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On weighted Ostrowski type inequalities in $L_1(a,b)$ spaces

Arif Rafiq[1,*] and Farooq Ahmad[2]

[1] *Mathematics Department, COMSATS Institute of Information Technology, Plot # 30, Sector H-8/1, Islamabad 44 000, Pakistan*
[2] *Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan 60 800, Pakistan*

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Abstract. The main aim of this paper is to establish weighted Ostrowski type inequalities for the product of two continuous functions whose derivatives are in $L_1(a,b)$ spaces. Our results also provide new weighted estimates on these inequalities.

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1. Introduction

In 1938, Ostrowski proved the following inequality ([7], see also [6, page 468]):

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\overset{0}{I}$ (interior of I), and let $a, b \in \overset{0}{I}$ with $a < b$. If $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\, \|f'\|_\infty , \qquad (1)$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In 2005, Pachpatte [9] established a new inequality of the type (1) involving two functions and their derivatives as given in the following theorem:

*Corresponding author. *Email addresses:* arafiq@comsats.edu.pk (A. Rafiq), farooqgujar@gmail.com (F. Ahmad)

Theorem 2. *Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions on $[a, b]$ and differentiable on (a, b), whose derivatives $f', g' : (a, b) \to \mathbb{R}$ are bounded on (a, b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, $\|g'\|_\infty := \sup_{t\in(a,b)} |g'(t)| < \infty$, then*

$$\left| f(x)\, g(x) - \frac{1}{2(b-a)} \left(g(x) \int_a^b f(y)\, dy + f(x) \int_a^b g(y)\, dy \right) \right|$$
$$\leq \frac{1}{2} \left(|g(x)|\, \|f'\|_\infty + |f(x)|\, \|g'\|_\infty \right) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a), \tag{2}$$

for all $x \in [a, b]$.

In [3], Dragomir and Wang established another Ostrowski like inequality for $\|.\|_1$ −norm as given in the following theorem:

Theorem 3. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a differentiable mapping on (a, b), whose derivative $f' : [a, b] \longrightarrow \mathbb{R}$ belongs to $\mathbf{L}_1(a, b)$. Then, we have the inequality:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left(\frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} \right) \|f'\|_1, \tag{3}$$

for all $x \in [a, b]$.

Mir and Arif obtained the inequality for $L_1(a, b)$ spaces [4], given in the form of the following theorem:

Theorem 4. *Let f, $g : [a, b] \to \mathbb{R}$ be continuous mappings on $[a, b]$ and differentiable on (a, b), whose derivatives $f', g' : (a, b) \to \mathbb{R}$ belong to $\mathbf{L}_1(a, b)$, i.e., $\|f'\|_1 = \int_a^b |f(t)|\, dt < \infty$, $\|g'\|_1 = \int_a^b |g(t)|\, dt < \infty$, then*

$$\left| f(x)\, g(x) - \frac{1}{2(b-a)} \left(g(x) \int_a^b f(y)\, dy + f(x) \int_a^b g(y)\, dy \right) \right|$$
$$\leq \frac{1}{2} \left(|g(x)|\, \|f'\|_1 + |f(x)|\, \|g'\|_1 \right) \left(\frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} \right), \tag{4}$$

for all $x, y \in [a, b]$.

In the last few years, the study of such inequalities has been the focus of many mathematicians and a number of research papers have appeared which deal with various generalizations, extensions and variants (see for example [2, 4, 6, 8] and references therein). Inspired and motivated by the research work going on related to inequalities $(1 - 4)$, we establish here new weighted Ostrowski type inequalities for the product of two continuous functions whose derivatives are in $\mathbf{L}_1(a, b)$. Our proofs are of independent interest and provide new estimates on these types of inequalities.

2. Main results

Let the weight $w : [a,b] \to [0,\infty)$ be non-negative, integrable and

$$\int_a^b w(t)\,dt < \infty.$$

The domain of w may be finite or infinite. We denote the zero moment as

$$m(a,b) = \int_a^b w(t)dt.$$

For any function $\phi \in L_1[a,b]$, we define $\|\phi\|_{w,1} = \int_a^b w(t)\,|\phi(t)|\,dt$ and $\|\phi\|_{w,1,[y,x]} = \int_y^x w(t)\,|\phi(t)|\,dt$ for all $y, x \in [a,b]$ and $y < x$.

Our main result is given in the following theorem:

Theorem 5. *Let f, $g : [a,b] \to \mathbb{R}$ be continuous mappings on $[a,b]$ and differentiable on (a,b) such that f' and g' belong to $\mathbf{L}_1(a,b)$. Let F and G be continuous mappings where $F(x) = \int_a^x w(t)\,f'(t)dt$ and $G(x) = \int_a^x w(t)\,g'(t)dt$. Then*

$$\begin{aligned}
&\left| F(x)G(x) - \frac{1}{2m(a,b)} \left(G(x) \int_a^b w(y)\,F(y)dy + F(x) \int_a^b w(y)\,G(y)dy \right) \right| \\
&\leq \frac{1}{2m(a,b)} \left[|G(x)| \int_a^b w(y)\,\|f'\|_{w,1,[y,x]}\,dy + |F(x)| \int_a^b w(y)\,\|g'\|_{w,1,[y,x]}\,dy \right] \\
&\leq \frac{\max\{|F(x)|, |G(x)|\}}{2m(a,b)} \int_a^b w(y) \left(\|f'\|_{w,1,[y,x]} + \|g'\|_{w,1,[y,x]} \right) dy, \qquad (5)
\end{aligned}$$

for all $x, y \in [a,b]$ and $y < x$.

Proof. For any $x \in [a,b]$, let $F(x) = \int_a^x w(t)\,f'(t)dt$ and $G(x) = \int_a^x w(t)\,g'(t)dt$, then we have the following identities

$$F(x) - F(y) = \int_a^x w(t)\,f'(t)dt - \int_a^y w(t)\,f'(t)dt = \int_y^x w(t)\,f'(t)dt. \qquad (6)$$

Similarly,

$$G(x) - G(y) = \int_y^x w(t)\,g'(t)dt. \qquad (7)$$

Multiplying both sides of (6) and (7) by $w(y)\,G(x)$ and $w(y)\,F(x)$ respectively and then adding, we get

$$\begin{aligned}&2F(x)G(x)w(y) - [G(x)w(y)\,F(y) + F(x)w(y)\,G(y)]\\&= G(x)w(y)\int_y^x w(t)\,f'(t)dt + F(x)w(y)\int_y^x w(t)\,g'(t)dt.\end{aligned} \tag{8}$$

Integrating both sides of (8) with respect to y over $[a,b]$ and rewriting, we have:

$$\begin{aligned}&F(x)G(x) - \frac{1}{2m(a,b)}\left(G(x)\int_a^b w(y)\,F(y)dy + F(x)\int_a^b w(y)\,G(y)dy\right)\\&= \frac{1}{2m(a,b)}\left[G(x)\int_a^b w(y)\left(\int_y^x w(t)\,f'(t)dt\right)dy\right.\\&\left.+ F(x)\int_a^b w(y)\left(\int_y^x w(t)\,g'(t)dt\right)dy\right],\end{aligned} \tag{9}$$

which implies

$$\begin{aligned}&\left|F(x)G(x) - \frac{1}{2m(a,b)}\left(G(x)\int_a^b w(y)\,F(y)dy + F(x)\int_a^b w(y)\,G(y)dy\right)\right|\\&\le \frac{1}{2m(a,b)}\left[|G(x)|\int_a^b w(y)\left|\int_y^x w(t)\,f'(t)dt\right|dy + |F(x)|\int_a^b w(y)\left|\int_y^x w(t)\,g'(t)dt\right|dy\right]\\&\le \frac{1}{2m(a,b)}\left[|G(x)|\int_a^b w(y)\,\|f'\|_{w,1,[y,x]}\,dy + |F(x)|\int_a^b w(y)\,\|g'\|_{w,1,[y,x]}\,dy\right].\end{aligned}$$

This completes the proof of the first part of inequality (5). Also

$$\begin{aligned}&\frac{1}{2m(a,b)}\left[|G(x)|\int_a^b w(y)\,\|f'\|_{w,1,[y,x]}\,dy + |F(x)|\int_a^b w(y)\,\|g'\|_{w,1,[y,x]}\,dy\right]\\&\le \frac{\max\{|F(x)|\,,|G(x)|\}}{2m(a,b)}\int_a^b w(y)\left(\|f'\|_{w,1,[y,x]} + \|g'\|_{w,1,[y,x]}\right)dy,\end{aligned}$$

which is the second inequality in (5). □

Remark 1. *Multiplying both sides of* (9) *by* $w(x)$, *then integrating with respect to* x *over* $[a,b]$ *and applying the properties of the modulus, we obtain the following* *weighted Grüss type inequality:*

$$\begin{aligned}
&\left|\frac{1}{m(a,b)}\int_a^b F(x)G(x)w(x)dx\right.\\
&\left.-\left(\frac{1}{m(a,b)}\int_a^b G(x)w(x)dx\right)\left(\frac{1}{m(a,b)}\int_a^b F(x)w\left(x\right)dx\right)\right|\\
&\leq \frac{1}{2m^2(a,b)}\int_a^b w(x)\max\left\{\left|F(x)\right|,\left|G(x)\right|\right\}\\
&\times\left(\int_a^b w\left(y\right)\left(\left\|f'\right\|_{w,1,[y,x]}+\left\|g'\right\|_{w,1,[y,x]}\right)dy\right)dx. \qquad (10)
\end{aligned}$$

A slight variant of Theorem 5 is embodied in the following theorem.

Theorem 6. *Under the assumptions of theorem* 5, *we have the inequality:*

$$\begin{aligned}
&\left|F\left(x\right)G(x)-\frac{1}{m(a,b)}F\left(x\right)\int_a^b G(y)w(y)dy-\frac{1}{m(a,b)}G(x)\int_a^b F\left(y\right)w(y)dy\right.\\
&\left.+\frac{1}{m(a,b)}\int_a^b F\left(y\right)G(y)w(y)dy\right|\\
&\leq \frac{1}{m(a,b)}\int_a^b w(y)\left\|f'\right\|_{w,1,[y,x]}\left\|g'\right\|_{w,1,[y,x]}dy. \qquad (11)
\end{aligned}$$

for all $x, y \in [a, b]$ *and* $y < x$.

Proof. From the hypothesis, identities (6) and (7) hold. Multiplying the left and right-hand sides of (6) and (7), we get

$$\begin{aligned}
&F\left(x\right)G(x)-F\left(x\right)G(y)-F\left(y\right)G(x)+F\left(y\right)G(y)\\
&=\int_y^x w(t)f'(t)dt\int_y^x w(t)g'(t)dt. \qquad (12)
\end{aligned}$$

Multiplying (12) by $w(y)$ and integrating the resultant with respect to y over $[a, b]$

and rewriting we have

$$F(x)G(x) - \frac{1}{m(a,b)}F(x)\int_a^b G(y)w(y)dy - \frac{1}{m(a,b)}G(x)\int_a^b F(y)w(y)dy$$
$$+\frac{1}{m(a,b)}\int_a^b F(y)G(y)w(y)dy$$
$$= \frac{1}{m(a,b)}\int_a^b w(y)\left(\int_y^x w(t)f'(t)dt\right)\left(\int_y^x w(t)g'(t)dt\right)dy, \tag{13}$$

which implies

$$\Bigg|F(x)G(x) - \frac{1}{m(a,b)}F(x)\int_a^b G(y)w(y)dy - \frac{1}{m(a,b)}G(x)\int_a^b F(y)w(y)dy$$
$$+\frac{1}{m(a,b)}\int_a^b F(y)G(y)w(y)dy\Bigg|$$
$$\leq \frac{1}{m(a,b)}\int_a^b w(y)\left(\int_y^x w(t)\left|f'(t)\right|dt\right)\left(\int_y^x w(t)\left|g'(t)\right|dt\right)dy$$
$$= \frac{1}{m(a,b)}\int_a^b w(y)\left\|f'\right\|_{w,1,[y,x]}\left\|g'\right\|_{w,1,[y,x]}dy.$$

This completes the proof. □

Remark 2. *Multiplying* (13) *by* $w(x)$*, then integrating both sides with respect to* x *over* $[a,b]$ *and applying the properties of modulus, we get*

$$\left|\frac{1}{m(a,b)}\int_a^b F(x)G(x)w(x)dx - \left(\frac{1}{m(a,b)}\int_a^b F(x)w(x)dx\right)\left(\frac{1}{m(a,b)}\int_a^b G(x)w(x)dx\right)\right|$$
$$\leq \frac{1}{2m^2(a,b)}\int_a^b w(x)\left(\int_a^b w(y)\left\|f'\right\|_{w,1,[y,x]}\left\|g'\right\|_{w,1,[y,x]}dy\right)dx, \tag{14}$$

for all $x, y \in [a,b]$ *and* $y < x$*. Inequality* (14) *is a modified Čebyšev inequality (see* [5, 297]*).*

Remark 3. *We note that the norms* $\|f'\|_{w,1,[y,x]}$ *and* $\|g'\|_{w,1,[y,x]}$ *are defined for all* x, $y \in [a,b]$ *and* $y < x$*, therefore we can recapture inequalities* (5), (10) *and* (14) *for* *the norm over [a,b] as follows:*

$$\left|F(x)G(x) - \frac{1}{2m(a,b)}\left(G(x)\int_a^b F(y)w(y)\,dy + F(x)\int_a^b G(y)w(y)\,dy\right)\right|$$
$$\le \frac{1}{2}\left(|G(x)|\,\|f'\|_{w,1} + |F(x)|\,\|g'\|_{w,1}\right), \tag{15}$$

$$\left|\frac{1}{m(a,b)}\int_a^b F(x)G(x)w(x)dx - \left(\frac{1}{m(a,b)}\int_a^b G(x)w(x)dx\right)\left(\frac{1}{m(a,b)}\int_a^b F(x)w(x)\,dx\right)\right|$$
$$\le \frac{\|f'\|_{w,1} + \|g'\|_{w,1}}{2m(a,b)}\int_a^b w(x)\max\{|F(x)|, |G(x)|\}\,dx, \tag{16}$$

and

$$\left|\frac{1}{m(a,b)}\int_a^b F(x)G(x)w(x)dx - \left(\frac{1}{m(a,b)}\int_a^b F(x)w(x)dx\right)\left(\frac{1}{m(a,b)}\int_a^b G(x)w(x)dx\right)\right|$$
$$\le \frac{1}{2}\|f'\|_{w,1}\|g'\|_{w,1}. \tag{17}$$

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