Extinction time for some nonlinear heat equations

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Abstract. *This paper concerns the study of the extinction time of the solution of the following initial-boundary value problem*

$$\begin{cases} u_t = \varepsilon L u(x,t) - f(u) \quad in \quad \Omega \times \mathbb{R}_+, \\ u(x,t) = 0 \quad on \quad \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x) > 0 \quad in \quad \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, ε is a positive parameter, $f(s)$ is a positive, increasing, concave function for positive values of s, $f(0) = 0$, $\int_0 \frac{ds}{f(s)} < +\infty$, L is an elliptic operator. We show that the solution of the above problem extincts in a finite time and its extinction time goes to that of the solution $\alpha(t)$ of the following differential equation

$$\alpha^{'}(t) = -f(\alpha(t)), \quad t > 0, \quad \alpha(0) = M,$$

as ε goes to zero, where $M = \sup_{x\in\Omega} u_0(x)$. We also extend the above result to other classes of nonlinear parabolic equations. Finally, we give some numerical results to illustrate our analysis.

Key words: *extinction, finite difference, nonlinear heat equations, extinction time*

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem

$$u_t = \varepsilon L u - f(u) \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{1}$$

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}_+, \tag{2}$$

$$u(x,0) = u_0(x) > 0 \quad \text{in} \quad \Omega, \tag{3}$$

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where ε is a positive parameter, $f(s)$ is a positive, increasing, concave function for the positive values of s, $f(0) = 0$, $\int_0 \frac{ds}{f(s)} < +\infty$. The operator L is defined as follows

$$Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}),$$

where $a_{ij} : \overline{\Omega} \to \mathbb{R}$, $a_{ij} \in C^1(\overline{\Omega})$, $a_{ij} = a_{ji}$, $1 \le i, j \le N$ and there exists a constant $C > 0$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge C\| \xi \|^2 \quad \forall\, x \in \overline{\Omega} \quad \forall\, \xi = (\xi_1, ..., \xi_N) \in \mathbb{R}^N,$$

where $\| \, . \, \|$ stands for the Euclidean norm of $\mathbb{R}^N$. The initial data $u_0 \in C^1(\overline{\Omega})$, $u_0(x) = 0$ on $\partial\Omega$, $u_0(x)$ is positive in Ω.

We need the following definition.

Definition 1.1. *We say that the solution u of (1)–(3) extincts in a finite time if there exists a finite time T such that $\|u(\cdot, t)\|_\infty > 0$ for $t \in [0, T)$ but $\|u(\cdot, t)\|_\infty = 0$ for $t \ge T$, where $\|u(\cdot, t)\|_\infty = \sup_{x\in\Omega} |u(x, t)|$. The time T is called the extinction time of the solution u.*

Solutions of nonlinear heat equations which extinct in a finite time have been the subject of investigation of many authors (see [3], [4], [8], [10], [11] and the references cited therein). The existence and uniqueness for the solution u of (1)–(3) have been proved. It is also shown that the solution u of (1)–(3) extincts in a finite time (see [3], [4]). In [6], some semidiscrete and discrete schemes have been used to study the phenomenon of extinction in the case where $N = 1$. Also in [5], one may find some results on extinction for elliptic equations in cylindrical domains. In this paper, we are interested in the asymptotic behavior as ε goes to zero of the extinction time. Our work was motived by the paper of Friedman and Lacey in [7] where they have considered the following initial-boundary value problem

$$u_t = \varepsilon\Delta u + g(u) \quad \text{in} \quad \Omega \times (0, T), \tag{4}$$

$$u(x, t) = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \tag{5}$$

$$u(x, 0) = u_0(x) \ge 0 \quad \text{in} \quad \Omega, \tag{6}$$

where $g(s)$ is a positive, increasing, convex function for the nonnegative values of s, $\int_0^{+\infty} \frac{ds}{g(s)} < +\infty$, the initial data u_0 is a continuous function in $\overline{\Omega}$. Under some additional conditions on the initial data, they have proved that if ε is small enough, then the solution u of (4)–(6) blows up in a finite time and its blow-up time tends to that of the solution $\alpha(t)$ of the differential equation defined as follows

$$\alpha^{'}(t) = g(\alpha(t)), \quad \alpha(0) = \sup_{x\in\Omega} u_0(x),$$

as ε goes to zero (we say that a solution blows up in a finite time if it reaches the value infinity in a finite time). The proof developed in [7] is based on the construction of upper and lower solutions and it is difficult to extend the above method to the problem described in (1)–(3). Nabongo and Boni have obtained in [13] an analogous result in the case of the phenomenon of extinction for stochastic differential equations. One may also consult the paper of Nabongo and Boni in [14] where a comparable result has been found in the context of the phenomenon of quenching (we say that a solution quenches in a finite time if it reaches a finite singular value in a finite time). In this paper, using a modification of Kaplan's method (see [9]) and a method based on the construction of upper solutions, we prove a similar result. Our paper is written in the following manner. In the next section, we show that when ε is small enough, then the solution u of (1)–(3) extincts in a finite time and its extinction time goes to that of the solution of a certain differential equation. We also extend the above result to other classes of parabolic problems in the third section. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. Extinction times

In this section, we show that the solution u of (1)–(3) extincts in a finite time and its extinction time goes to that of the solution of a certain differential equation as ε tends to zero. In order to facilitate our discussion, let us recall some results about the differential equations. Consider the solution $\beta(t)$ of the following differential equation

$$\begin{cases} \beta'(t) = -\beta^p(t), \quad t > 0, \\ \beta(0) = Q > 0, \end{cases}$$

with $p = const \in (0,1)$. The solution $\beta(t)$ is given explicitly by

$$\beta(t) = (Q^{1-p} - (1-p)t)_+^{\frac{1}{1-p}}, \quad t \geq 0,$$

where $(x)_+ = \max\{x, 0\}$. Thus, one sees that $\beta(t) > 0$ for $t \in [0, \frac{Q^{1-p}}{1-p})$ but $\beta(t) = 0$ for $t \geq \frac{Q^{1-p}}{1-p}$. In this case, we say that $\beta(t)$ extincts at the time $T_0 = \frac{Q^{1-p}}{1-p}$. More generally, let $\alpha(t)$ be the solution of the differential equation defined below

$$\begin{cases} \alpha'(t) = -f(\alpha(t)), \quad t > 0, \\ \alpha(0) = M, \end{cases}$$

where $M = \sup_{x \in \Omega} u_0(x) > 0$. It is not difficult to see that $\alpha(t) > 0$ for $t \in [0, T_0)$ but $\alpha(t) = 0$ for $t \geq T_0$, where $T_0 = \int_0^M \frac{ds}{f(s)}$. Hence, we discover that $\alpha(t)$ extincts at the time T_0.

Let us also recall an old result (see [2]). Let $a \in \Omega$ be such that $u_0(a) = M$ and consider the following eigenvalue problem

$$-L\psi = \lambda_\delta \psi \quad \text{in} \quad B(a,\delta), \tag{7}$$

$$\psi = 0 \quad \text{on} \quad \partial B(a,\delta), \tag{8}$$

$$\psi > 0 \quad \text{in} \quad B(a,\delta), \tag{9}$$

where $\delta > 0$, such that $B(a,\delta) = \{x \in \mathbb{R}^N \; ; \; \|x - a\| < \delta\} \subset \Omega$.

It is well known that the above problem admits a solution (ψ, λ_δ) such that $0 < \lambda_\delta \leq \frac{D}{\delta^2}$ where D is a positive constant which depends only on the upper bound of the coefficients of the operator L and the dimension N. We can normalize ψ so that $\int_{B(a,\delta)} \psi(x)dx = 1$.

Now, let us give our result on the extinction time.

Theorem 2.1. *Let u be the solution of (1)–(3). If*

$$\varepsilon < \min\left\{(M/2)^3, (Kdist(a, \partial\Omega))^3\right\},$$

then u extincts in a finite time and its extinction time T satisfies the following estimates

$$T_0 - AT_0\varepsilon^{1/3} + o(\varepsilon^{1/3}) \leq T \leq T_0,$$

where $T_0 = \int_0^M \frac{ds}{f(s)}$ is the extinction time of the solution $\alpha(t)$ of the differential equation defined below

$$\alpha^{'}(t) = -f(\alpha(t)), \quad t > 0, \quad \alpha(0) = M,$$

with $M = \sup_{x\in\Omega} u_0(x)$, $A = \frac{DK^2M}{f(M)}$ and K is an upper bound of the first derivatives of u_0.

Proof. Since the initial data $u_0(x)$ is nonnegative in Ω, owing to the maximum principle (see [15]), u is also nonnegative in $\Omega \times \mathbb{R}_+$. Introduce the function $z(x, t)$ defined as follows

$$z(x, t) = \alpha(t) \quad \text{in} \quad \overline{\Omega} \times \mathbb{R}_+.$$

A straightforward computation reveals that

$$z_t(x, t) = Lz(x, t) - f(z(x, t)) \quad \text{in} \quad \Omega \times \mathbb{R}_+,]]$$
$$z(x, t) \geq 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}_+,$$
$$z(x, 0) \geq u(x, 0) \quad \text{in} \quad \Omega.$$

According to the maximum principle, we have

$$0 \leq u(x, t) \leq z(x, t) = \alpha(t) \quad \text{in} \quad \Omega \times \mathbb{R}_+.$$

Since $\alpha(t)$ extincts at the time T_0, we deduce that u also extincts in a finite time at the time T which obeys the following estimate

$$T \leq T_0 = \int_0^M \frac{ds}{f(s)}. \tag{10}$$

Since $u_0 \in C^1(\overline{\Omega})$, from the mean value theorem and the triangle inequality, we have

$$u_0(x) \geq M - \varepsilon^{1/3} \quad \text{for} \quad x \in B(a, \delta) \subset \Omega,$$

where $\delta = \frac{\varepsilon^{1/3}}{K}$. Let w be the solution of the following initial-boundary value problem

$$w_t(x, t) = \varepsilon Lw(x, t) - f(w(x, t)) \quad \text{in} \quad B(a, \delta) \times \mathbb{R}_+,$$

$$w(x,t) = 0 \quad \text{on} \quad \partial B(a,\delta) \times \mathbb{R}_+,$$
$$w(x,0) = u_0(x) \quad \text{in} \quad B(a,\delta).$$

Owing to the maximum principle, w is nonnegative in $B(a,\delta) \times \mathbb{R}_+$ because the initial data is nonnegative in $B(a,\delta)$. Introduce the function $v(t)$ defined as follows

$$v(t) = \int_{B(a,\delta)} w\psi dx \quad \text{for} \quad t \in \mathbb{R}_+.$$

Take the derivative of v in t and use the definition of $v(t)$ given above to obtain

$$v^{'}(t) = \varepsilon \int_{B(a,\delta)} \psi L w dx - \int_{B(a,\delta)} f(w)\psi dx.$$

Applying Green's formula, we arrive at

$$v^{'}(t) = \varepsilon \int_{B(a,\delta)} w L\psi dx - \int_{B(a,\delta)} f(w)\psi dx.$$

It follows from (7) that

$$v^{'}(t) = -\varepsilon\lambda_\delta v(t) - \int_{B(a,\delta)} f(w)\psi dx. \tag{11}$$

Use Jensen's inequality to obtain

$$v^{'}(t) \geq -\varepsilon\lambda_\delta v(t) - f(v(t)),$$

which implies that

$$v^{'}(t) \geq -f(v(t))\left(1 + \frac{DK^2\varepsilon^{1/3}v(t)}{f(v(t))}\right),$$

because $0 < \lambda_\delta \leq \frac{D}{\delta^2} = \frac{DK^2}{\varepsilon^{2/3}}$. Since $f(0) = 0$ and $f(s)$ is a concave function for the positive values of s, we see that $\frac{f(s)}{s}$ is a decreasing function for the positive values of s. From (11), we find that the function $v(t)$ is nonincreasing for $t \geq 0$, which implies that $v(t) \leq v(0) \leq M$ for $t \geq 0$. We deduce that $\frac{f(v(t))}{v(t)} \geq \frac{f(M)}{M}$. Therefore, we have $\frac{v(t)}{f(v(t))} \leq \frac{M}{f(M)}$, which implies that

$$v^{'}(t) \geq -f(v(t))(1 + \frac{DK^2\varepsilon^{1/3}M}{f(M)}) \quad \text{for} \quad t \in \mathbb{R}_+.$$

We deduce that

$$v^{'}(t) \geq -(1 + \varepsilon^{1/3}A)f(v(t)) \quad \text{for} \quad t \in \mathbb{R}_+.$$

Let $\beta(t)$ be the solution of the following differential equation

$$\begin{cases} \beta^{'}(t) = -(1 + \varepsilon^{1/3}A)f(v(t)), & t > 0, \\ \beta(t) = v(0). \end{cases}$$

It is well known that $\beta(t)$ extincts at the time

$$T_* = \frac{1}{1+\varepsilon^{1/3}A}\int_0^{v(0)} \frac{ds}{f(s)}.$$

By the maximum principle (see [16]), we have $v(t) \geq \beta(t)$ for $t \in \mathbb{R}_+$. We deduce that $\sup_{x\in B(a,\delta)} |w(x,t)| \geq v(t) \geq \beta(t)$ for $t \in \mathbb{R}_+$. On the other hand, since u is nonnegative in $\Omega \times \mathbb{R}_+$, we deduce that

$$\begin{aligned}
&u_t(x,t) = \varepsilon Lu(x,t) - f(u(x,t)) \quad \text{in} \quad B(a,\delta)\times\mathbb{R}_+,\\
&u(x,t) \geq 0 \quad \text{on} \quad \partial B(a,\delta)\times\mathbb{R}_+,\\
&u(x,0) = u_0(x) \quad \text{in} \quad B(a,\delta).
\end{aligned}$$

It follows from the maximum principle that $u(x,t) \geq w(x,t)$ in $B(a,\delta)\times\mathbb{R}_+$. It is not difficult to see that

$$\|u(\cdot,t)\|_\infty \geq \sup_{x\in B(a,\delta)} |u(x,t)| \geq \sup_{x\in B(a,\delta)} |w(x,t)| \geq \beta(t) \quad \text{for} \quad t\in\mathbb{R}_+.$$

Since $\beta(t) > 0$ for $t \in [0, T_*)$, we see that

$$T \geq T_* = \frac{1}{1+\varepsilon^{1/3}A}\int_0^{v(0)} \frac{ds}{f(s)}. \tag{12}$$

Indeed, suppose that $T < T_*$. This implies that $\|u(\cdot,T)\|_\infty \geq \|w(\cdot,T)\|_\infty > 0$, which contradicts the fact that u extincts at the time T. Obviously $v(0) \geq M-\varepsilon^{1/3}$. Therefore, we have

$$\int_0^{v(0)} \frac{ds}{f(s)} \geq \int_0^{M-\varepsilon^{1/3}} \frac{ds}{f(s)} = \int_0^{M} \frac{ds}{f(s)} - \int_{M-\varepsilon^{1/3}}^{M} \frac{ds}{f(s)}.$$

On the other hand

$$\int_{M-\varepsilon^{1/3}}^{M} \frac{ds}{f(s)} \leq \frac{\varepsilon^{1/3}}{f(M-\varepsilon^{1/3})} \leq \frac{\varepsilon^{1/3}}{f(\frac{M}{2})},$$

because $f(s)$ is an increasing function for the positive values of s. We deduce that

$$\int_0^{v(0)} \frac{ds}{f(s)} \geq \int_0^{M} \frac{ds}{f(s)} - \frac{\varepsilon^{1/3}}{f(\frac{M}{2})}. \tag{13}$$

Apply Taylor's expansion to obtain

$$\frac{1}{1+\varepsilon^{1/3}A} = 1 - \varepsilon^{1/3}A + o(\varepsilon^{1/3}).$$

Use (10), (12), (13) and the above relation to complete the rest of the proof. □

3. Other extinction times

In this section, we extend the result of the previous section considering the following initial-boundary value problem

$$u_t = \varepsilon L\varphi(u) - f(u) \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{14}$$
$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}_+, \tag{15}$$
$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \tag{16}$$

where $\varphi(s)$ is a positive, increasing, concave function for the positive values of s. In addition $\frac{\varphi(s)}{f(s)}$ is an increasing function for the positive values of s. Using the methods developed in the proof of the above theorem, we prove the following.

Theorem 3.1. *Let u be the solution of (14)–(16). If*

$$\varepsilon < \min\left\{(M/2)^3, (Kdist(a, \partial\Omega))^3\right\},$$

then u extincts in a finite time T and its extinction time T obeys the following estimates

$$T_0 - AT_0\varepsilon^{1/3} + o(\varepsilon^{1/3}) \leq T \leq T_0,$$

where $T_0 = \int_0^M \frac{ds}{f(s)}$ *is the extinction time of the solution* $\beta(t)$ *of the differential equation defined below*

$$\beta^{'}(t) = -f(\beta(t)), \quad t > 0, \quad \beta(0) = M,$$

with $M = \sup_{x\in\Omega} u_0(x)$*,* $A = \frac{DK^2\varphi(M)}{f(M)}$ *and K is an upper bound of the first derivatives of* u_0*.*

4. Numerical results

In this section, we give some computational results to confirm the theory developed in the previous section. We consider the radial symmetric solution of the following initial-boundary value problem

$$u_t = \varepsilon\Delta u - u^p \quad \text{in} \quad B \times \mathbb{R}_+,$$
$$u(x,t) = 0 \quad \text{on} \quad S \times \mathbb{R}_+,$$
$$u(x,0) = u_0(x) \quad \text{in} \quad B,$$

where $B = \{x \in \mathbb{R}^N \; ; \; \|x\| < 1\}$, $S = \{x \in \mathbb{R}^N \; ; \; \|x\| = 1\}$. The above problem may be rewritten in the following form

$$u_t = \varepsilon(u_{rr} + \frac{N-1}{r}u_r) - u^p, \quad r \in (0,1), \quad t \in \mathbb{R}_+, \tag{17}$$
$$u_r(0,t) = 0, \quad u(1,t) = 0, \quad t \in \mathbb{R}_+, \tag{18}$$
$$u(r,0) = \varphi(r), \quad r \in (0,1). \tag{19}$$

Here, we take $\varphi(r) = a\sin(\pi r)$ with $a > 0$.

We start by the construction of some adaptive schemes as follows. Let I be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \le i \le I$ and approximate the solution u of (17)–(19) by the solution $U_h^{(n)} = (U_0^{(n)}, ..., U_I^{(n)})^T$ of the following explicit scheme

$$\begin{aligned}
\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} &= \varepsilon N \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} - (U_0^{(n)})^{p-1} U_0^{(n+1)}, \\
\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} &= \varepsilon\Big(\frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} + \frac{(N-1)}{ih} \frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h}\Big) \\
&\quad -(U_i^{(n)})^{p-1} U_i^{(n+1)}, 1 \le i \le I-1, \\
U_I^{(n)} &= 0, \\
U_i^{(0)} &= \varphi_i, \quad 0 \le i \le I,
\end{aligned}$$

$n \ge 0$. We also approximate the solution u of (17)–(19) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\begin{aligned}
\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} &= \varepsilon N \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} - (U_0^{(n)})^{p-1} U_0^{(n+1)}, \\
\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} &= \varepsilon\Big(\frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} + \frac{(N-1)}{ih} \frac{U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h}\Big) \\
&\quad -(U_i^{(n)})^{p-1} U_i^{(n+1)}, \quad 1 \le i \le I-1, \\
U_I^{(n+1)} &= 0, \\
U_i^{(0)} &= \varphi_i, \quad 0 \le i \le I,
\end{aligned}$$

$n \ge 0$. We take $\Delta t_n = \min\{\frac{h^2}{2N\varepsilon}, h^2 \|U_h^{(n)}\|_\infty^{p+1})\}$ for the explicit scheme and $\Delta t_n = h^2 \|U_h^{(n)}\|_\infty^{p+1}$ for the implicit scheme where $\|U_h^{(n)}\|_\infty = \sup_{0 \le i \le I} |U_i^{(n)}|$.

We remark that $\lim_{r\to 0} \frac{u_r(r,t)}{r} = u_{rr}(0,t)$. Hence, if $t = 0$, then we have $u_t(0,t) = \varepsilon N u_{rr}(0,t) - u^p(0,t)$. This remark has been used in the construction of our schemes when $i = 0$.

Let us notice that in the explicit scheme, the restriction on the time step ensures the nonnegativity of the discrete solution. For the implicit scheme, existence and nonnegativity are also guaranteed by standard methods (see, for instance [6]).

We need the following definition.

Definition 4.1. *We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme extincts in a finite time if* $\lim_{n\to+\infty} \|U_h^{(n)}\|_\infty = 0$ *and the series* $\sum_{n=0}^{+\infty} \Delta t_n$ *converges. The quantity* $\sum_{n=0}^{+\infty} \Delta t_n$ *is called the numerical extinction time of the solution* $U_h^{(n)}$.

In the following tables, in rows, we present the numerical extinction times, the number of iterations, CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128. We take for the numerical extinction time

$T^n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when $|T^{n+1} - T^n| \leq 10^{-16}$. The order($s$) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $a = \frac{1}{2}$, $N = 2$, $p = \frac{1}{2}$.
First case: $\varepsilon = \frac{1}{1000}$.

I	T^n	n	*CPU time*	s
16	1.408928	8774	20	-
32	1.405652	32185	140	-
64	1.404822	117263	1020	1.99
128	1.401235	457035	9840	2.11

Table 1. *Numerical extinction times, number of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method*

I	T^n	n	*CPU time*	s
16	1.408929	8774	25	-
32	1.405653	32185	128	-
64	1.404822	117263	1025	1.99
128	1.401207	437665	9785	2.12

Table 2. *Numerical extinction times, number of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method*

Second case: $\varepsilon = \frac{1}{4000}$.

I	T^n	n	*CPU time*	s
16	1.415965	8786	19	-
32	1.412814	32246	154	-
64	1.411936	117547	1069	1.84
128	1.412156	447615	9825	2.00

Table 3. *Numerical extinction times, number of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method*

I	T^n	n	*CPU time*	s
16	1.415966	8786	23	-
32	1.412815	32246	147	-
64	1.411936	117547	1079	1.84
128	1.412157	447615	998	2.00

Table 4. *Numerical extinction times, number of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method*

Numerical experiments for $a = \frac{1}{2}$, $N = 3$, $p = \frac{1}{2}$.
First case: $\varepsilon = \frac{1}{2000}$.

I	T^n	n	$CPU\ time$	s
16	1.413593	8782	20	-
32	1.410395	32225	140	-
64	1.409607	117459	1071	2.03
128	1.406313	449620	9839	2.07

Table 5. *Numerical extinction times, number of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method*

I	T^n	n	$CPU\ time$	s
16	1.413592	8782	22	-
32	1.410397	32225	71	-
64	1.409592	117459	1075	2.00
128	1.406309	449620	9981	2.03

Table 6. *Numerical extinction times, number of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method*

Third case: $\varepsilon = \frac{1}{6000}$**.**

I	T^n	n	$CPU\ time$	s
16	1.416759	8788	21	-
32	1.414623	32253	124	-
64	1.414129	117583	1102	2.12
128	1.414006	450367	10125	2.01

Table 7. *Numerical extinction times, number of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method*

I	T^n	n	$CPU\ time$	s
16	1.416762	8788	23	-
32	1.414625	32225	151	-
64	1.414133	127633	1187	2.13
128	1.414008	509093	11815	1.98

Table 8. *Numerical extinction times, number of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method*

Remark 4.1. *If we consider the problem (17)–(19) in the case where the initial data* $\varphi(r) = \frac{1}{2}\sin(r\pi)$ *and* $f(u) = \sqrt{u}$, *then we see that the extinction time of the solution* $\beta(t)$ *of the differential equation defined in Theorem 2.1 is equal* $\sqrt{2} \approx 1.4142$. *We observe from the above tables that when* ε *diminishes, the extinction time increases to 1.4142. This result does not surprise us because of the result established in Theorem 2.1.*

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