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Nonlinear functional integrodifferential evolution equations with nonlocal conditions in Banach spaces

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Abstract. In this paper, the Leray-Schauder Alternative is used to investigate the existence of mild solutions to first-order nonlinear functional integrodifferential evolution equations with nonlocal conditions in Banach spaces.
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1. Introduction

This paper is concerned with the existence of mild solutions for nonlinear functional integrodifferential evolution equations with nonlocal conditions. More precisely, we consider the following nonlocal Cauchy problem on a general Banach space X :

$$\begin{aligned} &x'(t) = A(t)x(t) + F\left(t, x(\sigma_1(t)), ..., x(\sigma_n(t)), \int_0^t h(t,s,x(\sigma_{n+1}(t)))ds\right), \quad t \in J, \\ &x(0) + g(x) = x_0, \end{aligned} \tag{1}$$

where $J = [0, b]$, the family $\{A(t) : 0 \leq t \leq b\}$ of an unbounded linear operator generates linear evolution systems. Nonlinear operators $F : J \times X^{n+1} \to X$, $h :$ $J \times J \times X$, $g : C(J,X) \to X$, $\sigma_i : J \to J, i = 1, ..., n+1$, are given functions.

The study of abstract nonlocal Cauchy problem was motivated by the paper of Byszewski and Lakshmikantham [4, 5]. In [5], the author has considered the existence and uniqueness of mild, strong, and classical solutions of the nonlocal Cauchy problem, where the operator $A(t) = A$ generates a strongly continuous semigroup. Subsequently, many papers have been interested in the nonlocal Cauchy problem and that stems mainly from the observation that nonlocal conditions are more realistic than the usual ones in treating physical problems, e.g. Byszewski and Akca [6], Ntouyas and Tsamatos [15], Lin and Liu [14], Ezzinbi and Fu [9], Liang *et al.* [13], Benchohra *et al.* [3], Ezzinbi and Liu [8], Aizicovici and Mckibben [1].

The fundamental tools used in the existence proofs of all above-mentioned works are essentially fixed-point arguments; Schauder's fixed point theorem [6], Leray-Schauder Alternative [15], the Sadovskii fixed point theorem [12], Schaefer' fixed point theorem [2].

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In this paper, we shall investigate the existence of mild solutions of a nonlocal Cauchy problem (1) in Banach spaces, by means of a different method, that is, by using the theory of evolution families, Banach's contraction principle and Leray-Schauder Alternative.

This paper will be organized as follows. In Section 2, we will briefly recall some preliminary facts which will be used in the paper. Section 3 is devoted to the existence of mild solutions of problem (1). Finally, a concrete example is presented in Section 4 to show the application of our main results.

2. Preliminaries

Throughout this section, for the family $\{A(t) : 0 \leq t \leq b\}$ of linear operators, we need the following assumptions (see [11]).

(I) The domain $D(A)$ of $\{A(t) : 0 \leq t \leq b\}$ is dense in the Banach space X and independent of t, $A(t)$ is a closed linear operator.

(II) For each $t \in J$, the resolvent $R(\lambda, A(t))$ exists for all λ with $\mathrm{Re}\lambda \geq 0$ and there exists $K > 0$ such that

$$\| R(\lambda, A(t)) \| \leq \frac{K}{(|\lambda| + 1)}.$$

(III) For any $t, s, \tau \in J$, there exists a $0 < \delta < 1$ and $K > 0$ such that

$$\| (A(t) - A(\tau))A^{-1}(s) \| \leq K|t - \tau|^{\delta},$$

and for each $t \in J$ and some $\lambda \in \rho(A(t))$, the resolvent $R(\lambda, A(t))$ set of $A(t)$ is a compact operator.

Definition 1 (see [16]). *A family of linear operators $\{U(t,s) : 0 \leq s \leq t \leq b\}$ on X is called an evolution system if the following two conditions are satisfied:*

(a) $U(t,s) \in L(X)$ the space of bounded linear transformations on X whenever $0 \leq s \leq t \leq b$ and for each $x \in X$ the mapping $(t,s) \to U(t,s)x$ is continuous;

(b) $U(t,s)U(s,\tau) = U(t,\tau)$ whenever $0 \leq \tau \leq s \leq t \leq b$.

Definition 2. *A continuous function $x(\cdot) : J \to X$ is said to be a mild solution to problem* (1) *if for all $x_0 \in X$, it satisfies the following integral equation*

$$x(t) = U(t,0)[x_0 - g(x)] + \int_0^t U(t,s)F\left(s, x(\sigma_1(s)), ..., x(\sigma_n(s)), \int_0^s h(s,\tau,x(\sigma_{n+1}(\tau)))d\tau\right)ds. \quad (2)$$

Lemma 1 (see [10]). *If conditions (I)-(III) are satisfied, then the family $\{A(t) : 0 \leq t \leq b\}$ generates a unique linear evolution system $\{U(t,s) : 0 \leq s \leq t \leq b\}$ is a compact linear operator on X whenever $t - s > 0 (0 \leq s < t \leq b)$.*

Lemma 2 (Leray-Schauder Nonlinear Alternative [7]). *Let X be a Banach space with $\Omega \subset X$ convex. Assume V is a relatively open subset of Ω with $0 \in V$ and $G : \overline{V} \to \Omega$ is a compact map. Then either*

(I) G has a fixed point in V, or

(II) there exists a point $v \in \partial V$ such that $v \in \lambda G(v)$ for some $\lambda \in (0,1)$.

Further we assume the following hypotheses:

(H1) $U(t,s)$ is a compact linear operator on X whenever $t-s>0$ and there exists a constant $M>0$, such that $\| U(t,s) \| \leq M, \quad 0 \leq s < t \leq b$.

(H2) The function $F : J \times X^{n+1} \to X$ is continuous and there exist constants $L>0, L_1 \geq 0$, such that for all $x_i, y_i \in X, i=1,...n+1$, we have

$$\| F(t,x_1,x_2,...,x_{n+1}) - F(t,y_1,y_2,...,y_{n+1}) \| \leq L\Big[\sum_{i=1}^{n+1} \| x_i - y_i \|\Big],$$

and

$$L_1 = \max_{t \in J} \| F(t,0,...,0) \| .$$

(H3) The function $h : J \times J \times X \to X$ is continuous and there exist constants $N>0, N_1 \geq 0$, such that for all $x,y \in X$,

$$\| h(t,s,x) - h(t,s,y) \| \leq N \| x-y \|,$$

and

$$N_1 = \max_{0 \leq s \leq t \leq b} \| h(t,s,0) \| .$$

(H4) $\sigma_i : J \to J, i=1,...,n+1$, are continuous functions such that $\sigma_i(t) \leq t, i = 1,...,n+1$.

(H5) (i)The function $g(\cdot) : C(J,X) \to X$ is continuous and there exists a $\delta \in (0,b)$ such that $g(\phi) = g(\psi)$ for any $\phi, \psi \in C := C(J,X)$ with $\phi = \psi$ on $[\delta, b]$.

(ii) There is a continuous nondecreasing function $\Lambda : [0,\infty) \to (0,\infty)$ such that

$$\| g(\phi) \| \leq \Lambda(\| \phi \|), \qquad \phi \in C.$$

(H6) There exists a constant $M^* > 0$ such that

$$\frac{M^*}{[M(\| x_0 \| + \Lambda(M^*)) + Mb(LbN_1 + L_1)]K_0} > 1,$$

where $K_0 = e^{ML(n+Nb)b}$.

3. Main result

Theorem 1. *Suppose that assumptions (H1)-(H6) are satisfied. Then the nonlocal Cauchy problem* (1) *has at least one mild solution on J.*

Proof. Consider the space $C := C(J,X)$ the Banach space of all continuous functions from J to X endowed with sup norm.

Let $L_0 := 2ML(n+Nb)$ and we introduce in the space C the equivalent norm defined as

$$\| \phi \|_V := \sup_{t \in J} e^{-L_0 t} \| \phi(t) \| .$$

Then, it is easy to see that $V := (C(J,X), \|\cdot\|_V)$ is a Banach space. Fix $v \in C$ and for $t \in J, \phi \in V$, we now define an operator

$$(Q_v\phi)(t) = U(t,0)[x_0 - g(v)] + \int_0^t U(t,s)F\left(s, \phi(\sigma_1(s)), ..., \phi(\sigma_n(s)), \int_0^s h(s,\tau,\phi(\sigma_{n+1}(\tau)))d\tau\right)ds. \quad (3)$$

Since $U(\cdot,0)(x_0 - g(v)) \in C(J,X)$, so it follows from (H1)-(H4) that $(Q_v\phi)(t) \in V$ for all $\phi \in V$. Let $\phi, \psi \in V$, we have

$$\begin{aligned}
& e^{-L_0 t} \parallel (Q_v\phi)(t) - (Q_v\psi)(t) \parallel \\
& \leq e^{-L_0 t} \int_0^t \left\| U(t,s)\left[F\left(s, \phi(\sigma_1(s)), ..., \phi(\sigma_n(s)), \int_0^s h(s,\tau,\phi(\sigma_{n+1}(\tau)))d\tau\right)\right.\right. \\
& \quad \left.\left. -F\left(s, \psi(\sigma_1(s)), ..., \psi(\sigma_n(s)), \int_0^s h(s,\tau,\psi(\sigma_{n+1}(\tau)))d\tau\right)\right]\right\| ds \\
& \leq ML \int_0^t e^{-L_0 t}\Big[\parallel \phi(\sigma_1(s)) - \psi(\sigma_1(s)) \parallel + \cdots + \parallel \phi(\sigma_n(s)) - \psi(\sigma_n(s)) \parallel \\
& \quad + \left\| \int_0^s h(s,\tau,\phi(\sigma_{n+1}(\tau)))d\tau - \int_0^s h(s,\tau,\psi(\sigma_{n+1}(\tau)))d\tau \right\| \Big] ds \\
& \leq ML \int_0^t e^{-L_0 t}\Big[\parallel \phi(s) - \psi(s) \parallel + \cdots + \parallel \phi(s) - \psi(s) \parallel \\
& \quad + N \int_0^s \parallel \phi(\sigma_{n+1}(\tau)) - \psi(\sigma_{n+1}(\tau)) \parallel d\tau \Big] ds \\
& \leq ML \int_0^t e^{-L_0 t}\Big[n \parallel \phi(s) - \psi(s) \parallel + Nb \parallel \phi(s) - \psi(s) \parallel \Big] ds \\
& \leq ML \int_0^t e^{L_0((s-t)}\Big[ne^{-L_0 s} \parallel \phi(s) - \psi(s) \parallel + Nb \sup_{s \in J} e^{-L_0 s} \parallel \phi(s) - \psi(s) \parallel \Big] ds \\
& \leq ML(n+Nb) \int_0^t e^{L_0(s-t)} ds \parallel \phi - \psi \parallel_V \\
& \leq \frac{ML(n+Nb)}{L_0} \parallel \phi - \psi \parallel_V, \quad t \in J,
\end{aligned}$$

which implies that

$$e^{-L_0 t} \parallel (Q_v\phi)(t) - (Q_v\psi)(t) \parallel \leq \frac{1}{2} \parallel \phi - \psi \parallel_V, \quad t \in J.$$

Thus

$$\parallel Q_v\phi - Q_v\psi \parallel_V \leq \frac{1}{2} \parallel \phi - \psi \parallel_V, \quad \phi, \psi \in V.$$

Therefore, Q_v is a strict contraction. By Banach's contraction principle we conclude that Q_v has a unique fixed point $\phi_v \in V$ and equation (3) has a unique mild solution on $[0,b]$. Set

$$\tilde{v}(t) := \begin{cases} v(t) & \text{if } t \in (\delta, b], \\ v(\delta) & \text{if } t \in [0, \delta]. \end{cases}$$

From (3), we have

$$\phi_{\tilde v}(t) = U(t,0)[x_0 - g(\tilde v)] + \int_0^t U(t,s)F\left(s, \phi_{\tilde v}(\sigma_1(s)), ..., \phi_{\tilde v}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde v}(\sigma_{n+1}(\tau))d\tau\right)ds. \tag{4}$$

Consider the map, $P : C_\delta = C([\delta, b], X) \to C_\delta$ defined by

$$(Pv)(t) = \phi_{\tilde v}(t), \quad t \in [\delta, b]. \tag{5}$$

We shall show that P satisfy all conditions of Lemma 2. The proof will be given in several steps.

Step 1. P maps bounded sets into bounded sets in C_δ.

Indeed, it is enough to show that there exists a positive constant $\mathcal{L}$ such that for each $v \in C_r(\delta) := \{\phi \in C_\delta; \sup\limits_{\delta \le t \le b} \| \phi(t) \| \le r\}$ one has $\| Pv \| \le \mathcal{L}$.

Let $v \in C_r(\delta)$, then for $t \in (0, b]$, we have

$$\begin{aligned}
\| \phi_{\tilde v}(t) \| \; &\le \| U(t,0)[x_0 - g(\tilde v)] \| \\
&\quad + \int_0^t \left\| U(t,s)F\left(s, \phi_{\tilde v}(\sigma_1(s)), ..., \phi_{\tilde v}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde v}(\sigma_{n+1}(\tau))d\tau\right)\right\| ds \\
&\le M[\| x_0 + g(\tilde v) \|] \\
&\quad + M\int_0^t \left[\left\| F\left(s, \phi_{\tilde v}(\sigma_1(s)), ..., \phi_{\tilde v}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde v}(\sigma_{n+1}(\tau))d\tau\right)\right.\right. \\
&\quad \left.\left. - F(s,0,...,0)\right\| + \| F(s,0,...,0) \|\right] ds \\
&\le M[\| x_0 \| + \| g(\tilde v) \|] + M\int_0^t \left\{ L\left[\| \phi_{\tilde v}(s) \| + \cdots + \| \phi_{\tilde v}(s) \| \right.\right. \\
&\quad \left.\left. + \int_0^s [\| h(s,\tau,\phi_{\tilde v}(\sigma_{n+1}(\tau)) - h(s,\tau,0) \| + \| h(s,\tau,0) \|] d\tau \right] + L_1 \right\} ds \\
&\le M[\| x_0 \| + \Lambda(r)] + M\int_0^t \left\{ L\left[n \| \phi_{\tilde v}(s) \| + b(N \| \phi_{\tilde v}(s) \| + N_1)\right] + L_1 \right\} ds \\
&\le M[\| x_0 \| + \Lambda(r)] + Mb(LbN_1 + L_1) + ML(n + Nb)\int_0^t \sup_{s \in (0,b]} \| \phi_{\tilde v}(s) \| \, ds.
\end{aligned}$$

Thus

$$\| Pv \| \le M[\| x_0 \| + \Lambda(r)] + Mb(LbN_1 + L_1) + ML(n + Nb)rb = \mathcal{L}.$$

Step 2. P maps bounded sets into equicontinuous sets of C_δ.

Let $\delta \le t_1 < t_2 \le b$, and $C_r(\delta)$ a bounded set as in Step 1. Let $v \in C_r(\delta)$, we have

$$
\begin{aligned}
&\| Pv(t_2) - Pv(t_1) \| \\
&\quad \leq \| [U(t_2,0) - U(t_1,0)][x_0 - g(\tilde{v})] \| \\
&\qquad + \int_0^{t_2} \left\| [U(t_2,s)F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right) \right\| ds \\
&\qquad - \int_0^{t_1} \left\| [U(t_1,s)F\left(s, \phi_{\tilde{v}}\sigma_1(s)), ..., \phi_{\tilde{v}}\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right) \right\| ds \\
&\quad \leq \| U(t_2,0) - U(t_1,0) \| [\| x_0 \| + \Lambda(r)] + \int_0^{t_1} \| U(t_2,s) - U(t_1,s) \| \\
&\qquad \left\| F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right) \right\| ds \\
&\qquad + M \int_{t_1}^{t_2} \left\| F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}_2}(\sigma_{n+1}(\tau)))d\tau\right) \right\| ds.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
&\left\| F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right) \right\| \\
&\quad \leq \left\| F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right) \right. \\
&\qquad\qquad \left. - F(s,0,...,0) \right\| + \| F(s,0,...,0) \| \\
&\quad \leq L\left[\| \phi_{\tilde{v}}(\sigma_1(s)) \| + \cdots + \| \phi_{\tilde{v}}(\sigma_n(s)) \| + \left\| \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau \right\| \right] + L_1 \\
&\quad \leq L\left[\| \phi_{\tilde{v}}(s) \| + \cdots + \| \phi_{\tilde{v}}(s) \| + \int_0^s [\| h(s,\tau,\phi_{\tilde{v}}(\tau)) - h(s,\tau,0) \| \right. \\
&\qquad \left. + \| h(s,\tau,0) \|]d\tau \right] + L_1 \\
&\quad \leq L\left[n \| \phi_{\tilde{v}}(s) \| + b[N \sup_{s\in[\delta,b]} \| \phi_{\tilde{v}}(s) \| + N_1] \right] + L_1 \\
&\quad \leq L\left[(n+Nb) \sup_{s\in[\delta,b]} \| \phi_{\tilde{v}}(s) \| + bN_1 \right] + L_1 \leq L[(n+Nb)r + bN_1] + L_1.
\end{aligned}
$$

We see that $\| Pv(t_2) - Pv(t_1) \|$ tend to zero independently of $v \in C_r(\delta)$ as $t_2 - t_1 \to 0$, since the compactness of $U(t,s)$ for $t - s > 0$ implies the continuity in the uniform operator topology. Thus the family of functions $\{(Pv) : v \in C_r(\delta)\}$ is equicontinuous on $[\delta, b]$.

Step 3. The set $\{P(v)(t) : v \in C_r(\delta)\}$ is relatively compact in C_δ.

Let $\delta < t \leq s \leq b$ be fixed and ε a real number satisfying $0 < \varepsilon < t$, for $v \in C_r(\delta)$, we define

$$
\begin{aligned}
(P_\varepsilon v)(t) &= U(t,0)[x_0 - g(\tilde{v})] \\
&\quad + \int_0^{t-\varepsilon} U(t,s)F\left(s, \phi_{\tilde{v}}\sigma_1(s)), ..., \phi_{\tilde{v}}\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right)ds \\
&= U(t,0)[x_0 - g(\tilde{v})] + U(t,t-\varepsilon) \\
&\quad \times \int_0^{t-\varepsilon} U(t-\varepsilon,s)F\left(s, \phi_{\tilde{v}}\sigma_1(s)), ..., \phi_{\tilde{v}}\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right)ds.
\end{aligned}
$$

Using the compactness of $U(t,s)$ for $t-s>0$, we obtain the set $\{(P_\varepsilon v)(t) : v \in C_r(\delta)\}$ is precompact $v \in C_r(\delta)$ for every $\varepsilon, 0 < \varepsilon < t$. Moreover, for every $v \in C_r(\delta)$ we have

$$
\begin{aligned}
&\| (Pv)(t) - (P_\varepsilon v)(t) \| \\
&\leq \int_{t-\varepsilon}^t \left\| U(t,s)F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right)\right\| ds \\
&\leq M\int_{t-\varepsilon}^t \left[\left\| F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}}(\sigma_{n+1}(\tau)))d\tau\right)\right\|\right] ds \\
&\leq M\int_{t-\varepsilon}^t \left[L(n+Nb)\sup_{s\in[\delta,b]} \| \phi_{\tilde{v}}(s) \| + LbN_1 + L_1\right] ds \\
&\leq M\int_{t-\varepsilon}^t [L(n+Nb)r + LbN_1 + L_1] ds.
\end{aligned}
$$

Therefore, there are precompact sets arbitrarily close to the set $\{(Pv) : v \in C_r(\delta)\}$. Hence the set $\{(Pv) : v \in C_r(\delta)\}$ is a precompact in C_δ.

Step 4. $P : C_\delta \to C_\delta$ is continuous.

From (4) and (H1)-(H5), we deduce that for $v_1, v_2 \in C_r(\delta),\ t \in (0,b]$,

$$
\begin{aligned}
&\| \phi_{\tilde{v}_1}(t) - \phi_{\tilde{v}_2}(t) \| \\
&\leq \| U(t,0)[g(\tilde{v}_1) - g(\tilde{v}_2)] \| \\
&\quad + \int_0^t \left\| U(t,s)\left[F\left(s, \phi_{\tilde{v}_1}(\sigma_1(s)), ..., \phi_{\tilde{v}_1}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}_1}(\sigma_{n+1}(\tau)))d\tau\right)\right.\right. \\
&\quad \left.\left. - F\left(s, \phi_{\tilde{v}_2}(\sigma_1(s)), ..., \phi_{\tilde{v}_2}(\sigma_n(s)), \int_0^s h(s,\tau,\phi_{\tilde{v}_2}(\sigma_{n+1}(\tau)))\right)\right]\right\| ds \\
&\leq M \| g(\tilde{v}_1) - g(\tilde{v}_2) \| \\
&\quad + M\int_0^t L\Big[\| \phi_{\tilde{v}_1}(\sigma_1(s)) - \phi_{\tilde{v}_2}(\sigma_1(s)) \| + \cdots + \| \phi_{\tilde{v}_1}(\sigma_n(s)) - \phi_{\tilde{v}_2}(\sigma_n(s)) \| \\
&\quad + \left\| \int_0^s h(s,\tau,\phi_{\tilde{v}_1}(\sigma_{n+1}(\tau)))d\tau)) - \int_0^s h(s,\tau,\phi_{\tilde{v}_2}(\sigma_{n+1}(\tau)))d\tau \right\|\Big] ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq M \parallel g(\tilde{v}_1) - g(\tilde{v}_2) \parallel \\
&\quad + ML \int_0^t \Bigg[\parallel \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \parallel + \cdots + \parallel \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \parallel \\
&\quad + N \int_0^s [\parallel \phi_{\tilde{v}_1}(\sigma_{n+1}(s)) - \phi_{\tilde{v}_2}(\sigma_{n+1}(s)) \parallel \Bigg] ds \\
&\leq M \parallel g(\tilde{v}_1) - g(\tilde{v}_2) \parallel + ML \int_0^t \Bigg[n \parallel \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \parallel \\
&\quad + Nb \parallel \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \parallel \Bigg] ds \\
&\leq M \parallel g(\tilde{v}_1) - g(\tilde{v}_2) \parallel + ML(n+Nb) \int_0^t \sup_{s \in (0,b]} \parallel \phi_{\tilde{v}_1}(s) - \phi_{\tilde{v}_2}(s) \parallel ds.
\end{aligned}
$$

Using again the Gronwall's inequality, that for t, v_1, v_2 as above

$$
\sup_{t \in (0,b]} \parallel \phi_{\tilde{v}_1}(t) - \phi_{\tilde{v}_2}(t) \parallel \leq M e^{ML(n+Nb)b} \parallel g(\tilde{v}_1) - g(\tilde{v}_2) \parallel,
$$

for all $t \in [0, b]$, which implies that

$$
\parallel Pv_1 - Pv_2 \parallel \leq M e^{[ML(n+Nb)+\beta]b} \parallel g(\tilde{v}_1) - g(\tilde{v}_2) \parallel .
$$

for all $t \in [\delta, b]$, $v_1, v_2 \in C_r(\delta)$. Therefore, P is continuous.

Step 5. We now show that there exists an open set $V^* \subseteq C_\delta$ with $v \notin \lambda Pv$ for $\lambda \in (0,1)$ and $v \in \partial V^*$. Let $\lambda \in (0,1)$ and let $v \in C_\delta$ be a possible solution of $v = \lambda P(v)$ for some $0 < \lambda < 1$. Thus, for each $t \in (0, b]$,

$$
\begin{aligned}
v(t) = \lambda \phi_{\tilde{v}}(t) = \lambda U(t,0)[x_0 - g(\tilde{v})] \\
+ \lambda \int_0^t U(t,s) F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s, \tau, \phi_{\tilde{v}}(\sigma_{n+1}(\tau)) d\tau \right) ds.
\end{aligned} \tag{6}
$$

This implies by (H1)-(H5) that for each $t \in J$ we have $\parallel v(t) \parallel \leq \parallel \phi_{\tilde{v}}(t) \parallel$ and

$$
\begin{aligned}
\parallel \phi_{\tilde{v}}(t) \parallel &\leq M[\parallel x_0 \parallel + g(\tilde{v}) \parallel] \\
&\quad + M \int_0^t \left[\left\| F\left(s, \phi_{\tilde{v}}(\sigma_1(s)), ..., \phi_{\tilde{v}}(\sigma_n(s)), \int_0^s h(s, \tau, \phi_{\tilde{v}}(\sigma_{n+1}(\tau)) d\tau \right) \right\| \right] ds \\
&\leq M[\parallel x_0 \parallel + \Lambda(\parallel \tilde{v} \parallel)] + Mb(LbN_1 + L_1) \\
&\quad + ML(n+Nb) \int_0^t \sup_{s \in (0,b]} \parallel \phi_{\tilde{v}}(s) \parallel ds.
\end{aligned}
$$

Making use of the Gronwall's inequality, such that

$$
\sup_{t \in (0,b]} \parallel \phi_{\tilde{v}}(t) \parallel \leq \Big[M(\parallel x_0 \parallel + \Lambda(\parallel \tilde{v} \parallel)) + Mb(LbN_1 + L_1) \Big] e^{ML(n+Nb)b},
$$

and the previous inequality holds. Consequently,

$$
\parallel v \parallel \leq \Big[M(\parallel x_0 \parallel + \Lambda(\parallel \tilde{v} \parallel)) + Mb(LbN_1 + L_1) \Big] e^{ML(n+Nb)b},
$$

and therefore

$$\frac{\| v \|}{[M(\| x_0 \| + \Lambda(\| \tilde{v} \|)) + Mb(LbN_1 + L_1)]K_0} \leq 1.$$

Then, by (H6), there exists M^* such that $\| v \| \neq M^*$. Set

$$V^* = \{v \in C([\delta, b], X); \sup_{\delta \leq t \leq b} \| v(t) \| < M^*\}.$$

As a consequence of Steps 1-4, together with the Arzela-Ascoli theorem it suffices to show that $P : \overline{V^*} \to C_\delta$ is a compact map.

From the choice of V^*, there is no $x \in \partial V^*$ such that $v \in \lambda Pv$ for $\lambda \in (0, 1)$. As a consequence of Lemma 2, we deduce that P has a fixed point $\overline{V^*}$. Let $x = \phi_{\tilde{v}_*}$. Then, we have

$$\begin{aligned} x(t) = {} & U(t,0)[x_0 - g(\tilde{v}_*)] \\ & + \int_0^t U(t,s) F\left(s, x(\sigma_1(s)), ..., x(\sigma_n(s)), \int_0^s k(s,\tau) h(\tau, x(\sigma_{n+1}(\tau))) d\tau\right) ds. \end{aligned} \tag{7}$$

Noting that $x = \phi_{\tilde{v}_*} = (P\tilde{v}_*)(t) = \tilde{v}_*$, $t \in [\delta, b]$. By (H5)(i), we obtain

$$g(x) = g(\tilde{v}_*).$$

This implies that x is Q has a fixed point in $\overline{V^*} \subset C(J, X)$. Hence, problem (1) has a mild solution and completes the proof of Theorem 1. □

Remark 1. *In [6], Byszewski and Akca discussed a related semilinear nonlocal problem when g is convex and compact on a given ball. In this paper, we consider the case when g is continuous but without imposing severe compactness conditions and convexity.*

Remark 2. *Condition (H5) on g in the above theorem is an extension of the corresponding conditions in paper [15], [13].*

4. Application

To illustrate the application of the obtained results of this paper, we study the following example in this section:

$$\begin{aligned} z_t(t,x) &= \frac{\partial^2}{\partial x^2} a_0(t,x) z(t,x) \\ & \quad + a_1(t) z(\sin t, x) + \sin z(t,x) + \frac{1}{1+t^2} \int_0^t a_2(s) z(\sin s, x) ds, \\ z(t,0) &= z(t,\pi) = 0, \\ z(0,x) &+ \int_\delta^1 [z(s,x) + \log(1 + |z(s,x)|)] ds = z_0(x), \quad 0 \leq t \leq 1, 0 \leq x \leq \pi, \end{aligned} \tag{8}$$

where $\delta > 0$, $z_0(x) \in X = L^2([0,\pi])$ and $z_0(0) = z_0(\pi) = 0$. Here, the function $a_0(t,x)$ is continuous and uniformly Hölder continuous in t.

Let $X = L^2([0,\pi])$ and the operators $A(t)$ be defined by

$$A(t)w = a_0(t,x)w^{''}$$

with the domain $D(A) = \{w \in X : w, w^{''}$ are absolutely continuous, $w^{''} \in X, w(0) = w(1) = 0\}$, then $A(t)$ generates an evolution system $U(t,s)$ satisfying assumptions (I)-(III)(see [11]).

We assume that the function $a_i(\cdot)$ is continuous on $[0,1]$, and $l_i = \sup\limits_{0\leq s\leq 1} |a_i(s)| < 1, i = 1,2$.

Define $F : [0,1]\times X\times X \to X, h : [0,1]\times[0,1]\times X \to X$ and $g : C([0,1],X) \to X$ by

$$F\biggl(t,z,\int_0^t h(t,s,z(\sigma(s)))ds\biggr)(x) = a_1(t)z(\sin t,x) + \sin z(t,x) + \frac{1}{1+t^2}\int_0^t a_2(s)z(\sin s,x)ds,$$

$$\int_0^t h(t,s,z(\sigma(s)))(x)ds = \frac{1}{1+t^2}\int_0^t a_2(s)z(\sin s,x)ds,$$

and

$$g(z)(x) = \int_\delta^1 [z(s,x) + \log(1+|z(s,x)|)]ds, \quad x \in C([0,1],X).$$

It is easy to see that with these choices, the assumptions (H1)-(H5) of Theorem 1 are satisfied. In particular, the constants are $L = 1 + l_1 + l_2, N = l_2$ and $L_1 = N_1 = 0$. If we assume that $Me^{M(1+l_1+l_2)(1+l_2)}(1-\delta) < \frac{1}{2}$ and choose the constant $M^* = \max\{8M \parallel x_0 \parallel e^{M(1+l_1+l_2)(1+l_2)}, 1\}$, then

$$1 > \frac{M[\parallel x_0 \parallel + (M^* + \log(1+M^*))(1-\delta)]e^{M(1+l_1+l_2)(1+l_2)}}{M^*}.$$

Now condition (H6) in Section 2 holds and hence by Theorem 1, we deduce that nonlocal Cauchy problem (8) has a mild solution on $[0,1]$.

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