On Wavelet –Based Algorithm for Solving Parabolic Differential Equations

F. Soesianto, Kais Ismail Ibraheem

Department of Electrical Engineering
Gadjah Mada University Yogyakarta, Indonesia
e-mail : kaisismail@yahoo.com

Abstract

In this paper an algorithm based on wavelet method is used to solve Parabolic Partial Differential Equations. We proceed to the numerical experiments to obtain a general idea of the feasibility of introducing wavelet method. This algorithm can be used to solve any parabolic Partial differential equation using arbitrary wavelet basis .For simplicity applying, the example of solving partial differential equation by Haar wavelets is given

Key words : finite difference, partial differential equations, Haar wavelet.

1. Introduction

Wavelets algorithms are developed in the late'80s with Daubechies fundamental on wavelet (Daubenchies, 1988),since then wavelet theories are successful in signal analysis, data compression, numerical analysis, and wide Wavelet –based algorithms for solving the variety of other theoretical and practical applications.

Wavelet basis function is localized in both frequency and time. There are many sets of wavelet basis functions (each wavelet basis is generated by translating and dilating a specific basis template, referred to as its mother wavelet). Analogously we can see that Fourier function have sine and cosine bases that are localized in frequency but not in time. As a review, we will discuss (Beykin's, 1992) who applied the wavelet algorithm on elliptic differential equations.
We have solving system of the linear equation in papers (Soesianto and Abdullah,2000), (Abdullahand kais ,2000)and (kais and Abdullah,2000).
Then we continue the work with Solving Parabolic Differential Equations using wavelet also.

2. Parabolic Differential Equations

The general second-order partial differential equation in terms of independent variables x and y is given by (2).The equation is repeated here ,except that y has been replaced by t.

$$A(x,t)\frac{\partial^2 y}{\partial x^2} + B(x,t)\frac{\partial^2 y}{\partial x \partial t} + C(x,t)\frac{\partial^2 y}{\partial t^2} + f(x,t,\frac{\partial y}{\partial x},\frac{\partial y}{\partial t}) = 0 \qquad (1)$$

This equation will be a parabolic equation if B^2-4AC=0.Parabolic equation are not defined in a closed domain but propagate in an open domain. for example the one dimensional heat –flow equation ,which describes heat flow assuming no energy generation ,is

$$K\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x} \qquad 0 < x < L \text{ and } t > 0 \qquad (2)$$

Where K is the thermal diffusivity and u is temperature of the material. Comparing (2), (1),we see that A,B, and C of (1) are K , 0 and o respectively so that the term if B^2-4AC of (1) is zero and the equation is parabolic.

In order to solve this equation, boundary conditions must be specified at x=0 and x=L and initial conditions when t=0 must also be given .To develop a finite difference solution we divide the spatial domain into n sections, each of length h, so that h=L/n ,and consider as many time steps as required ,each of duration k. A finite difference

$$\frac{d^2u}{dx^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \tag{3}$$

Approximation for (2) at node (i,j) is obtained by replacing $\frac{\partial^2 u}{\partial x^2}$ by the center difference approximation

And the $\frac{\partial u}{\partial x}$ by the forward difference approximation

Then $$\frac{du}{dx} = \frac{u_{i+1} - u_i}{h} \tag{4}$$

Or $$K\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right) = \frac{u_{i+1,j} - u_{i,j}}{k} \tag{5}$$

$$u_{i,j+1} = u_{i,j} + \alpha(u_{i-1,j} - 2u_{i,j} + ui + 1, j), \ \mathrm{i} = 0,1.....\mathrm{n}; \ \mathrm{j} = 0,1.. \tag{6}$$

In (6) $\alpha = Kk/h^2$. Node (i,j) is the point $x=ih$ and at time jk .Equation (6) allows us to determine $u_{i,j+1}$,i.e. u at time $j+1$ from values of u at time j. Value of $u_{i,0}$ are provided by Initial conditions, values of u_{0j} and u_{nj} are obtained from the boundary conditions. This method of solution is called the explicit method.

In the numeric solution of parabolic partial differential equation, solution stability and convergence are important. It can be proved that when using the explicit method we must make $\alpha \leq 0.5$ to ensure a steady decay of the entire solution. The requirement means that the grid separation in time must sometime be very small, necessitating a very large number of time steps.

An alternative finite difference approximation for (2) is obtained by considering node $(i,j+1)$. We again approximate $\frac{\partial^2 u}{\partial x^2}$ by the central difference approximation (3) but we approximate $\frac{\partial u}{\partial x}$ by the central difference approximation (5) given

$$K\left(\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}\right) = \frac{u_{i+1,j+1} - u_{i,j+1}}{k} \tag{7}$$

This equation is identical to (5) except that approximation is made at the $(j+i)$th time step instead of at the j^{th} time step .Rearranging (7) with $\alpha = Kk/h2$ gives

$$(1 + 2\alpha)u_{i,j+1} - \alpha(u_{i+1,j+1} + u_{i-1,j+1}) = u_{i,j} \tag{8}$$

Where $i=0,1,\ldots$ .The three variables on the left-hand side of this equation are unknown. However, if we have grid of $n+1$ spatial points, then at time $j+1$ there are $n-1$ unknown nodal values and two known boundary values. We can assemble the set of form (8) thus:

$$\begin{bmatrix} 1+2\alpha & -\alpha & 0 & \ldots & 0 \\ -\alpha & 1+2\alpha & -\alpha & \ldots & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha \ldots & . \\ . & . & . & \ldots & . \\ . & . & . & \ldots & . \\ . & . & . & \ldots & 0 \\ 0 & 0 & 0 & \ldots & -\alpha \\ 0 & 0 & 0 & \ldots & 1+2\alpha \end{bmatrix} \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ . \\ . \\ . \\ u_{n-2,j+1} \\ u_{n-1,j+1} \end{bmatrix} = \begin{bmatrix} u_{1,j} + \alpha u_0 \\ u_{2,j} \\ u_{3j} \\ . \\ . \\ . \\ u_{n-2,j} \\ u_{n-1,j} + \alpha u_n \end{bmatrix}$$

Theorem 1:
The forward-difference method has local truncation error $O(k+h^2)$; it is (conditionally) stable if and only if if $\alpha \leq 1/2$.
(Duchateau and Zachmann,1986)
Theorem 2:
The backward-difference method has local truncation error $O(k+h^2)$; it is (conditionally) stable .
(Duchateau and Zachmann,1986)

3. The Wavelet Method

Wavelet analysis is a rapidly developing area in the mathematical sciences, which is emerging an in independent field of investigation. Moreover, it has already created a common link between mathematicians and, for instance, electrical engineers, and has even drawn a great deal of attention from scientists and engineers in other disciplines. In physics and engineering, problems requiring numerical solution of differential equations rank among the most computing-intensive operation. New mathematical bases of their computation are always developed. Wavelet represents one which permit accurate representation of a variety of functions and operators without redundancy. Through the ability to represent local, high frequency information with localized basis elements, wavelets allow adaptation in a straightforward, consistent fashion. (Soesianto and Abdullah,2000),gives a brief presentation on wavelet and wavelet transforms.

The Haar wavelet is a common and simplest wavelet transformation, which can be used to transform values within a matrix. Haar transformation is a two-dimensional generalization of one-dimensional wavelet transform. The Haar transform uses the standard decomposition algorithm. To obtain the standard decomposition of a matrix, we first apply the one-dimensional wavelet transform to each row of value. This operation gives us an average value with detail coefficient for each row. Next we treat these transformed rows as if they were themselves an matrix and apply the one-dimensional transform to each column. The resulting values are all detail coefficients except for a single overall coefficient Stollintz, et al, (1996). An algorithm to compute the standard decomposition is given below

```
procedure StandardDecomposition (c: array [1 .. 2^j,1 .. 2^k] of reals)
        for row ←1 to 2^j do
          Decomposition(c[row, 1 .. 2^k])
        end for
        for col ←1 to 2^k do
          Decomposition(c[1 .. 2^j,col])
        end for
end procedure.
```

The corresponding reconstruction algorithm simply reserves the steps performed during decomposition:

```
procedure Standard Redecomposition (c: array [1 .. 2^j,1 .. 2^k] of reals)
        for col ←1 to 2^k do
          Redecomposition(c[1 .. 2^j,col])
        end for
        for row ←1 to 2^j do
          Redecomposition(c[row, 1 .. 2^k])
        end for
end procedure..
```

Algorithm Parabolic Differential Equations using wavelet

i. $$K\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial x}$$

ii. $$K\left(\frac{u_{i+1,j+1}-2u_{i,j+1}+u_{i-1,j+1}}{h^2}\right)=\frac{u_{i+1,j+1}-u_{i,j+1}}{k}$$

iii. let $$A=\begin{bmatrix} 1+2\alpha & -\alpha & 0 & \cdots & 0 \\ -\alpha & 1+2\alpha & -\alpha & \cdots & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & 0 \\ 0 & 0 & 0 & \cdots & -\alpha \\ 0 & 0 & 0 & \cdots & 1+2\alpha \end{bmatrix}$$ and

$$x=\begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-2,j+1} \\ u_{n-1,j+1} \end{bmatrix} \quad b=\begin{bmatrix} u_{1,j}+\alpha u_0 \\ u_{2,j} \\ u_{3j} \\ \cdot \\ \cdot \\ \cdot \\ u_{n-2,j} \\ u_{n-1,j}+\alpha u_n \end{bmatrix}$$

vi. Then

$$Ax=b$$
$$WAx=Wb$$
$$WAW^{-1}Wx=Wb$$
$$WAW'Wx=Wb$$
$$A_w x_w=b_w$$

where

$$A_w=WAW'$$
$$b_w=Wb$$
$$x_w=Wx$$

Not that u_0 and u_n are the known boundary condition ,assumed to be independent of time .By solving the equation above we determine $u_1,u_2,\ldots u_{n-1}$ at time step j+1 form $u_1,u_2,\ldots u_{n-1}$ at time step j.

3.1 Haar wavelet

$$\psi_H = \begin{cases} 1 \text{ for } 0 \le x \le \frac{1}{2} \\ -1 \text{ for } \frac{1}{2} \le x \le 1 \\ 0 \text{ otherwise} \end{cases}$$

Haar wavelet is defined as

Following Fourier, any wavelet could be used basis block to build any wave $f(x)$

$$f(x) = \sum_{j,k=-\infty}^{\infty} c_{j,k} \psi_{j,k}(x)$$

With

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k), \quad \textit{for all} \quad j,k \in z.$$

The coefficients $c_{j,k}$ are computable from :

$$c_{j,k} = \langle f, \psi_{j,k} \rangle$$

Also following the idea of Fourier transform W_ψ of any wave $f(x)$ can now be defined as following

$$(w_\varphi f)(b,a) = |a|^{\frac{-1}{2}} \int_{-\infty}^{\infty} f(x) \overline{\psi(\frac{x-b}{a})} \, dx$$

The coefficients $c_{j,k}$ are now computable following relation

$$c_{j,k} = (W_\psi f)\left(\frac{k}{2^j} - \frac{1}{2^j}\right)$$

(Bond and Vavasis,1994)

4. Numerical Result

Here the algorithm we built will be applied on an example, heat equation, and to show the time consumed by the calculations a comparison with the finite difference method is given.

Example

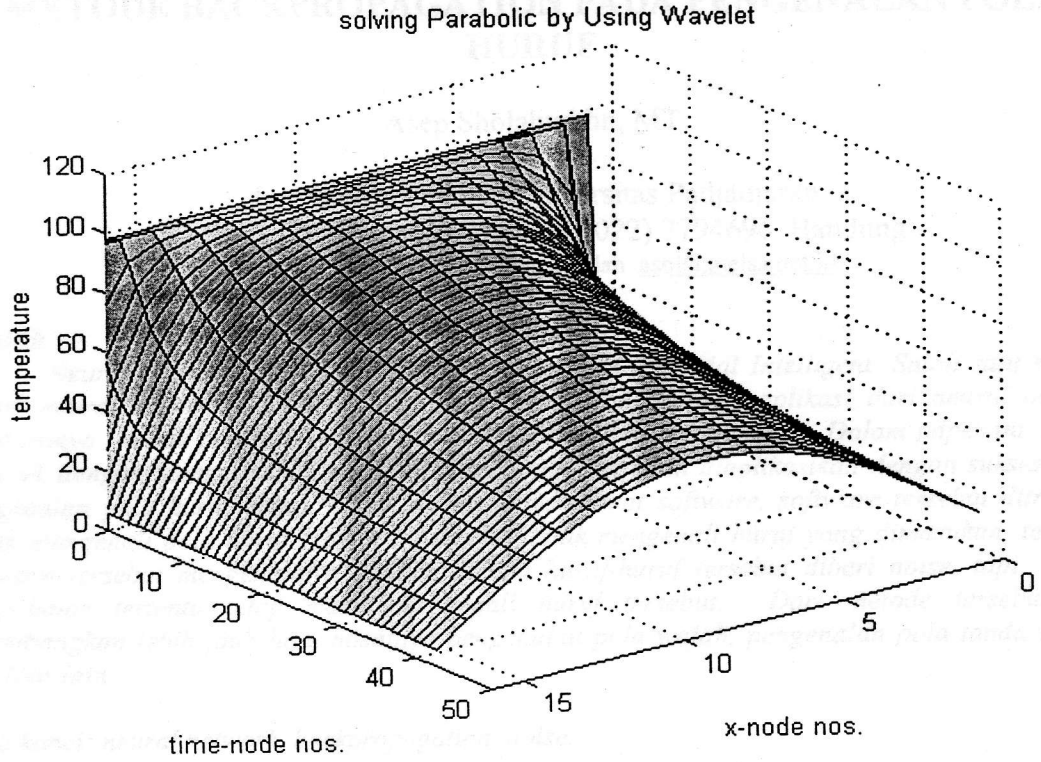
A brick wall with o.3 m thick, is initially at a uniform temperature of 100°C and its thermal conductivity, K=5*10^{-7} m^2/s. If the temperature of the both surfaces is suddenly lowered to 20°C and kept at this temperature. Plot the temperature distribution of the wall after 7.33 min (440s).

To study this problems we will use a mesh with 15 subintervals of x and 50 subdivisions of t. This corresponds to α=0.55.

Table 1 :

	Finite difference	Algorithm by using wavelet
Flops	1632819	32942
Time	5.4900	0.2200

Picture 1 :



5. Conclusion

The Wavelet Method for Solving Parabolic Differential Equations is introduced. The method examined for popular problem and the results are compared with results of the same problems solved by finite difference. From the table in our discussions we can conclude that:

1. The accuracy of both methods are the same
2. Wavelet method is able to solve Parabolic Differential Equations in shorter time and fewer flops.

6. References

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