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VORONOVSKAJA-TYPE THEOREM FOR CERTAIN GBS OPERATORS

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Abstract. In this paper we will demonstrate a Voronovskaja-type theorem and approximation theorem for GBS operator associated to a linear positive operator.

1. Introduction

In this section, we recall some notions and results which we will use in this article.

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}$, let $B_m : C([0,1]) \to C([0,1])$ the Bernstein operators, defined for any function $f \in C([0,1])$ by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.1}$$

where $p_{m,k}(x)$ are the fundamental polynomials of Bernstein, defined as follows

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \tag{1.2}$$

for any $x \in [0,1]$ and any $k \in \{0, 1, \dots, m\}$ (see [7,28]).

In 1932, E. Voronovskaja in the paper [31], proved the result contained in the following theorem.

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Theorem 1.1. *Let $f \in C([0,1])$ be a two times derivable function in the point $x \in [0,1]$. Then the equality*

$$\lim_{m\to\infty} m\left[(B_m f)(x) - f(x)\right] = \frac{x(1-x)}{2} f''(x) \tag{1.3}$$

holds.

Let $p \in \mathbb{N}_0$. For $m \in \mathbb{N}$, F. Schurer (see [26]) introduced and studied in 1962 the operators $\widetilde{B}_{m,p} : C([0,1+p]) \to C([0,1])$, named Bernstein-Schurer operators, defined for any function $f \in C([0,1+p])$ by

$$\left(\widetilde{B}_{m,p} f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.4}$$

where $\widetilde{p}_{m,k}(x)$ denotes the fundamental Bernstein-Schurer polynomials, defined as follows

$$\widetilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k} = p_{m+p,k}(x) \tag{1.5}$$

for any $x \in [0,1]$ and any $k \in \{0,1,\ldots,m+p\}$.

For $m \in \mathbb{N}$, let the operators $M_n : L_1([0,1]) \to C([0,1])$ defined for any function $f \in L_1([0,1])$ by

$$(M_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_0^1 p_{m,k}(t) f(t) dt, \tag{1.6}$$

for any $x \in [0,1]$.
These operators were introduced in 1967 by J. L. Durrmeyer in [11] and were studied in 1981 by M. M. Derriennic in [9].

For $m \in \mathbb{N}$, let the operators $K_m : L_1([0,1]) \to C([0,1])$ defined for any function $f \in L_1([0,1])$ by

$$(K_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t) dt, \tag{1.7}$$

for any $x \in [0,1]$.

The operators K_m, $m \in \mathbb{N}$, are named Kantorovich operators, introduced and studied in 1930 by L. V. Kantorovich (see [14]).

For the following construction see [18].

Define the natural number m_0 by

$$m_0 = \begin{cases} \max\{1, -[\beta]\}, & \text{if} \quad \beta \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1, 1-\beta\}, & \text{if} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.8}$$

For the real number β, we have that

$$m+\beta \geq \gamma_\beta \tag{1.9}$$

for any natural number m, $m \geq m_0$, where

$$\gamma_\beta = m_0+\beta = \begin{cases} \max\big\{1+\beta, \{\beta\}\big\}, & \text{if} \quad \beta \in \mathbb{R}\backslash\mathbb{Z} \\ \max\{1+\beta, 1\}, & \text{if} \quad \beta \in \mathbb{Z}. \end{cases} \tag{1.10}$$

For the real numbers α, β, $\alpha \geq 0$, we note

$$\mu^{(\alpha,\beta)} = \begin{cases} 1, & \text{if} \quad \alpha \leq \beta \\ 1+\dfrac{\alpha-\beta}{\gamma_\beta}, & \text{if} \quad \alpha > \beta. \end{cases} \tag{1.11}$$

For the real numbers α and β, $\alpha \geq 0$, we have that $1 \leq \mu^{(\alpha,\beta)}$ and

$$0 \leq \frac{k+\alpha}{m+\beta} \leq \mu^{(\alpha,\beta)} \tag{1.12}$$

for any natural number m, $m \geq m_0$ and for any $k \in \{0, 1, \ldots, m\}$.

For the real numbers α and β, $\alpha \geq 0$, m_0 and $\mu^{(\alpha,\beta)}$ defined by (1.8)-(1.11), let the operators $P_m^{(\alpha,\beta)} : C\big([0, \mu^{(\alpha,\beta)}]\big) \to C\big([0,1]\big)$, defined for any function $f \in C\big([0, \mu^{(\alpha,\beta)}]\big)$ by

$$\big(P_m^{(\alpha,\beta)} f\big)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.13}$$

for any natural number $m, m \geq m_0$ and for any $x \in [0,1]$.

These operators are named Stancu operators, introduced and studied in 1969 by D. D. Stancu in the paper [27]. In [27], the domain of definition of the Stancu operators is $C([0,1])$ and the numbers α and β verify the condition $0 \leq \alpha \leq \beta$.

In 1980, G. Bleimann, P. L. Butzer and L. Hahn introduced in [6] a sequence of linear positive operators $(L_m)_{m\geq 1}$, $L_m : C_B([0,\infty)) \to C_B([0,\infty))$, defined for any function $f \in C_B([0,\infty))$ by

$$(L_m f)(x) = \frac{1}{(1+x)^m} \sum_{k=0}^{m} \binom{m}{k} x^k f\left(\frac{k}{m+1-k}\right), \tag{1.14}$$

for any $x \in [0,\infty)$ and any $m \in \mathbb{N}$, where $C_B([0,\infty)) = \{f \mid f : [0,\infty) \to \mathbb{R},\ f$ bounded and continuous on $[0,\infty)\}$.

For $m \in \mathbb{N}$ consider the operators $S_m : C_2\left([0,\infty)\right) \to C\left([0,\infty)\right)$ defined for any function $f \in C_2\left([0,\infty)\right)$ by

$$\left(S_m f\right)(x) = e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{k}{m}\right), \tag{1.15}$$

for any $x \in [0,\infty)$, where $C_2([0,\infty)) = \Big\{ f \in C([0,\infty)) : \lim\limits_{x\to\infty} \dfrac{f(x)}{1+x^2}$ exists and is finite $\Big\}$.

The operators $(S_m)_{m\geq 1}$ are named Mirakjan-Favard-Szász operators and were introduced in 1941 by G. M. Mirakjan in [16].

They were intensively studied by J. Favard in 1944 in [12] and O. Szász in 1950 in [29].

Let for $m \in \mathbb{N}$ the operators $V_m : C_2([0,\infty)) \to C([0,\infty))$ be defined for any function $f \in C_2([0,\infty))$ by

$$(V_m f)(x) = (1+x)^{-m} \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x}{1+x} \right)^k f\left(\frac{k}{m} \right), \tag{1.16}$$

for any $x \in [0,\infty)$.

The operators $(V_m)_{m\geq 1}$ are named Baskakov operators and they were introduced in 1957 by V. A. Baskakov in [4].

W. Meyer-König and K. Zeller have introduced in [15] a sequence of linear and positive operators. After a slight adjustment given by E. W. Cheney and A. Sharma in [8], these operators take the form $Z_m : B([0,1)) \to C([0,1))$, defined for any function $f \in B([0,1))$ by

$$(Z_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k}{k} (1-x)^{m+1} x^k f\left(\frac{k}{m+k} \right), \tag{1.17}$$

for any $m \in \mathbb{N}$ and for any $x \in [0,1)$.

These operators are named the Meyer-König and Zeller operators.

Observe that $Z_m : C([0,1]) \to C([0,1])$, $m \in \mathbb{N}$.

In the paper [13], M. Ismail and C. P. May consider the operators $(R_m)_{m\geq 1}$.

For $m \in \mathbb{N}$, $R_m : C([0,\infty)) \to C([0,\infty))$ is defined for any function $f \in C([0,\infty))$ by

$$(R_m f)(x) = e^{-\frac{mx}{1+x}} \sum_{k=0}^{\infty} \frac{m(m+k)^{k-1}}{k!} \left(\frac{x}{1+x} \right)^k e^{-\frac{kx}{1+x}} f\left(\frac{k}{m} \right) \tag{1.18}$$

for any $x \in [0,\infty)$.

We consider $I \subset \mathbb{R}$, I an interval and we shall use the following functions sets: $E(I)$, $F(I)$ which are subsets of the set of real functions defined on I, $B(I) = \{ f \mid f : I \to \mathbb{R},\ f \text{ bounded on } I \}$, $C(I) = \{ f \mid f : I \to \mathbb{R},\ f \text{ continuous on } I \}$ and $C_B(I) = B(I) \cap C(I)$.

If $f \in B(I)$, then the first order modulus of smoothness of f is the function $\omega(f;\cdot) : [0,\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by

$$\omega(f;\delta) = \sup \{ |f(x') - f(x'')| : x', x'' \in I, |x' - x''| \leq \delta \}. \tag{1.19}$$

Let $I, J \subset \mathbb{R}$ intervals, $E(I \times J)$, $F(I \times J)$ which are subsets of the set of real functions defined on $I \times J$ and $L : E(I \times J) \to F(I \times J)$ be a linear positive operator.

The operator $UL : E(I \times J) \to F(I \times J)$ defined for any function $f \in E(I \times J)$, any $(x, y) \in I \times J$ by

$$(ULf)(x,y) = (L(f(x,*) + f(\cdot\,,y) - f(\cdot\,,*))\,(x,y) \tag{1.20}$$

is called GBS operator ("Generalized Boolean Sum" operator) associated to the operator L, where "$\cdot$" and "$*$" stand for the first and second variable (see [3]).

If $f \in E(I \times J)$ and $(x, y) \in I \times J$, let the functions $f_x = f(x, *)$, $f^y = f(\cdot\,, y) : I \times J \to \mathbb{R}$, $f_x(s,t) = f(x,t)$, $f^y(s,t) = f(s,y)$ for any $(s,t) \in I \times J$. Then, we can consider that f_x, f^y are functions of real variable, $f_x : J \to \mathbb{R}$, $f_x(t) = f(x,t)$ for any $t \in J$ and $f^y : I \to \mathbb{R}$, $f^y(s) = f(s,y)$ for any $s \in I$.

2. Preliminaries

For the following construction and result see [19] and [21], where $p_m = m$ for any $m \in \mathbb{N}$ or $p_m = \infty$ for any $m \in \mathbb{N}$.

Let I, J be intervals with $I \cap J \neq \emptyset$. For any $m \in \mathbb{N}$ and $k \in \{0, 1, ..., p_m\} \cap \mathbb{N}_0$ consider the functions $\varphi_{m,k} : J \to \mathbb{R}$ with the property that $\varphi_{m,k}(x) \geq 0$ for any $x \in J$ and the linear positive functionals $A_{m,k} : E(I) \to \mathbb{R}$.

Definition 2.1. For $m \in \mathbb{N}$ define the operator $L_m : E(I) \to F(J)$ by

$$(L_m f)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) A_{m,k}(f), \tag{2.1}$$

for any $f \in E(I)$ and $x \in J$.

Proposition 2.2. *The L_m, $m \in \mathbb{N}$ operators are linear and positive on $E(I \cap J)$.*

Definition 2.3. For $m \in \mathbb{N}$, let $L_m : E(I) \to F(J)$ be an operator defined in (2.1). For $i \in \mathbb{N}_0$, define $T^*_{m,i}$ by

$$\left(T^*_{m,i} L_m\right)(x) = m^i \left(L_m \psi_x^i\right)(x) = m^i \sum_{k=0}^{p_m} \varphi_{m,k}(x) A_{m,k}(\psi_x^i), \tag{2.2}$$

for any $x \in I \cap J$, where for $x \in I$, $\psi_x : I \to \mathbb{R}$, $\psi_x(t) = t - x$ for any $t \in I$.

In what follows $s \in \mathbb{N}_0$ is even and we suppose that the operators $(L_m)_{m \geq 1}$ verify the conditions: there exists, the smallest α_s, $\alpha_{s+2} \in [0, \infty)$ so that

$$\lim_{m \to \infty} \frac{\left(T^*_{m,j} L_m\right)(x)}{m^{\alpha j}} = B_j\,(x) \in \mathbb{R}, \tag{2.3}$$

$x \in I \cap J$, $j \in \{s, s+2\}$ and

$$\alpha_{s+2} < \alpha_s + 2. \tag{2.4}$$

Theorem 2.4. *Let $f : I \to \mathbb{R}$ be a function.*

If $x \in I \cap J$ and f is a s times differentiable function in x with $f^{(s)}$ continuous in x, then

$$\lim_{m\to\infty} m^{s-\alpha_s}\left[(L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!}\left(T^*_{m,i}L_m\right)(x)\right] = 0. \tag{2.5}$$

Assume that f is s times differentiable function on I, with $f^{(s)}$ continuous on I and there exists an interval $K \subset I \cap J$ such that there exist $m(s) \in \mathbb{N}$ and $k_j \in \mathbb{R}$ depending on K, so that for any $m \geq m(s)$ and any $x \in K$ we have

$$\frac{\left(T^*_{m,j}L_m\right)(x)}{m^{\alpha_j}} \leq k_j\,, \tag{2.6}$$

where $j \in \{s, s+2\}$. Then the convergence given in (2.5) *is uniform on K and*

$$\begin{aligned} m^{s-\alpha_s}&\left|(L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!}\left(T^*_{m,i}L_m\right)(x)\right| \\ &\leq \frac{1}{s!}\left(k_s + k_{s+2}\right)\omega\left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right), \end{aligned} \tag{2.7}$$

for any $x \in K$ and $m \geq m(s)$.

In the following we consider that

$$\left(T^*_{m,0}L_m\right)(x) = 1 \tag{2.8}$$

for any $x \in I \cap J$ and any $m \in \mathbb{N}$.

Corollary 2.5. *Let $f : I \to \mathbb{R}$ be a function. Assume that f is s times differentiable in $x \in I \cap J$ and $f^{(s)}$ is continuous in x. Then*

$$\lim_{m\to\infty}(L_m f)(x) = f(x) \tag{2.9}$$

if $s = 0$ and

$$\lim_{m\to\infty} m^{s-\alpha_s}\left[(L_m f)(x) - \sum_{i=0}^{s-1} \frac{f^{(i)}(x)}{m^i i!}\left(T^*_{m,i}L_m\right)(x)\right] = \frac{f^{(s)}(x)}{s!}B_s(x) \tag{2.10}$$

if $s \geq 2$.

If f is s times differentiable function on $I \cap J$, with $f^{(s)}$ continuous on $I \cap J$ and (2.6) *takes place for an interval $K \subset I \cap J$ then the convergence from* (2.9) *and* (2.10) *is uniform on K.*

From (2.3) and (2.8) it results that

$$\alpha_0 = 0 \tag{2.11}$$

and then

$$k_0 = 1. \tag{2.12}$$

COROLLARY 2.6. *Let $f : I \to \mathbb{R}$ be a function.*
If $x \in I \cap J$ and f is continuous in x, then

$$\lim_{m\to\infty}(L_m f)(x) = f(x). \tag{2.13}$$

Assume that f is continuous on I and there exists an interval $K \subset I \cap J$ such that there exists $m(0) \in \mathbb{N}$ and $k_2 \in \mathbb{R}$ depending on K, so that for any $m \geq m(0)$ and any $x \in K$ we have

$$\frac{\left(T^*_{m,2}L_m\right)(x)}{m^{\alpha_2}} \leq k_2. \tag{2.14}$$

Then the convergence given in (2.13) *is uniform on K and*

$$|(L_m f)(x) - f(x)| \leq (1 + k_2)\omega\left(f; \frac{1}{\sqrt{m^{2-\alpha_2}}}\right) \tag{2.15}$$

for any $x \in K$ and $m \in \mathbb{N}$, $m \geq m(0)$.

PROOF. It results from Theorem 2.4. □

For $m \in \mathbb{N}$, let the linear positive functionals $A^*_{m,k} : E(I \times I) \to \mathbb{R}$ with the property: if $(x, y) \in I \times I$, then

$$A^*_{m,k}(f) = A_{m,k}(F), \tag{2.16}$$

$$A^*_{m,k}(f_x) = A_{m,k}(f_x) \tag{2.17}$$

and

$$A^*_{m,k}(f^y) = A_{m,k}(f^y), \tag{2.18}$$

for any $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$, any $f \in E(I \times I)$, where we note $F : I \to \mathbb{R}$, $F(t) = f(t, t)$ for any $t \in I$.

Now, with the help of $(L_m)_{m\geq1}$ operators, we construct a sequence of bivariate operators. In the following let $\delta \in [0, 1]$.

DEFINITION 2.7. The operators $L^\delta_m : E(I \times I) \to F(J \times J)$, $m \in \mathbb{N}$, defined for any function $f \in E(I \times I)$ and any $(x, y) \in J \times J$ by

$$\left(L^\delta_m f\right)(x, y) = \sum_{k=0}^{p_m} \left(\delta\varphi_{m,k}(x) + (1 - \delta)\varphi_{m,k}(y)\right) A^*_{m,k}(f) \tag{2.19}$$

are named the bivariate operators of δL-type.

THEOREM 2.8. *The L^δ_m, $m \in \mathbb{N}$ operators are linear and positive on $E((I \times I) \cap (J \times J))$.*

PROOF. The proof follows immediately. □

3. Main results

Theorem 3.1. *Let* $f : I \times I \to \mathbb{R}$ *be a function.*

If f *is continuous in* (x, x), $(y, y) \in (I \times I) \cap (J \times J)$, *then*

$$\lim_{m\to\infty} \left(L_m^\delta f\right)(x, y) = \delta f(x, x) + (1 - \delta) f(y, y). \tag{3.1}$$

Assume that f *is continuous on* $I \times I$ *and there exists an interval* $K \subset I \cap J$ *such that there exists* $m(0) \in \mathbb{N}$ *and* $k_2 \in \mathbb{R}$ *depending on* K, *so that for any* $m \geq m(0)$ *and any* $x \in K$ *we have*

$$\frac{\left(T^*_{m,2} L_m\right)(x)}{m^{\alpha_2}} \leq k_2. \tag{3.2}$$

Then the convergence given in (3.1) *is uniform on* $K \times K$ *and*

$$\left|\left(L_m^\delta f\right)(x, y) - (\delta f(x, x) + (1 - \delta) f(y, y))\right| \leq (1 + k_2)\omega\left(F; \frac{1}{\sqrt{m^{2-\alpha_2}}}\right) \tag{3.3}$$

for any $(x, y) \in K \times K$ *and* $m \in \mathbb{N}$, $m \geq m(0)$.

Proof. If $m \in \mathbb{N}$, then

$$\left(L_m^\delta f\right)(x, y) = \delta \sum_{k=0}^{p_m} \varphi_{m,k}(x) A^*_{m,k}(f) + (1 - \delta) \sum_{k=0}^{p_m} \varphi_{m,k}(y) A^*_{m,k}(f)$$

and taking (2.16) into account, we obtain

$$\left(L_m^\delta f\right)(x, y) = \delta(L_m F)(x) + (1 - \delta)(L_m F)(y). \tag{3.4}$$

We have

$$\begin{aligned}
&\left|\left(L_m^\delta f\right)(x) - [\delta f(x, x) + (1 - \delta) f(y, y)]\right| \\
&= |\delta[(L_m F)(x) - F(x)] + (1 - \delta)[(L_m F)(y) - F(y)]| \\
&\leq \delta\, |(L_m F)(x) - F(x)| + (1 - \delta)\, |(L_m F)(y) - F(y)|
\end{aligned}$$

and apply the Corollary 2.6. □

Remark 3.2. In general, the sequence $\left(L_m^\delta f\right)_{m\geq 1}$, where $f : I \times I \to \mathbb{R}$ is doesn't converge to the function f.

Lemma 3.3. *Let the GBS operators* $\left(UL_m^\delta\right)_{m\geq 1}$ *associated to the* $\left(L_m^\delta\right)_{m\geq 1}$ *operators. If* $m \in \mathbb{N}$, $UL_m^\delta : E(I \times I) \to F(J \times J)$ *have the form*

$$\begin{aligned}
(UL_m^\delta f)(x, y) =& \delta\left[(L_m\, f_x)(x) + (L_m f^y)(x)\right] \\
&+ (1 - \delta)\left[(L_m\, f_x)(y) + (L_m f^y)(y)\right] - (L_m^\delta f)(x, y) \\
=& \delta\left[(L_m\, f_x)(x) + (L_m f^y)(x) - (L_m F)(x)\right] \\
&+ (1 - \delta)\left[(L_m\, f_x)(y) + (L_m f^y)(y) - (L_m F)(y)\right]
\end{aligned} \tag{3.5}$$

where $(x, y) \in J \times J$ *and* $f \in E(I \times I)$.

PROOF. It results from definition of GBS operator, (2.1), (2.16) - (2.18) and (3.4). ∎

THEOREM 3.4. *Let $f : I \times I \to \mathbb{R}$ be a function.*
If $(x, y) \in (I \times I) \cap (J \times J)$, the functions f_x, f^y and F are s times differentiable in x and y, the functions $\frac{\partial^s f_x}{\partial \tau^s}$, $\frac{\partial^s f^y}{\partial t^s}$ and $F^{(s)}$ are continuous in x and y, then

$$
\begin{aligned}
(3.6)\quad & \lim_{m\to\infty} m^{s-\alpha_s}\Bigg\{ \left(UL_m^{\delta} f\right)(x,y) \\
& - \sum_{i=0}^{s} \frac{1}{m^i i!}\Bigg[\delta\left(\frac{\partial^i f}{\partial \tau^i}(x,x) + \frac{\partial^i f}{\partial t^i}(x,y) - F^{(i)}(x)\right)\left(T^*_{m,i}L_m\right)(x) \\
& + (1-\delta)\left(\frac{\partial^i f}{\partial \tau^i}(x,y) + \frac{\partial^i f}{\partial t^i}(y,y) - F^{(i)}(y)\right)\left(T^*_{m,i}L_m\right)(y)\Bigg]\Bigg\} = 0.
\end{aligned}
$$

Assume that the functions f_x, f^y and F are s times differentiable on I for any $x, y \in I$, with $\frac{\partial^s f^y}{\partial \tau^s}$, $\frac{\partial^s f_x}{\partial t^s}$ and $F^{(s)}$ continuous on I for any $x, y \in I$ and there exists an interval $K \subset I \cap J$ such that there exist $m(s) \in \mathbb{N}$ and $k_j \in \mathbb{R}$ depending on K, so that for any $m \geq m(s)$ and any $x \in K$ we have

$$
(3.7)\qquad \frac{\left(T^*_{m,j}L_m\right)(x)}{m^{\alpha_j}} \leq k_j
$$

where $j \in \{s, s+s\}$. Then the convergence given in (3.6) *is uniform on $K \times K$ and*

$$
\begin{aligned}
(3.8)\quad & m^{s-\alpha_s}\Bigg| \left(UL_m^{\delta} f\right)(x,y) \\
& - \sum_{i=0}^{s} \frac{1}{m^i i!}\Bigg[\delta\left(\frac{\partial^i f}{\partial \tau^i}(x,x) + \frac{\partial^i f}{\partial t^i}(x,y) - F^{(i)}(x)\right)\left(T^*_{m,i}L_m\right)(x) \\
& + (1-\delta)\left(\frac{\partial^i f}{\partial \tau^i}(x,y) + \frac{\partial^i f}{\partial t^i}(y,y) - F^{(i)}(y)\right)\left(T^*_{m,i}L_m\right)(y)\Bigg]\Bigg| \\
& \leq \frac{1}{s!}(k_s + k_{s+2})\Bigg[\omega\left(\frac{\partial^s f_x}{\partial \tau^s}; \frac{1}{\sqrt{m^{2-\alpha_2}}}\right) + \omega\left(\frac{\partial^s f^y}{\partial t^s}; \frac{1}{\sqrt{m^{2-\alpha_2}}}\right) + \\
& + \omega\left(F^{(s)}; \frac{1}{\sqrt{m^{2-\alpha_2}}}\right)\Bigg]
\end{aligned}
$$

for any $x, y \in K$ and any $m \geq m(s)$.

PROOF. We use the (2.5) relation from Theorem 3.1 for the functions f_x, f^y and F and we obtain (3.8) relation. If we note by S the left member of (3.8) relation and taking (2.7) relation into account, we can write

$$\begin{aligned}
S =& m^{s-\alpha_s}\left|\delta\left\{\left[(L_m f_x)(x) - \sum_{i=0}^{s} \frac{1}{m^i i!}\frac{\partial^i f}{\partial \tau^i}(x,x)\left(T^*_{m,i}L_m\right)(x)\right]\right.\right.\\
&+\left[(L_m f^y)(x) - \sum_{i=0}^{s} \frac{1}{m^i i!}\frac{\partial^i f}{\partial t^i}(x,y)\left(T^*_{m,i}L_m\right)(x)\right]\\
&+\left.\left[\sum_{i=0}^{s} \frac{1}{m^i i!}F^{(i)}(x)\left(T^*_{m,i}L_m\right)(x) - (L_m F)(x)\right]\right\}\\
&+(1-\delta)\left\{\left[(L_m f_x)(y) - \sum_{i=0}^{s} \frac{1}{m^i i!}\frac{\partial^i f}{\partial \tau^i}(x,y)\left(T^*_{m,i}L_m\right)(y)\right]\right.\\
&+\left[(L_m f^y)(y) - \sum_{i=0}^{s} \frac{1}{m^i i!}\frac{\partial^i f}{\partial t^i}(y,y)\left(T^*_{m,i}L_m\right)(y)\right]\\
&+\left.\left.\left[\sum_{i=0}^{s} \frac{1}{m^i i!}F^{(i)}(y)\left(T^*_{m,i}L_m\right)(y) - (L_m F)(y)\right]\right\}\right|\\
\le& \delta\left[m^{s-\alpha_s}\left|(L_m f_x)(x) - \sum_{i=0}^{s} \frac{1}{m^i i!}\frac{\partial^i f}{\partial \tau^i}(x,x)\left(T^*_{m,i}L_m\right)(x)\right|\right.\\
&+m^{s-\alpha_s}\left|(L_m f^y)(x) - \sum_{i=0}^{s} \frac{1}{m^i i!}\frac{\partial^i f}{\partial t^i}(x,y)\left(T^*_{m,i}L_m\right)(x)\right|\\
&+\left.m^{s-\alpha_s}\left|(L_m F)(x) - \sum_{i=0}^{s} \frac{1}{m^i i!}F^{(i)}(x)\left(T^*_{m,i}L_m\right)(x)\right|\right]\\
&+(1-\delta)\left[m^{s-\alpha_s}\left|(L_m f_x)(y) - \sum_{i=0}^{s} \frac{1}{m^i i!}\frac{\partial^i f}{\partial \tau^i}(x,y)\left(T^*_{m,i}L_m\right)(y)\right|\right.\\
&+\left|(L_m f^y)(y) - \sum_{i=0}^{s} \frac{1}{m^i i!}\frac{\partial^i f}{\partial t^i}(y,y)\left(T^*_{m,i}L_m\right)(y)\right|\\
&+\left|(L_m F)(y) - \sum_{i=0}^{s} \frac{1}{m^i i!}F^{(i)}(y)\left(T^*_{m,i}L_m\right)(y)\right|\\
\le& \delta\left\{\frac{1}{s!}(k_s + k_{s+2})\left[\omega\left(\frac{\partial^s f_x}{\partial \tau^s};\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)+\right.\right.
\end{aligned}$$

$$+\,\omega\left(\frac{\partial^s f^y}{\partial t^s};\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)+\omega\left(F^{(s)};\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)\Bigg]\Bigg\}$$

$$+\,(1-\delta)\Bigg\{\frac{1}{s!}\left(k_s+k_{s+2}\right)\Bigg[\omega\left(\frac{\partial^s f_x}{\partial \tau^s};\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)$$

$$+\,\omega\left(\frac{\partial^s f^y}{\partial t^s};\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)+\omega\left(F^{(s)};\frac{1}{\sqrt{m^{2-\alpha_2}}}\right)\Bigg]\Bigg\},$$

from where we obtain (3.8) relation. From (3.8) the uniform convergence for (3.6) results. ∎

THEOREM 3.5. *Let $f : I \times I \to \mathbb{R}$ be a function.*

If $(x, y) \in (I \times I) \cap (J \times J)$, the functions f_x, f^y and F are s times differentiable in x and y, the functions $\frac{\partial^s f_x}{\partial \tau^s}$, $\frac{\partial^s f^y}{\partial t^s}$ and $F^{(s)}$ are continuous in x and y, then

$$\lim_{m\to\infty}\left(UL_m^{\delta} f\right)(x,y)=f(x,y) \tag{3.9}$$

if $s = 0$, and

$$\lim_{m\to\infty} m^{s-\alpha_s}\Bigg\{\left(UL_m^{\delta} f\right)(x,y)-\sum_{i=0}^{s-1}\frac{1}{m^i i!}\Bigg[\delta\Bigg(\frac{\partial^i f}{\partial \tau^i}(x,x) \tag{3.10}$$

$$+\frac{\partial^i f}{\partial t^i}(x,y)-F^{(i)}(x)\Bigg)\left(T^*_{m,i}L_m\right)(x)$$

$$+(1-\delta)\left(\frac{\partial^i f}{\partial \tau^i}(x,y)+\frac{\partial^i f}{\partial t^i}(y,y)-F^{(i)}(y)\right)\left(T^*_{m,i}L_m\right)(y)\Bigg]\Bigg\}$$

$$=\frac{1}{s!}\Bigg[\delta\left(\frac{\partial^s f}{\partial \tau^s}(x,x)+\frac{\partial^s f}{\partial t^s}(x,y)-F^{(s)}(x)\right)B_s(x)$$

$$+(1-\delta)\left(\frac{\partial^s f}{\partial \tau^s}(x,y)+\frac{\partial^s f}{\partial t^s}(y,y)-F^{(s)}(y)\right)B_s(y)\Bigg].$$

Assume that the functions f_x, f^y and F are s times differentiable on I for any $x, y \in I$, with $\frac{\partial^s f_x}{\partial \tau^s}$, $\frac{\partial^s f^y}{\partial t^s}$, $F^{(s)}$ continuous on I for any $x, y \in I$ and there exists an interval $K \subset I \cap J$ such that there exist $m(s) \in \mathbb{N}$ and $k_j \in \mathbb{R}$ depending on K so that for any $m \geq m(s)$ and any $x \in K$ we have

$$\frac{\left(T^*_{m,j}L_m\right)(x)}{m^{\alpha_j}}\leq k_j \tag{3.11}$$

where $j \in \{s, s+2\}$. Then the convergence given in (3.9) *and* (3.10) *is uniform on $K \times K$.*

Proof. It results from Theorem 3.4 and Corollary 2.5. □

Corollary 3.6. *Let $f : I \times I$ be a function.*

If $(x, y) \in (I \times I) \cap (J \times J)$, the functions f_x, f^y and F are continuous in x and y, then

$$\lim_{m\to\infty} \left(UL_m^\delta f\right)(x,y) = f(x,y). \tag{3.12}$$

Assume that the functions f_x, f^y and F are continuous on I for any $x, y \in I$ and there exists an interval $K \subset I \cap J$ such that there exist $m(0) \in \mathbb{N}$ and $k_2 \in \mathbb{R}$ depending on K, so that for any $m \in \mathbb{N}$, $m \geq m(0)$ and any $x \in K$ we have

$$\frac{\left(T_{m,2}^* L_m\right)(x)}{m^{\alpha_2}} \leq k_2. \tag{3.13}$$

Then the convergence given in (3.12) *is uniform on $K \times K$ and*

$$\begin{aligned} \left|\left(UL_m^\delta f\right)(x,y) - f(x,y)\right| \leq (1+k_2)\Bigg[&\omega\left(f_x; \frac{1}{\sqrt{m^{2-\alpha_2}}}\right) \\ &+ \omega\left(f^y; \frac{1}{\sqrt{m^{2-\alpha_2}}}\right) + \omega\left(F; \frac{1}{\sqrt{m^{2-\alpha_2}}}\right)\Bigg] \end{aligned} \tag{3.14}$$

for any $x, y \in K$ and any $m \geq m(s)$.

Proof. It results from Theorem 3.4 for $s = 0$. □

Corollary 3.7. *Let $f : I \times I \to \mathbb{R}$ be a function.*

If $(x, y) \in (I \times I) \cap (J \times J)$, the functions f_x, f^y and F are two times differentiable in x and y, the functions $\dfrac{\partial^2 f_x}{\partial \tau^2}$, $\dfrac{\partial^2 f^y}{\partial t^2}$ and F'' are continuous in x and y, then

$$\begin{aligned} \lim_{m\to\infty} m^{2-\alpha_2} \Bigg\{ &\left(UL_m^\delta f\right)(x,y) - f(x,y) \\ &- \frac{1}{m}\Bigg[\delta\left(\frac{\partial f}{\partial \tau}(x,x) + \frac{\partial f}{\partial t}(x,y) - F'(x)\right)\left(T_{m,1}^* L_m\right)(x) \\ &+ (1-\delta)\left(\frac{\partial f}{\partial \tau}(x,y) + \frac{\partial f}{\partial t}(y,y) - F'(y)\right)\left(T_{m,1}^* L_m\right)(y)\Bigg]\Bigg\} \\ =&\frac{1}{2}\Bigg[\delta\left(\frac{\partial^2 f}{\partial \tau^2}(x,x) + \frac{\partial^2 f}{\partial t^2}(x,y) - F''(s)\right)B_s(x) \\ &+ (1-\delta)\left(\frac{\partial^2 f}{\partial \tau^2}(x,y) + \frac{\partial^2 f}{\partial t^2}(y,y) - F''(y)\right)B_s(y)\Bigg]. \end{aligned} \tag{3.15}$$

Assume that the functions f_x, f^y *and* F *are two times differentiable on* I *for any* $x, y \in I$, *with* $\frac{\partial^2 f_x}{\partial \tau^2}$, $\frac{\partial^2 f^y}{\partial t^2}$ *and* F'' *continuous on* I *for any* $x, y \in I$ *and there exists an interval* $K \subset I \cap J$ *such that there exist* $m(2) \in \mathbb{N}$ *and* $k_j \in \mathbb{R}$ *depending on* K *so that for any* $m \geq m(2)$ *and any* $x \in K$ *we have*

$$\frac{\left(T^*_{m,j} L_m\right)}{m^{\alpha_j}} \leq k_j \tag{3.16}$$

where $j \in \{2, 4\}$. *Then the convergence given in* (3.15) *is uniform on* $K \times K$.

Proof. It results from Theorem 3.5 for $s = 2$. □

In the following, by particularization and applying Theorem 3.5, Corollary 3.6 and Corollary 3.7, we can obtain Voronovskaja's type theorem and approximation theorem for some known operators. Because every application is a simple substitute in the theorems of this section, we won't replace anything.

In Applications 3.1-3.4, let $p_m = m$, $\varphi_{m,k} = p_{m,k}$, where $m \in \mathbb{N}$, $k \in \{0, 1, \ldots, m\}$ and $K = [0, 1]$.

Application 3.1. If $I = J = [0, 1]$, $E(I) = F(J) = C([0, 1])$, $A_{m,k}(f) = f\left(\frac{k}{m}\right)$ where $m \in \mathbb{N}$, $k \in \{0, 1, \ldots, m\}$ and $f \in C([0, 1])$ then we obtain the Bernstein operators. We have $k_2 = \frac{5}{4}$, $k_4 = \frac{19}{16}$, $\left(T^*_{m,1} B_m\right)(x) = 0$, $x \in [0, 1]$, $m \in \mathbb{N}$ and $m(0) = m(2) = 1$ (see [19]).

If $\delta = \frac{1}{2}$ we obtain the GBS operators $\left(UB_m^{\frac{1}{2}}\right)_{m \geq 1}$ associated to the $\left(B_m^{\frac{1}{2}}\right)_{m \geq 0}$ operators, studied in the paper [2]. These operators do not satisfy the assumptions of Theorem A from paper [3]. There exists no satisfactory choice of δ_1 and δ_2 in Corollary 5 to express the degree of approximation of $\left(UB_m^{\frac{1}{2}}\right)_{m \geq 1}$ operators (see [3]).

Application 3.2. If $I = J = [0, 1]$, $E(I) = L_1([0, 1])$, $F(J) = C([0, 1])$, $A_{m,k}(f) = (m + 1) \int_0^1 p_{m,k}(t) f(t) dt$, where $m \in \mathbb{N}$, $k \in \{0, 1, \ldots, m\}$ and $f \in L_1([0, 1])$, then we obtain the Durrmeyer operators. In this case $k_2 = \frac{3}{2}$, $k_4 = \frac{7}{4}$, $\left(T^*_{m,1} M_m\right)(x) = \frac{m(1 - 2x)}{m + 2}$, $x \in [0, 1]$, $m \in \mathbb{N}$ and $m(0) = m(2) = 3$ (see [19]).

Application 3.3. If $I = J = [0,1]$, $E(I) = L_1([0,1])$, $F(J) = C([0,1])$, $A_{m,k}(f) = (m+1)\int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(t)dt$, where $m \in \mathbb{N}$, $k \in \{0,1,\dots,m\}$ and $f \in L_1([0,1])$, then we obtain the Kantorovich operators. We have $k_2 = 1$, $k_4 = \dfrac{3}{2}$, $\left(T^*_{m,1}K_m\right)(x) = \dfrac{m}{2(m+1)}(1-2x)$, $x \in [0,1]$, $m \in \mathbb{N}$ and $m(0) = m(2) = 3$ (see [19]).

Application 3.4. Let $\alpha, \beta \in \mathbb{R}$, $\alpha \geq 0$. If $I = [0, \mu^{(\alpha,\beta)}]$, $J = [0,1]$, $E(I) = C([0,\mu^{(\alpha,\beta)}])$, $F(J) = C([0,1])$, $A_{m,k} = f\left(\dfrac{k+\alpha}{m+\beta}\right)$, where $m \in \mathbb{N}$, $k \in \{0,1,\dots,m\}$ and $f \in C([0,\mu^{(\alpha,\beta)}])$, then we obtain the Stancu operators.

Application 3.5. Let $p \in \mathbb{N}_0$. If $I = [0,1+p]$, $J = [0,1]$, $E(I) = C([0,1+p])$, $F(J) = C([0,1])$, $K = [0,1]$, $\varphi_{m,k} = \widetilde{p}_{m,k} = p_{m+p,k}$, $A_{m,k}(f) = f\left(\dfrac{k}{m}\right)$, $p_m = m+p$, where $m \in \mathbb{N}$, $k \in \{0,1,\dots,m\}$ and $f \in C([0,1+p])$, then we obtain the Schurer operators.

In Applications 3.6–3.8 and Application 3.10 let $K = [0,b]$, $b > 0$.

Application 3.6. If $I = J = [0,\infty)$, $E(I) = F(J) = C([0,\infty))$, $\varphi_{m,k}(x) = \dbinom{m}{k} x^k \dfrac{1}{(1+x)^m}$ for any $x \in [0,\infty)$, $A_{m,k}(f) = f\left(\dfrac{k}{m+1-k}\right)$, $p_m = m$, where $m \in \mathbb{N}$, $k \in \{0,1,\dots,m\}$ and $f \in C([0,\infty))$, then we obtain the Bleimann-Butzer-Hahn operators. In this case $k_2 = 4b(1+b)^2$ (see [22] or [25]).

In Applications 3.7–3.10 let $p_m = \infty$ for any $m \in \mathbb{N}$.

Application 3.7. If $I = J = [0,\infty)$, $E(I) = C_2([0,\infty))$, $F(J) = C([0,\infty))$, $\varphi_{m,k}(x) = e^{-mx}\dfrac{(mx)^k}{k!}$ for any $x \in [0,\infty)$, $A_{m,k}(f) = f\left(\dfrac{k}{m}\right)$, where $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $f \in C_2([0,\infty))$, then we obtain the Mirakjan-Favard-Szász operators. We have $k_2 = b$, $k_4 = 3b^2 + b$, $\left(T^*_{m,1}S_m\right)(x) = 0$, $x \in [0,\infty)$, $m \in \mathbb{N}$ and $m(0) = m(2) = 1$ (see [21]).

Application 3.8. If $I = J = [0,\infty)$, $E(I) = C_2([0,\infty))$, $F(J) = C([0,\infty))$, $\varphi_{m,k}(x) = (1+x)^{-m}\dbinom{m+k-1}{k}\left(\dfrac{x}{1+x}\right)^k$ for any $x \in [0,\infty)$, $A_{m,k}(f) = f\left(\dfrac{k}{m}\right)$ where $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $f \in C_2([0,\infty))$, then we obtain the Baskakov operators. In this case $k_2 = b(1+b)$, $k_4 = 9b^4 + 18b^3 + 10b^2 + b$, $\left(T^*_{m,1}V_m\right)(x) = 0$, $x \in [0,\infty)$, $m \in \mathbb{N}$ and $m(0) = m(2) = 1$ (see [21]).

Application 3.9. If $I = J = K = [0,1]$, $E(I) = E(J) = C([0,1])$, $\varphi_{m,k}(x) = \binom{m+k}{k}(1-x)^{m+1}x^k$ for any $x \in [0,\infty)$, $A_{m,k}(f) = f\left(\frac{k}{m+k}\right)$, where $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $f \in C([0,1])$, then we obtain the Meyer-König and Zeller operators. We have $k_2 = 2$ (see [21]).

Application 3.10. If $I = J = [0,\infty)$, $E(I) = F(J) = C([0,\infty))$, $\varphi_{m,k}(x) = e^{-\frac{(m+k)x}{1+x}} \frac{m(m+k)^{k-1}}{k!}\left(\frac{x}{1+x}\right)^k$ for any $x \in [0,\infty)$, $A_{m,k}(f) = f\left(\frac{k}{m}\right)$ where $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $f \in C([0,\infty))$, then we obtain the Ismail-May operators. In this case $k_2 = b(1+b)^2$, $k_4 = b^2(1+b)^4 + 1$, $\left(T^*_{m,1}R_m\right)(x) = A_{m,1}(x) = 0$, $x \in [0,\infty)$, $m \in \mathbb{N}$, $m(0) = 1$ and $m(2) = m_2$ (see [24]).

References

[1] O. Agratini, *Aproximare prin operatori liniari*, Presa Universitară Clujeană, Cluj-Napoca, 2000 (in Romanian).

[2] I. Badea and D. Andrica, *Voronovskaja-type theorems for certain non-positive linear operator*, Rev. Anal. Num. Théor. Approx. **15** (1986), 95-103.

[3] C. Badea and C. Cottin, *Korovkin-type Theorems for Generalized Boolean Sum Operators*, Colloquia Mathematica Societatis János Bolyai, **58**, Approximation Theory, Kecskemét (Hungary) (1990), 51-67.

[4] V. A. Baskakov, *An example of a sequence of linear positive operators in the space of continuous functions*, Dokl. Acad. Nauk, SSSR, **113** (1957), 249-251.

[5] M. Becker and R. J. Nessel, *A global approximation theorem for Meyer-König and Zeller operators*, Math. Zeitschr. **160** (1978), 195-206.

[6] G. Bleimann, P. L. Butzer and L. A. Hahn, *Bernstein-type operator approximating continuous functions on the semi-axis*, Indag. Math. **42** (1980), 255-262.

[7] S. N. Bernstein, *Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités*, Commun. Soc. Math. Kharkow (**2**), **13** (1912-1913), 1-2.

[8] E. W. Cheney and A. Sharma, *Bernstein power series*, Canadian J. Math. **16** (1964), 241-252.

[9] M. M. Derriennic, *Sur l'approximation des fonctions intégrables sur* $[0,1]$ *par les polynômes de Bernstein modifiés*, J. Approx. Theory **31** (1981), 325-343.

[10] Z. Ditzian and V. Totik, Moduli of Smoothness, Springer Verlag, Berlin, 1987.

[11] J. L. Durrmeyer, Une formule d'inversion de la transformée de Laplace: Applications à la théorie des moments, Thèse de 3e cycle, Faculté des Sciences de l'Université de Paris, 1967.

[12] J. Favard, *Sur les multiplicateurs d'interpolation*, J. Math. Pures Appl. **23** (**9**) (1944), 219-247.

[13] M. Ismail and C. P. May, *On a family of approximation operators*, J. Math. Anal. Appl. **63** (1978), 446-462.

[14] L. V. Kantorovich, *Sur certain développements suivant les polynômes de la forme de S. Bernstein*, I, II, C. R. Acad. URSS (1930), 563-568, 595-600.

[15] W. Meyer-König and K. Zeller, *Bernsteinsche Potenzreihen*, Studia Math. **19** (1960), 89-94.

[16] G. M. Mirakjan, *Approximation of continuous functions with the aid of polynomials*, Dokl. Acad. Nauk SSSR **31** (1941), 201-205 (in Russian).
[17] M. W. Müller, Die Folge der Gammaoperatoren, Dissertation, Stuttgart, 1967.
[18] O.T . Pop, *New properties of the Bernstein-Stancu operators*, Anal. Univ. Oradea, Fasc. Matematica Tom **XI** (2004), 51-60.
[19] O. T. Pop, *The generalization of Voronovskaja's theorem for a class of linear and positive operators*, Rev. Anal. Num. Théor. Approx. **34** (2005), 79-91.
[20] O. T. Pop, *About a class of linear and positive operators*, Carpathian J. Math. **21** (2005), 99-108.
[21] O. T. Pop, *About some linear and positive operators defined by infinite sum*, Dem. Math. **39** (2006), 377-388.
[22] O. T. Pop, *About operator of Bleimann, Butzer and Hahn*, Anal. Univ. Timişoara **43** (2005), 117-127.
[23] O. T. Pop, *The generalization of Voronovskaja's theorem for a class of bivariate operators*, to appear in Stud. Univ. Babes-Bolyai Math.
[24] O. T. Pop, *The generalization of Voronovskaja's theorem for exponential operators*, Creative Math. & Inf. **16** (2007), 54-62.
[25] O. T. Pop, *About a general property for a class of linear positive operators and applications*, Rev. Anal. Num. Théor. Approx. **34** (2005), 175-180.
[26] F. Schurer, Linear positive operators in approximation theory, Math. Inst. Tech. Univ. Delft. Report, 1962.
[27] D. D. Stancu, *Asupra unei generalizări a polinoamelor lui Bernstein*, Studia Univ. Babeş-Bolyai, Ser. Math.-Phys. **14** (1969), 31-45 (in Romanian).
[28] D. D. Stancu, Gh. Coman, O. Agratini and R. Trîmbiţaş, Analiză numerică şi teoria aproximării, I, Presa Universitară Clujeană, Cluj-Napoca, 2001 (in Romanian).
[29] O. Szász, *Generalization of. S. N. Bernstein's polynomials to the infinite interval*, J. Research, National Bureau of Standards **45** (1950), 239-245.
[30] A.F. Timan, *Theory of Approximation of Functions of Real Variable*, New York, Macmillan Co. 1963, MR22#8257.
[31] E. Voronovskaja, *Détermination de la forme asymptotique d'approximation des fonctions par les polynômes de M. Bernstein*, C. R. Acad. Sci. URSS (1932), 79-85.

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