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THE STRUCTURES OF STANDARD $(\mathfrak{g}, K)$-MODULES OF $SL(3, \mathbf{R})$

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Abstract. We describe explicitly the structures of standard $(\mathfrak{g}, K)$-modules of $SL(3, \mathbf{R})$.

1. Introduction

For an admissible representation of a real reductive Lie group, the $(\mathfrak{g}, K)$-module structure is a fundamental data of those. As far as we know, for some 'small' reductive Lie groups G, the $(\mathfrak{g}, K)$-module structures of standard representations are completely described. For example, the description of them for $SL(2, \mathbf{R})$ is found in standard textbooks, and there are rather complete results for some groups of real rank 1, e.g. $SU(n, 1)$ by Kraljević [5] and $Spin(1, 2n)$ by Thieleker [12]. Moreover, in recent years, many authors give the explicit description of degenerate principal series representations of several groups, e.g. Fujimura [2], Howe and Tan [4], Lee [6], Lee and Loke [7]. However, for standard representations of Lie groups of higher rank, there are few references as far as the author knows. It seems to be difficult to describe the whole $(\mathfrak{g}, K)$-module structures of those representations, since their K-types are not multiplicity free. In the paper [11], the $(\mathfrak{g}, K)$-module structures of principal series representations of $Sp(2, \mathbf{R})$ are described by Oda. In a former paper [10], we extend the result for principal series representations of $Sp(3, \mathbf{R})$. The method in these papers is applicable to study of standard representations of other groups. In this paper, we use this method to study standard $(\mathfrak{g}, K)$-modules of $SL(3, \mathbf{R})$.

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Before describing the case of $SL(3,\mathbf{R})$, let us explain the problem in a more precise form for a general real semisimple Lie group G with its Lie algebra $\mathfrak{g}$. Fix a maximal compact subgroup K of G. Since standard $(\mathfrak{g},K)$-modules are realized as subspaces of $L^2(K)$ as a K-module, we can investigate those K-module structure by Peter-Weyl's theorem. In order to describe the action of $\mathfrak{g}$ or $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g}\otimes_{\mathbf{R}}\mathbf{C}$, it suffices to investigate the action of $\mathfrak{p}$ or $\mathfrak{p}_{\mathbf{C}}$, because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$. To study the action of $\mathfrak{p}_{\mathbf{C}}$, we compute the linear map $\Gamma_{\tau,i}$ defined as follows.

Let (π, H_π) be a standard representation of G with its subspace $H_{\pi,K}$ of K-finite vectors. For a K-type (τ, V_τ) of π and a K-homomorphism $\eta\colon V_\tau \to H_{\pi,K}$, we define a linear map $\tilde{\eta}\colon \mathfrak{p}_{\mathbf{C}}\otimes_{\mathbf{C}} V_\tau \to H_{\pi,K}$ by $X\otimes v \mapsto X\cdot\eta(v)$. Then $\tilde{\eta}$ is a K-homomorphism with $\mathfrak{p}_{\mathbf{C}}$ endowed with the adjoint action Ad of K. Let $V_\tau\otimes_{\mathbf{C}}\mathfrak{p}_{\mathbf{C}} \simeq \bigoplus_{i\in I} V_{\tau_i}$ be the decomposition into a direct sum of irreducible K-modules and fix ι_i an injective K-homomorphism from V_{τ_i} to $V_\tau\otimes_{\mathbf{C}}\mathfrak{p}_{\mathbf{C}}$ for each i. We define a linear map $\Gamma_{\tau,i}\colon \mathrm{Hom}_K(V_\tau, H_{\pi,K}) \to \mathrm{Hom}_K(V_{\tau_i}, H_{\pi,K})$ by $\eta \mapsto \tilde{\eta}\circ\iota_i$. These linear maps $\Gamma_{\tau,i}$ $(i\in I)$ characterize the action of $\mathfrak{p}_{\mathbf{C}}$. The goal of this paper is to give explicit expressions of ι_i and $\Gamma_{\tau,i}$ for any standard representation π of $G = SL(3,\mathbf{R})$. As a result, we obtain infinite number of 'contiguous relations', a kind of system of differential-difference relations among vectors in $H_\pi[\tau]$ and $H_\pi[\tau_i]$. Here $H_\pi[\tau]$ is τ-isotypic component of H_π. These are described in Proposition 3.2, Theorem 4.5 and 5.5. We remark that R. Howe give another description of $\Gamma_{\tau,i}$ in [3] when π is a principal series representation of $GL(3,\mathbf{R})$.

As an application, we can utilize the contiguous relations to obtain the explicit formulae of some spherical functions. In the paper [8], Manabe, Ishii and Oda give the explicit formulae of Whittaker functions for principal series representations of $SL(3,\mathbf{R})$ to solve the holonomic system of differential equations characterizing those functions, which is derived from the Capelli elements and the contiguous relations around minimal K-type. We can obtain the holonomic systems characterizing Whittaker functions for standard representations of $SL(3,\mathbf{R})$ induced from the maximal parabolic subgroup by using the result of this paper. We give the explicit formulae of Whittaker functions by solving this system in [9]. On the other hand, if we have the explicit formula of Whittaker function with a certain K-type, then we can give those with another K-type by using contiguous relations.

We give the contents of this paper. In Section 2, we recall the structure of $SL(3,\mathbf{R})$ and define standard representations. In Section 3, we introduce the standard basis of a finite dimensional irreducible representation of K and give explicit expressions of $\iota_i\colon V_{\tau_i} \to V_\tau\otimes_{\mathbf{C}}\mathfrak{p}_{\mathbf{C}}$. In Section 4, we introduce the general setting of this paper and give matrix representations of $\Gamma_{\tau,i}$ for principal series representations in Theorem 4.5. In Section 5, we give the matrix representations of $\Gamma_{\tau,i}$ for standard representations of $SL(3,\mathbf{R})$ induced from the maximal parabolic subgroup in Theorem 5.5. In Section 6, we give explicit expressions of the action of $\mathfrak{p}_{\mathbf{C}}$ in Proposition 6.2.

2. Preliminaries

2.1. *Groups and algebras.* We denote by $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ the ring of rational integers, the real number field and the complex number field, respectively. Let $\mathbf{Z}_{\geq 0}$ be the set of non-negative integers, 1_n the unit matrix of size n and $O_{m,n}$ the zero matrix of size $m \times n$ and E_{ij} the matrix of size 3 with 1 at (i, j)-th entry and 0 at other entries. We denote by δ_{ij} the Kronecker delta, i.e.

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

For a Lie algebra $\mathfrak{l}$, we denote by $\mathfrak{l}_{\mathbf{C}} = \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$ the complexification of $\mathfrak{l}$.

Let G be the special linear group $SL(3, \mathbf{R})$ of degree three and $\mathfrak{g}$ its Lie algebra. We define a Cartan involution θ of G by $G \ni g \mapsto {}^t g^{-1} \in G$. Here ${}^t g$ and g^{-1} means the transpose and the inverse of g, respectively. Then the maximal compact subgroup of G is given by

$$K = \{g \in G \mid \theta(g) = g\} = SO(3).$$

If we denote the differential of θ again by θ, then we have $\theta(X) = -{}^t X$ for $X \in \mathfrak{g}$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and the -1 eigenspaces of θ in $\mathfrak{g}$, respectively, that is,

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid {}^t X = -X\} = \mathfrak{so}(3), \qquad \mathfrak{p} = \{X \in \mathfrak{g} \mid {}^t X = X\}.$$

Then $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{g}$ has the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Put $\mathfrak{a}_0 = \{\mathrm{diag}(t_1, t_2, t_3) \mid t_i \in \mathbf{R}\ (1 \leq i \leq 3),\ t_1 + t_2 + t_3 = 0\}$. Then $\mathfrak{a}_0$ is a maximal abelian subalgebra of $\mathfrak{p}$. For each $1 \leq i \leq 3$, we define a linear form e_i on $\mathfrak{a}_0$ by $\mathfrak{a}_0 \ni \mathrm{diag}(t_1, t_2, t_3) \mapsto t_i \in \mathbf{C}$. The set Σ of the roots for $(\mathfrak{a}_0, \mathfrak{g})$ is given by $\Sigma = \Sigma(\mathfrak{a}_0, \mathfrak{g}) = \{e_i - e_j \mid 1 \leq i \neq j \leq 3\}$, and the subset $\Sigma^+ = \{e_i - e_j \mid 1 \leq i < j \leq 3\}$ forms a positive root system. For each $\alpha \in \Sigma$, we denote the root space by $\mathfrak{g}_\alpha$ and choose a root vector E_α in $\mathfrak{g}_\alpha$ by $E_{e_i - e_j} = E_{ij}$ $(1 \leq i \neq j \leq 3)$.

If we put $\mathfrak{n}_0 = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$, then $\mathfrak{g}$ has an Iwasawa decomposition $\mathfrak{g} = \mathfrak{n}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{k}$. Also we have $G = N_0 A_0 K$, where $N_0 = \exp(\mathfrak{n}_0)$ and $A_0 = \exp(\mathfrak{a}_0)$.

Let $\mathfrak{n}_1,\ \mathfrak{n}_2$ be subalgebras of $\mathfrak{n}_0$ defined by $\mathfrak{n}_1 = \mathfrak{g}_{e_1 - e_2} \oplus \mathfrak{g}_{e_1 - e_3},\ \mathfrak{n}_2 = \mathfrak{g}_{e_1 - e_3} \oplus \mathfrak{g}_{e_2 - e_3}$. We take a basis $\{H_1, H_2\}$ of $\mathfrak{a}_0$ by

$$H_1 = \mathrm{diag}(1, 0, -1), \qquad H_2 = \mathrm{diag}(0, 1, -1),$$

and set $H^{(1)} = 2H_1 - H_2,\ H^{(2)} = H_1 + H_2$. we define subalgebras $\mathfrak{a}_1,\ \mathfrak{a}_2$ of $\mathfrak{a}_0$ by $\mathfrak{a}_1 = \mathbf{R} \cdot H^{(1)},\ \mathfrak{a}_2 = \mathbf{R} \cdot H^{(2)}$. The group G has three non-trivial standard parabolic subgroups $P_0,\ P_1,\ P_2$ with Langland decompositions $P_i = N_i A_i M_i$ $(0 \leq i \leq 2)$ where

$$M_0 = \{\mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1 \varepsilon_2) \mid \varepsilon_i \in \{\pm 1\}\ (1 \leq i \leq 2)\},$$

$$M_1 = \left\{ \begin{pmatrix} \det(h)^{-1} & O_{1,2} \\ O_{2,1} & h \end{pmatrix} \middle| h \in SL^{\pm}(2,\mathbf{R}) \right\},$$
$$M_2 = \left\{ \begin{pmatrix} h & O_{2,1} \\ O_{1,2} & \det(h)^{-1} \end{pmatrix} \middle| h \in SL^{\pm}(2,\mathbf{R}) \right\},$$
$$A_i = \exp(\mathfrak{a}_i) \quad N_i = \exp(\mathfrak{n}_i) \quad (i = 1, 2).$$

Here $SL^{\pm}(2,\mathbf{R}) = \{g \in GL(2,\mathbf{R}) \mid \det(g) = \pm 1\}$. For $i = 1, 2$, let $\mathfrak{m}_i$ be a Lie algebra of M_i.

2.2. *Definition of the P_i-principal series representations of G.* For $0 \leq i \leq 2$, in order to define the P_i-principal series representation of G, we prepare the data (ν_i, σ_i) as follows.

For $\nu_0 \in \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}_0, \mathbf{C})$, we define a coordinate $(\nu_{0,1}, \nu_{0,2}) \in \mathbf{C}^2$ by $\nu_{0,i} = \nu_0(H_i)$ $(i = 1, 2)$. Then the half sum $\rho_0 = e_1 - e_3$ of the positive roots has coordinate $(\rho_{0,1}, \rho_{0,2}) = (2, 1)$. We define a quasicharacter $e^{\nu_0} \colon A_0 \to \mathbf{C}^{\times}$ by

$$e^{\nu_0}(a) = a_1^{\nu_{0,1}} a_2^{\nu_{0,2}}, \quad a = \mathrm{diag}(a_1, a_2, a_3) \in A_0.$$

We fix a character σ_0 of M_0. σ_0 is realized by $(\sigma_{0,1}, \sigma_{0,2}) \in \{0, 1\}^{\oplus 2}$ such that

$$\sigma_0(\mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1\varepsilon_2)) = \varepsilon_1^{\sigma_{0,1}} \varepsilon_2^{\sigma_{0,2}}, \quad \varepsilon_1, \varepsilon_2 \in \{\pm 1\}.$$

For each $i = 1, 2$, we identify $\nu_i \in \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}_i, \mathbf{C})$ with a complex number $\nu_i(H^{(i)}) \in \mathbf{C}$. Let ρ_i $(i = 1, 2)$ be the half sums of positive roots whose root spaces are contained in $\mathfrak{n}_i$, i.e. $\rho_1 = \frac{1}{2}(2e_1 - e_2 - e_3)$, $\rho_2 = \frac{1}{2}(e_1 + e_2 - 2e_3)$. Then both ρ_1 and ρ_2 are identified with 3. We fix a discrete series representation σ_i of $M_i \simeq SL^{\pm}(2,\mathbf{R})$ for $i = 1, 2$.

Definition 2.1. *For $0 \leq i \leq 2$, we define the P_i-principal series representation $\pi_{(\nu_i,\sigma_i)}$ of G by*

$$\pi_{(\nu_i,\sigma_i)} = \mathrm{Ind}_{P_i}^{G}(1_{N_i} \otimes e^{\nu_i + \rho_i} \otimes \sigma_i),$$

i.e. $\pi_{(\nu_i,\sigma_i)}$ is the right regular representation of G on the space $H_{(\nu_i,\sigma_i)}$ which is the completion of

$$H^{\infty}_{(\nu_i,\sigma_i)} = \left\{ f \colon G \to V_{\sigma_i} \text{ smooth} \;\middle|\; \begin{array}{l} f(namx) = e^{\nu_i + \rho_i}(a)\sigma_i(m)f(x) \\ \text{for } n \in N_i,\ a \in A_i,\ m \in M_i,\ x \in G \end{array} \right\}$$

with respect to the norm

$$\|f\|^2 = \int_K \|f(k)\|^2_{\sigma_i} dk.$$

Here V_{σ_i} is a representation space of σ_i and $\|\cdot\|_{\sigma_i}$ is its norm.

Remark 2.2. *The P_i-principal series representations are also called standard representations or generalized principal representations.*

3. Representations of $K = SO(3)$

3.1. *The spinor covering.* To describe the finite dimensional representations of $SO(3)$, the simplest way seems to be the one utilizing the double covering $\varphi\colon SU(2) = Spin(3) \to SO(3)$. We use the following realization introduced in [8].

We define $\varphi\colon SU(2) \to SO(3)$ by

$$\varphi(x) = \begin{pmatrix} p^2+q^2-r^2-s^2 & -2(ps-qr) & 2(pr+qs) \\ 2(ps+qr) & p^2-q^2+r^2-s^2 & -2(pq-rs) \\ -2(pr-qs) & 2(pq+rs) & p^2-q^2-r^2+s^2 \end{pmatrix}$$

for $x = \begin{pmatrix} p+\sqrt{-1}q & r+\sqrt{-1}s \\ -r+\sqrt{-1}s & p-\sqrt{-1}q \end{pmatrix} \in SU(2)$ $(p, q, r, s \in \mathbf{R})$. Then φ is surjective homomorphism whose kernel is given by $\{\pm 1_2\}$.

The differential $d\varphi\colon \mathfrak{su}(2) \to \mathfrak{so}(3)$ of φ is an isomorphism and it maps the basis

$$u_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

to $-2K_{23},\ 2K_{13}, -2K_{12}$. Here $K_{ij} = E_{ij} - E_{ji}$ $(1 \le i < j \le 3)$.

3.2. *Representations of* $SU(2)$. The set of equivalence classes of the finite dimensional continuous representations of $SU(2)$ is exhausted by the symmetric tensor product τ_l $(l \in \mathbf{Z}_{\ge 0})$ of the representation $SU(2) \ni g \mapsto (v \mapsto g \cdot v) \in GL(\mathbf{C}^2)$. We use the following realizations of those which are introduced in [8].

Let V_l be the subspace consisting of degree l homogeneous polynomials of two variables x, y in the polynomial ring $\mathbf{C}[x, y]$. For $g \in SU(2)$ with $g^{-1} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$ and $f(x, y) \in V_l$ we set

$$\tau_l(g) f(x, y) = f(ax + by, -\bar{b}x + \bar{a}y).$$

Passing to the Lie algebra $\mathfrak{su}(2)$, the derivation of τ_l, denoted by same symbol, is described as follows by using the standard basis $\{v_k = x^k y^{l-k} \mid 0 \le k \le l\}$ of V_l and the basis $\{u_1, u_2, u_3\}$ of $\mathfrak{su}(2)$. Namely, we have

$$\tau_l(H)v_k = (l-2k)v_k, \qquad \tau_l(E)v_k = -kv_{k-1}, \qquad \tau_l(F)v_k = (k-l)v_{k+1}.$$

Here $\{E,\ H,\ F\}$ is $\mathfrak{sl}_2$-triple defined by

$$H = -\sqrt{-1}u_1,\ E = \frac{1}{2}(u_2 - \sqrt{-1}u_3),\ F = -\frac{1}{2}(u_2 + \sqrt{-1}u_3) \in \mathfrak{su}(2)_{\mathbf{C}}.$$

The condition that τ_l defines a representation of $SO(3)$ by passing to the quotient with respect to $\varphi\colon SU(2) \to SO(3)$ is that $\tau_l(-1_2) = (-1)^l = 1$, i.e. l is even. For $l \in \mathbf{Z}_{\ge 0}$, we denote the irreducible representation of $SO(3)$ induced from (τ_{2l}, V_{2l}) again by (τ_{2l}, V_{2l}).

3.3. *The adjoint representation of K on $\mathfrak{p}_{\mathbf{C}}$.* It is known that $\mathfrak{p}_{\mathbf{C}}$ becomes a K-module via the adjoint action of K. Concerning this, we have the following lemma.

LEMMA 3.1. *Let $\{w_j \mid 0 \le j \le 4\}$ be the standard basis of (τ_4, V_4) and $\{X_j \mid 0 \le j \le 4\}$ be a basis of $\mathfrak{p}_{\mathbf{C}}$ defined as follows:*

$$X_0 = H_2 - \sqrt{-1}(E_{23} + E_{32}), \quad X_1 = -\frac{1}{2}\{\sqrt{-1}(E_{12} + E_{21}) + (E_{13} + E_{31})\},$$
$$X_2 = -\frac{1}{3}(2H_1 - H_2), \qquad X_3 = -\frac{1}{2}\{\sqrt{-1}(E_{12} + E_{21}) - (E_{13} + E_{31})\},$$
$$X_4 = H_2 + \sqrt{-1}(E_{23} + E_{32}).$$

Then via the isomorphism between V_4 and $\mathfrak{p}_{\mathbf{C}}$ as K-modules we have the identification $w_j = X_j$ $(0 \le j \le 4)$.

PROOF. By direct computation, we have Table. 1 of the adjoint actions of the basis $\{d\varphi(E),\ d\varphi(H),\ d\varphi(F)\}$ of $\mathfrak{k}_{\mathbf{C}}$ on the basis $\{X_j \mid 0 \le j \le 4\}$ of $\mathfrak{p}_{\mathbf{C}}$. Comparing the actions in the above with the actions in Subsection 3.2, we obtain the assertion. ∎

TABLE 1. The adjoint actions of $\mathfrak{k}_{\mathbf{C}}$ on $\mathfrak{p}_{\mathbf{C}}$.

	X_0	X_1	X_2	X_3	X_4
$d\varphi(H)$	$4X_0$	$2X_1$	0	$-2X_3$	$-4X_4$
$d\varphi(E)$	0	$-X_0$	$-2X_1$	$-3X_2$	$-4X_3$
$d\varphi(F)$	$-4X_1$	$-3X_2$	$-2X_3$	$-1X_4$	0

3.4. *Clebsch-Gordan coefficients for the representations of $\mathfrak{sl}(2, \mathbf{C})$ with respect to standard basis.* For later use, we consider the irreducible decomposition of $V_l \otimes_{\mathbf{C}} V_4$ as $\mathfrak{sl}(2, \mathbf{C}) = \mathfrak{su}(2)_{\mathbf{C}}$-modules for arbitrary non-negative integer l.

Generically, the tensor product $V_l \otimes_{\mathbf{C}} V_4$ has five irreducible components V_{l+4}, V_{l+2}, V_l, V_{l-2} and V_{l-4}. Here some components may vanish. We give an explicit expression of a nonzero $\mathfrak{sl}(2, \mathbf{C})$-homomorphism from each irreducible component to $V_l \otimes_{\mathbf{C}} V_4$ as follows.

PROPOSITION 3.2. *Let $\{v_k^{(l)} \mid 0 \le k \le l\}$ be the standard basis of V_l for $l \in \mathbf{Z}_{\ge 0}$. We put $v_k^{(l)} = 0$ when $k < 0$ or $k > l$.*

If V_{l+2m}-component of $V_l \otimes_{\mathbf{C}} V_4$ does not vanish, then we define linear maps $I^l_{2m} \colon V_{l+2m} \to V_l \otimes_{\mathbf{C}} V_4$ $(-2 \le m \le 2)$ by

$$I^l_{2m}(v_k^{(l+2m)}) = \sum_{i=0}^{4} A_{[l,2m;k,i]} \cdot v^{(l)}_{k+2-m-i} \otimes w_i.$$

Here the coefficients $A_{[l,2m;k,i]} = a(l, 2m; k, i)/d(l, 2m)$ *are defined by following formulae.*

FORMULA 1: *The coefficients of* $I_4^l\colon V_{l+4} \to V_l \otimes_{\mathbf{C}} V_4$ *are given as follows:*

$$\begin{aligned} a(l,4;k,0) =&(l+4-k)(l+3-k)(l+2-k)(l+1-k),\\ a(l,4;k,1) =&4(l+4-k)(l+3-k)(l+2-k)k,\\ a(l,4;k,2) =&6(l+4-k)(l+3-k)k(k-1),\\ a(l,4;k,3) =&4(l+4-k)k(k-1)(k-2),\\ a(l,4;k,4) =&k(k-1)(k-2)(k-3),\\ d(l,4) =&(l+4)(l+3)(l+2)(l+1). \end{aligned}$$

FORMULA 2: *The coefficients of* $I_2^l\colon V_{l+2} \to V_l \otimes_{\mathbf{C}} V_4$ *are given as follows:*

$$\begin{aligned} a(l,2;k,0) =&(l+2-k)(l+1-k)(l-k),\\ a(l,2;k,1) =&-(l+2-k)(l+1-k)(l-4k),\\ a(l,2;k,2) =&-3(l+2-k)(l-2k+2)k,\\ a(l,2;k,3) =&-(3l-4k+8)k(k-1),\\ a(l,2;k,4) =&-k(k-1)(k-2), \quad d(l,2) = (l+2)(l+1)l. \end{aligned}$$

FORMULA 3: *The coefficients of* $I_0^l\colon V_l \to V_l \otimes_{\mathbf{C}} V_4$ *are given as follows:*

$$\begin{aligned} &a(l,0;k,0) =(l-k)(l-1-k), && a(l,0;k,1) =-2(l-k)(l-2k-1),\\ &a(l,0;k,2) =(l^2-6kl+6k^2-l), && a(l,0;k,3) =2(l-2k+1)k,\\ &a(l,0;k,4) =k(k-1), && d(l,0) =l(l-1). \end{aligned}$$

FORMULA 4: *The coefficients of* $I_{-2}^l\colon V_{l-2} \to V_l \otimes_{\mathbf{C}} V_4$ *are given as follows:*

$$\begin{aligned} &a(l,-2;k,0) =(l-k-2), && a(l,-2;k,1) =-(3l-4k-6),\\ &a(l,-2;k,2) =3(l-2k-2), && a(l,-2;k,3) =-(l-4k-2),\\ &a(l,-2;k,4) =-k, && d(l,-2) =l-2. \end{aligned}$$

FORMULA 5: *The coefficients of* $I_{-4}^l\colon V_{l-4} \to V_l \otimes_{\mathbf{C}} V_4$ *are given as follows:*

$$\begin{aligned} &a(l,-4;k,0) =1, && a(l,-4;k,1) =-4, && a(l,-4;k,2) =6,\\ &a(l,-4;k,3) =-4, && a(l,-4;k,4) =1, && d(l,-4) =1. \end{aligned}$$

Then I_{2m}^l *is a generator of* $\mathrm{Hom}_{\mathfrak{sl}(2,\mathbf{C})}(V_{l+2m}, V_l \otimes_{\mathbf{C}} V_4)$*, which is unique up to scalar multiple.*

PROOF. We have

$$\begin{aligned} &(\tau_l \otimes \tau_4)(E) \circ I_{2m}^l(v_0^{(l+2m)})\\ &= \sum_{i=0}^{4} A_{[l,2m;0,i]} \cdot (\tau_l(E) v_{2-m-i}^{(l)}) \otimes w_i + \sum_{i=0}^{4} A_{[l,2m;0,i]} \cdot v_{2-m-i}^{(l)} \otimes (\tau_4(E) w_i) \end{aligned}$$

$$= -\sum_{i=0}^{4}((2-m-i)A_{[l,2m;0,i]} + (i+1)A_{[l,2m;0,i+1]}) \cdot v^{(l)}_{1-m-i} \otimes w_i.$$

Here we put $A_{[l,2m;0,5]} = 0$. By direct computation, we confirm

$$(2-m-i)A_{[l,2m;0,i]} + (i+1)A_{[l,2m;0,i+1]} = 0$$

for $-2 \leq m \leq 2$ and $0 \leq i \leq 4$. Hence

$$(\tau_l \otimes \tau_4)(E) \circ I^l_{2m}(v^{(l+2m)}_0) = 0.$$

Moreover, we have

$$(\tau_l \otimes \tau_4)(H) \circ I^l_{2m}(v^{(l+2m)}_0) = (l+2m)I^l_{2m}(v^{(l+2m)}_0),$$

since

$$\begin{aligned}(\tau_l \otimes \tau_4)(H)(v^{(l)}_i \otimes w_j) &= (\tau_l(H)v^{(l)}_i) \otimes w_j + v^{(l)}_i \otimes (\tau_4(H)w_j)\\ &= (l+4-2i-2j)v^{(l)}_i \otimes w_j.\end{aligned}$$

This means $I^l_{2m}(v^{(l+2m)}_0)$ is the highest weight vector of the V_{l+2m}-component of $V_l \otimes_{\mathbf{C}} V_4$ with respect to a Borel subalgebra $(\mathbf{C}\cdot H) \oplus (\mathbf{C}\cdot E)$ of $\mathfrak{sl}(2,\mathbf{C})$.

Therefore, in order to complete the proof, it suffices to confirm

$$(\tau_l \otimes \tau_4)(F) \circ I^l_{2m}(v^{(l+2m)}_k) = I^l_{2m} \circ \tau_{l+2m}(F)(v^{(l+2m)}_k)$$

for each $0 \leq k \leq l+2m$.

We confirm these equations by direct computation. □

The coefficients $A_{[l,2m;k,i]}$ in the above proposition satisfy the following relations.

LEMMA 3.3. *The coefficients $A_{[l,2m;k,i]}$ in Proposition 3.2 satisfy following relations:*

$$\begin{aligned}&A_{[l,2m;l+2m-k,0]} = (-1)^m A_{[l,2m;k,4]}, \quad A_{[l,2m;l+2m-k,2]} = (-1)^m A_{[l,2m;k,2]},\\ &3\{(k-m+1)A_{[l,2m;k,1]} + (l-k+m+1)A_{[l,2m;k,3]}\}\\ &= (ml+m^2+m-6)A_{[l,2m;k,2]}.\end{aligned}$$

for $-2 \leq m \leq 2$ and $0 \leq k \leq l+2m$.

PROOF. These are obtained by direct computation. □

3.5. *The contragradient representation of (τ_l, V_l).* We denote by (τ^*, V^*) the contragradient representation of (τ, V). Here we note that V_l^* is equivalent to V_l as $SU(2)$-modules, since the irreducible $l+1$-dimensional representation of $SU(2)$ is unique up to isomorphism.

LEMMA 3.4. *Let* $\{v_k^{(l)*} \mid 0 \le k \le l\}$ *be the dual basis of the standard basis* $\{v_k^{(l)} \mid 0 \le k \le l\}$. *Via the isomorphism between* V_l *and* V_l^* *as* K*-modules we have the identification*

$$v_k^{(l)} = (-1)^k \frac{(l-k)!k!}{l!} v_{l-k}^{(l)*}$$

for $0 \le k \le l$.

PROOF. We denote by $\langle , \rangle$ the canonical pairing on $V_l^* \otimes_{\mathbf{C}} V_l$.
Since

$$\langle \tau_l^*(H) v_k^{(l)*}, v_m^{(l)} \rangle = -\langle v_k^{(l)*}, \tau_l(H) v_m^{(l)} \rangle = (2m-l)\delta_{km} = (2k-l)\delta_{km},$$

we have $\tau_l^*(H) v_k^{(l)*} = (2k-l) v_k^{(l)*}$. Similarly, we obtain

$$\tau_l^*(E) v_k^{(l)*} = (k+1) v_{k+1}^{(l)*}, \qquad \tau_l^*(F) v_k^{(l)*} = (l-k+1) v_{k-1}^{(l)*}.$$

From these equations, the identification $v_0^{(l)} = v_l^{(l)*}$ determines the isomorphism in the statement. □

4. THE $(\mathfrak{g}, K)$-MODULE STRUCTURES OF THE P_0-PRINCIPAL SERIES REPRESENTATIONS

4.1. *The irreducible decomposition of* $\pi_{(\nu_0,\sigma_0)}|_K$ *as a* K*-module.* We set

$$L^2_{(M_0,\sigma_0)}(K) = \{f \in L^2(K) \mid f(mx) = \sigma_0(m) f(x) \text{ for a.e. } m \in M,\ x \in K\}$$

and give a K-module structure by the right regular action of K. Then the restriction map $r_K \colon H_{(\nu_0,\sigma_0)} \ni f \mapsto f|_K \in L^2_{(M_0,\sigma_0)}(K)$ is an isomorphism of K-modules.

$L^2(K)$ has a $K \times K$-bimodule structure by the two sided regular action:

$$((k_1, k_2) f)(x) = f(k_1^{-1} x k_2), \quad x \in K,\ f \in L^2(K),\ (k_1, k_2) \in K \times K.$$

Then we define a homomorphism $\Phi_l \colon V_{2l}^* \otimes_{\mathbf{C}} V_{2l} \to L^2(K)$ of $K \times K$-bimodules by

$$w \otimes v \mapsto (x \mapsto \langle w, \tau_{2l}(x) v \rangle).$$

Then Peter-Weyl's theorem tells that

$$\widehat{\bigoplus_{l \in \mathbf{Z}_{\ge 0}}} \Phi_l \colon \widehat{\bigoplus_{l \in \mathbf{Z}_{\ge 0}}} V_{2l}^* \otimes_{\mathbf{C}} V_{2l} \to L^2(K)$$

is an isomorphism as $K \times K$-bimodules. Here $\widehat{\bigoplus}$ means a Hilbert space direct sum.

Since $L^2_{(M_0,\sigma_0)}(K) \subset L^2(K)$, we have an irreducible decomposition of $L^2_{(M_0,\sigma_0)}(K)$:

$$L^2_{(M_0,\sigma_0)}(K) \simeq \widehat{\bigoplus_{l \in \mathbf{Z}_{\ge 0}}} (V_{2l}^*[\sigma_0]) \otimes_{\mathbf{C}} V_{2l}.$$

Here $V[\sigma_0]$ means the σ_0-isotypic component in $(\tau|_{M_0}, V)$ for a K-module (τ, V). Therefore we obtain an isomorphism

$$r_K^{-1} \circ \widehat{\bigoplus_{l \in \mathbf{Z}_{\geq 0}}} \Phi_l \colon \widehat{\bigoplus_{l \in \mathbf{Z}_{\geq 0}}} (V_{2l}^*[\sigma_0]) \otimes_{\mathbf{C}} V_{2l} \to H_{(\nu_0,\sigma_0)}.$$

Since M_0 is generated by the two elements

$$m_{0,1} = \mathrm{diag}(-1, 1, -1), \qquad m_{0,2} = \mathrm{diag}(1, -1, -1) \in M_0,$$

we note that $v \in V_{2l}[\sigma_0]$ if and only if

$$\tau_{2l}(m_{0,i})v = \sigma_0(m_{0,i})v = (-1)^{\sigma_{0,i}} v \qquad (i = 1, 2)$$

for $v \in V_{2l}$. From the definition of (τ_{2l}, V_{2l}) and

$$\varphi_1^{-1}(m_{0,1}) = \left\{ \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \quad \varphi_1^{-1}(m_{0,2}) = \left\{ \pm \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} \right\},$$

we have $\tau_{2l}(m_{0,1}) v_k^{(2l)} = (-1)^k v_{2l-k}^{(2l)}$ and $\tau_{2l}(m_{0,2}) v_k^{(2l)} = (-1)^{l-k} v_k^{(2l)}$. Hence we have

$$V_{2l}[\sigma_0] = \bigoplus_{k \in Z(\sigma_0;l)} \mathbf{C} \cdot (v_{2l-k}^{(2l)} + (-1)^{\varepsilon(\sigma_0;l)} v_k^{(2l)}),$$

where $\varepsilon(\sigma_0; l) \in \{0, 1\}$ such that $\varepsilon(\sigma_0; l) \equiv l - \sigma_{0,1} - \sigma_{0,2} \bmod 2$ and

$$Z(\sigma_0; l) = \begin{cases} \{k \in \mathbf{Z} \mid 0 \leq k \leq l,\ k \equiv l - \sigma_{0,2} \bmod 2\} & \text{if } \varepsilon(\sigma_0; l) = 0, \\ \{k \in \mathbf{Z} \mid 0 \leq k \leq l-1,\ k \equiv l - \sigma_{0,2} \bmod 2\} & \text{if } \varepsilon(\sigma_0; l) = 1. \end{cases}$$

We see that $\{v_{2l-k}^{(2l)*} + (-1)^{\varepsilon(\sigma_0;l)} v_k^{(2l)*} \mid k \in Z(\sigma_0; l)\}$ is the basis of $V_{2l}^*[\sigma_0]$, by using the identification $V_{2l}^* = V_{2l}$ in Lemma 3.4.

Now we define the elementary function $s(l; p, q) \in H_{(\nu_0,\sigma_0)}$ by

$$s(l; p, q) = r_K^{-1} \circ \Phi_l((v_{2l-p}^{(2l)*} + (-1)^{\varepsilon(\sigma_0;l)} v_p^{(2l)*}) \otimes v_q^{(2l)})$$

for $l \in \mathbf{Z}_{\geq 0}$, $p \in Z(\sigma_0; l)$ and $0 \leq q \leq 2l$.

For each $p \in Z(\sigma_0; l)$, we put $S(l; p)$ a column vector of degree $2l+1$ whose $q+1$-th component is $s(l; p, q)$, i.e. ${}^t(\ s(l;p,0),\ s(l;p,1),\ \cdots\ , s(l;p,2l)\ )$.

Moreover we denote by $\langle S(l; p) \rangle$ the subspace of $H_{(\nu_0,\sigma_0)}$ generated by the functions in the entries of the vector $S(l; p)$, i.e. $\langle S(l; p) \rangle = \bigoplus_{q=0}^{2l} \mathbf{C} \cdot s(l; p, q) \simeq V_{2l}$. Via the isomorphism between $\langle S(l; p) \rangle$ and V_{2l}, we identify $\{s(l; p, q) \mid 0 \leq q \leq 2l\}$ with the standard basis.

From above arguments, we obtain the following.

Proposition 4.1. *As an unitary representation of K, it has an irreducible decomposition:*

$$H_{(\nu_0,\sigma_0)} \simeq \widehat{\bigoplus_{l \in \mathbf{Z}_{\geq 0}}} (V_{2l}^*[\sigma_0]) \otimes_{\mathbf{C}} V_{2l}.$$

Then the τ_{2l}-isotypic component of $\pi_{(\nu_0,\sigma_0)}$ is given by

$$\bigoplus_{p\in Z(\sigma_0;l)} \langle S(l;p)\rangle.$$

COROLLARY 4.2. *The multiplicity $d(\sigma_0;l)$ of τ_{2l} in $\pi_{(\nu_0,\sigma_0),K}$ is given by*

$$d(\sigma_0;l) = \begin{cases} (l+2)/2 & \text{if } (\sigma_{0,1},\sigma_{0,2}) = (0,0) \text{ and } l \text{ is even,} \\ (l-1)/2 & \text{if } (\sigma_{0,1},\sigma_{0,2}) = (0,0) \text{ and } l \text{ is odd,} \\ l/2 & \text{if } (\sigma_{0,1},\sigma_{0,2}) \neq (0,0) \text{ and } l \text{ is even,} \\ (l+1)/2 & \text{if } (\sigma_{0,1},\sigma_{0,2}) \neq (0,0) \text{ and } l \text{ is odd.} \end{cases}$$

4.2. *General setting.* Let $H_{(\nu_i,\sigma_i),K}$ be the K-finite part of $H_{(\nu_i,\sigma_i)}$. In order to describe the action of $\mathfrak{g}$ or $\mathfrak{g}_{\mathbf{C}} = \mathfrak{g}\otimes_{\mathbf{R}}\mathbf{C}$, it suffices to investigate the action of $\mathfrak{p}$ or $\mathfrak{p}_{\mathbf{C}}$, because of the Cartan decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$.

For a K-type (τ_{2l}, V_{2l}) of $\pi_{(\nu_i,\sigma_i)}$ and a K-homomorphism $\eta\colon V_{2l} \to H_{(\nu_i,\sigma_i),K}$, we define a linear map

$$\tilde{\eta}\colon \mathfrak{p}_{\mathbf{C}} \otimes_{\mathbf{C}} V_{2l} \to H_{(\nu_i,\sigma_i),K}$$

by $X\otimes v \mapsto \pi_{(\nu_i,\sigma_i)}(X)\eta(v)$. Here we denote differential of $\pi_{(\nu_i,\sigma_i)}$ again by $\pi_{(\nu_i,\sigma_i)}$. Then $\tilde{\eta}$ is K-homomorphism with $\mathfrak{p}_{\mathbf{C}}$ endowed with the adjoint action Ad of K.

Since

$$V_{2l}\otimes_{\mathbf{C}}\mathfrak{p}_{\mathbf{C}} \simeq V_{2l}\otimes_{\mathbf{C}} V_4 \simeq \bigoplus_{-2\le m\le 2} V_{2(l+m)},$$

there are five injective K-homomorphisms

$$I^{2l}_{2m}\colon V_{2(l+m)} \to V_{2l}\otimes_{\mathbf{C}}\mathfrak{p}_{\mathbf{C}}, \qquad -2\le m\le 2$$

for general $l\in\mathbf{Z}_{\ge 0}$. Then we define $\mathbf{C}$-linear maps

$$\Gamma^i_{l,m}\colon \mathrm{Hom}_K(V_{2l}, H_{(\nu_i,\sigma_i),K}) \to \mathrm{Hom}_K(V_{2(l+m)}, H_{(\nu_i,\sigma_i),K}), \quad -2\le m\le 2$$

by $\eta\mapsto\tilde{\eta}\circ I^{2l}_{2m}$.

Now we settle the goal of this paper:

PROBLEM 4.1. (i) *Describe the injective K-homomorphism I^{2l}_{2m} in terms of the standard basis.*

(ii) *Determine the matrix representations of the linear homomorphisms $\Gamma^i_{l,m}$ with respect to the induced basis defined in the next subsection.*

We have already accomplished (i) in Proposition 3.2. We accomplish (ii) in Theorem 4.5 and 5.5. As a result, we obtain infinite number of 'contiguous relations', a kind of system of differential-difference relations among vectors in $H_{(\nu_i,\sigma_i)}[\tau_{2l}]$ and $H_{(\nu_i,\sigma_i)}[\tau_{2(l+m)}]$. Here $H_{(\nu_i,\sigma_i)}[\tau]$ is τ-isotypic component of $H_{(\nu_i,\sigma_i)}$.

4.3. *The canonical blocks of elementary functions.* Let $\eta\colon V_{2l} \to H_{(\nu_i,\sigma_i),K}$ be a non-zero K-homomorphism. Then we identify η with the column vector of degree $2l+1$ whose $q+1$-th component is $\eta(v_q^{(2l)})$ for $0 \le q \le 2l$, i.e. ${}^t(\ \eta(v_0^{(2l)}),\ \eta(v_1^{(2l)}),\ \cdots\ ,\eta(v_{2l}^{(2l)})\ )$.

By this identification, we identify $S(l;p)$ with the K-homomorphism

$$V_{2l} \ni v_q^{(2l)} \mapsto s(l;p,q) \in H_{(\nu_0,\sigma_0),K}, \quad 0 \le q \le 2l$$

for $p \in Z(\sigma_0;l)$. We note that $\{S(l;p) \mid p \in Z(\sigma_0;l)\}$ is a basis of the space $\mathrm{Hom}_K(V_{2l}, H_{(\nu_0,\sigma_0),K})$ and we call it *the induced basis from the standard basis.*

We define a certain matrix of elementary functions corresponding to the induced basis $\{S(l;p) \mid p \in Z(\sigma_0;l)\}$ of $\mathrm{Hom}_K(V_{2l}, H_{(\nu_0,\sigma_0),K})$ for each K-type τ_{2l} of our principal series representation $\pi_{(\nu_0,\sigma_0)}$.

Definition 4.3. *The following $(2l+1) \times d(\sigma_0;l)$ matrix $\mathbf{S}(\sigma_0;l)$ is called the canonical block of elementary functions for τ_{2l}-isotypic component of $\pi_{(\nu_0,\sigma_0)}$: When $(\sigma_{0,1},\sigma_{0,2}) = (0,0)$, we consider the matrix*

$$\mathbf{S}(\sigma_0;l) = \begin{cases} (\ S(l;0),\ S(l;2),\ S(l;4),\ \cdots\ ,S(l;l)\ ) & \text{if } l \text{ is even,} \\ (\ S(l;1),\ S(l;3),\ S(l;5),\ \cdots\ ,S(l;l-2)\ ) & \text{if } l \text{ is odd.} \end{cases}$$

When $(\sigma_{0,1},\sigma_{0,2}) = (1,0)$, we consider the matrix

$$\mathbf{S}(\sigma_0;l) = \begin{cases} (\ S(l;0),\ S(l;2),\ S(l;4),\ \cdots\ ,S(l;l-2)\ ) & \text{if } l \text{ is even,} \\ (\ S(l;1),\ S(l;3),\ S(l;5),\ \cdots\ ,S(l;l)\ ) & \text{if } l \text{ is odd.} \end{cases}$$

When $\sigma_{0,2} = 1$, we consider the matrix

$$\mathbf{S}(\sigma_0;l) = \begin{cases} (\ S(l;1),\ S(l;3),\ S(l;5),\ \cdots\ ,S(l;l-1)\ ) & \text{if } l \text{ is even,} \\ (\ S(l;0),\ S(l;2),\ S(l;4),\ \cdots\ ,S(l;l-1)\ ) & \text{if } l \text{ is odd.} \end{cases}$$

4.4. *The $\mathfrak{p}_{\mathbf{C}}$-matrix corresponding to I_{2m}^{2l}.* For two integers c_0, c_1 such that $c_0 \le c_1$ and a rational function $f(x)$ in the variable x, we denote by

$$\operatorname*{Diag}_{c_0 \le n \le c_1} (f(n))$$

the diagonal matrix of size $c_1 - c_0 + 1$ with an entry $f(n)$ at the $(n-c_0+1, n-c_0+1)$-th component. Let $\mathbf{e}_i^{(l)}$ $(0 \le i \le l)$ be the column unit vector of degree $l+1$ with its $i+1$-th component 1 and the remaining components 0. Moreover, let $\mathbf{e}_i^{(l)}$ be the column zero vector of degree $l+1$ when $i<0$ or $l<i$.

In this subsection, we define $\mathfrak{p}_{\mathbf{C}}$-matrix $\mathfrak{C}_{l,m}$ of size $(2(l+m)+1)\times(2l+1)$ corresponding to I_{2m}^{2l} with respect to the standard basis.

Let $\sum_{i=0}^{4} \iota_i^{(l,m)} \otimes X_i$ be the image of I_{2m}^{2l} under the composite of natural linear maps

$$\begin{aligned} \mathrm{Hom}_K(V_{2(l+m)}, V_{2l} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}) &\to \mathrm{Hom}_{\mathbf{C}}(V_{2(l+m)}, V_{2l} \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}) \\ &\simeq \mathrm{Hom}_{\mathbf{C}}(V_{2(l+m)}, V_{2l}) \otimes_{\mathbf{C}} \mathfrak{p}_{\mathbf{C}}. \end{aligned}$$

Then we define $\mathfrak{p}_{\mathbf{C}}$-matrix $\mathfrak{C}_{l,m} = \sum_{i=0}^{4} R(\iota_i^{(l,m)}) \otimes X_i$ where $R(\iota_i^{(l,m)})$ is the matrix representation of $\iota_i^{(l,m)}$ with respect to the standard basis. Explicit expression of the matrix $R(\iota_i^{(l,m)})$ of size $(2(l+m)+1) \times (2l+1)$ is given by

$$\left(O_{2(l+m)+1,m+2}, R(\iota_i^{(l,m)}), O_{2(l+m)+1,m+2}\right)$$
$$= \left(O_{2(l+m)+1,4-i},\ \operatorname*{Diag}_{0\le k\le 2(l+m)}(A_{[2l,2m;k,i]}),\ O_{2(l+m)+1,i}\right)$$

for $-2 \le m \le 2$ and $0 \le i \le 4$. Here we omit the symbol $O_{m,n}$ when $m = 0$ or $n = 0$.

For a column vector $\mathbf{v} = {}^t(v_0, v_1, \cdots, v_{2l}) \in (H_{(\nu_i,\sigma_i),K})^{\oplus 2l+1}$ which is identified with an element of $\mathrm{Hom}_K(V_{2l}, H_{(\nu_i,\sigma_i),K})$, we define $\mathfrak{C}_{l,m}\mathbf{v} \in (H_{(\nu_i,\sigma_i),K})^{\oplus 2(l+m)+1} \simeq \mathbf{C}^{2(l+m)+1} \otimes_{\mathbf{C}} H_{(\nu_i,\sigma_i),K}$ by

$$\mathfrak{C}_{l,m}\mathbf{v} = \sum_{\substack{0\le j\le 4\\ 0\le q\le 2l}} (R(\iota_j^{(l,m)}) \cdot \mathbf{e}_q^{(2l)}) \otimes (\pi_{(\nu_i,\sigma_i)}(X_j)v_q).$$

Here $R(\iota_j^{(l,m)}) \cdot \mathbf{e}_q^{(2l)}$ is the ordinal product of matrices $R(\iota_j^{(l,m)})$ and $\mathbf{e}_q^{(2l)}$.

From the definition of $\mathfrak{C}_{l,m}$, we note that the vector $\mathfrak{C}_{l,m}\mathbf{v}$ is identified with the image of $\mathbf{v}$ under $\Gamma_{l,m}^i$.

4.5. *The contiguous relations.*

LEMMA 4.4. *The standard basis X_i $(0 \le i \le 4)$ in $\mathfrak{p}_{\mathbf{C}}$ have the following expressions according to the Iwasawa decomposition $\mathfrak{g}_{\mathbf{C}} = \mathfrak{n}_{\mathbf{C}} \oplus \mathfrak{a}_{\mathbf{C}} \oplus \mathfrak{k}_{\mathbf{C}}$:*

$$\begin{aligned}
X_0 =& -2\sqrt{-1}E_{e_2-e_3} + H_2 + \sqrt{-1}K_{23},\\
X_1 =& -(E_{e_1-e_3} + \sqrt{-1}E_{e_1-e_2}) + \frac{1}{2}(K_{13} + \sqrt{-1}K_{12}),\\
X_2 =& -\frac{1}{3}(2H_1 - H_2),\\
X_3 =& (E_{e_1-e_3} - \sqrt{-1}E_{e_1-e_2}) - \frac{1}{2}(K_{13} - \sqrt{-1}K_{12}),\\
X_4 =& 2\sqrt{-1}E_{e_2-e_3} + H_2 - \sqrt{-1}K_{23}.
\end{aligned}$$

PROOF. We obtain the assertion immediately from Lemma 3.1. ∎

We give the matrix representation of $\Gamma_{l,m}^0$ with respect to the induced basis as follows.

THEOREM 4.5. *For $l \in \mathbf{Z}_{\ge 0}$, $-2 \le m \le 2$ such that $d(\sigma_0; l) > 0$ and $d(\sigma_0; l+m) > 0$, we have*

$$\mathfrak{C}_{l,m}\mathbf{S}(\sigma_0; l) = \mathbf{S}(\sigma_0; l+m) \cdot R(\Gamma_{l,m}^0) \tag{4.1}$$

with the matrix representation $R(\Gamma^0_{l,m}) \in M_{d(\sigma_0;l+m),d(\sigma_0;l)}(\mathbf{C})$ *of* $\Gamma^0_{l,m}$ *with respect to the induced basis* $\{S(l;p) \mid p \in Z(\sigma_0;l)\}$*:*

We give the explicit expressions of the matrix

$$\begin{pmatrix} O_{n(\sigma_0;l,m),d(\sigma_0;l)} \\ R(\Gamma^0_{l,m}) \end{pmatrix}$$

as follows:

When $\sigma_{0,2} = 0$ *and* $(m, \sigma_{0,1} + l) \in \{0, \pm 2\} \times (2\mathbf{Z})$,

$$\begin{pmatrix} \operatorname*{Diag}_{0 \le k \le d(\sigma_0;l)-1} \left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),-1]}\right) \\ O_{1,d(\sigma_0;l)} \end{pmatrix} + \begin{pmatrix} O_{1,d(\sigma_0;l)} \\ \operatorname*{Diag}_{0 \le k \le d(\sigma_0;l)-1} \left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),0]}\right) \end{pmatrix}$$
$$+ \begin{pmatrix} O_{2,d(\sigma_0;l)-1} & O_{2,1} \\ \operatorname*{Diag}_{0 \le k \le d(\sigma_0;l)-2} \left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),1]}\right) & \gamma^{(0)}_{[l,m;l,1]} \cdot e^{(d(\sigma_0;l)-2)}_{d(\sigma_0;l)-3} \end{pmatrix}.$$

When $\sigma_{0,2} = 0$ *and* $(m, \sigma_{0,1} + l) \in \{0, \pm 2\} \times (1 + 2\mathbf{Z})$,

$$\begin{pmatrix} \operatorname*{Diag}_{0 \le k \le d(\sigma_0;l)-1} \left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),-1]}\right) \\ O_{1,d(\sigma_0;l)} \end{pmatrix} + \begin{pmatrix} O_{1,d(\sigma_0;l)} \\ \operatorname*{Diag}_{0 \le k \le d(\sigma_0;l)-1} \left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),0]}\right) \end{pmatrix}$$
$$+ \begin{pmatrix} O_{2,d(\sigma_0;l)-1} & O_{2,1} \\ \operatorname*{Diag}_{0 \le k \le d(\sigma_0;l)-2} \left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),1]}\right) & O_{d(\sigma_0;l)-1,1} \end{pmatrix}.$$

When $\sigma_{0,2} = 0$, $(m, \sigma_{0,1} + l) \in \{\pm 1\} \times (2\mathbf{Z})$,

$$\operatorname*{Diag}_{0 \le k \le d(\sigma_0;l)-1} \left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),-1]}\right)$$
$$+ \begin{pmatrix} O_{1,d(\sigma_0;l)-1} & 0 \\ \operatorname*{Diag}_{0 \le k \le d(\sigma_0;l)-2} \left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),0]}\right) & O_{d(\sigma_0;l)-1,1} \end{pmatrix}$$
$$+ \begin{pmatrix} O_{2,d(\sigma_0;l)-2} & O_{2,1} & O_{2,1} \\ \operatorname*{Diag}_{0 \le k \le d(\sigma_0;l)-3} \left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),1]}\right) & O_{d(\sigma_0;l)-2,1} & -\gamma^{(0)}_{[l,m;l,1]} \cdot e^{(d(\sigma_0;l)-3)}_{d(\sigma_0;l)-3} \end{pmatrix}.$$

When $\sigma_{0,2} = 0$ *and* $(m, \sigma_{0,1} + l) \in \{\pm 1\} \times (1 + 2\mathbf{Z})$,

$$\begin{pmatrix} \operatorname*{Diag}_{0 \le k \le d(\sigma_0;l)-1} \left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),-1]}\right) \\ O_{2,d(\sigma_0;l)} \end{pmatrix} + \begin{pmatrix} O_{1,d(\sigma_0;l)} \\ \operatorname*{Diag}_{0 \le k \le d(\sigma_0;l)-1} \left(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),0]}\right) \\ O_{1,d(\sigma_0;l)} \end{pmatrix}$$

$$+\begin{pmatrix} O_{2,d(\sigma_0;l)} \\ \operatorname*{Diag}_{0\le k\le d(\sigma_0;l)-1}\Big(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),1]}\Big)\end{pmatrix}.$$

When $\sigma_{0,2}=1$,

$$\begin{pmatrix} \operatorname*{Diag}_{0\le k\le d(\sigma_0;l)-1}\Big(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),-1]}\Big) \\ O_{1,d(\sigma_0;l)}\end{pmatrix} + \begin{pmatrix} O_{1,d(\sigma_0;l)} \\ \operatorname*{Diag}_{0\le k\le d(\sigma_0;l)-1}\Big(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),0]}\Big)\end{pmatrix}$$

$$+\begin{pmatrix} O_{2,d(\sigma_0;l)-1} & O_{2,1} \\ \operatorname*{Diag}_{0\le k\le d(\sigma_0;l)-2}\Big(\gamma^{(0)}_{[l,m;2k+\delta(\sigma_0;l),1]}\Big) & (-1)^{\varepsilon(\sigma_0;l+m)}\gamma^{(0)}_{[l,m;l-1,1]}\cdot e^{(d(\sigma_0;l)-2)}_{d(\sigma_0;l)-2}\end{pmatrix}.$$

Here

$$\gamma^{(0)}_{[l,m;p,1]} = (\nu_{0,2}+\rho_{0,2}-l+p)A_{[2l,2m;2l-p+m-2,0]},$$

$$\gamma^{(0)}_{[l,m;p,0]} = -\frac{1}{3}\Big(2\nu_{0,1}-\nu_{0,2}+2\rho_{0,1}-\rho_{0,2}+lm-3+\frac{m(m+1)}{2}\Big)A_{[2l,2m;2l-p+m,2]},$$

$$\gamma^{(0)}_{[l,m;p,-1]} = (\nu_{0,2}+\rho_{0,2}+l-p)A_{[2l,2m;2l-p+m+2,4]},$$

$$n(\sigma_0;l,m)=\begin{cases}(2-m)/2 & \text{if } m\in\{0,\ \pm 2\},\\ (3-m)/2 & \text{if } (m,l+\sigma_{0,2})\in\{\pm1\}\times(2\mathbf{Z}),\\ (1-m)/2 & \text{if } (m,l+\sigma_{0,2})\in\{\pm1\}\times(1+2\mathbf{Z}),\end{cases}$$

and $\delta(\sigma_0;l)\in\{0,1\}$ *such that* $\delta(\sigma_0;l)\equiv l-\sigma_{0,2} \bmod 2$.

In the above equations, we put $A_{[2l,2m;k,i]}=0$ *for* $k<0$ *or* $k>2(l+m)$, *and omit the symbols* $\operatorname*{Diag}_{c\le n\le c-1}(f(n))$, $O_{0,n}$, $O_{m,0}$ *and* $\mathbf{e}_i^{(-1)}$.

Proof. Since

$$s(l;p,q)(1_3)=\langle (v^{(2l)*}_{2l-p}+(-1)^{\varepsilon(\sigma_0;l)}v^{(2l)*}_p), v^{(2l)}_q\rangle = \delta_{2l-p\,q}+(-1)^{\varepsilon(\sigma_0;l)}\delta_{pq},$$

we have

$$S(l;p)(1_3)=\mathbf{e}^{(2l)}_{2l-p}+(-1)^{\varepsilon(\sigma_0;l)}\mathbf{e}^{(2l)}_p. \tag{4.2}$$

Hence $S(l;p)(1_3)$ $(p\in Z(\sigma_0;l))$ are linearly independent over $\mathbf{C}$. Thus we note that it suffices to evaluate the both side of the equation (4.1) at $1_3\in G$.

First, we compute $\{\pi_{(\nu_0,\sigma_0)}(X_i)s(l;p,q)\}(1_3)$ for $0\le i\le 4$, $p\in Z(\sigma_0;l)$ and $0\le q\le 2l$. Since $\{s(l;p,q)\mid 0\le q\le 2l\}$ is the standard basis of $\langle S(l;p)\rangle$, we have

$$\{\pi_{(\nu_0,\sigma_0)}(\sqrt{-1}K_{23})s(l;p,q)\}(1_3)=(l-q)(\delta_{2l-p\,q}+(-1)^{\varepsilon(\sigma_0;l)}\delta_{pq}),$$

$$\{\pi_{(\nu_0,\sigma_0)}(K_{13}+\sqrt{-1}K_{12})s(l;p,q)\}(1_3)=-q(\delta_{2l-p+1\,q}+(-1)^{\varepsilon(\sigma_0;l)}\delta_{p+1\,q}),$$

$$\{\pi_{(\nu_0,\sigma_0)}(K_{13}-\sqrt{-1}K_{12})s(l;p,q)\}(1_3)=$$

$$= (2l-q)(\delta_{2l-p-1\,q} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{p-1\,q}).$$

Moreover, we obtain

$$\{\pi_{(\nu_0,\sigma_0)}(E_\alpha)s(l;p,q)\}(1_3) = 0 \qquad (\alpha \in \Sigma^+),$$

$$\begin{aligned}\{\pi_{(\nu_0,\sigma_0)}(H_i)s(l;p,q)\}(1_3) &= (\nu_{0,i}+\rho_{0,i})s(l;p,q)(1_3)\\ &= (\nu_{0,i}+\rho_{0,i})(\delta_{2l-p\,q} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{pq}) \quad (i=1,2),\end{aligned}$$

from the definition of principal series representation. From these computations and Iwasawa decomposition in Lemma 4.4, we obtain

$$\{\pi_{(\nu_0,\sigma_0)}(X_0)s(l;p,q)\}(1_3) = (\nu_{0,2}+\rho_{0,2}+l-q)(\delta_{2l-p\,q} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{pq}),$$

$$\{\pi_{(\nu_0,\sigma_0)}(X_1)s(l;p,q)\}(1_3) = -\frac{q}{2}(\delta_{2l-p+1\,q} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{p+1\,q}),$$

$$\{\pi_{(\nu_0,\sigma_0)}(X_2)s(l;p,q)\}(1_3)$$
$$= -\frac{1}{3}(2\nu_{0,1}-\nu_{0,2}+2\rho_{0,1}-\rho_{0,2})(\delta_{2l-p\,q} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{pq}),$$

$$\{\pi_{(\nu_0,\sigma_0)}(X_3)s(l;p,q)\}(1_3) = -\frac{2l-q}{2}(\delta_{2l-p-1\,q} - (-1)^{\varepsilon(\sigma_0;l)}\delta_{p-1\,q}),$$

$$\{\pi_{(\nu_0,\sigma_0)}(X_4)s(l;p,q)\}(1_3) = (\nu_{0,2}+\rho_{0,2}-l+q)(\delta_{2l-p\,q} + (-1)^{\varepsilon(\sigma_0;l)}\delta_{pq}).$$

We set

$$\pi_{(\nu_0,\sigma_0)}(X_i)S(l;p) = \sum_{0\le q\le 2l} \mathbf{e}_q^{(2l)} \otimes (\pi_{(\nu_0,\sigma_0)}(X_i)s(l;p,q)).$$

Then we obtain

$$\{\pi_{(\nu_0,\sigma_0)}(X_0)S(l;p)\}(1_3)$$
$$= (\nu_{0,2}+\rho_{0,2}-l+p)\mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0;l)}(\nu_{0,2}+\rho_{0,2}+l-p)\mathbf{e}_p^{(2l)},$$

$$\{\pi_{(\nu_0,\sigma_0)}(X_1)S(l;p)\}(1_3) = -\frac{2l-p+1}{2}\mathbf{e}_{2l-p+1}^{(2l)} - (-1)^{\varepsilon(\sigma_0;l)}\frac{p+1}{2}\mathbf{e}_{p+1}^{(2l)},$$

$$\{\pi_{(\nu_0,\sigma_0)}(X_2)S(l;p)\}(1_3)$$
$$= -\frac{1}{3}(2\nu_{0,1}-\nu_{0,2}+2\rho_{0,1}-\rho_{0,2})(\mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0;l)}\mathbf{e}_p^{(2l)}),$$

$$\{\pi_{(\nu_0,\sigma_0)}(X_3)S(l;p)\}(1_3) = -\frac{p+1}{2}\mathbf{e}_{2l-p-1}^{(2l)} - (-1)^{\varepsilon(\sigma_0;l)}\frac{2l-p+1}{2}\mathbf{e}_{p-1}^{(2l)},$$

$$\{\pi_{(\nu_0,\sigma_0)}(X_4)S(l;p)\}(1_3)$$
$$= (\nu_{0,2}+\rho_{0,2}+l-p)\mathbf{e}_{2l-p}^{(2l)} + (-1)^{\varepsilon(\sigma_0;l)}(\nu_{0,2}+\rho_{0,2}-l+p)\mathbf{e}_p^{(2l)}.$$

Let us compute $\{\mathfrak{C}_{l,m}S(l;p)\}(1_3)$. By the above equations, we have

$$\{\mathfrak{C}_{l,m}S(l;p)\}(1_3)$$
$$= \sum_{\substack{0\le i\le 4\\ 0\le q\le 2l}} (R(\iota_i^{(l,m)})\cdot \mathbf{e}_q^{(2l)}) \otimes \{(\pi_{(\nu_0,\sigma_0)}(X_i)s(l;p,q))\}(1_3)$$

$$= \sum_{0\le i\le 4} R(\iota_i^{(l,m)}) \cdot \{(\pi_{(\nu_0,\sigma_0)}(X_i)S(l;p))\}(1_3)$$
$$= R(\iota_0^{(l,m)}) \cdot \{(\nu_{0,2}+\rho_{0,2}-l+p)\mathbf{e}^{(2l)}_{2l-p} + (-1)^{\varepsilon(\sigma_0;l)}(\nu_{0,2}+\rho_{0,2}+l-p)\mathbf{e}^{(2l)}_{p}\}$$
$$+ R(\iota_1^{(l,m)}) \cdot \Big\{ -\frac{2l-p+1}{2}\mathbf{e}^{(2l)}_{2l-p+1} - (-1)^{\varepsilon(\sigma_0;l)}\frac{p+1}{2}\mathbf{e}^{(2l)}_{p+1}\Big\}$$
$$+ R(\iota_2^{(l,m)}) \cdot \Big\{ -\frac{1}{3}(2\nu_{0,1}-\nu_{0,2}+2\rho_{0,1}-\rho_{0,2})(\mathbf{e}^{(2l)}_{2l-p} + (-1)^{\varepsilon(\sigma_0;l)}\mathbf{e}^{(2l)}_{p})\Big\}$$
$$+ R(\iota_3^{(l,m)}) \cdot \Big\{ -\frac{p+1}{2}\mathbf{e}^{(2l)}_{2l-p-1} - (-1)^{\varepsilon(\sigma_0;l)}\frac{2l-p+1}{2}\mathbf{e}^{(2l)}_{p-1}\Big\}$$
$$+ R(\iota_4^{(l,m)}) \cdot \{(\nu_{0,2}+\rho_{0,2}+l-p)\mathbf{e}^{(2l)}_{2l-p} + (-1)^{\varepsilon(\sigma_0;l)}(\nu_{0,2}+\rho_{0,2}-l+p)\mathbf{e}^{(2l)}_{p}\}.$$

Since

$$R(\iota_i^{(l,m)})\mathbf{e}^{(2l)}_q = A_{[2l,2m;i+q+m-2,i]}\mathbf{e}^{(2(l+m))}_{i+q+m-2}, \quad -2\le m\le 2,$$

we obtain

(4.3) $\{\mathfrak{C}_{l,m}S(l;p)\}(1_3)$

$$= \sum_{-1\le i\le 1} \{\alpha_{[l,m;p,i]}\mathbf{e}^{(2(l+m))}_{2(l+m)-(p+m+2i)} + (-1)^{\varepsilon(\sigma_0;l)}\beta_{[l,m;p,i]}\mathbf{e}^{(2(l+m))}_{p+m+2i}\},$$

where

$$\begin{aligned}
\alpha_{[l,m;p,1]} =& (\nu_{0,2}+\rho_{0,2}-l+p)A_{[2l,2m;2l-p+m-2,0]},\\
\alpha_{[l,m;p,0]} =& -\frac{1}{3}(2\nu_{0,1}-\nu_{0,2}+2\rho_{0,1}-\rho_{0,2})A_{[2l,2m;2l-p+m,2]}\\
&-\frac{2l-p+1}{2}A_{[2l,2m;2l-p+m,1]} - \frac{p+1}{2}A_{[2l,2m;2l-p+m,3]},\\
\alpha_{[l,m;p,-1]} =& (\nu_{0,2}+\rho_{0,2}+l-p)A_{[2l,2m;2l-p+m+2,4]},\\
\beta_{[l,m;p,1]} =& (\nu_{0,2}+\rho_{0,2}-l+p)A_{[2l,2m;p+m+2,4]},\\
\beta_{[l,m;p,0]} =& -\frac{1}{3}(2\nu_{0,1}-\nu_{0,2}+2\rho_{0,1}-\rho_{0,2})A_{[2l,2m;p+m,2]}\\
&-\frac{p+1}{2}A_{[2l,2m;p+m,1]} - \frac{2l-p+1}{2}A_{[2l,2m;p+m,3]},\\
\beta_{[l,m;p,-1]} =& (\nu_{0,2}+\rho_{0,2}+l-p)A_{[2l,2m;p+m-2,0]}.
\end{aligned}$$

By the relations of the coefficients $A_{[2l,2m;k,i]}$ in Lemma 3.3, we see that

$$\alpha_{[l,m;p,i]} = (-1)^m\beta_{[l,m;p,i]} = \gamma^{(0)}_{[l,m;p,i]}, \quad -1\le i\le 1.$$

Therefore, (4.3) become

(4.4) $\{\mathfrak{C}_{l,m}S(l;p)\}(1_3)$

$$= \sum_{-1\le i\le 1} \gamma^{(0)}_{[l,m;p,i]}\{\mathbf{e}^{(2(l+m))}_{2(l+m)-(p+m+2i)} + (-1)^{\varepsilon(\sigma_0;l)+m}\mathbf{e}^{(2(l+m))}_{p+m+2i}\}.$$

From the equations (4.2), (4.4) and $\varepsilon(\sigma_0; l) + m \equiv \varepsilon(\sigma_0; l+m) \bmod 2$, we obtain the assertion. ∎

5. The $(\mathfrak{g}, K)$-module structures of the P_i-principal series representations for $i = 1, 2$

In this section, we set $i = 1$ or 2.

5.1. *The discrete series representations of* $SL^{\pm}(2,\mathbf{R})$. The set of equivalence classes of discrete series representations of $SL^{\pm}(2,\mathbf{R})$ is exhausted by the induced representation $D_k = \mathrm{Ind}_{SL(2,\mathbf{R})}^{SL^{\pm}(2,\mathbf{R})}(D_k^+)$. Here D_k^+ is the discrete series representation of $SL(2,\mathbf{R})$ with Blattner parameter k, i.e. the one whose minimal $SO(2)$-type is given by the character

$$SO(2) \ni \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \mapsto e^{\sqrt{-1}kt} \in \mathbf{C}^{\times}.$$

We denote by D_k^- the contragradient representation of D_k^+ and set $y_0 = \mathrm{diag}(1,-1) \in O(2)$. Then a discrete series representation D_k is uniquely determined by specifying the $SL(2,\mathbf{R})$-module structure together with the action of y_0. Since $D_k|_{SL(2,\mathbf{R})} = D_k^+ \oplus D_k^-$ and $D_k^+ \oplus D_k^-$ is infinitesimally equivalent with a subrepresentation of some principal series representation of $SL(2,\mathbf{R})$, we obtain the following realization of associated $(\mathfrak{sl}(2,\mathbf{C}), O(2))$-module of D_k:

$$V_{D_k,O(2)} = \bigoplus_{\alpha \in \mathbf{Z}_{\geq 0}} W_{k+2\alpha} \qquad \big(W_p = \mathbf{C}\cdot\chi_p + \mathbf{C}\cdot\chi_{-p}\big)$$

and

$$D_k(\kappa_t)\chi_p = e^{\sqrt{-1}pt}\chi_p \qquad D_k(y_0)\chi_p = \chi_{-p}, \qquad D_k(w)\chi_p = \sqrt{-1}p\chi_p,$$
$$D_k(x_+)\chi_p = (k+p)\chi_{p+2}, \quad D_k(x_-)\chi_p = (k-p)\chi_{p-2},$$

where

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ x_{\pm} = \begin{pmatrix} 1 & \pm\sqrt{-1} \\ \pm\sqrt{-1} & -1 \end{pmatrix} \in \mathfrak{sl}(2,\mathbf{C}),$$
$$\kappa_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in SO(2) \quad (t \in \mathbf{R}).$$

Here we denote differential of D_k again by D_k and the $O(2)$-finite part of V_{D_k} by $V_{D_k,O(2)}$. See [1, §2.5] for details.

5.2. *The irreducible decompositions of* $\pi_{(\nu_1,\sigma_1)}|_K$ *and* $\pi_{(\nu_2,\sigma_2)}|_K$ *as* K*-modules.* We identify M_i with $SL^{\pm}(2,\mathbf{R})$ by natural isomorphisms $m_i\colon SL^{\pm}(2,\mathbf{R}) \to M_i$ defined by

$$m_1(h) = \begin{pmatrix} \det(h)^{-1} & O_{1,2} \\ O_{2,1} & h \end{pmatrix}, \qquad m_2(h) = \begin{pmatrix} h & O_{2,1} \\ O_{1,2} & \det(h)^{-1} \end{pmatrix}$$

for $h \in SL^{\pm}(2, \mathbf{R})$. Then we may put $\sigma_i = D_k \circ m_i^{-1}$ for some $k \geq 2$.

We analyze the K-type of the representation space $H_{(\nu_i,\sigma_i)}$ of the P_i-principal series representation. The target V_{σ_i} of functions $\mathbf{f}$ in $H_{(\nu_i,\sigma_i)}$ has a decomposition:

$$V_{\sigma_i} = V_{D_k} = \widehat{\bigoplus_{\alpha \in \mathbf{Z}_{\geq 0}}} W_{k+2\alpha}.$$

Denote the corresponding decomposition of $\mathbf{f}$ by

$$\mathbf{f}(x) = \sum_{\alpha=0}^{\infty} (f_{k+2\alpha}(x) \otimes \chi_{k+2\alpha} + f_{-(k+2\alpha)}(x) \otimes \chi_{-(k+2\alpha)}).$$

From the definition of the space $H_{(\nu_i,\sigma_i)}$, we have

$$\mathbf{f}|_K(mx) = \sigma_i(m)\mathbf{f}|_K(x) \qquad (\text{a.e. } x \in K,\ m \in K_i = M_i \cap K \simeq O(2)).$$

For $m = m_i(\kappa_t),\ m_i(y_0)$, comparing the coefficients of χ_p in the left hand side with those in the right hand side, we have the equations

$$f_p|_K(m_i(\kappa_t)x) = e^{\sqrt{-1}pt} f_p|_K(x), \qquad f_p|_K(m_i(y_0)x) = f_{-p}|_K(x).$$

Moreover, from the equality of inner products

$$\int_K \|\mathbf{f}|_K(x)\|^2_{\sigma_i} dx = \sum_{\varepsilon \in \{\pm 1\},\ \alpha \in \mathbf{Z}_{\geq 0}} \left\{ \int_K \left| f_{\varepsilon(k+2\alpha)}|_K(x) \right|^2 dx \right\} \|\chi_{\varepsilon(k+2\alpha)}\|^2_{\sigma_i},$$

we have $f_p|_K \in L^2(K)$. Therefore $\mathbf{f}|_K$ belongs to

$$\widehat{\bigoplus_{\alpha \in \mathbf{Z}_{\geq 0}}} L^2_i(K; W_{k+2\alpha})$$

where

$$L^2_i(K; W_p) = \left\{ \mathbf{f} \colon K \to W_p \ \middle|\ \begin{array}{l} \mathbf{f}(x) = f(x) \otimes \chi_p + f(m_i(y_0)x) \otimes \chi_{-p}, \\ f \in L^2_{(K_i^\circ, \chi_p)}(K), \quad x \in K \end{array} \right\},$$

$$L^2_{(K_i^\circ, \chi_p)}(K) = \left\{ f \in L^2(K) \ \middle|\ \begin{array}{l} f(m_i(\kappa_t)x) = e^{\sqrt{-1}pt} f(x), \\ m_i(\kappa_t) \in K_i^\circ,\ x \in K \end{array} \right\}.$$

Here K_i° means the connected component of K_i, which is isomorphic to $SO(2)$. We easily see that the restriction map

$$r_K^{(i)} : H_{(\nu_i,\sigma_i)} \ni \mathbf{f} \mapsto \mathbf{f}|_K \in \widehat{\bigoplus_{\alpha \in \mathbf{Z}_{\geq 0}}} L^2_i(K; W_{k+2\alpha})$$

is a K-isomorphism.

By Peter-Weyl's theorem, we have the following irreducible decomposition of $L^2_{(K_i^\circ, \chi_p)}(K)$:

$$L^2_{(K_i^\circ, \chi_p)}(K) \simeq \widehat{\bigoplus_{l \in \mathbf{Z}_{\geq 0}}} (V^*_{2l}[\xi_{(i;-p)}]) \otimes_{\mathbf{C}} V_{2l}.$$

Here

$$\xi_{(i;p)}\colon K_i^\circ \ni m_i(\kappa_t) \mapsto e^{\sqrt{-1}pt} \in \mathbf{C}^\times$$

and $V[\xi_{(i;p)}]$ means the $\xi_{(i;p)}$-isotypic component in $(\tau|_{K_i^\circ}, V)$ for a K-module (τ, V).

In this section, we denote by $\{v_{1,q}^{(2l)} \mid 0 \le q \le 2l\}$ the standard basis of V_{2l}. We define an another basis $\{v_{2,q}^{(2l)} \mid 0 \le q \le 2l\}$ of V_{2l} by

$$v_{2,q}^{(2l)} = \tau_{2l}(u_c)v_{1,q}^{(2l)} = \frac{1}{2^l}(x+y)^q(-x+y)^{2l-q} \quad (0 \le q \le 2l)$$

where

$$u_c = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in SO(3).$$

We note that $v \in V_{2l}[\xi_{(i;-p)}]$ if and only if

$$\tau_{2l}(m_i(\kappa_t))v = \xi_{(i;-p)}(m_i(\kappa_t))v = e^{-\sqrt{-1}pt}v \qquad (t \in \mathbf{R})$$

for $v \in V_{2l}$. From the definition of (τ_{2l}, V_{2l}) and

$$\varphi^{-1}(m_1(\kappa_t)) = \varphi^{-1}(u_c^{-1}m_2(\kappa_t)u_c) = \left\{\pm\,\mathrm{diag}(e^{-\sqrt{-1}t/2}, e^{\sqrt{-1}t/2})\right\},$$

we have $\tau_{2l}(m_i(\kappa_t))v_{i,q}^{(2l)} = e^{\sqrt{-1}(q-l)t}v_{i,q}^{(2l)}$. Hence we have

$$V_{2l}[\xi_{(i;-p)}] = \begin{cases} \mathbf{C}\cdot v_{i,l-p}^{(2l)} & \text{if } -l \le p \le l, \\ 0 & \text{otherwise .} \end{cases}$$

By the identification $V_{2l}^* = V_{2l}$ in Lemma 3.4, we obtain

$$L^2_{(K_i^\circ,\chi_p)}(K) \simeq \widehat{\bigoplus_{l\in\mathbf{Z}_{\ge 0}}}(\mathbf{C}\cdot v_{i,l+p}^{(2l)*}) \otimes_{\mathbf{C}} V_{2l}.$$

Here we put $v_{i,l+p}^{(2l)*} = 0$ if $p < -l$ or $l < p$. Moreover, since

$$\varphi^{-1}(m_1(y_0)) = \left\{\pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\},$$
$$\varphi^{-1}(u_c^{-1}m_2(y_0)u_c) = \left\{\pm\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}\right\},$$

we have

$$\tau_{2l}^*(m_1(y_0)^{-1})v_{1,l+p}^{(2l)*} = (-1)^{l+p}v_{1,l-p}^{(2l)*}, \quad \tau_{2l}^*(m_2(y_0)^{-1})v_{2,l+p}^{(2l)*} = (-1)^{l}v_{2,l-p}^{(2l)*}.$$

For $0 \le p \le l-k$ such that $p \equiv l-k \bmod 2$, we define the elementary function $t_i(l;p,q) \in H_{(\nu_i,\sigma_i)}$ by

$$t_i(l;p,q) = r_K^{(i)-1}(\tilde{t}_i(l;p,q))$$

where

$$\tilde{t}_1(l;p,q)(x) = \langle v^{(2l)*}_{1,2l-p}, \tau_{2l}(x)v^{(2l)}_{1,q}\rangle \otimes \chi_{l-p} + (-1)^p \langle v^{(2l)*}_{1,p}, \tau_{2l}(x)v^{(2l)}_{1,q}\rangle \otimes \chi_{p-l},$$
$$\tilde{t}_2(l;p,q)(x) = \langle v^{(2l)*}_{2,2l-p}, \tau_{2l}(x)v^{(2l)}_{1,q}\rangle \otimes \chi_{l-p} + (-1)^l \langle v^{(2l)*}_{2,p}, \tau_{2l}(x)v^{(2l)}_{1,q}\rangle \otimes \chi_{p-l}.$$

Let $T_i(l;p)$ be a column vector of degree $2l+1$ with its $q+1$-th component $t_i(l;p,q)$, i.e. ${}^t(\ t_i(l;p,0),\ t_i(l;p,1),\ \cdots\ ,t_i(l;p,2l)\ )$.

Moreover we denote by $\langle T_i(l;p)\rangle$ the subspace of $H_{(\nu_i,\sigma_i)}$ generated by the functions in the entries of the vector $T_i(l;p)$, i.e.

$$\langle T_i(l;p)\rangle = \bigoplus_{q=0}^{2l} \mathbf{C}\cdot t_i(l;p,q) \simeq V_{2l}.$$

Via the isomorphism between $\langle T_i(l;p)\rangle$ and V_{2l}, we identify $\{t_i(l;p,q) \mid 0 \le q \le 2l\}$ with the standard basis.

From above arguments, we obtain the following.

PROPOSITION 5.1. *As an unitary representation of K, it has an irreducible decomposition:*

$$H_{(\nu_i,\sigma_i)} = \widehat{\bigoplus_{\substack{l\in\mathbf{Z}_{\ge 0},\ 0\le p\le l-k \\ p\equiv l-k \bmod 2}}} \langle T_i(l;p)\rangle$$

for $i = 1, 2$. Then the τ_{2l}-isotypic component of $\pi_{(\nu_i,\sigma_i)}$ is given by

$$\bigoplus_{\substack{0\le p\le l-k \\ p\equiv l-k \bmod 2}} \langle T_i(l;p)\rangle.$$

COROLLARY 5.2. *The multiplicity $d(\sigma_i;l)$ of τ_{2l} in $\pi_{(\nu_i,\sigma_i),K}$ is given by*

$$d(\sigma_i;l) = \begin{cases} (l-k+2)/2 & \text{if } k \le l \text{ and } l-k \text{ is even,} \\ (l-k+1)/2 & \text{if } k \le l \text{ and } l-k \text{ is odd,} \\ 0 & \text{if } k > l. \end{cases}$$

5.3. *The canonical blocks of elementary functions.* By the identification introduced in Subsection 4.3, we identify $T_i(l;p)$ with the K-homomorphism

$$V_{2l} \ni v^{(2l)}_{1,q} \mapsto t_i(l;p,q) \in H_{(\nu_i,\sigma_i),K}, \quad 0 \le q \le 2l$$

for $0 \le p \le l-k$ such that $p \equiv l-k \bmod 2$. We note that $\{T_i(l;p) \mid 0 \le p \le l-k,\ p \equiv l-k \bmod 2\}$ is a basis of $\mathrm{Hom}_K(V_{2l}, H_{(\nu_i,\sigma_i),K})$ and we call it *the induced basis from the standard basis.*

For each K-type τ_{2l} of our P_i-principal series representation $\pi_{(\nu_i,\sigma_i)}$, we define a certain matrix of elementary functions corresponding to the induced basis $\{T_i(l;p) \mid 0 \le p \le l-k,\ p \equiv l-k \bmod 2\}$ of $\mathrm{Hom}_K(V_{2l}, H_{(\nu_i,\sigma_i),K})$.

Definition 5.3. *For $l \in \mathbf{Z}_{\geq 0}$ such that $d(\sigma_i;l) > 0$, the following $(2l+1) \times d(\sigma_i;l)$ matrix $\mathbf{T}_i(\sigma_i;l)$ is called the canonical block of elementary functions for τ_{2l}-isotypic component of $\pi_{(\nu_i,\sigma_i)}$: When $l-k$ is even, we consider the matrix*

$$\mathbf{T}_i(\sigma_i;l) = (\ T_i(l;0),\ T_i(l;2),\ T_i(l;4),\ \cdots\ ,T_i(l;l-k)\ ).$$

When $l-k$ is odd, we consider the matrix

$$\mathbf{T}_i(\sigma_i;l) = (\ T_i(l;1),\ T_i(l;3),\ T_i(l;5),\ \cdots\ ,T_i(l;l-k)\ ).$$

5.4. *The contiguous relations.*

Lemma 5.4. *(i) The standard basis $\{X_j \mid 0 \leq j \leq 4\}$ of $\mathfrak{p}_{\mathbf{C}}$ have the following expressions according to the decomposition $\mathfrak{g}_{\mathbf{C}} = (\mathfrak{n}_{1,\mathbf{C}} \oplus \mathfrak{a}_{1,\mathbf{C}} \oplus \mathfrak{m}_{1,\mathbf{C}}) + \mathfrak{k}_{\mathbf{C}}$:*

$$\begin{aligned} X_0 =& m_1(x_-), \qquad X_1 = -(E_{e_1-e_3} + \sqrt{-1}E_{e_1-e_2}) + \frac{1}{2}(K_{13} + \sqrt{-1}K_{12}),\\ X_2 =& -\frac{1}{3}H^{(1)}, \quad X_3 = (E_{e_1-e_3} - \sqrt{-1}E_{e_1-e_2}) - \frac{1}{2}(K_{13} - \sqrt{-1}K_{12}),\\ X_4 =& m_1(x_+). \end{aligned}$$

(ii) The standard basis $\{X_j \mid 0 \leq j \leq 4\}$ of $\mathfrak{p}_{\mathbf{C}}$ have the following expressions according to the decomposition $\mathfrak{g}_{\mathbf{C}} = \mathrm{Ad}(u_c^{-1})(\mathfrak{n}_{2,\mathbf{C}} \oplus \mathfrak{a}_{2,\mathbf{C}} \oplus \mathfrak{m}_{2,\mathbf{C}}) + \mathfrak{k}_{\mathbf{C}}$:

$$\begin{aligned} X_0 =& -Ad(u_c^{-1})m(x_-),\\ X_1 =& Ad(u_c^{-1})(E_{e_1-e_3} - \sqrt{-1}E_{e_2-e_3}) - \frac{1}{2}(K_{13} + \sqrt{-1}K_{12}),\\ X_2 =& \frac{1}{3}Ad(u_c^{-1})H^{(2)},\\ X_3 =& -Ad(u_c^{-1})(E_{e_1-e_3} + \sqrt{-1}E_{e_2-e_3}) + \frac{1}{2}(K_{13} - \sqrt{-1}K_{12}),\\ X_4 =& -Ad(u_c^{-1})m(x_+), \end{aligned}$$

Proof. We obtain the assertion immediately from Lemma 3.1. □

We give the matrix representation of $\Gamma^i_{l,m}$ with respect to the induced basis as follows.

Theorem 5.5. *For $i = 1,2$ and $-2 \leq m \leq 2$, we have the following equation with the matrix representation $R(\Gamma^i_{l,m}) \in M_{d(\sigma_i;l+m),d(\sigma_i;l)}(\mathbf{C})$ of $\Gamma^i_{l,m}$ with respect to the induced basis $\{T_i(l;p) \mid 0 \leq p \leq l-k,\ p \equiv l-k \bmod 2\}$:*

$$\mathfrak{C}_{l,m}\mathbf{T}_i(\sigma_i;l) = \mathbf{T}_i(\sigma_i;l+m) \cdot R(\Gamma^i_{l,m}). \tag{5.1}$$

We give the explicit expressions of the matrix

$$\begin{pmatrix} O_{n(\sigma_i;l,m),d(\sigma_i;l)} \\ R(\Gamma^i_{l,m}) \end{pmatrix}$$

by

$$\begin{pmatrix} \underset{0\le j\le d(\sigma_i;l)-1}{\mathrm{Diag}}\Big(\gamma^{(i)}_{[l,m;2j+\delta(\sigma_i;l),-1]}\Big) \\ O_{1,d(\sigma_i;l)} \end{pmatrix} + \begin{pmatrix} O_{1,d(\sigma_i;l)} \\ \underset{0\le j\le d(\sigma_i;l)-1}{\mathrm{Diag}}\Big(\gamma^{(i)}_{[l,m;2j+\delta(\sigma_i;l),0]}\Big) \end{pmatrix}$$
$$+ \begin{pmatrix} O_{2,d(\sigma_i;l)-1} & O_{2,1} \\ \underset{0\le j\le d(\sigma_i;l)-2}{\mathrm{Diag}}\Big(\gamma^{(i)}_{[l,m;2j+\delta(\sigma_i;l),1]}\Big) & O_{d(\sigma_i;l)-1,1} \end{pmatrix}.$$

Here

$$\gamma^{(i)}_{[l,m;p,1]} =(-1)^{i+1}(k-l+p)A_{[2l,2m;2l-p+m-2,0]},$$
$$\gamma^{(i)}_{[l,m;p,0]} =\frac{(-1)^i}{3}\Big(\nu_i+\rho_i+lm-3+\frac{m(m+1)}{2}\Big)A_{[2l,2m;2l-p+m,2]},$$
$$\gamma^{(i)}_{[l,m;p,-1]} =(-1)^{i+1}(k+l-p)A_{[2l,2m;2l-p+m+2,4]},$$
$$n(\sigma_i;l,m)=\begin{cases} (2-m)/2 & \textit{if } m\in\{0,\ \pm 2\},\\ (3-m)/2 & \textit{if } (m,l-k)\in\{\pm1\}\times(2\mathbf{Z}),\\ (1-m)/2 & \textit{if } (m,l-k)\in\{\pm1\}\times(1+2\mathbf{Z}),\end{cases}$$

and $\delta(\sigma_i;l)\in\{0,1\}$ *such that* $\delta(\sigma_i;l)\equiv l-k \bmod 2$.

In the above equations, we put $A_{[2l,2m;p,j]}=0$ *for* $p<0$ *or* $p>2(l+m)$, *and omit the symbols* $\underset{c_0\le n\le c_1}{\mathrm{Diag}}(f(n))$ $(c_0>c_1)$, $O_{m,n}$ $(m\le 0$ *or* $n\le 0)$.

PROOF. By the similar computation in the proof of Theorem 4.5 using Lemma 5.4 (i), we obtain the assertion in the case of $i=1$. In the case of $i=2$, the value of $T_2(l;p)$ at $u_c\in G$ is given by

$$T_2(l;p)(u_c) =\mathbf{e}^{(2l)}_{2l-p}\otimes\chi_{l-p}+(-1)^l\mathbf{e}^{(2l)}_p\otimes\chi_{p-l}.$$

Thus, by the similar computation using Lemma 5.4 (ii), we also obtain the assertion in the case of $i=2$ evaluating the both side of the equation (5.1) at $u_c\in G$. ∎

6. THE ACTION OF $\mathfrak{p}_{\mathbf{C}}$

The linear map $\Gamma^i_{l,m}$ characterize the action of $\mathfrak{p}_{\mathbf{C}}$. In this section, we give the explicit description of the action of $\mathfrak{p}_{\mathbf{C}}$ on the elementary functions.

6.1. *The projectors for* $V_l\otimes_{\mathbf{C}}V_4$. For $-2\le m\le 2$, we describe a surjective $\mathfrak{sl}(2,\mathbf{C})$-homomorphism P^l_{2m} from $V_l\otimes_{\mathbf{C}}V_4$ to V_{l+2m} in terms of the standard basis as follows.

LEMMA 6.1. *Let* $\{v^{(l)}_q \mid 0\le q\le l\}$ *be the standard basis of* V_l *for* $l\in\mathbf{Z}_{\ge 0}$. *We put* $v^{(l)}_q=0$ *when* $q<0$ *or* $q>l$.

We define linear maps $P^l_{2m}\colon V_l\otimes_{\mathbf{C}}V_4\to V_{l+2m}$ $(-2\le m\le 2)$ *by*

$$P^l_{2m}(v^{(l)}_q\otimes w_r)=B_{[l,2m;q,r]}\cdot v^{(l+2m)}_{q+r+m-2},$$

when V_{l+2m}-component of $V_l \otimes_{\mathbf{C}} V_4$ does not vanish.

Here the coefficients $B_{[l,2m;q,r]} = b(l,2m;q,r)/d'(l,2m)$ are defined by following formulae.

Formula 1: *The coefficients of $P^l_4\colon V_l \otimes_{\mathbf{C}} V_4 \to V_{l+4}$ are given as follows:*

$$b(l,4;q,r) = 1 \quad (0 \le r \le 4), \qquad d'(l,4) = 1.$$

Formula 2: *The coefficients of $P^l_2\colon V_l \otimes_{\mathbf{C}} V_4 \to V_{l+2}$ are given as follows:*

$$b(l,2;q,0) = 4q, \quad b(l,2;q,1) = -(l-4q), \quad b(l,2;q,2) = -2(l-2q),$$
$$b(l,2;q,3) = -(3l-4q), \quad b(l,2;q,4) = -4(l-q), \quad d'(l,2) = l+4.$$

Formula 3: *The coefficients of $P^l_0\colon V_l \otimes_{\mathbf{C}} V_4 \to V_l$ are given as follows:*

$$b(l,0;q,0) = 6q(q-1), \qquad b(l,0;q,1) = -3q(l-2q+1),$$
$$b(l,0;q,2) = l^2 - 6lq + 6q^2 - l, \qquad b(l,0;q,3) = 3(l-2q-1)(l-q),$$
$$b(l,0;q,4) = 6(l-q)(l-q-1), \qquad d'(l,0) = (l+3)(l+2).$$

Formula 4: *The coefficients of $P^l_{-2}\colon V_l \otimes_{\mathbf{C}} V_4 \to V_{l-2}$ are given as follows:*

$$b(l,-2;q,0) = 4q(q-1)(q-2), \quad b(l,-2;q,1) = -q(q-1)(3l-4q+2),$$
$$b(l,-2;q,2) = 2q(l-2q)(l-q),$$
$$b(l,-2;q,3) = -(l-4q-2)(l-q)(l-q-1),$$
$$b(l,-2;q,4) = -4(l-q)(l-q-1)(l-q-2), \quad d'(l,-2) = (l+2)(l+1)l.$$

Formula 5: *The coefficients of $P^l_{-4}\colon V_l \otimes_{\mathbf{C}} V_4 \to V_{l-4}$ are given as follows:*

$$b(l,-4;q,0) = q(q-1)(q-2)(q-3),$$
$$b(l,-4;q,1) = -q(q-1)(q-2)(l-q),$$
$$b(l,-4;q,2) = q(q-1)(l-q)(l-q-1),$$
$$b(l,-4;q,3) = -q(l-q)(l-q-1)(l-q-2),$$
$$b(l,-4;q,4) = (l-q)(l-q-1)(l-q-2)(l-q-3),$$
$$d'(l,-4) = (l+1)l(l-1)(l-2).$$

Then P^l_{2m} is the generator of $\mathrm{Hom}_{\mathfrak{sl}(2,\mathbf{C})}(V_l \otimes_{\mathbf{C}} V_4, V_{l+2m})$ such that $P^l_{2m} \circ I^l_{2m} = \mathrm{id}_{V_{l+2m}}$.

Proof. The composite

$$V_l \otimes_{\mathbf{C}} V_4 \simeq V_l^* \otimes_{\mathbf{C}} V_4^* \simeq (V_l \otimes_{\mathbf{C}} V_4)^* \ni f \mapsto f \circ I^l_{2m} \in V^*_{l+2m} \simeq V_{l+2m}$$

is a surjective $\mathfrak{sl}(2,\mathbf{C})$-homomorphism from $V_l \otimes_{\mathbf{C}} V_4$ to V_{l+2m}, which is unique up to scalar multiple. Therefore we obtain the assertion from Proposition 3.2 and Lemma 3.4. □

6.2. *The action of $\mathfrak{p}_{\mathbf{C}}$ on the elementary functions.*

PROPOSITION 6.2. *(i) An explicit expression of the action of $\mathfrak{p}_{\mathbf{C}}$ on the basis $\{s(l;p,q) \mid l \geq 0,\ p \in Z(\sigma_0;l),\ 0 \leq q \leq 2l\}$ of $H_{(\nu_0,\sigma_0),K}$ is given by following equation:*

$$\begin{aligned}&\pi_{(\nu_0,\sigma_0)}(X_r)s(l;p,q)\\ &\quad = \sum_{\substack{-1\leq j\leq 1\\ -2\leq m\leq 2}} \gamma^{(0)}_{[l,m;p,j]} B_{[2l,2m;q,r]} s(l+m; p+m+2j, q+m+r-2).\end{aligned}$$

Here we put

$$\gamma^{(0)}_{[0,m;0,j]} = B_{[0,2m;0,r]} = 0 \text{ for } m<2, \quad \gamma^{(0)}_{[1,m;p,j]} = B_{[2,2m;q,r]} = 0 \text{ for } m<0,$$
$$s(l;p,q) = 0 \text{ whenever } p \leq l \text{ such that } p \notin Z(\sigma_0;l) \text{ or } q<0 \text{ or } q>2l,$$
$$s(l;p,q) = (-1)^{\varepsilon(\sigma_0;l)} s(l;2l-p,q) \text{ for } p>l.$$

(ii) For $i=1,2$, the explicit expression of the action of $\mathfrak{p}_{\mathbf{C}}$ on the basis $\{t_i(l;p,q) \mid l \geq k,\ 0 \leq p \leq l-k,\ p \equiv l-k \bmod 2,\ 0 \leq q \leq 2l\}$ of $H_{(\nu_i,\sigma_i),K}$ is given by following equation:

$$\begin{aligned}&\pi_{(\nu_i,\sigma_i)}(X_r)t_i(l;p,q)\\ &\quad = \sum_{\substack{-1\leq j\leq 1\\ -2\leq m\leq 2}} \gamma^{(i)}_{[l,m;p,j]} B_{[2l,2m;q,r]} t_i(l+m; p+m+2j, q+m+r-2)\end{aligned}$$

Here we put $t_i(l;p,q) = 0$ unless $0 \leq p \leq l-k$, $p \equiv l-k \bmod 2$ and $0 \leq q \leq 2l$.

PROOF. Since

$$\pi_{(\nu_0,\sigma_0)}(X_r)s(l;p,q) = \sum_{-2\leq m\leq 2} \Gamma^0_{l,m}(S(l;p)) \circ P^l_{2m}(v^{(2l)}_q \otimes X_r),$$
$$\pi_{(\nu_i,\sigma_i)}(X_r)t_i(l;p,q) = \sum_{-2\leq m\leq 2} \Gamma^i_{l,m}(T_i(l;p)) \circ P^l_{2m}(v^{(2l)}_q \otimes X_r) \quad (i=1,2),$$

we obtain the assertion from Theorem 4.5, 5.5 and Lemma 6.1. ∎

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