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ON HEREDITARY REFLEXIVITY OF TOPOLOGICAL VECTOR SPACES. THE DE RHAM COCHAIN AND CHAIN SPACES

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Dedicated to Professor Sibe Mardešić

Abstract. For proving reflexivity of the spaces of de Rham cohomology and homology of C^∞-manifolds the author considers the notion of hereditary reflexivity as well as the notion of dual hereditary reflexivity of locally convex topological vector spaces which is interesting in itself. Complete barrelled nuclear spaces with complete nuclear duals turn out to be hereditarily reflexive. The Pontryagin duality in locally convex topological spaces is also considered.

1. Introduction

Let $\Omega^p(M)$, $\Omega^p_c(M)$, $\Omega^c_p(M)$, $\Omega_p(M)$, $p \geq 0$, be the de Rham cochain complex of C^∞-differential forms, the de Rham cochain complex of C^∞-differential forms with compact supports and the de Rham chain complex of C^∞-differential forms with coefficients in Schwartz distributions with compact supports (currents with compact supports), the de Rham chain complex of C^∞-differential forms with coefficients in Schwartz distributions (currents) of a C^∞-manifold M, respectively.

The locally convex topology on $\Omega^p(M)$ is the topology of compact convergence for all derivatives, on $\Omega^p_c(M)$ it is the topology of the strict inductive

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limit $\lim\limits_{\to i} \Omega^p_{K_i}$ of the Fréchet spaces $\Omega^p_{K_i}$ of C^∞-differential forms whose supports are contained in compact subspaces $K_i \subseteq M$ such that $K_i \subseteq K_{i+1}$ and $\cup_{i=1}^\infty K_i = M$, on $\Omega^c_p(M)$ it is the strong topology on the space of all continuous linear forms on $\Omega^p(M)$ and on $\Omega_p(M)$ it is the strong topology on the space of all continuous linear forms on $\Omega^p_c(M)$, respectively. Recall that, by definition, a Fréchet space is a complete metrizable locally convex space, shortly denoted by F-space.

These spaces are Montel spaces, i.e., barrelled spaces in which every bounded set is relatively compact, hence reflexive (for the first time its reflexivity was proved by de Rham in [17] using a different argument), moreover, $(\Omega^p(M), \Omega^c_p(M))$ and $(\Omega^p_c(M), \Omega_p(M))$ are dual pairs in the strong topologies; its cohomology and homology $H^p(M)$, $H^p_c(M)$ and $H^c_p(M)$, $H_p(M)$ coincide with the usual cohomology, the cohomology of second type, the usual homology and the homology of second type of the C^∞-manifold M, respectively (de Rham theorem).

In this author's paper [10] it was shown that these cohomology and homology of the C^∞-manifold M are also Montel spaces and its topologies are induced by the topologies of cochain and chain spaces, respectively. One question remained open: is the strong topology on $H^c_p(M)$ naturally induced by the topology of $\Omega^c_p(M)$? It should not be difficult to give a positive answer to this question if one can prove that the Montel F-space $\Omega^p(M)$ and its dual, the Montel DF-space in the sense of Grothendieck $\Omega^c_p(M)$ are hereditarily reflexive. However, for this purpose the mentioned properties are not enough (there is an example of a Montel F-space whose separated quotient space is not reflexive and a closed subspace of its strong dual (which is also Montel) is not reflexive as well [1, c. 278]). Nevertheless, these spaces have additional properties: they are nuclear in the sense of Grothendieck and its hereditary reflexivity was already proved in [14]. But $\Omega^p_c(M)$ and $\Omega_p(M)$ are also nuclear spaces. Its hereditary reflexivity remained an unsolved problem ([16, p. 250]).

The purpose of this paper is a general study of the hereditary reflexivity problem of locally convex topological vector spaces and a proof of the hereditary reflexivity of barrelled dually nuclear and dually complete locally convex topological vector spaces. In particular, the gap in [10] is removed and a new proof of natural reflexivity of the de Rham cohomology and homology of a C^∞-manifold M is given. The problem of reflexivity of classical homology and cohomology spaces is raised. The Pontryagin duality is also considered in the case of locally convex topological vector spaces.

2. Hereditary reflexivity of locally convex topological vector spaces

In this paper we consider only *separated (i.e., Hausdorff) locally convex topological vector spaces* and we abbreviate this to *locally convex topological* *vector spaces*, if one does not need an additional assumption like a separated inductive limit, a separated quotient, etc. Let E be a locally convex topological vector space endowed with a topology ξ. The space E' of all in ξ continuous linear forms on E is called the topological dual (or conjugate) of E (in contrast to the algebraic dual E^* of E, i.e., the space of all linear forms on E). By the Hahn-Banach extension theorem for bounded linear forms, the spaces E and E' form a *dual pair* (E, E'), i.e.,

1) for each $x \neq 0$ in E there is some $x' \in E'$ with $< x, x' > \stackrel{def}{=} x'(x) \neq 0$;
2) for each $x' \neq 0$ in E' there is some $x \in E$ with $< x, x' > \neq 0$.

Recall that a set B in a topological vector space E over the field K ($\mathbb{R}$ or $\mathbb{C}$) is called bounded if it is absorbed by every neighborhood U of the origin, i.e., there exists some $\alpha > 0$ such that $B \subseteq \lambda U$, for all $\lambda \in K$ with $|\lambda| \geq \alpha$. In E' there are several distinct topologies which make E' a locally convex topological space: the *weak topology* $\sigma = \sigma(E', E)$, i.e., the topology of pointwise convergence or the topology of uniform convergence on all finite sets of E; the *compact-open topology* or the *topology of compact convergence* $\kappa = \kappa(E', E)$, i.e., the topology of uniform convergence on all compact sets of E; the *strong topology* $\beta = \beta(E', E)$, i.e., the topology of uniform convergence on all weakly bounded sets of E. In these cases we shall call E' the weak dual, the compact dual and the strong dual of E, respectively, and we shall denote the space E', endowed with these topologies, by E'_σ, E'_κ and E'_β, respectively. On the space E itself there are also two distinct topologies, consistent with the duality (E, E'), i.e., the set of all in these topologies continuous linear forms on E coincides with E'. One of them, the smallest — the weak topology $\sigma = \sigma(E, E')$, i.e., the topology of uniform convergence on all finite sets of E' and the second one, the finest — the Mackey topology $\tau = \tau(E, E')$, i.e., the topology of uniform convergence on all absolutely convex $\sigma(E', E)$-compact sets in E' (see in [18, Chapter 1, p. 4]); the space E, endowed with these topologies, will be denote by E_σ and E_τ, respectively. It is clear, that $\sigma \leq \xi \leq \tau$, moreover, the original topology $\xi = \xi(E, E')$ is the topology of uniform convergence on all equicontinuous sets of E'; if there is no danger of confusion, we shall denote by E the space in its original topology, but sometimes we will denote it by E_ξ. A space E, for which $E_\xi = E_\tau$, is usually called a *Mackey space*. On E there is also the strong topology $\beta(E, E')$, i.e., the topology of uniform convergence on all weakly bounded sets of E'; it is clear, that $\tau \leq \beta$.

By definition, a space E is reflexive in the topology ξ, if its strong bidual E'' is just E itself, algebraically as well as topologically, i.e., $E'' \stackrel{def}{=} (E'_\beta)'_\beta = E$. Actually, one says that the dual pair (E, E') is reflexive. Clearly, in this case $E_\xi = E_\tau = E_\beta$. It is well-known that there are different (but equivalent) conditions of reflexivity (see, e.g., [19, Chapter IV, Theorem 5.6, p. 145]). We shall here formulate some suitable conditions.

Proposition 2.1. *A locally convex space E with topology ξ is reflexive if and only if it is infrabarrelled in the topology ξ and it is quasi-complete in the weak topology $\sigma(E, E')$.*

Recall that, by a barrel in E we mean every closed absorbent absolutely convex set (a set A of a vector space E is called absorbent if for each $x \in E$ there is some $\lambda > 0$ such that $x \in \mu A$, for all μ with $|\mu| \geq \lambda$, see [18, Chapter I, p. 5]; that A is absolutely convex means that A is convex and balanced, ibid., p. 4). A locally convex topological vector space E is called infrabarrelled (respectively, barrelled), if every barrel which absorbs bounded sets (respectively, every barrel) is a neighborhood of the origin. A locally convex topological vector space E is called quasi-complete if every closed bounded set in E is complete.

Remark 2.2. Usually in the dual E' of a locally convex topological space E_ξ one distinguishes the inclusions $\mathcal{B}'_1 \subseteq \mathcal{B}'_2 \subseteq \mathcal{B}'_3 \subseteq \mathcal{B}'_4$, where $\mathcal{B}'_1$ denotes the family of all equicontinuous sets, $\mathcal{B}'_2$ the family of all sets with weakly compact closed absolutely convex envelope (see [18, Chapter I, p. 5]), $\mathcal{B}'_3$ the family of all strongly bounded sets, and $\mathcal{B}'_4$ the family of all weakly bounded sets.

There are examples of spaces E for which all these families are different (see [1, Chapter IV, § 3, Exercise 5]). The equality $\mathcal{B}'_1 = \mathcal{B}'_2$ characterizes Mackey spaces, the equality $\mathcal{B}'_1 = \mathcal{B}'_3$ characterizes infrabarrelled spaces, the equality $\mathcal{B}'_1 = \mathcal{B}'_4$ characterizes barrelled spaces; the equality $\mathcal{B}'_2 = \mathcal{B}'_3$ describes the isomorphism $E_\tau \subseteq (E'_\beta)'_\beta = E''$; the equality $\mathcal{B}'_2 = \mathcal{B}'_4$ says that $\beta(E, E') = \tau(E, E')$, i.e., the strong topology on E is consistent with duality; and at last, the equality $\mathcal{B}'_3 = \mathcal{B}'_4$ describes the isomorphism $E_\beta \subseteq (E'_\beta)'_\beta$. A sufficient condition for the last equality is quasi-completeness of E_ξ, this is why every infrabarrelled and quasi-complete space E is barrelled. In our definitions, instead of using the term barrelled, we use the term infrabarrelled, which is here equivalent to barrelled, in order to achieve symmetric formulations: quasi-complete and infra-barrelled. If we denote by $\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_3 \subseteq \mathcal{B}_4$, respectively, the family of all equicontinuous sets in E, the family of all sets in E with weakly compact closed absolutely convex envelope, the family of all bounded sets in E, then besides statements similar to the ones of above, the equality $\mathcal{B}_2 = \mathcal{B}_4$ characterizes the weak quasi-completeness of E, moreover, in this case the family $\mathcal{B}_2$ coincides with the family of all relatively $\sigma(E, E')$-compact sets in E.

Thus, reflective spaces differ from non-reflective spaces by the fact that for them *properties of equicontinuity, relative weak compactness, strong boundedness and weak boundedness are equivalent, for every subset of a strongly dual space as well as for every subset of the space itself.* The inverse statement is also true.

Definition 2.3. *A locally convex topological vector space E is called hereditarily reflexive if each of its closed subspaces and each of its separated quotient spaces are reflexive.*

Since quasi-completeness is inherited by closed subspaces and infrabarrelledness is inherited by separated quotients, taking into account that a closed subspace in the original topology is also closed in the weak topology, by Proposition 2.1, we obtain the following characterization of hereditary reflexivity.

Proposition 2.4. *A locally convex topological vector space E is hereditarily reflexive if and only if each of its closed subspace is infrabarrelled and each of its separated quotient spaces is quasi-complete in the weak topology.*

Definition 2.5. *A locally convex topological vector space E is called dually hereditarily reflexive if each of its closed subspaces and each of the closed subspaces of the strong dual E' are reflexive.*

Clearly, a dually hereditary reflexive space E is hereditarily reflexive and its strong dual E' is also hereditarily reflexive. Moreover, in this case there is a bijection between closed subspaces $L \subseteq E$ and its polars $L^o \subseteq E'$, and one can make the following identifications (algebraic and topological) $L = (E'_\beta/L^o)'_\beta$, $E/L = (L^o)'_\beta$, $L^o = (E/L)'_\beta$, $E'_\beta/L^o = L'_\beta$. (Recall that by a polar L^o of a subspace L in E one understands the set of all x' in E' such that $< x, x' >= 0$, for each x in L. Clearly, L^o is a closed subspace in E' and, if L is a closed subspace in E, then $(L^o)^o = L$.) Indeed, L^o and E/L form a dual pair $(L^o, E/L)$. Since the topology of a subspace L^o of E' is consistent with the duality $(L^o, E/L)$ (see [19, Chapter IV, § 4, Corollary 3 to Theorem 4.1, p. 135]) and by assumption, in this topology L^o is reflexive, then the strong dual E/L is also reflexive. In this case $\beta(E/L, L^o) = \tau(E/L, L^o)$ and the last assertion follows from the topology $\tau(E, E')$, i.e., it is the quotient topology of $\tau(E, E')$, and, by the reflexivity of E, $\tau(E, E')$ coincides with $\beta(E, E')$. Proof of the reflexivity of E'/L^o is analogous.

Remark 2.6. It is well-known that, in general, for a closed subspace L in E the Mackey topology $\tau(L, E'/L^o)$ is finer than the topology of a subspace induced by $\tau(E, E')$ but, as it was already mentioned, it is consistent with the duality $(L, E'/L^o)$. These topologies coincide if and only if each absolutely convex set in E'/L^o, which is compact in the weak topology $\sigma(E'/L^o)$, is the canonical image of the absolutely convex set of E', compact in the weak topology $\sigma(E', E)$ (see [1, Chapter IV, § 2, Exercise 5b]). By Remark 2.2, bounded sets and relatively weakly compact sets of reflexive spaces coincide, consequently, a dually hereditary reflexive space E and its strong dual E' have the following property: *every bounded set of the separated quotient space is the canonical image of a bounded set of the space itself.* The inverse statement is also true.

THEOREM 2.7. *A locally convex topological vector space E is dually hereditarily reflexive if and only if E and its strong dual E' have the following properties: each of their separated quotients is reflexive and every bounded set of their separated quotients is the canonical image of a bounded set.*

PROOF. Since the "only if" part was already proved above, we prove the "if" part. Since $E = E/0$ and $E' = E'/0$ and the assumptions of Theorem 2.7 are trivially fulfilled, E and E' are reflexive. One has to show that every closed subspace L in E and its polar L^o in E' are reflexive. But E'/L^o is reflexive, moreover, every absolutely convex set in it, which is compact in the weak topology, being bounded in the weak topology, is the canonical image of a bounded set in E', which is also closed in the weak topology and is absolutely convex. By the reflexivity of E', it is weakly compact, because it is quasi-complete. Thus, the topology of the strong dual L of E'/L^o coincides with the topology of a subspace. Proof of the reflexivity of L^o is analogous. ∎

From the above said we get directly.

PROPOSITION 2.8. *A locally convex topological vector space E is dually hereditarily reflexive if and only if each of its closed subspaces and each closed subspace of the strong dual E' are infrabarrelled.*

We get immediately that reflexive Banach spaces are dually hereditarily reflexive, which is well-known (see [1, Chapter IV, § 5, Proposition 1]).

Now we consider one more class of dually hereditarily reflexive spaces. It is known that any vector space E can be made into a locally convex space by taking as a base of neighborhoods of the origin the set of all absolutely convex absorbent sets. This is the finest locally convex topology on E. It is called *the finest locally convex topology* and it is usually denoted by ω while a space E endowed with this topology, is denoted by E_ω; the strong dual of E_ω is E^*.

THEOREM 2.9. *Every vector space endowed with the finest locally convex topology is dually hereditarily reflexive.*

PROOF. It is known that every vector space, endowed with the finest locally convex topology, is a Montel space (see [18, Chapter IV, Supplement 3, p. 54]), consequently, it is reflexive. Really, a Montel space is barrelled by definition and since each if its closed bounded sets is compact, it is also weakly compact, and hence complete, i.e., E is quasi-complete in the weak topology and we can apply Proposition 2.1. Moreover, the dual E^* endowed with strong topology $\beta(E^*, E)$, which coincides in this case with the weak topology $\sigma(E^*, E)$, is also a Montel space (see [1, Chapter IV, § 3, Proposition 7]). By Proposition 2.8, we have to show that every closed subspace L of E and its polar L^o in E^* are infrabarrelled.

Recall that if E is a real or complex vector space and K is its scalar field, then a locally convex topology μ in E is called *minimal* if its topology is minimal, i.e., if there exists no strictly coarser separated locally convex topology on E. If E' is the dual of E, then $\mu = \sigma(E, E')$ and $E = E'^*$. It is known that a separated locally convex vector space E is minimal if and only if it is isomorphic to the topological product $K^I = \prod_{i \in I} K_i$, where $K_i = K$. Moreover, every closed subspace of a minimal space is itself a minimal space and in any separated locally convex space F each minimal subspace E has a topological complement (see [1, Chapter IV, § 1, Exercise 13]).

It is known that if one endows E^* with the weak topology $\sigma(E^*, E)$, which coincides in our case with $\beta(E^*, E)$, then E^*_σ is isomorphic to some topological product K^I and, as it was said above, it is a minimal space. Moreover, every subspace of E endowed with topology $\sigma(E, E^*)$ is closed and has a topological complement (see [1, Chapter IV, § 1, Exercise 11]). Now let L be any closed subspace in E. Then, from the above said, L has a weak topological complement in E. But then its polar L^o in E^*, being a closed subspace, is a minimal space and also has a topological complement in the strong topology, which coincides with the weak topology. But E and E^* are reflexive and strongly dual to each other. Consequently, L has a topological complement in the finest locally convex topology on E. But E is barrelled in this topology, consequently, $L = E/L^\perp$, where $L^\perp$ is a topological complement of L, is also barrelled and hence infrabarrelled. On the other hand, any closed subspace L^o in E^*, by minimality of E^*, has a topological complement, consequently, $L^o = E^*/L^{o\perp}$ is also barrelled and hence, infrabarrelled. This completes the proof of Theorem 2.9. □

Corollary 2.10. *If E is a minimal space, then it is dually hereditarily reflexive.*

To prove the assertion one has to note that in this case $\tau(E', E) = \beta(E', E)$ and the Mackey topology $\tau(E', E)$ is the finest locally convex topology on E' (see [19, Chapter IV, Exercise 6b, p. 191]).

Corollary 2.11. *The singular chain complex $S_n(X;K)$ and the singular cochain complex $S^n(X;K)$ of an arbitrary topological space X with coefficients in a field K ($\mathbb{R}$ or $\mathbb{C}$), endowed with the finest topology and minimal locally convex topology, respectively, are dually hereditarily reflexive.*

For proving the assertion, one has to notice that the singular chain complex $S_n(X;K) = \sum_{i \in I} K_i$ and the singular cochain complex $S^n(X;K) = \prod_{i \in I} K_i$ and $(S_n(X;K))^* = S^n(X;K)$.

Corollary 2.12. *The singular homology $H_n^s(X;K)$ and the singular cohomology $H_s^n(X;K)$ of every topological space X with coefficients in a field K ($\mathbb{R}$ or $\mathbb{C}$), endowed with the finest and the minimal locally convex topologies, respectively, are dual to each other in the strong topologies and $H_n^s(X;K) = \sum_{j \in J} K_j$ and $H_s^n(X;K) = \prod_{j \in J} K_j$. Moreover, these topologies are naturally induced by topologies on $S_n(X;K)$ and $S^n(X;K)$, respectively.*

PROOF. If $S_n(X;K)$ is endowed with the finest locally convex topology, the linear mapping $d_n : S_n(X;K) \to S_{n-1}(X;K)$ is continuous and its dual $d^{n-1} : S^{n-1}(X;K) \to S^n(X;K)$ is continuous in the minimal topology. Moreover, since $B_{n-1}(X;K)$ is a closed subspace in $S_{n-1}(X;K)$ (recall that every subspace in the finest locally convex topology is closed), the dual mapping d^{n-1} of d_n is a homomorphism. Consequently, $B^n(X;K)$ is a minimal space and hence it has a topological complement and thus, is itself closed. So, the linear mapping d_n, which is the dual of d^{n-1}, is a weak homomorphism. Since every equicontinuous set of $S^n(X;K)$ lying in $B^n(X;K)$ is the canonical image of an equicontinuous set of $S^{n-1}(X;K)$ under the canonical mapping $S^{n-1}(X;K) \to B^n(X;K)$, d_n is a homomorphism in the strong topology (see [1, Chapter IV, § 4, Exercise 3]). Thus, we have shown that topologies on the set of singular (co)chains coincide, if they are the quotient topologies (co)chains modulo (co)cycles and if one considers them as topologies of subspaces. Further, since the singular cycles $Z_n(X;K)$ and the singular coboundaries $B^n(X;K)$ are polars to each other and the singular boundaries $B_n(X;K)$ and the singular cocycles $Z^n(X;K)$ are polars to each other and $S_n(X;K)$ and $S^n(X;K)$ are dually hereditarily reflexive, $Z_n(X;K)$ is the strong dual of $S^n(X;K)/B^n(X;K)$ and $B_n(X;K)$ is the strong dual of $S^n(X;K)/Z^n(X;K)$, in addition, the former is the finest topology and the latter is the minimal topology. Moreover, $Z_n(X;K)$ is itself dually hereditarily reflexive, consequently, $H_n(X;K) = Z_n(X;K)/B_n(X;K)$ is the strong dual of $H^n(X;K) = ker(S^n(X;K)/B^n(X;K) \to S^n(X;K)/Z^n(X;K))$. Analogously, $H^n(X;K) = Z^n(X;K)/B^n(X;K)$ is the strong dual of $H_n(X;K) = ker(S_n(X;K)/B_n(X;K) \to S_n(X;K)/Z_n(X;K))$. In both cases the homologies are endowed with the finest topology induced by the topology of the singular chains and the cohomologies are endowed with the minimal topology induced by the topology of the singular cochains. We immediately conclude that $H_n^s(X;K) = \sum_{j\in J} K_j$ and $H_s^n(X;K) = \prod_{j\in J} K_j$. This completes the proof of Corollary 2.12. □

COROLLARY 2.13. *The Massey cochain complex $C^n(X;K)$ and the chain complex $C_n(X;K)$ of any locally compact Hausdorff topological space X with coefficients in the field K ($\mathbb{R}$ or $\mathbb{C}$), endowed with the finest and minimal locally convex topologies, respectively, are dually hereditarily reflexive.*

PROOF. The proof is analogous since $C^n(X;K) = \sum_{i\in I} K_i$, $C_n(X;K) = \prod_{i\in I} K_i$ and $(C^n(X;K))^* = C_n(X;K)$. □

COROLLARY 2.14. *The Massey cohomology spaces $H^n(X;K)$ and the Massey homology spaces $H_n(X;K)$ of any locally compact Hausdorff topological space X with coefficients in a field K ($\mathbb{R}$ or $\mathbb{C}$), endowed with the finest and the minimal locally convex topologies, respectively, are dual to each other in the* *strong topologies,* $H^n(X;K) = \sum_{j\in J} K_j$ *and* $H_n(X;K) = \prod_{j\in J} K_j$. *Moreover, these topologies are naturally induced by the topologies on* $C^n(X;K)$ *and* $C_n(X;K)$*, respectively.*

PROOF. Proof is analogous to the proof of Corollary 2.12. □

Now let $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system of topological spaces and continuous mappings over a partially ordered directed set Λ and let $S_n(\underline{X};K) = (S_n(X_\lambda;K), p^*_{\lambda\lambda'}, \Lambda)$ and $S^n(\underline{X};K) = (S^n(X_\lambda), p^*_{\lambda\lambda'}, \Lambda)$ be the inverse and the direct systems of singular chains and singular cochains with coefficients in the field K ($\mathbb{R}$ or $\mathbb{C}$), respectively. Denote by Λ_m the set of all increasing $(m+1)$-tuples $\underline{\lambda} = (\lambda_0, ..., \lambda_m)$, $m \geq 0$, where $\lambda_i \in \Lambda$, $i = 0, 1, ..., m$, and $\lambda_0 \leq \lambda_1 \leq ... \leq \lambda_m$. Then the homotopy inverse limit

$$\bar{C}_n(\underline{X};K) = ((ho\lim_{\leftarrow\lambda} S_*(\underline{X};K)))_n = \prod_{m=0}^{\infty} \prod_{\underline{\lambda}\in\Lambda_m} S_{n+m}(X_{\lambda_0};K) \tag{2.1}$$

and the homotopy direct limit

$$\bar{C}^n(\underline{X};K) = ((ho\lim_{\rightarrow\lambda} S^*(\underline{X};K)))^n = \sum_{m=0}^{\infty} \sum_{\underline{\lambda}\in\Lambda_m} S^{n+m}(X_{\lambda_0};K), \tag{2.2}$$

endowed with the projective and the injective topologies, respectively, are reflexive and are strongly dual to each other, because they are, respectively, the topological product and the topological sum of reflexive spaces dual to each other (see [19, Chapter IV, § 5, 5.8, p. 146]).

The differential operator $\bar{d}_n : \bar{C}_n(\underline{X};K) \to \bar{C}_{n-1}(\underline{X};K)$ is defined in the following way:

$$(\bar{d}_n x)_{\underline{\lambda}} = d_n(x_{\underline{\lambda}}), m = 0, \tag{2.3}$$

$$(\bar{d}_n x)_{\underline{\lambda}} = (-1)^m d_{n+m}(x_{\underline{\lambda}}) + p^*_{\lambda_0\lambda_1} x_{\lambda_1\ldots\lambda_m} + \sum_{j=1}^{m} (-1)^j x_{\lambda_0\ldots\lambda_{j-1}\lambda_{j+1}\ldots\lambda_m}, m \geq 1; \tag{2.4}$$

and the differential operator $\bar{d}^n : \bar{C}^n(\underline{X};K) \to \bar{C}^{n+1}(\underline{X};K)$ is defined as

$$\begin{aligned}(\bar{d}^n x)_{\underline{\lambda}} &= (-1)^m d^{n+m}(x_{\underline{\lambda}}) + \sum_{\lambda\leq\lambda_0} p^*_{\lambda\lambda_0} x_{\lambda\lambda_0\ldots\lambda_m} \\ &+ \sum_{\lambda_0\leq\lambda\leq\lambda_m} (-1)^j x_{\lambda_0\ldots\lambda_{j-1}\lambda\lambda_j\ldots\lambda_m} + \sum_{\lambda_m\leq\lambda} x_{\lambda_0\ldots\lambda_m\lambda}, m \geq 0.\end{aligned} \tag{2.5}$$

In [8] it was shown that these linear mappings d_n and d^n are dual and, consequently, continuous (see [19, Chapter IV, 7.4, p. 158]).

QUESTION 2.1. *Are the reflexive spaces* $\bar{C}_n(\underline{X};K)$ *and* $\bar{C}^n(\underline{X};K)$ *hereditarily reflexive?*

We shall see below that every separated quotient space of $\bar{C}_n(\underline{X};K)$ and every closed subspace in $\bar{C}^n(\underline{X};K)$ are reflexive. The question, whether every closed subspace in $\bar{C}_n(\underline{X};K)$ and every separated quotient space of $\bar{C}^n(\underline{X};K)$ are reflexive, is open.

There is a real difficulty, because the topological product of finest topologies does not need to be the finest locally convex topology and the topological sum of minimal topologies need not be minimal. But even after a positive answer, the following question remains open.

QUESTION 2.2. *Are homologies $H_n(C_*(\underline{X};K))$ and cohomologies $H^n(C^*(\underline{X};K))$ reflexive and dual to each other? If yes, do they inherit topologies of their (co)chains?*

To prove this one has to show that the continuous linear mappings d_n and d^n are homomorphisms, for which it suffices to show that $\bar{B}_n(\underline{X};K)$ and $\bar{B}^n(\underline{X};K)$ are closed in their subspace topologies.

Analogous questions arises in the dual situation. If we denote by $\underline{X} = (X_\lambda, i_{\lambda\lambda'}, \Lambda)$ a direct system of compact Hausdorff topological spaces and continuous mappings over Λ and by $\bar{C}^n(\underline{X};K) = (C^n(X_\lambda;K), i^*_{\lambda\lambda'}, \Lambda)$ and $\bar{C}_n(\underline{X};K) = (C_n(X_\lambda;K), i^*_{\lambda\lambda'}, \Lambda)$ the inverse and the direct systems of the Massey cochains and the Massey chains with coefficients in the field K ($\mathbb{R}$ or $\mathbb{C}$), respectively, then the homotopical inverse limit

$$\bar{C}^n(\underline{X};K) = ((ho\lim_{\leftarrow\lambda} C^*(\underline{X};K)))^n = \prod_{m=0}^{\infty}\prod_{\underline{\lambda}\in\Lambda_m} C^{n-m}(X_{\lambda_0};K) \tag{2.6}$$

and the homotopical direct limit

$$\bar{C}_n(\underline{X};K) = ((ho\lim_{\rightarrow\lambda} C_*(\underline{X};K)))_n = \sum_{m=0}^{\infty}\sum_{\underline{\lambda}\in\Lambda_m} C_{n-m}(X_{\lambda_0};K), \tag{2.7}$$

endowed with the projective and the injective topologies, respectively, are reflexive and strongly dual to each other. The differential operators in these complexes are defined in the same way as above and they are continuous in the corresponding topologies.

If for any topological space X we consider an ANR-resolution in the sense of Mardešić ([12]), then the first construction defines the strong homology $\bar{H}_n(X;K)$ (for the theory of strong homology see [12]) and Čech cohomology $\check{H}^n(X;K)$ of X with coefficients in a field K; if one considers the direct system of all compact Hausdorff subsets in X, the second construction presents classical homology with compact supports $H^c_n(X;K)$ and the strong cohomology $\bar{H}^n(X;K)$ of X (for the theory of strong cohomology see [9]) with coefficients in a field K. Thus all questions which arose concern the topological vector space structures of classical (co)homologies of topological spaces with coefficients in the field K ($\mathbb{R}$ or $\mathbb{C}$).

My student Volodya Marchenko gave a positive answer to the second question in the case of inverse sequences $\underline{X} = (X_i, p_{ii+1}, \mathbb{N})$ (he studied reflexivity of the Čech homology and cohomology of metric compacta [11]).

The class of spaces considered above is a subclass of Montel spaces and as we have already seen, reflective spaces. Nevertheless, even a Montel F-space can have a non-reflexive separated quotient. There is an example of a Montel F-space E and its closed subspace L such that $E/L = l^1$, where l^1 is Banach space of absolutely summable sequences of real numbers, which is not weakly quasi-complete. So the polar L^o in the strongly dual DF-space E' is not infrabarrelled, consequently, it is not reflexive (see [1, Chapter V, § 5, Exercise 21]). In the next section we consider one more subclass of Montel spaces, which consists of dually hereditarily reflective spaces.

3. On hereditary reflexivity of nuclear locally convex spaces

The definition of a nuclear space is rather complicated and belongs to A. Grothendieck [4]. However, the class of nuclear locally convex topological vector spaces has a number of remarkable properties and, by its significance in Functional Analysis takes the same place as the class of normed spaces. If one takes into account that the intersection of these two classes is trivial, i.e., it coincides with finite dimensional vector spaces, which have a unique locally convex topological structure, then the properties of nuclear locally convex topological spaces, in some sense, are complementary to the properties of normed spaces. The most important for us here is that every bounded subset of a nuclear space is precompact. Second, that nuclearity is inherited by any closed subspace as well as by every separated quotient space. Third, that every complete nuclear space is an inverse limit of Hilbert spaces, endowed with projective topology. Crucial for our problems will be the property of complete nuclear spaces that the canonical mapping of a complete nuclear space E to its separated quotient E/L transfers the family of all bounded sets to the base of all bounded sets in E/L. Some authors (A. Pietsch [14]) tried to avoid the original Grothendieck definition of nuclear spaces via techniques of topological tensor products based at the study of absolutely summable mappings. Although for our purposes the above mentioned properties of nuclear spaces suffice, we shall give minimal data reductions (from [19, Chapter III, § 6 and § 7, p. 92–106]) concerning their structure because of the importance of the de Rham (co)chains we study here.

Recall the definition of the topological tensor product of locally convex vector spaces E and F over the same field K (as above, $K = \mathbb{R}$ or $\mathbb{C}$). Let E, F be vector spaces over K and let $B(E, F)$ be the vector space of all bilinear forms on $E \times F$. For each pair $(x, y) \in E \times F$, the mapping $f \mapsto f(x, y)$ is a linear form on $B(E, F)$ and hence an element $u_{x,y}$ of the algebraic dual $B(E, F)^*$. Clearly, the mapping $\varphi : (x, y) \mapsto u_{x,y}$ of the product $E \times F$ to

$B(E,F)^*$ is bilinear. The linear envelope of the set $\varphi(E \times F)$ in $B(E,F)^*$ is denoted by $E \otimes F$ and is called the *tensor product* of E and F; φ is called the *canonical bilinear mapping* of $E \times F$ to $E \otimes F$. Every element $u \in E \times F$ admits the following representation $u = \sum_i x_i \otimes y_i$. This representation u is not unique but one can assume that both sets $\{x_i\}$ and $\{y_i\}$ are linearly independent and consist of $r \geq 0$ elements. This number r is uniquely defined by u and is called the rank of u. The finest topology on $E \otimes F$, for which the canonical mapping φ is continuous, is called the *projective topology* of the tensor product. Such topology always exists; it is always separated and locally convex. In most cases the tensor product with the projective topology is an incomplete space. Denote by $E \widetilde{\otimes} F$ the completion of the tensor product with the projective topology.

Now we need the definition of a nuclear mapping of locally convex topological vector spaces. For our purposes we restrict ourselves to a simpler definition of a nuclear mapping $f : E \to F$ of Banach spaces E and F. Let $L(E,F)$ be the vector space of all linear continuous mappings endowed with the normed topology. This space is complete. Now let E' be the strong dual of a Banach space E. Every element $v \in E' \otimes F$ defines a linear mapping $u \in L(E,F)$. To be exact, if $v = \sum_{i=1}^{r} f_i \otimes y_i$, then $x \mapsto u(x) = \sum_{i=1}^{r} f_i(x) y_i$ and the mapping $v \mapsto u$ is even an algebraic isomorphism of $E' \otimes F$ in $L(E,F)$. Moreover, this embedding is continuous in the projective topology on $E \otimes F$ and the strong (i.e., normed) topology on $L(E,F)$. Since the latter is complete, the mapping $v \mapsto u$ has a continuous extension π on $E \widetilde{\otimes} F$ with values in $L(E,F)$. Linear mappings containing in the image of π are called *nuclear*. We need the following Proposition (see [19, Chapter III, Proposition 7.1, p. 99]).

Proposition 3.1. *A linear mapping $u \in L(E,F)$ is nuclear if and only if it is of the form*

$$x \mapsto u(x) = \sum_{i=1}^{\infty} \lambda_i f_i(x) y_i, \tag{3.1}$$

where $\sum_{i=1}^{\infty} |\lambda_i| < +\infty$, $\{f_i\}$ is an equicontinuous sequence in E', and $\{y_i\}$ is a sequence contained in an absolutely convex bounded set $B \subseteq F$.

Now let E be a locally convex topological vector space and $\{U_\alpha, \alpha \in A\}$ be a base of absolutely convex neighborhoods of the origin in E. The set A can be ordered by putting $\alpha \leq \alpha'$ whenever $U_{\alpha'} \subseteq U_\alpha$. Denote by q_α, the Minkowski functional of a neighborhood U_α, given by the formula $q_\alpha(x) = inf\{\mu > 0 : x \in \mu U_\alpha\}$, $x \in E$. Since U_α is a neighborhood of the origin, $q_\alpha : E \to \mathbb{R}$ is a continuous mapping and $V_\alpha = q_\alpha^{-1}(0)$ is a closed subspace in E. Put $E_\alpha = E/V_\alpha$, and denote by p_α, the canonical mapping of E onto E_α. If x_α

is an equivalent class $x \in E$ mod V_α, then $x_\alpha \mapsto ||x_\alpha|| = q_\alpha(x)$ is a norm on E_α, which induces a topology coarser than the quotient topology on E/V_α, consequently, $p_\alpha : E \to E_\alpha$ is continuous. If $\alpha \leq \alpha'$, then each equivalence class $x_{\alpha'}$ mod $V_{\alpha'}$ is contained in a unique equivalence class x_α mod V_α, because $V_{\alpha'} \subseteq V_\alpha$. The mapping $p_{\alpha\alpha'} : x_{\alpha'} \mapsto x_\alpha$ is linear and continuous, because $||x_\alpha|| \leq ||x_{\alpha'}||$, which is called the canonical mapping of $E_{\alpha'}$ to E_α. Moreover, for every $\alpha \leq \alpha' \leq \alpha''$, we have the equality $p_{\alpha\alpha'}p_{\alpha'\alpha''} = p_{\alpha\alpha''}$, and for each $\alpha \in A$, evidently, $p_{\alpha\alpha} = 1_{E_\alpha}$.

The inverse limit $F = \lim\limits_{\leftarrow\alpha}(E_\alpha, p_{\alpha\alpha'}, A)$ endowed with the projective topology is a separated locally convex topological vector space, everywhere dense in $F = \lim\limits_{\leftarrow\alpha}(E_\alpha, p_{\alpha\alpha'}, A)$, where E_α is the completion of the normed space E_α, and $p_{\alpha\alpha'} : E_{\alpha'} \to E_\alpha$ is a continuous extension of the mapping $p_{\alpha\alpha'} : E_{\alpha'} \to E_\alpha$, $\alpha \leq \alpha'$. If E is a complete space, then $E = \lim\limits_{\leftarrow\alpha}(E_\alpha, p_{\alpha\alpha'}, A) = \lim\limits_{\leftarrow\alpha}(E_\alpha, p_{\alpha\alpha'}, A)$. If E is not complete, then its completion is $E = \lim\limits_{\leftarrow\alpha}(E_\alpha, p_{\alpha\alpha'}, A)$.

Definition 3.2. *A locally convex topological vector space E is called nuclear, if for each of its absolutely convex neighborhoods U_α of the origin there exists an absolutely convex neighborhood $U_{\alpha'}$ of the origin, $\alpha \leq \alpha'$, such that the canonical mapping of Banach spaces $p_{\alpha\alpha'} : E_{\alpha'} \to E_\alpha$ is nuclear.*

Examples of nuclear spaces are (see, e.g., [19, Chapter III, § 8, p. 106–108; or [14], Chapter 6]): minimal spaces K^I; the space $\mathcal{D}_F(\mathbb{R}^n)$, $n \geq 1$, of all real (or complex) valued infinitely differentiable functions on $\mathbb{R}^n$, whose supports are contained in a compact subset F, endowed with the topology of uniform convergence of every derivative; the space $\mathcal{D}(\mathbb{R}^n)$, $n \geq 1$, which is a union of all $\mathcal{D}_F(\mathbb{R}^n)$, where F runs over the family of all compact subsets in $\mathbb{R}^n$), endowed with the strict inductive limit of topologies in $\mathcal{D}_F(\mathbb{R}^n)$, $n \geq 1$; the space $\mathcal{E}(\mathbb{R}^n)$, $n \geq 1$, of all real (or complex) infinitely differentiable functions on $\mathbb{R}^n$, endowed with the topology of compact convergence for all derivatives; the space $\mathcal{H}(D)$ of all functions holomorphic on a domain D in the complex plane, endowed with the topology of compact convergence; the space of Schwartz distributions $\mathcal{D}'(\mathbb{R}^n)$, endowed with the strong topology; the spaces of Schwartz distributions with compact supports $\mathcal{E}'(\mathbb{R}^n)$, endowed with the strong topology; the space $\mathcal{H}'(D)$, endowed with the strong topology; spaces $\Omega^p(M)$, $\Omega^p_c(M)$, $\Omega^c_p(M)$, $\Omega_p(M)$, $p \geq 0$, mentioned in the Introduction and many others (e.g., Köthe gestufter Raum or ladder space with some additional properties, see [19, Chapter III, Exercise 25, p. 120–121]).

From Proposition 3.1 and Definition 3.2 one obtains immediately the following properties of nuclear mappings and nuclear spaces.

Proposition 3.3. *Every nuclear mapping is compact, i.e., there exists a neighborhood of the origin, whose image under this mapping is relatively compact.*

Proposition 3.4. *Every bounded subset of a nuclear space is precompact.*

Proposition 3.5. *Every complete nuclear space E is isomorphic to the projective limit of some family (cardinality of A) of Hilbert spaces, i.e., $E = \varprojlim_{\alpha}(E_\alpha, p_{\alpha\alpha'}, A)$, where E_α is a Hilbert space. Moreover, for each $\alpha \le \alpha'$, the mapping $p_{\alpha\alpha'} : E_{\alpha'} \to E_\alpha$ is nuclear.*

The latter statement follows from the well-known fact that the composition of a nuclear mapping with a linear continuous mapping (and vice versa) is nuclear (see [19, Chapter III, Corollary 2 to 7.1, p. 100]).

Proposition 3.6. *Every subspace and every separated quotient space of a nuclear space is nuclear. The topological product of an arbitrary family of nuclear spaces is nuclear and the topological sum of countably many nuclear spaces is a nuclear space. Moreover, the projective limit of any family of nuclear spaces and the inductive limit of a sequence of nuclear spaces are nuclear.*

For a proof see [19, Chapter III, Theorem 7.4 and its Corollary, p. 103].

For nuclear spaces Propositions 2.1 and 2.4 are more symmetric.

Proposition 3.7. *A nuclear space E with the topology ξ is reflexive if and only if E is infrabarrelled and quasi-complete in ξ.*

Proof. Necessity follows immediately from Proposition 2.1, because a space, which is quasi-complete in the weak topology, is quasi-complete in any finer topology, in particular, in ξ. On the other hand, every space, which is nuclear and quasi-complete in ξ, is quasi-complete in the weak topology, because the closure of any bounded set is compact, consequently, it is compact in the weak topology and hence it is complete. □

Proposition 3.8. *A nuclear space E is hereditarily reflexive if and only if each of its closed subspaces is infrabarrelled and each of its separated quotients is quasi-complete.*

The property of all bounded sets of separated quotients, we used in Theorem 2.9, is crucial for us. These properties are fulfilled in every Banach space, more general, in any DF-space in the sense of Grothendieck [5].

Theorem 3.9. *Every bounded set in the separated quotient E/L of a complete nuclear space E is the canonical image of a bounded set in E.*

Proof. Let $F = E/L$ be the separated quotient space and let $\varphi : E \to F$ be the canonical mapping. Further, let B be a bounded set in F. We shall show that there exists a bounded set C in E such that $\varphi(C) = B$.

Represent, by Proposition 3.5, the complete nuclear space E as an inverse limit $\varprojlim_{\alpha}(E_\alpha, p_{\alpha\alpha'}, A)$ of Hilbert spaces $E_\alpha = (\widetilde{E/U_\alpha})$, endowed with the projective topology, where $\{U_\alpha, \alpha \in A\}$ is for this purpose an appropriate base of convex neighborhoods of the origin in E, and $p_{\alpha\alpha'}$, $\alpha \leq \alpha'$, are nuclear mappings. Denote by $p_\alpha : E \to E_\alpha$ the corresponding projection, for each $\alpha \in A$. Then $\{V_\alpha = \varphi(U_\alpha), \alpha \in A\}$ will be a base of convex neighborhoods of the origin in E/L with the same property. Indeed, we can identify $F_\alpha = \widetilde{(F/V_\alpha)}$ with the quotient E_α/L_α, where L_α is the closure in E_α of the set $p_\alpha(L)$ (see [19, Chapter III, Exercise 3, p. 116]). Since E_α is a Hilbert space, consider its decomposition in the topological sum $E_\alpha = L_\alpha \oplus L_\alpha^\perp$, where $L_\alpha^\perp$ is the orthogonal complement of L_α in E_α. Thus, F_α can be identified with $L_\alpha^\perp$, and F is everywhere dense in the projective topology of the inverse limit $F = \lim\limits_{\leftarrow\alpha}(L_\alpha^\perp, q_{\alpha\alpha'}, A)$, where the mapping $q_{\alpha\alpha'} : L_{\alpha'}^\perp \to L_\alpha^\perp$, $\alpha \leq \alpha'$, is induced by the nuclear mapping $p_{\alpha\alpha'} : E_{\alpha'} \to E_\alpha$ and it is itself nuclear. Really, by Proposition 3.1, $p_{\alpha\alpha'}$ has a form $\sum\limits_{i=1}^{\infty} \lambda_i f_i(x) y_i$, where $x \in E_{\alpha'}$, $y_i \in E_\alpha$, $f_i \in (E_{\alpha'})'$. Since the strong dual $(E_{\alpha'})'$ of $E_{\alpha'}$ is also a Hilbert space, there are a decomposition $f_i = f_i^1 + f_i^2$ in $(E_{\alpha'})' = (L_{\alpha'})' \oplus (L_{\alpha'}^\perp)'$ and a decomposition $y_i = y_i^1 + y_i^2$ in $E_\alpha = L_\alpha \oplus L_\alpha^\perp$ such that $f_i^1(L_{\alpha'}^\perp) = f_i^2(L_{\alpha'}) = 0$. Then the linear mapping $q_{\alpha\alpha'}$, $\alpha \leq \alpha$, has the form $\sum\limits_{i=1}^{\infty} \lambda_i f_i^2(x) y_i^2$, where $x \in L_{\alpha'}^\perp$ (see details in [19, Chapter III, Theorem 7.4, p. 103–105]).

Therefore, by completeness of E, the space F can be identified with a subset in E but, in general, not with a subspace with the topology induced by E. We denote this bijection by $\psi : F \to \psi(F) \subset E$. If B is bounded in F, then, for each $\alpha \in A$, the set $q_\alpha(B)$ is bounded in $F_\alpha = L_\alpha^\perp$ and hence, it is bounded in E_α. But then the set $C = \psi(B)$, having bounded projections $p_\alpha(C) = q_\alpha(B)$, for each $\alpha \in A$, is bounded. By construction, $\varphi(C) = B$. This completes the proof of Theorem 3.9. □

Remark 3.10. If $\psi(F)$ is topologically identified with F, then it is a topological complement of L in E, thus, E/L is a complete space (for example, when $E = K^I$). We shall see below that E/L can be incomplete, but even if it is a complete space, one cannot, in general, identify it with a subspace of E. It is known (see [19, Chapter IV, Exercise 12a, p. 192–193]) that in the complete nuclear space D_G there is a closed subspace L such that $(D_G)/L = K^I$. If L had a topological complement equal to K^I, then on the latter there should be a continuous norm (because such norm exists in D_G), but this is not true, because there is no continuous norm on K^I (see [19, Chapter IV, Exercise 6b, p. 191]).

Remark 3.11. Actually, a more general statement of Theorem 3.9 is valid: If E is represented as an inverse limit of spaces E_λ, endowed with the projective topology such that each closed subspace of E_λ, $\lambda \in \Lambda$, has a topological complement, then every bounded set in the separated quotient

E/L of the space E is the canonical image of a bounded set in E. In particular, this is true for spaces E, which are represented as inverse systems of spaces E_λ endowed with finest locally convex topologies; or when E is the topological product of Hilbert spaces and for closed subspaces of E.

As a Corollary of Theorem 3.9 we obtain the following propositions.

PROPOSITION 3.12. *A complete barrelled nuclear space is dually hereditarily reflexive if and only if it is hereditarily reflexive.*

PROOF. One has only to prove sufficiency. Let L be any closed subspace of E. It is reflexive by assumption. We shall prove that L^o is also reflexive. We need to show that L^o is infrabarrelled. Since E/L is reflexive by assumption, its strong dual L^o with the Mackey topology $\tau(L^o, E/L)$ is reflexive and thus, infrabarrelled. It remains to show that the topology $\tau(L^o, E/L)$ coincides with the topology of a subspace L^o in E'. By Remark 2.6, for this purpose it is sufficient to see that every absolute convex set in E/L, which is compact in the weak topology $\sigma(E/L, L^o)$, is the canonical mapping of an absolute convex set in E, which is compact in the weak topology $\sigma(E, E')$. Since E and E/L are reflexive, and by Remark 2.2, for reflexive spaces the properties of being weakly bounded, bounded and being relatively weakly compact coincide, we apply Theorem 3.9 to any absolute convex set B in E/L, which is weakly compact and hence weakly bounded, and we find a bounded set C in E, whose canonical image is the set B. Take now the closed absolute convex envelope C of the closure $\bar{C}$ in E, which is compact, because E is complete (see [19, Chapter IV, § 11, Theorem 11.4, p. 189]). Evidently, it is mapped onto B. ◻

PROPOSITION 3.13. *A complete barrelled nuclear space E, which has a complete nuclear strongly dual space E', is dually hereditarily reflexive if and only if for every closed subspace L in E the quotient spaces E/L and E'/L^o are reflexive.*

PROOF. One has to prove only sufficiency (necessity has been already shown above). For this purpose it is enough to show that L and L^o are infrabarrelled. One can show it as in the proof of Proposition 3.12, because, by assumption, E' is also complete and nuclear. E' is also barrelled, because $E/0$ is reflexive by assumption. ◻

The main result of this section is the following.

THEOREM 3.14. *Every complete barrelled nuclear space E, whose strong dual E' is complete and nuclear, is dually hereditarily reflexive.*

PROOF. By Theorem 3.9 and Proposition 3.6, it is enough to show that, for every closed subspace L of E, the quotients E/L and E'/L^o are quasi-complete. Note that they are barrelled, because barrelledness is inherited by separated quotients. From Proposition 3.6 we obtain that E is reflexive. Consequently, its strongly dual space E' is also reflexive and hence barrelled. Let B be a closed bounded set in E/L, which, by Proposition 3.4, is precompact. We shall prove that B is compact in E/L and thus, complete. Since B is the canonical image of a bounded set of C, which is actually closed in E, and by completeness of E and by Proposition 3.4, it is compact. Consequently, B is also compact and thus, complete. So, E/L is quasi-complete.

Proof of quasi-completeness of E'/L^o is analogous, because E' is barrelled. □

COROLLARY 3.15. *The spaces* $\mathcal{D}_F(\mathbb{R}^n)$, $\mathcal{D}(\mathbb{R}^n)$, $\mathcal{E}(\mathbb{R}^n)$, $\mathcal{H}(D)$, $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{E}'(\mathbb{R}^n)$, $\mathcal{H}'(D)$, $\Omega^p(M)$, $\Omega^p_c(M)$, $\Omega^c_p(M)$, $\Omega_p(M)$, $p \geq 0$, $n \geq 1$ *are dually hereditarily reflexive.*

REMARK 3.16. In [20] a closed subspace L in the complete barrelled nuclear space $\mathcal{D}(\mathbb{R}^1) = \lim\limits_{\to n} \mathcal{D}_{[-n,n]}(\mathbb{R}^1)$ is exhibited, which is not bornological (in other terminology, boundedly closed space, i.e., if every convex set in E absorbing all bounded sets is a neighborhood of the origin in E). On the other hand, the strict inductive limit $\lim\limits_{\to n} L \cap \mathcal{D}_{[-n,n]}(\mathbb{R}^1)$ is bornological (see [19, Chapter 2, Corollary 2 to Proposition 8.2, p. 62]). Thus, there are two different reflexive topologies on L: the first one is the topology of a subspace L of E and the second one is the topology of the strict inductive limit. For the first topology, the strongly dual E'/L^o is quasi-complete but incomplete and, for the second topology, the strongly dual is the completion $(\widetilde{E'/L^o})$ of E'/L^o. This example shows that there exist incomplete reflexive spaces, precisely E'/L^o (first example of an incomplete reflexive space was given in [7]). Since topology of the strict inductive limit is finer than the topology of a subspace on L, the identity mapping $L \to L$ is continuous but it is not an isomorphism and the inverse mapping with a closed graph is not continuous. This is one more example showing that the Closed graph theorem for the strict inductive limit of F-spaces as well as the Open mapping theorem cannot be improved by supposing merely barrelledness of another space (see details in [18, Chapter VI, Supplement 1, p. 123–124]). This example, as is was pointed out by D. A. Raykov (see [16, Appendix 2, p. 249]), shows that $\mathcal{D}'(\mathbb{R}^1)$ is not fully complete (for the definition of a fully complete space see [18, Chapter VI, § 2, p. 111]). We have shown a stronger result that this space is not hereditarily complete (it is a negative answer to Raykov's question). Analogous questions are open for the space $\mathcal{D}(\mathbb{R}^1)$ itself.

A positive answer to the following problem would generalize Theorem 3.14.

PROBLEM 3.1. *Let E be a quasi-complete infrabarrelled nuclear space, whose strong dual E' is nuclear. Is E dually hereditarily reflexive?*

It should be enough to prove Theorem 3.9 replacing the assumption of completeness by the assumption of quasi-completeness. On the other hand, the positive solution of this Problem will make all the above spaces *strongly hereditarily reflexive*, i.e., every separated quotient space of such a space is itself hereditarily reflexive (hereditary reflexivity of closed subspaces is evident).

4. Reflexivity of the de Rham cohomology and homology

The hereditary reflexivity of the de Rham cochains and chains (currents) of C^∞-manifolds has been shown above. For proving a natural reflexivity of the corresponding de Rham cohomology and homology, i.e., a reflexivity which is a consequence of the reflexivity of cochains and chains, one has to show, first of all, that coboundaries $B^n(M)$, $B^n_c(M)$ and boundaries $B_n(M)$, $B^c_n(M)$ are closed subspaces of $\Omega^n(M)$, $\Omega^n_c(M)$ and $\Omega^n(M)$, $\Omega^c_n(M)$, respectively. Cocycles $Z^n(M)$, $Z^n_c(M)$ and cycles $Z_n(M)$, $Z^c_n(M)$ are automatically closed, because the differential operators $d^n : \Omega^n(M) \to \Omega^{n+1}(M)$, $d^n_c : \Omega^n_c(M) \to \Omega^{n+1}_c(M)$ and $d_n : \Omega_n(M) \to \Omega_{n-1}(M)$, $d^c_n : \Omega^c_n(M) \to \Omega^c_{n-1}(M)$ are continuous (see [17, Chapter III, § 11]). Closedness of coboundaries and boundaries is a consequence of a rather non-trivial and fine de Rham theorem saying that $B^n(M)$, $B^n_c(M)$ and $B_n(M)$, $B^c_n(M)$ are polars of cycles and cocycles $Z^c_n(M)$, $Z_n(M)$ and $Z^n_c(M)$, $Z^n(M)$, respectively, (see [17, Chapter IV, § 18, Theorems 14, 17, 17', 18]). The fact that cycles and cocycles $Z^c_n(M)$, $Z_n(M)$ and $Z^n_c(M)$, $Z^n(M)$ are polars of coboundaries and boundaries $B^n(M)$, $B^n_c(M)$ and $B_n(M)$, $B^c_n(M)$, respectively, is a simple consequence of the facts that polars in $\Omega^n(M)$, $\Omega^n_c(M)$ and $\Omega_n(M)$, $\Omega^c_n(M)$ are null-subspaces and that d^c_n, d_n and d^n, d^n_c are dual, respectively.

Secondly, since coboundaries and boundaries $B^n(M)$, $B^n_c(M)$ and $B_n(M)$, $B^c_n(M)$ have two topologies: the quotient topology and the topology of a subspace in $\Omega^n(M)$, $\Omega^n_c(M)$ $\Omega^n(M)$, $\Omega^c_n(M)$, respectively, one has to show that these topologies coincide, in other words, one has to show that the differentials d^n, d^n_c and d_n, d^c_n are homomorphisms in the strong topologies. This is an immediately consequence of the following theorem.

Theorem 4.1. *Let $f : E \to F$ be a weak homomorphism of complete barrelled nuclear spaces with complete nuclear strongly dual spaces E' and F' such that $f(E)$ is closed in F. Then f is a strong homomorphism and its dual mapping $f' : F' \to E'$ is also a strong homomorphism.*

Proof. From the continuity of f we obtain the continuity of f'. From closedness of $f(E)$ in F we obtain that f' is a weak homomorphism. Since f is a homomorphism, the image $f'(F')$ is closed in E'. Put $L = kerf$ and $M = kerf'$. Then, by the hereditary reflexivity of E and F', the quotient E/L is reflexive and L^o is its strong dual ($L^o \subseteq E'$), besides $L^o = f'(F')$, and F'/M is also reflexive and M^o is its strong dual ($M^o \subseteq F$), besides $M^o = f(E)$ (see

[1, Chapter IV, § 4, Corollary to Proposition 2]). We shall prove that f is a strong homomorphism. It is known that f is a strong homomorphism if and only if every equicontinuous set in E', being contained in $f'(F')$, is the canonical image of an equicontinuous set in F' under the canonical mapping F' onto $L^o = f'(F') = F'/M$ (see [1, Chapter IV, § 4, Exercise 3a]). Since E' and L^o are reflexive, their equicontinuous sets coincide with weakly bounded sets in E' and L^o, respectively. Since the weak topology is inherited under separated quotients and closed subspaces, and f' is a weak homomorphism, weak bounded sets in L^o are the same in the subspace L^o of E', and in the quotient F'/M. By reflexivity of F'/M, weakly bounded sets coincide with strongly bounded sets. Since F' is a complete nuclear space, every strongly bounded set in F'/M is the canonical image of a strongly bounded set in F' under the canonical mapping F' onto F'/M, which, by reflexivity of F', coincides with an equicontinuous set in F'. The proof that f' is a strong homomorphism is analogous. □

Remark 4.2. Notice, that even for F-spaces E and F, in contrast to the case of Banach spaces (for which the same property of covering all bounded sets under canonical mappings onto the quotient spaces holds), in general, a strong topological homomorphism $f : E \to F$ does not guarantee a strong homomorphism $f' : F' \to E'$ (see [1, Chapter IV, § 4, Exercise 5b]).

Theorem 4.1 shows that if a complete barrelled nuclear space E with a complete nuclear strongly dual space E' is a (co)chain complex, with continuous (co)differentials $d : E \to E$, $d^2 = 0$, and with closed (co)boundaries B, then, first, E' is also a (co)chain complex with continuous (co)differentials $d' : E' \to E'$, $d'^2 = 0$, secondly, (co)homologies $H = Z/B$ are correctly defined with the quotient topology as well as with the topology of a subspace, since $H = ker(E/B \to E/Z)$. Similarly, for (co)homologies of E'. This is why one has to prove not only reflexivity of H but also coincidence of these two topologies on H, or, which is the same, that a natural linear mapping $Z \to H \subseteq E/B$ is a strong homomorphism. Since Z is itself reflexive, as we have shown above, Z/B is reflexive and its strongly dual space is a closed subspace H' of E'/Z^o with the induced topology. Similarly, H' as the quotient Z'/B' (here $'$ does not mean duality but refers to (co)cycles and (co)boundaries in E'), is reflexive and its strong dual as a closed subspace of E/B is H. In the general case, we cannot prove coincidence of these two topologies on H and H', but if E coincides with one of the following spaces $\Omega^*(M)$, $\Omega^*_c(M)$, $\Omega_*(M)$, $\Omega^c_*(M)$, then their (co)homologies are reflexive and the two topologies coincide.

Theorem 4.3. *The de Rham cohomology $H^n(M)$ and the de Rham cohomology with compact supports $H^n_c(M)$ of a C^∞-manifold M are reflexive and their strong duals are the de Rham homology with compact supports $H^c_n(M)$*

and the de Rham homology $H_n(M)$ of a C^∞-manifold M, respectively. Moreover, both topologies induced by the topologies of the corresponding spaces of (co)chains coincide.

PROOF. The spaces $\Omega^n(M)$ and $\Omega^c_n(M)$ are complete, barrelled, nuclear, and hence reflexive, and they are strong duals of each other (de Rham theorem). The coboundaries $B^n(M)$ and the boundaries with compact supports $B^c_n(M)$ are closed subspaces of $\Omega^n(M)$ and $\Omega^c_n(M)$, respectively, and the subspaces $Z^n(M)$, $B^n(M)$ and $B^c_n(M)$, $Z^c_n(M)$ are polars to each other, respectively (de Rham's result). Since, by Theorem 3.14, $\Omega^n(M)$ and $\Omega^c_n(M)$ are hereditarily reflexive, $Z^n(M)$, $B^n(M)$, $\Omega^n(M)/Z^n(M)$, $\Omega^n(M)/B^n(M)$ and Z^c_n, B^c_n, $\Omega^c_n(M)/Z^c_n(M)$, $\Omega_n(M)^c/B^c_n(M)$ are reflexive and their strong dual spaces are $\Omega^c_n(M)/B^c_n(M)$, $\Omega^c_n(M)/Z^c_n(M)$, $B^c_n(M)$, $Z^c_n(M)$ and $\Omega^n(M)/B^n(M)$, $\Omega^n(M)/Z^n(M)$, $B^n(M)$, $Z^n(M)$, respectively. The spaces $Z^n(M)$ and $Z^c_n(M)$ are themselves complete, barrelled and nuclear, but, in general, their strong duals can be incomplete, therefore, we can state only a reflexivity (not hereditary reflexivity) of quotients $H^n(M) = Z^n(M)/B^n(M)$ and $H^c_n(M) = Z^c_n(M)/B^c_n(M)$ and their strongly dual spaces are $H^c_n(M) = ker(\Omega^c_n(M)/B^c_n(M) \to \Omega^c_n(M)/Z^c_n(M))$ and $H^n(M) = ker(\Omega^n(M)/B^n(M) \to \Omega^n(M)/Z^n(M))$, respectively. Since the de Rham cohomology $H^n(M)$ coincides (up to an isomorphism) with the singular cohomology $H^n_s(M;\mathbb{R})$ and the de Rham homology with compact supports $H^c_n(M)$ coincides with the singular homology $H^s_n(M;\mathbb{R})$, as we have seen above, it is the minimal space and $H^c_n(M)$ is endowed with the finest locally convex topology. Moreover, $\sigma(H^n(M), H^c_n(M)) = \tau(H^n(M), H^c_n(M))$ and $\beta(H^c_n(M), H^n(M)) = \omega(H^n(M), H^c_n(M))$. Therefore, two topologies on $H^n(M)$ are consistent with duality $(H^n(M), H^c_n(M))$, and they are Mackey spaces and their strong topologies on the dual spaces will be the finest, consequently, they coincide on $H^c_n(M)$.

Proof of a natural reflexivity of $H^n_c(M)$ and $H_n(M)$ is analogous, because $H_n(M)$, $n \geq 0$, are minimal spaces. □

5. "Pontryagin duality" in locally convex topological vector spaces

It is known (see [1, Chapter IV, § 3, Exercise 1c]) that whenever E is an infrabarrelled space, then the strong topology on E' coincides with the topology of compact convergence if and only if E is a Montel space (because the closure of every bounded set is compact). Therefore, for Montel space E, not only the identities $E = (E'_\beta)'_\beta$ and $E' = ((E')'_\beta)'_\beta$ hold but also $E = (E'_\kappa)'_\kappa$ and $E' = ((E')'_\kappa)'_\kappa$, because E' is a Montel space too (see [1, Chapter, § 3, Proposition 7]). In this case one says that there is a "Pontryagin duality" between E and E' (see [1, p. 250]). This terminology goes back to the famous Pontryagin duality between locally compact Hausdorff groups G and their character groups Γ, i.e., the groups of all continuous homomorphisms $\Gamma = \{G \to \mathbf{T}\}$, $\mathbf{T} = \mathbb{R}/\mathbb{Z}$, endowed with the compact-open topology and in these topologies G and Γ are character groups of each other. Moreover, for any closed subgroup H of G and its annihilator Λ in Γ the groups G/H and H are topologically isomorphic to Λ and Γ/Λ, respectively, (see [15, Chapter 6, § 40, Theorems 53 and 54]; [13, Chapter 7, Theorem 27]). In our terminology Pontryagin duality coincides with dual hereditary reflexivity.

Kaplan ([6]) generalized Pontryagin duality to arbitrary topological Abelian groups considering character group endowed with the compact-open topology and in the case, when the repeated character group with the compact-open topology coincided with the original group including its topology, it was called reflexive. Although, the class of reflexive groups is not fully described, it contains not only all locally compact Hausdorff Abelian groups but also all Banach spaces considered as topological groups (see [21]). Kaplan has shown that if one has an arbitrary family of reflexive groups, then its direct product with the Tychonov topology is also reflexive. Moreover, its dual character group is the direct sum endowed with a special topology (see details in [13, Chapter 5, Remark to Theorem 17]). Hereditary reflexivity was not considered by Kaplan.

Since every quasi-complete and infrabarrelled nuclear space is Montel, we have shown above that every complete barrelled nuclear space E with complete nuclear strong dual space E' is hereditarily reflexive in the sense of the "Pontryagin duality", i.e., $E = (E'_\kappa)'_\kappa$ and $E' = ((E')'_\kappa)'_\kappa$. Thus, concerning the Pontryagin duality in locally convex topological vector spaces, like in the usual strong reflexivity cases, we shall discuss below their κ-reflexivity, hereditary κ-reflexivity and dual hereditary κ-reflexivity. One can raise a natural question: *Which further classes of locally convex topological vector spaces are κ-reflexive, hereditarily κ-reflexive and dually hereditarily κ-reflexive?*

From the well-known Banach and Dieudonné results (see [2] for a generalization of the Banach theorem) saying that for a Banach space E (respectively, for an F-space E) there exists a finest topology τ_f on E', which coincides with $\sigma(E', E)$ on every equicontinuous set and hence, by the Grothendieck theorem [3], has the property that $E = (E'_{\tau_f})'$, moreover, this topology τ_f coincides with the topology of compact convergence (see [19, Chapter IV, Corollary 2 to Theorem 6.3, p. 151]). Since every complete metric space E is a Mackey space, the family of all equicontinuous sets in E', by Remark 2.2, coincides with the family of all sets having weakly compact closed absolutely convex envelope. And since, for an F-space E, its dual space E' endowed with the topology of compact convergence is complete (see [1, Chapter IV, § 3, Exercise 20a]) and, for the complete space E', the closed absolutely convex envelope of any compact set is compact (see [19, Chapter IV, Theorem 11.5, p. 189]), the family of all closed equicontinuous sets in E' coincides with the set of all compact sets in E'_{τ_f} and hence $E = (E'_\kappa)'_\kappa$. Moreover, since an F-space E, evidently, is hereditarily complete and hereditarily barrelled and the topology of compact convergence in these cases is also hereditary (see [1, Chapter IV, § 3, Exercise 12a]), we have actually proved the following theorem.

THEOREM 5.1. *Every F-space, in particular, every Banach space is hereditarily κ-reflexive.*

REMARK 5.2. By completeness of E and E'_{τ_f}, the topology of compact convergence is inherited by the quotients E/L and E'_{τ_f}/L^o, respectively, and since E is metric and thus the topology of compact convergence is inherited by closed subspaces L, we can identify $E/L = ((E/L)'_\kappa)'_\kappa$, $L = (L'_\kappa)'_\kappa$ and $E'_{\tau_f}/L^o = ((E'_{\tau_f}/L^o)'_\kappa)'_\kappa$. For the identification $L^o = ((L^o)'_\kappa)'_\kappa$, it is necessary and sufficient that every compact set in E/L is the canonical image of some compact set in E under the canonical mapping E onto E/L. Thus, for an F-space to be dually hereditarily κ-reflexive, it is necessary and sufficient that the stated property be fulfilled for every closed subspace L, for example, it is fulfilled when E is a nuclear F-space.

Now from Theorem 5.1 we obtain immediately the following theorems.

THEOREM 5.3. *Let $E = \prod_{\gamma\in\Gamma} E_\gamma$ be a direct product of F-spaces E_γ, $\gamma \in \Gamma$, endowed with the Tychonov topology (projective topology). Then E is κ-reflexive and its dual E', endowed with the topology of compact convergence coincides with the direct sum $\sum_{\gamma\in\Gamma} E'_\gamma$, endowed with the topology of compact convergence.*

PROOF. The fact that the topological product E and the topological sum E' are dual to each other is a consequence of known theorems (see [18, Chapter V, Propositions 26 and 25, p. 93]): 1) The dual of the topological product $\prod_{\gamma\in\Gamma} E_\gamma$ is the direct sum $\sum_{\gamma\in\Gamma} E'_\gamma$ of the duals E'_γ of E_γ, moreover, if each E_γ is separated, and each E'_γ has the topology of $\mathcal{A}_\gamma$-convergence (where the sets of $\mathcal{A}_\gamma$ are supposed to be absolutely convex), then the direct sum topology on $\sum_{\gamma\in\Gamma} E'_\gamma$ is the topology of $\mathcal{A}$-convergence, where $\mathcal{A}$ is the set of all products $\prod_{\gamma\in\Gamma} A_\gamma$, with $A_\gamma \in \mathcal{A}_\gamma$, for each $\gamma \in \Gamma$; and 2) The dual of the topological sum $\sum_{\gamma\in\Gamma} E'_\gamma$ is the product $\prod_{\gamma\in\Gamma} E_\gamma$ of the duals E_γ of E'_γ, moreover, if each E'_γ is separated and each E_γ has the topology of $\mathcal{A}_\gamma$-convergence, then the product topology on $\prod_{\gamma\in\Gamma} E_\gamma$ is the topology of $\mathcal{A}'$-convergence, where $\mathcal{A}'$ is the set of all finite unions of sets of $\bigcup_{\gamma\in\Gamma} A'_\gamma$. In our case $\mathcal{A}_\gamma$ consists of all closed absolutely convex envelopes of compact sets in E_γ, which, as said above, are compact themselves, because of the completeness of E_γ. But the $\mathcal{A}_\gamma$-topology on E'_γ coincides with the κ-topology, because the passage to absolutely convex envelopes and their closures does not change the topology of compact convergence (see [1, Chapter III, § 3]). On the other hand, by the Tychonov theorem, the product $\prod_{\gamma\in\Gamma} A_\gamma$ of compact sets $A_\gamma \in \mathcal{A}_\gamma$ is compact and the family $\mathcal{A}$ is cofinal in the set of all compact sets in E, the $\mathcal{A}$-topology on E' also coincides with the κ-topology, because the $\mathcal{A}$-topology is not changed under the passage to the closures of all subsets of the elements in $\mathcal{A}$ (see [1, Chapter III, § 3]). On the other hand, $\mathcal{A}'$ consists of various compact sets in E'_γ. Since the family $\mathcal{A}'$ is cofinal in the set of all compact subsets in E' (in a separated topological direct sum the closed set A is compact if and only if it is contained in a finite sum of compact subsets of the E'_γ, see [18, Chapter V, Corollary to Proposition 24, p. 93]), the $\mathcal{A}'$-topology on E' also coincides with the κ-topology, because the $\mathcal{A}$-topology is not changed under the passage to closures of all subsets of the elements in $\mathcal{A}$ (see [1, Chapter III, § 3]). At last, since a topological product of Mackey spaces is a Mackey space (see [19, Chapter IV, Corollary 2 to Theorem 4.3, p. 137]) and the topological sum of separated complete spaces is a complete space (see [18, Chapter V, Proposition 23, p. 92]), the family of all closed equicontinuous sets in E', as we have seen above, coincides with the family of all compact sets in E', $E = (E')'_\kappa$. Evidently, $E' = ((E')'_\kappa)'_\kappa$, if one endows E' with the topology of direct sum. ∎

THEOREM 5.4. *Let $E = \sum_{\gamma\in\Gamma} E_\gamma$ be the topological sum of F-spaces E_γ, $\gamma \in \Gamma$. Then E is κ-reflexive and its dual space E', endowed with the topology of compact convergence, coincides with the topology of the direct product $\prod_{\gamma\in\Gamma} E'_\gamma$ of spaces E'_γ, $\gamma \in \Gamma$, endowed with the topology of compact convergence.*

PROOF. By the same two theorems, used in the proof of the previous theorem, the topological sum E and the topological product E' are dual to each other (one has to change only the place of primes in the formulae). In our case $\mathcal{A}_\gamma$ consists of various compact sets in E_γ. And since the family $\mathcal{A}$ is cofinal in the set of all compact subsets in E (in a separated topological sum a set A is compact if and only if it is contained in a finite sum of compact subsets of E_γ, see [18, Chapter V, Corollary to Proposition 24, p. 93]), the $\mathcal{A}$-topology on E' also coincides with the κ-topology, because the $\mathcal{A}$-topology is not changed under the passage to closures of all subsets of the elements in $\mathcal{A}$ (see [1, Chapter III, § 3]). On the other hand, $\mathcal{A}'_\gamma$ consists of various closed absolutely convex envelopes of compact sets in E'_γ, which, as said above, are compact themselves because of completeness of E'_γ.

At last, since the topological sum of Mackey spaces is a Mackey space (see [19, Chapter IV, Corollary 2 to Theorem 4.3, p. 138]) and the topological product of separated complete spaces is a complete space (see [18, Chapter V, Corollary 1 to Proposition 18, p. 88]), the family of all closed equicontinuous sets in E' coincides, as above, with the set of all compact sets in E', $E = (E')'_\kappa$. Evidently, $E' = ((E')'_\kappa)'_\kappa$, if one endows E with the topology of direct sum.

□

THEOREM 5.5. *Let $E = \lim\limits_{\to n} E_n$ be the strict inductive limit of a sequence of F-spaces. Then E is κ-reflexive and its dual space E', endowed with the topology of compact convergence, is the inverse limit $\lim\limits_{\leftarrow n} E'_n$, endowed with the projective topology.*

PROOF. It is known (see [18, Chapter V, Proposition 15, p. 85]) that the dual of the separated inductive limit $E = \lim\limits_{\to \gamma}(E_\gamma, u_{\gamma\gamma'}, \Gamma)$ is the projective limit $E' = \lim\limits_{\leftarrow \gamma}(E'_\gamma, u'_{\gamma\gamma'}, \Gamma)$ of the dual spaces E'_γ of E_γ. Moreover, if each E_γ is a separated space and each E'_α has the topology of $\mathcal{A}_\gamma$-convergence, then the projective topology of the inverse limit $\lim\limits_{\leftarrow \gamma}(E'_\gamma, u'_{\gamma\gamma'}, \Gamma)$ is the topology of $\mathcal{A}$-convergence, where $\mathcal{A}$ is the set of finite unions of sets of $\bigcup\limits_{\gamma\in\Gamma} u_\gamma(A_\gamma)$.

In our case, the strict inductive limit of a sequence of separated spaces is separated itself and $\mathcal{A}_n$ consists of various compact sets in E_n. And since the family $\mathcal{A}$ is cofinal in the set of all compact subsets in E (in the strict inductive limit a set A is compact if and only if it is contained in one of the spaces E_n, for some n, see [18, Chapter VII, Propositions 2, 4, pp. 128, 129]), the $\mathcal{A}$-topology on E' also coincides with the κ-topology.

It is also known (see [19, Chapter IV, Proposition 4.4, p. 139]) that if $E = \lim\limits_{\leftarrow \gamma}(E_\gamma, u_{\gamma\gamma'}, \Gamma)$ is a reduced projective limit (i.e., for each γ, the projection $u_\gamma(E)$ is dense in E_γ) of locally convex spaces, then the dual E', under its Mackey topology $\tau(E', E)$, can be identified with the inductive limit $E' = \lim\limits_{\to \gamma}(E'_\gamma, u'_{\gamma\gamma'}, \Gamma)$ of the family $(E'_\gamma, u'_{\gamma\gamma'}, \Gamma)$ with respect to the dual mappings $u'_{\gamma\gamma'}$ of $u_{\gamma\gamma'}$. In our case, since $u_{nn+1} : E_n \to E_{n+1}$ is a homomorphism, for each $n \in \mathbb{N}$, (from the assumption of strict inductive limit), the dual mapping $u'_{nn+1} : E'_{n+1} \to E'_n$ is surjective and hence, $E' = \lim\limits_{\leftarrow n} E'_n$ is a reduced limit, consequently, the dual of E' can be identified with the inductive limit E with the Mackey topology.

Since the strict inductive limit of Mackey spaces is a Mackey space (see [19, Chapter IV, Corollary 2 to Theorem 4.3, p. 138]) and the projective limit of separated complete spaces is a complete space (see [19, Chapter II, Proposition 5.3, p. 52]) and the family of all closed equicontinuous sets in E' coincides, as above, with the family of all compact sets in E', $E = (E')'_\kappa$. Evidently, $E' = ((E')'_\kappa)'_\kappa$, if one considers the projective topology in E'. This completes the proof of Theorem 5.5. Notice, that the strict inductive limit of

F-spaces is a complete space (see [18, Chapter VII, Proposition 3, p. 128]), therefore, in our case, E is a complete space. □

REMARK 5.6. From the proofs of Theorems 8, 9 and 10 one can see that F-spaces E_γ in these theorems can be replaced by complete κ-reflexive Mackey spaces E_γ such that E'_γ have also a complete Mackey topology $\tau(E'_\gamma, E_\gamma)$, in particular, the topology of compact convergence.

REMARK 5.7. As to hereditary κ-reflexivity as well as to dual hereditary κ-reflexivity, it was noted in Remark 5.2, since the topology of compact convergence is inherited by the quotients E/L and E'/L^o, respectively, that we can identify $E/L = ((E/L)'_\kappa)'_\kappa$ and $E'/L^o = ((E'/L^o)'_\kappa)'_\kappa$. For the identification $L = (L'_\kappa)'_\kappa$ and $L^o = ((L^o)'_\kappa)'_\kappa$ it is necessary and sufficient that every compact set in E/L be the canonical image of a compact set in E under the canonical mapping E onto E/L and every compact set in E'/L^o is the canonical image of a compact set in E' under the canonical mapping E' onto E'/L^o. For example, this is the case for complete barrelled nuclear spaces E, whose strongly dual spaces E' are also complete and nuclear.

REMARK 5.8. Notice, that every locally convex topological vector space E is reflexive in the weak topology, i.e., $E_\sigma = ((E_\sigma)'_\sigma)'_\sigma$. Moreover, E_σ is dually hereditarily reflexive, because the weak topology is inherited by closed subspaces an separated quotients. We can say the same about the Mackey topology, i.e., $E_\tau = ((E_\tau)'_\tau)'_\tau$. However, in general, E_τ is neither hereditarily τ-reflexive, nor dually hereditarily τ-reflexive, because the Mackey topology is inherited by separated quotients, but in general, it is not inherited by closed subspaces. If E is a metrizable space, then clearly, it is hereditarily τ-reflexive, but it is not dually hereditarily τ-reflexive. On the other hand, every infinite-dimensional vector space E can be endowed with a locally convex topology such that it is not κ-reflexive, precisely, endow E with the finest locally convex topology ω and then consider it with the weak topology $\sigma(E, E^*)$. Then $E_\sigma \neq ((E_\sigma)'_\beta)'_\beta = ((E_\sigma)'_\kappa)'_\kappa$. The latter equality is fulfilled, because, as it was already noticed, for Montel spaces the strong topology coincides with the topology of compact convergence.

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