Adaptive Mann iterations for nonlinear accretive and pseudocontractive operator equations

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Abstract. *We construct adaptive Mann iterations for finding fixed points of strongly pseudocontractive mappings and solving nonlinear strongly accretive (not necessary continuous) operator equation $Sx = f$ in p-smooth Banach spaces.*

Key words: *Mann iterations, strongly accretive mappings, strongly pseudocontractive mappings, p-smooth Banach spaces, Banach spaces smooth of power type*

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1. Introduction

We consider the problem of solving the equation

$$Sx = f \tag{1}$$

for a strongly accretive operator $S : X \to X$, where X is a Banach space. This problem is closely related to finding a fixed point of the (strongly pseudocontractive) operator $T : X \to X$ defined via $Tx = f + x - Sx$. The aim of this paper is to construct *problem-adapted* iterative solvers of equation (1) for a wide range of continuous and discontinuous operators S.

In the following, we review the general framework for these kinds of iterative solvers. It is well-known (cf. e.g. [2, 3, 5, 8, 16]) that Mann [12] and Ishikawa [11] iterations are well suited methods for finding the solutions of $Sx = f$. We recall that for a nonempty, convex subset C of a Banach space X and T a mapping of C into itself the sequence (x_n) is a Mann iteration if for some nonnegative sequence (α_n) the recursion

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n \qquad \text{with } x_0 \in C \tag{2}$$

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is satisfied. The sequence (x_n) is an Ishikawa iteration if for some nonnegative sequences (α_n) and (β_n) the recursion

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n \qquad \text{with } x_0 \in C \end{aligned}$$

is satisfied.

Recently Noor, Rassias and Huang [15] introduced a new three-step iterative method (known as Noor iteration) for solving nonlinear equations in Banach spaces. In the case where X is a Hilbert space such iterative methods were already considered in [13] and [14]. We recall that for a nonempty, convex subset C of a Banach space X and T a mapping of C into itself the sequence (x_n) is a Noor iteration if the recursion

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n \qquad \text{with } x_0 \in C \end{aligned}$$

is satisfied for some nonnegative sequences (α_n), (β_n) and (γ_n). In 2004 Rhoades and Soltuz [17] introduced a general multistep-iteration for solving nonlinear equations in Banach spaces. We recall again that for a nonempty, convex subset C of a Banach space X and T a mapping of C into itself the sequence (x_n) is a general multi-step iteration if for some integer $p \geq 2$ and some nonnegative sequences (α_n) and $(\beta_n^1), \dots, (\beta_n^{p-1})$ the recursion

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n^i &= (1-\beta_n^i)x_n + \beta_n^i T y_n^{i+1} \qquad \text{for } 1 \leq i \leq p-2. \\ y_n^{p-1} &= (1-\beta_n^{p-1})x_n + \beta_n^{p-1} T x_n \qquad \text{with } x_0 \in C \end{aligned}$$

is satisfied. It is clear that Mann iteration is a special case of Ishikawa iteration, which again is a special case of Noor iteration, which itself is a special case of general multi-step iteration.

It was shown that the iterations mentioned above are equivalent for a wide range of operators. Rhoades and Soltuz [17] have shown that for uniformly smooth X and continuous, strongly pseudocontractive operator S the Mann iteration for a certain choice of (α_n) converges to a solution of $Sx = f$ if and only if the general multi-step iteration converges to a solution of $Sx = f$ for (α_n), (β_n^i) satisfying

$$\lim_n \alpha_n = \lim_n \beta_n^i = 0 \qquad \text{and} \qquad \sum_n \alpha_n = \infty \tag{3}$$

where $\alpha_n \in (0,1)$, $\beta_n^i \in [0,1)$. Under some continuity assumptions similar results were also obtained for ψ-uniformly accretive and other operator types (cf. e.g. [18]).

For arbitrary Banach spaces Osilike [16] has shown that the Ishikawa iteration converges strongly to the solution of $Sx = f$, if $C = X$, S is strongly accretive, Lipschitz continuous and the sequences (α_n) and (β_n) are chosen such that

$$\lim_n \alpha_n = \lim_n \beta_n = 0 \qquad \text{and} \qquad \sum_n \alpha_n = \infty. \tag{4}$$

Generally speaking, the convergence analysis for all methods presented above depends heavily on

- the geometry of the Banach space X (and the subset C),
- additional properties of the accretive operator S, and
- properties of sequences (α_n) and (β_n^i).

The Banach spaces and properties of S are usually given by the underlying problem, e.g. regularization of some Cauchy-type partial differential equation. If some sequences (α_n) and (β_n^i) fulfill (3), then $c\cdot(\alpha_n)$ and $c\cdot(\beta_n^i)$ also fulfill condition (3) for any $c > 0$. In real-life applications the main problem is to choose numerically appropriate sequences $c \cdot (\alpha_n)$ and $c \cdot (\beta_n^i)$.

Chidume and Mutangadura [4] showed that Mann iterations cannot be extended to merely pseudocontractive or accretive mappings. They showed, that for this type of operators Ishikawa iterations are more appropriate.

However, multi-step iterations do not a priori perform better than Mann iterations. For example, for the operator $Tx = -x$ every Ishikawa iteration converges at best at the same rate as the related Mann iteration. To be precise: consider an operator of the type $Tx = tx$, where t is a negative number. Then T is Lipschitz continuous and strongly pseudocontractive (Definition 2.1). We further know that the only fixed point of T is $x^* = 0$. We start a Mann iteration (x_n^M) and a Noor iteration (x_n^N) with $x_0^M = x_0^N$ and the same choice of (α_n), then $\|x_n^N - x^*\| \leq \|x_n^M - x^*\|$ if and only if

$$\beta_n(t^2 - t)(1 + t\gamma_n) \leq 0.$$

For this special type of operators we can read this as: every Ishikawa iteration converges at best as fast as the related Mann iteration. Noor iteration may converge faster than the related Mann iteration. However, due to the usual assumption $\gamma_n \to 0$ Noor iteration may perform worse than the related Mann iteration for large values of n. In light of the equivalence results and the examples stated above we decided to consider only the simplest form of the iterations mentioned above, i.e. the Mann iteration.

The aim of this paper is to construct a *problem-adapted* sequence (α_n) such that the related Mann iteration converges for a wide range of discontinuous operators S. In particular, we shall not impose any conditions of the form (3) resp. (4) on (α_n). We will not impose any continuity conditions on the operator S and - in contrast to e.g. [15] - we will not require that $(I - S)$ has bounded range (if S is accretive) or that T has a bounded range (if T is pseudocontractive).

In the next section, we give the necessary theoretical tools and apply them in sections 3. and 4. to construct problem-adapted Mann iterations.

2. Preliminaries

For $p > 1$ the set-valued mapping $J_p : X \to 2^{X^*}$ given by

$$J_p(x) = \{x^* \in X^* \ : \ \langle x^*, x\rangle = \|x^*\|\|x\|, \|x^*\| = \|x\|^{p-1}\},$$

where X^* denotes a dual space of X and $\langle \cdot, \cdot \rangle$ the duality pairing is called the *duality mapping* of X. By j_p we denote a single-valued selection of J_p.

Definition 2.1. *A map $T : C \to C$, where C is a convex, nonempty subset of X, is called* strongly pseudocontractive*, if there exists a constant $k > 0$ such that for each x and y in X there is a $j_2(x-y) \in J_2(x-y)$ satisfying*

$$\langle Tx - Ty, j_2(x-y) \rangle \leq (1-k)\|x-y\|^2.$$

Definition 2.2. *An operator $S : X \to X$ is called* strongly accretive*, if there exists a constant $\kappa > 0$ such that for each x and y in X there exists a $j_2(x-y) \in J_2(x-y)$ satisfying*

$$\kappa \|x-y\|^2 \leq \langle Sx - Sy, j_2(x-y) \rangle.$$

Remark 2.3. *We remark that*

$$J_p(x) = \|x\|^{p-2} J_2(x) \qquad \text{for all } x \neq 0$$

[6, Th. I.4.7.f]. Therefore, a strongly pseudocontractive mapping is also characterized by the relation

$$\langle Tx - Ty, j_p(x-y) \rangle \leq (1-k)\|x-y\|^p$$

for arbitrary $p > 1$. Strongly accretive mappings can analogously be characterized by

$$\kappa \|x-y\|^p \leq \langle Sx - Sy, j_p(x-y) \rangle$$

for arbitrary $p > 1$.

Definition 2.4. *A Banach space X is said to be p-smooth, if there exists a positive constant G_p such that for all x and y in X*

$$\|x-y\|^p \leq \|x\|^p - p\langle J_p(x), y \rangle + G_p \|y\|^p.$$

Remark 2.5. *It is well known [19], [9, Ch. IV.], that a Banach space X is p-smooth, if and only if there exists a positive constant g_p such that the* modulus of smoothness *defined by*

$$\rho_X(\tau) = \frac{1}{2} \sup\{\|x+y\| + \|x-y\| - 2 : \|x\| = 1, \|y\| \leq \tau\}$$

fulfills

$$\rho_X(\tau) \leq g_p \tau^p.$$

Therefore, p-smooth spaces are sometimes referred to as Banach spaces smooth of power type p.

For the sequence spaces ℓ_p, Lebesgue spaces L_p and Sobolev spaces W_p^m it is known [10, 19] that

$$\ell_p, L_p, W_p^m \text{ with } 1 < p \leq 2 \text{ are p-smooth}$$

and

$$\ell_q, L_q, W_q^m \text{ with } 2 \le q < \infty \text{ are } 2\text{-smooth.}$$

For Hilbert spaces polarization identity $\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$ *asserts that every Hilbert space is* 2*-smooth.*

3. Adaptive Mann iterations on the whole space

In this section we construct adaptive Mann iterations for strongly pseudocontractive and strongly accretive operators mapping X to X.

Let X be a p-smooth Banach space, $T : X \to X$ strongly pseudocontractive and suppose T has a fixed point. If T has a fixed point, then by the strong pseudocontractivity this fixed point is unique. Assume that there exist two different fixed points x^* and y^*. Then

$$\begin{aligned} 0 < \|x^* - y^*\|^2 &= \langle x^* - y^*, j_2(x-y)\rangle \\ &= \langle Tx^* - Ty^*, j_2(x-y)\rangle \\ &\le (1-k)\|x^* - y^*\|^2. \end{aligned}$$

But this is a contradiction to the assumption that $k > 0$. We denote the unique fixed point of T by x^*.

Since X is p-smooth, for the Mann iterates as defined in (2) we get

$$\begin{aligned} \|x_{n+1} - x^*\|^p &= \|(1-\alpha_n)x_n - x^* + \alpha_n T x_n\|^p \\ &= \|x_n - x^* - \alpha_n(x_n - Tx_n)\|^p \\ &\le \|x_n - x^*\|^p - \alpha_n \cdot p\langle x_n - Tx_n, j_p(x_n - x^*)\rangle + G_p \cdot \alpha_n^p \|x_n - Tx_n\|^p. \end{aligned}$$

Consider

$$\begin{aligned} -\langle x_n - Tx_n, j_p(x_n - x^*)\rangle &= -\langle x_n - x^* - (Tx_n - Tx^*), j_p(x_n - x^*)\rangle \\ &= -\langle x_n - x^*, j_p(x_n - x^*)\rangle + \langle Tx_n - Tx^*, j_p(x_n - x^*)\rangle \\ &\le -\|x_n - x^*\|^p + (1-k)\|x_n - x^*\|^p \\ &= -k\|x_n - x^*\|^p. \end{aligned}$$

This implies the important inequality

$$\|x_{n+1} - x^*\|^p \le \|x_n - x^*\|^p - \alpha_n \cdot pk\|x_n - x^*\|^p + \alpha_n^p \cdot G_p \|x_n - Tx_n\|^p. \tag{5}$$

We see that the choice

$$\alpha_n = \left(\frac{k}{G_p} \cdot \frac{\|x_n - x^*\|^p}{\|x_n - Tx_n\|^p}\right)^{\frac{1}{p-1}} \tag{6}$$

minimizes the right-hand side in equation (5). We do not know the exact value of $\|x_n - x^*\|^p$. Assume that we know an upper estimate R_n on $\|x_n - x^*\|^p$, i.e.

$$\|x_n - x^*\|^p \le R_n.$$

Then we replace $\|x_n - x^*\|^p$ in (6) by $\beta_n R_n$. We shall impose additional conditions on β_n later, for the time being assume that $\beta_n \in (0,1)$. We introduce auxiliary variables

$$D_n := \left(\frac{k}{G_p} \cdot \frac{R_n}{\|x_n - Tx_n\|^p}\right)^{\frac{1}{p-1}} \qquad \text{and} \qquad q := \frac{p}{p-1}. \tag{7}$$

Then we can write

$$\alpha_n = D_n \beta_n^{q-1}.$$

By (5) we get for x_n with $\|x_n - x^*\|^p \geq \beta_n R_n$

$$\begin{aligned} \|x_{n+1} - x^*\|^p &\leq R_n - D_n\beta_n^{q-1} \cdot pk \cdot \beta_n R_n + \left(D_n\beta_n^{q-1}\right)^p \cdot G_p\|x_n - Tx_n\|^p. \\ &= R_n - pkD_n\beta_n^q R_n + kD_n\beta_n^q R_n, \end{aligned}$$

implying

$$\|x_{n+1} - x^*\|^p \leq (1 - pkD_n\beta_n^q + kD_n\beta_n^q)\, R_n. \tag{8}$$

If on the other hand $\|x_n - x^*\|^p < \beta_n R_n$, we conclude again with (5)

$$\|x_{n+1} - x^*\|^p \leq (\beta_n + kD_n\beta_n^q)\, R_n. \tag{9}$$

The main idea in our construction is that the optimal choice of β_n minimizes the worst case in the estimations in (8) and (9). Thus β_n minimizes

$$\max\{1 - pkD_n\beta_n^q + kD_n\beta_n^q, \beta_n + kD_n\beta_n^q\}.$$

for fixed p, k and D_n. The term $1 - pkD_n\beta_n^q + kD_n\beta_n^q$ is monotonically decreasing in β_n and $\beta_n + kD_n\beta_n^q$ is monotonically increasing in β_n. So we conclude that the above maximum is minimal, if

$$1 - pkD_n\beta_n^q + kD_n\beta_n^q = \beta_n + kD_n\beta_n^q$$

or equivalently

$$pkD_n\beta_n^q = 1 - \beta_n. \tag{10}$$

If β_n fulfills (10), then by (8) or (9) we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^p &\leq (\beta_n + kD_n\beta_n^q)\, R_n \\ &= \left(\beta_n + \tfrac{1}{p}(1 - \beta_n)\right) R_n \\ &= \left(\tfrac{1}{p} + \left(1 - \tfrac{1}{p}\right)\beta_n\right) R_n. \end{aligned}$$

Thus the number R_{n+1} defined by

$$R_{n+1} := \left(\tfrac{1}{p} + \left(1 - \tfrac{1}{p}\right)\beta_n\right) R_n \tag{11}$$

fulfills

$$\|x_{n+1} - x^*\|^p \leq R_{n+1} < R_n,$$

where the right-hand side is true due to

$$R_{n+1} = R_n \iff \beta_n = 1 \iff R_n = 0 \text{ or } \|x_n - Tx_n\| = 0 \iff x_n = x^*.$$

Here we implicitly assume that the iteration stops if x_n is a fixed point. Now we state the following algorithm.

Algorithm 3.1 [Adaptive Mann iteration].

(S_0) *Choose an arbitrary* $x_0 \in X$ *with* $\|x_0 - x^*\|^p \leq R_0$*. Set* $n = 0$*.*

(S_1) *Stop if* $Tx_n = x_n$*, else compute the unique positive solution* β_n *of the equation*

$$pk \left(\frac{k}{G_p} \cdot \frac{R_n}{\|x_n - Tx_n\|^p} \right)^{\frac{1}{p-1}} \cdot \beta_n^{\frac{p}{p-1}} = 1 - \beta_n.$$

(S_2) *Set*

$$\alpha_n = \frac{1}{pk} \cdot \frac{1 - \beta_n}{\beta_n}$$
$$R_{n+1} = \left(\tfrac{1}{p} + \left(1 - \tfrac{1}{p} \right) \beta_n \right) R_n.$$

(S_3) *Set*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T x_n.$$

(S_4) *Let* $n \leftarrow (n+1)$ *and go to step* (S_1)*.*

Theorem 3.2. *Let* X *be a* p*-smooth Banach space and* $T : X \to X$ *a strongly pseudocontractive map with constants* G_p *and* k *as in Definitions 2.4 and 2.1. Suppose* T *maps bounded sets on bounded sets and suppose* T *has a fixed point in* X*. Then the sequence* (x_n) *defined by the Mann iteration of Algorithm 3.1 converges strongly to the fixed point of* T*.*

Proof. The formulas in (S_1) and (S_2) follow directly from (6), (7), (10) and (11). We prove now the strong convergence. The sequence (R_n) is by construction bounded and monotonically decreasing, hence convergent. Therefore the sequence $(R_n - R_{n+1})$ is a zero sequence. We have

$$0 \leq R_n - R_{n+1} = \left(\left(1 - \tfrac{1}{p} \right) - \left(1 - \tfrac{1}{p} \right) \beta_n \right) R_n = \left(1 - \tfrac{1}{p} \right) (1 - \beta_n) R_n.$$

The sequence $(R_n - R_{n+1})$ is a zero sequence if $R_n \to 0$ or $\beta_n \to 1$. In the first case we get strong convergence due to $R_n \geq \|x_n - x^*\|^p \to 0$. On the other hand $\beta_n \to 1$, if and only if

$$\frac{R_n}{\|x_n - Tx_n\|^p} \to 0.$$

Again this is the case if $R_n \to 0$ or $\|x_n - Tx_n\| \to \infty$. We proved already that $R_n \to 0$ ensures a strong convergence. We have

$$\|x_n - x^*\|^p \leq R_n < \cdots < R_0.$$

Therefore the sequence (x_n) is bounded. The map T maps bounded sets on bounded sets, hence there exists a constant M, such that for all n

$$\|x_n - Tx_n\| \le \|x_n - x^*\| + \|Tx_n - Tx^*\| \le M < \infty.$$

Therefore $\|x_n - Tx_n\|$ cannot diverge. Altogether we proved $\|x_n - x^*\| \to 0$. □

It is well known that strongly pseudocontractive mappings are closely related to strongly accretive mappings.

Remark 3.3. *Let S be strongly accretive with constant k, i.e.*

$$k\|x-y\|^2 \le \langle Sx - Sy, j_2(x-y)\rangle. \tag{12}$$

For $f \in X$ we define the operator $T : X \to X$ by

$$T(x) = f + x - Sx$$

and have

$$\begin{aligned}\langle Tx - Ty, j_2(x-y)\rangle &= \langle (f + x - Sx) - (f + y - Sy), j_2(x-y)\rangle \\ &\le \|x-y\|^2 - k\|x-y\|^2.\end{aligned}$$

Thus T is strongly pseudocontractive with constant k. If S maps bounded sets on bounded sets then also T maps bounded sets on bounded sets. A solution of the equation

$$Sx = f$$

is a fixed point of T. As a consequence of Theorem 3.2 and 3.3 we get the following theorem:

Theorem 3.4. *Let X be a p-smooth Banach space with constant G_p (cf. Definition 2.4), $f \in X$ and $S : X \to X$ strongly accretive with constant k (cf.* (12)*). Suppose S maps bounded sets on bounded sets and suppose the equation*

$$Sx = f$$

has a solution. Then this solution is unique and the sequence (x_n) defined by the Mann iteration of Algorithm 3.1 with $T : X \to X$ and

$$T(x) = f + x - Sx$$

converges strongly to this unique solution.

Remark 3.5. *If in addition to strong pseudocontractivity in Theorem 3.2 we assume that T is Lipschitz continuous, then T maps bounded sets on bounded sets and the existence of the fixed point of T follows from [1, 7].*

Remark 3.6. *If in addition to strong accretivity in Theorem 3.4, we assume that S is Lipschitz continuous, then S maps bounded sets on bounded sets and the existence of the solution of $Sx = f$ follows from the fact that the related operator $T = f + I - S$ is also Lipschitz continuous and therefore has a fixed point.*

Corollary 3.7. *Let X be a p-smooth Banach space with constant G_p (cf. Definition 2.4), and $T : X \to X$ strongly pseudocontractive with constant k (cf. Definition 2.1), then the sequence (x_n) defined by the Mann iteration of Algorithm 3.1 converges strongly to the fixed point of T.*

Corollary 3.8. *Let X be a p-smooth Banach space with constant G_p (cf. Definition 2.4), $f \in X$ and $S : X \to X$ strongly accretive with constant k (cf. (12)), then the sequence (x_n) defined by the Mann iteration of Algorithm 3.1 with $T : X \to X$ and $T(x) = f + x - Sx$ converges strongly to the solution of $Sx = f$.*

4. Adaptive Mann iterations on convex subsets

Operator T is often only a self-mapping between some convex subset C of X. In this case we must alternate our Algorithm 3.1.

First we ensure that all iterates stay in the convex set C. We set

$$\alpha_n = \min\{1, \frac{1}{pk} \cdot \frac{1-\beta_n}{\beta_n}\}$$

where β_n is again the unique positive solution of (10).

Next we must choose an appropriate value for R_{n+1}. If $\alpha_n < 1$, then all computations in the last section are still valid and we can choose R_{n+1} as in (11). Therefore assume that $\alpha_n = 1$ resp.

$$\frac{1}{pk} \cdot \frac{1-\beta_n}{\beta_n} \geq 1. \tag{13}$$

Again we distinguish the cases $\|x_n - x^*\|^p \geq \beta_n R_n$ and $\|x_n - x^*\| < \beta_n R_n$. In case $\|x_n - x^*\|^p \geq \beta_n R_n$ we have

$$\|x_{n+1} - x^*\|^p \leq R_n - pk\beta_n R_n + G_p \|x_n - Tx_n\|^p.$$

Otherwise

$$\|x_{n+1} - x^*\|^p \leq \beta_n R_n + G_p \|x_n - Tx_n\|^p.$$

We see that

$$R_n - pk\beta_n R_n + G_p \|x_n - Tx_n\|^p \geq \beta R_n + G_p \|x_n - Tx_n\|^p$$

as this is equivalent to (13). Thus we set

$$R_{n+1} := (1 - pk\beta_n) R_n + G_p \|x_n - Tx_n\|^p.$$

Then we can state the following algorithm.

Algorithm 4.1 [Adaptive Mann iteration on convex sets].

(S_0) *Choose an arbitrary $x_0 \in C$ with $\|x_0 - x^*\|^p \leq R_0$. Set $n = 0$.*

(S_1) *Stop if $Tx_n = x_n$, else compute the unique positive solution β_n of the equation*

$$pk \left(\frac{k}{G_p} \cdot \frac{R_n}{\|x_n - Tx_n\|^p} \right)^{\frac{1}{p-1}} \cdot \beta_n^{\frac{p}{p-1}} = 1 - \beta_n.$$

(S_2) Set

$$\alpha_n = \min\{1, \frac{1}{pk} \cdot \frac{1-\beta_n}{\beta_n}\}$$

$$R_{n+1} = \begin{cases} \left(\frac{1}{p} + \left(1 - \frac{1}{p}\right)\beta_n\right) R_n & \text{if } \alpha_n < 1, \\ (1 - pk\beta_n)R_n + G_p\|x_n - Tx_n\|^p & \text{if } \alpha_n = 1. \end{cases}$$

(S_3) Set

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n.$$

(S_4) Let $n \leftarrow (n+1)$ and go to step (S_1).

Then we can prove to following theorem.

Theorem 4.2. *Let X be a p-smooth Banach space, C a closed convex subset of X and $T : C \to C$ a strongly pseudocontractive map with constants G_p and k as in Definitions 2.4 and 2.1. Suppose T maps bounded sets on bounded sets and suppose T has a fixed point in C. Then the sequence (x_n) defined by the Mann iteration of Algorithm 4.1 converges strongly to the fixed point of T.*

Proof. The proof is similar to the proof of Theorem 3.2 and we only give the main modifications. If $\alpha_n = 1$ then

$$\frac{1}{pk} \cdot \frac{1-\beta_n}{\beta_n} = D_n\beta_n^{q-1} \geq 1$$

where D_n is the same as in (7). Since $\beta_n < 1$, it follows that

$$\frac{k}{G_p}\frac{R_n}{\|x_n - Tx_n\|^p} \geq 1. \tag{14}$$

This implies

$$\begin{aligned} R_{n+1} &= (1 - pk\beta_n)R_n + G_p\|x_n - Tx_n\|^p \\ &\leq (1 - pk\beta_n + k\beta_n)R_n. \end{aligned}$$

Therefore

$$R_n - R_{n+1} = (p-1)k\beta_n R_n.$$

Then $(R_n - R_{n+1})$ is only a zero sequence, if $R_n \to 0$ or $\beta_n \to 0$. In the first case we already proved strong convergence. On the other hand, $\beta_n \to 0$, if and only if $R_n \to \infty$ or $\|x_n - Tx_n\| \to 0$. The sequence (R_n) is bounded and therefore cannot diverge. Assume $\|x_n - Tx_n\| \to 0$, then (x_n) converges to a fixed point $x^\dagger$ of T. The uniqueness of the fixed point forces $x^\dagger = x^*$. □

5. Conclusions

With the characteristic inequalities of Xu and Roach [19] the approach of Algorithms 3.1 and 4.1 can be easily extended to all uniformly smooth Banach spaces. The main ideas of Section 3. can still be carried out and the proofs are similar.

We remark that our approach can also be extended to set-valued strong pseudo-contractions and strongly accretive mappings without any difficulties.

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