Explanation of this revision

Paper number: RNC-06-0157

First of all, the authors would like to express their sincere thanks to the Editor and the anonymous reviewers for their helpful comments and suggestions. The explanation of the modifications as well as corrections in this revision can be arranged as follows (comment numbers are in 1:1 correspondence with the reviewers' comments).

Reply to Professor Andras Balogh

In this revision, all the comments from the reviewers have been carefully taken into account and thoroughly implemented.

Reply to Reviewer No. 1

Thank you very much.

Reply to Reviewer No. 2

(1) *Page 9, Remark 1*

Question: In your paper, to reduce the design conservatism, you used parameter-dependent Lyapunov matrix P. On the other hand, you introduced a constant matrix X. Can you make the matrix X also parameter-dependent?

Answer: The purpose of introducing a constant matrix X in Theorem 2 is to make a decoupling between the matrix function $P(r_{i,j})$ and the system dynamic matrices in Theorem 1. This decoupling technique enables us to obtain a more easily tractable condition for the filter synthesis. Therefore, the matrix X cannot be selected to be parameter-dependent. It seems that some conservatism will be introduced because of a constant matrix X, not a parameter-dependent one, but the motivation of doing this is to obtain a tractable filter synthesis condition (please refer to Remark 1 in Page 9).

(2) *Page 19, first paragraph*

Question: In the example section, can you give some performance evaluation figure for the H_∞ or other performance?

Answer: In the revised version, we have added a table to give the H_∞ and $L_2 - L_\infty$ performance evaluations, which further illustrates the effectiveness of the proposed design schemes (please refer to the example section).

$\mathcal{H}_\infty$ and l_2-l_∞ Filtering for Two-Dimensional Linear Parameter-Varying Systems

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Abstract

In this paper, the $\mathcal{H}_\infty$ and l_2-l_∞ filtering problem is investigated for two-dimensional (2-D) discrete-time linear parameter-varying (LPV) systems. Based on the well-known Fornasini-Marchesini local state-space (FMLSS) model, the mathematical model of 2-D systems under consideration is established by incorporating the parameter-varying phenomenon. The purpose of the problem addressed is to design full-order $\mathcal{H}_\infty$ and l_2-l_∞ filters such that the filtering error dynamics is asymptotic stable and the prescribed noise attenuation levels in $\mathcal{H}_\infty$ and l_2-l_∞ senses can be achieved, respectively. Sufficient conditions are derived for existence of such filters in terms of parameterized linear matrix inequalities (PLMIs), and the corresponding filter synthesis problem is then transformed into a convex optimization problem that can be efficiently solved by using standard software packages. A simulation example is exploited to demonstrate the usefulness and effectiveness of the proposed design method.

Keywords

Linear parameter-varying (LPV) systems; Parameterized linear matrix inequalities (PLMIs); $\mathcal{H}_\infty$ filtering; l_2-l_∞ filtering; Two-dimensional (2-D) systems

I. Introduction

It is well known that one of the fundamental problems in control systems and signal processing is the estimation of the state variables of a dynamic system through available noisy measurements, which is referred to as the filtering problem, see [1], [27], [29] and the references therein. In the past few decades, the $\mathcal{H}_\infty$ filtering problem has drawn particular attention, since $\mathcal{H}_\infty$ filters are insensitive to the exact knowledge of the statistics of the noise signals. To be specific, $\mathcal{H}_\infty$ filtering procedure ensures that the $\mathcal{L}_2$-induced gain from the noise input signals to the estimation error is less than a prescribed level, where the noise input is an arbitrary energy-bounded signal. Several methods have been proposed to solve the $\mathcal{H}_\infty$ filtering problem, see [3], [16], [26], [32], [33] for some recent publications. Other filtering methods for systems with partially known noise information are l_2-l_∞ filtering ($\mathcal{L}_2$-$\mathcal{L}_\infty$ filtering for continuous-time systems) [13], [18] and l_1 filtering ($\mathcal{L}_1$ filtering for continuous-time systems) [24], where the l_2-l_∞ and l_1 performances have different physical meanings when used as performance indices.

On the other hand, linear parameter-varying (LPV) systems are those systems dependent on unknown but measurable time-varying parameters, where the measurement of the time-varying parameters provides real-time information on the variations of the plant's characteristics [28]. LPV systems are ubiquitous in chemical processes, robotics systems, automotive systems and many manufacturing processes. The LPV systems theory has been motivated by the gain-scheduling approach for control of linear and nonlinear systems [21]. Generally speaking, there are two basic approaches to dealing with the analysis and design problems for LPV systems. One approach has been developed in [2] by assuming that the trajectory of the parameters is not known *a*

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priori although its value is known through real-time measurements. Therefore, in such a case, the state-space matrices are dependent continuously on the varying parameters, and the stability analysis and control synthesis problems have been tackled based on the notion of quadratic stability using a single quadratic Lyapunov function. An alternative approach has been proposed in [10] where the real uncertain parameters and their rates have been assumed to vary in some prescribed ranges and the state-space matrices are therefore allowed to depend affinely on the varying parameters. In this case, the stability analysis has been conducted based on the notion of affine quadratic stability using parameter-dependent quadratic Lyapunov functions [17]. Up to now, many important results have been reported for LPV systems. For instance, the controllability, stabilizability and stability analysis problems have been investigated in [22], [28], the stabilization and control problems have been solved in [8], [23], [25], the filtering problems have been studied in [5], [17], [29] and the model reduction problem has also been coped with in [9].

Recently, two-dimensional (2-D) systems have received considerable research attention since 2-D systems are capable of modeling a wide range of practical systems and have been successfully applied in image data processing and transmission, thermal processes, gas absorption, and water stream heating, etc. [14]. Great deals of publications have been available in the literature. To mention just a few, the stability and stabilization problems of 2-D systems have been investigated in [15], [19], the controller and filter design problems have been studied in [12], [30], [31], and the model approximation problem has been addressed in [11]. Inevitably, when 2-D system is applied in modeling real-time plants such as chemical process control, the system would be naturally dependent on unknown but measurable time-varying parameters. Therefore, *2-D LPV* systems emerge as a more reasonable description to account for the parameter drifting phenomenon, and have a great potential in engineering applications. To the best of the authors' knowledge, there have been very few results addressing the 2-D LPV systems due to the mathematical complexity, and both the filtering and control problems for 2-D LPV systems still remain open and challenging.

In this paper, we make an attempt to investigate the problems of $\mathcal{H}_\infty$ and l_2-l_∞ filtering for 2-D LPV discrete-time systems, where the mathematical model of 2-D systems is established upon the well-known Fornasini-Marchesini local state-space (FMLSS) model. Sufficient conditions are obtained for the existence of desired $\mathcal{H}_\infty$ and l_2-l_∞ filters in terms of parameterized linear matrix inequalities (PLMIs). Moreover, the decoupling technique by the introduction of an auxiliary slack variable [7] is applied such that, in the improved PLMI condition, the product terms no longer exist in our main results. On the other hand, such a decoupling method enables us to obtain a more tractable condition for the filter analysis and synthesis problems. The desired filter is then obtained by solving a convex optimization problem using the efficient interior-point optimization algorithms [6]. A numerical example is provided to demonstrate the effectiveness of the proposed controller design procedures.

The rest of this paper is organized as follows. The problems of $\mathcal{H}_\infty$ and l_2-l_∞ filtering for 2-D LPV systems are formulated in Section 2. Section 3 gives the main results of the $\mathcal{H}_\infty$ filtering problem. These obtained results are further extended to the l_2-l_∞ filtering in Section 4. Section 5 provides an illustrative example and we conclude this paper in Section 6.

Notations: The superscript "T" stands for matrix transposition; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{R}^{m\times n}$ is the set of all real matrices of dimension $m \times n$ and the notation $P > 0$ means that P is real symmetric and positive definite; I and 0 represent identity matrix and zero matrix respectively; $|\cdot|$ refers to the Euclidean vector norm; and $\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ denote the minimum and the maximum eigenvalues of a real symmetric matrix respectively. In symmetric block matrices or long matrix expressions, we use an asterisk $(*)$ to represent a term that is induced by symmetry and $\text{diag}\{\ldots\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. Sometimes, we omit the time argument for a function if no confusion arises.

II. Problem Formulation

Consider the following 2-D LPV system ($\mathcal{S}$) described by the FMLSS model:

$$\begin{aligned}\mathcal{S}: \qquad x_{i+1,j+1} &= A_1(r_{i,j+1})x_{i,j+1} + A_2(r_{i+1,j})x_{i+1,j} + B_1(r_{i,j+1})\omega_{i,j+1} + B_2(r_{i+1,j})\omega_{i+1,j} \\ y_{i,j} &= C(r_{i,j})x_{i,j} + D(r_{i,j})\omega_{i,j} \\ z_{i,j} &= L(r_{i,j})x_{i,j} \end{aligned} \tag{1}$$

where $x_{i,j} \in \mathbb{R}^n$ is the state vector; $y_{i,j} \in \mathbb{R}^m$ is the measured output; $z_{i,j} \in \mathbb{R}^q$ is the signal to be estimated and $\omega_{i,j} \in \mathbb{R}^l$ is the disturbance input which belongs to $l_2\{[0,\infty),[0,\infty)\}$. $A_1(r_{i,j+1})$, $A_2(r_{i+1,j})$, $B_1(r_{i,j+1})$, $B_2(r_{i+1,j})$, $C(r_{i,j})$, $D(r_{i,j})$ and $L(r_{i,j})$ are known matrix functions of a time-varying parameter vector $r_{i,j} \in \mathcal{F}_{\mathcal{P}}^v$, where $\mathcal{F}_{\mathcal{P}}^v$ is the set of allowable parameter trajectories, which is defined as $\mathcal{F}_{\mathcal{P}}^v \triangleq \left\{ r_{i,j} \in \mathcal{C}(\mathbb{R},\mathbb{R}^s) : r_{i,j}^k \in \mathcal{P},\ \left\| r_{i,j}^k \right\| \leq v_k,\ k = 1,2,\ldots,s,\ \forall\ i = 1,2,\ldots;\ j = 1,2,\ldots \right\}$, where $\mathcal{P}$ is a compact subset of $\mathbb{R}^s$, $\{v_k\}_{k=1}^s$ are nonnegative numbers and $v = [v_1, v_2, \ldots, v_s]^T$. In other words, we consider bounded parameter trajectories.

The boundary conditions are defined by

$$X^h(0) = \begin{bmatrix} x_{0,1}^T & x_{0,2}^T & \cdots \end{bmatrix}^T, \qquad X^v(0) = \begin{bmatrix} x_{1,0}^T & x_{2,0}^T & \cdots \end{bmatrix}^T.$$

Then, we make the following assumption on the boundary condition.

Assumption 1: The boundary condition is assumed to satisfy

$$\lim_{N\to\infty} \sum_{k=0}^{N} (|x_{0,k}|^2 + |x_{k,0}|^2) < \infty \tag{2}$$

The purpose of the filtering problem addressed in this paper is to design a full-order $\mathcal{H}_\infty$ or l_2-l_∞ filter for the system ($\mathcal{S}$) in (1) with the following form:

$$\begin{aligned}\mathcal{F}: \qquad \hat{x}_{i+1,j+1} &= A_{1F}(r_{i,j+1})\hat{x}_{i,j+1} + A_{2F}(r_{i+1,j})\hat{x}_{i+1,j} + B_{1F}(r_{i,j+1})y_{i,j+1} + B_{2F}(r_{i+1,j})y_{i+1,j} \\ \hat{z}_{i,j} &= C_F(r_{i,j})\hat{x}_{i,j} \\ \hat{x}_{i,j} &= 0 \quad \text{for } i = 0 \text{ or } j = 0 \end{aligned} \tag{3}$$

where $\hat{x}_{i,j} \in \mathbb{R}^n$ is the filter state, and the matrices $A_{1F}(r_{i,j+1})$, $A_{2F}(r_{i+1,j})$, $B_{1F}(r_{i,j+1})$, $B_{2F}(r_{i+1,j})$ and $C_F(r_{i,j})$ are filter parameters to be determined.

By defining $\xi_{i,j}^T \triangleq \begin{bmatrix} x_{i,j}^T & \hat{x}_{i,j}^T \end{bmatrix}^T$ and augmenting the model of ($\mathcal{S}$) to include the states of the filter, we can obtain the following filtering error system ($\mathcal{E}$):

$$\begin{aligned}\mathcal{E}: \qquad \xi_{i+1,j+1} &= \bar{A}_1(r_{i,j+1})\xi_{i,j+1} + \bar{A}_2(r_{i+1,j})\xi_{i+1,j} + \bar{B}_1(r_{i,j+1})\omega_{i,j+1} + \bar{B}_2(r_{i+1,j})\omega_{i+1,j} \\ \tilde{z}_{i,j} &= \bar{C}(r_{i,j})\xi_{i,j} \end{aligned} \tag{4}$$

where

$$\begin{aligned}
\bar{A}_1(r_{i,j+1}) &\triangleq \begin{bmatrix} A_1(r_{i,j+1}) & 0 \\ B_{1F}(r_{i,j+1})C(r_{i,j+1}) & A_{1F}(r_{i,j+1}) \end{bmatrix}, \\
\bar{A}_2(r_{i+1,j}) &\triangleq \begin{bmatrix} A_2(r_{i+1,j}) & 0 \\ B_{2F}(r_{i+1,j})C(r_{i+1,j}) & A_{2F}(r_{i+1,j}) \end{bmatrix}, \\
\bar{B}_1(r_{i,j+1}) &\triangleq \begin{bmatrix} B_1(r_{i,j+1}) \\ B_{1F}(r_{i,j+1})D(r_{i,j+1}) \end{bmatrix}, \quad \bar{B}_2(r_{i+1,j}) \triangleq \begin{bmatrix} B_2(r_{i+1,j}) \\ B_{2F}(r_{i+1,j})D(r_{i+1,j}) \end{bmatrix}, \\
\bar{C}(r_{i,j}) &\triangleq \begin{bmatrix} L(r_{i,j}) & -C_F(r_{i,j}) \end{bmatrix}
\end{aligned} \tag{5}$$

Before presenting the main objective of this paper, we first introduce the following definitions for the filtering error system $(\mathcal{E})$ in (4), which will be essential for our derivation.

Definition 1: The filtering error system $(\mathcal{E})$ in (4) with $\omega_{i,j} = 0$ is asymptotically stable if

$$\lim_{i+j\to\infty} |\xi_{i,j}|^2 = 0$$

for every boundary condition $\left(X^h(0), X^v(0)\right)$ satisfying Assumption 1.

Definition 2: Given a scalar $\gamma > 0$. The filtering error system $(\mathcal{E})$ in (4) is said to be asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level γ if it is asymptotically stable and, under zero initial and boundary conditions, $\|\tilde{z}\|_2 < \gamma \|\omega\|_2$ holds for all nonzero $\omega \triangleq \{\omega_{i,j}\} \in l_2\{[0,\infty),[0,\infty)\}$, where

$$\|\tilde{z}\|_2 \triangleq \sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |\tilde{z}_{i,j}|^2}, \qquad \|\omega\|_2 \triangleq \sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |\omega_{i,j}|^2}$$

Definition 3: Given a scalar $\gamma > 0$. The filtering error system $(\mathcal{E})$ in (4) is said to be asymptotically stable with an l_2-l_∞ disturbance attenuation level γ if it is asymptotically stable and, under zero initial and boundary conditions, $\|\tilde{z}\|_\infty < \gamma \|\omega\|_2$ holds for all nonzero $\omega \triangleq \{\omega_{i,j}\} \in l_2\{[0,\infty),[0,\infty)\}$ where

$$\|\tilde{z}\|_\infty \triangleq \sqrt{\sup_{\forall\, i,j} |\tilde{z}_{i,j}|^2}$$

We are now in a position to state the problem to be studied in this paper as follows: Determine the filter parameters $A_{1F}(r_{i,j+1})$, $A_{2F}(r_{i+1,j})$, $B_{1F}(r_{i,j+1})$, $B_{2F}(r_{i+1,j})$ and $C_F(r_{i,j})$ of the full-order $\mathcal{H}_\infty$ or l_2-l_∞ filter $(\mathcal{F})$ for the 2-D LPV system $(\mathcal{S})$, such that the following two requirements are satisfied:

1. The resulting filtering error dynamics $(\mathcal{E})$ is asymptotically stable;
2. The filtering error system $(\mathcal{E})$ ensures a noise attenuation level γ in an $\mathcal{H}_\infty$ or l_2-l_∞ sense.

III. $\mathcal{H}_\infty$ Filtering

A. Filter Analysis

In this section, we propose a sufficient condition for the solvability of the $\mathcal{H}_\infty$ filtering problem formulated in the previous section. First, we give the following theorem which will play a key role in the derivation of our main results.

Theorem 1: The filtering error system $(\mathcal{E})$ in (4) is asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level $\gamma > 0$ if there exist matrix functions $P(r_{i,j}) > 0$ and $Q(r_{i,j}) > 0$ such that the following PLMI holds:

$$\left[\begin{array}{ccccccc} -P(r_{i+1,j+1}) & 0 & 0 & P(r_{i+1,j+1})\bar{A}_1(r_{i,j+1}) & P(r_{i+1,j+1})\bar{A}_2(r_{i+1,j}) & P(r_{i+1,j+1})\bar{B}_1(r_{i,j+1}) & P(r_{i+1,j+1})\bar{B}_2(r_{i+1,j}) \\ * & -I & 0 & 0 & \bar{C}(r_{i+1,j}) & 0 & 0 \\ * & * & -I & \bar{C}(r_{i,j+1}) & 0 & 0 & 0 \\ * & * & * & Q(r_{i,j+1})-P(r_{i,j+1}) & 0 & 0 & 0 \\ * & * & * & * & -Q(r_{i+1,j}) & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -\gamma^2 I \end{array}\right] < 0 \qquad (6)$$

Proof: First, let us examine the asymptotic stability of the filtering error system $(\mathcal{E})$ in (4) with $\omega_{i,j} \equiv 0$. Notice that PLMI (6) implies $P(r_{i,j+1}) - Q(r_{i,j+1}) > 0$ ($\forall\ i,j = 1,2,\ldots$) and consider the following index:

$$\mathcal{I}_{i,j} \triangleq \xi_{i+1,j+1}^T P(r_{i+1,j+1})\xi_{i+1,j+1} - \tilde{\xi}^T \mathrm{diag}\left\{(P(r_{i,j+1}) - Q(r_{i,j+1})), Q(r_{i+1,j})\right\}\tilde{\xi} \qquad (7)$$

where $\tilde{\xi} \triangleq \left[\begin{array}{cc} \xi_{i,j+1}^T & \xi_{i+1,j}^T \end{array}\right]^T$, and $P(r_{i,j})$, $Q(r_{i,j})$ $(i,j = 1,2,\ldots)$ are symmetric positive definite matrix functions to be determined. Then, along the solution of the filtering error system $(\mathcal{E})$, we have

$$\begin{aligned} \mathcal{I}_{i,j} &= \left[\bar{A}_1(r_{i,j+1})\xi_{i,j+1} + \bar{A}_2(r_{i+1,j})\xi_{i+1,j}\right]^T P(r_{i+1,j+1})\left[\bar{A}_1(r_{i,j+1})\xi_{i,j+1} + \bar{A}_2(r_{i+1,j})\xi_{i+1,j}\right] \\ &\quad -\tilde{\xi}^T \mathrm{diag}\left\{(P(r_{i,j+1}) - Q(r_{i,j+1})), Q(r_{i+1,j})\right\}\tilde{\xi}, \end{aligned} \qquad (8)$$

and it follows that

$$\mathcal{I}_{i,j} = \tilde{\xi}^T \Psi \tilde{\xi} \qquad (9)$$

where

$$\Psi \triangleq \left[\begin{array}{cc} \bar{A}_1^T(r_{i,j+1})P(r_{i+1,j+1})\bar{A}_1(r_{i,j+1}) - P(r_{i,j+1}) + Q(r_{i,j+1}) & \bar{A}_1^T(r_{i,j+1})P(r_{i+1,j+1})\bar{A}_2(r_{i+1,j}) \\ * & \bar{A}_2^T(r_{i+1,j})P(r_{i+1,j+1})\bar{A}_2(r_{i+1,j}) - Q(r_{i+1,j}) \end{array}\right]$$

By Schur complement [6], PLMI (6) implies $\Psi < 0$. Then, for all $\tilde{\xi} \neq 0$, we have

$$\begin{aligned} & \frac{\xi_{i+1,j+1}^T P(r_{i+1,j+1})\xi_{i+1,j+1} - \tilde{\xi}^T \mathrm{diag}\left\{(P(r_{i,j+1}) - Q(r_{i,j+1})), Q(r_{i+1,j})\right\}\tilde{\xi}}{\tilde{\xi}^T \mathrm{diag}\left\{(P(r_{i,j+1}) - Q(r_{i,j+1})), Q(r_{i+1,j})\right\}\tilde{\xi}} \\ = \quad & \frac{-\tilde{\xi}^T(-\Psi)\tilde{\xi}}{\tilde{\xi}^T \mathrm{diag}\left\{(P(r_{i,j+1}) - Q(r_{i,j+1})), Q(r_{i+1,j})\right\}\tilde{\xi}} \\ \leq \quad & \frac{-\lambda_{\min}(-\Psi)}{\lambda_{\max}(\mathrm{diag}\left\{(P(r_{i,j+1}) - Q(r_{i,j+1})), Q(r_{i+1,j})\right\})} = \alpha - 1, \quad \forall\ i,j = 1,2,\ldots \end{aligned}$$

where

$$\alpha \triangleq 1 - \frac{\lambda_{\min}(-\Psi)}{\lambda_{\max}(\mathrm{diag}\left\{(P(r_{i,j+1}) - Q(r_{i,j+1})), Q(r_{i+1,j})\right\})}$$

Since

$$\frac{\lambda_{\min}(-\Psi)}{\lambda_{\max}(\mathrm{diag}\left\{(P(r_{i,j+1})-Q(r_{i,j+1})),Q(r_{i+1,j})\right\})}>0$$

we have $\alpha<1$. Obviously,

$$\alpha\geq\frac{\xi_{i+1,j+1}^{T}P(r_{i+1,j+1})\xi_{i+1,j+1}}{\tilde{\xi}^{T}\mathrm{diag}\left\{(P(r_{i,j+1})-Q(r_{i,j+1})),Q(r_{i+1,j})\right\}\tilde{\xi}}>0,$$

which means $\alpha\in(0,1)$ and α is independent of $\tilde{\xi}$. Therefore, we arrive at

$$\xi_{i+1,j+1}^{T}P(r_{i+1,j+1})\xi_{i+1,j+1}\leq\alpha\tilde{\xi}^{T}\mathrm{diag}\left\{(P(r_{i,j+1})-Q(r_{i,j+1})),Q(r_{i+1,j})\right\}\tilde{\xi}$$

or

$$\xi_{i+1,j+1}^{T}P(r_{i+1,j+1})\xi_{i+1,j+1}\leq\alpha\left\{\xi_{i,j+1}^{T}(P(r_{i,j+1})-Q(r_{i,j+1}))\xi_{i,j+1}+\xi_{i+1,j}^{T}Q(r_{i+1,j})\xi_{i+1,j}\right\}\tag{10}$$

Using relationship (10) and $P(r_{i,j+1})>Q(r_{i,j+1})$, it can be established that

$$\begin{array}{rcl}
\xi_{k+1,0}^{T}P(r_{k+1,0})\xi_{k+1,0} & = & \xi_{k+1,0}^{T}P(r_{k+1,0})\xi_{k+1,0}\\
\xi_{k,1}^{T}P(r_{k,1})\xi_{k,1} & \leq & \alpha\left\{\xi_{k-1,1}^{T}(P(r_{k-1,1})-Q(r_{k-1,1}))\xi_{k-1,1}+\xi_{k,0}^{T}Q(r_{k,0})\xi_{k,0}\right\}\\
 & \leq & \alpha\left\{\xi_{k-1,1}^{T}(P(r_{k-1,1})-Q(r_{k-1,1}))\xi_{k-1,1}+\xi_{k,0}^{T}P(r_{k,0})\xi_{k,0}\right\}\\
\xi_{k-1,2}^{T}P(r_{k-1,2})\xi_{k-1,2} & \leq & \alpha\left\{\xi_{k-2,2}^{T}(P(r_{k-2,2})-Q(r_{k-2,2}))\xi_{k-2,2}+\xi_{k-1,1}^{T}Q(r_{k-1,1})\xi_{k-1,1}\right\}\\
 & & \vdots\\
\xi_{1,k}^{T}P(r_{1,k})\xi_{1,k} & \leq & \alpha\left\{\xi_{0,k}^{T}(P(r_{0,k})-Q(r_{0,k}))\xi_{0,k}+\xi_{1,k-1}^{T}Q(r_{1,k-1})\xi_{1,k-1}\right\}\\
 & \leq & \alpha\left\{\xi_{0,k}^{T}P(r_{0,k})\xi_{0,k}+\xi_{1,k-1}^{T}Q(r_{1,k-1})\xi_{1,k-1}\right\}\\
\xi_{0,k+1}^{T}P(r_{0,k+1})\xi_{0,k+1} & = & \xi_{0,k+1}^{T}P(r_{0,k+1})\xi_{0,k+1}
\end{array}$$

which imply

$$\begin{array}{rcl}
\sum_{j=0}^{k+1}\xi_{k+1-j,j}^{T}P(r_{k+1-j,j})\xi_{k+1-j,j} & \leq & \alpha\sum_{j=0}^{k}\xi_{k-j,j}^{T}P(r_{k-j,j})\xi_{k-j,j}\\
 & & +\xi_{k+1,0}^{T}P(r_{k+1,0})\xi_{k+1,0}+\xi_{0,k+1}^{T}P(r_{0,k+1})\xi_{0,k+1}
\end{array}$$

Using the above relationship iteratively, we obtain

$$\begin{array}{rcl}
\sum_{j=0}^{k+1}\xi_{k+1-j,j}^{T}P(r_{k+1-j,j})\xi_{k+1-j,j} & \leq & \sum_{j=0}^{k}\alpha^{j}\left[\xi_{k+1-j,0}^{T}P(r_{k+1-j,0})\xi_{k+1-j,0}+\xi_{0,k+1-j}^{T}P(r_{0,k+1-j})\xi_{0,k+1-j}\right]\\
 & & +\alpha^{k+1}\xi_{0,0}^{T}P(r_{0,0})\xi_{0,0}\\
 & \leq & \sum_{j=0}^{k+1}\alpha^{j}\left[\xi_{k+1-j,0}^{T}P(r_{k+1-j,0})\xi_{k+1-j,0}+\xi_{0,k+1-j}^{T}P(r_{0,k+1-j})\xi_{0,k+1-j}\right]
\end{array}$$

Therefore, we have

$$\sum_{j=0}^{k+1}\left|\xi_{k+1-j,j}\right|^{2}\leq\kappa\sum_{j=0}^{k+1}\alpha^{j}\left\{\left|\xi_{k+1-j,0}\right|^{2}+\left|\xi_{0,k+1-j}\right|^{2}\right\}\tag{11}$$

where

$$\kappa\triangleq\frac{\max_{i,j}\lambda_{\max}(P(r_{i,j}))}{\min_{i,j}\lambda_{\max}(P(r_{i,j}))}$$

Now, by denoting $\mathcal{X}_k \triangleq \sum_{j=0}^{k} |\xi_{k-j,j}|^2$, it follows from the inequality (11) that

$$\begin{aligned}
\mathcal{X}_0 &\leq \kappa\left\{|\xi_{0,0}|^2 + |\xi_{0,0}|^2\right\} \\
\mathcal{X}_1 &\leq \kappa\left\{\alpha\left[|\xi_{0,0}|^2 + |\xi_{0,0}|^2\right] + \left[|\xi_{1,0}|^2 + |\xi_{0,1}|^2\right]\right\} \\
\mathcal{X}_2 &\leq \kappa\left\{\alpha^2\left[|\xi_{0,0}|^2 + |\xi_{0,0}|^2\right] + \alpha\left[|\xi_{1,0}|^2 + |\xi_{0,1}|^2\right] + \left[|\xi_{2,0}|^2 + |\xi_{0,2}|^2\right]\right\} \\
&\vdots \\
\mathcal{X}_N &\leq \kappa\left\{\alpha^N\left[|\xi_{0,0}|^2 + |\xi_{0,0}|^2\right] + \alpha^{N-1}\left[|\xi_{1,0}|^2 + |\xi_{0,1}|^2\right] + \cdots + \left[|\xi_{N,0}|^2 + |\xi_{0,N}|^2\right]\right\}
\end{aligned}$$

Summing up the both sides of the above inequality system yields

$$\begin{aligned}
\sum_{k=0}^{N}\mathcal{X}_k &\leq \kappa(1+\alpha+\cdots+\alpha^N)\left\{|\xi_{0,0}|^2 + |\xi_{0,0}|^2\right\} + \kappa(1+\alpha+\cdots+\alpha^{N-1}) \\
&\quad \times\left\{|\xi_{1,0}|^2 + |\xi_{0,1}|^2\right\} + \cdots + \kappa\left\{|\xi_{N,0}|^2 + |\xi_{0,N}|^2\right\} \\
&\leq \kappa(1+\alpha+\cdots+\alpha^N)\left\{|\xi_{0,0}|^2 + |\xi_{0,0}|^2\right\} + \kappa(1+\alpha+\cdots+\alpha^{N}) \\
&\quad \times\left\{|\xi_{1,0}|^2 + |\xi_{0,1}|^2\right\} + \cdots + \kappa(1+\alpha+\cdots+\alpha^N)\left\{|\xi_{N,0}|^2 + |\xi_{0,N}|^2\right\} \\
&= \kappa\frac{1-\alpha^N}{1-\alpha}\left\{\sum_{k=0}^{N}\left[|\xi_{k,0}|^2 + |\xi_{0,k}|^2\right]\right\}
\end{aligned}$$

Then, under Assumption 1, the right side of the above inequality is bounded for every boundary condition, which means $\lim_{k\to\infty}\mathcal{X}_k = 0$, that is, $|\xi_{i,j}|^2 \to 0$ as $i+j\to\infty$, and the filtering error system $(\mathcal{E})$ with $\omega_{i,j} = 0$ is guaranteed to be asymptotically stable.

Having dealt with the stability issue, we are now ready to establish the $\mathcal{H}_\infty$ performance for the filtering error system $(\mathcal{E})$ by assuming zero initial and boundary conditions, that is $\xi_{i,j} = 0$ for $i = 0$ or $j = 0$. Consider the following index:

$$\mathcal{J} \triangleq z^T z - \gamma^2\omega^T\omega + \mathcal{I}_{i,j} \tag{12}$$

where $z \triangleq \begin{bmatrix} z_{i,j+1}^T & z_{i+1,j}^T \end{bmatrix}^T$, $\omega \triangleq \begin{bmatrix} \omega_{i,j+1}^T & \omega_{i+1,j}^T \end{bmatrix}^T$, $\mathcal{I}_{i,j}$ is defined in (7) and has been further developed to (8). Then, along the solutions of the filtering error system $(\mathcal{E})$, we have

$$\begin{aligned}
\mathcal{J} &= \xi_{i,j+1}^T\bar{C}^T(r_{i,j+1})\bar{C}(r_{i,j+1})\xi_{i,j+1} + \xi_{i+1,j}^T\bar{C}^T(r_{i+1,j})\bar{C}(r_{i+1,j})\xi_{i+1,j} \\
&\quad -\gamma^2\omega_{i,j+1}^T\omega_{i,j+1} - \gamma^2\omega_{i+1,j}^T\omega_{i+1,j} + \mathcal{I}_{i,j} \\
&\triangleq \eta^T\Pi\eta
\end{aligned}$$

where $\eta \triangleq \begin{bmatrix} \xi_{i,j+1}^T & \xi_{i+1,j}^T & \omega_{i,j+1}^T & \omega_{i+1,j}^T \end{bmatrix}^T$ and

$$\Pi \triangleq \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} \\ * & \Pi_{22} & \Pi_{23} & \Pi_{24} \\ * & * & \Pi_{33} & \Pi_{34} \\ * & * & * & \Pi_{44} \end{bmatrix}$$

in which

$$\begin{aligned}
\Pi_{11} &\triangleq \bar{A}_1^T(r_{i,j+1})P(r_{i+1,j+1})\bar{A}_1(r_{i,j+1})+\bar{C}^T(r_{i,j+1})\bar{C}(r_{i,j+1})-P(r_{i,j+1})+Q(r_{i,j+1}),\\
\Pi_{12} &\triangleq \bar{A}_1^T(r_{i,j+1})P(r_{i+1,j+1})\bar{A}_2(r_{i+1,j}),\\
\Pi_{13} &\triangleq \bar{A}_1^T(r_{i,j+1})P(r_{i+1,j+1})\bar{B}_1(r_{i,j+1}),\\
\Pi_{22} &\triangleq \bar{A}_2^T(r_{i+1,j})P(r_{i+1,j+1})\bar{A}_2(r_{i+1,j})+\bar{C}^T(r_{i+1,j})\bar{C}(r_{i+1,j})-Q(r_{i+1,j}),\\
\Pi_{23} &\triangleq \bar{A}_2^T(r_{i+1,j})P(r_{i+1,j+1})\bar{B}_1(r_{i,j+1}),\\
\Pi_{33} &\triangleq \bar{B}_1^T(r_{i,j+1})P(r_{i+1,j+1})\bar{B}_1(r_{i,j+1})-\gamma^2 I,\\
\Pi_{14} &\triangleq \bar{A}_1^T(r_{i,j+1})P(r_{i+1,j+1})\bar{B}_2(r_{i+1,j}),\\
\Pi_{24} &\triangleq \bar{A}_2^T(r_{i+1,j})P(r_{i+1,j+1})\bar{B}_2(r_{i+1,j}),\\
\Pi_{34} &\triangleq \bar{B}_1^T(r_{i,j+1})P(r_{i+1,j+1})\bar{B}_2(r_{i+1,j}),\\
\Pi_{44} &\triangleq \bar{B}_2^T(r_{i+1,j})P(r_{i+1,j+1})\bar{B}_2(r_{i+1,j})-\gamma^2 I.
\end{aligned}$$

By Schur complement, PLMI (6) implies $\Pi < 0$, and we have $\mathcal{J} < 0$ for all $\eta \neq 0$, i.e.,

$$\xi_{i+1,j+1}^T P(r_{i+1,j+1})\xi_{i+1,j+1} < \tilde{\xi}^T \mathrm{diag}\left\{(P(r_{i,j+1})-Q(r_{i,j+1})), Q(r_{i+1,j})\right\}\tilde{\xi} - z^T z + \gamma^2\omega^T\omega$$

that is,

$$\begin{aligned}
\xi_{i+1,j+1}^T P(r_{i+1,j+1})\xi_{i+1,j+1} \;<\;& \xi_{i,j+1}^T(P(r_{i,j+1})-Q(r_{i,j+1}))\xi_{i,j+1}+\xi_{i+1,j}^T Q(r_{i+1,j})\xi_{i+1,j}\\
&-z_{i,j+1}^T z_{i,j+1}-z_{i+1,j}^T z_{i+1,j}+\gamma^2\omega_{i,j+1}^T\omega_{i,j+1}+\gamma^2\omega_{i+1,j}^T\omega_{i+1,j}
\end{aligned} \tag{13}$$

Using relationship (13), it can be established that

$$\begin{aligned}
\xi_{k+1,0}^T P(r_{k+1,0})\xi_{k+1,0} &= \xi_{k+1,0}^T P(r_{k+1,0})\xi_{k+1,0}\\
\xi_{k,1}^T P(r_{k,1})\xi_{k,1} &< \xi_{k-1,1}^T(P(r_{k-1,1})-Q(r_{k-1,1}))\xi_{k-1,1}+\xi_{k,0}^T Q(r_{k,0})\xi_{k,0}\\
&\quad -z_{k-1,1}^T z_{k-1,1}-z_{k,0}^T z_{k,0}+\gamma^2\omega_{k-1,1}^T\omega_{k-1,1}+\gamma^2\omega_{k,0}^T\omega_{k,0}\\
\xi_{k-1,2}^T P(r_{k-1,2})\xi_{k-1,2} &< \xi_{k-2,2}^T(P(r_{k-2,2})-Q(r_{k-2,2}))\xi_{k-2,2}+\xi_{k-1,1}^T Q(r_{k-1,1})\xi_{k-1,1}\\
&\quad -z_{k-2,2}^T z_{k-2,2}-z_{k-1,1}^T z_{k-1,1}+\gamma^2\omega_{k-2,2}^T\omega_{k-2,2}+\gamma^2\omega_{k-1,1}^T\omega_{k-1,1}\\
&\;\;\vdots\\
\xi_{1,k}^T P(r_{1,k})\xi_{1,k} &< \xi_{0,k}^T(P(r_{0,k})-Q(r_{0,k}))\xi_{0,k}+\xi_{1,k-1}^T Q(r_{1,k-1})\xi_{1,k-1}\\
&\quad -z_{0,k}^T z_{0,k}-z_{1,k-1}^T z_{1,k-1}+\gamma^2\omega_{0,k}^T\omega_{0,k}+\gamma^2\omega_{1,k-1}^T\omega_{1,k-1}\\
\xi_{0,k+1}^T P(r_{0,k+1})\xi_{0,k+1} &= \xi_{0,k+1}^T P(r_{0,k+1})\xi_{0,k+1}
\end{aligned}$$

which imply

$$\sum_{j=0}^{k+1}\xi_{k+1-j,j}^T P(r_{k+1-j,j})\xi_{k+1-j,j} < \sum_{j=0}^{k}\xi_{k-j,j}^T P(r_{k-j,j})\xi_{k-j,j} - 2\sum_{j=0}^{k} z_{k-j,j}^T z_{k-j,j} + 2\gamma^2\sum_{j=0}^{k}\omega_{k-j,j}^T\omega_{k-j,j}$$

Summing up both sides of the above inequality from $k=0$ to $k=N$, we have

$$\sum_{k=0}^{N}\sum_{j=0}^{k} z_{k-j,j}^T z_{k-j,j} < \gamma^2\sum_{k=0}^{N}\sum_{j=0}^{k}\omega_{k-j,j}^T\omega_{k-j,j} - \frac{1}{2}\sum_{j=0}^{N+1}\xi_{N+1-j,j}^T P(r_{N+1-j,j})\xi_{N+1-j,j}$$

and

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k} z_{k-j,j}^{T} z_{k-j,j} < \gamma^2 \sum_{k=0}^{\infty}\sum_{j=0}^{k} \omega_{k-j,j}^{T} \omega_{k-j,j}$$

which indicates that $\|z\|_2 < \gamma \|\omega\|_2$ for all nonzero $\omega \triangleq \{\omega_{i,j}\} \in l_2\{[0,\infty),[0,\infty)\}$. The proof is now complete. □

Remark 1: Note that there exist product terms between the matrix function $P(r_{i,j})$ and the system dynamic matrices in the PLMI condition proposed in Theorem 1, which will bring some difficulties in solving the filter synthesis problem. Applying the approach proposed in [4], in the following, we will make a decoupling between the matrix function $P(r_{i,j})$ and the system dynamic matrices by introducing a slack matrix variable. This decoupling technique enables us to obtain a more easily tractable condition for the filter synthesis, which leads to the result in the next theorem.

Theorem 2: The filtering error system $(\mathcal{E})$ in (4) is asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level $\gamma > 0$ if there exist matrix functions $P(r_{i,j}) > 0$, $Q(r_{i,j}) > 0$ and matrix X such that the following PLMI holds:

$$\begin{bmatrix} \Phi_{11} & 0 & 0 & X^T\bar{A}_1(r_{i,j+1}) & X^T\bar{A}_2(r_{i+1,j}) & X^T\bar{B}_1(r_{i,j+1}) & X^T\bar{B}_2(r_{i+1,j}) \\ * & -I & 0 & 0 & \bar{C}(r_{i+1,j}) & 0 & 0 \\ * & * & -I & \bar{C}(r_{i,j+1}) & 0 & 0 & 0 \\ * & * & * & \Phi_{44} & 0 & 0 & 0 \\ * & * & * & * & -Q(r_{i+1,j}) & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{14}$$

where

$$\Phi_{11} \triangleq P(r_{i+1,j+1}) - X - X^T, \quad \Phi_{44} \triangleq Q(r_{i,j+1}) - P(r_{i,j+1}).$$

Proof. All we need to do is to prove the equivalence between (6) and (14). First, if (6) holds, then (14) can be readily established by choosing $X = X^T = P(r_{i+1,j+1})$. On the other hand, if (14) holds, then $P(r_{i+1,j+1}) - X - X^T < 0$, which implies that X is nonsingular since $P(r_{i+1,j+1}) > 0$. In addition, we have $(X - P(r_{i+1,j+1}))^T P^{-1}(r_{i+1,j+1})\ (X - P(r_{i+1,j+1})) > 0$, which means $-X^T P^{-1}(r_{i+1,j+1})X < P(r_{i+1,j+1}) - X - X^T$. Therefore, the following PLMI holds:

$$\begin{bmatrix} -X^TP^{-1}(r_{i+1,j+1})X & 0 & 0 & X^T\bar{A}_1(r_{i,j+1}) & X^T\bar{A}_2(r_{i+1,j}) & X^T\bar{B}_1(r_{i,j+1}) & X^T\bar{B}_2(r_{i+1,j}) \\ * & -I & 0 & 0 & \bar{C}(r_{i+1,j}) & 0 & 0 \\ * & * & -I & \bar{C}(r_{i,j+1}) & 0 & 0 & 0 \\ * & * & * & \Phi_{44} & 0 & 0 & 0 \\ * & * & * & * & -Q(r_{i+1,j}) & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{15}$$

Performing a congruence transformation to (15) by $\mathrm{diag}\left\{X^{-1}P(r_{i+1,j+1}),\ I,\ I,\ I,\ I,\ I,\ I\right\}$ yields (6), and the proof is then completed. □

B. $\mathcal{H}_\infty$ Filter Synthesis

Now, we are in a position to give the result on the filter synthesis problem based on the improved PLMI condition proposed in Theorem 2. The following theorem gives a sufficient condition for the existence of such an $\mathcal{H}_\infty$ filter with the form of $(\mathcal{F})$ for the 2-D LPV system $(\mathcal{S})$.

Theorem 3: Consider the 2-D LPV system $(\mathcal{S})$ in (1). Given a scalar $\gamma > 0$, then there exists a full-order $\mathcal{H}_\infty$ filter in the form of $(\mathcal{F})$ such that the resulting filtering error system $(\mathcal{E})$ is asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation level γ if there exist matrix functions $\bar{P}_1(r_{i,j})$, $\bar{P}_2(r_{i,j})$, $\bar{P}_3(r_{i,j})$, $\bar{Q}_1(r_{i,j})$, $\bar{Q}_2(r_{i,j})$, $\bar{Q}_3(r_{i,j})$, $\bar{A}_{1F}(r_{i,j})$, $\bar{A}_{2F}(r_{i,j})$, $\bar{B}_{1F}(r_{i,j})$, $\bar{B}_{2F}(r_{i,j})$, $\bar{C}_F(r_{i,j})$ and matrices U, V, W such that the following PLMIs hold:

$$\left[\begin{array}{ccccc}
\bar{P}_1(r_{i+1,j+1}) - U^T - U & \bar{P}_{21}(r_{i+1,j+1}) - W^T - V & 0 & 0 & U^T A_1(r_{i,j+1}) + \bar{B}_{1F}(r_{i,j+1})C(r_{i,j+1}) \\
* & \bar{P}_{31}(r_{i+1,j+1}) - W^T - W & 0 & 0 & V^T A_1(r_{i,j+1}) + \bar{B}_{1F}(r_{i,j+1})C(r_{i,j+1}) \\
* & * & -I & 0 & 0 \\
* & * & * & -I & L(r_{i,j+1}) \\
* & * & * & * & \bar{Q}_1(r_{i,j+1}) - \bar{P}_1(r_{i,j+1}) \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & *
\end{array}\right.$$

$$\begin{array}{ccc}
\bar{A}_{1F}(r_{i,j+1}) & U^T A_2(r_{i+1,j}) + \bar{B}_{2F}(r_{i+1,j})C(r_{i+1,j}) & \bar{A}_{2F}(r_{i+1,j}) \\
\bar{A}_{1F}(r_{i,j+1}) & V^T A_2(r_{i+1,j}) + \bar{B}_{2F}(r_{i+1,j})C(r_{i+1,j}) & \bar{A}_{2F}(r_{i+1,j}) \\
0 & L(r_{i+1,j}) & -\bar{C}_F(r_{i+1,j}) \\
-\bar{C}_F(r_{i,j+1}) & 0 & 0 \\
\bar{Q}_2(r_{i,j+1}) - \bar{P}_2(r_{i,j+1}) & 0 & 0 \\
\bar{Q}_3(r_{i,j+1}) - \bar{P}_3(r_{i,j+1}) & 0 & 0 \\
* & -\bar{Q}_1(r_{i+1,j}) & -\bar{Q}_2(r_{i+1,j}) \\
* & * & -\bar{Q}_3(r_{i+1,j}) \\
* & * & * \\
* & * & *
\end{array}$$

$$\left.\begin{array}{cc}
U^T B_1(r_{i,j+1}) + \bar{B}_{1F}(r_{i,j+1})D(r_{i,j+1}) & U^T B_2(r_{i+1,j}) + \bar{B}_{2F}(r_{i+1,j})D(r_{i+1,j}) \\
V^T B_1(r_{i,j+1}) + \bar{B}_{1F}(r_{i,j+1})D(r_{i,j+1}) & V^T B_2(r_{i+1,j}) + \bar{B}_{2F}(r_{i+1,j})D(r_{i+1,j}) \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
-\gamma^2 I & 0 \\
* & -\gamma^2 I
\end{array}\right] < 0 \tag{16}$$

$$\bar{P}(r_{i,j}) = \begin{bmatrix} \bar{P}_1(r_{i,j}) & \bar{P}_2(r_{i,j}) \\ * & \bar{P}_3(r_{i,j}) \end{bmatrix} > 0 \tag{17}$$

$$\bar{Q}(r_{i,j}) = \begin{bmatrix} \bar{Q}_1(r_{i,j}) & \bar{Q}_2(r_{i,j}) \\ * & \bar{Q}_3(r_{i,j}) \end{bmatrix} > 0 \tag{18}$$

Moreover, the parameters of a desired $\mathcal{H}_\infty$ filter of the form (3) can be determined as follows:

$$\begin{bmatrix} A_{1F}(r_{i,j+1}) & B_{1F}(r_{i,j+1}) \\ A_{2F}(r_{i+1,j}) & B_{2F}(r_{i+1,j}) \\ C_F(r_{i,j}) & 0 \end{bmatrix} = \begin{bmatrix} W^{-T} & 0 & 0 \\ * & W^{-T} & 0 \\ * & * & I \end{bmatrix} \begin{bmatrix} \bar{A}_{1F}(r_{i,j+1}) & \bar{B}_{1F}(r_{i,j+1}) \\ \bar{A}_{2F}(r_{i+1,j}) & \bar{B}_{2F}(r_{i+1,j}) \\ \bar{C}_F(r_{i,j}) & 0 \end{bmatrix} \tag{19}$$

Proof. As mentioned in the proof of Theorem 2, since $P(r_{i,j}) > 0$, X is nonsingular if (14) holds. Now, partition X as

$$X = \begin{bmatrix} X_1 & X_2 \\ X_4 & X_3 \end{bmatrix}. \tag{20}$$

Without loss of generality, we assume that X_3 and X_4 are nonsingular (if not, X_3 and X_4 may be perturbed by matrices ΔX_3 and ΔX_4 respectively with sufficiently small norm such that $X_3 + \Delta X_3$ and $X_4 + \Delta X_4$ are nonsingular and satisfying (15)). Introduce the following matrices:

$$\begin{aligned} \Gamma &\triangleq \begin{bmatrix} I & 0 \\ 0 & X_3^{-1}X_4 \end{bmatrix},\ U \triangleq X_1,\ V \triangleq X_2X_3^{-1}X_4,\ W \triangleq X_4^TX_3^{-T}X_4, \\ \bar{P}(r_{i,j}) &\triangleq \Gamma^T P(r_{i,j})\Gamma = \begin{bmatrix} \bar{P}_1(r_{i,j}) & \bar{P}_2(r_{i,j}) \\ * & \bar{P}_3(r_{i,j}) \end{bmatrix} > 0, \\ \bar{Q}(r_{i,j}) &\triangleq \Gamma^T Q(r_{i,j})\Gamma = \begin{bmatrix} \bar{Q}_1(r_{i,j}) & \bar{Q}_2(r_{i,j}) \\ * & \bar{Q}_3(r_{i,j}) \end{bmatrix} > 0 \end{aligned} \tag{21}$$

and

$$\begin{bmatrix} \bar{A}_{1F}(r_{i,j+1}) & \bar{B}_{1F}(r_{i,j+1}) \\ \bar{A}_{2F}(r_{i+1,j}) & \bar{B}_{2F}(r_{i+1,j}) \\ \bar{C}_F(r_{i,j}) & 0 \end{bmatrix} \triangleq \begin{bmatrix} X_4^T & 0 & 0 \\ * & X_4^T & 0 \\ * & * & I \end{bmatrix} \begin{bmatrix} A_{1F}(r_{i,j+1}) & B_{1F}(r_{i,j+1}) \\ A_{2F}(r_{i+1,j}) & B_{2F}(r_{i+1,j}) \\ C_F(r_{i,j}) & 0 \end{bmatrix} \begin{bmatrix} X_3^{-1}X_4 & 0 \\ 0 & I \end{bmatrix} \tag{22}$$

Performing a congruence transformation to (14) by diagonal matrix $\mathrm{diag}\{\Gamma,\ I,\ I,\ \Gamma,\ \Gamma,\ I,\ I\}$, we have

$$\begin{bmatrix} \bar{P}(r_{i+1,j+1}) - \Gamma^T X\Gamma - \Gamma^T X^T\Gamma & 0 & 0 & \Gamma^T X^T \bar{A}_1(r_{i,j+1})\Gamma & \Gamma^T X^T \bar{A}_2(r_{i+1,j})\Gamma & \Gamma^T X^T \bar{B}_1(r_{i,j+1}) & \Gamma^T X^T \bar{B}_2(r_{i+1,j}) \\ * & -I & 0 & 0 & \bar{C}(r_{i+1,j})\Gamma & 0 & 0 \\ * & * & -I & \bar{C}(r_{i,j+1})\Gamma & 0 & 0 & 0 \\ * & * & * & \bar{Q}(r_{i,j+1}) - \bar{P}(r_{i,j+1}) & 0 & 0 & 0 \\ * & * & * & * & -\bar{Q}(r_{i+1,j}) & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{23}$$

in which

$$\begin{aligned}
\Gamma^T X^T \bar{A}_1(r_{i,j+1})\Gamma &= \begin{bmatrix} X_1^T A_1(r_{i,j+1}) + X_4^T B_{1F}(r_{i,j+1})C(r_{i,j+1}) & X_4^T A_{1F}(r_{i,j+1})X_3^{-1}X_4 \\ X_4^T X_3^{-T} X_2^T A_1(r_{i,j+1}) + X_4^T B_{1F}(r_{i,j+1})C(r_{i,j+1}) & X_4^T A_{1F}(r_{i,j+1})X_3^{-1}X_4 \end{bmatrix}, \\
\Gamma^T X^T \bar{A}_2(r_{i+1,j})\Gamma &= \begin{bmatrix} X_1^T A_2(r_{i+1,j}) + X_4^T B_{2F}(r_{i+1,j})C(r_{i+1,j}) & X_4^T A_{2F}(r_{i+1,j})X_3^{-1}X_4 \\ X_4^T X_3^{-T} X_2^T A_2(r_{i+1,j}) + X_4^T B_{2F}(r_{i+1,j})C(r_{i+1,j}) & X_4^T A_{2F}(r_{i+1,j})X_3^{-1}X_4 \end{bmatrix}, \\
\Gamma^T X^T \bar{B}_1(r_{i,j+1}) &= \begin{bmatrix} X_1^T B_1(r_{i,j+1}) + X_4^T B_{1F}(r_{i,j+1})D(r_{i,j+1}) \\ X_4^T X_3^{-T} X_2^T B_1(r_{i,j+1}) + X_4^T B_{1F}(r_{i,j+1})D(r_{i,j+1}) \end{bmatrix}, \\
\Gamma^T X^T \bar{B}_2(r_{i+1,j}) &= \begin{bmatrix} X_1^T B_2(r_{i+1,j}) + X_4^T B_{2F}(r_{i+1,j})D(r_{i+1,j}) \\ X_4^T X_3^{-T} X_2^T B_2(r_{i+1,j}) + X_4^T B_{2F}(r_{i+1,j})D(r_{i+1,j}) \end{bmatrix}, \\
\Gamma^T X^T \Gamma &= \begin{bmatrix} X_1^T & X_4^T X_3^{-1} X_4 \\ X_4^T X_3^{-T} X_2^T & X_4^T X_3^{-1} X_4 \end{bmatrix}, \quad \bar{C}(r_{i,j})\Gamma = \begin{bmatrix} L(r_{i,j}) & -C_F(r_{i,j})X_3^{-1}X_4 \end{bmatrix} \qquad (24)
\end{aligned}$$

Substituting (21)–(22) and (24) into (23) yields (16). On the other hand, (22) is equivalent to

$$\begin{aligned}
\begin{bmatrix} A_{1F}(r_{i,j+1}) & B_{1F}(r_{i,j+1}) \\ A_{2F}(r_{i+1,j}) & B_{2F}(r_{i+1,j}) \\ C_F(r_{i,j}) & 0 \end{bmatrix} &= \begin{bmatrix} X_4^{-T} & 0 & 0 \\ * & X_4^{-T} & 0 \\ * & * & I \end{bmatrix} \begin{bmatrix} \bar{A}_{1F}(r_{i,j+1}) & \bar{B}_{1F}(r_{i,j+1}) \\ \bar{A}_{2F}(r_{i+1,j}) & \bar{B}_{2F}(r_{i+1,j}) \\ \bar{C}_F(r_{i,j}) & 0 \end{bmatrix} \begin{bmatrix} X_4^{-1}X_3 & 0 \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} \left(X_4^{-1}X_3\right)^{-1} W^{-T} & 0 & 0 \\ * & \left(X_4^{-1}X_3\right)^{-1} W^{-T} & 0 \\ * & * & I \end{bmatrix} \\
&\quad \times \begin{bmatrix} \bar{A}_{1F}(r_{i,j+1}) & \bar{B}_{1F}(r_{i,j+1}) \\ \bar{A}_{2F}(r_{i+1,j}) & \bar{B}_{2F}(r_{i+1,j}) \\ \bar{C}_F(r_{i,j}) & 0 \end{bmatrix} \begin{bmatrix} X_4^{-1}X_3 & 0 \\ 0 & I \end{bmatrix} \qquad (25)
\end{aligned}$$

Then, it is noted that the filter matrices of (3) can be written as (25). This implies that $X_4^{-1}X_3$ can be viewed as a similarity transformation on the state-space realization of the filter and, as such, has no effect on the filter mapping from y to $\hat{z}$. Without loss of generality, we can set $X_4^{-1}X_3 = I$, thus obtain (19). Therefore, we can conclude that the filter in (3) can be constructed by (19). This completes the proof. □

Remark 2: Note that Theorem 3 provides a sufficient condition for the solvability of the $\mathcal{H}_\infty$ filtering problem for 2-D LPV system. Since the obtained condition is within the PLMIs framework, the desired filter can be determined by solving the following convex optimization problem:

$$\text{Minimize } \gamma^2 \qquad \text{subject to } (16)\text{–}(18). \qquad (26)$$

Remark 3: Notice that the PLMI condition (16) corresponds to an infinite-dimensional convex problem due to its parametric dependence. To convert it into a finite-dimensional optimization problem, by using the gridding technique, the parameter-dependent matrix function $\mathcal{Y}(r_{i,j}) \triangleq \{\bar{P}_1(r_{i,j}),\ \bar{P}_2(r_{i,j}),\ \bar{P}_3(r_{i,j}),\ \bar{Q}_1(r_{i,j}),\ \bar{Q}_2(r_{i,j}),\ \bar{Q}_3(r_{i,j}),\ \bar{A}_{1F}(r_{i,j}),\ \bar{A}_{2F}(r_{i,j}),\ \bar{B}_{1F}(r_{i,j}),\ \bar{B}_{2F}(r_{i,j}),\ \bar{C}_F(r_{i,j})\}$ that appears in (16) can be approximated using a finite set of basis functions [20]. That is, we can choose appropriate basis functions $\{f_k(r_{i,j})\}_{k=1}^{n_f}$ such that

$$\mathcal{Y}(r_{i,j}) = \sum\nolimits_{k=1}^{n_f} f_k(r_{i,j})\mathcal{Y}_k \qquad (27)$$

where $\mathcal{Y}_k \triangleq \{\bar{P}_{1k},\ \bar{P}_{2k},\ \bar{P}_{3k},\ \bar{Q}_{1k},\ \bar{Q}_{2k},\ \bar{Q}_{3k},\ \bar{A}_{1Fk},\ \bar{A}_{2Fk},\ \bar{B}_{1Fk},\ \bar{B}_{2Fk},\ \bar{C}_{Fk}\}$ denotes the vertices of $\mathcal{Y}(r_{i,j})$.

IV. l_2-l_∞ Filtering

A. Filter Analysis

In this section, the sufficient condition for the solvability of the l_2-l_∞ filtering problem formulated in the previous section is derived. First, we give the following theorem which will play a key role in the derivation of our main results.

Theorem 4: The filtering error system $(\mathcal{E})$ in (4) is asymptotically stable with an l_2-l_∞ disturbance attenuation level $\gamma > 0$ if there exist matrix functions $P(r_{i,j}) > 0$ and $Q(r_{i,j}) > 0$ such that the following PLMIs hold:

$$\begin{bmatrix} -P(r_{i+1,j+1}) & P(r_{i+1,j+1})\bar{A}_1(r_{i,j+1}) & P(r_{i+1,j+1})\bar{A}_2(r_{i+1,j}) & P(r_{i+1,j+1})\bar{B}_1(r_{i,j+1}) & P(r_{i+1,j+1})\bar{B}_2(r_{i+1,j}) \\ * & Q(r_{i,j+1}) - P(r_{i,j+1}) & 0 & 0 & 0 \\ * & * & -Q(r_{i+1,j}) & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{28}$$

$$\begin{bmatrix} -\frac{1}{2}\gamma^2 I & 0 & \bar{C}(r_{i,j+1}) & 0 \\ * & -\frac{1}{2}\gamma^2 I & 0 & \bar{C}(r_{i+1,j}) \\ * & * & -P(r_{i,j+1}) & 0 \\ * & * & * & -P(r_{i+1,j}) \end{bmatrix} < 0 \tag{29}$$

Proof. For the establishment of the asymptotic stability of the filtering error system $(\mathcal{E})$ in (4) with $\omega_{i,j} \equiv 0$, we refer the readers to the proof of Theorem 1. In the following, we shall develop an l_2-l_∞ performance for the filtering error system $(\mathcal{E})$. Consider the following index:

$$\mathcal{L} \triangleq -\omega^T\omega + \mathcal{I}_{i,j} \tag{30}$$

where $\tilde{z} \triangleq \begin{bmatrix} \tilde{z}_{i,j+1}^T & \tilde{z}_{i+1,j}^T \end{bmatrix}^T$, $\omega \triangleq \begin{bmatrix} \omega_{i,j+1}^T & \omega_{i+1,j}^T \end{bmatrix}^T$, $\mathcal{I}_{i,j}$ is defined in (7) and has been further developed to (8). Then, along the solution of the filtering error system $(\mathcal{E})$, we calculate that

$$\mathcal{L} = -\omega_{i,j+1}^T\omega_{i,j+1} - \omega_{i+1,j}^T\omega_{i+1,j} + \mathcal{I}_{i,j} = \eta^T\Sigma\eta$$

where $\eta \triangleq \begin{bmatrix} \xi_{i,j+1}^T & \xi_{i+1,j}^T & \omega_{i,j+1}^T & \omega_{i+1,j}^T \end{bmatrix}^T$ and

$$\Sigma \triangleq \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} \\ * & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} \\ * & * & \Sigma_{33} & \Sigma_{34} \\ * & * & * & \Sigma_{44} \end{bmatrix}$$

with

$$\begin{aligned}
\Sigma_{11} &\triangleq \bar{A}_1^T(r_{i,j+1})P(r_{i+1,j+1})\bar{A}_1(r_{i,j+1}) + Q(r_{i,j+1}) - P(r_{i,j+1}),\\
\Sigma_{12} &\triangleq \bar{A}_1^T(r_{i,j+1})P(r_{i+1,j+1})\bar{A}_2(r_{i+1,j}),\\
\Sigma_{13} &\triangleq \bar{A}_1^T(r_{i,j+1})P(r_{i+1,j+1})\bar{B}_1(r_{i,j+1}),\\
\Sigma_{14} &\triangleq \bar{A}_1^T r_{i,j+1})P(r_{i+1,j+1})\bar{B}_2(r_{i+1,j}),\\
\Sigma_{22} &\triangleq \bar{A}_2^T(r_{i+1,j})P(r_{i+1,j+1})\bar{A}_2(r_{i+1,j}) - Q(r_{i+1,j}),\\
\Sigma_{23} &\triangleq \bar{A}_2^T(r_{i+1,j})P(r_{i+1,j+1})\bar{B}_1(r_{i,j+1}),\\
\Sigma_{24} &\triangleq \bar{A}_2^T(r_{i+1,j})P(r_{i+1,j+1})\bar{B}_2(r_{i+1,j}),\\
\Sigma_{33} &\triangleq \bar{B}_1^T(r_{i,j+1})P(r_{i+1,j+1})\bar{B}_1(r_{i,j+1}) - I,\\
\Sigma_{34} &\triangleq \bar{B}_1^T(r_{i,j+1})P(r_{i+1,j+1})\bar{B}_2(r_{i+1,j}),\\
\Sigma_{44} &\triangleq \bar{B}_2^T(r_{i+1,j})P(r_{i+1,j+1})\bar{B}_2(r_{i+1,j}) - I.
\end{aligned}$$

By Schur complement, PLMI (28) implies $\Sigma < 0$, and then for $\eta \neq 0$, we have $\mathcal{L} < 0$, i.e.,

$$\xi_{i+1,j+1}^T P(r_{i+1,j+1})\xi_{i+1,j+1} < \tilde{\xi}^T \mathrm{diag}\left\{(P(r_{i,j+1}) - Q(r_{i,j+1})), Q(r_{i+1,j})\right\}\tilde{\xi} + \omega^T\omega,$$

that is,

$$\begin{aligned}
\xi_{i+1,j+1}^T P(r_{i+1,j+1})\xi_{i+1,j+1} \;&<\; \xi_{i,j+1}^T(P(r_{i,j+1}) - Q(r_{i,j+1}))\xi_{i,j+1} + \xi_{i+1,j}^T Q(r_{i+1,j})\xi_{i+1,j}\\
&\quad +\omega_{i,j+1}^T\omega_{i,j+1} + \omega_{i+1,j}^T\omega_{i+1,j}
\end{aligned} \tag{31}$$

Using the relationship (31), it can be seen that

$$\begin{aligned}
\xi_{k+1,0}^T P(r_{k+1,0})\xi_{k+1,0} &= \xi_{k+1,0}^T P(r_{k+1,0})\xi_{k+1,0}\\
\xi_{k,1}^T P(r_{k,1})\xi_{k,1} &< \xi_{k-1,1}^T(P(r_{k-1,1}) - Q(r_{k-1,1}))\xi_{k-1,1} + \xi_{k,0}^T Q(r_{k,0})\xi_{k,0}\\
&\quad +\omega_{k-1,1}^T\omega_{k-1,1} + \omega_{k,0}^T\omega_{k,0}\\
\xi_{k-1,2}^T P(r_{k-1,2})\xi_{k-1,2} &< \xi_{k-2,2}^T(P(r_{k-2,2}) - Q(r_{k-2,2}))\xi_{k-2,2} + \xi_{k-1,1}^T Q(r_{k-1,1})\xi_{k-1,1}\\
&\quad +\omega_{k-2,2}^T\omega_{k-2,2} + \omega_{k-1,1}^T\omega_{k-1,1}\\
&\;\;\vdots\\
\xi_{1,k}^T P(r_{1,k})\xi_{1,k} &< \xi_{0,k}^T(P(r_{0,k}) - Q(r_{0,k}))\xi_{0,k} + \xi_{1,k-1}^T Q(r_{1,k-1})\xi_{1,k-1}\\
&\quad +\omega_{0,k}^T\omega_{0,k} + \omega_{1,k-1}^T\omega_{1,k-1}\\
\xi_{0,k+1}^T P(r_{0,k+1})\xi_{0,k+1} &= \xi_{0,k+1}^T P(r_{0,k+1})\xi_{0,k+1}
\end{aligned}$$

which imply

$$\sum_{j=0}^{k+1}\xi_{k+1-j,j}^T P(r_{k+1-j,j})\xi_{k+1-j,j} < \sum_{j=0}^{k}\xi_{k-j,j}^T P(r_{k-j,j})\xi_{k-j,j} + 2\sum_{j=0}^{k}\omega_{k-j,j}^T\omega_{k-j,j}$$

Summing up both sides of the above inequality with respect to K from 0 to N, we have

$$\sum_{j=0}^{N+1}\xi_{N+1-j,j}^T P(r_{N+1-j,j})\xi_{N+1-j,j} < 2\sum_{k=0}^{N}\sum_{j=0}^{k}\omega_{k-j,j}^T\omega_{k-j,j} \tag{32}$$

Since the integer N can be taken arbitrarily, it is also true from (32) that:

$$\xi_{i,j+1}^T P(r_{i,j+1})\xi_{i,j+1} + \xi_{i+1,j}^T P(r_{i+1,j})\xi_{i+1,j} < 2\sum_{k=0}^{\infty}\sum_{l=0}^{k}\omega_{k-l,l}^T\omega_{k-l,l} = 2\left\|\omega\right\|_2^2 \quad \forall\; i,j=1,2,\ldots \tag{33}$$

On the other hand, by Schur complement, (29) yields

$$\begin{aligned} -P(r_{i,j+1}) + 2\gamma^{-2}\bar{C}^T(r_{i,j+1})\bar{C}(r_{i,j+1}) &< 0 \qquad (34)\\ -P(r_{i+1,j}) + 2\gamma^{-2}\bar{C}^T(r_{i+1,j})\bar{C}(r_{i+1,j}) &< 0 \qquad (35)\end{aligned}$$

Pre- and post- multiplying both sides of (34) with $\xi_{i,j+1}^T$ and its transpose, and pre- and post- multiplying both sides of (35) with $\xi_{i+1,j}^T$ and its transpose, we sum up both sides of two resultant inequalities and obtain

$$\begin{aligned} & 2\gamma^{-2}\xi_{i,j+1}^T\bar{C}^T(r_{i,j+1})\bar{C}(r_{i,j+1})\xi_{i,j+1} + 2\gamma^{-2}\xi_{i+1,j}^T\bar{C}^T(r_{i+1,j})\bar{C}(r_{i+1,j})\xi_{i+1,j} \\ < \;& \xi_{i,j+1}^T P(r_{i,j+1})\xi_{i,j+1} + \xi_{i+1,j}^T P(r_{i+1,j})\xi_{i+1,j}, \end{aligned}$$

that is,

$$2\gamma^{-2}\left(\tilde{z}_{i,j+1}^T\tilde{z}_{i,j+1} + \tilde{z}_{i+1,j}^T\tilde{z}_{i+1,j}\right) < \xi_{i,j+1}^T P(r_{i,j+1})\xi_{i,j+1} + \xi_{i+1,j}^T P(r_{i+1,j})\xi_{i+1,j}. \tag{36}$$

Considering (33) and (36), we have

$$|\tilde{z}_{i,j}|^2 = \tilde{z}_{i,j+1}^T\tilde{z}_{i,j+1} + \tilde{z}_{i+1,j}^T\tilde{z}_{i+1,j} < \gamma^2\left\|\omega\right\|_2^2 \quad \forall\; i,j=1,2,\ldots$$

Therefore, we conclude that

$$\left\|\tilde{z}\right\|_\infty^2 = \sup_{\forall\; i,j}|\tilde{z}_{i,j}|^2 < \gamma^2\left\|\omega\right\|_2^2$$

which implies $\left\|\tilde{z}\right\|_\infty < \gamma\left\|\omega\right\|_2$, that is, the l_2-l_∞ gain from ω to $\tilde{z}$ is less than γ. This completes the proof. □

Along the same line of the derivation in Theorem 2, we can obtain the following theorem for which the proof is omitted.

Theorem 5: The filtering error system $(\mathcal{E})$ in (4) is asymptotically stable with an l_2-l_∞ disturbance attenuation level $\gamma > 0$ if there exist matrix functions $P(r_{i,j}) > 0$, $Q(r_{i,j}) > 0$ and matrix X such that (29) and the following PLMI holds:

$$\begin{bmatrix} \Phi_{11} & X^T\bar{A}_1(r_{i,j+1}) & X^T\bar{A}_2(r_{i+1,j}) & X^T\bar{B}_1(r_{i,j+1}) & X^T\bar{B}_2(r_{i+1,j}) \\ * & \Phi_{44} & 0 & 0 & 0 \\ * & * & -Q(r_{i+1,j}) & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -I \end{bmatrix} < 0 \tag{37}$$

where Φ_{11} and Φ_{44} are defined in (14).

B. l_2-l_∞ Filter Synthesis

Now, we are in a position to give the result on the filter synthesis problem based on the improved PLMI condition proposed in Theorem 5. The following theorem gives a sufficient condition for the existence of such an l_2-l_∞ filter with the form of $(\mathcal{F})$ for the 2-D LPV system $(\mathcal{S})$.

Theorem 6: Consider the 2-D LPV system $(\mathcal{S})$ in (1). Given a scalar $\gamma > 0$, then there exists a full-order l_2-l_∞ filter in the form of $(\mathcal{F})$ such that the resulting filtering error system $(\mathcal{E})$ is asymptotically stable with an l_2-l_∞ disturbance attenuation level γ if there exist matrix functions $\bar{P}_1(r_{i,j})$, $\bar{P}_2(r_{i,j})$, $\bar{P}_3(r_{i,j})$, $\bar{Q}_1(r_{i,j})$,

$\bar{Q}_2(r_{i,j})$, $\bar{Q}_3(r_{i,j})$, $\bar{A}_{1F}(r_{i,j})$, $\bar{A}_{2F}(r_{i,j})$, $\bar{B}_{1F}(r_{i,j})$, $\bar{B}_{2F}(r_{i,j})$, $\bar{C}_F(r_{i,j})$ and matrices U, V, W such that the following PLMIs hold:

$$\begin{bmatrix} \bar{P}_1(r_{i+1,j+1})-U^T-U & \bar{P}_{21}(r_{i+1,j+1})-W^T-V & U^TA_1(r_{i,j+1})+\bar{B}_{1F}(r_{i,j+1})C(r_{i,j+1}) & \bar{A}_{1F}(r_{i,j+1}) & U^TA_2(r_{i+1,j})+\bar{B}_{2F}(r_{i+1,j})C(r_{i+1,j}) & \bar{A}_{2F}(r_{i+1,j}) & U^TB_1(r_{i,j+1})+\bar{B}_{1F}(r_{i,j+1})D(r_{i,j+1}) & U^TB_2(r_{i+1,j})+\bar{B}_{2F}(r_{i+1,j})D(r_{i+1,j}) \\ * & \bar{P}_{31}(r_{i+1,j+1})-W^T-W & V^TA_1(r_{i,j+1})+\bar{B}_{1F}(r_{i,j+1})C(r_{i,j+1}) & \bar{A}_{1F}(r_{i,j+1}) & V^TA_2(r_{i+1,j})+\bar{B}_{2F}(r_{i+1,j})C(r_{i+1,j}) & \bar{A}_{2F}(r_{i+1,j}) & V^TB_1(r_{i,j+1})+\bar{B}_{1F}(r_{i,j+1})D(r_{i,j+1}) & V^TB_2(r_{i+1,j})+\bar{B}_{2F}(r_{i+1,j})D(r_{i+1,j}) \\ * & * & \bar{Q}_1(r_{i,j+1})-\bar{P}_1(r_{i,j+1}) & \bar{Q}_2(r_{i,j+1})-\bar{P}_2(r_{i,j+1}) & 0 & 0 & 0 & 0 \\ * & * & * & \bar{Q}_3(r_{i,j+1})-\bar{P}_3(r_{i,j+1}) & 0 & 0 & 0 & 0 \\ * & * & * & * & -\bar{Q}_1(r_{i+1,j}) & -\bar{Q}_2(r_{i+1,j}) & 0 & 0 \\ * & * & * & * & * & -\bar{Q}_3(r_{i+1,j}) & 0 & 0 \\ * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{38}$$

$$\begin{bmatrix} -\frac{1}{2}\gamma^2 I & 0 & L(r_{i,j+1}) & -\bar{C}_F(r_{i,j+1}) & 0 & 0 \\ * & -\frac{1}{2}\gamma^2 I & 0 & 0 & L(r_{i+1,j}) & -\bar{C}_F(r_{i+1,j}) \\ * & * & -\bar{P}_1(r_{i,j+1}) & -\bar{P}_2(r_{i,j+1}) & 0 & 0 \\ * & * & * & -\bar{P}_3(r_{i,j+1}) & 0 & 0 \\ * & * & * & * & -\bar{P}_1(r_{i+1,j}) & -\bar{P}_2(r_{i+1,j}) \\ * & * & * & * & * & -\bar{P}_3(r_{i+1,j}) \end{bmatrix} < 0 \tag{39}$$

$$\bar{P}(r_{i,j}) = \begin{bmatrix} \bar{P}_1(r_{i,j}) & \bar{P}_2(r_{i,j}) \\ * & \bar{P}_3(r_{i,j}) \end{bmatrix} > 0 \tag{40}$$

$$\bar{Q}(r_{i,j}) = \begin{bmatrix} \bar{Q}_1(r_{i,j}) & \bar{Q}_2(r_{i,j}) \\ * & \bar{Q}_3(r_{i,j}) \end{bmatrix} > 0 \tag{41}$$

Moreover, a desired l_2-l_∞ filter is given in the form of (3) with parameters as follows:

$$\begin{bmatrix} A_{1F}(r_{i,j}) & B_{1F}(r_{i,j}) \\ A_{2F}(r_{i,j}) & B_{2F}(r_{i,j}) \\ C_F(r_{i,j}) & 0 \end{bmatrix} = \begin{bmatrix} W^{-T} & 0 & 0 \\ * & W^{-T} & 0 \\ * & * & I \end{bmatrix} \begin{bmatrix} \bar{A}_{1F}(r_{i,j}) & \bar{B}_{1F}(r_{i,j}) \\ \bar{A}_{2F}(r_{i,j}) & \bar{B}_{2F}(r_{i,j}) \\ \bar{C}_F(r_{i,j}) & 0 \end{bmatrix}. \tag{42}$$

Proof. The proof can be carried out by employing the same techniques used as in the proof of Theorem 3, and is thus omitted here. □

Remark 4: It should be pointed out that, in order to obtain (39), we can perform a congruence transformation to (29) by diag $\{I,\ I,\ \Gamma,\ \Gamma\}$ and consider (24) and (21)–(22).

Remark 5: Notice that Theorem 6 provides a sufficient condition for the solvability of the l_2-l_∞ filter problem for the 2-D LPV system. Since the obtained conditions are expressed in terms of PLMIs, the desired filter can be determined by solving the following convex optimization problem:

$$\text{Minimize } \gamma^2 \qquad \text{subject to } \textit{(38)–(41)} \tag{43}$$

V. An Illustrative Example

Consider 2-D LPV system $(\mathcal{S})$ in (1) with the following matrices:

$$\begin{aligned}
A_1(r_{i,j+1}) &= \begin{bmatrix} 0.2+0.1r^1_{i,j+1} & -0.5 \\ 0.5 & 0.2+0.1r^2_{i,j+1} \end{bmatrix}, \quad B_1(r_{i,j+1}) = \begin{bmatrix} 0.5+0.1r^1_{i,j+1} \\ 0.25 \end{bmatrix} \\
A_2(r_{i+1,j}) &= \begin{bmatrix} 0.25 & 0.1r^1_{i+1,j} \\ 0.05 & 0.4+0.1r^2_{i+1,j} \end{bmatrix}, \quad B_2(r_{i+1,j}) = \begin{bmatrix} 0 \\ 0.5+0.1r^1_{i+1,j} \end{bmatrix} \\
C(r_{i,j}) &= \begin{bmatrix} 2.0+0.2r^1_{i,j} & 1.0-0.1r^2_{i,j} \end{bmatrix}, \quad D(r_{i,j}) = 1.0-0.1r^1_{i,j} \\
L(r_{i,j}) &= \begin{bmatrix} 1.0+0.1r^1_{i,j} & 2.0-0.2r^2_{i,j} \end{bmatrix}
\end{aligned}$$

where $r^1_{i,j} = \sin(i+j)$ and $r^2_{i,j} = |\cos(5i+5j)|$ are two time-varying parameters. Let the disturbance input $\omega_{i,j}$ be

$$\omega_{i,j} = \begin{cases} 0.5, & 3 \le i,j \le 19 \\ 0, & \text{otherwise} \end{cases}$$

Our purpose hereafter is to design an $\mathcal{H}_\infty$ or l_2-l_∞ filter in the form of (3) such that the filtering error system $(\mathcal{E})$ is asymptotically stable with an $\mathcal{H}_\infty$ or l_2-l_∞ disturbance attenuation level γ. To solve the filters synthesis problem, we choose three basis functions in expansion (27) as follows:

$$f_1(r_{i,j}) = 1, \quad f_2(r_{i,j}) = r^1_{i,j}, \quad f_3(r_{i,j}) = r^2_{i,j}. \tag{44}$$

Gridding the parameter space uniformly using a 5×5 grid, solving the convex optimization problem of (26) by using the LMI-Toolbox in the Matlab and considering (19), we obtain that the minimum achievable (according to the feasibility of the LMI conditions) noise attenuation level for the full-order $\mathcal{H}_\infty$ filtering problem is $\gamma^* = 1.8469$ and the corresponding filter matrices as follows:

$$\begin{aligned}
A_{1F1} &= \begin{bmatrix} 0.0765 & -0.1711 \\ 0.0079 & -0.0399 \end{bmatrix}, \quad A_{1F2} = \begin{bmatrix} -0.0298 & -0.0246 \\ -0.0128 & 0.0062 \end{bmatrix} \\
A_{1F3} &= \begin{bmatrix} -0.1197 & -0.0623 \\ 0.0091 & 0.0598 \end{bmatrix}, \quad A_{2F1} = \begin{bmatrix} 0.0159 & -0.0120 \\ -0.1949 & 0.1517 \end{bmatrix} \\
A_{2F2} &= \begin{bmatrix} -0.0349 & 0.0242 \\ -0.0621 & -0.0105 \end{bmatrix}, \quad A_{2F3} = \begin{bmatrix} -0.0057 & 0.0011 \\ -0.0701 & 0.0207 \end{bmatrix} \\
B_{1F1} &= \begin{bmatrix} -0.0043 \\ -0.1489 \end{bmatrix}, \quad B_{1F2} = \begin{bmatrix} -0.0263 \\ 0.0046 \end{bmatrix}, \quad B_{1F3} = \begin{bmatrix} -0.0746 \\ -0.0160 \end{bmatrix} \\
B_{2F1} &= \begin{bmatrix} -0.0371 \\ -0.1055 \end{bmatrix}, \quad B_{2F2} = \begin{bmatrix} -0.0139 \\ -0.0203 \end{bmatrix}, \quad B_{2F3} = \begin{bmatrix} -0.0060 \\ -0.0565 \end{bmatrix} \\
C_{F1} &= \begin{bmatrix} -0.9559 & -2.0140 \end{bmatrix}, \quad C_{F2} = \begin{bmatrix} -0.1029 & -0.0133 \end{bmatrix} \\
C_{F3} &= \begin{bmatrix} -0.0658 & 0.2222 \end{bmatrix}.
\end{aligned}$$

By solving the convex optimization problem of (43) and considering (42), the minimum l_2-l_∞ attenuation performance obtained is $\gamma^* = 1.3632$, and the corresponding filter matrices are

$$\begin{aligned}
A_{1F1} &= \begin{bmatrix} -0.0652 & -0.1653 \\ -0.0110 & -0.0253 \end{bmatrix}, \quad A_{1F2} = \begin{bmatrix} -0.0021 & -0.0076 \\ 0.0063 & 0.0064 \end{bmatrix} \\
A_{1F3} &= \begin{bmatrix} -0.0378 & -0.0184 \\ 0.0061 & 0.0261 \end{bmatrix}, \quad A_{2F1} = \begin{bmatrix} 0.0067 & -0.0072 \\ -0.1121 & 0.0406 \end{bmatrix} \\
A_{2F2} &= \begin{bmatrix} -0.0265 & 0.0131 \\ -0.0158 & 0.0039 \end{bmatrix}, \quad A_{2F3} = \begin{bmatrix} 0.0012 & 0.0025 \\ -0.0121 & 0.0128 \end{bmatrix} \\
B_{1F1} &= \begin{bmatrix} -0.0610 \\ -0.0655 \end{bmatrix}, \quad B_{1F2} = \begin{bmatrix} -0.0064 \\ 0.0065 \end{bmatrix}, \quad B_{1F3} = \begin{bmatrix} -0.0276 \\ -0.0032 \end{bmatrix} \\
B_{2F1} &= \begin{bmatrix} -0.0251 \\ -0.0601 \end{bmatrix}, \quad B_{2F2} = \begin{bmatrix} -0.0112 \\ -0.0022 \end{bmatrix}, \quad B_{2F3} = \begin{bmatrix} -0.0015 \\ -0.0142 \end{bmatrix} \\
C_{F1} &= \begin{bmatrix} -0.9544 & -1.9419 \end{bmatrix}, \quad C_{F2} = \begin{bmatrix} -0.0950 & -0.0107 \end{bmatrix} \\
C_{F3} &= \begin{bmatrix} -0.0375 & 0.1738 \end{bmatrix}.
\end{aligned}$$

Then, from Remark 3, the corresponding $\mathcal{H}_\infty$ or l_2-l_∞ filter parameter matrices $\mathcal{X}(r_{i,j}) \triangleq \{A_{1F}(r_{i,j}), A_{2F}(r_{i,j}), B_{1F}(r_{i,j}, B_{2F}(r_{i,j}, C_F(r_{i,j})\}$ can be described by

$$\mathcal{X}(r_{i,j}) = \sum\nolimits_{k=1}^{3} f_k(r_{i,j})\mathcal{X}_k$$

where $\mathcal{X}_k \triangleq \{A_{1Fk}, A_{2Fk}, B_{1Fk}, B_{2Fk}, C_{Fk}\}$ denotes the vertices of $\mathcal{X}_k(r_{i,j})$, and $f_k(r_{i,j})$ have defined in (44).

In the following, we shall show the usefulness of the designed $\mathcal{H}_\infty$ and l_2-l_∞ filters by presenting simulation results. To show the asymptotic stability of the filtering error system, let the initial and boundary conditions be

$$x_{0,i} = x_{i,0} = \begin{cases} \begin{bmatrix} 1 & 1.5 \end{bmatrix}^T, & 0 \le i \le 15 \\ \begin{bmatrix} 0 & 0 \end{bmatrix}^T, & i > 15 \end{cases}$$

The $\mathcal{H}_\infty$ and l_2-l_∞ performances are summarized in Table 1. It can be seen that the achieved $\mathcal{H}_\infty$ and l_2-l_∞ gains less than the corresponding minimum feasible $\mathcal{H}_\infty$ performance $\gamma^* = 1.8469$ and l_2-l_∞ performance $\gamma^* = 1.3632$, respectively.

Performances	Minimum Feasible γ	Achieved Values
$\mathcal{H}_\infty$ performance	1.8469	1.6714
l_2-l_∞ performance	1.3632	1.2035

Table 1. Summary of the $\mathcal{H}_\infty$ and l_2-l_∞ performances

The state responses of the designed $\mathcal{H}_\infty$ filter are given in Figures 1 and 2, and Figure 3 shows the filtering error $\tilde{z}_{i,j}$ of the $\mathcal{H}_\infty$ filtering. Similarly, the state responses of the designed l_2-l_∞ filter are given in Figures 4 and 5, and Figure 6 shows the filtering error $\tilde{z}_{i,j}$ of the l_2-l_∞ case. It can be seen from Figures 3 and 6 that both the $\mathcal{H}_\infty$ and l_2-l_∞ filters guarantee that $\tilde{z}_{i,j}$ converges to zero under the above conditions.

VI. Conclusion

In this paper, the problems of $\mathcal{H}_\infty$ and l_2-l_∞ filtering for a class of 2-D LPV systems have been investigated. Some sufficient conditions have been proposed for the existences of $\mathcal{H}_\infty$ and l_2-l_∞ filters in terms of PLMIs, respectively. The designed $\mathcal{H}_\infty$ or l_2-l_∞ filter guarantees asymptotic stability and a prescribed $\mathcal{H}_\infty$ or l_2-l_∞ performance of the filtering error system, and the desired filters can be found by solving the corresponding convex optimization problems. An illustrative example has been presented to demonstrate the effectiveness of the proposed methods.

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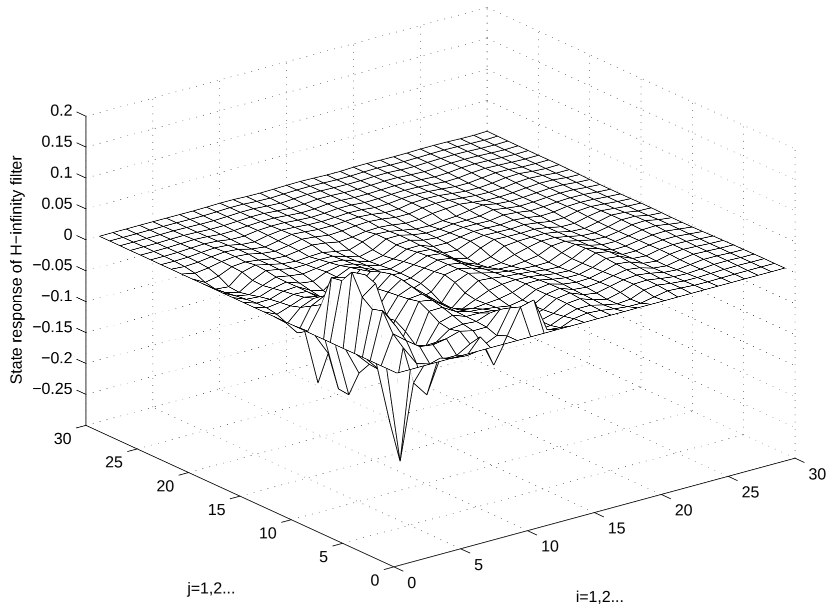

Fig. 1. State response of the $\mathcal{H}_\infty$ filter $\hat{x}_1(i,j)$

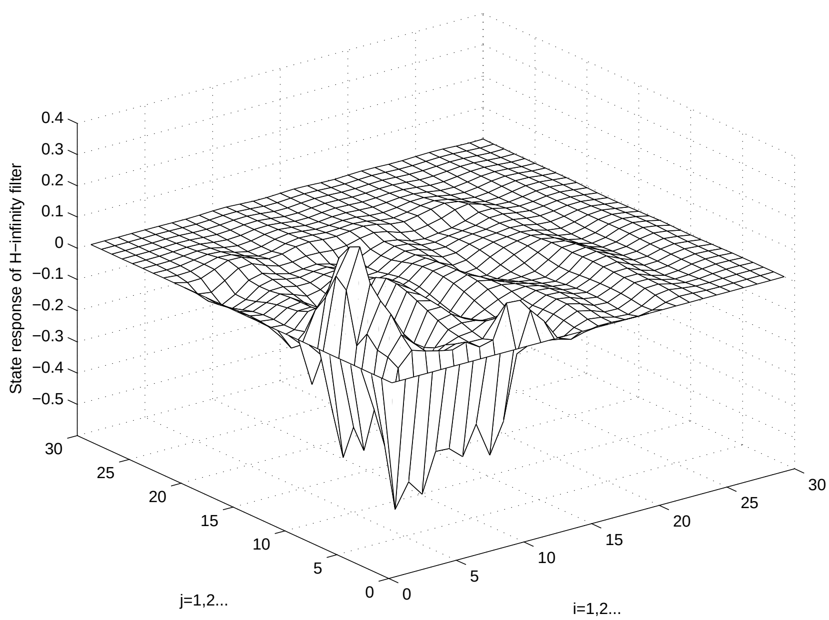

Fig. 2. State response of the $\mathcal{H}_\infty$ filter $\hat{x}_2(i,j)$

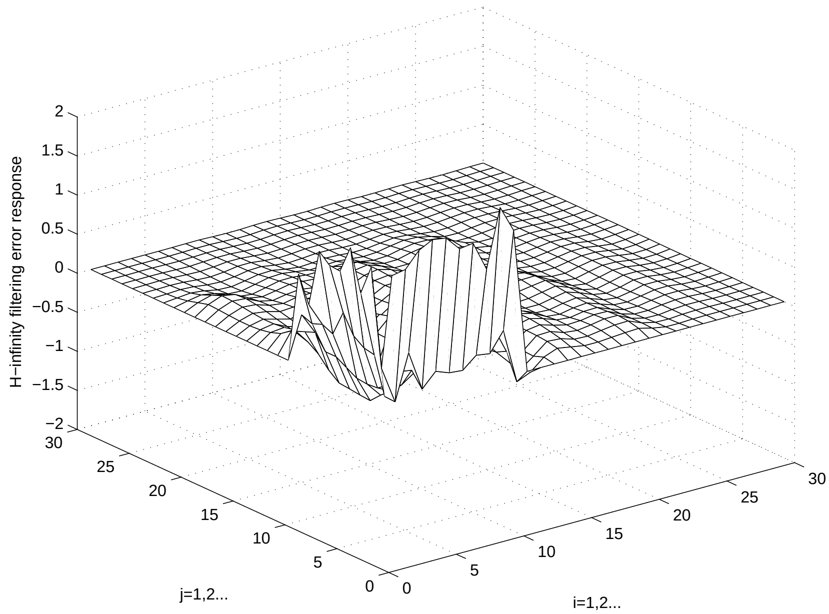

Fig. 3. The filtering error $\tilde{z}_{i,j}$ of the $\mathcal{H}_\infty$ case

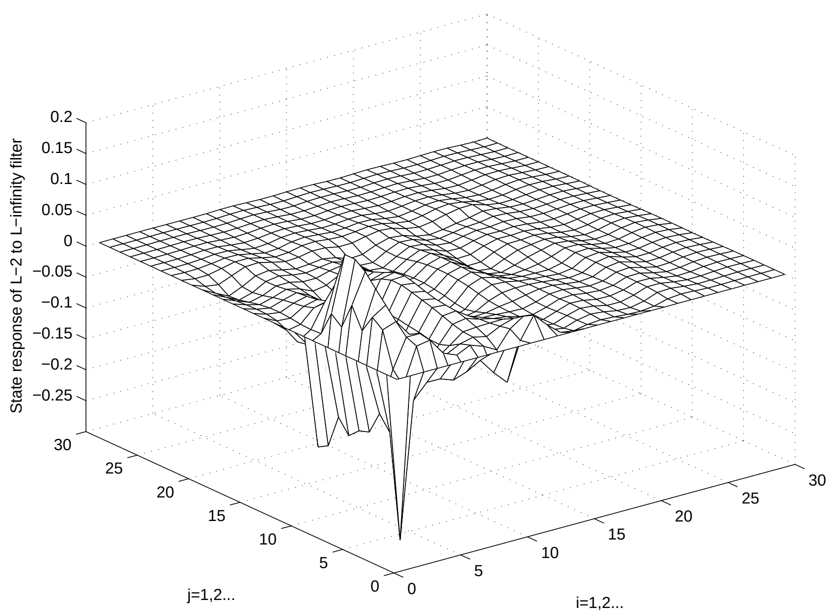

Fig. 4. State response of the l_2-l_∞ filter $\hat{x}_1(i,j)$

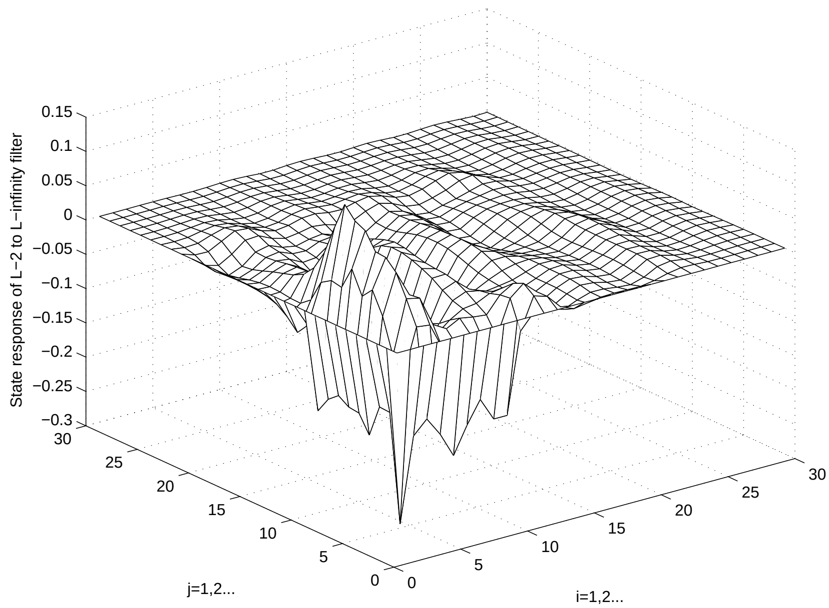

Fig. 5. State response of the l_2-l_∞ filter $\hat{x}_2(i,j)$

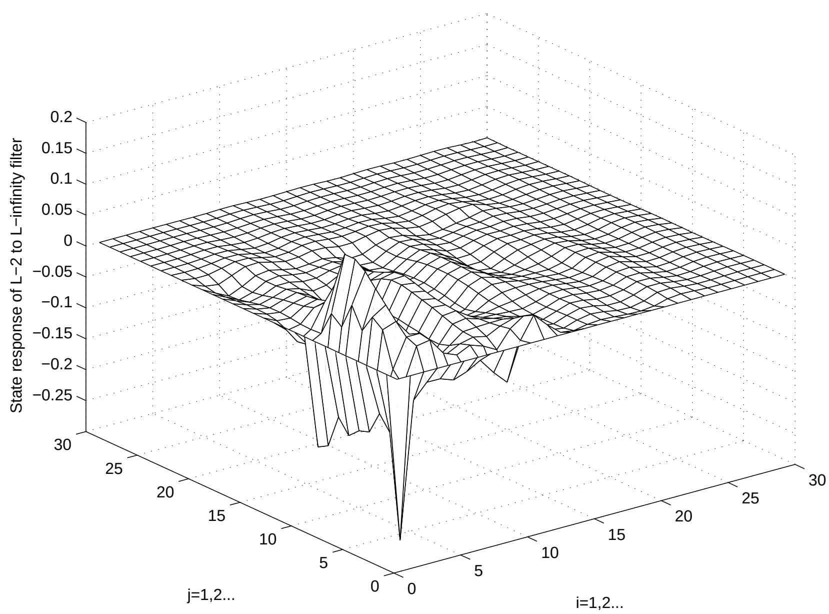

Fig. 6. The filtering error $\tilde{z}_{i,j}$ of the l_2-l_∞ case