Algebraic Analysis of Vertex-Distinguishing Edge-Colorings

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Vertex-distinguishing edge-colorings (vdec colorings) are a restriction of proper edge-colorings. These special colorings require that the sets of edge colors incident to every vertex be distinct. This is a relatively new field of study. We present a survey of known results concerning vdec colorings. We also define a new matrix which may be used to study vdec colorings, and examine its properties. We find several bounds on the eigenvalues of this matrix, as well as results concerning its determinant, and other properties. We finish by examining related topics and open problems.

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Chapter 1

Introduction

Graph colorings are among the most often explored topics in graph theory. They are particularly appealing because there are so many variations, each of which may be useful in describing physical systems (such as computer networks) or other combinatorial problems. This thesis is the result of the author's interest in a particular type of edge-coloring, as well as the algebraic aspects of graph coloring. The goal of this thesis is to develop some useful techniques and structures for the study of vertex-distinguishing edge-colorings, and to explore their properties.

1.1 Contributions and Definitions

The main contributions of this thesis are a survey of work done on vertex-distinguishing edge-colorings, development of matrices and related structures useful in the analysis of such colorings, and exploration of the properties and uses of these structures.

We will primarily focus on proper edge-colorings. Intuitively, a proper edge-coloring is an assignment of colors to the edges of a graph such that no adjacent edges share a color. Formally, we may define a proper edge-coloring as follows.

Definition 1.1.1. *Let $G = (V, E)$ be a graph and C be a set of colors. A* proper edge-coloring *π of G is a function $\pi : E \to C$ such that if $vu, vw \in E$ with $u \neq w$, then $\pi(vu) \neq \pi(vw)$.*

Throughout this thesis, every edge-coloring will be assumed to be proper, unless otherwise specified.

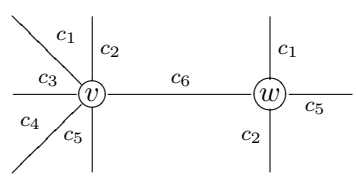
In addition, we assume that all graphs are simple undirected graphs. Other graph-theoretic terms will be as in [13]. We will usually work with a further restriction:

Definition 1.1.2. *A* vertex-distinguishing edge-colorable graph *(or* vdec graph*) is a (possibly disconnected) graph G with no isolated edges and at most one isolated vertex.*

Definition 1.1.2 is the standard definition of a vdec graph, and allows disconnected graphs. However, in this thesis all vdec graphs are assumed to be connected unless otherwise stated. In addition, a *nontrivial vdec graph* is a vdec graph with at least three vertices. The reason for these restrictions will soon become apparent.

Definition 1.1.3. *Let $G = (V, E)$ be a graph with a proper edge-coloring π. Let $v \in V$. Then the* incident color set *to v, denoted $S(v)$ is defined by $S(v) = \{\pi(vw) : vw \in E\}$.*

In other words, $S(v)$ is the set of colors used on edges incident to v. For clarity, we may specify a particular coloring π, using the notation $S_\pi(v)$. Note that, because G has a proper edge-coloring, $S(v)$ is never a multiset, and $|S(v)| = \deg(v)$. If two vertices v and w have distinct sets $S(v)$ and $S(w)$, they are said to be *distinguished*. For an example, see Figure 1.1.1.



$S(v) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ $\quad S(w) = \{c_1, c_2, c_5, c_6\}$

Figure 1.1.1: Incident color sets

Definition 1.1.4. *Let $G = (V, E)$ be a vdec graph. A* vertex-distinguishing edge-coloring *(or* vdec coloring*) C of G is a proper edge-coloring π of G such that, for all pairs of distinct vertices $v, w \in V$, $S_\pi(v) \neq S_\pi(w)$.*

If G has an isolated edge vw, then $S(v) = S(w)$ always, and G cannot have a vdec coloring. Similarly, if G has two isolated vertices v, w then $S(v) = S(w) = \emptyset$. However, every vdec graph must have a vdec coloring: simply assign a different color to each edge. Thus vdec graphs are exactly the

graphs which permit a vdec coloring. Figures 1.1.2 and 1.1.3 demonstrate vdec and non-vdec colorings of some graphs.

A coloring in which each edge is assigned a different color is a *maximum coloring*. A vdec coloring which uses the minimum number of edge colors among all vdec colorings of G is called a *minimum coloring*, and the number of colors required for a minimum coloring is denoted $\chi'_s(G)$.

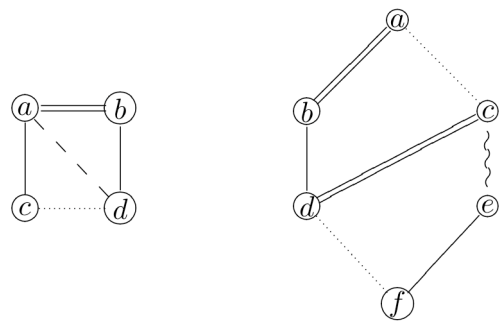


Figure 1.1.2: Two vdec colored graphs

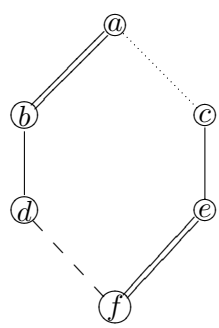


Figure 1.1.3: A non-vdec colored graph (b and e are not distinguished)

We often need to refer to the number of vertices of a given degree. We use the notation n_k to denote the number of vertices of degree k in a graph.

Finally, we often need to refer to the maximum or minimum degree of the vertices of a graph. We use Δ and δ, respectively, to denote these quantities. It will be assumed that Δ refers to the graph currently under discussion, unless this would be ambiguous. We then use $\Delta(G)$ to specify. Similarly, n and e will always be the number of vertices and edges, respectively, of the graph at hand.

1.2 Terminology Used in Other Works

The terminology used to describe vdec colorings is rather inconsistent. Our basic terminology comes from Burris and Schelp [10] who, among others, prefer the term "vertex-distinguishing edge-coloring," and refer to a graph with such a coloring as a vdec graph. There is no standard term to describe the coloring itself. The terms "Vertex-Distinguishing Proper (VDP)" and "vdec" have been used, with capitalization as presented here. More common is "strong edge-coloring" or simply "strong coloring" to refer to the same type of coloring. This, and the standard notation χ' for proper edge-colorings in turn provide the notation χ'_s. However, some papers use the term "vertex-distinguishing index" (vdi) or $\tilde{\chi}'$ for the same purpose. The "observability" of a graph is equivalent to being a vdec graph, and leads to the terminology $\mathrm{obs}(G)$, which is identical to $\chi'_s(G)$. "Irregular assignments" are a similar idea which explicitly admit improper edge-colorings.

One must be careful when searching for papers on the topic of strong edge-colorings. The term "strong edge-coloring" is also used to refer to proper edge-colorings, as opposed to general edge-colorings which permit adjacent edges to have the same color. Similarly, at least one paper uses "vertex-distinguishing edge-coloring" to permit improper edge-colorings.

In this work, we will use the term "vdec" to refer both to a graph which admits a vertex-distinguishing edge-coloring, and (more often) to refer to the coloring itself. The specific term "vdec graph" or "vdec coloring" will be used if there is any chance of ambiguity. We recognize the redundancy in the term "vdec coloring," but the extra word helps to clarify the meaning.

1.3 Outline

This thesis examines several main areas.

- Chapter 1 introduces vertex-distinguishing edge-colorings and presents definitions.
- Chapter 2 is a survey of previous work, including basic results, important conjectures and theorems, and techniques used in other works.
- Chapter 3 introduces the VCICM, a matrix useful in studying vdec colorings, and examines its determinant and related properties.
- Chapter 4 examines the eigenvalues and eigenvectors of the VCICM, and gives several tight bounds.

- Chapter 5 introduces some variations on the colorings examined in this thesis, states related results, and lists some related work.
- Chapter 6 presents some conclusions and remarks on future research directions.

Chapter 2

Survey of Results

The idea of a vdec coloring as we use it was first defined by Burris and Schelp in 1997 [10]. Both Burris and Schelp have expanded on the idea in several papers, and made the main conjectures and theorems seen in this chapter. Several authors have continued to refine bounds and produce new results for vdec colorings.

In this chapter, we introduce some basic results, most importantly the value of $\chi'_s(G)$ for some standard classes of graphs. We will see some important theorems and conjectures related to χ'_s, and finish with a summary of techniques and bounds related to these results.

2.1 Basic Results and Techniques

In their initial paper on the subject, Burris and Schelp provided values of χ'_s for many standard classes of graphs, along with basic techniques to color some of these graphs.

Since vdec colorings are proper edge-colorings, we deduce that $\chi'_s(G) \geq \Delta$: each edge incident to a maximum-degree vertex must have a different color. If there are two or more vertices of degree Δ, then $\chi'_s(G) \geq \Delta + 1$ (if there were only Δ colors, then all vertices of degree Δ would have the same incident color sets). Often, we show that a given coloring is minimum by showing that the coloring meets one of these lower bounds.

There are many standard classes of graphs for which exact results are known. These are versions of Propositions 8, 9, and 10 in Burris and Schelp's original paper [10], presented with proofs created by the author of this thesis:

- *Complete Graphs:* We will consider minimum colorings of complete

graphs in some detail, as they demonstrate some particularly useful coloring techniques.

Let $n \geq 3$. Then

$$\chi'_s(K_n) = \begin{cases} n & \text{if } n \text{ is odd} \\ n+1 & \text{if } n \text{ is even} \end{cases}$$

From above, $\chi'_s(K_n) \geq n$ for all n. This occurs because $\Delta = n - 1$ for all K_n, and there are always at least two maximum-degree vertices when $n \geq 2$. We only consider K_n where $n \geq 3$. There is only one vdec coloring of K_3, which is a maximum coloring. Thus, let $k \geq 3$. To color K_{2k-1} and K_{2k-2}, we require the following 1-factorization of K_{2k}: arrange $2k-1$ vertices in a cycle, and place the remaining vertex in the center of the cycle. Each 1-factor consists of an edge from the center vertex to one other vertex, along with all edges perpendicular to this edge (see Figure 2.1.1).

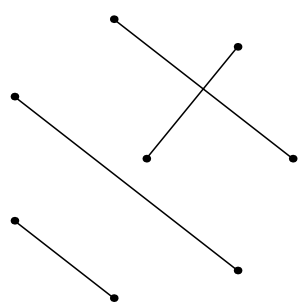


Figure 2.1.1: A sample 1-factor

Color each 1-factor with a different color. This results in a proper edge-coloring of K_{2k} with $2k-1$ colors, in which all vertices have the same color set. Now, delete one vertex v from the outside cycle. This removes a different color from each vertex, so all pairs of vertices are now distinguished. This colors K_{2k-1} with $2k-1$ colors. Because $\Delta = 2k-2$ and $n = 2k-1$, this must be a minimum coloring.

To color K_{2k-2}, begin with the minimum colored K_{2k-1} produced above, and remove one vertex w from the outside cycle, selected so that v and w are adjacent in the outside cycle. This guarantees that in these two deletion steps, we do not remove the same two colors from two vertices. Note that this is a nontrivial fact, but requires a

very long explanation which would not significantly clarify this section. Thus, all pairs of vertices are still distinguished. Since we began with $k \geq 3$, there is at least one edge of each color. We have not totally eliminated any color from the graph, and thus this is a vdec coloring for K_{2k-2} using $2k-1$ colors.

To show minimality of the above coloring, suppose there is a vdec coloring of K_{2k-2} using $2k-2$ colors. For each vertex v, treat $S(v)$ as a label on vertex v. Then there are $2k-2$ labels, each one a set of size $2k-3$, and each label must be distinct. There are exactly $\binom{2k-2}{2k-3} = 2k-2$ sets of size $2k-3$ with $2k-2$ colors. Thus there would be exactly enough labels to label each vertex with a different set. But, any one color c_i would appear in all but one set, that is, $2k-3$ sets. Since we are coloring edges, every color must appear in an even number of sets (twice for each edge on which it is used). But, $2k-3$ is odd, so a coloring with $2k-2$ colors is impossible. Thus $2k-1$ colors give a minimum coloring of K_{2k-2}.

Using sets $S(v)$ to label vertices is a useful technique later developed by Balister, Riordan, and Schelp [6]. This will be treated in more detail in Section 2.3.2.

- *Complete Bipartite Graphs*
  - $\chi'_s(K_{1,n}) = n$: There is only one proper edge-coloring of a star, and, for $n \geq 2$, this is also a vdec coloring. Stars often provide extremal examples related to vdec colorings.
  - $\chi'_s(K_{m,n}) = n+1$, where $n > m > 1$: Arrange the vertices in two columns. Let our color set be $C = \{c_1, \ldots, c_{n+1}\}$. We assign colors to the edges incident to the ith vertex of degree n as follows, listing the color for the uppermost edge first and moving downwards. The top vertex has, in order, colors $(c_1, \ldots, c_n)$. The second has $(c_2, \ldots, c_{n+1})$. The third has $(c_3, \ldots, c_{n+1}, c_1)$, and so on, "wrapping" colors in this way. This guarantees that the vertices of degree n are all distinguished. The vertices of degree m have color sets $\{c_1, c_2, \ldots, c_m\}, \{c_2, \ldots, c_{m+1}\}$, etc. Figure 2.1.2 gives an example of this coloring for $K_{3,4}$.
  - $\chi'_s(K_{n,n}) = n+2$, where $n \geq 2$, and let (A, B) be the bipartition of $G = K_{n,n}$. In vertex class A, consider the ith vertex. We assign colors to the edges incident to this ith vertex from the following set, starting at the top and moving downwards: $(c_i, c_{i+1}, \ldots)$,

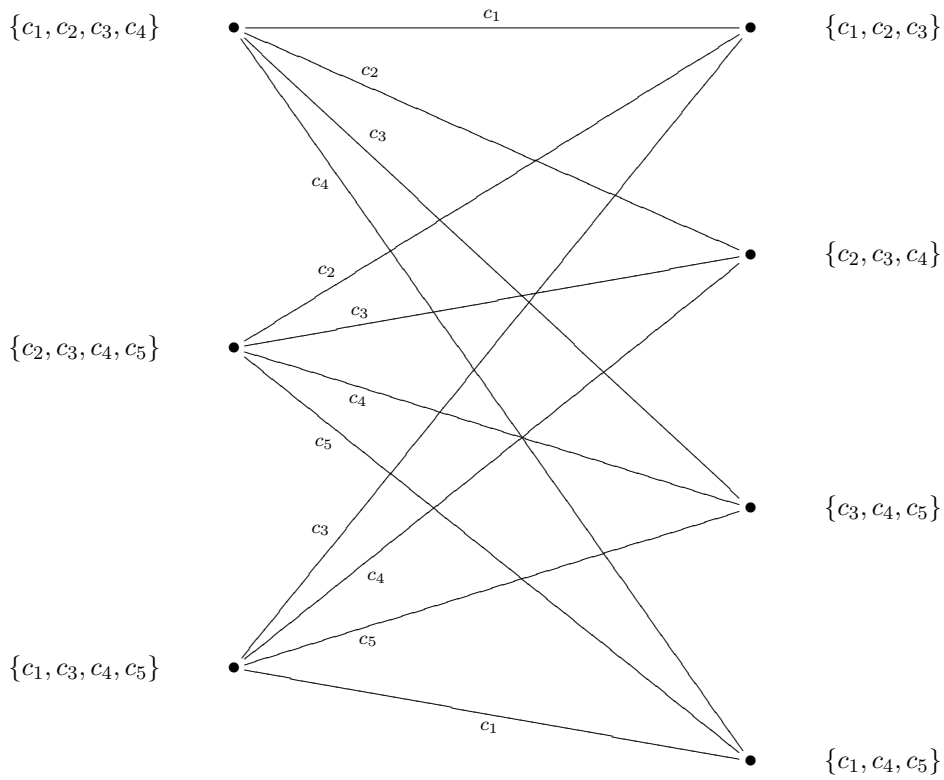


Figure 2.1.2: Minimum vdec coloring of $K_{3,4}$

wrapping around if necessary (as above). But, assign the nth vertex in the class the colors $(c_{n+1}, c_{n+2}, c_1, \ldots, c_{n-2})$. This distinguishes all vertices in A. The ith vertex in B has an incident color set containing exactly the same colors as the ith vertex in A, except for one element. The last vertex in each color class is similarly distinguished.

For $K_{m,n}$ and $K_{n,n}$, we have at least two vertices of maximum degree, so $\chi'_s(G) \geq n+1$. For $K_{n,n}$ in particular, $n+1$ colors give at most $\binom{n+1}{n} = n+1$ color sets, not enough to give each of the $2n$ vertices a distinct set. Thus, these are minimum values of χ'_s.

- *Paths*: Let $n \geq 3$, and set j to be the smallest positive integer such that $\binom{j}{2} \geq n-2$. Then

$$\chi'_s(P_n) = \begin{cases} j+1, & \text{if } j \text{ is odd and } n = (j^2-j+4)/2 \text{ or} \\ & j \text{ is even and } n > (j^2-2j+6)/2 \\ j, & \text{otherwise} \end{cases}$$

- *Cycles*: Let $n \geq 3$ and set j to be the smallest positive integer such that $\binom{j}{2} \geq n$. Then

$$\chi'_s(C_n) = \begin{cases} j+1, & \text{if } j \text{ is odd and } \binom{j}{2} - 2 \leq n \leq \binom{j}{2} - 1 \text{ or} \\ & j \text{ is even and } n > (j^2-2j)/2 \\ j, & \text{otherwise} \end{cases}$$

  For both P_n and C_n, the $j+1$ appears for parity reasons, as in coloring K_{2k-2}.

Note that for both P_n and C_n, we choose (approximately) the smallest set of colors such that there are enough subsets to color each maximum-degree vertex with a different set. This will prove to be a useful general technique.

There are several common classes of graphs which have not been treated in previous papers. We provide some new results, by giving exact values of χ'_s for two more common classes of graphs.

- *Triangle Graphs*: The triangle graph T_n on n vertices is a set of $n-2$ triangles all sharing one edge (see Figure 2.1.3 for an example). Equivalently, it is $K_{2,n-2}$ with an edge added. For $n \geq 3$,

$$\chi'_s(T_n) = n.$$

  First note that $\Delta(T_n) = n-1$ and there are two maximum degree

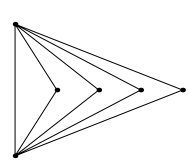


Figure 2.1.3: T_6

  vertices, so n is the minimum number of colors possible. Let our colors

be $C = \{c_1, \ldots, c_n\}$. We will assign color sets to each vertex, and there is a clear way to turn these into corresponding edge colors. Assign to one vertex of degree $n-1$ the set $\{c_1, \ldots, c_{n-1}\}$. Assign to the other vertex of degree $n-1$ the set $\{c_1, c_3, \ldots, c_n\}$. The color sets on the degree 2 vertices will be $\{c_2, c_3\}, \{c_3, c_4\}, \ldots, \{c_{n-1}, c_n\}$.

- *Wheel Graphs*: The wheel graph W_n on n vertices is C_{n-1} with one additional vertex, connected to every vertex on the cycle (see Figure 2.1.4 for an example).

$$\chi'_s(W_n) = \begin{cases} n-1, & n \geq 5 \\ 5, & n = 4 \end{cases}$$

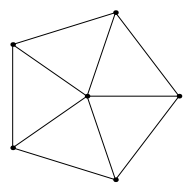


Figure 2.1.4: W_6

For $n \geq 5$, assign colors $c_1, c_2, \ldots, c_{n-1}$ clockwise to the edges incident to the center "hub" vertex. For edges on the cycle, beginning with the edge between edges colored c_1 and c_2, assign colors $c_3, c_4, \ldots, c_{n-1}, c_1, c_2$ clockwise around the cycle. Thus the cycle vertices have color sets $\{c_1, c_2, c_3\}$, $\{c_2, c_3, c_4\}$, $\ldots$, $\{c_{n-1}, c_1, c_2\}$. Since $\Delta(W_n) = n-1$ and there is only one such vertex, this must be minimum. If $n = 4$, we have $\chi'_s(K_4) = 5$.

2.2 Some Important Results

A number of the colorings in the previous section demonstrate important techniques and results concerning χ'_s. This section will present some results which generalize these techniques.

A common thread among many of the minimality proofs is choosing the smallest number of colors such that we can label all vertices with a different set of the appropriate size. Given a vdec coloring, we can label every vertex v with $S(v)$, and the labels are all different. It is not necessarily true that any labeling of the vertices of G with sets of size $\deg(v)$ corresponds to a vdec coloring, or even a proper edge-coloring.

Let G be a vdec graph. Clearly, if G has a vdec coloring using j colors, then we must have $\binom{j}{k} \geq n_k$ for $\delta \leq k \leq \Delta$. If not, any proper edge-coloring with j colors must have two vertices of degree k (for some k) sharing the same color set. As we saw for K_{2k}, sometimes we must add an additional color for parity reasons. This leads to a natural conjecture of Burris and Schelp:

Conjecture 2.2.1. *(Burris and Schelp [10]) Let G be a vdec graph. Let k be the minimum integer such that $\binom{k}{j} \geq n_j$ for $\delta \leq j \leq \Delta$. Then $\chi'_s(G) = k$ or $k+1$.*

Some sources use $1 \leq j \leq \Delta$, which is equivalent (if $n_j = 0$, then $\binom{k}{j} \geq n_j$ always). In fact, j need only range over the degree sequence of G. This conjecture is still unproven, although it has been shown to be true for many classes of graphs. It is true for all of the standard graphs in Section 2.1, as well as for graphs which are a disjoint union of cycles and paths [3], graphs of large maximum degree [5], regular graphs with small components [6], and several others. Many of these are considered in Section 2.3.

Burris and Schelp [10] made an additional conjecture about χ'_s, which has proven more tractable:

Theorem 2.2.2. *(Bazgan, Harkat-Benhamdine, and Li [7]) If G is a vdec graph, then $\chi'_s(G) \leq |V(G)| + 1$.*

Theorem 2.2.2 was proved by Bazgan, Harkat-Benhamdine, and Li in 1999 [7]. Its bound on $\chi'_s(G)$ is tight in some cases. For example, $\chi'_s(K_{2k}) = 2k+1$ as above. However, it is a poor approximation for many other cases. A particular example is $K_{n,n}$, for which $\chi'_s(K_{n,n}) = n+2$, which is quite far from $|V| + 1 = 2n+1$.

Finally, Balister looked at vdec colorings of random graphs, and gave a strong bound in his main theorem:

Theorem 2.2.3. *(Balister [2]) If G is a random graph on n vertices with edge probability $p = p(n)$, and if $\frac{pn}{\log n}$, $\frac{(1-p)n}{\log n} \to \infty$ as $n \to \infty$, then the probability that $\chi'_s(G) = \Delta$ goes to 1 as $n \to \infty$.*

In other words, "almost all" graphs have $\chi'_s(G) = \Delta$.

2.3 Upper Bounds on χ'_s

The results of the previous section give bounds on χ'_s for general graphs. Many other bounds have been given, often improving on these results for specific classes of graphs.

The first upper bound on χ'_s to be proved was presented by Burris and Schelp in their initial paper on vdec colorings [10]. For any vdec graph G, this bound is:

$$\chi'_s(G) \leq (\Delta + 1)(5 + 2\max\{n_j^{1/j}\}_{j=1}^{\Delta}).$$

This is related to a lower bound presented in the same paper:

$$\chi'_s(G) \geq \max\{(k!n_k)^{1/k} + (k-1)/2 : 1 \leq k \leq \Delta\}.$$

The proof of the upper bound relies on finding matchings of a graph and bounding the number of different colors which must be used on each matching. The lower bound comes from a counting argument based on Conjecture 2.2.1. Any graph given a vdec coloring with j colors must have $\binom{j}{k} \geq n_k$. Using this fact and the fact that an arithmetic mean is always at least as large as the corresponding geometric mean, we have the bound.

The same paper presented a tighter bound for trees. For any tree $T \neq K_2$,

$$\chi'_s(T) \leq \max\{n_1 + 1, 6.35n_2^{1/2}, 21\}.$$

There have been many bounds related to specific classes of graphs. Favaron, Li, and Schelp [15] presented a bound shortly after the first paper appeared on this subject. Their main theorem applies to a graph G with minimum degree $\delta \geq 5$ and maximum degree $\Delta < ((2c-1)n - 4)/3$, where c is a constant with $\frac{1}{2} < c \leq 1$. If this is the case, then

$$\chi'_s(G) \leq \lceil cn \rceil.$$

Bazgan, et. al. [8], provided an upper bound applying to any vdec graph with n vertices and minimum degree $\delta(G) > n/3$:

$$\chi'_s(G) \leq \Delta + 5.$$

Taczuk and Woźniak [21] have found a bound for several classes of cubic graphs. "Ladder graphs" consist of two identical cycles C_n with corresponding vertices connected by an edge. Let G be a ladder on $n = 2k$ vertices. If j is the smallest integer such that $\binom{j}{3} \geq n$, then $\chi'_s(G) \leq j + 1$. This satisfies Conjecture 2.2.1. In the same paper, the authors proved that $\chi'_s(pK_4)$ satisfies a similar bound. Note that pK_4 means p disjoint copies of K_4. Let j be the smallest integer such that $\binom{j}{3} \geq 4p$, then $\chi'_s(pK_4) \leq j + 1$. This also satisfies the conjecture.

Finally, Dedó, et. al. [12] also bounded χ'_s for several cubic graphs. A Fibonacci string is a binary string without two consecutive ones. The nth

Fibonacci number is exactly the number of Fibonacci strings of length n. A Fibonacci cube Γ_n is a graph whose vertices are all Fibonacci strings of order n, and vertices are adjacent if their strings differ in exactly one position. Similarly, a Lucas string is a binary string without either two consecutive ones, or an initial and terminal one. A Lucas cube L_n is defined similarly to a Fibonacci cube. Then, $\chi'_s(\Gamma_n) = n$ if $n \geq 4$. Similarly, $\chi'_s(L_n) = n$ if $n \geq 2$. These also satisfy Conjecture 2.2.1.

There are also many bounds for variations on χ'_s, which will be considered in Chapter 5.

2.3.1 Line Graphs and Packing

Balister, Bollobás, and Schelp [3] provided a good upper bound on χ'_s and some new techniques for graphs with $\Delta = 2$, i.e. a disjoint union of paths and cycles. Let G be such a graph which is vdec, and choose k as small as possible such that $n_1(G) \leq k$ and $n_2(G) \leq \binom{k}{2}$. Then:

$$k \leq \chi'_s(G) \leq k + 5. \tag{2.1}$$

The paper also gives exact values of χ'_s for unions of only paths or only cycles, which confirm Conjecture 2.2.1.

This same paper introduces a new technique of using line graphs and then packing paths and cycles into K_n. The line graph $L(G)$ of a graph G has one vertex for each edge of G. Vertices of $L(G)$ are adjacent if their associated edges in G are adjacent. Thus, a proper edge-coloring of G is equivalent to a proper vertex-coloring of $L(G)$. This is especially useful for graphs with $\Delta = 2$, because $L(P_n) = P_{n-1}$ and $L(C_n) = C_n$.

A *packing* of a graph G_1 into another graph G_2 is a mapping from the vertices of G_1 to the vertices of G_2 such that edges are preserved and the induced map from edges to edges is an injection. In other words, we "fold" G_1 into G_2 so that no two edges are folded onto each other, but several vertices may be mapped to the same vertex.

Consider a graph G with n vertices and $\Delta = 2$, and a complete graph K_n with each vertex colored a different color. Then packing $L(G)$ into this K_n produces a proper vertex-coloring of $L(G)$, which in turn corresponds to a proper edge-coloring of G. Note that the circuit and path components of $L(G)$ are mapped to edge-disjoint paths and circuits in K_n. If we have path components in $L(G)$, we additionally require that their endpoints in the packing be distinct. Thus, the resulting edge-coloring is also vdec. Packing results can produce bounds for $\chi'_s(G)$, such as Equation (2.1).

While very useful when $\Delta = 2$, the line graphs of more complex graphs are not as well-behaved. There may be other specific classes of graphs with relatively simple line graphs which would yield to this method. For example, trees have line graphs that are complete graphs connected by edges or vertices. Thus far, no such possibilities have been examined.

2.3.2 Labeling Vertices With Sets

One method used in many of the results for standard classes of graphs provides a starting point for another useful technique. Conjecture 2.2.1 implies that if there are enough subsets of a color set to assign every vertex a distinct set, we can find a corresponding vdec coloring. Many of the standard classes of graphs can be colored by such a method (especially complete bipartite graphs, cycles, and paths). This is not guaranteed to work in general, as some labellings may not correspond to proper edge-colorings. Balister, Riordan, and Schelp [6] expanded this idea with a great deal of success.

Balister, et. al. define a number of parameters for vdec graphs. Here, $\oplus$ is the symmetric difference operator, which produces the set of items which appear in exactly an odd number of its operands.

- Parameter $k(G)$ is the minimum integer k such that $\binom{k}{j} \geq n_j$ for all j with $\delta \leq j \leq \Delta$. This is equivalent to the definition of k in Conjecture 2.2.1, and thus $\chi'_s(G) \geq k(G)$.

- Parameter $k'(G)$ is the minimum k such that, there exist distinct sets $S(v) \subseteq \{1, \ldots, k\}$ with $|S(v)| = \deg(v)$, for all $v \in V$, such that $\oplus_v S(v) = \emptyset$. This corresponds to the fact (used in the coloring of K_{2k}) that each color must appear in an even number of sets.

- Parameter $k''(G)$ is the minimum k such that there exist distinct sets $S(v) \subseteq \{1, \ldots, k\}$ with $|S(v)| = \deg(v)$, for all $v \in V$, and with the additional condition that all $X \subseteq V$ have $|\oplus_{v \in X} S(v)| \leq C$, where C is the number of edges between X and $\overline{X}$. That is, for any subset of vertices X, if a given color is not used an even number of times in color sets within X, then the color must be used at least once on an edge leaving X.

Thus we have $\chi'_s(G) \geq k''(G) \geq k'(G) \geq k(G)$. Balister, et. al. showed that $\chi'_s(G) = k''(G)$ for all simple vdec graphs with at most 11 vertices, and all 3-regular graphs with at most 22 vertices. They also conjecture that this holds in general, as follows:

Conjecture 2.3.1. *(Balister, Riordan, Schelp [6]) For any vdec graph G, $\chi'_s(G) = k''(G)$.*

There are several results in [6] bounding values of k' and k'', and also showing that $k'' \leq k' + 1$. These results are used to show that Conjecture 2.2.1 holds for d-regular graphs with $d - 2$ edge-disjoint 1-factors, at most $\binom{k}{d}$ vertices, and each component G_i satisfying $|V(G_i)| \leq \frac{3(k-1)}{4(d-1)}$.

The idea of labeling vertices with color sets arises naturally when studying vdec colorings. The method given in [6] is very promising, and provides a likely direction for further study.

Chapter 3

The VCICM and its Determinant

The techniques of spectral graph theory may be applied to extract a great deal of information about a graph from its adjacency matrix and incidence matrix. It is natural to consider how this idea may be extended to graph colorings. Some information about the vertex- or edge-colorability of a graph may be obtained from the adjacency matrix, but these are not sufficient for the additional conditions imposed by vertex-distinguishing edge-colorings. In this chapter, we present a new matrix useful in the study of vdec colorings, develop it using standard matrices, and examine some of its properties. In particular, we study its determinant and give some properties of the determinant for general colorings. We also present a variety of constructions which affect the determinant in predictable ways.

3.1 Vertex Color Incidence Count Matrix

Our primary interest in vdec colorings is whether the sets $S(v)$ and $S(w)$ are distinct for each pair of vertices v and w. Thus, a natural value to consider is $c = |S(v) \cap S(w)|$. If $|S(v)| = |S(w)| = c$, then v and w are not distinguished. Yet, in all other cases, the vertices are distinguished. We can encode this information for an entire graph into a matrix.

Definition 3.1.1. *Let $G = (V, E)$ be a graph with a proper edge-coloring. The* Vertex Color Incidence Count Matrix (VCICM) *of G with given coloring π is a matrix M with rows and columns indexed by V and*

$$M_{ij} = |S(i) \cap S(j)|.$$

In other words, each entry in M is the number of edge colors in common between a pair of vertices.

A few properties of M are immediately available:

- M is symmetric: $S(i) \cap S(j) = S(j) \cap S(i)$.
- M has only nonnegative integer entries.
- The diagonal elements of M give the degree sequence of G: $S(i) \cap S(i) = S(i)$.

In addition, we can inspect the VCICM to determine whether a given coloring is vdec. In particular, if $\deg(v) = \deg(w)$ and $M_{vw} = \deg(v)$, then vertices v and w are not distinguished. This is particularly easy for k-regular graphs, in which the first condition is always satisfied. In this case the coloring is vdec if and only if every off-diagonal element of M is strictly less than k.

	a	b	c	d
a	3	2	1	2
b	2	2	1	1
c	1	1	2	2
d	2	1	2	3

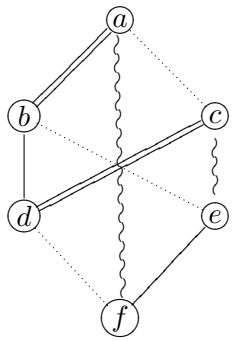
	a	b	c	d
a	3	2	2	3
b	2	2	2	2
c	2	2	2	2
d	3	2	2	3

Figure 3.1.1: Two proper edge-colorings of C_4 with a chord

Example 3.1.2. *Let $G = C_4$ with a chord. Figure 3.1.1 shows two proper edge-colorings of G, one vdec, one not vdec. Below each is its respective VCICM. Note that the (d, a) entry of the second matrix (among others) is equal to the (a, a) and (d, d) entries, and that vertices a and d are not distinguished.*

Even for non-regular graphs, we can use the idea of examining entries of the VCICM to give a strong condition on when a coloring is vdec.

Theorem 3.1.3. *Let G be any graph with a proper edge-coloring π and VCICM M. Coloring π is vdec if and only if no two columns of M are equal.*



	a	b	c	d	e	f
a	3	2	**3**	2	2	2
b	2	3	2	**3**	1	1
c	**3**	2	3	2	2	2
d	2	**3**	2	3	2	2
e	2	1	2	2	3	**3**
f	2	1	2	2	**3**	3

Figure 3.1.2: VCICM for a non-vdec colored 3-regular graph

Proof. First, suppose π is not vdec, then there must be two vertices v, w such that $S(v) = S(w)$. Thus for any other vertex $z \neq v, w$, $|S(v) \cap S(z)| = |S(w) \cap S(z)|$. In particular, this means that columns v and w of M are identical.

Second, suppose that two columns v and w of M, corresponding to vertices a and b of G (respectively), are equal. Let $v = (M_{1a}, \ldots, M_{na})$ and $w = (M_{1b}, \ldots, M_{nb})$. Note that $M_{aa} = \deg a$, and that $M_{bb} = \deg b$. Since the two columns are identical, we must have $M_{ab} = M_{aa}$ and $M_{ba} = M_{bb}$. Since M is symmetric, we must also have $M_{ab} = M_{ba}$. Then, $M_{aa} = M_{ab} = M_{ba} = M_{bb}$, so $\deg a = \deg b$. Thus, $M_{ab} = S(a) \cap S(b) = \deg a = \deg b$. Therefore a and b are not distinguished, and π is not vdec. $\square$

Theorem 3.1.3 lets us examine the VCICM and know exactly whether the associated coloring is vdec or not. We will make use of it in Section 3.3, in conjunction with the determinant. To do so, we must first develop another way to define the VCICM.

3.1.1 Another Way to Define the VCICM

It is natural to define the VCICM as in Definition 3.1.1. However, we can define the matrix equivalently in terms of other matrices. This method reveals more information about the properties of the VCICM. To re-define the VCICM, we first must define the unsigned incidence matrix:

Definition 3.1.4. *Let $G = (V, E)$ be a graph. The* unsigned incidence matrix (IM) *B of G is a matrix with rows indexed by V, columns indexed by E, and*

$$B_{ve} = \begin{cases} 1, & \textit{if vertex } v \textit{ is incident to edge } e \\ 0, & \textit{otherwise} \end{cases}$$

Second, we need a new matrix. This matrix indicates which colors are used on which edges.

Definition 3.1.5. *Let $G = (V, E)$ be a graph with an edge-coloring using colors $C = \{c_1, \ldots, c_k\}$. The* arc color matrix (ACM) *A is a matrix with rows indexed by E, columns indexed by C, and*

$$A_{ec} = \begin{cases} 1, & \textit{if edge } e \textit{ is colored with color } c \\ 0, & \textit{otherwise} \end{cases}$$

Example 3.1.6. *The IM for the graphs in Example 3.1.2 is given by:*

	ab	ac	ad	bd	cd
a	1	1	1	0	0
b	1	0	0	1	0
c	0	1	0	0	1
d	0	0	1	1	1

The ACM's for the graphs in Example 3.1.2 are given by:

	—	=	−−	···
ab	0	1	0	0
ac	1	0	0	0
ad	0	0	1	0
bd	1	0	0	0
cd	0	0	0	1

	—	=	−−
ab	0	1	0
ac	1	0	0
ad	0	0	1
bd	1	0	0
cd	0	1	0

Let $G = (V, E)$ be a graph with a proper edge-coloring using colors $C = \{c_1, \ldots, c_k\}$. Let B and A be, respectively, the IM and ACM of G. Consider the $|V| \times |C|$ matrix $F = B \cdot A$. The (v, c) entry of F is the dot

product of row v of B and column c of A. This dot product sums up the values in column c which correspond to edges incident to vertex v. That is, F_{vc} is the number of edges incident to v which are colored with color c. Since G has a proper edge-coloring, F is a $(0,1)$ matrix. Thus each row of F is a $(0,1)$ vector indicating which colors are used to color edges incident to v.

Example 3.1.7. *The F matrices for the graphs in Example 3.1.2 are given by:*

	—	=	−−	···
a	1	1	1	0
b	1	1	0	0
c	1	0	0	1
d	1	0	1	1

	—	=	−−
a	1	1	1
b	1	1	0
c	1	1	0
d	1	1	1

Now let $M = F \cdot F^T$. This will be a $|V| \times |V|$ matrix whose vw entry is the dot product of row v of F and row w of F. Thus the dot product $v \cdot w$ is a sum in which a 1 appears for every color c such that both v and w are incident to an edge of color c. So, M_{vw} is exactly the number of colors shared between $S(v)$ and $S(w)$, or $|S(v) \cap S(w)|$. This proves the following result:

Lemma 3.1.8. *Let G be a vdec graph with an edge-coloring π. Let A be the ACM of G, and let B be the IM of G. Let $M = BA(BA)^T$. Then M is exactly equal to the VCICM of G with π.*

3.2 Structure of the VCICM

The VCICM has interesting structural properties for various graphs and colorings. For specific colorings and graphs, we may write the VCICM in terms of other well-known matrices.

Maximum colorings will be of interest in future sections. These are particularly interesting in terms of the VCICM, because they minimize the entries in the matrix. Maximum coloring also often offer VCICM's which are much easier to analyze than other colorings. Let G be a vdec graph with a maximum coloring. Let M be the VCICM of this graph coloring. As always, the diagonal will be the degree sequence of G. Since every edge has a distinct color, each vertex will share exactly one color with each adjacent

vertex, and no colors with non-adjacent vertices. In other words,

$$M_{vw} = \begin{cases} \deg v, & \text{if } v = w \\ 1, & \text{if } vw \in E \\ 0, & \text{otherwise} \end{cases}$$

This is very similar to the Laplacian matrix of the graph.

The *Laplacian* of a graph is defined in terms of its signed incidence matrix. Let G be a graph with incidence matrix B. Obtain the signed incidence matrix C by arbitrarily changing one 1 in each column to a -1. This corresponds to arbitrarily directing the edges of G, giving an orientation of G. Then the Laplacian is $L = CC^T$, and does not depend on the orientation. In particular, L has diagonal elements equal to the degree sequence of G, and off-diagonal entry $M_{vw} = -1$ if $vw \in E$ and 0 otherwise. Thus the maximum colored VCICM corresponds to the Laplacian of G with positive off-diagonal entries. In other words,

$$M = L + 2A,$$

where A is the adjacency matrix of G. However, the Laplacian may be written as

$$L = D - A.$$

Here D is a diagonal matrix with entries equal to the degree sequence of G. This gives us the following result, which will prove extremely useful in Chapter 4, to find the eigenvalues of M.

Lemma 3.2.1. *Let G be a graph with a maximum coloring. Fix an arbitrary ordering of the vertices V of G. Let A be the adjacency matrix of G, and D a diagonal matrix whose entries give the degree sequence of G (both indexed by V in the above ordering). Let M be the VCICM of G. Then $M = D + A$.*

Lemma 3.2.1 will be useful in Chapter 4, in finding the eigenvalues of certain VCICM's.

3.3 Determinant and Rank

The first major property of the VCICM which we will examine is its determinant and, by extension, its rank. The determinant of the adjacency matrix of a graph G has been found to give some structural properties of G (see [18] and [9]). This inspires our study of the determinant of the VCICM. We find that some properties of a proper edge-coloring are given by the determinant

of the VCICM, and that the determinant may be explicitly obtained for several classes of graphs.

As mentioned in Section 3.1, the VCICM is a nonnegative integer matrix. Let G be a graph with a proper edge-coloring, VCICM M, IM B and ACM A. We can write $M = (BA)(BA)^T$. Let u be any real vector with entries indexed by $|V|$, and consider $u^T M u$. We have

$$\begin{aligned} u^T M u &= u^T BA(BA)^T u \\ &= (u^T BA)(u^T BA)^T \\ &= \|u^T BA\| \\ &\geq 0. \end{aligned}$$

This proves the following observation:

Observation 3.3.1. *Let M be the VCICM of a graph G with a proper edge-coloring. Then M is positive semidefinite.*

In consequence, $\det(M) \geq 0$. This also implies that all eigenvalues of M are nonnegative, a fact which we will use extensively in the next chapter.

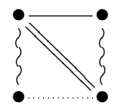
From Section 3.1, Theorem 3.1.3 gives us a strong condition for vdec colorings. It would be much more useful, however, to be able to determine if a coloring is vdec by using more standard functions. We can do so by expressing one part of Theorem 3.1.3 in terms of the determinant. If M has two identical columns, then $\det M = 0$. Thus we have the following result:

Corollary 3.3.2. *Let G be any graph with a proper edge-coloring and VCICM M. If $\det(M) \neq 0$ then the coloring of G is vdec.*

In addition, if $\det(M) > 0$ then M is positive definite, which also means that it has no zero eigenvalues.

Unfortunately, Corollary 3.3.2 weakens Theorem 3.1.3, since the converse is no longer necessarily true. Consider Figure 3.3.1. This graph has a vdec coloring, but its VCICM is singular.

Theorem 3.1.3 helps us determine when a coloring is vdec, in a manner which is easy to check by hand. Corollary 3.3.2 puts this property in terms of the determinant, which we can more easily study using our knowledge of the VCICM. As a result, we are interested in knowing when the determinant of the VCICM is positive, as this guarantees a vdec coloring. By extension, we are interested in knowing when the VCICM is singular. Unfortunately, the VCICM may be singular for reasons other than having two identical columns, so this makes our study less precise. However, we are also interested in the



$$\det \begin{bmatrix} 3 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 3 & 2 \\ 1 & 1 & 2 & 2 \end{bmatrix} = 0$$

Figure 3.3.1: A vdec colored graph with a singular VCICM

determinant of the VCICM for theoretical purposes, and to give a more complete description of the VCICM and its properties.

Using functions written in *Mathematica*, we have created vdec colorings for a large variety of vdec graphs, and found the VCICM's of each graph coloring. See Section 4.5 for more details of the code generating these colorings. In many cases, M is singular. Some of this can be explained by analyzing the matrix definition. We can write the VCICM M as a product of the IM B and the ACM A in this way:

$$M = B(AA^T B^T).$$

Also,

$$\text{rk } (XY) \leq \min\{\text{rk } X, \text{rk } Y\}.$$

for matrices X, Y. We know that rk $B = |V| - c_0$, where c_0 is the number of bipartite connected components of G. (See [16], pp. 175-178.) Thus if G is bipartite, rk $M \leq$ rk $B < |V|$, so $\det M = 0$. Similarly, if rk $A < |V|$ then $\det M = 0$. This happens if fewer than $|V|$ colors are used in the edge-coloring.

As mentioned previously, this does not cover all cases in which G may be vdec colored but has $\det M = 0$. There are cases in which G is not bipartite and at least $|V|$ colors are used, but a linear dependence among the columns of BA makes M singular. Unfortunately, we have not found any simple combinatorial conditions which determine when this will happen.

We have also observed some general trends in the determinant of the VCICM. Let G be a vdec graph, and consider the determinants of VCICM's M for various vdec colorings of G. Then in general:

- Colorings with fewer colors generally produce a larger determinant. This is because fewer colors mean that each vertex (on average) shares

more colors which each other vertex, so each entry in the VCICM is larger.

- However, there is not a strictly monotonic relation between number of colors and $\det M$. There are typically many different colorings with a fixed number of colors, each of which usually gives a different determinant. It is quite possible for a coloring with k colors to have a smaller determinant than one with $k+1$ colors.

- The maximum coloring is unique, and the determinant it gives seems to be a lower bound on the determinant given by other colorings of the same graph.

Finally, we usually assume that all vdec graphs are connected. However, we may treat disconnected graphs when examining the determinant of the VCICM. Suppose a vdec graph G has several components $G_i, 1 \leq i \leq k$, and that G is given a vdec coloring in which no color is used in two different components. Write $M(G_i)$ for the VCICM of G_i. Then the VCICM of G is a block diagonal matrix with the VCICM's of each G_i as the blocks:

$$M(G) = \begin{bmatrix} M(G_1) & 0 & 0 & 0 \\ 0 & M(G_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & M(G_k) \end{bmatrix}$$

The determinant of a block diagonal matrix is the product of the determinants of its blocks, so $\det M = (\det M(G_1))(\det M(G_2)) \cdots (\det M(G_k))$. Thus we may, in general, look at only individual components of a disconnected graph.

3.3.1 Value of $\det M$ for specific graphs and colorings

Every vdec graph has many possible colorings, so in general it is difficult to find a simple expression for the determinant of the VCICM. However, maximum colorings (in which every edge of the graph has a different color) are more manageable. We now present some of these results, and a conjecture based on their pattern.

Let $G = C_k$, $k \geq 3$, be given a maximum coloring. The VCICM for G

is:

$$M = \begin{bmatrix} 2 & 1 & 0 & & 1 \\ 1 & 2 & 1 & \cdots & 0 \\ 0 & 1 & 2 & & 0 \\ & \vdots & & \ddots & 1 \\ 1 & 0 & 0 & 1 & 2 \end{bmatrix}$$

We will show that

$$\det M = \begin{cases} 4, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases} \tag{3.1}$$

The proof uses induction on n for several smaller matrices, which in turn form the cofactor expansion of $\det M$. We know that C_{2k} is bipartite, so $\det M = 0$ always in this case. Thus we assume that G has $2k+1$ vertices.

Let M be the VCICM of C_{2k+1}. Then M may be written as above. Finding the determinant by cofactor expansion along the top row, we have

$$\det M = 2 \begin{bmatrix} 2 & 1 & & 0 \\ 1 & 2 & \cdots & 0 \\ & \vdots & \ddots & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 & \cdots & 0 \\ 0 & 2 & & 0 \\ & \vdots & \ddots & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & \cdots & 0 \\ 0 & 1 & & 0 \\ & \vdots & \ddots & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

We need some notation to simplify this. Let

$$X_k = \begin{bmatrix} 2 & 1 & & 0 \\ 1 & 2 & \cdots & 0 \\ & \vdots & \ddots & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \text{ from the } (1,1)\text{-cofactor of } M$$

$$Y_k = \begin{bmatrix} 1 & 1 & \cdots & 0 \\ 0 & 2 & & 0 \\ & \vdots & \ddots & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \text{ from the } (1,2)\text{-cofactor of } M$$

$$Z_k = \begin{bmatrix} 1 & 2 & \cdots & 0 \\ 0 & 1 & & 0 \\ & \vdots & \ddots & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ from the } (1,2k+1)\text{-cofactor of } M$$

In each, the subscript k indicates that the matrix has dimensions $k \times k$.

Suppose M has dimensions $2k+1 \times 2k+1$. We may factor M as follows:

$$\det M = 2 \det X_{2k} - \det Y_{2k} + \det Z_{2k}.$$

Now, we expand each smaller matrix by cofactors. Let n be any positive integer. To begin, $\det X_n$ may be expanded along the top row. This yields

$$\det X_n = 2 \det X_{n-1} - \det X_{n-2}.$$

Note that we have combined two steps to obtain the final term. Similarly, $\det Y_n$ may be expanded along the left column, giving

$$\det Y_n = \det X_{n-1} + (-1)^{n+1}.$$

Note that we have combined two steps to obtain the final term. This comes from the fact that the $(2k, 1)$-cofactor of Y_{2k+1} has a single 1 in its top row. Expanding along this, we get an upper-triangular matrix with all 1's on the diagonal.

Also, $\det Z_n$ may be expanded along the left column, giving

$$\det Z_n = 1 + (-1)^{n+1} X_{n-1}.$$

We will now perform several simultaneous inductions to find the determinants of some of these matrices. First, $\det X_n = n+1$. We need two matrices as our basis cases:

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 = n+1 \quad \text{and} \quad \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4 = n+1.$$

Second, $\det Y_n = n + (-1)^{n+1}$. As a basis case,

$$\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 = n-1.$$

Suppose that these hypotheses hold for all $k < n$. Then from the equations above,

$$\begin{aligned} \det X_n &= 2X_{n-1} - X_{n-2} = 2n - (n-1) = n+1 \\ \det Y_n &= X_{n-1} + (-1)^{n+1} = n + (-1)^{n+1} \end{aligned}$$

Thus by induction, our equations are correct. Substituting into the above equation for $\det M$ with $n = 2k+1$, we have

$$\det M = 2n - (n-1+(-1)^n) + 1 + (-1)^n(n-1) = 2n - n + 1 + 1 + 1 - n + 1 = 4$$

Thus $\det M = 4$ for all C_{2k+1}.

We may find a similar result for complete graphs. Let $G = K_n$ with a maximum coloring. Then

$$\det M = 4\binom{n-1}{2}(n-2)^{n-2}.$$

This may be proved much as Equation (3.1) was proved, by expanding a general VCICM for a maximum colored K_n by cofactors. Several inductive arguments give the determinants of the cofactors, and thus this result.

If $G = P_k$, $k \geq 2$ with a maximum coloring, then G is bipartite. Thus

$$\det M = 0.$$

This also holds for all $K_{m,n}$.

Note that all maximum colorings thus far have $\det M = 4k$ for some non-negative integer k. This is true in general, and can be generalized somewhat depending on properties of the coloring.

Theorem 3.3.3. *Let G be a vdec graph with a proper edge-coloring and VCICM M. Suppose that each color is used on a multiple of $j \geq 1$ edges. Then $\det M = 4j^2k$ for some integer $k \geq 0$.*

Before we prove Theorem 3.3.3, we require several supporting lemmas.

Lemma 3.3.4. *Let $j \in \mathbb{Z}$. Let M be a square $(0,1)$ matrix such that the number of ones in each row is an integer multiple of j. Then $\det M = kj$ for some $k \in \mathbb{Z}$.*

Proof. Note that distinct rows may have different numbers of ones, as long as each has a multiple of j ones.

Let $c_1, \ldots, c_n$ represent the columns of M. The determinant of M does not change if we replace column c_i with $c_i + c_j, i \neq j$. Applying this repeatedly, we obtain a matrix A whose first column is the sum of all columns of M:

$$A = [c_1 + c_2 + \ldots + c_n,\ c_2,\ \cdots,\ c_n].$$

Thus

$$\det M = \det A.$$

However, each entry of the column $c = c_1 + \ldots + c_n$ must be an integer multiple of j. This happens because the ith entry of c is the sum of the

entries in row i of M, which is a multiple of j. Thus $(c_1 + \ldots + c_n)/j$ is a vector with integer entries. Define matrix B as follows:

$$B = \left[\frac{1}{j}\left(c_1 + c_2 + \ldots + c_n\right),\ c_2,\ \cdots,\ c_n\right].$$

Note that B is still an integer matrix, so $\det B \in \mathbb{Z}$. We also know that multiplying one column of a matrix by a scalar multiplies the determinant by the same amount. Thus:

$$\det M = \det A = j \det B.$$

Therefore, $\det M$ is an integer multiple of j. $\square$

Suppose we find a submatrix of a matrix M by taking only rows $i_1, \ldots, i_k$ and only columns $j_1, \ldots, j_l$. We will denote this $l \times k$ submatrix by $M\binom{i_1,\ldots,i_k}{j_1,\ldots,j_l}$. The following theorem proves to be very useful in our proof:

Theorem 3.3.5. *(Binet-Cauchy) Suppose that A and B are $p \times q$ and $q \times p$ matrices, respectively, with $p \leq q$. Let $i_1, \ldots, i_p$ be distinct indices. Then*

$$\det AB = \sum_{j_1 < j_2 < \cdots < j_p} \det A \begin{pmatrix} i_1, i_2, \ldots, i_p \\ j_1, j_2, \ldots, j_p \end{pmatrix} \det B \begin{pmatrix} j_1, j_2, \ldots, j_p \\ i_1, i_2, \ldots, i_p \end{pmatrix}.$$

Note that, since A has p rows and B has p columns, we take all rows of A and all columns of B. This theorem, with the above lemma, allow us to prove Theorem 3.3.3.

Proof. (Theorem 3.3.3) We know that the VCICM M can be written as $M = F^T F$. Let G have n vertices, and c colors used on its edges. We may assume that F is a $n \times c$ matrix in which $n \leq c$. If not, then rk $M < n$, so $\det M = 0$. Thus, we may apply the Binet-Cauchy theorem to M. Let n be the number of vertices in G. The determinants within the summation in the Binet-Cauchy theorem are corresponding submatrices of F^T and F. Since the determinant of the transpose of a matrix is the same as the determinant of the original matrix, we simplify the Binet-Cauchy theorem to the following:

$$\det M = \sum_{\substack{S:\ S \text{ is an } n \times n \\ \text{submatrix of } F}} (\det S)^2. \tag{3.2}$$

However, each $n \times n$ submatrix S of F has some special properties. In particular, S is a $(0, 1)$ square matrix whose columns are indexed by colors

and rows are indexed by vertices. The columns represent a subset of the colors used in G, but the rows cover every vertex. The (v, c) entry is 1 if vertex v is incident to an edge of color c, and 0 otherwise. Thus, each column has a multiple of $2j$ ones: each edge is incident to two vertices, so if j edges use color c_i, then $2j$ vertices are incident to an edge of color c_i.

By Lemma 3.3.4, $|S|$ is a multiple of $2j$, so $|S|^2$ is a multiple of $4j^2$. Thus $4j^2$ is a factor of every term in the summation in Equation (3.2). Thus, $\det M = 4j^2k$ for some integer $k \geq 0$, as required. □

As an immediate corollary, we have:

Corollary 3.3.6. *For every vdec graph with a proper edge-coloring, the VCICM M of G satisfies* $\det M = 4k$ *for some nonnegative integer k.*

For most colorings, this is the most we can say. However, Theorem 3.3.3 also gives some more insight into the colorings for which M is singular. Since each term of the summation in the proof of Theorem 3.3.3 is squared, the terms are all nonnegative. Thus, the VCICM is singular only if every term of the summation is exactly zero. Equivalently, every $n \times n$ submatrix of F must be singular.

How may this happen? Let S be an $n \times n$ submatrix of F. The columns of S are indexed by a subset C' of the colors used on G. Note that if G is colored with more than n colors, not all colors used on G appear in C'. Suppose there is a vertex v of G such that $S(v) \cap C' = \emptyset$. Thus, we have a zero row, and S is singular. In addition, the columns of S may be linearly dependent in other ways. Unfortunately, we have not found any simple conditions which determine when this happens.

The next section gives a construction which affects the determinant of the VCICM in a predictable way.

3.3.2 Appending Trees

We begin by showing that appending a leaf to a graph G affects the VCICM in a predictable way. Let G be a vdec graph with a vdec coloring and VCICM M. Create G' from G by appending a leaf: add a new vertex w and connect the vertex to any existing vertex v of G. Color this new edge with some color not used in the rest of the graph. This operation does not change the determinant of the VCICM, as proved in the following result:

Lemma 3.3.7. *Let G be a nontrivial vdec graph with a vdec coloring and VCICM M. Let G' be G with a leaf appended. Let M' be the VCICM of G'. Then G' is vdec, the coloring of G' is vdec, and* $\det M = \det M'$.

Proof. Let w be the new leaf and v be the vertex of G to which it was attached. Since G is nontrivial and vdec, G' must have no isolated edges or vertices. Also, edge vw uses a new color, so v is still distinguished from all other vertices. Similarly, w must be distinguished, because no other leaves could use its color.

By appending a leaf w to G at vertex v, the VCICM gains one row and one column. Because the color of wv is not used elsewhere, w will share no colors with most vertices, and exactly one color with itself and v. In addition, the degree of vertex v is increased by one, so M_{vv} increases. Thus, M' can be written as:

$$M' = \begin{bmatrix} 1 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & & & & & \\ \vdots & & & & & \\ 1 & & & M^+ & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix},$$

where M^+ is equal to M, except $M^+_{vv} = M_{vv} + 1$.

We find $\det M'$ by expanding by cofactors along the left column. This gives $\det M' = \det M^+ + (-1)^{1+i} \det N$, where i is the position of row v and N is M' with row v removed.

$$N = \left[\frac{0 \cdots 1 \cdots 0}{[M - \text{row } v]} \right]$$

But, $\det N$ can be found by expanding along the top row, which gives us $\det N = (-1)^{1+i-1} \det C$ where C is M with row and column v removed, that is, the (v, v)-cofactor of M.

We now have $\det M' = \det M^+ - \det C$. Suppose that we found $\det M^+$ by expanding along row v and $\det M$ by expanding along the same row. The only term which differs between these two expansions is the cofactor of (v, v). This cofactor is C in both cases, but its coefficient is exactly one larger in $\det M^+$. So, $\det M' = \det M^+ - \det C = \det M$. □

This can be generalized by repeatedly appending edges of new colors, as in the following theorem:

Theorem 3.3.8. *(Appending Construction) Let G be a nontrivial vdec graph with a vdec coloring and VCICM M. Let T be a maximum colored tree which does not use any colors used in G. Obtain G' by identifying one*

vertex v of G with one vertex w of T. Let M' be the VCICM of G'. Then G' is vdec, the coloring of G' is vdec, and $\det M = \det M'$.

Proof. Repeatedly append edges of T to G, beginning with one edge incident to w and working outwards. At each step, we have appended a single leaf, which keeps the determinant unchanged as in Lemma 3.3.7. Thus $\det M = \det M'$. Similarly, the graph remains nontrivial, and each step has a vdec coloring, so the coloring of the final graph is also vdec, with a vdec coloring. $\square$

Chapter 4

VCICM Eigenvalues

In Chapter 3, we introduced the VCICM and examined its determinant. In this chapter, we continue the exploration of the VCICM's properties by examining its eigenvalues. We begin with some basic properties of the eigenvalues and eigenvectors of the VCICM. We also examine the link between the VCICM and the adjacency matrix, which provides several bounds and related results for specific graphs. After examining the values of the eigenvalues and eigenvectors for some standard graphs, we present several upper and lower bounds on the largest eigenvalue of the VCICM.

Many of these bounds will involve the number of colors used to color a graph. One motivation for investigating these bounds is to use them to find bounds on $\chi'_s(G)$. The relationship of the eigenvalues of the VCICM to some bounds on $\chi'_s(G)$ is discussed in Section 4.9. Unfortunately, our investigation mostly fails in finding good bounds on $\chi'_s(G)$. However, we do provide a thorough basis for future work.

We also frequently examine the eigenvalues of maximum colorings. Naturally, we already know the number of colors used in these colorings. However, the VCICM's of these colorings are often the most accessible VCICM's. They also admit analysis in some interesting ways, which is of theoretical interest.

4.1 Notation

For a vdec graph G with n vertices, we denote the eigenvalues of the VCICM as $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$. (The initial strict inequality will be justified shortly.) The degree sequence of G will be denoted $d_1 \geq d_2 \geq \cdots \geq d_n$ in nonincreasing order, with repeated degrees included. As before, the number

of vertices of degree k will be n_k.

We will often make use of the vector consisting entirely of 1's. We denote this by $\vec{1}$, and its dimension is n unless otherwise stated. Similarly, I is the $n \times n$ identity matrix and J is the $n \times n$ matrix of all 1's, unless their size is otherwise stated.

Finally, we denote the number of edges in a graph by $e = |E|$. Later bounds will use this value very often.

4.2 Basic Properties

Let G be a vdec graph with a proper edge-coloring and VCICM M. We know that

$$\det M = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Therefore, $\det M > 0$ if and only if $\lambda_i > 0$ for all i. In other words, we have:

Corollary 4.2.1. *If 0 is not an eigenvalue of M, then the coloring of G is vdec.*

Let G be given any proper edge-coloring. Every vertex shares at least one color with each of its neighbors. Thus if ij is an edge of G, then $M_{ij} \geq 1$ for every edge-coloring of G. The *underlying directed graph* X of M is a graph on V containing edge ij if and only if $M_{ij} \neq 0$. The underlying directed graph of M thus contains a subgraph X which is a directed version of G. That is, for every edge uv of G, X contains two directed arcs uv and vu. We always assume that G is connected, so X will be strongly connected. Such a graph is called *irreducible*, and thus the Perron-Frobenius Theorem applies to M. (See [16], pp. 175-178.)

Theorem 4.2.2. *(Perron-Frobenius) Suppose A is a real nonnegative $n \times n$ matrix whose underlying directed graph X is strongly connected. Then the largest eigenvalue λ_1 of A is real and simple. If x is an eigenvector of λ_1, then no entries of x are zero, and all have the same sign.*

We have several other useful properties:

- As a consequence of the Perron-Frobenius Theorem, the largest eigenvalue λ_1 of M is always positive, real, and simple. Because $\lambda_1 > 0$ and all entries of M are positive, all entries of the eigenvector x corresponding to λ_1 will be strictly positive as well.

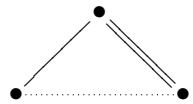
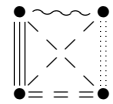
- Since M is a real symmetric matrix, eigenvectors corresponding to different eigenvalues will be orthogonal. In particular, if v is the eigenvector corresponding to λ_1, then all other eigenvectors of M will be orthogonal to v.

- We usually assume that all vdec graphs are connected. However, we may look at the eigenvalues of disconnected graphs easily. As with the determinant, we can analyze disconnected graphs in terms of their components. Let G be a vdec graph with a vdec coloring, and components $G_i, 1 \leq i \leq k$. Suppose each color is used in at most one component. Let $M(G_i)$ be the VCICM of component G_i. Then as in Section 3.2, the VCICM M of G is a block diagonal matrix consisting of the VCICM's of each component. If v is an eigenvector of one such component's VCICM, we may add an appropriate number of zeroes to v to make it an eigenvector of M with the same eigenvalue. In addition, each pair of eigenvectors so derived from different components must be orthogonal: none of their nonzero elements coincide. Similarly, two eigenvectors from the same component with different eigenvalues will be orthogonal. Thus we have the correct number of eigenvectors for M, so every eigenvector of M must correspond to an eigenvector of some component. The eigenvalues of M are then exactly the eigenvalues of its components.

Example 4.2.3. *Let G_1 be a minimum-colored K_3, and G_2 a minimum-colored K_4, using different colors for each minimum coloring. Let G be the union of G_1 and G_2. The VCICM's of these graphs are given in Figure 4.2.1.*

Consider each graph individually. The eigenvalues of $M(G_1)$ are $(4, 1, 1)$, with corresponding eigenvectors $(1, 1, 1)$, $(-1, 0, 1)$, and $(-1, 1, 0)$. The eigenvalues of $M(G_2)$ are $(8, 2, 2, 0)$ with corresponding eigenvectors $(1, 1, 1, 1)$, $(0, -1, 0, 1)$, $(-1, 0, 1, 0)$, and $(-1, 1, -1, 1)$.

Now, consider the graph union $G = G_1 \cup G_2$. Thus, the eigenvalues of $M(G)$ are $(8, 4, 2, 2, 1, 1, 0)$, which is the disjoint union of the eigenvalues of each submatrix. The eigenvector corresponding to 8 is exactly the eigenvector for $M(G_2)$ corresponding to eigenvalue 8, with zeroes added at the beginning: $(0, 0, 0, 1, 1, 1, 1)$. Note that we add three zeroes, and that $M(G_1)$ is 3×3. Similarly, one eigenvector corresponding to 1 is $(-1, 0, 1, 0, 0, 0, 0)$. The other eigenvectors similarly have zeroes inserted at the beginning and end as appropriate.

$G_1 =$  $G_2 =$ 

$$M(G_1) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad M(G_2) = \begin{bmatrix} 3 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 1 & 2 & 3 \end{bmatrix}$$

$$M(G) = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 & 3 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 2 & 1 & 2 & 3 \end{bmatrix}$$

Figure 4.2.1: VCICM's for the union of a minimum colored K_3 and K_4

4.3 Eigenvalues and Eigenvectors of Standard Graphs

This section contains data on the eigenvalues and eigenvectors of the VCICM's of various standard graphs. We present values for the maximum colorings of all such graphs, as well as for some other colorings. Frequently, it is very difficult to obtain exact values for the eigenvalues of the VCICM's of colorings more complex than a maximum coloring. Most eigenvalue results will be written in the form

$$(\lambda_1^{\langle m_1 \rangle}, \lambda_2^{\langle m_2 \rangle}, \ldots)$$

in nonincreasing order. Here, m_i is the multiplicity of λ_i. Usually, only the eigenvector corresponding to the largest eigenvalue will be presented.

- *Complete Graphs:* For a maximum coloring, the eigenvalues of K_n are $(2n-2^{\langle 1 \rangle}, n-2^{\langle n-1 \rangle})$. The eigenvector corresponding to $2n-2$ is $\vec{1}$.

  For a minimum coloring, the largest eigenvalue is $\lambda_1 = (n-1)^2$ if n is odd, and $(n-2)^2+n$ if n is even. In either case, the eigenvector corresponding to λ_1 is $\vec{1}$.

- *Complete Bipartite Graphs:* For a maximum coloring, the eigenvalues of $K_{i,j}$ are $(i+j^{\langle 1 \rangle}, j^{\langle i-1 \rangle}, i^{\langle j-1 \rangle}, 0)$ The eigenvector corresponding to

$i+j$ is $(i,\ldots,i,j,\ldots,j)$ in which i appears j times, and j appears i times.

- *Cycles:* For a maximum coloring, the eigenvalues of C_n are
$$\left\{2+2\cos\left(\frac{2k\pi}{n}\right) \mid k=1,2,\ldots,n\right\}$$
The largest eigenvalue is always 4 with multiplicity 2 and eigenvector $\vec{1}$. If n is even, the eigenvalue 0 has multiplicity 2. Otherwise, all other eigenvalues have multiplicity 1. This result uses techniques from Section 4.4, and the eigenvalues of the adjacency matrix of C_n given in [20].

- *Paths:* Even the largest eigenvalue of most paths is difficult to state in general. Using Equation (4.10) (from below), we have $\lambda_1 \leq 2\Delta$, so $\lambda_1 \leq 4$. In fact, as n increases, λ_1 approaches 4. The eigenvectors are similarly unpleasant.

- *Wheels:* Let
$$k = \frac{2n-2}{6-n+\sqrt{n^2-8n+32}}.$$
Then the largest eigenvalue of a maximum colored W_n is
$$\lambda = 5 + \frac{1}{k}$$
with eigenvector
$$(1,k,\ldots,k).$$
We will present the derivation of this eigenvalue and eigenvector. The technique used is to guess an eigenvector, and use it to find an eigenvalue. Using *Mathematica*, we have seen a pattern in the largest eigenvector of W_n: specifically, for some value $k<1$, the eigenvector appears to be $(1,k,\ldots,k)$. Supposing that this is true in general, we will attempt to find k. Suppose $v=(1,k,\ldots,k)$ is the eigenvalue associated with λ.

The VCICM of a maximum colored W_n has a nice structure. In general, it is:
$$\begin{bmatrix} n-1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 1 & 0 & 0 & & 1 \\ 1 & 1 & 3 & 1 & 0 & & 0 \\ 1 & 0 & 1 & 3 & 1 & & 0 \\ & & \vdots & & & \ddots & \\ 1 & 1 & 0 & 0 & & 1 & 3 \end{bmatrix}$$

One row has diagonal entry $n-1$ and all ones following. This corresponds to the "hub" of the wheel. Each other row has diagonal entry 3, a 1 in the column corresponding to the hub, and two other 1's. We can extract a value for k by considering the dot product of v with two rows. First, the row corresponding to the hub:

$$\begin{aligned}\lambda &= (n-1,1,\ldots,1)\cdot(1,k,\ldots,k)\\ &= (n-1)+(n-1)k\\ &= (n-1)(k+1).\end{aligned}$$

Second, a generic row corresponding to a "spoke":

$$\begin{aligned}\lambda k &= (1,0,\ldots,0,1,3,1)\cdot(1,k,\ldots,k)\\ &= 1+k+3k+k\\ &= 1+5k.\end{aligned}$$

Putting these together, we have

$$(n-1)(k-1) = \frac{1+5k}{k}.$$

Solving, we find that

$$k = \frac{-(n-6)\pm\sqrt{n^2-8n+32}}{2n-2}.$$

We take only the positive term, as we are looking for a positive simple eigenvalue. Substituting into our expression for λ, we find

$$\lambda = 5+\frac{1}{k}.$$

Checking, this does indeed give an eigenvector and eigenvalue of M. The eigenvalue is positive, and the associated eigenvector is entirely positive. For a square real symmetric matrix (such as M), the largest eigenvalue has a single (all positive) eigenvector v, and all other eigenvectors are orthogonal to v. Thus, all other eigenvectors must have a 0 or negative term. So, $(1,k,\ldots,k)$ must be the single eigenvalue associated with the largest eigenvalue of M. Thus λ must be the largest eigenvalue of M.

Similarly, the largest eigenvalue for the minimum coloring of W_n given in Section 2.1 is $\lambda = n+8$ with associated eigenvector $(1,\frac{3}{n-1},\ldots,\frac{3}{n-1})$. This may be found using an argument similar to the above.

- *Triangle Graphs:* Let

$$k = \frac{n + 2 + \sqrt{(n+6)(n-2)}}{2}.$$

The largest eigenvalue for a maximum colored T_n is

$$\lambda = 2k + 2.$$

The associated eigenvector is

$$(k, k, 1, \ldots, 1).$$

This may be found by using an argument similar to that presented for wheel graphs.

- *Hypercubes:* The hypercube Q_n on $n = 2^k$ vertices is a k-regular graph which may be represented as follows: each vertex is a distinct binary string of length k. Vertices are adjacent if their binary strings differ in exactly one position. The eigenvalues of the adjacency matrix A are of the form $k - 2i$, for $i = 0, \ldots, k$. The multiplicity of $k - 2i$ is $\binom{k}{i}$. Using the techniques from Section 4.4, the eigenvalues of the VCICM of a maximum colored Q_n are $2k - 2i$, for $i = 0, \ldots, k$ with $2k - 2i$ having multiplicity $\binom{k}{i}$ [20].

4.4 Results from the Adjacency Matrix

The adjacency matrix is the focus of much of spectral graph theory. Let $G = (V, E)$ be a graph. Then the *adjacency matrix* A of G has rows and columns indexed by V. Entry A_{ij} is 1 if i and j are adjacent, and 0 otherwise.

Let A be the adjacency matrix of G, and let D be a diagonal matrix containing the degree sequence of G. The eigenvalues of D are given by the degrees of vertices of G. From Lemma 3.2.1, we know that the VCICM of a maximum colored graph G can be written as

$$M = D + A.$$

If $D = kI$ (i.e. if G is regular), then the eigenvalues of M will be the sum of the eigenvalues of D and A. To prove this, let v be an eigenvector of A corresponding to eigenvalue α. Note that v is also an eigenvector of I, so it is an eigenvector of kI. Then:

$$(A + D)v = (A + kI)v = (\alpha v + kv) = (\alpha + k)v.$$

This allows us to use many tools from spectral graph theory to analyze the eigenvalues of a regular maximum colored graph. This connection to the adjacency matrix gives some insight into the structure of the VCICM, and provides an interesting reason to study maximum colorings.

We denote the eigenvalues of A as $\alpha_1 > \ldots \geq \alpha_n$, in nonincreasing order. Repeated eigenvalues will be included. Much is known about the eigenvalues of A. We make use of many results from Godsil and Royle [16]. In particular, the largest eigenvalue of A is simple and positive.

Let G be a k-regular graph with a maximum coloring and VCICM M. Using this notation, we can write the eigenvalues of M in terms of the eigenvalues of A, as follows:

Lemma 4.4.1. *Let G be a k-regular graph with a maximum coloring and VCICM M. Let α_i be the eigenvalues of A and λ_i the eigenvalues of M, all in nonincreasing order. Then*

$$\lambda_i = k + \alpha_i, \text{ for all } 1 \leq i \leq n.$$

This gives us some simple bounds on the eigenvalues. The largest eigenvalue of A is $k = \Delta$, so

$$\lambda_1 = 2k. \tag{4.1}$$

The eigenvector corresponding to this is $\vec{1}$. We can also bound in the other direction. For example, we know that $\lambda_i \geq 0$, so

$$\alpha_i \geq -k. \tag{4.2}$$

This bound on α_n is tight for some graphs, such as C_4. If M is positive definite, the inequality will be strict.

4.4.1 Strongly Regular Graphs

A *strongly regular graph* or *SRG* is a regular graph which satisfies additional restrictions: for any pair of distinct vertices v, w, the number of common neighbors of v and w depends only on whether v and w are adjacent. An SRG is usually denoted by its parameters $(n, k; a, c)$, in which n is the order of the graph, k is the regularity, a is the number of common neighbors of adjacent vertices, and c is the number of common neighbors of nonadjacent vertices. SRG's have a great deal of structure, and their eigenvalues are completely determined by their parameters. All SRG's have exactly three eigenvalues. We follow the notation in [16] and denote the eigenvalues as follows.

$$
\begin{array}{lcll}
k & & & \text{the simple largest eigenvalue} \\
\theta & = & \frac{(a-c)+\sqrt{(a-c)^2+4(k-c)}}{2} & \text{with multiplicity } -\frac{(n-1)\tau+k}{\theta-\tau} \\
\tau & = & \frac{(a-c)-\sqrt{(a-c)^2+4(k-c)}}{2} & \text{with multiplicity } \frac{(n-1)\theta+k}{\theta-\tau}
\end{array}
$$

Thus, the eigenvalues of the VCICM of a maximum colored SRG are completely determined:

$$
\begin{array}{rl}
2k & \text{the simple largest eigenvalue} \\
\theta+k & \text{with multiplicity } -\frac{(n-1)\tau+k}{\theta-\tau} \\
\tau+k & \text{with multiplicity } \frac{(n-1)\theta+k}{\theta-\tau}
\end{array}
$$

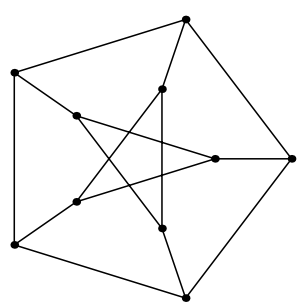


Figure 4.4.1: The Petersen graph

Example 4.4.2. *The Petersen graph (see Figure 4.4.1) is an SRG with parameters* $(10, 3; 0, 1)$*. The eigenvalues of its adjacency matrix are thus* 3 *(simple),* 1 *(multiplicity 5), and* -2 *(multiplicity 4). If it is maximum colored, the eigenvalues of its VCICM* M *are* 6*,* 4*, and* 1 *with the same respective multiplicities.*

Other common examples of SRG's include the line graphs of K_n and $K_{n,n}$. Additionally, graphs arising from Latin Squares and Orthogonal Arrays are strongly regular, as are incidence graphs from certain designs. Of particular interest is the fact that only one class of SRG's has non-integral eigenvalues. These are exactly the Conference graphs, which occur only when the multiplicities of τ and θ are equal. If G is an SRG but not a Conference graph, then all of its eigenvalues are integral (See [16], Chapter 10). This transfers directly to vdec colorings of these graphs:

Observation 4.4.3. *The eigenvalues of the VCICM M of a maximum vdec coloring of an SRG G will always be integral if the multiplicities of τ and θ are different.*

We have very detailed knowledge of the eigenvalues of SRG's, and we also know that the determinant of the VCICM M is zero exactly when some eigenvalue of M is zero. Thus, it is natural to use our knowledge of SRG's to determine when this happens.

Lemma 4.4.4. *Let G be a strongly regular vdec graph with parameters $(n, k; a, c)$. Let G be given a maximum coloring, and let M be the VCICM of G. Then* $\det M = 0$ *if and only if* $G = K_{n-k-1,\ldots,n-k-1}$.

Proof. Suppose G is $K_{n-k-1,\ldots,n-k-1}$, with m vertex classes. Then G is an SRG, with parameters $(n, k; -n + 2k + 2, k)$. Using our equations for τ and θ from above, one of the eigenvalues of G is 0, so $\det M = 0$.

Now suppose $\det M = 0$. This portion of the proof will use many identities from Godsil and Royle ([16], Chapter 10). We have some $\lambda_i = 0$, so the corresponding $\alpha_i = -k$ for the adjacency matrix A. Thus, it must be that $\tau = -k$ is one of the eigenvalues of SRG G. We have $\theta\tau = c - k$, so $\theta = -(c - k)/k$ (we may assume that $k \neq 0$ or else the graph is empty). Thus $\theta = 1 - c/k$. In addition, we have that $\theta + \tau = a - c$, so $\theta = a - c + k$.

We know that θ must be integral unless G is a Conference graph. We also have two expressions for θ. Equating them, we have:

$$\begin{aligned} 1 - \frac{c}{k} &= a - c + k \\ k - c &= ka - kc + k^2 \\ k(1 - a + c - k) &= c \end{aligned}$$

We also have two useful facts. First, $a, c, k \geq 0$. Second, we always have $k \geq c$. This happens because k is the number of neighbors of each vertex, and c is the number of neighbors in common with another, non-adjacent vertex. Thus we cannot have $c > k$, so $c - k \leq 0$. There are several possibilities:

- *Case 1:* $c = 0$. We may assume that $k \neq 0$, so we must have $1 - a + c - k = 0$. Then $1 = a + k - c$. We also know that $c - k \leq 0$, so we have two subcases:
  - *Case 1a:* $c - k = 0$. Then $k = c$, so $a = 1$. This gives $\theta = 0$. However, from $\theta + \tau = a - c$ we have $-k = 1 - k$ which is impossible.

- *Case 1b:* $c-k<0$. Since a is nonnegative, we must have $c-k=-1$, and thus $a=0$. From $\theta=a-c+k$, we get $\theta=1$. But then the equation $\theta\tau=c-k$ gives us $1\cdot(-k)=-1$, so $k=1$. Thus G would consist only of disconnected edges, meaning that it is not vdec. Thus this case is also impossible.

- *Case 2:* $c>0$. Then $1-a+c-k>0$. We know that $c-k\leq 0$, so we must have $c=k$. Thus $a<1$, so $a=0$. This gives $\theta=0$.

Thus we have θ integral, in particular $\theta=0$. From the only case which does not give a contradiction, we have found that the eigenvalues of our SRG are:

$$\theta=-k,\ \tau=0,\ \text{and } k.$$

Thus our SRG has parameters

$$(n,k;0,k).$$

We have a simple set of equations to find the parameters of the complement of a graph. Using these, the complement of any graph with these parameters has parameters

$$(n,\overline{k};\overline{a},\overline{c})=(n,n-k-1;n-k-2,n-2k).$$

Note that $\overline{a}=\overline{k}-1$. By Godsil and Royle's Lemma 10.1.1 [16], this graph must be isomorphic to $mK_{\overline{n}-\overline{k}}=mK_{k+1}$ for some $m>1$. Thus the original graph was isomorphic to a complete multipartite graph $K_{n-k-1,\ldots,n-k-1}$ on n vertices. Thus our result is proved for all cases. □

Thus, if G is an SRG, M has zero as an eigenvalue exactly when G is a complete multipartite graph. As a corollary, $\det M>0$ for all other vdec SRG's with maximum colorings.

4.4.2 Complements

The complement of a vdec graph may or may not be vdec. If it is, we may give it a maximum coloring and algebraically determine the eigenvalues of its VCICM.

The complement $\overline{G}$ of G has edges exactly where G does not have edges. The vertices of the complement have degrees $\overline{d_i}=n-d_i$. Thus, the VCICM $\overline{M}$ has 0's exactly where the original VCICM M had 1's off-diagonal, and vice-versa. Corresponding diagonal entries sum to $n-1$. Thus we have

$$M+\overline{M}=J+(n-2)I.$$

We can use this to characterize the eigenvalues of the VCICM of the complement of a regular vdec graph:

Lemma 4.4.5. *Let G be a k-regular vdec graph such that $\overline{G}$ is also vdec. Let G be given a maximum coloring, and VCICM M. Let $\overline{G}$ be the complement of G, with a maximum coloring and VCICM $\overline{M}$. Then the eigenvalues $\overline{\lambda}_i$ of $\overline{M}$ are exactly*

$$\overline{\lambda}_1 = 2n - 2 - \lambda_1$$

and

$$\overline{\lambda}_i = n - \lambda_{n-i+1} - 2 \quad \textit{for } 2 \leq i \leq n-1.$$

Proof. We know that M has eigenvector $\vec{1}$ corresponding to $\lambda_1 = k$, so let v be another eigenvector of M with eigenvalue λ. Then v is orthogonal to $\vec{1}$, and thus $Jv = 0$. We can write

$$(M + \overline{M})v = (J + (n-2)I)v = (n-2)v. \tag{4.3}$$

Similarly,

$$(M + \overline{M})v = \lambda v + \overline{M}v. \tag{4.4}$$

Combining Equations (4.3) and (4.4), we have

$$\overline{M}v = (n - \lambda - 2)v.$$

Thus, for each eigenvalue λ_i of M, $n - \lambda_i - 2$ is an eigenvalue of $\overline{M}$ with the same eigenvector. We could restate this as:

$$\overline{\lambda}_i = n - \lambda_{n-i+1} - 2.$$

This takes into account the fact that the largest eigenvalue of M will give the smallest eigenvalue of $\overline{M}$, and so on.

For the largest eigenvalue λ_1, the associated eigenvector is $\vec{1}$. Although $\vec{1}$ is not orthogonal to the rows of J, we may apply the same argument. This gives an extra $n\vec{1}$ term, which in turn gives:

$$\overline{\lambda}_1 = 2n - \lambda_1 - 2. \qquad \square$$

If G is not regular, we can follow a similar argument to find a slightly weaker result:

Lemma 4.4.6. *Let G be any vdec graph such that $\overline{G}$ is also vdec. Let G be given a maximum coloring, with VCICM M. Let $\overline{G}$ be the complement of G, with a maximum coloring and VCICM $\overline{M}$. Let θ_i be the eigenvalues of $M - J$. Then the eigenvalues $\overline{\lambda_i}$ of $\overline{M}$ are exactly*

$$\overline{\lambda_i} = n - \theta_{n-i+1} - 2, \ \textit{for } 1 \leq i \leq n.$$

If G is regular, all but the largest eigenvector will be orthogonal to the rows of J, so the eigenvalues we obtain will be the same as above.

This raises a natural question: if G is vdec, when is $\overline{G}$ vdec? Note that this may not necessarily include only regular graphs. We must have at most one isolated vertex in $\overline{G}$, so there must be at most one vertex of degree $n-1$ in G. There must be no isolated edges, so there must be no pair of vertices v, w in G such that $vw \notin E(G)$ and $\deg v = \deg w = n-2$.

Thus, very many of our standard graphs have vdec complements. In particular, all of the following are vdec graphs whose complements are also vdec:

- Cycles C_n with $n \geq 5$.
- Paths P_n with $n \geq 4$.
- Complete bipartite graphs $K_{m,n}$ with $m, n \geq 3$.
- Wheels W_n with $n \geq 5$.

Note that this does not depend on G being connected.

We also have some interesting bounds on the values of the eigenvalues of the VCICM of the complement graph. For example, for a k-regular graph whose complement is vdec, we know that $\overline{\lambda}_i \geq 0$, so

$$\lambda_i \leq n-2, \text{ for } 2 \leq i \leq n-1.$$

Note that this is tight for maximum colorings of K_n and T_n, among others.

4.5 A Note on Eigenvalue Plots

In the next sections, we will see many plots which demonstrate the tightness of various bounds on the largest eigenvalue of the VCICM. This section explains these plots, and the method used to generate them.

Figure 4.5.1 demonstrates a typical plot.

The plot represents eigenvalues of various colorings of a single graph G. Each point on the plot represents the largest eigenvalue of the VCICM of a particular coloring of G. The horizontal axis represents the number of colors used in the coloring, and the vertical axis represents the largest eigenvalue of the VCICM. Note that for a given number of colors, there are many different colorings, and thus many different eigenvalues. The line running below (or above) these points is the bound currently being examined.

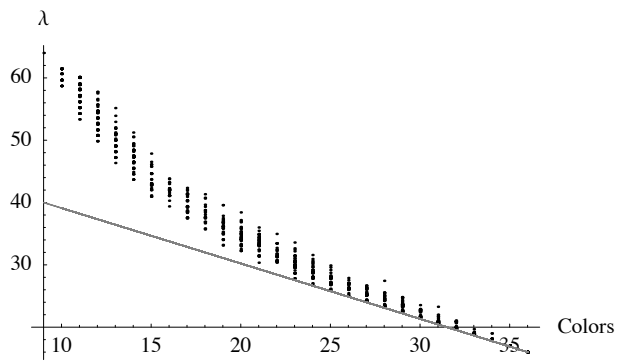


Figure 4.5.1: A sample eigenvalue plot

The data points correspond to different graph colorings, so we must have a method to obtain these colorings. This may be split up into several related functions:

Function: RandomEdgeColor

Input: A graph G, coloring size parameter $0 < s \leq 1$.

For $i = 1$ to $|E|$

(a) If edge e_i may be properly colored with an existing color, randomly select one such color c_j.

(b) Generate a new unused color c_k.

(c) Randomly select $0 \leq r \leq 1$ with uniform distribution.

i. If c_j exists and $r > s$, color e_i with c_j.

ii. Otherwise, color e_i with c_k.

Output: G with a proper edge-coloring.

In words, the RandomEdgeColor function colors a graph one edge at a time. Its goal is to create a proper edge-coloring, not necessarily a vdec coloring. At each step, it determines which colors may be used to color the edge e_i so as to maintain a proper edge-coloring. It also generates a new, previously unused color. The parameter s determines which is used. If $s = 1$, we get a maximum coloring. If s is near 0, we get a near-minimum edge-coloring. Thus, the parameter may be used to control the number of colors used in the coloring.

Function: RandomVDECColor

Input: A vdec graph G, coloring size parameter $0 < s \leq 1$.

1. RandomEdgeColor G with s.
2. If the coloring of G is not vdec, go to 1.

Output: G with a vdec coloring.

The RandomVDECColor function relies on the fact that the RandomEdgeColor function often produces vdec colorings. This function keeps trying to color G until one coloring is vdec. The input graph must be vdec, so this is likely to happen eventually. The larger the value of s, the faster this will happen.

To generate plots, we wish to ensure a wide range of colorings. This may be done by repeatedly calling RandomVDECColor with a range of s values. This will force colorings with different numbers of colors, giving a representative sample of the eigenvalues. This method has been used to generate all eigenvalue plots in this thesis. Thus, each plot may be taken to be representative of the possible vdec colorings of G. Of course, not every possible coloring may appear on the plot.

A typical plot has 1000 data points (some may coincide). The s range is from .05 to .95 in steps of .1, with 100 colorings per s-value. This guarantees several near-minimum colorings, and a maximum coloring, as well as many other colorings. Some plots will also include an upper line of data points, which come from the "largest" possible coloring for a given number of colors. These colorings give the VCICM with the largest average row sum, which will prove important. These may not be vdec colorings, but they are proper edge-colorings.

4.6 Lower Bounds in General

This section presents many lower bounds on the largest eigenvalue of the VCICM. These bounds are tight for many graphs and colorings. Some depend on specific graphs or colorings, but many work for any vdec graph with any vdec coloring. In this and following sections, we will use $\lambda = \lambda_1$ to denote the largest eigenvalue. The strictly positive eigenvector associated with λ will be denoted v, with entries $(v_1, \ldots, v_n)$. Unless otherwise noted, v will be normalized, so that its largest entry is 1.

We will begin with a quick estimate of the eigenvalue. Let G be a vdec graph with any proper edge-coloring, and let M be its VCICM. We may

write the dot product of one row of M with v as:

$$\begin{aligned} \lambda v_i &= M_{ii}v_i + \sum_{j=1, j\neq i}^{n} M_{ij}v_j \\ &> M_{ii}v_i. \end{aligned}$$

This estimate is valid because each term in the summation is at least zero. At least one off-diagonal element is nonzero in the VCICM, thus making the inequality strict. However, we made no assumptions about M_{ii}. This gives us a simple lower bound on λ:

Lower Bound 4.6.1. *Let G be a vdec graph with VCICM M. Then $\lambda > \Delta$.*

Proof. In the VCICM, the diagonal entries are equal to the degrees of vertices of G, including vertices of maximum degree. Using the above bounding argument, we may divide by $v_i \neq 0$ and take M_{ii} to represent a vertex of maximum degree. This gives the result. □

We may generalize this result to any real nonnegative symmetric square matrix M which has an entirely positive eigenvector. For example, any nonnegative symmetric matrix for which the Perron-Frobenius theorem holds will suffice, although the inequality may not be strict. If each row has at least one nonzero off-diagonal element, then the inequality will remain strict. This will happen because the estimate involves removing the summation, which includes a term for each off-diagonal entry. The matrix satisfies the Perron-Frobenius Theorem, so the entries in v will be entirely positive. Thus, at least one term of the summation is nonzero.

In fact, we know of no graphs with $\lambda < \Delta + 1$. However, $\lambda = \Delta + 1$ is tight. The star graph on n vertices has exactly one vdec coloring, which has $\lambda = \Delta + 1$ and associated eigenvector $(1, \frac{1}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1})$. A proof that $\lambda \geq \Delta + 1$ is, as yet, unknown.

Conjecture 4.6.2. *Let G be a vdec graph with VCICM M. Then $\lambda \geq \Delta+1$.*

4.7 Average Row Sum Lower Bounds

Our most fruitful lower bounds will come from the average row sum of the VCICM.

Let M be a $n \times m$ matrix. Let s_i be the sum of the entries in row i of M, $1 \leq i \leq n$. Then the *average row sum* of M is

$$a = \frac{s_1 + \ldots + s_n}{n}.$$

Each s_i can be written in terms of the entries of M, in this fashion:

$$s_i = M_{i1} + M_{i2} + \ldots + M_{in}$$

But, M is symmetric. If we add up all row sums of M, entry M_{ij} will appear twice if $i \neq j$, and once if $i = j$. Thus the average row sum may be written as

$$a = \frac{M_{11} + M_{22} + \ldots + M_{nn} + 2(M_{12} + \ldots + M_{n-1,n})}{n}. \tag{4.5}$$

In examining the eigenvalues of M, it is logical to consider the dot product of an eigenvector v of M with a row of M. This gives us an equation for the eigenvalue λ, in terms of the entries of the matrix and the eigenvector. We obtain n such equations, one for each row of the matrix. Adding these equations together and simplifying, we obtain an expression giving $n\lambda$ in terms of the entries of M and v in which each diagonal entry of M will appear once, and each off-diagonal entry will appear twice. This is similar to the average row sum of M. This similarity gives rise to a useful bound:

Theorem 4.7.1. *Let M be a real nonnegative symmetric $n \times n$ matrix for which the Perron-Frobenius theorem holds. Let a be the average row sum of M. Then the largest eigenvalue λ of M is simple, with positive eigenvector v, and $\lambda \geq a$.*

Proof. Since M fits the assumptions of the Perron-Frobenius theorem, it has a simple real largest eigenvalue λ with a strictly positive eigenvector v. Equation 4.5 gives the average row sum written in terms of the entries of the matrix.

Setting this aside for a moment, consider the dot product of one row of M with v. For row i of M, we have the relation

$$\lambda v_i = M_{i,1}v_1 + \ldots + M_{i,n}v_n.$$

Recall that M has nonnegative entries, and v has strictly positive real entries. Thus we may divide by v_i, giving us:

$$\lambda = M_{ii} + \frac{M_{i1}v_1 + \ldots + M_{i,i-1}v_{i-1} + M_{i,i+1}v_{i+1} + \ldots + M_{in}v_n}{v_i}.$$

Now, add up the above equation for all rows of M. We get:

$$\begin{aligned} n\lambda &= M_{11} + M_{22} + \ldots + M_{nn} \\ &\quad + \frac{M_{1,2}v_2 + \ldots + M_{1n}v_n}{v_1} + \ldots + \frac{M_{n1}v_1 + \ldots + M_{n,n-1}v_{n-1}}{v_n}. \end{aligned}$$

Each M_{ij} with $i \neq j$ appears twice due to the symmetry of M: once with coefficient $\frac{v_i}{v_j}$ and once with coefficient $\frac{v_j}{v_i}$. Combining these terms, we have M_{ij} with coefficient $\frac{v_i^2+v_j^2}{v_i v_j}$. Thus, we may rewrite the above sum as

$$n\lambda = M_{11} + M_{22} + \ldots + M_{nn} + \frac{v_1^2 + v_2^2}{v_1 v_2} M_{12} + \ldots + \frac{v_{n-1}^2 + v_n^2}{v_{n-1} v_n} M_{n-1,n}. \quad (4.6)$$

Now, note that $\frac{v_i^2+v_j^2}{v_i v_j}$ is actually twice the ratio of the arithmetic mean and geometric mean of v_i^2 and v_j^2. Since the geometric mean is always less than or equal to the arithmetic mean, this ratio must be at least 1, so twice the ratio is at least 2. Thus, we may estimate the sum in Equation (4.6) as:

$$n\lambda \geq M_{11} + M_{22} + \ldots + M_{nn} + 2M_{12} + \ldots + 2M_{n-1,n}$$

or equivalently,

$$\lambda \geq \frac{M_{11} + M_{22} + \ldots + M_{nn} + 2M_{12} + \ldots + 2M_{n-1,n}}{n}. \quad (4.7)$$

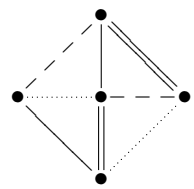
The right hand side of Equation (4.7) is exactly the expression for a in Equation (4.5), so we have $\lambda \geq a$. □

In this section, we will mainly bound the average row sum of the VCICM, using information about the associated graph and coloring. We will then use Theorem 4.7.1 to bound the largest eigenvalue of the same VCICM's.

Note that this theorem applies to many more matrices than the VCICM's of graphs. The result was inspired by the uses of average row sums in eigenvalue interlacing arguments [16]. It also applies to the adjacency matrix of any graph G, as used in [11]. For adjacency matrices, this tells us that the largest eigenvalue is bounded below by the average of the degree sequence of G, also called the "mean valency" of G. The mean valency is a well studied use of average row sums.

Also note that $\frac{v_i^2+v_j^2}{v_i v_j} = 2$ when $v_i = v_j$. This happens for every i, j when $v = \vec{1}$, which in turn happens only when every row of G has exactly the same sum. Thus the bound is tight for all maximum colored regular graphs. It is also tight for certain other colorings, such as the minimum coloring given for K_n in Section 2.1, and for any other coloring in which the rows of G have the same sum.

Example 4.7.2. *We have two examples for the average row sum bound in Theorem 4.7.1.*



$$M = \begin{bmatrix} 4 & 3 & 3 & 3 & 3 \\ 3 & 3 & 2 & 2 & 2 \\ 3 & 2 & 3 & 2 & 2 \\ 3 & 2 & 2 & 3 & 2 \\ 3 & 2 & 2 & 2 & 3 \end{bmatrix}$$

Figure 4.7.1: A minimum colored W_4 and its VCICM

1. *Let G be a minimum-colored K_4 with VCICM M, as in Figure 4.2.1. The average row sum is $a = (8+8+8+8)/4 = 8$. The largest eigenvalue is $\lambda = 8$, so $\lambda = a$. Note that $\vec{1}$ is an eigenvector for this graph and coloring.*

2. *Next, let G be a minimum-colored W_4 with VCICM M, as in Figure 4.7.1. The average row sum is $a = (16+12+12+12+12)/5 = 12.8$. The largest eigenvalue is $\lambda = 13$, so the bound inequality is strict: $\lambda > a$.*

4.7.1 Two Simple Bounds

Theorem 4.7.1 is particularly useful, as it converts a difficult problem (finding the largest eigenvalue of a matrix) into a much easier problem (bounding the average row sum of a matrix). The VCICM has a great deal of structure, so we may bound the average row sum very tightly. Maximum colorings in particular make the average row sum easy to bound, because there are exactly d_i ones in each row. A simple example of this comes from maximum colored graphs.

Lower Bound 4.7.3. *Let G be a vdec graph with any proper edge-coloring and VCICM M. Then the average row sum a of M satisfies*

$$a \geq \frac{4e}{n}.$$

Proof. Suppose G is maximum colored. Then the average row sum will be

$$\begin{aligned} a &= \frac{2(d_1 + \ldots + d_n)}{n} \\ &\geq \frac{2(2e)}{n} \\ &= \frac{4e}{n}. \end{aligned}$$

Here, we use the fact that $d_1 + \ldots + d_n = 2e$. If G is not maximum colored, then each row sum will be larger, and the estimate on the average row sum will be strict. □

Using Theorem 4.7.1, we immediately bound the largest eigenvalue λ:

Corollary 4.7.4. *Let G be a vdec graph with any proper edge-coloring and VCICM M.*

$$\lambda \geq \frac{4e}{n}.$$

Note that every d_i is at least δ, so we may simplify this bound further:

Corollary 4.7.5. *Let G be a vdec graph with any proper edge-coloring and VCICM M. Then $a \geq 2\delta$ and therefore $\lambda \geq 2\delta$.*

This bound is better than $\lambda > \Delta$ if the degrees of G are closely clustered. It is tight for regular maximum colored graphs.

Example 4.7.6. *(Good and bad cases for two bounds)*

1. *Let $G = W_5$ with a maximum coloring. The degrees of G are relatively close. From the bound $\lambda > \Delta$, we have $\lambda > 4$. From the bound $\lambda \geq 2\delta$, we have $\lambda \geq 6$. The actual value of λ is $\lambda = \frac{1}{2}(9 + \sqrt{17}) \approx 6.56$. Note that the average row sum is $a = 6.4$.*

2. *Now, let $G = K_{1,5}$, the star on 6 vertices, with a maximum coloring. From $\lambda > \Delta$, we have $\lambda > 5$. From $\lambda \geq 2\delta$, we have $\lambda \geq 2$. The actual value of λ is $\lambda = 6$. Here the average row sum is about is $a = 3.33$.*

Lower Bound 4.7.3 lends itself to simplification if we know the degree sequence of a graph, or if we may estimate e easily. Thus, we may obtain better bounds. For example:

- If G is a tree, $a \geq \frac{4(n-1)}{n} = 4 - \frac{4}{n}$.
- If G is k-regular, $a \geq \frac{2nk}{n} = 2k$.

- If G is complete, $a \geq \frac{2n(n-1)}{n} = 2(n-1)$.

We may use Theorem 4.7.1 to obtain eigenvalue estimates from all of the above average row sum bounds as well. These bounds are not particularly tight, although (as usual), maximum colored regular graphs will make some tight.

4.7.2 Simple Color Bound

Note that none of our bounds thus far depend on the number of colors actually used in the graph. Most work for maximum vdec colorings, and are less tight for colorings with fewer colors. We remedy this situation by considering the number of colors actually used in a graph coloring. We will do this by changing a graph's coloring in a predictable manner, and examining the corresponding change in the VCICM's row sums.

Lemma 4.7.7. *Let G be a vdec graph with proper edge-coloring π. Suppose G has VCICM M with average row sum a. Suppose color c_j is used on k edges, and c_i is used on exactly one edge. Further suppose that no edges of color c_i and c_j are adjacent. Change the edge of color c_i to color c_j to obtain a new coloring π'. Let the VCICM of π' be M', with average row sum a'. Then*

$$a' - a \geq \frac{8k}{n}.$$

Proof. Let $C_i = \{v \in V : \text{ there exists } e = vw \in E \text{ such that } \pi(e) = c_i\}$, and let $C_j = \{v \in V : \text{ there exists } e = vw \in E \text{ such that } \pi(e) = c_j\}$. In other words, C_i contains all vertices incident to edges using color c_i in coloring π, and C_j contains all vertices incident to edges using color c_j in coloring π. Note that the set of all edges using color c_j in coloring π' is exactly $C_i \cup C_j$. Let $f = uv$ be the single edge of color c_i in π. Then $C_i = \{u, v\}$ and $u, v \notin C_j$.

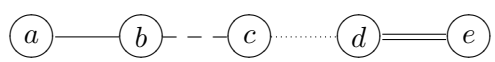
Let $w \in C_j$. We know that $w \neq u, v$. We also know that $S_{\pi'}(v) \cap S_{\pi'}(w)$ contains exactly one more color than $S_\pi(v) \cap S_\pi(w)$, specifically color c_j. The reason is as follows: in coloring π, no vertices in C_i are incident to an edge of color c_j. However, in coloring π', all vertices in C_i do share color c_j with every vertex of C_j, as well as all colors they formerly shared. Thus $|S_{\pi'}(v) \cap S_{\pi'}(w)| = |S_\pi(v) \cap S_\pi(w)| + 1$.

Thus, we obtain M' by adding 1 to each entry M_{ab} of M such that $a \in C_j$ and $b = u$ or v. We will count changed entries as follows. For the row of M corresponding to u, we add one to each entry corresponding to an element of C_j. There are k rows of color j in coloring π, so $|C_j| = 2k$. Thus, we add

$2k$ to the total row sum. We do similarly for the row of M corresponding to v, thus adding a total of $4k$ to the total row sum, between these two vertices. In addition, each row of M corresponding to some $w \in C_j$ must have one added to each entry corresponding to u and v, thus adding 2 total. This adds 2 for each of $2k$ rows, for a total of $4k$ more added to the total row sum. Thus, we add a total of $8k$ to the total row sum, or $8k/n$ to the average row sum. □

Observation 4.7.8. *Let G be a vdec graph with any proper edge-coloring π. If we begin with a maximum coloring of G, we may repeatedly change the color of individual edges so that we obtain a coloring isomorphic to π. In particular, we need change the color of each edge at most once: change it to the color which it has in coloring π. Any proper edge-coloring of any graph may be obtained by this method. This allows us to use Lemma 4.7.7 to estimate the average row sum of the VCICM of any coloring.*

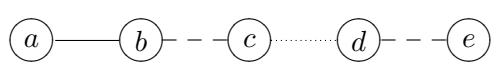
Example 4.7.9. *Consider a maximum colored path P_5:*



The VCICM of this maximum colored graph is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Suppose we change edge de to have the same color as edge bc. Note that this coloring is no longer vdec, but the change in the average row sum will be the same whether the coloring is vdec or not. The new coloring is:



The VCICM has eight changed entries, which are shown in bold:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & \mathbf{1} & \mathbf{1} \\ 0 & 1 & 2 & \mathbf{2} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{2} & 2 & 1 \\ 0 & \mathbf{1} & \mathbf{1} & 1 & 1 \end{bmatrix}$$

The average row sum of the original matrix is $16/5 = 3.2$, while the average row sum of the changed matrix is $24/5 = 4.8$.

We may use Lemma 4.7.7 and Observation 4.7.8 to find the following bound:

Lower Bound 4.7.10. *(Simple color bound) Let G be a vdec graph with a vdec coloring π using $c \leq e$ colors. Let M be the VCICM of G with π, and let a be the average row sum of M. Then $a \geq \frac{4}{n}(3e - 2c)$.*

Proof. Suppose we begin with a maximum coloring of G, with edges $e_1, \ldots, e_c$ colored to agree with π, such that no two of $e_1, \ldots, e_c$ have the same color. Thus all colors used in π are also used in this maximum coloring. Lower Bound 4.7.3 implies that the average row sum for this VCICM is at least $4e/n$. One by one, change the colors of the remaining edges to their respective colors in π. Each time we make such a change, there is already at least one edge with that new color. By Lemma 4.7.7, the average row sum increases by at least $8/n$. We change exactly $e - c$ edges, so the total change in average row sum is at least $(8/n)(e - c)$.

Thus the average row sum for the VCICM of π is at least

$$\frac{4e}{n} + \frac{8}{n}(e - c) = \frac{4}{n}(3e - 2c),$$

which is as we claimed. □

As usual, Theorem 4.7.1 gives us an eigenvalue bound based on Lower Bound 4.7.10.

Corollary 4.7.11. *Let G be a vdec graph with a vdec coloring π using $c \leq e$ colors. Then $\lambda \geq \frac{4}{n}(3e - 2c)$.*

If $c = e$, we obtain the same bound as in Lower Bound 4.7.3. Figures 4.7.2 and 4.7.3 are plots comparing eigenvalues of many vdec colorings of a graph G to the bound values. Further details of the generation of these plots may be found in Section 4.5.

We see from the figures that the bound is much tighter for colorings using more colors, than for those using fewer colors. In particular, the eigenvalues for K_9 begin a steep ascent away from the bound values as we move left from about $c = 18$. This is the point at which no color is used on only one edge any more. When this happens, we will actually be adding more than 8 to the total row sum. For the purposes of this bound, we ignore this possibility at the cost of loosening the bound. However, we will include it in the next bound, to obtain a much tighter value.

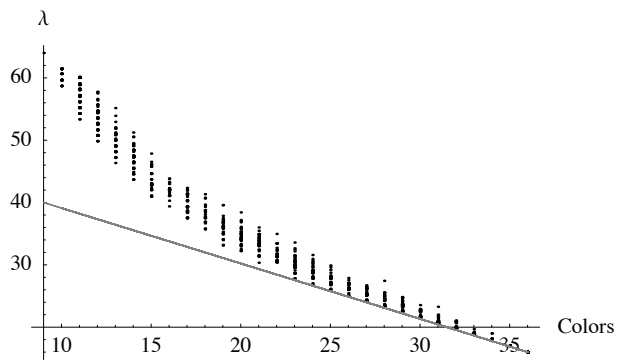


Figure 4.7.2: Simple color bound values for K_9

4.7.3 Matching Color Bound

To produce a better bound, we must consider the parameter k in Lemma 4.7.7. In Lower Bound 4.7.10, we always assume $k = 1$, but it is often larger. We must consider how many times an edge is changed to a color which has already been used, and how many times it may have been used.

Our goal is to construct an auxiliary coloring π' of G. Although we do have a particular coloring π of G in mind, we are not trying to construct π. Rather, we are trying to find a coloring π' of G with c colors, whose VCICM has the lowest average row sum among all colorings of G with c colors. By finding the average row sum of this VCICM, we bound the average row sum of the VCICM of G with coloring π.

Let π' be a proper edge-coloring of G using c colors, whose VCICM has the smallest possible average row sum. Suppose that we constructed π' by giving G a maximum coloring, and then changing one edge at a time to its color in π'. We must change $e - c$ edges: c edges begin with the correct color (in the maximum coloring), and the rest must be changed to their color in π'. We may determine the change in average row sum (from maximum coloring to π') by counting the number of times each color is used in π'. For example, if color c_i is used 3 times in π', then one edge f already had color c_i in the maximum coloring. Using Lemma 4.7.10 with $k = 1$, we change another edge g to color c_i. Finally, using Lemma 4.7.10 with $k = 2$, we change the final edge h to color c_i. Thus, the total added to the row sum is $8 + 16 = 24$. This happens regardless of when we change f and g to color c_i, or in which order.

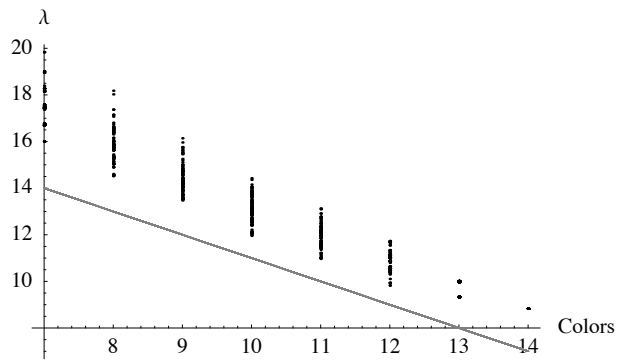


Figure 4.7.3: Simple color bound values for W_8

Suppose that one color c_i is used $m+1$ times in π', while another color c_j is used $m-1$ times in π'. Using Lemma 4.7.10 m times with $k = 1, 2, \ldots, m$ respectively, the edges of color c_i contribute $8 + 16 + \ldots + 8m$ to the total row sum, while the edges of color c_j contribute $8 + 16 + \ldots + 8(m-2)$. If instead we change one edge of color c_i to color c_j, so that both colors are on $m-1$ edges, each contributes $8 + 16 + \ldots + 8(m-1)$ to the total row sum. Thus the average row sum would be 8 less. Therefore, we may assume that all colors in π' are used either m or $m+1$ times, for some integer m.

We must determine m. The value of m comes from distributing the colors of π' as evenly as possible throughout the graph. To see this, we set up a system of simultaneous equations. We wish to find m such that x colors are used on m edges, and y colors are used on $m+1$ edges. Thus we have:

$$\begin{aligned} mx + (m+1)y &= e \\ x + y &= c \end{aligned}$$

Solving this system, we find that $m = \frac{e-y}{c}$. But, m must be an integer. We only have control over the choice of y. Thus, we pick the smallest integer y such that m is an integer. Then $0 \leq y \leq c-1$, and so we may rewrite

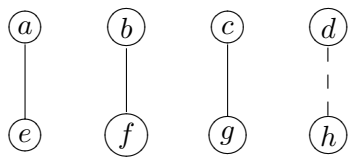
$$m = \left\lfloor \frac{e}{c} \right\rfloor.$$

Then we have

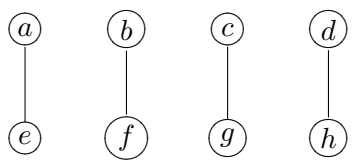
$$y = e - c\left\lfloor \frac{e}{c} \right\rfloor.$$

As a result, we also know $m+1$ and $x = c - y$.

Thus, the minimum average row sum may be obtained in a coloring π' in which $x = c(\lfloor e/c \rfloor + 1) - e$ colors are used on $m = \lfloor e/c \rfloor$ edges, and $y = e - c\lfloor e/c \rfloor$ colors are used on $\lfloor e/c \rfloor + 1$ times.



$$
\begin{array}{c|ccccccccccc}
 & \cdots & a & b & c & d & e & f & g & h & \cdots \\
\hline
\vdots & & & & & \vdots & & & & & \\
a & & 9 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & \\
b & & 1 & 9 & 1 & 0 & 1 & 1 & 1 & 0 & \\
c & & 1 & 1 & 9 & 0 & 1 & 1 & 1 & 0 & \\
d & \cdots & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 1 & \cdots \\
e & & 1 & 1 & 1 & 0 & 9 & 1 & 1 & 0 & \\
f & & 1 & 1 & 1 & 0 & 1 & 9 & 1 & 0 & \\
g & & 1 & 1 & 1 & 0 & 1 & 1 & 9 & 0 & \\
h & & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 9 & \\
\vdots & & & & & \vdots & & & & &
\end{array}
$$



$$
\begin{array}{c|ccccccccccc}
 & \cdots & a & b & c & d & e & f & g & h & \cdots \\
\hline
\vdots & & & & & \vdots & & & & & \\
a & & 9 & 1 & 1 & \mathbf{1} & 1 & 1 & 1 & \mathbf{1} & \\
b & & 1 & 9 & 1 & \mathbf{1} & 1 & 1 & 1 & \mathbf{1} & \\
c & & 1 & 1 & 9 & \mathbf{1} & 1 & 1 & 1 & \mathbf{1} & \\
d & \cdots & \mathbf{1} & \mathbf{1} & \mathbf{1} & 9 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & \cdots \\
e & & 1 & 1 & 1 & \mathbf{1} & 9 & 1 & 1 & \mathbf{1} & \\
f & & 1 & 1 & 1 & \mathbf{1} & 1 & 9 & 1 & \mathbf{1} & \\
g & & 1 & 1 & 1 & \mathbf{1} & 1 & 1 & 9 & \mathbf{1} & \\
h & & \mathbf{1} & \mathbf{1} & \mathbf{1} & 1 & \mathbf{1} & \mathbf{1} & \mathbf{1} & 9 & \\
\vdots & & & & & \vdots & & & & &
\end{array}
$$

Figure 4.7.4: Changing one edge to be the same color as three other edges

Example 4.7.12. *Figure 4.7.4 shows the effect of changing one edge to be the same color as three other edges. For simplicity, we show only the relevant subgraph of some graph G and submatrix of the VCICM. The diagonal entries are set at 9, although the degrees of these vertices may be arbitrary. We also assume that none of the displayed vertices share any other colors with each other. This could happen; the entries in each example would just be correspondingly larger, while still differing by the same amount.*

By changing edge dh to the same color as the other three edges, we add exactly 24 to the row sum in the submatrix. Changed entries are shown in bold. If any other edges share the new color, the row sum will increase by more than 24. Thus, the total row sum increases by at least 24.

We may use these values to obtain the following bound:

Lower Bound 4.7.13. *(Matching color bound) Let G be a vdec graph with any vdec coloring π using $c \leq e$ colors. Let M be the VCICM of G with π, and let a be the average row sum of M. Then*

$$a \geq \frac{4}{n}\left(e + \left\lfloor \frac{e}{c} \right\rfloor \left(2e - c - c\left\lfloor \frac{e}{c} \right\rfloor\right)\right).$$

Proof. We will find a coloring π' which uses c colors, and whose VCICM has the minimum row sum among all colorings using c colors. Based on the argument presented above, coloring π' will have $e - c\lfloor e/c \rfloor$ colors used on $\lfloor e/c \rfloor + 1$ edges, and $c\lfloor e/c \rfloor + c - e$ colors used on $\lfloor e/c \rfloor$ edges. Note that this totals to c colors and e edges.

Using Lemma 4.7.10 repeatedly, we can find the total row sum of M, the VCICM of G with coloring π'. The row sum of the VCICM of G with a maximum coloring is $4e$. Thus the average row sum of M is:

$$\begin{aligned} a &= \frac{4e}{n} + \frac{8}{n}\left(e - c\left\lfloor \frac{e}{c} \right\rfloor\right)\left(1 + 2 + \ldots + \left\lfloor \frac{e}{c} \right\rfloor\right) \\ &+ \frac{8}{n}\left(c\left\lfloor \frac{e}{c} \right\rfloor + c - e\right)\left(1 + 2 + \ldots + \left\lfloor \frac{e}{c} \right\rfloor - 1\right) \end{aligned}$$

Evaluating the sums of the form $1 + 2 + \ldots + \left\lfloor \frac{e}{c} \right\rfloor$ and simplifying, we have:

$$\begin{aligned} a &= \frac{4}{n}\Bigg(e + 2\left(e - c\left\lfloor \frac{e}{c} \right\rfloor\right)\frac{\left\lfloor \frac{e}{c} \right\rfloor\left(\left\lfloor \frac{e}{c} \right\rfloor + 1\right)}{2} \\ &+ 2\left(c\left\lfloor \frac{e}{c} \right\rfloor + c - e\right)\frac{\left\lfloor \frac{e}{c} \right\rfloor\left(\left\lfloor \frac{e}{c} \right\rfloor - 1\right)}{2}\Bigg) \end{aligned}$$

Simplifying further, we have:

$$\begin{aligned} a &= \frac{4}{n}\left(e + \left\lfloor \frac{e}{c} \right\rfloor\left(\left(e - c\left\lfloor \frac{e}{c} \right\rfloor\right)\left(\left\lfloor \frac{e}{c} \right\rfloor + 1\right) + \left(c\left\lfloor \frac{e}{c} \right\rfloor + c - e\right)\left(\left\lfloor \frac{e}{c} \right\rfloor - 1\right)\right)\right) \\ &= \frac{4}{n}\left(e + \left\lfloor \frac{e}{c} \right\rfloor\left(\left(e - c\left\lfloor \frac{e}{c} \right\rfloor\right)\left(\left(\left\lfloor \frac{e}{c} \right\rfloor + 1\right) - \left(\left\lfloor \frac{e}{c} \right\rfloor - 1\right)\right) + c\left(\left\lfloor \frac{e}{c} \right\rfloor - 1\right)\right)\right) \\ &= \frac{4}{n}\left(e + \left\lfloor \frac{e}{c} \right\rfloor\left(2\left(e - c\left\lfloor \frac{e}{c} \right\rfloor\right) + c\left(\left\lfloor \frac{e}{c} \right\rfloor - 1\right)\right)\right) \end{aligned}$$

This simplifies to the final form of the bound:

$$\lambda \geq a = \frac{4}{n}\left(e + \left\lfloor \frac{e}{c} \right\rfloor\left(2e - c - c\left\lfloor \frac{e}{c} \right\rfloor\right)\right) \tag{4.8}$$

□

At the cost of loosening the bound further, we may simplify Equation (4.8) and remove some of the floors. In particular, the floor inside the innermost term may be removed, maintaining the inequality, to get

$$a \geq \frac{4}{n}\left(e + \left\lfloor \frac{e}{c} \right\rfloor (e - c)\right).$$

We must make additional estimates to remove the other floors. However, we may remove the floors and note that this adds at most $(c-1)/c$ to the term. Thus we may replace e with $e - c + 1$ and simplify to obtain

$$a \geq \frac{4}{n}(c - e - 1) + \frac{4}{cn}(e^2 + e) \tag{4.9}$$

Equation (4.9) is nice, but Equation (4.8) is tighter. Consequently, will usually Equation (4.8).

A direct corollary of Lower Bound 4.7.13 and Theorem 4.7.1 is the following eigenvalue estimate:

Corollary 4.7.14. *Let G be a vdec graph with any vdec coloring and VCICM M. Then*

$$\lambda \geq \frac{4}{n}\left(e + \left\lfloor \frac{e}{c} \right\rfloor \left(2e - c - c\left\lfloor \frac{e}{c} \right\rfloor\right)\right).$$

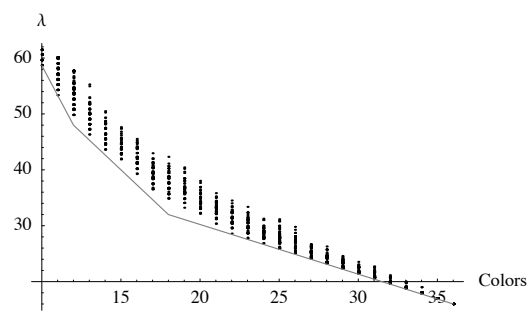


Figure 4.7.5: Matching color bound values for K_9

Example 4.7.15. *Figure 4.7.5 shows eigenvalues for various colorings of K_9. Note that the bound line follows the sharp upward trend of the eigenvalues to the left. Each change in the slope of the bound line corresponds to a point at which the minimum number of edges per color increases.*

Figure 4.7.6 shows eigenvalues for various colorings of the complete bipartite graph $K_{6,4,2}$. Note that the bound is less sharp for non-regular graphs, but still follows the upward trend of eigenvalues.

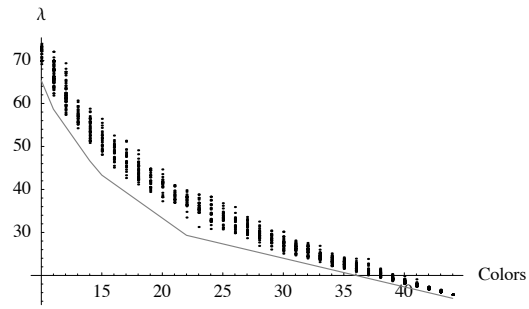


Figure 4.7.6: Matching color bound values for $K_{6,4,2}$

This bound is made tighter than the previous bound by considering the number of each color in a particular graph coloring. However, we generalize greatly and look only at the total number of colors. The final lower bound in this chapter will depend on the number and arrangement of colors used in a graph, and will prove to be much tighter than previous results.

4.7.4 Color Count Lower Bound

Our previous bounds depended on only the graph in question, or the number of colors used on that graph. We will now consider a different sort of argument, which takes into account the arrangement of each different color in a graph. This will give the exact value of the average row sum.

We will make use of a particular decomposition of the VCICM. Consider a particular color i used to color G. Let $G[i]$ be the subgraph of G consisting only of edges of color i. Since we have a proper edge-coloring of G, $G[i]$ is a matching of G. Consider an entry of M corresponding to two vertices with degree at least 1 in $G[i]$. These two vertices will share color i. We can think of this as adding 1 to the corresponding entry of M. Let $A(i)$ be the adjacency matrix of $G[i]$. Then we can write

$$M = D + \sum_i A(i).$$

We must add D to obtain the correct diagonal, as adjacency matrices have all zeroes on the diagonal. Thus, the average row sum of M is the sum of the average row sums of each $A(i)$, plus the average of the degrees of vertices of G. To simplify, let $A'(i)$ be $A(i)$, with $A'_{vv} = 1$ if vertex v is incident to

color i. Then

$$M = \sum_i A'(i).$$

We will use this idea in the following theorem, which gives an exact value for the average row sum:

Theorem 4.7.16. *(Color count) Let G be a vdec graph with a proper edge-coloring π, and VCICM M. Let a be the average row sum of M. Suppose G is colored with colors $C = \{1, \ldots, k\}$, and let c_i represent the number of edges colored with color i. Then*

$$a = \frac{4}{n}\left(c_1^2 + \ldots + c_k^2\right).$$

Proof. Consider one row of $A'(i)$, indexed by vertex v. If v is incident to an edge of color i, then this contributes $2c_i$ 1's to the total row sum of $A(i)$. That is, vertex v shares color i with $2c_i - 1$ other vertices, plus itself. Note that we multiply by 2, because each edge consists of two vertices, and no two edges of color i are adjacent. This is true for all $2c_i$ vertices which are incident to an edge of color i.

Thus, the edges of color i contribute $(2c_i)^2 = 4c_i^2$ to the total row sum of $A(i)$. There are no other nonzero entries in $A(i)$, so the average row sum of $A(i)$ is $4c_i^2/n$. Repeating this process for all k colors, we have counted every entry in the VCICM. Thus we deduce that the average row sum is at least

$$a = \frac{4}{n}\left(c_1^2 + \ldots + c_k^2\right). \qquad \square$$

Again, we obtain an eigenvalue estimate from Theorem 4.7.16 and Theorem 4.7.1:

Corollary 4.7.17. *(Color count bound) Let G be a vdec graph with a proper edge-coloring and VCICM M. Suppose G is colored with colors $C = \{1, \ldots, k\}$, and let c_i represent the number of edges colored with color i. Then*

$$\lambda \geq \frac{4}{n}\left(c_1^2 + \ldots + c_k^2\right).$$

If G has a maximum coloring, this bound becomes the familiar $\lambda \geq \frac{4e}{n}$. If G has a coloring such that $\vec{1}$ is an eigenvector, then the bound is tight. In other cases, the bound is still very close to tight.

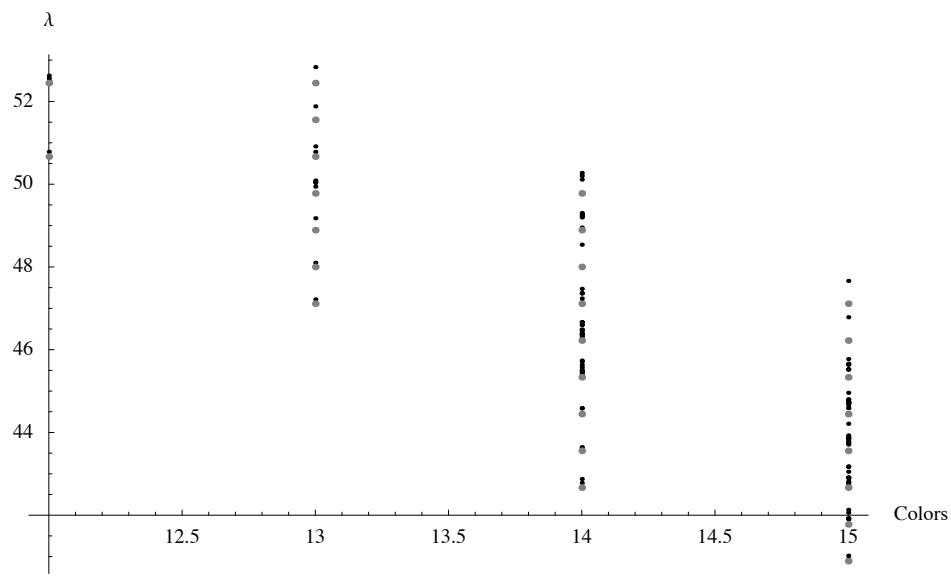


Figure 4.7.7: Detail of color count bound values for K_9

Example 4.7.18. *It is difficult to demonstrate the color count bound, as there are many different lower bounds for a given number of colors. Figure 4.9.1 (see the end of this Chapter) shows the full eigenvalue plot of K_9 with the color count bound. Figure 4.7.7 shows a detail of the eigenvalue plot for K_9. The grey, slightly larger points represent the values of the color count bound. Note that each bound value has a "tail" of eigenvalues above it, and that the bound is generally tighter for lower eigenvalues. In some cases, the bound is exact, or close enough that the associated eigenvalue point is covered by the bound point. Figure 4.7.8 shows the bound for a non-regular graph, W_8.*

There are several interesting properties of the color count bound, which we examine in the following points:

- We always have $c_1 + \ldots + c_k = e$, with $c_i \leq \nu$. Here ν is the size of a maximum matching of G. This gives a feasibility condition for colorings of G.

- For a given number of colors, there may be several different bound values. This depends on the distribution of colors. For example: suppose

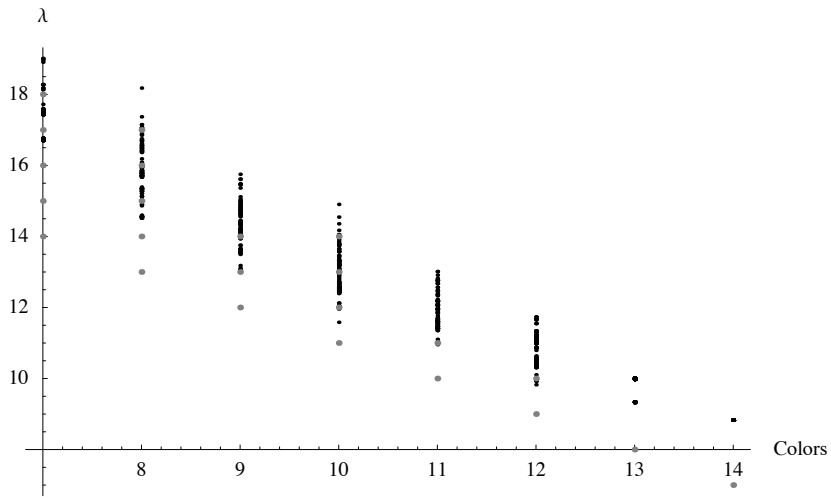


Figure 4.7.8: Color count bound values for W_8

we have 7 edges. We may have 2 edges colored red, 2 blue, 2 green, and 1 yellow. Then the bound is $\frac{2}{7}(2^2 + 2^2 + 2^2 + 1^2) = \frac{26}{7}$. However, we may instead have 3 red, 2 blue, 1 green, and 1 yellow. This gives a bound of $\frac{2}{7}(2^3 + 2^2 + 1^2 + 1^2) = 4$. Each of these uses the same number of colors. This accounts for the multiple bound values in a single vertical column of the eigenvalue plots. It also makes the bound much tighter for some colorings.

- Similarly, different numbers of colors may still produce the same bound value. Suppose we have 14 edges, as in W_8. With 9 colors, we may have 3 red, 2 each of blue, green, and yellow, and one each of five different shades of grey. This gives a bound of 13. We may also have 8 colors: two each of red, green, blue, yellow, purple, and orange, and one each of two shades of grey. This also gives a bound of 13.

- The above points illustrate a general idea: there is a minimum distance between different bound values for the same number of colors. Suppose we have two colorings π_1, π_2 of the same graph, using the same number of colors. The smallest possible difference in arrangement of colors between them is as follows. Change a single edge in coloring π_1 to a color already used in π_1, but in such a way that the total number of colors remains constant. Suppose that we list the number of edges of

each color in π_1. This list is $(c_1, \ldots, c_k)$. Then the related list for π_2 may be represented as $(c_1, \ldots, c_i - 1, \ldots, c_j + 1, \ldots, c_k)$. The bound values are then

$$\begin{aligned} b &= \frac{4}{n}(c_1^2 + \ldots + c_i^2 + \ldots + c_j^2 + \ldots + c_k^2) \\ b' &= \frac{4}{n}(c_1^2 + \ldots + (c_i - 1)^2 + \ldots + (c_j + 1)^2 + \ldots + c_k^2) \end{aligned}$$

Taking the difference, we have:

$$\begin{aligned} b' - b &= \frac{4}{n}\left((c_1^2 + \ldots + (c_i - 1)^2 + \ldots + (c_j + 1)^2 + \ldots + c_k^2)\right. \\ &\quad \left. - (c_1^2 + \ldots + c_i^2 + \ldots + c_j^2 + \ldots + c_k^2)\right) \\ &= \frac{4}{n}\left((c_i - 1)^2 + (c_j + 1)^2 - c_i^2 - c_j^2\right) \\ &= \frac{4}{n}\left(-2c_i + 1 + 2c_j + 1\right) \\ &= \frac{8}{n}\left(c_j - c_i + 1\right). \end{aligned}$$

This leaves two cases. If $c_j = c_i - 1$, the bounds are equal. However, substituting these values into the original bounds, we see that the distribution of colors must be exactly the same. If $c_j \neq c_i - 1$, then we have an integer multiple of $\frac{8}{n}$ between two "consecutive" bounds. An example of this is in the previous point. Often the distance is exactly $\frac{8}{n}$, but not always. For example, three colors with distribution $(4, 2, 2)$ and three colors with distribution $(5, 2, 1)$ have bound distance $3 \cdot \frac{8}{n}$.

- For a given number of colors, the smallest bound comes from the most even distribution of colors. For example: for a graph with 5 colors, 10 edges, and 5 vertices, the smallest bound comes from distributing two edges per color, giving $\frac{4}{5}(5 \cdot 2^2) = 16$. The largest bound comes from coloring as many edges as possible with the same color. If we ignore proper colorings and take a strictly algebraic view, we may have one color on 6 edges, and each of the 4 remaining edges colored differently. This gives a bound of $\frac{4}{5}(6^2 + 4 \cdot 1^2) = 32$. Taking into account that we always want proper colorings, the maximum value will be lower. We will consider this case in Section 4.8. In general, the smallest lower bound value becomes $4e^2/nk$, when each $c_i = e/k$. The largest value is then $4e^2/n$, which occurs when $c_1 = e$ and $k = 1$.

4.8 Upper Bounds

Good upper bounds for $\lambda_1 = \lambda$ are difficult to obtain, compared to lower bounds. This section contains several reasonably good bounds, as well as some ideas for future bounds.

Let G be a vdec graph with VCICM M. The simplest bound for general graphs comes from the observation that the eigenvector v corresponding to λ is entirely positive. We may assume that v is normalized, so that each entry v_i satisfies $0 < v_i \leq 1$ and at least one entry has $v_i = 1$. Thus v may, at "largest" be $\vec{1}$. Consider the dot product of v with each row of M. If $v = \vec{1}$, then Mv consists of the row sums of M. Thus, λ is at most the largest row sum of M. If M corresponds to a maximum coloring, this corresponds to

$$\lambda \leq 2\Delta. \tag{4.10}$$

This also allows us to bound the largest eigenvalue of any maximum colored graph:

$$2\delta \leq \lambda \leq 2\Delta.$$

If the graph is not regular, both inequalities are strict. In particular, a k-regular graph has $\delta = k = \Delta$. A maximum colored non-regular graph will have rows with different row sums, thus $\vec{1}$ will not be an eigenvector. Hence, the largest eigenvalue will be less than the maximum row sum, so $\lambda < 2\Delta$. Likewise, $2\delta \leq \lambda$ comes from Corollary 4.7.5, in which we suppose that all vertices of G have degree δ. If some vertices have degree not equal to δ, the inequality will be strict.

For a general vdec coloring, each entry of the row corresponding to a vertex of degree Δ is at most $\Delta - 1$. (If two vertices share Δ colors, they must not be distinguished.) Thus,

$$\lambda \leq \Delta + (n-1)(\Delta - 1) = n\Delta - n + 1. \tag{4.11}$$

Each of these bounds is tight for regular graphs, as we saw in Section 4.4. As with lower bounds, we now derive some bounds based on the number of colors used to color the graph.

4.8.1 Basic Color Bound

From the beginning of this section, we know that λ is at most the maximum row sum. For a maximum coloring, this becomes $\lambda \leq 2\Delta$. Suppose we change one edge of a maximum coloring to another, already used, coloring. This adds at most 2 to the maximum row sum: we may add 1 in two entries,

one for each end of the changed edge. Note that if no edge of this new color is incident to the vertex corresponding to the row with largest sum, we might not add anything at all.

If we again change one edge to an already used color, we again add at most 2 to the maximum row sum. Thus, in the worst case, we add 2 every time we change an edge and no more. Hence, when G is colored with c colors, we have added at most $2(e-c)$ to the maximum row sum, giving us this bound:

Upper Bound 4.8.1. *(Basic Color Bound) Let G be a VDEC graph with a proper edge-coloring and VCICM M. Then*

$$\lambda \leq 2(\Delta + e - c).$$

Note that $c \geq \Delta$ always, so we may simplify this bound as

$$\lambda \leq 2e.$$

Both of these apply to all edge-colorings, not just maximum-colored graphs.

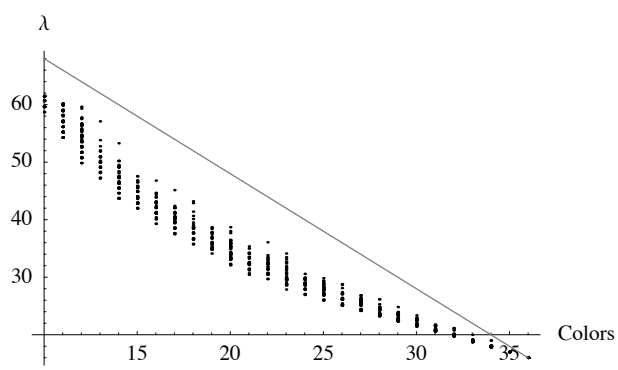


Figure 4.8.1: Basic color bound values for K_9

Example 4.8.2. *Figure 4.8.1 demonstrates the basic color bound for various colorings of K_9. Note that this, like the simple color bound, is linear and does not account for the sharp upward turn of the eigenvalues to the left. It is very close for minimum and maximum colorings, however.*

Figure 4.8.2 demonstrates that this bound is considerably less tight for non-regular graphs.

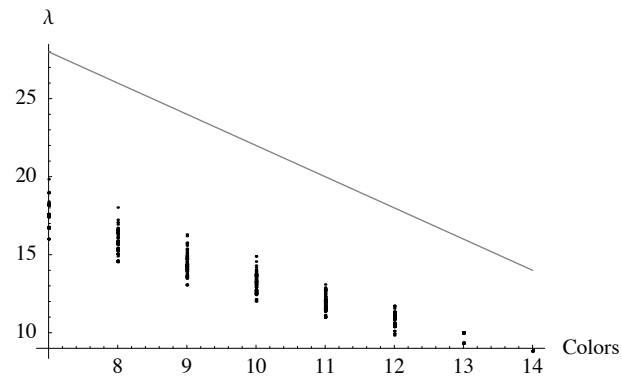


Figure 4.8.2: Basic color bound values for W_8

4.8.2 Average Row Sums, again

We would very much like to find a way to use average row sums to find upper bounds. Ultimately, we will find that this is not as easily accessible as it is for lower bounds. However, we will obtain some very tight upper bounds on the average row sum of the VCICM. Let G be a vdec graph with a proper edge-coloring. Let a be the average row sum of the VCICM M of G with this coloring. We can write $\lambda = \frac{\lambda}{a}a$. Thus, if we can find an upper bound on λ/a, we can bound λ from above. Unfortunately, these bounds are not particularly good (in fact, no better than any upper bound we use for λ). However, they do produce some interesting expressions for upper bounds, which we will examine. They also provide an opportunity to study the ratio λ/a, which gives a measure of the error in our previous bounds. The most interesting results in this section will be upper bounds on the average row sum of the VCICM.

We have many bounds for λ, and by extension, many bounds for the average row sum a. We begin with a simple bound on λ/a, which we then extend to much better bounds.

Lemma 4.8.3. *Let G be a vdec graph with a vdec coloring. Let M be the VCICM of G with this coloring, with largest eigenvalue λ and average row sum a. Then*

$$\frac{\lambda}{a} \leq \frac{n}{2}.$$

Proof. Using results from previous sections, we have $\lambda \leq 2e$ and $a \geq \frac{4e}{n}$.

Thus we have the inequality:

$$\frac{\lambda}{a} \leq \frac{2e}{\frac{4e}{n}} = \frac{n}{2}. \qquad \square$$

This is a poor bound in general, which is not unexpected, since our bound depends only on n. However, the bound is very simple, and provides a method for finding better bounds.

By estimating λ and a more tightly, we can produce a better bound. In particular, we have the following bound:

Lemma 4.8.4. *Let G be a vdec graph with a vdec coloring. Let M be the VCICM of G with this coloring, with largest eigenvalue λ and average row sum a. Then*

$$\frac{\lambda}{a} \leq \frac{cn}{2e}.$$

Proof. We begin with the bound $\lambda \leq 2e$ from above. Similarly, we know that $a \geq \frac{4}{cn}(e^2 + e - c)$, which is the simplified form of the matching color bound. Thus we have

$$\frac{\lambda}{a} \leq \frac{2e}{\frac{4}{cn}(e^2 + e - c)}.$$

Note that $e - c \geq 0$, so we may remove that term and maintain the inequality. Thus, we have

$$\frac{\lambda}{a} \leq \frac{2e}{\frac{4}{cn}e^2} \leq \frac{cn}{2e}. \qquad \square$$

This improves on previous estimates, while still having a relatively simple formulation. Note that $\frac{c}{e} \leq 1$, with equality only when we have a maximum coloring. Thus, this bound is much better for most minimum colorings. Recall that the ratio λ/a indicates how close our estimates on λ and a are. Thus, small values are best. In particular, $\frac{\lambda}{a} \geq 1$ with equality in some cases. Values near 1 show that λ is very near to a, so our eigenvalue estimates are close to tight.

- Let $G = K_{2k-1}$. The first bound gives us $\frac{\lambda}{a} \leq \frac{n}{2}$. For a maximum coloring, the second bound gives us the same value. However, for a minimum coloring, $\frac{\lambda}{a} \leq \frac{n^2}{n(n-1)} = \frac{n}{n-1}$ which is very close to 1. Thus we have $\lambda \leq \frac{n}{n-1}(n-1)^2 = n(n-1)$. The actual value of λ is $(n-1)^2$.

- Let G be a tree. Then the second bound gives us $\frac{\lambda}{a} \leq \frac{cn}{2(n-1)}$, which is near $\frac{c}{2}$, especially for large n. Here it is harder to bound a, but for a maximum coloring we have $\lambda \leq 2e$.

- Let G be given a minimum coloring. We know that $\chi'_s(G) \leq n+1$, so $\lambda \leq \frac{n(n+1)}{2e}$. This is of the proper order (for a random graph, e is quadratic in terms of n), so this bound value will be near 1. If G is a complete graph, we get $\lambda \leq \frac{n+1}{n-1}$.

In looking for upper bounds on $\lambda = (\lambda/a)a$, we must also bound the average row sum a. Most bounds come from the fact that an entry of the VCICM is bounded by the degrees of the vertices corresponding to the row and column of the entry. Let $G = (V, E)$ be a vdec graph with a vdec coloring, and let $v, w \in V$. Let M be the VCICM of G. Then

$$M_{vw} \leq \max\{\deg(v), \deg(w)\}. \tag{4.12}$$

If $\deg(v) = \deg(w)$, then the inequality must be strict.

We may assume that the rows of M are arranged in nonincreasing order, by the degree of the corresponding vertex. Thus $M_{11} = d_1, M_{22} = d_2$, etc. Then in Equation (4.12), we have $M_{vw} \leq \deg(w)$ (assuming that $\deg(w) \leq \deg(v)$). We may thus bound the average row sum a, as follows:

Lemma 4.8.5. *Let G be a vdec graph with a proper edge-coloring and VCICM M. Let a be the average row sum of M. Then*

$$a \leq \frac{2}{n}\left(e + (n-1)d_n + (n-2)d_{n-1} + \ldots + d_2\right).$$

Proof. We begin with Equation (4.12). In the VCICM M, every off-diagonal element in the row and column corresponding to a vertex of degree d_n will be at most d_n. There are $2(n-1)$ such elements. Similarly, all other elements in a row and column corresponding to a vertex of degree d_{n-1} will be at most d_{n-1}, and there are $2(n-2)$ such elements. (There are two elements already counted for d_n). Continuing in this fashion, we have:

$$\begin{aligned} a &\leq \frac{2e + 2(n-1)d_n + 2(n-2)d_{n-1} + \ldots + 2d_2}{n} \\ &= \frac{2}{n}\left(e + (n-1)d_n + (n-2)d_{n-1} + \ldots + d_2\right). \end{aligned}$$

Note that the term $2e$ comes from the sum of the diagonal entries, the values of which are all known. □

Lemma 4.8.5 leads to several other estimates. If G is k-regular, then each off-diagonal entry is at most $k-1$, so we have:

Corollary 4.8.6. *If G is k-regular, then the average row sum a of M satisfies*

$$a \leq k + (k-1)(n-1).$$

Proof. From Lemma 4.8.5, we have:

$$\begin{aligned} a &\leq \frac{2}{n}\left(\frac{nk}{2} + (k-1)((n-1)+(n-2)+\ldots+1)\right) \\ &= \frac{2}{n}\left(\frac{nk}{2} + (k-1)\frac{n(n-1)}{2}\right) \\ &= k + (k-1)(n-1). \end{aligned}$$

□

This bound is sometimes tight. For example, the minimum coloring given for K_{2k+1} in Section 2.1 satisfies this bound with equality. In general, colorings which are close to minimum will give average row sums closer to this bound.

Using the estimate of Corollary 4.8.6, we can estimate λ for k-regular graphs:

Upper Bound 4.8.7. *The largest eigenvalue λ of M satisfies $\lambda \leq c + \frac{c}{k}(k-1)(n-1)$.*

Proof. From Corollary 4.8.6, we have:

$$\begin{aligned} \lambda &\leq \frac{\lambda}{a}a \\ &\leq \frac{cn}{2e}(k + (k-1)(n-1)) \\ &= \frac{cn}{kn}(k + (k-1)(n-1)) \\ &= c + \frac{c}{k}(k-1)(n-1). \end{aligned}$$

□

This bound is fairly tight for near-minimum colorings, but very poor for near-maximum colorings. This is no surprise, as our maximum row sum estimate does not depend on the number of colors – thus it is best for a minimum number of colors, which produces a large row sum.

We conclude this section with a considerably better upper bound on average row sums, which may be used with some of the bounds found in

this section. As with lower bounds on the average row sum, we will consider extreme colorings. That is, for a graph G and a coloring π with c colors, we wish to find a coloring π' of G which uses c colors, and whose VCICM has the largest average row sum of all such colorings. Let G be a vdec graph with a maximum coloring, and let M be the VCICM of G. As before, changing one edge to a different previously used color c_i will increase the total row sum by 8. Thus, we add at most $8/n$ to the average row sum (or 8 to the total row sum), by changing one edge. If we change another edge to c_i, we will add 16 to the total row sum, and thus $16/n$ to the average row sum. Again, this is the largest change we can make.

We will continue in this fashion. Let ν be the size of a maximum matching in G. Then we formulate the change in the average row sum as follows:

Lemma 4.8.8. *Let G be a vdec graph with a proper edge-coloring. Suppose G has VCICM M with average row sum a. Select a color c_i which is used on only one edge e, and change as many edges as possible to color c_i, while maintaining a proper edge-coloring. Let the VCICM of this new coloring be M', with average row sum a'. Then*

$$a' - a \leq \frac{4}{n}\nu(\nu - 1).$$

Proof. We know that the edges of color c_i form a partial matching in G. Thus, at most ν edges may have color c_i. Before changing the colors, one edge already had color c_i, and thus we may change at most $\nu - 1$ edges to color c_i. Any further changes would make the edge-coloring improper.

Changing one edge to color c_i adds at most 8 to the total row sum. Changing a second edge adds at most 16, and so on, always assuming that the edges being changed had colors which were used only once. This avoids reducing the row sum at all. Thus, changing as many edges as possible adds at most $8 + 16 + 24 + \ldots + 8(\nu - 1)$ to the total row sum. Factoring out the 8 and collapsing the sum into a simple formula, we have the result. □

Note that, similar to our work in Section 4.7.2, we are bounding the average row sum by finding a coloring whose VCICM has the largest possible average row sum. Repeatedly applying Lemma 4.8.8, we find the following bound:

Lemma 4.8.9. *Let G be a vdec graph with a vdec coloring. Let M be the VCICM of G, with average row sum a. Let ν be the maximum size of a*

matching in G. Then

$$\begin{aligned} a &\leq \frac{4}{n}\left(e + \left\lfloor \frac{e-c}{\nu-1} \right\rfloor (\nu-1)\nu \right. \\ &\quad \left. + \left(e - c - \left\lfloor \frac{e-c}{\nu-1} \right\rfloor (\nu-1)\right)\left(e - c - \left\lfloor \frac{e-c}{\nu-1} \right\rfloor (\nu-1) + 1\right)\right). \end{aligned}$$

Proof. We begin with a maximum coloring of G, which has average row sum $\frac{4e}{n}$. Repeatedly apply Lemma 4.8.8, each time using a different edge whose color is used only once. Each time, we add at most $(4/n)\nu(\nu-1)$ to the total row sum. This maximizes the change in average row sum.

Each time we apply Lemma 4.8.8, $\nu - 1$ colors are totally removed from G. Thus, we may continue for k iterations, where k satisfies $e - k(\nu-1) \geq c$. So far, we have

$$\frac{1}{n}\left(4e + \left\lfloor \frac{e-c}{\nu-1} \right\rfloor (4\nu(\nu-1))\right).$$

However, this only accounts for changing $\nu - 1$ edges at a time. We may have up to $\nu - 2$ additional edges to change before arriving at the correct number of colors. In fact, the exact number of edges we have to change is $e - c - \left\lfloor \frac{e-c}{\nu-1} \right\rfloor (\nu-1)$. Thus, a preliminary form of our bound is

$$\begin{aligned} a &\leq \frac{1}{n}\left(4e + \left\lfloor \frac{e-c}{\nu-1} \right\rfloor (4\nu(\nu-1)) \right. \\ &\quad \left. + \left(8 + 16 + \ldots + 8\left(e - c - \left\lfloor \frac{e-c}{\nu-1} \right\rfloor (\nu-1)\right)\right)\right). \end{aligned}$$

Factoring out 8's in the third term, we see a sum of consecutive integers. Simplifying, this gives us the final form of the bound:

$$\begin{aligned} a &\leq \frac{4}{n}\left(e + \left\lfloor \frac{e-c}{\nu-1} \right\rfloor (\nu-1)\nu \right. \\ &\quad \left. + \left(e - c - \left\lfloor \frac{e-c}{\nu-1} \right\rfloor (\nu-1)\right)\left(e - c - \left\lfloor \frac{e-c}{\nu-1} \right\rfloor (\nu-1) + 1\right)\right). \end{aligned}$$

The coloring π' produced in this manner must have a VCICM with a maximal average row sum. If some coloring π_2' had a larger row sum, we have several possibilities. First, some color in π_2' may be used on at least $\nu + 1$ edges, and thus we have an improper edge-coloring, which we do not allow. If this is not the case, then suppose some color in π_2' is used on fewer than ν edges. If more than one such color exists, then the average row sum of

the VCICM of π_2' must be lower than the average row sum of the VCICM of π'. If only one such color exists, then all other colors in both colorings are used on ν edges, so the same number of edges are available to have the final remaining color. Thus, the VCICM's of these two colorings must have the same average row sums. $\square$

We can find colorings which make this bound tight for almost all values of c. It is sometimes impossible to color a graph in the manner described above, for small values of c or graphs with few edges. This happens due to the difficulty of repeatedly finding large matchings in such graphs. The following example demonstrates the tightness of this bound.

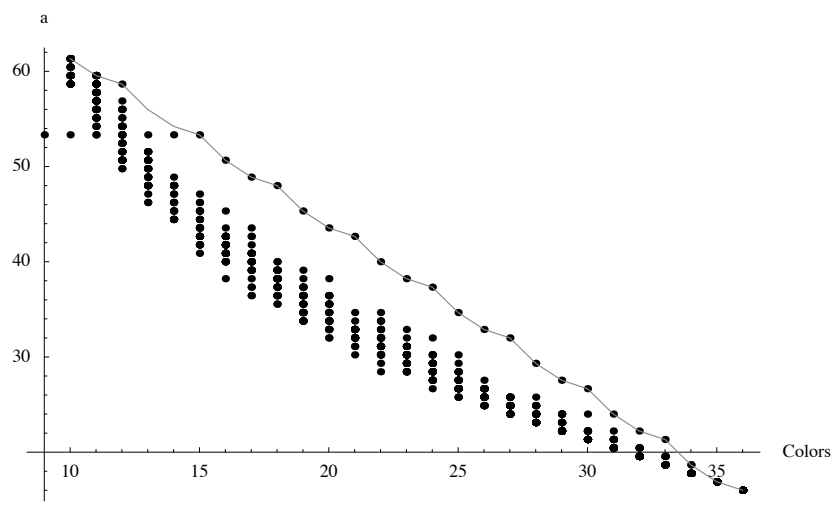


Figure 4.8.3: Average row sum upper bound for K_9

Example 4.8.10. *Figures 4.8.3 and 4.8.4 show the average row sum upper bounds for K_9 and W_8, respectively. Note that the bound line is tight for most colorings. The space between the bound line arises from the fact that the coloring algorithm does not tend to produce colorings with large induced matchings. The bound values on the bound line arise from specially checking for colorings of maximum average row sum. This plot compares the number of colors used to the average row sum of the VCICM, not the largest eigenvalue (as in most previous plots).*

These results now give us an upper bound on λ:

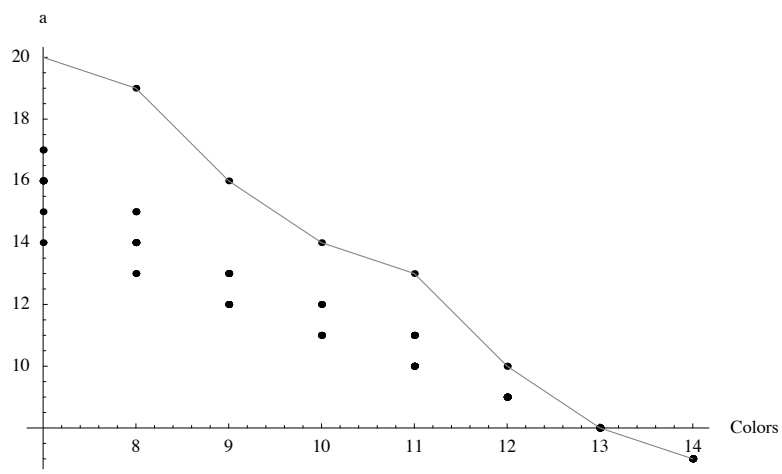


Figure 4.8.4: Average row sum upper bound for W_8

Upper Bound 4.8.11. *Let G be a vdec graph with a vdec coloring and VCICM M. Then*

$$\begin{aligned}\lambda \;\leq\; & \frac{2c}{e}\left(e + \left\lfloor \frac{e-c}{\nu-1} \right\rfloor (\nu-1)\nu \right. \\ & \left. + \left(e - c - \left\lfloor \frac{e-c}{\nu-1} \right\rfloor (\nu-1)\right)\left(e - c - \left\lfloor \frac{e-c}{\nu-1} \right\rfloor (\nu-1) + 1\right)\right).\end{aligned}$$

Proof. The result may be obtained by combining the bound on the ratio $\frac{\lambda}{a}$ from Lemma 4.8.4 with the bound on the average row sum from Lemma 4.8.9. □

Unfortunately, as mentioned previously, our upper bounds which use the average row sum are not very good. Future work in this area could produce better upper bounds based on the average row sums. In addition, this bound depends on repeatedly finding large matchings in a graph. For most graphs, the size of a maximum matching decreases quickly when a large matching is removed, which makes our bound less tight. The bound could be algorithmically improved: after coloring a group of edges, we remove all edges of the new color from the graph, and find a maximum matching in the resulting graph. We have implemented this algorithm, which produces somewhat tighter bounds, especially for non-regular graphs.

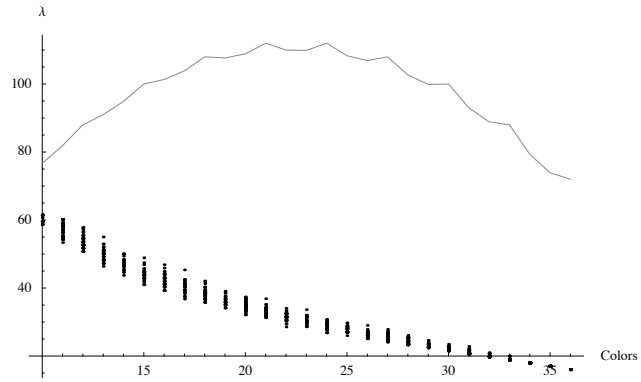


Figure 4.8.5: Eigenvalue upper bound for K_9

Example 4.8.12. *Figure 4.8.5 shows the upper average row sum bound on λ. This is clearly not optimal, despite the fact that the bound on the average row sum itself is tight.*

4.9 A note on bounds for $\chi'_s(G)$

We have many lower and upper bounds on a and λ, several of which involve c, the number of colors used to color the graph. Suppose we are given a VCICM M for an unknown coloring. Here, we know e and n, and we can calculate the largest eigenvalue λ for M. If we have an inequality giving a lower bound on λ in terms of e, n, and c, we may use this to obtain an upper bound on $c = \chi'_s(G)$ in terms of e, n and λ. Conversely, suppose that we are given a minimum coloring of a graph G. Thus, we know e, n, and c. If we have an inequality giving an upper bound on $\chi'_s(G)$ which uses e, n, and λ, then we obtain a lower bound on λ which uses $\chi'_s(G), e$, and n. Likewise, we may find lower bounds on $\chi'_s(G)$ or upper bounds on λ.

Unfortunately, our current bounds do not produce useful bounds on χ'_s. For example, consider the bound $\lambda \leq 2(\Delta + e - c)$. Solving for c, we have

$$c \leq 2\Delta + 2e - \frac{\lambda}{2}.$$

This bound is not good. Note that $c \leq 2e$ always, and we still have another (always positive) term 2Δ. This produces a bound on c which is very inexact (usually greater than $2e$). However, this bound may be useful if one is given

the VCICM of an unknown coloring, from which one can calculate λ exactly. If one takes the approach of finding extremal VCICM's with the goal of examining the c values associated with them, this bound may prove to be useful.

We do have one bound which may be useful: the color count bound. We solve for the terms involving c_i to find:

$$c_1^2 + \ldots + c_k^2 \leq \frac{n\lambda}{4}.$$

The c_i's must satisfy two conditions: first, $c_1 + \ldots + c_k = e$. Second, $k = c$, the number of colors used to color G. This alone may provide a basis for future bounds. We now present a looser bound based on these ideas. The smallest value of $c_1^2 + \ldots + c_k^2$ comes when $c_i \approx e/k$ for all i. Thus, we can estimate:

$$\begin{aligned} k\left(\frac{e}{k}\right)^2 &\leq \frac{n\lambda}{4} \\ \frac{e^2}{k} &\leq \frac{n\lambda}{4} \\ \frac{4e^2}{n\lambda} &\leq k. \end{aligned}$$

The above estimate gives us a lower bound on $k = c$, if λ is already known. For example, given a VCICM of K_6 with $\lambda = 16$, we know that $k \geq 4 \cdot 15^2/(6 \cdot 16) = 9.375$. However, we know that $\chi_s'(K_6) = 7$, so this VCICM does not represent a minimal coloring of K_6.

Suppose instead that we rewrite this bound as:

$$\frac{4e^2}{nk} \leq \lambda.$$

We know that a minimal coloring of K_6 has $k = 7$. Thus, we have $4 \cdot 15^2/(6 \cdot 7) = 21.4286$, so the largest eigenvalue of a minimum coloring of k must be at least 21.4286. This illustrates a general use of our bounds: if we know an extremal value of one of $\chi_s'(G)$ or λ, we obtain an estimate on the extremal value of the other. One may begin by finding extremal VCICM's, whose eigenvalues then make these bounds give extremal k values.

Unfortunately, these bounds are generally quite poor. The method we used to obtain these bounds could lead to much better bounds, with further work.

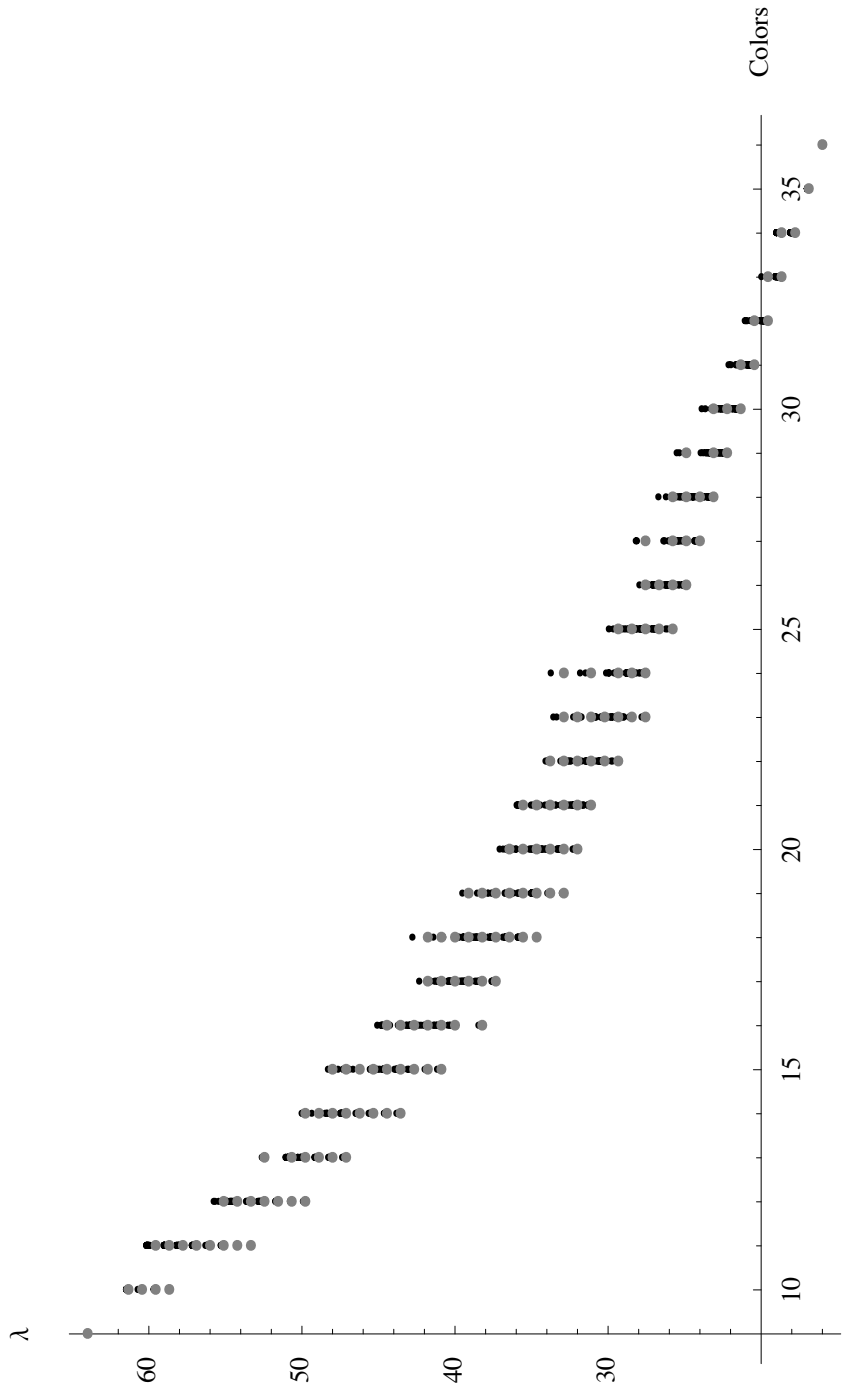


Figure 4.9.1: Color count bound values for K_9

Chapter 5

Related Topics

There are some topics related to vdec colorings which, while we have not studied them in depth, are still interesting. In particular, adjacent vdec colorings are a well-studied subtopic of vdec colorings. We will also consider some other uses of such colorings, and generalizations of the idea of a vdec coloring.

5.1 Adjacent Vertex-Distinguishing Edge-Colorings

The vdec colorings which we have studied require *every* pair of vertices to be distinguished. A logical restriction is to require only adjacent vertices to be distinguished. This is known as an adjacent vertex-distinguishing edge-coloring, or avd coloring.

Definition 5.1.1. *Let $G = (V, E)$ be a vdec graph with a proper edge-coloring. The coloring is called* adjacent vertex-distinguishing (avd) *if $S(v) \neq S(w)$ for all vertices v, w with $vw \in E$.*

Note that a graph with more than one isolated vertex could still admit an avd coloring, but we still must have no isolated edges. We may ignore the isolated vertices, and assume that these graphs are vdec.

Every vdec coloring is an avd coloring, but not necessarily vice-versa. These avd colorings have been studied in much the same way as vdec colorings. Since they are proper edge-colorings, we may use the VCICM to analyze them. In fact, all bounds in the previous chapter which depend only on G having a proper edge-coloring will hold as well for avd colorings.

As with vdec coloring, avd colorings have been called by many names, mostly related to other names for vdec colorings. For example, an equivalent idea is that of an "adjacent strong edge-coloring," denoted "ASEC." In addition, the term "neighbour-distinguishing" is used in some papers. However, this is also used to refer to improper edge-colorings in some cases. We use the (more or less common) notation of $\chi'_a(G)$ to denote the minimum number of colors needed to give G an avd coloring.

We begin with a few simple facts about avd colorings. Several of these are notes or theorems from the first paper to introduce avd colorings, written by Zhang, Liu, and Wang in [22]. In each, let G be a vdec graph.

- We always have $\chi'_a(G) \geq \Delta$.
- If no two adjacent vertices of G have the same degree, then $\chi'_a(G) = \Delta$ (Theorem 8).
- If two vertices of maximum degree are adjacent, then $\chi'_a(G) \geq \Delta + 1$ (Theorem 7).
- If G has components $G_1, \ldots, G_k$, then

$$\chi'_a(G) = \max\{\chi'_a(G_1), \ldots, \chi'_a(G_k)\}.$$

As with vdec colorings, values of χ'_a are known for many standard graphs. The techniques used to find these colorings are similar to those used for vdec colorings, although often we may put the structure of a graph to better use: if a graph has structure such that we can obtain a great deal of data about the neighbors of any given vertex, this often makes it easier to find a good avd coloring. For example:

- *Cycles:* For a cycle C_n,

$$\chi'_a(C_n) = \begin{cases} 3, & \text{if } n \equiv 0 \pmod 3 \\ 4, & \text{if } n \not\equiv 0 \pmod 3 \text{ and } n \neq 5 \\ 5, & \text{if } n = 5 \end{cases}$$

- *Complete Graphs:* For a complete graph K_n,

$$\chi'_a(K_n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ n+1, & \text{if } n \text{ is even} \end{cases}$$

  Note that this is the same as for vdec colorings, because each pair of vertices is adjacent.

- *Complete Bipartite Graphs:*

$$\chi'_a(K_{m,n}) = \begin{cases} n, & \text{if } n > m > 0 \\ n+2, & \text{if } m = n \text{ and } m \geq 2 \end{cases}$$

  If $n > m > 0$, we may assign the same color sets to all vertices in one vertex class, and similarly for the other class. If $n = m$, we require at least $n+1$ colors, or else all vertices would have the same incident color sets. As with vdec colorings, $n+1$ colors do not give enough incident color sets, for parity reasons. Thus, we have $\chi'_a(K_{n,n}) = n+2$, much like $\chi'_s(K_{n,n})$.

- *Trees:* Let T be a vdec tree. If any two vertices of maximum degree are adjacent, then $\chi'_a(T) = \Delta + 1$. Otherwise, $\chi'_a(T) = \Delta$. The proof of this is given in [22].

As with vdec colorings, there is one main conjecture about the value of χ'_a, and many upper bounds:

Conjecture 5.1.2. *(Zhang, Liu, and Wang in [22]) Let G be a connected vdec graph on at least 3 vertices, with $G \neq C_5$. Then*

$$\Delta \leq \chi'_a(G) \leq \Delta + 2.$$

This is in the spirit of Theorem 2.2.2. There are examples for $\chi'_a(G) = \Delta, \Delta+1$, and $\Delta+2$. For example, the star graph has $\chi'_a(S_n) = \Delta$. We also know that $\chi'_a(K_{2k+1}) = 2k+1 = \Delta+1$, and $\chi'_a(K_{2k}) = 2k+1 = \Delta+2$.

Balister, Győri, Lehel, and Schelp [4] proved several upper bounds which support Conjecture 5.1.2. The following are Theorems 1.1 through 1.3 from their paper:

- If G is a graph with no isolated edges and $\Delta = 3$, then $\chi'_a(G) \leq 5$. Note that this is consistent with Conjecture 5.1.2.

- If G is a bipartite graph with no isolated edges, then $\chi'_a(G) \leq \Delta + 2$. This is also consistent with Conjecture 5.1.2.

- If G is a k-chromatic graph with no isolated edges, then $\chi'_a(G) \leq \Delta + O(\log k)$.

The proof for graphs with $\Delta = 3$ illustrates an interesting technique. The graphs are colored with elements of the Klein IV group $K = \mathbb{Z}_2 \times \mathbb{Z}_2$, plus the color 5. The group elements are $\{0, a, b, c\}$, with the property that

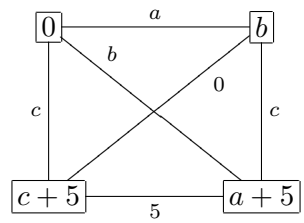


Figure 5.1.1: K_4 colored with the Klein IV group

$x + x = 0$ for all x, and $a + b = c$. Each vertex is then assigned a label equal to the sum of the incident edge colors. Thus, different labels correspond to different incident color sets. Then, the sum of labels on adjacent vertices is examined. If the sum is zero, the adjacent vertices must have the same label, and thus are not distinguished.

Example 5.1.3. *Figure 5.1.1 shows K_4 with edges colored from the Klein IV group, plus color 5. The vertices are labeled with the sum of the colors on their incident edges. Note that we have a proper edge-coloring, and that the labels on adjacent vertices are distinct. Thus, we have an avd coloring. It also happens to be a vdec coloring.*

Other bounds for χ'_a include a recent bound from Hatami:

Theorem 5.1.4. *(Hatami [19]) If G has no isolated edges, and $\Delta > 10^{20}$, then $\chi'_a(G) \leq \Delta + 300$.*

This asymptotically improves on the bound $\chi'_a(G) \leq \Delta + O(\log k)$. Similarly, Greenhill and Ruciński [17] proved the following theorem:

Theorem 5.1.5. *(Greenhill and Ruciński [17]) If $d \geq 4$, then almost all d-regular graphs G satisfy $\chi'_a(G) \leq \left\lceil \frac{3d}{2} \right\rceil$.*

This proves Conjecture 5.1.2 for almost all 4-regular graphs. Note that here, as in Theorem 2.2.3, "almost all" means that, as $n \to \infty$, the probability that a random d-regular graph on n vertices has this property tends to 1.

Finally, Edwards, Horňák, and Woźniak proved Conjecture 5.1.2 for many bipartite graphs:

Theorem 5.1.6. *(Edwards, Horňák, and Woźniak [14]) Let G be a planar bipartite graph with $\Delta \geq 12$. Then $\chi'_a(G) \leq \Delta + 1$.*

Paper [14] also provided further generalizations and specializations of avd and vdec colorings. In particular, it introduces the idea of an f-distinguishing coloring, in which $f(x)$ is a function which returns a set $S(x)$ of vertices. A coloring is said to be f-distinguishing if, for all x, $S(x) \neq S(y)$ for all $y \in f(x)$. In particular, if $f(x) = V \setminus \{x\}$, we have an equivalent definition of a vdec coloring. If $f(x) = \{y : y \text{ is a neighbor of } x\}$, we have an *avd* coloring.

5.2 Induced Colorings

One of the techniques mentioned in the previous section was to color the edges of a graph with elements of a certain group (in this case, Klein IV), and label each vertex with the sum of the incident edge "colors." This idea of an "induced coloring" has been used in several other papers to good effect.

In particular, this is used when allowing improper edge-colorings. If the multiset of colors incident to one vertex is distinct from the multiset of colors incident to any adjacent vertex, this is called (unfortunately) a vertex-distinguishing edge-coloring [1]. It is in fact a variation on the avd coloring, which allows improper colorings. Addario-Berry, et. al., used positive integers as their edge colors, and were able to prove that every graph with no isolated edge has such a coloring using at most 30 colors, $\{1, \ldots, 30\}$. It is also conjectured that at most 3 integer colors are needed for any such graph.

Instead of using consecutive integers (in which two vertices may have the same sum, but different incident color sets, if we are not careful), we could use prime powers. For example, labeling edges with powers of two will induce a binary number on each vertex. Thus two vertices have the same label if and only if their incident color sets are identical.

General integer colorings have an interesting application to multigraphs. Every simple graph must have at least two vertices of the same degree, but this is not necessarily true for multigraphs. We may restrict the coloring so that all pairs of vertices are distinguished. Then, we can consider the numeric "color" on each edge to represent the number of parallel edges into which it must be split, so that no two vertices have the same degree in the resulting multigraph. A minimum coloring in this way will produce a graph satisfying this condition, with the smallest possible number of edges.

Induced colorings are useful, since they turn an edge-coloring problem into a vertex-coloring problem, in a structured way. An induced coloring which gives a vertex-coloring will indicate if any two vertices were not dis-

tinguished in the original coloring. If any two vertices have the same color, they were not distinguished in the original edge-coloring. For avd colorings, we need only look at the neighbors of each vertex.

We further generalize the idea of the vdec coloring by defining a new type of coloring, the "adjacent edge-distinguishing vertex-coloring" (aed coloring). Let $G = (V, E)$ be a simple, undirected graph with a proper vertex-coloring π. Let $e = vw \in E$. Define $S(e) = \{\pi(v), \pi(w)\}$. Then the coloring of G is adjacent edge-distinguishing (aed) if $S(e) \neq S(f)$ for any adjacent edges $e, f \in E$. Since π is a proper vertex-coloring, $S(e)$ is never a multiset.

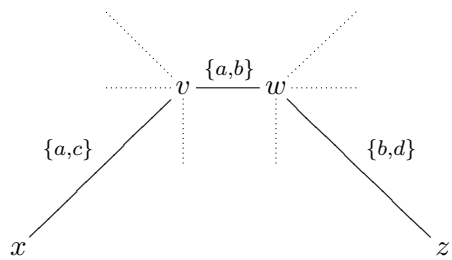
This relates to induced colorings in an interesting way. The induced edge-coloring of a vertex-coloring is found in a way parallel to the induced vertex-coloring: for each edge e, label the edge with $S(e)$. These labels can be seen as edge colors on G. Since the induced coloring given by any avd coloring is proper, so too the induced edge-coloring given by any aed coloring is an avd coloring.

Theorem 5.2.1. *Let G be a vdec graph with a proper edge-coloring. If the coloring of G is aed, then the induced edge-coloring will be avd.*

Proof. Suppose otherwise. Then either the induced edge-coloring is not a proper edge-coloring, or two adjacent vertices have the same labels.

Suppose that the induced edge-coloring is not proper. Then two adjacent edges e, f have $S(e) = S(f)$. We know that $e = uv$ and $f = vw$ share one vertex, and that the original vertex-coloring was proper. Thus, the colors of u and w must be the same. But then, edges e and f were not distinguished in the original vertex-coloring.

Instead, suppose that two adjacent vertices v, w have the same labels. First, note that $\deg(v) = \deg(w)$, or else they must be distinguished. Their labels in the original aed coloring were a, b respectively, with $a \neq b$. Since G is vdec, edge vw is not isolated, so both v and w have another edge incident to them. Since v and w have the same labels, for each edge incident to v with a particular label, there is an edge incident to w with the same label. Suppose that we have two edges vx and wz, respectively, which have the same labels. Let the colors on x and z in the original aed coloring be c and d, respectively.



Thus, we must have $\{a, c\} = \{b, d\}$. We know that $a \neq b$, so we must have $a = d$. Thus $b = c$. But then, edges xv and vw were not distinguished in the original coloring, so the coloring was not aed. This is a contradiction. Thus, the induced edge-coloring on G must be an avd coloring. $\square$

This provides interesting possibilities for analyzing avd colorings in terms of vertex-colorings. In particular, if we can generate aed colorings, we can automatically find avd colorings as well. Generating aed colorings with few colors would produce avd colorings with fewer colors, as well. This is an interesting area for future study. As aed colorings are not directly related to the primary focus of this thesis, we will not pursue this line of thought in this work.

Chapter 6

Future Work and Conclusions

Vertex-distinguishing edge-colorings and the related fields mentioned in the previous chapter present many open problems and conjectures. While there are many bounds on χ'_s, χ'_a, λ, and a, there are few exact results. The VCICM presents some new possibilities for bounding these values, as well as discovering new properties of vdec colorings.

There are many open problems in the area of vertex-distinguishing edge-colorings. This section will enumerate some directions for future study.

- The biggest open question is Burris and Schelp's unsolved conjecture (here, Conjecture 2.2.1) on χ'_s. Even a bound of the form $\chi'_s(G) \leq k+c$ where c is a constant, perhaps depending on Δ or n, would provide progress.

- In a similar vein, the conjecture of Zhang, Liu, and Wang (Conjecture 5.1.2) is still open. There are some constant bounds on χ'_a, and tighter results would also be welcome. In addition, most bounds are valid only for graphs with very large numbers of vertices. Results for graphs with smaller numbers of vertices would be useful.

- Theorem 3.1.3 gives a strong condition for determining if a coloring is vdec, using the VCICM. It warrants much more study on its own. Similarly, Corollary 3.3.3 provides some information about the determinant of the VCICM, but there seems to be much more structure to this matrix. Further information about its values would potentially be of use.

- Better upper bounds on the largest eigenvalue λ of the VCICM are a useful direction for further research. Our current best bounds are not particularly tight. In particular, it would be useful to find a bound similar to the "color count (lower) bound," which takes into account not just the number of colors, but their distribution. We have some excellent bounds, both upper and lower, on the average row sum of the VCICM. Any bounds making use of these could potentially be very sharp.

- It seems likely that reasonably good bounds on χ'_s could arise from some of the eigenvalue bounds presented here. As many of these bounds depend on the number of colors c used to color the graph, it is possible to solve the bound inequalities for c. These bounds are often very poor, but future work could improve that.

- Finally, as mentioned in Section 5.2, induced colorings may provide interesting information about vdec and avd colorings. In particular, we may find aed vertex-colorings, and look at their induced avd colorings. We know a great deal about vertex-colorings, so this may provide insight into the structure of avd colorings.

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