Huygens' Principle for Relativistic Wave Equations on Petrov Type III Space-Times

by

Fernando Deeke Sasse

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Abstract

This thesis makes a contribution to the solution of Hadamard's problem for some relativistic wave equations on Petrov type III space-times. For the conformally invariant scalar wave equation we show that if any one of the spin coefficients α, β, π or Ricci spin coefficient Φ_{11} vanishes in an appropriate null tetrad, then Huygens' principle is not satisfied on Petrov type III space-times. We also show that the corresponding problem for the non-self-adjoint scalar wave equation can be reduced to the conformally invariant scalar equation case. Finally, we prove that there are no Petrov type III space-times on which either the conformally invariant scalar equation, Weyl's neutrino equation or Maxwell's equations satisfy Huygens' principle in the strict sense. In order to obtain the above results we have employed Newman-Penrose spin coefficient formalism and Penrose's two-component spinor formalism, together with their implementations available in the computer algebra system Maple, to determine the components of the tensorial relations given by the imposition of Huygens' principle. The resulting system of polynomial equations is then analysed by using a variant of Buchberger's algorithm, available in Maple.

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To
ELISA AND MARCUS

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Chapter 1

Introduction

In this Thesis we are concerned with the validity of Huygens' principle for wave propagation in curved space-times. Probably the first systematic study of wave propagation through a medium was made by Christiaan Huygens [46] in his "Traité de la Lumière", in 1690. The term "Huygens' principle", however, was not clear until its precise mathematical meaning, within the context of the theory of hyperbolic partial differential equations, was given by Hadamard [42] in 1923. His formulation, given in his Yale lectures on the Cauchy problem is the following:

(A) (major premise). The action of phenomena produced at the instant $t = 0$ on the state of matter at the later time $t = t_0$ takes place by the mediation of every intermediate instant $t = t'$, i.e. (assuming $0 < t' < t_0$), in order to find out what takes place for $t = t_0$, we can deduce from the state at $t = 0$ the state at $t = t'$ and, from the latter, the required state at $t = t_0$.

(B) (minor premise). If, at the instant $t = 0$, or more precisely, in the short interval $-\epsilon \leq t \leq 0$, we produce a luminous disturbance localized in the immediate neighbourhood of O, the effect at the subsequent instant $t = t'$ is localized in a very thin spherical shell with centre O and radius ct', where c is the velocity of the light.

(C) (conclusion). In order to calculate the effect of the initial phenomenon produced at O at $t = 0$, we may replace it by a proper system of disturbances taking place at $t = t'$ and distributed over the surface of the sphere with centre O and radius ct'.

Premise (A) can be considered as a statement of the causality principle. Premise (C) is the principle of superposition of secondary waves, or diffusion of waves, which holds for all kinds of wave propagation. What was considered by Hadamard as the proper Huygens' principle, or "Huygens' principle in the narrow sense" is premise (B). Unlike the diffusion of waves, premise (B), which essentially asserts that signals emitted sharply will propagate sharply, is a very exceptional phenomenon, since a small perturbation in the wave equation will destroy the property.

Huygens' principle is more precisely formulated using the concept of Cauchy's problem. Let us consider the general second order hyperbolic equation in n dimensions, with C^∞ coefficients:

$$P[u] := \sum_{i,j=0}^{n} g^{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=0}^{n} A^i(x)\frac{\partial u}{\partial x^i} + B(x)u = f(x)\,. \tag{1.1}$$

The Cauchy problem for (1.1) is the problem of determining a solution which assumes given values of u and its normal derivative on a given space-like $(n-1)$-dimensional manifold S. These given values are called Cauchy data. The first general solution to Cauchy's problem for (1.1)was given by Hadamard [42]. He has shown that in general, for the second order hyperbolic equation (1.1) the solution $u(\xi)$, at fixed point ξ, depends on the data in the interior of the region defined by $S \cap C^-(\xi)$, where S is the initial surface and $C^-(\xi)$ is the past characteristic conoid with vertex at ξ. If the solution depends only on the data in an arbitrarily small neighborhood of $S \cap C^-(\xi)$ for every Cauchy problem and for every point of the manifold, we say that the equation satisfies Huygens' principle, or that it is a Huygens' equation. The simplest examples of Huygens' equations are the ordinary wave equations, also called trivial equations:

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_N\right) u(t, x^1, \ldots, x^N) = 0\,, \tag{1.2}$$

where

$$\Delta_N := \frac{\partial^2}{\partial (x^1)^2} + \cdots \frac{\partial^2}{\partial (x^N)^2}\,, \tag{1.3}$$

and $N := n - 1$. For the initial data

$$u(0, x) = 0\,, \qquad u_t(0, x) = \delta(x)\,, \tag{1.4}$$

the fundamental solutions have the form

$$G_2 = \frac{1}{2\pi}\frac{H(t^2 - |x|^2)}{\sqrt{t^2 - |x|^2}} \qquad N = 2\,, \tag{1.5}$$

$$G_3 = \frac{1}{2\pi}\delta(t^2 - |x|^2) \qquad N = 3\,, \tag{1.6}$$

where H is the Heaviside function, defined by $H(t) = 1$ if $t \geq 0$, and $H(t) = 0$ if $t < 0$, and δ is the Dirac delta functional. Equation (1.6) shows that for $N = 3$ the support of the fundamental solution is located only on the characteristic cone surface, i.e., for $t > 0$ the solution will always stay concentrated on the surface of a sphere of radius $|x| = t$, propagating sharply. Thus, Huygens' principle is satisfied in this case. For $N = 2$, (1.5) shows that the support of the fundamental solution is contained in the interior of the characteristic cone and, for $t > 0$, the solution is concentrated on the surface of a disk $|x| \leq t$. Propagation here is not sharp in general. Once a signal has reached a point in space, it persists there indefinitely as a reverberation, thus violating Huygens' principle. Hadamard also showed [42] that in order that Huygens' principle be valid for (1.1) it is necessary that $n \geq 4$ be even. For (1.1) it can be shown [42] [32] that for n even, the forward and backward fundamental solutions $G_{\pm}$ of the operator P split in two parts as follows:

$$G_{\pm}(x) = G_{\pm}^{sing}(x) + G_{\pm}^{reg}(x)\,, \tag{1.7}$$

where

$$\text{supp}G_{\pm}^{sing}(x) = C^{\pm}(x)\,, \tag{1.8}$$

and

$$G_{\pm}^{reg}(\xi) = \int_{D_{\pm}(\xi)} K(x,\xi) f(\xi) d^n\xi\,, \tag{1.9}$$

where K is a smooth kernel. It is from this "tail term" K that the necessary conditions for the validity of Huygens' principle can be obtained. In general there are some coefficients of (1.1) which make this kernel vanish at all points. For n odd however, it can be shown that such decomposition is impossible. An example is given by (1.5), which is singular on C^-.

Two equations of the form (1.1) are said to be *equivalent* if they are related by one or a combination of the following transformations, called *trivial transformations* which preserve the Huygens' character of the differential equation:

(a) a transformation of coordinates;
(b) multiplication of both sides of the equation by a non-vanishing factor $e^{-2\phi}$, where $\phi(x)$ is an arbitrary function of the coordinates (this transformation induces a conformal transformation of the metric);
(c) replacement of the unknown u by λu, where $\lambda(x)$ is a non-vanishing functions of the coordinates.

Hadamard's problem is the problem of finding all Huygens' equations. For the scalar wave equation it can be formulated as follows:

Determine explicitly all equivalence classes of Huygens' equations modulo the trivial transformations, on the set of all second order linear hyperbolic partial differential equations of the form (1.1).

Although this problem has been studied extensively, it is still far from being solved. A conjecture that R. Courant [26] attributed to Hadamard states that the only Huygens' operators are the trivial ones. This is known as *Hadamard's conjecture.* Mathisson [56] Hadamard [43] and Asgeirsson [8] proved this conjecture for the case of four-dimensional Minkowskian space-times. They have considered hyperbolic equations of the form (1.1) with $g^{ij} = const.$ and proved that they satisfy Huygens' principle if and only if they are equivalent to the ordinary wave equation, i.e., $A^i = 0$, $B = 0$. However, Stellmacher [76] showed in 1953 that this conjecture is not valid for Minkowski spaces with N odd and $N \geq 5$. These non-trivial hyperbolic equations are given by

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_N\right) u + \frac{2}{(x^1)^2} u = 0\,, \qquad \left(\frac{\partial^2}{\partial t^2} - \Delta_N\right) u + \frac{2}{(t)^2} u = 0\,. \tag{1.10}$$

Stellmacher also proved that for $N = 5$ the above equations exhaust all possible cases for non-trivial Huygens' equations of the form

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_5\right) u + B(t, x^1, \ldots, x^5) u = 0\,. \tag{1.11}$$

Further generalizations for Minkowski spaces were obtained by Stellmacher, Lagnese, [77] [51] [52] [53] [54], Berest and Veselov [13] [15] [14].

For many years there was a suspicion that the four-dimensional Minkowski space, and the Riemannian space-times conformally related to it, were the only instances of space-times in which Huygens' principle for the second order linear hyperbolic operator is valid. However, in 1965 Günther [40] established the existence of non-conformally flat space-times for which the scalar wave equation (1.1), with $A^i = 0$ and $B = 0$, is a Huygens' equation. These space-times are described by the following metric:

$$ds^2 = 2dx^0 dx^1 - a_{ij} dx^i dx^j \qquad (i, j = 2, 3)\,. \tag{1.12}$$

The symmetric matrix (a_{ij}) is positive definite with components which are functions only of x^0. This family of metrics had been previously studied by Petrov [68], who classified it as a maximum mobility space-time of type T_2. This metric had also been studied by Ehlers and Kundt [31] in a coordinate system where it has the form

$$ds^2 = 2dv[du + (Dz^2 + \overline{D}\overline{z}^2 + ez\overline{z})dv] - 2dzd\overline{z}\,, \tag{1.13}$$

where $D = D(v)$ and $e = e(v) = \overline{e}$. This is the exact plane gravitational wave solution of the vacuum or Einstein-Maxwell equations. Up to now these are the only known nontrivial Huygens' equations for $n = 4$. In particular, McLenaghan [58] proved the following result:

The general scalar wave equation on space-times that are conformally related to an empty space-time ($R_{ab} = 0$), satisfies Huygens' principle only if it is equivalent to the plane-wave metric (1.13).

Thus, Hadamard's problem was solved for this case. One special aspect of the space-times with metric (1.12) is that their conformal group is *nontrivial* in the sense that in any space-time which is conformally equivalent to (1.12), the conformal group is wider than the group of isometric motions [49]. It was shown by Ibragimov [47] [48] [49] that in space-times with non-trivial conformal group, conformal invariance of the associated scalar operator implies the validity of Huygens' principle.

By imposing that the tail term on the elementary solution vanishes, Hadamard was able to establish a necessary and sufficient condition for the validity of Huygens' principle, for the scalar wave equation. From this criteriion, necessary conditions, involving the coefficients of the equation, were obtained, allowing deeper investigations about the nature of Huygens' equations. These conformally invariant, symmetric, trace-free tensorial expressions are the result of work of several researchers (see Section 2.3.2). By considering these necessary conditions, Carminati and McLenaghan [18] have outlined a program for the solution of Hadamard's problem in four dimensions based on the Petrov classification of the Weyl conformal curvature tensor, in the respective five disjoint cases. This seems to be a very effective way to deal with the problem, since the Petrov type is invariant under a general conformal transformation.

Hadamard's problem for (1.1) was completely solved for Petrov type N space-times by Carminati, McLenaghan and Walton [19] [20] [61], and the following result was found:

Every Petrov type N space-time on which equation (1.1) satisfies Huygens' principle is conformally related to an exact plane wave space-time (1.13).

Equation (1.1) with $A^i = 0$ and $B = R/6$, where R is the Ricci scalar associated to the metric g_{ij}, is known as the *conformally invariant scalar wave equation.* Equation (1.1) with $A^i \not\equiv 0$ is called *non-self-adjoint equation.* In Petrov type D space-times the following result was obtained by Carminati, McLenaghan and Williams [21] [62] and Wünsch [87]:

There exist no Petrov type D space-times on which the conformally invariant scalar wave equation satisfies Huygens' principle.

In all other cases the results are partial, but there are indications that the following *conjecture* may be true [19] [20] [23]:

Every space-time on which the conformally invariant scalar wave equation satisfies Huygens' principle is conformally related to the plane-wave space-time (1.13) or is conformally flat.

In this Thesis we are also concerned with Hadamard's problem formulated for Maxwell's equations for the electromagnetic field [40]:

$$d\omega = 0\,, \qquad \delta\omega = 0\,, \tag{1.14}$$

where ω is the 2-form

$$\omega_2 = \frac{1}{2}\,\omega_{ij}\,dx^i \wedge dx^j\,. \tag{1.15}$$

The symbols d and δ denote the exterior differentiation and codifferentiation operators, respectively. The initial value problem for (1.14) was studied by Duff [30] in 1953, using the Riesz kernel formalism [71], and Lichnerowicz [55] in 1961, using the distribution formalism. Günther [40] used the first method to establish the first necessary condition for the validity of Huygens' principle, which is the vanishing of Bach's tensor.

We are also concerned with Weyl's neutrino equation. It can be expressed in a unified form with Maxwell's equations in the following form [85]:

$$(M\varphi)_{A_1\ldots A_{m-1}\dot{X}_0} := \nabla^K{}_{\dot{X}_0}\varphi_{KA_1\cdots A_{m-1}} = 0\,, \tag{1.16}$$

where φ is a symmetric $(m,0)$-spinor field, and $\nabla^K{}_{\dot{X}}$ is the covariant derivative on 2-spinors. For $m = 1$ we have the Weyl neutrino equation and for $m = 2$ we have the homogeneous Maxwell's equations.

Günther and Wünsch [39] [41] [82] [84] [85] [86] have formulated Cauchy's problem for these equations and several other spinor field equations by defining the Riesz spinor kernel. By using Hadamard's criteria for the resulting representation formulas, necessary conditions for the validity of Huygens' principle were obtained for Maxwell's equations and Weyl's neutrino equation [39] [41] [82] [84] [85] [36] [3]. Using these conditions, Carminati and McLenaghan [21], Wünsch [87], and McLenaghan and Williams [62] obtained the following result:

There exist no Petrov type D space-times on which Maxwell's equations (1.14) or Weyl's neutrino equation (1.16) satisfy Huygens' principle.

For Maxwell's equations and Weyl's neutrino equation on Petrov type N space-times, Hadamard's problem was solved recently by Wünsch [89]:

Every Petrov type N space-time or C-space-time[1] *on which Maxwell's equations or Weyl's neutrino equation satisfy Huygens' principle is conformally related to the exact plane-wave space-time (1.13).*

In this Thesis we make contributions to the study of Hadamard's problem for the non-self-adjoint and self-adjoint scalar wave equations, Maxwell's equations and Weyl's neutrino equation on Petrov type III space-times. A physical example of a Petrov type III space-time was given by Robinson and Trautman [73]. For this purpose we have followed a research program started by Carminati and McLenaghan [18] which consists in analysing the necessary conditions for each Petrov type, using the tools of two-component spinor formalism of Penrose [67] [69] and the spin-coefficient formalism of Newman and Penrose [64], [24] (see Appendix C). Essentially, the analysis starts with the use of the conformal invariance of the problem and a convenient choice of local basis (dyads) to simplify the necessary conditions. The tensorial expressions for these conditions must be converted first to their corresponding spinor form and then decomposed in a dyad basis.

The task of representing tensorial expressions in a spinor dyad basis involves lengthy calculations, especially when the expressions contain derivatives of third order of the Weyl tensor. However, using the Maple computer algebra package `NPspinor`, developed by Czapor and McLenaghan [27] [29], we were able to do these computations very efficiently. After the conversion to the dyad form is completed, we still have to face the problem of using the expressions obtained from the necessary conditions together with the NP Ricci identities, NP Bianchi identities and NP commutation relations to obtain useful Pfaffian and polynomial relations. Although the expressions involved are in general very large, `NPspinor` provides efficient tools for their manipulation.

The final problem is to solve the resulting (very large, in general) polynomial system, involving the NP scalars. For this purpose we have used with success the Maple package `grobner`, developed by Czapor [28], which makes use of Buchbergers' algorithm to find Gröbner bases for systems of polynomial equations (see Appendix A). Our experience shows that a direct application of Buchberger's algorithm to our problem, using the procedure `gbasis` from `grobner` or the package Gb [34] [33], in total degree or lexicographical ordering, is not practical. However, the procedure `gsolve` from `grobner` has proved to be very successfull in this case. This procedure computes a collection of reduced lexicographic Gröbner bases corresponding to a set of polynomials. The system corresponding to the set is first subdivided by factorization. Then a variant of Buchberger's algorithm which factors all intermediate results is applied to each subsystem. The result is a list of reduced subsystems whose roots are those of the original system, but whose variables

[1] C-space-times are defined by the property $\nabla^a C_{abcd} = 0$

have been successively eliminated and separated as far as possible. This means that instead of trying to find a Gröbner basis, the package attempts to factor the polynomials that form the system after each step of the reduction algorithm.

Next we summarize the contents of the chapters that form this Thesis.

In Chapter 2 we present a review for the Cauchy problem and Huygens' principle for the second order linear partial differential equations of normal hyperbolic type in curved space. In Section 2.1 we present some basic definitions. Section 2.2 is based on Friedlander's book [32], and McLenaghan's works [57] [60], and summarizes the procedures for the construction of the fundamental solution of (1.1) and the representation formula. This approach is based on the work of Hadamard [42], although the theory of scalar distributions is used. As we shall see, he fundamental solution consists of the sum of two terms: a singular part, with support on the characteristic cone, and a regular part, with support in the interior of the cone. By imposing that the regular part vanish, we can obtain necessary and sufficient conditions for the validity of Huygens' principle. Finally, in Section 2.3 we present a review based on McLenaghan's fundamental paper [60] showing how to derive explicit necessary conditions from the expansion of the diffusion kernel in a Taylor series in normal coordinates. The form of the known necessary conditions is presented.

Chapter 3 is dedicated to the study of Hadamard's problem for the conformally invariant scalar equation (also called the self-adjoint scalar equation) on Petrov type III space-times. Carminati and McLenaghan [22], using the two- and four- index necessary conditions and the Newman-Penrose formalism have proved the following result:

If any one of the following three conditions

$$\Psi_{ABCD;E\dot{E}}\,\iota^A\iota^B\iota^D\iota^E\bar{o}^{\dot{E}} = 0\,, \tag{1.17}$$

$$\Psi_{ABCD;E\dot{E}}\,\iota^A\iota^B o^D o^E\bar{\iota}^{\dot{E}} = 0\,, \tag{1.18}$$

$$\Psi_{ABCD;E\dot{E}}\,\iota^A\iota^B\iota^D o^E\bar{o}^{\dot{E}} = 0\,, \tag{1.19}$$

is satisfied, then there exist no Petrov type III space-times on which the conformally invariant scalar wave equation (1.1) satisfies Huygens' principle.

This is the same as stating that, with the particular choice of tetrad and conformal gauge used to deal with this problem, space-times for which any one of the NP coefficients, α, β or π is zero, violate Huygens' principle. Using the six-index necessary condition of Rinke and Wünsch [72], we have found a system of polynomials involving α, β, π and the constant NP Ricci component Φ_{11}. Although the complete solution of the problem was not yet achieved, we give a step further towards its solution by proving that, *On Petrov type III space-times, the validity of Huygens' principle for the conformally invariant equation (3.1) implies that* $\Phi_{AB\dot{A}\dot{B}}o^A\iota^B\bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}} \neq 0$.

Chapter 4 is dedicated to the non-self-adjoint scalar equation. Here we have used a five-index condition derived Anderson and McLenaghan [5] [6], extending the results found recently by the same authors [7]. Our main result can be stated in the following way:

If a non-self-adjoint scalar equation satisfies Huygens' principle on Petrov type III space-times, then it must be equivalent to the conformally invariant scalar wave equation.

In Chapter 5 we study Hadamard's problem for Maxwell's equations and Weyl's neutrino equation on Petrov type III space-times. In Sections 5.2 and 5.3 we give a review based mainly on Wünsch and Günther's papers [39] [41] [82] [84] [85]. We show how Riesz kernel theory can be used for Maxwell's equations and some spinor equations, to determine necessary and sufficient conditions for the validity of Huygens' principle. We also show how the theory of conformally invariant, rational integral, metric differential concomitants [78] [70] [82] can provide a very elegant tool in the determination of the necessary conditions [36] [37] [3]. In Section 5.4 we describe some important results for the Hadamard problem and present the known necessary conditions. Finally, in Section 5.6 we use the five-index necessary condition derived by Alvarez, Gerlach and Wünsch [3] [4] [36] to prove the main result of the Chapter, which solves Hadamard's problem in this case [63]:

There exist no Petrov type III space-times on which Maxwell's equations or Weyl's equation satisfy Huygens' principle.

In Chapter 6 we present the conclusions of this work and prospects for future research on this subject.

Appendix A is a brief introduction to the theory of Gröbner bases.

Appendix B makes a brief description of the two-component spinor formalism of Penrose [67] [69] and the spin-coefficient formalism of Newman and Penrose [64], [24].

Appendix C lists the NP field equations and commutation relations.

Appendix D lists the conventions used in this Thesis.

Appendix E gives a list of some Maple codes used to perform the NP calculations.

Chapter 2

Huygens' Principle for the Scalar Wave Equation

2.1 Definitions

Let us consider the general linear second-order hyperbolic equation with C^∞ coefficients:

$$P[u] := \sum_{i,j=0}^{n} g^{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=0}^{n} \dot{A}^i(x)\frac{\partial u}{\partial x^i} + \dot{B}(x)u = 0\,. \tag{2.1}$$

This equation can be interpreted as a generalized wave equation in a pseudo-Riemannian space $\mathcal{M}^n$ with a metric defined by the principal part of the operator P:

$$\mathrm{d}s^2 = g_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j\ , \tag{2.2}$$

where $\|g_{ij}(x)\| =: g = \|g^{ij}(x)\|^{-1}$. The hyperbolic character of (2.1) implies that the metric (2.2) has a Lorentzian signature $(+,-,\cdots,-)$. The operator P can be expressed in covariant form:

$$P = \Box + A^k(x)\nabla_k + B(x)\,, \tag{2.3}$$

where

$$\Box := g^{ij}\nabla_i\nabla_j = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}g^{ij}\frac{\partial}{\partial x^j}\right)\,, \tag{2.4}$$

with $g := det(g_{ij}(x))$, is the Laplace-Beltrami operator defined on $\mathcal{M}^n$, and ∇_k is the covariant derivative defined in terms of the metric (2.2).

The *Cauchy problem* for the wave equation $Pu = f$, where f is C^∞, consists of finding a C^2 solution in a neighborhood of a $(n-1)$-dimensional space-like hyper surface S, given u and ∇u on S. We denote a point in $\mathcal{M}^n$ by $x = (x^0, x^1, \ldots, x^n)$.

The considerations here will be local. This means that we will be restricted to an open connected neighborhood $\Omega \in \mathcal{M}^n$ of a point x_0. The null (characteristic) conoid through any point $x_0 \in \Omega$ will be denoted by $C(x_0)$. It is defined as the set of points x such that $\Gamma(x_0, x) = 0$, where $\Gamma(x_0, x)$ is the square of the geodesic distance of x from x_0. The null cone is separated into two null semi-conoids $C^+(x_0)$ and $C^-(x_0)$, called respectively the *future (or forward)* and *past (or backward)·conoid,* respectively, such that $C(x_0) = C^+(x_0) \cup C^-(x_0)$ and $C^+(x_0) \cap C^-(x_0) = \{x_0\}$. The corresponding open subsets of Ω bounded by $C^\pm(x_0)$ are denoted by $D^\pm(x_0)$ and satisfy

$$D^+(x_0) \cup D^-(x_0) = \{x \in \Omega, \Gamma(x_0, x) > 0\}\,. \tag{2.5}$$

The closure of $D^\pm(x_0)$, $\overline{D^\pm(x_0)})$, is called the *future (past) emission* of x_0 and is denoted by $J^\pm(x_0)$.

Let C_0^k denote the class of all functions in C^k with compact support. The members of C_0^∞ are called *test functions* and will be denoted by ϕ. A *distribution* $\langle f, \phi\rangle$ is defined as a continuous linear functional on C_0^∞. The symbol $\mathcal{D}'(\Omega)$ denotes the vector space of distributions in Ω.

It is useful to reformulate Huygens' principle, in the sense of Hadamard's premise (B), in a more general way. The homogeneous equation $Pu = 0$ satisfies Huygens' principle if $u(x)$ depends only on the Cauchy data in an arbitrarily small neighborhood of $C^-(x) \cap S$ (the intersection between the backward characteristic conoid with vertex at x and the initial surface S), for arbitrary Cauchy data on S, arbitrary S, and for all points x in the future of S.

The *forward (backward) fundamental solution* of the operator P is defined as a distribution $G^\pm \in \mathcal{D}'(\Omega)$ such that

$$P[G^\pm(x, x_0)] = \delta_{x_0}(x)\ , \tag{2.6}$$

$$\operatorname{supp}\, G^\pm(x, x_0) \subseteq \overline{D^\pm(x_0)}\ , \tag{2.7}$$

where $\delta_{x_0}(x)$ is a Dirac delta-measure on $\mathcal{M}^n$ concentrated at the point x_0.

In Subsection 2.2.5 it will be shown that *the operator P, defined by (2.3), is a Huygens' operator in a connected open set $\Omega \subset \mathcal{M}^n$ if and only if for every point $x_0 \in \Omega$ we have*

$$\operatorname{supp}\, G^\pm(x, x_0) \subseteq C^\pm(x_0)\,. \tag{2.8}$$

2.2 Fundamental solutions and representation formula

2.2.1 The ordinary wave equation

Let us consider the ordinary wave equation on a four-dimensional Minkowskian space-time,

$$\Box u = (\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)u = f\,, \tag{2.9}$$

where $\partial_i = \frac{\partial}{\partial x^i}$, and x denotes a point on $\mathbf{R}^4$, with Cartesian coordinates $(t, X) = (x^0, x^1, x^2, x^3)$. This can be also considered, in a fixed coordinate system, as a partial differential equation on $\mathbf{R}^4$.

A fundamental solution of (2.9) is a distribution G such that $\Box G(x) = \delta(x)$. This means that

$$\langle G, \Box\phi\rangle = \langle \Box G, \phi\rangle = \phi(0), \quad \textit{for all } \phi \in C_0^\infty(\mathbf{R}^4). \tag{2.10}$$

Next we shall show how to construct the fundamental solution for the ordinary wave equation (2.9). First, we need to enunciate a lemma from distribution theory [32]:

Lemma 2.1 *Let $\rho(x) \in C_0^\infty$, $x \in \mathbf{R}$, be such that*

$$\rho(x) \geq 0, \quad supp\, \rho \subset \{x;\ |x| \leq 1\}, \quad \int \rho\, dx = 1. \tag{2.11}$$

Suppose that $f(x) \in C^k$, and put

$$f_\epsilon(x) = \epsilon^{-n} \int f(y)\, \rho\left(\frac{x-y}{\epsilon}\right) dy, \tag{2.12}$$

where ϵ is a positive number and $y \in \mathbf{R}$. Then $f_\epsilon \in C^\infty$, and its support is contained in an ϵ-neighborhood of supp f. Moreover, if $\epsilon \to 0$, then $\partial^\alpha f_\epsilon \to \partial^\alpha f$ for all α with $|\alpha| \leq k$, uniformly in any compact set.

We are now ready to prove the following theorem:

Theorem 2.1 *Let r denote the Euclidean norm of $X \in \mathbf{R}^3$. The two distributions*

$$G^+ = \frac{\delta(t-r)}{4\pi r}, \qquad G^- = \frac{\delta(t+r)}{4\pi r}, \tag{2.13}$$

defined by

$$\langle G^+, \phi\rangle = \int \frac{\phi(r, X)}{4\pi r}\, dX, \quad \langle G^-, \phi\rangle = \int \frac{\phi(-r, X)}{4\pi r}\, dX, \tag{2.14}$$

where dX is the volume element on $\mathbf{R}^3$, are fundamental solutions of the wave equation (2.9).

Proof: If $\psi(t) \in C_0^\infty(\mathbf{R})$, then the function $(X,t) \to \psi(t-r)/r$ is locally integrable, and so can be identified with the distribution

$$\langle \frac{\psi(t-r)}{r}, \phi \rangle = \int \frac{\psi(t-r)}{r} \phi(t,X)\, dt\, dX. \tag{2.15}$$

Thus,

$$\langle \Box \frac{\psi(t-r)}{r}, \phi \rangle = \langle \frac{\psi(t-r)}{r}, \Box \phi \rangle = \lim_{\epsilon \to 0} \int_{r \geq \epsilon} \frac{\psi(t-r)}{r} \Box \phi \, dt\, dX. \tag{2.16}$$

Since $\Box[\psi(t-r)/r] = 0$ on $r \geq \epsilon$, the last integral can be written in the form

$$\int_{r\geq\epsilon} \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial X^2} \right) \left[\frac{\psi(t-r)}{r} \phi \right] dt\, dX = \int_{r\geq\epsilon} \left\{ \frac{\partial}{\partial t} \left[\frac{\psi(t-r)}{r} \frac{\partial \phi}{\partial t} + \frac{\psi'(t-r)}{r} \phi \right] - \right.$$

$$\left. \nabla_X \cdot \left[\frac{\psi(t-r)}{r} \nabla_X \phi - \phi \nabla_X \left(\frac{\psi(t-r)}{r} \right) \right] \right\} dt\, dX. \tag{2.17}$$

The integral of the $\partial/\partial t$ term is zero, since $\psi(t)$ has compact support. The remaining terms are, according to the divergence theorem, given by

$$\int dt \int_{r=\epsilon} \left[\frac{\psi(t-r)}{r} \frac{\partial \phi}{\partial r} - \phi \frac{\partial}{\partial r} \frac{\psi(t-r)}{r} \right] r^2\, dw, \tag{2.18}$$

where dw is the surface element on S^2. This can also be written as

$$\int dt \int \phi(t, \epsilon\theta)\, \psi(t-\epsilon)\, dw + O(\epsilon), \quad \theta \in S^2. \tag{2.19}$$

When $\epsilon \to 0$ we get

$$\langle \Box \frac{\psi(t-r)}{r}, \phi \rangle = 4\pi \int dt\, \psi(t)\, \phi(t,0) = 4\pi \langle \psi(t) \otimes \delta(X), \phi(t,X) \rangle. \tag{2.20}$$

Hence,

$$\Box \frac{\psi(t-r)}{r} = 4\pi \delta(t) \otimes \delta(X). \tag{2.21}$$

Let us take $\psi(t) \geq 0$, and

$$\psi(t) \geq 0, \quad supp\, \psi \subset \{t;\ |t| \leq 1\}, \quad \int \psi\, dt = 1. \tag{2.22}$$

Then, by Lemma 2.1, there exists a function $\psi_\nu \in C^\infty$, where ν is a positive number, that tends to the distribution $\delta(t)$ when $\nu \to 0$. Replacing ψ by ψ_ν in (2.15), and changing the variables of integration on the right-hand-side, yields

$$\langle \frac{\psi_\nu(t-r)}{r}, \phi \rangle = \int \psi_\nu(t)\, dt \int \frac{\phi(t+r,X)}{r}\, dX\,. \tag{2.23}$$

Also, from Lemma 2.1,

$$\lim_{\nu\to\infty}\langle\frac{\psi_\nu(t-r)}{r},\phi\rangle=\int\frac{\phi(r,X)}{r}\,dX=\langle G^+,\phi\rangle. \tag{2.24}$$

On the other hand, since $\psi_\nu\otimes\delta(x)\to\delta(t)\otimes\delta(X)=\delta(x)$, as $\nu\to\infty$, we have from (2.21)

$$\lim_{\nu\to\infty}\Box\frac{\psi_\nu(t-r)}{r}=\Box\frac{\delta_\nu(t-r)}{r}=\Box G^+=\delta(x). \tag{2.25}$$

The proof for the advanced potential of ϕ is similar.

The support of $G^\pm$ is the future (past) null semi-cone $C^\pm(0)=\{(t,X);\ \pm t= r=|X|\}$ with vertex on the origin of the coordinate system.

2.2.2 Manifestly invariant derivation of the fundamental solutions of the ordinary wave equation

The square of the geodesic distance of a point $x=(t,X)$ from the origin is

$$\gamma(x)=\eta_{ij}x^ix^j=t^2-|X|^2, \tag{2.26}$$

and $D^+(0)$ will denote the interior of the future null semi-cone $C^+(0)$:

$$D^+(0)=\{x=(t,X)\,;\,t>|X|\}. \tag{2.27}$$

Since $\nabla\gamma\neq 0$ in $D^+(0)$, a theorem of distribution theory [1] guarantees that we can define a distribution $\delta_+(\gamma-\epsilon)\in\mathcal{D}'(D^+(0))$, where ϵ is a positive number, by

$$\langle\delta_+(\gamma-\epsilon),\phi\rangle=\int_{\gamma=\epsilon}\phi\,\mu_\gamma,\qquad \phi\in C_0^\infty(D^+(0)), \tag{2.28}$$

where μ_γ is the Leray form, such that $d\gamma\wedge\mu_\gamma=dx$, with dx being the volume element on $\mathcal{M}^4$. We shall denote the support of $\delta_+(\gamma-\epsilon)$, which is the hyperboloid $\{x,\ \gamma(x)=\epsilon\}$, by Σ_ϵ^+. We can extend $\delta_+(\gamma-\epsilon)$ to $\mathcal{D}'(M^4)$ by setting it to zero in $\left\{x=(t,X),t<\sqrt{|X|^2+\epsilon}\right\}$, so that, instead of (2.28) we have

$$\langle\delta_+(\gamma-\epsilon),\phi\rangle=\int_{\Sigma_\epsilon^+}\phi\,\mu_\gamma,\qquad \phi\in C_0^\infty(M^4), \tag{2.29}$$

A similar expression for $\delta_-(\gamma-\epsilon)$, with support on the lower sheet Σ_ϵ^- of $\gamma(x)=\epsilon$ can be obtained.

[1] [32], p. 69

Since, from (2.26), $d\gamma = 2tdt - 2XdX$, in order to satisfy $d\gamma \wedge \mu_\gamma = dx$, we can take $\mu_\gamma = dX/2t$. Thus

$$\langle \delta_\pm(\gamma - \epsilon), \phi \rangle = \int_{\Sigma_\epsilon^\pm} \phi\, \mu_\gamma = \frac{1}{2} \int \frac{\phi(\pm\sqrt{|X|^2 + \epsilon}, X)}{\sqrt{|X|^2 + \epsilon}} dX. \tag{2.30}$$

When $\epsilon \to 0$,

$$\langle \delta_\pm(\gamma), \phi \rangle = \frac{1}{2} \int \frac{\phi(\pm|X|, X)}{|X|} dX \tag{2.31}$$

Comparing this to (2.14) we obtain the Lorentz invariant forms of $G^\pm$:

$$G^\pm = \frac{1}{2\pi} \delta^\pm(\gamma) = \lim_{\epsilon \to 0} \delta_\pm(\gamma - \epsilon). \tag{2.32}$$

In the next section we deal with the general second order linear hyperbolic scalar equation, using the basic ideas that were presented above for the ordinary wave equation.

2.2.3 Singular part of the fundamental solution

A method for finding necessary and sufficient conditions for the validity of Huygens' principle, in terms of the coefficients of P, was provided by Hadamard [42]. Using the initial value formulation he determined a fundamental (elementary) solution for the problem. Thus, the conditions can be obtained by making vanish those parts of the solution which do not propagate along the characteristic surfaces of the differential equation. In what follows we shall summarize this procedure using the modern approach of distribution theory on a curved space-time [57] [32].

Suppose that the connected open set $\Omega \subset \mathcal{M}^4$ is a *geodesically convex domain*, i.e., any two points x_0 and x in Ω are joined by a unique geodesic in Ω. This implies that Ω is time-orientable. In what follows, x_0 is considered to be fixed. Let us now consider the scalar differential operator (2.3) defined on Ω. The fundamental solution of P is a distribution $\langle G_{x_0}(x), \phi(x) \rangle$ in $\mathcal{D}'(\Omega)$, such that

$$P\, G_{x_0}(x) = \delta_{x_0}(x)\,. \tag{2.33}$$

This means that

$$\langle PG_{x_0}, \phi \rangle = \langle G_{x_0}, {}^t P\phi \rangle = \phi(x_0), \quad \phi \in C_0^\infty, \tag{2.34}$$

where ${}^tP$ is the adjoint of P,

$${}^tP\phi = \Box\phi - \nabla_i(A^i\phi) + B\phi. \tag{2.35}$$

The study of the Minkowski case suggests that the fundamental solutions of P have the form $U\delta_\pm(\Gamma)$, where U is a C^∞ function which depends on x and x_0.

Since we can always construct a local coordinate system which is normal and Minkowskian at x_0, defined for all geodesically convex neighborhoods Ω, the distributions $\delta_\pm(\Gamma) = \lim_{\epsilon\to 0}\delta_\pm(\Gamma-\epsilon)$ exist. However, since this distribution has support on the null cone, and since we expect that Huygens' principle is not satisfied in the general case, we could also guess that this is only part of the solution. In fact, as we shall see, this is the singular part.

Let us first examine the effect of P on $U\delta_+(\Gamma)$. Using the chain rule for distributions, which states that for a distribution $f(S)$, $S \in C_0^\infty(\Omega)$, $\nabla f(S) = f'(S)\nabla S$, we find

$$\begin{aligned} P\left(U\delta_+(\Gamma-\epsilon)\right) &= P(U)\delta_+(\Gamma-\epsilon) + 4\Gamma U\delta_+''(\Gamma-\epsilon) \\ &+ \left(2\nabla_i\Gamma\,\nabla^i U + (\Box\Gamma + A^i\,\nabla_i\Gamma)U\right)\delta_+'(\Gamma-\epsilon), \end{aligned} \tag{2.36}$$

Since $t\delta''(t) + 2\delta'(t) = 0$, we have

$$(\Gamma-\epsilon)\delta_+''(\Gamma-\epsilon) = -2\delta_+'(\Gamma-\epsilon), \tag{2.37}$$

or

$$4\Gamma\delta_+''(\Gamma-\epsilon) = 4\epsilon\delta_+''(\Gamma-\epsilon) - 8\delta_+'(\Gamma-\epsilon). \tag{2.38}$$

So, (2.36) can be written as

$$\begin{aligned} P\left(U\delta_+(\Gamma-\epsilon)\right) &= P(U)\delta_+(\Gamma-\epsilon) + 4\,\epsilon\,U\delta_+''(\Gamma-\epsilon) \\ &+ \left(2\nabla_i\Gamma\,\nabla^i U + (\Box\Gamma + A^i\,\nabla_i\,\Gamma - 8)U\right)\delta_+'(\Gamma-\epsilon). \end{aligned} \tag{2.39}$$

Since $\delta_+'(\Gamma-\epsilon)$ does not tend to a limit when $\epsilon\to 0$, we choose U as a solution of

$$2\nabla_i\Gamma\,\nabla^i U + (\Box\Gamma + A^i\,\nabla_i\Gamma - 8)U = 0. \tag{2.40}$$

If $s \to x(s)$ is a geodesic such that $x(0) = x_0$, s being an affine parameter, then [2]

$$\nabla\Gamma = 2\,s\,\frac{dx}{ds}, \tag{2.41}$$

along this geodesic. We can then see that (2.40) is in fact a first order ordinary differential equation whose characteristic curves are the geodesics through x:

$$4\,s\,\frac{dU}{ds} + (\Box\Gamma + A^i\,\nabla_i\Gamma - 8)U = 0. \tag{2.42}$$

[2] see [32], p. 17.

The general solution for this equation is

$$U(x_0, x) = exp\left[-\frac{1}{4}\int_0^{s(x)} (\Box\Gamma + A^i\nabla_i\Gamma - 8)\,\frac{dt}{t}\right], \tag{2.43}$$

where the integration is carried out along the geodesic x_0x. We made the constant of integration equal to one, so that $U(x_0, x_0) = 1$. To obtain a more explicit form of this integral we introduce local coordinates that are normal through x_0. Then, with x_0 having coordinates $x = 0$, we obtain

$$g_{ij}(x)\,x^j = g_{ij}(0)\,x^j, \qquad \Gamma = g_{ij}(0)x^i x^j\,. \tag{2.44}$$

Thus

$$\nabla^i\Gamma = 2\,g^{ij}(x)\,g_{jk}(0)\,x^k = 2\,g^{ij}(x)\,g_{jk}(x)\,x^k = 2\,x^i, \tag{2.45}$$

$$\begin{aligned}\Box\Gamma = \nabla^i\nabla_i\Gamma = \nabla_i(2x^i) &= \frac{2}{\sqrt{-g}}\partial_i(\sqrt{-g}\,x^i) = \frac{2}{\sqrt{-g}}\left[\sqrt{-g}\,\delta_i^i + x^i\,\partial_i\sqrt{-g}\right]\\ &= 8 + 2\,x^i\,\partial_i(\log(\sqrt{-g})\,.\end{aligned} \tag{2.46}$$

Multiplying (2.40) by $(-g)^{1/4}$ gives

$$\begin{aligned}&(-g)^{1/4}\left[4x^i\nabla_iU + (2x^i\partial_i(\log(\sqrt{-g}) + A_i2x^i)U\right]\\ &= 2(2\,(-g)^{1/4}x^i\nabla_iU + \frac{x^i\partial_i\sqrt{-g}}{\sqrt{-g}}(-g)^{1/4} + A_ix^iU) = 0\,,\end{aligned} \tag{2.47}$$

or

$$x^i\partial_i\left[(-g)^{1/4}U\right] + \frac{1}{2}A_ix^i(-g)^{1/4}U = 0\,. \tag{2.48}$$

If x is replaced by rx, $0 \le r \le 1$, we get

$$\frac{\partial}{\partial r}\left[(-g)^{1/4}U|_{rx}\right] = -\frac{1}{2}A_ix^i(-g)^{1/4}U|_{rx}\,, \tag{2.49}$$

or

$$\frac{\partial}{\partial r}\left(\log\left|(-g)^{1/4}U|_{rx}\right|\right) = -\frac{1}{2}A_ix^i. \tag{2.50}$$

Then, since $U(x_0) = U(0) = 1$, the solution of (2.39) can be written as

$$U(x) = \left|\frac{g(0)}{g(x)}\right|^{1/4} exp\left[-\frac{1}{2}\int_0^1 A_i(rx)\,x^i\,dr\right]. \tag{2.51}$$

As the second member is a C^∞ function of x, and the exponential map depends smoothly on x_0, $U \in C_0^\infty(\Omega \times \Omega)$.

Now let us investigate the term that remains in (2.39):

$$P\,(U\,\delta_+(\Gamma-\epsilon)) = P(U)\delta_+(\Gamma-\epsilon) + 4\epsilon U\delta_+''(\Gamma-\epsilon). \tag{2.52}$$

Since the limit $\delta_+(\Gamma - \epsilon) \to \delta_+(\Gamma)$ as $\epsilon \to 0$ exists, we also have, by continuity of the map $P : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$, $P(U\,\delta_+(\Gamma - \epsilon)) \to P(\delta_+(\Gamma))$. In normal coordinates, $\Gamma(x, x_0) = \gamma(x) = \eta_{ij}\,x^i x^j$. Thus $4\,\epsilon\,U\,\delta''_+(\Gamma - \epsilon) = 4\,\epsilon\,U\,\delta''_+(\gamma - \epsilon)$. In distributional notation,

$$\langle 4\,\epsilon\,U\,\delta''_+(\Gamma - \epsilon), \phi\rangle = 4\,\epsilon\langle\delta''_+(\gamma - \epsilon), U(x)\,\phi(x)\,\sqrt{-g}\,\rangle. \tag{2.53}$$

Using the chain rule for distributions, the relation $t\,\delta''(t) + 2\,\delta'(t) = 0$, and (2.32), we find

$$\lim_{\epsilon\to 0} 4\,\epsilon\delta''_+(\gamma - \epsilon) = 2\,\pi\delta(x). \tag{2.54}$$

Then (2.53) tends to $2\,\pi\,U(0)\phi(0)\sqrt{-g}$ when $\epsilon \to 0$,. But $U(0) = 1$ and $g(0) = 1$. Thus, from (2.52) we obtain, as $\epsilon \to 0$,

$$P\,(U\,\delta_+(\Gamma)) = P(U)\,\delta_+(\Gamma) + 2\,\pi\delta_{x_0}. \tag{2.55}$$

Similarly,

$$P\,(U\,\delta_-(\Gamma)) = P(U)\,\delta_-(\Gamma) + 2\,\pi\delta_{x_0}. \tag{2.56}$$

If $P(U) = 0$ on $C^\pm(x_0)$ then $U(x, x_0)\,\delta_\pm(\Gamma(x, x_0))$ is a fundamental solution of P, with support on $C^\pm(x_0)$. If $P(U) \neq 0$, the term $P(U)\delta_\pm(\Gamma)$ in (2.55-2.56) has to be eliminated, i.e., we have to find the regular part of the fundamental solution.

2.2.4 The general equation - regular part of the fundamental solution

Using the Heaviside function $H(t)$, and the fact that $\nabla_x\Gamma \neq 0$ in $D^+(x_0)$, we can define the distribution $H_+(\Gamma - \epsilon)$, $\epsilon > 0$, extended to $\mathcal{D}'(\Omega)$ (cf. eq. 2.28), by

$$\langle H_+(\Gamma - \epsilon),\ \phi\rangle = \langle H(t),\ \int_{\Sigma_t^+} \phi\,\mu_\Gamma(x, x_0)\rangle, \quad \phi \in C_0^\infty(\Omega), \tag{2.57}$$

where $\Sigma_t^+ = \{x;\ x \in D^+(x_0),\ \Gamma(x, x_0) - \epsilon = t > 0\}$, and μ_Γ is the Leray form such that $d_x\Gamma(x, x_0) \wedge \mu_\Gamma = \mu(x)$ (μ is the invariant volume element). When $\epsilon \to 0$ this distribution tends to the characteristic function $J^+(x_0)$. Thus

$$H_+(\Gamma) = 1\ \ if\ \ x \in J^+(x_0), \quad H_+(\Gamma) = 0\ \ if\ \ x \notin J^+(x_0). \tag{2.58}$$

Similarly, $H_-(\Gamma)$ is the characteristic function of $J^-(x_0)$. We can now begin to construct the complete fundamental solution of P in the form $U\delta_\pm(\Gamma) + W\,H_\pm(\Gamma)$, $W \in C^\infty(\Omega\times\Omega)$. The first step will be the investigation of the action of P on $W\,H_+(\Gamma)$. Since $H'(t) = \delta(t)$ and $\lim_{\epsilon\to 0}\delta'(\Gamma - \epsilon)$ does not exist, we first calculate $P(W\,H_+(\Gamma - \epsilon))$, $\epsilon > 0$. Using the chain rule for distributions we find

$$\begin{aligned} &P(W\,H_+(\Gamma - \epsilon)) = H_+(\Gamma - \epsilon)\,P(W) + \\ &+ \Big[2\nabla_i\Gamma\,\nabla^i W + (\Box\Gamma + A^i\nabla_i\Gamma)W\Big]\,\delta_+(\Gamma - \epsilon) + 4\,W\,\Gamma\delta'_+(\Gamma - \epsilon). \end{aligned} \tag{2.59}$$

Since $\lim_{\epsilon\to 0}\epsilon\Gamma\delta'_+(\Gamma-\epsilon)=0$ [3], we have, using $t\delta'(t)+\delta(t)=0$,

$$\lim_{\epsilon\to 0}\Gamma\delta'_+(\Gamma-\epsilon)=\lim_{\epsilon\to 0}\left(\epsilon\delta'_+(\Gamma-\epsilon)-\delta_+(\Gamma-\epsilon)\right)=-\delta_+(\Gamma). \tag{2.60}$$

Hence,

$$P(W\,H_+(\Gamma))=P(W)\,H_+(\Gamma)+\left[2\nabla_i\Gamma\,\nabla^iW+(\Box\Gamma+A^i\nabla_i\Gamma-4)\,W\right]\delta_+(\Gamma). \tag{2.61}$$

We now prove that there exists a $V_0\in C^\infty(\Omega\times\Omega)$ such that

$$2\nabla_i\Gamma\,\nabla^iV_0+(\Box\Gamma+A^i\nabla_i\Gamma-4)\,V_0=-PU. \tag{2.62}$$

Dividing the above equation by U, and using (2.40) to eliminate $\nabla_i\Gamma\,\nabla^iU$, we get

$$2\,\nabla_i\Gamma\,\nabla^i\left(\frac{V_0}{U}\right)+4\,\frac{V_0}{U}=-\frac{PU}{U}. \tag{2.63}$$

Consider this equation in local coordinates. Let $[0,1]\ni r\to z(r)=h(x_0,r\theta(x,x_0))$ be the solutions of the differential equations of the geodesic x_0x with initial values $z(0)=x_0$, $\partial z(0)/\partial r=\theta$. Then, on this geodesic,

$$\nabla^i\Gamma\Big|_{x=z(r)}=2\,r\frac{\partial z^i}{\partial r}. \tag{2.64}$$

Then

$$2\,\nabla_i\Gamma\,\nabla^i\left(\frac{V_0}{U}\right)=4\,r\frac{\partial z^i}{\partial r}\nabla^i\left(\frac{V_0}{U}\right)\Big|_{x=z(r)}=\frac{\partial}{\partial r}\left(\frac{V_0}{U}\right), \tag{2.65}$$

and (2.63) becomes

$$\frac{\partial}{\partial r}\left(\frac{r\,V_0}{U}\Big|_{x=z(r)}\right)=-\frac{1}{4}\,\frac{PU}{U}\Big|_{x=z(r)}. \tag{2.66}$$

The bounded solution for this equation is

$$V_0|_{x=z(r)}=-\frac{1}{4r}\,U\Big|_{x=z(r)}\int_0^r\frac{PU}{U}\Big|_{x=z(s)}ds, \tag{2.67}$$

or, since $z(1)=x$,

$$V_0(x,x_0)=\frac{1}{4}U(x,x_0)\int_0^1\frac{PU}{U}\Big|_{x=z(s)}ds. \tag{2.68}$$

Thus, if $W=V_0$ on $C^+(x_0)$, (2.61) and (2.62) imply

$$P(W\,H_+(\Gamma))=P(W)H^+(\Gamma)-P(U)\,\delta_+(\Gamma). \tag{2.69}$$

[3] See [32], p.133

Adding this to (2.55) we get

$$P(U\,\delta_+(\Gamma) + W\,H_+(\Gamma)) = P(W)\,H_+(\Gamma) + 2\pi\,\delta_{x_0}. \tag{2.70}$$

Similarly,

$$P(U\,\delta_-(\Gamma) + W\,H_-(\Gamma)) = P(W)\,H_-(\Gamma) + 2\pi\,\delta_{x_0}. \tag{2.71}$$

This result shows that we can obtain the fundamental solution of P if we can solve the characteristic initial value problem

$$PV^\pm = 0,\ x \in D^\pm(x_0); \quad V^\pm = V_0,\ x \in C^\pm(x_0). \tag{2.72}$$

In fact, for $W = V^\pm$, (2.42) and (2.43) imply that

$$G^\pm_{x_0} = \frac{1}{2\pi}\left(U\delta_\pm(\Gamma) + V^\pm\,H_\pm(\Gamma)\right) \tag{2.73}$$

is a fundamental solution of P, whose support is contained in $J^\pm(x_0)$. It can be shown that this problem is well posed [32].

We shall prove now that the characteristic initial value problem (2.72) can be formally solved by the series

$$\sum_{\nu=0}^{\infty} V_\nu \frac{\Gamma^\nu}{\nu!}, \tag{2.74}$$

where V_ν, $\nu \geq 1$, satisfy an appropriate set of transport equations. Applying P to (2.74) yields

$$\begin{aligned} P\left(V_\nu \frac{\Gamma^\nu}{\nu!}\right) &= P(V_\nu)\frac{\Gamma^\nu}{\nu!} + \left[2\,\nabla_i\Gamma\,\nabla^i V_\nu + (\Box\Gamma + A^i\,\nabla_i\Gamma)V_\nu\right]\frac{\Gamma^{\nu-1}}{(\nu-1)!} \\ &+4\,\Gamma\,V_\nu(\nu-1)\frac{\Gamma^{\nu-2}}{(\nu-1)!} \\ &= P(V_\nu)\frac{\Gamma^\nu}{\nu!} + \left[2\,\nabla_i\Gamma\,\nabla^i V_\nu + (\Box\Gamma + A^i\,\nabla_i\Gamma + 4\nu - 4)V_\nu\right]\frac{\Gamma^{\nu-1}}{(\nu-1)!}\,. \end{aligned} \tag{2.75}$$

If we suppose that V_ν can be chosen so that

$$2\,\nabla_i\Gamma\,\nabla^i V_\nu + (\Box\Gamma + A^i\,\nabla_i\Gamma + 4\nu - 4)V_\nu = -PV_{\nu-1}\ \ (\nu = 1, 2, \ldots)\,, \tag{2.76}$$

then

$$P\left(V_\nu \frac{\Gamma^\nu}{\nu!}\right) = P(V_\nu)\frac{\Gamma^\nu}{\nu!} - P(V_{\nu-1})\frac{\Gamma^{\nu-1}}{(\nu-1)!}\,. \tag{2.77}$$

These equations have the same form as (2.61) and (2.62). If we eliminate the term $\Box\Gamma + A^i\nabla_i\Gamma$ in (2.76), using (2.40), we get

$$2\,\nabla_i\Gamma\,\nabla^i V_\nu + \left(-2\,\nabla_i\Gamma\,\frac{\nabla^i U}{U} + 4\nu + 4\right)V_\nu = -P\,V^{\nu-1}. \tag{2.78}$$

In local coordinates, with

$$\nabla^i\Gamma\Big|_{x=z(r)} = 2\,r\,\frac{\partial z^i}{\partial r}, \tag{2.79}$$

(2.78) assumes the form

$$4r\frac{\partial V_\nu}{\partial r}\Big|_{x=z(r)} + \left(-\frac{4r}{U}\frac{\partial u}{\partial r}\Big|_{x=z(r)} + 4\nu + 4\right) V_\nu = -P\,V_{\nu-1}, \tag{2.80}$$

or

$$r\frac{\partial}{\partial r}\left(\frac{V_\nu}{U}\Big|_{x=z(r)}\right) + (\nu+1)\frac{V_\nu}{U}\Big|_{x=z(r)} = -\frac{1}{4}\frac{P\,V_{\nu-1}}{U}\Big|_{x=z(r)}. \tag{2.81}$$

If we multiply this by r^ν, we obtain

$$\frac{\partial}{\partial r}\left(r^{\nu+1}\frac{V_\nu}{U}\Big|_{x=z(r)}\right) = \frac{1}{4}r^\nu\frac{P\,V_{\nu-1}}{U}\Big|_{x=z(r)}. \tag{2.82}$$

Assuming that $V_{\nu-1}$ has already been determined, we can solve this for V_ν, and determine the only solution that remains bounded when $r \to 0$:

$$\frac{V_\nu}{U}\Big|_{x=z(r)} = -\frac{1}{4r^{\nu+1}}\int_0^r \frac{P\,V_{\nu-1}}{U}\Big|_{x=z(s)} s^\nu\,ds\,. \tag{2.83}$$

Putting $r = 1$ we find

$$V_\nu(y,x) = -\frac{1}{4}U(y,x)\int_0^1 \frac{P\,V_{\nu-1}}{U}\Big|_{x=z(s)} s^\nu\,ds\,. \tag{2.84}$$

In the analytical case, i.e., when the space-time has an analytical structure and analytic metric, and the coefficients of P are analytic, the series (2.74) converges to a solution of the characteristic initial problem (2.72), for sufficient small $|\Gamma(x,x_0)|$ [4] In the general C^∞ case the series does not converge. However, by multiplying each term of the series by a suitable factor, we can convert it into a convergent series. The sum is a C^∞ function $\tilde{V}$ such that $\tilde{V} = V_0$ on $C(x_0)$, and $P\tilde{V}$ vanishes to all orders on $C(x_0)$. This is formally expressed in the following lemma [32]:

Lemma 2.2 *Let $\sigma(t) \in C_0^\infty(\mathbf{R})$ be such that $0 \le \sigma(t) \le 1$, that $\sigma(t) = 1$ for $|t| \le 1/2$, and that $\sigma(t) = 0$ for $|t| \ge 1$. Then there exists a sequence of positive numbers k_1, k_2, ..., strictly increasing and tending to infinity, such that the series*

$$\sum_{\nu=1}^{\infty} V_\nu \frac{\Gamma^\nu}{\nu!}\sigma(k_\nu\Gamma) \tag{2.85}$$

[4] see [32] p.141.

converges to a function $S(x, x_0) \in C^\infty(\Omega \times \Omega)$, *uniformly in every compact subset of* $\Omega \times \Omega$. *Also,* $S \to 0$ *when* $\Gamma \to 0$, *and*

$$\lim_{\Gamma \to 0} \Gamma^{-\mu} \left(S - \sum_{\nu=1}^{\mu} \frac{\Gamma^\nu}{\nu!} \right) = 0 . \quad (\mu = 1, 2 \ldots) . \tag{2.86}$$

The last relation implies that the series (2.74) is an asymptotic expansion of S, when $\Gamma \to 0$. Applying (2.70) to a test function $\phi \in C_0^\infty(\Omega)$, we get, using (2.33),

$$\phi(x_0) + \int_{J^+(x_0)} K(x_0, x)\, \phi(x)\, \mu(x) = \frac{1}{2\pi} \int_{C^+(x_0)} U(x_0, x)\, {}^tP\phi(x)\, \mu_\Gamma(x)$$
$$+\frac{1}{2\pi} \int_{J^+(x_0)} W(x_0, x)\, {}^tP\phi(x)\, \mu(x), \tag{2.87}$$

where $K = 1/(2\pi) P W(x_0, x)$, $W = V_0$ when $x \in C^+(x_0)$, and μ_Γ is the Leray form such that $d_x\Gamma(x, x_0) \wedge \mu_\Gamma(x) = \mu(x)$. This can be viewed as an integral equation for ϕ. If this can be solved, then we can define the linear form ${}^tP\phi \to \phi(x_0)$, which will be a fundamental solution of P, if it is a distribution.

The next step is to build the fundamental solutions in a causal domain. First we introduce the concept of a *parametrix* [55]: A C^∞ parametrix of P is a distribution $\tilde{G_{x_0}}$ such that

$$P\tilde{G_{x_0}} - \delta_{x_0} \in C^\infty(\Omega) . \tag{2.88}$$

It can be proved that it has the form [5]

$$\tilde{G}^{\pm}_{x_0} = \frac{1}{2\pi} \left(U\delta_+(\Gamma) + \tilde{V} H_+(\Gamma) \right) , \tag{2.89}$$

where

$$\tilde{V} = V_0 + S = V_0 + \sum_{\nu=1}^{\infty} V_\nu \frac{\Gamma^\nu}{\nu!} \sigma(k_\nu \Gamma) . \tag{2.90}$$

A *causal domain* is a connected open set Ω_0 if (i) there is a geodesically convex domain Ω such that $\Omega \subset \Omega_0$, and (ii) for all pairs of points x, x_0 in Ω_0, $J^+(x_0) \cap J^-(x)$ is a compact subset of Ω, or void. Condition (ii) is a simplified version of the condition of global hyperbolicity, suitable for domains that satisfy (i). In a causal domain an important theorem can be established:

Theorem 2.2 Theorem 2.4 *Let* Ω_0 *be a causal domain, and let* $K(x, x_0)$ *be a function defined on* $\Omega_0 \times \Omega_0$ *whose support is contained in the closed set*

$$\Delta^+ = \left\{ (x, x_0);\ (x, x_0) \in \Omega_0 \times \Omega_0,\ x \in J^+(x_0) \right\} ,$$

[5] see [32], p.144

and which is continuous on Δ^+. *Then there exists a function* $L(x, x_0)$, *also supported in* Δ^+ *and continuous on* Δ^+, *such that, for every* $\phi \in C_0^\infty(\Omega_0)$ *the equation*

$$\phi(x_0) + \int K(x, x_0)\, \phi(x)\, \mu(x) = \psi(x_0) \tag{2.91}$$

implies that

$$\psi(x_0) + \int L(x, x_0)\, \psi(x)\, \mu(x) = \psi(x_0). \tag{2.92}$$

Also, if $K \in C^\infty(\Omega_0 \times \Omega_0)$, *then* $L \in C^\infty(\Omega_0 \times \Omega_0)$.

The proof can be found in [32] (p.151).

The fundamental solution to P in a causal domain can then be given by the following theorem:

Theorem 2.3 *Let* Ω_0 *be a causal domain, and let* $P = \Box + A^i \nabla_i + B$, *where* A *and* B *are in* $C_0^\infty(\Omega_0)$, *be a scalar differential operator defined on* Ω_0. *Then* P *has a fundamental solution* $G^\pm_{x_0}$ *in* Ω_0, *such that* $PG^\pm_{x_0} = \delta_{x_0}$ *and supp* $G^\pm_{x_0} \subset J^\pm(x_0)$, *given by*

$$G^+_{x_0} = \frac{1}{2\pi}\left(U\, \delta^\pm(\Gamma) + V^\pm\right), \tag{2.93}$$

where supp $V^\pm \subset \Delta^\pm$ *and* $V^\pm \in C^\infty(\Delta^\pm)$. *Moreover, when* $\Gamma \to 0$ *for* $(x, x_0) \in \Delta^\pm$, *we have the following asymptotic expansion for* $V^\pm$:

$$V^\pm \sim \sum_{\nu=0}^{\infty} V_\nu \frac{\Gamma^\nu}{\nu!}. \tag{2.94}$$

The proof of the first part consists of taking $W = \tilde{V}$ in (2.87). Then, using the parametrix properties (2.88), (2.89) and (2.90), we use Theorem 1.6 to solve the corresponding integral equation. It has the form

$$\phi(x_0) = \frac{1}{2\pi}\int_{C^+(x_0)} U(x, x_0)\, {}^tP\phi(x)\, \mu\, \Gamma(x) + \frac{1}{2\pi}\int V^+(x, x_0)\, {}^tP\phi(x)\, \mu(x). \tag{2.95}$$

Then, using (2.34), equation (2.93) follows.
The two fundamental solutions $G^\pm_{x_0}$ can now be used to build the theory of wave equations in the curved space-time. [6]

The distribution $G^+_{x_0}$ can be considered as the field of a point source at x_0, acting on a previously undisturbed background. This fundamental solution consists of two parts:

[6] The proof of existence and uniqueness for these solutions can be found in [32], p.167.

(i) a singular part $U\delta_+(\Gamma)$ that is a measure supported on the future semi-cone with vertex $C^+(x_0)$;
(ii) a regular part V^+, which is a function $x \to V^+(x, x_0)$, with support contained in $J^+(x_0)$, and $V^+ \in C^\infty(J^+(x_0))$.

The regular part, also called the tail term of the fundamental solution, does not appear in the Minkowskian case, where $G^+_{x_0}$ is "sharp", with support only in $C^+(x_0)$.

2.2.5 Representation formula

In order to state more precisely the necessary and sufficient conditions for the validity of Huygens' principle, we need the representation formula for the solutions of the scalar wave equation [55] [57]. At this point we change some of our conventions to those of McLenaghan [57] [60]. He defines the fundamental solution of the scalar operator P as the scalar distribution satisfying

$$ {}^tP(G_{x_0}) = \delta_{x_0} \tag{2.96}$$

instead of $P(G_{x_0}) = \delta_{x_0}$. Thus, instead of (2.68) we must have

$$V_0(x, x_0) = \frac{1}{4}U(x, x_0) \int_0^1 \frac{{}^tPU}{U}\bigg|_{x=x(s)} ds. \tag{2.97}$$

We must also change A_i by $-A_i$, and the form of the expressions in the previous sections remains valid.

The reason for this is that later we shall use Green's theorem, applied to P and ${}^tP$, in order to obtain a representation formula. This is the method used by McLenaghan [57]. The explicit assumption that Ω is a causal domain is not necessary in the process, although we still keep it. Let S be a non-compact space-like 3-manifold contained in a causal domain Ω_0, and let x_0 be a point in $J^+ \setminus S$. Consider the operators P and ${}^tP$, defined in (2.3) and (2.35). For the functions u and v in $C^2(\Omega_0)$, we can use Green's theorem [7] to form the expression:

$$vP(u) - u\,{}^tP(v) = \delta[udv - vdu - \alpha uv], \tag{2.98}$$

where $\alpha = A_i\, dx^i$, and δT is the coderivative of the tensor T, defined in local coordinates by

$$(\delta T)^{j_1 \dots j_s}{}_{i_1 \dots i_r} = -\nabla_i T^{i j_1 \dots j_s}{}_{i_1 \dots i_r}. \tag{2.99}$$

[7] See [32], p. 64.

Lichnerowicz [55] defined the *biscalar fundamental solutions* (also called elementary kernels) $G^{\pm}(x_0, x) \in \Omega_0 \times \Omega_0$ in terms of the fundamental solutions $G^{\pm}_{x_0}(x)$ as follows: For $\phi \in C^{\infty}(\Omega \times \Omega)$,

$$\langle G^{\pm}(x_0, x), \phi(x_0, x)\rangle = \langle G_{x_0}{}^{\pm}(x), \phi(x_0, x)\rangle . \tag{2.100}$$

Thus, (2.96) can be written as

$${}^{t}P_x(G^{\pm}(x_0, x)) = \delta(x_0, x) , \tag{2.101}$$

where $\delta(x_0, x)$ is the biscalar distribution defined by

$$\langle \delta(x_0, x), \phi(x_0, x)\rangle = \langle \delta_{x_0}(x), \phi(x_0, x)\rangle = \int_{\Omega_0} \phi(x_0, x_0)\mu(x_0) . \tag{2.102}$$

Let us consider the inhomogeneous equation $Pu = \Box u + A^k \nabla_k u + Bu = f$. Substituting $v = G^-(x_0, x)$ in (2.98), and using (2.101), gives

$$G^-(x_0, x)f(x) - u(x)\delta(x_0, x) = \delta_x A^-(u) , \tag{2.103}$$

where

$$A^-(u) = -G'^-(x_0, x)dx u(x) + u'(x)G^-(x_0, x)dx - \alpha(x)u(x)G^-(x_0, x). \tag{2.104}$$

Let D^+ denotes the future of the initial surface S, and $H^+(S, x)$ be the characteristic function of D^+. Since the support of A^- is in $D^-(x_0)$, the function

$$\phi(x_0) = \int_{\Omega_0} *\delta_x A^-(u)H^+(S, x) \tag{2.105}$$

is a well-defined distribution, because the integrand has compact support. Thus, for a scalar $\theta(x_0)$ defined in Ω_0,

$$\begin{aligned}\langle \phi(x_0), \theta(x_0)\rangle &= \int_{D_S^+} *\delta_x \langle A^-(u), \theta(x_0)\rangle . \\ &= -\int_{\partial D_S^+} *\langle A^-(u), \theta(x_0)\rangle = -\langle \int_S *A^-(u), \theta(x_0)\rangle ,\end{aligned} \tag{2.106}$$

where Stokes' theorem was used, and S, the boundary of D_S^+ in Ω_0, is oriented in the future direction. On the other hand, from (2.103),

$$\begin{aligned}\langle \phi(x_0), \theta(x_0)\rangle &= \langle \int_{\Omega_0} \left(G^-(x_0, x)P_x(u) - u(x)\delta(x_0, x)\right) H^+(S, x)\mu(x) \, , \theta(x_0)\rangle \\ &= \langle \int_{\Omega_0} G^-(x_0, x)f^+(x)\mu(x), \theta(x_0)\rangle \; - \langle u^+(x_0), \theta(x_0)\rangle ,\end{aligned} \tag{2.107}$$

where

$$f^+(x) = f(x)H^+(S,x), \quad u^+(x_0) = f(x_0)H^+(S,x_0)\,. \tag{2.108}$$

Thus, from (2.107) and (2.106), we obtain the retarded weak solution of Cauchy's problem:

$$u^+ = \int_{\Omega_0} G^-(x_0,x)f^+(x)\mu(x) + \int_S *A^-(u)\,. \tag{2.109}$$

The advanced solution can be obtained from the above by interchanging + and −.

If we now substitute the expression (2.93) in (2.109), considering the homogeneous case $f = 0$, we get the formal solution:

$$\begin{aligned} u(x_0) = \int_S * &\left[(u\nabla_i U - U\nabla_i u + uV^-\nabla_i\Gamma - uUA_i)\,\delta_-(\Gamma(x_0,x))\right. \\ &\left. +uU\nabla_i\Gamma\delta'_-(\Gamma(x_0,x)) + u\nabla_i V^- - V^-\nabla_i u - uV^-A_i\right] dx^i\,. \end{aligned} \tag{2.110}$$

For signal propagation to be sharp, $u(x_0)$ can depend only on the Cauchy data in an arbitrarily small neighborhood of $S \cap C^-(x_0)$. It is clear that a sufficient condition for the validity of Huygens' principle is that $V^-(x_0,x) = 0$ for any x_0 and all $x \in D^-(x_0)$. We can show that this is also a necessary condition. Therefore, in order that the solution (2.110) be independent of the Cauchy data on $S \cap D^-(x_0)$, we must have

$$\int_S *(u\nabla_i V^- - V^-\nabla_i u - uV^-\nabla_i A)dx^i = 0\,. \tag{2.111}$$

We now choose coordinates $(x^0,x^1,x^2,x^3) = (t,X)$, such that, at each point on $S \cap D^-(x_0)$, x^0 is normal to S. Then, since $dx^0 = dt = 0$ on S, the above equation becomes:

$$\int_{S\cap D^-(x_0)} (u\partial_t V^- - V^-\partial_t u - uV^-\partial_t A)\sqrt{-g}dx^1 \wedge dx^2 \wedge dx^3 = 0. \tag{2.112}$$

But this equation must be valid for any initial condition on S. Since u and u_t can be chosen independently, we take $u = 0$ on S. Then (2.112) becomes

$$\int_{S\cap D^-(x_0)} V^-\partial_t u\sqrt{-g}dx^1 \wedge dx^2 \wedge dx^3 = 0\,. \tag{2.113}$$

Thus we must have $V^- = 0$ on $S \cap D^-(x_0)$. Since this must be true for all S and x the necessary and sufficient condition for the validity of Huygens' principle for the retarded Cauchy problem is

$$V^-(x_0,x) = 0\,, \qquad \forall x_0, \quad \forall x \in D^-(x_0)\,. \tag{2.114}$$

Similarly, for the advanced solution,

$$V^+(x_0,x) = 0\,, \qquad \forall x_0, \quad \forall x \in D^+(x_0)\,. \tag{2.115}$$

This condition is not very useful, however. A more convenient necessary and sufficient condition is

$$
{}^{t}PU = 0, \quad x \in C(x_0) = C^{+}(x_0) \cup C^{-}(x_0). \tag{2.116}
$$

To prove this, consider first the case when $V^{\pm} = 0$. Then, since $U \neq 0$ in general, we have from (2.68) and (2.72),

$$
\int_0^1 \frac{{}^{t}PU}{U} ds = 0 \,, \quad x \in C(x_0) \tag{2.117}
$$

which implies ${}^{t}PU = 0$ on $C(x_0)$. Conversely, suppose ${}^{t}PU = 0$ on $C(x_0)$ holds. Then, by (2.68) and (2.72), $V^{\pm}(x_0, x) = 0$ for $x \in C^{\pm}(x_0)$. Since this must hold for any x_0, and since $V^{\pm}$ is continuous, (2.114-2.117) follows.

2.3 Necessary conditions for the validity. of Huygens' principle

2.3.1 The trivial transformations

As we shall show in this section, the Huygens' character of the second order differential operators (2.3) is preserved under the following local transformations, called *trivial (or elementary) transformations*:

(a) a transformation of coordinates,
(b) the multiplication of $Pu = 0$ by a factor $e^{2\varphi(x)}$ (equivalent to a conformal transformation of the metric),
(c) replacing the dependent variable u by $\lambda(x)u$, where $\lambda(x)$ is a nowhere vanishing function.

We need to define the following tensors on $\mathcal{M}^4$, which will be useful in the subsequent analysis:

$$
H_{ij} := A_{[i,j]} \,, \tag{2.118}
$$

$$
C_{ijkl} := R_{ijkl} - 2g_{[i[l}L_{j]k]} \,, \tag{2.119}
$$

$$
S_{ijk} := L_{i[j;k]} \,, \tag{2.120}
$$

$$
L_{ij} := -R_{ij} + \frac{R}{6} g_{ij} \,. \tag{2.121}
$$

Here $A_i := g_{ik}A^k$, R_{abcd} denotes the Riemann tensor, C_{ijkl} the Weyl tensor, $R_{ij} := g^{cd}R_{cijd}$, the Ricci tensor, and $R := g^{ij}R_{ij}$ the Ricci scalar associated to the metric

g_{ij}. We shall now, following McLenaghan [60], use the trivial transformations to find convenient parameters φ and λ that simplify the process of obtaining the explicit forms of the necessary conditions for the validity of Huygens' principle. These will be obtained, for general space-times, from a covariant Taylor expansion of the diffusion kernel. We shall first show that the Huygens' character of the operator P is preserved under Hadamard's trivial transformations.

We shall also use an additional transformation defined by Hadamard [43], that we shall denominate as (bc):

> (bc) *Replacement of the function u in $Pu = 0$ by $\lambda\, u$, ($\lambda(x) \neq 0$), and simultaneous multiplication of the equation by* λ^{-1}.

In what follows, we shall use the notation $\tilde{a}$ to refer to a quantity obtained by applying (b) to a. The symbol $\overline{a}$ denotes the effect of both (b) and (bc). The transformation (b) and (bc) transform the differential operator $P[u]$ into a similar operator $\overline{P}[u]$, as we shall see next. The specialization for $n = 4$ will be made later.

Transformation (b) induces a conformal transformation on the metric:

$$\tilde{g}^{ij} = e^{-2\varphi} g^{ij}\,, \qquad \tilde{g}_{ij} = e^{2\varphi} g_{ij}\,, \tag{2.122}$$

from which one obtains the following transformation laws

$$\widetilde{\left\{ {i \atop j\,k} \right\}} = \left\{ {i \atop j\,k} \right\} + 2\delta^i_{(j}\varphi_{k)} - g_{jk}\varphi^i \tag{2.123}$$

where $\varphi_i = \partial_i \varphi$ and $\varphi^i = g^{ij}\varphi_j$. Thus, applying (b) and (bc) to Pu we get

$$\overline{P}u = \lambda^{-1} e^{-2\varphi} P[\lambda u] = \tilde{g}^{ij}\tilde{\nabla}_{ij}u + \overline{A}^i\nabla_i + \overline{B}u, \tag{2.124}$$

so we must have

$$\overline{A}_i = A_i + 2\nabla_i(\log\lambda) - (n-2)\nabla_i\varphi\,, \tag{2.125}$$

$$\overline{B} = e^{-2\varphi}(B + \lambda^{-1}\Box\lambda + A^i\nabla_i(\log\lambda). \tag{2.126}$$

Let us define

$$F := {}^tP\,, \tag{2.127}$$

where ${}^tP$ denotes the adjoint operator of P, defined in (2.35). Then, according to (2.124), we have

$$\overline{F}v = \lambda e^{-n\varphi} F[\lambda^{-1} e^{(n-2)\varphi} v]. \tag{2.128}$$

We define the fundamental solutions $\overline{G}^{\pm}_{x_0}$ of $\overline{F}v$ by

$$\overline{F}[\overline{G}^{\pm}_{x_0}(x)] = \overline{\delta}_{x_0}(x)\,, \tag{2.129}$$

where $\overline{\delta}_{x_0}(x) := e^{-n\varphi}\delta_{x_0}(x)$. From these two equations we then get

$$F[\overset{\circ}{\lambda}\,\lambda^{-1}e^{(n-2)\varphi}\overline{G}^{\pm}_{x_0}(x)] = \delta_{x_0}(x)\,, \tag{2.130}$$

where $\overset{\circ}{\lambda} := \lambda(x_0)$. Thus by uniqueness of the elementary solutions we have

$$\overline{G}^{\pm}_{x_0}(x) = \lambda\,\overset{\circ}{\lambda}^{-1}\;e^{-(n-2)\varphi}G^{\pm}_{x_0}(x)]. \tag{2.131}$$

Thus supp $\overline{G}^{\pm}(x,x_0) \subseteq J^{\pm}(x_0)$ provided supp $G^{\pm}(x,x_0) \subseteq J^{\pm}(x_0)$. We can now obtain the transformation properties of $[U]$, $V^{\pm}$, and $[F(U)]$, where the brackets [ ] denote the restriction of the enclosed function to $C(x_0) = C^{+}(x_0) \cup C^{-}(x_0)$. $[F(U)]$ is called the diffusion kernel. Thus, from (2.93), we have, for $n = 4$:

$$\overline{U}\delta^{\pm}(\tilde{\Gamma}(x,x_0)) + \overline{V}^{\pm}(x,x_0) = \overset{\circ}{\lambda}^{-1}\;\lambda e^{-2\varphi}[U\delta^{\pm}(\Gamma(x,x_0)) + V^{\pm}(x,x_0)]\,. \tag{2.132}$$

We must now find a relation between $\delta(\Gamma)$ and $\delta(\tilde{\Gamma})$. Since $\Gamma = 0$ if and only if $\tilde{\Gamma} = 0$, we write $\tilde{\Gamma}$ in the form:

$$\tilde{\Gamma} = a_1\Gamma + a_2\Gamma^2 + \ldots\;, \tag{2.133}$$

where the a_i are functions of x_0 and x to be determined. Since

$$\nabla_i\Gamma\,\nabla^i\Gamma = 4\Gamma, \quad \tilde{\nabla}_i\tilde{\Gamma}\,\tilde{\nabla}^i\tilde{\Gamma} = 4\tilde{\Gamma}\,, \tag{2.134}$$

we obtain, on substituting the series in the second of the equations above, and equating coefficients of equal powers of Γ:

$$s\frac{da_1}{ds} + a_1 = e^{2\varphi}\,. \tag{2.135}$$

The solution to this equation, regular at $s = 0$, is

$$a_1 = \frac{1}{s}\int_0^{s(x)} e^{2\varphi}dt\,, \tag{2.136}$$

where the integration is carried along the null geodesic x_0x with respect to an affine parameter. Since

$$\delta(\Gamma) = \left.\frac{d\tilde{\Gamma}}{d\Gamma}\right|_{\Gamma=0}\delta(\tilde{\Gamma})\,, \tag{2.137}$$

we obtain from (2.132),

$$\delta(\bar{\Gamma}) = a_1^{-1}\delta(\Gamma)\,. \tag{2.138}$$

Thus, from (2.132) we get

$$[\overline{U}] = \overset{\circ}{\lambda}^{-1} a_1[\lambda e^{-2\varphi}U]\,, \tag{2.139}$$

and

$$\overline{V}^{\pm} = \overset{\circ}{\lambda}^{-1} \lambda e^{-2\varphi}V^{\pm}\,. \tag{2.140}$$

If we differentiate (2.97) we get

$$\frac{d}{ds}\left[\frac{sV^{\pm}}{U}\right] = -\left[\frac{F(U)}{4U}\right]. \tag{2.141}$$

A similar expression must hold for the transformed operator

$$\frac{d}{d\bar{s}}\left[\frac{\bar{s}\overline{V}^{\pm}}{\overline{U}}\right] = -\left[\frac{\overline{F}(\overline{U})}{4\overline{U}}\right] \tag{2.142}$$

where $\bar{s}$ is an affine parameter along the generators of $\bar{C}(x_0) = C(x_0)$ (null geodesics are preserved under conformal transformations), related to s along a fixed null geodesic by

$$\bar{s} = \int_0^s e^{2\varphi}dt\,. \tag{2.143}$$

Substituting (2.139), (2.140) and (2.143) into (2.142) gives

$$e^{-2\varphi}\frac{d}{ds}\left[\frac{sV^{\pm}}{U}\right] = -\left[\frac{\overline{F}(\overline{U})}{4\overset{\circ}{\lambda}^{-1} \lambda e^{-2\varphi}Ua_1}\right]. \tag{2.144}$$

Comparing this equation to (2.141) we find

$$\overline{F}(\overline{U}) = \lambda^{-1}a_1\left[\lambda e^{-4\varphi}F(U)\right], \tag{2.145}$$

which is the transformation law for the diffusion kernel. Thus, the necessary and sufficient condition (2.116) is invariant under the trivial transformations.

We can now specify a choice for transformations (b) and (bc) that will simplify future calculations. We begin with transformation (bc). From (2.43) and (2.125) we find

$$\overline{U} = exp\left\{-\frac{1}{2}\int_0^{s(x)} \nabla^i(\log\lambda)\,\nabla_i\Gamma\frac{dt}{t}\right\}U. \tag{2.146}$$

Since

$$\nabla^i\Gamma = 2t\frac{dx^i}{dt}, \tag{2.147}$$

(2.146) simplifies to

$$\overline{U} = \overset{\circ}{\lambda}^{-1} \lambda U. \tag{2.148}$$

We can now express U in a form different from (2.43), using the identity [74]

$$\Box \Gamma = 8 + \nabla^i \nabla_i (\log \rho) \,, \tag{2.149}$$

where ρ is given by

$$\rho = 16\sqrt{g(x)g(x_0)} \left(det \left| \frac{\partial^2 \Gamma}{\partial x^i \partial x^j} \right| \right) . \tag{2.150}$$

Substituting (2.149) into (2.43) gives

$$U = \frac{1}{\sqrt{\rho}} \exp \left\{ \frac{1}{4} \int_0^s A^i \nabla_i \frac{dt}{t} \right\} . \tag{2.151}$$

If we choose, for a given point x_0

$$\lambda(x) = \exp \left\{ -\frac{1}{4} \int_0^s A^i \nabla_i \frac{dt}{t} , \right\} \tag{2.152}$$

then $\overset{\circ}{\lambda} = 1$. Consequently, from (2.148) and (2.151) we find

$$\overline{U} = \frac{1}{\sqrt{\rho}} . \tag{2.153}$$

Using this in (2.46) we find

$$\int_0^{s(x)} \overline{A}^i \nabla_i \Gamma \frac{dt}{t} = 0 \tag{2.154}$$

for all $x \in \Omega$. Then

$$\overline{A}^i \nabla_i \Gamma = 0. \tag{2.155}$$

To choose a transformation (b) we first note that, under (b), (2.121) transforms as

$$\bar{L}_{ij} = L_{ij} - 2\nabla_j \varphi_i + 2\varphi_i \varphi_j - g_{ij} \varphi_k \varphi^k \, . \tag{2.156}$$

Following Günther [38], at any point x_0 we can choose the derivatives of ϕ at x_0 such that

$$\overset{\circ}{\bar{L}}_{ij} = 0 \,, \tag{2.157}$$

$$\overset{\circ}{\bar{L}}_{(ij;k)} = 0 \,, \tag{2.158}$$

$$\overset{\circ}{\bar{L}}_{(ij;kl)} = 0\,, \tag{2.159}$$

$$\vdots$$

where $\overset{\circ}{\bar{L}}_{ij}= \bar{L}_{ij}(x_0)$, and so on. From now on we assume that the chosen transformations (b) and (bc) have been made and drop tildes and bars.

Contracting (2.157), and using the definition of L_{ij} given in (2.121), we find at x_0:

$$\overset{\circ}{R}= 0\,, \quad \overset{\circ}{R}_{ij}= 0\,, \quad \overset{\circ}{C}_{ijkl}=\overset{\circ}{R}_{ijkl}\,. \tag{2.160}$$

On the other hand, from the Ricci identity we find

$$\overset{\circ}{R}_{ij;kl}=\overset{\circ}{R}_{ij;lk}\,, \quad \overset{\circ}{L}_{ij;kl}=\overset{\circ}{L}_{ij;lk}\,. \tag{2.161}$$

From (2.158), we get the following

$$\overset{\circ}{R}_{,i}= 0\,, \quad \overset{\circ}{L}_{ij;k}= -\overset{\circ}{R}_{ij;k}\,, \quad \Box\, \overset{\circ}{R}= 0\,, \tag{2.162}$$

$$\Box\, \overset{\circ}{R}_{ij}= -\frac{5}{3}\, \overset{\circ}{R}_{;ij}= \Box\, \overset{\circ}{L}_{ij} \quad \Box\, \overset{\circ}{L}_{ij;k}= \frac{4}{3}\,, \tag{2.163}$$

where $S_{ijk} := L_{i[j;k]}$. From (2.159) we get

$$\overset{\circ}{L}_{ij;kl}= \frac{5}{3}\, \overset{\circ}{S}_{(ij)(k;l)} -\frac{1}{3}\, \overset{\circ}{S}_{(kl)(i;j)}\,. \tag{2.164}$$

It still remains to choose transformation (a). A convenient choice is a system of normal coordinates x^i around the point x_0. We can use (2.51) and (2.155) to find the form of U in these coordinates [8]:

$$U \doteq \left(\frac{\overset{\circ}{g}}{g}\right)^{1/4}. \tag{2.165}$$

From this equation we can obtain

$$U_{,k} \doteq -\frac{1}{4} U g^{ij} g_{ij,k}\,, \quad \Box U = -\frac{1}{4} U\gamma\,, \tag{2.166}$$

where

$$\gamma :\doteq (g^{ij}g^{kl}g_{kl,i})_{,j} + \frac{1}{4} g^{ij}g_{ij,k}g^{kl}g^{mn}g_{mn,k}\,. \tag{2.167}$$

If we define

$$\sigma := -\frac{4F(U)}{U}\,, \tag{2.168}$$

[8] $\doteq$ means equality only in a normal coordinate system

then, in the normal coordinates, using (2.35), we find

$$\sigma \doteq \gamma + A^k g^{ij} g_{ij,k} + 4A^i{}_{,i} - 4B\,. \tag{2.169}$$

From (2.165) we see that $U(x_0, x) \neq 0$ for x in the convex neighborhood Ω. Then $[\sigma] = 0 \Leftrightarrow [F(U)] = 0$. Thus, Huygens' principle is valid if and only if

$$[\sigma(x_0, x)] = 0\,, \tag{2.170}$$

for any point x_0 in the manifold.

2.3.2 Expansion of the diffusion kernel and necessary conditions

The next step consists in obtaining a covariant Taylor expansion of σ about an arbitrarily chosen point x_0. The calculations are lengthy and we shall present only the method and some results. Details can be found in [57] [58] [60]. The terms to expand are those that define σ: g_{ij}, g^{ij}, A^i, and B. This expansion is made in a normal coordinate system x^i, with origin in x_0. The covariant expansion of g_{ij} and g^{ij}, to sixth and fourth order respectively was obtained by McLenaghan [58], using the methods of Herglotz [44] and Günther [38]. When the obtained results are substituted in (2.167), and the conformal transformation (2.157)-(2.159) is assumed, we obtain γ, expanded terms of the tensor of Riemann (and its derivatives), the Weyl tensor, and the metric tensor, all evaluated at x_0, and in terms of the coordinates x^i. For example, to second order we have

$$\gamma[0] \doteq 0\,,\quad \gamma[1] \doteq 0\,,\quad \gamma[2] \doteq \frac{6}{5}(\mathring{R}_{ij;kl} - \frac{2}{9}\,\mathring{C}_{mijn}\mathring{C}^{\;m}{}_{kl}{}^{n})\;\mathring{g}^{\,(ij}x^{kl)}\,. \tag{2.171}$$

For A^i, an expansion is performed about x_0, to fifth order, using the condition $A^i x_i = 0$ that comes from (2.45), (2.51) and (2.155). Then, in order to make the expansion covariant, partial derivatives must be expressed in terms of covariant derivatives. The result, to second order, is

$$A_i \doteq \mathring{H}_{ij}\, x^j + \frac{2}{3}\,\mathring{H}_{ij;k}\, x^k x^j\,, \tag{2.172}$$

$$A^i{}_{,i} \doteq \frac{2}{3}\,\mathring{H}^{\,j}{}_{k;j} x^k + \frac{3}{4}\left(\mathring{H}^{\,k}{}_{(k;jl)} - \mathring{C}^{\,k}{}_{(kj}{}^{m}\,\mathring{H}_{|m|l)}\right) x^j x^l\,, \tag{2.173}$$

$$A_i A^i \doteq \mathring{H}_{ki}\mathring{H}^{\,k}{}_j\, x^i x^j\,. \tag{2.174}$$

The expansion for the scalar B is trivial, since in normal coordinates (x^i) we have the following property for scalars:

$$B_{;i_1\ldots i_n} x^{i_1}\ldots x^{i_n} \doteq B_{,i_1\ldots i_n} x^{i_1}\ldots x^{i_n}\,. \tag{2.175}$$

Then, to second order,

$$B \doteq \overset{\circ}{B} + \overset{\circ}{B}_{;i}\, x^i + \frac{1}{2}\, \overset{\circ}{B}_{;ij}\, x^i x^j\,. \tag{2.176}$$

Now, on $C(x_0)$,

$$\begin{aligned} \sigma &\doteq \overset{\circ}{\sigma} + \overset{\circ}{\sigma}_{,i}\, x^i + \frac{1}{2}\, \overset{\circ}{\sigma}_{,ij}\, x^i x^j + \ \ldots = 0\,, \\ &\doteq \overset{\circ}{\sigma} + \overset{\circ}{\sigma}_{;i}\, x^i + \frac{1}{2}\, \overset{\circ}{\sigma}_{;ij}\, x^i x^j + \ \ldots = 0\,. \end{aligned} \tag{2.177}$$

Thus, the necessary conditions are given by

$$\overset{\circ}{\sigma} = 0\,, \ \ \overset{\circ}{\sigma}_{,i} = 0\,, \ \ TS(\overset{\circ}{\sigma}_{;ij}) = 0\,, \ldots\,, \tag{2.178}$$

where $TS(\ )$ denotes the operation which forms the trace-free symmetric tensor from the enclosed tensor.

Order of magnitude [0]:
According to (2.169) we have

$$\overset{\circ}{\sigma} = -4\, \overset{\circ}{B}\ . \tag{2.179}$$

which implies

$$\overset{\circ}{B} = 0\,. \tag{2.180}$$

This is valid only for our special choice of trivial transformations. So, we have to find a quantity that reduces to B in x_0, when the special trivial transformations are taken, and that satisfies, in general, a relation invariant under trivial transformations. The required quantity is the Cotton invariant denoted by $\mathcal{C}$. In fact, under trivial transformations we have $\overline{\mathcal{C}} = e^{-2\varphi}\mathcal{C}$. Thus, the first necessary condition is

$$(I) \qquad\qquad \mathcal{C} := B - \frac{1}{2}A^i{}_{;i} - \frac{1}{4}A_iA^i - \frac{1}{6}R = 0. \tag{2.181}$$

Order of magnitude [1]:
Since,

$$\left\{ {i \atop j\, i} \right\} \doteq \frac{1}{2} g_{ik,j} g^{ik}\,, \tag{2.182}$$

we have

$$-2A^i{}_{;i} + g^{ik} g_{ik,j} A^j \doteq -2A^i{}_{,i}\,. \tag{2.183}$$

Using (2.181) and (2.183) we can write σ in the form:

$$\sigma \doteq \gamma - \frac{2}{3}R + 2A^i{}_{,i} - A^iA_i\,. \tag{2.184}$$

By differentiating the equation above, evaluating at x_0, and using (2.173), we obtain

$$\overset{\circ}{\sigma}_{,i} = 2\,\overset{\circ}{A}{}^{k}{}_{,ki} = \frac{4}{3}\,\overset{\circ}{H}{}^{k}{}_{i;k} = 0\,, \tag{2.185}$$

in the special gauge. But this is valid also in general, since, under trivial transformations,

$$\bar{\nabla}_k \overline{H}^k{}_i = e^{-2\varphi} \nabla_k H^k{}_i\,. \tag{2.186}$$

Thus, the second condition is given by

$$(II) \qquad H^k{}_{a;k} = 0\,, \qquad H_{ab} := A_{[a;b]}\,. \tag{2.187}$$

The third, fourth and fifth conditions are calculated in the same way and are given respectively by

$$(III) \qquad S_{abk;}{}^{k} - \frac{1}{2} C^k{}_{ab}{}^{l} L_{kl} = -5\left(H_{ak}H_b{}^k - \frac{1}{4} g_{ab} H_{kl} H^{kl}\right)\,, \tag{2.188}$$

$$(IV) \qquad TS(3S_{abk}H^k{}_c + C^k{}_{ab}{}^l H_{ck;l}) = 0\,, \tag{2.189}$$

$$\begin{aligned} (V) \qquad & TS\Big(3C^k{}_{ab}{}^{l}{}_{;}{}^{m} C_{kcdl;m} + 8C^k{}_{ab}{}^{l}{}_{;c} S_{kld} + 40 S_{ab}{}^k S_{cdk} - 8C^k{}_{ab}{}^l S_{klc;d} \\ & -24 C^k{}_{ab}{}^l S_{cdk;l} + 4C^k{}_{ab}{}^l C_l{}^m{}_{ck} L_{dm} + 12 C^k{}_{ab}{}^l C^m{}_{cdl} L_{km} + 12 H_{kc;de} H^k{}_f \\ & -16 H_{kc;d} H^k{}_{e;f} - 84 H^k{}_c C_{kdel} H^l{}_f - 18 H_{kc} H^k{}_d L_{ef}\Big) = 0\,, \end{aligned} \tag{2.190}$$

$$\begin{aligned} (VI) \qquad & TS\Big(36 C^k{}_{ab}{}^l C_{lcdm;k} H^m{}_e - 6C^k{}_{ab}{}^{l}{}_{;c} C_{lde}{}^m H_{km} - 138 S_{ab}{}^k C_{kcdl} H^l{}_e \\ & +6 S_{abk} H^k{}_{c;de} + 6 C^k{}_{ab}{}^{l}{}_{;c} H_{kd;le} - 24 S_{abk;c} H^k{}_{d;e} \\ & +12 C^k{}_{ab}{}^l L_{kc} H_{ld;e} - 9 C^k{}_{ab}{}^{l}{}_{;c} L_{kd} H_{le} - 9 S_{abk} L_{cd} H^k{}_e\Big) = 0\,. \end{aligned} \tag{2.191}$$

A seventh necessary condition, valid for the self-adjoint scalar equation, has the following form:

$$(VII) \quad TS\left(Q^{(1)}_{abcdef} - 10 Q^{(2)}_{abcdef} + 4 Q^{(3)}_{abcdef} + 5 Q^{(4)}_{abcdef} + Q^{(5)}_{abcdef}\right) = 0\,, \tag{2.192}$$

where

$$\begin{aligned} Q^{(1)}_{abcdef} \;=\; & 3C^k{}_{ab}{}^{l}{}_{;}{}^{m}{}_{c} C_{kdel;mf} + C^k{}_{ab}{}^{l}{}_{;cd}(10 S_{kle;f} + 6 S_{efk;l}) + 64 S_{abk;c} S_{de}{}^k{}_{;f} \\ & -C^k{}_{ab}{}^l\,(3C^m{}_{cdk;ef} L_{lm} + 5 C_{kcdl;me} L^m{}_f + 7 C^m{}_{cdk;le} L_{mf} \\ & +13 S_{klc;d} L_{ef} + 12 S_{cdk;l} L_{ef} + 71 S_{cdk;e} L_{lf})\,, \end{aligned} \tag{2.193}$$

$$
\begin{aligned}
Q^{(2)}_{abcdef} \;=\;& C^k{}_{ab}{}^l{}_{;c}\left(S_{kld;ef} + 3S_{dek;lf} + 2S_{abk;cd}S_{ef}{}^k - 5S_{abk}S_{cd}L_{ef}\right) \\
& -\frac{1}{2}C^k{}_{ab}{}^l{}_{;c}\,(2C^m{}_{kld;e}L_{mf} + 3C^m{}_{dek;l}L_{mf} + S_{kld}L_{ef} \\
& +3C_{kde}{}^m{}_{;f}L_{lm} + 15S_{dek}S_{lf}) - C^k{}_{ab}{}^l\left(C_{kcd}{}^m{}_{;e}L_{(lm;f)}\right. \\
& \left.+S_{cdk}L_{(le;f)} - \frac{1}{12}R_{;c}C_{kdel;f}\right),
\end{aligned} \tag{2.194}
$$

$$
\begin{aligned}
Q^{(3)}_{abcdef} \;=\;& -C^k{}_{ab}{}^l\,(2C_k{}^{mn}{}_eC_{lnmd;ef} - 10C^{mn}_{cd}C_{kefl;mn} + 20C_{lcd}{}^mS_{kme;f}) \\
& -5C_k{}^{mn}{}_aC_{lmnb}C^k{}_{cd}{}^l{}_{;ef} + C^k{}_{ab}{}^l\,(7C_k{}^{mn}{}_cC_{lmnd}L_{ef} \\
& -10C_{kefl}C^m{}_{cd}{}^nL_{mn}),
\end{aligned} \tag{2.195}
$$

$$
\begin{aligned}
Q^{(4)}_{abcdef} \;=\;& -C^k{}_{ab}{}^l\,(2C_k{}^{mn}{}_{c;d}C_{lmnd;ef} + 54C_{lcd}{}^m{}_{;e}S_{kmf} + 74C_{lcd}{}^m{}_{;k}S_{efm} \\
& -\frac{76}{3}C_{ckl}{}^m{}_{;d}S_{efm} - \frac{404}{3}S_{cdk}S_{efl}) + 6C_k{}^{mn}{}_aC^k{}_{bc;d}C_{lefm;n},
\end{aligned} \tag{2.196}
$$

$$
Q^{(5)}_{abcdef} = -C^k{}_{ab}{}^lC_{lcd}{}^mL_{km}L_{ef} + \frac{1}{6}C^k{}_{ab}{}^lC_{kcdl}(87L^m{}_eL_{mf} + 19RL_{ef}). \tag{2.197}
$$

Condition I was already known by Cotton in 1900 [25]. He proved that the necessary and sufficient conditions for the $n-$dimensional general wave equation $Pu = 0$ to be equivalent to the $n-$dimensional ordinary wave equation are

$$
C_{ijkl} = 0\,, \tag{2.198}
$$

$$
H_{ij} := A_{[i,j]} = 0\,, \tag{2.199}
$$

$$
C := B - \frac{1}{2}A^i{}_{;i} - \frac{1}{4}A_iA^i - \frac{n-2}{4(n-1)}R = 0. \tag{2.200}
$$

Hölder [45] also found condition I in the case $A^a = B = 0$. Mathisson [56], Hadamard [43], and Asgeirson [8] obtained conditions I, II and III in the case g^{ij} constant. Conditions I to IV were obtained in the general case by Günther [38]. Condition V was obtained by McLenaghan [58] in the case $R_{ab} = 0$, and by Wünsch [81] for $A^a = 0$. Condition VI was found by Anderson and McLenaghan [6]. In the general case, condition V was obtained by McLenaghan [60]. Condition VII was obtained by Rinke and Wünsch [72] for $A^a = 0$. The term on the right side of condition III is also called *Bach tensor*:

$$
C_{ab} := S_{abk;}{}^k \;-\; \frac{1}{2}C^k{}_{ab}{}^lL_{kl}\,. \tag{2.201}
$$

Günther [39] observed that conditions II and III display some interesting formal analogies. If we consider A_a as a four-potential then (2.187) represents the homogeneous Maxwell equations. On the right side of (2.188) we would have the electromagnetic energy-momentum tensor. The tensor on the left side of (2.188), called *Bach tensor* and denoted by C_{ab}, can be obtained from the variational principle applied to the conformally invariant integral $\int C^{ijkl} C_{ijkl}\, dx$ [9] and has null divergence, $C^{ij}{}_{;j} = 0$. Thus (2.188) is analogue to Einstein's field equations, but it is conformally invariant.

It is expected that the use of a finite number of necessary conditions would be sufficient to solve Hadamard's problem, but this is not yet proved.

2.4 Necessary conditions in NP form

We can use the relations (B.37)-(B.40), given in Appendix B to convert the necessary conditions $II - VII$, given respectively by (2.181), (2.187), (2.188), (2.189), (2.190), (2.191) and (2.192) to their spinor forms.

For condition II,

$$H^k{}_{a;k} = 0\,, \tag{2.202}$$

we obtain, using the correspondence (B.38),

$$\phi^K{}_{A;K\dot{K}}\varepsilon^{\dot{K}}_{\dot{A}} + \overline{\phi}^{\dot{K}}{}_{\dot{A};K\dot{K}}\varepsilon^K{}_A = -\phi^K{}_{A;K\dot{A}} - \overline{\phi}^{\dot{K}}{}_{\dot{A};A\dot{K}} = 0\,. \tag{2.203}$$

Since $H_{ab} = A_{[a,b]}$, we have

$$H_{[ab;c]} = 0\,, \tag{2.204}$$

or, in spinor form,

$$\begin{aligned} &\phi_{AB;C\dot{C}}\varepsilon_{\dot{A}\dot{B}} + \overline{\phi}_{\dot{A}\dot{B};\dot{C},C}\varepsilon_{AB} + \phi_{CA;B\dot{B}}\varepsilon_{\dot{C}\dot{A}} + \overline{\phi}_{\dot{C}\dot{A};\dot{B}B}\varepsilon_{CA} \\ &+\phi_{BC;A\dot{A}}\varepsilon_{\dot{B}\dot{C}} + \overline{\phi}_{\dot{B}\dot{C};\dot{A}A}\varepsilon_{BC} = 0\,. \end{aligned} \tag{2.205}$$

Multiplying this equation by $\varepsilon^{BC}\varepsilon_{\dot{B}\dot{A}}$ we find

$$\phi_{AB;}{}^B{}_{\dot{C}} - \overline{\phi}_{\dot{B}\dot{C};}{}^{\dot{B}}{}_A = 0\,, \tag{2.206}$$

which can also be written as

$$\phi_{AK;}{}^K{}_{\dot{A}} - \overline{\phi}_{\dot{A}\dot{K};}{}^{\dot{K}}{}_A = 0\,. \tag{2.207}$$

By comparing this equation with (2.203) we find

$$(IIs) \qquad \phi_{AK;}{}^K{}_{\dot{A}} = 0\,. \tag{2.208}$$

The four dyad components of this equation are the NP source-free Maxwell equations:

$$\left.\begin{aligned} D\phi_1 - \bar{\delta}\phi_0 &= (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 + (\rho - 2\epsilon)\phi_2\,, \\ D\phi_2 - \bar{\delta}\phi_1 &= -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\epsilon)\phi_2\,, \\ \delta\phi_1 - \Delta\phi_0 &= (\mu - 2\gamma)\phi_0 + 2\tau\phi_1 - \sigma\phi_2\,, \\ \delta\phi_2 - \Delta\phi_1 &= -\nu\phi_0 + 2\mu\phi_1 + (\tau - 2\beta)\phi_2\,. \end{aligned}\right\} \tag{2.209}$$

For condition III a direct application of the correspondence relations yields

$$\Psi_{ABKL;}{}^{K}{}_{\dot{A}}{}^{L}{}_{\dot{B}} + \overline{\Psi}_{\dot{A}\dot{B}\dot{K}\dot{L};}{}^{\dot{K}}{}_{A}{}^{\dot{L}}{}_{B} + \Psi_{AB}{}^{KL}\Phi_{KL\dot{A}\dot{B}} + \overline{\Psi}_{\dot{A}\dot{B}}{}^{\dot{K}\dot{L}}\Phi_{\dot{K}\dot{L}AB} + 10\phi_{AB}\overline{\phi}_{\dot{A}\dot{B}} = 0\,. \tag{2.210}$$

Instead of (2.210) we shall use a stronger form of this condition, obtained by McLenaghan and Williams [62]:

$$(IIIs) \qquad \nabla^{K}{}_{\dot{A}}\nabla^{L}{}_{\dot{B}}\Psi_{ABLK} + \Phi^{KL}{}_{\dot{A}\dot{B}}\Psi_{ABKL} + 5\phi_{AB}\overline{\phi}_{\dot{A}\dot{B}} = 0\,. \tag{2.211}$$

While the original necessary condition (2.210) is Hermitian, (2.211) is complex. Its dyad components can be obtained now by using the NPspinor package. The conversion of the remaining conditions to the respective spinor form and then the determination of the dyad components is better done automatically by defining templates in the `NPspinor` package (see Appendix E). In the definition of the templates for the necessary conditions that involve the trace-free symmetric part of a tensor, we use the following theorem (see [79] for the proof):

Theorem 2.4 *If $X_{r_1\cdots r_p}$ is a real trace-free symmetric tensor and*

$$X_{r_1\cdots r_p} \leftrightarrow X_{A_1\cdots A_p\dot{B}_1\cdots\dot{B}_p}\,, \tag{2.212}$$

then

$$X_{A_1\cdots A_p\dot{B}_1\cdots\dot{B}_p} = X_{(A_1\cdots A_p)(\dot{B}_1\cdots\dot{B}_p)}\,. \tag{2.213}$$

Chapter 3

The Conformally Invariant Scalar Wave Equation

3.1 Previous results

This chapter is devoted to the solution of Hadamard's problem for the conformally invariant scalar wave equation,

$$\Box u + \frac{1}{6} R u = 0 \,, \tag{3.1}$$

Petrov type III space-times are characterized by the existence of a spinor field o^A satisfying

$$\Psi_{ABCD} o^C o^D = 0 \,, \quad \Psi_{ABCD} o^D \neq 0. \tag{3.2}$$

Such a spinor field is called a *repeated principal spinor* of the Weyl spinor and is determined by the latter up to an arbitrary variable complex factor. Let ι^A be any spinor field satisfying

$$o_A \iota^A = 1 \,. \tag{3.3}$$

The ordered set o_A, ι_A, called a dyad, defines a basis for the 1-spinor fields on $\mathcal{M}^4$.

The main results, obtained by Carminati and McLenaghan for Petrov type III space-times, using conditions III and V, can be stated as follows [22]:

Theorem 3.1 *The validity of Huygens' principle for the conformally invariant scalar wave equation (3.1), on any Petrov type III space-time implies that the space-time is* *conformally related to one in which every repeated principal spinor field o_A of the Weyl spinor is recurrent, that is*

$$o_{A;B\dot{B}} = o_A I_{B\dot{B}}\,, \tag{3.4}$$

where $I_{B\dot{B}}$ is a 2-spinor, and

$$\Psi_{ABCD;E\dot{E}}\,\iota^A\iota^B\iota^C o^D o^E \bar{o}^{\dot{E}} = 0\,, \tag{3.5}$$

$$R = 0\,, \qquad \Phi_{AB\dot{A}\dot{B}} o^A o^B = 0\,. \tag{3.6}$$

Theorem 3.2 *If any one of the following three conditions*

$$\Psi_{ABCD;E\dot{E}}\,\iota^A\iota^B\iota^D\iota^E\bar{o}^{\dot{E}} = 0\,, \tag{3.7}$$

$$\Psi_{ABCD;E\dot{E}}\,\iota^A\iota^B o^D o^E \bar{\iota}^{\dot{E}} = 0\,, \tag{3.8}$$

$$\Psi_{ABCD;E\dot{E}}\,\iota^A\iota^B\iota^D o^E \bar{o}^{\dot{E}} = 0\,, \tag{3.9}$$

is satisfied, then there exist no Petrov type III space-times on which the conformally invariant scalar wave equation (3.1) satisfies Huygens' principle.

For the sake of completeness, in the remaining part of this Section we shall obtain the results of Carminati and McLenaghan [22], that lead to the proof of the above theorems. A generalization of their result will be obtained subsequently.

In Petrov type III space-times the Weyl spinor has the form [1]:

$$\Psi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\beta_{D)}\,, \tag{3.10}$$

where α_A and β_A are the principal spinors. If we choose the spin basis such that α_A is proportional to the dyad basis spinor o_A and β_A proportional to ι_A, we obtain from (B.42):

$$\Psi_{ABCD} = -4\Psi_3 o_{(ABC}\iota_{D)}\,. \tag{3.11}$$

Using (B.87) we can see that we can choose the tetrad such that $\Psi_3 = -1$, so that

$$\Psi_{ABCD} = 4 o_{(ABC}\iota_{D)}\,. \tag{3.12}$$

Condition (3.12) determines uniquely the spinor dyad $\{o_a\,,\,\iota_A\}$

In order to make this expression conformally invariant, it is clear from (B.104) that we must set $r = -1$.

[1] See Appendix B

We now start examining the consequences that follow from the necessary conditions III and V. This can be done by running the appropriate codes listed in the Appendix E, which use the package NPspinor in Maple [29]. The input of these codes consists of the tensorial expressions of the necessary conditions III and V and the output consists of the independent dyad components obtained by contracting the resulting dyad expressions with appropriate products of o^A, ι^A and their complex conjugates. The conversion from the tensorial form to the spinorial one is done automatically by defining templates in the beginning of the programs. The conversion to the respective dyad form is carried out by using expressions (B.42), (B.44) and (B.48).

It follows from the conformal invariance of conditions III and V [60] [82] that each dyad equation must be individually invariant under the conformal transformation (B.100). We begin by substituting the expression (3.12) for Ψ_{ABCD} in condition V. The contraction of the resulting dyad expression with $o^{ABCD}\bar{\iota}^{\dot{A}\dot{B}\dot{C}\dot{D}}$ and $o^{ABC}\iota^{D}\bar{o}^{\dot{A}\dot{B}\dot{C}}\iota^{\dot{D}}$ implies, respectively:

$$\kappa = 0\,, \qquad \sigma = 0\,. \tag{3.13}$$

These conditions, which are invariant under general dyad transformations (B.65) and conformal transformations (B.100), imply that the repeated principal null congruence of C_{abcd} defined by the principal null vector field l^a, is geodesic and shear-free. We use now the conformal freedom to restrict ρ. From (B.101) we obtain

$$\bar{\tilde{\rho}} + \tilde{\rho} = e^{-2\phi}(\bar{\rho} + \rho - 2\mathrm{D}\phi)\,. \tag{3.14}$$

Thus, by imposing $\mathrm{D}\phi = (\bar{\rho} + \rho)/2$, we obtain $\bar{\tilde{\rho}} + \tilde{\rho} = 0$ or, dropping tildes,

$$\rho = -\bar{\rho}\,. \tag{3.15}$$

By contracting $\iota^{(A}o^{B)}\bar{o}^{\dot{A}\dot{B}}$ with III, and $\iota^{(AB}o^{CD)}\bar{\iota}^{(\dot{A}\dot{B}}\bar{o}^{\dot{C}\dot{D})}$ with V, we get, respectively:

$$-\Phi_{00} + 6\rho\epsilon - 4\,\rho^2 + 2\,\epsilon\bar{\epsilon} - 2\,\bar{\epsilon}\,r + 2\,\mathrm{D}\rho - 2\,\epsilon^2 - 2\,\mathrm{D}\epsilon = 0\,, \tag{3.16}$$

$$15\,\rho^2 + 10\rho\bar{\epsilon} - 10\,\epsilon\rho - 10\epsilon\bar{\epsilon} - \Phi_{00} - \mathrm{D}\epsilon - \epsilon^2 - \mathrm{D}\bar{\epsilon} - \bar{\epsilon}^2 = 0\,. \tag{3.17}$$

Adding (3.16) to its complex conjugate, and subtracting the result from (3.17), we obtain

$$19\rho^2 - 12\bar{\epsilon}\epsilon - 14\rho(\epsilon + \bar{\epsilon}) = 0\,. \tag{3.18}$$

Solving this equation for ρ and using (3.15), we obtain

$$\rho = 0\,, \qquad \epsilon = 0\,. \tag{3.19}$$

From the Newman-Penrose Ricci identities [2] (NP1), (NP3), (NP4), (NP5), (NP11), (NP21) and from the contraction of condition *III* with $\iota^{AB}\bar{o}^{\dot{A}\dot{B}}$ and $\iota^{(A}o^{B)}\bar{\iota}^{(\dot{A}}\bar{o}^{\dot{B})}$ we have, respectively,

$$\Phi_{00}=0\,,\quad \Phi_{01}=0\,,\quad \mathrm{D}\alpha=0\,,\quad \mathrm{D}\beta=0\,,\quad \mathrm{D}\Phi_{11}=0\,,\quad \mathrm{D}\pi=0\,,\quad \mathrm{D}\tau=0\,. \tag{3.20}$$

We still have enough conformal freedom, preserving (3.19), to set

$$\tau=0\,. \tag{3.21}$$

According to (B.101), this further restriction is possible if there exists a solution for the following system of partial differential equations:

$$\mathrm{D}\phi=0\,,\qquad \delta\phi=\tau\,,\qquad \bar{\delta}\phi=\bar{\tau}\,. \tag{3.22}$$

In order to establish that this system has a solution we must show that the integrability conditions for (3.22) are satisfied. They are given by the following NP commutation relations applied on ϕ:

$$[\delta,\,\mathrm{D}]=(\bar{\alpha}+\beta-\bar{\pi})\mathrm{D}\,, \tag{3.23}$$

$$[\bar{\delta},\,\mathrm{D}]=(\bar{\alpha}+\beta-\pi)\mathrm{D}\,, \tag{3.24}$$

$$[\bar{\delta}\,,\,\delta]=(\bar{\mu}-\mu)\mathrm{D}+(\alpha-\bar{\beta})\delta+(\beta-\bar{\alpha})\bar{\delta}\,. \tag{3.25}$$

It can be easily verified that the above conditions are satisfied when applied to ϕ.

From (NP16) and (NP17) we now obtain, using (3.21),

$$\Phi_{02}=0\,,\qquad \Lambda=0\,. \tag{3.26}$$

From Bianchi identities (NP23) and (NP22) we also have

$$\mathrm{D}\Phi_{11}\;=\;0\,,\qquad \delta\Phi_{11}\;=\;0\,. \tag{3.27}$$

The expressions in (3.20) which are not conformally invariant can also be recovered.

The above results can be summarized as follows: Necessary conditions *III* and *V* imply that there exists a dyad $\{o_A,\iota_A\}$ and a conformal transformation ϕ such that

$$\kappa\;=\;\sigma\;=\;\rho\;=\;\tau\;=\;\epsilon\;=\;0\,, \tag{3.28}$$

$$\Psi_0\;=\;\Psi_1\;=\;\Psi_2\;=\;\Psi_4\;=\;0\,,\;\Psi_3\;=\;-1\,, \tag{3.29}$$

$$\Phi_{00}\;=\;\Phi_{01}\;=\;\Phi_{02}\;=\;\Lambda\;=\;0\,, \tag{3.30}$$

$$\mathrm{D}\alpha\;=\;\mathrm{D}\beta\;=\;\mathrm{D}\pi\;=\;0\,, \tag{3.31}$$

$$\delta\Phi_{11}\;=\;\mathrm{D}\Phi_{11}\;=\;0\,. \tag{3.32}$$

[2] see Appendix C

We notice that the expressions (3.29) determine the tetrad uniquely. On the other hand, conditions (3.28) are invariant under any conformal transformation satisfying

$$\mathrm{D}\phi = 0\,, \qquad \delta\phi = 0\,, \tag{3.33}$$

which implies that we still have some conformal freedom. Under (3.33) the transformation law for Φ_{11}, given by (B.107), becomes

$$\tilde{\Phi}_{11} = e^{-2\phi}\Phi_{11}\,. \tag{3.34}$$

Thus, we can choose ϕ such that

$$\Phi_{11} = c\,, \tag{3.35}$$

where c is a constant. Conditions (3.33) are satisfied in view of (3.32). Although it simplifies calculations, this last specification of the conformal gauge is not strictly necessary for our proof, i.e., our results do not depend on the Pfaffian $\Delta\Phi_{11}$.

3.2 Main Theorem

Carminati and McLenaghan [22] used the above conditions to prove that Huygens' principle is not satisfied if any of the spin coefficients α, β or π vanish. We shall extend the proof for the case in which $\alpha\beta\pi \neq 0$ and $\Phi_{11} = 0$, i.e., we shall prove the following theorem:

Theorem 3.3 (Main Theorem) *Let $\mathcal{M}^4$ be any space-time which admits a spinor dyad with the properties*

$$o_{A;B\dot{B}} = o_A I_{B\dot{B}}\,, \tag{3.36}$$

where $I_{B\dot{B}}$ is a 2-spinor, and

$$\Psi_{ABCD;E\dot{E}}\,\iota^A\iota^B\iota^C o^D o^E \bar{o}^{\dot{E}} = 0\,, \tag{3.37}$$

$$R = 0\,, \qquad \Phi_{AB\dot{A}\dot{B}}o^A o^B = 0\,. \tag{3.38}$$

Then the validity of Huygens' principle for the conformally invariant equation (3.1) implies that

$$\Phi_{AB\dot{A}\dot{B}}o^A\iota^B\bar{o}^{\dot{A}}\bar{\iota}^{\dot{B}} \neq 0\,. \tag{3.39}$$

3.3 Proof of the Main Theorem

In what follows we assume that $\alpha\beta\pi \neq 0$, since the case in which this is not true was already considered in [22].

By contracting condition III with $\iota^A o^B \bar{\iota}^{\dot{A}\dot{B}}$ we get

$$\delta\beta = -\beta(\overline{\alpha} + \beta)\,. \tag{3.40}$$

From the Bianchi identities, using the above conditions, we obtain

$$D\Phi_{12} = 2\overline{\pi}\Phi_{11}\,, \tag{3.41}$$
$$D\Phi_{22} = -2(\beta + \overline{\beta}) + 2\Phi_{21}\overline{\pi} + 2\Phi_{12}\pi\,, \tag{3.42}$$
$$\delta\Phi_{12} = 2\overline{\alpha} + 4\overline{\pi} + 2\overline{\lambda}\Phi_{11} - 2\overline{\alpha}\Phi_{12}\,, \tag{3.43}$$
$$\overline{\delta}\Phi_{12} = -2\beta + 2\overline{\mu}\Phi_{11} - 2\overline{\beta}\Phi_{12}\,. \tag{3.44}$$

From the Ricci identities we get the following relevant Pfaffians:

$$D\gamma = \overline{\pi}\alpha + \beta\pi + \Phi_{11}\,, \tag{3.45}$$
$$\mathrm{D}\lambda = (1/2)\delta\alpha - (11/2)\overline{\beta}\alpha + \pi^2 - 2\pi\alpha - 11\pi\overline{\beta} - (3/2)\alpha^2\,, \tag{3.46}$$
$$\delta\overline{\pi} = D\overline{\lambda} - \overline{\pi}^2 - \overline{\pi}\overline{\alpha} + \overline{\pi}\beta\,, \tag{3.47}$$
$$D\overline{\nu} = \Delta\overline{\pi} + \overline{\pi}\overline{\mu} + \overline{\lambda}\pi + \overline{\pi}\overline{\gamma} - \overline{\pi}\gamma - 1 + \Phi_{12}\,, \tag{3.48}$$
$$\delta\alpha = \overline{\delta}\beta + \alpha\overline{\alpha} + \beta\overline{\beta} - 2\beta\alpha + \Phi_{11}\,, \tag{3.49}$$
$$\delta\pi = D\mu - \overline{\pi}\pi + \pi\overline{\alpha} - \beta\pi\,. \tag{3.50}$$

We can obtain useful integrability conditions for the above Pfaffians, by using NP commutation relations. By substituting them in the commutator expression $[\delta, D]\Phi_{22} - [\Delta, D]\Phi_{12}$, we get

$$\delta\overline{\beta} = -2\Phi_{11} - \overline{\beta}\overline{\alpha} - 4\overline{\beta}\overline{\pi} - 2D\overline{\mu} - \beta\overline{\beta} + 2\overline{\pi}\pi. \tag{3.51}$$

By contracting condition V with $\iota^{ABCD}\bar{\iota}^{\dot{A}\dot{B}}\bar{o}^{\dot{C}\dot{D}}$, we get

$$20\overline{\beta}\pi + 12\overline{\beta}\alpha + 6\pi\alpha + 3\alpha^2 + \overline{\delta}\alpha + 2\overline{\delta}\pi + \overline{\delta\beta} + \overline{\beta}^2 = 0\,. \tag{3.52}$$

By substituting (3.35) into this equation we get

$$\delta(2\overline{\pi} + \overline{\alpha}) = -20\overline{\pi}\beta - 11\beta\overline{\alpha} - 6\overline{\pi}\overline{\alpha} - 3\overline{\alpha}^2\,. \tag{3.53}$$

From (3.49), (3.50) and (3.35) we obtain:

$$\delta(2\pi + \alpha) = 2\pi\overline{\alpha} + \alpha\overline{\alpha} - 6\beta\pi - 3\beta\alpha - \Phi_{11}\,. \tag{3.54}$$

By contracting condition V with $\iota^{ABC}{}_{o}{}^{D}\bar{\iota}^{\dot{A}\dot{B}\dot{C}}\bar{o}^{\dot{D}}$, we find

$$\begin{aligned}&-6\,\delta\pi\;-15\,\alpha\overline{\pi}-10\,\alpha\overline{\alpha}-68\,\pi\overline{\pi}-15\,\pi\overline{\alpha}-3\,\delta\,\alpha\;-126\,\overline{\beta}\,\beta\\&+5\,\mathrm{D}\,\overline{\gamma}\;+10\,\mathrm{D}\,\overline{\mu}\;-24\,\overline{\beta}\,\overline{\alpha}-3\,\overline{\delta}\overline{\alpha}-6\,\overline{\delta}\,\overline{\pi}\;-15\,\delta\,\overline{\beta}\;-3\,\overline{\beta}\,\overline{\pi}\\&+5\,\mathrm{D}\,\gamma\;+10\,\mathrm{D}\,\mu\;-15\,\overline{\delta}\beta-24\,\beta\,\alpha-3\,\beta\,\pi-4\,\Phi_{11}=0\,.\end{aligned}\tag{3.55}$$

Using (3.45), (3.50) and the complex conjugate of (3.51), we get

$$9\Phi_{11}+10\overline{\beta}\overline{\pi}+5[\mathrm{D}\mu+\mathrm{D}\overline{\mu}]-2\overline{\pi}\alpha-12\beta\overline{\beta}-2\alpha\overline{\alpha}-16\overline{\pi}\pi+10\beta\pi-2\pi\overline{\alpha}=0\,.\tag{3.56}$$

On the other hand, the NP commutator $[\overline{\delta},\delta](\alpha+2\pi)=(\alpha-\overline{\beta})\delta(\alpha+2\pi)+(-\overline{\alpha}+\beta)\overline{\delta}(\alpha+2\pi)$, yields the following expression

$$\begin{aligned}&2\pi\beta\overline{\beta}+22\pi\overline{\beta}\overline{\alpha}+43\pi\overline{\beta}\overline{\pi}-22\overline{\pi}\pi^2+\overline{\beta}\mathrm{D}\mu+22\pi\mathrm{D}\overline{\mu}+12\alpha\overline{\beta}\overline{\alpha}\\&+6\overline{\beta}\Phi_{11}-12\alpha\overline{\pi}\pi+11\alpha\Phi_{11}+18\pi\Phi_{11}+24\overline{\pi}\overline{\beta}\alpha+12\alpha\mathrm{D}\overline{\mu}=0\,.\end{aligned}\tag{3.57}$$

Eliminating $\mathrm{D}\overline{\mu}$ between (3.56) and (3.57), and solving for $\mathrm{D}\mu$, we get

$$\begin{aligned}\mathrm{D}\mu=-\frac{1}{5}\Big(&108\,\pi\Phi_{11}-44\,\pi^2\overline{\alpha}-24\,\overline{\pi}\,\alpha^2-68\,\pi\alpha\overline{\alpha}\\&144\,\alpha\beta\overline{\beta}+53\,\alpha\Phi_{11}-274\,\pi\beta\overline{\beta}+120\,\pi\beta\alpha-24\,\alpha^2\overline{\alpha}\\&-242\,\overline{\pi}\,\pi^2+220\,\beta\pi^2-176\,\alpha\overline{\pi}\,p-60\,\alpha\overline{\beta}\,\overline{\alpha}-30\,\overline{\beta}\,\Phi_{11}\\&+5\,\pi\overline{\pi}\overline{\beta}-110\,\overline{\alpha}\pi\overline{\beta}\Big)\,/(-\overline{\beta}+12\,\alpha+22\,\pi)\,,\end{aligned}\tag{3.58}$$

where we have assumed that the denominator of the expression above, given by

$$d_1:=-\overline{\beta}+12\,\alpha+22\,\pi\,,\tag{3.59}$$

is non-zero. The case $d_1=0$ will be considered later.

Substituting expression (3.58) for $\mathrm{D}\mu$ into (3.56) we obtain

$$\begin{aligned}\mathrm{D}\overline{\mu}=-\frac{1}{5}\Big(&90\pi\Phi_{11}+12\beta\overline{\beta}^2-10\overline{\beta}^2\overline{\pi}+55\alpha\Phi_{11}+122\alpha\overline{\beta}\overline{\pi}-110\overline{\pi}\pi^2-60\alpha\overline{\pi}\pi\\&+62\alpha\overline{\beta}\overline{\alpha}+21\overline{\beta}\Phi_{11}+231\pi\overline{\beta}\overline{\pi}+112\overline{\alpha}\pi\overline{\beta}\Big)\,/(-\overline{\beta}+12\,\alpha+22\,\pi)\,.\end{aligned}\tag{3.60}$$

One side relation can now be obtained by subtracting the complex conjugate of (3.58) from (3.60). We obtain:

$$\begin{aligned}S_1:=\frac{1}{5}(&720\,\overline{\alpha}^2\,\overline{\beta}\,\alpha+2904\,\pi^2\,\overline{\pi}^2-12\,\beta^2\,\overline{\beta}^2+288\,\overline{\alpha}^2\,\alpha^2+528\,\pi^2\,\overline{\alpha}^2+528\,\overline{\pi}^2\,\alpha^2\\&+2420\,\overline{\alpha}\,\pi\,\overline{\beta}\,\overline{\pi}+3056\,\overline{\alpha}\,\pi\,\beta\,\overline{\beta}+2888\,\overline{\alpha}\,\alpha\,\overline{\pi}\,\pi+1320\,\overline{\alpha}\,\pi\,\beta\,\alpha+1320\,\overline{\pi}\,\beta\,\alpha^2\\&+1606\,\overline{\alpha}\,\beta\,\overline{\beta}\,\alpha+5802\,\overline{\pi}\,\pi\,\beta\,\overline{\beta}+1320\,\overline{\pi}\,\overline{\alpha}\,\overline{\beta}\,\alpha+2420\,\overline{\pi}\,\beta\,\alpha\,\pi+1320\,\pi\,\overline{\beta}\,\overline{\alpha}^2\end{aligned}$$

$$+3056\,\overline{\pi}\,\beta\,\alpha\,\overline{\beta} + 2552\,\pi\,\overline{\pi}^2\,\alpha + 305\,\beta\,\Phi_{11}\,\alpha + 570\,\beta\,\Phi_{11}\,\overline{\pi} - 51\,\beta\,\Phi_{11}\,\overline{\beta}$$
$$+24\,\overline{\alpha}\,\Phi_{11}\,\alpha - 86\,\overline{\alpha}\,\Phi_{11}\,\pi + 305\,\overline{\alpha}\,\Phi_{11}\,\overline{\beta} + 816\,\pi\,\alpha\,\overline{\alpha}^2 + 2552\,\overline{\alpha}\,\overline{\pi}\,\pi^2$$
$$+816\,\overline{\alpha}\,\overline{\pi}\,\alpha^2 - 86\,\overline{\pi}\,\alpha\,\Phi_{11} - 396\,\overline{\pi}\,\pi\,\Phi_{11} + 720\,\beta\,\overline{\alpha}\,\alpha^2 + 570\,\overline{\pi}\,\overline{\beta}\,\Phi_{11}\Big/$$
$$((-12\,\alpha - 22\,\pi + \overline{\beta})\,(12\,\overline{\alpha} + 22\,\overline{\pi} - \beta)) = 0\,. \tag{3.61}$$

We notice that (3.58) and (3.60) have the same denominator. Thus, if we keep these expressions for $\mathrm{D}\mu$ and $\mathrm{D}\overline{\mu}$, the Pfaffians $\delta\overline{\beta}$, $\delta\alpha$, $\delta\pi$, given by (3.51), (3.47) and (3.50), respectively, and their complex conjugates, also have the same denominator. This procedure is crucial to keep the expressions to be obtained from the integrability conditions within a reasonable size. Except for $\overline{\delta}\alpha$, all Pfaffians involving δ, $\overline{\delta}$, applied to α, β, π are explicitly determined.

The following expression for $\overline{\delta}\alpha$ can be obtained from the NP commutator $[\overline{\delta},\delta]\overline{\beta} = (\overline{\mu}-\mu)\mathrm{D}\overline{\beta} + (\alpha-\overline{\beta})\delta\overline{\beta} + (\beta-\overline{\alpha})\overline{\delta}\overline{\beta}$:

$$\overline{\delta}\alpha = (2\pi\mathrm{D}\overline{\mu} - 2\overline{\delta}(\mathrm{D}\overline{\mu}) - 3\overline{\pi}\alpha^2 - 8\overline{\beta}\mathrm{D}\overline{\mu} - 11\alpha\overline{\beta}\overline{\pi} - 2\overline{\pi}\pi^2$$
$$-4\alpha\overline{\pi}\pi - 4\overline{\beta}\Phi_{11} - 14\overline{\pi}\overline{\beta}\overline{\pi})/\overline{\pi}\,. \tag{3.62}$$

By substituting (3.58) and (3.60) into this equation we get:

$$\overline{\delta}\alpha := -\frac{1}{5}(19519\,\Phi_{11}\overline{\beta}^2\,\alpha + 8570\,\Phi_{11}\overline{\beta}\,\alpha^2 + 1950\,\Phi_{11}\,\pi\,\alpha^2 + 35850\,\Phi_{11}\,\pi\,\overline{\beta}^2$$
$$+2900\,\Phi_{11}\,\pi^2\,\overline{\beta} - 3180\,\pi^2\,\overline{\beta}^2\,\overline{\pi} - 210\,\alpha^3\,\overline{\beta}\,\overline{\pi} + 150\,\alpha^3\,\overline{\beta}\,\overline{\alpha} - 1307\,\alpha^2\,\overline{\beta}^2\,\overline{\pi}$$
$$-180\,\alpha^2\,\beta\,\overline{\beta}^2 + 628\,\alpha^2\,\overline{\beta}^2\,\overline{\alpha} - 1280\,\pi^2\,\overline{\beta}^2\,\overline{\alpha} + 1950\,\overline{\alpha}\,\overline{\beta}^3\,\alpha + 4668\,\alpha\,\overline{\beta}^3\,\beta$$
$$+3520\,\overline{\beta}^3\,\overline{\alpha}\,\pi + 8160\,\beta\,\pi\,\overline{\beta}^3 + 330\,\overline{\beta}^3\,\overline{\pi}\,\pi + 975\,\Phi_{11}\,\alpha^3 - 860\,\overline{\beta}^3\,\Phi_{11}$$
$$+17950\,\Phi_{11}\,\overline{\beta}\,\alpha\,\pi - 420\,\alpha^2\,\overline{\beta}\,\overline{\pi}\,\pi + 300\,\alpha^2\,\overline{\beta}\,\pi\,\overline{\alpha} - 4116\,\alpha\,\overline{\beta}^2\,\overline{\pi}\,\pi$$
$$+588\,\alpha\,\pi\,\overline{\alpha}\,\overline{\beta}^2 + 175\,\overline{\beta}^3\,\alpha\,\overline{\pi})\Big/(\overline{\beta}^2\,\overline{\pi} - 14\,\alpha\,\overline{\beta}\,\overline{\pi} + 10\,\overline{\beta}\,\alpha\,\overline{\alpha} + 20\,\overline{\beta}\,\pi\,\overline{\alpha}$$
$$+65\,\alpha\,\Phi_{11} - 12\,\beta\,\overline{\beta}^2 + 130\,\pi\,\Phi_{11} - 31\,\overline{\beta}\,\Phi_{11} - 28\,\overline{\beta}\,\overline{\pi}\,\pi - 6\,\overline{\beta}^2\,\overline{\alpha})\,, \tag{3.63}$$

where, for now, we assume that the denominator in the expression above:

$$d_2 := \overline{\beta}^2\,\overline{\pi} - 14\,\alpha\,\overline{\beta}\,\overline{\pi} + 10\,\overline{\beta}\,\alpha\,\overline{\alpha} + 20\,\overline{\beta}\,\pi\,\overline{\alpha} + 65\,\alpha\,\Phi_{11} - 12\,\beta\,\overline{\beta}^2$$
$$+130\,\pi\,\Phi_{11} - 31\,\overline{\beta}\,\Phi_{11} - 28\,\overline{\beta}\,\overline{\pi}\,\pi - 6\,\overline{\beta}^2\,\overline{\alpha}\,, \tag{3.64}$$

is non-zero.

Contracting condition VII with $\iota^{ABCDE}{}_{o}{}^{F}\overline{o}^{\dot{B}\dot{C}\dot{D}}\overline{\iota}^{\dot{A}\dot{E}\dot{F}}$ yields

$$VI_{13} := 648\,\overline{\beta}^2\,\alpha\,\overline{\pi} - 165\,\pi\,\overline{\pi}\,\overline{\delta\beta} - 40\,\pi\,\mathrm{D}\Phi_{21} + 1557\,\delta\overline{\beta}\,\alpha\,\pi$$

$$
\begin{aligned}
&+270\,\alpha\bar{\beta}\,\delta\pi + 144\,\delta\alpha\,\alpha\bar{\beta} + 378\,\alpha\bar{\beta}\,\mathrm{D}\bar{\gamma} + 2058\,\alpha^2\,\bar{\pi}\,\pi \\
&+570\,\mathrm{D}\mu\,\overline{\delta\beta} + 630\,\mathrm{D}\mu\,\bar{\beta}^2 + 21\,\mathrm{D}\lambda\,\bar{\delta}\bar{\alpha} - 201\,\mathrm{D}\bar{\gamma}\,\bar{\delta}\pi \\
&-36\,\pi^2\,\bar{\delta}\bar{\alpha} - 954\,\bar{\delta}\beta\,\bar{\beta}^2 - 864\,\bar{\delta}\beta\,\overline{\delta\beta} - 864\,\beta\,\bar{\beta}^3 - 132\,\bar{\delta}\alpha\,\bar{\delta}\bar{\alpha} \\
&+1116\,\alpha^3\,\bar{\alpha} - 366\,\alpha^2\,\bar{\delta}\bar{\alpha} - 900\,\alpha\bar{\beta}\,\delta\bar{\beta} - 20\,\alpha\,\mathrm{D}\Phi_{21} + 1454\,\Phi_{11}\,\pi\,\bar{\beta} \\
&+771\,\bar{\delta}\bar{\pi}\,\pi\,\bar{\beta} + 423\,\bar{\delta}\alpha\,\bar{\alpha}\,\pi + 939\,\bar{\alpha}\,\alpha\,\bar{\delta}\pi - 1038\,\mathrm{D}\bar{\mu}\,\alpha\,\pi \\
&-90\,\lambda\,\mathrm{D}(\,\mathrm{D}\bar{\mu}\,) + 45\,\pi\,\bar{\delta}(\,\delta\bar{\beta}\,) - 15\,\pi\,\bar{\delta}(\,\mathrm{D}\bar{\gamma}\,) - 30\,\pi\,\bar{\delta}(\,\mathrm{D}\bar{\mu}\,) \\
&+84\,\alpha\bar{\beta}\,\bar{\delta}\bar{\pi} - 774\,\bar{\delta}\alpha\,\bar{\beta}\,\bar{\pi} - 570\,\overline{\delta\beta}\,\bar{\alpha}\,\pi + 180\,\bar{\delta}\alpha\,\beta\,\bar{\beta} \\
&+135\,\mathrm{D}(\,\delta\bar{\beta}\,)\,\lambda - 45\,\lambda\,\mathrm{D}(\,\mathrm{D}\bar{\gamma}\,) - 186\,\bar{\delta}\bar{\beta}\;\alpha\,\bar{\pi} + 570\,\alpha\bar{\beta}\,\mathrm{D}\bar{\mu} \\
&+30\,\mathrm{D}\lambda\,\alpha\,\bar{\pi} - 39\,\mathrm{D}\lambda\,\bar{\alpha}\,\pi - 78\,\mathrm{D}\lambda\,\bar{\pi}\,\pi - 744\,\bar{\alpha}\,\alpha\,\overline{\delta\beta} - 924\,\bar{\alpha}\,\alpha^2\,\bar{\beta} \\
&-387\,\pi^2\,\bar{\alpha}\bar{\beta} - 21\,\mathrm{D}\lambda\,\bar{\alpha}\,\alpha + 351\,\mathrm{D}\bar{\gamma}\,\pi\,\bar{\beta} - 1773\,\bar{\beta}\,\bar{\pi}\,\bar{\delta}\pi \\
&+216\,\bar{\alpha}\,\pi\,\bar{\delta}\pi + 870\,\mathrm{D}\mu\,\alpha\bar{\beta} - 198\,\bar{\delta}\bar{\alpha}\,\alpha\bar{\beta} + 762\,\bar{\delta}\alpha\,\bar{\alpha}\,\alpha \\
&+378\,\beta\,\alpha\,\overline{\delta\beta} + 828\,\alpha^2\,\bar{\beta}\,\beta - 1674\,\alpha\bar{\beta}\,\bar{\delta}\beta - 594\,\beta\,\bar{\beta}\,\overline{\delta\beta} + 636\,\pi^2\,\bar{\alpha}\,\alpha \\
&+2289\,\alpha^2\,\pi\,\bar{\alpha} + 36\,\mathrm{D}\lambda\,\bar{\alpha}\bar{\beta} - 660\,\alpha\,\pi\,\bar{\delta}\bar{\alpha} + 1773\,\bar{\beta}^2\,\bar{\pi}\,\pi - 555\,\mathrm{D}\bar{\gamma}\,\alpha\,\pi \\
&+2187\,\beta\,\pi\,\bar{\beta}^2 + 2619\,\beta\,\pi\,\overline{\delta\beta} - 888\,\bar{\alpha}\bar{\beta}^2\,\alpha - 126\,\bar{\delta}\alpha\,\bar{\alpha}\bar{\beta} \\
&+3681\,\beta\,\pi\,\alpha\bar{\beta} - 2055\,\alpha\,\pi\,\bar{\alpha}\bar{\beta} - 6951\,\bar{\beta}\,\bar{\pi}\,\alpha\,\pi - 1476\,\alpha\bar{\beta}^2\,\beta \\
&+150\,\bar{\delta}\alpha\,\bar{\pi}\,\pi - 63\,\mathrm{D}\lambda\,\bar{\beta}\,\bar{\pi} - 30\,\bar{\beta}^2\,\bar{\alpha}\,\pi + 934\,\Phi_{11}\,\alpha\,\pi - 465\,\pi^2\,\bar{\beta}\,\bar{\pi} \\
&-1608\,\alpha\,\pi\,\bar{\delta}\bar{\pi} + 63\,\mathrm{D}\lambda\,\beta\,\bar{\beta} + 20\,\bar{\beta}\,\mathrm{D}\Phi_{21} + 567\,\pi^2\,\beta\,\bar{\beta} \\
&-426\,\bar{\delta}\bar{\pi}\,\bar{\delta}\pi + 18\,\mathrm{D}\lambda\,\mathrm{D}\bar{\mu} - 39\,\pi^2\,\mathrm{D}\bar{\gamma} + 39\,\pi^3\,\bar{\alpha} - 78\,\pi^2\,\mathrm{D}\bar{\mu} \\
&+78\,\pi^3\,\bar{\pi} + 9\,\mathrm{D}\lambda\,\mathrm{D}\bar{\gamma} + 324\,\delta\alpha\,\overline{\delta\beta} + 324\,\delta\alpha\,\bar{\beta}^2 \\
&-189\,\bar{\delta}\bar{\alpha}\,\bar{\delta}\pi + 276\,\overline{\delta\beta}\,\mathrm{D}\gamma + 306\,\bar{\beta}^2\,\mathrm{D}\gamma + 432\,\alpha^3\,\bar{\pi} \\
&+1080\,\alpha^2\,\delta\bar{\beta} - 144\,\bar{\delta}\alpha\,\bar{\delta}\bar{\pi} + 360\,\bar{\delta}\alpha\,\delta\bar{\beta} - 372\,\alpha^2\,\bar{\delta}\bar{\pi} \\
&+630\,\overline{\delta\beta}\,\delta\pi + 630\,\bar{\beta}^2\,\delta\pi + 528\,\mathrm{D}\bar{\mu}\,\pi\,\bar{\beta} + 408\,\bar{\pi}\,\pi\,\bar{\delta}\pi \\
&+246\,\alpha\,\bar{\pi}\,\bar{\delta}\pi + 639\,\bar{\delta}\bar{\alpha}\,\pi\,\bar{\beta} + 156\,\alpha\bar{\beta}\,\mathrm{D}\gamma - 2592\,\alpha^2\,\bar{\beta}\,\bar{\pi} \\
&+204\,\bar{\delta}\alpha\,\alpha\,\bar{\pi} + 117\,\beta\,\bar{\beta}\,\bar{\delta}\pi - 657\,\delta\bar{\beta}\,\pi\,\bar{\beta} + 1128\,\pi^2\,\alpha\,\bar{\pi} \\
&-108\,\bar{\delta}\alpha\,\mathrm{D}\bar{\gamma} - 240\,\bar{\delta}\alpha\,\mathrm{D}\bar{\mu} - 720\,\alpha^2\,\mathrm{D}\bar{\mu} - 324\,\alpha^2\,\mathrm{D}\bar{\gamma} \\
&+567\,\delta\bar{\beta}\,\bar{\delta}\pi - 30\,\mathrm{D}\lambda\,\bar{\delta}\bar{\pi} - 27\,\mathrm{D}\lambda\,\delta\bar{\beta} + 117\,\pi^2\,\delta\bar{\beta} \\
&-378\,\mathrm{D}\bar{\mu}\,\bar{\delta}\pi - 102\,\Phi_{11}\,\mathrm{D}\lambda + 182\,\Phi_{11}\,\pi^2 + 366\,\Phi_{11}\,\bar{\delta}\pi \\
&-600\,\Phi_{11}\,\overline{\delta\beta} - 600\,\Phi_{11}\,\bar{\beta}^2 + 1008\,\Phi_{11}\,\alpha^2 + 336\,\Phi_{11}\,\bar{\delta}\alpha + 1704\,\Phi_{11}\,\alpha\bar{\beta} \\
&+30\,\pi\,\mathrm{D}(\,\bar{\delta}\bar{\gamma}\,) - 90\,\pi\,\delta(\,\overline{\delta\beta}\,) + 60\,\pi\,\mathrm{D}(\,\bar{\delta}\bar{\mu}\,) - 90\,\mathrm{D}(\,\overline{\delta\beta}\,)\,\mu \\
&-90\,\pi\,\mathrm{D}(\,\Delta\bar{\beta}\,) - 660\,\bar{\beta}\,\pi\,\mathrm{D}\gamma - 270\,\alpha\,\bar{\delta}(\,\delta\bar{\beta}\,) - 30\,\alpha\,\bar{\delta}(\,\bar{\delta}\bar{\alpha}\,) \\
&-60\,\alpha\,\bar{\delta}(\,\bar{\delta}\bar{\pi}\,) + 30\,\bar{\beta}\,\bar{\delta}(\,\mathrm{D}\gamma\,) + 60\,\bar{\beta}\,\bar{\delta}(\,\mathrm{D}\mu\,) + 90\,\alpha\,\bar{\delta}(\,\mathrm{D}\bar{\gamma}\,) \\
&+180\,\alpha\,\bar{\delta}(\,\mathrm{D}\bar{\mu}\,) - 180\,\bar{\beta}\,\bar{\delta}(\,\delta\pi\,) - 90\,\bar{\beta}\,\bar{\delta}(\,\delta\alpha\,) + 45\,\pi\,\bar{\delta}(\,\bar{\delta}\bar{\alpha}\,)
\end{aligned}
$$

$$
\begin{aligned}
&+90\,\pi\,\overline{\delta}(\,\overline{\delta\pi}\,) + 30\,\overline{\beta}\,\mathrm{D}(\,\overline{\delta}\gamma\,) + 60\,\overline{\beta}\,\mathrm{D}(\,\overline{\delta}\mu\,) + 15\,\lambda\,\mathrm{D}(\,\overline{\delta\alpha}\,)\\
&+30\,\lambda\,\mathrm{D}(\,\overline{\delta\pi}\,) + 60\,\overline{\alpha}\,\overline{\delta}(\,\overline{\delta}\alpha\,) - 90\,\overline{\beta}\,\overline{\delta}(\,\overline{\delta}\beta\,) - 90\,\overline{\beta}\,\mathrm{D}(\,\Delta\alpha\,)\\
&-180\,\overline{\beta}\,\mathrm{D}(\,\Delta\pi\,) + 180\,\beta\,\overline{\delta}(\,\overline{\delta\beta}\,) + 120\,\overline{\alpha}\,\overline{\delta}(\,\overline{\delta}\pi\,) + 60\,\gamma\,\mathrm{D}(\,\overline{\delta}\alpha\,)\\
&+120\,\gamma\,\mathrm{D}(\,\overline{\delta}\pi\,) - 180\,\overline{\alpha}\,\overline{\beta}\,\overline{\delta}\pi - 60\,\overline{\beta}\,\mathrm{D}\mu\,\pi - 315\,\overline{\beta}\,\overline{\delta}\beta\,\pi\\
&+180\,\overline{\beta}\,\delta\pi\,\pi \;=\; 0\,. \qquad (3.65)
\end{aligned}
$$

The second-order terms $\mathrm{D}(\delta\overline{\gamma})$, $\mathrm{D}(\overline{\delta}\gamma)$, $\mathrm{D}(\delta\overline{\mu})$, $\mathrm{D}(\Delta\overline{\beta})$, $\mathrm{D}(\Delta\alpha)$, $\mathrm{D}(\Delta\pi)$, can be expressed in terms of known Pfaffians and $\overline{\delta}\alpha$, by using the NP commutation relations involving each pair of operators. After the substitutions we obtain

$$
\begin{aligned}
VII_{13} :=\; &\frac{2}{5}(760\,\overline{\beta}^2\,\overline{\pi}\,\pi\,\overline{\delta}\alpha + 9600\,\overline{\beta}\,\Phi_{11}\,\overline{\delta}\alpha\,\pi + 25040\,\overline{\beta}\,\overline{\delta}\alpha\,\overline{\alpha}\,\pi^2\\
&-23440\,\overline{\beta}\,\overline{\pi}\,\pi^2\,\overline{\delta}\alpha + 2400\,\beta\,\pi\,\overline{\delta}\alpha\,\overline{\beta}^2 - 240\,\overline{\beta}^2\,\overline{\delta}\alpha\,\overline{\alpha}\,\pi\\
&+2880\,\alpha\,\overline{\beta}^2\,\beta\,\overline{\delta}\alpha + 8600\,\alpha\,\Phi_{11}\,\overline{\delta}\alpha\,\pi + 360\,\alpha\,\overline{\beta}^2\,\overline{\pi}\,\overline{\delta}\alpha\\
&+600\,\alpha\,\overline{\delta}\alpha\,\overline{\alpha}\,\overline{\beta}^2 + 24880\,\alpha\,\overline{\beta}\,\overline{\delta}\alpha\,\overline{\alpha}\,\pi - 22160\,\alpha\,\overline{\beta}\,\overline{\pi}\,\pi\,\overline{\delta}\alpha\\
&+8940\,\alpha\,\Phi_{11}\,\overline{\delta}\alpha\,\overline{\beta} - 5280\,\alpha^2\,\overline{\beta}\,\overline{\pi}\,\overline{\delta}\alpha + 6240\,\overline{\beta}\,\overline{\alpha}\,\alpha^2\,\overline{\delta}\alpha - 9885300\,\alpha\,\Phi_{11}\,\pi^2\,\overline{\beta}\\
&+113040\,\alpha\,\beta\,\overline{\beta}^4 - 1200\,\Phi_{11}\,\overline{\delta}\alpha\,\alpha^2 + 48400\,\alpha\,\pi^3\,\Phi_{11} + 40200\,\Phi_{11}\,\pi\,\alpha^3\\
&+22400\,\Phi_{11}\,\overline{\delta}\alpha\,\pi^2 - 720\,\overline{\beta}^3\,\beta\,\overline{\delta}\alpha - 300\,\overline{\delta}\alpha\,\overline{\alpha}\,\overline{\beta}^3 + 142635\,\alpha\,\overline{\beta}^3\,\Phi_{11}\\
&-910512\,\alpha^2\,\overline{\beta}^3\,\beta - 351912\,\overline{\beta}^2\,\alpha^3\,\overline{\pi} - 5085800\,\overline{\beta}\,\pi^3\,\Phi_{11} - 1467420\,\Phi_{11}\,\overline{\beta}\,\alpha^3\\
&-1760\,\Phi_{11}\,\overline{\delta}\alpha\,\overline{\beta}^2 - 3600\,\Phi_{11}\,\alpha^4 + 35640\,\overline{\beta}^3\,\alpha^2\,\overline{\pi} - 360060\,\overline{\alpha}\,\overline{\beta}^3\,\alpha^2\\
&+120000\,\Phi_{11}\,\pi^2\,\alpha^2 - 633396\,\Phi_{11}\,\overline{\beta}^2\,\alpha^2 + 8640\,\alpha^3\,\beta\,\overline{\beta}^2 - 15840\,\alpha^4\,\overline{\beta}\,\overline{\pi}\\
&-155352\,\alpha^3\,\overline{\beta}^2\,\overline{\alpha} + 18720\,\alpha^4\,\overline{\beta}\,\overline{\alpha} + 176880\,\pi^2\,\overline{\beta}^3\,\overline{\pi} + 205920\,\beta\,\pi\,\overline{\beta}^4\\
&+85800\,\overline{\beta}^4\,\overline{\alpha}\,\pi + 47100\,\overline{\alpha}\,\overline{\beta}^4\,\alpha + 227170\,\Phi_{11}\,\pi\,\overline{\beta}^3 - 932620\,\Phi_{11}\,\pi^2\,\overline{\beta}^2\\
&-2702400\,\pi^2\,\beta\,\overline{\beta}^3 - 1041760\,\pi^3\,\overline{\beta}^2\,\overline{\alpha} - 2977440\,\pi^3\,\overline{\beta}^2\,\overline{\pi} - 1064800\,\pi^2\,\overline{\beta}^3\,\overline{\alpha}\\
&-835840\,\alpha^2\,\pi\,\overline{\alpha}\,\overline{\beta}^2 - 6584360\,\Phi_{11}\,\overline{\beta}\,\alpha^2\,\pi + 74640\,\alpha^3\,\overline{\beta}\,\pi\,\overline{\alpha}\\
&-66480\,\alpha^3\,\overline{\beta}\,\overline{\pi}\,\pi + 75120\,\pi^2\,\overline{\beta}\,\alpha^2\,\overline{\alpha} - 70320\,\pi^2\,\alpha^2\,\overline{\beta}\,\overline{\pi} - 2193544\,\alpha^2\,\overline{\beta}^2\,\overline{\pi}\,\pi\\
&+7200\,\alpha^2\,\beta\,\pi\,\overline{\beta}^2 - 4461264\,\alpha\,\pi^2\,\overline{\beta}^2\,\overline{\pi} - 3128928\,\alpha\,\beta\,\pi\,\overline{\beta}^3\\
&-1237920\,\alpha\,\overline{\beta}^3\,\overline{\alpha}\,\pi + 161720\,\alpha\,\overline{\beta}^3\,\overline{\pi}\,\pi - 1662504\,\alpha\,\Phi_{11}\,\pi\,\overline{\beta}^2\\
&-1572528\,\alpha\,\pi^2\,\overline{\beta}^2\,\overline{\alpha} - 25\,\overline{\beta}^4\,\Phi_{11})\,/(\,-\overline{\beta} + 12\,\alpha + 22\,\pi\,)^2 \;=\; 0\,. \qquad (3.66)
\end{aligned}
$$

Solving this equation for $\overline{\delta}\alpha$ we get

$$
\begin{aligned}
\overline{\delta}\alpha := &-(227170\,\pi\,\overline{\beta}^3\,\Phi_{11} + 40200\,\pi\,\alpha^3\,\Phi_{11} + 176880\,\pi^2\,\overline{\beta}^3\,\overline{\pi} + 18720\,\overline{\beta}\,\alpha^4\,\overline{\alpha}\\
&-1467420\,\overline{\beta}\,\alpha^3\,\Phi_{11} - 5085800\,\overline{\beta}\,\pi^3\,\Phi_{11} + 85800\,\pi\,\overline{\beta}^4\,\overline{\alpha} - 1064800\,\pi^2\,\overline{\beta}^3\,\overline{\alpha}\\
&+8640\,\overline{\beta}^2\,\alpha^3\,\beta - 932620\,\pi^2\,\overline{\beta}^2\,\Phi_{11} - 15840\,\overline{\beta}\,\alpha^4\,\overline{\pi} + 205920\,\pi\,\overline{\beta}^4\,\beta
\end{aligned}
$$

$$-2977440\,\pi^3\overline{\beta}^2\,\overline{\pi} - 1041760\,\pi^3\overline{\alpha}\,\overline{\beta}^2 - 2702400\,\pi^2\beta\,\overline{\beta}^3 - 360060\,\alpha^2\overline{\beta}^3\,\overline{\alpha}$$
$$-351912\,\alpha^3\overline{\beta}^2\,\overline{\pi} - 155352\,\alpha^3\overline{\beta}^2\,\overline{\alpha} - 633396\,\alpha^2\overline{\beta}^2\,\Phi_{11} - 910512\,\alpha^2\beta\,\overline{\beta}^3$$
$$+35640\,\alpha^2\overline{\beta}^3\,\overline{\pi} + 47100\,\overline{\beta}^4\,\alpha\,\overline{\alpha} + 142635\,\overline{\beta}^3\,\alpha\,\Phi_{11} + 113040\,\overline{\beta}^4\,\alpha\,\beta$$
$$-3600\,\alpha^4\,\Phi_{11} - 1572528\,\pi^2\overline{\beta}^2\,\alpha\,\overline{\alpha} - 1662504\,\pi\,\overline{\beta}^2\,\alpha\,\Phi_{11} + 161720\,\pi\,\overline{\beta}^3\,\alpha\,\overline{\pi}$$
$$-835840\,\pi\,\overline{\beta}^2\,\alpha^2\,\overline{\alpha} - 1237920\,\pi\,\overline{\beta}^3\,\alpha\,\overline{\alpha} - 4461264\,\pi^2\alpha\,\overline{\beta}^2\,\overline{\pi}$$
$$-2193544\,\alpha^2\pi\,\overline{\beta}^2\,\overline{\pi} + 7200\,\overline{\beta}^2\,\alpha^2\,\pi\,\beta - 3128928\,\overline{\beta}^3\,\alpha\,\pi\,\beta$$
$$-9885300\,\overline{\beta}\,\pi^2\,\alpha\,\Phi_{11} + 75120\,\overline{\beta}\,\pi^2\,\alpha^2\,\overline{\alpha} - 70320\,\overline{\beta}\,\pi^2\,\overline{\pi}\,\alpha^2$$
$$-6584360\,\overline{\beta}\,\alpha^2\,\pi\,\Phi_{11} + 74640\,\overline{\beta}\,\alpha^3\,\pi\,\overline{\alpha} - 66480\,\overline{\beta}\,\alpha^3\,\overline{\pi}\,\pi + 48400\,\pi^3\,\alpha\,\Phi_{11}$$
$$+120000\,\alpha^2\,\pi^2\,\Phi_{11} - 25\,\overline{\beta}^4\,\Phi_{11}\Big)\Big/\big(-1760\,\overline{\beta}^2\,\Phi_{11} - 300\,\overline{\beta}^3\,\overline{\alpha} - 720\,\beta\,\overline{\beta}^3$$
$$+360\,\overline{\beta}^2\,\alpha\,\overline{\pi} - 240\,\overline{\beta}^2\,\overline{\alpha}\,\pi + 2400\,\beta\,\pi\,\overline{\beta}^2 + 600\,\overline{\alpha}\,\overline{\beta}^2\,\alpha + 2880\,\alpha\,\overline{\beta}^2\,\beta$$
$$+760\,\overline{\beta}^2\,\overline{\pi}\,\pi - 5280\,\alpha^2\,\overline{\beta}\,\overline{\pi} - 22160\,\overline{\beta}\,\overline{\pi}\,\alpha\,\pi + 25040\,\pi^2\,\overline{\alpha}\,\overline{\beta}$$
$$+8940\,\overline{\beta}\,\alpha\,\Phi_{11} + 9600\,\overline{\beta}\,\pi\,\Phi_{11} + 6240\,\overline{\alpha}\,\alpha^2\,\overline{\beta} - 23440\,\pi^2\,\overline{\beta}\,\overline{\pi}$$
$$+24880\,\alpha\,\pi\,\overline{\alpha}\,\overline{\beta} + 22400\,\pi^2\,\Phi_{11} - 1200\,\alpha^2\,\Phi_{11} + 8600\,\pi\,\alpha\,\Phi_{11}\big)\,, \tag{3.67}$$

where the denominator of (3.67),

$$\begin{aligned} d_3 \;:=\; & -1760\,\overline{\beta}^2\,\Phi_{11} - 300\,\overline{\beta}^3\,\overline{\alpha} - 720\,\beta\,\overline{\beta}^3 + 2880\,\alpha\,\overline{\beta}^2\,\beta \\ & +360\,\overline{\beta}^2\,\alpha\,\overline{\pi} - 240\,\overline{\beta}^2\,\overline{\alpha}\,\pi + 2400\,\beta\,\pi\,\overline{\beta}^2 + 600\,\overline{\alpha}\,\overline{\beta}^2\,\alpha \\ & +760\,\overline{\beta}^2\,\overline{\pi}\,\pi - 5280\,\alpha^2\,\overline{\beta}\,\overline{\pi} - 22160\,\overline{\beta}\,\overline{\pi}\,\alpha\,\pi + 25040\,\pi^2\,\overline{\alpha}\,\overline{\beta} \\ & +8940\,\overline{\beta}\,\alpha\,\Phi_{11} + 9600\,\overline{\beta}\,\pi\,\Phi_{11} + 6240\,\overline{\alpha}\,\alpha^2\,\overline{\beta} - 23440\,\pi^2\,\overline{\beta}\,\overline{\pi} \\ & +24880\,\alpha\,\pi\,\overline{\alpha}\,\overline{\beta} + 22400\,\pi^2\,\Phi_{11} - 1200\,\alpha^2\,\Phi_{11} + 8600\,\pi\,\alpha\,\Phi_{11}\,, \end{aligned} \tag{3.68}$$

is supposed to be non-zero for now.

Subtracting (3.63) from (3.67) and taking the numerator

$$N_1 := 684288\,\overline{\beta}^5\,\beta^2\,\alpha + 286000\,\pi^3\,\alpha\,\Phi_{11}^2$$
$$-7290900\,\pi^2\overline{\beta}^2\,\Phi_{11}^2 - 7913100\,\alpha^3\,\Phi_{11}^2\,\overline{\beta} + 165600\,\overline{\beta}^5\,\overline{\alpha}^2\,\alpha$$
$$-655680\,\pi^3\overline{\beta}^3\,\overline{\alpha}^2 + 295488\,\alpha^3\overline{\beta}^3\,\overline{\pi}^2 - 205632\,\alpha^3\overline{\beta}^3\,\overline{\alpha}^2 - 582090\,\pi\,\overline{\beta}^3\,\Phi_{11}^2$$
$$-1157435\,\overline{\beta}^2\,\Phi_{11}^2\,\alpha^2 - 1517760\,\pi^2\overline{\beta}^4\,\overline{\alpha}^2 - 447840\,\overline{\beta}^4\,\overline{\alpha}^2\,\alpha^2$$
$$+299000\,\pi^2\alpha^2\,\Phi_{11}^2 + 123540\,\overline{\beta}^5\,\Phi_{11}\,\beta + 303600\,\pi\,\overline{\beta}^5\,ac^2 + 51450\,\overline{\beta}^5\,\overline{\alpha}\,\Phi_{11}$$
$$-126720\,\pi^2\overline{\beta}^4\,\overline{\pi}^2 - 30643000\,\pi^3\overline{\beta}\,\Phi_{11}^2 - 23040\,\overline{\beta}^4\,\overline{\pi}^2\,\alpha^2 + 78000\,\pi\,\alpha^3\,\Phi_{11}^2$$
$$+3111840\,\pi^3\overline{\beta}^3\,\overline{\pi}^2 - 361718\,\overline{\beta}^3\,\Phi_{11}^2\,\alpha + 25\,\overline{\beta}^5\,\overline{\pi}\,\Phi_{11} + 1296000\,\pi\,\overline{\beta}^5\,\beta^2$$
$$-14398260\,\pi\,\overline{\beta}^2\,\Phi_{11}\,\overline{\alpha}\,\alpha^2 - 57600\,\overline{\beta}^5\,\overline{\alpha}\,\alpha\,\overline{\pi} + 781920\,\overline{\beta}^4\,\overline{\alpha}\,\alpha^2\,\overline{\pi}$$
$$+89856\,\alpha^3\overline{\beta}^3\,\overline{\alpha}\,\overline{\pi} + 301945\,\overline{\beta}^4\,\Phi_{11}^2 - 138240\,\overline{\beta}^5\,\beta\,\alpha\,\overline{\pi}$$

$$
\begin{aligned}
&+1886976\,\overline{\beta}^4\,\beta\,\alpha^2\,\overline{\pi} - 1119744\,\overline{\beta}^4\,\beta\,\overline{\alpha}\,\alpha^2 + 682560\,\overline{\beta}^5\,\overline{\alpha}\,\alpha\,\beta \\
&+972820\,\overline{\beta}^4\,\overline{\alpha}\,\alpha\,\Phi_{11} - 3755902\,\overline{\beta}^3\,\overline{\alpha}\,\alpha^2\,\Phi_{11} - 266205\,\overline{\beta}^4\,\Phi_{11}\,\alpha\,\overline{\pi} \\
&+1749108\,\overline{\beta}^4\,\Phi_{11}\,\alpha\,\beta + 3627158\,\overline{\beta}^3\,\Phi_{11}\,\alpha^2\,\overline{\pi} - 3766320\,\overline{\beta}^3\,\beta\,\alpha^2\,\Phi_{11} \\
&+599880\,\alpha^3\,\Phi_{11}\,\overline{\beta}^2\,\overline{\pi} - 3021720\,\alpha^3\,\Phi_{11}\,\overline{\alpha}\,\overline{\beta}^2 + 6871680\,\pi^2\,\overline{\beta}^4\,\overline{\pi}\,\beta \\
&-14919600\,\pi^2\,\overline{\beta}^3\,\Phi_{11}\,\beta - 13552520\,\pi^2\,\overline{\beta}^3\,\Phi_{11}\,\overline{\alpha} - 1389792\,\pi^2\,\overline{\beta}^3\,\overline{\alpha}^2\,\alpha \\
&+4357632\,\pi^2\,\overline{\beta}^3\,\overline{\pi}^2\,\alpha + 2915280\,\pi^2\,\overline{\beta}^4\,\overline{\alpha}\,\overline{\pi} + 13578100\,\pi^2\,\overline{\beta}^3\,\Phi_{11}\,\overline{\pi} \\
&-3718080\,\pi^2\,\overline{\beta}^4\,\overline{\alpha}\,\beta - 60662200\,\pi^2\,\overline{\beta}\,\Phi_{11}^2\,\alpha - 713952\,\pi^2\,\overline{\beta}^3\,\overline{\pi}\,\alpha\,\overline{\alpha} \\
&-8470120\,\pi^2\,\overline{\beta}^2\,\Phi_{11}\,\overline{\pi}\,\alpha - 22343840\,\pi^2\,\overline{\beta}^2\,\Phi_{11}\,\alpha\,\overline{\alpha} + 46000\,\pi^2\,\alpha^2\,\Phi_{11}\,\overline{\alpha}\,\overline{\beta} \\
&-64400\,\pi^2\,\alpha^2\,\Phi_{11}\,\overline{\beta}\,\overline{\pi} - 26400\,\pi^2\,\alpha\,\Phi_{11}\,\beta\,\overline{\beta}^2 - 105600\,\pi\,\overline{\beta}^5\,\overline{\alpha}\,\overline{\pi} \\
&+1806460\,\pi\,\overline{\beta}^4\,\Phi_{11}\,\overline{\alpha} - 38468250\,\pi\,\overline{\beta}\,\Phi_{11}^2\,\alpha^2 - 111360\,\pi\,\overline{\beta}^4\,\overline{\pi}^2\,\alpha \\
&+3379920\,\pi\,\overline{\beta}^4\,\Phi_{11}\,\beta - 474200\,\pi\,\overline{\beta}^4\,\Phi_{11}\,\overline{\pi} - 253440\,\pi\,\overline{\beta}^5\,\overline{\pi}\,\beta \\
&+1268640\,\pi\,\overline{\beta}^5\,\overline{\alpha}\,\beta - 1647600\,\pi\,\overline{\beta}^4\,\overline{\alpha}^2\,\alpha - 6010420\,\pi\,\overline{\beta}^2\,\Phi_{11}^2\,\alpha \\
&-939744\,\pi\,\overline{\beta}^3\,\overline{\alpha}^2\,\alpha^2 + 1993248\,\pi\,\overline{\beta}^3\,\overline{\pi}^2\,\alpha^2 + 3021840\,\pi\,\overline{\beta}^4\,\overline{\alpha}\,\overline{\pi}\,\alpha \\
&+51264\,\pi\,\overline{\beta}^3\,\overline{\pi}\,\overline{\alpha}\,\alpha^2 + 14037554\,\pi\,\overline{\beta}^3\,\Phi_{11}\,\alpha\,\overline{\pi} - 14255872\,\pi\,\overline{\beta}^3\,\Phi_{11}\,\overline{\alpha}\,\alpha \\
&-15000840\,\pi\,\overline{\beta}^3\,\Phi_{11}\,\alpha\,\beta - 592500\,\pi\,\overline{\beta}^2\,\Phi_{11}\,\alpha^2\,\overline{\pi} - 4074624\,\pi\,\overline{\beta}^4\,\overline{\alpha}\,\alpha\,\beta \\
&+7198848\,\pi\,\overline{\beta}^4\,\overline{\pi}\,\alpha\,\beta - 14400\,\pi\,\alpha^2\,\Phi_{11}\,\beta\,\overline{\beta}^2 - 16800\,\pi\,\alpha^3\,\Phi_{11}\,\overline{\beta}\,\overline{\pi} \\
&+12000\,\pi\,\alpha^3\,\Phi_{11}\,\overline{\alpha}\,\overline{\beta} - 9855600\,\pi^3\,\overline{\beta}^2\,\Phi_{11}\,\overline{\pi} - 11178800\,\pi^3\,\overline{\beta}^2\,\Phi_{11}\,\overline{\alpha} \\
&-929760\,\pi^3\,\overline{\beta}^3\,\overline{\pi}\,\overline{\alpha} + 44000\,\pi^3\,\alpha\,\Phi_{11}\,\overline{\alpha}\,\overline{\beta} - 61600\,\pi^3\,\alpha\,\Phi_{11}\,\overline{\beta}\,\overline{\pi} = 0\,. \qquad (3.69)
\end{aligned}
$$

At this point we suppose that $\Phi_{11} = 0$. In this case, N_1 surprisingly factors in the following form :

$$N_1 := -12\overline{\beta}\,p_1\,p_2\,, \qquad (3.70)$$

where

$$p_1 := 12\,\beta\,\overline{\beta} + 2\,\pi\,\overline{\alpha} + 2\,\alpha\,\overline{\alpha} + 5\,\overline{\beta}\,\overline{\alpha} + 6\,\overline{\pi}\,\pi + 2\,\overline{\pi}\,\alpha\,, \qquad (3.71)$$

$$
\begin{aligned}
p_2 \;:=\; & 1188\,\overline{\beta}\,\beta\,\alpha + 240\,\alpha\,\overline{\beta}\,\overline{\pi} + 440\,\overline{\beta}\,\overline{\pi}\,\pi - 1265\,\overline{\beta}\,\pi\,\overline{\alpha} \\
& -2250\,\overline{\beta}\,\beta\,\pi + 6830\,\pi^2\,\overline{\alpha} + 7647\,\alpha\,\pi\,\overline{\alpha} + 2142\,\alpha^2\,\overline{\alpha} \\
& -690\,\overline{\beta}\,\alpha\,\overline{\alpha} - 10805\,\overline{\pi}\,\pi^2 - 11529\,\alpha\,\overline{\pi}\,\pi - 3078\,\overline{\pi}\,\alpha^2\,. \qquad (3.72)
\end{aligned}
$$

Let us consider first the case in which $p_2 = 0$. Applying $\overline{\delta}$ to (3.72) and solving for $\overline{\delta}\alpha$, we obtain:

$$
\begin{aligned}
\overline{\delta}\alpha := -(&690120\,\overline{\beta}\,\beta\,\alpha^3 - 177100\,\overline{\beta}^3\,\overline{\alpha}\,\pi + 2475000\,\overline{\beta}\,\alpha\,\pi^2\,\beta - 97175\,\overline{\alpha}\,\overline{\beta}^3\,\alpha \\
&-186390\,\alpha\,\overline{\beta}^3\,\beta + 1716210\,\overline{\alpha}\,\alpha^2\,\overline{\beta}^2 - 1131915\,\alpha^2\,\overline{\beta}^2\,\overline{\pi} - 4470219\,\overline{\beta}\,\alpha^3\,\overline{\alpha}
\end{aligned}
$$

$$-3875990\,\pi^2\overline{\beta}^2\,\overline{\pi}-341280\,\beta\,\pi\,\overline{\beta}^3+5791820\,\pi^2\,\overline{\alpha}\,\overline{\beta}^2+9784170\,\alpha^3\,\overline{\pi}\,\pi$$
$$+2573586\,\alpha^2\,\beta\,\overline{\beta}^2+8639361\,\alpha^3\,\overline{\beta}\,\overline{\pi}+8903160\,\pi^2\,\beta\,\overline{\beta}^2-28683640\,\pi^3\,\overline{\beta}\,\overline{\alpha}$$
$$-11970480\,\pi^2\,\alpha^2\,\overline{\alpha}+18687060\,p^2\,\overline{\pi}\,\alpha^2-6346350\,\alpha^3\,\pi\,\overline{\alpha}+9567000\,\beta\,\pi\,\alpha\,\overline{\beta}^2$$
$$+2615220\,\overline{\beta}\,\beta\,\pi\,\alpha^2+33800\,\overline{\beta}^3\,\alpha\,\overline{\pi}+61600\,\overline{\beta}^3\,\overline{\pi}\,\pi+90203190\,\pi^2\,\alpha\,\overline{\beta}\,\overline{\pi}$$
$$-1119420\,\alpha^4\,\overline{\alpha}-4188240\,\overline{\beta}^2\,\overline{\pi}\,\alpha\,\pi+6302070\,\alpha\,\overline{\beta}^2\,\pi\,\overline{\alpha}$$
$$-24866544\,\overline{\beta}\,\alpha^2\,\pi\,\overline{\alpha}-46210320\,\overline{\beta}\,\alpha\,\pi^2\,\overline{\alpha}+48333948\,\alpha^2\,\overline{\beta}\,\overline{\pi}\,\pi$$
$$+56154880\,\pi^3\,\overline{\beta}\,\overline{\pi}+1705860\,\alpha^4\,\overline{\pi}-7513000\,\pi^3\,\alpha\,\overline{\alpha}+11885500\,\overline{\pi}\,\pi^3\,\alpha\Big)\Big/$$
$$\Big((-12\alpha-22\pi+bc)(115\overline{\beta}\overline{\alpha}+126\beta\overline{\beta}-40\overline{\beta}\overline{\pi}+783\overline{\pi}\alpha-1634\pi\overline{\alpha}$$
$$+1448\pi\overline{\pi}-921\alpha\overline{\alpha})\Big)\,, \tag{3.73}$$

where the denominator of the expression above, given by

$$\begin{aligned} d_4 \;:=\; & (-12\alpha-22\pi+\overline{\beta})(115\overline{\beta}\overline{\alpha}+126\beta\overline{\beta}-40\overline{\beta}\overline{\pi}+783\overline{\pi}\alpha-1634\pi\overline{\alpha} \\ & +1448\pi\overline{\pi}-921\alpha\overline{\alpha})\,, \end{aligned} \tag{3.74}$$

is assumed to be non-zero, for now.

Here N_1, and all equations obtained by comparing different expressions for $\bar{\delta}\overline{\alpha}$, are polynomials in three complex variables α, β and π. One complex variable can be eliminated by introducing the following new variables:

$$x_1 := \frac{\alpha}{\pi}\,, \qquad x_2 := \frac{\beta}{\overline{\pi}}\,. \tag{3.75}$$

In what follows we first prove that the necessary conditions imply that both x_1 and x_2 are constants. Then, later, we shall prove that this leads to a contradiction.

In the new variables defined by (3.75), (3.72) assumes the form

$$\begin{aligned} p_2 = & -2250\,\overline{x_2}\,x_2-1188\,\overline{x_2}\,x_2\,x_1-3078\,x_1^2-11529\,x_1-10805 \\ & +7647\,x_1\,\overline{x_1}+6830\,+2142\,x_1^2\,\overline{x_1}\overline{x_1}-1265\,\overline{x_2}\,\overline{x_1}+440\,\overline{x_2}-690\,\overline{x_2}\,x_1\,\overline{x_1} \\ & +240\,x_1\,\overline{x_2}+2142\,x_1^2\,\overline{x_1}=0\,. \end{aligned} \tag{3.76}$$

Subtracting (3.73) from (3.67) (with $\Phi_{11}=0$), and taking the numerator, gives

$$\begin{aligned} N_2 := & -75130000\,\overline{x_1}^2\,x_1-568034312\,\overline{x_2}\,x_1\,\overline{x_1}^2-162662000\,x_1^2\,\overline{x_1}^2 \\ & -263829680\,\overline{x_2}\,\overline{x_1}^2+69828000\,\overline{x_1}\,\overline{x_2}\,x_2\,x_1+91299060\,\overline{x_2}^2 \\ & -105963000\,\overline{x_2}\,x_2\,x_1+328900\,\overline{x_2}^4\,\overline{x_1}^2-3248115\,\overline{x_2}^3-97977600\,x_1^4 \\ & -12927600\,x_1^5-278399100\,x_1^3+37400\,\overline{x_2}^4-16070400\,x_1^4\,\overline{x_2}\,x_2 \\ & -277285752\,x_1^2\,\overline{x_2}^2\,x_2+6646212\,x_1^2\,\overline{x_2}^3\,x_2+78647544\,x_1^4\,\overline{x_2}\,\overline{x_1} \end{aligned}$$

$$
\begin{aligned}
&-48437136\, x_1^3 \overline{x_2}^2 x_2 + 15461670\, x_1 \overline{x_1}\, \overline{x_2}^3 - 39517398\, x_1^3 \overline{x_2}^2 \overline{x_1} \\
&+12081744\, \overline{x_1} x_1^3 \overline{x_2}^2 x_2 + 593329572\, x_1^3 \overline{x_2}\, \overline{x_1} + 4160700\, x_1^2 \overline{x_2}^3 \overline{x_1} \\
&-221403987\, x_1^2 \overline{x_2}^2 \overline{x_1} - 42924720\, \overline{x_1}\, \overline{x_2}^3 x_2 - 459117172\, \overline{x_1}^2 \overline{x_2} x_1^2 \\
&+1679968716\, \overline{x_1}\, \overline{x_2} x_1^2 + 112691520\, \overline{x_1}\, \overline{x_2}^2 x_2 \\
&+163297776\, \overline{x_1}\, \overline{x_2}^2 x_2 x_1 - 413617894\, \overline{x_1}\, \overline{x_2}^2 x_1 + 125141836\, \overline{x_1}^2 \overline{x_2}^2 x_1 \\
&+14366310\, \overline{x_1}\, \overline{x_2}^3 - 169509600\, x_1^2 \overline{x_2} x_2 - 15940800\, x_1^2 \overline{x_2}^2 x_2^2 \\
&+114001200\, x_1^2 \overline{x_2} x_2 \overline{x_1} + 62078400\, x_1^3 \overline{x_2} x_2 \overline{x_1} \\
&+11275200\, x_1^4 \overline{x_2} x_2 \overline{x_1} - 45650088\, x_1 \overline{x_1}\, \overline{x_2}^3 x_2 - 213660\, \overline{x_2}^4 x_1 x_2 \\
&+77697000\, \overline{x_1} x_1^2 \overline{x_2}^2 x_2 - 12134448\, \overline{x_1} x_1^2 \overline{x_2}^3 x_2 \\
&-165106878\, \overline{x_1}^2 x_1^3 \overline{x_2} + 66914052\, \overline{x_1}^2 x_1^2 \overline{x_2}^2 + 383646000\, x_1^3 \overline{x_1} \\
&+136587600\, x_1^4 \overline{x_1} - 1405233644\, x_1^2 \overline{x_2} - 495836310\, x_1^3 \overline{x_2} \\
&+78021681\, x_1^2 \overline{x_2}^2 - 257647540\, \overline{x_1}\, \overline{x_2}^2 - 44763192\, x_2^2 \overline{x_2}^3 x_1 \\
&-14850000\, x_2^2 x_1 \overline{x_2}^2 - 121050\, \overline{x_2}^4 x_1 \overline{x_1} - 11901168\, x_1^2 \overline{x_2}^3 x_2^2 \\
&-4276800\, x_1^3 \overline{x_2}^2 x_2^2 - 22290588\, \overline{x_1}^2 x_1^4 \overline{x_2} + 11929896\, \overline{x_1}^2 x_1^3 \overline{x_2}^2 \\
&-3427140\, \overline{x_1}^2 x_1^2 \overline{x_2}^3 - 65630268\, x_1^4 \overline{x_2} + 13886406\, x_1^3 \overline{x_2}^2 \\
&-939210\, x_1^2 \overline{x_2}^3 - 11744220\, \overline{x_1}^2 \overline{x_2}^3 + 78035680\, \overline{x_1}^2 \overline{x_2}^2 + 20400\, \overline{x_2}^4 x_1 \\
&+1533600\, x_2^2 \overline{x_2}^4 + 146164809\, \overline{x_2}^2 x_1 + 750960\, x_1 \overline{x_2}^4 x_2 \overline{x_1} \\
&+179400\, \overline{x_1}^2 \overline{x_2}^4 x_1 - 90396000\, x_1^3 \overline{x_2} x_2 - 12687610\, x_1 \overline{x_1}^2 \overline{x_2}^3 \\
&+25076106\, \overline{x_2}^3 x_1 x_2 - 528822552\, \overline{x_2}^2 x_1 x_2 - 3492935\, \overline{x_2}^3 x_1 \\
&+23644920\, \overline{x_2}^3 x_2 - 397830\, \overline{x_2}^4 x_2 - 336008520\, \overline{x_2}^2 x_2 - 6426000\, \overline{x_1}^2 x_1^5 \\
&-131992500\, x_1^3 \overline{x_1}^2 + 18230400\, x_1^5 \overline{x_1} + 824256\, x_1 \overline{x_2}^4 x_2^2 \\
&+1395480\, \overline{x_2}^4 \overline{x_1} x_2 - 221925\, \overline{x_2}^4 \overline{x_1} - 47574000\, x_1^4 \overline{x_1}^2 - 166397000\, x_1 \\
&-42109200\, x_2^2 \overline{x_2}^3 - 836819360\, \overline{x_2} - 1770562898\, x_1 \overline{x_2} + 224037000\, x_1 \overline{x_1} \\
&+478803300\, x_1^2 \overline{x_1} - 351507100\, x_1^2 + 999889240\, \overline{x_2}\, \overline{x_1} \\
&+2115717928\, \overline{x_2} x_1 \overline{x_1} = 0\,.
\end{aligned}
\tag{3.77}
$$

We now apply the Maple procedure gsolve to the system of polynomial equations $\{p_2 = 0,\ N_2 = 0\}$. In the algorithm, the variables x_1, x_2 and their complex conjugates, $\overline{x_1}$ and $\overline{x_2}$, are treated as independent variables. In the subsequent analysis we use the fact that they are complex conjugates of each other.

In the present case, despite the size of one the equations, the set of solutions was promptly obtained and is given as follows:

$$
G_1 := [\,-8 + 23\,\overline{x_1}, 11\,\overline{x_2} + 8, 66\,x_1 + 125\,]\,, \tag{3.78}
$$

$$G_2 := [\,9108\,x_2\,\overline{x_2} + 247, -8 + 23\,\overline{x_1}, 66\,x_1 + 125\,]\,, \tag{3.79}$$

$$G_3 := [\,828\,x_2\,\overline{x_2} + 75\,\overline{x_2} + 77, -8 + 23\,\overline{x_1}, 66\,x_1 + 125\,]\,, \tag{3.80}$$

$$G_4 := [\,271\,x_1 + 138\,x_2\,\overline{x_2} + 517, -8 + 23\,\overline{x_1}\,]\,, \tag{3.81}$$

$$G_5 := [\,36\,x_2\,\overline{x_2} + 7 - 5\,\overline{x_1}, 6\,x_1 + 11\,]\,, \tag{3.82}$$

$$\begin{aligned}
G_6 := [&671514624\,x_2^2\,\overline{x_2}^2 + 488374272\,x_2^2\,\overline{x_2} - 220446720\,\overline{x_1}\,x_2\\
&+35785728\,\overline{x_2}^2\,x_2 + 88473600\,x_2\,\overline{x_2} - 27979776\,x_1\,x_2 + 69101568\,x_2\\
&-167878656\,\overline{x_2}^2\,\overline{x_1}^2 + 181020672\,\overline{x_1}^2\,\overline{x_2} - 26599040\,\overline{x_1}^2\\
&+73852416\,\overline{x_2}^2\,\overline{x_1} - 132857600\,\overline{x_1}\,\overline{x_2} + 23168456\,\overline{x_1} - 26978094\,x_1\\
&-49722705 - 2204136\,x_1^2 + 48043776\,x_1\,\overline{x_2} + 111324800\,\overline{x_2} - 7645440\,\overline{x_2}^2,\\
&847872\,\overline{x_1}\,\overline{x_2}\,x_2 - 294912\,x_2\,\overline{x_2} + 423936\,\overline{x_1}^2\,\overline{x_2} - 382720\,\overline{x_1}^2\\
&-218112\,\overline{x_1}\,\overline{x_2} + 323928\,\overline{x_1} - 54450\,x_1 - 169493 + 24576\,\overline{x_2},\\
&139392\,\overline{x_2}^3\,x_2 + 202752\,\overline{x_2}^2\,x_2 + 73728\,x_2\,\overline{x_2} + 69696\,\overline{x_2}^3\,\overline{x_1}\\
&+38456\,\overline{x_2}^2\,\overline{x_1} - 54656\,\overline{x_1}\,\overline{x_2} - 33280\,\overline{x_1} - 4224\,x_1 + 10432 - 11616\,x_1\,\overline{x_2}\\
&+22544\,\overline{x_2} + 2827\,\overline{x_2}^2 - 11616\,\overline{x_2}^3 - 7986\,\overline{x_2}^2\,x_1, 304128\,x_1\,x_2\,\overline{x_2}\\
&+576000\,x_2\,\overline{x_2} + 66240\,\overline{x_1}\,\overline{x_2} - 59800\,\overline{x_1} - 191598\,x_1 - 279575 - 17424\,x_1^2\\
&+198000\,x_1\,\overline{x_2} + 351960\,\overline{x_2}, 192\,x_1\,\overline{x_1} - 282\,x_1 - 505 + 280\,\overline{x_1}]\,.
\end{aligned} \tag{3.83}$$

Using the fact that the pairs (x_1, x_2) and $(\overline{x_1}, \overline{x_2})$ are complex conjugates of each other, we conclude that the sets G_1 to G_5 provide solutions which are either impossible or imply that x_1 is constant. Only G_6 is not immediately trivial. Its smallest term is:

$$192\,x_1\,\overline{x_1} - 282\,x_1 - 505 + 280\,\overline{x_1} = 0\,. \tag{3.84}$$

Subtracting (3.84) from its complex conjugate we obtain the conclusion that x_1 is real, which implies that it must be constant. Let us consider now the case

$$p_1 = 12x_2\overline{x_2} + 6 + 2x_1 + 2\overline{x_1} + 2x_1\overline{x_1} + 5\overline{x_1}\overline{x_2} = 0\,. \tag{3.85}$$

We now use the side relation S_1 given by (3.61). Its numerator takes the form:

$$\begin{aligned}
num(S_1) := &-6\,x_2^2\,\overline{x_2}^2 + 1210\,x_2\,x_1 + 1276\,x_1 + 1276\,\overline{x_1} + 360\,x_2\,x_1^2\,\overline{x_1}\\
&+2901\,x_2\,\overline{x_2} + 1528\,x_1\,x_2\,\overline{x_2} + 660\,x_1\,\overline{x_2}\,\overline{x_1} + 660\,x_2\,x_1^2 + 408\,x_1\,\overline{x_1}^2\\
&+660\,\overline{x_1}^2\,\overline{x_2} + 408\,\overline{x_1}\,x_1^2 + 264\,\overline{x_1}^2 + 144\,x_1^2\,\overline{x_1}^2 + 1452 + 264\,x_1^2\\
&+1210\,\overline{x_1}\,\overline{x_2} + 1444\,\overline{x_1}\,x_1 + 1528\,x_2\,\overline{x_1}\,\overline{x_2} + 660\,x_2\,\overline{x_1}\,x_1\\
&+803\,x_2\,x_1\,\overline{x_2}\,\overline{x_1} + 360\,x_1\,\overline{x_2}\,\overline{x_1}^2\,.
\end{aligned} \tag{3.86}$$

Applying gsolve to p_1 and $num(S_1)$ we obtain the following system of equations:

$$s_1 := 6x_2\overline{x_2} + 31x_1\overline{x_1} + 56x_1 + \overline{x_1} + 3 = 0, \tag{3.87}$$

$$s_2 := 72x_2x_1\overline{x_1} + 132x_1x_2 + 31x_1\overline{x_1}^2 + 56\overline{x_1}x_1 + \overline{x_1}^2 + 3\overline{x_1} = 0\,, \tag{3.88}$$

$$s_3 := \overline{x_1}\overline{x_2} - 22x_1 - 12\overline{x_1}x_1 = 0\,. \tag{3.89}$$

Subtracting (3.87) from its complex conjugate yields $\overline{x_1} = x_1$. Subtracting (3.89) from its complex conjugate now gives $x_2 = \overline{x_2} = 12x_1 + 22$. Substituting these relations back in (3.87) and (3.88) results in a system with no solution.

Let us consider now the cases where each of the denominators d_2, d_3 and d_4, that appeared in the preceeding equations are zero.

(i) $\mathbf{d_2 = 0}$

In this case, from (3.64) and (3.75), we have

$$d_2 := -12\,\overline{x_2}\,x_2 - 6\,\overline{x_2}\,\overline{x_1} - 14\,x_1 + 10\,x_1\,\overline{x_1} + \overline{x_2} + 20\,\overline{x_1} - 28 = 0. \tag{3.90}$$

This implies that the numerator of (3.63) is also zero:

$$\begin{aligned} n_2 := {} & -1280\,\overline{x_2}\,\overline{x_1} + 175\,\overline{x_2}^2\,x_1 + 628\,x_1^2\,\overline{x_2}\,\overline{x_1} + 150\,x_1^3\,\overline{x_1} \\ & -210\,x_1^3 + 3520\,\overline{x_2}^2\,\overline{x_1} - 3180\,\overline{x_2} + 8160\,x_2\,\overline{x_2}^2 + 1950\,\overline{x_1}\,\overline{x_2}^2\,x_1 \\ & -1307\,\overline{x_2}\,x_1^2 + 4668\,x_1\,\overline{x_2}^2\,x_2 + 330\,\overline{x_2}^2 + 300\,x_1^2\,\overline{x_1} - 420\,x_1^2 \\ & +588\,x_1\,\overline{x_1}\,\overline{x_2} - 4116\,\overline{x_2}\,x_1 - 180\,x_1^2\,x_2\,\overline{x_2} = 0\,. \end{aligned} \tag{3.91}$$

Applying gsolve to these two equations we obtain the following equivalent set of equations:

$$1152\overline{x_2}x_2 + 576\overline{x_1}\overline{x_2} - 520\overline{x_1} + 163 - 66x_1 - 96\overline{x_2} = 0\,, \tag{3.92}$$

$$280\overline{x_1} - 505 + 192x_1\overline{x_1} - 282x_1 = 0\,. \tag{3.93}$$

From the real part of (3.93) we obtain that $\overline{x_1} = x_1$ and thus

$$192x_1^2 - 2x_1 - 505 = 0\,. \tag{3.94}$$

Applying gsolve to (3.94) and (3.92) we find that $\overline{x_2} = x_2$, which implies that $x_2 = const.$, $x_1 = const.$

(ii) $\mathbf{d_3 = 0}$

Here, from (3.68) and (3.75) we have

$$\begin{aligned} d_3 := {} & -15\,\overline{x_2}^2\,\overline{x_1} - 36\,x_2\,\overline{x_2}^2 + 18\,x_1\,\overline{x_2} - 12\,\overline{x_1}\,\overline{x_2} + 120\,x_2\,\overline{x_2} + 30\,\overline{x_2}\,x_1\,\overline{x_1} \\ & +144\,x_1\,x_2\,\overline{x_2} + 38\,\overline{x_2} - 264\,x_1^2 - 1108\,x_1 + 1252\,\overline{x_1} + 312\,x_1^2\,\overline{x_1} - 1172 \\ & +1244\,x_1\,\overline{x_1} = 0\,. \end{aligned} \tag{3.95}$$

The numerator of (3.67) must also be zero, i.e.,

$$
\begin{aligned}
n_3 := {} & 51480\,\overline{x_2}^3\,x_2 - 1115316\,x_1\,\overline{x_2} - 548386\,x_1^2\,\overline{x_2} + 40430\,\overline{x_2}^2\,x_1 \\
& +8910\,x_1^2\,\overline{x_2}^2 - 87978\,x_1^3\,\overline{x_2} + 18660\,x_1^3\,\overline{x_1} - 675600\,x_2\,\overline{x_2}^2 \\
& -260440\,\overline{x_1}\,\overline{x_2} + 18780\,x_1^2\,\overline{x_1} - 266200\,\overline{x_2}^2\,\overline{x_1} + 21450\,\overline{x_2}^3\,\overline{x_1} \\
& +4680\,x_1^4\,\overline{x_1} - 90015\,x_1^2\,\overline{x_2}^2\,\overline{x_1} - 38838\,x_1^3\,\overline{x_2}\,\overline{x_1} + 2160\,\overline{x_2}\,x_1^3\,x_2 \\
& +1800\,\overline{x_2}\,x_1^2\,x_2 - 309480\,\overline{x_2}^2\,x_1\,\overline{x_1} - 208960\,\overline{x_2}\,x_1^2\,\overline{x_1} \\
& -393132\,\overline{x_2}\,x_1\,\overline{x_1} + 44220\,\overline{x_2}^2 - 17580\,x_1^2 - 16620\,x_1^3 - 3960\,x_1^4 \\
& -744360\,\overline{x_2} + 11775\,\overline{x_2}^3\,x_1\,\overline{x_1} - 782232\,\overline{x_2}^2\,x_1\,x_2 + 28260\,\overline{x_2}^3\,x_1\,x_2 \\
& -227628\,x_1^2\,x_2\,\overline{x_2}^2 = 0\,.
\end{aligned}
\tag{3.96}
$$

Applying `gsolve` to d_3 and n_3 given above we obtain the following sets of solutions

$$
\begin{aligned}
g_1 \;:=\; & [36\,x_2\,\overline{x_2}^2 + 216\,x_2\,\overline{x_2} + 15\,\overline{x_2}^2\,\overline{x_1} + 82\,\overline{x_1}\,\overline{x_2} - 48\,\overline{x_1} + 4\,\overline{x_2} + 24, \\
& 7 + 3\,x_1] \qquad (3.97) \\
g_2 \;:=\; & [232848\,x_2\,\overline{x_1} + 13860\,x_2\,\overline{x_2} - 232848\,x_2 + 80443\,\overline{x_1}^2 + 13872\,\overline{x_1} \\
& +8400\,\overline{x_2} - 94315, 1452\,x_2\,\overline{x_2}^2 + 29252\,\overline{x_1} + 880\,\overline{x_2}^2 + 3343\,\overline{x_2} - 29252, \\
& -16\,\overline{x_2} + 11\,\overline{x_1}\,\overline{x_2} - 84\,\overline{x_1} + 84, 5 + 6\,x_1]\,, \qquad (3.98) \\
g_3 \;:=\; & [49680\,\overline{x_2}\,\overline{x_1}\,x_2 + 1867536\,\overline{x_1}\,x_1^2\,x_2 + 6305904\,x_1\,\overline{x_1}\,x_2 + 5454900\,x_2\,\overline{x_1} \\
& -17280\,x_2\,\overline{x_2} - 2081376\,x_1^2\,x_2 - 7018704\,x_1\,x_2 - 5900400\,x_2 + 20700\,\overline{x_2}\,\overline{x_1}^2 \\
& +666990\,\overline{x_1}^2\,x_1^2 + 2171355\,\overline{x_1}^2\,x_1 + 1836400\,\overline{x_1}^2 - 1267005\,x_1\,\overline{x_1} \\
& -7200\,\overline{x_1}\,\overline{x_2} - 471870\,x_1^2\,\overline{x_1} - 782825\,\overline{x_1} - 1349040\,x_1 - 318240\,x_1^2 \\
& -1401200, -2831220\,x_1^3\,\overline{x_1} - 12488256\,x_1^3\,x_2 + 4001940\,x_1^3\,\overline{x_1}^2 \\
& -1909440\,x_1^3 + 11205216\,x_1^3\,x_2\,\overline{x_1} - 11594880\,x_1^2 - 13346640\,x_1^2\,\overline{x_1} \\
& +20750580\,\overline{x_1}^2\,x_1^2 - 65764224\,x_1^2\,x_2 + 59057424\,\overline{x_1}\,x_1^2\,x_2 \\
& +104387400\,x_1\,\overline{x_1}\,x_2 - 20709225\,x_1\,\overline{x_1} + 36279765\,\overline{x_1}^2\,x_1 - 23246640\,x_1 \\
& -115160400\,x_1\,x_2 + 61987500\,x_2\,\overline{x_1} - 15413200 - 67050000\,x_2 \\
& -10555975\,\overline{x_1} + 21429800\,\overline{x_1}^2, 46332\,x_2\,\overline{x_2}^2 + 196560\,x_2\,\overline{x_2} \\
& +19305\,\overline{x_2}^2\,\overline{x_1} + 96849\,\overline{x_1}\,\overline{x_2} + 199728\,x_1^2 - 179208\,x_1^2\,\overline{x_1} \\
& +24354\,x_1\,\overline{x_2} + 1133088\,x_1 - 914928\,x_1\,\overline{x_1} + 38214\,\overline{x_2} - 1051354\,\overline{x_1} \\
& +1435744, 46332\,x_1\,x_2\,\overline{x_2} + 87750\,x_2\,\overline{x_2} + 4140\,\overline{x_1}\,\overline{x_2} - 121734\,x_1^2 \\
& +133398\,x_1^2\,\overline{x_1} + 23760\,x_1\,\overline{x_2} - 365673\,x_1 + 434271\,x_1\,\overline{x_1} + 43560\,\overline{x_2} \\
& +367280\,\overline{x_1} - 264005, 10512\,x_1^2 - 9432\,x_1^2\,\overline{x_1} + 1170\,\overline{x_2}\,x_1\,\overline{x_1} \\
& -1440\,x_1\,\overline{x_2} + 35448\,x_1 - 31848\,x_1\,\overline{x_1} - 2640\,\overline{x_2} + 1965\,\overline{x_1}\,\overline{x_2} - 27550\,\overline{x_1}
\end{aligned}
$$

$$+29800] \,. \tag{3.99}$$

The only set whose incompatibility is not evident is the third one, g_3. The system is given by

$$\begin{aligned}
F_1 \;:=\;& 49680\,\overline{x_2}\,\overline{x_1}\,x_2 + 1867536\,\overline{x_1}\,x_1^2\,x_2 + 6305904\,x_1\,\overline{x_1}\,x_2 + 5454900\,x_2\,\overline{x_1} \\
& -17280\,x_2\,\overline{x_2} - 2081376\,x_1^2\,x_2 - 7018704\,x_1\,x_2 - 5900400\,x_2 + 20700\,\overline{x_2}\,\overline{x_1}^2 \\
& +666990\,\overline{x_1}^2\,x_1^2 + 2171355\,\overline{x_1}^2\,x_1 + 1836400\,\overline{x_1}^2 - 1267005\,x_1\,\overline{x_1} \\
& -7200\,\overline{x_1}\,\overline{x_2} - 471870\,x_1^2\,\overline{x_1} - 782825\,\overline{x_1} - 1349040\,x_1 - 318240\,x_1^2 \\
& -1401200 = 0\,, \qquad (3.100)\\
F_2 \;:=\;& -2831220\,x_1^3\,\overline{x_1} - 12488256\,x_1^3\,x_2 + 4001940\,x_1^3\,\overline{x_1}^2 - 1909440\,x_1^3 \\
& +11205216\,x_1^3\,x_2\,\overline{x_1} - 11594880\,x_1^2 - 13346640\,x_1^2\,\overline{x_1} + 20750580\,\overline{x_1}^2\,x_1^2 \\
& -65764224\,x_1^2\,x_2 + 59057424\,\overline{x_1}\,x_1^2\,x_2 + 104387400\,x_1\,\overline{x_1}\,x_2 \\
& -20709225\,x_1\,\overline{x_1} + 36279765\,\overline{x_1}^2\,x_1 - 23246640\,x_1 - 115160400\,x_1\,x_2 \\
& +61987500\,x_2\,\overline{x_1} - 15413200 - 67050000\,x_2 - 10555975\,\overline{x_1} \\
& +21429800\,\overline{x_1}^2 = 0\,, \qquad (3.101)\\
F_3 \;:=\;& 46332\,x_2\,\overline{x_2}^2 + 196560\,x_2\,\overline{x_2} + 19305\,\overline{x_2}^2\,\overline{x_1} + 96849\,\overline{x_1}\,\overline{x_2} + 199728\,x_1^2 \\
& -179208\,x_1^2\,\overline{x_1} + 24354\,x_1\,\overline{x_2} + 1133088\,x_1 - 914928\,x_1\,\overline{x_1} + 38214\,\overline{x_2} \\
& -1051354\,\overline{x_1} + 1435744 = 0\,, \qquad (3.102)\\
F_4 \;:=\;& 46332\,x_1\,x_2\,\overline{x_2} + 87750\,x_2\,\overline{x_2} + 4140\,\overline{x_1}\,\overline{x_2} - 121734\,x_1^2 + 133398\,x_1^2\,\overline{x_1} \\
& +23760\,x_1\,\overline{x_2} - 365673\,x_1 + 434271\,x_1\,\overline{x_1} + 43560\,\overline{x_2} + 367280\,\overline{x_1} \\
& -264005 = 0\,, \qquad (3.103)\\
F_5 \;:=\;& 10512\,x_1^2 - 9432\,x_1^2\,\overline{x_1} + 1170\,\overline{x_2}\,x_1\,\overline{x_1} - 1440\,x_1\,\overline{x_2} + 35448\,x_1 \\
& -31848\,x_1\,\overline{x_1} - 2640\,\overline{x_2} + 1965\,\overline{x_1}\,\overline{x_2} - 27550\,\overline{x_1} + 29800 = 0\,. \qquad (3.104)
\end{aligned}$$

Applying $\overline{\delta}$ to d_3, given by (3.95), and solving for $\overline{\delta}\alpha$ we obtain

$$\begin{aligned}
\overline{\delta}\alpha \;:=\;& -(28080\,\overline{\beta}\,\alpha^3\,\beta - 644600\,\overline{\pi}\,\pi^3\,\alpha + 688600\,\pi^3\,\alpha\,\overline{\alpha} + 66000\,\pi^2\,\overline{\beta}\,\beta\,\alpha \\
& +900\,\overline{\beta}^4\,\beta + 375\,\overline{\beta}^4\,\overline{\alpha} - 11970\,\pi\,\overline{\beta}^3\,\overline{\alpha} + 184168\,\pi^2\,\overline{\beta}^2\,\overline{\pi} - 132284\,\pi^2\,\overline{\beta}^2\,\overline{\alpha} \\
& -37344\,\pi\,\overline{\beta}^3\,\beta + 1062440\,\pi^2\,\alpha^2\,\overline{\alpha} - 2955536\,\pi^3\,\overline{\beta}\,\overline{\pi} + 2766408\,\pi^3\,\overline{\alpha}\,\overline{\beta} \\
& +177648\,\pi^2\,\beta\,\overline{\beta}^2 - 982120\,\pi^2\,\overline{\pi}\,\alpha^2 - 2480\,\pi\,\overline{\beta}^3\,\overline{\pi} - 23512\,\alpha^2\,\overline{\beta}^2\,\overline{\alpha} \\
& +545580\,\alpha^3\,\pi\,\overline{\alpha} - 397614\,\alpha^3\,\overline{\beta}\,\overline{\pi} + 409686\,\alpha^3\,\overline{\beta}\,\overline{\alpha} + 78876\,\alpha^2\,\beta\,\overline{\beta}^2 \\
& +49193\,\alpha^2\,\overline{\beta}^2\,\overline{\pi} - 497700\,\alpha^3\,\overline{\pi}\,\pi - 7155\,\overline{\beta}^3\,\alpha\,\overline{\alpha} - 21552\,\overline{\beta}^3\,\alpha\,\beta \\
& +93240\,\alpha^4\,\overline{\alpha} - 83880\,\alpha^4\,\overline{\pi} + 4384668\,\pi^2\,\overline{\beta}\,\alpha\,\overline{\alpha} + 190260\,\pi\,\overline{\beta}^2\,\alpha\,\overline{\pi} \\
& +2319796\,\pi\,\overline{\beta}\,\alpha^2\,\overline{\alpha} - 115006\,\pi\,\overline{\beta}^2\,\alpha\,\overline{\alpha} - 4541152\,\pi^2\,\alpha\,\overline{\beta}\,\overline{\pi}
\end{aligned}$$

$$
\begin{aligned}
&-2326802\,\alpha^2\,\pi\,\overline{\beta}\,\overline{\pi} + 87480\,\overline{\beta}\,\alpha^2\,\pi\,\beta + 243240\,\overline{\beta}^2\,\alpha\,\pi\,\beta - 1345\,\overline{\beta}^3\,\overline{\pi}\,\alpha) \\
&/\,\Big(5(-12\alpha - 22\pi + \overline{\beta})(36\overline{\beta}\overline{\alpha} + 84\beta\overline{\beta} - \overline{\beta}\overline{\pi} + 26\overline{\pi}\alpha - 8\pi\overline{\alpha} \\
&+64\overline{\pi}\pi + 2\alpha\overline{\alpha})\Big)\ ,
\end{aligned}
\tag{3.105}
$$

where the denominator, given by

$$
d_5 := 5(-12\alpha - 22\pi + \overline{\beta})(36\overline{\beta}\overline{\alpha} + 84\beta\overline{\beta} - \overline{\beta}\overline{\pi} + 26\overline{\pi}\alpha - 8\pi\overline{\alpha} + 64\overline{\pi}\pi + 2\alpha\overline{\alpha})\ , \tag{3.106}
$$

is assumed to be non-zero for now.

Subtracting this expression from (3.63) and taking the numerator, we get, in the variables defined by (3.75):

$$
\begin{aligned}
N_3 \ := \ & 33445536\,\overline{x_2}^2\,\overline{x_1}\,x_2 - 230934\,\overline{x_2}^3\,x_1 - 7320000\,x_1^4\,\overline{x_1}^2 - 222048\,\overline{x_2}^3 \\
&+47400\,\overline{x_2}^4\,\overline{x_1}^2 + 15950\,\overline{x_2}^4\,\overline{x_1} + 811536\,\overline{x_2}^3\,x_2 - 157828632\,x_1\,\overline{x_2} \\
&-119432728\,x_1^2\,\overline{x_2} + 11283368\,\overline{x_2}^2\,x_1 + 9900\,x_2\,\overline{x_2}^5\,\overline{x_1} + 32172800\,x_1\,\overline{x_1} \\
&+10800\,x_2^2\,\overline{x_2}^5 - 111188912\,\overline{x_2}\,\overline{x_1}^2\,x_1 + 5761996\,x_1^2\,\overline{x_2}^2 \\
&-40209168\,x_1^3\,\overline{x_2} + 49253600\,x_1^3\,\overline{x_1} - 36104928\,x_2\,\overline{x_2}^2 \\
&+137812704\,\overline{x_1}\,\overline{x_2} + 64980800\,x_1^2\,\overline{x_1} - 30351096\,\overline{x_2}^2\,\overline{x_1} \\
&+877252\,\overline{x_2}^3\,\overline{x_1} + 16608000\,x_1^4\,\overline{x_1} + 30537208\,\overline{x_2}^2\,\overline{x_1}^2\,x_1 - 900\,x_2\,\overline{x_2}^5 \\
&-28082000\,\overline{x_1}^2\,x_1^2 - 55553440\,\overline{x_2}\,\overline{x_1}^2 - 13772000\,\overline{x_1}^2\,x_1 - 18048800\,x_1 \\
&-22269136\,x_1^2\,\overline{x_2}^2\,\overline{x_1} + 71015616\,x_1^3\,\overline{x_2}\,\overline{x_1} - 2908800\,\overline{x_2}\,x_1^3\,x_2 \\
&-7382400\,\overline{x_2}\,x_1^2\,x_2 - 45102916\,\overline{x_2}^2\,x_1\,\overline{x_1} + 210053312\,\overline{x_2}\,x_1^2\,\overline{x_1} \\
&+277320136\,\overline{x_2}\,x_1\,\overline{x_1} + 7374120\,\overline{x_2}^2 - 35932400\,x_1^2 - 26826800\,x_1^3 \\
&-8904000\,x_1^4 - 83936136\,\overline{x_2}\,\overline{x_1}^2\,x_1^2 - 78277568\,\overline{x_2} + 2250\,\overline{x_2}^5\,\overline{x_1}^2 \\
&-375\,\overline{x_2}^5\,\overline{x_1} + 811142\,\overline{x_2}^3\,x_1\,\overline{x_1} - 51957168\,\overline{x_2}^2\,x_1\,x_2 + 623352\,\overline{x_2}^3\,x_1\,x_2 \\
&-24682752\,x_1^2\,x_2\,\overline{x_2}^2 + 518400\,\overline{x_2}^2\,x_2^2\,x_1^3 + 1382400\,\overline{x_2}^2\,x_2^2\,x_1^2 \\
&+792000\,\overline{x_2}^2\,x_2^2\,x_1 - 3479668\,\overline{x_2}^3\,\overline{x_1}^2\,x_1 - 28348656\,x_1^3\,\overline{x_1}^2\,\overline{x_2} \\
&+14811556\,x_1^2\,\overline{x_1}^2\,\overline{x_2}^2 + 2150\,\overline{x_2}^4 + 8685\,x_1\,\overline{x_2}^4\,\overline{x_1} \\
&+2381796\,x_1^3\,\overline{x_1}^2\,\overline{x_2}^2 + 23520\,x_1\,\overline{x_2}^4\,\overline{x_1}^2 - 21494000\,x_1^3\,\overline{x_1}^2 \\
&+2102400\,x_1^5\,\overline{x_1} - 1108800\,x_1^5 - 936000\,x_1^5\,\overline{x_1}^2 + 982260\,x_1^3\,\overline{x_2}^2 \\
&-5082912\,x_1^4\,\overline{x_2} - 12915120\,x_2\,\overline{x_2}^3\,\overline{x_1} + 4521600\,x_1^3\,\overline{x_2}\,\overline{x_1}\,x_2 \\
&+22815360\,x_1^2\,\overline{x_2}^2\,\overline{x_1}\,x_2 - 5887200\,x_1\,x_2\,\overline{x_2} + 1170\,\overline{x_2}^4\,x_1 \\
&-3858624\,\overline{x_2}^2\,x_1^3\,x_2 + 101088\,\overline{x_2}^3\,x_1^2\,x_2 + 9753600\,x_1^2\,\overline{x_2}\,\overline{x_1}\,x_2 \\
&-60066\,x_1^2\,\overline{x_2}^3 - 3416384\,\overline{x_2}^3\,\overline{x_1}^2 + 52344\,x_2\,\overline{x_2}^4 + 9046656\,x_1^4\,\overline{x_2}\,\overline{x_1} \\
&-3650904\,x_1^3\,\overline{x_2}^2\,\overline{x_1} + 133488\,\overline{x_2}^4\,x_1\,x_2^2 - 13932864\,\overline{x_2}^3\,x_1\,x_2^2
\end{aligned}
$$

$$
\begin{aligned}
&-3773952\, x_1^2\, x_2^2\, \overline{x_2}^3 - 12947904\, x_2^2 \overline{x_2}^3 + 237312\, \overline{x_2}^4\, x_2^2 \\
&+28044\, x_1\, \overline{x_2}^4\, x_2 + 203736\, x_2\, \overline{x_2}^4\, \overline{x_1} - 345600\, \overline{x_2}\, x_1^4\, x_2 \\
&+3640896\, x_1^3\, \overline{x_2}^2\, \overline{x_1}\, x_2 + 691200\, x_1^4\, \overline{x_2}\, \overline{x_1}\, x_2 - 3519936\, x_2\, \overline{x_2}^3\, x_1^2\, \overline{x_1} \\
&+6943200\, \overline{x_2}\, x_1\, \overline{x_1}\, x_2 + 107676\, x_2\, x_1\, \overline{x_2}^4\, \overline{x_1} - 13461720\, x_2\, \overline{x_2}^3\, x_1\, \overline{x_1} \\
&+47820240\, \overline{x_2}^2\, x_1\, x_2\, \overline{x_1} - 3616992\, x_1^4\, \overline{x_1}^2\, \overline{x_2} - 885414\, \overline{x_2}^3\, \overline{x_1}^2\, x_1^2 \\
&+20887648\, \overline{x_2}^2\, \overline{x_1}^2 + 183120\, \overline{x_2}^3\, x_1^2\, \overline{x_1} = 0\,.
\end{aligned}
\tag{3.107}
$$

Applying gsolve to N_3, given by (3.107), and d_3, given by (3.95), we now obtain the sets of solutions:

$$h_1 := [\,4\, x_1 + 12\, x_2\, \overline{x_2} + 8 + 5\, \overline{x_2}, \overline{x_1} - 1\,]\,, \tag{3.108}$$

$$h_2 := [\,18\, x_2\, \overline{x_2} + 9\, \overline{x_2} + 7, \overline{x_1} - 1, -3\, \overline{x_2} + 10 + 12\, x_1\,]\,, \tag{3.109}$$

$$
\begin{aligned}
h_3 := {} & [-23218 + 84708\, x_2\, \overline{x_2} + 33819\, \overline{x_2} + 22950\, x_2\, \overline{x_2}^2 + 10125\, \overline{x_2}^2, \overline{x_1} - 1\,, \\
& -3\, \overline{x_2} + 10 + 12\, x_1]\,,
\end{aligned}
\tag{3.110}
$$

$$h_4 := [\,\overline{x_1} - 1, 7 + 3\, x_1, \overline{x_2} + 6\,]\,, \tag{3.111}$$

$$h_5 := [\,108\, x_2 + 49\, \overline{x_1} - 2, 7 + 3\, x_1, \overline{x_2} + 6\,]\,, \tag{3.112}$$

$$
\begin{aligned}
h_6 := {} & [16236\, x_2 + 4515\, \overline{x_1} - 122\, x_1 + 2461, 451\, x_1\, \overline{x_1} - 391\, x_1 + 833\, \overline{x_1} - 678\,, \\
& \overline{x_2} + 6]\,,
\end{aligned}
\tag{3.113}
$$

$$h_7 := [\,7 + 3\, x_1, \overline{x_2} + 6\,]\,, \tag{3.114}$$

$$
\begin{aligned}
h_8 := {} & [15300\, x_1\, x_2\, \overline{x_2} + 26868\, x_2\, \overline{x_2} - 360\, x_1\, \overline{x_2} - 340\, x_1\, \overline{x_1} + 12275\, \overline{x_1}\, \overline{x_2} \\
& +678\, x_1^2\, \overline{x_1} - 3602\, \overline{x_1} - 664\, x_1 + 6915\, \overline{x_2}\, x_1\, \overline{x_1} + 102\, x_1^2 - 600\, \overline{x_2} - 1646, \\
& 6375\, \overline{x_2}^2\, \overline{x_1} + 54200\, \overline{x_1}\, \overline{x_2} + 14910\, \overline{x_2}\, x_1\, \overline{x_1} - 530060\, x_1\, \overline{x_1} \\
& -129888\, x_1^2\, \overline{x_1} - 546508\, \overline{x_1} + 15300\, x_2\, \overline{x_2}^2 - 18550\, \overline{x_2} + 56472\, x_2\, \overline{x_2} \\
& -9090\, x_1\, \overline{x_2} + 491516 + 468244\, x_1 + 112608\, x_1^2, -84456\, x_1^3 + 97416\, x_1^3\, \overline{x_1} \\
& -499212\, x_1^2 + 570492\, x_1^2\, \overline{x_1} + 6894\, x_1^2\, \overline{x_2} - 10674\, \overline{x_2}\, x_1^2\, \overline{x_1} - 987180\, x_1 \\
& +405\, \overline{x_2}^2\, x_1\, \overline{x_1} + 24390\, x_1\, \overline{x_2} - 41400\, \overline{x_2}\, x_1\, \overline{x_1} + 1107060\, x_1\, \overline{x_1} \\
& -270\, \overline{x_2}^2\, x_1 + 709812\, \overline{x_1} - 40106\, \overline{x_1}\, \overline{x_2} - 651912 + 810\, \overline{x_2}^2\, \overline{x_1} \\
& +21536\, \overline{x_2} - 450\, \overline{x_2}^2]\,.
\end{aligned}
\tag{3.115}
$$

The only set that is not obviously incompatible is (3.115), which forms the following system:

$$
\begin{aligned}
K_1 := {} & 15300\, x_1\, x_2\, \overline{x_2} + 26868\, x_2\, \overline{x_2} - 360\, x_1\, \overline{x_2} - 340\, x_1\, \overline{x_1} + 12275\, \overline{x_1}\, \overline{x_2} \\
& +678\, x_1^2\, \overline{x_1} - 3602\, \overline{x_1} - 664\, x_1 + 6915\, \overline{x_2}\, x_1\, \overline{x_1} + 102\, x_1^2 - 600\, \overline{x_2} \\
& -1646 = 0\,,
\end{aligned}
\tag{3.116}
$$

$$
\begin{aligned}
K_2 \;&:=\; 6375\,\overline{x_2}^2\,\overline{x_1} + 54200\,\overline{x_1}\,\overline{x_2} + 14910\,\overline{x_2}\,x_1\,\overline{x_1} - 530060\,x_1\,\overline{x_1} \\
&\quad -129888\,x_1^2\,\overline{x_1} - 546508\,\overline{x_1} + 15300\,x_2\,\overline{x_2}^2 - 18550\,\overline{x_2} + 56472\,x_2\,\overline{x_2} \\
&\quad -9090\,x_1\,\overline{x_2} + 491516 + 468244\,x_1 + 112608\,x_1^2 = 0\,, \qquad (3.117)\\
K_3 \;&:=\; -84456\,x_1^3 + 97416\,x_1^3\,\overline{x_1} - 499212\,x_1^2 + 570492\,x_1^2\,\overline{x_1} + 6894\,x_1^2\,\overline{x_2} \\
&\quad -10674\,\overline{x_2}\,x_1^2\,\overline{x_1} - 987180\,x_1 + 405\,\overline{x_2}^2\,x_1\,\overline{x_1} + 24390\,x_1\,\overline{x_2} \\
&\quad -41400\,\overline{x_2}\,x_1\,\overline{x_1} + 1107060\,x_1\,\overline{x_1} - 270\,\overline{x_2}^2\,x_1 + 709812\,\overline{x_1} \\
&\quad -40106\,\overline{x_1}\,\overline{x_2} - 651912 + 810\,\overline{x_2}^2\,\overline{x_1} + 21536\,\overline{x_2} - 450\,\overline{x_2}^2 = 0\,. \qquad (3.118)
\end{aligned}
$$

Applying gsolve to K_3 and F_5, given respectively by (3.118) and (3.104), we obtain

$$
\begin{aligned}
m_1 \;&:=\; [\,23\,\overline{x_1} + 17, 13\,\overline{x_2} - 24, 78\,x_1 + 131\,]\,, \qquad (3.119)\\
m_2 \;&:=\; [\,\overline{x_1} - 1, 13\,\overline{x_2} + 44, 78\,x_1 + 131\,]\,, \qquad (3.120)\\
m_3 \;&:=\; [\,1035\,\overline{x_1} + 2368, 450\,\overline{x_2} - 2927, 78\,x_1 + 131\,]\,, \qquad (3.121)\\
m_4 \;&:=\; [\,\overline{x_2} + 6, 7 + 3\,x_1\,]\,, \qquad (3.122)\\
m_5 \;&:=\; [\overline{x_1} - 1, 3\,\overline{x_2} - 10 - 12\,x_1,\, 252279144\,x_1^3 \\
&\quad +1392655797\,x_1^2 + 2509004949\,x_1 + 1475636585]\,, \qquad (3.123)\\
m_6 \;&:=\; [\,\overline{x_1} - 1, 3\,\overline{x_2} - 10 - 12\,x_1\,]\,, \qquad (3.124)\\
m_7 \;&:=\; [\,\overline{x_2} + 6, 7 + 3\,x_1\,]\,, \qquad (3.125)\\
m_8 \;&:=\; [1788390861\,\overline{x_1} - 1589992456, 11543996797 + 2140199424\,\overline{x_2}, \\
&\quad 1536\,x_1 + 3515]\,, \qquad (3.126)\\
m_9 \;&:=\; [1125\,\overline{x_1}\,\overline{x_2} - 63294\,\overline{x_1}\,x_1 - 109126\,\overline{x_1} + 1980\,\overline{x_2}\,x_1 - 35100\,x_1^2 - 81996\,x_1 \\
&\quad +4020\,\overline{x_2} - 32944, 124398\,\overline{x_1}\,x_1^2 - 165168\,x_1^2 + 423963\,\overline{x_1}\,x_1 - 601128\,x_1 \\
&\quad +359684\,\overline{x_1} - 547064, -10530\,x_1^3 + 594\,\overline{x_2}\,x_1^2 - 66915\,x_1^2 + 2619\,\overline{x_2}\,x_1 \\
&\quad -140031\,x_1 + 2787\,\overline{x_2} - 96734]\,. \qquad (3.127)
\end{aligned}
$$

The non-trivial system here is m_9, with equations

$$
\begin{aligned}
D_1 \;&:=\; 1125\,\overline{x_1}\,\overline{x_2} - 63294\,x_1\,\overline{x_1} - 109126\,\overline{x_1} + 1980\,x_1\,\overline{x_2} - 35100\,x_1^2 - 81996\,x_1 \\
&\quad -32944 + 4020\,\overline{x_2} = 0\,, \qquad (3.128)\\
D_2 \;&:=\; 124398\,x_1^2\,\overline{x_1} - 165168\,x_1^2 + 423963\,x_1\,\overline{x_1} - 601128\,x_1 + 359684\,\overline{x_1} \\
&\quad -547064 = 0\,, \qquad (3.129)\\
D_3 \;&:=\; -10530\,x_1^3 - 66915\,x_1^2 + 594\,x_1^2\,\overline{x_2} + 2619\,x_1\,\overline{x_2} - 140031\,x_1 - 96734 \\
&\quad +2787\,\overline{x_2} = 0\,. \qquad (3.130)
\end{aligned}
$$

If we now apply gsolve to D_1, $\overline{D_2}$, D_3, K_1, $\overline{K_1}$ and K_3, we find that there are no solutions.

When the denominator d_5, given by (3.106), is zero, we have, in terms of the variables defined in (3.75),

$$d_5 := (22-\overline{x_2}+12\,x_1)\,(26\,x_1+2\,x_1\,\overline{x_1}+64-8\,\overline{x_1}+36\,\overline{x_1}\,\overline{x_2}+84\,x_2\,\overline{x_2}-\overline{x_2}) = 0\,. \quad (3.131)$$

Applying `gsolve` to d_3 (cf. (3.95)), $\overline{d_3}$, d_5, $\overline{d_5}$, and F_5 (cf. (3.104)), we find again that the system allows no solution.

(iii) $\mathbf{d_4 = 0}$

From (3.74) and (3.75) we have

$$d_4 := 115\overline{x_2}\,\overline{x_1} + 126x_2\overline{x_2} - 40\overline{x_2} + 783x_1 - 1634\overline{x_1} + 1448 - 921x_1\overline{x_1} = 0\,. \quad (3.132)$$

The numerator of (3.73) now satisfies:

$$\begin{aligned} n_4 := {} & 11970480\,\overline{x_1}\,x_1^2 + 7513000\,x_1\,\overline{x_1} + 28683640\,\overline{x_2}\,\overline{x_1} \\ & -90203190\,x_1\,\overline{x_2} - 18687060\,x_1^2 - 2475000\,x_2\,\overline{x_2}\,x_1 \\ & +46210320\,\overline{x_1}\,\overline{x_2}\,x_1 - 56154880\,\overline{x_2} + 3875990\,\overline{x_2}^2 \\ & -9784170\,x_1^3 - 61600\,\overline{x_2}^3 - 1705860\,x_1^4 - 5791820\,\overline{x_2}^2\,\overline{x_1} \\ & +341280\,\overline{x_2}^3\,x_2 + 177100\,\overline{x_2}^3\,\overline{x_1} + 1119420\,x_1^4\,\overline{x_1} \\ & -8903160\,x_2\,\overline{x_2}^2 + 6346350\,x_1^3\,\overline{x_1} - 48333948\,\overline{x_2}\,x_1^2 \\ & +4188240\,\overline{x_2}^2\,x_1 - 33800\,\overline{x_2}^3\,x_1 + 1131915\,x_1^2\,\overline{x_2}^2 \\ & -8639361\,\overline{x_2}\,x_1^3 - 690120\,\overline{x_2}\,x_1^3\,x_2 + 97175\,\overline{x_2}^3\,\overline{x_1}\,x_1 \\ & -6302070\,\overline{x_2}^2\,x_1\,\overline{x_1} - 2615220\,\overline{x_2}\,x_1^2\,x_2 \\ & +24866544\,\overline{x_2}\,x_1^2\,\overline{x_1} - 9567000\,\overline{x_2}^2\,x_1\,x_2 \\ & +186390\,\overline{x_2}^3\,x_1\,x_2 - 2573586\,\overline{x_2}^2\,x_1^2\,x_2 \\ & -1716210\,\overline{x_2}^2\,x_1^2\,\overline{x_1} + 4470219\,\overline{x_2}\,x_1^3\,\overline{x_1} - 11885500\,x_1 = 0\,. \end{aligned} \quad (3.133)$$

Applying `gsolve` to system of polynomials formed by $d_4 = 0$, $n_4 = 0$ and $p_2 = 0$ (cf. (3.76)) we obtain a set of solutions where x_1 and x_2 are constants.

(iv) $\mathbf{x_1}$, $\mathbf{x_2}$ = const.

Let us consider now the case in which x_1 and x_2 are constants. We must then have $\delta x_2 = 0$, or

$$\overline{\delta}\alpha := -11\overline{\beta}\alpha - 4\pi\alpha - 18\pi\overline{\beta} - 3\alpha^2\,. \quad (3.134)$$

Substituting this in $\overline{\delta}x_1 = 0$ we obtain, in the coordinates $\{x_1, x_2\}$

$$x_1^2 + 5x_2x_1 + 2x_1 + 9x_2 = 0\,. \quad (3.135)$$

From $\delta x_1 = 0$ we now obtain

$$(x_1+2)(6x_1+11)(2x_1\overline{x_1}+2x_1+5\overline{x_2}\overline{x_1}+12x_2\overline{x_2}+2x_1+6)\,. \tag{3.136}$$

From $\overline{\delta} x_2 = 0$ we get

$$\begin{aligned}&60x_2x_1^2-6x_2^2\overline{x_2}^2+5x_2\overline{x_2}^2-31x_1x_2\overline{x_1}\overline{x_2}+78x_2\overline{x_2}x_1+110x_2x_1\\&+151x_2\overline{x_2}-56x_2\overline{x_2}\overline{x_1}+24x_1^2+24\overline{x_1}x_1^2+132+44\overline{x_1}\\&+60\overline{x_1}\overline{x_2}x_1+68x_1\overline{x_1}+116x_1+110x_2\overline{x_1}=0\,.\end{aligned} \tag{3.137}$$

Applying gsolve to (3.135), (3.136), their complex conjugates and (3.137), we find that this system has no solutions.

(v) $\mathbf{d_1 = 0}$

Let us consider finally the case in which the denominator of $D\mu$, given by (3.58), is zero. Here we shall suppose that Φ_{11} is not necessarily zero, so we can illustrate how is the form of the equation that appear when we try to solve this problem in the general case. According to (3.59) and (3.75),

$$d_1 := 22+12x_1-\overline{x_2}=0\,. \tag{3.138}$$

From (3.58) we obtain

$$\begin{aligned}E_1 :=\ &-53\,x_1\,\phi_{11}+242-220\,x_2+24\,\overline{x_1}\,x_1^2+30\,\overline{x_2}\,\phi_{11}+274\,x_2\,\overline{x_2}\\&-5\,\overline{x_2}+110\,\overline{x_2}\,\overline{x_1}+24\,x_1^2+44\,\overline{x_1}+144\,x_1\,x_2\,\overline{x_2}-108\,\phi_{11}\\&-120\,x_1x_2+68\,x_1\,\overline{x_1}+176\,x_1+60\,\overline{x_1}\,\overline{x_2}\,x_1=0\,,\end{aligned} \tag{3.139}$$

where ϕ_{11} is defined as follows:

$$\phi_{11} := \frac{\Phi_{11}}{\pi\overline{\pi}}\,. \tag{3.140}$$

Applying $\overline{\delta}$ to f_1, using (3.35), (3.47) and (3.45), and solving for $\overline{\delta}\alpha$, we get

$$\overline{\delta}\alpha = 120\overline{\beta}\alpha+66\pi\alpha+220\pi\overline{\beta}+33\alpha^2-\overline{\beta}^2\,. \tag{3.141}$$

By applying δ to d_1, now using (3.49), (3.50), (3.51) and (3.56) and solving for $\mathrm{D}\mu$, we get

$$\mathrm{D}\mu = (4\overline{\pi}\alpha+29\beta\overline{\beta}+64\alpha\overline{\alpha}-370\beta\pi+32\overline{\pi}\pi-190\beta\alpha+114\pi\overline{\alpha}+5\overline{\beta}\overline{\alpha}-68\Phi_{11})/20\,. \tag{3.142}$$

Subtracting $D\overline{\mu}$, given by (3.56), from the complex conjugate of (3.142), gives

$$\begin{aligned}E_2 \ :=\ &20\,\phi_{11}-2\,x_2\,\overline{x_2}-22\,x_1-24\,x_1\,\overline{x_1}+66\,x_2+35\,x_1\,x_2-22\,\overline{x_1}\\&+35\,\overline{x_2}\,\overline{x_1}+66\,\overline{x_2}=0\,.\end{aligned} \tag{3.143}$$

Applying $\overline{\delta}$ to (3.143) gives

$$\begin{aligned} E_3 \;:=\; & 1168\, x_1 \phi_{11} + 3980\, x_2 x_1^2 - 264 + 14520\, x_2 - 3584\, \overline{x_1}\, x_1^2 - 255\, \overline{x_2}\, \phi_{11} \\ & -2838\, x_2 \overline{x_2} - 9482\, x_1 \overline{x_1} - 24200\, \overline{x_2} - 12870\, \overline{x_2}\, \overline{x_1} - 12870\, x_1 \overline{x_2} \\ & -3784\, x_1^2 - 1210\, \overline{x_2}^2 - 4928\, \overline{x_1} - 1514\, x_1 x_2 \overline{x_2} + 20\, x_2 \overline{x_2}^2 - 625\, \overline{x_2}^2\, \overline{x_1} \\ & +2046\, \phi_{11} + 15180\, x_1 x_2 - 7480\, x_1 - 6835\, \overline{x_1}\, \overline{x_2}\, x_1 = 0\,. \end{aligned} \tag{3.144}$$

Applying gsolve to d_1, $\overline{f_1}$, E_1, $\overline{E_1}$, E_2 and E_3, we find that this system has no solution.

Thus, for Huygens' principle to be satisfied on Petrov type III space-times we must have $\Phi_{11} \neq 0$. Using Theorem 3.1, the Main Theorem, which states this result in a conformally invariant way, is proved.

3.4 Discussion

In order to obtain the above result [3]. we have used the six-index necessary condition VII, derived by Rinke and Wünsch [72], which had not been employed in [22]. It was essential for the proof that $D\mu$ and $D\overline{\mu}$ could be expressed in such a way that they both have the same denominator. As a result the integrability conditions obtained from the commutation relations turned out to be relatively simple. The most general case, in which Φ_{11}, α, β and π are not necessarily zero, has been reduced to a system of polynomial equations in one real and two complex variables. These polynomial equations are obtained by applying the operator $\overline{\delta}$ to N_1, given in (3.69), and then solving for $\overline{\delta}\alpha$. A new side relation can then be obtained by subtracting this $\overline{\delta}\alpha$ from (3.63). Using the variables defined in (3.75) and (3.140) we obtain equations in one real and two complex variables. The system formed by the side relation $S_1 = 0$ (cf. (3.61)) and $N_1 = 0$ was not yet solved in the general case in which $\Phi_{11} \not\equiv 0$, but its size is relatively small. If any of these variables can be shown to be zero, then Hadamard's problem will be solved for this case.

[3] In a private communication, S. Czapor has informed us that he has found the same result using Gröbner bases defined on a prime field (unpublished).

Chapter 4

The Non-Self-Adjoint Scalar Wave Equation

4.1 Previous results and Main Theorem

In this chapter we consider the general linear second-order hyperbolic equation, in four space-time dimensions, with C^∞ coefficients:

$$\Box u + A^k(x)\nabla_k u + B(x)u = 0\,. \tag{4.1}$$

This equation is also referred as the *non-self-adjoint* scalar equation.

McLenaghan and Walton [61] have shown that any non-self-adjoint equation (4.1) on any Petrov type N space-time satisfies Huygens' principle if and only if it is equivalent to a scalar equation with $A^i = 0$ and $B = 0$, on an space-time corresponding to the exact plane-wave metric (1.12). For the case of Petrov type III space-times, Anderson, McLenaghan and Walton ([7]) have proved the following theorems:

Theorem 4.1 *The validity of Huygens' principle for any non-self-adjoint scalar wave equation (4.1) on any Petrov type III space-time implies that the space-time is conformally related to one in which every repeated null vector field of the Weyl tensor l_a, is recurrent, i.e.,*

$$l_{[i}l_{j;k]} = 0\,. \tag{4.2}$$

Theorem 4.2 *There exist no non-self-adjoint Huygens' equation (4.1) on any Petrov type III space-time for which the following conditions hold*

$$\left.\begin{array}{l}\Psi_{ABCD;E\dot{E}}\iota^A\iota^B\iota^C o^D\iota^E\bar{o}^{\dot{E}} = 0\,,\\ \Psi_{ABCD;E\dot{E}}\iota^A\iota^B\iota^C o^D o^E\bar{o}^{\dot{E}} = 0\,,\\ \Psi_{ABCD;E\dot{E}}\iota^A\iota^B\iota^C\iota^D o^E\bar{o}^{\dot{E}} = 0\,.\end{array}\right\} \tag{4.3}$$

In this chapter we show that the restrictions imposed by these two theorems can be removed thus obtaining a stronger result, which can be stated as follows:

Theorem 4.3 (Main Theorem) *If a non-self-adjoint scalar wave equation of the form (4.1) satisfies Huygens' principle on any Petrov type III space-time, then it must be equivalent to a conformally invariant scalar wave equation*

$$g^{ab}\nabla_a\nabla_b u + \frac{1}{6}Ru = 0\,. \tag{4.4}$$

4.2 Proof of the Main Theorem

In the first part of this proof we get the same result obtained by Anderson et al. [7]. The procedure is, however, slightly different. The final part, which consists in the proof of the Main Theorem, is our original contribution.

We shall prove first, using the necessary conditions II to VI (see (2.187) - (2.191)), that the assumption $H_{ij} := A_{[i;j]} \neq 0$ leads to a contradiction. In terms of the dyad components of the Maxwell tensor H_{ab}, this is the same as proving that the necessary conditions imply $\phi_0 = \phi_1 = \phi_2 = 0$. Finally we invoke a lemma by Günther [38] that states that every equation of the form (4.1) for which $A_{[a,b]} := 0$ is related by a trivial transformation to one for which $A_i = 0$. It then follows from condition I (2.181) that $B = R/6$.

In this Chapter we shall use a notation for the dyad components of the necessary conditions in the form X_{ab}, where X is the Roman numeral corresponding to the necessary condition, a denotes the number of indices corresponding to the dyad spinor ι and b the number of dotted indices corresponding to the dyad spinor $\bar{\iota}$.

As a starting point we use condition IV. The contraction with $o^{ABC}\bar{o}^{(\dot{A}\dot{B}}\bar{\iota}^{\dot{C})}$ gives

$$IV_{32} = \kappa\overline{\phi}_0 = 0\,. \tag{4.5}$$

Let us suppose that $\kappa \neq 0$. Then $\phi_0 = 0$ and

$$IV_{31} = \kappa\phi_1 = 0\,, \tag{4.6}$$

implies $\phi_1 = 0$. It then follows from

$$IV_{21} = \kappa(3\overline{\phi}_2 + \phi_2) = 0 \tag{4.7}$$

that $\phi_2 = 0$ and $\phi_{AB} = 0$, which contradicts the initial assumption. Thus, we must have

$$\kappa = 0\,. \tag{4.8}$$

From III_{22} we then have immediately that

$$\phi_0 = 0\,. \tag{4.9}$$

This implies

$$\sigma\phi_1 = 0\,. \tag{4.10}$$

If $\sigma \neq 0$ then $\phi_1 = 0$, and

$$III_{20} = 9\sigma\overline{\phi}_2 - \sigma\phi_2 = 0 \tag{4.11}$$

implies $\phi_2 = 0$, which again contradicts our initial assumption. Thus, we obtain

$$\sigma = 0\,. \tag{4.12}$$

We now have

$$II_{11} = \mathrm{D}\phi_1 - 2\rho\phi_1 = 0\,, \tag{4.13}$$

and

$$IV_{21} = \mathrm{D}\phi_1 + \rho\phi_1 + 6(\overline{\epsilon} - \overline{\rho}) = 0\,, \tag{4.14}$$

and thus

$$\phi_1(2\overline{\epsilon} - 2\overline{\rho} + \rho) = 0\,. \tag{4.15}$$

If we suppose that $\phi_1 \neq 0$, then we get

$$\overline{\epsilon} = \overline{\rho} - \frac{1}{2}\rho\,. \tag{4.16}$$

Since, according to (B.101) and (B.113), the equations (4.8) (4.9) (4.12) are conformally invariant, we use this freedom to impose again condition (3.15), i.e., $\rho = -\overline{\rho}$, so (4.16) implies that

$$\epsilon = \frac{3}{2}\rho\,. \tag{4.17}$$

From the NP equation (C1) we now obtain

$$\mathrm{D}\rho = -\rho^2 + \Phi_{00}\,. \tag{4.18}$$

Adding (4.18) to its complex conjugate we obtain

$$\Phi_{00} = -\rho^2 . \tag{4.19}$$

Substituting these results in V_{11} we obtain

$$\phi_1\overline{\phi}_1\rho^2 = 0 . \tag{4.20}$$

This implies that $\rho = \epsilon = \Phi_{00} = 0$. Now, substituting these results into NP equations (NP1), (NP3) and (NP11) we obtain $D\tau = D\beta = \Phi_{01} = 0$. From III_{11} this would imply finally that $\phi_1 = 0$, which contradicts the initial assumption, which must then be false and therefore

$$\phi_1 = 0 . \tag{4.21}$$

The conditions II_{00} and II_{01} now become, respectively,

$$\delta\phi_2 = \phi_2(\tau - 2\beta) , \tag{4.22}$$

$$D\phi_2 = \phi_2(\rho - 2\epsilon) . \tag{4.23}$$

Substituting these equations into IV_{10} we get

$$\overline{\delta}\phi_2 = (-6\overline{\beta} - 2\alpha + 6\overline{\tau})\phi_2 + (2\alpha + 4\pi + \overline{\tau})\overline{\phi}_2 , \tag{4.24}$$

Equation (4.19) is still valid:

$$\Phi_{00} = -\rho^2 . \tag{4.25}$$

From IV_{11}, NP1 and III_{12} we now have, respectively,

$$s_1 := (3\rho + 2\overline{\epsilon})\phi_2 + (-3\rho + 2\epsilon)\overline{\phi}_2 = 0 , \tag{4.26}$$

$$D\rho = (\epsilon + \overline{\epsilon})\rho . \tag{4.27}$$

$$D\epsilon = -\frac{3}{2}\rho^2 + 4\epsilon\rho + \epsilon\overline{\epsilon} - \epsilon^2 . \tag{4.28}$$

Substituting the previous Pfaffians into V_{22} we obtain

$$s_2 := 4\phi_2\overline{\phi}_2\rho^2 + 24\epsilon\overline{\epsilon} + 28\epsilon\rho - 41\rho^2 - 28\epsilon\rho = 0 . \tag{4.29}$$

Applying the operator D to (4.29) yield a third side relation:

$$s_3 := \overline{\epsilon}\rho^2 + \epsilon\rho^2 = 0 . \tag{4.30}$$

Let us suppose that $\rho \neq 0$. Then, from (4.30) it follows that $\bar{\epsilon} = -\epsilon$. This implies that, from (4.27), that $D\rho = 0$. By subtracting (4.28) from its complex conjugate we get $D\epsilon = 0$. Then, (NP3), (NP11), (NP4) and (NP5) give, respectively,

$$D\tau = (\tau + \bar{\pi})\rho + 2\epsilon\tau + \Phi_{01}\,, \tag{4.31}$$

$$\delta\rho = \rho(\bar{\alpha} + \beta + 2\tau) + \Phi_{01}\,, \tag{4.32}$$

$$D\alpha = \bar{\delta}\epsilon + (\epsilon - \rho)\beta + (\bar{\pi} - \bar{\alpha})\epsilon\,, \tag{4.33}$$

$$D\beta = \delta\epsilon + (\epsilon - \rho) + (\bar{\pi} - \bar{\lambda})\epsilon\,. \tag{4.34}$$

Since III_{11} has the form

$$D\beta + \delta\epsilon + \rho(2\tau - 5\beta) - \Phi_{01} + \epsilon(\beta - \bar{\alpha} - 4\tau - \bar{\pi}) = 0\,, \tag{4.35}$$

we can now determine the Pfaffians $D\alpha$, $D\beta$ and $\delta\epsilon$:

$$\begin{aligned} D\alpha = -\frac{1}{2\phi_2}(&\rho\alpha\bar{\phi}_2 + 3\rho\overline{\beta\phi_2} - 6\rho\bar{\tau}\bar{\phi}_2 - 21\phi_2\rho\bar{\beta} + 26\phi_2\rho\bar{\tau} \\ &-3\phi_2\rho\alpha - 2\phi_2\Phi_{10} + 14\epsilon\phi_2\bar{\beta} - 16\epsilon\phi_2\bar{\tau} + 2\epsilon\phi_2\alpha - 10\pi\epsilon\bar{\phi}_2 \\ &-2\bar{\beta}\epsilon\bar{\phi}_2 + 2\epsilon\alpha\bar{\phi}_2 + 10\rho\pi\bar{\phi}_2 + \Phi_{10}\bar{\phi}_2)\,, \end{aligned} \tag{4.36}$$

$$D\beta = \rho(-\tau + 2\beta) + \epsilon(\bar{\pi} + 2\tau) + \frac{1}{2}\Phi_{01}\,, \tag{4.37}$$

$$\delta\epsilon = \rho(\tau + 3\beta) + \epsilon(2\tau - \beta + \bar{\alpha}) + \frac{1}{2}\Phi_{01}\,. \tag{4.38}$$

IV_{11} has the form

$$(\phi_2 - \bar{\phi}_2)(3\rho - 2\epsilon) = 0\,. \tag{4.39}$$

Also from (NP11), III_{11} and III_{12}, we have, respectively

$$\delta\rho = \rho(\bar{\alpha} + \beta + 2\tau) + \Phi_{01}\,, \tag{4.40}$$

$$\delta\epsilon = \epsilon(2\tau - \beta - \tau + \bar{\alpha}) + 3\rho\beta + \frac{1}{2}\Phi_{01}\,, \tag{4.41}$$

$$(2\epsilon - \rho)(2\epsilon - 3\rho) = 0\,. \tag{4.42}$$

Regarding (4.42), let us consider first the case $\epsilon = \rho/2$. Then, from (4.39), $\phi_2 = \bar{\phi}_2$. The expression $\delta(\epsilon - \rho/2) = 0$, using (4.40) and (4.41), gives

$$\rho(2\beta - \tau) = 0\,, \tag{4.43}$$

i.e., $\tau = 2\beta$. The conditions s_1, V_{23} and III_{02} now become, respectively,

$$\phi_2(2\beta + \bar{\pi}) = 0\,, \tag{4.44}$$

$$\rho^2\phi_2(5\alpha - 21\bar{\beta}) = 0\,, \tag{4.45}$$

$$D\pi = -\rho\pi - \Phi_{10}\,. \tag{4.46}$$

i.e., $\overline{\pi} = -2\beta$ and $\overline{\beta} = (5/21)\alpha$. Applying D to (4.45) and solving for Φ_{10} gives

$$\Phi_{10} = 2\rho\overline{\beta}\,. \tag{4.47}$$

V_{12} and V_{22} give, respectively,

$$\rho\alpha(-11 + 4{\phi_2}^2) = 0\,, \tag{4.48}$$
$$\epsilon^2(-19 + 4{\phi_2}^2) = 0\,. \tag{4.49}$$

This implies $\alpha = \beta = \pi = \tau = \Phi_{10} = 0$ and $-19 + 4{\phi_2}^2 = 0$. From V_{20}, now we have $\epsilon\overline{\lambda} = 0$, and from (NP16), $\Phi_{02} = -2\epsilon\overline{\lambda}$, so $\Phi_{02} = 0$.
The remaining conditions III are III_{00} and III_{01} are given by

$$2\epsilon\overline{\nu} + 2\delta\gamma + 5\delta\mu + 2\Phi_{12} + 5{\phi_2}^2 = 0\,, \tag{4.50}$$
$$\epsilon(-7\mu + o\mu - 3\gamma + \overline{\gamma}) + 2\Phi_{11} - \Delta\epsilon + 2\mathrm{D}\mu + \mathrm{D}\gamma = 0\,. \tag{4.51}$$

Almost all Pfaffians in (4.50) and (4.51) can be determined from NP equations. From (NP6), (NP8), (NP15) and (NP17) we obtain, respectively,

$$\mathrm{D}\gamma = \Delta\epsilon - \epsilon(\gamma + \overline{\gamma}) - \Lambda + \Phi_{11}\,, \tag{4.52}$$
$$\mathrm{D}\mu = -2\epsilon\mu + 2\Lambda\,, \tag{4.53}$$
$$\delta\gamma = -\epsilon\overline{\nu} + \Phi_{12}\,, \tag{4.54}$$
$$\Delta\epsilon = \epsilon(\overline{\mu} + \gamma + \overline{\gamma}) - \Lambda\,. \tag{4.55}$$

Thus, (4.50) and (4.51) have the following form, respectively:

$$5{\phi_2}^2 + 4\Phi_{12} - \frac{8}{3}\epsilon\overline{\nu} + 4\delta\mu = 0\,, \tag{4.56}$$
$$\epsilon(-11\mu - 4\gamma + \overline{\mu}) + 3\Phi_{11} + 3\Lambda = 0\,. \tag{4.57}$$

From (NP21) and (NP27) we obtain

$$\mathrm{D}\Phi_{11} = -2\,(\overline{\mu} - 2\mu - 2\gamma - 2\overline{\gamma})\,\epsilon^2(-4\Delta\epsilon - 6\Phi_{11})\epsilon\,, \tag{4.58}$$
$$\mathrm{D}\Lambda = -\frac{4}{3}\,(2\gamma + 2\overline{\gamma} - \mu - \overline{\mu})\,\epsilon^2 - \frac{1}{3}\mathrm{D}\Phi_{11} + \frac{8}{3}\Delta\epsilon\,. \tag{4.59}$$

Applying D to (4.57) three times successively, using (4.58) and (4.59), we obtain

$$F_1 := -\epsilon(17\mu + 3\overline{\mu}) + 6\Lambda + 8\Phi_{11} = 0\,, \tag{4.60}$$
$$F_2 := \epsilon(-33\mu + \overline{\mu}) + 16\Lambda + 18\Phi_{11} = 0\,, \tag{4.61}$$
$$F_3 := -\epsilon(69\mu + 3\overline{\mu}) + 28\Lambda + 38\Phi_{11} = 0\,. \tag{4.62}$$

The solutions for the system formed by these last three equations, together with (4.57) are easily obtained by gsolve and yield

$$\gamma = \frac{3}{2}\mu\,, \tag{4.63}$$
$$\Phi_{11} = \Lambda = \epsilon\mu\,, \tag{4.64}$$
$$\overline{\mu} = -\mu\,. \tag{4.65}$$

From (NP13),

$$\delta\mu = -4\epsilon\nu - 1 - \Phi_{21}\,. \tag{4.66}$$

Using (4.65) and (4.66), (4.56) now becomes

$$-2\epsilon\overline{\nu} + 1 + 3\Phi_{12} + 5{\phi_2}^2\,. \tag{4.67}$$

Solving this equation for $\overline{\nu}$ and substituting in (NP18) we find

$$4 + 24\Phi_{21} + 45{\phi_2}^2 = 0\,. \tag{4.68}$$

Solving this for Φ_{21} and substituting in (NP9) we get

$${\phi_2}^2 + 4 = 0\,, \tag{4.69}$$

which is impossible.

Let us consider again (4.42) in the case $\rho = (2/3)\epsilon$. From III_{11}, III_{02} and V_{22}, we have, respectively,

$$\Phi_{01} = -\frac{2}{3}\tau\epsilon\,, \tag{4.70}$$
$$\mathrm{D}\pi = -\frac{2}{3}\epsilon\left(2\alpha + \frac{4}{3}\overline{\tau} + \Phi_{10} + 7\pi\right)\,, \tag{4.71}$$
$$\epsilon^2(-11 + 4\phi_2\overline{\phi}_2) = 0\,. \tag{4.72}$$

By using (4.71) and (4.72) in V_{12} to eliminate $\mathrm{D}\pi$ and $\overline{\phi}_2$ we get

$$t_1 := (252\overline{\tau} - 328\overline{\beta} + 64\pi){\phi_2}^2 + 242\alpha + 484\pi + 121\overline{\tau} = 0\,. \tag{4.73}$$

By applying D to the previous equation two times successively we obtain (after discarding factors in ϵ):

$$t_2 := (700\overline{\tau} + 232\pi - 984\overline{\beta} + 64\alpha){\phi_2}^2 + 363\overline{\tau} + 726\alpha + 1452\pi = 0\,, \tag{4.74}$$
$$\begin{aligned} t_3 &:= (1864\overline{\tau} + 808\pi - 2952\overline{\beta} + 424\alpha){\phi_2}^2 \\ &+1089\overline{\tau} + 2178\alpha + 4356\pi = 0\,. \end{aligned} \tag{4.75}$$

By applying δ to (4.72) we obtain a fourth equation in the same variables:

$$t_4 = (308\overline{\tau} - 353\overline{\beta} - 88\alpha)\phi_2{}^2 + 1221\overline{\tau} + 242\alpha + 484\pi\,. \tag{4.76}$$

By applying **gsolve** to the set $\{t_1, t_2, t_3, t_4\}$ we find that

$$\alpha = \beta = \pi = \tau = \Phi_{01} = 0\,. \tag{4.77}$$

Conditions III_{00} and III_{01} now become

$$2\epsilon\overline{\nu} + 6\delta\gamma + 8\delta\mu + 6\Phi_{12} + 15\phi_2\overline{\phi}_2 = 0\,, \tag{4.78}$$

$$\epsilon(7\mu - \overline{\mu} + 3\gamma - \overline{\gamma}) + 6\Phi_{11} + \Delta\epsilon + 6\mathrm{D}\mu + 3\mathrm{D}\gamma = 0\,. \tag{4.79}$$

From (NP6), (NP8), (NP15) remain the same as (4.52), (4.53) and (4.54) while (NP17), (NP21) and (NP27) imply

$$\Delta\epsilon = \epsilon(\overline{\mu} + \gamma + \overline{\gamma}) - 3\Lambda\,, \tag{4.80}$$

$$\mathrm{D}\Phi_{11} = -\frac{2}{9}\left(\overline{\mu} - 2\mu - 2\gamma - 2\overline{\gamma}\right)\epsilon^2 - \left(\frac{4}{9}\Delta\epsilon + 2\Phi_{11}\right)\epsilon\,, \tag{4.81}$$

$$\mathrm{D}\Lambda = -\frac{4}{27}\left(2\gamma + 2\overline{\gamma} - \mu - \overline{\mu}\right)\epsilon^2 - \frac{1}{3}\mathrm{D}\Phi_{11} + \frac{8}{27}\epsilon\Delta\epsilon\,. \tag{4.82}$$

Expression (4.79) now becomes

$$\epsilon(3\mu - 5\overline{\mu} + 4\gamma) - 2\Lambda + 9\Phi_{11} = 0\,. \tag{4.83}$$

Applying D to this equation two times consecutively, we get

$$\epsilon(-3\mu + 7\overline{\mu}) + 6\Lambda + 24\Phi_{11} = 0\,, \tag{4.84}$$

$$\epsilon(13\overline{\mu} + 19\mu) + 48\Lambda - 66\Phi_{11} = 0\,. \tag{4.85}$$

By applying **grobner** to (4.83), (4.84) and its complex conjugate, and (4.85) we obtain

$$\overline{\mu} = -\mu\,, \tag{4.86}$$

$$\gamma = -\frac{5}{2}\mu\,. \tag{4.87}$$

Using (4.86) we obtain from (NP13),

$$\delta\mu = \frac{4}{3}\epsilon\overline{\nu} + 1 + \Phi_{12}\,. \tag{4.88}$$

Equation (4.65) now becomes

$$\frac{2}{3}\epsilon\overline{\nu} - 1 + \Phi_{12} + 5\phi_2\overline{\phi}_2 = 0\,. \tag{4.89}$$

Solving this equation for $\bar{\nu}$ and substituting it in (NP18), using (4.87) (4.88), we obtain

$$\Phi_{21} = \frac{2}{5} - \frac{5}{4}\phi_2\overline{\phi}_2 . \tag{4.90}$$

Substituting in (NP9) we get

$$\frac{6}{5} - \frac{5}{2}\phi_2\overline{\phi}_2 = 0 , \tag{4.91}$$

which contradicts (4.72). Thus,

$$\rho = 0 , \tag{4.92}$$

and we now go back to (4.30). From III_{11} and III_{12} we have, and

$$\Phi_{01} = \epsilon = 0 . \tag{4.93}$$

We notice now that, according to (B.101) and (B.113), equations (4.8), (4.12), (4.21), (4.8), (4.92), (4.93) are conformally invariant. Thus, we have exactly the same case that was already discussed, for the self-adjoint scalar equation, in Chapter 3 (cf. (3.21) and the discussion that follows). Thus, the remaining gauge freedom is used to set

$$\tau = 0 . \tag{4.94}$$

From the NP equations we now obtain immediately:

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = \Lambda = 0 , \tag{4.95}$$

and

$$\mathrm{D}\alpha = \mathrm{D}\beta = \mathrm{D}\Phi_{11} = 0 . \tag{4.96}$$

Thus, we have proved the lemmas of Anderson et al. [7]:

Lemma 4.1 *For the non-self-adjoint scalar wave equation of the form (4.1), on Petrov type III space-times, the necessary conditions II, III, IV, V and VI together with the assumption that the Maxwell spinor ϕ_{AB} is nonzero, imply that there exists a spinor dyad $\{o_A, \iota_A\}$ and a conformal transformation such that*

$$\left.\begin{array}{l} \kappa = \sigma = \rho = \tau = \epsilon = 0 , \\ \Psi_0 = \Psi_1 = \Psi_4 = 0 , \ \Psi_3 = -1 , \\ \Phi_{00} = \Phi_{01} = \Phi_{02} = \Lambda = 0 , \\ D\alpha = D\beta = D\Phi_{11} = 0 . \end{array}\right\} \tag{4.97}$$

This proves Theorem 4.1, as can be verified from the following lemma, proved by Carminati and McLenaghan [22]:

Lemma 4.2 *If, for any space-time, there exists a spinor dyad $\{o_A, \iota_A\}$ and a conformal transformation such that*

$$\left.\begin{array}{l}\kappa = \sigma = \rho = \tau = \epsilon = 0\,,\\ \Psi_0 = \Psi_1 = \Psi_4 = 0\,,\ \Psi_3 = -1\,,\\ \Phi_{00} = \Phi_{01} = \Phi_{02} = \Lambda = 0\,,\end{array}\right\} \tag{4.98}$$

then every repeated principal null vector field of the Weyl tensor is recurrent.

We now proceed with the proof of the Main Theorem, using the necessary conditions conditions (4.97). Let us assume initially that $\alpha\beta\pi \neq 0$.

From II_{01}, II_{00}, III_{10}, (NP6) and (NP25) we have, respectively,

$$\mathrm{D}\phi_2 = 0\,, \tag{4.99}$$

$$\delta\phi_2 = -2\phi_2\beta\,, \tag{4.100}$$

$$\delta\beta = -\beta(\overline{\alpha} + \beta)\,, \tag{4.101}$$

$$\mathrm{D}\gamma = \alpha\overline{\pi} + \beta\pi + \Phi_{11}\,, \tag{4.102}$$

$$\delta\Phi_{21} = 2(\alpha + 2\pi + \lambda\Phi_{11} - \alpha\Phi_{21})\,. \tag{4.103}$$

By adding (NP22) to the complex conjugate of (NP23), and solving for $\mathrm{D}\Phi_{12}$ we get

$$\mathrm{D}\Phi_{12} = 2\overline{\pi}\Phi_{11}\,, \tag{4.104}$$

and

$$\delta\Phi_{11} = 0\,. \tag{4.105}$$

Subtracting (NP24) from (NP29) and solving for $\overline{\delta}\Phi_{12}$ we obtain

$$\overline{\delta}\Phi_{12} = 2(-\beta + \Phi_{11}\overline{\mu} - \overline{\beta}\Phi_{12})\,, \tag{4.106}$$

Adding (NP24) to two times (NP29) and solving for $\mathrm{D}\Phi_{22}$ we get

$$\mathrm{D}\Phi_{22} = 2(-\beta - \overline{\beta} + \Phi_{21}\overline{\pi} + \Phi_{12}\pi)\,. \tag{4.107}$$

By substituting (4.100) into IV_{10} we find

$$\overline{\delta}\phi_2 = 2(3\overline{\beta}\phi_2 - \alpha\phi_2 + 2\pi\overline{\phi}_2 + \alpha\overline{\phi}_2)\,. \tag{4.108}$$

Using (4.107) and (4.108), V_{20} and VI_{03} can be written respectively as

$$\begin{aligned}&-24\overline{\phi}_2\phi_2\beta^2 + 24\overline{\pi}\overline{\alpha} + 80\overline{\pi}\beta + {\phi_2}^2(-6\overline{\alpha}\overline{\pi} + 18\beta\overline{\pi} + 9\beta\overline{\alpha} - \epsilon\overline{\alpha}^2 - 2\delta\overline{\pi} - \delta\overline{\alpha})\\ &+12\overline{\alpha}^2 + 4\delta\overline{\alpha} + 8\delta\overline{\pi} + 44\beta\overline{\alpha} = 0\,,\end{aligned} \tag{4.109}$$

$$\beta(\phi_2(\delta\overline{\alpha} + 2\delta\overline{\pi} + 3\overline{\alpha}^2 - 18\overline{\pi}\beta + 6\overline{\alpha\pi} - 9\overline{\alpha}\beta) + 24\overline{\phi}_2\beta^2) = 0\,. \tag{4.110}$$

Eliminating $\delta(\overline{\alpha} + 2\pi)$ from (4.109) and (4.110) we find

$$\frac{\beta^2}{\phi_2{}^2 - 4}\left(19\overline{\pi}\phi_2 + 10\overline{\alpha}\phi_2 - 12\beta\overline{\phi}_2\right) = 0\,. \tag{4.111}$$

The denominator $-\phi_2{}^2 + 4$ in the expression above must be nonzero, since ϕ_2 cannot be constant. Otherwise, from (4.100), we would have $\beta = 0$.

In order to determine further side relations we still need to find the Pfaffians $\delta\alpha$, $\delta\overline{\beta}$, and $\delta\pi$, in terms of $\overline{\delta}\alpha$. From (NP6), (NP7), (NP8), (NP9) and (NP12) we have

$$\mathrm{D}\gamma = \alpha\overline{\pi} + \beta\pi + \Phi_{11}\,, \tag{4.112}$$

$$\overline{\delta}\pi = \mathrm{D}\lambda - \pi^2 - \pi\alpha + \pi\overline{\beta}\,, \tag{4.113}$$

$$\delta\pi = \mathrm{D}\mu - \pi\overline{\pi} + \pi\overline{\alpha} - \beta\pi\,, \tag{4.114}$$

$$\mathrm{D}\nu = \Delta\pi + \pi\mu + \pi\mu + \overline{\pi}\lambda + \pi\gamma - \pi\overline{\gamma} - 1 + \Phi_{21}\,, \tag{4.115}$$

$$\delta\alpha = \overline{\delta}\beta + \alpha\overline{\alpha} + \beta\overline{\beta} - 2\overline{\beta}\overline{\alpha} + \Phi_{11}\,. \tag{4.116}$$

Using (4.104) and (4.107), we now evaluate the NP commutator $[\delta, \mathrm{D}]\Phi_{22} - [\Delta, \mathrm{D}]\Phi_{12}$ to obtain

$$\begin{aligned}&-2\delta\overline{\beta} + \mathrm{D}\Delta\Phi_{12} + 2\overline{\pi}\Phi_{11}\overline{\gamma} + 2\beta\overline{\beta} + 4\beta^2 + 2\pi\overline{\lambda}\Phi_{11} - 6\beta\Phi_{21}\overline{\pi} + 4\pi\overline{\pi} + 2\pi\overline{\alpha}\\&+ 4\beta\overline{\alpha} - \mathrm{D}\delta\Phi_{22} - 4\overline{\alpha}\Phi_{12}\pi - 2\beta\Phi_{12}\pi - 2\overline{\alpha}\Phi_{21}\overline{\pi} + 2\overline{\pi}\Phi_{11}\gamma - 2\overline{\pi}\Phi_{11}\overline{\mu}\\&+ 4\overline{\pi}\Phi_{11}\mu + 2\overline{\pi}\Phi_{12}\pi + 2\Phi_{21}\delta\overline{\pi} + 2\Phi_{12}\delta\pi - 2\Phi_{11}\Delta\overline{\pi} + 2\overline{\alpha}\overline{\beta} - 6\overline{\pi}\overline{\beta}\\&+ 2\Phi_{21}\overline{\pi}^2 + 2\overline{\pi}\Phi_{12}\overline{\beta} = 0\,.\end{aligned} \tag{4.117}$$

From $[\Delta, \mathrm{D}]\Phi_{12}$ we obtain

$$\begin{aligned}\mathrm{D}\Delta\Phi_{12} = {}& 2\Phi_{11}\Delta\overline{\pi} - 2\overline{\pi}\Phi_{11}\gamma - 2\overline{\pi}\Phi_{11}\overline{\gamma} + 2\pi\overline{\alpha} + 4\pi\overline{\pi} + 2\pi\overline{\lambda}\Phi_{11} - 2\overline{\alpha}\Phi_{12}\pi\\&- 2\overline{\pi}\beta + 2\overline{\pi}\Phi_{11}\overline{\mu} - 2\overline{\pi}\Phi_{12}\overline{\beta}.\end{aligned} \tag{4.118}$$

From (NP26),

$$\delta\Phi_{22} = \Delta\Phi_{12} + 2\overline{\gamma} + 4\overline{\mu} - 2\overline{\nu}\Phi_{11} + 2\overline{\lambda}\Phi_{21} + 2\Phi_{12}\overline{\gamma} + 2\Phi_{12}\mu - 2\Phi_{22}\beta - 2\Phi_{22}\overline{\alpha}\,. \tag{4.119}$$

Substituting (4.119), (4.118), (4.102), (4.112), (4.113), (4.114) and (4.115) into (4.117) we obtain

$$\delta\overline{\beta} = -\overline{\alpha}\overline{\beta} - 4\overline{\pi}\overline{\beta} - 2\mathrm{D}\overline{\mu} - \beta\overline{\beta} + 2\pi\overline{\pi} - 2\Phi_{11}\,. \tag{4.120}$$

Using (4.99), (4.100), (4.108), (4.114), (4.116) and (4.120) in the commutator $[\overline{\delta}, \delta]\phi_2$, and solving for $\mathrm{D}\mu$ we obtain

$$\mathrm{D}\mu = -\frac{1}{12\overline{\phi}_2}(8\overline{\phi}_2\alpha\overline{\pi} + 24\overline{\phi}_2\beta\pi - 8\phi_2\alpha\overline{\pi} + 10\overline{\phi}_2\Phi_{11} + 12\beta\overline{\phi}_2\alpha - 12\overline{\alpha}\phi_2\overline{\beta}$$
$$+4\overline{\phi}_2\pi\overline{\pi} + 4\overline{\alpha}\overline{\phi}_2\alpha - 4\overline{\alpha}\phi_2\alpha - 2\phi_2\Phi_{11} - 24\phi_2\overline{\pi}\overline{\beta} + 8\overline{\phi}_2\pi\overline{\alpha}\,.) \tag{4.121}$$

The explicit form of $\mathrm{D}\lambda$ can be obtained by substituting (4.113) into (4.109) and (4.110):

$$\mathrm{D}\lambda = \frac{1}{2(-\overline{\phi}_2{}^2+4)}\left(\overline{\phi}_2{}^2(3\alpha^2 - 2\pi^2 + 4\alpha\pi - 16\overline{\beta}\pi - 9\overline{\beta}\alpha + \overline{\delta}\alpha)\right.$$
$$\left.+8\pi^2 - 16\pi\alpha - 12\alpha^2 + 24\overline{\phi}_2\phi_2\overline{\beta}^2 - 88\pi\overline{\beta} - 44\overline{\beta}\alpha - 4\overline{\delta}\alpha)\right)\,. \tag{4.122}$$

From V_{12} we get

$$\mathrm{D}\pi = 0\,. \tag{4.123}$$

We have now determined all Pfaffians needed for finding new side relations, using integrability conditions. Before we proceed let us go back to (4.111) and introduce a further simplification by expressing ϕ_2 in terms of $\overline{\phi}_2$,

$$\phi_2 = \frac{\overline{\phi}_2}{12\overline{\beta}}(19\pi + 10\alpha)\,. \tag{4.124}$$

Substituting (4.124) into the numerator of the complex conjugate of (4.111), we find

$$S_1 := 361\pi\overline{\pi} + 190\alpha\overline{\pi} + 190\pi\overline{\alpha} + 100\alpha\overline{\alpha} - 144\beta\overline{\beta} = 0\,. \tag{4.125}$$

From the NP commutator $[\overline{\delta}\,,\,\delta](\alpha + 2\pi)$ we get

$$\begin{aligned}&(-820\,\alpha^2\,\Phi_{11} + 95\,\pi^2\,\Phi_{11} + 2680\,\alpha^3\,\overline{\alpha} - 5448\,\alpha\overline{\beta}\,\Phi_{11} - 10236\,\pi\overline{\beta}\,\Phi_{11}\\&+ 5360\,\alpha^3\,\overline{\pi} - 9648\,\overline{\beta}\,\alpha^2\,\beta - 38016\,\pi^2\,\beta\,\overline{\beta} - 13926\,\pi^2\,\overline{\beta}\,\overline{\alpha} - 2508\,\pi^2\,\overline{\pi}\,\overline{\beta}\\&- 3816\,\overline{\beta}\,\alpha^2\,\overline{\alpha} - 720\,\alpha^2\,\overline{\pi}\,\overline{\beta} + 61628\,\alpha\,\pi^2\,\overline{\pi} + 40128\,\pi^3\,\overline{\pi} + 20064\,\pi^3\,\overline{\alpha}\\&+ 30814\,\pi^2\,\alpha\,\overline{\alpha} + 15752\,\pi\,\alpha^2\,\overline{\alpha} - 38376\,\pi\,\overline{\beta}\,\alpha\,\beta + 31504\,\pi\,\alpha^2\,\overline{\pi}\\&- 14592\,\alpha\,\pi\,\overline{\beta}\,\overline{\alpha} - 2688\,\pi\,\alpha\,\overline{\pi}\,\overline{\beta} - 1508\,\pi\,\alpha\,\Phi_{11})\Big/(19\,\pi + 10\,\alpha) = 0\,.\end{aligned} \tag{4.126}$$

It follows from (4.124) that the numerator $19\,\pi + 10\,\alpha$ in the preceeding equation must be non-zero. Solving this equation for Φ_{11} we obtain

$$\begin{aligned}\Phi_{11} = {}&(2680\,\alpha^3\,\overline{\alpha} + 5360\,\alpha^3\,\overline{\pi} - 9648\,\overline{\beta}\,\alpha^2\,\beta - 38016\,\pi^2\,\beta\,\overline{\beta} - 13926\,\pi^2\,\overline{\beta}\,\overline{\alpha}\\&- 2508\,\pi^2\,\overline{\pi}\,\overline{\beta} - 3816\,\overline{\beta}\,\alpha^2\,\overline{\alpha} - 720\,\alpha^2\,\overline{\pi}\,\overline{\beta} + 61628\,\alpha\,\pi^2\,\overline{\pi} + 40128\,\pi^3\,\overline{\pi}\\&+ 20064\,\pi^3\,\overline{\alpha} + 30814\,\pi^2\,\alpha\,\overline{\alpha} + 15752\,\pi\,\alpha^2\,\overline{\alpha} - 38376\,\pi\,\overline{\beta}\,\alpha\,\beta + 31504\,\pi\,\alpha^2\,\overline{\pi}\\&- 14592\,\alpha\,\pi\,\overline{\beta}\,\overline{\alpha} - 2688\,\pi\,\alpha\,\overline{\pi}\,\overline{\beta})\Big/\\&(820\,\alpha^2 - 95\,\pi^2 + 5448\,\overline{\beta}\,\alpha + 10236\,\pi\,\overline{\beta} + 1508\,\pi\,\alpha)\,,\end{aligned} \tag{4.127}$$

where, for now, we assume that the denominator in the expression above,

$$d_1 := 820\alpha^2 - 95\pi^2 + 5448\overline{\beta}\alpha + 10236\pi\overline{\beta} - 1508\pi\alpha\,, \tag{4.128}$$

is non-zero.

Evaluating $\delta\phi_2 + 2\phi_2\beta = 0$ (cf. Eq. (4.104)), using (4.124), Rand solving for Φ_{11} we find

$$\begin{aligned}\Phi_{11} = {} & (700\,\alpha^2\,\overline{\alpha} + 5300\,\alpha\,\pi\,\overline{\pi} + 2650\,\pi\,\alpha\,\overline{\alpha} + 2508\,\pi^2\,\overline{\alpha} + 1400\,\alpha^2\,\overline{\pi} \\ & - 2520\,\overline{\beta}\,\alpha\,\beta - 4752\,\pi\,\beta\,\overline{\beta} - 1650\,\pi\,\overline{\beta}\,\overline{\alpha} - 132\,\pi\,\overline{\pi}\,\overline{\beta} + 5016\,\pi^2\,\overline{\pi} \\ & - 900\,\overline{\beta}\,\alpha\,\overline{\alpha} - 72\,\alpha\,\overline{\pi}\,\overline{\beta}) \Big/ (-437\,\pi + 372\,\overline{\beta} - 230\,\alpha)\,,\end{aligned} \tag{4.129}$$

where we assume, for now, that

$$d_2 := 437\pi - 372\overline{\beta} + 230\alpha \neq 0\,. \tag{4.130}$$

Evaluating the commutators $[\overline{\delta}\,,\,\delta]\overline{\beta}$ and $[\overline{\delta}\,,\,\delta]\overline{\phi}_2$, and solving each one for $\overline{\delta}\alpha$ we find, respectively,

$$\begin{aligned}\overline{\delta}\alpha = {} & (-308\,\pi\,\alpha\,\Phi_{11} + 8016\,\pi\,\alpha^2\,\overline{\pi} + 234\,\overline{\beta}\,\alpha^2\,\beta + 3972\,\pi\,\alpha^2\,\overline{\alpha} + 78\,\overline{\beta}\,\alpha^2\,\overline{\alpha} \\ & + 759\,\alpha\,\overline{\beta}\,\Phi_{11} - 29\,\alpha^2\,\Phi_{11} + 1440\,\alpha^3\,\overline{\pi} + 702\,\alpha^3\,\overline{\alpha} + 1386\,\pi\,\overline{\beta}\,\Phi_{11} \\ & - 513\,\pi^2\,\Phi_{11} + 1296\,\pi\,\overline{\beta}\,\alpha\,\beta + 528\,\pi^2\,\overline{\beta}\,\overline{\alpha} + 14872\,\alpha\,\pi^2\,\overline{\pi} + 7436\,\pi^2\,\alpha\,\overline{\alpha} \\ & + 1584\,\pi^2\,\beta\,\overline{\beta} + 4598\,\pi^3\,\overline{\alpha} + 9196\,\pi^3\,\overline{\pi} + 432\,\alpha\,\pi\,\overline{\beta}\,\overline{\alpha} - 108\,\pi\,\alpha^2\,\beta - 54\,\alpha^3\,\beta) \\ & \Big/ (12\,\pi\,\overline{\alpha} + 6\,\alpha\,\overline{\alpha} + 36\,\beta\,\pi + 18\,\beta\,\alpha + 3\,\Phi_{11})\,,\end{aligned} \tag{4.131}$$

$$\overline{\delta}\alpha = -\frac{6\,\alpha^2\,\overline{\pi} - 572\,\pi\,\overline{\pi}\,\overline{\beta} - 286\,\pi\,\overline{\beta}\,\overline{\alpha} - 157\,\overline{\beta}\,\alpha\,\overline{\alpha} + 3\,\alpha^2\,\overline{\alpha} - 314\,\alpha\,\overline{\pi}\,\overline{\beta}}{\overline{\alpha} + 2\,\overline{\pi}}\,, \tag{4.132}$$

where we assume, for the moment, that

$$d_3 := 4\pi\overline{\alpha} + 2\alpha\overline{\alpha} + 12\beta\pi + 6\beta\alpha + \Phi_{11} \neq 0\,, \tag{4.133}$$

$$d_4 := \alpha + 2\pi \neq 0\,. \tag{4.134}$$

By subtracting (4.131) from (4.132) and solving for Φ_{11} we have

$$\begin{aligned}\Phi_{11} = {} & (-864\,\overline{\beta}\,\alpha^2\,\overline{\alpha} + 7436\,\pi^2\,\alpha\,\overline{\alpha} - 3168\,\alpha\,\pi\,\overline{\beta}\,\overline{\alpha} + 4008\,\pi\,\alpha^2\,\overline{\alpha} + 4598\,\pi^3\,\overline{\alpha} \\ & - 2904\,\pi^2\,\overline{\beta}\,\overline{\alpha} + 720\,\alpha^3\,\overline{\alpha} + 9196\,\pi^3\,\overline{\pi} - 8712\,\pi^2\,\beta\,\overline{\beta} + 14872\,\alpha\,\pi^2\,\overline{\pi} \\ & - 2592\,\overline{\beta}\,\alpha^2\,\beta + 8016\,\pi\,\alpha^2\,\overline{\pi} + 1440\,\alpha^3\,\overline{\pi} - 9504\,\pi\,\overline{\beta}\,\alpha\,\beta) \Big/ \\ & (-288\,\overline{\beta}\,\alpha + 20\,\alpha^2 - 528\,\pi\,\overline{\beta} + 513\,\pi^2 + 308\,\pi\,\alpha)\,,\end{aligned} \tag{4.135}$$

where the denominator of (4.135),

$$d_5 := -288\,\overline{\beta}\,\alpha + 20\,\alpha^2 - 528\,\pi\,\overline{\beta} + 513\,\pi^2 + 308\,\pi\,\alpha\,, \tag{4.136}$$

is assumed to be nonzero for the moment.

Subtracting (4.127) from (4.129) and taking the numerator we find

$$\begin{aligned}
S_2 := {} & 39914208\,\pi^2\,\alpha^2\,\overline{\pi} + 19957104\,\pi^2\,\alpha^2\,\overline{\alpha} + 35332704\,\pi^3\,\overline{\pi}\,\overline{\beta} \\
& + 12279168\,\pi^3\,\overline{\beta}\,\overline{\alpha} + 1190400\,\overline{\alpha}\,\alpha^4 - 12773376\,\overline{\beta}^2\,\overline{\alpha}\,\pi\,\alpha - 3483648\,\overline{\beta}^2\,\overline{\alpha}\,\alpha^2 \\
& + 1200960\,\overline{\beta}\,\overline{\alpha}\,\alpha^3 - 11708928\,\overline{\beta}^2\,\overline{\alpha}\,\pi^2 - 20739456\,\pi\,\overline{\beta}\,\alpha^2\,\beta \\
& + 8008704\,\alpha^2\,\pi\,\overline{\beta}\,\overline{\alpha} + 30335616\,\pi\,\alpha^2\,\overline{\pi}\,\overline{\beta} + 56708640\,\pi^2\,\alpha\,\overline{\pi}\,\overline{\beta} \\
& - 32440608\,\pi^2\,\overline{\beta}\,\alpha\,\beta - 16161552\,\pi^3\,\beta\,\overline{\beta} + 8022720\,\pi\,\alpha^3\,\overline{\alpha} + 5408640\,\alpha^3\,\overline{\pi}\,\overline{\beta} \\
& + 16045440\,\pi\,\alpha^3\,\overline{\pi} + 8529708\,\pi^4\,\overline{\alpha} + 43221504\,\alpha\,\pi^3\,\overline{\pi} + 21610752\,\pi^3\,\alpha\,\overline{\alpha} \\
& + 17059416\,\pi^4\,\overline{\pi} + 2380800\,\alpha^4\,\overline{\pi} + 17343792\,\alpha\,\pi^2\,\overline{\beta}\,\overline{\alpha} - 10139904\,\overline{\beta}^2\,\alpha^2\,\beta \\
& - 124416\,\alpha^2\,\overline{\pi}\,\overline{\beta}^2 - 418176\,\pi^2\,\overline{\pi}\,\overline{\beta}^2 - 456192\,\pi\,\alpha\,\overline{\pi}\,\overline{\beta}^2 - 4285440\,\overline{\beta}\,\alpha^3\,\beta \\
& - 37407744\,\pi\,\overline{\beta}^2\,\alpha\,\beta - 34499520\,\pi^2\,\beta\,\overline{\beta}^2 = 0\,.
\end{aligned} \tag{4.137}$$

By subtracting (4.129) from (4.135) and taking the numerator, we find the third side relation:

$$\begin{aligned}
S_3 := {} & -9374472\,\pi^2\,\alpha^2\,\overline{\pi} - 4687236\,\pi^2\,\alpha^2\,\overline{\alpha} + 6137076\,\pi^3\,\overline{\pi}\,\overline{\beta} + 5150178\,\pi^3\,\overline{\beta}\,\overline{\alpha} \\
& - 179600\,\overline{\alpha}\,\alpha^4 - 2128896\,\overline{\beta}^2\,\overline{\alpha}\,\pi\,\alpha - 580608\,\overline{\beta}^2\,\overline{\alpha}\,\alpha^2 + 686160\,\overline{\beta}\,\overline{\alpha}\,\alpha^3 \\
& - 1951488\,\overline{\beta}^2\,\overline{\alpha}\,\pi^2 + 4189824\,\pi\,\overline{\beta}\,\alpha^2\,\beta + 4040184\,\alpha^2\,\pi\,\overline{\beta}\,\overline{\alpha} \\
& + 5272368\,\pi\,\alpha^2\,\overline{\pi}\,\overline{\beta} + 9852984\,\pi^2\,\alpha\,\overline{\pi}\,\overline{\beta} + 8913384\,\pi^2\,\overline{\beta}\,\alpha\,\beta \\
& + 6244920\,\pi^3\,\beta\,\overline{\beta} - 1505080\,\pi\,\alpha^3\,\overline{\alpha} + 940320\,\alpha^3\,\overline{\pi}\,\overline{\beta} - 3010160\,\pi\,\alpha^3\,\overline{\pi} \\
& - 3295930\,\pi^4\,\overline{\alpha} - 12877972\,\alpha\,\pi^3\,\overline{\pi} - 6438986\,\pi^3\,\alpha\,\overline{\alpha} - 6591860\,\pi^4\,\overline{\pi} \\
& - 359200\,\alpha^4\,\overline{\pi} + 7909932\,\alpha\,\pi^2\,\overline{\beta}\,\overline{\alpha} - 1689984\,\overline{\beta}^2\,\alpha^2\,\beta - 20736\,\alpha^2\,\overline{\pi}\,\overline{\beta}^2 \\
& - 69696\,\pi^2\,\overline{\pi}\,\overline{\beta}^2 - 76032\,\pi\,\alpha\,\overline{\pi}\,\overline{\beta}^2 + 646560\,\overline{\beta}\,\alpha^3\,\beta - 6234624\,\pi\,\overline{\beta}^2\,\alpha\,\beta \\
& - 5749920\,\pi^2\,\beta\,\overline{\beta}^2 = 0\,.
\end{aligned} \tag{4.138}$$

We can eliminate one complex variable by defining new variables x_1 and x_2 by

$$x_1 := \frac{\alpha}{\pi}\,, \tag{4.139}$$

$$x_2 := \frac{\beta}{\overline{\pi}}\,. \tag{4.140}$$

The side relations now assume the form (modulo non-zero factors)

$$S_1 := 361 + 190\,x_1 + 190\,\overline{x_1} + 100\,x_1\,\overline{x_1} - 144\,x_2\,\overline{x_2} = 0\,, \tag{4.141}$$

$$\begin{aligned}S_2 := &-1663092\, x_1^2\, \overline{x_1} - 99200\, \overline{x_1}\, x_1^4 - 2527968\, x_1^2\, \overline{x_2} + 975744\, \overline{x_2}^2\, \overline{x_1} \\ &- 1023264\, \overline{x_2}\, \overline{x_1} + 2874960\, x_2\, \overline{x_2}^2 - 4725720\, x_1\, \overline{x_2} - 198400\, x_1^4 \\ &- 3326184\, x_1^2 - 1337120\, x_1^3 + 34848\, \overline{x_2}^2 - 3601792\, x_1 - 710809\, \overline{x_1} \\ &- 2944392\, \overline{x_2} + 844992\, \overline{x_2}^2\, x_1^2\, x_2 + 1064448\, \overline{x_2}^2\, \overline{x_1}\, x_1 \\ &+ 290304\, \overline{x_2}^2\, \overline{x_1}\, x_1^2 - 100080\, \overline{x_2}\, \overline{x_1}\, x_1^3 + 1728288\, \overline{x_2}\, x_1^2\, x_2 \\ &- 667392\, x_1^2\, \overline{x_2}\, \overline{x_1} + 2703384\, \overline{x_2}\, x_1\, x_2 + 1346796\, x_2\, \overline{x_2} - 668560\, x_1^3\, \overline{x_1} \\ &- 450720\, x_1^3\, \overline{x_2} - 1800896\, x_1\, \overline{x_1} + 10368\, x_1^2\, \overline{x_2}^2 + 38016\, x_1\, \overline{x_2}^2 \\ &- 1445316\, x_1\, \overline{x_2}\, \overline{x_1} + 357120\, \overline{x_2}\, x_1^3\, x_2 \\ &+ 3117312\, \overline{x_2}^2\, x_1\, x_2 - 1421618 = 0\,, \end{aligned} \tag{4.142}$$

$$\begin{aligned}S_3 := &\ 2343618\, x_1^2\, \overline{x_1} + 89800\, \overline{x_1}\, x_1^4 - 2636184\, x_1^2\, \overline{x_2} + 975744\, \overline{x_2}^2\, \overline{x_1} \\ &- 2575089\, \overline{x_2}\, \overline{x_1} + 2874960\, x_2\, \overline{x_2}^2 + 3295930 - 4926492\, x_1\, \overline{x_2} + 179600\, x_1^4 \\ &+ 4687236\, x_1^2 + 1505080\, x_1^3 + 34848\, \overline{x_2}^2 + 6438986\, x_1 + 1647965\, \overline{x_1} \\ &- 3068538\, \overline{x_2} + 844992\, \overline{x_2}^2\, x_1^2\, x_2 + 1064448\, \overline{x_2}^2\, \overline{x_1}\, x_1 \\ &+ 290304\, \overline{x_2}^2\, \overline{x_1}\, x_1^2 - 343080\, \overline{x_2}\, \overline{x_1}\, x_1^3 - 2094912\, \overline{x_2}\, x_1^2\, x_2 \\ &- 2020092\, x_1^2\, \overline{x_2}\, \overline{x_1} - 4456692\, \overline{x_2}\, x_1\, x_2 - 3122460\, x_2\, \overline{x_2} + 752540\, x_1^3\, \overline{x_1} \\ &- 470160\, x_1^3\, \overline{x_2} + 3219493\, x_1\, \overline{x_1} + 10368\, x_1^2\, \overline{x_2}^2 + 38016\, x_1\, \overline{x_2}^2 \\ &- 3954966\, x_1\, \overline{x_2}\, \overline{x_1} - 323280\, \overline{x_2}\, x_1^3\, x_2 + 3117312\, \overline{x_2}^2\, x_1\, x_2 = 0\,. \end{aligned} \tag{4.143}$$

Applying gsolve to the set of equations formed by S_2, S_3, their complex conjugates, and S_1, we find the the only possible solution for which $x_1 \neq 0$ and $x_2 \neq 0$ is given by

$$324x_2\overline{x_2} - 1 = 0\,, \qquad 6x_1 + 11 = 0\,, \qquad 6\overline{x_1} + 11 = 0\,. \tag{4.144}$$

By substituting (4.144) into any of the previous expressions for Φ_{11} we find that $\Phi_{11} = 0$. Using this and $\pi = -\alpha\, 6/11$ in V_{11} one gets

$$-1761\overline{\beta}\alpha\overline{\alpha} + 5\alpha^2\overline{\alpha} + 3267\beta\overline{\beta}^2 - 5445\beta\alpha\dot{\overline{\beta}} = 0\,. \tag{4.145}$$

It is easy to verify that (4.145) and the first equation in (4.42), now given in the form $1089\beta\overline{\beta} - \alpha\overline{\alpha} = 0$, imply that $\alpha = \beta = 0$.

Let us consider first the cases in which each one of the denominators d_1, d_2, d_3, d_4 and d_5, given respectively by (4.128), (4.130), (4.133), (4.134) and (4.136), is zero.

(i) $\mathbf{d_1 = 0}$

From (4.128) we have, in terms of the variables x_1 and x_2:

$$820\, x_1^2 + 1508\, x_1 + 5448\, \overline{x_2}\, x_1 - 95 + 10236\, \overline{x_2} = 0\,. \tag{4.146}$$

Since the numerator of (4.127) must be zero too, we have

$$\begin{aligned}
&2680\, x_1^3 + 1340\, x_1^3\, \overline{x_1} - 1908\, \overline{x_2}\, \overline{x_1}\, x_1^2 + 7876\, x_1^2\, \overline{x_1} \\
&- 360\, \overline{x_2}\, x_1^2 + 15752\, x_1^2 - 4824\, \overline{x_2}\, x_2\, x_1^2 + 30814\, x_1 - 1344\, \overline{x_2}\, x_1 \\
&+ 15407\, x_1\, \overline{x_1} - 19188\, \overline{x_2}\, x_2\, x_1 - 7296\, \overline{x_2}\, \overline{x_1}\, x_1 + 20064 + 10032\, \overline{x_1} \\
&- 19008\, \overline{x_2}\, x_2 - 6963\, \overline{x_2}\, \overline{x_1} - 1254\, \overline{x_2} = 0\,. \qquad (4.147)
\end{aligned}$$

Applying `gsolve` to the set of equations consisting of (4.146), (4.147), their complex conjugates, and $S_1 = 0$ (cf. eq. (4.141)) we find that all possible solutions require that $x_2 = 0$.

(ii) $\mathbf{d_2 = 0}$

From (4.130) we have

$$230\, x_1 + 437 - 372\, \overline{x_2} = 0\,. \qquad (4.148)$$

Since the numerator of (4.130) must also vanish we have

$$\begin{aligned}
&350\, x_1^2\, \overline{x_1} + 700\, x_1^2 + 1325\, x_1\, \overline{x_1} - 36\, \overline{x_2}\, x_1 + 2650\, x_1 \\
&- 450\, \overline{x_2}\, \overline{x_1}\, x_1 - 1260\, \overline{x_2}\, x_2\, x_1 - 825\, \overline{x_2}\, \overline{x_1} - 66\, \overline{x_2} - 2376\, \overline{x_2}\, x_2 + 2508 \\
&+ 1254\, \overline{x_1} = 0\,. \qquad (4.149)
\end{aligned}$$

Applying `gsolve` to the set of equations consisting of (4.148), (4.149), their complex conjugates, and $S_1 = 0$ we find again that all solutions require that $x_2 = 0$.

(iii) $\mathbf{d_3 = 0}$

Solving $d_3 = 0$ for Φ_{11}, and using the variables x_1 and x_2, we find

$$\Phi_{11} = -2\pi\overline{\pi}(x_1 + 2)(3x_2 + \overline{x_1})\,. \qquad (4.150)$$

By subtracting (4.150) from its complex conjugate we obtain

$$E_1 := -6x_2 - 3x_1x_2 - 2\overline{x_1} + 6\overline{x_2} + 3\overline{x_2x_1} + 2x_1 = 0\,. \qquad (4.151)$$

Subtracting (4.150) from (4.127) and taking the numerator, we get

$$\begin{aligned}
E_2 &:= 246x_2x_1^2 + 268x_1^2 + 216x_1^2\overline{x_1}^2 + 477x_1x_2 + 1152\overline{x_2}x_1x_2 + 692x_1\overline{x_1} \\
&-36\overline{x_2}x_1 + 1066x_1 + 354\overline{x_2}x_1\overline{x_1} + 2232x_2\overline{x_2} + 518\overline{x_1} - 66\overline{x_2} + 1056 \\
&+711\overline{x_2x_1} - 30x_2 = 0\,. \qquad (4.152)
\end{aligned}$$

An additional side relation is obtained by subtracting the complex conjugate of (4.150) from (4.129) and taking the numerator of the resulting expression:

$$\begin{aligned} E_3 &:= 144x_2\overline{x_2} + 2691x_1x_2 + 2622x_2 + 144\overline{x_2}x_1x_2 + 690x_2x_1^2 + 78x_2x_1\overline{x_1} \\ &-120x_1^2\overline{x_1} - 428x_1\overline{x_1} + 81x_2x_1 - 380\overline{x_1} - 2508 + 66\overline{x_2} - 2650x_1 \\ &+36\overline{x_2}x_1 - 700x_1^2 = 0\,. \end{aligned} \tag{4.153}$$

Applying **grobner** to the set consisting of E_2, its complex conjugate, E_1, E_3 and S_1 we find that this system admits no solution.

(iv) $\mathbf{d_4 = 0}$

When the denominator of (4.132) is zero, $d_4 := \alpha + 2\pi = 0$, its numerator must be zero, implying that $\alpha = \overline{\beta}(443/3)$. This, on the other hand implies immediately, from $S_1 = 0$ (cf. (4.125), that $\beta = 0$.

(v) $\mathbf{d_5 = 0}$

In terms of variables x_1 and x_2 the equations for the numerator and denominator are, in this case, given respectively by

$$(11 + 6x_1)^2(38 + 19\overline{x_1}`12\overline{x_2}\overline{x_1} + 10x_1\overline{x_1} + 20x_1 - 36x_2\overline{x_2}) = 0\,, \tag{4.154}$$

$$308x_1 + 513 + 20x_1^2 - 528\overline{x_2} - 288\overline{x_2}x_1 = 0\,. \tag{4.155}$$

Applying **gsolve** to the polynomial system defined by the system of polynomials defined by (4.154), (4.155),their complex cojugates, and $S_1 = 0$, we find that there are no possible solutions.

Next, we shall prove that if either $\alpha = 0$ or $\beta = 0$ or $\pi = 0$ then we must have $\phi_2 = 0$. We notice that this proof is concerned with a case that is more general than that one considered by Anderson et al. [7]. They have used the assumption $\alpha = \beta = \pi = 0$ as the starting point.

Let us begin with the case $\alpha = 0$.

(i) $\boldsymbol{\alpha = 0}$.

In this case, from (4.101) and (4.116), we have

$$\overline{\delta\beta} = -\overline{\beta}^2\,, \tag{4.156}$$

$$\overline{\delta}\beta = -\beta\overline{\beta} - \Phi_{11}\,. \tag{4.157}$$

From the commutator $[\overline{\delta}\,,\,\delta]\overline{\beta}$ we now obtain

$$\beta\Phi_{11} = 0\,. \tag{4.158}$$

Let us suppose first that $\beta = 0$, then $\Phi_{11} = 0$, from (4.157). From (NP25), $\pi = 0$. From II_{00} and IV_{10} we get

$$\delta\phi_2 = 0\,, \qquad \overline{\delta}\phi_2 = 0\,. \tag{4.159}$$

From the NP equations,

$$\mathrm{D}\gamma = \mathrm{D}\lambda = \mathrm{D}\mu = \mathrm{D}\Phi_{12} = \mathrm{D}\Phi_{22} = \delta\Phi_{12} = \delta\Phi_{21} = 0\,, \tag{4.160}$$

$$\delta\gamma = \Phi_{12}\,,\ \overline{\delta}\gamma = 1\,,\ \mathrm{D}\nu = -1 + \Phi_{21}\,. \tag{4.161}$$

An interesting relation comes from III_{00}:

$$\delta(2\gamma + 4\mu) + 2\Phi_{12} + 5\phi_2\overline{\phi}_2 = 0\,. \tag{4.162}$$

Thus, we must now determine $\delta\mu$. From the commutators $[\overline{\delta}\,,\,\Delta]$, $[\delta\,,\,\Delta]$, we obtain

$$\overline{\delta}(\Delta\phi_2) = 0\,, \qquad \overline{\delta}(\Delta\phi_2) = 0\,. \tag{4.163}$$

From IV_{00},

$$\Delta(\phi_2 + \overline{\phi}_2) + 12\phi_2\overline{\mu} + 12\overline{\phi}_2\mu + 6\overline{\gamma}\phi_2 + 6\gamma\overline{\phi}_2 + 2\phi_2\gamma + 2\overline{\phi}_2\overline{\gamma} = 0\,. \tag{4.164}$$

Applying δ to this equation, using (4.163), (4.161) and (4.160), we obtain

$$6\phi_2\delta\overline{\mu} + 6\overline{\phi}\delta\mu + \phi_2\Phi_{12} + 3\overline{\phi}_2\Phi 12 - 3\phi_2 - \overline{\phi}_2 = 0\,. \tag{4.165}$$

On the other hand, from V_{10},

$$\delta(41\overline{\mu} - 8\mu) + 16\Phi_{12} + 42\overline{\phi}_2{}^2 + 15\phi_2\overline{\phi}_2\Phi_{12} + 3\phi_2\overline{\phi}_2 - 20 = 0\,. \tag{4.166}$$

From these two equations we can determine $\delta\mu$:

$$\delta\mu = \frac{1}{24(\overline{\phi}_2 + 5\phi_2)}\left(28\overline{\phi}_2\Phi_{21} + 45\overline{\phi}_2{}^2\phi_2\Phi_{21} + 126\phi_2{}^2\overline{\phi}_2 + 9\phi_2\overline{\phi}_2{}^2 - 60\phi_2\Phi_{21} + 20\phi_2\right), \tag{4.167}$$

where, obviously, $\overline{\phi}_2 + 5\phi_2 \neq 0$. Substituting this Pfaffian in (4.162) we obtain

$$52\phi_2\Phi_{12} + 60\overline{\phi}_2\Phi_{12} + 45\phi_2{}^2\overline{\phi}_2\Phi_{21} + 276\phi_2\overline{\phi}_2{}^2 + 39\phi_2{}^2\overline{\phi}_2 + 20\overline{\phi}_2 = 0\,. \tag{4.168}$$

Solving this equation for Φ_{12} we obtain

$$\Phi_{12} = -\frac{\phi_2(276\phi_2\overline{\phi}_2 + 39\overline{\phi}_2{}^2 + 20)}{52\overline{\phi}_2 + 60\phi_2 + 45\phi_2\phi_2{}^2}\,, \tag{4.169}$$

where, for now, we suppose that the denominator of the previous equation,

$$e_1 := 52\overline{\phi}_2 + 60\phi_2 + 45\phi_2\phi_2{}^2\,, \tag{4.170}$$

is nonzero. The Pfaffian $\delta\lambda$ is now determined, since from (NP13),

$$\delta\lambda = \bar{\delta}\mu + \Phi_{21} + 1\,. \tag{4.171}$$

Now, from VI_{01} we have

$$\begin{aligned}
&(-16380\,\phi_2{}^3\,\Delta\phi_2\bar{\phi}_2{}^3 + 36180\,\phi_2{}^4\,\Delta\phi_2\bar{\phi}_2{}^2 + 11160\,\phi_2{}^3\,\Delta\bar{\phi}_2\bar{\phi}_2{}^3\\
&- 3120\,\phi_2\,\Delta\bar{\phi}_2\bar{\phi}_2^3 - 15390\,\phi_2{}^4\,\bar{\mu}\,\bar{\phi}_2{}^3 + 20250\,\phi_2{}^5\,\bar{\phi}_2{}^4\,\bar{\gamma}\\
&+ 79944\,\phi_2{}^2\,\gamma\,\bar{\phi}_2{}^3 + 339048\,\phi_2{}^3\,\mu\,\bar{\phi}_2{}^2 + 44064\,\phi_2{}^2\,\bar{\phi}_2\,\bar{\gamma}\\
&- 119832\,\phi_2{}^3\,\bar{\gamma}\,\bar{\phi}_2{}^2 + 15030\,\phi_2{}^4\,\bar{\phi}_2{}^3\,\bar{\gamma} + 72360\,\phi_2{}^5\,\gamma.\bar{\phi}_2{}^2\\
&+ 310112\,\phi_2{}^2\,\gamma\,\bar{\phi}_2 + 111780\,\phi_2{}^4\,\mu\,\bar{\phi}_2{}^3 - 6240\,\phi_2\,\Delta\phi_2\,\bar{\phi}_2{}^3\\
&+ 4800\,\phi_2\,\Delta\phi_2\bar{\phi}_2 + 480\,\phi_2\,\Delta\bar{\phi}_2\bar{\phi}_2 + 279800\,\phi_2{}^3\,\gamma\,\bar{\phi}_2{}^2\\
&- 10125\,\phi_2{}^4\,\Delta\phi_2\bar{\phi}_2{}^4 + 248400\,\phi_2{}^4\,\bar{\phi}_2\,\mu + 573728\,\phi_2{}^2\,\bar{\phi}_2\,\mu\\
&+ 42832\,\phi_2{}^2\,\Delta\phi_2\bar{\phi}_2{}^2 - 15600\,\phi_2\,\bar{\mu}\,\bar{\phi}_2{}^2 + 4680\,\phi_2{}^4\,\Delta\bar{\phi}_2\,\bar{\phi}_2{}^2\\
&+ 10125\,\phi_2{}^5\,\Delta\bar{\phi}_2\bar{\phi}_2{}^3 + 37260\,\phi_2{}^2\,\bar{\phi}_2{}^5\,\lambda - 16568\,\phi_2{}^2\,\bar{\phi}_2{}^3\,\bar{\gamma}\\
&- 79164\,\phi_2{}^3\,\bar{\mu}\,\bar{\phi}_2{}^2 + 97056\,\phi_2\,\bar{\phi}_2{}^4\,\lambda - 97200\,\phi_2{}^5\,\bar{\gamma}\,\bar{\phi}_2{}^2 - 4680\,\phi_2\,\gamma\,\bar{\phi}_2{}^4\\
&+ 220416\,\phi_2{}^4\,\gamma\,\bar{\phi}_2 - 59256\,\phi_2{}^4\,\bar{\mu}\,\bar{\phi}_2 - 12064\,\phi_2{}^2\,\Delta\bar{\phi}_2\,\bar{\phi}_2{}^2\\
&- 11340\,\phi_2{}^5\,\Delta\bar{\phi}_2\bar{\phi}_2 + 25920\,\phi_2\,\bar{\gamma}\,\bar{\phi}_2{}^2 - 30510\,\phi_2^3\,\gamma\,\bar{\phi}_2{}^4\\
&- 10530\,\phi_2{}^3\,\mu\,\bar{\phi}_2{}^4 - 14040\,\phi_2\,\mu\,\bar{\phi}_2{}^4 + 43740\,\phi_2{}^3\,\bar{\phi}_2{}^4\,\lambda + 39600\,\phi_2{}^4\,\gamma\,\bar{\phi}_2{}^3\\
&+ 279840\,\phi_2\,\mu\,\bar{\phi}_2{}^2 + 136872\,\phi_2{}^2\,\mu\,\bar{\phi}_2{}^3 + 45408\,\phi_2{}^3\,\Delta\phi_2\,\bar{\phi}_2\\
&+ 22320\,\phi_2{}^3\,\bar{\phi}_2{}^4\,\bar{\gamma} + 62400\,\bar{\phi}_2{}^3\,\lambda + 18720\,\phi_2{}^3\,\bar{\gamma} - 18720\,\phi_2{}^3\,\bar{\mu}\\
&+ 158080\,\phi_2{}^3\,\gamma + 287040\,\phi_2{}^3\,\mu - 13104\,\phi_2{}^4\,\Delta\bar{\phi}_2 + 416\,\phi_2{}^2\,\Delta\bar{\phi}_2\\
&+ 4160\,\phi_2{}^2\,\Delta\phi_2 - 22872\,\phi_2{}^3\,\Delta\bar{\phi}_2\bar{\phi}_2 - 13500\,\phi_2{}^2\,\Delta\phi_2\bar{\phi}_2{}^4\\
&- 37260\,\phi_2{}^5\,\bar{\mu}\,\bar{\phi}_2{}^2 - 96120\,\phi_2{}^4\,\bar{\phi}_2\,\bar{\gamma} - 6240\,\phi_2\,\bar{\phi}_2{}^4\,\bar{\gamma} - 35120\,\phi_2{}^2\,\bar{\mu}\,\bar{\phi}_2\\
&+ 58320\,\phi_2{}^3\,\bar{\phi}_2{}^2\,\lambda - 20520\,\phi_2{}^2\,\bar{\mu}\,\bar{\phi}_2{}^3 + 144684\,\phi_2{}^2\,\bar{\phi}_2{}^3\,\lambda + 59280\,\phi_2{}^2\,\bar{\phi}_2\,\lambda\\
&+ 123376\,\phi_2\,\bar{\phi}_2{}^2\,\lambda + 147360\,\phi_2\,\gamma\,\bar{\phi}_2{}^2 - 20250\,\phi_2{}^5\,\gamma\,\bar{\phi}_2{}^4)\Big/\\
&\Big((\,52\,\phi_2 + 60\,\bar{\phi}_2 + 45\,\phi_2{}^2\,\bar{\phi}_2\,)\,(\,52\,\bar{\phi}_2 + 60\,\phi_2 + 45\,\phi_2\,\bar{\phi}_2{}^2\,)\Big) = 0\,.
\end{aligned} \tag{4.172}$$

Applying δ to (4.172) and using (4.163), (4.167), (4.168), (4.169), and (4.171) we find

$$\begin{aligned}
&-(134325\,\phi_2{}^5\,\bar{\phi}_2{}^4 + 361260\,\phi_2{}^5\,\bar{\phi}_2{}^2 - 164700\,\phi_2{}^4\,\bar{\phi}_2{}^5 + 1121145\,\phi_2{}^4\,\bar{\phi}_2{}^3\\
&\quad + 260172\,\phi_2{}^4\,\bar{\phi}_2 - 67335\,\phi_2^3\,\bar{\phi}_2{}^4 + 116516\,\phi_2{}^3\,\bar{\phi}_2{}^2 + 28080\,\phi_2{}^3\\
&\quad - 25020\,\phi_2{}^2\,\bar{\phi}_2{}^5 - 246324\,\phi_2{}^2\,\bar{\phi}_2{}^3 - 64672\,\phi_2{}^2\,\bar{\phi}_2 - 126084\,\bar{\phi}_2{}^4\,\phi_2\\
&\quad - 182784\,\phi_2\,\bar{\phi}_2{}^2 - 93600\,\bar{\phi}_2^3)\Big/((\,52\,\bar{\phi}_2 + 60\,\phi_2 + 45\,\phi_2\,\bar{\phi}_2{}^2\,)\\
&\quad (\,52\,\phi_2 + 60\,\bar{\phi}_2 + 45\,\phi_2{}^2\,\bar{\phi}_2\,)\,) = 0\,.
\end{aligned} \tag{4.173}$$

Defining two new real variables, y_1 and y_2, by

$$\phi_2 =: y_1 + iy_2 , \tag{4.174}$$

we find that the real and imaginary parts of (4.173) can be written respectively as

$$\begin{aligned} N_r := &-y_1(527792\, y_1{}^2 y_2^2 - 1390050\, y_1{}^6 + 121500\, y_1{}^6 y_2{}^2 + 182250\, y_1{}^4 y_2{}^4 \\ &+ 121500\, y_1{}^2 y_2{}^6 + 30375\, y_2{}^8 + 30375\, y_1{}^8 - 4280\, y_1{}^4 + 312976\, y_1{}^2 + 50896\, y_2{}^2 \\ &- 45090\, y_2{}^6 - 1480230\, y_1{}^2 y_2{}^4 - 2825190\, y_1{}^4 y_2{}^2 + 532072\, y_2^4) \end{aligned}$$

$$\begin{aligned} N_i := &\, y_2(1521608\, y_1{}^4 - 3568\, y_2{}^2 - 23416\, y_2{}^4 + 802200\, y_2{}^6 + 299025\, y_2{}^8 \\ &+ 1196100\, y_1{}^6 y_2{}^2 + 1196100\, y_1{}^2 y_2{}^6 + 5496840\, y_1{}^4 y_2{}^2 + 2347320\, y_1{}^6 \\ &+ 483152\, y_1{}^2 + 3951720\, y_1{}^2 y_2{}^4 + 1498192\, y_1{}^2 y_2{}^2 + 299025\, y_1{}^8 \\ &+ 1794150\, y_1{}^4 y_2{}^4) \end{aligned} \tag{4.175}$$

Applying `gsolve` to the system of equations formed by (4.175) and (4.175) we find the set of five possible solutions:

$$R_1 := [\, y_2, y_1\,] , \tag{4.176}$$

$$R_2 := [\, y_1, -3568 + 802200\, y_2{}^4 - 23416\, y_2{}^2 + 299025\, y_2{}^6\,] , \tag{4.177}$$

$$R_3 := [\, y_2, 30375\, y_1{}^6 - 4280\, y_1{}^2 + 312976 - 1390050\, y_1{}^4\,] , \tag{4.178}$$

$$R_4 := [\, 13896\, y_2{}^2 + 24660\, y_1{}^2 + 1211, 44562960\, y_1{}^4 - 49781304\, y_1{}^2 + 54289\,] , \tag{4.179}$$

$$R_5 := [4676101217279600303041737074 1397\, y_2^2 \tag{4.180}$$

$$\begin{aligned} &+ 1361206104888872599141168449555840\, y_1{}^4 \\ &- 1129797024894241533886199705407151\, y_1{}^2 \\ &+ 214500301038606097487489610812056, 7846163568700057338480 0\, y_1{}^6 \\ &- 950851288449202001315 85\, y_1{}^4 + 3281840643279201136524 7\, y_1{}^2 \\ &- 3624591018009119098115] . \end{aligned} \tag{4.181}$$

The solution R_1 is of course a direct contradiction and R_4 is impossible, since it does not admit any real solution for y_2. The remaining solutions require further analysis, since they can lead to real solutions. Using the fact that ϕ_2 is constant we have $\Delta\phi_2 = 0$. The condition V_{00} now becomes

$$\begin{aligned} &-10\Phi_{22} - 9\phi_2\overline{\phi}_2\overline{\gamma}^2 - 3\phi_2\overline{\phi}_2\Delta\overline{\gamma} - 12\lambda\overline{\lambda} - 8\Delta\mu - 9\overline{\phi}_2\phi_2\gamma^2 - 24\mu\gamma \\ &- 24\overline{\mu}\overline{\gamma} - 96\overline{\gamma}\mu - 56\gamma\overline{\gamma} - 96\gamma\overline{\mu} - 172\mu\overline{\mu} - 8\Delta\overline{\mu} - 4\Delta\overline{\gamma} \\ &- 4\Delta\gamma - 12\overline{\gamma}^2 - 12\gamma^2 + 9\phi_2\overline{\phi}_2\Phi_{22} - 3\overline{\phi}_2\phi_2\Delta\gamma + 10\phi_2\overline{\phi}_2\overline{\gamma}\gamma = 0 . \end{aligned} \tag{4.182}$$

Applying δ to this equation, using (4.119), (4.161), (4.167), and the second order Pfaffians $\delta(\Delta\gamma)$, $\delta(\Delta\overline{\gamma})$, $\delta(\Delta\mu)$, $\delta(\Delta\overline{\mu})$, obtained from the corresponding commutators we find

$$\begin{aligned}
&-18560\,\mu\,\phi_2\,\overline{\phi}_2 - 8128\,\gamma\,\phi_2\,\overline{\phi}_2 - 49920\,\phi_2{}^2\,\overline{\mu} + 31200\,\mu\,\phi_2{}^2 - 128960\,\overline{\mu}\,\overline{\phi}_2{}^2 \\
&- 102272\,\overline{\gamma}\,\phi_2\,\overline{\phi}_2 + 90960\,\overline{\phi}_2\,\mu\,\phi_2{}^3 + 24960\,\gamma\,\phi_2{}^2 - 192064\,\phi_2\,\overline{\mu}\,\overline{\phi}_2 \\
&+ 1008900\,\phi_2{}^3\,\overline{\mu}\,\overline{\phi}_2{}^3 - 419232\,\phi_2\,\overline{\mu}\,\overline{\phi}_2{}^3 + 677700\,\phi_2{}^4\,\overline{\mu}\,\overline{\phi}_2{}^2 \\
&+ 94736\,\phi_2{}^2\,\overline{\mu}\,\overline{\phi}_2{}^2 + 562080\,\phi_2{}^3\,\overline{\mu}\,\overline{\phi}_2 + 589680\,\mu\,\overline{\phi}_2{}^4 + 288288\,\gamma\,\overline{\phi}_2{}^4 \\
&+ 104832\,\overline{\gamma}\,\overline{\phi}_2{}^4 - 1025280\,\phi_2{}^2\,\mu\,\overline{\phi}_2{}^4 - 352080\,\phi_2{}^2\,\gamma\,\overline{\phi}_2^4 \\
&- 266220\,\phi_2{}^2\,\overline{\mu}\,\overline{\phi}_2{}^4 - 459240\,\phi_2{}^2\,\overline{\gamma}\,\overline{\phi}_2{}^4 + 90720\,\phi_2\,\overline{\gamma}\,\overline{\phi}_2{}^5 \\
&+ 510300\,\phi_2\,\mu\,\overline{\phi}_2{}^5 + 249480\,\phi_2\,\gamma\,\overline{\phi}_2{}^5 + 49680\,\phi_2\,\overline{\lambda}\,\overline{\phi}_2{}^3 \\
&- 391500\,\phi_2{}^3\,\overline{\gamma}\,\overline{\phi}_2{}^5 - 641925\,\phi_2{}^3\,\mu\,\overline{\phi}_2{}^5 - 278640\,\phi_2{}^3\,\overline{\lambda}\,\overline{\phi}_2{}^3 \\
&- 236925\,\phi_2{}^5\,\overline{\lambda}\,\overline{\phi}_2{}^3 - 36450\,\phi_2{}^3\,\gamma\,\overline{\phi}_2{}^5 + 65520\,\phi_2{}^4\,\overline{\lambda} + 40500\,\phi_2{}^4\,\gamma\,\overline{\phi}_2{}^4 \\
&- 24960\,\phi_2{}^2\,\overline{\gamma} + 56700\,\phi_2{}^5\,\overline{\lambda}\,\overline{\phi}_2 + 298350\,\phi_2{}^4\,\overline{\gamma}\,\overline{\phi}_2{}^4 + 508275\,\phi_2{}^4\,\overline{\mu}\,\overline{\phi}_2{}^4 \\
&- 32400\,\phi_2\,\lambda\,\delta\overline{\lambda}\,\overline{\phi}_2{}^3 - 24300\,\phi_2{}^3\,\lambda\,\delta\overline{\lambda}\,\overline{\phi}_2{}^3 - 32400\,\phi_2{}^3\,\lambda\,\delta\overline{\lambda}\,\overline{\phi}_2 \\
&- 37440\,\phi_2{}^2\,\delta\overline{\lambda}\,\lambda - 56160\,\phi_2{}^2\,\lambda\,\delta\overline{\lambda}\,\overline{\phi}_2{}^2 - 75648\,\phi_2\,\lambda\,\delta\overline{\lambda}\,\overline{\phi}_2 \\
&- 37440\,\lambda\,\delta\overline{\lambda}\,\overline{\phi}_2{}^2 - 39520\,\mu\,\overline{\phi}_2{}^2 - 25792\,\gamma\,\overline{\phi}_2{}^2 - 69888\,\overline{\gamma}\,\overline{\phi}_2{}^2 \\
&- 56160\,\overline{\lambda}\,\overline{\phi}_2{}^2 + 374616\,\overline{\lambda}\,\overline{\phi}_2{}^2\,\phi_2{}^2 - 94272\,\overline{\lambda}\,\overline{\phi}_2\,\phi_2 + 350928\,\overline{\phi}_2{}^2\,\overline{\gamma}\,\phi_2{}^2 \\
&+ 365256\,\overline{\lambda}\,\phi_2{}^3\,\overline{\phi}_2 - 275768\,\overline{\phi}_2{}^2\,\mu\,\phi_2{}^2 - 378544\,\gamma\,\overline{\phi}_2{}^2\,\phi_2{}^2 \\
&+ 318600\,\mu\,\phi_2\,\overline{\phi}_2{}^3 - 787680\,\mu\,\phi_2{}^3\,\overline{\phi}_2{}^3 - 47856\,\gamma\,\phi_2\,\overline{\phi}_2{}^3 + 12060\,\gamma\,\phi_2{}^3\,\overline{\phi}_2{}^3 \\
&+ 18480\,\overline{\phi}_2\,\gamma\,\phi_2{}^3 + 54000\,\overline{\phi}_2{}^2\,\gamma\,\phi_2{}^4 + 52576\,\overline{\gamma}\,\phi_2\,\overline{\phi}_2{}^3 + 150540\,\overline{\gamma}\,\phi_2{}^3\,\overline{\phi}_2{}^3 \\
&+ 437040\overline{\phi}_2\overline{\gamma}\phi_2{}^3 + 397800\overline{\phi}_2{}^2\overline{\gamma}\phi_2{}^4 + 6480\overline{\lambda}\phi_2{}^4\overline{\phi}_2{}^2 - 39520\overline{\lambda}\phi_2{}^2 = 0\,.
\end{aligned} \tag{4.183}$$

On the other hand, substituting all calculated Pfaffians into VI_{01}, and taking its complex conjugate, we have

$$\begin{aligned}
&1676700\,\overline{\phi}_2{}^3\,\phi_2{}^7\,\delta\overline{\lambda} + 7854912\,\overline{\phi}_2\,\phi_2{}^5\,\delta\overline{\lambda} + 3499200\,\overline{\phi}_2{}^4\,\phi_2{}^2\,\delta\overline{\lambda} \\
&+ 10485120\,\overline{\phi}_2{}^2\,\phi_2{}^2\,\delta\overline{\lambda} + 14381280\,\overline{\phi}_2{}^3\,\phi_2{}^3\,\delta\overline{\lambda} \\
&+ 3556800\,\overline{\phi}_2{}^3\,\phi_2\,\delta\overline{\lambda} + 11020860\,\overline{\phi}_2{}^3\,\phi_2{}^5\,\delta\overline{\lambda} + 5248800\,\overline{\phi}_2{}^4\,\phi_2{}^4\,\delta\overline{\lambda} \\
&+ 10159552\,\overline{\phi}_2\,\phi_2{}^3\,\delta\overline{\lambda} + 1968300\,\overline{\phi}_2{}^4\,\phi_2{}^6\,\delta\overline{\lambda} \\
&+ 18898848\,\overline{\phi}_2{}^2\,\phi_2{}^4\,\delta\overline{\lambda} + 6305040\,\overline{\phi}_2{}^2\,\phi_2{}^6\,\delta\overline{\lambda} + 872384\,\phi_2{}^2\,\overline{\phi}_2{}^2 \\
&+ 3149056\,\phi_2\,\overline{\phi}_2{}^3 - 386880\,\overline{\phi}_2\,\phi_2{}^3 + 14430928\,\phi_2{}^3\,\overline{\phi}_2{}^3 + 14177712\,\phi_2{}^2\,\overline{\phi}_2{}^4 \\
&+ 96384\,\phi_2{}^4\,\overline{\phi}_2{}^2 + 1822080\,\overline{\phi}_2{}^4 - 121680\,\phi_2{}^5\,\overline{\phi}_2 + 38361720\,\phi_2{}^4\,\overline{\phi}_2{}^4 \\
&+ 4074000\,\phi_2{}^5\,\overline{\phi}_2{}^3 + 46800\,\phi_2{}^6\,\overline{\phi}_2{}^2 - 18083520\,\overline{\phi}_2{}^6 + 75951540\,\overline{\phi}_2{}^5\,\phi_2{}^3 \\
&- 18712224\,\phi_2\,\overline{\phi}_2{}^5 + 30853440\,\overline{\phi}_2{}^6\,\phi_2{}^2 + 561600\,\phi_2{}^6\,\overline{\phi}_2{}^4 + 7275825\,\phi_2{}^5\,\overline{\phi}_2{}^5
\end{aligned}$$

$$+ 29205900\,\phi_2{}^4\,\overline{\phi}_2{}^6 + 26268300\,\phi_2{}^3\,\overline{\phi}_2{}^7 - 15649200\,\phi_2\,\overline{\phi}_2{}^7$$
$$+ 3244800\,\phi_2{}^4\,\delta\overline{\lambda} + 2794500\,\overline{\phi}_2{}^7\,\phi_2{}^5 + 394875\,\phi_2{}^6\,\overline{\phi}_2{}^6 = 0\,. \tag{4.184}$$

Solving (4.184) for $\delta\overline{\lambda}$ we find

$$\begin{aligned}\delta\overline{\lambda} = &-\frac{1}{4}\overline{\phi}_2\,(\,52\,\overline{\phi}_2 + 60\,\phi_2 + 45\,\phi_2\,\overline{\phi}_2{}^2\,)(8775\,\overline{\phi}_2{}^3\,\phi_2{}^5 + 780\,\phi_2^5\,\overline{\phi}_2\\
&+ 62100\,\phi_2{}^4\,\overline{\phi}_2{}^4 + 68745\,\phi_2{}^4\,\overline{\phi}_2{}^2 - 2028\,\phi_2{}^4 + 577260\,\phi_2{}^3\,\overline{\phi}_2{}^3\\
&+ 3364\,\phi_2{}^3\,\overline{\phi}_2 + 583740\,\phi_2{}^2\,\overline{\phi}_2{}^4 + 242436\,\phi_2{}^2\,\overline{\phi}_2{}^2 - 6448\,\phi_2{}^2\\
&+ 11088\,\phi_2\,\overline{\phi}_2{}^3 + 20128\,\phi_2\,\overline{\phi}_2 - 347760\,\overline{\phi}_2{}^4 + 35040\,\overline{\phi}_2{}^2)\Big/\\
&\Big((\,243\,\phi_2\,\overline{\phi}_2{}^2 + 247\,\overline{\phi}_2 + 207\,\phi_2{}^2\,\overline{\phi}_2 + 300\,\phi_2\,)\,(\,52\,\phi_2\\
&+60\,\overline{\phi}_2 + 45\,\phi_2{}^2\,\overline{\phi}_2\,)^2\Big)\,,\end{aligned} \tag{4.185}$$

where the denominator in the above expression,

$$e_2 := \; 207\,\phi_2{}^2\,\overline{\phi}_2 + 300\,\phi_2 + 243\,\phi_2\,\overline{\phi}_2{}^2 + 247\,\overline{\phi}_2\,, \tag{4.186}$$

is supposed to be non-zero for now.

Substituting the previous expression for $\delta\overline{\lambda}$ into (4.183) gives

$$\begin{aligned}&(3678289920\,\overline{\phi}_2{}^4\,\overline{\lambda}\,\phi_2{}^3 + 10028770056\,\overline{\phi}_2{}^4\,\overline{\lambda}\,\phi_2{}^5 - 832291200\,\overline{\phi}_2{}^4\,\overline{\lambda}\,\phi_2\\
&+ 4499533800\,\overline{\phi}_2{}^7\,\overline{\mu}\,\phi_2{}^6 - 4369507968\,\overline{\phi}_2{}^2\,\overline{\lambda}\,\phi_2{}^3 + 12393977472\,\overline{\phi}_2{}^3\,\overline{\lambda}\,\phi_2{}^4\\
&- 7019449875\,\overline{\phi}_2{}^8\,\mu\,\phi_2{}^7 + 4378738500\,\overline{\phi}_2{}^8\,\lambda\,\phi_2{}^5 + 7440174000\,\overline{\phi}_2{}^8\,\mu\,\phi_2{}^3\\
&- 3129310080\,\overline{\phi}_2{}^3\,\overline{\lambda}\,\phi_2{}^2 + 284244480\,\overline{\phi}_2{}^5\,\lambda - 2821029120\,\overline{\phi}_2{}^7\,\lambda\\
&- 2689598080\,\overline{\phi}_2\,\overline{\lambda}\,\phi_2{}^4 + 14364078336\,\overline{\phi}_2{}^2\,\overline{\lambda}\,\phi_2{}^5 + 96675840\,\overline{\phi}_2{}^2\,\lambda\,\phi_2{}^3\\
&+ 2474297280\,\overline{\phi}_2\,\overline{\lambda}\,\phi_2{}^8 - 4281052500\,\overline{\phi}_2{}^8\,\overline{\gamma}\,\phi_2{}^7 - 398580750\,\overline{\phi}_2{}^8\,\gamma\,\phi_2{}^7\\
&+ 702921984\,\overline{\phi}_2{}^3\,\lambda\,\phi_2{}^2 - 4716046800\,\overline{\phi}_2{}^8\,\overline{\gamma}\,\phi_2{}^5 + 2779130250\,\overline{\phi}_2{}^6\,\overline{\gamma}\,\phi_2{}^9\\
&+ 724334400\,\overline{\phi}_2{}^6\,\overline{\lambda}\,\phi_2{}^3 - 616512000\,\overline{\lambda}\,\phi_2{}^5 - 3519320400\,\overline{\phi}_2{}^6\,\overline{\lambda}\,\phi_2{}^5\\
&+ 486720000\,\mu\,\phi_2{}^5 - 529869600\,\overline{\phi}_2{}^7\,\lambda\,\phi_2{}^2 + 16160180400\,\overline{\phi}_2{}^7\,\mu\,\phi_2{}^2\\
&+ 7900532640\,\overline{\phi}_2{}^7\,\gamma\,\phi_2{}^2 + 2872920960\,\overline{\phi}_2{}^7\,\overline{\gamma}\,\phi_2{}^2 + 6502854960\,\overline{\phi}_2{}^7\,\gamma\,\phi_2{}^4\\
&- 8069978160\,\overline{\phi}_2{}^7\,\overline{\gamma}\,\phi_2{}^4 + 444301200\,\overline{\phi}_2{}^7\,\mu\,\phi_2{}^4 - 3881487600\,\overline{\phi}_2{}^7\,\overline{\mu}\,\phi_2{}^4\\
&- 389376000\,\overline{\gamma}\,\phi_2{}^5 + 19537872300\,\overline{\phi}_2{}^7\,\lambda\,\phi_2{}^4 + 1546846875\,\overline{\phi}_2{}^7\,\lambda\,\phi_2{}^6\\
&+ 389376000\,\gamma\,\phi_2{}^5 - 11094486975\,\overline{\phi}_2{}^5\,\overline{\lambda}\,\phi_2{}^8 + 52722252000\,\overline{\phi}_2{}^5\,\overline{\mu}\,\phi_2{}^6\\
&- 2206956375\,\overline{\phi}_2{}^5\,\overline{\lambda}\,\phi_2{}^{10} + 29141337600\,\overline{\phi}_2{}^5\,\overline{\mu}\,\phi_2{}^8\\
&+ 12991387500\,\overline{\phi}_2{}^5\,\overline{\gamma}\,\phi_2{}^8 - 8936242380\,\overline{\lambda}\,\phi_2{}^6\,\overline{\phi}_2{}^5 + 1685520900\,\overline{\phi}_2{}^5\,\gamma\,\phi_2{}^8\\
&- 7337239200\,\overline{\phi}_2{}^5\,\mu\,\phi_2{}^8 + 21603880800\,\overline{\phi}_2{}^5\,\lambda\,\phi_2{}^4 - 18317517120\,\overline{\gamma}\,\phi_2{}^5\,\overline{\phi}_2{}^6\end{aligned}$$

$$
\begin{aligned}
&- 731354688\,\overline{\phi_2}^{5}\,\lambda\,\phi_2^{2} + 1947248100\,\overline{\phi_2}^{5}\,\lambda\,\phi_2^{6} + 8028651960\,\overline{\phi_2}^{6}\,\overline{\mu}\,\phi_2^{5}\\
&- 4867824600\,\gamma\,\phi_2^{5}\,\overline{\phi_2}^{6} + 19316050560\,\overline{\phi_2}^{6}\,\lambda\,\phi_2^{3} + 11570922900\,\overline{\phi_2}^{6}\,\lambda\,\phi_2^{5}\\
&- 5928159744\,\overline{\phi_2}^{6}\,\lambda\,\phi_2 + 146893500\,\overline{\phi_2}^{6}\,\lambda\,\phi_2^{7} - 2567246400\,\overline{\phi_2}^{6}\,\gamma\,\phi_2^{7}\\
&- 1339403850\,\overline{\phi_2}^{6}\,\overline{\gamma}\,\phi_2^{7} - 6501294900\,\overline{\phi_2}^{6}\,\overline{\lambda}\,\phi_2^{7} - 2590774875\,\overline{\phi_2}^{6}\,\overline{\lambda}\,\phi_2^{9}\\
&+ 4734581625\,\overline{\phi_2}^{6}\,\overline{\mu}\,\phi_2^{9} + 26937297225\,\overline{\phi_2}^{6}\,\overline{\mu}\,\phi_2^{7}\\
&- 33739432200\,\overline{\phi_2}^{6}\,\mu\,\phi_2^{7} - 10057782960\,\overline{\phi_2}^{6}\,\overline{\mu}\,\phi_2^{3} + 377257500\,\overline{\phi_2}^{6}\,\gamma\,\phi_2^{9}\\
&+ 1523301120\,\overline{\phi_2}^{2}\,\gamma\,\phi_2^{7} + 2497681440\,\overline{\phi_2}^{2}\,\mu\,\phi_2^{7} + 16577516160\,\overline{\phi_2}^{2}\,\overline{\gamma}\,\phi_2^{7}\\
&+ 1986087600\,\overline{\phi_2}^{2}\,\overline{\lambda}\,\phi_2^{9} + 2560374144\,\overline{\phi_2}^{3}\,\lambda\,\phi_2^{4} + 56936537424\,\overline{\phi_2}^{3}\,\overline{\mu}\,\phi_2^{6}\\
&+ 724194000\,\overline{\phi_2}^{3}\,\lambda\,\phi_2^{6} + 1482397200\,\overline{\phi_2}^{3}\,\gamma\,\phi_2^{8} + 2630917260\,\overline{\phi_2}^{3}\,\overline{\lambda}\,\phi_2^{8}\\
&+ 13723246800\,\overline{\phi_2}^{3}\,\overline{\gamma}\,\phi_2^{8} + 847292400\,\overline{\phi_2}^{3}\,\mu\,\phi_2^{8}\\
&+ 21679488000\,\overline{\phi_2}^{3}\,\overline{\mu}\,\phi_2^{8} + 528160500\,\overline{\phi_2}^{3}\,\overline{\lambda}\,\phi_2^{10} + 7553665440\,\overline{\phi_2}^{4}\,\lambda\,\phi_2^{5}\\
&+ 107406000\,\overline{\phi_2}^{4}\,\lambda\,\phi_2^{7} - 446400720\,\overline{\phi_2}^{4}\,\gamma\,\phi_2^{7} + 30743713080\,\overline{\phi_2}^{4}\,\overline{\gamma}\,\phi_2^{7}\\
&+ 53308125\,\overline{\phi_2}^{8}\,\lambda\,\phi_2^{7} - 1911187200\,\overline{\phi_2}^{4}\,\phi_2\,\overline{\mu} - 5068372500\,\overline{\phi_2}^{4}\,\overline{\lambda}\,\phi_2^{9}\\
&+ 6312775500\,\overline{\phi_2}^{4}\,\overline{\mu}\,\phi_2^{9} + 63950936940\,\overline{\phi_2}^{4}\,\overline{\mu}\,\phi_2^{7}\\
&- 20686398840\,\overline{\phi_2}^{4}\,\mu\,\phi_2^{7} + 503010000\,\overline{\phi_2}^{4}\,\gamma\,\phi_2^{9} + 3705507000\,\overline{\phi_2}^{4}\,\overline{\gamma}\,\phi_2^{9}\\
&- 69638400\,\overline{\phi_2}\,\lambda\,\phi_2^{4} + 893917440\,\overline{\phi_2}\,\gamma\,\phi_2^{6} - 21902400\,\overline{\phi_2}\,\lambda\,\phi_2^{6}\\
&+ 2176012800\,\overline{\phi_2}\,\mu\,\phi_2^{6} + 6212194560\,\overline{\phi_2}\,\overline{\gamma}\,\phi_2^{6} - 1632960\,\overline{\phi_2}^{2}\,\lambda\,\phi_2^{5}\\
&+ 23745424320\,\overline{\phi_2}^{2}\,\overline{\mu}\,\phi_2^{7} + 8424000\,\overline{\phi_2}^{2}\,\lambda\,\phi_2^{7} + 13238941920\,\mu\,\phi_2^{3}\,\overline{\phi_2}^{4}\\
&+ 26519546880\,\mu\,\phi_2^{3}\,\overline{\phi_2}^{6} + 8739057600\,\mu\,\overline{\phi_2}^{6}\,\phi_2 + 4272428160\,\gamma\,\overline{\phi_2}^{6}\,\phi_2\\
&+ 1553610240\,\overline{\gamma}\,\overline{\phi_2}^{6}\,\phi_2 + 7806681792\,\gamma\,\phi_2^{2}\,\overline{\phi_2}^{5} - 3640228416\,\gamma\,\phi_2^{3}\,\overline{\phi_2}^{4}\\
&+ 12207060288\,\gamma\,\phi_2^{3}\,\overline{\phi_2}^{6} + 2993647488\,\overline{\gamma}\,\phi_2^{2}\,\overline{\phi_2}^{5} + 4438741696\,\overline{\gamma}\,\phi_2^{3}\,\overline{\phi_2}^{4}\\
&+ 550667232\,\overline{\gamma}\,\phi_2^{3}\,\overline{\phi_2}^{6} - 915985408\,\gamma\,\phi_2^{2}\,\overline{\phi_2}^{3} - 3671296512\,\overline{\gamma}\,\phi_2^{2}\,\overline{\phi_2}^{3}\\
&- 283148032\,\gamma\,\phi_2^{3}\,\overline{\phi_2}^{2} - 4614637568\,\overline{\gamma}\,\phi_2^{3}\,\overline{\phi_2}^{2} - 1494014080\,\mu\,\phi_2^{2}\,\overline{\phi_2}^{3}\\
&- 726592640\,\mu\,\phi_2^{3}\,\overline{\phi_2}^{2} + 643069440\,\overline{\phi_2}\,\gamma\,\phi_2^{4} - 12704389312\,\overline{\phi_2}^{3}\,\gamma\,\phi_2^{4}\\
&- 7326124704\,\gamma\,\phi_2^{4}\,\overline{\phi_2}^{5} + 12362181888\,\overline{\gamma}\,\phi_2^{4}\,\overline{\phi_2}^{3} - 1720888656\,\overline{\gamma}\,\phi_2^{4}\,\overline{\phi_2}^{5}\\
&- 2365309440\,\overline{\phi_2}\,\overline{\gamma}\,\phi_2^{4} - 3362952032\,\mu\,\phi_2^{4}\,\overline{\phi_2}^{3} - 13957246440\,\mu\,\phi_2^{4}\,\overline{\phi_2}^{5}\\
&+ 672796800\,\overline{\phi_2}\,\mu\,\phi_2^{4} - 585686400\,\mu\,\overline{\phi_2}^{4}\,\phi_2 - 382237440\,\gamma\,\overline{\phi_2}^{4}\,\phi_2\\
&- 1035740160\,\overline{\gamma}\,\overline{\phi_2}^{4}\,\phi_2 + 22333540320\,\mu\,\phi_2^{2}\,\overline{\phi_2}^{5} + 819227136\,\overline{\phi_2}^{4}\,\lambda\,\phi_2\\
&+ 7557189120\,\overline{\phi_2}\,\overline{\mu}\,\phi_2^{6} + 1322697600\,\overline{\phi_2}^{8}\,\overline{\gamma}\,\phi_2^{3} - 5523939180\,\overline{\lambda}\,\phi_2^{7}\,\overline{\phi_2}^{4}\\
&+ 7524157500\,\overline{\gamma}\,\phi_2^{6}\,\overline{\phi_2}^{5} + 12714309504\,\overline{\lambda}\,\phi_2^{7}\,\overline{\phi_2}^{2}\\
&- 63430361280\,\mu\,\phi_2^{6}\,\overline{\phi_2}^{5} - 11223922140\,\gamma\,\phi_2^{6}\,\overline{\phi_2}^{5} + 4920984768\,\overline{\phi_2}^{4}\,\lambda\,\phi_2^{3}\\
&+ 3637418400\,\overline{\phi_2}^{8}\,\gamma\,\phi_2^{3} - 15520642992\,\mu\,\phi_2^{6}\,\overline{\phi_2}^{3} - 6465614256\,\overline{\phi_2}^{3}\,\gamma\,\phi_2^{6}
\end{aligned}
$$

$$
\begin{aligned}
&+ 37715656752\,\overline{\gamma}\,\phi_2{}^6\,\overline{\phi}_2{}^3 - 29677061475\,\overline{\phi}_2{}^7\,\mu\,\phi_2{}^6 - 384365250\,\overline{\phi}_2{}^7\,\overline{\gamma}\,\phi_2{}^8 \\
&+ 1022112000\,\overline{\lambda}\,\phi_2{}^7 - 3779136000\,\overline{\phi}_2{}^8\,\mu\,\phi_2{}^5 + 103335750\,\overline{\phi}_2{}^7\,\gamma\,\phi_2{}^8 \\
&+ 5557987125\,\overline{\phi}_2{}^7\,\overline{\mu}\,\phi_2{}^8 + 6759979200\,\overline{\phi}_2\,\overline{\lambda}\,\phi_2{}^6 - 13987736100\,\overline{\phi}_2{}^7\,\overline{\gamma}\,\phi_2{}^6 \\
&+ 2196622800\,\overline{\phi}_2{}^8\,\gamma\,\phi_2{}^5 + 8263069200\,\overline{\phi}_2{}^8\,\lambda\,\phi_2{}^3 + 2515322160\,\overline{\phi}_2{}^5\,\overline{\lambda}\,\phi_2{}^4 \\
&- 82555200\,\overline{\phi}_2{}^5\,\overline{\lambda}\,\phi_2{}^2 - 4629678912\,\overline{\phi}_2{}^2\,\gamma\,\phi_2{}^5 - 5979531375\,\overline{\phi}_2{}^7\,\mu\,\phi_2{}^8 \\
&- 4882550400\,\overline{\phi}_2{}^8\,\lambda\,\phi_2 - 41778749448\,\mu\,\phi_2{}^5\,\overline{\phi}_2{}^4 - 6092022400\,\overline{\mu}\,\phi_2{}^4\,\overline{\phi}_2{}^3 \\
&- 18991709568\,\overline{\mu}\,\phi_2{}^3\,\overline{\phi}_2{}^4 - 8452359072\,\overline{\mu}\,\overline{\phi}_2{}^5\,\phi_2{}^4 \\
&+ 35157088368\,\overline{\mu}\,\overline{\phi}_2{}^4\,\phi_2{}^5 - 8093255040\,\overline{\mu}\,\overline{\phi}_2{}^5\,\phi_2{}^2 \\
&+ 12348779904\,\overline{\mu}\,\overline{\phi}_2{}^2\,\phi_2{}^5 - 4535930880\,\overline{\mu}\,\overline{\phi}_2\,\phi_2{}^4 \\
&- 8675612416\,\overline{\mu}\,\overline{\phi}_2{}^2\,\phi_2{}^3 - 6824030720\,\overline{\mu}\,\phi_2{}^2\,\overline{\phi}_2{}^3 \\
&+ 18639723240\,\overline{\lambda}\,\phi_2{}^6\,\overline{\phi}_2{}^3 + 15570182592\,\overline{\phi}_2{}^2\,\overline{\gamma}\,\phi_2{}^5 \\
&+ 21946320672\,\overline{\gamma}\,\phi_2{}^5\,\overline{\phi}_2{}^4 + 377257500\,\overline{\phi}_2{}^9\,\lambda\,\phi_2{}^6 - 2254031550\,\overline{\phi}_2{}^7\,\gamma\,\phi_2{}^6 \\
&- 818215200\,\overline{\phi}_2{}^2\,\mu\,\phi_2{}^5 - 19233376128\,\gamma\,\phi_2{}^5\,\overline{\phi}_2{}^4 - 778752000\,\overline{\mu}\,\phi_2{}^5 \\
&- 47010632580\,\mu\,\phi_2{}^5\,\overline{\phi}_2{}^6 + 3546220500\,\overline{\phi}_2{}^9\,\lambda\,\phi_2{}^4 - 2112642000\,\overline{\phi}_2{}^9\,\lambda\,\phi_2{}^2) \,/ \\
&\Big((52\,\phi_2 + 60\,\overline{\phi}_2 + 45\,\phi_2{}^2\,\overline{\phi}_2\,)^2\,(52\,\overline{\phi}_2 + 60\,\phi_2 + 45\,\phi_2\,\overline{\phi}_2{}^2\,) \\
&(2\phi_2{}^2\overline{\phi}_2 + 300\phi_2 + 243\phi_2\overline{\phi}_2{}^2 + 247\overline{\phi}_2)\phi_2\Big) = 0\,. \qquad (4.187)
\end{aligned}
$$

Applying δ to this equation we finally obtain

$$
\begin{aligned}
&441591750\,\phi_2{}^7\,\overline{\phi}_2{}^6 + 310645125\,\phi_2{}^8\,\overline{\phi}_2{}^5 - 22136490\,\phi_2{}^8\,\overline{\phi}_2{}^3 + 17583744\,\phi_2\,\overline{\phi}_2{}^2 \\
&- 3498368\,\phi_2{}^2\,\overline{\phi}_2 + 16398720\,\overline{\phi}_2{}^3 - 3560128\,\phi_2{}^3 - 393120\,\phi_2{}^8\,\overline{\phi}_2 \\
&- 177632000\,\phi_2{}^3\,\overline{\phi}_2{}^2 - 107857904\,\phi_2{}^4\,\overline{\phi}_2 - 52284384\,\phi_2{}^2\,\overline{\phi}_2{}^3 \\
&- 112661856\,\overline{\phi}_2{}^4\,\phi_2 - 105529032\,\phi_2{}^7\,\overline{\phi}_2{}^2 + 988474185\,\phi_2{}^7\,\overline{\phi}_2^4 \\
&- 88746576\,\phi_2{}^6\,\overline{\phi}_2 - 1530578808\,\overline{\phi}_2{}^4\,\phi_2{}^5 + 667433538\,\overline{\phi}_2{}^3\,\phi_2^6 \\
&- 190683720\,\overline{\phi}_2{}^6\,\phi_2{}^5 + 804787650\,\overline{\phi}_2{}^5\,\phi_2{}^6 + 332813600\,\phi_2{}^5\,\overline{\phi}_2{}^2 \\
&+ 1312034112\,\overline{\phi}_2{}^5\,\phi_2{}^2 + 686129256\,\overline{\phi}_2{}^4\,\phi_2{}^3 - 200584704\,\overline{\phi}_2{}^3\,\phi_2{}^4 \\
&- 2318632020\,\overline{\phi}_2{}^5\,\phi_2{}^4 - 303342408\,\overline{\phi}_2{}^6\,\phi_2{}^3 + 35471520\,\overline{\phi}_2{}^6\,\phi_2 \\
&- 162751680\,\overline{\phi}_2{}^5 - 48740016\,\phi_2{}^5 + 1022112\,\phi_2{}^7 = 0\,. \qquad (4.188)
\end{aligned}
$$

In terms of the real variables y_1 and y_2, defined in (4.174), the real and imaginary parts of (4.188) have the form

$$
\begin{aligned}
L_r := y_1(&269797096\,y_1{}^4\,y_2^2 - 5466377538\,y_2{}^8 + 2096123712\,y_1{}^2\,y_2{}^2 \\
&- 3591041850\,y_1{}^8 + 4213949265\,y_1{}^8\,y_2{}^2 + 697196970\,y_1{}^6\,y_2{}^4 \\
&- 7387688430\,y_1{}^4\,y_2{}^6 - 2462001075\,y_2{}^{10} - 7913378835\,y_1{}^2\,y_2{}^8 - 490343625\,y_2{}^{12}
\end{aligned}
$$

$$+ 3270840750\,{y_1}^{10}\,{y_2}^2 + 5070650625\,{y_1}^8\,{y_2}^4 + 2618932500\,{y_1}^6\,{y_2}^6$$
$$- 1142251875\,{y_1}^4\,{y_2}^8 + 1580441625\,{y_1}^{10} + 752236875\,{y_1}^{12}$$
$$- 1699481250\,{y_1}^2\,{y_2}^{10} + 26923968\,{y_1}^2 - 625815584\,{y_2}^4 - 661927840\,{y_1}^4$$
$$- 24430400\,{y_2}^2 - 6465455704\,{y_1}^2\,{y_2}^4 - 4722528648\,{y_2}^6 + 2078139320\,y_1^6$$
$$- 18320579712\,{y_1}^2\,{y_2}^6 - 23807908980\,{y_1}^4\,{y_2}^4 - 14544748656\,{y_1}^6\,{y_2}^2)\,, \qquad (4.189)$$

$$L_i := y_2(-6970927880\,{y_1}^4\,{y_2}^2 - 287858646\,{y_2}^8 - 1381203968\,{y_1}^2\,{y_2}^2$$
$$+ 3169984050\,{y_1}^8 + 15739890435\,{y_1}^8\,{y_2}^2 + 24103263150\,{y_1}^6\,{y_2}^4$$
$$+ 16372561590\,{y_1}^4\,{y_2}^6 - 15139305\,{y_2}^{10} + 4143838095\,{y_1}^2\,{y_2}^8$$
$$+ 130946625\,{y_2}^{12} + 6998582250\,{y_1}^{10}\,{y_2}^2 + 14390004375\,{y_1}^8\,{y_2}^4$$
$$+ 15044737500\,{y_1}^6\,{y_2}^6 + 8177101875\,{y_1}^4\,{y_2}^8 + 3850211475\,{y_1}^{10}$$
$$+ 1373527125\,{y_1}^{12} + 2028260250\,{y_1}^2\,{y_2}^{10} - 80958656\,{y_1}^2 - 16139904\,{y_2}^4$$
$$+ 459122560\,{y_1}^4 - 1123264\,{y_2}^2 - 2499935176\,{y_1}^2\,{y_2}^4 - 32733656\,{y_2}^6$$
$$- 4438311192\,{y_1}^6 + 3988582704\,{y_1}^2\,{y_2}^6 + 12035885076\,{y_1}^4\,{y_2}^4$$
$$+ 10929427776\,{y_1}^6\,{y_2}^2)\,. \qquad (4.190)$$

Applying gsolve now to the set of equations $\{N_r\,,\, N_i\,,\, L_r\,,\, L_i\}$ we find that the only possible solutions are given by

$$T_1 := [y_1\,,\, y_2]\,, \qquad (4.191)$$

$$T_2 := [24600y_1^2 + 13896y_2^2 + 1211\,, 990288y_2^4 + 2135768y_2^2 + 182405]\,. \qquad (4.192)$$

Obviously, the set T_2 admits no real solution.

Let us now suppose that the denominator of (4.169) is zero:

$$e_1 := 2\overline{\phi}_2 + 60\phi_2 + 45\phi_2{\phi_2}^2 = 0\,. \qquad (4.193)$$

This also implies that the numerator of (4.169) is zero:

$$f_1 := 276\phi_2\overline{\phi}_2 + 39{\overline{\phi}_2}^2 + 20 = 0\,. \qquad (4.194)$$

In terms of the real variables (4.174) the real part of (4.194) have the form

$$\Re(f_1) = 315y_1^2 + 237y_2^2 + 20 = 0\,, \qquad (4.195)$$

which leads to a contradiction, since it admits no real solution.

Let us consider now the case when the denominator factor e_2 of (4.185) is zero:

$$e_2 := \ 207\,{\phi_2}^2\,\overline{\phi}_2 + 300\,\phi_2 + 243\,\phi_2\,{\overline{\phi}_2}^2 + 247\,\overline{\phi}_2 = 0\,. \qquad (4.196)$$

In this case the numerator of (4.185) is also zero:

$$f_2 := 8775\,\overline{\phi}_2{}^3\,\phi_2{}^5 + 780\,\phi_2{}^5\,\overline{\phi}_2 \tag{4.197}$$

$$+ 62100\,\phi_2{}^4\,\overline{\phi}_2{}^4 + 68745\,\phi_2{}^4\,\overline{\phi}_2{}^2 - 2028\,\phi_2{}^4 + 577260\,\phi_2{}^3\,\overline{\phi}_2{}^3$$
$$+ 3364\,\phi_2{}^3\,\overline{\phi}_2 + 583740\,\phi_2{}^2\,\overline{\phi}_2{}^4 + 242436\,\phi_2{}^2\,\overline{\phi}_2{}^2 - 6448\,\phi_2{}^2$$
$$+ 11088\,\phi_2\,\overline{\phi}_2{}^3 + 20128\,\phi_2\,\overline{\phi}_2 - 347760\,\overline{\phi}_2{}^4 + 35040\,\overline{\phi}_2{}^2 = 0\,. \tag{4.198}$$

The real and imaginary parts of (4.196) are given, respectively, by

$$\Re(e_2) := y_1\,(\,450\,y_1{}^2 + 450\,y_2{}^2) = 0\,, \tag{4.199}$$
$$\Im(e_2) := -y_2\,(\,36\,y_1{}^2 + 36\,y_2{}^2) = 0\,. \tag{4.200}$$

The real part of (4.198) has the form

$$\Re(f_2) := 2380365\,y_1{}^4\,y_2{}^2 + 1075395\,y_1{}^2\,y_2{}^4 + 62875\,y_1{}^8 + 61325\,y_2{}^8 + 1230525\,y_1{}^6$$
$$- 74445\,y_2{}^6 + 246850\,y_1{}^2\,y_2{}^6 - 8464\,y_2{}^2 + 48720\,y_1{}^2 + 2583600\,y_1{}^2\,y_2{}^2$$
$$- 92900\,y_1{}^4 - 121804\,y_2{}^4 + 249950\,y_1{}^6\,y_2{}^2 + 372600\,y_1{}^4\,y_2{}^4 = 0\,. \tag{4.201}$$

The application of `gsolve` to the system of equations formed by (4.199), (4.200) and (4.201) shows that the only possible solution is $y_1 = 0$, $y_2 = 0$, which is a contradiction.

Let us now go back to (4.158) and consider the case in which $\Phi_{11} = 0$. We assume that $\beta \neq 0$, since the case $\beta = 0$, $\Phi_{11} = 0$ was already considered.

Suppose initially that $\pi \neq 0$. From II_{11}, IV_{10} and III_{10} we have

$$\delta\phi_2 = 2\phi_2\beta\,, \tag{4.202}$$
$$\delta\overline{\phi}_2 = -6\overline{\phi}_2\beta + 4\overline{\pi}\phi_2\,, \tag{4.203}$$
$$\delta\beta = \beta^2\,. \tag{4.204}$$

From the NP equations,

$$\mathrm{D}\mu = \delta\pi + \pi\overline{\pi} + \beta\pi\,, \tag{4.205}$$
$$\delta\overline{\beta} = -\beta\overline{\beta}\,, \tag{4.206}$$
$$\mathrm{D}\gamma = \beta\pi\,, \tag{4.207}$$
$$\delta\Phi_{21} = -2\overline{\beta} - 2\Phi_{21}\beta \tag{4.208}$$
$$\delta\Phi_{12} = 4\overline{\pi}\,. \tag{4.209}$$

From the commutator $[\overline{\delta}\,,\,\delta]\Phi_{21}$ we find

$$\delta\pi = -3\beta\pi\,. \tag{4.210}$$

From the commutator $[\overline{\delta}\,,\,\delta]\phi_2$ we now find

$$\pi(3\beta\overline{\phi}_2 - 2\overline{\pi}\phi_2) = 0\,. \tag{4.211}$$

Since we are assuming $\pi \neq 0$, we can solve this equation for $\overline{\phi}_2$. Substituting in V_{20} we find

$$\overline{\pi}\beta = 0\,, \tag{4.212}$$

which contradicts the assumption.

Let us now consider the case in which $\beta \neq 0$ and $\pi = 0$. From V_{20}, we find

$$\overline{\phi}_2\phi_2\beta^2 = 0\,, \tag{4.213}$$

a contradiction.

(ii) $\pi = 0$.

Here we consider $\alpha \neq 0$, since the case $\alpha = 0$, $\pi = 0$ was examined before. Let us assume first that $\beta \neq 0$. Then, from (4.100) we must have $\phi_2{}^2 - 4 \neq 0$. Equations (4.111) and (4.125) now imply, respectively,

$$-6\phi_2\overline{\beta} + 5\overline{\phi}_2\alpha = 0\,, \tag{4.214}$$

$$-36\beta\overline{\beta} + 25\alpha\overline{\alpha} = 0\,. \tag{4.215}$$

From V_{20},

$$\begin{aligned}
&-6\,\overline{\phi}_2\,\phi_2\,\delta\beta - 6\,\phi_2\,\overline{\phi}_2\,\delta\overline{\alpha} - 15\,\phi_2\,\delta\overline{\phi}_2\overline{\alpha} + 24\,\delta\beta + 24\,\delta\overline{\alpha}\\
&- 9\,\overline{\phi}_2\,\delta\phi_2\beta + 19\,\phi_2\,\delta\overline{\phi}_2\beta + 13\,\overline{\phi}_2\,\delta\phi_2\overline{\alpha} + 8\,\delta\phi_2\delta\overline{\phi}_2\\
&+ 32\,\phi_2\,\beta\,\overline{\phi}_2\,\overline{\alpha} + 72\,\overline{\alpha}^2 + 24\,\beta^2 - 3\,\phi_2\,\delta(\,\delta\overline{\phi}_2\,) - 3\,\overline{\phi}_2\,\delta(\,\delta\phi_2)\\
&+ 288\,\beta\,\overline{\alpha} - 18\,\phi_2\,\overline{\phi}_2\,\overline{\alpha}^2 - 6\,\overline{\phi}_2\,\phi_2\,\beta^2 = 0\,.
\end{aligned} \tag{4.216}$$

Substituting (4.100), (4.101) and (4.108) into (4.216), and solving the complex conjugate for $\overline{\delta}\alpha$ we obtain

$$\overline{\delta}\alpha = \frac{-24\phi_2\overline{\phi}_2\overline{\beta}^2 - 3\overline{\phi}_2{}^2\alpha^2 + 44\overline{\beta}\alpha + 9\overline{\phi}^2\overline{\beta}\alpha + 12\alpha^2}{\overline{\phi}_2{}^2 - 4}\,. \tag{4.217}$$

Solving (4.214) for $\overline{\phi}_2$ and substituting into (4.217) we find

$$\overline{\delta}\alpha = -\alpha(11\overline{\beta} + 3\alpha)\,. \tag{4.218}$$

However, from (4.132),

$$\overline{\delta}\alpha = \alpha(157\overline{\beta} - 3\alpha)\,, \tag{4.219}$$

implying $\overline{\beta}\alpha = 0$, a contradiction.

Let us suppose now that $\beta = 0$. From III_{01},

$$\mathrm{D}\gamma + 2\Phi_{11} + 2\mathrm{D}\mu + \delta\alpha - \alpha\overline{\alpha} = 0 . \tag{4.220}$$

From the NP equations

$$\delta\alpha = \alpha\overline{\alpha} + \Phi_{11} , \qquad \mathrm{D}\gamma = \Phi_{11} , \qquad \mathrm{D}\mu = 0 . \tag{4.221}$$

Substituting these Pfaffians into (4.220) we find

$$\Phi_{11} = 0 . \tag{4.222}$$

The commutator $[\overline{\delta}\,,\,\delta]\overline{\phi}_2$ (cf. (4.132)) gives

$$\alpha\overline{\alpha}(\phi_2 - \overline{\phi}_2) = 0 , \tag{4.223}$$

i.e., $\phi_2 = \overline{\phi}_2$. Equation (4.219) becomes

$$\overline{\delta}\alpha = -3\alpha^2 , \tag{4.224}$$

and from V_{11},

$$\alpha\overline{\alpha}({\phi_2}^2 - 4) = 0 . \tag{4.225}$$

From III_{00}, and (NP15),

$$\delta\mu = -\Phi_{12} - \mu\overline{\alpha} - \frac{5}{4}{\phi_2}^2 , \tag{4.226}$$

$$\overline{\delta}\overline{\gamma} = -\overline{\gamma}\alpha + \overline{\alpha}\overline{\lambda} + \Phi_{21} . \tag{4.227}$$

Since ϕ_2 is constant V_{02} now becomes

$$24\,\alpha + 10\,\alpha^2\,\overline{\mu} - 5\,\overline{\alpha}\,\lambda\,\alpha + 2\,\overline{\delta}\alpha\,\overline{\mu} - \overline{\gamma}\,\overline{\delta}\alpha + 3\,\overline{\delta}\overline{\gamma}\alpha - 2\,\Phi_{21}\,\alpha + 2\,\overline{\delta}\overline{\alpha}\lambda$$
$$+\,4\,\overline{\delta}\overline{\mu}\,\alpha = 0 . \tag{4.228}$$

Using (4.224), (4.226) and (4.227), this equation becomes

$$\alpha\phi_2(-24 + 3\Phi_{21} + 4{\phi_2}^2) = 0 , \tag{4.229}$$

or, since ${\phi_2}^2 = 4$,

$$\alpha(-8 + 3\Phi_{21}) = 0 . \tag{4.230}$$

However, from the NP equations,

$$\overline{\delta}\Phi_{21} = -2\alpha(-1 + \Phi_{21}) = 0 , \tag{4.231}$$

so $\alpha = 0$, which is a contradiction.

(iii) $\beta = 0$.

We now suppose that $\alpha \neq 0$, since the case $\alpha = 0$, $\beta = 0$ was already considered. From the NP equations

$$\delta\alpha = \alpha\overline{\alpha} + \Phi_{11}\,, \tag{4.232}$$
$$\delta\pi = \mathrm{D}\mu - \pi\overline{\pi} + \pi\overline{\alpha}\,, \tag{4.233}$$
$$\mathrm{D}\gamma = \alpha\overline{\pi} + \Phi_{11}\,. \tag{4.234}$$

From (4.120),

$$\mathrm{D}\mu = \pi\overline{\pi} - \Phi_{11}\,. \tag{4.235}$$

Thus, II_{00} and IV_{10} now become

$$\delta\phi_2 = 0\,, \tag{4.236}$$
$$\delta\overline{\phi}_2 = -2\overline{\phi}_2\overline{\alpha} + 4\overline{\pi}\phi_2 + 2\phi_2\overline{\alpha}\,. \tag{4.237}$$

Thus, using these Pfaffians in the commutator $[\overline{\delta}\,,\,\delta]\phi_2$ we find

$$\begin{aligned} g_1 \;:=\; & 2\,\overline{\alpha}\,\phi_2\,\alpha - 4\,\overline{\alpha}\,\pi\,\overline{\phi}_2 - 2\,\overline{\alpha}\,\overline{\phi}_2\,\alpha - \phi_2\,\Phi_{11} + 8\,\pi\,\overline{\pi}\,\phi_2 + 4\,\pi\,\phi_2\,\overline{\alpha} \\ & -\overline{\phi}_2\,\Phi_{11} + 4\,\alpha\,\overline{\pi}\,\phi_2 = 0\,. \end{aligned} \tag{4.238}$$

From V_{20} and V_{12} we have, respectively

$$\begin{aligned} & -16\,{\phi_2}^2\,\overline{\alpha}\,\overline{\pi} - 12\,{\phi_2}^2\,\overline{\alpha}^2 + 32\,\mathrm{D}\overline{\lambda} - 32\,\overline{\pi}^2 + 64\,\overline{\alpha}\,\overline{\pi} + 16\,\delta\overline{\alpha} + 48\,\overline{\alpha}^2 \\ & - 8\,{\phi_2}^2\,\mathrm{D}\overline{\lambda} + 8\,{\phi_2}^2\,\overline{\pi}^2 - 4\,{\phi_2}^2\,\delta\overline{\alpha} = 0\,, \end{aligned} \tag{4.239}$$

$$\begin{aligned} & 4\,\phi_2\,\overline{\delta}\alpha\,\overline{\alpha} - 8\,\pi^2\,\phi_2\,\overline{\alpha} - 6\,\overline{\delta}\alpha\,\overline{\phi}_2\,\overline{\alpha} + 16\,\mathrm{D}\lambda\,\overline{\pi}\,\phi_2 + 8\,\mathrm{D}\lambda\,\phi_2\,\overline{\alpha} \\ & + 4\,\overline{\pi}\,\overline{\phi}_2\,\overline{\delta}\alpha + 8\,\overline{\pi}\,\phi_2\,\overline{\delta}\alpha - 12\,\overline{\alpha}\,\overline{\phi}_2\,\mathrm{D}\lambda + 8\,\overline{\pi}\,\overline{\phi}_2\,\mathrm{D}\lambda - 16\,\pi^2\,\overline{\pi}\,\phi_2 \\ & - 2\,\Phi_{11}\,\overline{\phi}_2\,\alpha - 26\,\Phi_{11}\,\overline{\phi}_2\,\pi - 30\,\alpha^2\,\overline{\phi}_2\,\overline{\alpha} - 12\,\overline{\pi}\,\overline{\phi}_2\,\alpha^2 + 28\,\pi^2\,\overline{\alpha}\,\overline{\phi}_2 \\ & + 24\,\overline{\pi}\,\overline{\phi}_2\,\pi^2 + 6\,\Phi_{11}\,\phi_2\,\alpha + 24\,\phi_2\,\alpha^2\,\overline{\alpha} + 48\,\overline{\pi}\,\phi_2\,\alpha^2 - 4\,\pi\,\phi_2\,\Phi_{11} \\ & + 6\,\overline{\alpha}\,\pi\,\phi_2\,\alpha + 12\,\overline{\pi}\,\pi\,\phi_2\,\alpha - 16\,\overline{\pi}\,\alpha\,\pi\,\overline{\phi}_2 \\ & -38\,\pi\,\overline{\phi}_2\,\overline{\alpha}\,\alpha + \mathrm{D}(\Delta\overline{\phi}_2)\,\alpha = 0\,, \end{aligned} \tag{4.240}$$

where the second order Pfaffian $\mathrm{D}(\Delta\overline{\phi}_2)$ can be determined from the NP commutator $[\Delta\,,\,\mathrm{D}]\overline{\phi}_2$:

$$\mathrm{D}(\Delta\overline{\phi}_2) = -2\overline{\alpha}\pi\overline{\phi}_2 + 4\pi\overline{\pi}\phi_2 + 2\pi\phi_2\overline{\alpha}\,. \tag{4.241}$$

Solving the complex conjugate of (4.239) for Dλ and substituting it into (4.240) we obtain

$$\begin{aligned} g_2 := & -16\,\overline{\pi}\,\alpha\,\pi\,\overline{\phi}_2 - 8\,p\,\overline{\phi}_2\,\overline{\alpha}\,\alpha + 16\,\overline{\pi}\,\overline{\phi}_2\,\pi^2 + 8\,\pi^2\,\overline{\alpha}\,\overline{\phi}_2 - 6\,\alpha^2\,\overline{\phi}_2\,\overline{\alpha} \\ & - 12\,\overline{\pi}\,\overline{\phi}_2\,\alpha^2 - \Phi_{11}\,\overline{\phi}_2\,\alpha - 13\,\Phi_{11}\,\overline{\phi}_2\,\pi + 12\,\overline{\pi}\,\phi_2\,\alpha^2 - 2\,\pi\,\phi_2\,\Phi_{11} \\ & - 4\,\overline{\alpha}\,\pi\,\phi_2\,\alpha - 8\,\overline{\pi}\,\pi\,\phi_2\,\alpha + 6\,\phi_2\,\alpha^2\,\overline{\alpha} + 3\,\Phi_{11}\,\phi_2\,\alpha = 0\,. \end{aligned} \tag{4.242}$$

Applying gsolve to g_1, g_2, and their complex conjugates, we obtain the following set of possible solutions:

$$K_1 := [\,121\,\Phi_{11} + 24\,\overline{\alpha}\,\alpha, \phi_2 - \overline{\phi}_2, 4\,\overline{\alpha} + 11\,\overline{\pi}, 4\,\alpha + 11\,\pi\,]\,, \tag{4.243}$$

$$K_2 := [\,\Phi_{11}, \overline{\alpha} + 2\,\overline{\pi}, \alpha + 2\,\pi\,]\,, \tag{4.244}$$

$$K_3 := [\,\Phi_{11}, \phi_2\,\alpha - 2\,\pi\,\overline{\phi}_2 - \overline{\phi}_2\,\alpha, \overline{\alpha}\,\pi + 2\,\pi\,\overline{\pi} + \alpha\,\overline{\pi}\,]\,. \tag{4.245}$$

If $\phi_2 = \overline{\phi}_2$ we have from (4.238) and (4.239) that $\pi\phi_2 = 0$, so $\pi = \alpha = 0$, a contradiction.

If $\alpha + 2\pi = 0$ and $\Phi_{11} = 0$, we obtain the corresponding forms of $\delta\overline{\phi}_2$, $\delta\alpha$, Dμ, calculate the the second order Pfaffian D$(\Delta\phi_2)$ from the respective commutator, and substitute them into V_{11} to find $\alpha = \pi = 0$.

What remains is solution K_3. If $\pi = 0$ we have immediately that $\phi_2 - \overline{\phi}_2 = 0$, a case that was considered above. So, taking $\pi \neq 0$, we introduce the variable $x_1 := \alpha/\pi$. Thus this solution implies

$$\Phi_{11} = 0\,, \tag{4.246}$$

$$x_1 + \overline{x_1} + 2 = 0\,, \tag{4.247}$$

$$x_1(\phi_2 - \overline{\phi}_2) - 2\overline{\phi}_2 = 0\,. \tag{4.248}$$

Thus, we can eliminate x_1 in terms of ϕ_2:

$$x_1 = \frac{2\overline{\phi}_2}{\phi_2 - \overline{\phi}_2}\,. \tag{4.249}$$

At this point, the simplest attempt to close this case would consist in using (4.247) and (4.248) to eliminate $\overline{x_1}$ and x_1 from V_{11}:

$$\frac{\pi\overline{\pi}({\phi_2}^2{\overline{\phi}_2}^2 - 6\phi_2\overline{\phi}_2 + {\overline{\phi}_2}^2 + {\phi_2}^2)}{(-\phi_2 + \overline{\phi}_2)^2}\,. \tag{4.250}$$

Since the case $\phi_2 - \overline{\phi}_2 = 0$ was already considered, we find that ϕ_2 must satisfy the side relation

$${\phi_2}^2{\overline{\phi}_2}^2 - 6\phi_2\overline{\phi}_2 + {\overline{\phi}_2}^2 + {\phi_2}^2 = 0\,. \tag{4.251}$$

However, this equation is not restrictive enough. It is also recurrent under further applications of δ to it. Thus, we still need one more side relation. We observe that V_{02} has the form

$$\begin{aligned}
&4\,\overline{\delta}\overline{\mu}\,\overline{\delta}\phi_2 + 2\,\overline{\delta}\overline{\gamma}\,\overline{\delta}\phi_2 + 2\,\overline{\delta}(\,\delta\overline{\phi}_2\,)\,\lambda - 2\,\Phi_{21}\,\overline{\delta}\phi_2 + 2\,\overline{\delta}\alpha\,\Delta\overline{\phi}_2 \\
&+ 2\,\overline{\delta}(\,\overline{\delta}\phi_2\,)\,\overline{\mu} + 7\,\alpha^2\,\Delta\overline{\phi}_2 + \overline{\delta}(\,\Delta\overline{\phi}_2\,)\,\alpha + 48\,\alpha\,\phi_2 - 3\,\overline{\gamma}\,\overline{\delta}(\,\overline{\delta}\phi_2\,) \\
&+ 92\,\pi\,\phi_2 + 4\,\overline{\delta}\pi\,\Delta\overline{\phi}_2 + 8\,\overline{\delta}\overline{\mu}\,\phi_2\,\alpha - 14\,\overline{\gamma}\,\phi_2\,\alpha^2 - 2\,\overline{\alpha}\,\lambda\,\overline{\delta}\phi_2 \\
&+ 14\,\alpha^2\,\overline{\phi}_2\,\overline{\gamma} + 12\,\pi\,\alpha\,\Delta\overline{\phi}_2 + 20\,\phi_2\,\alpha^2\,\overline{\mu} + 4\,\phi_2\,\overline{\delta}\alpha\,\overline{\mu} + 14\,\alpha\,\overline{\delta}\phi_2\,\overline{\mu} \\
&- 13\,\overline{\gamma}\,\alpha\,\overline{\delta}\phi_2 - 6\,\overline{\gamma}\,\phi_2\,\overline{\delta}\alpha + 2\,\overline{\phi}_2\,\overline{\delta}\overline{\gamma}\,\alpha - 3\,\lambda\,\delta\overline{\phi}_2\,\alpha - 4\,\Phi_{21}\,\phi_2\,\alpha \\
&+ 4\,\overline{\phi}_2\,\overline{\delta}\overline{\alpha}\,\lambda + 4\,\overline{\delta}\overline{\gamma}\,\phi_2\,\alpha + 8\,\overline{\delta}\pi\,\overline{\phi}_2\,\overline{\gamma} + 4\,\overline{\delta}\alpha\,\overline{\phi}_2\,\overline{\gamma} - 4\,\overline{\pi}\,\lambda\,\overline{\delta}\phi_2 \\
&+ 6\,\pi\,\overline{\phi}_2\,\Phi_{21} - 4\,\overline{\alpha}\,\lambda\,\phi_2\,\alpha - 8\,\overline{\pi}\,\lambda\,\phi_2\,\alpha - 6\,\lambda\,\overline{\phi}_2\,\overline{\alpha}\alpha \\
&+ 24\,\pi\,\alpha\,\overline{\phi}_2\,\overline{\gamma} = 0\,.
\end{aligned} \tag{4.252}$$

Let us determine all Pfaffians appearing in the expression above, beside those found previously. From (NP15), (NP7) and the commutator $[\overline{\delta}\,,\,\Delta]\overline{\phi}_2$ we have, respectively:

$$\overline{\delta}\overline{\gamma} = -\overline{\gamma}\alpha + \overline{\alpha}\lambda + \Phi_{21}\,, \tag{4.253}$$

$$\overline{\delta}\pi = \mathrm{D}\lambda - \pi^2 - \alpha\pi\,, \tag{4.254}$$

$$\overline{\delta}(\Delta\overline{\phi}_2) = -\alpha\Delta\overline{\phi}_2 - 2\lambda\overline{\phi}_2\overline{\alpha} + 4\lambda\overline{\pi}\phi_2 + 2\lambda\overline{\alpha}\phi_2\,. \tag{4.255}$$

The Pfaffians $\mathrm{D}\lambda$ and $\mathrm{D}\overline{\mu}$ can be eliminated using (4.239) and (4.235), respectively. The terms containing $\overline{\delta}\alpha$ and $\Delta\overline{\phi}_2$ cancel themselves. Finally, from III_{00},

$$\delta\mu = \frac{1}{2}\left(-\Phi_{12} - \gamma\overline{\alpha} - \delta\gamma + \alpha\overline{\lambda} - 2\mu\overline{\alpha} - \frac{5}{2}\phi_2\overline{\phi}_2 + 2\pi\overline{\lambda}\right)\,. \tag{4.256}$$

After all substitutions, (4.252) takes the form

$$\begin{aligned}
&-10\,\pi\,\phi_2\,\overline{\phi}_2^2 - 5\,\phi_2\,\alpha\,\overline{\phi}_2^2 + 8\,\overline{\alpha}\,\lambda\,\pi\,\overline{\phi}_2 - 5\,\pi\,\overline{\phi}_2\,\Phi_{21} + 4\,\lambda\,\overline{\phi}_2\,\overline{\alpha}\,\alpha \\
&- 3\,\overline{\phi}_2\,\alpha\,\Phi_{21} + 16\,\overline{\pi}\,\lambda\,\pi\,\overline{\phi}_2 + 8\,\overline{\pi}\,\lambda\,\overline{\phi}_2\,\alpha + 46\,\phi_2\,\pi + 24\,\phi_2\,\alpha \\
&-8\,\overline{\pi}\,\lambda\,\phi_2\,\alpha - 4\,\overline{\alpha}\,\lambda\,\phi_2\,\alpha = 0\,.
\end{aligned} \tag{4.257}$$

Using (4.247) and (4.249) this equation assumes the form

$$\frac{\pi\,(\,10\,\phi_2^2\,\overline{\phi}_2^2 - 46\,\phi_2^2 + 5\,\phi_2\,\overline{\phi}_2\,\Phi_{21} - 2\,\phi_2\,\overline{\phi}_2 + \Phi_{21}\,\overline{\phi}_2^2\,)}{\phi_2 - \overline{\phi}_2} = 0\,. \tag{4.258}$$

From (NP25),

$$\overline{\delta}\Phi_{21} = 2(\alpha + 2\pi - \alpha\Phi_{21})\,. \tag{4.259}$$

Applying $\overline{\delta}$ to (4.258), using (4.236), (4.237), (4.259), (4.247) and (4.249), we obtain

$$\frac{\overline{\phi}_2\pi(\phi_2 - \overline{\phi}_2\Phi_{21})(5\phi_2 + \overline{\phi}_2)}{\phi_2 - \overline{\phi}_2} = 0\,. \tag{4.260}$$

Thus,

$$\Phi_{21} = \frac{\phi_2}{\overline{\phi}_2}\,. \tag{4.261}$$

Substituting this expression into (4.258) we get, finally,

$$\phi_2(41\phi_2 + \overline{\phi}_2) = 0\,, \tag{4.262}$$

i.e., $\phi_2 = 0$, a contradiction.

Thus, we have proved that the assumption that $A_{[i,j]} \neq 0$ leads to a contradiction. So, the necessary conditions for the validity of Huygens' principle imply that we must have $H_{ij} := A_{[i,j]} = 0$. The last step of the proof requires the use of the following lemma [38]:

Lemma 4.3 *Every scalar wave equation of the form (4.1) for which $A_{[i,j]} = 0$, is related by a trivial transformation to one for which $A_i = 0$.*

The proof consists in observing that $A_{[i,j]} = 0$ implies that the differential form $A = A_i dx^i$ is closed. Thus there exists locally a function h such that $A = dh$. It follows that for transformation (bc) (cf. Subsection 2.3.1) with $\lambda = exp(-h/2)$ one has, according to (2.125), $\overline{A_i} = 0$.

From necessary condition I (cf. (2.181)) we now have $B = R/6$, so the wave equation is conformally invariant.

Thus, the Main Theorem is proved.

Chapter 5

Maxwell's Equations and Weyl's Neutrino Equation

5.1 Introduction

In this chapter we consider the problem of determining the validity of Huygens' principle for Maxwell's equations and Weyl's neutrino equation in Petrov type III space-times. We start by reviewing the theory of the initial value problem for both cases and the determination of the necessary conditions. The review will follow the papers of Günther and Wünsch [39] [41] [82] [84] [85]. Finally we shall prove the main result of this chapter, that states that there are no Petrov type III space-times on which Maxwell's equations or Weyl's neutrino equation satisfy Huygens' principle [63].

5.2 Huygens' principle for Maxwell's equations

The problem of the validity of Huygens' principle for the solutions of Maxwell's equations for p-forms in a curved space-time was considered for the first time in 1965 by Günther [39]. These equations are given by

$$d\omega_p = 0\,, \qquad \delta\omega_p = 0\,, \tag{5.1}$$

where ω_p is the p-form

$$\omega_p = (1/p!)\,\omega_{i_1\ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}\,. \tag{5.2}$$

The symbols d and δ denote the exterior differentiation and codifferentiation operators, respectively, given by

$$(d\omega_p)_{i_1\dots i_{p+1}} = (p+1)\partial_{[i_1}\omega_{i_2\dots i_{p+1}]}\,, \tag{5.3}$$

and

$$(\delta\omega_p)_{i_1\dots i_{p-1}} = -\nabla^k\omega_{ki_1\dots i_{p-1}}\,. \tag{5.4}$$

For $p=2$ and dimension $n=4$ we obtain the Maxwell's equations for the electromagnetic field:

$$\partial_{[a}\omega_{bc]} = 0\,, \qquad \nabla^j\omega_{ja} = 0\,. \tag{5.5}$$

The Laplace-Beltrami operator in this case is defined as

$$\Box = -(\delta\, d + d\,\delta). \tag{5.6}$$

The initial value problem for (5.1) was studied by Duff [30] in 1953, using the Riesz kernel formalism [71] , and by Lichnerowicz [55] in 1961, using the distribution formalism. Günther used the first method to establish the first necessary condition for the validity of Huygens' principle for (5.1) with $n=4$ and $p=2$, which is the vanishing of Bach's tensor (2.201).

5.2.1 The Riesz kernel

Let $\Gamma(x,\xi)$ denote the square of the geodesic distance between two points x and ξ (cf. (2.134) in a n-dimensional pseudo-Riemannian space $\mathcal{M}^n$. The Riesz kernel is defined by the double p-tensor:

$$V_p(x,\xi,\lambda) = \frac{\Gamma^{(\lambda-n)/2}}{2^{\lambda-1}\pi\boldsymbol{\Gamma}\left(\frac{\lambda}{2}\right)}\sum_{k=0}^{\infty}\frac{V_k^{(p)}(x,\xi)\Gamma^k}{2^k\boldsymbol{\Gamma}\left(\frac{\lambda-2}{2}+\mathrm{k}\right)}\,, \tag{5.7}$$

where λ is a complex parameter, $\boldsymbol{\Gamma}(\lambda)$ is the Euler gamma function [1] analytically continued to the complex plane, and $V_k^{(p)}$ are double p-differential forms given by

$$V_k^{(p)}(x,\xi) = A_{ki_1\cdots i_p,\alpha_1\cdots\alpha_p}(x,\xi)\; dx^{i_1}\wedge\cdots dx^{i_p}d\xi^{\alpha_1}\tilde{\wedge}\cdots\tilde{\wedge}d\xi^{\alpha_p}\,. \tag{5.8}$$

The symbols $A_{ki_1\cdots i_p,\alpha_1\cdots\alpha_p}$ denote the components of the double p-form $V_k^{(p)}$ and the symbol $\tilde{}$ is used to denote operations relative to the variables ξ^α. It can be shown [30] that V_p must satisfy

$$\Box V_p(x,\xi,\lambda+2) = V_p(x,\xi,\lambda)\,. \tag{5.9}$$

[1] we denote the Euler gamma function by a boldfaced gamma to distinguish it from the square of the geodesic distance

Substituting (5.7) into (5.9) and equalling the coefficients of the powers of Γ to zero, we obtain the following recursive system:

$$g^{ij}\nabla_i\Gamma\,\nabla_j V_0^{(p)} + LV_0^{(p)} = 0\,, \tag{5.10}$$

$$g^{ij}\nabla_i\Gamma\,\nabla_j V_k^{(p)} + (L+2k)V_k^{(p)} = -\Box\, V_{k-1}^{(p)}\,, \quad k = 1, 2, \dots\,, \tag{5.11}$$

where the covariant derivatives are relative to x^a, and $\nabla_a V_k^p$ is the differential form whose coefficients are given by $\nabla_a A_{k i_1\cdots i_p,\alpha_1\cdots\alpha_p}(x\,\xi)$ (with $A_{k\dots,\dots}$ considered as a covariant tensor of rank p). $L(x,\xi)$ is given by

$$L(x,\xi) := \frac{1}{2}\nabla^i\nabla_i\Gamma - n\,, \qquad L(\xi,\xi) = 0\,. \tag{5.12}$$

Equations (5.10) and (5.11) can be considered as ordinary differential equations (transport equations) along the geodesic line between x and ξ [71] [42] [32]. For $p = 0$, $V_0^{(0)}$ is uniquely determined by (5.10) and the initial condition $V_0^{(0)}(\xi,\xi) = 1$. Notice that $V_0^{(0)}$ can be identified with the U defined for the self-adjoint scalar equation ($A^i = 0$) (see (2.40)). It can be shown that for $V_0^{(p)}$ we have the following initial condition:

$$A_{0 i_1\cdots i_p,\alpha_1\cdots\alpha_p}(\xi,\xi) = \frac{1}{(p!)^2}\begin{vmatrix} g_{i_1\alpha_1} & \cdots & g_{i_1\alpha_p} \\ \vdots & & \vdots \\ g_{i_p\alpha_1} & \cdots & g_{i_p\alpha_p}\end{vmatrix}(\xi)\,, \tag{5.13}$$

while $V_k^{(p)}$, $k \geq 1$ is determined by (5.11), together with the requirements for regularity at $x = \xi$. This shows that (5.11) admits a solution system $V_k^{(p)}(x,\xi)$, which is analytic in x and ξ. When x and ξ are in a sufficiently small neighborhood $U(\xi)$ and $\Re(\lambda) \geq \mu$ the series in (5.7) converges absolutely and is regular in x, ξ and λ. Thus $V_p(x,\xi,\lambda)$ for $x, \xi \in U$, $\Gamma > 0$, and λ arbitrary, is an analytic function in x, ξ and λ. If $\Re(\lambda - n) > 0$, then $V_p(x,\xi,\lambda)$ is also regular on the characteristic surface, $\Gamma(x,\xi) = 0$. If, however, $\Re(\lambda - n) < 0$, then $V_p(x,\xi,\lambda)$ is singular for $\Gamma(x,\xi) = 0$. One exception occurs for $\lambda = n - 2q$, $q = 1, 2, \dots$, when $V_p(x,\xi,n-2q)$ is regular for all $x, \xi \in U$.

The Riesz kernel also satisfies the following useful relations [30]:

$$V_p(x,\xi,\lambda) = V_p(\xi,x,\lambda)\,, \qquad V_k^{(p)}(x,\xi) = V_k^{(p)}(\xi,x)\,, \tag{5.14}$$

$$V_p(x,\xi,\lambda) = -*\bar{*}V_{4-p}(x,\xi,\lambda)\,, \tag{5.15}$$

$$dV_p(x,\xi,\lambda) = -\bar{\delta}V_{p+1}(\xi,x,\lambda)\,, \tag{5.16}$$

$$\delta V_p(x,\xi,\lambda) = -\bar{d}V_{p-1}(\xi,x,\lambda)\,. \tag{5.17}$$

5.2.2 Cauchy's problem for Maxwell's equations

Let us consider an $n-1$-dimensional space-like surface S of class C^∞, with parametric representation

$$x^a = x^a(u^i) \qquad i = 1, 2, \ldots, n-1 \,. \tag{5.18}$$

Using this parametrized representation for the coefficients of a form ω_p, we obtain the tangential part of ω_p, denoted by $(\omega_p)_S$, given in terms of the variables u^i and differentials dx^a. If we substitute $dx^a = \frac{\partial x^a}{\partial u^i} du^i$ into $(\omega_p)_S$ we obtain a form $(\omega_p)_{S,dS}$ in the variables u^i and differentials du^i. The class of these forms is denoted by $M(S)$. The initial value problem for Maxwell's equations is the problem of finding a form ω_p, $1 \le p \le n-1$, satisfying (5.1) for given $(\omega_p)_S$. This problem is also equivalent to the following: For given $(\omega_p)_S$ find a form ω_p, $1 \le p \le n-1$, satisfying

$$\Box \omega_p = 0 \tag{5.19}$$

and

$$(d\omega_p)_S = 0 \,, \qquad (\delta\omega_p)_S = 0 \,. \tag{5.20}$$

The coefficients of $(\omega_p)_S$ (the initial data) are not arbitrary, but on account of (5.20) must satisfy

$$(d\omega_p)_{S,dS} = d\{(\omega_p)_{S,dS}\} = 0 \,, \tag{5.21}$$

$$(d * \omega_p)_{S,dS} = d\{(*\omega_p)_{S,dS}\} = 0 \,. \tag{5.22}$$

The initial value problem can now be stated in the following manner:

Given on $M(S)$ the $p-$form $\Theta_p = (\omega_p)_{S,dS}$ and a $(n-p)$-form $\Pi_{n-p} = (*\omega_p)_{S,dS}$, with $d\Theta_p = 0$, $d\Pi_{n-p} = 0$, find a $p-$form ω_p satisfying the equations

$$\Box \omega_p = 0 \,, \qquad (d\omega_p)_S = 0 \,, \qquad (\delta\omega_p)_S = 0 \,. \tag{5.23}$$

It has been shown by Duff [30] that the above problem has a unique solution.

Now we can state Huygens' principle as follows:

Huygens' principle is said to be valid for Maxwell's equations (5.1) if for any space-like surface S and any point x, $\omega_p(x) = 0$ for all permissible choices for the initial data on S, with support in $D^{\pm}(x) \cap S$.

5.2.3 Necessary and sufficient condition

For the initial value problem, let us choose x in a neighborhood of S, $supp\,\Theta_p$, $supp\,\Pi_{n-p} \in D(x) \cap S$, Then, from the Riesz integration method the following representation is found:

$$\omega_p(x) = -\int_{D(x)\cap S} \tilde{\delta}\, V_p(x,\xi,2)\,\bar{\wedge}\,\Pi_{n-p} + \int_{D(x)\cap S} \Theta_p\,\bar{\wedge}\,\bar{*}\,\tilde{d}\,V_p(x,\xi,2)\,, \tag{5.24}$$

where the integration variable is ξ. Employing the above representation, with

$$\Theta_p = d\Theta_{p-1}\,, \qquad \Pi_{n-p} = d\Pi_{n-p-1}\,, \tag{5.25}$$

and using partial integration, Günther [39] proved that a necessary and sufficient condition for the validity of Huygens' principle Maxwell's equations (5.1) of order p, $1 \le p \le n-1$ is

$$d\delta V_p(x,\xi,2) = 0\,, \tag{5.26}$$

for all x and ξ. Condition (5.26) is also equivalent to

$$\delta d V_p(x,\xi,2) = 0\,, \qquad d\tilde{d}V_{p-1}(x,\xi,2) = 0\,, \qquad d\tilde{\delta}V_{p+1}(x,\xi,2) = 0\,. \tag{5.27}$$

A detailed proof can be found in [39].

From this result Günther [39] obtained the following immediate consequences:

(i) If n is odd Huygens' principle is never valid for (5.1).
(ii) If n is even and the metric is Minkowskian then Huygens' principle is valid for Maxwell's equations (5.1) for any order p, satisfying $1 \le p \le n-1$.
(iii) If n is even and the space-time is conformally flat, then Huygens' principle is valid for Maxwell's equations (5.1) of order $p = n/2$.

The essential reason for the restriction in **(iii)** is that Maxwell's equations (5.1), for n even, are conformally invariant only for $p = n/2$[39].

5.2.4 Necessary conditions for $n = 4$ and $p = 2$

When $n = 4$ and $p = 2$, equation (5.1) represents Maxwell's equations for the electromagnetic field and Huygens' principle is satisfied if the metric is conformally flat. Explicit necessary conditions for the validity of Huygens' principle can now be obtained from (5.26) or (5.27). Günther [39] considered the second equation in (5.27) :

$$K(x,\xi) := \left[d\,\tilde{d}\,V_1(x,\xi,2)\right]_{x=\xi} = 0\,, \tag{5.28}$$

with

$$V_1(x,\xi,2) = \frac{1}{4\pi}\sum_{k=0}^{\infty}\frac{V^{(1)}_{k+1}(x,\xi)\Gamma^k}{2^k k!}\,. \tag{5.29}$$

The coefficients $A_{k}{}_{i,\alpha}(x,\xi)$ of $V^{(1)}_k$ can be developed into a Taylor series around ξ:

$$A_{0}{}_{i,\alpha}(x,\xi) = a_{i,\alpha}(\xi) + a_{i,\alpha|i_1}q^{i_1} + a_{i,\alpha|i_1 i_2}q^{i_1}q^{i_2} + \cdots\,, \tag{5.30}$$

$$A_{1}{}_{i,\alpha}(x,\xi) = b_{i,\alpha}(\xi) + b_{i,\alpha|i_1}q^{i_1} + b_{i,\alpha|i_1 i_2}q^{i_1}q^{i_2} + \cdots\,, \tag{5.31}$$

$$A_{2}{}_{i,\alpha}(x,\xi) = c_{i,\alpha}(\xi) + c_{i,\alpha|i_1}q^{i_1} + c_{i,\alpha|i_1 i_2}q^{i_1}q^{i_2} + \cdots\,, \tag{5.32}$$

where $q^i = x^i - \xi^i$ and the following notation is used:

$$B_{\alpha\cdots|i_1\cdots i_r}(\xi) := \left[\frac{1}{r!}\partial_{i_1}\cdots\partial_{i_r}B_{\alpha\cdots}(x,\xi)\right]\,. \tag{5.33}$$

As in the case for the scalar equation, covariant Taylor expansions are here determined in a system of normal coordinates x^i, with origin at a fixed point ξ_0. The equalities that are valid only in this coordinate system are denoted by $\stackrel{*}{=}$. Thus

$$q^i \stackrel{*}{=} x^i\,, \qquad \Gamma(x,\xi_0) \stackrel{*}{=} \mathring{g}_{ij}\,x^i x^j\,, \qquad \mathring{g}_{ij} \stackrel{*}{=} g_{ij}(\xi_0)\,, \tag{5.34}$$

$$L(x,\xi_0) \stackrel{*}{=} \frac{1}{2}g^{ij}x^k g_{ij,k}\,. \tag{5.35}$$

From (5.10) a differential equation for $A_{0}{}_{i,\alpha}(x,\xi)$ can be obtained in normal coordinates. Using the initial condition $A_{0}{}_{i,\alpha}(\xi,\xi) = g_{i\alpha}(\xi)$, from (5.13), the coefficients $a_{i,\alpha}\,, a_{i\alpha|i_1}\,,\ldots$ can be specified in terms of the metric and its derivatives. In order to expand (5.29) up to three terms, we need to find $b_{i,\alpha}$, $b_{i\alpha|i_1}$, $b_{i\alpha|i_1 i_2}$ and $c_{i,\alpha}$.

Using (5.11), a differential recurrence relation between $A_{1}{}_{i,\alpha}(x_1,\xi)$ and $A_{0}{}_{i,\alpha}(x_1,\xi)$ is induced. Since $V^{(1)}_0$ does not appear in (5.29), we need only the first four terms in the expansion (5.30). The recurrence relation allows the determination of $b_{i,\alpha}$, $b_{i\alpha|i_1}$, $b_{i\alpha|i_1 i_2}$ and $c_{i,\alpha}(\xi)$ in terms of the metric and its derivatives. The calculations are lengthy, requiring the use of the conformal invariance of certain tensors. Details can be found in Günther's paper [39]. The result is

$$\left[4\pi d\bar{d}V_1(x,\xi,2)\right]_{x=\xi=\xi_0} \stackrel{*}{=} -\frac{1}{20}\,\mathring{g}_{i\alpha}\left(\mathring{L}_{j\beta;l}{}^{l} - \mathring{L}_{;\beta j}\right)dx^j\wedge dx^i\bar{\wedge}d\xi^\alpha\,. \tag{5.36}$$

This implies that $\mathring{L}_{ab} = 0$ and so, the conformally invariant Bach's tensor, $C_{ab} := L_{a[b;c]} - \frac{1}{2}C^k{}_{ab}{}^l L_{kl}$, vanishes.

Thus, we have the following result [39]:

A necessary condition for the validity of Huygens' principle for the homogeneous Maxwell equations for the electromagnetic field is the vanishing of Bach's tensor.

5.2.5 Determining necessary conditions using conformal invariance of tensors

The vanishing of Bach's tensor and other higher order necessary conditions for the validity of Huygens' principle for Maxwell's equations [2] can be determined by using general properties of tensors which are symmetric, trace-free and conformally invariant. We shall summarize the method by following the paper of Wünsch [82]. We write $K(x,\xi)$ (cf. (5.28)) as

$$K(x,\xi) = K_{ij\alpha\beta}dx^i \wedge dx^j \, \bar{d}\xi^\alpha \,\bar{\wedge}\, \bar{d}\xi^\beta\,, \tag{5.37}$$

and

$$K_{ij\alpha\beta|i_1\cdots i_r}(\xi) = \left[\frac{1}{r!}\partial_{i_1}\cdots\partial_{i_r}K_{ij\alpha\beta}(x,\xi)\right]_{x=\xi}, \tag{5.38}$$

$$K_{ij\alpha\beta;i_1\cdots i_r}(\xi) = \left[\frac{1}{r!}\nabla_{i_r}\cdots\nabla_{i_1}K_{ij\alpha\beta}(x,\xi)\right]_{x=\xi}, \quad r\in\mathrm{N}\,. \tag{5.39}$$

Thus, from (5.28), Huygens' principle is satisfied when $K_{ij\alpha\beta|i_1\cdots i_r}(\xi) = 0$, $\forall r\in \mathrm{N}$. Let $\mathcal{G}$ denote the set of all metric tensors $\mathrm{g}\in \mathrm{C}^\infty$ defined on $\mathcal{M}^4$,

$$\mathcal{G}_s := \begin{cases} \mathcal{G}\,, & \text{for } s=0 \\ \left\{\mathrm{g}\in\mathcal{G}|\forall r\in\{0,1,\ldots,s-1\}\forall\xi\in\mathcal{M}^4 : K_{ij\alpha\beta|i_1\cdots i_r}(\xi)=0\right\} & \text{for } s\in\mathrm{N}\backslash\{0\}\,. \end{cases} \tag{5.40}$$

Wünsch [83] proved that $(\mathcal{M}^4,\mathcal{G}_s)$ defines a conformal class of Riemannian spaces. It can also be shown that for $s\in\mathrm{N}$ and $\mathrm{g}\in\mathcal{G}_s$ we have

$$K_{ij\alpha\beta|i_1\cdots i_s} = K_{ij\alpha\beta;i_1\cdots i_s}\,. \tag{5.41}$$

We now define an additional tensor

$$E_{j\beta i_1\cdots i_s} := \begin{cases} K_{ij}{}^{i}{}_{\beta;i_1\cdots i_s}\,, & \text{if } s \text{ is even}\,, \\ *K_{ij}{}^{i}{}_{\beta;i_1\cdots i_s}\,, & \text{if } s \text{ is odd}\,. \end{cases} \tag{5.42}$$

In [82] Wünsch proved the following facts:

1. For $s\in\mathrm{N}$ and $\mathrm{g}\in\mathcal{G}_s$ we have

$$K_{ij}{}^{i}{}_{;i_1\cdots i_s} = 0 \Leftrightarrow E_{j\beta i_1\cdots i_s} = 0\,, \tag{5.43}$$

2. $E_{j\beta i_1\cdots i_s}$ in the class $(\mathcal{M}^4,\mathcal{G}_s)$, are conformally invariant tensors with weight -1.
3. $E_{j\beta i_1\cdots i_s}$ is symmetric:

$$E_{j\beta i_1\cdots i_s} = E_{(j\beta i_1\cdots i_s)}\,, \tag{5.44}$$

[2] From now on, by Maxwell's equations we mean the physically relevant equations (5.1), with $n=4$ and $p=2$.

4. $E_{i_1 i_2 \cdots i_s}$ is trace-free:

$$g^{j\beta} E_{j\beta i_1 \cdots i_{s-2}} = 0\,. \tag{5.45}$$

As consequence, the following result follows:

Let $E_{j\beta i_1 \cdots i_s}$, $s \in \mathbf{N}$, be symmetric, trace-free, in the class $(\mathcal{M}^4, \mathcal{G})$ of conformally invariant tensors of weight -1. Then Huygens' principle is valid for Maxwell's equations (5.5) when the following equation is satisfied:

$$E_{i_1 i_2 \cdots i_{s-2}} = 0\,, \qquad \forall s \in \mathbf{N}\,. \tag{5.46}$$

This result enables us to determine explicit necessary conditions, by studying the possible representations of $E_{i_1 \cdots i_r}$. It was shown by Wünsch [83] that if $E_{i_1 \cdots i_s}$ is a rational integral, symmetric, trace-free tensor, in the class $(\mathcal{M}^4, \mathcal{G}_s)$ of conformally invariant tensors of weight -1, then the following properties must be satisfied:

$$E_{i_1 \cdots i_r} = 0 \qquad \text{for } s = 0, 1, 3\,, \tag{5.47}$$

$$E_{i_1 i_2} = \alpha_0 C_{i_1 i_2}\,, \tag{5.48}$$

$$E_{i_1 i_2 i_3 i_4} = \sum_{\nu=1}^{3} \alpha_\nu TS\left(\overset{\nu}{N}_{i_1 i_2 i_3 i_4}\right)\,, \qquad \alpha_\nu \in \mathbf{R}\,, \tag{5.49}$$

where

$$\begin{aligned} \overset{1}{N}_{i_1 i_2 i_3 i_4} \;:=\;& \nabla^a C^b{}_{i_1 i_2}{}^c \nabla_a C_{b i_3 i_4 c} - 4 C^a{}_{i_1 i_2}{}^b \left(2\nabla_a S_{i_3 i_4 b} + C_{a i_3 i_4}{}^c L_{bc}\right) \\ & + 16 S_{i_1 i_2 a} S_{i_3 i_4}{}^a\,, \end{aligned} \tag{5.50}$$

$$\begin{aligned} \overset{2}{N}_{i_1 i_2 i_3 i_4} \;:=\;& 2\nabla_{i_1} C^a{}_{i_2 i_3}{}^b S_{a i_4 b} + C^a{}_{i_1 i_2}{}^b \left(2\nabla_{i_3} S_{a b i_4} - C_a{}^c{}_{b i_3} L_{c i_4}\right) \\ & + 2 S_{i_1 i_2 a} S_{i_3 i_4}{}^a\,, \end{aligned} \tag{5.51}$$

$$\overset{3}{N}_{i_1 i_2 i_3 i_4} := C^a{}_{i_1 i_2}{}^b C^c{}_{i_3 i_4}{}^d C_{abcd}\,, \tag{5.52}$$

where

$$S_{abc} := L_{a[b;c]}\,. \tag{5.53}$$

In order to find the coefficients α_ν we first define classes of "test metrics":

$$\mathcal{H}_1 := \left\{ \mathrm{g} \in \mathcal{G} \,|\, \nabla_a R_{bcde} = 0\,, \quad R_{ab} = \frac{1}{4} g_{ab} R \right\} \quad \text{(Einstein's space)}, \tag{5.54}$$

$$\mathcal{H}_2 := \{ \mathrm{g} \in \mathcal{G} \,|\, R_{ab} = 0 \}\,. \tag{5.55}$$

These classes are such that $\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{G}_1$. Furthermore, with

$$\overset{\nu}{M}_{i_1 i_2 i_3 i_4} := TS\left(\overset{\nu}{N}_{i_1 i_2 i_3 i_4}\right), \tag{5.56}$$

we define, for any null vector k^a, the scalars

$$\overset{\nu}{M}(k) := \overset{\nu}{M}_{i_1 \cdots i_4}\, k^{i_1 \cdots i_4} \qquad (\nu - 1, 2, 3). \tag{5.57}$$

By developing $E_{i_1 \cdots i_4} k^{i_1 \cdots i_4}$, with respect to a normal coordinate system, in Taylor series we obtain, for g $\in \mathcal{H}_2$ [41],

$$E_{i_1 \cdots i_4} k^{i_1 \cdots i_4} = \frac{1}{336} \overset{1}{M}(k). \tag{5.58}$$

In the class $\mathcal{H}_2$, $\overset{2}{M}(k)$ is zero, but not $\overset{1}{M}(k)$ and $\overset{3}{M}(k)$, in general. Thus,

$$\alpha_1 = \frac{1}{336}, \qquad \alpha_3 = 0. \tag{5.59}$$

Calculating now $E_{i_1 \cdots i_4} k^{i_1 \cdots i_4}$ for g $\in \mathcal{H}_2$, and using (5.59), Wünsch [82] has shown that

$$\alpha_2 = -\frac{1}{105}. \tag{5.60}$$

Thus, from these results and from (5.46), we obtain two necessary conditions for the validity of Huygens' principle:

$$C_{i_1 i_2} = 0, \qquad 5\, \overset{1}{M}_{i_1 i_2 i_3 i_4} - 16\, \overset{2}{M}_{i_1 i_2 i_3 i_4} = 0. \tag{5.61}$$

The generating set for $s = 5$ was found by Gerlach and Wünsch [36]. The necessary condition for the validity of Huygens' principle in this case was partially determined by Alvarez and Wünsch [3]. We shall present the explicit form of this condition later.

5.3 Huygens' principle for some spinor equations

Using Riesz' integration method, Wünsch [84] [85] [86] has formulated the Cauchy problem for several *generalized* spinor equations. In many cases, he found a representation formula, and was able to formulate the corresponding necessary and sufficient conditions for the validity of Huygens' principle. We shall follow Wünsch's papers and Alvarez's thesis [3] to make a brief summary in this section. Complete details can be found in these papers. As we shall see, the spinor approach can deal not only with the Weyl neutrino equation, but also with Maxwell's equations.

5.3.1 Definitions

Let $S = S(\mathcal{M}^4)$ be a complex vector fiber bundle on $\mathcal{M}^4$, whose fiber on each point $x \in \mathcal{M}^4$ is a two-dimensional vector space S_x over the complex field $\mathbf{C}$. Let S^* be the dual vector fiber bundle to S, and $\overline{S}$ ($\overline{S^*}$) the antidual vector fiber bundle to S (S^*). The elements of

$$S^{pq}_{nm} := \overset{p}{\otimes} S \overset{p}{\otimes} \overline{S} \overset{n}{\otimes} S^* \overset{m}{\otimes} \overline{S^*}, \qquad (p, q, n, m \in \mathbf{N}) \tag{5.62}$$

are called spinors of type $\binom{pq}{nm}$. In other words, a spinor at $x \in \mathcal{M}^4$ is defined to be a point of S^{pq}_{nm} lying in the fiber over x. A smooth spinor field ϱ is a (smooth) cross section of S^{pq}_{nm}. The coordinates of a spinor field with respect to a basis of S^{pq}_{nm} are designated by

$$\varrho_{A_1 \cdots A_n \dot{X}_1 \cdots \dot{X}_m}{}^{B_1 \cdots B_p \dot{Y}_1 \cdots \dot{Y}_q}, \tag{5.63}$$

where the indices can take vales 0 and 1.

Spinors with $p + q + n + m = 1$ are called 1-spinors and spinors of type $\binom{00}{nm}$ are called (n, m)-spinors. The set of all C^∞ (n, m)-spinors on $\mathcal{M}^4$ is designated by $S_{n,m}$. The set of all C^∞ symmetric (n, m)-spinor fields on $\mathcal{M}^4$ is designated by $\mathcal{S} = \{\varrho \in S_{n,m} | \varrho_{A_1 \cdots A_n \dot{X}_1 \cdots \dot{X}_m} = \varrho_{(A_1 \cdots A_n)(\dot{X}_1 \cdots \dot{X}_m)}\}$.

We define the first order differential operators $M : \mathcal{S}_{nm} \to \mathcal{S}_{n-1,m+1}$, and $\bar{M} : \bar{\mathcal{S}}_{nm} \to \bar{\mathcal{S}}_{n-1,m+1}$ by

$$(M\varphi)_{A_1 \cdots A_{n-1} \dot{X}_0 \cdots \dot{X}_m} := \nabla_{\dot{X}_0}{}^K \varphi_{K A_1 \cdots A_{n-1} \dot{X}_1 \cdots \dot{X}_m}, \tag{5.64}$$

$$(\bar{M}\varrho)_{A_0 \cdots A_{n-1} \dot{X}_1 \cdots \dot{X}_m} := \nabla_{A_0}{}^{\dot{K}} \varrho_{A_1 \cdots A_{n-1} \dot{K} \dot{X}_1 \cdots \dot{X}_m}, \tag{5.65}$$

where $n \in \mathbf{N}\backslash\{0\}$ $m \in \mathbf{N}$, $\varphi \in \mathcal{S}_{n,m}$ and $\varrho \in \mathcal{S}_{n-1,m+1}$.

The differential operators of second order $Q : \mathcal{S}_{n,m} \to \mathcal{S}_{n,m}$, and $\bar{Q} : \mathcal{S}_{n-1,m+1} \to \mathcal{S}_{n-1,m+1}$ are defined by

$$Q := \bar{M}M, \qquad \bar{Q} := M\bar{M}, \tag{5.66}$$

where $n \in \mathbf{N}\backslash\{0\}$ $m \in \mathbf{N}$.

Using the operator M we can express some particular field equations. Let us consider

$$M\varphi = 0. \tag{5.67}$$

For $\varphi \in \mathcal{S}_{1,m}$, we have

$$(M\varphi)_{\dot{X}_0 \cdots \dot{X}_m} := \nabla_{\dot{X}_0}{}^K \varphi_{K \dot{X}_1 \cdots \dot{X}_m} = 0, \tag{5.68}$$

called the *generalized Weyl equation.* The relativistic equation for a massless field of spin s is given by:

$$(M\varphi)_{A_1\cdots A_{n-1}} := \nabla_{\dot{X}_0}{}^{K}\varphi_{KA_1\cdots A_{n-1}} = 0\,, \tag{5.69}$$

where $\varphi \in \mathcal{S}_{n,0}$, and $s = n/2$. In particular, for $n = 1$ ($\varphi \in S_{1,0}$) we have the *Weyl neutrino equation*:

$$(M\varphi)_{\dot{X}} := \nabla_{\dot{X}}{}^{K}\varphi_K = 0\,. \tag{5.70}$$

For $n = 2$, ($\varphi \in S_{2,0}$), we have the *homogeneous Maxwell equations* for the electromagnetic field:

$$(M\varphi)_{A\dot{X}} := \nabla_{\dot{X}}{}^{K}\varphi_{KA} = 0\,. \tag{5.71}$$

The Riesz spinor kernels $V^Q_{nm}(x,\xi,\lambda)$ with respect to the operator Q are spinors on $S_{n,m}(x,\xi) \otimes S_{m,n}(x,\xi)$, dependent of a complex parameter λ, and for all $x\,,\xi \in \Omega$, (where Ω is a geodesically convex domain), which satisfy the relation [3]

$$QV^Q_{nm}(x,\xi,\lambda+2) = V^Q_{nm}(x,\xi,\lambda)\,, \tag{5.72}$$

with $V^Q_{nm}(x,\xi,0) = 0$. For ξ and x in a sufficiently small neighborhood $V^Q_{nm}(x,\xi,\lambda)$ is analytic. Thus, it can be represented as a power series in Γ [30] [75] [84]:

$$V^Q_{nm}(x,\xi,\lambda) = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}\,{}_{k}V^Q_{nm}(x,\xi)\Gamma^{\frac{\lambda}{2}+k-2}}{(-2)^{\lambda/2-1}\pi\boldsymbol{\Gamma}\left(\frac{\lambda}{2}\right)\boldsymbol{\Gamma}\left(\frac{\lambda}{2}+\mathbf{k}-\mathbf{1}\right)}\,. \tag{5.73}$$

The spinor coefficients ${}_{k}V^Q_{nm}$ must then satisfy the transport equations [71] [32] [75] [80]

$$\nabla^j\Gamma\nabla_j\,{}_{k}V^Q_{nm} + (L+2k)\,{}_{k}V^Q_{nm} = \begin{cases} 0 & \text{for } k=0\,, \\ -Q\,{}_{k-1}V^Q_{n} & \text{for } k \geq 0\,, \end{cases} \tag{5.74}$$

where $L = L(x,\xi)$ is given by (5.12).

The coefficient ${}_{0}V^Q_{nm}$ satisfies the initial condition

$$ {}_{0}V^Q_{nm}(\xi,\xi) = \Sigma_{nm}\,, \tag{5.75}$$

where

$$(\Sigma_{nm})_{A_1\cdots A_n\dot{X}_1\cdots\dot{X}_m\hat{A}_1\cdots\hat{A}_n\hat{\dot{X}}_1\cdots\hat{\dot{X}}_m} := \varepsilon_{A_1\hat{A}_1}\varepsilon_{A_n\hat{A}_n}\varepsilon_{X_1\hat{X}_1}\varepsilon_{X_m\hat{X}_m}\,, \tag{5.76}$$

with hatted indices referring to ξ. The coefficients ${}_{k}V^Q_{nm}$, $k \geq 1$, are determined by the second equation in (5.74) and by the the condition that each ${}_{k}V_{n}$ remains bounded when $x \to \xi$. If $\Gamma > 0$ and $x\,,\ \xi \in \Omega$ the series (5.73) converges absolutely and the Riesz kernels $V^Q_{nm}(x,\xi,\lambda)$ are analytic in x, ξ, λ.

[3] The operator Q and all derivatives in the following formulas refer to ξ.

Defining the Riesz spinor kernels $V^{\bar{Q}}_{nm}(x,\xi,\lambda)$ on $S_{n-1,m+1}\otimes S_{n-1,m+1}$ the above relations are again valid and the following symmetry relations, obtained by Wünsch [84] are satisfied:

$$V^{Q}_{nm}(x,\xi,\lambda)=(-1)^{n+m}V^{Q}_{nm}(\xi,x,\lambda)\,, \tag{5.77}$$
$$V^{\bar{Q}}_{nm}(x,\xi,\lambda)=(-1)^{n+m}V^{\bar{Q}}_{nm}(\xi,x,\lambda)\,, \tag{5.78}$$
$$M_x V^{Q}_{nm}(x,\xi,\lambda)=(-1)^{n+m}\tilde{M}_\xi V^{\bar{Q}}_{n-1m+1}(\xi,x,\lambda)\,. \tag{5.79}$$

5.3.2 Cauchy's Problem

Let us consider a three-dimensional space-like surface $F\subset\mathcal{M}^4$ of class C^∞, with parametric representation

$$x^\alpha=x^a(u^\alpha)\qquad(a=0,1,2,3\,;\quad i=0,1,2)\,. \tag{5.80}$$

If φ is a spinor field on $\mathcal{M}^4$, then $\overset{o}{\varphi}:=\varphi_{|F}$ on F is induced by the parametrization (5.80). The spinor field on F defined from $\mathcal{S}_{mn}$ is designated by $\mathcal{S}_{mn}(F)$. The *Cauchy Problem* for the equation $M\varphi=0$, $\varphi\in\mathcal{S}_{n,m}$ is the following:

Find a solution $\varphi\in\mathcal{S}_{n,m}$ of $M\varphi=0$, for which $\overset{o}{\varphi}$ on F is prescribed.

In order to obtain a representation for the solution of Cauchy's problem the use of Green's formulas for spinor fields is necessary. Let Ω be a bounded domain of $\mathcal{M}^4$ with smooth boundary $\partial\Omega$. Then, for φ, $\varrho\in\mathcal{S}_{n,m}$ we define the scalar product

$$\langle\varphi,\varrho\rangle=\int_\Omega(\varphi,\varrho)(x)dV_x\,, \tag{5.81}$$

where dV_x is the invariant volume element in $x\in\Omega$, and

$$(\varphi,\varrho)=\varphi_{A_1\cdots A_n\dot{X}_1\cdots\dot{X}_m}\varrho^{A_1\cdots A_n\dot{X}_1\cdots\dot{X}_m}\,. \tag{5.82}$$

For $\varphi\in\mathcal{S}_{n,m}$ and $\varrho\in\mathcal{S}_{n-1,m+1}$ we have

$$(M\varphi,\varrho)-(\varphi,\tilde{M}\varrho)=\nabla_{\dot{K}}{}^{K}(\varphi_{KA_1\cdots A_{n-1}\dot{X}_1\cdots\dot{X}_m}\varphi^{A_1\cdots A_{n-1}\dot{X}_1\cdots\dot{X}_m})\,. \tag{5.83}$$

Integrating the above expression and using Gauss' theorem, the Green's formula can be obtained [84]:

$$\langle M\varphi,\varrho\rangle_\Omega-\langle\varphi,\tilde{M}\varrho\rangle_\Omega=-\int_{\partial\Omega}(\varphi\mathrm{n},\varrho)dS_x\,, \tag{5.84}$$

where

$$(\varphi n)_{A_1\cdots A_{n-1}\dot{X}_0\cdots\dot{X}_m} := \eta^K{}_{\dot{X}_0}\varphi_{KA_1\cdots A_{n-1}\dot{X}_1\cdots\dot{X}_m}\,, \tag{5.85}$$

$$\eta^K{}_{\dot{X}_0} := \sigma_i{}^K{}_{\dot{X}_0}\eta^i\,. \tag{5.86}$$

Here η^i is the interior unit normal vector of $\partial\Omega$ and dS_x is the invariant element of area of $\partial\Omega$.

Following the Riesz method, using (5.84) with $\varrho = M\overset{sQ}{V}_{nm} := MV^Q_{(nm)}$, $\Omega = D_F(\xi) := D^-(\xi)\cap F$, Wünsch obtained, by analytic continuation of λ, as solution of the Cauchy's problem in the point ξ, the following representation formula [84] [30] [75]:

$$\varphi(\xi) = \underset{\lambda\to 2}{\text{an. cont.}} \int_{F_\xi} (M\overset{sQ}{V}_{nm}(x,\xi,\lambda), \overset{o}{\varphi}\mathrm{n})dS_x\,, \tag{5.87}$$

where $F_\xi := D(\xi)\cap S$.

On the other hand, *Huygens' principle is valid for the equations (5.68), (5.70) or (5.71) if and only if, for arbitrarily chosen consistent Cauchy data which has support inside F_ξ, the solution $\varphi(\xi)$ of the enunciated Cauchy problem at any point ξ vanishes.*

Wünsch [84] has proved using (5.87) that for Weyl's neutrino equation, i.e. $n = 1$, Huygens' principle is valid if and only if for all x and ξ:

$$M\overset{sQ}{V}_{1m}(x,\xi,2) = 0\,, \qquad m\in\mathbf{N}\,. \tag{5.88}$$

If one defines

$$W^{(m)}(x,\xi) := \pi M\overset{sQ}{V}_{1m}(x,\xi,2)\,, \qquad m\in\mathbf{N}\,, \tag{5.89}$$

the criteria for validity of Huygens' principle become the following: *A necessary and sufficient condition for the validity of Huygens' principle for equations (5.68) and (5.70) is*

$$W^0(x,\xi) = 0 \qquad \textit{(Weyl's neutrino equation)}\,, \tag{5.90}$$

$$W^m(x,\xi) = 0 \qquad \textit{(Generalized Weyl's equation)}\,, \tag{5.91}$$

for all x, ξ.

Let

$$Y(x,\xi) := \pi MM\overset{sQ}{V}_{20}(x,\xi,2)\,. \tag{5.92}$$

It was shown in [84], using (5.87), that: *a necessary and sufficient condition for the validity of Huygens' principle for the homogeneous Maxwell equations is given by*

$$Y(x,\xi) = 0\,, \tag{5.93}$$

for all x, ξ.

It can be shown that the conditions (5.90), (5.91), and (5.92) are conformally invariant [85]. The following relations are then satisfied [84]:

$$W^{(0)}(x,\xi)=0 \Longleftrightarrow W^{(0)}_{\dot{X}\hat{B}}(x,\xi)\Big|_{\Gamma(x,\xi)=0} = \nabla^{B}{}_{\dot{X}} \underset{1}{V}{}^{Q}{}_{B\hat{B}}(x,\xi)$$
$$-\underset{2}{V}{}^{Q}{}_{B\hat{B}}(x,\xi)\nabla^{B}{}_{\dot{X}}\Gamma(x,\xi)=0\,, \tag{5.94}$$

$$W^{(m)}(x,\xi)=0 \Longleftrightarrow W^{(m)}_{\hat{B}\dot{X}_0\cdots\dot{X}_m\dot{X}_0\cdots\dot{X}_m}(x,\xi)\Big|_{\Gamma(x,\xi)=0} =$$
$$= \nabla^{B}{}_{\dot{X}_0} \underset{1}{V}{}^{Q}{}_{B\hat{B}X_1\cdots X_m\dot{X}_1\cdots\dot{X}_m}(x,\xi) -$$
$$\overset{*}{\underset{2}{V}}{}^{Q}{}_{B\hat{B}\dot{X}_1\cdots\dot{X}_m\dot{X}_1\cdots\dot{X}_m}(x,\xi)\nabla^{B}{}_{\dot{X}_0}P(x,\xi)=0\,. \tag{5.95}$$

Let us consider the case of Weyl's neutrino equation. In terms of the spinor field $W^{(0)}_{\dot{X}\hat{B}}(x,\xi)$, we define for $r \in \mathrm{N}$ the trace-free symmetric tensor

$$W_{i_1\cdots i_r}(\xi) := TS\left[\frac{1}{(r-1)!}\sigma_{i_r}{}^{\hat{B}\dot{X}}(x)\nabla_{i_1}\cdots\nabla_{i_{r-1}}W^{(0)}_{\dot{X}\hat{B}}(x,\xi)\right]_{x=\xi}\,. \tag{5.96}$$

From (5.90) it follows that *Weyl's equation (5.70) satisfies Huygens' principle if and only if for* $k \in \mathrm{N}\backslash 0$ *and for all* $\xi \in \mathcal{M}^4$ *we have*

$$W_{i_1\cdots i_r}(\xi)=0\,. \tag{5.97}$$

Let us define, (cf. (5.40)):

$$\mathcal{A}_k := \begin{cases} \mathcal{G}\,, & \text{for } k=1\,,\\ \{\mathrm{g}\in\mathcal{G} \mid \forall r\in\{1,\ldots,k-1\}\forall\xi\in\mathcal{M}^4 : W_{i_1\cdots i_r}(\xi)=0\} & \text{for } k\in\mathrm{N}\backslash\{0\}\,. \end{cases} \tag{5.98}$$

In [85] Wünsch has proved that:

> *The tensors* $\overset{r}{W}_{i_1\cdots i_k} := (i)^k W_{i_1\cdots i_k}$ *($k \in \mathrm{N}\backslash\{0\}$) in* $\mathcal{A}_k$ *are real, symmetric, trace-free, integer rational and conformally invariant with weight -1.*

For the homogeneous Maxwell's equations (5.71) we define for $r \in \mathrm{N}\backslash\{0,1\}$,

$$Y_{i_1\cdots i_r}(x,\xi) := TS\left[\frac{1}{(r-2)!}\sigma_{i_r}{}^{\dot{A}\dot{X}}(x)\sigma_{i_{r-1}}{}^{\hat{B}\dot{Y}}(x)\nabla_{i_1}\cdots\nabla_{i_{r-2}}Y_{\dot{X}\dot{Y}\hat{A}\hat{B}}(x,\xi)\right]_{x=\xi}\,, \tag{5.99}$$

$$\mathcal{H}_k := \begin{cases} \mathcal{G}\,, & \text{for } k=2\,,\\ \{\mathrm{g}\in\mathcal{G} \mid \forall r\in\{2,\ldots,k-1\}\forall\xi\in\mathcal{M}^4 : Y_{i_1\cdots i_r}(\xi)=0\} & \text{for } k>2\,. \end{cases} \tag{5.100}$$

Thus, *Huygens' principle is valid for Maxwell's equations (5.71) if and only if, for* $k \in \mathbb{N}\backslash\{0,1\}$ *and for all* $\xi \in \mathcal{M}^4$ *we have*

$$Y_{i_1 \cdots i_k}(\xi) = 0. \tag{5.101}$$

For this case, Wünsch [85] has proved that:

> *The tensors* $E_{i_1 \cdots i_k} := (i)^k Y_{i_1 \cdots i_k}$ $(k \in \mathbb{N}\backslash\{0,1\})$ *in* $\mathcal{H}_k$ *are real, trace-free, symmetric, integer rational and conformally invariant with weight -1.*

5.3.3 Necessary conditions

We now refer to the discussion in Subsection 5.2.5. The results described in the preceeding Section imply that the tensors $W_{i_1 \cdots i_r}$ and $Y_{i_1 \cdots i_r}$ must be decomposable as in (5.48) and (5.49). The vanishing of Bach's tensor as a necessary condition for the validity of Huygens' principle, for both Weyl's and Maxwell's equations, follows immediately. The coefficients α_ν can be determined by using the transport equations (5.74) and (5.75) together with the necessary and sufficient conditions (5.96) and (5.99). Taylor series expansions of tensors expressed in a normal coordinate system can be applied, in a procedure similar to that explained in Chapter 2 for the scalar wave equation. Using the weak field space-time as a test metric, Wünsch [85] found the coefficients of the expansion (5.49) for $W_{i_1 \cdots i_4}$, to be $\alpha_1 = -1/2520$, $\alpha_2 = 13/20160$, $\alpha_3 = 0$.

For $r = 5$ a five-index necessary condition is obtained. Using the results of Gerlach and Wünsch [36], Alvarez [3] was able to show that

$$\overset{r}{W}_{i_1 \cdots i_5} = \lambda_0 \overset{1}{T}_{i_1 \cdots i_5} + \lambda_1 \overset{2}{T}_{i_1 \cdots i_5} + \lambda_2 \overset{3}{T}_{i_1 \cdots i_5}\,, \tag{5.102}$$

where λ_0, λ_1 and λ_2 are fixed real numbers, and the $\overset{\nu}{T}_{i_1 \cdots i_5}$ are trace-free, symmetric, conformally invariant tensors given by

$$\begin{aligned} \overset{1}{T}_{i_1 \cdots i_5} = TS[&4\,{}^{*}C^{k}{}_{i_1 i_2}{}^{l} C^{u}{}_{d i_5 l;u k i_3} - 6\,{}^{*}C^{k}{}_{i_2 i_3}{}^{l}{}_{;i_1} C^{u}{}_{i_4 i_5 l;uk} + \\ &{}^{*}C^{k}{}_{i_1 i_2}{}^{l}{}_{;}{}^{n} C_{k i_4 i_5 l;n i_3} + 5\,{}^{*}C^{k}{}_{i_1 i_2}{}^{l} C^{n}{}_{i_4 i_5 l;i_3} L_{kn} + 4\,{}^{*}C^{k}{}_{i_1 i_2}{}^{l} C^{n}{}_{i_3 i_4 l;k} L_{i_5 n} \\ &+ 4\,{}^{*}C^{k}{}_{i_1 i_2}{}^{l} C^{u}{}_{k l i_3;u} L_{i_4 i_5} - 21\,{}^{*}C^{k}{}_{i_1 i_2}{}^{l} C^{u}{}_{i_3 i_4 k;u} L_{i_5 l} \\ &+ 26\,{}^{*}C^{u}{}_{i_1 i_2}{}^{k}{}_{;u} C^{v}{}_{i_4 i_5 k;v i_3}]\,, \end{aligned} \tag{5.103}$$

$$\overset{2}{T}_{i_1 \cdots i_5} := TS[-12\,{}^{*}C^{k}{}_{i_1 i_2}{}^{l} C_{k i_3 i_4}{}^{n} C^{u}{}_{l n i_5;u} - C_{k}{}^{nh}{}_{i_1} C_{l n h i_2}\,{}^{*}C^{k}{}_{i_4 i_5}{}^{l}{}_{;i_3}] \tag{5.104}$$

$$\overset{3}{T}_{i_1 \cdots i_5} := TS[-8\,{}^{*}C^{k}{}_{i_1 i_2}{}^{l} C_{k i_3 i_4}{}^{n} C^{u}{}_{l n i_5;u} +{}^{*} C_{k}{}^{nh}{}_{i_1} C_{l i_2 n h} C^{k}{}_{i_4 i_5}{}^{l}{}_{;i_3}] \tag{5.105}$$

A similar relation is valid for $Y_{i_1\cdots i_5}$. The constant λ_0 was determined in [3] and is nonzero for both cases. Thus the necessary conditions for validity of Huygens' principle for the Maxwell's equations (5.71) and Weyl's neutrino equation (5.70) are given by [4]

$$(III) \qquad C_{i_1 i_2} = 0\,, \tag{5.106}$$

$$(V') \qquad k_1 \overset{1}{M}_{i_1 i_2 i_3 i_4} - k_2 \overset{1}{M}_{i_1 i_2 i_3 i_4} = 0. \tag{5.107}$$

$$(VI') \qquad \overset{1}{T}_{i_1\cdots i_5} + \sigma_1 \overset{2}{T}_{i_1\cdots i_5} + \sigma_2 \overset{3}{T}_{i_1\cdots i_5} = 0\,, \tag{5.108}$$

where

$$\overset{\nu}{M}_{i_1 i_2 i_3 i_4} = TS\left(\overset{\nu}{M}_{i_1 i_2 i_3 i_4}\right)\,, \tag{5.109}$$

$(k_1, k_2) = (5, 16)$ corresponds to Maxwell's equations, $(k_1, k_2) = (8, 13)$ to Weyl's neutrino equation. The real numbers σ_1 and σ_2 are fixed in each case. We note here that necessary condition III for the conformally invariant scalar wave equation (3.1) is given by (5.106) with $(k_1, k_2) = (3, 4)$.

5.4 Previous results

Applying condition VI' to a Petrov type N space-time Alvarez and Wünsch [3] [4] obtained the following result.

> *Every Petrov type N space-time on which Maxwell's equations or Weyl's neutrino equation satisfy Huygens' principle is conformally related to the generalized plane wave metric space-time of McLenaghan and Leroy [59].*

Later, Wünsch [89] (see also [88]) solved Hadamard's problem in this case, obtaining the following result:

> *Every Petrov type N space-time or C-space-time[5] on which Maxwell's equations or Weyl's neutrino equation satisfy Huygens' principle is conformally related to the exact plane-wave space-time (1.12).*

In Petrov type D space-times Hadamard's problem was solved by Carminati and McLenaghan [22], Wünsch [87], and McLenaghan and Williams [62]. The result can be stated as follows:

[4] In our conventions, the Riemann tensor, Ricci tensor and the Ricci scalar have opposite sign to those used by Wünsch and collaborators [85] [3] [4].

[5] C-space-times are defined by the property $\nabla^a C_{abcd} = 0$.

There exist no Petrov type D space-times on which Maxwell's equations or Weyl's neutrino equation satisfy Huygens' principle.

5.5 Main Theorem

The main results obtained by Carminati and McLenaghan [22] for the conformally invariant scalar equation on Petrov type III space-times as stated in Theorems 3.1 and 3.2 of Chapter 3 are also valid for Maxwell's equations and Weyl's neutrino equation.

Maxwell's and Weyl's equations in Petrov type III space-times was stated in Theorems 3.1 and 3.2 of Chapter 3, also valid for the conformally invariant scalar equation.

It will be proved here that conditions (3.7), (3.8) and (3.9) are superfluous, i.e., they are consequences of the necessary conditions for the validity of Huygens' principle. The main result of this Chapter is the proof of the following Theorem:

Theorem 5.1 (Main Theorem) *There exist no Petrov type III space-times on which Maxwell's equations (5.71) or Weyl's equation (5.70) satisfy Huygens' principle.*

5.6 Proof of the Main Theorem

We shall use here the same method employed on Chapters 3 and 4. The explicit form of the necessary conditions is obtained by first converting the spinorial expressions to the dyad form and then contracting them with appropriate products of o^A and ι^A and their complex conjugates. The templates used for obtaining the dyad components of the necessary conditions are given in the Appendix E.

It was shown in [22] that there exists a dyad o_A, ι_A and a conformal transformation such that

$$\kappa = \sigma = \rho = \tau = \epsilon = 0\,, \tag{5.110}$$

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0, \quad \Psi_3 = -1\,, \tag{5.111}$$

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = \Lambda = 0\,, \tag{5.112}$$

$$D\alpha = D\beta = 0\,. \tag{5.113}$$

$$\Phi_{11} = c\,, \tag{5.114}$$

where c is a constant. The proof is similar to that illustrated in Chapter 4, when we treated the self-adjoint scalar equation, except for the fact that the paper by Carminati and McLenaghan [22] considers all three cases at once. Here we shall consider the cases for Maxwell's equations and Weyl's neutrino equation separately.

5.6.1 Maxwell's equations

We start by supposing that $\alpha\beta\pi \neq 0$. By contracting condition III with $\iota^{AB}\overline{o}^{\dot{A}\dot{B}}$ and $\iota^{A}o^{B}\overline{\iota}^{\dot{A}\dot{B}}$ (where the notation $o_{A_1\cdots A_p} = o_{A_1}\cdots o_{A_p}$, etc. has been used) we get, respectively,

$$D\pi = 0\,, \tag{5.115}$$

$$\delta\beta = -\beta(\overline{\alpha}+\beta)\,. \tag{5.116}$$

From the Bianchi identities, using the above conditions, we obtain

$$D\Phi_{12} = 2\overline{\pi}\Phi_{11}\,, \tag{5.117}$$

$$D\Phi_{22} = -2(\beta+\overline{\beta}) + 2\Phi_{21}\overline{\pi} + 2\Phi_{12}\pi\,, \tag{5.118}$$

$$\delta\Phi_{12} = 2\overline{\alpha} + 4\overline{\pi} + 2\overline{\lambda}\Phi_{11} - 2\overline{\alpha}\Phi_{12}\,, \tag{5.119}$$

$$\overline{\delta}\Phi_{12} = -2\beta + 2\overline{\mu}\Phi_{11} - 2\overline{\beta}\Phi_{12}\,. \tag{5.120}$$

$$\begin{aligned}\delta\Phi_{22} &= \Delta\Phi_{12} + 2\overline{\gamma} + 4\overline{\mu} - 2\overline{\nu}\Phi_{11} + 2\overline{\lambda}\Phi_{21}\\ &+2\Phi_{12}\mu - 2\Phi_{22}\beta - 2\Phi_{22}\overline{\alpha} + 2\overline{\gamma}\Phi_{12}\,.\end{aligned} \tag{5.121}$$

From the Ricci identities we get

$$D\gamma = \overline{\pi}\alpha + \beta\pi + \Phi_{11}\,, \tag{5.122}$$

$$\delta\overline{\pi} = D\overline{\lambda} - \overline{\pi}^2 - \overline{\pi}\overline{\alpha} + \overline{\pi}\beta\,, \tag{5.123}$$

$$D\overline{\nu} = \Delta\overline{\pi} + \overline{\pi}\overline{\mu} + \overline{\lambda}\pi + \overline{\pi}\overline{\gamma} - \overline{\pi}\gamma - 1 + \Phi_{12}\,, \tag{5.124}$$

$$\delta\alpha = \overline{\delta}\beta + \alpha\overline{\alpha} + \beta\overline{\beta} - 2\beta\alpha + \Phi_{11}\,, \tag{5.125}$$

$$\delta\pi = D\mu - \overline{\pi}\pi + \pi\overline{\alpha} - \beta\pi\,, \tag{5.126}$$

We can obtain useful integrability conditions for the above identities by using the NP commutation relations. Using (5.117), (5.118), (5.121), (5.116), (5.122), (5.123), (5.124) and (5.126) in the commutator expression $[\delta, D]\Phi_{22} - [\Delta, D]\Phi_{12}$, we obtain

$$\delta\overline{\beta} = -2\Phi_{11} - \overline{\beta}\overline{\alpha} - 4\overline{\beta}\overline{\pi} - 2D\overline{\mu} - \beta\overline{\beta} + 2\overline{\pi}\pi. \tag{5.127}$$

From now on we shall consider both Maxwell and Weyl cases separately. We begin with Maxwell's equations, i.e., $(k_1, k_2) = (5, 16)$ in (5.107). By contracting condition V with $\iota^{ABCD}\overline{\iota}^{\dot{A}\dot{B}}\overline{o}^{\dot{C}\dot{D}}$, we get

$$62\overline{\beta}\pi + 40\overline{\beta}\alpha + 6\pi\alpha + 3\alpha^2 + \overline{\delta}\alpha + 2\overline{\delta}\pi + \overline{\delta}\overline{\beta} + \overline{\beta}^2 = 0\,. \tag{5.128}$$

Substituting (5.116) in this equation we get

$$\delta(2\overline{\pi}+\overline{\alpha}) = -62\overline{\pi}\beta - 39\beta\overline{\alpha} - 6\overline{\pi}\overline{\alpha} - 3\overline{\alpha}^2 \,. \tag{5.129}$$

From (5.125), (5.126) and (5.127) we obtain

$$\delta(2\pi+\alpha) = 2\pi\overline{\alpha} + \alpha\overline{\alpha} - 6\beta\pi - 3\beta\alpha - \Phi_{11} \,. \tag{5.130}$$

Contracting condition V with $\iota^{ABC}o^{D}\overline{\iota}^{\dot{A}\dot{B}\dot{C}}\overline{o}^{\dot{D}}$, using (5.122), (5.126) and the complex conjugate of (5.127), leads to the real expression:

$$\begin{aligned}&148\Phi_{11} + 152\overline{\beta}\overline{\pi} + 76[D\mu + D\overline{\mu}] - 8\overline{\pi}\alpha - 104\beta\overline{\beta} - 8\alpha\overline{\alpha} - 232\overline{\pi}\pi \\ &+152\beta\pi - 8\pi\overline{\alpha} = 0 \,.\end{aligned} \tag{5.131}$$

Using (5.129), (5.130), (5.116), (5.125), (5.126), (5.127) in $[\overline{\delta},\delta](\alpha+2\pi) = (\alpha-\overline{\beta})\delta(\alpha+2\pi) + (-\overline{\alpha}+\beta)\overline{\delta}(\alpha+2\pi)$, we obtain

$$\begin{aligned}&-80\,\alpha\overline{\beta}\overline{\pi} - 16\overline{\beta}\beta\,\pi - 40\overline{\beta}\alpha\,\overline{\alpha} - 8\overline{\beta}\mathrm{D}\mu - 64\,\pi\,\mathrm{D}\overline{\mu} - 40\,\alpha\,\mathrm{D}\overline{\mu} - 39\,\alpha\,\Phi_{11} \\ &- 20\overline{\beta}\Phi_{11} + 64\overline{\pi}\pi^2 - 64\overline{\beta}\pi\,\overline{\alpha} - 60\,\pi\,\Phi_{11} + 40\,\alpha\,\overline{\pi}\,\pi - 120\overline{\beta}\overline{\pi}\,\pi = 0 \,.\end{aligned} \tag{5.132}$$

By eliminating $\mathrm{D}\overline{\mu}$ from (5.131) and (5.132), we find

$$\begin{aligned}\mathrm{D}\mu = \frac{1}{152}\Big[&-1520\alpha\beta\pi + 208\alpha\overline{\alpha}\pi - 152\overline{\beta}\overline{\pi}\pi + 1040\beta\overline{\beta}\alpha + 760\overline{\beta}\alpha\overline{\alpha} + 1968\overline{\beta}\beta\pi \\ &+1216\overline{\beta}\pi\overline{\alpha} - 1228\pi\Phi_{11} + 1688\alpha\pi\overline{\pi} - 739\alpha\Phi_{11} + 80\alpha^2\overline{\alpha} + 380\overline{\beta}\Phi_{11} + 80\overline{\pi}\alpha^2 \\ &+2496\overline{\pi}\pi^2 - 2432\beta\pi^2 + 128\pi^2\overline{\alpha}\Big]/(-\overline{\beta}+5\alpha+8\pi)\,,\end{aligned} \tag{5.133}$$

where we are assuming, for now, that the denominator of (5.133),

$$v_1 := \overline{\beta} - 5\alpha - 8\pi \,, \tag{5.134}$$

is non-zero.

By substituting (5.133) into (5.131), we find an expression for $D\overline{\mu}$:

$$\begin{aligned}D\overline{\mu} = \tfrac{1}{152}\Big[&776\alpha\overline{\beta}\overline{\alpha} - 304\overline{\beta}^2\overline{\pi} + 208\overline{\beta}^2\beta - 1216\overline{\pi}\pi^2 + 741\alpha\Phi_{11} + 1332\pi\overline{\beta}\overline{\alpha} \\ &+84\overline{\beta}\Phi_{11} + 1536\alpha\overline{\pi}\overline{\beta} - 760\alpha\overline{\pi}\pi + 2744\overline{\pi}\pi\overline{\beta} + 1140\pi\Phi_{11}\Big]/(\overline{\beta}-5\alpha-8\pi)\,.\end{aligned} \tag{5.135}$$

We notice that (5.133) and (5.135) have the same denominator. So, in what follows we shall use the Pfaffians $\delta\alpha$, $\delta\pi$, and $\delta\overline{\beta}$, given by (5.125), (5.126) and (5.127), respectively, and their complex conjugates, in such a way that they have all the same denominator. This procedure simplifies the expressions to be obtained from the integrability conditions.

Converting (5.108) to its spinorial form in the dyad basis, and contracting with $\iota^{AB}{}_{o}{}^{CDE}\bar{\iota}^{\dot{A}\dot{B}\dot{C}\dot{D}\dot{E}}$ gives

$$\begin{aligned}&-14\overline{\alpha}\delta\overline{\pi} - 12\,\overline{\alpha}^3 - \delta(\,\delta\overline{\alpha}) - 2\,\delta(\,\delta\overline{\pi}) + \delta(\,\delta\beta) - 8\,\beta\,\overline{\alpha}^2 + 14\overline{\alpha}\delta\beta\\ &+ 20\overline{\alpha}\beta^2 + 42\,\beta^2\,\overline{\pi} - 14\,\beta\,\delta\overline{\pi} - 6\,\beta\,\delta\overline{\alpha} + 28\overline{\pi}\delta\beta + 2\,\overline{\lambda}\,\mathrm{D}\overline{\pi}\\ &+ \overline{\lambda}\,\mathrm{D}\overline{\alpha} - 10\overline{\alpha}\delta\overline{\alpha} - 24\,\overline{\alpha}^2\,\overline{\pi} - 6\,\overline{\pi}\,\delta\overline{\alpha} + 2\,\beta\,\delta\beta - 14\,\beta\,\overline{\alpha}\,\overline{\pi} = 0\,. \end{aligned} \tag{5.136}$$

Substituting the expression (5.116) for $\delta\beta$ into (5.136), we get

$$\begin{aligned}&-14\overline{\alpha}\,\delta\overline{\pi} - 12\overline{\alpha}^3 - \delta(\delta(\overline{\alpha} + 2\overline{\pi})) - 21\beta\overline{\alpha}^2 + 7\overline{\lambda}\beta^2 + 14\beta^2\overline{\pi}\\ &-7\beta\delta(\overline{\alpha} + 2\overline{\pi}) - 42\beta\overline{\alpha}\overline{\pi} - (10\overline{\alpha} + 6\overline{\pi})\delta\overline{\alpha} - 24\overline{\alpha}^2\overline{\pi} = 0\,. \end{aligned} \tag{5.137}$$

We observe that the terms (5.104) and (5.105) do not contribute to this component. Using (5.129) to eliminate $\delta\overline{\pi}$ from this equation, we have

$$\beta(192\overline{\pi}\beta + 121\beta\overline{\alpha} - \delta\overline{\alpha} - 3\overline{\alpha}^2) = 0\,. \tag{5.138}$$

Solving (5.138) for $\delta\overline{\alpha}$ yields

$$\delta\overline{\alpha} = 192\beta\overline{\pi} - 3\overline{\alpha}^2 + 121\beta\overline{\alpha}\,. \tag{5.139}$$

Now, from (5.129),

$$\delta\overline{\pi} = -127\overline{\pi}\beta - 80\beta\overline{\alpha} - 3\overline{\pi}\overline{\alpha}\,. \tag{5.140}$$

We have now all the Pfaffians we need to complete the proof. The integrability conditions provided by the NP commutation relations can now be used.

Let us onsider the NP commutator $[\overline{\delta}, \delta]\alpha = (\overline{\mu} - \mu)D\alpha + (\alpha - \overline{\beta})\delta(\alpha) + (-\overline{\alpha} + \beta)\overline{\delta}\alpha$. Using the Pfaffians calculated previously, and solving for Φ_{11}, we obtain

$$\begin{aligned}\Phi_{11} :=\; &-8\overline{\beta}(-1381888\overline{\pi}\pi^3 + 62208\,\pi^3\,\overline{\alpha} - 62928\overline{\beta}\pi^2\,\overline{\alpha} - 76544\,\beta\,\pi^2\,\overline{\beta}\\ &+ 145232\,\alpha\pi^2\,\overline{\alpha} - 2556128\,\overline{\pi}\,\pi^2\,\alpha + 26450\,\alpha^3\,\overline{\alpha} + 65550\overline{\pi}\overline{\beta}\,\alpha^2\\ &- 325050\overline{\pi}\alpha^3 - 26650\,\beta\overline{\beta}\alpha^2 + 19760\,\pi\,\beta\,\overline{\beta}^2 + 12350\,\alpha\,\beta\,\overline{\beta}^2 - 24225\,\overline{\alpha}\overline{\beta}\alpha^2\\ &+ 180576\overline{\beta}\overline{\pi}\,\pi^2 + 9025\,\overline{\alpha}\,\overline{\beta}^2\,\alpha + 14440\,\pi\,\overline{\beta}^2\,\overline{\alpha} - 1577860\overline{\pi}\pi\,\alpha^2\\ &+ 108790\,\pi\,\alpha^2\,\overline{\alpha} - 90480\,\beta\,\pi\,\overline{\beta}\,\alpha - 78090\,\pi\overline{\beta}\alpha\,\overline{\alpha} + 217740\,\pi\overline{\pi}\alpha\,\overline{\beta})\\ &\Big/(-214700\,\alpha^3 + 10868\,\pi\,\overline{\beta}^2 + 772320\,\pi^2\,\overline{\beta} + 1020276\,\pi\overline{\beta}\alpha + 4085\,\overline{\beta}^2\,\alpha\\ &+ 335985\overline{\beta}\alpha^2 - 1158240\,\pi\,\alpha^2 - 2048352\,\pi^2\,\alpha - 1191680\,\pi^3)\,, \end{aligned} \tag{5.141}$$

where the denominator of the expression above,

$$\begin{aligned}v_2 \;:=\; &-214700\,\alpha^3 + 10868\,\pi\,\overline{\beta}^2 + 772320\,\pi^2\,\overline{\beta} + 1020276\,\pi\overline{\beta}\alpha + 4085\,\overline{\beta}^2\,\alpha\\ &+ 335985\overline{\beta}\alpha^2 - 1158240\,\pi\,\alpha^2 - 2048352\,\pi^2\,\alpha - 1191680\,\pi^3\,, \end{aligned} \tag{5.142}$$

is assumed to be non-zero for the moment.

On the other hand, the commutator $[\bar{\delta}, \delta](\alpha+\bar{\beta}) = (\mu - \bar{\mu})(\mathrm{D}(\alpha+\bar{\beta}) + (\alpha - \bar{\beta})(\delta(\alpha+\bar{\beta}) + (-\bar{\alpha}+\beta)\bar{\delta}(\alpha+\bar{\beta})$ gives the following expression for Φ_{11}:

$$\begin{aligned}\Phi_{11} := 8\bar{\beta}(&-8056\bar{\beta}\pi\,\bar{\alpha} - 5330\bar{\beta}\beta\,\alpha - 172736\bar{\pi}\pi^2 + 7776\,\pi^2\,\bar{\alpha} - 9568\bar{\beta}\beta\,\pi \\ &+ 13294\,\alpha\,\pi\,\bar{\alpha} - 211556\,\alpha\,\bar{\pi}\,\pi - 5035\bar{\beta}\alpha\,\bar{\alpha} + 5290\,\alpha^2\,\bar{\alpha} + 20672\bar{\beta}\bar{\pi}\,\pi \\ &+ 12920\bar{\pi}\alpha\,\bar{\beta} - 65010\bar{\pi}\alpha^2)\Big/(148960\,\pi^2 + 10412\,\bar{\beta}^2 - 13047\bar{\beta}\alpha \\ &+42940\,\alpha^2 + 162944\,\pi\,\alpha - 18564\,\pi\bar{\beta})\,,\end{aligned} \tag{5.143}$$

where we assume, for now, that

$$\begin{aligned}v_3 \;:=\; &148960\,\pi^2 + 10412\,\bar{\beta}^2 - 13047\bar{\beta}\alpha + 42940\,\alpha^2 + 162944\,\pi\,\alpha \\ &-18564\,\pi\bar{\beta} \neq 0\,.\end{aligned} \tag{5.144}$$

Using the fact that $\bar{\delta}(\Phi_{11}) = 0$, we obtain, from (5.143), a third expression for Φ_{11}:

$$\begin{aligned}\Phi_{11} := -8\bar{\beta}(&-6145239989312\,\alpha\,\bar{\pi}\,\pi^2\,\bar{\beta}^2 - 7649757648780\,\alpha^3\bar{\pi}\pi\,\bar{\beta} \\ &- 16323677160464\,\alpha^2\bar{\pi}\pi^2\,\bar{\pi} - 15274850502912\,\alpha\bar{\pi}\pi^3\,\bar{\beta} \\ &- 26601021440\,\bar{\beta}^3\,\alpha\bar{\alpha}\pi - 3675701240760\,\alpha^2\bar{\pi}\pi\,\bar{\beta}^2 \\ &- 1635701635136\,\bar{\beta}^2\,\alpha\bar{\alpha}\pi^2 - 1040399202440\,\alpha^2\bar{\alpha}\pi\,\bar{\beta}^2 \\ &+ 13162176133400\,\alpha^2\bar{\alpha}\pi^2\,\bar{\pi} + 5632482563850\,\alpha^3\bar{\alpha}\pi\,\bar{\beta} \\ &- 1208861900450\,\alpha^5\,\bar{\alpha} + 49609754350\,\bar{\beta}^2\,\beta\,\alpha^3 + 4338917366784\bar{\beta}\beta\,\pi^3\,\alpha \\ &+ 5225292181050\bar{\pi}\alpha^5 + 1952913512680\,\bar{\beta}\,\beta\,\pi\,\alpha^3 + 265058659320\,\bar{\beta}^2\,\beta\,\pi\,\alpha^2 \\ &+ 4375379783424\bar{\beta}\beta\,\pi^2\,\alpha^2 - 265004094784\,\bar{\beta}^3\,\beta\,\pi^2 \\ &- 30330200378072\,\alpha^3\bar{\alpha}\pi^2 + 905003032125\,\alpha^4\bar{\alpha}\bar{\beta} - 9564050953390\,\alpha^4\bar{\alpha}\pi \\ &- 8524238850\,\alpha^2\bar{\alpha}\bar{\beta}^3 + 469041329536\,\bar{\beta}^2\,\beta\,\pi^2\,\alpha - 220969445675\,\alpha^3\bar{\alpha}\bar{\beta}^2 \\ &- 329130147840\,\bar{\beta}^3\,\beta\,\pi\,\alpha + 13687783600768\,\alpha\,\pi^3\bar{\alpha}\bar{\beta} \\ &+ 724772046800\,\alpha\bar{\pi}\pi\,\bar{\beta}^3 + 4980319900\,\bar{\beta}^4\,\alpha\,\bar{\alpha} \\ &+ 180931104170496\,\alpha^2\bar{\pi}\pi^3 + 39628423187260\,\alpha^4\bar{\pi}\pi \\ &+ 119915073751888\,\alpha^3\bar{\pi}\pi^2 + 5344911334400\,\pi^4\bar{\alpha}\bar{\beta} \\ &- 20739582848\,\pi^2\bar{\alpha}\bar{\beta}^3 - 48191081692160\,\alpha^2\,\pi^3\,\bar{\alpha} + 40801870077952\,\bar{\pi}\,\pi^5 \\ &- 5268590200832\bar{\pi}\pi^4\,\bar{\beta} + 592713574016\bar{\pi}\pi^2\,\bar{\beta}^3 - 38360907652096\,\pi^4\bar{\alpha}\alpha \\ &- 858758431040\,\pi^3\bar{\alpha}\bar{\beta}^2 + 24894675520\,\bar{\beta}^4\,\beta\,\pi + 7968511840\,\bar{\beta}^4\,\pi\,\bar{\alpha} \\ &+ 136081885849600\bar{\pi}\pi^4\,\alpha - 3416931669632\bar{\pi}\pi^3\,\bar{\beta}^2 \\ &- 102469061500\,\bar{\beta}^3\,\beta\,\alpha^2 + 274949282816\,\bar{\beta}^2\,\beta\,\pi^3 - 1329967209650\,\bar{\pi}\,\alpha^4\,\bar{\beta}\end{aligned}$$

$$
\begin{aligned}
&+ 221453789400\overline{\pi}\alpha^2\overline{\beta}^3 - 730927683900\overline{\pi}\alpha^3\overline{\beta}^2 + 325649974650\,\overline{\beta}\beta\,\alpha^4 \\
&+ 15559172200\,\overline{\beta}^4\beta\,\alpha - 12779688800\overline{\pi}\alpha\,\overline{\beta}^4 - 20447502080\,\overline{\beta}^4\,\overline{\pi}\,\pi \\
&+ 1606267826176\overline{\beta}\pi^4\,\beta - 12237656133632\,\pi^5\,\overline{\alpha}\Big) \Big/ \Big((7684576\,\pi^2\,\overline{\beta} \\
&+ 9460852\,\pi\overline{\beta}\alpha - 24320\,\pi\,\overline{\beta}^2 + 133000\,\overline{\beta}^2\,\alpha + 2915745\overline{\beta}\alpha^2 \\
&- 194465152\,\pi^2\,\alpha - 123050612\,\pi\,\alpha^2 - 102596352\,\pi^3 - 25995895\,\alpha^3) \\
&(\,10412\,\overline{\beta}^2 + 162944\,\pi\,\alpha - 13047\,\overline{\beta}\,\alpha - 18564\,\pi\,\overline{\beta} + 42940\,\alpha^2 \\
&+148960\,\pi^2\,)\Big)\;,
\end{aligned}
\tag{5.145}
$$

where we assume that

$$
\begin{aligned}
v_4 \;:=\; &(7684576\,\pi^2\,\overline{\beta} + 9460852\,\pi\overline{\beta}\alpha - 24320\,\pi\,\overline{\beta}^2 + 133000\,\overline{\beta}^2\,\alpha + 2915745\overline{\beta}\alpha^2 \\
&- 194465152\,\pi^2\,\alpha - 123050612\,\pi\,\alpha^2 - 102596352\,\pi^3 - 25995895\,\alpha^3) \\
&(\,10412\,\overline{\beta}^2 + 162944\,\pi\,\alpha - 13047\,\overline{\beta}\,\alpha - 18564\,\pi\,\overline{\beta} + 42940\,\alpha^2 \\
&+148960\,\pi^2\,) \neq 0\,.
\end{aligned}
\tag{5.146}
$$

The next step consists in proving that (5.141), (5.143), and (5.145) imply that α, β and π are proportional to each other. In order to get a system with only two complex variables, instead of three, new variables are defined as follows:

$$
x_1 := \frac{\alpha}{\pi}, \quad x_2 := \frac{\beta}{\overline{\pi}}. \tag{5.147}
$$

By subtracting (5.141) from (5.143), taking the numerator and dividing by $(8+5x_1-\overline{x_2})(5776\overline{x_2}^2\pi^4\overline{\pi})$, we obtain

$$
\begin{aligned}
N_1 := &\, 178100\,x_1\,x_2\,\overline{x_2}^2 + 284960\,x_2\,\overline{x_2}^2 + 208240\,\overline{x_2}^2\,\overline{x_1} + 130150\,\overline{x_1}\,\overline{x_2}^2\,x_1 \\
&+ 109825\,\overline{x_1}\,\overline{x_2}\,x_1^2 + 252850\,x_2\,\overline{x_2}\,x_1^2 + 523744\,x_2\,\overline{x_2} + 3451480\,x_1\,\overline{x_2} \\
&+ 341900\,\overline{x_2}\,x_1\,\overline{x_1} + 265888\,\overline{x_2}\,\overline{x_1} + 735800\,x_2\,\overline{x_2}\,x_1 + 2915264\,\overline{x_2} \\
&+ 1018400\,\overline{x_2}\,x_1^2 - 879008\,\overline{x_1} + 18263488 - 408050\,x_1^3\,\overline{x_1} + 4864450\,x_1^3 \\
&+ 35335248\,x_1 + 22731900\,x_1^2 - 2101032\,x_1\,\overline{x_1} - 1622550\,x_1^2\,\overline{x_1} = 0\,.
\end{aligned}
\tag{5.148}
$$

By subtracting (5.143) from (5.145), taking the numerator and dividing by $(8+5x_1-\overline{x_2})(5776\overline{x_2}\pi^6\overline{\pi})$, we get

$$
\begin{aligned}
N_2 := &-11651821200\,x_1^3 - 26531539120\,x_1^2 - 10132263424 - 26800626944\,x_1 \\
&+ 242619584\,\overline{x_2} + 593671488\,\overline{x_2}^2 + 2256829184\,\overline{x_1} - 35155250\,\overline{x_2}^2\,x_2\,x_1{}^2 \\
&+ 21550100\,\overline{x_2}^3\,x_2\,x_1 - 128589500\,\overline{x_2}\,x_2\,x_1^3 + 11036720\,\overline{x_2}^3\,\overline{x_1} \\
&+ 21755825\,\overline{x_2}^2\,x_1^2\,\overline{x_1} + 130839250\,x_1^3\,\overline{x_2} - 216631750\,\overline{x_2}\,x_1^2\,\overline{x_1} \\
&- 17700400\,\overline{x_2}^3\,x_1 + 372958500\,x_1^4\,\overline{x_1} + 220600700\,\overline{x_2}^2\,x_1^2 + 34480160\,x_2\,\overline{x_2}^3
\end{aligned}
$$

$$
\begin{aligned}
&- 1915591500\, x_1^4 - 28320640\, \overline{x_2}^3 + 6897950\, \overline{x_2}^3 x_1 \overline{x_1} + 724005800\, \overline{x_2}^2 x_1 \\
&- 112640320\, x_1 x_2 \overline{x_2}^2 - 90878112\, x_2 \overline{x_2}^2 + 59839936\, \overline{x_2}^2 \overline{x_1} \\
&+ 72209280\, \overline{x_1}\,\overline{x_2}^2 x_1 - 1002142990\, \overline{x_1}\,\overline{x_2} x_1^2 - 585808600\, x_2 \overline{x_2} x_1^2 \\
&- 448045312\, x_2 \overline{x_2} + 632690016\, x_1 \overline{x_2} - 1546809184\, \overline{x_2} x_1 \overline{x_1} - 796702240\, \overline{x_2}\,\overline{x_1} \\
&- 887976960\, x_2 \overline{x_2} x_1 + 509836460\, \overline{x_2} x_1^2 + 2328634300\, x_1^3 \overline{x_1} \\
&+ 5728848896\, x_1 \overline{x_1} + 5470002280\, x_1^2 \overline{x_1} = 0 .
\end{aligned}
\tag{5.149}
$$

At this point we shall consider x_1, x_2, $\overline{x_1}$, $\overline{x_2}$ as independent variables, and apply `gsolve` to the polynomial system formed by the polynomials N_1, N_2. The four possible solutions are given by

$$
V_1 := [\,42 - 3\,\overline{x_1} + 65\, x_2 \overline{x_2}, 8 + 5\, x_1\,]\,,
$$

$$
\begin{aligned}
V_2 := [&205049562510\, \overline{x_1} x_2 - 2072817918600\, x_2 \overline{x_2} + 529175067720\, x_2 \\
&+ 5001500073283\, \overline{x_1}^2 - 3239213905470\, \overline{x_1} + 3029503111800\, \overline{x_2} \\
&- 26163100475032, 23707187714600\, x_2 \overline{x_2}^2 - 12070111345240\, x_2 \overline{x_2} \\
&- 15004500219849\, \overline{x_1} - 34648966659800\, \overline{x_2}^2 + 11975391986580\, \overline{x_2} \\
&+ 41386627076564, -1113092 - 431311\, \overline{x_1} + 1909780\, \overline{x_2} + 954890\, \overline{x_2}\,\overline{x_1}. \\
&205\, x_1 + 368]\,,
\end{aligned}
$$

$$
\begin{aligned}
V_3 := [&2175607695654600868570\, \overline{x_1} x_2 - 2444290609441941712429 25\, x_2 \overline{x_2} x_1 \\
&- 3620164563375434326179 20\, x_2 \overline{x_2} - 2549200439513642036395 0\, x_2 x_1 \\
&- 3377755223900242846024 0\, x_2 - 3522103199771706262971 90\, x_1 \overline{x_1}^2 \\
&- 5275150331854002380123 71\, \overline{x_1}^2 - 3726097736978679899400 85\, x_1 \overline{x_1} \\
&- 5687582663580095969926 94\, \overline{x_1} + 3572424736876684041242 75\, x_1 \overline{x_2} \\
&+ 5291009746471788630569 60\, \overline{x_2} + 4211651961630108156296 50\, x_1 \\
&+ 6135236945699030503343 20, 74421671368200\, x_2 \overline{x_2}^2 \\
&- 372108356841000\, x_2 \overline{x_2} x_1 - 595373370945600\, x_2 \overline{x_2} \\
&+ 202992871981785\, x_1 \overline{x_1} + 309188233840256\, \overline{x_1} - 108770135076600\, \overline{x_2}^2 \\
&+ 408707348737250\, x_1 \overline{x_2} + 731690446259960\, \overline{x_2} - 363026773505180\, x_1 \\
&- 558465136160528, \\
&139740\, \overline{x_2}\,\overline{x_1} - 497365\, x_1 \overline{x_1} - 799324\, \overline{x_1} + 279480\, \overline{x_2} - 1187280\, x_1 - 1879248\,, \\
&43975\, x_1^2 + 137900\, x_1 + 107824]\,,
\end{aligned}
$$

$$
\begin{aligned}
V_4 := [&15138500\, \overline{x_1} x_2 \overline{x_2} - 200682625\, x_2 x_1^2 \overline{x_1} - 589775940\, x_2 x_1 \overline{x_1} \\
&- 425769864\, \overline{x_1} x_2 + 30277000\, x_2 \overline{x_2} - 677199250\, x_2 x_1^2 - 2061417280\, x_2 x_1 \\
&- 1553711328\, x_2 - 34073270\, \overline{x_2}\,\overline{x_1}^2 - 2202762525\, x_1^2 \overline{x_1}^2 - 6761833290\, x_1 \overline{x_1}^2 \\
&- 5193018500\, \overline{x_1}^2 - 13933181776 - 6178746050\, x_1^2 - 18555219840\, x_1
\end{aligned}
$$

$$
\begin{aligned}
&- 180544080\,\overline{x_2} - 17865753288\,\overline{x_1} - 158418580\,\overline{x_2}\,\overline{x_1} - 23445716600\,x_1\overline{x_1}\\
&- 7696469075\,x_1^2\overline{x_1}, 38482345375\,x_1^3\overline{x_1} + 30893730250\,x_1^3\\
&+ 11013812625\,x_1^3\overline{x_1}^2 + 1003413125\,x_2x_1^3\overline{x_1} + 3385996250\,x_2x_1^3\\
&+ 51211999525\,x_1^2\overline{x_1}^2 + 16053417900\,x_2x_1^2 + 4651759450\,x_2{x_1}^2\overline{x_1}\\
&+ 177479507850\,x_1^2\overline{x_1} + 140245780600\,x_1^2 + 212140488080\,x_1\\
&+ 7133355840\,x_2x_1\overline{x_1} + 272934958520\,x_1\overline{x_1} + 25260582880\,x_2x_1\\
&+ 79415365840\,x_1\overline{x_1}^2 + 106967929856 + 13183919424\,x_2 + 41078949112\,\overline{x_1}^2\\
&+ 139995956352\,\overline{x_1} + 3612843312\,\overline{x_1}x_2, 427238747000\,x_2\overline{x_2}^2\\
&- 80593740760\,x_2\overline{x_2} - 961615825940\,\overline{x_2}^2\,\overline{x_1} + 72041680708\,\overline{x_2}\,\overline{x_1}\\
&+ 31587988349750\,x_1^2 + 9360849735875\,x_1^2\overline{x_1} + 90434478667240\,x_1\\
&+ 26404464403460\,x_1\overline{x_1} + 549197297800\,x_1\overline{x_2} + 64702426970096\\
&+ 18636073790528\,\overline{x_1} + 1397883090496\,\overline{x_2} - 2547657512880\,\overline{x_2}^2,\\
&31185310\,x_2\overline{x_2}x_1 + 52924196\,x_2\overline{x_2} - 6814654\,\overline{x_2}\,\overline{x_1} - 716849880\,x_1^2\\
&- 440552505\,x_1^2\overline{x_1} - 2156806048\,x_1 - 1352366658\,x_1\overline{x_1} - 45578530\,x_1\overline{x_2}\\
&- 1620399064 - 1038603700\,\overline{x_1} - 90980056\,\overline{x_2}, 505750\,x_1^2 + 149875\,x_1^2\overline{x_1}\\
&+ 1539520\,x_1 + 440460\,x_1\overline{x_1} + 116450\,\overline{x_2}x_1\overline{x_1} + 232900\,x_1\overline{x_2} + 1160352\\
&+ 317976\,\overline{x_1} + 372640\,\overline{x_2} + 186320\,\overline{x_2}\,\overline{x_1}] .
\end{aligned}
$$

The only sets where the solutions $x_1 = \mathit{const.}$ and $x_2 = \mathit{const.}$ are not obvious are V_1, and V_4. For V_1, if we substitute $x_1 = -8/5$ into N_1 we get $195x_2\overline{x_2} + 702/5 = 0$, which is incompatible with the first equation of this set.

We consider now the fourth and fifth equations in set V_4, given respectively by

$$
\begin{aligned}
g_1 :=\ & 31185310\,\overline{x_2}x_2x_1 + 52924196\,\overline{x_2}x_2 - 6814654\,\overline{x_2}\,\overline{x_1}\\
&- 716849880\,{x_1}^2 - 440552505\,{x_1}^2\overline{x_1} - 1352366658\,x_1\overline{x_1}\\
&- 2156806048\,x_1 - 45578530\,x_1\overline{x_2} - 1038603700\,\overline{x_1}\\
&- 1620399064 - 90980056\,\overline{x_2} = 0\,,
\end{aligned}
\tag{5.150}
$$

$$
\begin{aligned}
g_2 :=\ & 505750\,{x_1}^2 + 149875\,{x_1}^2\overline{x_1} + 440460\,x_1\overline{x_1}\\
&+ 116450\,\overline{x_2}x_1\overline{x_1} + 1539520\,x_1 + 232900\,x_1\overline{x_2}\\
&+ 317976\,\overline{x_1} + 186320\,\overline{x_2}\,\overline{x_1} + 1160352 + 372640\,\overline{x_2} = 0\,.
\end{aligned}
\tag{5.151}
$$

Solving (5.151) for $\overline{x_2}$, we get

$$
\begin{aligned}
\overline{x_2} = & -(505750\,{x_1}^2 + 149875\,{x_1}^2\overline{x_1} + 440460\,x_1\overline{x_1} + 1539520\,x_1\\
&+317976\,\overline{x_1} + 1160352)\Big/(116450\,x_1\overline{x_1}\\
&+232900\,x_1 + 186320\,\overline{x_1} + 372640\,)\,.
\end{aligned}
\tag{5.152}
$$

By decomposing x_1 into real and imaginary parts, $x_1 := a_1 + ib_1$, where a_1 and b_1 are real constants, we obtain

$$\begin{aligned} x_2 := &-\frac{1}{23290}(946210\,a_1^2 + 1011500\,i\,a_1\,b_1 - 65290\,b_1^2 \\ &+ 149875\,a_1^3 + 149875\,i\,a_1^2\,b_1 + 149875\,a_1\,b_1^2 + 149875\,i\,b_1^3 \\ &+1857496\,a_1 + 1221544\,i\,b_1 + 1160352) \\ &\Big/(5\,a_1^2 + 5\,b_1^2 + 18\,a_1 + 2\,i\,b_1 + 16\,)\,. \end{aligned} \tag{5.153}$$

Note that the denominator in (5.153) is non-zero, for otherwise $b_1 = 0$ and a_1 is $-8/5$ or -2, in which case the numerator is not zero. The imaginary and real parts of the numerator of (5.150) are given, respectively, by

$$\begin{aligned} \Im(g_1) = &-92472504314674800\,a_1^3\,b_1 - 3396515814215625\,a_1^4\,b_1^3 \\ &- 12887111896092160\,b_1 - 22335692616810000\,5a_1^3\,b_1^3 \\ &- 8853268391445500\,b_1^5 - 44681861502535500\,a_1^4\,b_1 \\ &- 58888845411599360\,a_1\,b_1 - 103665520621538240\,a_1^2\,b_1 \\ &- 11167846308405000\,a_1^5\,b_1 - 11167846308405000\,a_1\,b_1^5 \\ &- 1132171938071875\,a_1^6\,b_1 - 3396515814215625\,a_1^2\,b_1^5 \\ &- 53535129893981000\,a_1^2\,b_1^3 - 55036870101954800\,a_1\,b_1^3 \\ &-20188750190495040\,b_1^3 - 1132171938071875\,b_1^7 = 0\,, \end{aligned} \tag{5.154}$$

$$\begin{aligned} \Re(g_1) = &-118070804784230912\,a_1 - 256990350929161280\,a_1^3 \\ &- 236342536550411968\,a_1^2 - 164607606502680600\,a_1^4 \\ &- 62274474933156500\,a_1^5 - 12908018823713750\,a_1^6 \\ &- 1132171938071875\,a_1^7 - 14331668411316200\,b_1^4 \\ &- 111592846552991680\,a_1\,b_1^2 - 141503640701276800\,a_1^2\,b_1^2 \\ &- 34722066950419648\,b_1^2 - 1740172515308750\,b_1^6 \\ &- 26445881822066500\,a_1\,b_1^4 - 16388363854331250\,a_1^2\,b_1^4 \\ &- 3396515814215625\,a_1^3\,b_1^4 - 88720356755223000\,a_1^3\,b_1^2 \\ &- 27556210162736250\,a_1^4\,b_1^2 - 3396515814215625\,a_1^5\,b_1^2 \\ &-1132171938071875\,b_1^6\,a_1 - 24562542110062592 = 0\,. \end{aligned} \tag{5.155}$$

Applying `gsolve` to these two equations we get

$$\begin{aligned} [[b_1,\ &118070804784230912\,a_1 + 256990350929161280\,a_1^3 \\ &+ 236342536550411968\,a_1^2 + 164607606502680600\,a_1^4 \end{aligned}$$

$$
\begin{array}{l}
+\, 62274474933156500\, a_1{}^5 + 12908018823713750\, a_1{}^6 \\
+\, 1132171938071875\, a_1{}^7 + 24562542110062592], \\
[\, 16\, b_1{}^2 + 1, 4\, a_1 + 7\,], [\, 148225\, b_1{}^2 + 40804, 385\, a_1 + 818\,], \\
[\, 265225\, b_1{}^2 + 6084, 952 + 515\, a_1\,], [6717299609435847\backslash \\
94824830992201165680405029801540622591163634175\backslash \\
80757295393684357497056747230537452850103356992\backslash \\
21375 b_1{}^2 + 652465671642132707467065583137156167\!7\backslash \\
59961810634274525286395504982544348989403683041\backslash \\
758465403674359064954389129142030000 a_1{}^4 + 1268436\backslash \\
36848768527485589126510676549749821344243259063\backslash \\
17578586444019188819023711307002067527298652615\backslash \\
32491267281522597038 a_1 + 44225855545491775278 47\backslash \\
32263068743767829966110341826186726255851063327\backslash \\
89904909811686411081682293895778361714099659320\backslash \\
2350 a_1{}^3 + 1123721128597758577721050280791677130 0\backslash \\
12135639384027821315066763423010850458738004101\backslash \\
56475983332270554798910024595921249 5 a_1{}^2 + 536656\backslash \\
60951906154805215031233483426225190352037534112\backslash \\
39243178045826084009946615806771758683781318083\backslash \\
3697523140762769524 4, \\
4271837519053269047352867237340429575683921365062 5\, a_1{}^5 \\
+\, 361406063293826379400562992657674998947543835431875 a_1{}^4 \\
+1222753876336528955590266313149671000600007216479875 a_1{}^3 \\
+2067983846571524351461108235292920431994899305750505 a_1{}^2 \\
+1748279636334904363175342610403332595088199218681388 a_1 \\
+\, 591032130843345547761619408868305384589251879895804]]\,.
\end{array}
$$

This implies that a_1 and b_1 must be constants, and so x_1 and x_2 are also constants.

Let us consider now the special cases in which each one of the denominators v_1, v_2, v_3 and v_4, in the previous used expressions for Φ_{11}, is zero.

(i) $\mathbf{v_1 = 0}$

In terms of the variables x_1 and x_2, $v_1 = 0$ (cf. (5.134)) assumes the form

$$8 - \overline{x_2} + 5x_1 = 0\,. \tag{5.156}$$

Applying δ to this equation, using (5.116), (5.139) and (5.140), we get

$$34x_2\overline{x_1} - x_2^2 + 56x_2 + 15\overline{x_1}^2 + 24\overline{x_1} = 0\,. \tag{5.157}$$

Thus, the only solution for these equations is

$$x_1 = -3/2, \quad x_2 = 1/2\,. \tag{5.158}$$

Since the numerator on the right side of (5.133) must be zero, we obtain, solving for Φ_{11},

$$\Phi_{11} = -\frac{86}{141}\pi\overline{\pi}\,. \tag{5.159}$$

Applying δ to this equation, using (5.140), we obtain

$$\delta\pi = -\pi\overline{\pi} = -\frac{141}{86}\Phi_{11}\,. \tag{5.160}$$

Applying δ to this equation we get

$$\overline{\delta}\pi = \pi^2\,. \tag{5.161}$$

If we calculate the NP commutator $[\overline{\delta}, \delta]\pi$, using these Pfaffians and the values given in (5.158), we find no solution other than $\pi = 0$, which is a contradiction to our initial assumptions.

(ii) $\mathbf{v_2 = 0}$

In this case, from (5.142) we have

$$\begin{aligned}
&10868\,\overline{x_2}^2 + 772320\,\overline{x_2} + 1020276\,x_1\overline{x_2} + 4085\,\overline{x_2}^2 x_1 \\
&- 1158240\,x_1^2 - 2048352\,x_1 - 214700\,x_1^3 \\
&-1191680 + 335985\,\overline{x_2}x_1^2 = 0\,,
\end{aligned} \tag{5.162}$$

Applying $\overline{\delta}$ to (5.162) we obtain

$$\begin{aligned}
&63729588\,\overline{x_2}x_1^2 - 670120\,x_1^2\overline{x_2}^2 + 12371205\,x_1^3\overline{x_2} - 617652\,\overline{x_2}^3 + 18435168\,x_1^2 \\
&+ 108263616\,x_1\overline{x_2} + 10424160\,x_1{}^3 + 10725120\,x_1 - 1048608\,\overline{x_2}^2 + 60746496\,\overline{x_2} \\
&- 383325\,\overline{x_2}^3 x_1 - 1749232\,\overline{x_2}^2 x_1 + 1932300\,x_1^4 = 0\,.
\end{aligned} \tag{5.163}$$

Applying $\overline{\delta}$ again on (5.163) leads to

$$\begin{aligned}
&-71849032\,\overline{x_2}^3 x_1 - 2917197592\,x_1^2\overline{x_2}^2 + 2126429184\,\overline{x_2}x_1^2 \\
&+ 3683108352\,x_1\overline{x_2} + 357185400\,x_1^3\overline{x_2} - 128701440\,x_1^2 - 125089920\,x_1{}^4 \\
&- 67408896\,\overline{x_2}^3 - 18590290\,x_1^2\overline{x_2}^3 + 6696360\,\overline{x_2}^4 + 4179810\,x_1\overline{x_2}^4 \\
&- 221222016\,x_1^3 - 2418547200\,\overline{x_2}^2 + 2059223040\,\overline{x_2} - 22411650\,x_1^4\overline{x_2} \\
&- 614629870\,x_1^3\overline{x_2}^2 - 23187600\,x_1^5 - 4605932928\,\overline{x_2}^2 x_1 = 0\,.
\end{aligned} \tag{5.164}$$

Using `gsolve` on (5.162), (5.163) and (5.164) gives the empty set solution.

(iii) $\mathbf{v_3 = 0}$

From (5.144) we have

$$148960 + 42940\, x_1^2 + 10412\, \overline{x_2}^2 - 18564\, \overline{x_2} - 13047\, x_1 \overline{x_2} + 162944\, x_1 = 0\,, \tag{5.165}$$

Applying $\overline{\delta}$ twice to (5.165) gives

$$\begin{aligned} &-128832\, \overline{x_2}^2 - 101344\, \overline{x_2}^2 x_1 - 2591852\, \overline{x_2} x_1^2 - 8248048\, x_1 \overline{x_2} - 257640\, x_1^3 \\ &- 6550592\, \overline{x_2} - 977664\, x_1^2 - 20824\, \overline{x_2}^3 - 893760\, x_1 = 0\,, \end{aligned} \tag{5.166}$$

and

$$\begin{aligned} &62472\, \overline{x_2}^4 - 60094064\, \overline{x_2} x_1^2 + 35714228\, x_1^2 \overline{x_2}^2 + 2832764\, x_1^3 \overline{x_2} \\ &- 2838720\, \overline{x_2}^3 + 8043840\, x_1^2 - 210698752\, x_1 \overline{x_2} + 8798976\, x_1^3 \\ &- 171601920\, \overline{x_2} - 1690904\, \overline{x_2}^3 x_1 + 86775744\, \overline{x_2}^2 \\ &+111204048\, \overline{x_2}^2 x_1 + 2318760\, x_1^4 = 0\,. \end{aligned} \tag{5.167}$$

Using `gsolve` on (5.165), (5.166) and (5.167) we obtain the empty set solution.

(iv) $\mathbf{v_4 = 0}$

From (5.146),

$$\begin{aligned} &(148960 + 42940\, x_1^2 + 10412\, \overline{x_2}^2 - 18564\, \overline{x_2} - 13047\, x_1 \overline{x_2} + 162944\, x_1) \\ &(-24320\, \overline{x_2}^2 + 7684576\, \overline{x_2} + 9460852\, x_1 \overline{x_2} + 2915745\, \overline{x_2} x_1^2 - 194465152\, x_1 \\ &- 123050612\, x_1^2 - 102596352 - 25995895\, x_1^3 + 133000\, \overline{x_2}^2 x_1) = 0\,. \end{aligned} \tag{5.168}$$

We observe now that one of the factors in d_3 is d_2. Thus, if $d_3 = 0$, we have to consider only the expression

$$\begin{aligned} &133000\, \overline{x_2}^2 x_1 - 24320\, \overline{x_2}^2 + 7684576\, \overline{x_2} + 9460852\, x_1 \overline{x_2} + 2915745\, \overline{x_2} x_1^2 \\ &- 194465152\, x_1 - 123050612\, x_1^2 - 102596352 - 25995895\, x_1^3 = 0\,. \end{aligned} \tag{5.169}$$

By applying $\overline{\delta}$ twice to this equation, we obtain

$$\begin{aligned} &1923742456\, \overline{x_2} x_1^2 - 54838615\, x_1^2 \overline{x_2}^2 + 387128860\, x_1^3 \overline{x_2} + 28673280\, \overline{x_2}^3 \\ &+ 1750186368\, x_1^2 + 3181762656\, x_1 \overline{x_2} + 1107455508\, x_1^3 + 923367168\, x_1 \\ &- 143083296\, \overline{x_2}^2 + 1751900928\, \overline{x_2} + 17772600\, \overline{x_2}^3 x_1 - 175990444\, \overline{x_2}^2 x_1 \\ &+ 233963055\, x_1^4 = 0\,, \end{aligned} \tag{5.170}$$

and

$$\begin{aligned}
&3071183072\,\overline{x_2}^3\,x_1 - 65381238064\,x_1^2\overline{x_2}^2 + 363466389312\,\overline{x_2}x_1^2 \\
&+ 414477092352\,x_1\overline{x_2} + 141775882688\,x_1^3\overline{x_2} - 11080406016\,x_1^2 \\
&- 13289466096\,x_1^4 + 2839158528\,\overline{x_2}^3 + 811332320\,x_1^2\overline{x_2}^3 - 315187200\,x_2^4 \\
&- 196695600\,x_1\overline{x_2}^4 - 21002236416\,x_1^3 - 58327724544\,\overline{x_2}^2 \\
&+ 20770389380\,x_1^4\overline{x_2} - 13320040240\,x_1^3\overline{x_2}^2 - 2807556660\,x_1^5 \\
&- 106950649152\,\overline{x_2}^2\,x_1 + 177286496256\,\overline{x_2} = 0\,. \qquad (5.171)
\end{aligned}$$

By applying gsolve to (5.170) and (5.171) we obtain again the empty solution set.

We consider now the case in which x_1 and x_2 are constants. From $\overline{\delta}x_1 = \overline{\delta}(\alpha/\pi) = 0$, and $\delta x_2 = \delta(\beta/\overline{\pi}) = 0$ we get, respectively,

$$31\,x_1 + 10\,{x_1}^2 + 2 = 0\,, \qquad (5.172)$$

and

$$63\,x_2 + 40\,x_2\,\overline{x_1} + \overline{x_1} = 0\,. \qquad (5.173)$$

Thus, we have two solutions, given by $x_1 = x_2 = -8/5$ and $x_1 = -3/2$, $x_2 = 1/2$. The first solution is impossible, since these values do not satisfy the equation $N_1 = 0$. The second one is included the case $5x_1 + \overline{x_2} - 8 = 0$, which was already considered.

Before analysing the cases in which any one of the spin coefficients α, β or π is zero, let us study the case of Weyl's neutrino equation.

5.6.2 Weyl's neutrino equation

For the Weyl neutrino equation (5.70) the proof follows the same steps used in Maxwell's equations case. The numerical coefficients are different, however, since in condition V, we now have $k_1 = 8$ and $k_2 = 13$. Condition VI' remains the same. Supposing that $\alpha\beta\pi \neq 0$, the corresponding Pfaffians are now given by

$$\begin{aligned}
&\delta\beta = -\beta(\overline{\alpha} + \beta)\,, && (5.174)\\
&\delta\overline{\beta} = -2\Phi_{11} - \overline{\beta}\overline{\alpha} - 4\overline{\beta}\overline{\pi} - 2D\overline{\mu} - \beta\overline{\beta} + 2\overline{\pi}\pi\,, && (5.175)\\
&\delta\pi = D\mu - \overline{\pi}\pi + \pi\overline{\alpha} - \beta\pi\,, && (5.176)\\
&\delta\alpha = \overline{\delta}\beta + \alpha\overline{\alpha} + \beta\overline{\beta} - 2\beta\alpha + \Phi_{11}\,, && (5.177)\\
&\mathrm{D}\overline{\mu} := \frac{1}{109}\left(38\overline{\pi}\alpha + 242\beta\overline{\beta} + 38\alpha\overline{\alpha} + 346\pi\overline{\alpha} - 199\Phi_{11}\right) - 2\beta\pi \\
&\qquad -2\overline{\beta}\overline{\pi} - \mathrm{D}\mu\,, && (5.178)
\end{aligned}$$

$$\begin{aligned}
\mathrm{D}\mu := & -\frac{1}{218}(17480\,\pi^2\,\overline{\alpha} + 9728\overline{\pi}\alpha^2 - 49684\,\pi\,\Phi_{11} + 109020\overline{\pi}\pi^2 \\
& +116988\,\pi\,\beta\,\overline{\beta} + 9728\,\alpha^2\,\overline{\alpha} + 78152\,\alpha\overline{\pi}\pi - 55808\,\alpha\,\beta\,\pi - 25111\,\alpha\,\Phi_{11} \\
& +13952\overline{\beta}\Phi_{11} - 2834\overline{\beta}\overline{\pi}\,\pi + 27208\,\alpha\,\pi\,\overline{\alpha} + 27904\overline{\beta}\alpha\,\overline{\alpha} + 50140\overline{\beta}\pi\,\overline{\alpha} - \\
& -100280\,\beta\,\pi^2 + 61952\,\alpha\,\beta\,\overline{\beta}) \Big/ (-128\,\alpha - 230\,\pi + 13\overline{\beta})\,, \qquad (5.179)
\end{aligned}$$

$$\overline{\delta}\pi = -3\pi\alpha - \frac{901}{19}\pi\overline{\beta} - \frac{512}{19}\overline{\beta}\alpha\,, \qquad (5.180)$$

$$\overline{\delta}\alpha = -3\alpha^2 + \frac{787}{19}\overline{\beta}\alpha + \frac{1380}{19}\overline{\beta}\alpha\,. \qquad (5.181)$$

For now we suppose that the denominator of (5.179)

$$w_1 := -128\,\alpha - 230\,\pi + 13\,\overline{\beta}\,, \qquad (5.182)$$

is non-zero.

By evaluating the NP commutator $[\overline{\delta}, \delta]\alpha$ and solving for Φ_{11}, we find

$$\begin{aligned}
\Phi_{11} := & -\frac{4}{19}\overline{\beta}\,(64\,\alpha + 115\,\pi)(-70557184\,\alpha^2\,\overline{\alpha} + 284828160\overline{\pi}\alpha^2 - 30870762\overline{\pi}\alpha\,\overline{\beta} \\
& + 198927872\,\alpha\,\beta\,\overline{\beta} - 208191208\,\alpha\,\pi\,\overline{\alpha} + 1075093784\,\alpha\overline{\pi}\pi \\
& + 94822806\overline{\beta}\alpha\,\overline{\alpha} + 372750664\,\pi\,\beta\,\overline{\beta} - 32576830\,\beta\,\overline{\beta}^2 - 14673035\,\overline{\beta}^2\,\overline{\alpha} \\
& + 174461040\overline{\beta}\pi\,\overline{\alpha} - 148196124\,\pi^2\,\overline{\alpha} - 63509940\overline{\beta}\overline{\pi}\,\pi + 1015943576\overline{\pi}\pi^2) \\
& \Big/ (6639868416\,\pi\,\alpha^2 + 12728096988\,\pi^2\,\alpha - 3519721740\,\pi^2\,\overline{\beta} - 320122100\,\pi\,\overline{\beta}^2 \\
& - 1512677760\overline{\beta}\alpha^2 - 160164491\,\overline{\beta}^2\,\alpha - 4676894514\,\pi\,\alpha\,\overline{\beta} \\
& + 1134018560\,\alpha^3 + 8011469480\,\pi^3)\,, \qquad (5.183)
\end{aligned}$$

where, for now, we assume that the denominator of (5.183),

$$\begin{aligned}
w_2 \;:=\; & 6639868416\,\pi\,\alpha^2 + 12728096988\,\pi^2\,\alpha - 3519721740\,\pi^2\,\overline{\beta} - 320122100\,\pi\,\overline{\beta}^2 \\
& - 1512677760\overline{\beta}\alpha^2 - 160164491\,\overline{\beta}^2\,\alpha - 4676894514\,\pi\,\alpha\,\overline{\beta} + 8011469480\,\pi^3 \\
& + 1134018560\,\alpha^3\,, \qquad (5.184)
\end{aligned}$$

is non-zero.

From the commutator $[\overline{\delta}, \delta](\alpha + \overline{\beta})$ we obtain

$$\begin{aligned}
\Phi_{11} := & -\frac{4}{19}\overline{\beta}(186375332\,\pi\,\beta\,\overline{\beta} - 35278592\,\alpha^2\,\overline{\alpha} + 49969088\overline{\beta}\alpha\,\overline{\alpha} \\
& + 537546892\,\alpha\overline{\pi}\pi - 104095604\,\alpha\,\pi\,\overline{\alpha} - 23139610\overline{\beta}\overline{\pi}\,\pi - 12877696\overline{\pi}\alpha\,\overline{\beta} \\
& + 507971788\overline{\pi}\pi^2 + 142414080\,\overline{\pi}\,\alpha^2 - 74098062\,\pi^2\,\overline{\alpha} + 89788205\overline{\beta}\pi\,\overline{\alpha} \\
& + 99463936\,\alpha\,\beta\,\overline{\beta}) \Big/ (3956264\,\overline{\beta}^2 + 34832476\,\pi^2 + 8859520\,\alpha^2\overline{\beta} \\
& - 1909041\overline{\beta}\alpha + 35954522\,\pi\,\alpha - 2471658\,\pi)\,, \qquad (5.185)
\end{aligned}$$

where the denominator of (5.185),

$$
\begin{aligned}
w_3 \;:=\;& 3956264\,\overline{\beta}^2 + 34832476\,\pi^2 + 8859520\,\alpha^2 - 2471658\,\pi\,\overline{\beta} \\
& -1909041\,\overline{\beta}\alpha + 35954522\,\pi\,\alpha\,,
\end{aligned}
\tag{5.186}
$$

is assumed to be non-zero for now. Applying $\overline{\delta}$ to (5.185) and solving for Ψ_{11} leads to

$$
\begin{aligned}
\Phi_{11} := \frac{4}{19}\overline{\beta}(& -1344269327096344151932\,\pi^2\beta\,\overline{\beta}^3 - 1040327843533920802704 8\,\pi^5\,\overline{\alpha} \\
& + 599232924501638284 00\,\pi\beta\,\overline{\beta}^4 - 1477114920115465297524\,\pi\beta\,\overline{\beta}^3\,\alpha \\
& + 280677400132209377280\,\overline{\pi}\alpha^5 - 3747835103813576 4224\,\alpha^3\overline{\alpha}\overline{\beta}^2 \\
& - 1598908524070446875 52\,\alpha^2\overline{\alpha}\overline{\beta}^3 - 1946766823604491226559 2\,\alpha^3\overline{\alpha}\pi^2 \\
& + 473040134074265051136\,\alpha^4\overline{\alpha}\overline{\beta} - 5632169965798660707328\,\alpha^4\overline{\alpha}\pi \\
& + 13306080518774467840\,\overline{\beta}^4\,\alpha\,\overline{\alpha} - 6384731356034140878432\,\alpha^2\overline{\pi}\pi^3 \\
& + 1311340403820242086912\,\alpha^4\overline{\pi}\pi + 500972089824147807112\,\alpha^3\overline{\pi}\pi^2 \\
& - 3384854522745310612 8476\,\alpha^2\,\pi^3\,\overline{\alpha} - 6161759722991919800\,\overline{\beta}^4\overline{\pi}\pi \\
& - 535463117601593042154 8\,\overline{\beta}^2\overline{\pi}\pi^3 + 393214960941764596310\,\overline{\beta}^3\overline{\pi}\pi^2 \\
& + 1161779084905798045 44\,\overline{\pi}\alpha^2\,\overline{\beta}^3 - 34291532371433292 80\,\overline{\pi}\alpha\,\overline{\beta}^4 \\
& - 847472365589974615296\,\overline{\pi}\alpha^3\,\overline{\beta}^2 + 231619251011756130304\,\overline{\pi}\alpha^4\,\overline{\beta} \\
& - 1199440811552213716630 4\,\overline{\pi}\pi^4\,\alpha + 4127830918305528790888\,\overline{\pi}\pi^4\,\overline{\beta} \\
& - 520382783165765812955\,\pi^2\overline{\alpha}\overline{\beta}^3 - 2959394098295627861288 8\,\pi^4\overline{\alpha}\alpha \\
& + 2780732489078980454348\,\pi^4\overline{\alpha}\overline{\beta} - 1490037506143535071 58\,\pi^3\overline{\alpha}\overline{\beta}^2 \\
& + 23909363432172871900\,\overline{\beta}^4\,\pi\,\overline{\alpha} - 406196834518165360384\,\alpha^2\,\beta\,\overline{\beta}^3 \\
& + 33348614928786826240\,\alpha\beta\,\overline{\beta}^4 + 1058764894625310146 56\,\alpha^3\,\beta\,\overline{\beta}^2 \\
& + 581251655030130180096\,\alpha^4\,\beta\,\overline{\beta} - 655826674254976090112\,\alpha^5\,\overline{\alpha} \\
& + 7419075098951138158536\,\pi^3\,\beta\overline{\beta}\alpha + 1539963724455861237676\,\pi^2\,\beta\,\overline{\beta}^2\,\alpha \\
& + 714092390250930442752\,\pi\beta\,\overline{\beta}^2\,\alpha^2 + 3501348512586350593024\,\pi\beta\overline{\beta}\alpha^3 \\
& + 7746506116538053739160\,\pi^2\,\beta\overline{\beta}\alpha^2 \\
& + 6947187936974535067874\,\alpha^2\overline{\alpha}\pi^2\,\overline{\beta} \\
& + 296781034284494801 5872\,\alpha^3\overline{\alpha}\pi\,\overline{\beta} - 1753566779206719240 92\,\alpha^2\overline{\alpha}\pi\,\overline{\beta}^2 \\
& - 277120019536861321340\,\overline{\beta}^2\,\alpha\,\overline{\alpha}\,\pi^2 - 576908206919856353633\,\overline{\beta}^3\,\alpha\overline{\alpha}\pi \\
& - 4702384568989017941628\,\alpha^2\overline{\pi}\pi\,\overline{\beta}^2 + 427589853234427405106\,\alpha\overline{\pi}\pi\,\overline{\beta}^3 \\
& + 8256906521350382449848\,\alpha\overline{\pi}\pi^3\,\overline{\beta} \\
& + 6102451801889024027752\,\alpha^2\overline{\pi}\pi^2\,\overline{\beta} \\
& + 1966547466224183147520\,\alpha^3\overline{\pi}\pi\,\overline{\beta}
\end{aligned}
$$

$$
\begin{aligned}
&- 869306282765366875980\,\alpha\overline{\pi}\pi^2\,\overline{\beta}^2 \\
&+ 7192813273579142396128\,\alpha\pi^3\overline{\alpha}\overline{\beta} + 2573820923374354496128\overline{\beta}\pi^4\,\beta \\
&+ 1075278308022562464392\,\pi^3\,\beta\,\overline{\beta}^2 - 6444060548819418029648\overline{\pi}\pi^5)\,\Big/ \\
&((1055170442911\,\pi\overline{\beta}^2 + 1078454503804\,\pi^2\overline{\beta} + 1512507879634\,\pi\,\alpha\,\overline{\beta} \\
&+ 502505920896\overline{\beta}\alpha^2 + 539884980544\,\overline{\beta}^2\,\alpha + 20438706838844\,\pi^2\,\alpha \\
&+ 11874643647996\,\pi^3 + 2258636415872\,\alpha^3 + 11752612613194\,\pi\,\alpha^2) \\
&(3956264\,\overline{\beta}^2 + 34832476\,\pi^2 + 8859520\,\alpha^2 - 2471658\,\pi\overline{\beta} - 1909041\overline{\beta}\alpha \\
&+ 35954522\,\pi\,\alpha))\,,
\end{aligned}
\tag{5.187}
$$

where the denominator of the expression above, given by

$$
\begin{aligned}
w_4 \;:=\; &(1055170442911\,\pi\overline{\beta}^2 + 1078454503804\,\pi^2\overline{\beta} + 1512507879634\,\pi\,\alpha\,\overline{\beta} \\
&+ 502505920896\overline{\beta}\alpha^2 + 539884980544\,\overline{\beta}^2\,\alpha + 20438706838844\,\pi^2\,\alpha \\
&+ 11874643647996\,\pi^3 + 2258636415872\,\alpha^3 + 11752612613194\,\pi\,\alpha^2) \\
&(3956264\,\overline{\beta}^2 + 34832476\,\pi^2 + 8859520\,\alpha^2 - 2471658\,\pi\overline{\beta} - 1909041\overline{\beta}\alpha \\
&+ 35954522\,\pi\,\alpha)\,,
\end{aligned}
\tag{5.188}
$$

is assumed to be non-zero.

Using again the new variables defined on (5.147) we find that the numerator of the expression resulting from the subtraction of (5.183) from (5.185), modulo non-zero factors, is given by

$$
\begin{aligned}
W_1 := {} & 26702241280\,x_1\,x_2\,\overline{x_2}^2 + 12027042560\,\overline{x_1}\,\overline{x_2}^2\,x_1 + 47980589800\,x_2\,\overline{x_2}^2 \\
&+ 21611092100\,\overline{x_2}^2\,\overline{x_1} + 8984800256\,\overline{x_1}\,\overline{x_2}\,x_1^2 + 31980778048\,\overline{x_2}\,x_1^2 \\
&+ 18846422525\,\overline{x_2}\,\overline{x_1} + 35403422912\,x_1^2\,x_2\,\overline{x_2} + 117520024900\,\overline{x_2} \\
&+ 122867909195\,x_1\,\overline{x_2} + 26633006800\,\overline{x_2}\,x_1\,\overline{x_1} + 76765656110\,x_2\,\overline{x_2} \\
&+ 106782873769\,x_1\,x_2\,\overline{x_2} + 261675596240 + 485314284486\,x_1 \\
&+ 58433053760\,x_1^3 - 67281983133\,x_1^2\,\overline{x_1} + 293978412579\,x_1^2 \\
&- 15664790464\,\overline{x_1}\,x_1^3 - 94729861629\,x_1\,\overline{x_1} - 43762238110\,\overline{x_1} = 0\,.
\end{aligned}
\tag{5.189}
$$

Subtracting (5.187) from (5.185) and taking the numerator we obtain

$$
\begin{aligned}
W_2 := {} & -1601969338273040\,\overline{x_2} + 4129050225093160\,\overline{x_1} - 17022062106400\,\overline{x_2}^2\,\overline{x_1} \\
&- 2049538760980992\,x_1^3 - 566063835068160\,x_2\,\overline{x_2}^2 \\
&- 1290515732341388\,\overline{x_2}\,x_1^2 + 1813947885483370\,\overline{x_2}^2\,x_1 \\
&- 156307460657920\,x_1^2\,x_2\,\overline{x_2}^2 - 529932588630016\,x_1^3\,x_2\,\overline{x_2}
\end{aligned}
$$

$$
\begin{aligned}
&- 2395601461620224\, x_1^2 x_2 \overline{x_2} - 592506195135140\, x_1 x_2 \overline{x_2}^2 \\
&+ 215914322990080\, x_1 x_2 \overline{x_2}^3 - 12298734418880\, \overline{x_2}^2 \overline{x_1} x_1^2 \\
&+ 86149705857280\, \overline{x_2}^3 \overline{x_1} x_1 - 324490247237888\, \overline{x_2}\,\overline{x_1} x_1^3 \\
&- 3566306629981524\, x_1 x_2 \overline{x_2} - 2497770555691740\, x_1 \overline{x_2} \\
&- 1751778168157160\, x_2 \overline{x_2} - 1145153147020940\, \overline{x_2}\,\overline{x_1} \\
&+ 9538495990898584\, x_1 \overline{x_1} + 8328991417137380\, x_1^2 \overline{x_1} \\
&- 2262375560761580\, \overline{x_2} x_1 \overline{x_1} + 1681954895431200\, \overline{x_2}^2 \\
&- 1829880263729056\, x_1 + 3261696353289728\, \overline{x_1} x_1^3 \\
&+ 387971049122800\, x_2 \overline{x_2}^3 + 154800252712300\, \overline{x_2}^3 \overline{x_1} \\
&+ 483642598244352\, x_1^4 \overline{x_1} - 220462420144384\, x_1^3 \overline{x_2} \\
&+ 488572339166080\, x_1^2 \overline{x_2}^2 - 22201920565760\, \overline{x_2}^3 x_1 \\
&- 396075553505280\, x_1^4 - 39894076016600\, \overline{x_2}^3 + 150798528554560 \\
&- 31572436015965\, \overline{x_1}\,\overline{x_2}^2 x_1 - 1487221068155276\, \overline{x_1}\,\overline{x_2} x_1^2 \\
&- 3469302732542492\, x_1^2 = 0\,.
\end{aligned}
\tag{5.190}
$$

By applying gsolve to (5.189) and (5.190) we obtain

$$G_1 := [\, -969\, \overline{x_1} + 7744\, x_2 \overline{x_2} + 1911,\, 64\, x_1 + 115\,]\,,$$

$$
\begin{aligned}
G_2 \quad := \quad & [702595747667968\, x_2 + 121561751175031\, \overline{x_1} - 134682583774537, \\
& 169\, \overline{x_2} + 128,\, 832\, x_1 + 1559]\,,
\end{aligned}
$$

$$
\begin{aligned}
G_3 := {} & [4763826238429598822020\, x_2 + 2088112373321932882471\, \overline{x_1} \\
& + 7425894734976201158 66,\, 102349\, \overline{x_2} - 1005670,\, 7873\, x_1 + 6290]\,,
\end{aligned}
$$

$$
\begin{aligned}
G_4 := {} & [4853169289024\, x_1^3 + 16104186410816\, x_1^3 \overline{x_1} + 38184678219776\, x_2 x_1^3 \\
& + 86310739986951\, x_1^2 \overline{x_1} + 26904390691527\, x_1^2 + 203221914331392\, x_2 x_1^2 \\
& + 49611268000368\, x_1 + 360358810420950\, x_1 x_2 + 154128880390023\, x_1 \overline{x_1} \\
& + 30431388478120 + 212900193245300\, x_2 + 91706289015320\, \overline{x_1}, \\
& -230 - 128\, x_1 + 13\, \overline{x_2}]\,,
\end{aligned}
$$

$$
\begin{aligned}
G_5 := {} & [4037246221032260175082572\, x_2 + 1708477115667069714218 98\, \overline{x_2}\,\overline{x_1} \\
& + 1842223470946382876251169\, \overline{x_1} - 2151796111124306284725932\, \overline{x_2} \\
& + 294782176497660105361 34, \\
& 30653197266757\, \overline{x_2}^2 - 7333203052239\, \overline{x_2} + 91133703096833, \\
& 7873\, x_1 + 6290]\,,
\end{aligned}
$$

$$
\begin{aligned}
G_6 := [&4323109461237319842355422221345280\, x_2 \\
&+ 4618347102653829689252912715436595 2\, \overline{x_2} x_1 \overline{x_1} \\
&+ 7983262305827308990555231953450560 0\, \overline{x_2}\,\overline{x_1} \\
&+ 3082609378732226156458171777886394 9\, x_1 \overline{x_1} \\
&+ 5388143033496377820259013883407247 0\, \overline{x_1} \\
&+ 3765365407055730619177291822688409 6\, x_1 \overline{x_2} \\
&+ 7222703023035253600929279390717056 0\, \overline{x_2} \\
&+ 612427389403827899065561094706957\, x_1 \\
&+ 2335214054314487710589163609138390, 8021313391208\, \overline{x_2}^2 \\
&- 3870574900377\, x_1 \overline{x_2} - 5011279180026\, \overline{x_2} + 9575660143104\, x_1 \\
&+ 14886269629872, 129759808\, x_1^2 + 457429931\, x_1 + 402632890] ,
\end{aligned}
$$

$$
\begin{aligned}
G_7 := [&1164217719808\, x_2 \overline{x_2} + 3442198165120\, x_2 x_1^2 + 13969446387182\, x_1 x_2 \\
&+ 13533496732756\, x_2 + 545200873048\, \overline{x_2} x_1 \overline{x_1} + 1171997712964\, \overline{x_2}\,\overline{x_1} \\
&+ 1466225999873\, x_1^2 \overline{x_1} + 5848154092029\, x_1 \overline{x_1} + 5675239070344\, \overline{x_1} \\
&+ 616344189185\, x_1^2 + 3286184711720\, x_1 - 4334043693096\, \overline{x_2} \\
&- 2200889444520\, x_1 \overline{x_2} + 4110217929624, 220300682567680\, x_2 x_1^3 \\
&+ 93838463991872\, x_1^3 \overline{x_1} + 39446028107840\, x_1^3 - 140856924449280\, \overline{x_2} x_1^2 \\
&+ 177545178666675\, x_1^2 + 34892855875072\, x_1^2 \overline{x_1}\,\overline{x_2} \\
&+ 1306843564119808\, x_2 x_1^2 + 576543920419059\, x_1^2 \overline{x_1} \\
&- 531669354282872\, x_1 \overline{x_2} + 1142523519936231\, x_1 \overline{x_1} \\
&+ 102957422665864\, \overline{x_2} x_1 \overline{x_1} + 254458848112632\, x_1 \\
&+ 2541402784559210\, x_1 x_2 - 502417624470008\, \overline{x_2} \\
&+ 73287873790092\, \overline{x_2}\,\overline{x_1} + 112380655902536 + 736100107004696\, \overline{x_1} \\
&+ 1622978569720508\, x_2, 3956264\, \overline{x_2}^2 + 34832476 + 8859520\, x_1^2 \\
&- 2471658\, \overline{x_2} - 1909041\, x_1 \overline{x_2} + 35954522\, x_1] ,
\end{aligned}
$$

$$
\begin{aligned}
G_8 := [&608541253301109086673\, \overline{x_1} x_2 - 189357473777142747648 0\, x_2 \overline{x_2} \\
&+ 748235809613808093864\, x_2 - 717730821827652203347\, \overline{x_1}^2 \\
&- 180021573512234092653 9\, \overline{x_1} + 1705782201794095825920\, \overline{x_2} \\
&- 303698104523960879024, 1016217137316961648640 0\, x_2 \overline{x_2}^2 \\
&- 485717567524017811840 0\, x_2 \overline{x_2} - 2153192465482956610041\, \overline{x_1} \\
&- 915435272459081154560 0\, \overline{x_2}^2 + 4356900576465488213760\, \overline{x_2} \\
&+ 563782884805837289012, \\
&-5684169941\, \overline{x_1} + 15668257280\, \overline{x_2} - 6989007688 + 7834128640\, \overline{x_2}\,\overline{x_1}, \\
&832\, x_1 + 1559] ,
\end{aligned}
$$

$$
\begin{aligned}
G_9 := [&340576064287880626663147342529479895807249 01428\,\overline{x_1}\,x_2\\
&+ 1189094466646830891540308518501127951317064268800\,x_1\,x_2\,\overline{x_2}\\
&+ 1956615635119345231267189995466941947416744478720\,x_2\,\overline{x_2}\\
&+ 1112976027371166045326606804622128105239044935968\,x_1\,x_2\\
&+ 1969029949528043861810432486908073579520105553960\,x_2\\
&+ 267790460084507616578085547631039975448516085824\,x_1\,\overline{x_1}^2\\
&+ 430130557723693693939253804346911726108144402641\,\overline{x_1}^2\\
&+ 975915158101472824368468507141365476792688611 36\,x_1\,\overline{x_1}\\
&+ 744436412874896960603703081505337098043676164 48\,\overline{x_1}\\
&- 1071167742681855927090029987740685509864132275200\,x_1\,\overline{x_2}\\
&- 1762571109322385373620857103354517952631612794880\,\overline{x_2}\\
&- 1236094110677986125167928907954767973155881869708\,x_1\\
&- 2191948479304771236829317467690244724888782628780,\\
&2051400193244229127976607360 0\,x_2\,\overline{x_2}^2\\
&- 20198401902712409875461980160 0\,x_1\,x_2\,\overline{x_2}\\
&- 36294003418936361494970745600 0\,x_2\,\overline{x_2}\\
&+ 70819418588411562406146864696\,x_1\,\overline{x_1}\\
&+ 12220159051799628549312431234 88\,\overline{x_1}\\
&- 18479555459803386359458694400\,\overline{x_2}^2\\
&+ 12090891863440580186952982624 0\,x_1\,\overline{x_2}\\
&+ 22328392560888376897661925952 0\,\overline{x_2}\\
&- 18543741798739275278297336209\,x_1 - 309989801922004656690462906 70,\\
&17259564660\,\overline{x_2}\,\overline{x_1} - 67579394104\,x_1\,\overline{x_1} - 129104492662\,\overline{x_1}\\
&+ 34519129320\,\overline{x_2} - 340905273759\,x_1 - 611541656670,\\
&129759808\,x_1^2 + 457429931\,x_1 + 402632890]\,,
\end{aligned}
$$

$$
\begin{aligned}
G_{10} := [&7447690047469100\,\overline{x_1}\,x_2\,\overline{x_2} + 10072797662497856\,x_2\,x_1^2\,\overline{x_1}\\
&+ 30187987711437699\,x_1\,x_2\,\overline{x_1} + 21615932628713280\,\overline{x_1}\,x_2\\
&+ 14895380094938200\,x_2\,\overline{x_2} + 19346866846955648\,x_2\,x_1^2\\
&+ 58539752116856582\,x_1\,x_2 + 42274669373988480\,x_2\\
&- 2990121038695700\,\overline{x_2}\,\overline{x_1}^2 - 6899169505349702 4\,x_1^2\,\overline{x_1}^2\\
&- 22897580765258510 1\,x_1\,\overline{x_1}^2 - 19151147974864352 0\,\overline{x_1}^2\\
&- 84111156108619456 0 - 31977375100488281 6\,x_1^2\\
&- 10360033986808925 94\,x_1 - 25378636471710600\,\overline{x_2}\\
&- 80400987372342048 0\,\overline{x_1} - 18669560313246700\,\overline{x_2}\,\overline{x_1}\\
&- 97678037390328613 1\,x_1\,\overline{x_1} - 29823002347764358 4\,x_1^2\,\overline{x_1},
\end{aligned}
$$

$$
\begin{aligned}
&10072797662497856\, x_2 x_1^3 \overline{x_1} - 298230023477643584\, x_1^3 \overline{x_1} \\
&+ 19346866846955648\, x_2 x_1^3 - 68991695053497024\, x_1^3 \overline{x_1}^2 \\
&- 319773751004882816\, x_1^3 - 1487784358371461151\, x_1^2 \overline{x_1} \\
&- 348909525470751281\, x_1^2 \overline{x_1}^2 - 1577701651433863034\, x_1^2 \\
&+ 38235478794520579\, x_2 x_1^2 \overline{x_1} + 73996603694063622\, x_2 x_1^2 \\
&- 590857363509776915\, x_1 \overline{x_1}^2 - 2484379550399181610\, x_1 \overline{x_1} \\
&- 2603145849422208520\, x_1 + 45734114097650550\, x_1 x_2 \overline{x_1} \\
&+ 89044012802799340\, x_1 x_2 + 33774631063430400\, x_2 \\
&- 1439493309495798400 - 1391014344034870800\, \overline{x_1} . \\
&- 335461621166647400\, \overline{x_1}^2 + 17269683250934400\, \overline{x_1} x_2, \\
&6893085186209601441920\, x_2 \overline{x_2}^2 + 36347153899567955649920\, x_2 \overline{x_2} \\
&- 27674566080809380678400\, \overline{x_2}^2 \overline{x_1} - 41160443800662115740500\, \overline{x_2} \overline{x_1} \\
&- 15550881368829020327975 68\, x_1^2 - 8096446972051748511640 96\, x_1^2 \overline{x_1} \\
&- 27477025120447842767377 09\, x_1 \overline{x_1} + 1247829183760538636147 20\, x_1 \overline{x_2} \\
&- 5231459772963769369377242\, x_1 - 2335350482507069487515200\, \overline{x_1} \\
&+ 1031583796440071888998 00\, \overline{x_2} - 440286467902230940918 7200 \\
&- 1174438664836721958336 00\, \overline{x_2}^2, 392792077300480\, x_1 x_2 \overline{x_2} \\
&+ 313814577190400\, x_2 \overline{x_2} + 157374791510300\, \overline{x_2} \overline{x_1} \\
&+ 7956462676222848\, x_1^2 + 3631141844920896\, x_1^2 \overline{x_1} \\
&+ 12051358297504479\, x_1 \overline{x_1} - 353837491121920\, x_1 \overline{x_2} \\
&+ 25875506363560382\, x_1 + 10079551565718080\, \overline{x_1} + 32057112659000\, \overline{x_2} \\
&+ 21132353181745280, 1562218112\, x_1^2 + 813356864\, x_1^2 \overline{x_1} \\
&+ 2437615431\, x_1 \overline{x_1} + 1205250560\, x_1 \overline{x_2} + 4726959758\, x_1 \\
&+ 602625280\, \overline{x_2} x_1 \overline{x_1} + 1745440320\, \overline{x_1} + 2165684600\, \overline{x_2} + 3413589120 \\
&+ 1082842300\, \overline{x_2} \overline{x_1}] .
\end{aligned}
$$

Among these sets of solutions, the only ones which are not trivially impossible, or do not imply x_1 and x_2 be constants, are G_7 and G_{10}. Let us take the third equation in G_7:

$$
\begin{aligned}
f_1 := {} & 3956264\, \overline{x_2}^2 - 2471658\, \overline{x_2} - 1909041\, x_1 \overline{x_2} + 8859520\, x_1^2 \\
& 35954522\, x_1 + 34832476 = 0 .
\end{aligned} \tag{5.191}
$$

Applying $\overline{\delta}$ twice successively to this equation, using the Pfaffians calculated previously, we obtain respectively:

$$
\begin{aligned}
f_2 := {} & 1079685917\, \overline{x_2} x_1^2 + 252496320\, x_1^3 + 3781787431\, x_1 \overline{x_2} + 1024703877\, x_1^2 \\
& + 3287720348\, \overline{x_2} + 87748156\, \overline{x_2}^2 x_1 + 90137805\, \overline{x_2}^2 + 37584508\, \overline{x_2}^3 \\
& + 992725566\, x_1 = 0 ,
\end{aligned} \tag{5.192}
$$

$$\begin{aligned} f_3 := {} & 3794884235\, x_1^3 \overline{x_2} + 13984191503\, x_1^2 \overline{x_2}^2 + 2272466880\, x_1^4 + 8934530094\, x_1^2 \\ & - 917394512\, \overline{x_2}^3 x_1 + 40424847312\, \overline{x_2}^2 - 72805233934\, x_1 \overline{x_2} \\ & + 9222334893\, x_1^3 - 11338514006\, \overline{x_2} x_1^2 + 47275438170\, \overline{x_2}^2 x_1 \\ & + 112753524\, \overline{x_2}^4 - 1918581915\, \overline{x_2}^3 - 72103225320\, \overline{x_2} = 0\,. \end{aligned} \tag{5.193}$$

Applying `gsolve` to these three equations we obtain $x_2 = 0$, a contradiction.

Let us now consider G_{10}. The fourth and fifth equations of this set are given respectively by

$$\begin{aligned} h_1 := {} & 1205250560\, x_1 \overline{x_2} + 602625280\, \overline{x_2} x_1 \overline{x_1} + 2165684600\, \overline{x_2} \\ & + 1082842300\, \overline{x_2}\,\overline{x_1} + 1562218112\, x_1^2 + 813356864\, x_1^2 \overline{x_1} + 4726959758\, x_1 \\ & + 2437615431\, x_1 \overline{x_1} + 1745440320\, \overline{x_1} + 3413589120 = 0\,, \end{aligned} \tag{5.194}$$

$$\begin{aligned} h_2 := {} & 392792077300480\, x_1 x_2 \overline{x_2} + 313814577190400\, x_2 \overline{x_2} \\ & + 157374791510300\, \overline{x_2}\,\overline{x_1} - 353837491121920\, x_1 \overline{x_2} \\ & + 32057112659000\, \overline{x_2} + 7956462676222848\, x_1^2 + 3631141844920896\, x_1^2 \overline{x_1} \\ & + 25875506363560382\, x_1 + 12051358297504479\, x_1 \overline{x_1} \\ & + 10079551565718080\, \overline{x_1} + 21132353181745280 = 0\,. \end{aligned} \tag{5.195}$$

Applying `gsolve` to (5.194), (5.195) and their complex conjugates we find the four formally possible solutions, not listed here due their huge size, with coefficients having up to a thousand digits. They can be easily computed, however, by running the code listed in the Appendix E. They all imply that at most x_1 and x_2 are constants.

We shall consider now the cases where each of the denominators that appeared in the equations above is zero.

(i)$\mathbf{w_1 = 0}$

In terms of variables x_1 and x_2 we have from (5.182),

$$-128x_1 + 13\overline{x_2} - 230 = 0\,. \tag{5.196}$$

Applying $\overline{\delta}$ twice to this equation, using (5.174), (5.181) and (5.189), we obtain

$$883\overline{x_2}x_1 + 384x_1^2 + 1610\overline{x_2} + 690x_1 - 13\overline{x_2} = 0\,, \tag{5.197}$$

$$\begin{aligned} & -72841\overline{x_2}^2 x_1 + 92014\overline{x_2}x_1^2 - 131330\overline{x_2}^2 - 21888x_1^3 + 429410\overline{x_2}x_1 \\ & -39330x_1^2 + 476100\overline{x_2} + 247\overline{x_2}^3 = 0. \end{aligned} \tag{5.198}$$

Applying `gsolve` to (5.196), (5.197) and (5.198) we find that we must have $x_2 = 0$.

(ii)$\mathbf{w_2 = 0}$

From (5.184) we have

$$\begin{aligned}
&1134018560\, x_1^3 + 6639868416\, x_1^2 + 12728096988\, x_1 - 3519721740\, \overline{x_2} \\
&- 320122100\, \overline{x_2}^2 - 1512677760\, \overline{x_2} x_1^2 - 160164491\, \overline{x_2}^2\, x_1 \\
&- 4676894514\, \overline{x_2} x_1 + 8011469480 = 0\,. \qquad (5.199)
\end{aligned}$$

By applying $\overline{\delta}$ to this equation twice, we obtain, respectively,

$$\begin{aligned}
&-1369961281080\, x_1 - 6430475744520\, \overline{x_2} x_1 - 3248076062982\, \overline{x_2} x_1^2 \\
&+ 81643014022\, \overline{x_2}^2\, x_1 - 1135417499136\, x_1^3 - 4090228161000\, \overline{x_2} \\
&- 2176504584948\, x_1^2 + 57571701013\, \overline{x_2}^2\, x_1^2 + 43939311441\, \overline{x_2}^3\, x_1 \\
&- 521008666752\, \overline{x_2} x_1^3 + 79567654320\, \overline{x_2}^3 - 193917173760\, x_1^4 \\
&- 44701140780\, \overline{x_2}^2 = 0\,, \qquad (5.200)
\end{aligned}$$

$$\begin{aligned}
&13059027509688\, x_1^3 + 8219767686480\, x_1^2 - 49751225470800\, \overline{x_2} \\
&+ 59461698073800\, \overline{x_2}^2 + 6812504994816\, x_1^4 - 410251659210\, \overline{x_2}^4 \\
&+ 1838793243264\, x_1^4 \overline{x_2} - 227972148945\, \overline{x_2}^4\, x_1 + 11422677583148\, x_1^3 \overline{x_2}^2 \\
&+ 5129406799620\, \overline{x_2}^3 + 1095258633485\, \overline{x_2}^3\, x_1^2 + 1163503042560\, x_1^5 \\
&+ 4820826637652\, \overline{x_2}^3\, x_1 + 60294205229306\, \overline{x_2}^2\, x_1^2 \\
&+ 104573868807840\, \overline{x_2}^2\, x_1 - 23114065312032\, \overline{x_2} x_1^2 \\
&- 68556933929400\, \overline{x_2} x_1 + 3098494712094\, \overline{x_2} x_1^3 = 0\,. \qquad (5.201)
\end{aligned}$$

By applying `gsolve` to (5.199), (5.200) and (5.201) we find immediately that $\overline{x_2} = 0$.

(iii)$\mathbf{w_3 = 0}$

From (5.186) we have

$$\begin{aligned}
&-1909041\, \overline{x_2} x_1 - 2471658\, \overline{x_2} + 3956264\, \overline{x_2}^2 + 34832476 + 8859520\, x_1^2 \\
&+ 35954522\, x_1 = 0\,. \qquad (5.202)
\end{aligned}$$

Again, by applying $\overline{\delta}$ to this equation we get

$$\begin{aligned}
&87748156\, \overline{x_2}^2\, x_1 + 1079685917\, \overline{x_2} x_1^2 + 90137805\, \overline{x_2}^2 \\
&+ 3781787431\, \overline{x_2} x_1 + 37584508\, \overline{x_2}^3 + 992725566\, x_1 + 3287720348\, \overline{x_2} \\
&+ 252496320\, x_1^3 + 1024703877\, x_1^2 = 0\,, \qquad (5.203)
\end{aligned}$$

and

$$\begin{aligned}
&-72805233934\,\overline{x_2}x_1 - 11338514006\,\overline{x_2}x_1^2 + 47275438170\,\overline{x_2}^2 x_1 \\
&+ 9222334893\,x_1^3 - 72103225320\,\overline{x_2} + 8934530094\,x_1^2 + 40424847312\,\overline{x_2}^2 \\
&+ 13984191503\,\overline{x_2}^2 x_1^2 - 917394512\,\overline{x_2}^3 x_1 + 3794884235\,\overline{x_2}x_1^3 \\
&- 1918581915\,\overline{x_2}^3 + 112753524\,\overline{x_2}^4 + 2272466880\,x_1^4 = 0\,. \qquad (5.204)
\end{aligned}$$

By applying **gsolve** to these three equations we find again that $x_2 = 0$.

$(\mathbf{iv})\mathbf{w_4} = \mathbf{0}$

From (5.188),

$$\begin{aligned}
&(11874643647996 + 11752612613194\,x_1^2 + 20438706838844\,x_1 \\
&+ 1512507879634\,\overline{x_2}x_1 + 2258636415872\,x_1^3 + 502505920896\,\overline{x_2}x_1^2 \\
&+ 1078454503804\,\overline{x_2} + 1055170442911\,\overline{x_2}^2 + 539884980544\,\overline{x_2}^2 x_1) \\
&(-1909041\,\overline{x_2}x_1 - 2471658\,\overline{x_2} + 3956264\,\overline{x_2}^2 + 34832476 + 8859520\,x_1^2 \\
&+ 35954522\,x_1) = 0\,. \qquad (5.205)
\end{aligned}$$

We observe here again that the second factor of (5.205) is proportional to (5.202). Thus, only the first factor needs to be considered. Applying $\bar{\delta}$ to the first factor of (5.205) we get

$$\begin{aligned}
&-106871792831964\,x_1 - 20327727742848\,x_1^4 - 183948361549596\,x_1^2 \\
&- 204828754890972\,\overline{x_2} - 352155985333632\,x_1\overline{x_2} - 7151232439104\,\overline{x_2}^3 x_1 \\
&- 203691384240944\,\overline{x_2}x_1^2 - 990832120537\,\overline{x_2}^2 x_1 - 39554229871616\,x_1^3\overline{x_2} \\
&- 105773513518746\,x_1^3 - 2331389501632\,\overline{x_2}^2 x_1^2 + 6494485393044\,\overline{x_2}^2 \\
&- 12934935407511\,\overline{x_2}^3 = 0\,. \qquad (5.206)
\end{aligned}$$

Applying **gsolve** to (5.205) and (5.206) we find that all possible solutions require that both x_1 and x_2 are constants.

When x_1 and x_2 are constants we have

$$\bar{\delta}x_1 = \bar{\delta}(\alpha/\pi) = \frac{4}{19}x_2\pi(64x_1 + 115)(2x_1 + 3) = 0\,, \qquad (5.207)$$

$$\delta x_2 = \delta(\beta/\bar{\pi}) = \frac{2}{19}(19\overline{x_1} + 441x_2 + 256x_2\overline{x_1}) = 0\,. \qquad (5.208)$$

From (5.207) we have $x_1 = -115/64$ or $x_1 = -3/2$. In both cases, we can obtain the corresponding value for x_2 from (5.208), substitute into $M_1 = 0$ (cf. (5.189)) and find that this would imply $\pi = 0$, a contradiction to our initial assumptions.

The analysis of the case $\alpha\beta\pi = 0$ follows exactly the steps described in [22] for the self-adjoint scalar equation. Instead of repeating them again we present the proof for the Weyl and Maxwell cases in the form of a Maple code in Appendix E.

5.7 Discussion

We have proved Theorem 5.1 (Main Theorem): *there exist no Petrov type III space-times on which Maxwell equations or Weyl neutrino equations satisfy Huygens' principle.* The use of the necessary five-index condition VI' determined by Gerlach, Alvarez and Wünsch [36] [3] [4] was essential for the solution of this problem. Its conversion, from tensorial to spinor dyad form was possible thanks to the Maple package `NPspinor`. The polynomial system obtained from integrability conditions was simplified using the procedure `gsolve`, from Maple's package `grobner`. Since a direct application of the algorithm seems impossible, due to the large size of the polynomial system, a "divide and conquer" approach was applied with success to solve this problem, i.e., we took pairs of polynomials, decomposing the problem into several smaller, manageable parts.

Chapter 6

Conclusion

In this Thesis we have studied three problems concerning Hadamard's problem in Petrov type III space-times.

In Chapter 3 we have considered the conformally invariant (self-adjoint) scalar wave equation in four dimensions:

$$\Box u + \frac{1}{6} R u = 0 , \tag{6.1}$$

In this case we have proved the following result:

Let $\mathcal{M}^4$ be any space-time which admits a spinor dyad with the properties

$$o_{A;B\dot{B}} = o_A I_{B\dot{B}} , \tag{6.2}$$

where $I_{B\dot{B}}$ is a 2-spinor, and

$$\Psi_{ABCD;E\dot{E}} \iota^A \iota^B \iota^C o^D o^E \bar{o}^{\dot{E}} = 0 , \tag{6.3}$$

$$R = 0 , \qquad \Phi_{AB\dot{A}\dot{B}} o^A o^B = 0 . \tag{6.4}$$

Then the validity of Huygens' principle for the conformally invariant equation (6.1) implies that

$$\Phi_{AB\dot{A}\dot{B}} o^A \iota^B \bar{o}^{\dot{A}} \bar{\iota}^{\dot{B}} \neq 0 . \tag{6.5}$$

Besides the necessary conditions III and V (cf. (2.188) and (2.190)), that were already used by Carminati and McLenaghan [22], we have used the six-index necessary condition VII derived by Rinke and Wünsch [72] (cf. (2.192)). Although we have not solved polynomial system involving α, β, π and Φ_{11} in the general case, the analysis offers evidence that these necessary conditions are enough to settle the problem for Petrov type III space-times.

In Chapter 4 we have studied the non-self-adjoint scalar wave equation:

$$\Box u + A^k(x)\nabla_k u + B(x)u = 0\,. \tag{6.6}$$

In this case, using the necessary conditions II, III, IV, V and VI (cf.(2.187) - (2.191)) we have obtained the following result:

If a non-self-adjoint scalar wave equation (6.6) satisfies Huygens' principle on any Petrov type III space-time, then it must be equivalent to the self-adjoint invariant scalar wave equation (6.6) with $A^i = 0$ and $B = R/6$.

In Chapter 5 we have considered Hadamard's problem for the homogeneous Maxwell's equations, and Weyl's neutrino equation, given respectively by

$$d\omega = 0\,, \qquad \delta\omega = 0\,, \tag{6.7}$$

$$\nabla^{K\dot{A}}\varphi_K = 0\,. \tag{6.8}$$

We have solved Hadamard's problem for this case by proving the following statement:

There exist no Petrov type III space-times on which Maxwell's equations (6.7) or Weyl's equation (6.8) satisfy Huygens' principle

In order to obtain this result, we have used necessary conditions III, V' and VI' (cf. (5.106), (5.107), (5.108)).

An essential tool in our calculations was the Maple package `NPspinor` [27] [29], used to convert tensorial expressions into their corresponding spinorial forms , and also for manipulating the Newman-Penrose (NP) expressions. The Maple package `grobner` [28] was used to study the polynomial systems that would be obtained from the necessary conditions, the NP field equations and commutation relations. We have verified that the procedure `gsolve`, from `grobner`, which attempts to factor the polynomial system after each reduction step of Buchberger's algorithm, was more useful than procedures or programs that try to find a Gröbner basis. Moreover, we have used a "divide and conquer" approach, dealing with pairs of equations and then analysing separately the several sets of possible (smaller) results together with the polynomials that had been left out.

We believe that Hadamard's problem can finally be solved for the scalar wave equation, on Petrov type III space-times, by using the methods employed on Chapter 3, or using heuristic Gröbner basis methods. Hadamard's problem for equations (6.1), (6.6) (6.7) and (6.8) on Petrov type II space-times has been partially studied [23] and is good area for the application of the techniques developed in this Thesis.

Appendix A

Gröbner Bases

In this Appendix we present a summary based on the references [35], [1], [28] [2].

A.1 Basic definitions

Definition A.1 A *commutative ring* is a set $(\mathsf{R}, +, \cdot)$ with the two binary operations $(+)$ and multiplication $(\cdot)$ defined on R such that $(\mathsf{R}, +)$ is a commutative group, $(\mathsf{R}, \cdot)$ is commutative and associative, and the distributive law $a \cdot (b + c) = ab + ac$ holds $\forall\, a, b, c, \in \mathsf{R}$.

Definition A.2 A *field* $(\mathsf{K}, +, \cdot)$ is a commutative ring in which every nonzero element has a multiplicative inverse.

Definition A.3 Let N denote the non-negative integers. Let $\alpha = (\alpha_1, ..., \alpha_n)$ be a power vector in N^n, and let $x_1, x_2, ..., x_n$ be any n variables. Then a *monomial* x^α is defined as the product $\mathsf{x}^\alpha = \mathsf{x}_1^{\alpha_1} \cdot \mathsf{x}_2^{\alpha_2} \cdots \mathsf{x}_n^{\alpha_n}$. The *total degree* of the monomial x^α is defined as $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Definition A.4 A multivariate *polynomial* f in $x_1, x_2, ..., x_n$ with coefficients in a field K is a finite linear combination $f(x_1, x_2, ..., x_n) = \sum_\alpha a_\alpha x^\alpha$, of monomials x^α and coefficients $a_\alpha \in \mathsf{K}$. The *total degree* of the polynomial f is defined as the maximum $|\alpha|$ such that $a_\alpha \neq 0$.

We denote the set of all multivariate polynomials in $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n)$ with coefficients in a field by $\mathsf{K}[\mathbf{x}]$. It can be shown that $\mathsf{K}[\mathbf{x}]$ forms a commutative ring. It is called a *polynomial ring.*

Definition A.5 A nonempty subset I of a noncommutative ring K is called an *ideal* of K if, $\forall\, x, y \in I$ and $r \in \mathsf{K}$,

(i) $x - y \in I$,
(ii) $x \cdot r$ and $r \cdot x \in I$.

Definition A.6 Let $F = \{f_1, ..., f_s\}$ be a set of multivariate polynomials. Then the ideal generated by F, denotated by $I = \langle F \rangle = \langle f_1, ..., f_s \rangle$, is given by

$$\langle f_1, ..., f_s \rangle = \{\sum_{i=1}^{s} h_i f_i \; : \; h_1, \ldots, h_s \in \mathsf{K}[x_1, \ldots, x_n]\}.$$

The polynomials $f_1, ..., f_s$ are said to form a *basis* for the ideal they generate and, since F is finite, we say that the ideal is finitely generated. The two polynomials are said to be equivalent with respect to an ideal if their difference belongs to the ideal.

Theorem A.7 (Hilbert Basis Theorem). In the ring $\mathsf{K}[x_1, \ldots, x_n]$ the following properties are satisfied:

(i) If I is any ideal of $\mathsf{K}[x_1, \ldots, x_n]$, then there exist polynomials $f_1, ..., f_s \in \mathsf{K}[x_1, \ldots, x_n]$ such that $I = \langle f_1, ..., f_s \rangle$.
(ii) If $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ is an ascending chain of ideals of $\mathsf{K}[x_1, \ldots, x_n]$, then there exists N such that $I_N = I_{N+1} = I_{N+2} = \cdots$.

Thus, according to Hilbert Basis Theorem, every ideal is finitely generated. Every ideal can be generated by different bases, since we can always add any linear combination of the generators, or suppress one of them if it is a linear combination of the others. In general, it is a difficult problem to decide whether a given polynomial is a member of an arbitrary ideal. This problem can be considered as an instance of the "zero-equivalence" problem. For example, deciding if a polynomial $h \in I = \langle f_1, ..., f_s \rangle$ is the same as deciding if h simplifies to 0 with respect to the side relations $f_1 = 0, \ldots, f_s = 0$. Also, the problem of solving a polynomial system of equations $f_1 \doteq 0, \ldots, f_s = 0$, where each $f_i \in \mathsf{K}[x_1, \ldots, x_n]$, is equivalent to finding a "reduced" basis for the ideal $\langle f_1, ..., f_s \rangle$, i.e., a basis in which the system assumes a simpler form.

What the Gröbner bases theory provides is an algorithm that leads to the determination of a standard basis for a polynomial ideal, where this standard or reduced basis always exists and from which the existence and uniqueness of solutions (or even the solutions themselves) may be easily determined.

A.2 Monomial ordering

As we shall see, the computation of Gröbner bases is very sensitive to the choice of the monomial ordering. Let us consider polynomials in $\mathsf{K}[x_1, \ldots, x_n]$. For the variables x_i we assume the ordering $x_1 > x_2 > \ldots > x_{n-1} > x_n$.

Definition A.8 An *admissible total ordering* $>$ on N^n is defined by the two conditions:

1. $\forall \alpha \in N^n$, $\alpha > 0$.
2. $\forall \alpha, \beta, \gamma \in N^n$, $\alpha > \beta \Rightarrow \alpha + \gamma > \beta + \gamma$.

An admissible ordering establishes a one-to-one correspondence between N^n and the monomials $x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ in $K[x_1, \dots, x_n]$, i.e., if $>$ is an admissible ordering on N^n then $>$ is an ordering on the monomials, $\alpha > \beta \Rightarrow x^\alpha > x^\beta$. Among the several different monomial orderings we consider only the three most important ones:

Definition A.9 Let α and β be in N^n. We define the following monomial orderings:

1. *Pure lexicographic order* (plex): $\alpha >_{plex} \beta \iff$ the left-most nonzero entry in $\alpha - \beta$ is positive.
2. *Graded lexicographic order* (grlex): $\alpha >_{grlex} \beta \iff |\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and $\alpha >_{plex} \beta$.
3. *Graded reverse lexicographic order* (grevlex): $\alpha >_{grevlex} \beta \iff |\alpha| > |\beta|$ or $|\alpha| = |\beta|$ and the right-most nonzero entry in $\alpha - \beta$ is negative.

The orderings grlex and grevlex are also called *total degree ordering.* In Maple the grevlex order is referred by tdeg.

Definition A.10 Given a particular admissible ordering $>$ and a nonzero polynomial $h \in K[x_1, \dots, x_n]$, we define:

-*Multidegree* of h: $multideg(h) = \max(\alpha \in N^n,\ a_\alpha \neq 0)$.
- *Leading monomial* of f : $LM(f) = x^{multideg(f)}$.
- *Leading coefficient* of f : $LC(f) = a_{multideg(f)}$.
- *Leading term* of f : $LT(f) = LC(f) \cdot LM(f)$.

A.3 Polynomial reduction

The reduction method for the case of multivariate polynomials is a generalization of the reduction process known for linear and univariate polynomials (long division), normally used for solving systems of equations. The basic idea consists of the following: when dividing f by $f_1, \dots, f_s$, we have to cancel terms of f using the leading terms of the f_is (so the new terms which are introduced are smaller than the cancelled terms) and continue this process until it cannot be done any more.

Definition A.11 Given f, g, h in $K[x_1, \ldots, x_n]$, with $g \neq 0$, $a \in K - 0$, we say that f reduces to h modulo g in one step, denoted by $f \xrightarrow{g} h$, if and only if $LM(g)$ divides a nonzero term ax^α of f and

$$h = f - \frac{ax^\alpha}{LT(g)} g\,.$$

Example A.12 Consider the polynomials $f = 6x^4 + 13x^3 - 6x + 1$, $g = 3x^2 + 5x - 1$. Then, if we decide to reduce the first term of f we get $f \xrightarrow{g} h$, where

$$h = f - \frac{6x^4}{3x^2} g = 3x^3 + 2x^2 - 6x + 1\,.$$

If we start by reducing the term of degree 3 of f, we obtain

$$h = f - \frac{13x^3}{3x^2} g = 6x^4 - \frac{65}{3}x^2 - \frac{5}{3}x + 1$$

In both cases we could continue the reduction until we get 0, since in fact g divides f.

Example A.13 Let $f = 6x^2y - x + 4y^3 - 1$ and $g = 2xy + y^3$. If we use the *plex* ordering with $x > y$, then $LT(g) = 2xy$, $ax^\alpha = 6x^2y$, and $f \xrightarrow{g} h$, where

$$h = f - \frac{6x^2y}{2xy} g = -3xy^3 - x + 4y^3 - 1\,.$$

We can continue the process to get

$$f \xrightarrow{g} -3xy^3 - x + 4y^3 - 1 \xrightarrow{g} -x + \frac{3}{2}y^5 + 4y^3 - 1\,.$$

Since no term in the last polynomial is divisible by $LT(g) = 2xy$, the process cannot continue. If we use *grlex* ordering, with $x > y$, then $LT(g) = y^3$, and $ax^\alpha = 4y^3$ and $f \xrightarrow{g} h$, where

$$h = f - \frac{4y^3}{y^3} g = 6x^2y - 8xy - x - 1\,.$$

Here h cannot be reduced further since it does not contain any term which is divisible by $LT(g) = y^3$.

In the multivariate case we usually have to make reductions modulo many polynomials at a time. Thus, the following Definition is necessary:

Definition A.14 Let f, h, and $f_1, \ldots, f_s$ be polynomials in $K[x_1, \ldots, x_n]$, with $f_i \neq 0$, $(1 \leq i \leq s)$ and let $F = \{f_1, \ldots, f_s\}$. We say that f reduces to h modulo F, denoted $f \xrightarrow{F}{}_{+} h$, if and only if there exist a sequence of indices $i_1, i_2, \ldots, i_t \in 1, \ldots, s$ and a sequence of polynomials $h_1, \ldots, h_{t-1} \in K[x_1, \ldots, x_n]$ such that

$$f \xrightarrow{f_{i_1}} h_1 \xrightarrow{f_{i_2}} h_2 \xrightarrow{f_{i_3}} \cdots \xrightarrow{f_{i_{t-1}}} h_{t-1} \xrightarrow{f_{i_t}} h\,.$$

Example A.15 Let $F = \{f_1, f_2\}$, where $f_1 = yx - y$, $f_2 = y^2 - x$, and $f = y^2x$. Using *grlex* order we have $f \xrightarrow{F}_+ x$, since $y^2x \xrightarrow{f_1} y^2 \xrightarrow{f_2} x$.

Definition A.16 A polynomial r is said to be reduced with respect to a set of non-zero polynomials $F = f_1, ..., f_s$ if $r = 0$ or no power product that appears in r is divisible by any one of the $LT(f_i)$, $i = 1, ..., s$, i.e., r cannot be reduced modulo F.

Definition A.17 If $f \xrightarrow{F}_+ r$ and r is reduced with respect to F, then we call r a *remainder* for f with respect to F. The polynomial r is also called a normal form of f.

Usually an ordering among the polynomials in the set $f_1, ..., f_s$ is taken by choosing i to be the smallest integer such that $LT(f_i)$ divides $LT(f)$ [28]. The multivariate division procedure described above can be presented in the following algorithmic form.

> **Input:** $f, f_1, \ldots, f_s \in \mathsf{K}[\mathsf{x}_1, \ldots, \mathsf{x}_n]$ with $f_i \neq 0\ (1 \leq i \leq s)$
> **Output:** $u_1, ..., u_s, r$ such that $f = u_1 f_1 + \cdots + u_s f_s + r$ and
> r is reduced with respect to $\{f_1, \ldots, f_s\}$ and
> $\max(LM(u_1)LM(f_1), \ldots, LM(u_s)LM(f_s), LM(r)) = LM(f)$
> **Initial Values:** $u_1 := 0,\ u_2 := 0, \ldots, u_s := 0,\ r := 0,\ h := f$.
> **While** $h \neq 0$ do
>   **If** there exists i such that $LM(f_i)$ divides $LM(h)$ **then**
>   choose i least such that $LM(f_i)$ divides $LM(h)$
> $u_i := u_i + \frac{LT(h)}{LT(f_i)}$
> $h := h - \frac{LT(h)}{LT(f_i)} f_i$
>   else
> $r := r + LT(h)$
> $h := h - LT(h)$

This algorithm provides the proof for the following Theorem.

Theorem A.18 Given a set of non-zero polynomials $F = \{f_1, ..., f_s\}$ and f in $\mathsf{K}[\mathsf{x}_1, \ldots, \mathsf{x}_n]$, then $\exists u_1, ..., u_s, r \in \mathsf{K}[\mathsf{x}_1, \ldots, \mathsf{x}_n]$ such that $f = u_1 f_1 + \cdots + u_s f_s + r$ and either $r = 0$ or r is a completely reduced polynomial.

With f written as in the Theorem A.18, we have $f - r \in \langle f_1, ..., f_s \rangle$. Therefore, if $r = 0$, then f is in $\langle f_1, ..., f_s \rangle$. However, the converse is not necessarily true, i.e., we may have $f \in \langle f_1, ..., f_s \rangle$ and $r \neq 0$, as we can see in the following example:

Example A.19 Let $f = y^2x - x$, $F = \{f_1, f_2\}$, $I = \langle f_1, f_2 \rangle$, where $f_1 = yx - y$, $f_2 = y^2 - x$. Using *grlex* term with $y > x$, we obtain, according to the Division Algorithm, $f \xrightarrow{f_1} y^2 - x \xrightarrow{f_2} 0$. Thus $f \xrightarrow{F}_+ 0$, and indeed, $f = yf_1 + f_2$, so $f \in I$. However, if we start the reduction process with f_2 instead of f_1, we get $f \xrightarrow{F}_+ x^2 - x$, so the remainder of the division of f by $\{f_2, f_1\}$ is nonzero, but $f \in \langle f_1, f_2 \rangle$. This fact shows that the Multivariate Division Algorithm is of limited usefulness. In the next sections we describe how Gröbner-Buchberger theory deals with these difficulties.

A.4 Gröbner bases

Definition A.20 A set of non-zero polynomials $G = \{g_1, ..., g_t\}$ contained in an ideal I is called a *Gröbner basis* for I if and only if for all non-zero $f \in I$, there exists $i \in \{1, ..., t\}$ such that $LT(g_i)$ divides $LT(f)$. In other words, if G is a Gröbner basis for I, then there are no non-zero polynomials in I reduced with respect to G.

There are other useful characterizations of a Gröbner basis, which will be presented in the next Theorem. We first need the following definition:

Definition A.22 For a subset S of $\mathsf{K}[\mathsf{x}_1, \ldots, \mathsf{x}_n]$, the *leading term ideal* of S is defined as the ideal

$$Lt(S) = \langle LT(s) | s \in S \rangle . \tag{A.1}$$

Theorem A.23 Let I be a non-zero ideal of $\mathsf{K}[\mathsf{x}_1, \ldots, \mathsf{x}_n]$. The following are equivalent for a set of non-zero polynomials $G = \{g_1, \ldots, g_t\} \subseteq I$:

(i) G is a Gröbner basis for I,
(ii) $f \in I \iff f \xrightarrow{G}_+ 0$,
(iii) $f \in I \iff f = \sum_{i=1}^{t} h_i g_i$, with $LM(f) = \max_{1 \leq i \leq t}(LM(h_i) \cdot LM(g_i))$,
(iv) $Lt(G) = Lt(I)$.

Proof:

(i) $\Longrightarrow$ (ii). Let $f \in \mathsf{K}[\mathsf{x}_1, \ldots, \mathsf{x}_n]$. Then, by Theorem A.18, there exists $r \in \mathsf{K}[\mathsf{x}_1, \ldots, \mathsf{x}_n]$, reduced to G, such that $f \xrightarrow{G}_+ r$. Thus $f - r \in I$ and so $f \in I$ if and only if $r \in I$. If $r = 0$ then, of course, $f \in I$. Conversely, if $f \in I$ and $r \neq 0$ then $r \in I$ and, by (i), $\exists i \in \{1, ..., t\}$ such that $LM(g_i)$ divides $LM(r)$. this contradicts the fact that r is reduced with respect to G. Thus $r = 0$ and $f \xrightarrow{G}_+ 0$.
(ii) $\Longrightarrow$ (iii). Follows immediately from Theorem A.18.
(iii) $\Longrightarrow$ (iv). Clearly, $Lt(G) \subseteq Lt(I)$. To prove the reverse inclusion, we notice that from (iii) it follows that $LT(f) = \sum_i LT(h_i) \, LT(g_i)$, where the sum is over all i such that $LM(f) = LM(H_i) \, LM(g_i)$. This implies that for all $f \in I$, $LT(f) \in Lt(G)$ and thus, $Lt(I) \subseteq Lt(G)$.

(iv) $\Longrightarrow$ (i). For $f \in I$ we have $LT(f) \in Lt(G)$ and thus $LT(f) = \sum_{i=1}^{t} h_i LT(g_i)$ for some $h_i \in \mathsf{K}[x_1, \ldots, x_n]$. Expanding the right side of this equation it is clear that each term is divisible by some $LM(g_i)$. Thus $LT(f)$ is also divisible by some $LM(g_i)$, as required.

Corollary A.24 If $G = \{g_1, ..., g_i\}$ is a Gröbner basis for the ideal I, then $I = \langle g_1, ..., g_t \rangle$.

Proof: We must have $\langle g_1, ..., g_t \rangle \subseteq I$, since each g_i is in I. For the reverse inclusion, let $f \in I$. Then, by Theorem A.3, $f \xrightarrow{G}_+ 0$, which implies that $f \in \langle g_1, ..., g_t \rangle$.

Let us consider now the special case of ideals generated by monomials.

Lemma A.25 Let I be an ideal generated by a set S of nonzero terms, and let $f \in \mathsf{K}[x_1, \ldots, x_n]$. Then f is in I if and only if for every monomial X appearing in f there exists a monomial $Y \in S$ such that Y divides X. Moreover, there exists a finite subset S_0 of S such that $I = \langle S_0 \rangle$.

Proof: If $f \in I$ then $f = \sum_{i=1}^{l} h_i X_i$, where $h_i \in \mathsf{K}[x_1, \ldots, x_n]$ and $X_i \in S$, for $i = 1, ..., l$. Thus, every term of the left-hand side of this equation must be divisible by some $X_i \in S$. Conversely, if for every term X appearing in f there exists a term $Y \in \langle S \rangle$ such that Y divides X. For the last statement we note that, according to the Hilbert Basis Theorem (Theorem A.7), I has a finite generating set. By the first part of the the lemma, each term of each member of this generating set is divisible by an element od S. Thus, the finite set S_0 of such divisors is a generating set for $\langle I \rangle$.

Corollary A.26 Every non-zero ideal I of $\mathsf{K}[x_1, \ldots, x_n]$ has a Gröbner basis.

Proof: According to Lemma A.25, the leading term ideal $Lt(I)$ has a finite generating set of the form $\{LT(g_1), ..., LT(g_t)\}$, with $g_1, ..., g_t \in I$. If we let $G = \{g_1, ..., g_t\}$, then $Lt(G) = Lt(I)$. Thus, by Theorem A.23, G is a Gröbner basis.

In order to make use of a shorter terminology, from now on we say that a subset $G = \{g_1, ..., g_t\}$ of $\mathsf{K}[x_1, \ldots, x_n]$ is a *Gröbner basis* if and only if it is a Gröbner basis for the ideal $\langle G \rangle$ it generates.

Theorem A.27 Let $G = \{g_1, ..., g_t\}$ be a set of non-zero polynomials in $\mathsf{K}[x_1, \ldots, x_n]$. Then G is a Gröbner basis if and only if $\forall f \in \mathsf{K}[x_1, \ldots, x_n]$, the remainder of the division of f by G is unique.

The proof for this Theorem is straightforward but long, and can be found in [1].

Example A.28 Let us consider Example A.19 again, where $f = yx^2 - x$, $f_1 = yx - x$, $f_2 = y^2 - x$, and $F = \{f_1, f_2\}$. We have already verified that $f \xrightarrow{F}_+ 0$ or $f \xrightarrow{F}_+ x^2 - x$, depending of the order in which f_1 and f_2 are taken to make the reduction. Thus, according to Theorem A.27, F is not a Gröbner basis. There is another way to prove this. Since $f = yf_1 + f_2 \in \langle f_1, f_2 \rangle$, and $f \xrightarrow{F}_+ x^2 - x$, we have $x^2 - x \in \langle f_1, f_2 \rangle$. However, $LM(x^2 - x) = x^2$ is not divisible by either $LM(f1) = xy$ or $LM(f_2) = y^2$. Thus, by Definition A.20, F is not a Gröbner basis.

A.5 Buchberger's algorithm

The preceding section proved the existence of a Gröbner basis for an ideal I. We shall now be concerned with a method to find Gröbner bases. As we have already seen in Example A.19, the multivariate algorithm alone has limitated usefulness, since we can have $f \in I = \langle f_1, \ldots, f_s \rangle$, where the leading power products of f are not divisible by any $LM(f_i)$. Let us analyse this ambiguity in a more systematic way. Namely, in the division of f by $f_1, \ldots, f_s$, it may happen that some term X appearing in F is divisible by both $LM(f_i)$ and $LM(f_j)$, for $i \neq j$. If we reduce f using f_i we get the polynomial $h_1 = f - \frac{X}{LT(f_i)} f_i$, and if we reduce f using f_j, we get $h_2 = f - \frac{X}{LT(f_j)} f_j$. Thus, the introduced ambiguity is given by $h_2 - h_1 = \frac{X}{LT(f_i)} f_i - \frac{X}{LT(f_j)} f_j$. SInce X must be also divisible by $L = LCM(LM(f_i), LM(f_j))$, where LCM denotes the "least common product", we can rewrite this equation as $h_2 - h_1 = \frac{X}{L}(\frac{L f_i}{LT(f_i)} - \frac{L f_j}{LT(f_j)}) = \frac{X}{L} S(f_i, f_j)$, where $S(f_i, f_j) = \frac{L f_i}{LT(f_i)} - \frac{L f_j}{LT(f_j)}$. This leads us to the following definition.

Definition A.29 Let $0 \neq f, g \in \mathsf{K}[\mathsf{x}_1, \ldots, \mathsf{x}_n]$, and let $L = LCM(LM(f), LM(g))$. We define the *S-polynomial* of f and g as

$$S(f_i, f_j) := \frac{L f_i}{LT(f_i)} - \frac{L f_j}{LT(f_j)}. \tag{A.2}$$

Theorem A.30 Let $G = \{g_1, \ldots, g_t\}$ be a set of non-zero polynomials in $\mathsf{K}[\mathsf{x}_1, \ldots, \mathsf{x}_n]$. Then G is a Gröbner basis for the ideal $I = \langle g_1, \ldots, g_t \rangle$ if and only if for all $i \neq j$,

$$S(f_i, f_j) \xrightarrow{G}_+ 0. \tag{A.3}$$

In order to prove this theorem, we need to introduce the following lemma:

Lemma A.31 Let $f_1, \ldots, f_s \in \mathsf{K}[\mathsf{x}_1, \ldots, \mathsf{x}_n]$ be such that $LM(f_i) = X \neq 0$, $\forall i = 1, \ldots, s$. Let $f = \sum_{i=1}^{s} c_i f_i$, with $c_i \in \mathsf{K}$, $\mathsf{i} = 1, \ldots, \mathsf{s}$. If $LM(f) < X$, then f is a linear combination, with coefficients in K, of $S(f_i, f_j)$, $1 \leq i \leq j \leq s$.

Proof: Let $f_1 = a_i X +$lower terms, $a_i \in \mathsf{K}$. Then the hypothesis asserts that $\sum_{i=1}^{s} c_i a_i = 0$, since $\sum_{i=1}^{s} c_i (a_i X + \text{lower terms})$ and $LM(f) < X$. Since $LM(f_i) = LM(f_j) = X$, the

S-polynomials will take the form $S(f_i, f_j) := \frac{1}{a_i} f_i - \frac{1}{a_j} f_j$. Thus

$$f = c_1 f_1 + \cdots + c_s f_s = c_1 a_1 (\frac{1}{a_1} f_1) + \cdots + c_s a_s (\frac{1}{a_s} f_s)$$
$$= c_1 a_1 (\frac{1}{a_1} f_1) - \frac{1}{a_2} f_2) + (c_1 a_1 + c_2 a_2)(\frac{1}{a_2} f_2) - \frac{1}{a_3} f_3) + \cdots$$
$$+(c_1 a_1 + \cdots + c_{s-1} a_{s-1})(\frac{1}{a_{s-1}} f_{s-1}) - \frac{1}{a_s} f_s) + (c_1 a_1 + \cdots c_s a_s)\frac{1}{a_s} f_s$$
$$= c_1 a_1 S(f_1, f_2) + (c_1 a_1 + c_2 a_2) S(f_2, f_3) + \cdots + (c_1 a_1 + \cdots + c_{s-1} a_{s-1}) S(f_{s-1}, f_s).$$

Proof of Theorem A.30: If $G = g_1, ..., g_t$ is a Gröbner basis for $I = \langle g_1, ..., g_t \rangle$, then, by Theorem A.23 (ii), $S(g_i, g_j) \xrightarrow{G}_+ 0$, $\forall i \neq j$, since $S(g_i, g_j) \in I$. Conversely, let us assume that $S(g_i, g_j) \xrightarrow{G}_+ 0$, $\forall i \neq j$. Let $f \in I$. Then it can be written as $f = \sum_{i=1}^t h_i g_i$, with $X = \max_{1 \leq i \leq t}(LM(h_i) LM(g_i))$. According to Theorem A.23 (iii), if we prove that $X = LM(f)$, then we are done. Let us assume, on the contrary, that $LM(f) < X$. Let $S = \{i | LM(h_i) LM(g_i) = X\}$. For $i \in S$ we can write $h_i = c_1 X_i +$ lower terms. Define $g = \sum_{i \in S} c_i X_i g_i$. Then $LM(X_i g_i) = X$, $\forall i \in S$, and $LM(g) < X$ (since $LM(f) < X$). By Lemma A.31, there exist $d_{ij} \in \mathsf{K}$ such that $g = \sum_{i,j \in S, i \neq j} d_{ij} S(X_i g_i, X_j, g_j)$. Since $X = LCM(LM(X_i g_i), LM(X_j g_j))$, we have

$$S(X_i g_i, X_j, g_j) = \frac{X}{LT(X_i g_i)} X_i g_i - \frac{X}{LT(X_j g_j)} X_j g_j$$
$$= \frac{X}{LT(g_i)} g_i - \frac{X}{LT(g_j)} g_j = \frac{X}{X_{ij}} S(g_i, g_j),$$

where $X_{ij} = LCM(LM(g_i), LM(g_j))$. Since $S(g_i, g_j) \xrightarrow{G}_+ 0$, we have $S(X_i g_i, X_j g_j) \xrightarrow{G}_+ 0$. Thus, by Theorem A.23 (iii) we can write

$$S(X_i g_i, X_j g_j) = \sum_{\nu=1}^t h_{ij\nu} g_\nu \tag{A.4}$$

where

$$\max_{1 \leq \nu \leq t}(LT(h_{ij\nu}) LT(g_\nu) = LT(S(X_i g_i, X_j g_j) < \max(LT(X_i g_i), LT(X_j g_j)) = X. \tag{A.5}$$

However,

$$f = \sum_{i=1}^t h_i g_i = \sum_{i=1, i \notin S}^t (\text{lower terms}) g_i + \sum_{i \in S} c_i X_i g_i,$$

where

$$\sum_{i \in S} c_i X_i g_i = g = \sum_{i,j \in S, i \neq j} d_{ij} S(X_i g_i, X_j, g_j) = \sum_{\nu=1}^t \sum_{i,j \in S, i \neq j} d_{ij} h_{ij\nu} g_\nu. \tag{A.6}$$

Thus,

$$f = \sum_{i=1}^{t} h'_j g_j \,. \tag{A.7}$$

Thus, from (A.4), (A.5), (A.6) and (A.7), we obtain, finally,

$$\max_{1 \le j \le t}(LT(H'_j)LT(g_j)) < X \,, \tag{A.8}$$

which is a contradiction.

Buchberger's Theorem provides a strategy for computing Gröbner bases: reduce the S-polynomials and, if a remainder is non-zero, add this remainder to the list of polynomials in the generating set. Continue this until there are enough polynomials to make all S-polynomials reduce to zero. In other words, given a set $F = \{f_1, ..., f_t\}$, $f_j \in \mathsf{K}[x_1, ..., x_s]$, we start testing F by checking whether $S(f_i, f_j) \xrightarrow{F}_+ 0$, $\forall f_i,\ f_j \in F$, $i \neq j$. If we find a pair (f_i, f_j) such that $S(f_i, f_j) \xrightarrow{F}_+ r \neq 0$, then $\langle F \rangle = \langle F, r \rangle$ (since $S(f_i, f_j) \in \langle F \rangle$), and $S(f_i, f_j) \xrightarrow{F \cup \{r\}}_+ 0$. The procedure is repeated for all pairs (f_i, f_j) formed in the updated generating set, until the process terminates.

Example A.32 Let $G = \{g_1, g_2\}$ with $g_1 = 4x^2z - 7y^2$, $g_2 = xyz^2 + 3xz^4$. We use plex ordering with $x > y > z$. Thus, in the first step we get

$$S(g_1, g_2) = 12x^2z^4 - 7y^3z \xrightarrow{G}_+ -21z^3y^2 - 7y^3z = g_3 \,.$$

The generating set is now $G = \{g_1, g_2, g_3\}$. The S-polynomials to be considered now are $S(g_1, g_3)$ and $S(g_2, g_3)$. If we pick the first pair we get $S(g_1, g_3) \xrightarrow{G}_+ 49y^5 + 1323z^6y^2 = g_4$, and G is updated to $G = \{g_1, g_2, g_3, g_4\}$. The S-polynomials to be considered now are $S(g_2, g_3)\,, S(g_1, g_4)\,, S(g_2, g_4)$, and $S(g_3, g_4)$. If we compute each of these S-polynomials and reduce them modulo G, we find that they all reduce to zero. Hence, $G = \{g_1, g_2, g_3, g_4\}$ is a Gröbner basis. Buchberger's Algorithm for computing Gröbner basis is as follows:

Input: $F = \{f, , f_1, ..., f_s\} \subseteq \mathsf{K}[x_1, \ldots, x_n]$ with $f_i \neq 0\,(1 \le i \le s)$
Output: $G := \{g_1, ..., g_t\}$, a Gröbner basis for $\langle f_1, ..., f_s \rangle$
Initialization:$G := F$, $M := \{\{f_i, f_j\} | f_i \neq f_j \in G\}$
While $M \neq 0$ **do**
Choose any $\{f, g\} \in M$
$M := M - \{\{f, g\}\}$
$S(f, g) \xrightarrow{G}_+ h$
If $h \neq 0$ **then**
$M ; = M \cup \{\{u, h\} | \forall u \in G\}$
$G := G \cup \{h\}$

We can show that this algorithm terminates. Let us suppose the contrary. Then, as the algorithm progresses, we obtain a strictly increasing infinite sequence $G_1 \subsetneq G_2 \subsetneq \cdots$. Each G_i is obtained from G_{i-1}, by adding some $h \in I$ to G_{i-1}, where h is the non-zero reduction, with respect to G_{i-1}, of an S-polynomial formed by a pair of elements of G_{i-1}. Since h is reduced with respect to G_{i-1}, we have $LT(h) \notin Lt(G_{i-1})$. Thus, $Lt(G_1) \subsetneq Lt(G_2) \subsetneq \cdots$. This is a strictly ascending chain of ideals which contradicts the Hilbert Basis Theorem.

Example A.33 Let $F = \{g_1, g_2, g_3\}$, where $g_1 = x^2z - y^2$, $g_2 = yz^2 + z$, $g_3 = y - z$. We use plex ordering with $x > y > z$.
Initialization:$G = F = \{g_1, g_2, g_3\}$, $M = \{(g_1, g_2), (g_1, g_3), \{g_2, g_3)\}$.
First pass through the **while** loop:
Choose the pair (g_1, g_2)
$M = \{(g_1, g_2), (g_2, g_3)\}$
$S(g_1, g_2) = x^2z + y^3z \xrightarrow{G}_+ z^4 + z^2 = h \neq 0$
$M = \{(g_1, g_3), (g_2, g_3), (g_4, g_1), (g_4, g_2), (g_4, g_3)\}$
$G = \{g_1, g_2, g_3, g_4\}$
Second pass through the **while** loop:
Choose the pair (g_1, g_3)
$M = \{(g_2, g_3), (g_4, g_1), (g_4, g_2), (g_4, g_3)\}$
$S(g_1, g_3) = xz^2 - y^3 \xrightarrow{G}_+ 0 = h$
Choose the pair (g_2, g_3)
$M = \{(g_4, g_1), (g_4, g_2), (g_4, g_3)\}$
$S(g_2, g_3) = z + z^3 \xrightarrow{G}_+ z^3 + z = h \neq 0$
$M = \{(g_4, g_1), (g_4, g_2), (g_4, g_3), (g_5, g_1, (g_5, g_2), (g_5, g_3), (g_5, g_4)\}$
$G = \{g_1, g_2, g_3, g_4, g_5\}$
Third pass through the **while** loop:
All pairs of M reduce to zero modulo G
$M = 0$
$G = \{g_1, g_2, g_3, g_4, g_5\}$
where $g_1 = x^2z - y^2$, $g_2 = yz^2 + z$, $g_3 = y - z$, $g_4 = z^4 + z^2$, $g_5 = z^3 + z$.

From this example we observe the following facts:

1. The computed Gröbner basis G is not reduced:

   -$g_4 = zg_5$, and therefore the ideal generated by G does not change. Thus g_4 can be eliminated.
   -$LT(g_3)$ divides $LT(g_2)$, so g_3 can also be eliminated.

2. There are some reductions of S-polynomials that are unnecessary. For example, the

fact that $LT(g_1) = x^2z$ and $LT(g_3) = y$ are relatively prime assures us that $S(g_1, g_3)$ reduces to zero without carrying out the reduction.

These facts give a hint for some improvements to Buchberger's algorithm which will be be described in the next section.

A.6 Improved Gröbner bases

As we saw in the previous section, Buchberger's algorithm consists basically of two steps: the computation of S-polynomials and their reduction. A possible problem that might arise is the possibly very large number of S-polynomials that must be computed and reduced. As the computation progresses the number of polynomials in the basis gets larger, and therefore, each time a new polynomial is added to the basis, the number of S-polynomials to compute also increases. Since the proportion of S-polynomials which reduce to zero eventually increases as we get far in the algorithm, a huge amount of computation might be performed for very little gain, since few new polynomials are added to the basis. Indeed, at some point, the computation of S-polynomials and their reductions are useless except for the fact that they verify that we really have a Gröbner basis. In order to improve this situation one has to find a way to predict which S-polynomials reduce to zero without actually computing or reducing them.

A starting point towards the desired improvements is given by the following theorem:

Theorem A.34 Let f, $g \in \mathsf{K}[\mathsf{x}_1, ..., \mathsf{x}_s]$ be non-zero polynomials. Then the following statements are equivalent:

> (i)$LCM(LM(f), LM(g)) = LM(f)LM(g)$, i.e., $LM(f)$ and $LM(g)$ are relatively prime.
> (ii)$S(f, g) \xrightarrow{G}_+ 0$.
> In particular, $\{f, g\}$ is a Gröbner basis if and only if $LM(f)$ and $LM(g)$ are relatively prime.

A proof for this theorem can be find in [1], p.125, although it is formulated there in a slightly different form.

This theorem gives a criterion for a priori zero reduction: during Buchberger's algorithm, whenever f and g are such that $LM(f)$ and $LM(g)$ are relatively prime, it is not necessary to compute $S(f, g)$, since it will reduce to zero, and therefore will not create a new polynomial in the basis. The next criterion is based on the following theorem.

Theorem A.35 G is a Gröbner basis if and only if $\forall f, g \in G$ either

(i) $S(f,g) \xrightarrow{G}_+ 0$, or (ii) $\exists\, h \in G$, $f \neq g$, such that
$LM(h)$ divides $LCM(LM(f), LM(g))$,
$S(f,h) \xrightarrow{G}_+ 0$, $S(g,h) \xrightarrow{G}_+ 0$.

The proof is given in [35] , p.444 .

This theorem implies that if there is an element f_k of the basis such that $LM(f_k)$ divides $LCM(LM(f_i, LM(f_j))$ and if $S(f_i, f_k)$ and $S(f_j, f_k)$ have already been considered, then $S(f_i, f_j)$ reduces to zero and can be ignored.

A third criterion was proposed by Buchberger and Winkler [17] : In the process of selecting a pair $\{f_i, f_j\}$, choose one such that $LCM(LM(f_i; LM(f_j))$ has the minimal degree among the pairs.

Another useful improvement is the reduction criterion divides $LM(g_j)$, then g_i can be deleted from the basis. We can carry this out by reducing all polynomials in the basis with respect to each other. Each time a new polynomial is adjoined to the basis, all the other polynomials may be reduced using the new polynomial. Such reductions initiate a whole cascade of reductions and cancellations. If this process is carried out systematically, the resulting basis will be reduced.

We can summarize these criteria as follows:

$Criterion1\,(f_i, f_j) \Longleftrightarrow LCM(LM(f_i, LM(f_j))$ is of minimal degree.
$Criterion2\,(f_i, f_j) \Longleftrightarrow LCM(LM(f_i, LM(f_j)) \neq LM(f_i) \cdot LM(f_j)$.
$Criterion3\,(f_i, f_j, G) \Longleftrightarrow \exists f_k \in G$ such that $f_i \neq f_k \neq f_j$, $LCM(LM(f_i, LM(f_j))$ is a multiple of $LM(f_k)$, and $S(f_j, f_k)$ have already been considered.
$Criterion4\,(G) \Longleftrightarrow$ keep G reduced.

The modified Buchberger's algorithm is given below.

Input: $F = \{f, , f_1, ..., f_s\} \subseteq \mathsf{K}[\mathsf{x}_1, \ldots, \mathsf{x}_n]$ with $f_i \neq 0\,(1 \leq i \leq s)$
Output: $G := \{g_1, ..., g_t\}$, a Gröbner basis for $\langle f_1, ..., f_s \rangle$
Initialization:$G := F$, reduce(G) $M := \{\{f_i, f_j\} | f_i \neq f_j \in G\,, 1 \leq i \leq j \leq\}\,, t = s$
While $M \neq 0$ **do**
    $\{f_i, f_j\}$:= a pair in M satisfying $Criterion1(f_i, f_j) = true$
    If ($Criterion2(f_i, f_j) = true$ **and** $Criterion3(f_i, f_j, G) = false$) **then**
        $S(f_i, f_j) \xrightarrow{G}_+ h$
        If $h \neq 0$ **then**
            $t := t + 1$
            $f_t := h$
            $M := Mv \cup \{\{f_i, f_t\} \forall 1 \leq i \leq t - 1\}$
            $G := G \cup \{f_t\}$
        $M := M - \{\{f_1, f_j\}\}$
        $G := Reduce(G)$

Example A.36 Let us consider again Example A.33, where $F = \{g_1, g_2, g_3\}$, $g_1 = x^2z - y^2$, $g_2 = yz^2 + z$, $g_3 = y - z$, using plex ordering with $x > y > z$.
Initialization: $G = F = \{g_1, g_2, g_3\}$. Since $LM(g_3)$ divides $LM(g_2)$, we detect that $g_2 \xrightarrow{\{g_1, g_3\}}_+ g_4 = z^3 + z$. Thus, $G = \{g_1, g_3, g_4\}$ and $M = \{(g_1, g_3), (g_1, g_4), (g_3, g_4)\}$.
First pass through the **while** loop:
Since $LCM(LT(g_1), LT(g_3)) = 1$ has the minimum degree, we choose (g_1, g_3). However, $LT(g_1)$ and $LT(g_3)$ are relatively prime. Thus, by *Criterion2*, this pair can be ignored. For the same reason, the next pair, (g_3, g_4), can also be deleted, and the only remaining pair is (g_1, g_4). Thus $S(g_1, g_4) = x^2z + y^2z^2 \xrightarrow{G}_+ 0$. The algorithm then stops, and the Gröbner basis for the ideal generated by F is, therefore, $G = \{g_1, g_3, g_4\} = \{x^2z - z^2, y - z, z^3 + z\}$.

Thus, in comparison with its predecessor, the improved version of Buchberger's algorithm provided a reduced Gröbner basis, and instead of ten reductions we needed only three.

Appendix B

Spinor Formalism

B.1 The NP formalism

We shall use here the two-component spinor formalism of Penrose [67] [69] [24] and the spin-coefficient formalism of Newman and Penrose [64], whose conventions [1] we follow. In the spinor formalism tensor and spinor indices are related by the complex connection quantities $\sigma_a{}^{A\dot{A}}$ $(a = 0, \ldots, 3\,; A = 0, 1)$ which are usually chosen to be Hermitian in the spinor indices A, $\dot{A}$, and satisfy

$$\sigma_a{}^{A\dot{A}}\sigma^a{}_{B\dot{B}} = \delta^A_B \delta^{\dot{A}}_{\dot{B}}\,, \qquad \sigma^a{}_{A\dot{B}}\sigma_b{}^{A\dot{B}} = \delta^a{}_b\,. \tag{B.1}$$

In these equations spinor indices have been lowered by the Levi-Civita symbols defined by

$$\varepsilon_{AB} = \varepsilon^{AB} = \varepsilon_{[AB]}\,,\ \varepsilon_{01} = 1\,, \tag{B.2}$$

with the conventions

$$\xi_A = \xi^B \varepsilon_{BA}\,, \qquad \xi^B = \varepsilon^{BA}\xi_A\,. \tag{B.3}$$

Thus

$$\varepsilon^{AC}\varepsilon_{BC} = \delta^A_B = -\delta_B{}^A\,, \tag{B.4}$$

To every tensor $T_{c\ldots}{}^{ab\cdots}$ we can therefore associate an equivalent spinor defined by the relation:

$$T_{C\dot{C}\ldots}{}^{AB\dot{A}\dot{B}\cdots} = \sigma_a{}^{A\dot{A}}\sigma_b{}^{B\dot{B}}\sigma^c{}_{C\dot{C}}\cdots T_{c\ldots}{}^{ab\cdots}\,. \tag{B.5}$$

It can be verified that the spinor equivalent of a real tensor is Hermitian.

[1] See the conventions in Appendix D.

The correspondence between tensors and spinors will be denoted in the following way:

$$T_{c\cdots}{}^{ab\cdots} \leftrightarrow T_{C\dot{C}\cdots}{}^{AB\dot{A}\dot{B}\cdots}. \tag{B.6}$$

The invariant associated with a 4-vector ξ^a, expressed in terms of the metrics g_{ab}, ε_{AB} and $\varepsilon_{\dot{A}\dot{B}}$ of the space-time and spinor space, has the form

$$\xi_a\xi^a = g_{ab}\xi^a\xi^b = \xi_{A\dot{B}}\xi^{A\dot{B}} = \varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}}\xi^{A\dot{B}}\xi^{C\dot{D}}. \tag{B.7}$$

Using $\xi^{A\dot{B}} = \sigma_a{}^{A\dot{B}}\xi^a$ in the above equation we obtain

$$g_{ab} = \varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}}\sigma_a{}^{A\dot{C}}\sigma_b{}^{B\dot{D}}. \tag{B.8}$$

B.2 The dyad formalism

We can set up, at each point of space-time, an *orthonormal dyad basis* $\zeta_{(a)}{}^A$ and $\zeta_{(\dot{a})}{}^{\dot{A}}$ (a, $\dot{a}$ = 0,1 and A, $\dot{A}$ = 0,1) for spinors in the same way as we set up an orthonormal tetrad basis $e^b{}_{(a)}$ (a, b = 0,1,2,3) for tensors in a tetrad formalism. The dyad indices are included in parentheses. It is convenient, however, to define special symbols for the two basis spinors in the following way

$$\zeta_{(0)}{}^A = o^A, \qquad \zeta_{(1)}{}^A = \iota^A. \tag{B.9}$$

The condition of orthonormality is given by

$$o_A\iota^A = \varepsilon_{AB}o^A\iota^B = o^0\iota^1 - o^1\iota^0 = -o^A\iota_A = 1. \tag{B.10}$$

This implies that

$$\varepsilon_{AB}\zeta_{(a)}{}^A\zeta_{(b)}{}^B = \zeta_{(a)B}\zeta_{(b)}{}^B = -\zeta_{(a)}{}^B\zeta_{(b)B} = \varepsilon_{(a)(b)} \tag{B.11}$$

and

$$\varepsilon^{(a)(b)}\zeta^A{}_{(a)}\zeta^B{}_{(b)} = \zeta^A{}_{(0)}\zeta^B{}_{(1)} - \zeta_{(1)}{}^A\zeta_{(0)}{}^B = o^A\iota^B - \iota^B\iota^A = \varepsilon^{AB}. \tag{B.12}$$

As in the tetrad formalism, we can project any spinor ξ_A onto the dyad basis:

$$\xi_{(a)} = \xi_A\zeta^A{}_{(a)}, \tag{B.13}$$

or

$$\xi_{(0)} = \xi_A o^A, \qquad \xi_{(1)} = \xi_A\iota^A. \tag{B.14}$$

The spinors o^A and ι^A and their complex conjugates determine the null tetrad $(\mathbf{l}\,\mathbf{n}\,\mathbf{m}\,\overline{\mathbf{m}})$ by the correspondence

$$l^a \leftrightarrow o^A\overline{o}^{\dot{B}}\,, \quad m^a \leftrightarrow o^A\overline{\iota}^{\dot{B}}\,, \quad \overline{m^a} \leftrightarrow \iota^A\overline{o}^{\dot{B}}\,, \quad n^a \leftrightarrow \iota^A\overline{\iota}^{\dot{B}}\,. \tag{B.15}$$

These null vectors satisfy the orthogonality conditions

$$l^a n_a = o^A\overline{o}^{\dot{B}}\iota_A\overline{\iota}_{\dot{B}} = 1\,, \quad m^a\overline{m}_a = o^A\overline{\iota}^{\dot{B}}\iota_A\overline{o}_{\dot{B}} = -1\,, \tag{B.16}$$

while all the remaining inner products are zero. We can determine, from (B.15), the Hermitian representation for the matrices $\sigma_a{}^{A\dot{B}}$ and $\sigma^a{}_{A\dot{B}}$. In a dyad basis we have

$$\sigma^a{}_{(0)(\dot{0})} = \sigma^a{}_{A\dot{B}}\zeta^A{}_{(0)}\overline{\zeta}^{\dot{B}}{}_{(\dot{0})} = \sigma^a{}_{A\dot{B}}o^A\overline{o}^{\dot{B}} = l^a\,, \tag{B.17}$$

$$\sigma^a{}_{(0)(\dot{1})} = \sigma^a{}_{A\dot{B}}\zeta^A{}_{(0)}\overline{\zeta}^{\dot{B}}{}_{(\dot{1})} = \sigma^a{}_{A\dot{B}}o^A\overline{\iota}^{\dot{B}} = m^a\,, \tag{B.18}$$

$$\sigma^a{}_{(1)(\dot{0})} = \sigma^a{}_{A\dot{B}}\zeta^A{}_{(1)}\overline{\zeta}^{\dot{B}}{}_{(\dot{0})} = \sigma^a{}_{A\dot{B}}\iota^A\overline{o}^{\dot{B}} = \overline{m}^a\,, \tag{B.19}$$

$$\sigma^a{}_{(1)(\dot{1})} = \sigma^a{}_{A\dot{B}}\zeta^A{}_{(1)}\overline{\zeta}^{\dot{B}}{}_{(\dot{1})} = \sigma^a{}_{A\dot{B}}\iota^A\overline{\iota}^{\dot{B}} = n^a\,. \tag{B.20}$$

Thus

$$\sigma^a{}_{(A)(\dot{B})} = \begin{pmatrix} l^a & m^a \\ \overline{m}^a & n^a \end{pmatrix}\,, \quad \sigma_a{}^{(A)(\dot{B})} = \begin{pmatrix} n^a & -\overline{m}^a \\ -m^a & l^a \end{pmatrix}\,. \tag{B.21}$$

The metric tensor, according to (B.8) and (B.21), is given by

$$g_{ab} = \varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}}\sigma_a{}^{A\dot{C}}\sigma_b{}^{B\dot{D}} = 2n_{(a}l_{b)} - 2m_{(a}\overline{m}_{b)}\,. \tag{B.22}$$

Thus, the null vectors $(l^a, n^a, m^a, \overline{m}^a)$, determined from the dyads, are the same as those originally defined in the Newman-Penrose (NP) spin coëfficient formalism.

B.3 Covariant derivative of spinors

The covariant differentiation of spinor fields can be uniquely defined by the following postulates:

(i) It must satisfy the correspondence relations

$$\nabla_a \leftrightarrow \nabla_{A\dot{B}}\,, \tag{B.23}$$

$$\nabla_a X_b = X_{b;a} \leftrightarrow \nabla_{A\dot{B}}X_{C\dot{D}} = X_{C\dot{D};A\dot{B}}\,, \tag{B.24}$$

or

$$X_{C\dot{D};A\dot{B}} = \sigma^j{}_{C\dot{D}}\sigma^i{}_{A\dot{B}}X_{j;i}\,. \tag{B.25}$$

(ii) The covariant differentiation of spinor fields satisfies the Leibnitz rule:

$$\nabla_{A\dot{B}}\left(S^{\cdots}{}_{\cdots}\times T^{\cdots}{}_{\cdots}\right)=T^{\cdots}{}_{\cdots}\nabla_{A\dot{B}}\left(S^{\cdots}{}_{\cdots}\right)+S^{\cdots}{}_{\cdots}\nabla_{A\dot{B}}\left(T^{\cdots}{}_{\cdots}\right). \tag{B.26}$$

where $S^{\cdots}{}_{\cdots}$ are any two spinor fields.

(iii) The operator $\nabla_{A\dot{B}}$ is real, i.e.,

$$\nabla_{A\dot{B}}=\overline{\nabla}_{\dot{A}B}. \tag{B.27}$$

In a way analogous to the case of the tetrad formalism, we define the *intrinsic derivative* of the dyadic component $\xi_{(c)}$ of a spinor along the "direction" $(a)(\dot{b})$ by

$$\xi_{(c)|(a)(\dot{b})}=\zeta_{(c)}{}^{C}\xi_{C;A\dot{B}}\zeta_{(a)}{}^{A}\zeta_{(\dot{b})}{}^{\dot{B}}, \tag{B.28}$$

or

$$\xi_{(c)|A\dot{B}}=\zeta_{(c)}{}^{C}\xi_{C;A\dot{B}}. \tag{B.29}$$

From (B.25), (B.1) and the Leibnitz rule it follows that

$$\sigma^{a}{}_{C\dot{D};A\dot{B}}=0,\qquad \sigma_{a}{}^{C\dot{D}}{}_{;A\dot{B}}=0. \tag{B.30}$$

Also, from (B.24), the Leibnitz rule for tensor and spinor fields, and (B.8) it follows that

$$\varepsilon_{CD;A\dot{B}}=0. \tag{B.31}$$

The *spin coefficients*, $\Gamma_{(a)(b)(c)(\dot{d})}$, are defined in the dyad formalism by

$$\Gamma_{(a)(b)(c)(\dot{d})}:=[\zeta_{(a)F}]_{;C\dot{D}}\zeta_{(b)}{}^{F}\zeta_{(c)}{}^{C}\zeta_{(\dot{d})}{}^{\dot{D}}, \tag{B.32}$$

or

$$\Gamma_{(a)(b)CD}:=[\zeta_{(a)F}]_{;C\dot{D}}\zeta_{(b)}{}^{F}. \tag{B.33}$$

The first two indices of the above spin coefficients are symmetric, as it can be immediately verified by applying the covariant derivative to (B.11) and by using the definition (B.33) together with (B.31). We now use the preceeding definitions to find the explicit expression of intrinsic derivative of the dyadic components of spinors of first rank, in terms of the spin coefficients. Thus,

$$\xi_{(a)|B\dot{C}}=\xi_{(a);B\dot{C}}=[\xi_{A}\zeta_{(a)}{}^{A}]_{;B\dot{C}}-\xi_{A}[\zeta_{(a)}{}^{A}]_{;B\dot{C}}. \tag{B.34}$$

The first term inside the square brackets is the scalar $\xi_{(a)}$. To find an expression for the second term we notice that, using (B.11), we obtain from (B.33):

$$[\zeta_{(a)E}]_{;C\dot{D}}=-\zeta^{(b)}{}_{E}\Gamma_{(b)(a)C\dot{D}}=\zeta_{(b)E}\Gamma^{(b)}{}_{(a)C\dot{D}}. \tag{B.35}$$

Thus

$$\xi_{(a)|B\dot{C}} = \xi_{(a),B\dot{C}} - \xi_A \Gamma^{(d)}{}_{(a)B\dot{C}} \zeta_{(d)}{}^A = \xi_{(a),B\dot{C}} + \Gamma_{(d)(a)B\dot{C}} \zeta^{(d)} . \tag{B.36}$$

Since the spin coefficients are symmetric in the two first indices, there are twelve independent components. To these coefficients are assigned special symbols described in the Table B.3.

	$(a)(b)$	00	01	11
$(c)(\dot{d})$				
$0\dot{0}$		κ	ϵ	π
$1\dot{0}$		ρ	α	λ
$0\dot{1}$		σ	β	μ
$1\dot{1}$		τ	γ	ν

Table B.1: NP spin coefficients

It can be verified that these definitions of the spin coefficients agree with those of the Ricci spin-rotation γ_{abc} defined in the NP formalism [24].

B.4 The basic equations of the NP formalism

The spinor equivalents of the tensors (2.119), (2.118), (2.121) and (2.120), which appear in the necessary conditions $I - VII$ are given by [69]

$$C_{abcd} \leftrightarrow \Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{D}\dot{C}} + \overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}\varepsilon_{AB}\varepsilon_{DC} , \tag{B.37}$$

$$H_{ab} \leftrightarrow 2(\phi_{AB}\varepsilon_{\dot{A}\dot{B}} + \overline{\phi}_{\dot{A}\dot{B}}\varepsilon_{AB} , \tag{B.38}$$

$$L_{ab} \leftrightarrow 2(\Phi_{AB\dot{A}\dot{B}} - \Lambda\varepsilon_{AB}\varepsilon_{\dot{A}\dot{B}}) , \tag{B.39}$$

$$S_{abc} \leftrightarrow \Psi^D{}_{ABC;D\dot{A}}\varepsilon_{\dot{C}\dot{B}} + \overline{\Psi}^{\dot{D}}{}_{\dot{A}\dot{B}\dot{C};\dot{D}A}\varepsilon_{CB} , \tag{B.40}$$

where $\Psi_{ABCD} = \Psi_{(ABCD)}$, is the Weyl spinor and $\Lambda := (1/24)R$. $\phi_{AB} = \phi_{(AB)}$ is called the Maxwell spinor, and $\Phi_{AB\dot{A}\dot{B}} = \Phi_{(AB)(\dot{A}\dot{B})} = \overline{\Phi}_{AB\dot{A}\dot{B}}$ is called the trace free Ricci spinor (since it is the spinor equivalent of the trace-free Ricci tensor $R_{ab} - \frac{1}{4}Rg_{ab}$). As we shall see, the basic equations of the NP formalism can be expressed in terms of spinors defined above and their derivatives.

The NP components of the Weyl tensor are defined as follows

$$\left.\begin{aligned}
\Psi_0 &:= \Psi_{(0)(0)(0)(0)} = \zeta_{(0)}{}^A\zeta_{(0)}{}^B\zeta_{(0)}{}^C\zeta_{(0)}{}^D\Psi_{ABCD} = \Psi_{ABCD}o^{ABCD}\,,\\
\Psi_1 &:= \Psi_{(0)(0)(0)(1)} = \zeta_{(0)}{}^A\zeta_{(0)}{}^B\zeta_{(0)}{}^C\zeta_{(1)}{}^D\Psi_{ABCD} = \Psi_{ABCD}o^{ABC}\iota^D\,,\\
\Psi_2 &:= \Psi_{(0)(0)(1)(1)} = \zeta_{(0)}{}^A\zeta_{(0)}{}^B\zeta_{(1)}{}^C\zeta_{(1)}{}^D\Psi_{ABCD} = \Psi_{ABCD}o^{AB}\iota^{CD}\,,\\
\Psi_3 &:= \Psi_{(0)(1)(1)(1)} = \zeta_{(0)}{}^A\zeta_{(1)}{}^B\zeta_{(1)}{}^C\zeta_{(1)}{}^D\Psi_{ABCD} = \Psi_{ABCD}o^{A}\iota^{BCD}\,,\\
\Psi_4 &:= \Psi_{(1)(1)(1)(1)} = \zeta_{(1)}{}^A\zeta_{(1)}{}^B\zeta_{(1)}{}^C\zeta_{(1)}{}^D\Psi_{ABCD} = \Psi_{ABCD}\iota^{ABCD}\,,
\end{aligned}\right\} \tag{B.41}$$

or

$$\begin{aligned}
\Psi_{ABCD} = {} & \Psi_0\iota_{ABCD} - 4\Psi_1 o_{(A}\iota_{BCD)} + 6\Psi_2 o_{(AB}\iota_{CD)}\\
& -4\Psi_3 o_{(ABC}\iota_{D)} + \Psi_4 o_{ABCD}\,.
\end{aligned} \tag{B.42}$$

In (B.41) and (B.42) we have used the notation $o_{AB\ldots} := o_A o_B \cdots$.

The NP components of the trace-free Ricci spinor are given by

$$\left.\begin{aligned}
\Phi_{00} &:= \Phi_{AB\dot A\dot B}o^{AB}\bar o^{\dot A\dot B}\,, & \Phi_{01} &:= \Phi_{AB\dot A\dot B}o^{AB}\bar o^{\dot A}\bar\iota^{\dot B}\,,\\
\Phi_{02} &:= \Phi_{AB\dot A\dot B}o^{AB}\bar\iota^{\dot A\dot B}\,, & \Phi_{11} &:= \Phi_{AB\dot A\dot B}o^{A}\iota^{B}\bar o^{\dot A}\bar\iota^{\dot B}\,,\\
\Phi_{12} &:= \Phi_{AB\dot A\dot B}o^{A}\iota^{B}\bar\iota^{\dot A\dot B}\,, & \Phi_{22} &:= \Phi_{AB\dot A\dot B}\iota^{AB}\bar\iota^{\dot A\dot B}\,,
\end{aligned}\right\} \tag{B.43}$$

or, equivalently,

$$\begin{aligned}
\Phi_{AB\dot A\dot B} = {} & \Phi_{22}o_{AB}\bar o_{\dot A\dot B} - 2\Phi_{21}o_{AB}\bar o_{(\dot A}\bar\iota_{\dot B)} - 2\Phi_{12}o_{(A}\iota_{B)}\bar o_{\dot A\dot B}\\
& +\Phi_{20}o_{AB}\bar\iota_{\dot A\dot B} + \Phi_{02}\iota_{AB}\bar\iota_{\dot A\dot B} + 4\Phi_{11}o_{(A}\iota_{B)}\bar o_{(\dot A}\bar\iota_{\dot B)}\\
& -2\Phi_{10}o_{(A}\iota_{B)}\bar o_{(\dot A}\bar\iota_{\dot B)} - 2\Phi_{01}\iota_{AB}\bar\iota_{\dot A\dot B}\,.
\end{aligned} \tag{B.44}$$

For the Maxwell spinor ϕ_{AB} we have

$$\left.\begin{aligned}
\phi_0 &= \phi_{(0)(0)} = \zeta_{(0)}{}^A\zeta_{(0)}{}^B\phi_{AB} = o^{AB}\phi_{AB}\,,\\
\phi_1 &= \phi_{(0)(1)} = \phi_{(1)(0)} = \zeta_{(0)}{}^A\zeta_{(0)}{}^B\phi_{AB} = o^{A}\iota^{B}\phi_{AB}\,,\\
\phi_2 &= \phi_{(1)(1)} = \zeta_{(1)}{}^A\zeta_{(1)}{}^B\phi_{AB} = \iota^{AB}\phi_{AB}\,,
\end{aligned}\right\} \tag{B.45}$$

or

$$\phi_{AB} = \phi_0\iota_{AB} - 2\phi_1 o_{(A}\iota_{B)} + \phi_2 o_{AB}\,. \tag{B.46}$$

The covariant derivatives of the dyad basis spinors can be obtained from (B.33), together with definition of the NP spin coefficient symbols defined on Table B.3:

$$o_{A;B\dot B} = o_A I_{B\dot B} + \iota_A II_{B\dot B}\,, \quad \iota_{A;B\dot B} = o_A III_{B\dot B} - \iota_A I_{B\dot B}\,, \tag{B.47}$$

where

$$\left.\begin{aligned}
I_{B\dot B} &:= \gamma o_B\bar o_{\dot B} - \alpha o_B\bar\iota_{\dot B} - \beta\iota_B\bar o^{\dot B} + \varepsilon\iota_B\bar\iota_{\dot B}\,,\\
II_{B\dot B} &:= -\tau o_B\bar o_{\dot B} + \rho o_B\bar\iota_{\dot B} + \sigma\iota_B\bar o^{\dot B} - \kappa\iota_B\bar\iota_{\dot B}\,,\\
III_{B\dot B} &:= \nu o_B\bar o_{\dot B} - \lambda o_B\bar\iota_{\dot B} - \mu\iota_B\bar o^{\dot B} + \pi\iota_B\bar\iota_{\dot B}\,.
\end{aligned}\right\} \tag{B.48}$$

The NP differential operators can be defined as being the dyad components of $\partial_{A\dot{B}}$:

$$\left.\begin{aligned}
\mathrm{D} &:= \partial_{(0)(\dot{0})} = \zeta_{(0)}{}^{A}\overline{\zeta}_{(\dot{0})}{}^{\dot{B}}\partial_{A\dot{B}} = l^a\partial_a\,,\\
\delta &:= \partial_{(0)(\dot{1})} = \zeta_{(0)}{}^{A}\overline{\zeta}_{(\dot{1})}{}^{\dot{B}}\partial_{A\dot{B}} = m^a\partial_a\,,\\
\overline{\delta} &:= \partial_{(1)(\dot{0})} = \zeta_{(1)}{}^{A}\overline{\zeta}_{(\dot{0})}{}^{\dot{B}}\partial_{A\dot{B}} = \overline{m}^a\partial_a\,,\\
\Delta &:= \partial_{(1)(\dot{1})} = \zeta_{(1)}{}^{A}\overline{\zeta}_{(\dot{1})}{}^{\dot{B}}\partial_{A\dot{B}} = n^a\partial_a\,.
\end{aligned}\right\} \tag{B.49}$$

These operators give rise to the following commutation relations

$$[\Delta, D] = (\gamma+\overline{\gamma})D + (\epsilon+\overline{\epsilon})\Delta - (\overline{\tau}+\pi)\delta - (\tau+\overline{\pi})\overline{\delta}\,, \tag{B.50}$$

$$[\delta, D] = (\overline{\alpha}+\beta-\overline{\pi})D + \kappa\Delta - (\overline{\rho}+\epsilon-\overline{\epsilon})\delta + \sigma\overline{\delta}\,, \tag{B.51}$$

$$[\delta, \Delta] = -\overline{\nu}D + (\tau-\overline{\alpha}-\beta)\Delta + (\mu-\gamma+\overline{\gamma})\delta + \overline{\lambda}\overline{\delta}\,, \tag{B.52}$$

$$[\delta, \overline{\delta}] = (\overline{\mu}-\mu)D + (\overline{\rho}-\rho)\Delta + (\alpha-\overline{\beta})\delta + (\beta-\overline{\alpha})\overline{\delta}\,. \tag{B.53}$$

The Ricci identity

$$X_{ab;[cd]} = \frac{1}{2}(R^{k}{}_{acd}X_{kb} + R^{k}{}_{bcd}X_{ak})\,, \tag{B.54}$$

has the following spinor form

$$\nabla_{(B\dot{A}}\nabla_{A)}{}^{\dot{A}}\xi_C = -\Psi_{ABCD}\xi^D + 2\Lambda\xi_{(A}\epsilon_{B)C}\,, \tag{B.55}$$

$$\nabla^{C}{}_{(\dot{B}}\nabla_{C\dot{A})}\xi_A = \Phi_{AB\dot{A}\dot{B}}\xi^B\,, \tag{B.56}$$

where ξ^A is an arbitrary 1-spinor. It is easier to count independent components in the spinor formalism than in tetrads or coordinates. Identity (B.55) clearly contains six complex components, while identity (B.56) has six real ones.

The Bianchi identity

$$R_{ab[dc;e]} = 0\,, \tag{B.57}$$

has the spinor form

$$\nabla^{A}{}_{\dot{A}}\Psi_{ABCD} = \nabla_{(B}{}^{\dot{E}}\Phi_{CD)\dot{A}\dot{E}}\,, \tag{B.58}$$

$$\nabla^{B\dot{B}}\Phi_{AB\dot{A}\dot{B}} = -3\nabla_{A\dot{A}}\Lambda\,. \tag{B.59}$$

Clearly, equation (B.58) possesses eight complex components and (B.59) three real components.

The Ricci and Bianchi identities can be written explicitly in terms of twenty nine dyadic components, by using (B.42), (B.44), (B.47) and (B.49). They are listed in Appendix C. Together with the commutation relations (B.50- B.53), they constitute the basic equations of the Newman-Penrose formalism. In general, however, it is not clear what these equations are for and in what sense they replace Einstein's equation or are equivalent to it [24].

B.5 Petrov classification

Let $\varphi_{A_1A_2\ldots A_p} = \varphi_{(A_1A_2\ldots A_p)}$ be a symmetric p-spinor, and let ζ^{A_i} be an arbitrary 1-spinor. Then the expression

$$\varphi(\zeta) := \varphi_{A_1A_2\ldots A_p}\zeta^{A_1}\cdots\zeta^{A_p} \tag{B.60}$$

is a homogeneous polynomial of degree p in ζ^1 and ζ^2. By the Fundamental Theorem of Algebra, this polynomial can be factored into p linear factors:

$$\varphi(\zeta) = (\overset{1}{\alpha}_{A_1}\,\zeta^{A_1})\cdots(\overset{p}{\alpha}_{A_p}\,\zeta^{A_p})\,. \tag{B.61}$$

Thus,

$$(\varphi_{A_1A_2\ldots A_p} - \overset{1}{\alpha}_{A_1}\cdots\overset{p}{\alpha}_{A_p})\zeta^{A_1}\cdots\zeta^{A_p} = 0\,. \tag{B.62}$$

Since ζ^{A_i} is arbitrary, we have

$$\varphi_{A_1\ldots A_p} = \overset{1}{\alpha}_{(A_1}\cdots\overset{p}{\alpha}_{A_p)}\,, \tag{B.63}$$

i.e., a p-spinor is decomposable into p 1-spinors. Since the Weyl spinor is totally symmetric we must have

$$\Psi_{ABCD} = \overset{1}{\alpha}_{(A}\overset{2}{\alpha}_{B}\overset{3}{\alpha}_{C}\overset{4}{\alpha}_{D)}\,, \tag{B.64}$$

Each of the 1-spinors $\overset{r}{\alpha}_A$ is called a principal spinor of Ψ_{ABCD} and their corresponding null vectors are the principal null vectors (or directions). Space-times in which all four principal null directions of the Weyl spinor are not distinct are called degenerate or algebraically special. The Petrov type is defined according to the following table:

Petrov type	*Form of* Ψ_{ABCD}	*Eq. satisfied by* Ψ_{ABCD}
I	$\overset{1}{\alpha}_{(A}\overset{2}{\alpha}_{B}\overset{3}{\alpha}_{C}\overset{4}{\alpha}_{D)}$	$\Psi_{ABCD}\,\overset{1}{\alpha}{}^{B}\,\overset{1}{\alpha}{}^{C}\,\overset{1}{\alpha}{}^{D} = \lambda\,\overset{1}{\alpha}_{A}$
II	$\overset{1}{\alpha}_{(A}\overset{1}{\alpha}_{B}\overset{2}{\alpha}_{C}\overset{3}{\alpha}_{D)}$	$\Psi_{ABCD}\,\overset{1}{\alpha}{}^{C}\,\overset{1}{\alpha}{}^{D} = \lambda\,\overset{1}{\alpha}_{A}\overset{1}{\alpha}_{B}$
D	$\overset{1}{\alpha}_{(A}\overset{1}{\alpha}_{B}\overset{2}{\alpha}_{C}\overset{2}{\alpha}_{D)}$	$\Psi_{ABCD}\,\overset{1}{\alpha}{}^{C}\,\overset{1}{\alpha}{}^{D} = \lambda\,\overset{1}{\alpha}_{A}\overset{1}{\alpha}_{B}$
III	$\overset{1}{\alpha}_{(A}\overset{1}{\alpha}_{B}\overset{1}{\alpha}_{C}\overset{2}{\alpha}_{D)}$	$\Psi_{ABCD}\,\overset{1}{\alpha}{}^{D} = \lambda\,\overset{1}{\alpha}_{A}\overset{1}{\alpha}_{B}\overset{1}{\alpha}_{C}$
N	$\overset{1}{\alpha}_{(A}\overset{1}{\alpha}_{B}\overset{1}{\alpha}_{C}\overset{1}{\alpha}_{D)}$	$\Psi_{ABCD} = \overset{1}{\alpha}_{A}\overset{1}{\alpha}_{B}\overset{1}{\alpha}_{C}\overset{1}{\alpha}_{D}$

Table B.2: Petrov classification

where the 1-spinors $\overset{1}{\alpha}_A$, $\overset{2}{\alpha}_A$, $\overset{3}{\alpha}_A$ and $\overset{4}{\alpha}_A$ are distinct, and $\lambda \neq 0$.

B.6 Dyad transformations

In establishing the necessary conditions for the validity of Huygens' principle, it is usual to take advantage of the freedom to choose a null basis that is best suited for the calculations. The proofs for the results presented in this and in the next Section can be found in [79] [61]. The tetrads transform as follows:

$$l' = e^{a} l\,, \quad m' = e^{ib}(m + \bar{q} l) \quad n' = e^{-a}(n + qm + \overline{qm} + q\bar{q} l)\,, \tag{B.65}$$

where a and b are real-valued functions and q is a complex-valued function. The corresponding transformation of the spinor dyad $\{o, \iota\}$ is given by

$$o' = e^{w/2} o\,, \quad \iota' = e^{-w/2}(\iota + q o)\,. \tag{B.66}$$

where $w = a + ib$. This induces the following transformations on the covariant derivatives of the dyad:

$$o'_{A;B\dot{B}} = e^{w/2}(o_{A;B\dot{B}} + \frac{1}{2} o_A w_{B,\dot{B}})\,, \tag{B.67}$$

$$\iota'_{A;B\dot{B}} = e^{w/2}\left(\iota_{A;B\dot{B}} + q o_{A;B\dot{B}} + o_A(q_{B\dot{B}} - \frac{1}{2} q w_{B\dot{B}}) - \frac{1}{2}\iota_A w_{B\dot{B}}\right)\,. \tag{B.68}$$

Combining (B.67), (B.68) with (B.47), we find

$$I'_{B\dot{B}} = I_{B\dot{B}} + \frac{1}{2} w_{B\dot{B}} - q II_{B\dot{B}}\,, \tag{B.69}$$

$$II'_{B\dot{B}} = e^{w} II_{B\dot{B}}\,, \tag{B.70}$$

$$III'_{B\dot{B}} = e^{-w}(III_{B\dot{B}} + 2q_{B\dot{B}} - q^2 II_{B\dot{B}} + q_{B\dot{B}})\,. \tag{B.71}$$

The NP spin coefficients can now be obtained by contracting the above equations, using (B.48), with the appropriate pairs of basis spinors. The result is as follows:

$$\kappa' = e^{(3w+\bar{w})/2}\kappa\,, \tag{B.72}$$

$$\sigma' = e^{(3w-\bar{w})/2}(\sigma + \bar{q}\kappa)\,, \tag{B.73}$$

$$\rho' = e^{(w+\bar{w})/2}(\rho + q\kappa)\,, \tag{B.74}$$

$$\tau' = e^{(w-\bar{w})/2}(\tau + q\sigma + \bar{q}\rho + q\bar{q}\kappa)\,, \tag{B.75}$$

$$\begin{aligned}\nu' = e^{-(3w+\bar{w})/2}\Big(&\nu + \Delta q + q(\delta q + \mu + 2\gamma) + \bar{q}(\lambda + \bar{\delta} q) \\ &+ q\bar{q}(Dq + \pi + 2\alpha) + q^2(\tau + 2\beta) + q^2\bar{q}(\rho + 2\epsilon) + q^3\bar{q}\kappa\Big)\,,\end{aligned} \tag{B.76}$$

$$\lambda' = e^{(w-3\bar{w})/2}\left(\lambda + \bar{\delta} q + q(Dq + \pi + 2\alpha) + q^2(\rho + 2\epsilon) + q^3\kappa\right)\,, \tag{B.77}$$

$$\mu' = e^{-(w+\bar{w})/2}\left(\mu + \delta q + 2q\beta + q^2\sigma + \bar{q}(D(q + \pi) + 2q\bar{q}\epsilon + q^2\bar{q}\kappa\right)\,, \tag{B.78}$$

$$\pi' = e^{(w-\overline{w})/2}\left(\pi + 2q\epsilon + q^2\kappa + Dq\right), \tag{B.79}$$

$$\gamma' = e^{-(w-\overline{w})/2}\left(\gamma + \frac{1}{2}\Delta w + q(\tau + \beta + \frac{1}{2}\delta w) + \overline{q}(\alpha + \frac{1}{2}\overline{\delta} w)\right.$$
$$\left. + q\overline{q}(\rho + \epsilon + \frac{1}{2}Dw) + q^2\sigma + q^2\overline{q}\kappa\right), \tag{B.80}$$

$$\alpha' = e^{(w-\overline{w})/2}\left(\beta + \frac{1}{2}\delta w + q\sigma + \overline{q}(\epsilon + \frac{1}{2}Dw) + q\overline{q}\kappa\right), \tag{B.81}$$

$$\beta' = e^{(w-\overline{w})/2}\left(\beta + \frac{1}{2}\delta w) + q\sigma + \overline{q}(\epsilon + \frac{1}{2}Dw) + q\overline{q}\kappa\right), \tag{B.82}$$

$$\epsilon' = e^{(w+\overline{w})/2}\left(\epsilon + \frac{1}{2}Dw + q\kappa\right). \tag{B.83}$$

The transformation laws for the NP components of the Weyl spinor are obtained by contracting Ψ_{ABCD} with the transformed basis spinors $o^{A'}$ and $\iota^{A'}$. The result is:

$$\Psi_0' = e^{2w}\Psi_0\,, \tag{B.84}$$
$$\Psi_1' = e^{2w}(\Psi_1 + q\Psi_0)\,, \tag{B.85}$$
$$\Psi_2' = \Psi_2 + 2q\Psi_1 + q^2\Psi_0\,, \tag{B.86}$$
$$\Psi_3' = e^{-w}(\Psi_3 + 3q\Psi_2 + 3q^2\Psi_1 + q^3\Psi_0)\,, \tag{B.87}$$
$$\Psi_4' = e^{-2w}(\Psi_4 + 4q\Psi_3 + 6q^2\Psi_2 + 4q^3\Psi_1 + q^4\Psi_0)\,. \tag{B.88}$$

The NP components of the trace-free Ricci spinor are obtained by contracting $\Phi_{AB\dot{A}\dot{B}}$ with the basis spinors $o^{A'}$, $\iota^{B'}$ and its complex conjugates:

$$\Phi_{00}' = e^{w+\overline{w}}\Phi_{00}\,, \tag{B.89}$$
$$\Phi_{01}' = e^{w}(\Phi_{01} + \overline{q}\Phi_{00})\,, \tag{B.90}$$
$$\Phi_{02}' = e^{w-\overline{w}}(\Phi_{02} + 2\overline{q}\Phi_{01} + \overline{q}^2\Phi_{00})\,, \tag{B.91}$$
$$\Phi_{11}' = \Phi_{11} + q\Phi_{01} + \overline{q}\Phi_{10} + q\overline{q}\Phi_{00}\,, \tag{B.92}$$
$$\Phi_{12}' = e^{-\overline{w}}(\Phi_{12} + q\Phi_{02} + 2\overline{q}\Phi_{11} + 2q\overline{q}\Phi_{01} + q^2\Phi_{10} + q\overline{q}\Phi_{00})\,, \tag{B.93}$$
$$\Phi_{22}' = e^{-(w+\overline{w})}(\Phi_{22} + 2q\Phi_{12} + 2\overline{q}\Phi_{21} + 4q\overline{q}\Phi_{11} + q^2\Phi_{02} \tag{B.94}$$
$$\overline{q}^2\Phi_{20} + 2q^2\overline{q}\Phi_{01} + 2q\overline{q}^2\Phi_{10} + q^2\overline{q}^2\Phi_{00}\,. \tag{B.95}$$

The NP operators transform as follows:

$$D' = o^{A'}\overline{o}^{\dot{A}'}\nabla_{A\dot{A}} = e^{(w+\overline{w})/2}D\,, \tag{B.96}$$
$$\delta' = o^{A'}\overline{\iota}^{\dot{A}'}\nabla_{A\dot{A}} = e^{(w-\overline{w})/2}(\delta + \overline{q}D)\,, \tag{B.97}$$
$$\Delta' = o^{A'}\overline{o}^{\dot{A}'}\nabla_{A\dot{A}} = e^{-(w+\overline{w})/2}(\Delta + \overline{q}\overline{\delta} + q\delta + q\overline{q}D)\,. \tag{B.98}$$

B.7 Conformal transformations

The trivial transformation corresponding to the conformal transformation of the metric:

$$\tilde{g}_{ab} = e^{2\phi} g_{ab} \,, \tag{B.99}$$

is induced by the following transformation of the null tetrad [79] [61]:

$$\tilde{l}_a = e^{r\phi} l_a \qquad \tilde{n}_a = e^{(2-r)\phi} n_a \,, \qquad \tilde{m}_a = e^{\phi} m_a \,. \tag{B.100}$$

The transformation laws for the spin coefficients now are:

$$\left.\begin{array}{ll}
\tilde{\alpha} = e^{-\phi}\left(\alpha + \frac{1}{2}(r-2)\right) \,, & \tilde{\beta} = e^{-\phi}\left(\beta + \frac{r}{2}\delta\phi\right) \,, \\
\tilde{\sigma} = e^{(r-2)\phi}\sigma \,, & \tilde{\epsilon} = e^{(r-2)\phi}\left(\epsilon + \frac{r}{2}D\phi\right) \,, \\
\tilde{\pi} = e^{-\phi}\left(\pi + \bar{\delta}\phi\right) \,, & \tilde{\lambda} = e^{-r\phi}\lambda \,, \\
\tilde{\gamma} = e^{-r\phi}\left(\gamma + \frac{1}{2}(r-2)\Delta\phi\right) \,, & \tilde{\nu} = e^{(1-2r)\phi}\nu \,, \\
\tilde{\tau} = e^{-\phi}\left(\tau - \delta\phi)\right) \,, & \tilde{\mu} = e^{-r\phi}\left(\mu + \Delta\phi\right) \,, \\
\tilde{\kappa} = e^{-r\phi}\kappa \,, & \tilde{\rho} = e^{(r-2)\phi}\left(\rho - D\phi\right) \,. \\
\tilde{\gamma} = e^{-r\phi}\left(\gamma + \frac{1}{2}(r-2)\Delta\phi\right) \,, &
\end{array}\right\} \tag{B.101}$$

where r is a parameter associated to the transformation of the basis spinors in the following way:

$$\tilde{o}^A = e^{s\phi} o^A \,, \;\; \tilde{\iota}^A = e^{-s\phi}\iota^A \,, \tag{B.102}$$

$$s := \frac{r-1}{2} \,. \tag{B.103}$$

The Weyl spinor components transform as follows:

$$\left.\begin{array}{lll}
\tilde{\Psi}_0 = e^{2(r-2)\phi}\Psi_0 \,, & \tilde{\Psi}_1 = e^{(r-3)\phi}\Psi_1 \,, & \tilde{\Psi}_2 = e^{-2\phi}\Psi_2 \,, \\
\tilde{\Psi}_3 = e^{-(r+1)\phi}\Psi_3 & \tilde{\Psi}_4 = e^{-2r\phi}\Psi_4 \,. &
\end{array}\right\} \tag{B.104}$$

For the NP components of the trace-free Ricci spinor $\Phi_{AB\dot{A}\dot{B}}$ we get

$$\tilde{\Phi}_{00} = e^{2(r-2)\phi}\left(\Phi_{00} - D^2\phi + (D\phi)^2 + (\epsilon + \bar{\epsilon})D\phi - \bar{\kappa}\delta\phi - \kappa\bar{\delta}\phi\right) \,, \tag{B.105}$$

$$\begin{aligned}
\tilde{\Phi}_{01} = e^{(r-3)\phi}\Bigg(&\Phi_{01} - \frac{1}{2}(D\delta\phi + \delta D\phi) - \frac{1}{2}(\bar{\epsilon} - \epsilon + \bar{\rho})\delta\phi \\
&+ \frac{1}{2}(\bar{\pi} + \bar{\alpha} + \beta)D\phi - \frac{1}{2}\kappa\Delta\phi - \frac{1}{2}\sigma\bar{\delta}\phi + \delta\phi D\phi\Bigg) \,,
\end{aligned} \tag{B.106}$$

$$\tilde{\Phi}_{11} = e^{-2\phi}\left(\Phi_{11} - \frac{1}{4}(D\Delta\phi + \Delta D\phi + \delta\bar{\delta}\phi + \bar{\delta}\delta\phi)\right.$$

$$-\frac{1}{4}(\epsilon+\bar{\epsilon}+\rho+\bar{\rho})\Delta\phi+\frac{1}{4}(\bar{\pi}+\bar{\alpha}-\tau-\beta)\bar{\delta}\phi$$
$$+\frac{1}{4}(\pi+\alpha-\bar{\tau}-\bar{\beta})\delta\phi\Big) \tag{B.107}$$

$$\tilde{\Phi}_{12}=e^{(r-3)\phi}\left(\Phi_{12}-\frac{1}{2}(\Delta\delta\phi+\delta\Delta\phi)+\frac{1}{2}(\mu+\gamma-\bar{\gamma})\delta\phi\right.$$
$$\left.-\frac{1}{2}(\bar{\alpha}+\tau+\beta)\Delta\phi+\delta\phi D\phi+\frac{1}{2}\bar{\nu}D\phi+\frac{1}{2}\overline{\lambda\delta\phi}\right), \tag{B.108}$$

$$\tilde{\Phi}_{22}=e^{2(r-2)\phi}\left(\Phi_{22}-\Delta^2\phi+(\Delta\phi)^2-(\gamma+\bar{\gamma})\Delta\phi+\nu\bar{\delta}\phi+\bar{\nu}\delta\phi\right). \tag{B.109}$$

The Maxwell tensor H_{ab} is invariant under conformal transformations [60]:

$$\tilde{H}_{ab}=H_{ab}. \tag{B.110}$$

Thus, using the correspondence (B.38),

$$\tilde{\phi}_{AB}\varepsilon_{\dot{A}\dot{B}}+\bar{\tilde{\phi}}_{\dot{A}\dot{B}}\varepsilon_{AB}=\tilde{\sigma}^a{}_{A\dot{A}}\tilde{\sigma}^b{}_{B\dot{B}}\tilde{H}_{ab}=e^{-2\phi}(\phi_{AB}\varepsilon_{\dot{A}\dot{B}}+\bar{\phi}_{\dot{A}\dot{B}}\varepsilon_{AB}), \tag{B.111}$$

or

$$\tilde{\phi}_{AB}=e^{-2\phi}\phi_{AB}. \tag{B.112}$$

Thus, the NP components of the Maxwell spinor transform as follows:

$$\left.\begin{aligned}\tilde{\phi}_0&=\tilde{\phi}_{AB}\tilde{o}^A\tilde{o}^B=e^{-2\phi}\phi_{AB}e^{(r-1)\phi}o^Ao^B=e^{(r-3)\phi}\phi_0,\\ \tilde{\phi}_1&=e^{-2\phi}\phi_1,\\ \tilde{\phi}_2&=e^{-(r+1)\phi}\phi_2.\end{aligned}\right\} \tag{B.113}$$

Appendix C

Newman-Penrose Field Equations

C.1 Bianchi identities

$$
\begin{aligned}
&(NP1)\ \ \mathrm{D}\rho - \overline{\delta}\kappa = \rho^2 + \sigma\overline{\sigma} + (\epsilon + \overline{\epsilon})\rho - \overline{\kappa}\tau - (3\alpha + \overline{\beta} - \pi)\kappa + \Phi_{00},\\
&(NP2)\ \ \mathrm{D}\sigma - \delta\kappa = (\rho + \overline{\rho})\sigma + (3\epsilon - \overline{\epsilon})\sigma - (\tau - \overline{\pi} + \overline{\alpha} + 3\beta)\kappa + \Psi_0,\\
&(NP3)\ \ \mathrm{D}\tau - \Delta\kappa = (\tau + \overline{\pi})\rho + (\overline{\tau} + \pi)\sigma + (\epsilon - \overline{\epsilon})\tau - (3\gamma + \overline{\gamma})\kappa + \Psi_1 + \Phi_{01},\\
&(NP4)\ \ \mathrm{D}\alpha - \overline{\delta}\epsilon = (\rho + \overline{\epsilon} - 2\epsilon)\alpha + \beta\overline{\sigma} - \overline{\beta}\epsilon - \kappa\lambda - \overline{\kappa}\gamma + (\epsilon + \rho)\pi + \Phi_{10},\\
&(NP5)\ \ \mathrm{D}\beta - \delta\epsilon = (\alpha + \pi)\sigma + (\overline{\rho} - \overline{\epsilon})\beta - (\mu + \gamma)\kappa + (\overline{\pi} - \overline{\alpha})\epsilon + \Psi_1,\\
&(NP6)\ \ \mathrm{D}\gamma - \Delta\epsilon = (\tau + \overline{\pi})\alpha + (\overline{\tau} + \pi)\beta - (\epsilon + \overline{\epsilon})\gamma - (\gamma + \overline{\gamma})\epsilon + \tau\pi - \nu\kappa\\
&\qquad\qquad + \Psi_2 - \Lambda + \Phi_{11},\\
&(NP7)\ \ \mathrm{D}\lambda - \overline{\delta}\pi = \rho\lambda + \overline{\sigma}\mu + \pi^2 + (\alpha - \overline{\beta})\pi - \nu\overline{\kappa} + (\overline{\epsilon} - 3\epsilon)\lambda + \Phi_{20},\\
&(NP8)\ \ \mathrm{D}\mu - \delta\pi = \overline{\rho}\mu + \sigma\lambda + \pi\overline{\pi} - (\epsilon + \overline{\epsilon})\mu - (\overline{\alpha} - \beta)\pi - \nu\kappa + \Psi_2 + 2\Lambda,\\
&(NP9)\ \ \mathrm{D}\nu - \Delta\pi = (\overline{\tau} + \pi)\mu + (\tau + \overline{\pi})\lambda + (\gamma - \overline{\gamma})\pi - (3\epsilon + \overline{\epsilon})\nu + \Psi_3 + \Phi_{21},\\
&(NP10)\ \ \Delta\lambda - \overline{\delta}\nu = (\overline{\gamma} - 3\gamma - \mu - \overline{\mu})\lambda + (3\alpha + \overline{\beta} + \pi - \overline{\tau})\nu - \Psi_4,\\
&(NP11)\ \ \delta\rho - \overline{\delta}\sigma = (\overline{\alpha} + \beta)\rho - (3\alpha - \overline{\beta})\sigma + (\rho - \overline{\rho})\tau + (\mu - \overline{\mu})\kappa - \Psi_1 + \Phi_{01},
\end{aligned}
$$

$$
\begin{aligned}
&(NP12)\ \ \delta\alpha - \overline{\delta}\beta = \mu\rho - \sigma\lambda + \alpha\overline{\alpha} + \beta\overline{\beta} - 2\alpha\beta + (\rho - \overline{\rho})\gamma + (\mu - \overline{\mu})\epsilon\\
&\qquad\qquad - \Psi_2 + \Lambda + \Phi_{11},\\
&(NP13)\ \ \delta\lambda - \overline{\delta}\mu = (\rho - \overline{\rho})\nu + (\mu - \overline{\mu})\pi + (\alpha + \overline{\beta})\mu + (\overline{\alpha} - 3\beta)\lambda - \Psi_3 + \Phi_{21},\\
&(NP14)\ \ \delta\nu - \Delta\mu = \mu^2 + \lambda\overline{\lambda} + (\gamma + \overline{\gamma})\mu - \overline{\nu}\pi + (\tau - \overline{\alpha} - 3\beta)\nu + \Phi_{22},\\
&(NP15)\ \ \delta\gamma - \Delta\beta = (\tau - \overline{\alpha} - \beta)\gamma + \mu\tau - \sigma\nu - \epsilon\overline{\nu} - (\gamma - \overline{\gamma} - \mu)\beta + \alpha\overline{\lambda} + \Phi_{12},\\
&(NP16)\ \ \delta\tau - \Delta\sigma = \mu\sigma + \overline{\lambda}\rho + (\tau - \overline{\alpha} + \beta)\tau - (3\gamma - \overline{\gamma})\sigma - \kappa\overline{\nu} + \Phi_{02},\\
&(NP17)\ \ \Delta\rho - \overline{\delta}\tau = -\rho\overline{\mu} - \sigma\lambda + (\gamma + \overline{\gamma})\rho - (\overline{\tau} + \alpha - \overline{\beta})\tau + \nu\kappa - \Psi_2 - 2\Lambda,\\
&(NP18)\ \ \Delta\alpha - \overline{\delta}\gamma = (\epsilon + \rho)\nu - (\tau + \beta)\lambda + (\overline{\gamma} - \overline{\mu})\alpha + (\overline{\beta} - \overline{\tau})\gamma - \Psi_3 .
\end{aligned}
$$

C.2 Ricci identities

$$(NP19)\ \bar\delta\Psi_0 - D\Psi_1 + D\Phi_{01} - \delta\Phi_{00} = (4\alpha-\pi)\Psi_0 - 2(2\rho+\epsilon)\Psi_1 + 3\kappa\Psi_2 + (\bar\pi - 2\bar\alpha - 2\beta)\Phi_{00} + 2(\epsilon+\bar\rho)\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \bar\kappa\Phi_{02},$$

$$(NP20)\ \Delta\Psi_0 - \delta\Psi_1 + D\Phi_{02} - \delta\Phi_{01} = (4\gamma-\mu)\Psi_0 - 2(2\tau+\beta)\Psi_1 + 3\sigma\Psi_2 - \bar\lambda\Phi_{00} + 2(\bar\pi-\beta)\Phi_{01} + 2\sigma\Phi_{11} + (2\epsilon - 2\bar\epsilon + \bar\rho)\Phi_{02} - 2\kappa\Phi_{12},$$

$$(NP21)\ 3\bar\delta\Psi_1 - 3D\Psi_2 + 2D\Phi_{11} - 2\delta\Phi_{10} + \bar\delta\Phi_{01} - \Delta\Phi_{00} = 3\lambda\Psi_0 - 9\rho\Psi_2 + 6(\alpha-\pi)\Psi_1 + 6\kappa\Psi_3 + (\bar\mu - 2\mu - 2\gamma - 2\bar\gamma)\Phi_{00} + (2\alpha + 2\pi + 2\bar\tau)\Phi_{01} + 2(\tau - 2\bar\alpha + \bar\pi)\Phi_{10} + 2(2\bar\rho - \rho)\Phi_{11} + 2\sigma\Phi_{20} - \bar\sigma\Phi_{02} - 2\bar\kappa\Phi_{12} - 2\kappa\Phi_{21},$$

$$(NP22)\ 3\Delta\Psi_1 - 3\delta\Psi_2 + 2D\Phi_{12} - 2\delta\Phi_{11} + \bar\delta\Phi_{02} - \Delta\Phi_{01} = 3\nu\Psi_0 + 6(\gamma-\mu)\Psi_1 - 9\tau\Psi_2 + 6\sigma\Psi_3 - \bar\nu\Phi_{00} + 2(\bar\mu - \mu - \gamma)\Phi_{01} - 2\bar\lambda\Phi_{10} + 2(\tau + 2\bar\pi)\Phi_{11} + (2\alpha + 2\pi + \bar\tau - 2\bar\beta)\Phi_{02} + (2\bar\rho - 2\rho - 4\bar\epsilon)\Phi_{12} + 2\sigma\Phi_{21} - 2\kappa\Phi_{22},$$

$$(NP23)\ 3\bar\delta\Psi_2 - 3D\Psi_3 + D\Phi_{21} - \delta\Phi_{20} + 2\bar\delta\Phi_{11} - 2\Delta\Phi_{10} = 6\lambda\Psi_1 - 9\pi\Psi_2 + 6(\epsilon-\rho)\Psi_3 + 3\kappa\Psi_4 - 2\nu\Phi_{00} + 2(\bar\mu - \mu - 2\bar\gamma)\Phi_{10} + (2\pi + 4\bar\tau)\Phi_{11} + (2\beta + 2\tau + \bar\pi - 2\bar\alpha)\Phi_{20} - 2\bar\sigma\Phi_{12} + 2(\bar\rho - \rho - \epsilon)\Phi_{21} - \bar\kappa\Phi_{22} + 2\lambda\Phi_{01},$$

$$(NP24)\ 3\Delta\Psi_2 - 3\delta\Psi_3 + D\Phi_{22} - \delta\Phi_{21} + 2\bar\delta\Phi_{12} - 2\Delta\Phi_{11} = 6\nu\Psi_1 - 9\mu\Psi_2 + 6(\beta-\tau)\Psi_3 + 3\sigma\Psi_4 - 2\nu\Phi_{01} - 2\bar\nu\Phi_{10} + 2(2\bar\mu - \mu)\Phi_{11} + 2\lambda\Phi_{02} - \bar\lambda\Phi_{20} + 2(\pi + \bar\tau - 2\bar\beta)\Phi_{12} + 2(\beta + \tau + \bar\pi)\Phi_{21} + (\bar\rho - 2\epsilon - 2\bar\epsilon - 2\rho)\Phi_{22},$$

$$(NP25)\ \bar\delta\Psi_3 - D\Psi_4 + \bar\delta\Phi_{21} - \Delta\Phi_{20} = 3\lambda\Psi_2 - 2(\alpha + 2\pi)\Psi_3 + (4\epsilon - \rho)\Psi_4 - 2\nu\Phi_{10} + 2\lambda\Phi_{11} + (2\gamma - 2\bar\gamma + \bar\mu)\Phi_{20} + 2(\bar\tau - \alpha)\Phi_{21} - \bar\sigma\Phi_{22},$$

$$(NP26)\ \Delta\Psi_3 - \delta\Psi_4 + \bar\delta\Phi_{22} - \Delta\Phi_{21} = 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + (4\beta - \tau)\Psi_4 - 2\nu\Phi_{11} - \bar\nu\Phi_{20} + 2\lambda\Phi_{12} + 2(\gamma + \bar\mu)\Phi_{21} + (\bar\tau - 2\bar\beta - 2\alpha)\Phi_{22},$$

$$(NP27)\ D\Phi_{11} - \delta\Phi_{10} - \bar\delta\Phi_{01} + \Delta\Phi_{00} + 3D\Lambda = (2\gamma - \mu + 2\bar\gamma - \bar\mu)\Phi_{00} + (\pi - 2\alpha - 2\bar\tau)\Phi_{01} + (\bar\pi - 2\bar\alpha - 2\tau)\Phi_{10} + 2(\rho + \bar\rho)\Phi_{11} + \bar\sigma\Phi_{02} + \sigma\Phi_{20} - \bar\kappa\Phi_{12} - \kappa\Phi_{21},$$

$$(NP28)\ D\Phi_{12} - \delta\Phi_{11} - \bar\delta\Phi_{02} + \Delta\Phi_{01} + 3\delta\Lambda = (2\gamma - \mu - 2\bar\mu)\Phi_{01} + \bar\nu\Phi_{00} - \bar\lambda\Phi_{10} + 2(\bar\pi - \tau)\Phi_{11} + (\pi + 2\bar\beta - 2\alpha - \bar\tau)\Phi_{02} + (2\rho + \bar\rho - 2\bar\epsilon)\Phi_{12} + \sigma\Phi_{21} - \kappa\Phi_{22},$$

$$(NP29)\ D\Phi_{22} - \delta\Phi_{21} - \bar\delta\Phi_{12} + \Delta\Phi_{11} + 3\Delta\Lambda = \nu\Phi_{01} + \bar\nu\Phi_{10} - 2(\mu + \bar\mu)\Phi_{11} - \lambda\Phi_{02} - \bar\lambda\Phi_{20} + (2\pi - \bar\tau + 2\bar\beta)\Phi_{12} + (2\beta - \tau + 2\bar\pi)\Phi_{21} + (\rho + \bar\rho - 2\epsilon - 2\bar\epsilon)\Phi_{22}.$$

C.3 NP commutation relations

$$
\begin{aligned}
&\bar{\delta}\delta - \delta\bar{\delta} = (-\mu + \bar{\mu})\mathrm{D} + (-\rho + \bar{\rho})\Delta + (-\bar{\alpha} + \beta)\bar{\delta}\,, \\
&\bar{\delta}\Delta - \Delta\bar{\delta} = \nu\Delta + (\bar{\tau} - \alpha - \bar{\beta})\Delta + \lambda\delta + (\bar{\mu} + \gamma + \bar{\gamma})\,, \\
&\bar{\delta}\mathrm{D} - \mathrm{D}\bar{\delta} = (\alpha + \bar{\beta} - \pi)\mathrm{D} + \bar{\kappa}\Delta - \bar{\sigma}\delta - (\rho - \epsilon + \bar{\epsilon})\bar{\delta}\,, \\
&\Delta\mathrm{D} - \mathrm{D}\Delta = (\gamma + \bar{\gamma})\mathrm{D} + (\epsilon + \bar{\epsilon})\Delta - (\bar{\tau} + \pi)\delta - (\tau + \bar{\pi})\delta\,.
\end{aligned}
$$

Appendix D

Notation and Conventions

D.1 Symbols and tensors

$\mathcal{M}^n$,	n-dimensional pseudo-Riemannian space
$a, b, \ldots,\ i, j, \ldots,\ i_1, i_2 \ldots$,	natural basis, range $0 \ldots n$
g_{ab} ,	metric tensor with signature $(+ - \cdots -)$
∇_a *or* $_{;a}$,	components of the metric covariant derivative
C_0^k ,	class of all functions in C^k with compact support
$\mathcal{G}$,	set of all metric tensors of class C^∞
$\mathcal{D}'$,	vector space of distributions on the set $\Omega \in \mathcal{M}^n$
Ω ,	open connected neighbourhood in $\mathcal{M}^n$
$\Gamma(x, \xi)$,	square of the geodesic distance of x from ξ
$\Gamma(\gamma)$,	Euler gamma function
$C(\xi)$,	null conoid with vertex at ξ
$C^\pm(\xi)$,	future (past) null semi-conoid with vertex at ξ
$D^\pm(\xi)$,	open subsets bounded by $C^\pm(\xi)$
$J^\pm(\xi) := \overline{D^\pm(\xi)}$,	closure of $D^\pm(\xi)$
$\Box := g^{ij}\nabla_i\nabla_j = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}g^{ij}\frac{\partial u}{\partial x^j}\right)$,	Laplace-Beltrami operator on $\mathcal{M}^n$
$\omega_p := (1/p!)\,\omega_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$,	p-form
$(d\omega)_{i_1 \ldots i_{p+1}} := (p+1)\partial_{[i_1}\omega_{i_2 \ldots i_{p+1}]}$,	exterior differentiation
$(\delta\omega_p)_{i_1 \ldots i_{p-1}} := -\nabla^k \omega_{k i_1 \ldots i_{p-1}}$,	exterior codifferentiation
$TS(\cdots)$,	trace-free symmetric part of the enclosed tensor

$X_{a;bc} - X_{a;cb} = R_{cba}{}^{k} X_k$,	Ricci identity
$R_{ab[cd;e]} = 0$,	Bianchi identity
$R_{ab} := R_{ak}{}^{k}{}_{b}$,	Ricci tensor
$R := R_a{}^a$,	Ricci scalar
$L_{ab} := -R_{ab} + \frac{1}{6} g_{ab} R$,	
$C_{abcd} := R_{abcd} + g_{b[d} L_{c]a} - g_{a[d} L_{c]b}$,	Weyl tensor
$S_{abc} := L_{a[b;c]} = C^{k}{}_{acb;k}$,	
$C_{ab} := S_{abk;}{}^{k} - \frac{1}{2} C^{k}{}_{ab}{}^{l} L_{kl}$,	Bach tensor
$e_{abcd} = e_{[abcd]}$, $e_{0123} := \sqrt{-det(g_{ab})}$,	Levi-Civita pseudotensor
${}^{*}C_{abcd} := \frac{1}{2} e_{ab}{}^{kl} C_{klcd}$,	left dual of C_{abcd}

D.2 Correspondence between tensors and spinors

$$\varepsilon_{AB} = \varepsilon_{[AB]} , \ \varepsilon_{01} = 1 ,$$
$$\varepsilon^{AC} \varepsilon_{BC} = \delta^A_B = -\delta_B^A ,$$
$$X_A = X^B \varepsilon_{BA} , \qquad X^B = \varepsilon^{BA} X_A ,$$
$$C_{abcd} \leftrightarrow \Psi_{ABCD} \varepsilon_{\dot{A}\dot{B}} \varepsilon_{\dot{D}\dot{C}} + \overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} \varepsilon_{AB} \varepsilon_{DC} ,$$
$${}^{*}C_{abcd} \leftrightarrow -i \Psi_{ABCD} \varepsilon_{\dot{A}\dot{B}} \varepsilon_{\dot{D}\dot{C}} + i \overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} \varepsilon_{AB} \varepsilon_{DC} ,$$
$$H_{ab} \leftrightarrow 2(\phi_{AB} \varepsilon_{\dot{A}\dot{B}} + \overline{\phi}_{\dot{A}\dot{B}} \varepsilon_{AB}) ,$$
$$S_{abc} \leftrightarrow \Psi^{D}{}_{ABC;;D\dot{A}} \varepsilon_{\dot{C}\dot{B}} + \overline{\Psi}^{D}{}_{\dot{A}\dot{B}\dot{C};\dot{D}A} \varepsilon_{CB} ,$$
$$L_{ab} \leftrightarrow 2(\Phi_{AB\dot{A}\dot{B}} - \Lambda \varepsilon_{AB} \varepsilon_{\dot{A}\dot{B}}) , \ \Lambda := (1/24) R .$$

Appendix E

Maple Codes

The codes described below were used to make obtain most of the results of this Thesis. They are applications of the Maple packages `NPspinor` written by Czapor [27] [29] and `grobner`, also by Czapor [28]. These codes should run under the versions 5.2 or 5.3 of Maple. The package `NPspinor` is not present in the latest version, 5.4, but it should appear again in the following one, as an updated package built by Holly and McLenaghan [1].

The files must be run interactively, since certain calculations can easily exceed Maple memory and/or system memory.

E.1 Heading and templates

```
################################################################################

#Call NPspinor package and unprotect the name "gc" (garbage collector)
with(NPspinor):
unprotect('gc'):
maplec:=op(gc):
gc:='gc':
gbar:=gc:
gbarc:=g:

#Define spinor symbols
template[loeps]:=(loeps[A,B]=o[A]*i[B]-i[A]*o[B]):
ex1:=psi[A,B,C,Z]*conj(loeps[A,B]*loeps[Z,C]):
template[CW]:=CW[A,Ac,B,Bc,C,Cc,Z,Zc]=ex1+conj(ex1):
# contract(dyad(eps[Ac,Cc]*loepsc[Ac,Cc])):
ex2:=-I*ex1:
template[SCW]:=SCW[A,Ac,B,Bc,C,Cc,Z,Zc]=ex2+conj(ex2):
template[LL]:=LL[A,Ac,B,Bc]=2*(phi[A,B,Ac,Bc]-L*loeps[A,B]*loepsc[Ac,Bc]):
template[SS]:=SS[A,Ac,B,Bc,C,Cc]=-
```

[1] private communication

```
del(CW[J,Jc,A,Ac,B,Bc,C,Cc],U,Uc)*eps[U,J]*eps[Uc,Jc]:
template[pp]:=pp[A,B]=p0*i[A]*i[B]-p1*(o[A]*i[B]+o[B]*i[A])+p2*o[A]*o[B]:
template[HH]:=HH[A,Ac,B,Bc]=pp[A,B]*loepsc[Ac,Bc]+ppc[Ac,Bc]*loeps[A,B]:
##########################################################################

#Name of the file: headingIII

#Zero spinors
W0:=0: W0c:=0: W1:=0: W1c:=0: W2:=0: W2c:=0: W4:=0: W4c:=0:
W3:=-1: W3c:=-1:

k:=0: kc:=0: sc:=0: s:=0: rc:=0 : r:=0: e:=0: ec:=0: t:=0: tc:=0:
R00:=0:R01:=0:R02:=0:R20:=0:R10:=0:
L:=0:
D(p):=0:D(pc):=0:D(a):=0:D(ac):=0:D(b):=0:D(bc):=0:
D(R11):=0: X(R11):=0: Y(R11):=0: V(R11):=0:
##########################################################################
```

E.2 Self-adjoint scalar equation

```
#Components of condition VIs (Rinke-Wunsch)
#Name of the file: VIs

read heading;
read headingIII;

#First term, Q1
Q11:=3*del(CW[R,Rc,A,Ac,B,Bc,Q,Qc],G,Gc)*eps[K,R]*eps[Kc,Rc]*eps[H,Q]*eps[Hc,Qc]*
eps[M,G]*eps[Mc,Gc]:
Q11:=contract(dyad(Q11)):
Q11:=rewrite(Q11,A1dum,A1exp):
A1dum:=A1exp:
Q11:=contract(dyad(Q11)):
Q11:=del(Q11,C,Cc):
Q11:=contract(dyad(Q11)):
Q11:=rewrite(Q11,B2dum,B2exp):
B2dum:=B2exp:
Q11:=contract(dyad(Q11)):

Q112:=del(CW[K,Kc,D,Dc,E,Ec,H,Hc],M,Mc):
Q112:=contract(dyad(Q112)):
Q112:=contract(dyad(Q112)):
Q112:=rewrite(Q112,C3dum,C3exp):
C3dum:=C3exp:
Q112:=del(Q112,F,Fc):
Q112:=contract(dyad(Q112)):
Q112:=rewrite(Q112,C4dum,C4exp):
C4dum:=C4exp:
Q112:=contract(dyad(Q112)):
Q11:=contract(dyad(Q11*Q112)):
Q11:=rewrite(Q11,A2dum,A2exp):
A2dum:=A2exp:
Q11:=contract(dyad(Q11)):
save(Q11,generQ11):

Q12:=del(CW[V,Vc,A,Ac,B,Bc,R,Rc],C,Cc)*eps[K,V]*
eps[Kc,Vc]*eps[H,R]*eps[Hc,Rc]:
Q12:=contract(dyad(Q12)):
Q12:=contract(dyad(Q12)):
Q12:=del(Q12,D,Dc):
Q12:=rewrite(Q12,D2dum,D2exp):
D2dum:=D2exp:
Q121:=10*del(SS[K,Kc,H,Hc,E,Ec],F,Fc)+6*del(SS[E,Ec,F,Fc,K,Kc],H,Hc):
```

```
Q121:=contract(dyad(Q121)):
Q121:=contract(dyad(Q121)):
Q121:=contract(dyad(Q121)):
Q12:=contract(Q12*Q121):
save(Q12,generQ12):

Q13:=64*del(SS[A,Ac,B,Bc,K,Kc],C,Cc):
Q13:=contract(dyad(Q13)):
Q13:=contract(dyad(Q13)):
Q13:=contract(dyad(Q13)):
Q13:=rewrite(Q13,E1dum,E1exp):
E1dum:=E1exp:
Q131:=del(SS[D,Dc,E,Ec,H,Hc],F,Fc)*eps[K,H]*eps[Kc,Hc]:
Q131:=contract(dyad(Q131)):
Q131:=contract(dyad(Q131)):
Q131:=contract(dyad(Q131)):
Q131:=rewrite(Q131,E3dum,E3exp):
E3dum:=E3exp:

Q13:=contract((Q13*Q131)):
save(Q13,generQ13):
Q14:=-CW[H,Hc,A,Ac,B,Bc,G,Gc]*eps[K,H]*eps[Kc,Hc]*eps[V,G]*eps[Vc,Gc]:
Q14:=contract(dyad(Q14)):
Q14:=contract(dyad(Q14)):
QW:='':
save(QW,generQW):
Q141:=3*del(CW[T,Tc,C,Cc,D,Dc,K,Kc],E,Ec)*eps[M,T]*eps[Mc,Tc]:
Q141:=contract(dyad(Q141)):
Q141:=contract(dyad(Q141)):
Q141:=rewrite(Q141,E5dum,E5exp):
E5dum:=E5exp:
Q141:=del(Q141,F,Fc)*LL[V,Vc,M,Mc]:
Q141:=contract(dyad(Q141)):
Q141:=contract(dyad(Q141)):
Q141:=contract(dyad(Q141)):
Q141:=rewrite(Q141,E6dum,E6exp):
E6dum:=E6exp:
Q14:=contract(Q14*Q141):
save(Q14,generQ14):

Q15:=5*del(CW[K,Kc,C,Cc,D,Dc,V,Vc],M,Mc):
Q15:=contract(dyad(Q15)):
Q15:=contract(dyad(Q15)):
Q15:=rewrite(Q15,E61dum,E61exp):
E61dum:=E61exp:
Q15:=del(Q15,E,Ec)*LL[R,Rc,F,Fc]*eps[M,R]*eps[Mc,Rc]:
Q15:=contract(dyad(Q15)):
Q15:=rewrite(Q15,E7dum,E7exp):
E7dum:=E7exp:
Q15:=contract(dyad(Q15*QW)):
save(Q15,generQ15):

Q16:=7*del(CW[R,Rc,C,Cc,D,Dc,K,Kc],V,Vc)*eps[M,R]*eps[Mc,Rc]:
Q16:=contract(dyad(Q16)):
Q16:=contract(dyad(Q16)):
Q16:=del(Q16,E,Ec)*LL[M,Mc,F,Fc]:
Q16:=contract(dyad(Q16));
Q16:=rewrite(Q16,F11dum,F11exp):
F11dum:=F11exp:
Q16:=contract(Q16*QW):
Q16:=contract(dyad(Q16));
save(Q16, generQ16):

Q17:=13*del(SS[K,Kc,V,Vc,C,Cc],D,Dc):
Q17:=contract(dyad(Q17)):
```

```
Q17:=contract(dyad(Q17)):
Q17:=contract(dyad(Q17)):
Q17:=rewrite(Q17,F3dum,F3exp):
F3dum:=F3exp:
Q17:=contract(Q17*dyad(dyad(LL[E,Ec,F,Fc]))*QW):
save(Q17,generQ17):

Q18:=12*del(SS[C,Cc,D,Dc,K,Kc],V,Vc):
Q18:=contract(dyad(Q18)):
Q18:=contract(dyad(Q18)):
Q18:=contract(dyad(Q18)):
Q18:=contract(dyad(Q18)):
Q18:=rewrite(Q18,F41dum,F41exp):
F41dum:=F41exp:
Q18:=contract(Q18*dyad(dyad(LL[E,Ec,F,Fc]))*QW):
save(Q18,generQ18):

Q19:=71*del(SS[C,Cc,D,Dc,K,Kc],E,Ec):
Q19:=contract(dyad(Q19)):
Q19:=contract(dyad(Q19)):
Q19:=contract(dyad(Q19)):
Q19:=rewrite(Q19,F5dum,F5exp):
F5dum:=F5exp:
Q19:=contract(Q19*dyad(dyad(LL[V,Vc,F,Fc]))*QW):

save(Q19,generQ19):

#Second term, Q2

Q21:=del(SS[K,Kc,V,Vc,D,Dc],E,Ec):
Q21:=contract(dyad(Q21)):
Q21:=contract(dyad(Q21)):
Q21:=contract(dyad(Q21)):
Q21:=expand("):
Q21:=del(Q21,F,Fc):
Q21:=rewrite(Q21,F7dum,F7exp):
F7dum:=F7exp:
Q211:=del(CW[R,Rc,A,Ac,B,Bc,G,Gc],C,Cc)*eps[K,R]*
eps[Kc,Rc]*eps[V,G]*eps[Vc,Gc]:
Q211:=contract(dyad(Q211)):
QW2:=contract(dyad(Q211)):
save(QW2,generQW2):
Q21:=contract(Q21*QW2):
save(Q21,generQ21):

Q22:=3*del(SS[D,Dc,E,Ec,K,Kc],V,Vc):
Q22:=contract(dyad(Q22)):
Q22:=contract(dyad(Q22)):
Q22:=contract(dyad(Q22)):
Q22:=rewrite(Q22,G2dum,G2exp):
G2dum:=G2exp:
Q22:=del(Q22,F,Fc):
Q22:=contract(Q22*QW2):
save(Q22,generQ22):

Q23:=2*del(SS[A,Ac,B,Bc,K,Kc],C,Cc):
Q23:=contract(dyad(Q23));
Q23:=contract(dyad(Q23));
Q23:=contract(dyad(Q23));
Q23:=expand("):
Q23:=del(Q23,D,Dc):
Q23:=rewrite(Q23,G5dum,G5exp):
G5dum:=G5exp:
QQ23:=dyad(dyad(dyad(SS[E,Ec,F,Fc,R,Rc])))*eps[K,R]*eps[Kc,Rc]:
QQ23:=rewrite(QQ23,G51dum,G51exp):
```

```
G51dum:=G51exp:
Q23:=contract(Q23*QQ23):

Q24:=-5*SS[A,Ac,B,Bc,K,Kc]*eps[K,R]*eps[Kc,Rc]:
Q24:=contract(dyad(Q24)):
Q24:=contract(dyad(Q24)):
Q24:=contract(dyad(Q24)):
Q24:=Q24*SS[C,Cc,D,Dc,R,Rc]*LL[E,Ec,F,Fc]:
Q24:=contract(dyad(Q24)):
Q24:=contract(dyad(Q24)):
Q24:=contract(dyad(Q24)):

QW3:=-(1/2)*del(CW[R,Rc,A,Ac,B,Bc,G,Gc],C,Cc)*eps[K,R]*
eps[Kc,Rc]*eps[V,G]*eps[Vc,Gc]:
QW3:=contract(dyad(QW3)):
QW3:=contract(dyad(QW3)):

Q25:=2*del(CW[R,Rc,K,Kc,V,Vc,D,Dc],E,Ec)*
eps[M,R]*eps[Mc,Rc]*LL[M,Mc,F,Fc]:
save(QW3,generQW3):
Q25:=contract(dyad(Q25)):
Q25:=contract(dyad(Q25)):
Q25:=contract(Q25*QW3):
save(Q25,generQ25):

Q26:=3*del(CW[R,Rc,D,Dc,E,Ec,K,Kc],V,Vc)*
eps[M,R]*eps[Mc,Rc]*LL[M,Mc,F,Fc]:
Q26:=contract(dyad(Q26)):
Q26:=contract(dyad(Q26)):
Q26:=contract(Q26*QW3):
save(Q26,generQ26):

Q27:=contract(dyad(SS[K,Kc,V,Vc,D,Dc]*LL[E,Ec,F,Fc])):
Q27:=contract(dyad(Q27)):
Q27:=contract(dyad(Q27)):
Q27:=contract(Q27*QW3):
save(Q27,generQ27):

Q28:=3*del(CW[K,Kc,D,Dc,E,Ec,R,Rc],F,Fc)*
eps[M,R]*eps[Mc,Rc]*LL[V,Vc,M,Mc]:
Q28:=contract(dyad(Q28)):
Q28:=contract(dyad(Q28)):
Q28:=contract(Q28*QW3):
save(Q28,generQ28):

Q29:=15*SS[D,Dc,E,Ec,K,Kc]*LL[V,Vc,F,Fc]:
Q29:=contract(dyad(Q29)):
Q29:=contract(dyad(Q29)):
Q29:=contract(dyad(Q29)):
Q29:=contract(Q29*QW3):
Q29:=contract(dyad(Q29)):
Q29:=contract(dyad(Q29)):
save(Q29,generQ29):

QW4:=-CW[G,Gc,A,Ac,B,Bc,R,Rc]*eps[K,G]*
eps[Kc,Gc]*eps[V,R]*eps[Vc,Rc]:
QW4:=dyad("):
QW4:=dyad("):
save(QW4,generQW4);

Q210:=del(CW[K,Kc,C,Cc,D,Dc,H,Hc],E,Ec)*eps[M,H]*eps[Mc,Hc]:
Q210:=contract(dyad(Q210)):
Q210:=contract(dyad(Q210)):
QQ210:=(1/6)*(del(LL[V,Vc,M,Mc],F,Fc)+
del(LL[M,Mc,F,Fc],V,Vc)+del(LL[F,Fc,V,Vc],M,Mc)
```

```
+del(LL[F,Fc,M,Mc],V,Vc)+del(LL[V,Vc,F,Fc],M,Mc)
+del(LL[M,Mc,V,Vc],F,Fc)):
QQ210:=contract(dyad(QQ210)):
QQ210:=contract(dyad(QQ210)):
Q210:=contract(dyad(Q210*QQ210*QW4)):
save(Q210,generQ210):

Q211:=(1/6)*(del(LL[V,Vc,E,Ec],F,Fc)+
del(LL[E,Ec,F,Fc],V,Vc)+del(LL[F,Fc,V,Vc],E,Ec)
+del(LL[F,Fc,E,Ec],V,Vc)+del(LL[V,Vc,F,Fc],E,Ec)
+del(LL[E,Ec,V,Vc],F,Fc)):
Q211:=contract(dyad(Q211)):
Q211:=contract(dyad(Q211)):
Q211:=rewrite(Q211,G8dum,G8exp):
G8dum:=G8exp:
QQ211:=dyad(SS[C,Cc,D,Dc,K,Kc]):
QQ211:=dyad(QQ211):
QQ211:=dyad(QQ211):
QQ211:=rewrite(QQ211,H2dum,H2exp):
H2dum:=H2exp:
Q211:=contract(QQ211*Q211*QW4):
save(Q211,generQ211):

#Third term, Q3

Q31:=2*del(CW[V,Vc,X,Xc,M,Mc,D,Dc],E,Ec):
Q31:=contract(dyad(Q31)):
Q31:=contract(dyad(Q31)):
Q31:=contract(dyad(Q31)):
Q31:=del(Q31,F,Fc):
Q31:=contract(dyad(Q31)):
Q31:=contract(dyad(Q31)):
Q31:=contract(dyad(Q31)):
Q31:=rewrite(Q31,H1dum,H1exp):
H1dum:=H1exp:
Q311:=CW[K,Kc,R,Rc,H,Hc,C,Cc]*eps[M,R]*
eps[Mc,Rc]*eps[X,H]*eps[Xc,Hc]:
Q31:=contract(dyad(Q31*Q311*QW4)):
Q31:=contract(dyad(Q31));
save(Q31,generQ31):

Q32:=-10*del(CW[K,Kc,E,Ec,F,Fc,V,Vc],M,Mc):
Q32:=contract(dyad(Q32)):
Q32:=contract(dyad(Q32)):
Q32:=del(Q32,X,Xc):
Q32:=rewrite(Q32,H32dum,H32exp):
H32dum:=H32exp:
Q321:=dyad(CW[H,Hc,C,Cc,D,Dc,U,Uc]*eps[M,H]*
eps[Mc,Hc]*eps[X,U]*eps[Xc,Uc]):
Q321:=contract(dyad(Q321)):
Q32:=contract(Q32*Q321*QW4):
save(Q32,generQ32):

Q33:=20*CW[V,Vc,C,Cc,D,Dc,H,Hc]*eps[M,H]*eps[Mc,Hc]*
del(SS[K,Kc,M,Mc,E,Ec],F,Fc):
Q33:=contract(dyad(Q33)):
Q33:=contract(dyad(Q33)):
Q33:=contract(dyad(Q33)):
Q33:=rewrite(Q33,K1dum,K1exp):
K1dum:=K1exp:
Q33:=contract(Q33*QW4):
save(Q33,generQ33):


Q34:=-5*del(CW[H,Hc,C,Cc,D,Dc,X,Xc],E,Ec)*eps[K,H]*
```

```
eps[Kc,Hc]*eps[V,X]*eps[Vc,Xc]:
Q34:=contract(dyad(Q34)):
Q34:=del(Q34,F,Fc):
Q34:=rewrite(Q34,K3dum,K3exp):
K3dum:=K3exp:
Q341:=CW[V,Vc,M,Mc,G,Gc,B,Bc]*CW[K,Kc,Y,Yc,U,Uc,A,Ac]*
eps[M,Y]*eps[Mc,Yc]*eps[G,U]*eps[Gc,Uc]:
Q341:=contract(dyad(Q341)):
Q341:=contract(dyad(Q341)):
Q34:=contract(Q341*Q34):
Q34:=contract(dyad(")):
Q34:=contract(dyad(")):
Q34:=contract(dyad(")):
Q34:=contract(dyad(")):
save(Q34,generQ34):

Q35:=7*CW[K,Kc,Y,Yc,U,Uc,C,Cc]*eps[M,Y]*
eps[Mc,Yc]*eps[G,U]*eps[Gc,Uc]*
CW[V,Vc,M,Mc,G,Gc,D,Dc]:
Q35:=contract(dyad(Q35)):
Q35:=contract(dyad(Q35)):
Q351:=LL[E,Ec,F,Fc]*(-QW4):
Q351:=contract(dyad(Q351)):
Q351:=contract(dyad(Q351)):
Q35:=contract(Q35*Q351):
Q35:=contract(dyad(")):
save(Q35,generQ35):

Q36:=-10*CW[K,Kc,E,Ec,F,Fc,V,Vc]*CW[U,Uc,C,Cc,D,Dc,X,Xc]*
eps[M,U]*eps[Mc,Uc]*eps[Y,X]*eps[Yc,Xc]:
Q36:=contract(dyad(Q36)):
Q36:=contract(dyad(Q36)):
Q361:=LL[M,Mc,Y,Yc]*(-QW4):
Q361:=contract(dyad(Q361)):
Q361:=contract(dyad(Q361)):
Q36:=contract(Q36*Q361):
Q36:=contract(dyad(Q36)):
Q36:=contract(dyad(Q36));
save(Q36,generQ36):

#Fourth term, Q4
Q41:=3*del(CW[K,Kc,U,Uc,X,Xc,C,Cc],D,Dc)*
eps[M,U]*eps[Mc,Uc]*eps[Y,X]*eps[Yc,Xc]:
Q41:=contract(dyad(Q41)):
Q41:=contract(dyad(Q41)):
Q41:=rewrite(Q41,K62dum,K62exp):
K62dum:=K62exp:
Q411:=del(CW[V,Vc,M,Mc,Y,Yc,E,Ec],F,Fc):
Q411:=contract(dyad(Q411)):
Q411:=contract(dyad(Q411)):
Q411:=rewrite(Q411,K63dum,K63exp):
K63dum:=K63exp:
Q41:=contract(Q41*Q411*QW4):
save(Q41,generQ41):

Q42:=54*del(CW[V,Vc,C,Cc,D,Dc,X,Xc],E,Ec)*eps[M,X]*eps[Mc,Xc]:
Q42:=contract(dyad(Q42)):
Q42:=contract(dyad(Q42)):
Q42:=rewrite(Q42,J2dum,J2exp):
J2dum:=J2exp:
Q421:=SS[K,Kc,M,Mc,F,Fc]*QW4:
Q421:=contract(dyad(Q421)):
Q421:=contract(dyad(Q421)):
Q421:=contract(dyad(Q421)):
Q421:=rewrite(Q421,J3dum,J3exp):
```

```
J3dum:=J3exp:
Q42:=contract(Q42*Q421):
save(Q42,generQ42):

Q43:=74*del(CW[V,Vc,C,Cc,D,Dc,X,Xc],K,Kc)*
eps[M,X]*eps[Mc,Xc]*SS[E,Ec,F,Fc,M,Mc]*QW4:
Q43:=contract(dyad(Q43)):
Q43:=contract(dyad(Q43)):
Q43:=contract(dyad(Q43)):
save(Q43,generQ43):

Q44:=(-76/3)*del(CW[C,Cc,K,Kc,V,Vc,X,Xc],D,Dc)*
SS[E,Ec,F,Fc,M,Mc]*eps[M,X]*eps[Mc,Xc]*QW4:
Q44:=contract(dyad(Q44)):
Q44:=contract(dyad(Q44)):
Q44:=contract(dyad(Q44)):
save(Q44,generQ44):

Q45:=(-404/3)*SS[C,Cc,D,Dc,K,Kc]:
Q45:=contract(dyad(Q45)):
Q45:=contract(dyad(Q45)):
Q45:=contract(dyad(Q45)):
Q45:=rewrite(Q45,J7dum,J7exp):
J7dum:=J7exp:
Q451:=SS[E,Ec,F,Fc,V,Vc]*QW4:
Q451:=contract(dyad(Q451)):
Q451:=contract(dyad(Q451)):
Q451:=contract(dyad(Q451)):
Q451:=rewrite(Q451,J8dum,J8exp):
J8dum:=J8exp:
Q45:=contract(Q45*Q451):
save(Q45,generQ45):

Q46:=6*del(CW[U,Uc,B,Bc,C,Cc,X,Xc],D,Dc)*eps[K,U]*
eps[Kc,Uc]*eps[V,X]*eps[Vc,Xc]:
Q46:=contract(dyad(Q46)):
Q46:=contract(dyad(Q46)):
Q461:=del(CW[V,Vc,E,Ec,F,Fc,M,Mc],G,Gc)*
CW[K,Kc,Y,Yc,U,Uc,A,Ac]*eps[M,Y]*eps[Mc,Yc]*eps[G,U]*eps[Gc,Uc]:
Q461:=contract(dyad(Q461)):
Q461:=contract(dyad(Q461)):
Q461:=rewrite(Q461,L1dum,L1exp):
L1dum:=L1exp:
Q46:=contract(Q46*Q461):
save(Q46,generQ46):

#Fifth term, Q5
Q51:=25*(-QW4)*CW[V,Vc,C,Cc,D,Dc,X,Xc]*eps[M,X]*eps[Mc,Xc]:
Q51:=contract(dyad(Q51)):
Q51:=contract(dyad(Q51)):
Q511:=LL[K,Kc,M,Mc]*LL[E,Ec,F,Fc]:
Q511:=contract(dyad(Q511)):
Q511:=contract(dyad(Q511)):
Q51:=contract(Q51*Q511):
save(Q51,generQ51):

Q52:=(1/6)*(-QW4)*CW[K,Kc,C,Cc,D,Dc,V,Vc]*87*
LL[X,Xc,E,Ec]*eps[M,X]*eps[Mc,Xc]*LL[M,Mc,F,Fc]:
Q52:=contract(dyad(Q52)):
Q52:=contract(dyad(Q52)):
save(Q52,generQ52):

#We have to break the contractions into parts in order
# to save computer memory
```

```
Q1:=Q11+Q12+Q13+Q14+Q15+Q16+Q17+Q19:
Q2:=Q21+Q22+Q23+Q24+Q25+Q26+Q27+Q29+Q210+Q211:
Q3:=Q31+Q32+Q33+Q34+Q35+Q36:
Q4:=Q41+Q42+Q43+Q44+Q45+Q56:
Q5:=Q51+Q52:

f:=eps[N,A]*eps[S,B]*eps[P,C]*eps[Z,D]*eps[Q,E]*eps[W,F]:
fc:=conj(f):

co1s:=i[N]*i[S]*i[P]*i[Q]*i[Z]*i[W]:
co2:=i[N]*i[S]*i[P]*i[Q]*i[Z]*o[W]:
co3:=i[N]*i[S]*i[P]*i[Q]*o[Z]*o[W]:
co4:=i[N]*i[S]*i[P]*o[Q]*o[Z]*o[W]:
co5:=i[N]*i[S]*o[P]*o[Q]*o[Z]*o[W]:
co6:=i[N]*o[S]*o[P]*o[Q]*o[Z]*o[W]:
co7s:=o[N]*o[S]*o[P]*o[Q]*o[Z]*o[W]:

co2s:=symm(co2,[N,S,P,Q,Z,W]):
co3s:=symm(co3,[N,S,P,Q,Z,W]):
co4s:=symm(co4,[N,S,P,Q,Z,W]):
co5s:=symm(co5,[N,S,P,Q,Z,W]):
co6s:=symm(co6,[N,S,P,Q,Z,W]):

co1sc:=conj(co1s):
co2sc:=conj(co2s):
co3sc:=conj(co3s):
co4sc:=conj(co4s):
co5sc:=conj(co5s):
co6sc:=conj(co6s):
co7sc:=conj(co7s):

c:=array(1..28):

c[1]:=co1s*co1sc:
c[2]:=co1s*co2sc:
c[3]:=co1s*co3sc:
c[4]:=co1s*co4sc:
c[5]:=co1s*co5sc:
c[6]:=co1s*co6sc:
c[7]:=co1s*co7sc:

c[8]:=co2s*co2sc:
c[9]:=co2s*co3sc:
c[10]:=co2s*co4sc:
c[11]:=co2s*co5sc:
c[12]:=co2s*co6sc:
c[13]:=co2s*co7sc:

c[14]:=co3s*co3sc:
c[15]:=co3s*co4sc:
c[16]:=co3s*co5sc:
c[17]:=co3s*co6sc:
c[18]:=co3s*co7sc:

c[19]:=co4s*co4sc:
c[20]:=co4s*co5sc:
c[21]:=co4s*co6sc:
c[22]:=co4s*co7sc:

c[23]:=co5s*co5sc:
c[24]:=co5s*co6sc:
c[25]:=co5s*co7sc:

c[26]:=co6s*co6sc:
c[27]:=co6s*co7sc:

c[28]:=co7s*co7sc:

save(c,csix);
```

```
sixQ1:=array(1..28):
for j from 1 to 28 do
sixQ1[j]:=expand(contract(Q1*c[j]*fc*fc)):
od:
sixQ2:=array(1..28):
for j from 1 to 28 do
sixQ2[j]:=expand(contract(Q2*c[j]*fc*fc)):
od:
sixQ3:=array(1..28):
for j from 1 to 28 do
sixQ3[j]:=expand(contract(Q3*c[j]*fc*fc)):
od:
sixQ4:=array(1..28):
for j from 1 to 28 do
sixQ4[j]:=expand(contract(Q4*c[j]*fc*fc)):
od:
sixQ5:=array(1..28):
for j from 1 to 28 do
sixQ5[j]:=expand(contract(Q5*c[j]*fc*fc)):
od:

six:=array(1..28):
for j from 1 to 28 do
six[j]:=primpart(factor(sixQ11[j]+sixQ12[j]+sixQ13[j]+sixQ14[j]+sixQ15[j]
+sixQ16[j]+sixQ17[j]+sixQ18[j]+sixQ19[j]-10*(sixQ21[j]
+sixQ22[j]+sixQ23[j]+sixQ24[j]+sixQ25[j]+sixQ26[j]
+sixQ27[j]+sixQ28[j]+sixQ29[j]+sixQ210[j]+sixQ211[j])+4*(sixQ31[j]+
sixQ32[j]+sixQ33[j]+sixQ34[j]+sixQ35[j]
+sixQ36[j])+5*(sixQ41[j]+sixQ42[j]+sixQ43[j]
+sixQ44[j]+sixQ45[j]+
sixQ46[j])+sixQ51[j]+sixQ52[j]));
od:
save(six,'six.array'):
s10:=six[10]:
save(s10,six10);
#########################################################################

#Solving for the case R11=0
#Name of the file: solveself0
R11:=0:

read 'tt.array';
X(b):=solve(tt[4],X(b));
Y(bc):=conj(X(b));
eq30:=tt[1]/2;
eq31:=tt[2];

eqns();eqns();
D(R12):=solve(eq22+conj(eq23),D(R12));
D(R21):=conj(D(R12));
D(g):=solve(eq6,D(g));
X(R12):=solve(conj(eq25),X(R12));
Y(R12):=solve(expand(eq24-eq29)/3,Y(R12));
X(R21):=conj(Y(R12));
Y(R21):=conj(X(R12));
D(R22):=solve(conj(eq24)+2*eq29,D(R22));
D(R12):=solve(eq22+conj(eq23),D(R12));
D(R21):=conj(D(R12));
X(a):=solve(eq12,X(a));
Y(ac):=conj(X(a));
X(l):=solve(eq13,X(l));
Y(lc):=conj(X(l));
Y_X(R12);
";
eq32:=expand(");
```

```
eq36:=conj(eq26);

X(R22):=solve(eq36,X(R22));
Y(R22):=conj(Y(R22));
D(n):=solve(eq9,D(n));
D(nc):=conj(D(n));
X(p):=solve(eq8,X(p));
Y(pc):=conj(X(p));
Y(p):=solve(eq7,Y(p));
X(pc):=conj(Y(p));
D(gc):=conj(D(g));
eq38:=normal(X_D(R22)-V_D(R12));
X(bc):=solve(eq38,X(bc));
Y(b):=conj(X(bc));
V(b):=solve(eq15,V(b));
V(bc):=conj(V(b));
X(m):=solve(eq30,X(m));
Y(mc):=conj(X(m));

# Condition V
t5:=array(1..15):
read `t5sca.array`;
#Scalar field equations
#k1:=3;k2:=4;
for j from 1 to 15 do
t5[j]:=normal(t5[j]);
od:
eq39:=normal(t5[1]);
eq40:=normal(t5[5]);
eq41:=normal(t5[9]);
eq42:=normal(t5[15]);
D(lc):=solve(eq41,D(lc));
D(l):=conj(D(lc));

Y_D(m):
D(Y(m)):=solve(",D(Y(m)));
Y_D(gc):":":":":
D(Y(gc)):=solve(",D(Y(gc)));
Y_D(g):":":":":
D(Y(g)):=solve(",D(Y(g)));
V_D(p) :
D(V(p)):=solve(",D(V(p)));
V_D(a):":":
D(V(a)):=solve(",D(V(a)));

read six10:
s10:":":
s10:=factor(s10);

solve(eq39,D(mc)):":":
D(mc):=factor(");

Y_X(a+2*p):":":":":
eq43:=factor(");
normal(eq43):":":":
solve(",D(m)):
D(m):=factor(");
normal(D(mc)-conj(D(m))):
":":":":
s1:=normal(");
ns1:=numer(")/2;

read six10:
s10:":":
s10:=factor(s10);
Ya6:=solve(s10,Y(a));
```

```
Y_X(bc):":":":":":":":":normal("):
Ya1:=normal(solve(",Y(a))):

Y_X(b):":":":":":":":":normal("):
normal(solve(",X(ac))):
Ya2:=conj("):
eq1:=normal(Ya1-Ya6);
neq1:=numer("):
neq1:=factor("):
neq1:=op(5,neq1);

#Y(neq1) yields neq1 again
Y(neq1):":":":":
eq60:=normal(");
solve(eq60,Y(a)):
Ya4:=";
eq4:=normal(Ya1-Ya4);
neq4:=factor(numer("));
a:=x1*p:
ac:=conj("):
b:=x2*pc:
bc:=conj("):

neq1:=factor(neq1)/(-pc*p^2);
neq1c:=conj(");
ns1:=factor(ns1)/(p^2*pc^2);
neq4:=factor(neq4)/(2*p^5*pc^2);
with(grobner);
F1:=[neq1,neq4]:
G1:=gsolve(F1);
##########################################################################
```

E.3 Non-self-adjoint scalar equation

```
#Components of condition II (on-self-adjoint scalar equation)
#Name of the file: IInon
read heading;

Q2:=del(pp[K,A],X,Ac)*eps[K,X]:
Q2:=contract(dyad(")):
Q2:=contract(dyad(")):

f:=eps[G,A]:
fc:=conj("):

c[1]:=o[G]*oc[Gc]:
c[2]:=o[G]*ic[Gc]:
c[3]:=oc[Gc]*i[G]:
c[4]:=i[G]*ic[Gc]:

for j from 1 to 4 do
QQ1[j]:=contract(dyad(Q2*f*fc*c[j]));
od;
save(QQ1,'QQ1.array'):
##########################################################################

#condition III
#Name of the file: IIInon
read heading;

T21:=del(psi[A,B,K,V],J,Ac)*eps[K,J]:
T21:=contract(dyad(")):
```

```
T21:=contract(dyad(")):
T21:=del(",H,Bc)*eps[V,H]:
T21:=contract(dyad(")):

T22:=psi[A,B,M,N]*eps[K,M]*eps[V,N]*phi[K,V,Ac,Bc]
+5*pp[A,B]*ppc[Ac,Bc]:
T22:=contract(dyad(")):
T22:=contract(dyad(")):

T2:=T21+T22:

f:=eps[M,A]*eps[N,B]:
fc:=conj("):

co1s:=i[M]*i[N]:
co2s:=symm(i[M]*o[N],[M,N]):
co3s:=o[M]*o[N]:

co1co:=conj(co1s):
co2co:=conj(co2s):
co3co:=conj(co3s):

c[1]:=co1s*co1co:
c[2]:=co1s*co2co:
c[3]:=co1s*co3co:

c[4]:=co2s*co1co:
c[5]:=co2s*co2co:
c[6]:=co2s*co3co:

c[7]:=co3s*co1co:
c[8]:=co3s*co2co:
c[9]:=co3s*co3co:

QQ2:=array(1..9):
for j from 1 to 9 do
QQ2[j]:=factor(contract(dyad(c[j]*T2*f*fc)));
od;

save(QQ2,'QQ2.array'):
##########################################################################

#Condition IV
#Name of the file: IVnon
read heading;

Q3:=3*SS[A,Ac,B,Bc,K,Kc]*HH[G,Gc,C,Cc]*eps[K,G]*eps[Kc,Gc]+
CW[M,Mc,A,Ac,B,Bc,N,Nc]*eps[K,M]*eps[Kc,Mc]*eps[V,N]*eps[Vc,Nc]*
del(HH[C,Cc,K,Kc],V,Vc):

Q3:=contract(dyad(")):
Q3:=contract(dyad(")):
Q3:=contract(dyad(")):
Q3:=factor("):

f:=eps[H,A]*eps[X,B]*eps[Y,C]:
fc:=conj("):

co1:=o[H]*o[X]*o[Y]:
co2:=symm(o[H]*o[X]*i[Y],[H,X,Y]):
co3:=symm(o[H]*i[X]*i[Y],[H,X,Y]):
co4:=i[H]*i[X]*i[Y]:

co1c:=conj(co1):
co2c:=conj(co2):
co3c:=conj(co3):
co4c:=conj(co4):
```

```
c[1]:=co1*co1c:
c[2]:=co1*co2c:
c[3]:=co1*co3c:
c[4]:=co1*co4c:

c[5]:=co2*co2c:
c[6]:=co2*co3c:
c[7]:=co2*co4c:
c[8]:=co3*co3c:
c[9]:=co3*co4c:

c[10]:=co4*co4c:

QQ3:=array(1..10):
for j from 1 to 10 do
QQ3[j]:=factor(contract(dyad(c[j]*Q3*f*fc)));
od;

save(QQ3,'QQ3.array'):
############################################################################

#Condition V
#Name of the file: Vnon

read heading;

Q41:= 3*del(CW[K,Kc,C,Cc,D,Dc,V,Vc],M,Mc)*
del(CW[W,Wc,E,Ec,F,Fc,G,Gc],H,Hc)*eps[K,W]*eps[Kc,Wc]*eps[V,G]*eps[Vc,Gc]*
eps[M,H]*eps[Mc,Hc]:
Q41:=contract(dyad("')):
Q41:=factor(contract(dyad("))):
save(Q41,Q41self):
1;
Q42:=8*del(CW[W,Wc,C,Cc,D,Dc,M,Mc],E,Ec)*eps[K,W]*
eps[Kc,Wc]*eps[V,M]*eps[Vc,Mc]*SS[K,Kc,V,Vc,F,Fc]:
Q42:=contract(dyad(")):
Q42:=contract(dyad(")):
Q42:=contract(dyad(")):
Q42:=factor("):
save(Q42,Q42self):
2;
Q43:=40*SS[C,Cc,D,Dc,H,Hc]*eps[K,H]*eps[Kc,Hc]:
Q43:=contract(dyad(")):
Q43:=contract(dyad(")):
Q43:=contract(dyad(")):
Q43:=contract(dyad(Q43*SS[E,Ec,F,Fc,K,Kc])):
Q43:=contract(dyad(")):
Q43:=contract(dyad(")):
Q43:=factor("):
save(Q43,Q43self):
3;

Q44:=-8*CW[W,Wc,C,Cc,D,Dc,M,Mc]*eps[K,W]*eps[Kc,Wc]*eps[V,M]*eps[Vc,Mc]:
Q44:=contract(dyad(")):
Q44:=contract(dyad(")):
Q441:=del(SS[K,Kc,V,Vc,E,Ec],F,Fc):
Q441:=contract(dyad(")):
Q441:=contract(dyad(")):
Q441:=contract(dyad(")):
Q441:=rewrite(Q441,F11dum,F11exp):
F11dum:=F11exp:
Q44:=contract(Q441*Q44):
Q44:=factor("):
save(Q44,Q44self):
4;
Q45:=-24*CW[W,Wc,C,Cc,D,Dc,M,Mc]*eps[K,W]*eps[Kc,Wc]*eps[V,M]*eps[Vc,Mc]:
```

```
Q45:=contract(dyad(")):
Q45:=contract(dyad(")):
Q451:=del(SS[E,Ec,F,Fc,K,Kc],V,Vc):
Q451:=contract(dyad(")):
Q451:=contract(dyad(")):
Q451:=contract(dyad(")):
Q451:=rewrite(Q451,F12dum,F12exp):
F12dum:=F12exp:
Q45:=contract(Q45*Q451):
Q45:=factor("):
save(Q45,Q45self):
5;
Q46:=4*CW[W,Wc,C,Cc,D,Dc,X,Xc]*eps[K,W]*eps[Kc,Wc]*eps[V,X]*eps[Vc,Xc]:
Q46:=contract(dyad(")):
Q46:=contract(dyad(")):
Q461:=CW[V,Vc,H,Hc,E,Ec,K,Kc]*eps[M,H]*eps[Mc,Hc]:
Q461:=contract(dyad(")):
Q461:=contract(dyad(")):
Q462:=LL[F,Fc,M,Mc]:
Q462:=contract(dyad(")):
Q462:=contract(dyad(")):
Q46:=contract(Q46*Q462*Q461):
Q46:=factor("):
save(Q46,Q46self):
6;
Q47:=12*CW[W,Wc,C,Cc,D,Dc,M,Mc]*eps[K,W]*eps[Kc,Wc]*eps[V,M]*eps[Vc,Mc]*
CW[H,Hc,E,Ec,F,Fc,V,Vc]*eps[M,H]*eps[Mc,Hc]:
Q47:=contract(dyad(")):
Q47:=contract(dyad(")):
Q471:=LL[K,Kc,M,Mc]:
Q471:=contract(dyad(")):
Q471:=contract(dyad(")):
Q47:=contract(Q471*Q47):
Q47:=factor("):
save(Q47,Q47self):
7;
Q48:=del(12*HH[K,Kc,C,Cc],D,Dc):
Q48:=contract(dyad(")):
Q48:=contract(dyad(")):
Q48:=del(Q48,E,Ec):
Q48:=rewrite(Q48,F13dum,F13exp):
F13dum:=F13exp:
Q481:=HH[H,Hc,F,Fc]*eps[K,H]*eps[Kc,Hc]:
Q481:=contract(dyad(")):
Q481:=contract(dyad(")):
Q48:=contract(Q48*Q481):
Q48:=factor("):
save(Q48,Q48self):
8;
Q49:=-16*del(HH[K,Kc,C,Cc],D,Dc):
Q49:=contract(dyad(")):
Q49:=contract(dyad(")):
Q49:=Q49*del(HH[M,Mc,E,Ec],F,Fc)*eps[K,M]*eps[Kc,Mc]:
Q49:=contract(dyad(")):
Q49:=contract(dyad(")):
Q49:=factor("):
save(Q49,Q49self):
9;
Q410:=-84*HH[M,Mc,C,Cc]*eps[K,M]*eps[Kc,Mc]:
Q410:=contract(dyad(")):
Q410:=contract(dyad(")):
Q410:=Q410*CW[K,Kc,D,Dc,E,Ec,V,Vc]:
Q410:=contract(dyad(")):
Q410:=contract(dyad(")):
Q410:=Q410*HH[H,Hc,F,Fc]*eps[V,H]*eps[Vc,Hc]:
```

```
Q410:=contract(dyad(")):
Q410:=contract(dyad(")):
Q410:=factor("):
save(Q410,Q410self):
10;

Q411:=-18*HH[K,Kc,C,Cc]*HH[G,Gc,D,Dc]*eps[K,G]*eps[Kc,Gc]*LL[E,Ec,F,Fc]:
Q411:=contract(dyad(")):
Q411:=contract(dyad(")):
Q411:-factor("):
save(Q411,Q411self):
11;

Q4:=Q41+Q42+Q43+Q44+Q45+Q46+Q47+Q48+Q49+Q410+Q411:

f:=eps[S,C]*eps[Sc,Cc]*eps[N,D]*eps[Nc,Dc]*
eps[P,E]*eps[Pc,Ec]*eps[Q,F]*eps[Qc,Fc] :

co1:=i[S]*i[N]*i[P]*o[Q]:
co2:=i[S]*i[N]*o[P]*o[Q]:
co3:=i[S]*o[N]*o[P]*o[Q]:
co4s:=o[S]*o[N]*o[P]*o[Q]:
co5s:=i[S]*i[N]*i[P]*i[Q]:

co1s:=symm(co1,[N,S,P,Q]):
co2s:=symm(co2,[N,S,P,Q]):
co3s:=symm(co3,[N,S,P,Q]):

co1co:=conj(co1s):
co2co:=conj(co2s):
co3co:=conj(co3s):
co4co:=conj(co4s):
co5co:=conj(co5s):
c[1]:=co1s*co1co:
c[2]:=co1s*co2co:
c[3]:=co1s*co3co:
c[4]:=co1s*co4co:
c[5]:=co1s*co5co:

c[6]:=co2s*co2co:
c[7]:=co2s*co3co:
c[8]:=co2s*co4co:
c[9]:=co2s*co5co:

c[10]:=co3s*co3co:
c[11]:=co3s*co4co:
c[12]:=co3s*co5co:

c[13]:=co4s*co4co:
c[14]:=co4s*co5co:

c[15]:=co5s*co5co:

QQ4:=array(1..15):

for j from 1 to 15 do
QQ4[j]:=expand(contract(dyad(Q4*c[j]*f)));
od;

save(QQ4,'QQ4.array');
############################################################################

#Condition VI
#Name of the file: VInon

read heading;

Q51:=36*CW[W,Wc,A,Ac,B,Bc,G,Gc]*eps[K,W]*eps[Kc,Wc]*eps[V,G]*eps[Vc,Gc]*
```

```
del(CW[V,Vc,C,Cc,D,Dc,M,Mc],K,Kc)*HH[J,Jc,E,Ec]*eps[M,J]*eps[Mc,Jc]:
Q51:=contract(dyad("")):
Q51:=contract(dyad("")):
Q51:=contract(dyad("")):
Q51:=factor("):
save(Q51,Q51self):
1;
Q52:=-6*del(CW[W,Wc,A,Ac,B,Bc,G,Gc],C,Cc)*
eps[K,W]*eps[Kc,Wc]*eps[V,G]*eps[Vc,Gc]*
CW[V,Vc,D,Dc,E,Ec,J,Jc]*eps[M,J]*eps[Mc,Jc]*HH[K,Kc,M,Mc]:
Q52:=contract(dyad("")):
Q52:=contract(dyad("")):
Q52:=contract(dyad("")):
Q52:=factor("):
save(Q52,Q52self):
2;
Q53:=-138*CW[K,Kc,C,Cc,D,Dc,V,Vc]:
Q53:=contract(dyad("")):
Q53:=contract(dyad("")):
Q53:=Q53*HH[G,Gc,E,Ec]*eps[V,G]*eps[Vc,Gc]:
Q53:=contract(dyad("")):
Q53:=contract(dyad("")):
Q531:=contract(dyad(SS[A,Ac,B,Bc,W,Wc]*eps[K,W]*eps[Kc,Wc])):
Q531:=contract(dyad("")):
Q531:=contract(dyad("")):
Q53:=contract(dyad(Q531*Q53)):
Q53:=contract(dyad("")):
Q53:=factor("):
save(Q53,Q53self):
3;

Q54:=6*del(HH[X,Xc,C,Cc],D,Dc)*eps[K,X]*eps[Kc,Xc]:
Q54:contract(dyad("")):
Q54:factor(contract(dyad(""))):
Q54:contract(dyad("")):
Q54:=dyad("):
Q54:=rewrite(Q54,F12dum,F12exp):
F12dum:=F12exp:
Q54:=del(Q54,E,Ec):
Q54:=rewrite(Q54,F13dum,F13exp):
F13dum:=F13exp:
Q541:=SS[A,Ac,B,Bc,K,Kc]:
Q541:=factor(contract(dyad(""))):
Q541:=contract(dyad("")):
Q54:=Q54*Q541:
Q54:contract(dyad("")):
Q54:contract(dyad("")):
Q54:=factor("):
save(Q54,Q54self):
4;
Q55:=6*del(CW[W,Wc,A,Ac,B,Bc,G,Gc],C,Cc)*
eps[K,W]*eps[Kc,Wc]*eps[V,G]*eps[Vc,Gc]:
Q55:=contract(dyad("")):
Q55:=contract(dyad("")):

Q551:=del(HH[K,Kc,D,Dc],V,Vc):
Q551:=contract(dyad("")):
Q551:=contract(dyad("")):
Q551:=del(Q551,E,Ec):
Q551:=rewrite(Q551,F14dum,F14exp):
F14dum:=F14exp:
Q55:=contract(Q551*Q55):
Q55:=factor("):
save(Q55,Q55self):
5;
```

```
Q56:=-24*del(SS[A,Ac,B,Bc,K,Kc],C,Cc):
Q56:=contract(dyad("")):
Q56:=contract(dyad("")):
Q56:=contract(dyad("")):
length(Q56):
Q56:=rewrite(Q56,F16dum,F16exp):
F16dum:=F16exp:
length(Q56):
Q561:=del(HH[X,Xc,D,Dc],E,Ec)*eps[K,X]*eps[Kc,Xc]:
Q561:=contract(dyad("")):
Q561:=contract(dyad("")):
Q56:=contract(Q56*Q561):
Q56:=factor("):
save(Q56,Q56self):
6;
Q57:=12*CW[W,Wc,A,Ac,B,Bc,G,Gc]*eps[K,W]*
eps[Kc,Wc]*eps[V,G]*eps[Vc,Gc]*LL[K,Kc,C,Cc]*del(HH[V,Vc,D,Dc],E,Ec):
Q57:=contract(dyad("")):
Q57:=contract(dyad("")):
Q57:=factor("):
save(Q57,Q57self):
7;
Q58:=-9*del(CW[W,Wc,A,Ac,B,Bc,G,Gc],C,Cc)*
eps[K,W]*eps[Kc,Wc]*eps[V,G]*eps[Vc,Gc]*LL[K,Kc,D,Dc]*HH[V,Vc,E,Ec]:
Q58:=contract(dyad("")):
Q58:=contract(dyad("")):
Q58:=factor("):
save(Q58,Q58self):
8;
Q59:=-9*SS[A,Ac,B,Bc,K,Kc]*LL[C,Cc,D,Dc]*HH[G,Gc,E,Ec]*eps[K,G]*eps[Kc,Gc]:
Q59:=contract(dyad("")):
Q59:=contract(dyad("")):
Q59:=contract(dyad("")):
Q59:=factor("):
save(Q59,Q59self):
9;

f:=eps[S,A]*eps[Sc,Ac]*eps[N,B]*eps[Nc,Bc]*eps[P,C]*eps[Pc,Cc]*eps[Q,D]*
eps[Qc,Dc]*eps[Z,E]*eps[Zc,Ec] :

co1s:=i[N]*i[S]*i[P]*i[Q]*i[Z]:
co2:=i[N]*i[S]*i[P]*i[Q]*o[Z]:
co3:=i[N]*i[S]*i[P]*o[Q]*o[Z]:
co4:=i[N]*i[S]*o[P]*o[Q]*o[Z]:
co5:=i[N]*o[S]*o[P]*o[Q]*o[Z]:
co6s:=o[N]*o[S]*o[P]*o[Q]*o[Z]:

co2s:=symm(co2,[N,S,P,Q,Z]):
co3s:=symm(co3,[N,S,P,Q,Z]):
co4s:=symm(co4,[N,S,P,Q,Z]):
co5s:=symm(co5,[N,S,P,Q,Z]):

co1sc:=conj(co1s):
co2sc:=conj(co2s):
co3sc:=conj(co3s):
co4sc:=conj(co4s):
co5sc:=conj(co5s):
co6sc:=conj(co6s):

c:=array(1..21):

c[1]:=co1s*co1sc:
c[2]:=co1s*co2sc:
c[3]:=co1s*co3sc:
c[4]:=co1s*co4sc:
c[5]:=co1s*co5sc:
```

```
c[6]:=co1s*co6sc:

c[7]:=co2s*co2sc:
c[8]:=co2s*co3sc:
c[9]:=co2s*co4sc:
c[10]:=co2s*co5sc:
c[11]:=co2s*co6sc:

c[12]:=co3s*co3sc:
c[13]:=co3s*co4sc:
c[14]:=co3s*co5sc:
c[15]:=co3s*co6sc:

c[16]:=co4s*co4sc:
c[17]:=co4s*co5sc:
c[18]:=co4s*co6sc:

c[19]:=co5s*co5sc:
c[20]:=co5s*co6sc:

c[21]:=co6s*co6sc:

QQ5:=array(1..21):
for j from 1 to 21 do
QQ51[j]:=expand(contract(dyad(Q51*c[j]*f)));
QQ52[j]:=expand(contract(dyad(Q52*c[j]*f)));
QQ53[j]:=expand(contract(dyad(Q53*c[j]*f)));
QQ54[j]:=expand(contract(dyad(Q54*c[j]*f)));
QQ55[j]:=expand(contract(dyad(Q55*c[j]*f)));
QQ56[j]:=expand(contract(dyad(Q56*c[j]*f)));
QQ57[j]:=expand(contract(dyad(Q57*c[j]*f)));
QQ58[j]:=expand(contract(dyad(Q58*c[j]*f)));
QQ59[j]:=expand(contract(dyad(Q59*c[j]*f)));
QQ5[j]:=factor(QQ51[j]+QQ52[j]+QQ53[j]+QQ55[j]+
QQ55[j]+QQ56[j]+QQ57[j]+QQ58[j]+QQ59[j]):
od;
save(QQ5,'QQ5.array');
##########################################################################

#Building and solving the polynomial system for a,b and p
#Name of the file: solvenon

read heading;
W0c:=0: W2c:=0: W1c:=0: W4c:=0: W0:=0: W2:=0: W4:=0 :
W1:=0:
W3:=-1: W3c:=-1:
k:=0:kc:=0:sc:=0: s:=0: rc:=0 : r:=0: e:=0:ec:=0:t:=0:tc:=0:
R00:=0:R01:=0:R02:=0:R20:=0:R10:=0:
L:=0:
D(a):=0:D(b):=0: D(ac):=0:D(bc):=0: D(p):=0: D(pc):=0:
X(R11):=0;Y(R11):=0;D(R11):=0;V(R11):=0;
D(p2):=0: D(p2c):=0:
p0:=0:p0c:=0:p1:=0:p1c:=0:

read 'QQ1.array':
X(p2):=solve(QQ1[4],X(p2));
Y(p2c):=conj("):

read 'QQ2.array':
X(b):=solve(QQ2[4],X(b)):
Y(bc):=conj("):

eqns(): eqns():
D(R12):=solve(eq22+conj(eq23),D(R12));
D(R21):=conj(D(R12));
X(R12):=solve(conj(eq25),X(R12));
Y(R12):=solve(expand(eq24-eq29)/3,Y(R12));
```

```
X(R21):=conj(Y(R12));
Y(R21):=conj(X(R12));
D(R22):=solve(conj(eq24)+2*eq29,D(R22));
D(g):=solve(eq6,D(g));
D(gc):=conj("):
D(n):=solve(eq9,D(n));
D(nc):=conj(D(n));
X(a):=solve(eq12,X(a));
Y(ac):=conj(X(a));
X(l):=solve(eq13,X(l));
Y(lc):=conj(X(l));
Y_X(R12):":":
eq32:=expand(");
eq36:=conj(eq26);
X(R22):=solve(eq36,X(R22));
Y(R22):=conj(");
X(p):=solve(eq8,X(p));
Y(pc):=conj(X(p));
Y(p):=solve(eq7,Y(p));
X(pc):=conj(Y(p));
eq38:=factor(X_D(R22)-V_D(R12));
X(bc):=solve(eq38,X(bc));
Y(b):=conj(X(bc));
V(b):=solve(eq15,V(b));
V(bc):=conj(V(b));
X(m):=solve(QQ2[1],X(m));
Y(mc):=conj(X(m));

read 'QQ3.array':
eq40:=QQ3[9];
eq41:=QQ3[10];
X(p2c):=solve(eq40,X(p2c));
Y(p2):=conj(");
Y_X(p2):":
eq44:=factor(rhs(")-lhs("));
D(mc):=solve(eq44,D(mc));
D(m):=conj(");

read 'QQ4.array':
for j from 1 to 15 do
QQ4[j]:=factor(QQ4[j]);
od;
D(lc):=solve(QQ4[9],D(lc));
D(l):=conj(");

Y_X(a+2*p):":":":":
eq45:=factor(lhs(")-rhs(")):

read 'QQ5.array':
for j from 1 to 21 do
QQ5[j]:=factor(QQ5[j]);
od;
QQ5[4]:":":":":":
eq46:=factor(");
eq47:=conj(");
eq48:=op(3,eq46);
eq49:=op(3,eq47);
Xp2:=X(p2);
p2:=solve(eq48,p2);
factor(X(p2)-Xp2):":":":
eq50:=factor(");
eq51:=numer(eq49)/p2c;

factor(Y_X(a+2*p)):":":":":":":
```

```
eq53:=factor(lhs(")-rhs("));

Y_X(bc):":":":":":":factor("):
eq54:=factor(lhs(")-rhs("));
Ya1:=solve(",(Y(a)));
Y_X(p2c):":":":":":":factor("):
eq55:=factor(lhs(")-rhs("));
Ya2:=solve(",Y(a));
eq56:=factor(Ya1-Ya2);

ex1:=solve(eq53,R11);
ex2:=solve(eq50,R11);
ex3:=solve(eq56,R11);

eq1:=factor(ex1-ex2);
eq2:=factor(ex2-ex3);

neq1:=numer(eq1);
neq2:=numer(eq2);

a:=x1*p:
ac:=conj("):
b:=x2*pc:
bc:=conj("):

ne1:=factor(neq1)/(-12*p^4*pc);
ne2:=factor(neq2)/(-2*p^4*pc);
ne3:=factor(eq51)/(p*pc);
ne1c:=conj(ne1);
ne2c:=conj(ne2);

with(grobner);
X:=[x1,x2,x1c,x2c]:

F1:=[ne1,ne1c,ne2,ne2c,ne3]:

R1:=gsolve(F1);
###########################################################################
```

E.4 Maxwell's equations and Weyl's neutrino equation

```
#Bachs' tensor, condition III for the self-adjoint scalar equation,
#Maxwell equations and Weyl-neutrino equation
#Name of the file: IIIs
read heading ;

t1:= del(psi[A,B,K,G],C,Ac)*eps[K,C]:
t1:=contract(dyad(t1)):
t1:=rewrite(t1,F1dum,F1exp):
F1dum:=F1exp:
t1:=del(t1,J,Bc)*eps[G,J]:
t1:=contract(dyad(t1)):
t1:=rewrite(t1,F2dum,F2exp):
F2dum:=F2exp:
t2:=psi[A,B,C,G]*phi[K,F,Ac,Bc]*eps[K,C]*eps[F,G]:
t2:=contract(dyad(t2));
t3:=conj(t2);
t4:=conj(t1);

#We take here only the first two terms. Thus the total expression will
#be complex (not Hermitian)
```

```
TT:=t1+t2;

f:=eps[N,A]*eps[M,B]:
TTu:=contract(dyad(TT*f*conj(f)));
save(TTu,'TTu.res'):
co1s:=i[M]*i[N]:
co2s:=symm(i[M]*o[N],[M,N]):
co3s:=o[M]*o[N]:

co1co:=conj(co1s):
co2co:=conj(co2s):
co3co:=conj(co3s):

c[1]:=co1s*co1co:
c[2]:=co1s*co2co:
c[3]:=co1s*co3co:

c[4]:=co2s*co1co:
c[5]:=co2s*co2co:
c[6]:=co2s*co3co:

c[7]:=co3s*co3co:
c[8]:=co3s*co3co:
c[9]:=co3s*co3co:

tt:=array(1..9):
for j from 1 to 9 do
tt[j]:=factor(contract(dyad(c[j]*TTu)));
od;
save(tt,'tt.array');
##############################################################################

#Condition Vs
#Name of the file: Vs
#Self-adjoint scalar equation: k1:=3, k2:=4
#Maxwell equations: k1:=5, k2:=16
#Weyl-neutrino equation: k1=5, k2= 16

read heading;

T1:=k1*del(psi[A,B,C,F],K,Kc)*del(psic[Ac,Bc,Cc,Fc],H,Hc)
*eps[K,H]*eps[Kc,Hc]:
T1:=contract(dyad(T1)):
T1:=contract(dyad(T1)):

T2:= del(psic[Bc,Cc,Fc,Gc],Mc,K)*eps[Gc,Mc]:
T2:=contract(dyad(T2)):
T2:=T2*del(psi[M,A,B,C],F,Ac)*eps[K,M]:
T2:=contract(dyad(T2)):
T2:=rewrite(T2,F2dum,F2exp):
F2dum:=F2exp:
T2:=k2*T2:
T3:=conj(T2):

T4:=del(psic[Bc,Cc,Fc,Kc],Hc,F)*eps[Kc,Hc]:
T4:=contract(dyad(T4)):
T4:=rewrite(T4,H2dum,H2exp):
H2dum:=H2exp:
T4:=T4*del(psi[A,B,C,K],H,Ac)*eps[K,H]:
T4:=expand("):
T4:=contract(dyad(T4)):
T4:=rewrite(T4,H1dum,H1exp):
H1dum:=H1exp:
T4:=-2*(8*k1-k2)*T4:

T5:=del(psic[Ac,Bc,Cc,Vc],Hc,K)*eps[Vc,Hc]:
```

```
T5:=contract(dyad(T5)):
T5:=rewrite(T5,K1dum,K1exp):
K1dum:=K1exp:
T5:=del(T5,F,Fc):
T5:=contract(dyad(T5)):
T5:=rewrite(T5,K2dum,K2exp):
K2dum:=K2exp:
T5:=T5*psi[H,A,B,C]*eps[K,H]:
T5:=contract(dyad(T5)):
T5:=rewrite(T5,K3dum,K3exp):
K3dum:=K3exp:
T5:=-k2*T5:
T6:=conj(T5):

T7:=del(psic[Ac,Bc,Cc,Vc],Hc,F)*eps[Vc,Hc]:
T7:=contract(dyad(T7)):
T7:=rewrite(T7,J1dum,J1exp):
J1dum:=J1exp:
 T7:=del(T7,K,Fc):
T7:=contract(dyad(T7)):
T7:=rewrite(T7,J2dum,J2exp):
 J2dum:=J2exp:
T7:=T7*psi[H,A,B,C]*eps[K,H]:
T7:=contract(dyad(T7)):
T7:=rewrite(T7,J3dum,J3exp):
J3dum:=J3exp:
T7:=4*k1*T7:
T8:=conj(T7):

T9:=psi[V,A,B,C]*eps[K,V]*phi[F,K,Kc,Ac]*psic[Bc,Cc,Fc,Vc]*eps[Kc,Vc]:
T9:=contract(dyad(T9)):
T9:=rewrite(T9,M1dum,M1exp):
M1dum:=M1exp:
T9:=2*(k2-4*k1)*T9:
T10:=-2*(4*k1+k2)*L*psi[A,B,C,F]*psic[Ac,Bc,Cc,Fc]:
T10:=contract(dyad(T10)):
TT:=T1+T2+T3+T4+T5+T6+T7+T8+T9+T10:

f:=eps[S,A]*eps[Sc,Ac]*eps[N,B]*eps[Nc,Bc]*eps[P,C]*eps[Pc,Cc]*eps[Q,F]*
eps[Qc,Fc] :
TTu:=TT*f:

co1:=i[S]*i[N]*i[P]*o[Q]:
co2:=i[S]*i[N]*o[P]*o[Q]:
co3:=i[S]*o[N]*o[P]*o[Q]:
co4s:=o[S]*o[N]*o[P]*o[Q]:
co5s:=i[S]*i[N]*i[P]*i[Q]:

co1s:=symm(co1,[N,S,P,Q]):
co2s:=symm(co2,[N,S,P,Q]):
co3s:=symm(co3,[N,S,P,Q]):

co1co:=conj(co1s):
co2co:=conj(co2s):
co3co:=conj(co3s):
co4co:=conj(co4s):
co5co:=conj(co5s):

c[1]:=co1s*co1co:
c[2]:=co1s*co2co:
c[3]:=co1s*co3co:
c[4]:=co1s*co4co:
c[5]:=co1s*co5co:

c[6]:=co2s*co2co:
c[7]:=co2s*co3co:
c[8]:=co2s*co4co:
```

```
c[9]:=co2s*co5co:

c[10]:=co3s*co3co:
c[11]:=co3s*co4co:
c[12]:=co3s*co5co:

c[13]:=co4s*co4co:
c[14]:=co4s*co5co:

c[15]:=co5s*co5co:

save(TTu,Vten);

t5:=array(1..15):
for j from 1 to 15 do
t5[j]:=expand(contract(dyad(TTu*c[j])));
od;
save(t5,`t5sca.array`);
##########################################################################

#Five-index necessary condition of Alvarez-Wunsch,
#Maxwell equations and Weyl-neutrino equations
#Condition VIs
#Name of the file: VImaxweyl

read heading;
read headingIII;

#Term 1 of eq (12):

T1:=del(CW[J,Jc,D,Dc,E,Ec,P,Pc],U,Uc)*eps[U,J]*eps[Uc,Jc]:
T1:=contract(dyad(T1)):
T1:=contract(dyad(T1)):
T1:=rewrite(T1,A1dum,A1exp):
A1dum:=A1exp:
T1:=del(T1,K,Kc):
T1a:=T1:
T1:=rewrite(T1,A2dum,A2exp):
A2dum:=A2exp:
T1:=del(T1,C,Cc):
T1:=rewrite(T1,A3dum,A3exp):
A3dum:=A3exp:
T1:=SCW[M,Mc,A,Ac,B,Bc,Q,Qc]*eps[K,M]
*eps[Kc,Mc]*eps[P,Q]*eps[Pc,Qc]*T1:
T1:=contract(dyad(T1)):
T1:=contract(dyad(T1)):
length(T1);
save(T1,T1res):

#Other terms of eq. (12)
# 2. Terms (13) and (14):

T2:=del(SCW[N,Nc,B,Bc,C,Cc,Q,Qc],A,Ac)
*eps[K,N]*eps[Kc,Nc]*eps[P,Q]*eps[Pc,Qc]:
T2:=contract(dyad(T2)):
T2:=contract(dyad(T2)):
T1a:=rewrite(T1,B2dum,B2exp):
B2dum:=B2exp:
T2:=T2*T1a:
T2:=contract(dyad(T2)):
T2:=contract(dyad(T2)):
save(T2,T2res):

T3:=del(CW[J,Jc,D,Dc,E,Ec,K,Kc],V,Vc)*eps[V,J]*eps[Vc,Jc]:
T3:=contract(dyad(T3)):
T3:=contract(dyad(T3)):
T3:=del(T3,C,Cc):
```

```
T3:=rewrite(T3,C2dum,C2exp):
C2dum:=C2exp:
T3:=del(SCW[H,Hc,A,Ac,B,Bc,Q,Qc],U,Uc)
*eps[U,H]*eps[Uc,Hc]*eps[K,Q]*eps[Kc,Qc]*T3:
T3:=contract(dyad(T3)):
T3:=contract(dyad(T3)):
save(T3,T3res):

T4:=del(CW[K,Kc,D,Dc,E,Ec,P,Pc],H,Hc):
T4:=contract(dyad(T4)):
T4:=contract(dyad(T4)):
T4:=rewrite(T4,D1dum,D1exp):
D1dum:=D1exp:
T4:=del(T4,C,Cc):
T4:=rewrite(T4,D2dum,D2exp):
D2dum:=D2exp:
T4a:=del(SCW[G,Gc,A,Ac,B,Bc,J,Jc],U,Uc)*eps[K,G]
*eps[Kc,Gc]*eps[P,J]*eps[Pc,Jc]*eps[H,U]*eps[Hc,Uc]:
T4a:=contract(dyad(T4a)):
T4a:=contract(dyad(T4a)):
T4a:=rewrite(T4a,D3dum,D3exp):
D3dum:=D3exp:
T4:=T4a*T4:
T4:=contract(dyad(T4)):
T4:=contract(dyad(T4)):
save(T4,T4res):

T5:=del(CW[J,Jc,D,Dc,E,Ec,P,Pc],C,Cc)*eps[H,J]*eps[Hc,Jc]*LL[K,Kc,H,Hc]:
T5:=contract(dyad(T5)):
T5:=contract(dyad(T5)):
T5:=rewrite(T5,E1dum,E1exp):
E1dum:=E1exp:
T5a:=SCW[G,Gc,A,Ac,B,Bc,U,Uc]*eps[K,G]*eps[Kc,Gc]*eps[P,U]*eps[Pc,Uc]:
T5a:=contract(dyad(T5a)):
T5a:=contract(dyad(T5a)):
T5:=T5*T5a:
T5:=contract(dyad(T5)):
T5:=contract(dyad(T5)):
save(T5,T5res):

T6:=del(CW[J,Jc,C,Cc,D,Dc,P.Pc],K,Kc)*eps[H,J]*eps[Hc,Jc]*LL[E,Ec,H,Hc]:
T6:=contract(dyad(T6)):
T6:=contract(dyad(T6)):
T6:=T5a*T6:
T6:=contract(dyad(T6)):
T6:=contract(dyad(T6)):
save(T6,T6res):

T7:=del(CW[J,Jc,K,Kc,P,Pc,C,Cc],U,Uc)*eps[U,J]*eps[Uc,Jc]*LL[D,Dc,E,Ec]:
T7:=contract(dyad(T7)):
T7:=contract(dyad(T7)):
T7:=T5a*T7:
T7:=contract(dyad(T7)):
T7:=contract(dyad(T7)):
save(T7,T7res):

T8:=del(CW[J,Jc,C,Cc,D,Dc,K,Kc],U,Uc)*eps[U,J]
*eps[Uc,Jc]*LL[E,Ec,P,Pc]:
T8:=contract(dyad(T8)):
T8:=contract(dyad(T8)):
T8:=T5a*T8:
T8:=contract(dyad(T8)):
T8:=contract(dyad(T8)):
save(T8,T8res):
```

```
t3a:=-SCW[J,Jc,A,Ac,B,Bc,V,Vc]*eps[K,J]*eps[Kc,Jc]
*eps[P,V]*eps[Pc,Vc]*
CW[K,Kc,C,Cc,D,Dc,Y,Yc]*eps[H,Y]*eps[Hc,Yc]*
del(CW[M,Mc,P,Pc,H,Hc,E,Ec],U,Uc)*eps[U,M]*eps[Uc,Mc]:
t3a:=contract(dyad(t3a)):
t3a:=contract(dyad(t3a)):

t3b:=-CW[K,Kc,Y,Yc,X,Xc,A,Ac]*eps[H,Y]*eps[Hc,Yc]*eps[J,X]*eps[Jc,Xc]*
CW[P,Pc,H,Hc,J,Jc,B,Bc]*
del(SCW[F,Fc,D,Dc,E,Ec,U,Uc],C,Cc)*eps[P,U]*eps[Pc,Uc]*
eps[K,F]*eps[Kc,Fc]:
t3b:=contract(dyad(t3b)):
t3b:=contract(dyad(t3b)):

t3:=12*t3a+t3b:
save(t3,t3res):

t4a:=SCW[K,Kc,Y,Yc,X,Xc,A,Ac]*eps[G,Y]*
eps[Gc,Yc]*eps[J,X]*eps[Jc,Xc]*
CW[P,Pc,B,Bc,G,Gc,J,Jc]*del(CW[F,Fc,D,Dc,E,Ec,U,Uc],C,Cc)
*eps[P,U]*eps[Pc,Uc]*eps[K,F]*eps[Kc,Fc]:
t4a:=contract(dyad(t4a)):
t4a:=contract(dyad(t4a)):
t4:=8*t3a+t4a:
save(t4,t4res):

# 3. Raising indices and summing (building eq(8)):

f:=eps[N,A]*eps[S,B]*eps[P,C]*eps[Z,D]*eps[Q,E];
fc:=conj("):

TTu1:=4*T1-6*T2+26*T3+T4+5*T5+4*T6+4*T7-21*t8:

#Calculating components

co1:=i[N]*i[S]*i[P]*i[Q]*o[Z];
co2:=i[N]*i[S]*i[P]*o[Q]*o[Z];
co3:=i[N]*i[S]*o[P]*o[Q]*o[Z];
co4:=i[N]*o[S]*o[P]*o[Q]*o[Z];
co5s:=i[N]*i[S]*i[P]*i[Q]*i[Z];
co6s:=o[N]*o[S]*o[P]*o[Q]*o[Z];

co1s:=symm(co1,[N,S,P,Q,Z]):
co2s:=symm(co2,[N,S,P,Q,Z]):
co3s:=symm(co3,[N,S,P,Q,Z]):
co4s:=symm(co4,[N,S,P,Q,Z]):

co1co:=conj(co1s):
co2co:=conj(co2s):
co3co:=conj(co3s):
co4co:=conj(co4s):
co5co:=conj(co5s):
co6co:=conj(co6s):

c[1]:=co1s*co1co:
c[2]:=co1s*co2co:
c[3]:=co1s*co3co:
c[4]:=co1s*co4co:
c[5]:=co1s*co5co:
c[6]:=co1s*co6co:

c[7]:=co2s*co2co:
c[8]:=co2s*co3co:
c[9]:=co2s*co4co:
c[10]:=co2s*co5co:
c[11]:=co2s*co6co:
```

```
c[12]:=co3s*co3co:
c[13]:=co3s*co4co:
c[14]:=co3s*co5co:
c[15]:=co3s*co6co:

c[16]:=co4s*co4co:
c[17]:=co4s*co5co:
c[18]:=co4s*co6co:

c[19]:=co5s*co5co:
c[20]:=co5s*co6co:


c[21]:=co6s*co6co:

f:=eps[N,A]*eps[S,B]*eps[P,C]*eps[Z,D]*eps[Q,E];
fc:=conj("):

TT:=array(1..21):
for j from 1 to 21 do
TT1[j]:=expand(contract(dyad(T1*f*fc*c[j])));
TT2[j]:=expand(contract(dyad(T2*f*fc*c[j])));
TT3[j]:=expand(contract(dyad(T3*f*fc*c[j])));
TT4[j]:=expand(contract(dyad(T4*f*fc*c[j])));
TT5[j]:=expand(contract(dyad(T5*f*fc*c[j])));
TT6[j]:=expand(contract(dyad(T6*f*fc*c[j])));
TT7[j]:=expand(contract(dyad(T7*f*fc*c[j])));
TT8[j]:=expand(contract(dyad(T8*f*fc*c[j])));
tt3[j]:=expand(contract(dyad(t3*f*fc*c[j])));
tt4[j]:=expand(contract(dyad(t4*f*fc*c[j])));
TT[j]:=factor(4*TT1[j]-6*TT2[j]+26*TT3[j]+TT4[j]+5*TT5[j]+4*TT6[j]
+4*TT7[j]-21*TT8[j]+alpha1*tt3[j]+alpha2*tt4[j]);
od;
save(TT,'TT.array'):
##########################################################################

#Determination of Pfaffians for the
# Maxwell equations and Weyl-neutrino equation
#Case a*b*p<>0
#Name of the file: Pfaweylmax

read heading;
read headingIII;

#Condition III (Bach tensor)
read 'tt.array';
X(b):=solve(tt[4],X(b));
Y(bc):=conj(X(b));
eq30:=tt[1]/2;
eq31:=tt[2];
eqns();eqns();

D(R12):=solve(eq22+conj(eq23),D(R12));
D(R21):=conj(D(R12));
D(g):=solve(eq6,D(g));
D(gc):=conj("):
(R12):=solve(conj(eq25),X(R12));
Y(R12):=solve(expand(eq24-eq29)/3,Y(R12));
X(R21):=conj(Y(R12));
Y(R21):=conj(X(R12));
D(R22):=solve(conj(eq24)+2*eq29,D(R22));
D(R12):=solve(eq22+conj(eq23),D(R12));
D(R21):=conj(D(R12));
X(a):=solve(eq12,X(a));
Y(ac):=conj(X(a));
D(n):=solve(eq9,D(n));
```

```
D(nc):=conj(D(n));
X(p):=solve(eq8,X(p));
Y(pc):=conj(X(p));
Y(p):=solve(eq7,Y(p));
X(pc):=conj(Y(p));

eq32:=factor(X_D(R22)-V_D(R12));
X(bc):=solve(eq32,X(bc));
Y(b):=conj(X(bc));

#E.g.
#Maxwell equations
#Condition V
read 't5max.array';
k1:=5;k2:=16;

eq33:=factor(t5[1]);
eq34:=factor(t5[9]);

D(lc):=solve(eq40,D(lc));
D(l):=conj(");

solve(eq33,D(mc)):":":
D(mc):=factor(");

Y_X(a+2*p):":":":
eq34:=factor(");
solve(",D(m)):":":
D(m):=factor(");

D(mc)-conj(D(m)):":":":
s1:=factor(");

#read component TT[14] of condition VIs
read TT14;
eq35:=factor(t14);
X(ac):=solve(eq46,X(ac));
Y(a):=conj(X(ac));
X(pc):":":
X(pc):=factor(");
Y(p):=conj(");
###########################################################################

#Using the Pfaffians to build and solve polynomial systems for a,b and p.
#Case a*b*p<>0
#Maxwell equations
#Name of the file: solvemax

read heading;
read headingIII;

Y(a) := 192*p*bc+121*bc*a-3*a^2;
Y(b) := -b*bc+2*pc*p-2*R11-2*D(m)-4*b*p-b*a;
X(a) := Y(b)+a*ac+b*bc-2*b*a+R11;
X(b) := -b*ac-b^2;
X(p) := D(m)-pc*p+p*ac-b*p;
Y(p) := -127*p*bc-80*bc*a-3*p*a;
D(m) := 1/152*(-1520*a*b*p+1040*bc*b*a
-1228*p*R11+208*a*p*ac+1688*a*pc*p+2496
*pc*p^2-2432*b*p^2-739*a*R11+80*a^2*ac
+380*bc*R11-152*bc*pc*p+1968*bc*b*p+760*
bc*a*ac+1216*bc*p*ac+128*p^2*ac+80*pc*a^2)/(-bc+5*a+8*p);
D(mc) := 2/19*pc*a+26/19*b*bc+2/19*a*ac
+58/19*pc*p-2*b*p+2/19*p*ac-37/19*R11-D(m)-2*bc*pc;

X(bc):=conj(Y(b)):
Y(bc):=conj(X(b)):
```

```
X(pc):=conj(Y(p)):
Y(pc):=conj(X(p)):
X(ac):=conj(Y(a)):
Y(ac):=conj(X(a)):

Y_X(a):":":":
solve(",R11):
ex1:=factor(");

Y_X(a+bc):":":":
solve(",R11):
ex2:=factor(");

Y(ex2):":":":
solve(",R11):
ex3:=factor(");

#Denominators

d1:= denom(ex1);
d2:=denom(ex2);
d3:=denom(ex3);

Y(d1):":
Yd1:=factor(");
Y(Yd1):":":
YYd1:=factor(");
with(grobner);
F1:=[d1,Yd1,YYd1]:
R1:=gsolve(F1);

Y(d2):":
Yd2:=factor(");
Y(Yd2):":":
YYd2:=factor(");
F2:=[d2,Yd2,YYd2]:
R2:=gsolve(F2);

Y(d3):":
Yd3:=factor(");
F3:=[d3,Yd3]:
R3:=gsolve(F3);

#New variables
a:=x1*p:
ac:=conj("):
b:=x2*pc:
bc:=conj("):

N1:=factor(numer(ex2-ex1)/(-5776*x2c^2*p^4*pc*(8+5*x1-x2c)));
N2:=factor(numer(ex2-ex3)/(5776*x2c*p^6*pc*(8+5*x1-x2c)));

F4:=[N1,N2]:
R4:=gsolve(F4);
#########################################################################

#Using the Pfaffians to find polynomial relations between a,b and p.
#Weyl neutrino equation
#Case a*b*p<>0
#Name of the file: solveweyl

read heading;
read headingIII;

Y(a):=787/19*bc*a-3*a^2+1380/19*p*bc;
Y(b):=-b*bc+2*pc*p-2*R11-b*a-4*b*p-2*D(m);
X(a) := Y(b)+a*ac+b*bc-2*b*a+R11;
X(p) := D(m)-pc*p+p*ac-b*p;
X(b) := -b*ac-b^2;
```

```
Y(p):=-3*p*a-901/19*p*bc-512/19*bc*a;
D(m):=-1/218*(17480*p^2*ac+9728*pc*a^2-
49684*p*R11+109020*pc*p^2+61952
*a*b*bc+116988*p*b*bc+9728*a^2*ac+
78152*a*pc*p-25111*a*R11-55808
*a*b*p+50140*bc*p*ac+13952*bc*R11-
2834*bc*pc*p+27208*a*p*ac+
27904*bc*a*ac-100280*b*p^2)/(-128*a-230*p+13*bc);
D(mc):=38/109*pc*a+242/109*b*bc+38/109*a*ac-
2*b*p+346/109*pc*p+38/109*p
*ac-199/109*R11-D(m)-2*bc*pc;

X(bc):=conj(Y(b)):
Y(bc):=conj(X(b)):
X(pc):=conj(Y(p)):
Y(pc):=conj(X(p)):
X(ac):=conj(Y(a)):
Y(ac):=conj(X(a)):

Y_X(a):":":":
solve(",R11):
ex1:=factor(");

Y_X(a+bc):":":":
solve(",R11):
ex2:=factor(");

Y(ex2):":":":
solve(",R11):
ex3:=factor(");

#Denominators

d1:= denom(ex1);
d2:=denom(ex2);
d3:=denom(ex3);

Y(d1):":
Yd1:=factor(");
Y(Yd1):":":
YYd1:=factor(");
with(grobner);
F1:=[d1,Yd1,YYd1]:
R1:=gsolve(F1);

Y(d2):":
Yd2:=factor(");
Y(Yd2):":":
YYd2:=factor(");
F2:=[d2,Yd2,YYd2]:
R2:=gsolve(F2);

Y(d3):":
Yd3:=factor(");
F3:=[d3,Yd3]:
R3:=gsolve(F3);

#New variables
a:=x1*p:
ac:=conj("):
b:=x2*pc:
bc:=conj("):

N1:=factor(numer(ex2-ex1)/(-95048*x2c^2*p^4*pc*(230+128*x1-13*x2c)));
N2:=factor(numer(ex2-ex3)/(47524*x2c*p^6*pc*(230+128*x1-13*x2c)));

with(grobner):
F2:=[N1,N2]:
```

```
R2:=gsolve(F2);

for j from 1 to nops(R2) do
G[j]:=op(j,R2);
od;

#G[10]
g1:=op(5,G[10]);
g2:=op(4,G[10]);
g1c:=conj(g1):
g2c:=conj(Xg2):

H1:=[g1,g2,g1c,g2c]:
X:={x1,x2,x1c,x2c}:
R3:=gsolve(H1);

#Analysis of the solutions
for j from 1 to nops(R3) do
H[j]:=op(j,R3);
od;

#G[7]
f1:=op(3,G[7]);
f1c:=conj("):

#f1:=8859520*x1^2+35954522*x1-1909041*x1*x2c+3956264*x2c^2-2471658*
#x2c+34832476
#Apply Y over f1
#f2:=-1079685917*bc*a^2-252496320*a^3-3781787431*a*p*bc-1024703877*p*
#a^2-3287720348*p^2*bc-87748156*bc^2*a-90137805*p*bc^2-37584508*
#bc^3-992725566*p^2*a
#Y(f2)
#f3:=3794884235*bc*a^3+13984191503*bc^2*a^2+2272466880*a^4-917394512*
#bc^3*a+40424847312*p^2*bc^2-72805233934*p^2*a*bc+9222334893*p*a^
#3-11338514006*a^2*p*bc+47275438170*bc^2*a*p+112753524*bc^4-
#1918581915*bc^3*p-72103225320*p^3*bc+8934530094*p^2*a^2

H3:=[f1,f2,f3]:
gsolve(H3):

######################################################################
#Conditions for a=b=p=0
#Maxwell equations
#Name of the file: solvemax0

read heading;
read headingIII;

a:=0:ac:=0:b:=0:bc:=0:p:=0:pc:=0: R11:=0:
read 'tt.array';
for j from 1 to 9 do
t3[j]:=factor(tt[j]);
od;
eq30:=tt[1]/2;
eq31:=tt[2];
#Newman-Penrose equations
eqns();eqns();

D(R12):=solve(eq28,D(R12));
D(R21):=conj(D(R12));
Y(R21):=solve(eq25,Y(R21));
X(R12):=conj("):
Y(R12):=solve(expand(eq24-eq29)/3,Y(R12));
X(R21):=conj("):
D(R22):=solve(conj(eq24)+2*eq29,D(R22));
Y(R22):=solve(eq26,Y(R22));
X(R22):=conj("):
```

```
D(m):=solve(eq8,D(m));
D(mc):=conj("):
X(m):=solve(eq30,X(m));
Y(mc):=conj("):
X(l):=solve(eq13,X(l));
Y(lc):=conj(X(l));
X(g):=solve(eq15,X(g));
Y(gc):=conj("):
Y(g):=solve(eq18,Y(g));
X(gc):=conj("):
D(g):=solve(eq6,D(g));
D(gc):=conj("):
D(l):=solve(eq7,D(l));
D(lc):=conj("):

#Maxwell equations
# Condition Vs
k1:=5;k2:=16;
read `t5max.array`;
for j from 1 to 15 do
tt5[j]:=factor(t5[j]):
od:
eq32:=primpart(factor(t5[5]));
eq33:=primpart(factor(t5[15]));

X(mc):=solve(eq32,X(mc));
Y(m):=conj("):

#Determination of second order Pfaffians

X_V(m):":":
X(V(m)):=solve(",X(V(m)));
Y(V(mc)):=conj("):

X_V(mc):":":
X(V(mc)):=solve(",X(V(mc)));
Y(V(m)):=conj("):

X_V(g):":":
X(V(g)):=solve(",X(V(g)));
Y(V(gc)):=conj("):

X_V(gc):":":
X(V(gc)):=solve(",X(V(gc)));
Y(V(g)):=conj("):


X_V(R12):":
X(V(R12)):=solve(",X(V(R12)));
Y(V(R21)):=conj("):

X_V(R21):":
X(V(R21)):=solve(",X(V(R21)));
Y(V(R12)):=conj("):

X(eq33):":":
eq34:=factor(");
X(lc):=solve(eq34,X(lc));
Y(l):=conj("):

Y_X(l):":":":
eq35:=(factor(lhs(")-rhs(")));

X(eq35):":":
eq36:=numer(factor("));
eq37:=numer(eq35);
```

```
V(R21):=V21;
V(R12):=V12;

with(grobner):
F1:=[eq44,eq45]:
vars:={R21,R12,m,mc,V12,V21,g,gc,l,lc}:
non:={l,lc}:
R1:=gsolve(F1,vars,non);
for j from 1 to nops(R1) do
r[j]:=op(j,R1);
od;
f1:=op(4,r[1]);

g1:=op(2,r[2]);
g2:=op(3,r[2]);

V12:=solve(g2,V12);
V21:=conj("):
eq38:=factor(g1);
eq39:=numer(")/(-50*lc);

R21:=x+I*y:
R12:=x-I*y:

eq40:=factor(evalc(Re(eq39)));
y:=w^(1/2);
solve(eq47,w);
##########################################################################

#Conditions for a=b=p=0
#Weyl neutrino equation
#Name of the file: solveweyl0

read heading;
read headingIII;

a:=0:ac:=0:b:=0:bc:=0:p:=0:pc:=0: R11:=0:
read `tt.array`;
for j from 1 to 9 do
t3[j]:=factor(tt[j]);
od;
eq30:=tt[1]/2;
eq31:=tt[2];
#Newman-Penrose equations
eqns();eqns();

D(R12):=solve(eq28,D(R12));
D(R21):=conj(D(R12));
Y(R21):=solve(eq25,Y(R21));
X(R12):=conj("):
Y(R12):=solve(expand(eq24-eq29)/3,Y(R12));
X(R21):=conj("):
D(R22):=solve(conj(eq24)+2*eq29,D(R22));
Y(R22):=solve(eq26,Y(R22));
X(R22):=conj("):
D(m):=solve(eq8,D(m));
D(mc):=conj("):
X(m):=solve(eq30,X(m));
Y(mc):=conj("):
X(l):=solve(eq13,X(l));
Y(lc):=conj(X(l));
X(g):=solve(eq15,X(g));
Y(gc):=conj("):
Y(g):=solve(eq18,Y(g));
X(gc):=conj("):
```

```
D(g):=solve(eq6,D(g));
D(gc):=conj("):
D(l):=solve(eq7,D(l));
D(lc):=conj("):

#Weyl-neutrino equation
# Condition Vs
k1:=8;k2:=13;
read `t5max.array`;
for j from 1 to 15 do
tt5[j]:=factor(t5[j]):
od:
eq32:=primpart(factor(t5[5]));
eq33:=primpart(factor(t5[15]));

X(mc):=solve(eq32,X(mc));
Y(m):=conj("):

#Determination of second order Pfaffians

X_V(m):":":
X(V(m)):=solve(",X(V(m)));
Y(V(mc)):=conj("):

X_V(mc):":":
X(V(mc)):=solve(",X(V(mc)));
Y(V(m)):=conj("):

X_V(g):":":
X(V(g)):=solve(",X(V(g)));
Y(V(gc)):=conj("):

X_V(gc):":":
X(V(gc)):=solve(",X(V(gc)));
Y(V(g)):=conj("):


X_V(R12):":
X(V(R12)):=solve(",X(V(R12)));
Y(V(R21)):=conj("):

X_V(R21):":
X(V(R21)):=solve(",X(V(R21)));
Y(V(R12)):=conj("):

X(eq33):":":
eq34:=factor(");
X(lc):=solve(eq34,X(lc));
Y(l):=conj("):

Y_X(l):":":":
eq35:=(factor(lhs(")-rhs(")));

X(eq35):":":
eq36:=numer(factor("));
eq37:=numer(eq35);

V(R21):=V21;
V(R12):=V12;

with(grobner):
F1:=[eq44,eq45]:
vars:={R21,R12,m,mc,V12,V21,g,gc,l,lc}:
non:={l,lc}:
R1:=gsolve(F1,vars,non);
for j from 1 to nops(R1) do
r[j]:=op(j,R1);
```

```
od;
f1:=op(4,r[1]);

g1:=op(2,r[2]);
g2:=op(3,r[2]);

V12:=solve(g2,V12);
V21:=conj("):
eq38:=factor(g1);
eq39:=numer(")/(-36*lc);

R21:=x+I*y:
R12:=x-I*y:

eq40:=factor(evalc(Re(eq39)));
y:=w^(1/2);
solve(eq40,w);
###########################################################################
```

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