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ON THE ORDER STRUCTURE ON THE SET OF COMPLETELY MULTI-POSITIVE LINEAR MAPS ON C^*-ALGEBRAS

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Abstract. In this paper we characterize the order relation on the set of all completely n-positive linear maps on C^*-algebras in terms of the representation associated to each completely n-positive linear map given by Suen's construction.

1. Introduction and Preliminaries

Completely positive linear maps are an often used tool in operator algebras theory and quantum information theory [1, 3, 5, 7, 10].

In the mathematical framework of quantum information theory, all admissible devices are modelled by the so-called quantum operations (that is, completely positive linear maps on the algebra of observables (C^*-algebra) of the physical system under consideration). A good analysis of completely multi-positive maps between C^*-algebras involves understanding and solving certain problems in quantum information theory and understanding the infinite dimensional non-commutative structure of topological $*$-algebras [2, 5, 7, 10]. The theorems on the structure of completely linear maps and Radon-Nikodym type theorems for completely positive linear maps are an extremely powerful and veritable tool for problems involving characterization and comparison of quantum operations.

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Given a C^*-algebra A and a positive integer n, we denote by $M_n(A)$ the C^*-algebra of all $n \times n$ matrices over A with the algebraic operations and the topology obtained by regarding it as a direct sum of n^2 copies of A.

DEFINITION 1.1. *A linear map $\rho\colon A \to B$ between two C^*-algebras is completely positive if the linear maps $\rho^{(n)}\colon M_n(A) \to M_n(B)$ defined by*

$$\rho^{(n)}([a_{ij}]_{i,j=1}^n) = [\rho(a_{ij})]_{i,j=1}^n$$

are positive for any positive integer n.

DEFINITION 1.2. *Let A and B be two C^*-algebras. An $n \times n$ matrix $[\rho_{ij}]_{i,j=1}^n$ of linear maps from A to B can be regarded as a linear map ρ from $M_n(A)$ to $M_n(B)$ defined by*

$$\rho([a_{ij}]_{i,j=1}^n) = [\rho_{ij}(a_{ij})]_{i,j=1}^n.$$

We say that $[\rho_{ij}]_{i,j=1}^n$ is a completely n-positive linear map from A to B if ρ is a completely positive linear map from $M_n(A)$ to $M_n(B)$.

We shall denote by $CP_\infty(A,B)$ the set of all completely positive linear maps from A to B and by $CP_\infty^n(A,B)$ the set of all completely n-positive linear maps from A to B.

In [9], Suen showed that any completely n-positive linear map from a C^*-algebra A to $L(H)$, the C^*-algebra of all bounded linear operators on a Hilbert space H, is of the form $[V^*T_{ij}\Phi(\cdot)V]_{i,j=1}^n$, where Φ is a representation of A on a Hilbert space K, $V \in L(H,K)$ and $[T_{ij}]_{i,j=1}^n$ is a positive element in $M_n(\Phi(A)')$ ($\Phi(A)'$ denotes the commutant of $\Phi(A)$ in $L(K)$).

THEOREM 1.3 ([9, 4]). *Let A be a C^*-algebra, let H be a Hilbert space and let $\rho = [\rho_{ij}]_{i,j=1}^n$ be a completely n-positive linear map from A to $L(H)$. Then there is a representation Φ_ρ of A on a Hilbert space H_ρ, $V_\rho \in L(H,H_\rho)$ and a positive element $T^\rho = [T_{ij}^\rho]_{i,j=1}^n$ in $M_n(\Phi_\rho(A)')$ with $\sum_{k=1}^n T_{kk}^\rho = n\,id_{L(H_\rho)}$ such that:*

i. $\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in H\}$ *spans a dense subspace in H_ρ;*
ii. $\rho_{ij}(a) = V_\rho^* T_{ij}^\rho \Phi_\rho(a) V_\rho$, *for all $a \in A$ and for all $i,j = 1,\dots,n$.*

The quadruple $(\Phi_\rho, H_\rho, V_\rho, T^\rho)$ will be called the Suen's construction associated with ρ and it is unique up to unitary equivalence [4, Theorem 2.3].

REMARK 1.4. The triple $(\Phi_\rho, H_\rho, V_\rho)$ is the Stinespring representation associated with $\widetilde{\rho} = \frac{1}{n}\sum_{k=1}^n \rho_{kk}$ (see, [4, the proof of Theorem 2.3]).

In this paper we characterize the order relation on the set of all completely n-positive linear maps on C^*-algebras in terms of the representation associated to each completely n-positive linear map given by Suen's construction [9].

We also give sufficient conditions for that a completely n-positive linear map from a unital C^*-algebra A to $L(H)$ to be an extreme point in the set of all completely n-positive linear maps $[\rho_{ij}]_{i,j=1}^n$ from A to $L(H)$ such that $[\rho_{ij}(1_A)]_{i,j=1}^n = T^0$ for some $T^0 \in M_n(L(H))$.

2. The main results

Let $\rho = [\rho_{ij}]_{i,j=1}^n$ be an element in $CP_\infty^n(A, L(H))$ and let $(\Phi_\rho, H_\rho, V_\rho, T^\rho)$ be the construction associated to ρ given by Theorem 1.3.

Lemma 2.1. *Let $S = [S_{ij}]_{i,j=1}^n$ be a positive element in $M_n(\Phi_\rho(A)')$. The map $\rho_S = [\rho_{S_{ij}}]_{i,j=1}^n$ from $M_n(A)$ to $M_n(L(H))$ defined by*

$$\rho_S([a_{ij}]_{i,j=1}^n) = [V_\rho^* S_{ij} \Phi_\rho(a_{ij}) V_\rho]_{i,j=1}^n$$

is a completely n-positive linear map from A to $L(H)$.

Proof. It is not difficult to see that ρ_S is an $n \times n$ matrix of linear maps from A to $L(H)$ whose (i,j)-entry is the linear map $\rho_{S_{ij}}$ from A to $L(H)$ defined by $\rho_{S_{ij}}(a) = V_\rho^* S_{ij} \Phi_\rho(a) V_\rho$ for all $a \in A$ and for all $i, j = 1, \ldots, n$.

To show that ρ_S is a completely n-positive linear map from A to $L(H)$ it is sufficient to show that $\Gamma(\rho_S) \in CP_\infty(A, M_n(L(H)))$, where Γ is the map from $CP_\infty^n(A, B)$ onto $CP_\infty(A, M_n(B))$ defined by $\Gamma([\rho_{ij}]_{i,j=1}^n)(a) = [\rho_{ij}(a)]_{i,j=1}^n$ for all $a \in A$ [2, Theorem 1.4]. For this, let m be a positive integer, $a_1, \ldots, a_m \in A, \xi_1 = (\xi_1^i)_{i=1}^n, \ldots, \xi_m = (\xi_m^i)_{i=1}^n \in H^n$. Then we have

$$\begin{aligned}
\sum_{k,l=1}^m \langle \Gamma(\rho_S)(a_l^* a_k)\xi_k, \xi_l \rangle &= \sum_{k,l=1}^m \left\langle [V_\rho^* S_{ij} \Phi_\rho(a_l^* a_k) V_\rho]_{i,j=1}^n (\xi_k^i)_{i=1}^n, (\xi_l^i)_{i=1}^n \right\rangle \\
&= \sum_{k,l=1}^m \sum_{i,j=1}^n \left\langle V_\rho^* S_{ij} \Phi_\rho(a_l^* a_k) V_\rho \xi_k^j, \xi_l^i \right\rangle \\
&= \sum_{i,j=1}^n \left\langle S_{ij} \sum_{k=1}^m \Phi_\rho(a_k) V_\rho \xi_k^j, \sum_{k=1}^m \Phi_\rho(a_k) V_\rho \xi_k^i \right\rangle \\
&= \left\langle [S_{ij}]_{i,j=1}^n (\sum_{k=1}^m \Phi_\rho(a_k) V_\rho \xi_k^i)_{i=1}^n, (\sum_{k=1}^m \Phi_\rho(a_k) V_\rho \xi_k^i)_{i=1}^n \right\rangle \geq 0
\end{aligned}$$

since $[S_{ij}]_{i,j=1}^n$ is a positive element in $M_n(\Phi_\rho(A)')$. From this fact we conclude that $\Gamma(\rho_S) \in CP_\infty(A, M_n(L(H)))$ and the lemma is proved. ∎

Remark 2.2. It is not difficult to check that:

1. $\rho_{T^\rho} = \rho$;
2. $\rho_{\alpha S} = \alpha \rho_S$, for all positive numbers α and for all positive elements S in $M_n(\Phi_\rho(A)')$;
3. $\rho_{S_1 + S_2} = \rho_{S_1} + \rho_{S_2}$, for all positive elements S_1, S_2 in $M_n(\Phi_\rho(A)')$;

4. $\rho_{S_1} \leq \rho_{S_2}$ if and only if $S_1 \leq S_2$, where S_1, S_2 are positive elements in $M_n(\Phi_\rho(A)')$.

Let $\rho, \theta \in CP^n_\infty(A, L(H))$. We say that ρ *dominates* θ, and we write $\theta \leq \rho$, if $\rho - \theta \in CP^n_\infty(A, L(H))$.

For $\rho \in CP^n_\infty(A, L(H))$, we put:

$$[0, \rho] = \left\{\theta = [\theta_{ij}]^n_{i,j=1} \in CP^n_\infty(A, L(H)); \theta \leq \rho\right\}$$

and

$$[0, T^\rho] = \left\{S = [S_{ij}]^n_{i,j=1} \in M_n(\Phi_\rho(A)'); 0 \leq S \leq T^\rho\right\}.$$

THEOREM 2.3. *The map $S \longrightarrow \rho_S$ is an affine order isomorphism from $[0, T^\rho]$ to $[0, \rho]$.*

PROOF. By Lemma 2.1 and Remark 2.2, the map $S \longrightarrow \rho_S$ from $[0, T^\rho]$ to $[0, \rho]$ is well-defined and moreover, it is affine.

To show that the map is injective, let $S = [S_{ij}]^n_{i,j=1}$ be an element in $[0, T^\rho]$ such that $\rho_S = 0$. Then $[\rho_{S_{ij}}]^n_{i,j=1} = 0$, that is $V_\rho^* S_{ij} \Phi_\rho(a) V_\rho = 0$, for all $a \in A$ and for all $i, j = 1, \ldots, n$.

For each $a, b \in A, \xi, \eta \in H$ and $i, j = 1, \ldots, n$, we have

$$\begin{aligned} \langle S_{ij}\Phi_\rho(a)V_\rho\xi, \Phi_\rho(b)V_\rho\eta\rangle &= \left\langle V_\rho^*\Phi_\rho(b)^* S_{ij}\Phi_\rho(a)V_\rho\xi, \eta\right\rangle \\ &= \left\langle V_\rho^*\Phi_\rho(b^*) S_{ij}\Phi_\rho(a)V_\rho\xi, \eta\right\rangle \\ &= \left\langle V_\rho^* S_{ij}\Phi_\rho(b^*a)V_\rho\xi, \eta\right\rangle = 0. \end{aligned}$$

From this fact and taking into account that $\left\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in H\right\}$ spans a dense subspace of H_ρ, we conclude that $S_{ij} = 0$. Hence $S = 0$ and the map $S \longrightarrow \rho_S$ is injective.

It remains to show that the map $S \longrightarrow \rho_S$ from from $[0, T^\rho]$ to $[0, \rho]$ is surjective.

Let $\sigma = [\sigma_{kl}]^n_{k,l=1}$ be an element in $[0, \rho]$. By [4, the proof of Theorem 2.3] (see also [6]), $\frac{1}{n}\sigma_{kk}, \frac{1}{2}\widetilde{\sigma} \pm \frac{1}{n}\text{Re}\sigma_{kl}, \frac{1}{2}\widetilde{\sigma} \pm \frac{1}{n}\text{Im}\sigma_{kl} \in [0, \widetilde{\rho}]$, where $\widetilde{\rho} = \frac{1}{n}\sum\limits_{j=1}^{n}\rho_{jj}$ and $\widetilde{\sigma} = \frac{1}{n}\sum\limits_{j=1}^{n}\sigma_{jj}$, for all $k, l = 1, \ldots, n$ with $k \neq l$. Let $(\Phi_\rho, H_\rho, V_\rho, T^\rho)$ be the Suen's construction associated with ρ. By Remark 1.4, $(\Phi_\rho, H_\rho, V_\rho)$ is the Stinespring representation of A associated with $\widetilde{\rho}$. Then by [1, Theorem 1.4.6], for each $j = 1, \ldots, n$, there is a positive element $S_{jj} \in \Phi_\rho(A)'$ such that

$$\sigma_{jj}(a) = V_\rho^* S_{jj}\Phi_\rho(a)V_\rho$$

for all $a \in A$ and for all $k, l = 1, \ldots, n$ with $k \neq l$, there are two positive elements $S^1_{kl}, S^2_{kl} \in \Phi_\rho(A)'$ such that

$$\frac{n}{2}\widetilde{\sigma}(a) + (\text{Re}\sigma_{kl})(a) = V_\rho^* S^1_{kl}\Phi_\rho(a)V_\rho$$

and

$$\frac{n}{2}\widetilde{\sigma}(a) + (\mathrm{Im}\sigma_{kl})(a) = V_\rho^* S_{kl}^2 \Phi_\rho(a) V_\rho$$

for all $a \in A$.

From these relations, we deduce that $\sigma_{kl}(a) = V_\rho^* S_{kl}\Phi_\rho(a)V_\rho$ for all $a \in A$, where

$$S_{kl} = S_{kl}^1 + \mathrm{i}S_{kl}^2 - \frac{1+\mathrm{i}}{2}\sum_{j=1}^{n} S_{jj}.$$

Clearly $S = [S_{ij}]_{i,j=1}^n \in M_n(\Phi_\rho(A)')$. Moreover, S is positive (see, for example, [4, the proof of Theorem 2.3]) and $\sigma = \rho_S$. Since $\sigma \leq \rho$, by Remark 2.2, $S \in [0, T^\rho]$ and the theorem is proved. ∎

Definition 2.4. *Let A be a C^*-algebra and let H be a Hilbert space. A completely n-positive linear map $\rho = [\rho_{ij}]_{i,j=1}^n$ from A to $L(H)$ is said to be pure if for every completely n-positive linear map $\theta = [\theta_{ij}]_{i,j=1}^n \in [0, \rho]$, there is a positive number α such that $\theta = \alpha\rho$.*

Proposition 2.5. *Let $\rho = [\rho_{ij}]_{i,j=1}^n$ be an element in $CP_\infty^n(A, L(H))$. Then ρ is pure if and only if $[0, T^\rho] = \{\alpha T^\rho; 0 \leq \alpha \leq 1\}$.*

Proof. First we suppose that ρ is pure. Let $S = [S_{ij}]_{i,j=1}^n$ be an element in $[0, T^\rho]$. By Theorem 2.3, $\rho_S \in [0, \rho]$ and since ρ is pure, there is a positive number α such that $\rho_S = \alpha\rho$. From this fact, Remark 2.2 and Theorem 2.3, we deduce that $S = \alpha T^\rho$ for some $0 \leq \alpha \leq 1$.

Conversely, suppose that $[0, T^\rho] = \{\alpha T^\rho; 0 \leq \alpha \leq 1\}$. Let $\theta = [\theta_{ij}]_{i,j=1}^n$ be an element in $[0, \rho]$. By Theorem 2.3, there is $S \in [0, T^\rho]$ such that $\rho_S = \theta$ and since $S = \alpha T^\rho$ for some positive number α, $\theta = \alpha\rho$ and the proposition is proved. ∎

Let A be a unital C^*-algebra, let H be a Hilbert space and $\rho = [\rho_{ij}]_{i,j=1}^n \in CP_\infty^n(A, L(H))$. We denote by $CP_\infty^n(A, L(H), T^0)$, where

$$T^0 = \mathrm{diag}(V_\rho^*, \ldots, V_\rho^*)T^\rho \mathrm{diag}(V_\rho, \ldots, V_\rho),$$

the set of all completely n-positive linear maps $\sigma = [\sigma_{ij}]_{i,j=1}^n$ from A to $L(H)$ such that $\sigma_{ij}(1_A) = (T^0)_{ij}$, for all $i, j = 1, \ldots, n$. Clearly, $CP_\infty^n(A, L(H), T^0)$ is a convex set.

Proposition 2.6. *Let $\rho = [\rho_{ij}]_{i,j=1}^n$ be an element in $CP_\infty^n(A, L(H), T^0)$ and let P_{H_0} be the projection on the closed subspace H_0 of H_ρ generated by $\{V_\rho\xi; \xi \in H\}$. If the map $S \longrightarrow diag(P_{H_0}, \ldots, P_{H_0})S\,diag(P_{H_0}, \ldots, P_{H_0})$ from $M_n(\Phi_\rho(A)')$ to $M_n(L(H_\rho))$ is injective then ρ is an extreme point in $CP_\infty^n(A, L(H), T^0)$.*

PROOF. Let θ, σ be elements in $CP^n_\infty(A, L(H), T^0)$ and $\alpha \in (0,1)$ such that $\alpha\theta + (1-\alpha)\sigma = \rho$. Since $\alpha\theta \in [0, \rho]$, by Theorem 2.3 there is a positive element S in $M_n(\Phi_\rho(A)')$ such that $\alpha\theta = \rho_S$. Then

$$\begin{aligned}\left\langle P_{H_0}(S_{ij} - \alpha T^\rho_{ij})P_{H_0}V_\rho\xi, V_\rho\eta\right\rangle &= \langle S_{ij}V_\rho\xi, V_\rho\eta\rangle - \alpha\left\langle T^\rho_{ij}V_\rho\xi, V_\rho\eta\right\rangle \\ &= \alpha\left\langle\theta_{ij}(1_A)\xi, \eta\right\rangle - \alpha\left\langle\rho_{ij}(1_A)\xi, \eta\right\rangle = 0,\end{aligned}$$

for all $\xi, \eta \in H$ and for all $i, j = 1, \ldots, n$.

From this fact we deduce that $P_{H_0}(S_{ij} - \alpha T^\rho_{ij})P_{H_0} = 0$ for all $i, j = 1, \ldots, n$ and since the map $S \longrightarrow \mathrm{diag}(P_{H_0}, \ldots, P_{H_0})S\mathrm{diag}(P_{H_0}, \ldots, P_{H_0})$ from $M_n(\Phi_\rho(A)')$ to $M_n(L(H_\rho))$ is injective, $S = \alpha T^\rho$. Thus we showed that $\theta = \rho$ and so ρ is an extreme point in $CP^n_\infty(A, L(H), T^0)$. □

By Remark 1.4, $(\Phi_\rho, H_\rho, V_\rho)$ is the Sinespring representation of A associated to $\widetilde{\rho}$. If $\rho = [\rho_{ij}]^n_{i,j=1} \in CP^n_\infty(A, L(H), T^0)$, then

$$\widetilde{\rho}(1_A) = \frac{1}{n}\sum_{k=1}^{n}\rho_{kk}(1_A) = \frac{1}{n}\sum_{k=1}^{n}V^*_\rho T_{kk}V_\rho = V^*_\rho V_\rho,$$

and by [1, Theorem 1.4.6], $\widetilde{\rho}$ is an extreme point in $CP_\infty(A, L(H), V^*_\rho V_\rho)$ if and only if the map $S \longrightarrow P_{H_0}SP_{H_0}$ from $\Phi_\rho(A)'$ to $L(H_\rho)$ is injective.

COROLLARY 2.7. *Let $\rho = [\rho_{ij}]^n_{i,j=1}$ be an element in $CP^n_\infty(A, L(H), T^0)$. If $\widetilde{\rho} = \frac{1}{n}\sum_{k=1}^{n}\rho_{kk}$ is an extreme point in $CP_\infty(A, L(H), V^*_\rho V_\rho)$, then ρ is an extreme point in $CP^n_\infty(A, L(H), T^0)$.*

PROOF. Since $\widetilde{\rho}$ is an extreme point in the set $CP_\infty(A, L(H), V^*_\rho V_\rho)$, the map $S_0 \longrightarrow P_{H_0}S_0P_{H_0}$ from $\Phi_\rho(A)'$ to $L(H_\rho)$ is injective [1, Theorem 1.4.6], and so the map $S \longrightarrow \mathrm{diag}(P_{H_0}, \ldots, P_{H_0})S\mathrm{diag}(P_{H_0}, \ldots, P_{H_0})$ is injective. From this fact and Proposition 2.6, we deduce that ρ is an extreme point in $CP^n_\infty(A, L(H), T^0)$. □

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