

On the existence of mild solutions of a nonconvex evolution inclusion

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Abstract. *We prove a Filippov type existence theorem for mild solutions of a nonconvex evolution inclusion by applying the contraction principle in the space of selections of the multifunction instead of the space of solutions.*

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1. Introduction

In this paper we study differential inclusions of the form

$$x'(t) \in A(t)x(t) + \int_0^t K(t,s)F(s,x(s))ds, \quad x(0) = x_0, \quad (1.1)$$

where $F : [0, T] \times X \rightarrow \mathcal{P}(X)$ is a set-valued map Lipschitzian with respect to the second variable, X is a separable Banach space, $A(t)$ is the infinitesimal generator of a strongly continuous evolution system of a two parameter family $\{G(t, \tau), t \geq 0, \tau \geq 0\}$ of bounded linear operators of X into X , $D = \{(t, s) \in [0, T] \times [0, T]; t \geq s\}$, $K(.,.) : D \rightarrow \mathbf{R}$ is continuous and $x_0 \in X$.

Existence of mild solutions of problem (1.1) have been obtained in [2-7,16] etc. via fixed point techniques. The aim of our paper is to provide a Filippov type result concerning the existence of solutions to problem (1.1). We recall that for a differential inclusion defined by a Lipschitzian set-valued map with nonconvex values, Filippov's theorem ([13]) consists in proving the existence of a solution starting from a given "quasi" or "almost" solution.

Our approach is different from the ones in [2-7,16] and consists in applying the contraction principle in the space of selections of the multifunction instead of the space of solutions. In addition, as usual at a Filippov existence type theorem, our result provides an estimate between the starting "quasi" solution and the solution of the differential inclusion.

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The idea of applying the set-valued contraction principle due to Covitz and Nadler ([12]) in the space of derivatives of the solutions belongs to Tallos ([14,17]) and it was already used for obtaining similar results for other classes of differential inclusions ([9-11]).

The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove the main result.

2. Preliminaries

Let denote by I the interval $[0, T]$, $T > 0$ and let X be a real separable Banach space with the norm $|\cdot|$ and with the corresponding metric $d(\cdot, \cdot)$.

In the sequel $\{A(t); t \in I\}$ is the infinitesimal generator of the strongly continuous evolution system $G(t, s)$, $0 \leq s \leq t \leq T$.

Recall that a family of bounded linear operators $G(t, s)$ on X , $0 \leq s \leq t \leq T$ depending on two parameters is said to be a strongly continuous evolution system if there are fulfilled the following conditions: $G(s, s) = I$, $G(t, r)G(r, s) = G(t, s)$ for $0 \leq s \leq r \leq t \leq T$ and $(t, s) \rightarrow G(t, s)$ is strongly continuous for $0 \leq s \leq t \leq T$, i.e. $\lim_{t \rightarrow s, t > s} G(t, s)x = x$ for all $x \in X$.

In what follows we are concerned with the evolution inclusion

$$x'(t) \in A(t)x(t) + \int_0^t K(t, s)F(s, x(s))ds, \quad x(0) = x_0, \quad (2.1)$$

where $F : I \times X \rightarrow \mathcal{P}(X)$ is a set-valued map, X is a separable Banach space, $A(t)$ is the infinitesimal generator of a strongly continuous evolution system of a two parameter family $\{G(t, \tau), t \geq 0, \tau \geq 0\}$ of bounded linear operators of X into X , $D = \{(t, s) \in I \times I; t \geq s\}$, $K(\cdot, \cdot) : D \rightarrow \mathbf{R}$ is continuous and $x_0 \in X$.

A continuous mapping $x(\cdot) \in C(I, X)$ is called a *mild solution* of problem (2.1) if there exists a (Bochner) integrable function $f(\cdot) \in L^1(I, X)$ such that

$$f(t) \in F(t, x(t)) \quad a.e. (I) \quad (2.2)$$

$$x(t) = G(t, 0)x_0 + \int_0^t G(t, \tau) \int_0^\tau K(\tau, s)f(s)dsd\tau \quad \forall t \in I. \quad (2.3)$$

In this case we shall call $(x(\cdot), f(\cdot))$ a *trajectory-selection pair* of (2.1).

We note that relation (2.3) can be rewritten as

$$x(t) = G(t, 0)x_0 + \int_0^t U(t, s)f(s)ds \quad \forall t \in I, \quad (2.4)$$

where $U(t, s) = \int_s^t G(t, \tau)K(\tau, s)d\tau$.

In what follows the following conditions are satisfied.

Hypothesis 2.1.

- (i) $\{A(t); t \in I\}$ is the infinitesimal generator of the strongly continuous evolution system $G(t, s)$, $0 \leq s \leq t \leq T$.

(ii) $F(.,.) : I \times X \rightarrow \mathcal{P}(X)$ has nonempty closed values and for every $x \in X$, $F(.,x)$ is measurable.

(iii) There exists $L(.) \in L^1(I, \mathbf{R}_+)$ such that for almost all $t \in I$, $F(t,.)$ is $L(t)$ -Lipschitz in the sense that

$$d_H(F(t,x), F(t,y)) \leq L(t)|x - y| \quad \forall x, y \in X,$$

where $d_H(A, B)$ is the Hausdorff distance between $A, B \subset X$

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \quad d^*(A, B) = \sup\{d(a, B); a \in A\}.$$

(iv) $d(0, F(t, 0)) \leq L(t) \quad a.e. (I)$

Let $m(t) = \int_0^t L(u)du$, denote $M := \sup_{t,s \in I} |G(t,s)|$ and $M_0 := \sup_{(t,s) \in D} |K(t,s)|$ and remark that $|U(t,s)| \leq MM_0(t-s) \leq MM_0T$.

Given $\alpha \in \mathbf{R}$ we consider on $L^1(I, X)$ the following norm

$$|f|_1 = \int_0^T e^{-\alpha m(t)} |f(t)| dt, \quad f \in L^1(I, X),$$

which is equivalent with the usual norm on $L^1(I, X)$.

Consider the following norm on $C(I, X) \times L^1(I, X)$

$$|(x, f)|_{C \times L} = |x|_C + |f|_1 \quad \forall (x, f) \in C(I, X) \times L^1(I, X),$$

where, as usual, $|x|_C = \sup_{t \in I} |x(t)| \quad \forall x \in C(I, X)$.

Finally we recall some basic results concerning set valued contractions that we shall use in the sequel.

Let (Z, d) be a metric space and consider a set valued map T on Z with nonempty closed values in Z . T is said to be a λ -contraction if there exists $0 < \lambda < 1$ such that:

$$d_H(T(x), T(y)) \leq \lambda d(x, y) \quad \forall x, y \in Z.$$

If Z is complete, then every set valued contraction has a fixed point, i.e. a point $z \in Z$ such that $z \in T(z)$ ([12]).

We denote by $Fix(T)$ the set of all fixed point of the multifunction T . Obviously, $Fix(T)$ is closed.

Proposition 2.2. ([15]) *Let Z be a complete metric space and suppose that T_1, T_2 are λ -contractions with closed values in Z . Then*

$$d_H(Fix(T_1), Fix(T_2)) \leq \frac{1}{1-\lambda} \sup_{z \in Z} d_H(T_1(z), T_2(z)).$$

3. The main result

We are ready now to present an existence theorem concerning mild solutions for the Cauchy problem (2.1).

Theorem 3.1. *Let Hypothesis 2.1 be satisfied and let $\alpha > M_0MT$ and let $y(\cdot)$ be a mild solution of the problem*

$$y'(t) = A(t)y(t) + \int_0^t K(t,s)g(s)ds, \quad y(0) = y_0,$$

where $g(\cdot) \in L^1(I, X)$ and there exists $p(\cdot) \in L^1(I, \mathbf{R})$ such that

$$d(g(t), F(t, y(t))) \leq p(t), \quad \text{a.e. } (I).$$

Then for every $\varepsilon > 0$ there exists $x(\cdot)$ a mild solution of (2.1) satisfying for all $t \in I$

$$\begin{aligned} |x(t) - y(t)| &\leq \left(M + \frac{M^2M_0T}{\alpha - MM_0T}e^{\alpha m(t)}\right)|x_0 - y_0| \\ &\quad + \frac{\alpha MM_0Te^{\alpha m(t)}}{\alpha - MM_0T} \int_0^T e^{-\alpha m(s)}p(s)ds + \varepsilon. \end{aligned} \quad (3.1)$$

Proof. Let us consider $x_0 \in X, f(\cdot) \in L^1(I, X)$ and define the following set valued maps

$$M_{x_0, f}(t) = F(t, G(t, 0)x_0 + \int_0^t U(t, s)f(s)ds), \quad t \geq 0, \quad (3.2)$$

$$T_{x_0}(f) = \{\phi(\cdot) \in L^1(I, X); \quad \phi(t) \in M_{x_0, f}(t) \quad \text{a.e. } (I)\}. \quad (3.3)$$

We shall prove first that $T_{x_0}(f)$ is nonempty and closed for every $f \in L^1(I, X)$. The fact that the set valued map $M_{x_0, f}(\cdot)$ is measurable is well known. For example, the map $t \rightarrow G(t, 0)x_0 + \int_0^t U(t, s)f(s)ds$ can be approximated by step functions and we can apply Theorem III. 40 in [8]. Since the values of F are closed and X is separable with the measurable selection theorem (Theorem III.6 in [8]) we infer that $M_{x_0, f}(\cdot)$ admits a measurable selection ϕ . According to Hypothesis 2.1 one has

$$\begin{aligned} |\phi(t)| &\leq d(0, F(t, 0)) + d_H(F(t, 0), F(t, x(t))) \leq L(t)(1 + |x(t)|)] \\ &\leq L(t)(1 + M|x_0| + \int_0^t MM_0(t-s)|f(s)|ds). \end{aligned}$$

Thus integrating by parts we obtain

$$\begin{aligned} \int_0^T e^{-\alpha m(t)}|\phi(t)|dt &\leq \int_0^T e^{-\alpha m(t)}L(t)(1 + M|x_0| + \int_0^t MM_0(t-s)|f(s)|ds)dt \\ &\leq \frac{1 + M|x_0|}{\alpha} + \frac{MM_0T|f|_1}{\alpha}. \end{aligned}$$

Hence, if $\phi(\cdot)$ is a measurable selection of $M_{x_0, f}(\cdot)$, then $\phi(\cdot) \in L^1(I, X)$ and thus $T_{x_0}(f) \neq \emptyset$.

The set $T_{x_0}(f)$ is closed. Indeed, if $\phi_n \in T_{x_0}(f)$ and $|\phi_n - \phi|_1 \rightarrow 0$ then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(t) \rightarrow \phi(t)$ for a.e. $t \in I$, and we find that $\phi \in T_{x_0}(f)$.

The next step of the proof will show that $T_{x_0}(\cdot)$ is a contraction on $L^1(I, X)$.

Let $f, g \in L^1(I, X)$ be given, $\phi \in T_{x_0}(f)$ and let $\delta > 0$. Consider the following set valued map

$$G(t) = M_{x_0, g}(t) \cap \{x \in X; |\phi(t) - x| \leq L(t) \left| \int_0^t U(t, s)(f(s) - g(s))ds \right| + \delta\}.$$

Since

$$\begin{aligned} d(\phi(t), M_{x_0, g}(t)) &\leq d(F(t, G(t, 0))x_0 + \int_0^t U(t, s)f(s)ds, \\ &F(t, G(t, 0))x_0 + \int_0^t U(t, s)g(s)ds) \leq L(t) \left| \int_0^t U(t, s)(f(s) - g(s))ds \right| \end{aligned}$$

we deduce that $G(\cdot)$ has nonempty closed values. Moreover, according to Proposition III.4 in [8], $G(\cdot)$ is measurable. Let $\psi(\cdot)$ be a measurable selection of $G(\cdot)$. It follows that $\psi \in T_{x_0}(g)$ and

$$\begin{aligned} |\phi - \psi|_1 &= \int_0^T e^{-\alpha m(t)} |\phi(t) - \psi(t)| dt \\ &\leq \int_0^T e^{-\alpha m(t)} L(t) \left(\int_0^t M M_0(t-s) |f(s) - g(s)| ds \right) dt + \int_0^T \delta e^{-\alpha m(t)} dt \\ &\leq \frac{M M_0 T}{\alpha} |f - g|_1 + \delta \int_0^T e^{-\alpha m(t)} dt. \end{aligned}$$

Since δ was arbitrary, we deduce that

$$d(\phi, T_{x_0}(g)) \leq \frac{M M_0 T}{\alpha} |f - g|_1.$$

Replacing f by g we obtain

$$d(T_{x_0}(f), T_{x_0}(g)) \leq \frac{M M_0 T}{\alpha} |f - g|_1,$$

hence $T_{x_0}(\cdot)$ is a contraction on $L^1(I, X)$.

We consider next the following set-valued maps

$$\tilde{F}(t, x) = F(t, x) + p(t)B, \quad (t, x) \in I \times X,$$

$$\tilde{M}_{y_0, f}(t) = \tilde{F}(t, G(t, 0))y_0 + \int_0^t U(t, s)f(s)ds, \quad t \in I, y_0 \in X,$$

$$\tilde{T}_{y_0}(f) = \{\phi(\cdot) \in L^1(I, X); \quad \phi(t) \in \tilde{M}_{y_0, f}(t) \quad a.e. (I)\}, \quad f \in L^1(I, X),$$

where B denotes the closed unit ball in X . Obviously, $\tilde{F}(\cdot, \cdot)$ satisfies Hypothesis 2.1.

Repeating the previous step of the proof we obtain that $\tilde{T}_{y_0}(\cdot)$ is also a $\frac{MM_0T}{\alpha}$ -contraction on $L^1(I, X)$ with closed nonempty values.

We prove next the following estimate

$$d_H(T_{x_0}(f), \tilde{T}_{y_0}(f)) \leq \frac{M}{\alpha}|x_0 - y_0| + \int_0^T e^{-\alpha m(t)} p(t) dt, \quad \forall f(\cdot) \in L^1(I, X). \quad (3.4)$$

Let $\phi \in T_{x_0}(f)$, $\delta > 0$ and, for $t \in I$, define

$$G_1(t) = \tilde{M}_{y_0, f}(t) \cap \{z \in X; |\phi(t) - z| \leq L(t)|G(t, 0)| \cdot |x_0 - y_0| + p(t) + \delta\}$$

With the same arguments used for the set-valued map $G(\cdot)$, we deduce that $G_1(\cdot)$ is measurable with nonempty closed values. Let $\psi(\cdot)$ be a measurable selection of $G_1(\cdot)$. It follows that $\psi(\cdot) \in \tilde{T}_{y_0}(f)$ and one has

$$\begin{aligned} |\phi - \psi|_1 &= \int_0^T e^{-\alpha m(t)} |\phi(t) - \psi(t)| dt \leq \int_0^T e^{-\alpha m(t)} [L(t)|G(t, 0)| \cdot |x_0 - y_0| \\ &\quad + p(t) + \delta] dt \leq \frac{M}{\alpha}|x_0 - y_0| + \int_0^T e^{-\alpha m(t)} p(t) dt + \delta \int_0^T e^{-\alpha m(t)} p(t) dt. \end{aligned}$$

Since $\delta > 0$ was arbitrary, as above, we obtain (3.4). Applying Proposition 2.2 we get

$$d_H(\text{Fix}(T_{x_0}), \text{Fix}(\tilde{T}_{y_0})) \leq \frac{M}{\alpha - MM_0T}|x_0 - y_0| + \frac{\alpha}{\alpha - M_0MT} \int_0^T e^{-\alpha m(t)} p(t) dt.$$

Since $g(\cdot) \in \text{Fix}(\tilde{T}_{y_0})$ it follows that there exists $f(\cdot) \in \text{Fix}(T_{x_0})$ such that for any $\varepsilon > 0$

$$\begin{aligned} |g - f|_1 &\leq \frac{M}{\alpha - MM_0T}|x_0 - y_0| + \frac{\alpha}{\alpha - MM_0T} \int_0^T e^{-\alpha m(t)} p(t) dt \\ &\quad + \frac{\varepsilon}{MM_0T e^{\alpha m(T)}}. \end{aligned} \quad (3.5)$$

We define $x(t) = G(t, 0)x_0 + \int_0^t U(t, s)f(s)ds$, $t \in I$ and we have

$$|x(t) - y(t)| \leq M|x_0 - y_0| + MM_0T e^{\alpha m(t)} |f - g|_1.$$

Combining the last inequality with (3.5) we obtain (3.1).

Remark 3.2. If $A(t) \equiv A$ and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{G(t); t \geq 0\}$ from X to X then problem (1.1) reduces to the problem

$$x'(t) \in Ax(t) + \int_0^t K(t, s)F(s, x(s))ds, \quad x(0) = x_0, \quad (3.6)$$

well known ([1-7, 16] etc.) as an integrodifferential inclusion.

Obviously, a similar result to the one in Theorem 3.1 may be obtained for problem (3.6).

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