GLASNIK MATEMATIČKI
Vol. 43(63)(2008), 121 – 136

APPROXIMATION AND MODULI OF FRACTIONAL ORDERS IN SMIRNOV-ORLICZ CLASSES

Ramazan Akgün and Daniyal M. Israfilov
Balikesir University, Turkey and Institute of Math. and Mech. NAS, Azerbaijan

Abstract. In this work we investigate the approximation problems in the Smirnov-Orlicz spaces in terms of the fractional modulus of smoothness. We prove the direct and inverse theorems in these spaces and obtain a constructive descriptions of the Lipschitz classes of functions defined by the fractional order modulus of smoothness, in particular.

1. Preliminaries and introduction

A function $M(u) : \mathbb{R} \to \mathbb{R}^{+}$ is called an N-function if it admits of the representation

$$M(u) = \int_{0}^{|u|} p(t)\, dt,$$

where the function $p(t)$ is right continuous and nondecreasing for $t \geq 0$ and positive for $t > 0$, which satisfies the conditions

$$p(0) = 0, \quad p(\infty) := \lim_{t\to\infty} p(t) = \infty.$$

The function

$$N(v) := \int_{0}^{|v|} q(s)\, ds,$$

where

$$q(s) := \sup_{p(t)\leq s} t, \quad (s \geq 0)$$

2000 *Mathematics Subject Classification.* 30E10, 46E30, 41A10, 41A25.
Key words and phrases. Orlicz space, Smirnov-Orlicz class, Dini-smooth curve, direct theorems, inverse theorems, fractional modulus of smoothness.

is defined as complementary function of M.

Let Γ be a rectifiable Jordan curve and let $G := int\Gamma$, $G^- := ext\Gamma$, $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$, $\mathbb{T} := \partial\mathbb{D}$, $\mathbb{D}^- := ext\mathbb{T}$. Without loss of generality we may assume $0 \in G$. We denote by $L^p(\Gamma)$, $1 \le p < \infty$, the set of all measurable complex valued functions f on Γ such that $|f|^p$ is Lebesgue integrable with respect to arclength. By $E^p(G)$ and $E^p(G^-)$, $0 < p < \infty$, we denote the Smirnov classes of analytic functions in G and G^-, respectively. It is well-known that every function $f \in E^1(G)$ or $f \in E^1(G^-)$ has a nontangential boundary values a.e. on Γ and if we use the same notation for the nontangential boundary value of f, then $f \in L^1(\Gamma)$.

Let M be an N-function and N be its complementary function. By $L_M(\Gamma)$ we denote the linear space of Lebesgue measurable functions $f : \Gamma \to \mathbb{C}$ satisfying the condition

$$\int_\Gamma M[\alpha|f(z)|]\,|dz| < \infty$$

for some $\alpha > 0$.

The space $L_M(\Gamma)$ becomes a Banach space with the norm

$$\|f\|_{L_M(\Gamma)} := \sup\left\{\int_\Gamma |f(z)\,g(z)|\,|dz| : g \in L_N(\Gamma),\ \rho(g;N) \le 1\right\},$$

where

$$\rho(g\,;N) := \int_\Gamma N[|g(z)|]\,|dz|.$$

The norm $\|\cdot\|_{L_M(\Gamma)}$ is called Orlicz norm and the Banach space $L_M(\Gamma)$ is called Orlicz space. Every function in $L_M(\Gamma)$ is integrable on Γ [18, p. 50], i.e.

$$L_M(\Gamma) \subset L^1(\Gamma).$$

An N-function M satisfies the Δ_2-condition if

$$\limsup_{x\to\infty}\frac{M(2x)}{M(x)} < \infty.$$

The Orlicz space $L_M(\Gamma)$ is reflexive if and only if the N-function M and its complementary function N both satisfy the Δ_2-condition [18, p. 113].

Let Γ_r be the image of the circle $\gamma_r := \{w \in \mathbb{C} : |w| = r,\ 0 < r < 1\}$ under some conformal mapping of $\mathbb{D}$ onto G and let M be an N-function.

The class of functions f analytic in G and satisfying

$$\sup_{0<r<1}\int_{\Gamma_r} M[|f(z)|]\,|dz| \le c < \infty$$

with c independent of r, will be called Smirnov-Orlicz class and denoted by $E_M(G)$. In the similar way $E_M(G^-)$ can be defined. Let

$$\tilde{E}_M\left(G^-\right) := \left\{f \in E_M\left(G^-\right) : f(\infty) = 0\right\}.$$

If $M(x) = M(x,p) := x^p$, $1 < p < \infty$, then the Smirnov-Orlicz class $E_M(G)$ coincides with the usual Smirnov class $E^p(G)$.

Every function in the class $E_M(G)$ has [13] the non-tangential boundary values a.e. on Γ and the boundary function belongs to $L_M(\Gamma)$.

Let

$$S[f] := \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

be Fourier series of a function $f \in L^1(\mathsf{T})$ where $\mathsf{T} := [-\pi, \pi]$, $\int_{\mathsf{T}} f(x)\,dx = 0$, so that $c_0 = 0$.

For $\alpha > 0$, the α-th integral of f is defined by

$$I_\alpha(x,f) := \sum_{k\in\mathbb{Z}^*} c_k (ik)^{-\alpha} e^{ikx},$$

where

$$(ik)^{-\alpha} := |k|^{-\alpha} e^{(-1/2)\pi i\alpha \operatorname{sign} k} \text{ and } \mathbb{Z}^* := \{\pm1, \pm2, \pm3, \ldots\}.$$

It is known [24, V. 2, p. 134] that

$$f_\alpha(x) := I_\alpha(x,f)$$

exist a.e. on T, $f_\alpha \in L^1(\mathsf{T})$ and $S[f_\alpha] = f_\alpha(x)$.

For $\alpha \in (0,1)$ let

$$f^{(\alpha)}(x) := \frac{d}{dx} I_{1-\alpha}(x,f)$$

if the right hand side exist.

We set

$$f^{(\alpha+r)}(x) := \left(f^{(\alpha)}(x)\right)^{(r)} = \frac{d^{r+1}}{dx^{r+1}} I_{1-\alpha}(x,f),$$

where $r \in \mathbb{Z}^+ := \{1, 2, 3, \ldots\}$.

Throughout this work by c, c_1, c_2,..., we denote the constants which are different in different places.

1.1. *Moduli of smoothness of fractional order.* Suppose that $x, h \in \mathbb{R} := (-\infty, \infty)$ and $\alpha > 0$. Then, by [16, Theorem 11, p. 135] the series

$$\Delta_h^\alpha f(x) := \sum_{k=0}^{\infty} (-1)^k C_k^\alpha\, f(x + (\alpha - k)h), \quad f \in L_M(\mathsf{T}),$$

converges absolutely a.e. on T [16, p. 135]. Hence $\Delta_h^\alpha f(x)$ measurable and by [16, Theorem 10, p. 134]

$$\|\Delta_h^\alpha f\|_{L_M(\mathsf{T})} \le C(\alpha)\, \|f\|_{L_M(\mathsf{T})},$$

with

$$C(\alpha) := \sum_{k=0}^{\infty} |C_k^\alpha| < \infty.$$

The quantity $\Delta_h^\alpha f(x)$ will be called the α-th difference of f at x, with increment h. If $\alpha \in \mathbb{Z}^+$ the above cited α-th difference is coincides with usual forward difference. Namely,

$$\Delta_h^\alpha f(x) := \sum_{k=0}^{\alpha} (-1)^k C_k^\alpha \; f(x + (\alpha - k) h) = \sum_{k=0}^{\alpha} (-1)^{\alpha-k} C_k^\alpha \; f(x + kh),$$

for $\alpha \in \mathbb{Z}^+$. For $\alpha > 0$ we define the α-th modulus of smoothness of a function $f \in L_M(\mathsf{T})$ as

$$\omega_\alpha(f,\delta)_M := \sup_{|h| \le \delta} \|\Delta_h^\alpha f\|_{L_M(\mathsf{T})}, \quad \omega_0(f,\delta)_M := \|f\|_{L_M(\mathsf{T})}.$$

Remark 1.1. The modulus of smoothness $\omega_\alpha(f,\delta)_M$ has the following properties.

(i) $\omega_\alpha(f,\delta)_M$ is non-negative and non-decreasing function of $\delta \ge 0$,

(ii) $\lim\limits_{\delta \to 0^+} \omega_\alpha(f,\delta)_M = 0$,

(iii) $\omega_\alpha(f_1 + f_2, \cdot)_M \le \omega_\alpha(f_1, \cdot)_M + \omega_\alpha(f_2, \cdot)_M$.

Let

$$E_n(f)_M := \inf_{T \in \mathcal{T}_n} \|f - T\|_{L_M(\mathsf{T})}, \quad f \in L_M(\mathsf{T}),$$

where $\mathcal{T}_n$ is the class of trigonometric polynomials of degree not greater than $n \ge 1$.

The proofs of following direct and inverse theorems are similar to the appropriate theorems from [21], where the approximation problems are investigated in Lebesgue spaces $L^p(\mathsf{T})$, $1 \le p < \infty$.

Theorem 1.2. *Let $L_M(\mathsf{T})$ be a reflexive Orlicz space and let M be an N-function. Then*

$$E_n(f)_M \le C_1(\alpha)\, \omega_\alpha(f, 1/n)_M, \quad n = 1, 2, \ldots$$

Theorem 1.3. *Let $L_M(\mathsf{T})$ be a reflexive Orlicz space and let M be an N-function. Then*

$$\omega_\alpha(f, 1/n)_M \le \frac{C_2(\alpha)}{n^\alpha} \sum_{\nu=0}^{n} (\nu + 1)^{\alpha - 1} E_\nu(f)_M, \quad n = 1, 2, \ldots$$

1.2. *Modulus of smoothness of fractional order in Smirnov-Orlicz classes.* Let $w = \varphi(z)$ and $w = \varphi_1(z)$ be the conformal mappings of G^- and G onto $\mathbb{D}^-$ normalized by the conditions

$$\varphi(\infty) = \infty, \qquad \lim_{z \to \infty} \varphi(z)/z > 0,$$

and

$$\varphi_1(0) = \infty, \qquad \lim_{z\to 0} z\varphi_1(z) > 0,$$

respectively. We denote by ψ and ψ, the inverse of φ and φ_1, respectively.

Since Γ is rectifiable, we have $\varphi' \in E^1(G^-)$ and $\psi' \in E^1(\mathbb{D}^-)$, and hence the functions φ' and ψ' admit nontangential limits almost everywhere (a.e.) on Γ and on $\mathbb{T}$ respectively, and these functions respectively belong to $L^1(\Gamma)$ and $L^1(\mathbb{T})$ (see, for example [7, p. 419]).

Let $f \in L^1(\Gamma)$. Then, the functions f^+ and f^- defined by

$$f^+(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{\varsigma - z}d\varsigma, \qquad z \in G,$$

$$f^-(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{\varsigma - z}d\varsigma, \qquad z \in G^-,$$

are analytic in G and G^-, respectively and $f^-(\infty) = 0$.

Let h be a function continuous on T. Its modulus of continuity is defined by

$$\omega(t,h) := \sup\{|h(t_1) - h(t_2)| : t_1, t_2 \in \mathsf{T},\ |t_1 - t_2| \le t\}, \qquad t \ge 0.$$

The function h is called Dini-continuous if

$$\int_0^c \frac{\omega(t,h)}{t}dt < \infty, \quad c > 0.$$

A curve Γ is called Dini-smooth [17, p. 48] if it has a parametrization

$$\Gamma : \varphi_0(\tau), \quad \tau \in \mathsf{T}$$

such that $\varphi_0'(\tau)$ is Dini-continuous and $\varphi_0'(\tau) \neq 0$.

If Γ is Dini-smooth, then [23]

$$0 < c_3 < |\psi'(w)| < c_4 < \infty, \quad 0 < c_5 < |\varphi'(z)| < c_6 < \infty, \tag{1.1}$$

where the constants c_3, c_4 and c_5, c_6 are independent of $|w| \ge 1$ and $z \in \overline{G^-}$, respectively.

Let Γ be a Dini-smooth curve and let $f_0 := f \circ \psi$, $f_1 := f \circ \psi_1$ for $f \in L_M(\Gamma)$. Then from (1.1), we have $f_0 \in L_M(\mathbb{T})$ and $f_1 \in L_M(\mathbb{T})$ for $f \in L_M(\Gamma)$. Using the nontangential boundary values of f_0^+ and f_1^+ on $\mathbb{T}$ we define

$$\omega_{\alpha,\Gamma}(f,\delta)_M := \omega_\alpha\left(f_0^+,\delta\right)_M, \qquad \delta > 0$$
$$\tilde{\omega}_{\alpha,\Gamma}(f,\delta)_M := \omega_\alpha\left(f_1^+,\delta\right)_M, \qquad \delta > 0$$

for $\alpha > 0$.

We set

$$E_n(f,G)_M := \inf_{P\in\mathcal{P}_n} \|f - P\|_{L_M(\Gamma)}, \quad \tilde{E}_n\left(g,G^-\right)_M := \inf_{R\in\mathcal{R}_n} \|g - R\|_{L_M(\Gamma)},$$

where $f \in E_M(G)$, $g \in E_M(G^-)$, $\mathcal{P}_n$ is the set of algebraic polynomials of degree not greater than n and $\mathcal{R}_n$ is the set of rational functions of the form

$$\sum_{k=0}^{n} \frac{a_k}{z^k}.$$

Let Γ be a rectifiable Jordan curve, $f \in L^1(\Gamma)$ and let

$$(S_\Gamma f)(t) := \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma \setminus \Gamma(t,\epsilon)} \frac{f(\varsigma)}{\varsigma - t} d\varsigma, \qquad t \in \Gamma$$

be Cauchy's singular integral of f at the point t. The linear operator S_Γ, $f \mapsto S_\Gamma f$ is called the Cauchy singular operator.

If one of the functions f^+ or f^- has the non-tangential limits a. e. on Γ, then $S_\Gamma f(z)$ exists a.e. on Γ and also the other one has the nontangential limits a. e. on Γ. Conversely, if $S_\Gamma f(z)$ exists a.e. on Γ, then both functions f^+ and f^- have the nontangential limits a.e. on Γ. In both cases, the formulae

$$f^+(z) = (S_\Gamma f)(z) + f(z)/2, \qquad f^-(z) = (S_\Gamma f)(z) - f(z)/2, \tag{1.2}$$

and hence

$$f = f^+ - f^- \tag{1.3}$$

holds a.e. on Γ (see, e.g., [7, p. 431]).

In this work we investigate the approximation problems in the Smirnov-Orlicz spaces in terms of the fractional modulus of smoothness. We prove the direct and inverse theorems in these spaces and obtain a constructive descriptions of the Lipschitz classes of functions defined by the fractional order modulus of smoothness, in particular.

In the spaces $L^p(\mathsf{T})$, $1 \le p < \infty$, these problems were studied in the works [21] and [3].

In terms of the usual modulus of smoothness, these problems in the Lebesgue and Smirnov spaces defined on the complex domains with the various boundary conditions were investigated by Walsh-Russel [22], Al'per [1], Kokilashvili [14, 15], Andersson [2], Israfilov [9, 10, 11], Cavus-Israfilov [4] and other mathematicians.

2. Main results

The following direct theorem holds.

Theorem 2.1. *Let Γ be a Dini-smooth curve and $L_M(\Gamma)$ be a reflexive Orlicz space on Γ. If $\alpha > 0$ and $f \in L_M(\Gamma)$ then for any $n = 1, 2, 3, \ldots$ there is a constant $c_7 > 0$ such that*

$$\|f - R_n(\cdot, f)\|_{L_M(\Gamma)} \le c_7 \left\{\omega_{\alpha,\Gamma}(f, 1/n)_M + \tilde{\omega}_{\alpha,\Gamma}(f, 1/n)_M\right\},$$

where $R_n(\cdot, f)$ is the nth partial sum of the Faber-Laurent series of f.

From this theorem we have the following corollaries.

COROLLARY 2.2. *Let G be a finite, simply connected domain with a Dini-smooth boundary Γ and let $L_M(\Gamma)$ be a reflexive Orlicz space on Γ. If $\alpha > 0$ and $S_n(f,\cdot) := \sum_{k=0}^{n} a_k \Phi_k$ is the nth partial sum of the Faber expansion of $f \in E_M(G)$, then for every $n = 1, 2, 3, \ldots$*

$$\| f - S_n(f,\cdot)\|_{L_M(\Gamma)} \le c_8 \; \omega_{\alpha,\Gamma}(f, 1/n)_M \,,$$

with some constant $c_8 > 0$ independent of n.

COROLLARY 2.3. *Let Γ be a Dini-smooth curve. If $\alpha > 0$ and $f \in \tilde{E}_M(G^-)$, then for every $n = 1, 2, 3, \ldots$ there is a constant $c_9 > 0$ such that*

$$\|f - R_n(\cdot, f)\|_{L_M(\Gamma)} \le c_9 \; \tilde{\omega}_{\alpha,\Gamma}(f, 1/n)_M \,,$$

where $R_n(\cdot, f)$ as in Theorem 2.1.

The following inverse theorem holds.

THEOREM 2.4. *Let G be a finite, simply connected domain with a Dini-smooth boundary Γ and let $L_M(\Gamma)$ be a reflexive Orlicz space on Γ. If $\alpha > 0$, then*

$$\omega_{\alpha,\Gamma}(f, 1/n)_M \le \frac{c_{10}}{n^{\alpha}} \sum_{k=0}^{n} (k+1)^{\alpha-1} E_k(f, G)_M \,, \quad n = 1, 2, \ldots$$

with a constant $c_{10} > 0$ depending only on M and α.

COROLLARY 2.5. *Under the conditions of Theorem 2.4, if*

$$E_n(f, G)_M = \mathcal{O}\left(n^{-\sigma}\right), \qquad \sigma > 0, \qquad n = 1, 2, 3, \ldots,$$

then for $f \in E_M(G)$ and $\alpha > 0$

$$\omega_{\alpha,\Gamma}(f, \delta)_M = \begin{cases} \mathcal{O}(\delta^{\sigma}) & , \alpha > \sigma; \\ \mathcal{O}\left(\delta^{\sigma} \left|\log \frac{1}{\delta}\right|\right) & , \alpha = \sigma; \\ \mathcal{O}(\delta^{\alpha}) & , \alpha < \sigma. \end{cases}$$

DEFINITION 2.6. *For $0 < \sigma < \alpha$ we set*

$$\overset{*}{Lip}\,\sigma(\alpha, M) := \left\{ f \in E_M(G) : \omega_{\alpha,\Gamma}(f, \delta)_M = \mathcal{O}(\delta^{\sigma}) \,, \quad \delta > 0 \right\},$$

$$\widetilde{Lip}\,\sigma(\alpha, M) := \left\{ f \in \tilde{E}_M\left(G^-\right) : \tilde{\omega}_{\alpha,\Gamma}(f, \delta)_M = \mathcal{O}(\delta^{\sigma}) \,, \quad \delta > 0 \right\}.$$

COROLLARY 2.7. *Under the conditions of Theorem 2.4, if $0 < \sigma < \alpha$ and*

$$E_n(f, G)_M = \mathcal{O}\left(n^{-\alpha}\right), \; n = 1, 2, 3, \ldots,$$

then $f \in Lip^* \sigma(\alpha, M)$.

COROLLARY 2.8. *Let $0 < \sigma < \alpha$ and let the conditions of Theorem 2.4 be fulfilled. Then the following conditions are equivalent.*

(a) $f \in Lip^* \sigma(\alpha, M)$

(b) $E_n(f, G)_M = \mathcal{O}(n^{-\sigma})$, $\quad n = 1, 2, 3, \ldots$.

Similar results hold also in the class $\tilde{E}_M(G^-)$.

THEOREM 2.9. *Let* Γ *be a Dini-smooth curve and* $L_M(\mathbb{T})$ *be a reflexive Orlicz space. If* $\alpha > 0$ *and* $f \in \tilde{E}_M(G^-)$*, then*

$$\tilde{\omega}_{\alpha,\Gamma}(f,1/n)_M \leq \frac{c_{11}}{n^\alpha} \sum_{k=0}^{n} (k+1)^{\alpha-1} \tilde{E}_k\left(f,G^-\right)_M, \quad n = 1,2,3,\ldots,$$

with a constant $c_{11} > 0$.

COROLLARY 2.10. *Under the conditions of Theorem 2.9, if*

$$\tilde{E}_n\left(f,G^-\right)_M = \mathcal{O}\left(n^{-\sigma}\right), \quad \sigma > 0, \quad n = 1,2,3,\ldots,$$

then for $f \in \tilde{E}_M(G^-)$ *and* $\alpha > 0$

$$\tilde{\omega}_{\alpha,\Gamma}(f,\delta)_M = \begin{cases} \mathcal{O}\left(\delta^\sigma\right) & ,\ \alpha > \sigma; \\ \mathcal{O}\left(\delta^\sigma \left|\log \frac{1}{\delta}\right|\right) & ,\ \alpha = \sigma; \\ \mathcal{O}\left(\delta^\alpha\right) & ,\ \alpha < \sigma. \end{cases}$$

COROLLARY 2.11. *Under the conditions of Theorem 2.9, if* $0 < \sigma < \alpha$ *and*

$$\tilde{E}_n\left(f,G^-\right)_M = \mathcal{O}\left(n^{-\sigma}\right), \quad n = 1,2,3,\ldots,$$

then $f \in \widetilde{Lip}\,\sigma(\alpha, M)$.

COROLLARY 2.12. *Let* $0 < \sigma < \alpha$ *and the conditions of Theorem 2.9 be fulfilled. Then the following conditions are equivalent.*

(a) $f \in \widetilde{Lip}\,\sigma(\alpha, M)$,
(b) $\tilde{E}_n(f,G^-)_M = \mathcal{O}(n^{-\sigma})$, $\quad n = 1,2,3,\ldots$.

2.1. *Some auxiliary results.*

LEMMA 2.13. *Let* $L_M(\mathbb{T})$ *be a reflexive Orlicz space. Then* $f^+ \in E_M(\mathbb{D})$ *and* $f^- \in E_M(\mathbb{D}^-)$ *for every* $f \in L_M(\mathbb{T})$.

PROOF. We claim that for every $f \in L_M(\mathbb{T})$ there exists a $p \in (1,\infty)$ such that $f \in L^p(\mathbb{T})$. Indeed, by Corollaries 4 and 5 of [18, p. 26] there exist some x_0, $c_{12} > 0$ and $p > 1$ such that

$$c_{13}^p |f|^p \leq \frac{1}{c_{12}} M(c_{13}|f|) \tag{2.1}$$

holds for $|f| \geq x_0$ and some $c_{13} > 0$.

Hence, using

$$\int_{\mathbb{T}} |f(z)|^p |dz| = \int_{\Gamma_0} |f(z)|^p |dz| + \int_{\mathbb{T}\setminus\Gamma_0} |f(z)|^p |dz|$$

with $\Gamma_0 := \{z \in \mathbb{T} : |f| \geq x_0\}$, from (2.1) we get that

$$\begin{aligned}\int_{\mathbb{T}} |f(z)|^p |dz| &\leq \frac{1}{c_{12}c_{13}^p} \int_{\Gamma_0} M(c_{13}|f(z)|)|dz| + \int_{\mathbb{T}\backslash\Gamma_0} |f(z)|^p |dz| \\ &\leq c_{14} \int_{\mathbb{T}} M(c_{13}|f(z)|)|dz| + x_0^p mes(\mathbb{T}\backslash\Gamma_0) < \infty\end{aligned}$$

and therefore $f \in L^p(\mathbb{T})$. Since $1 < p < \infty$, this implies [8] that $f^+ \in E^p(\mathbb{D})$, $f^- \in E^p(\mathbb{D}^-)$ and hence $f^+ \in E^1(\mathbb{D})$, $f^- \in E^1(\mathbb{D}^-)$.

Since $f^+ \in E^1(\mathbb{D})$ it can be represented by the Poisson integral of its boundary function. Hence, taking $z := re^{ix}$, $(0 < r < 1)$ we have

$$M\left[\left|f^+(z)\right|\right] = M\left[\frac{1}{2\pi}\left|\int_0^{2\pi} f^+\left(e^{iy}\right) P_r(x-y)\,dy\right|\right].$$

Now, using Jensen integral inequality [24, V:1, p.24] we get

$$\begin{aligned}M\left[\left|f^+(z)\right|\right] &\leq M\left[\frac{\int_0^{2\pi}\left|f^+\left(e^{iy}\right)\right| P_r(x-y)\,dy}{\int_0^{2\pi} P_r(x-y)\,dy}\right] \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} M\left[\left|f^+\left(e^{iy}\right)\right|\right] P_r(x-y)\,dy,\end{aligned}$$

and therefore

$$\begin{aligned}\int_{\gamma_r} M\left[\left|f^+(z)\right|\right]|dz| &\leq \int_{\gamma_r}\frac{1}{2\pi}\int_0^{2\pi} M\left[\left|f^+\left(e^{iy}\right)\right|\right] P_r(x-y)\,dy\,|dz| \\ &= \int_0^{2\pi}\frac{1}{2\pi}\int_0^{2\pi} M\left[\left|f^+\left(e^{iy}\right)\right|\right] P_r(x-y)\,dyrdx \\ &= \int_0^{2\pi} M\left[\left|f^+\left(e^{iy}\right)\right|\right]\left\{\frac{1}{2\pi}\int_0^{2\pi} P_r(x-y)\,dx\right\} rdy \\ &= \int_0^{2\pi} M\left[\left|f^+\left(e^{iy}\right)\right|\right] rdy < \int_0^{2\pi} M\left[\left|f^+\left(e^{ix}\right)\right|\right] dx.\end{aligned}$$

Taking into account the relations

$$f^+\left(e^{ix}\right) = (1/2)\,f\left(e^{ix}\right) + (S_{\mathbb{T}}f)\left(e^{ix}\right) = (1/2)\left\{f\left(e^{ix}\right) + 2\,(S_{\mathbb{T}}f)\left(e^{ix}\right)\right\},$$

we have

$$\begin{aligned} M\left[\left|f^{+}\left(e^{ix}\right)\right|\right] &= M\left[\frac{1}{2}\left|f\left(e^{ix}\right)+2\left(S_{\mathbb{T}}f\right)\left(e^{ix}\right)\right|\right] \\ &\leq M\left[\frac{1}{2}\left\{\left|f\left(e^{ix}\right)\right|+2\left|\left(S_{\mathbb{T}}f\right)\left(e^{ix}\right)\right|\right\}\right] \\ &\leq \frac{1}{2}\left\{M\left[\left|f\left(e^{ix}\right)\right|\right]+M\left[2\left|\left(S_{\mathbb{T}}f\right)\left(e^{ix}\right)\right|\right]\right\} \\ &\leq \frac{1}{2}\left\{M\left[\left|f\left(e^{ix}\right)\right|\right]+M\left[2x_0\right]+c_{15}M\left[\left|\left(S_{\mathbb{T}}f\right)\left(e^{ix}\right)\right|\right]\right\} \end{aligned}$$

for some $x_0 > 0$ and hence

$$\begin{aligned} &\int_{\gamma_r} M\left[\left|f^{+}(z)\right|\right]|dz| \\ &\quad< \frac{1}{2}\int_0^{2\pi}\left\{M\left[\left|f\left(e^{ix}\right)\right|\right]+M\left[2x_0\right]+c_{16}M\left[\left|\left(S_{\mathbb{T}}f\right)\left(e^{ix}\right)\right|\right]\right\}dx \\ &\quad= \frac{1}{2}\int_0^{2\pi}M\left[\left|f\left(e^{ix}\right)\right|\right]dx+c_{17}\int_0^{2\pi}M\left[\left|\left(S_{\mathbb{T}}f\right)\left(e^{ix}\right)\right|\right]dx+M\left[2x_0\right]\pi. \end{aligned}$$

On the other hand [19]

$$\left\|S_{\mathbb{T}}f\right\|_{L_M(\mathbb{T})} \leq c_{18}\left\|f\right\|_{L_M(\mathbb{T})}$$

which implies that

$$\int_0^{2\pi} M\left[\left|\left(S_{\mathbb{T}}f\right)\left(e^{ix}\right)\right|\right]dx \leq c_{19} < \infty$$

and then

$$\begin{aligned} \int_{\gamma_r} M\left[\left|f^{+}(z)\right|\right]|dz| &< \frac{1}{2}\int_0^{2\pi}M\left[\left|f\left(e^{ix}\right)\right|\right]dx+c_{20} \\ &= c_{21}\left(1/2\right)\int_{\mathbb{T}}M\left[\left|f(w)\right|\right]|dw|+c_{20} < \infty. \end{aligned}$$

Finally, we have $f^{+} \in E_M(\mathbb{D})$. Similar result also holds for f^{-}. □

Using Theorem 1.2 and the method, applied for the proof of the similar result in [4], we have

Lemma 2.14. *Let an N-function M and its complementary function both satisfy the Δ_2 condition. Then there exists a constant $c_{22} > 0$ such that for every $n = 1, 2, 3, \ldots$*

$$\left\| g(w) - \sum_{k=0}^{n} \alpha_k w^k \right\|_{L_M(\mathbb{T})} \leq c_{22}\ \omega_\alpha (g, 1/n)_M\,, \quad \alpha > 0$$

where α_k, $(k = 0, 1, 2, 3, \ldots)$ are the kth Taylor coefficients of $g \in E_M(\mathbb{D})$ at the origin.

We know [20, pp. 52, 255] that

$$\frac{\psi'(w)}{\psi(w) - z} = \sum_{k=0}^{\infty} \frac{\Phi_k(z)}{w^{k+1}}, \qquad z \in G, \quad w \in \mathbb{D}^-$$

and

$$\frac{\psi_1'(w)}{\psi_1(w) - z} = \sum_{k=1}^{\infty} \frac{F_k(1/z)}{w^{k+1}}, \quad z \in G^-, \quad w \in \mathbb{D}^-,$$

where $\Phi_k(z)$ and $F_k(1/z)$ are the *Faber polynomials* of degree k with respect to z and $1/z$ for the continuums $\overline{G}$ and $\overline{\mathbb{C}} \backslash G$, with the integral representations [20, pp. 35, 255]

$$\Phi_k(z) = \frac{1}{2\pi i} \int\limits_{|w|=R} \frac{w^k \psi'(w)}{\psi(w) - z} dw, \quad z \in G, \quad R > 1$$

$$F_k(1/z) = \frac{1}{2\pi i} \int\limits_{|w|=1} \frac{w^k \psi_1'(w)}{\psi_1(w) - z} dw, \quad z \in G^-,$$

and

$$\Phi_k(z) = \varphi^k(z) + \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\varphi^k(\varsigma)}{\varsigma - z} d\varsigma, \quad z \in G^-, \quad k = 0, 1, 2, ..., \tag{2.2}$$

$$F_k(1/z) = \varphi_1^k(z) - \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\varphi_1^k(\varsigma)}{\varsigma - z} d\varsigma, \quad z \in G \setminus \{0\}\,. \tag{2.3}$$

We put

$$a_k := a_k(f) := \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{f_0(w)}{w^{k+1}} dw, \quad k = 0, 1, 2, ...,$$

$$\tilde{a}_k := \widetilde{a}_k(f) := \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{f_1(w)}{w^{k+1}} dw, \quad k = 1, 2, ...$$

and correspond the series

$$\sum_{k=0}^{\infty} a_k \Phi_k(z) + \sum_{k=1}^{\infty} \tilde{a}_k F_k(1/z)$$

with the function $f \in L^1(\Gamma)$, i.e.,

$$f(z) \sim \sum_{k=0}^{\infty} a_k \Phi_k(z) + \sum_{k=1}^{\infty} \tilde{a}_k F_k(1/z).$$

This series is called the *Faber-Laurent* series of the function f and the coefficients a_k and $\widetilde{a}_k$ are said to be the *Faber-Laurent coefficients* of f.

Let $\mathcal{P}$ be the set of all polynomials (with no restrictions on the degree), and let $\mathcal{P}(\mathbb{D})$ be the set of traces of members of $\mathcal{P}$ on $\mathbb{D}$.

We define two operators $T : \mathcal{P}(\mathbb{D}) \to E_M(G)$ and $\widetilde{T} : \mathcal{P}(\mathbb{D}) \to \widetilde{E}_M(G^-)$ as

$$T(P)(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{P(w)\psi'(w)}{\psi(w) - z} dw, \quad z \in G$$

$$\widetilde{T}(P)(z) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{P(w)\psi_1'(w)}{\psi_1(w) - z} dw, \quad z \in G^-.$$

It is readily seen that

$$T\left(\sum_{k=0}^{n} b_k w^k\right) = \sum_{k=0}^{n} b_k \Phi_k(z) \text{ and } \quad \widetilde{T}\left(\sum_{k=0}^{n} d_k w^k\right) = \sum_{k=0}^{n} d_k F_k(1/z).$$

If $z' \in G$, then

$$T(P)(z') = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{P(w)\psi'(w)}{\psi(w) - z'} dw = \frac{1}{2\pi i} \int_{\Gamma} \frac{(P \circ \varphi)(\varsigma)}{\varsigma - z'} d\varsigma = (P \circ \varphi)^+(z'),$$

which, by (1.2) implies that

$$T(P)(z) = S_\Gamma(P \circ \varphi)(z) + (1/2)(P \circ \varphi)(z)$$

a. e. on Γ.

Similarly taking the limit $z'' \to z \in \Gamma$ over all nontangential paths outside Γ in the relation

$$\widetilde{T}(P)(z'') = \frac{1}{2\pi i} \int_{\Gamma} \frac{P(\varphi_1(\varsigma))}{\varsigma - z''} d\varsigma = [(P \circ \varphi_1)]^-(z''), \qquad z'' \in G^-$$

we get

$$\widetilde{T}(P)(z) = -(1/2)(P \circ \varphi_1)(z) + S_\Gamma(P \circ \varphi_1)(z)$$

a.e. on Γ.

By virtue of the Hahn-Banach theorem, we can extend the operators T and $\tilde{T}$ from $\mathcal{P}(\mathbb{D})$ to the spaces $E_M(\mathbb{D})$ as a linear and bounded operator.

Then for these extensions $T : E_M(\mathbb{D}) \to E_M(G)$ and $\tilde{T} : E_M(\mathbb{D}) \to \tilde{E}_M(G^-)$ we have the representations

$$T(g)(z) = \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{g(w)\,\psi'(w)}{\psi(w) - z} dw, \quad z \in G, \quad g \in E_M(\mathbb{D}),$$

$$\tilde{T}(g)(z) = \frac{1}{2\pi i} \int\limits_{\mathbb{T}} \frac{g(w)\,\psi_1'(w)}{\psi_1(w) - z} dw, \quad z \in G^-, \quad g \in E_M(\mathbb{D}).$$

The following lemma is a special case of Theorem 2.4 of [12].

Lemma 2.15. *If Γ is a Dini-smooth curve and $E_M(G)$ is a reflexive Smirnov-Orlicz class, then the operators*

$$T : \; E_M(\mathbb{D}) \to E_M(G) \quad \text{and} \quad \tilde{T} : \; E_M(\mathbb{D}) \to \tilde{E}_M\left(G^-\right)$$

are one-to-one and onto.

3. Proofs of the results

Proof of Theorem 2.1. Since $f(z) = f^+(z) - f^-(z)$ a.e. on Γ, considering the rational function

$$R_n(z, f) := \sum_{k=0}^{n} a_k \Phi_k(z) + \sum_{k=1}^{n} \tilde{a}_k F_k(1/z),$$

it is enough to prove inequalities

$$\left\| f^-(z) + \sum_{k=1}^{n} \tilde{a}_k F_k(1/z) \right\|_{L_M(\Gamma)} \leq c_{23}\; \tilde{\omega}_{\alpha,\Gamma}(f, 1/n)_M \tag{3.1}$$

and

$$\left\| f^+(z) - \sum_{k=0}^{n} a_k \Phi_k(z) \right\|_{L_M(\Gamma)} \leq c_{24}\; \omega_{\alpha,\Gamma}(f, 1/n)_M\,. \tag{3.2}$$

Let $f \in L_M(\Gamma)$. Then $f_1, f_0 \in L_M(\mathbb{T})$. We take $z' \in G \setminus \{0\}$. Using (2.3) and

$$f(\varsigma) = f_1^+(\varphi_1(\varsigma)) - f_1^-(\varphi_1(\varsigma)) \quad \text{a.e. on } \Gamma \tag{3.3}$$

we obtain that

$$\sum_{k=1}^{n} \tilde{a}_k F_k(1/z') = \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z') - \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\left(\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(\varsigma) - f_1^+(\varphi_1(\varsigma))\right)}{\varsigma - z'} d\varsigma$$
$$- f_1^-(\varphi_1(z')) - f^-(z')\,.$$

Taking the limit as $z' \to z$ along all non-tangential paths inside of Γ, we obtain

$$\sum_{k=1}^{n} \tilde{a}_k F_k(1/z) = \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z) - \frac{1}{2}\left(\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z) - f_1^+(\varphi_1(z))\right)$$
$$-S_\Gamma\left[\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k - \left(f_1^+ \circ \varphi_1\right)\right] - f_1^-(\varphi_1(z)) - f^+(z)$$

a.e. on Γ.

Using (1.3), (3.3), Minkowski's inequality and the boundedness of S_Γ we get

$$\left\| f^-(z) + \sum_{k=1}^{n} \tilde{a}_k F_k(1/z') \right\|_{L_M(\Gamma)} = \left\| \frac{1}{2}\left(\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z) - f_1^+(\varphi_1(z))\right) - S_\Gamma\left[\sum_{k=1}^{n} \tilde{a}_k \varphi_1^k - \left(f_1^+ \circ \varphi_1\right)\right](z) \right\|_{L_M(\Gamma)}$$

$$\le c_{25} \left\| \sum_{k=1}^{n} \tilde{a}_k \varphi_1^k(z) - f_1^+(\varphi_1(z)) \right\|_{L_M(\Gamma)} \le c_{26} \left\| f_1^+(w) - \sum_{k=1}^{n} \tilde{a}_k w^k \right\|_{L_M(\Gamma)}.$$

On the other hand, the Faber-Laurent coefficients $\tilde{a}_k$ of the function f and the Taylor coefficients of the function f_1^+ at the origin are coincide. Then taking Lemma 2.14 into account, we conclude that

$$\left\| f^- + \sum_{k=1}^{n} \tilde{a}_k F_k(1/z') \right\|_{L_M(\Gamma)} \le c_{27}\ \tilde{\omega}_{\alpha,\Gamma}(f, 1/n)_M,$$

and (3.1) is proved.

The proof of relation (3.2) goes similarly; we use the relations (2.2) and

$$f(\varsigma) = f_0^+(\varphi(\varsigma)) - f_0^-(\varphi(\varsigma)) \quad \text{a.e. on } \Gamma$$

instead of (2.3) and (3.3), respectively. ∎

Proof of Theorem 2.4. Let $f \in E_M(G)$. Then we have $T\left(f_0^+\right) = f$. Since the operator $T : E_M(\mathbb{D}) \to E_M(G)$ is linear, bounded, one-to-one and onto, the operator $T^{-1} : E_M(G) \to E_M(\mathbb{D})$ is linear and bounded. We take a $p_n^* \in \mathcal{P}_n$ as the best approximating algebraic polynomial to f in $E_M(G)$. Then $T^{-1}(p_n^*) \in \mathcal{P}_n(\mathbb{D})$ and therefore

$$E_n\left(f_0^+\right)_M \le \left\| f_0^+ - T^{-1}(p_n^*) \right\|_{L_M(\mathbb{T})} = \left\| T^{-1}(f) - T^{-1}(p_n^*) \right\|_{L_M(\mathbb{T})}$$

(3.4)
$$= \left\| T^{-1}(f - p_n^*) \right\|_{L_M(\mathbb{T})} \le \left\| T^{-1} \right\| \left\| f - p_n^* \right\|_{L_M(\Gamma)} = \left\| T^{-1} \right\| E_n(f, G)_M,$$

because the operator T^{-1} is bounded. From (3.4) we have

$$\omega_{\alpha,\Gamma}(f,1/n)_M = \omega_\alpha\left(f_0^+,1/n\right)_M \leq \frac{c_{28}}{n^\alpha}\sum_{k=0}^{n}(k+1)^{\alpha-1}E_k\left(f_0^+\right)_M$$

$$\leq \frac{c_{28}\left\|T^{-1}\right\|}{n^\alpha}\sum_{k=0}^{n}(k+1)^{\alpha-1}E_k(f,G)_M, \quad \alpha>0,\, n=1,2,\ldots$$

and the proof is completed. □

Proof of Theorem 2.9. Let $f \in \tilde{E}_M(G^-)$. Then $\tilde{T}\left(f_1^+\right) = f$. By Lemma 2.15 the operator $\tilde{T}^{-1} : \tilde{E}_M(G^-) \to E_M(\mathbb{D})$ is linear and bounded. Let $r_n^* \in \mathcal{R}_n$ be a function such that $\tilde{E}_n(f,G^-)_M = \|f - r_n^*\|_{L_M(\Gamma)}$. Then $\tilde{T}^{-1}(r_n^*) \in \mathcal{P}_n(\mathbb{D})$ and therefore

$$E_n\left(f_1^+\right)_M \leq \left\|f_1^+ - \tilde{T}^{-1}(r_n^*)\right\|_{L_M(\mathbb{T})} = \left\|\tilde{T}^{-1}(f) - \tilde{T}^{-1}(r_n^*)\right\|_{L_M(\mathbb{T})}$$

(3.5)

$$= \left\|\tilde{T}^{-1}(f - r_n^*)\right\|_{L_M(\mathbb{T})} \leq \left\|\tilde{T}^{-1}\right\|\|f - r_n^*\|_{L_M(\Gamma)} = \left\|\tilde{T}^{-1}\right\|\tilde{E}_n\left(f,G^-\right)_M.$$

From (3.5) we conclude

$$\tilde{\omega}_{\alpha,\Gamma}(f,1/n)_M = \omega_\alpha\left(f_1^+,1/n\right)_M \leq \frac{c_{29}}{n^\alpha}\sum_{k=0}^{n}(k+1)^{\alpha-1}E_k\left(f_1^+\right)_M$$

$$\leq \frac{c_{29}\left\|\tilde{T}^{-1}\right\|}{n^\alpha}\sum_{k=0}^{n}(k+1)^{\alpha-1}\tilde{E}_k\left(f,G^-\right)_M, \quad \alpha>0, \; n=1,2,\ldots$$

the required result. □

Acknowledgements.

Authors are indebted to referees for constructive discussions on the results obtained in this paper.

References

[1] S. Ya. Al'per, *Approximation in the mean of analytic functions of class E^p*, in: Investigations on the modern problems of the function theory of a complex variable. Gos. Izdat. Fiz.-Mat. Lit., Moscow, 1960, 272-286 (in Russian).

[2] J. E. Andersson, *On the degree of polynomial approximation in $E^p(D)$*, J. Approx. Theory **19** (1977), 61-68.

[3] P. L. Butzer, H. Dyckoff, E. Görlicz and R. L. Stens, *Best trigonometric approximation, fractional derivatives and Lipschitz classes*, Can. J. Math. **24** (1977), 781-793.

[4] A. Çavuş and D. M. Israfilov, *Approximation by Faber-Laurent rational functions in the mean of functions of the class $L_p(\Gamma)$ with $1 < p < \infty$*, Approx. Theory Appl. **11** (1995), 105-118.

[5] P. L. Duren, Theory of H^p spaces, Academic Press, 1970.

[6] D. Gaier, Lectures on Complex Approximation, Birkhäuser, 1987.

[7] G. M. Goluzin, Geometric theory of functions of a complex variable, Translation of Mathematical Monographs Vol. **26**, R. I.: AMS, Providence, 1969.

[8] V. P. Havin, *Continuity in L_p of an integral operator with the Cauchy kernel*, Vestnik Leningrad Univ. **22** (1967), 103 (russian, english summary).

[9] D. M. Israfilov, *Approximate properties of the generalized Faber series in an integral metric*, Izv. Akad. Nauk Az. SSR, Ser. Fiz.-Tekh. Math. Nauk **2** (1987), 10-14 (in Russian).

[10] D. M. Israfilov, *Approximation by p-Faber polynomials in the weighted Smirnov class $E^p(G,\omega)$ and the Bieberbach polynomials*, Constr. Approx. **17** (2001), 335-351.

[11] D. M. Israfilov, *Approximation by p-Faber-Laurent rational functions in the weighted Lebesgue spaces*, Czechoslovak Math. J. **54** (2004), 751-765.

[12] D. M. Israfilov and R. Akgün, *Approximation in weighted Smirnov-Orlicz classes*, J. Math. Kyoto Univ. **46** (2006), no:4, 755-770.

[13] V. M. Kokilashvili, *On analytic functions of Smirnov-Orlicz classes*, Studia Math. **31** (1968), 43-59.

[14] V. M. Kokilashvili, *Approximation of analytic functions of class E^p*, Proceedings of Math. Inst. of Tbilisi, **39** (1968), 82-102, (In Russian).

[15] V. M. Kokilashvili, *A direct theorem on mean approximation of analytic functions by polynomials*, Soviet Math. Dokl. **10** (1969), 411-414.

[16] I. P. Natanson, Teoriya funktsii veshestvennoy peremennoy, Moscow-Leningrad, 1974.

[17] Ch. Pommerenke, Boundary Behavior of Conformal Maps, Berlin, Springer-Verlag (1992).

[18] M. M. Rao and Z. D. Ren, Theory of Orlicz Spaces, Marcel Dekker, New York, 1991.

[19] R. Ryan, *On the conjugate functions in Orlicz space*, Pacific J. Math. **13** (1963), 1371-1377.

[20] P. K. Suetin, Series of Faber polynomials, Gordon and Breach Science Publishers, Amsterdam, 1998.

[21] R. Taberski, *Differences, moduli and derivatives of fractional orders*, Comment. Math. **19** (1977), 389-400.

[22] J. L. Walsh and H. G. Russel, *Integrated continuity conditions and degree of approximation by polynomials or by bounded analytic functions*, Trans. Amer. Math. Soc. **92** (1959), 355-370.

[23] S. E. Warschawski, *Über das ranverhalten der Ableitung der Abildungsfunktion bei Konformer Abbildung*, Math. Z. **35** (1932), 321-456.

[24] A. Zygmund, Trigonometric series, V: I and II, Cambridge, 1959.

R. Akgün
Balikesir University
Faculty of Science and Art
Department of Mathematics
10145, Balikesir
Turkey
E-mail: `rakgun@balikesir.edu.tr`

D. M. Israfilov
Institute of Math. and Mech. NAS Azerbaijan
F. Agayev Str. 9
Baku
Azerbaijan
E-mail: `mdaniyal@balikesir.edu.tr`

Received: 20.4.2007.