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Proximal methods for equilibrium problem in Hilbert spaces*

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Abstract. *An approximate procedure for solving equilibrium problems is proposed and its convergence is established under suitable conditions.*

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1. Introduction and preliminaries

Throughout this paper, we always assume that H is a real Hilbert space, C is a nonempty closed convex subset of H and $\phi : C \times C \to R$ is a real functional with $\phi(x, x) = 0$ for all $x \in C$. The "so called" *equilibrium problem for functional* ϕ is to find a point $x_* \in C$ such that

$$\phi(x_*, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

Denote the set of solutions of the equilibrium problem (1.1) by $EP(\phi)$.

Equilibrium problem theory has emerged as an interesting branch of applicable mathematics. This theory has become a rich source of inspiration and motivation for the study of a large number of problems arising in economics, optimization and operations research in a general and unified way.

Special Examples of Equilibrium Problem (1.1)

(1) If $\phi(x, y) = f(x) - f(y), \quad \forall x, y \in C,$ where $f : C \to \Re$ is a real function, then equilibrium problem (1.1) reduces to the following *minimization problem subject to implicit constraints*:

$$\text{find} \quad x_* \in C \quad \text{such that} \quad f(x_*) \leq f(x), \quad \forall x \in C. \tag{1.2}$$

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(2) If $\phi(x,y) = \sup_{\zeta\in B(x)}\langle \zeta,\ y-x\rangle$, where $B : C \to 2^H$ is a set-valued maximal monotone operator, then the equilibrium problem (1.1) is equivalent to the following *monotone inclusion problem*, i.e.,

$$\text{find} \quad x_* \in C \text{ such that } \quad 0 \in B(x_*). \tag{1.3}$$

(3) If $B = T + N_C$, where $T : C \to H$ is a single-valued mapping and N_C is the normal cone to C, then the inclusion problem (1.3) is reduced to the classical variational inequality problem, i.e.,

$$\text{find} \quad x_* \in C \quad \text{such that} \quad \langle T(x_*), x - x_* \rangle \geq 0, \quad \forall x \in C. \tag{1.4}$$

(4) In particular, if C is a closed convex cone, then the variational inequality problem (1.4) is equivalent to the well-known *complementarity problem* of mathematical programming:

$$\text{find} \quad x_* \in C \quad \text{such that} \quad T(x_*) \in C^* \textit{ and } \langle T(x_*),\ x_*\rangle = 0, \tag{1.5}$$

where $C^* = \{x \in H,\ \langle x,\ y\rangle \geq 0,\ \ \forall y \in C\}$.

(5) Let $P : C \to C$ be a given mapping. If $\phi(x,y) = \langle x - Px,\ y - x\rangle$, then the equilibrium problem (1.1) is equivalent to finding a fixed point $x_* \in C$ of P.

(6) Let I be a finite index set. For each $i \in I$, let C_i be a given set, $f_i : C \to \Re$ be a given function with $C := \prod_{i\in I} C_i$. For $x = (x_i)_{i\in I} \in C$, we define $x^i := (x_j)_{j\in I, j\neq i}$. The point $x_* = (x_{*i})_{i\in I} \in C$ is called a *Nash equilibrium*, if for each $i \in I$ the following inequalities hold:

$$f_i(x_*) \leq f_i(x_*^i, y_i), \quad \forall y_i \in C_i. \tag{1.6}$$

Let us define $\phi : C \times C \to \Re$ by

$$\phi(x,y) = \sum_{i\in I}(f_i(x^i, y_i) - f_i(x)). \tag{1.7}$$

Then $x_* \in C$ is a Nash equilibrium if and only if x_* is a solution of equilibrium problem (1.1).

(7) Let $P,\ Q$ be two closed convex subsets of H, $C = Q \times P$ and $L : C \to C$ be a convex-concave function. A point $(x_*, p_*) \in C$ is called a *saddle point* for the function L, if the following condition is satisfied:

$$L(x_*, p) \leq L(x_*, p_*) \leq L(x, p_*), \quad \forall x \in Q,\ p \in P. \tag{1.8}$$

Letting $\phi(w,v) = L(z,p) - L(x,y)$, where $w = (z,y)$ and $v = (x,p)$, it follows that the *saddle point problem* (1.8) is equivalent to equilibrium problem (1.1) and their sets of solutions coincide.

There are a substantial number of papers on results for solving equilibrium problems based on different relaxed notions and various compactness assumptions. But up to now only few iterative methods to solve such problems have been done (see, Antipin and Flam [1], Blum and Oettli [2], Moudafi [7], Moudafi et al. [8], Combettes and Hirstoaga [4], Suzuki [9], Takahashi and Takahashi [10]).

Motivated and inspired by numerical methods developed by Antipin and Moudafi [1, 7, 8] for optimization and monotone inclusion and the researches of Combettes-Histoaga [4], Takahashi and Takahashi [10], the purpose of this paper is, by using viscosity approximation methods, to consider a class of equilibrium problem which includes variational inequalities as well as complementarity problems, convex optimization, saddle point problem, problems of finding a zero of a maximal monotone operator and Nash equilibria problems as special cases. Then we propose and investigate iterative methods for such problems.

For the purpose, first, we recall some definitions, lemmas and notations.

In the sequel, we use $x_n \rightharpoonup x$ and $x_n \to x$ to denote the weak convergence and strong convergence of the sequence $\{x_n\}$ in H, respectively.

In a Hilbert space H, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$||x - P_C x|| \le ||x - y||, \ \ \forall y \in C.$$

Such a mapping P_C from H onto C is called the *metric projection.* We know that P_C is nonexpansive. Further, for any $x \in H$ and $z \in C$,

$$z = P_C(x) \Leftrightarrow \langle x - z, \ z - y\rangle \ge 0, \ \ \forall y \in C.$$

For solving the equilibrium problem (1.1), let us assume that ϕ satisfies the following conditions:

(A1) $\phi(x, x) = 0, \ \forall x \in C$;

(A2) ϕ is monotone, i.e.,

$$\phi(x, y) + \phi(y, x) \le 0, \ \ \forall x, y \in C;$$

(A3) for any $x, y, z \in C$ the functional $x \mapsto \phi(x, y)$ is upper-semicontinuous, i.e.,

$$\lim_{t\to 0+} \phi(tz + (1 - t)x, \ y) \le \phi(x, y), \ \ \forall x, \ y, \ z \in C;$$

(A4) $y \mapsto \phi(x, y)$ is convex and lower semi-continuous.

The following lemmas will be needed in proving our main results:

Lemma 1.1 [2]. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, $\phi : C \times C \to R$ be a functional satisfying the conditions (A1)-(A4), then, for any given $x \in H$ and $r > 0$, there exists $z \in C$ such that*

$$\phi(z, y) + \frac{1}{r}\langle y - z, \ z - x\rangle \ge 0, \ \ \forall y \in C.$$

Lemma 1.2 [4]. *Let all the conditions in Lemma 1.1 are satisfied. For any $r > 0$ and $x \in C$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : \phi(z, y) + \frac{1}{r}\langle y - z,\ z - x\rangle \geq 0,\ \ \forall y \in C\},\ \ x \in H.$$

Then the following holds:

(1) T_r is single-values;

(2) T_r is firmly nonexpansive, i.e.,

$$||T_r x - T_r y||^2 \leq \langle T_r x - T_r y,\ x - y\rangle,\ \ \forall x, y \in H,$$

and so $||T_r x - T_r y|| \leq ||x - y||,\ \ \forall x, y \in H$.

(3) $F(T_r) = EP(\phi),\ \ \forall r > 0$;

(4) $EP(\phi)$ is a closed and convex set.

Lemma 1.3 [6]. *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n + c_n,\ \ \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $\{\lambda_n\}$ is a sequence in (0, 1) with $\sum_{n=0}^{\infty} \lambda_n = \infty$, $b_n = \circ(\lambda_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.4 [5]. *Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X and $T : C \to X$ be a nonexpansive mapping with a fixed point, then $I - T$ is demiclosed in the sense that if $\{x_n\}$ is a sequence in C and if $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, then $(I - T)x = 0$.*

Lemma 1.5 [3]. *Let E ba a real Banach space, $J : E \to 2^{E^*}$ be the normalized duality mapping and x, y be any given points in E, then the following conclusion holds:*

$$||x + y||^2 \leq ||x||^2 + 2\langle y,\ j(x + y)\rangle,\ \ \forall j(x + y) \in J(x + y).$$

Especially, if $E = H$ is a real Hilbert space, then

$$||x + y||^2 \leq ||x||^2 + 2\langle y,\ x + y\rangle,\ \ \forall x, y \in H.$$

2. Main results

In this section, we shall prove our main theorems in this paper:

Theorem 2.1. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, $\phi : C \times C \to R$ be a functional satisfying the conditions (A1)- (A4), $T : C \to H$ be a nonexpansive mapping with $F(T) \bigcap EP(\phi) \neq \emptyset$ and $f : H \to H$ be a contraction mapping with a contractive constant $\alpha \in (0, 1)$. Let $\{\alpha_n\}$ be a sequence in (0, 1] and $\{r_n\} \subset (0, \infty)$ be a real sequence satisfying the following conditions:*

(i) $\alpha_n \to 0$; $\sum_{n=0}^{\infty} \alpha_n = \infty$; $|1 - \frac{\alpha_n}{\alpha_{n+1}}| \to 0$ *as* $n \to \infty$

(ii) $0 < r < r_n$ *for all* $n \geq 0$ *and* $\sum_{n=0}^{\infty} |r_n - r_{n+1}| < \infty$,

where r *is a positive constant. For any given* $x_0 \in H$, *let* $\{x_n\}$ *and* $\{u_n\}$ *be the sequences defined by*

$$\begin{cases} \phi(u_n, y) + \dfrac{1}{r_n}\langle y - u_n\, u_n - x_n\rangle \geq 0, \ \ \forall y \in C, \\ x_{n+1} = \alpha_n f(u_n) + (1 - \alpha_n)Tu_n, \end{cases} \tag{2.1}$$

for all $n \geq 0$. *Then* $x_n \to x_* \in F(T) \bigcap EP(\phi)$, *where* $x_* = P_{F(T)\cap EP(\phi)} f(x_*)$.

Proof. First we point out that the sequences $\{x_n\}$ and $\{u_n\}$ generated by (2.1) are well-defined. Indeed, it follows from Lemma 1.1 that for given $x_0 \in H$, there exists $u_0 \in C$ such that

$$\phi(u_0, y) + \frac{1}{r_0}\langle y - u_0\, u_0 - x_0\rangle \geq 0, \ \ \forall y \in C,$$

Define $x_1 \in H$ by

$$x_1 = \alpha_0 f(u_0) + (1 - \alpha_0)Tu_0.$$

By Lemma 1.1 again, there exists $u_1 \in C$ such that

$$\phi(u_1, y) + \frac{1}{r_1}\langle y - u_1\, u_1 - x_1\rangle \geq 0, \ \ \forall y \in C,$$

Continuing this way, the sequences $\{x_n\}$ and $\{u_n\}$ are obtained.

We divide the proof of Theorem 2.1 into six steps:

(I) First we prove that the mapping $P_{F(T)\cap EP(\phi)} f : H \to C$ has a unique fixed point. In fact, since $f : H \to H$ is a contraction and $P_{F(T)\cap EP(\phi)} : H \to F(T) \cap EP(\phi)$ is also a contraction, we have

$$||P_{F(T)\cap EP(\phi)} f(x) - P_{F(T)\cap EP(\phi)} f(y)|| \leq \alpha ||x - y||, \ \ \forall x, y \in H.$$

Therefore, there exists a unique $x_* \in C$ such that $x_* = P_{F(T)\cap EP(\phi)} f(x_*)$.

(II) Now we prove that the sequences $\{x_n\}$ and $\{u_n\}$ are bounded in H and C, respectively. In fact, from the definition of T_r in Lemma 1.2, we know that $u_n = T_{r_n} x_n$. Therefore, for any $p \in F(T) \bigcap EP(\phi)$, we have

$$||u_n - p|| = ||T_{r_n} x_n - T_{r_n} p|| \leq ||x_n - p||. \tag{2.2}$$

Therefore, it follows from (2.1) and (2.2) that

$$\begin{aligned} ||x_{n+1} - p|| &= ||\alpha_n (f(u_n) - p) + (1 - \alpha_n)(Tu_n - p)|| \\ &\leq \alpha_n ||f(u_n) - f(p)|| + \alpha_n ||f(p) - p|| + (1 - \alpha_n)||Tu_n - p|| \\ &\leq \alpha_n \alpha ||u_n - p|| + \alpha_n ||f(p) - p|| + (1 - \alpha_n)||u_n - p|| \\ &\leq (1 - \alpha_n(1 - \alpha))||x_n - p|| + \alpha_n ||f(p) - p|| \\ &\leq \max\{||x_n - p||, \ \frac{||f(p) - p||}{1 - \alpha}\} \\ &\leq \cdots \\ &\leq \max\{||x_0 - p||, \ \frac{||f(p) - p||}{1 - \alpha}\}. \end{aligned}$$

This implies that $\{x_n\}$ is a bounded sequence in H. By (2.2), we know that $\{u_n\}$ is a bounded sequence in C and so $\{Tu_n\}$, $\{f(u_n)\}$ both are bounded sequences in H. Let

$$M = \sup_{n\geq 0}\{||u_n - x_n|| + ||x_n - y||^2 + ||f(u_n)|| + T(u_n)||\}, \tag{2.3}$$

where $y \in H$ is some given point.

(III) Now, we make an estimation for the sequence $\{||u_{n+1} - u_n||\}$. By the definition of T_r, $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$. Hence we have

$$\phi(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1},\ u_{n+1} - x_{n+1}\rangle \geq 0,\ \ \forall y \in C; \tag{2.4}$$

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n,\ u_n - x_n\rangle \geq 0,\ \ \forall y \in C; \tag{2.5}$$

Take $y = u_{n+1}$ in (2.5) and $y = u_n$ in (2.4). Then, adding the resulting inequalities and noting the condition (A2), we have

$$\langle u_{n+1} - u_n,\ \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\rangle \geq 0$$

and hence

$$\langle u_{n+1} - u_n,\ u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\rangle \geq 0.$$

This implies that

$$\begin{aligned}||u_{n+1} - u_n||^2 &\leq \langle u_{n+1} - u_n,\ x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\rangle\\ &\leq ||u_{n+1} - u_n||\{||x_{n+1} - x_n|| + |1 - \frac{r_n}{r_{n+1}}| \cdot ||u_{n+1} - x_{n+1}||\}\end{aligned}$$

Thus, by the condition (iii), we have

$$\begin{aligned}||u_{n+1} - u_n|| &\leq ||x_{n+1} - x_n|| + |1 - \frac{r_n}{r_{n+1}}|||u_{n+1} - x_{n+1}||\\ &\leq ||x_{n+1} - x_n|| + \frac{1}{r}|r_{n+1} - r_n| \cdot M.\end{aligned} \tag{2.6}$$

(IV) Now we prove that $||Tu_n - u_n|| \to 0$. In fact, it follows from (2.1) and (2.6) that

$$\begin{aligned}&||x_{n+1} - x_n||\\ &= ||\alpha_n f(u_n) + (1 - \alpha_n)Tu_n - \alpha_{n-1}f(u_{n-1}) - (1 - \alpha_{n-1})Tu_{n-1}||\\ &= ||\alpha_n f(u_n) - \alpha_n f(u_{n-1}) + \alpha_n f(u_{n-1}) - \alpha_{n-1}f(u_{n-1})\\ &\quad + (1 - \alpha_n)Tu_n - (1 - \alpha_n)Tu_{n-1} + (1 - \alpha_n)Tu_{n-1} - (1 - \alpha_{n-1})Tu_{n-1}||\\ &\leq \alpha_n||f(u_n) - f(u_{n-1})|| + 2|\alpha_n - \alpha_{n-1}|M + (1 - \alpha_n)||Tu_n - Tu_{n-1}||\\ &\leq \alpha_n\alpha||u_n - u_{n-1}|| + 2|\alpha_n - \alpha_{n-1}|M + (1 - \alpha_n))||u_n - u_{n-1}||\\ &\leq (1 - \alpha_n(1 - \alpha)||u_n - u_{n-1}|| + 2|\alpha_n - \alpha_{n-1}|M\\ &\leq (1 - \alpha_n(1 - \alpha)||x_n - x_{n-1}|| + \frac{1}{r}|r_n - r_{n-1}|M + 2\alpha_n|1 - \frac{\alpha_{n-1}}{\alpha_n}|M.\end{aligned}$$

By condition (i), (ii) and Lemma 1.3 we have

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0, \ \ as\ n \to \infty. \tag{2.7}$$

It follows from (2.6), (2.7) and the condition (ii) that

$$||u_{n+1} - u_n|| \to 0, \quad \text{as} \ \ n \to \infty. \tag{2.8}$$

Since $\alpha_n \to 0$ and $\{f(u_n)\}$, $\{Tu_n\}$ both are bounded, from (2.7), we have

$$\begin{aligned} ||x_n - Tu_n|| &\le ||x_n - x_{n+1}|| + ||x_{n+1} - Tu_n| \\ &\le ||x_n - x_{n+1}|| + \alpha_n ||f(u_n) - Tu_n|| \to 0. \end{aligned} \tag{2.9}$$

Furthermore, for any $p \in F(T) \bigcap EP(\phi)$, from Lemma 1.2, we have

$$\begin{aligned} ||u_n - p||^2 &= ||T_{r_n} x_n - T_{r_n} p||^2 \\ &\le \langle T_{r_n} x_n - T_{r_n} p,\ x_n - p\rangle \\ &= \langle u_n - p,\ x_n - p\rangle \\ &= \frac{1}{2}\{||u_n - p||^2 + ||x_n - p||^2 - ||x_n - u_n||^2\}. \end{aligned}$$

Hence we have

$$||u_n - p||^2 \le ||x_n - p||^2 - ||x_n - u_n||^2. \tag{2.10}$$

From the convexity of function $x \mapsto ||x||^2$ and (2.10), we have

$$\begin{aligned} ||x_{n+1} - p||^2 &\le \alpha_n ||f(u_n) - p||^2 + (1 - \alpha_n)||Tu_n - p||^2 \\ &\le \alpha_n ||f(u_n) - p||^2 + (1 - \alpha_n)||u_n - p||^2 \\ &\le \alpha_n ||f(u_n) - p||^2 + (1 - \alpha_n)\{||x_n - p||^2 - ||x_n - u_n||^2\} \end{aligned}$$

and so

$$\begin{aligned} &(1 - \alpha_n)||x_n - u_n||^2 \\ &\le ||x_n - p||^2 - ||x_{n+1} - p||^2 + \alpha_n ||f(u_n) - p||^2 \\ &\le (||x_n - p|| - ||x_{n+1} - p||)(||x_n - p|| + ||x_{n+1} - p||) + \alpha_n ||f(u_n) - p||^2 \\ &\le (||x_n - x_{n+1}||)(||x_n - p|| + ||x_{n+1} - p||) + \alpha_n ||f(u_n) - p||^2, \end{aligned}$$

Since $\alpha_n \to 0$, $\{x_n\}$ and $\{f(u_n)\}$ are bounded and $||x_n - x_{n+1}|| \to 0$, we have

$$||x_n - u_n|| \to 0 \quad \text{as} \ \ n \to \infty. \tag{2.11}$$

Therefore from (2.9) we have

$$||Tu_n - u_n|| \le ||Tu_n - x_n|| + ||x_n - u_n|| \to 0. \tag{2.12}$$

The desired conclusion is proved.

(V) Now, we prove that

$$\limsup_{n\to\infty} \langle f(x_*) - x_*,\ x_n - x_*\rangle \le 0,$$

where $x_* = P_{F(T)\cap EP(\phi)} f(x_*)$.

In fact, we can choose a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that

$$\lim_{n_j\to\infty} \langle f(x_*) - x_*,\ x_{n_j} - x_*,\ \rangle = \limsup_{n\to\infty} \langle f(x_*) - x_*,\ x_n - x_*\rangle. \tag{2.13}$$

Since $\{u_{n_j}\}$ is bounded, without loss of generality, we can assume that $u_{n_j} \rightharpoonup w \in C$. By (2.12), $||Tu_{n_j} - u_{n_j}|| \to 0$. It follows from the demiclosed principle (see Lemma 1.4) that $Tw = w$ and $Tu_{n_j} \rightharpoonup w$.

Next, we prove that $w \in F(T) \cap EP(\phi)$. For the purpose it is sufficient to prove that $w \in EP(\phi)$. In fact, since $u_n = T_{r_n} x_n$, we have

$$\phi(u_n, y) + \frac{1}{r_n} \langle y - u_n,\ u_n - x_n\rangle \geq 0,\ \ \forall y \in C.$$

It follows from the condition (A2) that

$$\frac{1}{r_n} \langle y - u_n,\ u_n - x_n\rangle \geq \phi(y, u_n)$$

and so

$$\langle y - u_{n_j},\ \frac{u_{n_j} - x_{n_j}}{r_{n_j}}\rangle \geq \phi(y, u_{n_j}). \tag{2.14}$$

Since $\frac{||u_{n_j} - x_{n_j}||}{r_{n_j}} \leq \frac{||u_{n_j} - x_{n_j}||}{r} \to 0$ and $u_{n_j} \rightharpoonup w$, by virtue of the condition (A4), we have

$$\liminf_{n_j\to\infty} \phi(y, u_{n_j}) \leq \lim_{n_j\to\infty} \langle y - u_{n_j},\ \frac{u_{n_j} - x_{n_j}}{r_{n_j}}\rangle = 0,$$

i.e.,

$$\phi(y, w) \leq 0,\ \ \forall y \in C. \tag{2.15}$$

For any $t \in (0, 1)$ and $y \in C$, let $y_t = ty + (1-t)w$. Then $y_t \in C$ and so we have $\phi(y_t, w) \leq 0$. It follows from the conditions (A1), (A4) and (2.15) that

$$\begin{aligned} 0 &= \phi(y_t, y_t) \\ &\leq t\phi(y_t, y) + (1-t)\phi(y_t, w) \\ &\leq t\phi(y_t, y). \end{aligned}$$

This implies that $\phi(y_t, y) \geq 0$ for all $t \in (0, 1)$. Letting $t \to 0+$, by the condition (A3), we have

$$\phi(w, y) \geq 0,\ \ \forall y \in C.$$

This shows that $w \in EP(\phi)$ and so $w \in F(T) \cap EP(\phi)$.

Since $x_* = P_{F(T)\cap EP(\phi)} f(x_*)$, $u_{n_j} \rightharpoonup w$ and $||u_n - x_n|| \to 0$ (see (2.11)), we have

$$\begin{aligned} \limsup_{n\to\infty} \langle f(x_*) - x_*,\ x_n - x_*\rangle &= \lim_{n_j\to\infty} \langle f(x_*) - x_*,\ x_{n_j} - x_*\rangle \\ &= \lim_{n_j\to\infty} \langle f(x_*) - x_*,\ u_{n_j} - (u_{n_j} - x_{n_j}) - x_*\rangle \qquad (2.16) \\ &= \langle f(x_*) - x_*,\ w - x_*\rangle \leq 0. \end{aligned}$$

The desired conclusion is proved.

(VI) Finally, we prove that $x_n \to x_*$ as $n \to \infty$. In fact, it follows form (2.1) and Lemma 1.5 that

$$\begin{aligned}||x_{n+1} - x_*||^2 &= ||\alpha_n(f(u_n) - x_*) + (1-\alpha_n)(Tu_n - x_*)||^2 \\ &\le (1-\alpha_n)^2||Tu_n - x_*||^2 + 2\alpha_n\langle f(u_n) - x_*,\ x_{n+1} - x_*\rangle \\ &\le (1-\alpha_n)^2||u_n - x_*||^2 + 2\alpha_n\langle f(u_n) - f(x_*) + f(x_*) - x_*,\ x_{n+1} - x_*\rangle \\ &\le (1-\alpha_n)^2||u_n - x_*||^2 + 2\alpha_n\alpha||u_n - x_*||\cdot||x_{n+1} - x_*|| \\ &\quad + 2\alpha_n\langle f(x_*) - x_*, ,\ x_{n+1} - x_*\rangle \\ &\le (1-\alpha_n)^2||u_n - x_*||^2 + \alpha_n\alpha\{||u_n - x_*||^2 + ||x_{n+1} - x_*||^2\} \\ &\quad + 2\alpha_n\langle f(x_*) - x_*, ,\ x_{n+1} - x_*\rangle\end{aligned}$$

and so, from (2.2),

$$\begin{aligned}||x_{n+1} - x_*||^2 &\le \frac{(1-\alpha_n)^2 + \alpha_n\alpha}{1-\alpha_n\alpha}||u_n - x_*||^2 + \frac{2\alpha_n\langle f(x_*) - x_*,\ x_{n+1} - x_*\rangle}{1-\alpha_n\alpha} \\ &\le \frac{(1-\alpha_n)^2 + \alpha_n\alpha}{1-\alpha_n\alpha}||x_n - x_*||^2 + \frac{2\alpha_n\langle f(x_*) - x_*,\ x_{n+1} - x_*\rangle}{1-\alpha_n\alpha}.\end{aligned} \tag{2.17}$$

Since $\alpha_n \to 0$, for any $\varepsilon > 0$, there exists a nonnegative integer n_0 such that $1 - \alpha\alpha_n > \frac{1}{2}$ for all $n \ge n_0$. Note that

$$\begin{aligned}\frac{(1-\alpha_n)^2 + \alpha_n\alpha}{1-\alpha_n\alpha} &\le \frac{1-\alpha_n+\alpha_n^2}{1-\alpha_n\alpha} \\ &\le (1-\alpha_n(1-\alpha)) + \frac{\alpha_n^2}{1-\alpha_n\alpha} \\ &\le (1-\alpha_n(1-\alpha)) + 2\alpha_n^2, \ \ \forall n \ge n_0.\end{aligned} \tag{2.18}$$

Thus, substituting (2.18) into (2.17) and noting (2.3), we have

$$\begin{aligned}||x_{n+1} - x_*||^2 &\le (1-\alpha_n(1-\alpha))||x_n - x_*||^2 + 2\alpha_n^2 M \\ &\quad + \frac{2\alpha_n\langle f(x_*) - x_*, ,\ x_{n+1} - x_*\rangle}{1-\alpha_n\alpha}, \ \ \forall n \ge n_0,\end{aligned} \tag{2.19}$$

where $M = \sup_{n\ge 0}||x_n - x_*||^2$.

Let

$$\gamma_n = \max\{0,\ \ \langle f(x_*) - x_*,\ x_{n+1} - x_*\rangle\}, \ \ \forall n \ge 0.$$

Then $\gamma_n \ge 0, \ \forall n \ge 0$.

Next, we prove that

$$\gamma_n \to 0. \tag{2.20}$$

In fact, it follows from (2.16) that for any given $\varepsilon > 0$, there exists $n_1 \ge n_0$ such that

$$\langle f(x_*) - x_*, ,\ x_{n+1} - x_*\rangle < \varepsilon, \ \ \forall n \ge n_1.$$

and so we have

$$0 \le \gamma_n < \varepsilon, \ \ \forall n \ge n_1.$$

By the arbitrariness of $\varepsilon > 0$, we get $\gamma_n \to 0$. By virtue of $\{\gamma_n\}$, we can rewrite (2.19) as follows:

$$||x_{n+1} - x_*||^2 \\ \leq (1 - \alpha_n(1 - \alpha))||x_n - x_*||^2 + 2\alpha_n^2 M + 4\alpha_n\gamma_n, \ \ \forall n \geq n_1. \tag{2.21}$$

Therefore, taking $a_n = ||x_n - x_*||^2$, $\lambda_n = \alpha_n(1-\alpha)$, $b_n = 2\alpha_n^2 M + 4\alpha_n\gamma_n$, $c_n = 0$, by Lemma 1.4 and the conditions (i)–(ii), the sequence $x_n \to x_*$ as $n \to \infty$. This completes the proof. □

From Theorem 2.1, we can obtain the following results:

Theorem 2.2. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, $T : C \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : H \to H$ be a contraction mapping with a contractive constant $\alpha \in (0,1)$. Let $\{\alpha_n\}$ be a sequence in [0, 1] satisfying the following conditions:*

$$\alpha_n \to 0; \ \sum_{n=0}^{\infty} \alpha_n = \infty; \ \ |1 - \frac{\alpha_n}{\alpha_{n+1}}| \to 0.$$

For any $x_0 \in H$, let $\{x_n\}$ be the sequences defined by

$$x_{n+1} = \alpha_n f(u_n) + (1-\alpha_n)T(u_n), \ \ \forall n \geq 0, \tag{2.22}$$

where $u_n = P_C x_n$ for all $n \geq 0$ and P_C is the metric projection from H onto C. Then $x_n \to x_* \in F(T)$ as $n \to \infty$, where $x_* = P_{F(T)} f(x_*)$.

Proof. Taking $\phi(x,y) = 0$ for all $x, y \in C$ and $\{r_n\} = 1$ for all $n \geq 1$ in Theorem 2.1, then we have

$$\langle y - u_n, \ u_n - x_n\rangle \geq 0, \ \ \forall y \in C.$$

This implies that $u_n = P_C x_n$. Therefore, the conclusion of Theorem 2.2 can be obtained from Theorem 2.1 immediately. □

Theorem 2.3. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, $\phi : C \times C \to R$ be a functional satisfying the conditions (A1)-(A4) such that $EP(\phi) \neq \emptyset$ and $f : H \to H$ be a contraction mapping with a contractive constant $\alpha \in (0,1)$. Let $\{\alpha_n\}$ be a sequence in (0, 1] and $\{r_n\} \subset (0,\infty)$ be a real sequence satisfying the following conditions:*

(i) $\alpha_n \to 0$; $\sum_{n=0}^{\infty} \alpha_n = \infty$; $|1 - \frac{\alpha_n}{\alpha_{n+1}}| \to 0$;

(ii) $0 < r < r_n$ *for all* $n \geq 0$ *and* $\sum_{n=0}^{\infty} |r_n - r_{n+1}| < \infty$.

For any $x_0 \in H$, let $\{x_n\}$ and $\{u_n\}$ be the sequences defined by

$$\begin{cases} \phi(u_n, y) + \dfrac{1}{r_n}\langle y - u_n \ u_n - x_n\rangle \geq 0, \ \ \forall y \in C, \\ x_{n+1} = \alpha_n f(u_n) + (1-\alpha_n)u_n, \end{cases} \tag{2.23}$$

for all $n \geq 0$. Then $x_n \to x_* \in EP(\phi)$ as $n \to \infty$, where $x_* = P_{EP(\phi)} f(x_*)$.

Proof. Taking $T = I$ in Theorem 2.1, then $F(T) = H$ and so $P_{F(T)\cap EP(\phi)} = P_{EP(\phi)}$. Therefore, the conclusion of Theorem 2.3 can be obtained from Theorem 2.1. □

References

[1] A. S. Antipin, S. Flam, *Equilibrium programming using proximal-like algorithms*, Math. Program. **78**(1997), 29–41.

[2] E. Blum, S. Oettli, *From optimization and variational inequalities to equilibrium problems*, The Math. Student **63**(1994), 123–145.

[3] S. S. Chang, *Chidume's open questions and approximation solutions multivalued strongly accretive mapping equations in Banach spaces*, J. Math. Anal. Appl. **216**(1997), 94-111.

[4] P. L. Combettes, S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear and Convex Anal. **6**(2005), 117–136.

[5] K. Goebel, W. A. Kirk, *Topics in Metric Fixed Point Theory*, in: Cambridge Studies in Advanced Mathematics **28**(1990), Cambridge Univ. Press.

[6] L. S. Liu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl. **194**(1995), 114–125.

[7] A. Moudafi, *Second-order differential proximal methods for Equilibrium problems*, J. Inequal. in Pure and Applied Math. **4**(2003), 1–7.

[8] A. Moudafi, M. Thera, *Proximal and dynamical approaches to Equilibrium problems*, Lecture Notes in Econom. and Math. Systems, Springer-Verlag, Berlin, 187–201.

[9] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for oneparameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl. **305**(2005), 227–229.

[10] S. Takahashi, W. Takahashi, *Viscosity approximation methods for Equilibrium problems in Hilbert spaces*, J. Math. Anal. Appl. (2006), doi:10.1016/j.jmaa.2006.08. 036.

[11] H. K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. **298**(2004), 279-291.