$  vertices and  $n + 1 = 3$  cliques. Recall that  $B$  is the  $3 \times 6$  clique-incidence matrix, while  $C$  is defined as follows. Let  $w$  be a vertex of  $D$  and  $K$  a clique of  $D$ . Let  $c$  be the entry in the column for  $w$  and the row for  $K$  in  $C$ .

1. If  $w \notin K$  then  $c = M - j$ , where  $M$  is some number bigger than  $|V| + N = 3 + 5 = 8$ , and  $j$  is the index for the column  $w$ .
2. If  $w \in K$  then  $c$  is the height of  $w$  in the linear order of  $K$ .

Finally, we let  $\vec{b}$  be the vector of all 1s.

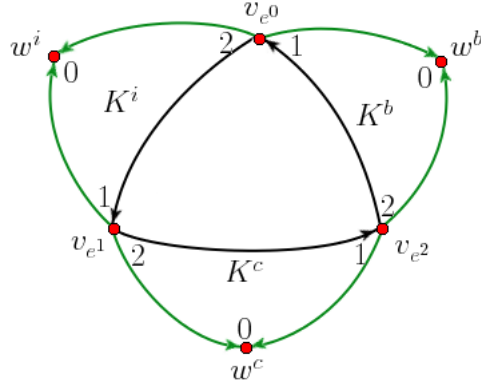


Figure A.3: Adding extra vertices for each clique of  $D$

For the directed graph of Figure A.3, here are matrices  $B, C$  that satisfy these conditions:

$$B = \left[ \begin{array}{c|ccc|ccc} & w^i & w^b & w^c & e^0 & e^1 & e^2 \\ \hline K^i & 1 & & & 1 & & 1 \\ K^b & & 1 & & 1 & 1 & \\ K^c & & & 1 & & 1 & 1 \end{array} \right]$$

$$C = \left[ \begin{array}{c|ccc|ccc} & w^i & w^b & w^c & e^0 & e^1 & e^2 \\ \hline K^i & 0 & 8 & 7 & 2 & 5 & 1 \\ K^b & 9 & 0 & 7 & 1 & 2 & 4 \\ K^c & 9 & 8 & 0 & 6 & 1 & 2 \end{array} \right]$$

We will find a stable fractional matching in  $H$  by finding a basis which is feasible for  $B$  and ordinal for  $C$  and an associated basic feasible solution  $\vec{x} \in R_+^6$  to  $B\vec{x} = \vec{b}$ . Then if we restrict  $\vec{x}$  to the hyperedges of  $H$ ,  $\vec{x}|_E$  will be a fractional stable matching.

## A.1 Scarf's Extension Algorithm

The first step is to find two bases  $J_B = \{w^i, j_1, j_2\}$  and  $J_C = \{j_0, j_1, j_2\}$  with  $j_0 \neq w^i$  where  $J_B$  is feasible for  $B$  and  $J_C$  is ordinal for  $C$ . The first is easy to find (the first 3 columns of  $B$  will do).

To find a basis which is ordinal for  $C$ , we start with the columns  $\{1, 2\}$  and add the column  $h$  from  $\{e^0, e^1, e^2\}$  for which  $c_{w^i h}$  is maximum. This is column  $e^1$ .

Hence our initial bases are

$$\begin{aligned} J_B &= \{w^i, w^b, w^c\} \\ J_C &= \{e^1, w^b, w^c\}. \end{aligned}$$

### Step 1: Add $e^1$ to $J_B$ .

$$RREF(B|\vec{b}) = [B|\vec{b}] = \left[ \begin{array}{c|ccc|ccc|c} & w^i & w^b & w^c & e^0 & e^1 & e^2 & \vec{b} \\ \hline K^i & 1 & & & 1 & & 1 & 1 \\ K^b & & 1 & & 1 & 1 & & 1 \\ K^c & & & 1 & & 1 & 1 & 1 \end{array} \right]$$

Note that our initial basis consists of the columns  $\{w^i, w^b, w^c\}$ , and so to find the lexicographical minimum, we will always look at the entries in these columns. To determine which column leaves the basis, we want to find the lexicographical minimum of the following 3-tuple:

$$\begin{array}{c|c} - & w^i \\ \hline \frac{(1,0,1,0)}{1} & w^b \\ \frac{(1,0,0,1)}{1} & w^c \end{array}$$

Column  $w^c$  leaves the basis. Swap columns  $w^c$  and  $e^1$ .

$$[B|\vec{b}] = \left[ \begin{array}{c|ccc|ccc|c} & w^i & w^b & e^1 & e^0 & w^c & e^2 & \vec{b} \\ \hline K^i & 1 & & & 1 & & 1 & 1 \\ K^b & & 1 & 1 & 1 & & & 1 \\ K^c & & & 1 & & 1 & 1 & 1 \end{array} \right]$$

The new pair of bases is

$$\begin{aligned} J_B &= \{w^i, w^b, e^1\} \\ J_C &= \{w^c, w^b, e^1\}. \end{aligned}$$

## Step 2: Remove $w^c$ from $J_C$

In doing so column  $e^1$  now has 2 row minimizers,  $u = K^i$  (old) and  $K^c$  (new).

$$C = \left[ \begin{array}{c|ccc|ccc} & w^i & [w^b] & w^c & e^0 & [e^1] & e^2 \\ \hline K^i & 0 & 8 & 7 & 2 & [5] & 1 \\ K^b & 9 & [0] & 7 & 1 & 2 & 4 \\ K^c & 9 & 8 & 0 & 6 & [1] & 2 \end{array} \right]$$

Next we examine all columns  $h$  with  $c_{ih} > \min\{c_{ij} | j = j_1, j_2\}$  for all  $i \neq u$ . These are  $\{w^i, e^0, e^2\}$ . Of those, we pick one that maximizes  $c_{uh} = c_{K^i h}$ . Hence column  $e^0$  enters the basis.

$$\begin{aligned} J_B &= \{w^i, w^b, e^1\} \\ J_C &= \{e^0, w^b, e^1\} \end{aligned}$$

## Step 3: Add $e^0$ to $J_B$ .

$$RREF(B|\vec{b}) = \left[ \begin{array}{ccc|ccc|c} & w^i & w^b & e^1 & e^0 & w^c & e^2 & \vec{b} \\ \hline K^i & 1 & & & 1 & & 1 & 1 \\ K^b & & 1 & & 1 & -1 & -1 & 0 \\ K^c & & & 1 & & 1 & 1 & 1 \end{array} \right]$$

To determine which column leaves the basis, we want to find the lexicographical minimum of the following 3-tuple:

$$\begin{array}{c|c} \frac{(1,1,0,0)}{(0,0,1,-1)} & w^i \\ \frac{1}{-} & w^b \\ & w^c \end{array}$$

Column  $w^b$  leaves the basis. Swap columns  $w^b$  and  $e^0$ .

$$[B|\vec{b}] = \left[ \begin{array}{c|ccc|ccc|c} & w^i & e^0 & e^1 & w^b & w^c & e^2 & \vec{b} \\ \hline K^i & 1 & 1 & & & & 1 & 1 \\ K^b & & 1 & & 1 & -1 & -1 & 0 \\ K^c & & & 1 & & 1 & 1 & 1 \end{array} \right]$$

The new pair of bases is

$$\begin{aligned} J_B &= \{w^i, e^0, e^1\} \\ J_C &= \{w^b, e^0, e^1\}. \end{aligned}$$

#### Step 4: Remove $w^b$ from $J_C$

In doing so column  $e^0$  now has 2 row minimizers,  $u = K^i$  (old) and  $K^b$  (new).

$$C = \left[ \begin{array}{c|ccc|ccc} & w^i & w^b & w^c & [e^0] & [e^1] & e^2 \\ \hline K^i & 0 & 8 & 7 & [2] & 5 & 1 \\ K^b & 9 & 0 & 7 & [1] & 2 & 4 \\ K^c & 9 & 8 & 0 & 6 & [1] & 2 \end{array} \right]$$

Next we examine all columns  $h$  with  $c_{ih} > \min \{c_{ij} | j = j_1, j_2\}$  for all  $i \neq u$ . These are  $\{w^i, e^2\}$ . Of those, we pick one that maximizes  $c_{uh} = c_{K^i h}$ . Hence column  $e^2$  enters the basis.

$$\begin{aligned} J_B &= \{w^i, e^0, e^1\} \\ J_C &= \{e^2, e^0, e^1\} \end{aligned}$$

#### Step 5: Add $e^2$ to $J_B$ .

$$RREF(B|\vec{b}) = \left[ \begin{array}{c|ccc|ccc|c} & w^i & e^0 & e^1 & w^b & w^c & e^2 & \vec{b} \\ \hline K^i & 1 & & & -1 & 1 & 2 & 1 \\ K^b & & 1 & & 1 & -1 & -1 & 0 \\ K^c & & & 1 & & 1 & 1 & 1 \end{array} \right]$$

To determine which column leaves the basis, we want to find the lexicographical minimum of the following 3-tuple:

$$\begin{array}{c|c} \frac{(1,1,-1,1)}{2} & w^i \\ - & w^b \\ \frac{(1,0,0,1)}{1} & w^c \end{array}$$

Column  $w^i$  leaves the basis, and we are done. To find the associated solution, we first swap columns  $w^i$  and  $e^2$ .

$$[B|\vec{b}] = \left[ \begin{array}{c|ccc|ccc|c} & e^2 & e^0 & e^1 & w^b & w^c & w^i & \vec{b} \\ \hline K^i & 2 & & & -1 & 1 & 1 & 1 \\ K^b & -1 & 1 & & 1 & -1 & & 0 \\ K^c & 1 & & 1 & & 1 & 1 & 1 \end{array} \right]$$

Finally, row reduce  $[B|\vec{b}]$ :

$$RREF(B|\vec{b}) = \left[ \begin{array}{c|ccc|ccc|c} & e^2 & e^0 & e^1 & w^b & w^c & w^i & \vec{b} \\ \hline K^i & 1 & & & -0.5 & 0.5 & 0.5 & 0.5 \\ K^b & & 1 & & 0.5 & -0.5 & 0.5 & 0.5 \\ K^c & & & 1 & 0.5 & 0.5 & -0.5 & 0.5 \end{array} \right]$$

So the final solution is:

$$\alpha = \begin{array}{c|c} 0 & w^i \\ 0 & w^b \\ 0 & w^c \\ 0.5 & e^0 \\ 0.5 & e^1 \\ 0.5 & e^2 \end{array} .$$

And so restricting to the edges of  $H$ , we get a fractional stable matching where each edge has weight  $\frac{1}{2}$ . This is the same as the solution we found in Section 8.2.1.

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