ON PRIMAL AND WEAKLY PRIMAL IDEALS OVER COMMUTATIVE SEMIRINGS

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Abstract. Since the theory of ideals plays an important role in the theory of semirings, in this paper we will make an intensive study of the notions of primal and weakly primal ideals in commutative semirings with an identity 1. It is shown that these notions inherit most of the essential properties of the primal and weakly primal ideals of a commutative ring with non-zero identity. Also, the relationship among the families of weakly prime ideals, primal ideals and weakly primal ideals of a semiring R is considered.

1. Introduction

The concept of semirings was introduced by H. S. Vandiver in 1935 and has since then been studied by many authors (see, for example, [2, 3, 11, 12, 13]). Primal ideals in a commutative ring with non-zero identity have been introduced and studied by Ladislas Fuchs in [8] (also see [9]). Weakly prime ideals in a commutative ring have been introduced and studied by D. D. Anderson and E. Smith in [1]. Also, weakly primal ideals in a commutative ring have been introduced and studied in [7]. This paper is concerned with generalizing some results of primal ideals and weakly primal ideals from commutative rings theory to commutative semirings theory.

For the sake of completeness, we state some definitions and notations used throughout. A commutative semiring R is defined as an algebraic system $(R, +, .)$ such that $(R, +)$ and $(R, .)$ are commutative semigroups, connected by $a(b + c) = ab + ac$ for all $a, b, c \in R$, and there exists $0 \in R$ such that $r + 0 = r$ and $r0 = 0r = 0$ for each $r \in R$. In this paper all semirings considered will be assumed to be commutative semirings. A semiring R is

said to be a semidomain if whenever $ab = 0$ $(a, b \in R)$, then either $a = 0$ or $b = 0$. A semifield is a semiring in which non-zero elements form a group under multiplication.

A subset I of a semiring R will be called an ideal if $a, b \in I$ and $r \in R$ implies $a + b \in I$ and $ra \in I$. A subtractive ideal (= k-ideal) K is an ideal such that if $x, x + y \in K$ then $y \in K$ (so $\{0\}$ is a k-ideal of R). A prime ideal of R is a proper ideal P of R in which $x \in P$ or $y \in P$ whenever $xy \in P$. So P is prime if and only if A and B are ideals in R such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$ where $AB = \{ab : a \in A \text{ and } b \in B\}$ (see [3, Theorem 5]). A primary ideal of P of R is a proper ideal of R such that, if $xy \in P$ and $x \notin P$, then $y^n \in P$ for some positive integer n.

Let R be a semiring. We define a proper ideal A of R to be weakly prime if $0 \neq ab \in A$ implies $a \in A$ or $b \in A$. So a prime ideal is a weakly prime. However, since 0 is always weakly prime (by definition), a weakly prime ideal need not be prime (see [1]).

2. Primal ideals

An ideal I of a semiring R is called primal if the elements of R that are not prime to I form an ideal. Here an element $r \in R$ is called prime to I if $rs \in I$ $(s \in R)$ implies $s \in I$, that is, $(I :_R r) = I$ (see [8]).

LEMMA 2.1. *Let I be a proper ideal of a semiring R, and let P be the set of elements of R that are not prime to I. Then the following hold:*

(i) *$I \subseteq P$.*
(ii) *If P is an ideal of R, then P is prime.*

PROOF. (i) Let $r \in I$. We can assume that $r \neq 0$ (since $0 \in P$). As $0 \neq r = 1_R r \in I$ with $1_R \notin I$, we must have r is not prime to I; hence $I \subseteq P$.

(ii) Let $a, b \in R - P$. Then $(I : a) = (I : b) = I$. It suffices to show that $(I : ab) \subseteq I$. Suppose that $r \in (I : ab)$, so $rab \in I$; hence $ra \in (I : b) = I$. It follows that $r \in (I : a) = I$; thus $I = (I : ab)$. Therefore, $ab \notin P$, as needed. □

If I is a primal ideal of R, then by Lemma 2.1, P is a prime ideal of R, call the adjoint prime ideal P of I. In this case we also say that I is a P-primal ideal.

PROPOSITION 2.2. *Let Q be a P-primary ideal of a semiring R. Then Q is a P-primal ideal of R.*

PROOF. It suffices to show that the set of elements of R that are not prime to Q is just P. Assume that a is an element of R such that is not prime to Q, so $Q \subsetneq (Q : a)$; hence there is an element $b \in (Q : a)$ with $b \notin Q$ and $ab \in Q$; thus Q primary gives $a \in P$. On the other hand, if $a \notin P$, then $(Q : a) = Q$ since Q is primary, and the proof is complete. □

Let R be a given semiring, and let S be the set of all multiplicatively cancelable elements of R (so $1 \in S$). Clearly, the set S is multiplicatively closed. Define a relation $\sim$ on $R \times S$ as follows: for $(a, s), (b, t) \in R \times S$, we write $(a, s) \sim (b, t)$ if and only if $ad = bc$. Then $\sim$ is an equivalence relation on $R \times S$. For $(a, s) \in R \times S$, denote the equivalence class of $\sim$ which contains (a, s) by a/s, and denote the set of all equivalence classes of $\sim$ by R_S. Then R_S can be given the structure of a commutative semiring under operations for which $a/s + b/t = (ta + sb)/st$, $(a/s)(b/t) = (ab)/st$ for all $a, b \in R$ and $s, t \in S$. This new semiring R_S is called the semiring of fractions of R with respect to S; its zero element is $0/1$, its multiplicative identity element is $1/1$ and each element of S has a multiplicative inverse in R_S (see [12, 13, 15]). It is easy to see that if R is a semidomain and $S = R - \{0\}$, then R_S is a semifield.

Throughout this paper we shall assume unless otherwise stated, that **S** is the set of all multiplicatively cancelable elements of a semiring R. Now suppose that I is an ideal of a semiring R. The ideal generated by I in R_S, that is, the set of all finite sums $s_1 a_1 + \cdots + s_n a_n$ where $a_i \in R_S$ and $s_i \in I$, is called the extention of I to R_S, and it is denoted by IR_S. Again, if J is an ideal of R_S then by the contraction of J in R we mean $J \cap R = \{r \in R : r/1 \in J\}$, which is clearly an ideal of R.

LEMMA 2.3. *Assume that I, J and K are ideals of a semiring R and let L be an ideal of the semiring R_S. Then the following hold:*

(i) $x \in IR_S$ *if and only if, it can be written in the form $x = a/c$ for some $a \in I$ and $c \in S$;*
(ii) $(L \cap R)R_S = L$;
(iii) $(I \cap J)R_S = (IR_S) \cap (JR_S)$;
(iv) *If P is a prime ideal of R with $P \cap S = \emptyset$, then PR_S is a prime ideal of R_S and $(PR_S) \cap R = P$;*
(v) *If Q is a prime ideal of R_S, then $Q \cap R$ is a prime ideal of R.*

PROOF. (i) Let $x \in IR_S$. Then there are elements $t_i \in R_S$ and $a_i \in I$ such that $x = \sum_{i=1}^{n} t_i a_i$. Since $t_i \in R_S$, $t_i = r_i/c_i$ where $r_i \in R$ and $c_i \in S$. Put $c_1 c_2 \cdots c_n = c$, then with suitable elements $e_1, \ldots, e_n$ of R we have $t_i = e_i/c$, and therefore $x = (e_1 a_1 + \cdots + e_n a_n)/c = a/c$ where $a \in I$. This proves half of the lemma, and the remaining half is obvious.

(ii) It suffices to prove that $L \subseteq (L \cap R)R_S$. Let $x \in L$. Then $x = r/s$ for some $r \in R$ and $s \in S$, so $r = sx \in L \cap R$ and, accordingly, $x = (1/s)r \in (L \cap R)R_S$, as required.

(iii) Let $K = I \cap J$. Since $KR_S \subseteq (IR_S) \cap (JR_S)$ is trivial, we will prove the reverse inclusion. Let $x \in (IR_S) \cap (JR_S)$. Then by (i), there exist $s, t \in S$, $a \in I$ and $b \in J$ such that $x = a/s = b/t$, so $at = bs \in I \cap J$; hence $x = a/s = (at)/(ts) \in KR_S$, so we have equality.

(iv) Suppose that $x = a/s$ and $y = b/t$ are two elements of R_S with the properties that $xy \in PR_S$ and $x \notin PR_S$. Then $a \notin P$ and by (i), $xy = (ab)/(st) = c/u$, where $c \in P$ and $u \in S$, consequently, $cst = abu \in P$, which shows that $b \in P$, because neither a nor u is in P, so $y \in PR_S$. Thus PR_S is prime. In order to prove that $(PR_S) \cap R = P$, it will be enough to show that $(PR_S) \cap R \subseteq P$. Let $z \in (PR_S) \cap R$, then $z/1 = d/w$, where $d \in P$ and $w \in S$; hence $wz \in P$ and $w \notin P$, consequently $z \in P$, as required.

(v) Suppose that $ab \in Q \cap R$, where $a, b \in R$. Then $(a/1)(b/1) = (ab)/1 \in Q$, so Q prime gives either $a/1 \in Q$ or $b/1 \in Q$; hence either $a \in Q \cap R$ or $b \in Q \cap R$, as needed. $\qquad\square$

THEOREM 2.4. *Let R be a semiring. If J is a P-primal ideal of the semiring R_S, then $J \cap R$ is a $(P \cap R)$-primal ideal of R.*

PROOF. Since P is prime in R_S, $P \cap R$ is prime in R by Lemma 2.3 (v). It only remains to show that $P \cap R$ is exactly the set of elements that are not prime to $J \cap R$. First, let $a \notin P \cap R$. Then $a/1 \notin P$, and so $(J :_{R_S} a/1) = J$. It follows that $(J \cap R :_R a) = J \cap R$; hence a is prime to $J \cap R$. Next, let $a \in P \cap R$; so $a/1 \in P$. Then $(a/1)(b/s) \in J$ for some $b/s \in R_S - J$; then by Lemma 2.3 (ii), $b/1 \notin J$. Since $(a/s)(b/1) \in J$ implies $ab \in J \cap R$, we conclude that a is not prime to $J \cap R$. $\qquad\square$

PROPOSITION 2.5. *Let R be a semiring. Then the following hold:*
(i) *If I is a P-primal ideal of R with $P \cap S = \emptyset$, then $IR_S \cap R = I$;*
(ii) *If I is a P-primal ideal of R with $P \nsubseteq S$, then $I \subsetneq IR_S \cap R$.*

PROOF. (i) The inclusion $I \subseteq IR_S \cap R$ being obvious, pick an $a \in IR_S \cap R$. We can write $a/1 = r/s$ for some $r \in I$ and $s \in S$. Therefore, $sa = r \in I$. Since $s \notin P$, we must have s is prime to I; hence $a \in I$, as required.

(ii) By assumption, there is an element $x \in P - S$, so $(I \subsetneq I : x)$. Let $y \in (I : x) - I$. Then $xy \in I$ implies $y \in IR_S \cap R$, showing that $I \subsetneq IR_S \cap R$. $\qquad\square$

THEOREM 2.6. *Let I be an ideal of a semiring R such that $(I : a) = P$ is a prime ideal of R for some $a \in R - I$. If $P \cap S = \emptyset$, then $IR_S \cap R$ is a P-primal ideal of R.*

PROOF. Set $J = IR_S \cap R$. First, we show that $(J : a) = P$. If $t \in P = (I : a)$, then $at \in I \subseteq J$; hence $t \in (J : a)$. For the other containment, assume that $y \in (J : a)$, so $ay \in J$; thus $(ay)/1 = b/s$ for some $b \in I$ and $s \in S$. It follows that $ys \in (I : a) = P$, so $y \in P$ since P is prime. As $a \notin J$, no element contained in P is prime to J. It remains to show that every $x \notin P$ is prime to J. Suppose $y \in (J : x)$. Then $(xy)/1 = c/t$ for some $c \in I$ and $t \in S$, so $xyt = c \in I$; hence $y \in IR_S \cap R$ since P is prime. Thus $(J : x) = J$, as needed. $\qquad\square$

Let R be a semiring. We say that R is a valuation semiring if its ideals are linearly ordered by inclusion. We say that R is a S-valuation semiring if R_S is a valuation semiring. An ideal I of a semiring R is irreducible if, for ideals J and K of R, $I = J \cap K$ implies that either $I = J$ or $I = K$.

LEMMA 2.7. *If R is a semiring, then every irreducible ideal of R is primal.*

PROOF. Assume that I is an irreducible ideal of R and let P be the set of elements of R that are not prime to I; we show that P is an ideal of R. Let $a, b \in P$. Then $I \subsetneq (I : a)$ and $I \subsetneq (I : b)$. Then I irreducible gives $I \subsetneq (I : a) \cap (I : b) \subseteq (I : a + b)$; hence $a + b \in P$. Finally, if $r \in R$, then $I \subsetneq (I : a) \subseteq (I : ra)$ shows that ra is not prime to I, as required. $\square$

THEOREM 2.8. *Let I be a P-primal ideal of a S-valuation semiring R with $P \cap S = \emptyset$. Then I is irreducible.*

PROOF. If $I = J \cap K$ for ideals J, K of R, then Lemma 2.3 gives $IR_S = (JR_S) \cap (KR_S)$. Since R_S is a valuation semiring, either $IR_S = JR_S$ or $IR_S = KR_S$. If the first alternative holds, then by Propostion 2.5, $J \subseteq (JR_S) \cap R = (IR_S) \cap R = I$; hence $J = I$, and the proof is complete. $\square$

THEOREM 2.9. *Let I be an ideal of a valuation semiring R. Then I is a primal ideal of R with adjoint prime ideal $P = \{r \in R : rI \subsetneq I\}$.*

PROOF. Let $I = J \cap K$ for ideals J, K of R. Then either $J \subseteq K$ or $K \subseteq J$; hence either $I = J$ or $I = K$. Therefore, I is a primal ideal of R by Lemma 2.7. It is easy to see that P is an ideal of R. Let $ab \in P$ with $a \notin P$, where $a, b \in R$. Then $aI = I$, so $bI = abI \subsetneq I$; hence $b \in P$. Thus P is prime. It remains to show that P is the set of elements of R that are not prime to I. If $c \in P$, then $cI \subsetneq I \subseteq (I : c)$. Suppose that $I = (I : c)$ and let $x \in (I : c)$. Then $cx \in I$, so $x \in cI$; hence $I = cI$ which is a contradiction. Thus c is not prime to I, as needed. $\square$

3. WEAKLY PRIMAL IDEALS

Let I be an ideal of a semiring R. An element $a \in R$ is called weakly prime to I if $0 \neq ra \in I$ ($r \in R$) implies that $r \in I$, and let $p(I)$ be the set of elements of R that are not weakly prime to I. 0 is always weakly prime to I. A proper ideal I of R is called weakly primal if the set $P = p(I) \cup \{0\}$ form an ideal: this ideal is called the weakly adjoint ideal P of I. Let R be a commutative semiring which is not a semidomain. Then 0 is a 0-weakly primal ideal of R (by definition), so a weakly primal ideal need not be primal.

CONVENTION. Let R be a commutative semiring, I an ideal of R and A a subset of R. We say that A satisfies $(*)$ if A is exactly the set of elements of R that are not weakly prime to I. We use the notation A^* to refer to the non-zero elements of A.

THEOREM 3.1. *Assume that A is a proper k-ideal of a semiring R and let P be a proper ideal of R. Then the following statements are equivalent.*

(i) *A is a P-weakly primal ideal of R.*

(ii) *For $a \notin P^*$, $(A :_R a) = A \cup (0 :_R a)$, and for $a \in P^*$, $A \cup (0 :_R a) \subsetneqq (A :_R a)$.*

(iii) *For $a \notin P^*$, $(A :_R a) = A$ or $(A :_R a) = (0 :_R a)$, and for $a \in P^*$, $A \subsetneqq (A :_R a)$ and $(0 :_R a) \subsetneqq (A :_R a)$.*

PROOF. $(i) \Rightarrow (ii)$ Let A be a P-weakly primal ideal of R. Then P^* satisfies $(*)$. Suppose that $a \notin P^*$ and let $b \in (A :_R a)$. If $ab \neq 0$, then a weakly prime to A gives $b \in A$. If $ab = 0$, then $b \in (0 :_R a)$. So $(A :_R a) \subseteq A \cup (0 :_R a)$. As the reverse containment holds for any ideal A, we have equality. Now suppose that $a \in P^*$. Then there exists $s \in R - A$ such that $0 \neq as \in A$; hence $s \in (A :_R a) - (A \cup (0 :_R a))$, as needed.

$(ii) \Rightarrow (iii)$ Let $a \notin P^*$. By [5, Lemma 2.1 and Lemma 2.2], if an ideal of R is the union of two k-ideals, then it is equal to one of them. Moreover, if $a \in P^*$, then by (ii), we must have $A \subsetneqq (A :_R a)$ and $(0 :_R a) \subsetneqq (A :_R a)$.

$(iii) \Rightarrow (i)$ By (iii), P^* satisfies $(*)$. Thus A is P-weakly primal. □

REMARK 3.2. Assume that A is a P-weakly primal ideal of R and let $0 \neq a \in A$. As $0 \neq 1_R a \in I$ with $1_R \notin I$, we must have a is not weakly prime to A; hence $I \subseteq P$. It follows that if A is a 0-weakly primal ideal of R, then $I = 0$.

THEOREM 3.3. *Let R be a semiring. Then every k-weakly prime ideal of R is weakly primal.*

PROOF. Let A be a k-weakly prime ideal of R. We can assume that $A \neq 0$. It is enough to show that P^* satisfies $(*)$. Let $0 \neq a \in A$. Then $0 \neq a = 1_R a \in A$ with $1_R \notin A$ gives a is not weakly prime to A. On the other hand, if $a \notin A^*$, then a is weakly prime to A by [5, Proposition 2.3]. Thus A is weakly primal. □

THEOREM 3.4. *Let A be a k-ideal of a semiring R. If A is a P-weakly primal ideal R, then P is a weakly prime ideal of R.*

PROOF. Suppose that $a, b \notin P$; we show that either $ab = 0$ or $ab \notin P$. We can assume that $ab \neq 0$. Let $0 \neq rab \in A$ for some $r \in R$. Then Theorem 2.1 (ii) gives $0 \neq ra \in (A :_R b) = A \cup (0 :_R b)$ where $ra \notin (0 :_R b)$; hence $0 \neq ra \in A$. Thus, $0 \neq r \in (A :_R a) = A \cup (0 :_R a)$, so $r \in I$; hence ab is weakly prime to A, as required. □

THEOREM 3.5. *Let A be a k-weakly primal ideal of a semiring R with the weakly adjoint ideal P. If P is not a prime ideal of R, then $A^2 = 0$, $AP = 0$, $A\sqrt{0} = 0$ and $\sqrt{A} = \sqrt{0}$.*

PROOF. By Remark 2.2, Proposition 2.4 and [5, Theorem 2.6 and Theorem 2.12], we must have $A^2 \subseteq P^2 = 0$, $AP \subseteq P^2 = 0$ and $A\sqrt{0} \subseteq P\sqrt{0} = 0$. Finally, since $\sqrt{0} \subseteq \sqrt{A}$ is trivial, we will prove the reverse inclusion. As $A^2 = 0$, we get $A \subseteq \sqrt{0}$; hence $\sqrt{A} \subseteq \sqrt{0}$, as needed.   □

THEOREM 3.6. *Let A and B be k-weakly primal ideals of a semiring R with the weakly adjoint ideals P and Q respectively. If P and Q are not prime ideals of R, $AB = 0$.*

PROOF. By Remark 2.2, Proposition 2.4 and [5, Theorem 2.13], we must have $AB \subseteq PQ = 0$.   □

An ideal I of a semiring R is called a partitioning ideal (= Q-ideal) if there exists a subset Q of R such that

(1) $R = \cup\{q + I : q \in Q\}$
(2) If $q_1, q_2 \in Q$ then $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ if and only if $q_1 = q_2$.

Let I be a Q-ideal of a semiring R and let $R/I = \{q + I : q \in Q\}$. Then R/I forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows: $(q_1 + I) \oplus (q_2 + I) = q_3 + I$ where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$.

$$(q_1 + I) \odot (q_2 + I) = q_4 + I$$

where $q_4 \in Q$ is the unique element such that $q_1 q_2 + I \subseteq q_4 + I$. This semiring R/I is called the quotient semiring of R by I. By definition of Q-ideal, there exists a unique $q' \in Q$ such that $0 + I \subseteq q' + I$. Then $q' + I$ is a zero element of R/I. Clearly, if R is commutative, then so is R/I (see [10, 11]).

THEOREM 3.7. *Let J be a Q-ideal of a semiring R, I a proper k-ideal of R and J a weakly prime ideal of R with $J \subseteq I$. Then I is a weakly primal ideal of R if and only if I/J is a weakly primal ideal of R/J. In particular, there is a bijective correspondence between the weakly primal ideals of R containing J and the weakly primal ideals of R/J.*

PROOF. First suppose that I is a P-weakly primal ideal ideal of R with $J \subseteq I$. Then by Remark 3.2 , Theorem 3.4 and [6, Corollary 2.9 (i)], P/J is a weakly prime ideal of R/J. It suffices to show that $(P/J)^*$ satisfies $(*)$. Let q' be the unique element in Q such that $q' + J$ is the zero in R/J. Let $a + J \in (P/J)^*$ where $a \in P \cap Q$ by [6, Proposition 2.2]. Since $0 + J \subseteq q' + J \neq a + J$, we must have $a \neq 0$ and a is not weakly prime to I; hence there exists $r \in R - I$ such that $0 \neq ra \in I$. If $0 \neq ra \in J$, then J weakly prime gives $r \in J$ which is a contradiction since $r \notin I$. So $0 \neq ra \notin J$. There is an element $q_1 \in Q$ such that $r \in q_1 + J$, so $r = q_1 + c$ for some $c \in J$; hence $0 \neq aq_1 \notin J$. It follows that $0 \neq (q_1 + J) \odot (a + J) \in I/J$ with $q_1 + J \notin I/J$, so $a + J$ is not weakly prime to I/J. Now assume that $b + J \neq q' + J$ is not weakly prime to I/J where $b \in Q$. Then there exists $c + J \in R/J - I/J$ such

that $0 \neq (c + J) \odot (b + J) = q_2 + J \in I/J$ where $q_2 \in Q \cap I$ is the unique element such that $bc + J \subseteq q_2 + J$; hence $cb \in I$ with $c \notin I$. So $b \neq 0$ is not weakly prime to I. Therefore, $b + J \in (P/J)^*$, and the proof is complete.

Second, suppose that I/J is a P/J-weakly primal ideal of R/J; we show that I is a P-weakly primal ideal of R. By Theorem 3.4 and [6, Theorem 2.8 (ii)], P is a weakly prime ideal of R. It is enough to show that P^* satisfies $(*)$. Let $a \in P^*$. By Remark 3.2, we can assume that $a \notin J$. There is an element $q_3 \in Q$ such that $a \in q_1 + J$, so $a = q_3 + d$ for some $d \in J$. As J is a weakly prime ideal and $0 \neq q_3 + J \in P/J$, there exists $r + J \in R/J - I/J$ such that $0 \neq (q_3 + J) \odot (r + J) = q_4 + J \in I/J$ where $q_4 \in Q \cap I$ is the unique element such that $q_3 r + J \subseteq q_4 + J$; hence $0 \neq ra \in I$ with $r \notin I$. Thus a is not weakly prime to I. Now assume that a is not weakly prime to I (so $a \neq 0$); we show that $a \in P$. We can assume that $a \notin I$. Then there is an element $r \in R - I$ such that $0 \neq ra \in I$. There are elements $q_5, q_6 \in Q$ such that $a \in q_5 + J$ and $r \in q_6 + J$, so $a = q_5 + e$ and $r = q_6 + f$ for some $e, f \in J$. Then $0 \neq ef \in I$. Therefore, J weakly prime ideal gives $0 \neq q_7 + J = (q_5 + J) \odot (q_6 + J) \in I/J$ where $q_7 \in Q \cap I$ is the unique element such that $q_5 q_6 + J \subseteq q_7 + J$ with $q_6 + J \notin I/J$; hence $a + J = q_5 + e + J \in (P/J)^*$ since I/J is P/J-weakly primal. Thus $a \in P$, as required. $\square$

LEMMA 3.8. *Let I and A be k-ideals of a semiring R. Then the following hold:*

(i) *If I is a P-weakly primal ideal of R with $P \cap S = \emptyset$ and $0 \neq a/s \in IR_S$, then $a \in I$;*

(ii) *If A is a weakly prime ideal of R with $A \cap S = \emptyset$ and $0 \neq a/s \in AR_S$, then $a \in A$.*

PROOF. (i) Assume that $0 \neq a/s \in IR_S$ but $a \notin I$. Then $a/s = r/t$ for some $r \in I$ and $t \in S$ by Lemma 2.3, so $0 \neq ta = sr \in I$ with $a \notin I$; hence $t \in S$ is not weakly prime to I which is a contradiction. Thus $a \in I$. (ii) follows from (i) and Theorem 3.3. $\square$

PROPOSITION 3.9. *Assume that I is a k-ideal of a semiring R and let I be a P-weakly primal such that $P \cap S = \emptyset$. Then the following hold:*

(i) *IR_S is a PR_S-weakly primal ideal of R_S;*

(ii) *$I = (IR_S) \cap R$.*

PROOF. (i) It suffices to show that $(PR_S)^*$ satisfies $(*)$. Suppose that $a/s \in (PR_S)^*$. Then Lemma 3.8 gives $0 \neq a \in P$; so there exists $r \in R - I$ such that $0 \neq ra \in I$. As $(ra)/s \neq 0/1$, we must have $0 \neq (r/1)(a/s) \in IR_S$ where $r/1 \notin IR_S$ by Lemma 3.8; hence a/s is not weakly prime to IR_S. On the other hand, assume that a/s is not weakly prime to IR_S. Then there exists $r/t \in R_S - IR_S$ such that $0 \neq (a/s)(r/t) \in IR_S$, so $0 \neq ra \in I$ with $r \notin I$ by Lemma 3.8; hence $0 \neq a \in P$. Thus $a/s \in (PR_S)^*$, as needed.

(ii) It suffices to prove that $(IR_S) \cap R \subseteq I$. Let $a \in (IR_S) \cap R$. Then $a/1 \in IR_S$, so $a \in I$ by Lemma 2.9, as needed. $\qquad\square$

PROPOSITION 3.10. *Assume that R is a semiring and let Q be a weakly prime ideal of R_S. Then $Q \cap R$ is a weakly prime ideal of R.*

PROOF. If $0 \neq ab \in Q \cap R$, then $0 \neq (ab)/1 = (a/1)(b/1) \in Q$; hence Q weakly prime gives either $a/1 \in Q$ or $b/1 \in Q$. Thus either $a \in Q \cap R$ or $b \in Q \cap R$, as required. $\qquad\square$

PROPOSITION 3.11. *Assume that R is a semiring and let I be a Q-weakly primal ideal of R_S. Then $I \cap R$ is a $Q \cap R$-weakly primal ideal of R.*

PROOF. By Proposition 3.10 and Lemma 2.3 (ii), we must have $P = Q \cap R$ is a weakly prime ideal of R and $I = (I \cap R)R_S$. It only remains to show that P^* satisfies $(*)$. First suppose that $a \in P^*$. Then $0 \neq a/1 \in Q$, so there exists $r/s \in R_S - (I \cap R)R_S$ such that $0 \neq (r/s)(a/1) \in I$; hence $0 \neq ra \in I \cap R$ by Lemma 2.3 with $r \notin I \cap R$. It follows that a is not weakly prime to $I \cap R$. Now assume that b is not weakly prime to $I \cap R$. Then there is an element $s \notin I \cap R$ with $0 \neq sb \in I \cap R$. So $0 \neq (s/1)(b/1) \in I$ with $s/1 \notin I$ gives $b/1 \in Q$; hence $b \in P$, and the proof is complete. $\qquad\square$

THEOREM 3.12. *Let P be a k-weakly prime ideal of a semiring R with $P \cap S = \emptyset$. Then there exists a one-to-one correspondence between the P-weakly primal ideals of R and the PR_S-weakly primal ideals of R_S.*

PROOF. This follows from Propositions 3.9, 3.10, 3.11 and [14, Lemma 5.24]. $\qquad\square$

PROPOSITION 3.13. *Assume that I is a k-ideal of a semiring R and let I be a P-weakly primal ideal of R such that $I^2 \neq 0$. If P is a prime ideal of R, then I is primal.*

PROOF. It suffices to show that P is exactly the set of elements non-prime to I. If $a \in P$, then a is not prime to I. Now assume that a is not prime to I; we show that $a \in P$. Then there is an element $r \in R - I$ such that $ra \in I$. If $0 \neq ra \in I$, then a is not weakly prime to I; hence $a \in P$. So assume that $ra = 0$. First suppose that $aI \neq 0$, say $ar_0 \neq 0$ where $r_0 \in I$. Then $0 \neq a(r + r_0) = ar_0 \in I$ with $r + r_0 \notin I$ since I is k-ideal; hence $a \in P$. So we can assume that $aI = 0$. If $rI \neq 0$, then there exists $c \in I$ such that $rc \neq 0$. Then $0 \neq (a + c)r \in I$ with $r \notin I$ gives $a + c \in I$; hence $a \in I \subseteq P$ since I is a k-ideal. So we can assume that $rI = 0$. Since $I^2 \neq 0$, there exists $a_0b_0 \in I$ with $a_0b_0 \neq 0$. Then $0 \neq a_0b_0 = (a + a_0)(r + b_0) \in I$ with $r + b_0 \notin I$, so $a + a_0 \in P$. Hence $a \in P$, as required. $\qquad\square$

THEOREM 3.14. *Let A be a k-weakly primal ideal of a semiring R that is not primal. Then $A^2 = 0$. In particular, $\sqrt{A} = \sqrt{0}$*

PROOF. Let A be a P-weakly primal ideal of R. If P is not prime then $A^2 = 0$ by Theorem 3.5. If P is prime, then $A^2 = 0$ by Proposition 3.13. ☐

PROPOSITION 3.15. *Let I be a k-ideal over a semidomain R. The I is primal if and only if it is weakly primal.*

PROOF. We can assume that $I \neq 0$. Suppose that I is a P-primal ideal of R. It suffices to show that P^* satisfies (∗). Suppose that $a \in P^*$. Then I primal gives $I \subsetneq (I :_R a)$. Assume that a is weakly prime to I and let $0 \neq b \in (I :_R a)$. Since $0 \neq ab \in I$, we must have $b \in I$; hence $I = (I :_R a)$ which is a contradiction. On the other hand, if a is not weakly prime to I, then $a \neq 0$ and a is not prime to I; hence $a \in P^*$, and the proof is complete.

Conversely, assume that I is a P-weakly primal ideal of R. By Theorem 3.4, P is weakly prime, so P is a prime ideal of R since R is a semidomain. It is enough to show that P is exactly the set of elements of R that are not prime to I. Clearly, 0 is not prime to I and $0 \in P$. Let $0 \neq a \in P$. Then a is not weakly prime to I; hence it is not prime to I. On the other hand, suppose that a ia not prime to I. We can assume that $a \neq 0$. Then there exists $r \in R - I$ such that $0 \neq ra \in I$, so a is not weakly prime to I; hence $a \in P$, as needed. ☐

THEOREM 3.16. *Let R be a semidomain. Then the following hold:*

(i) *Every primary ideal (so prime ideal) of R is weakly primal;*

(ii) *Every irreducible ideal of R is weakly primal;*

(iii) *If R is a valuation semidomain, then every proper ideal is weakly primal;*

(iv) *If R is a S-valuation semidomain and I is P-weakly primal ideal with $P \cap S = \emptyset$, then I is irreducible.*

PROOF. This follows from Proposition 2.2, Lemma 2.7, Theorem 2.8, Theorem 2.9 and Proposition 3.15. ☐

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