

EMBEDDINGS OF LINEAR ARRAYS, RINGS AND 2-D MESHES ON EXTENDED LUCAS CUBE NETWORKS

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Abstract

A Fibonacci string is a length n binary string containing no two consecutive 1s. Fibonacci cubes (FC), Extended Fibonacci cubes (ELC) and Lucas cubes (LC) are subgraphs of *hypercube* defined in terms of Fibonacci strings. All these cubes were introduced in the last ten years as models for interconnection networks and shown that their network topology possesses many interesting properties that are important in parallel processor network design and parallel applications. In this paper, we propose a new family of Fibonacci-like cube, namely *Extended Lucas Cube* (ELC). We address the following network simulation problem : Given a linear array, a ring or a two-dimensional mesh; how can its nodes be assigned to ELC nodes so as to keep their adjacent nodes near each other in ELC ? We first show a simple fact that there is a Hamiltonian path and cycle in any ELC. We prove that any linear array and ring network can be embedded into its corresponding optimum ELC (the smallest ELC with at least the number of nodes in the ring) with dilation 1, which is optimum for most cases. Then, we describe dilation 1 embeddings of a class of meshes into their corresponding optimum ELC.

Keywords : (Extended) Fibonacci cube, Extended Lucas cube, Fibonacci number, Hamiltonian path, Hamiltonian cycle, linear array, ring, mesh, network.

1. Introduction

A widely studied network topology is the *Hypercube* [1]. The hypercube provides a rich network structure which permits many other topologies to be efficiently emulated. Most other popular networks are easily embedded into a hypercube. For example a 8 node linear array, 2x4 mesh and 8 node ring may be embedded into a 8 node hypercube as shown in Figure 1. It is well known that any hypercube is Hamiltonian. Clearly, there must be an embedding of a linear array and ring into Hypercube. Furthermore, the number of nodes 2^n in an n -dimensional hypercube $H(n)$ grows rapidly as n increases [2]. This limits considerably the choice of the number of nodes in the graph. Figure 1 shows Hypercube of dimension n , $n = 0, 1, \dots, 4$.

The *Fibonacci cube* (FC) proposed by Hsu [3] is a special subcube of a hypercube based on Fibonacci numbers.

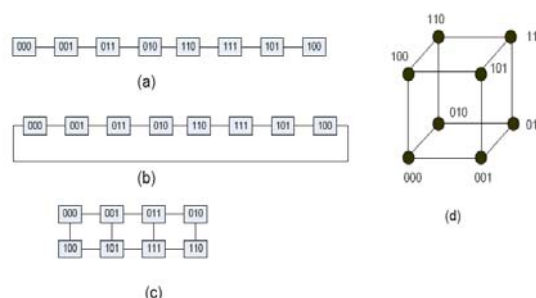


Fig 1. a) Linear Array-8 node, b) Ring-8 node, c) 2-D Mesh(2x4), d) Hypercube-8 node

Because of the rich properties of Fibonacci numbers, this network shows interesting properties. It has been shown that the diameter and node degree of the Fibonacci

cube network with N nodes are $O(\log N)$, which are similar to those of the Hypercube of $O(N)$ nodes. The Fibonacci cube has a very attractive recurrent structure, as it can be decomposed into networks which are also Fibonacci cubes. It has also been shown that the Fibonacci cube contains about 1/5 fewer edges than the hypercube for the same number of nodes. The size of the Fibonacci cube does not increase as fast as the hypercube with many faulty nodes.

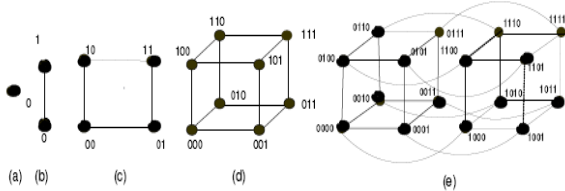


Fig.2 . a) H(0), b) H(1), c)H(2), d) H(3), e) H(4)

Throughout the paper, the term “graph” and term “network” means the same and are used interchangeably. Most Fibonacci cube are not Hamiltonian. In fact, Cong et al, showed that less than 1/3 of Fibonacci cube are Hamiltonian. Furthermore, Wu [1] proposed the *Extended Fibonacci Cube* (EFC), which is based on the same sequence as the Fibonacci cube; however its initial conditions are defined differently. It has been shown that EFC is Hamiltonian and maintains virtually all the desirable properties of FC. It has been shown that EFC is superior to FC in terms of various structural properties. All EFC are Hamiltonian. While Munarini et al proposed an other topology called *Lucas Cube* (LC), which is an induced subgraph of FC. LC is more sparse than both FC and EFC. LC are not Hamiltonian, however, almost all of the LC have a Hamiltonian path. The existence of a Hamiltonian path or cycle in cubes is a proof that linear array and ring can be embedded into the cubes. Network embedding has been studied extensively [4]. Thus far, there is a big gap between other networks (such as hypercube, star graph, ets) and the FC in relation to the availability of routing and network embedding results. The problem of embedding meshes into the Fibonacci cube was studied. It was shown that the Fibonacci cube contains two small meshes. This result did not take into consideration the problem of embedding meshes into their optimum Fibonacci cube, which is a more interesting problem. However, furthermore Cong showed that the meshes can be embedded into their optimum Fibonacci cube.

In this paper, we propose a new family of Fibonacci-like cube, namely *Extended Lucas Cube* (ELC). ELC is a network model obtained from EFC by removing nodes which are not Lucas string. Thus ELC is induced subgraph of EFC. In [5] can be found about Lucas string.

An embedding is defined as a mapping function g , which maps the vertices of a guest graph G to the vertices of a host graph H . The dilation of an embedding g is the maximum distance in H between images of the adjacent nodes in G . The expansion of an embedding g is the ratio of the number of nodes in H to the number of the nodes in G . The dilation is a lower bound for communication delay when H simulates G . The expansions are measurements for processor utilization of H . The problem of interest is : how to map the nodes of a ring and a two dimensional mesh to the nodes of its corresponding optimum ELC (the smallest ELC with at least as many nodes as the ring or mesh), on a one-to-on basis, so that dilation is kept minimum.

2. Network Topology

A Fibonacci cube of dimension n , denoted by $FC(n)$, is an undirected graph of f_n (which is the n -th Fibonacci number) nodes, each is labeled by a distinct $(n-2)$ -bit binary number such that no two 1's occur consecutively. Two nodes in $FC(n)$ are connected by an edge if and if their labels differ in exactly one bit position. Figure 3 shows $FC(3)$, $FC(4)$, $FC(5)$ and $FC(6)$. We use $|FC(n)|$ to denote the number of nodes in $FC(n)$. The following are some basic properties of $FC(n)$:

1. Assume $FC(n) = (V(n), E(n))$. $FC(n-1) = (V(n-1), E(n-1))$, and $FC(n-2) = (V(n-2), E(n-2))$, then $V(n) = 0V(n-1) \cup 10V(n-2)$. The initial conditions are $V(2) = \{0\}$ and $V(3) = \{0, 1\}$. For example, $V(5) = 0\{00, 01, 10\} \cup 10\{0, 1\} = \{000, 001, 010, 100, 101\}$. In other words, $FC(n)$ can be decomposed into two subgraphs that are isomorphic to $FC(n-1)$ and $FC(n-2)$, respectively, for $n \geq 4$. Figure 3 illustrates $FC(n)$, $n=3, 4, 5, 6$.
2. Let $H(n)$ denote the hypercube of dimension n . Then $FC(n)$ is a subgraph of $H(n-2)$ and $H(n)$ is a subgraph of $FC(2n+1)$.

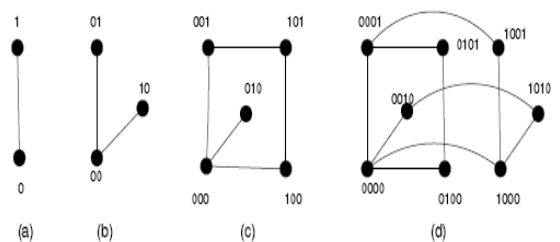


Fig. 3. a) FC(3), b) FC(4), c) FC(5), d) FC(6)

An extended Fibonacci cube of dimension n , denoted by $EFC(n)$, is an undirected graph of F_n (which is $2 \cdot f_{n-1}$) nodes [6], each is labeled by a distinct $(n-2)$ -bit binary number. Two nodes in $FC(n)$ are connected by an edge if

and if their labels differ in exactly one bit position. Figure 4 shows EFC(3), EFC(4), EFC(5) and EFC(6). We use $|EFC(n)|$ to denote the number of nodes in EFC(n). The following are some basic properties of EFC(n):

1. The two initial values are 2 and 4.
2. Assume $EFC(n) = (V_1(n), E_1(n))$. $EFC(n-1) = (V_1(n-1), E_1(n-1))$, and $EFC(n-2) = (V_1(n-2), E_1(n-2))$, then $V_1(n) = 0V_1(n-1) \cup 10V_1(n-2)$. The initial conditions are $V(3) = \{0, 1\}$ and $V(4) = \{00, 10, 11, 01\}$. For example, $V(5) = 0\{00, 10, 11, 01\} \cup 10\{0, 1\} = \{000, 010, 011, 001, 100, 101\}$. In other words, EFC(n) can be decomposed into two subgraphs that are isomorphic to EFC(n-1) and EFC(n-2), respectively, for $n \geq 5$. Figure 4 illustrates EFC(n), $n=3, 4, 5, 6$.

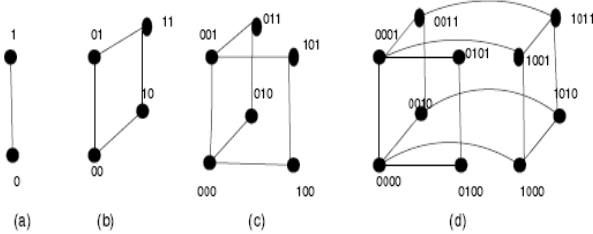


Fig. 4. a) EFC(3), b) EFC(4), c) EFC(5), d) EFC(6)

An extended Lucas cube of dimension n, denoted by ELC(n), is an undirected graph of C_n (which is $2 \cdot f_n \times 2 \cdot f_{n-2}$) nodes [7], each is labeled by a distinct $(n-2)$ -bit binary number. Two nodes in ELC(n) are connected by an edge if and if their labels differ in exactly one bit position. Figure 5 shows ELC(3), ELC(4), ELC(5) and ELC(6). We use $|ELC(n)|$ to denote the number of nodes in ELC(n).

Assume $ELC(n) = (V_2(n), E_2(n))$. $ELC(n-1) = (V_2(n-1), E_2(n-1))$, and $ELC(n-2) = (V_2(n-2), E_2(n-2))$, then $V_2(n) = 0V_1(n-1) \cup 10V_1(n-3)0$. For example, $V(5) = 0\{00, 10, 11, 01\} \cup 10\{0\}0 = \{000, 010, 011, 001, 100\}$. In other words, ELC(n) can be decomposed into two subgraphs that are isomorphic to ELC(n-1) and ELC(n-3), respectively, for $n \geq 5$. Figure 5 illustrates ELC(n), $n=3, 4, 5, 6$.

In this paper, we first show that there exists a Hamiltonian cycle in any ELC. Thus, there is a dilation 1 embedding of a linear array and ring into its corresponding optimum ELC.

Then, we consider the problem of simulating extended Fibonacci meshes on EFC. Finally using these results, we describe a dilation 1 embedding of the extended Fibonacci meshes (An $m \times k$ mesh is a extended Fibonacci mesh iff $m = f_i$ and $k = F_j$ for some i and j) into their optimum Extended Lucas cube.

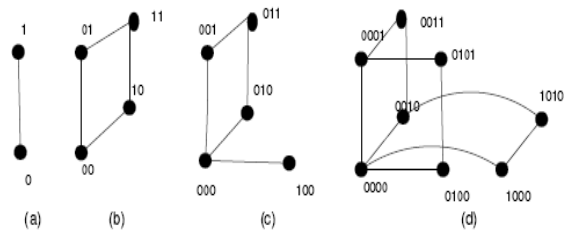


Fig. 5. a) ELC(3), b) ELC(4), c) ELC(5), d) ELC(6)

3. Embeddings of Linear Arrays and Rings

Lemma 1: For any $n \geq 2$, $n \neq 3$, ELC(n) is a Hamiltonian graph.

Proof: For $n=2$, ELC(n) is a Hamiltonian graph; it has a Hamiltonian cycle : $00 \rightarrow 10 \rightarrow 11 \rightarrow 01$. For $n=3$, ELC(3) is not a Hamiltonian graph. However, it has a Hamiltonian path : $100 \rightarrow 000 \rightarrow 001 \rightarrow 011 \rightarrow 010$. Using the induction technique can be proved that ELC(n) is a Hamiltonian graph. For each ELC(n), $n \geq 4$, a Hamiltonian path of type $000^{n-2} \rightarrow 100^{n-2} \rightarrow G \rightarrow 010^{n-2}$ can be constructed, where 0^{n-2} is a sequence of 0 of length $n-2$ (could be empty). G is a sequence of adjacent 0 of length $n-2$ and 010^{n-2} , respectively. For $n = 4$, a Hamiltonian can be obviously constructed in ELC(4); that is $0000 \rightarrow 1000 \rightarrow 1010 \rightarrow 0010 \rightarrow 0011 \rightarrow 0001 \rightarrow 0101 \rightarrow 0100$. Since the first and last vertices of this Hamiltonian path are adjacent, this Hamiltonian path is also a Hamiltonian cycle. As it can be seen in [7], ELC(n) contains two disjoint subgraphs are isomorphic to EFC(n-1) and EFC(n-3) respectively. In addition, the code for EFC(n) is two bit longer than the one for EFC(n-2). That the theorem holds for $n = 2, 3, 4$ is evident. Furthermore, based on those above, with using the induction technique it can be proved that ELC(n) is a Hamiltonian graph. \square

Theorem 2: A linear array can be embedded into its corresponding optimum ELC with dilation 1.

Proof: Immediately from Lemma 1.

Theorem 3: Ring network $R(b)$ can be embedded into ELC(n) with dilation 1, $b = 2 \cdot f_n \times 2 \cdot f_{n-2}$

Proof: Immediately from Lemma 1. \square

4. Embeddings of Extended Fibonacci Meshes

In this section, we first consider the problem of simulating extended Fibonacci meshes on EFCs, then the problem of simulating extended Fibonacci meshes on ELCs

Theorem 4 : An $f_n \times F_{n+1}$ extended Fibonacci mesh can be embedded into its corresponding optimum extended Fibonacci cube EFC(2n) with dilation 1.

Proof : we know that a linear array with f_n nodes, denoted by $LA(f_n)$, can be embedded into $FC(n)$ with dilation 1. The 2-D mesh $f_n \times F_{n+1}$ can be considered as matrix with f_n rows and F_{n+1} columns. The basic idea of embedding here is to label columns according to the embedding of $LA(F_{n+1})$ to $EFC(n+1)$ and to label rows according to the embedding of $LA(f_n)$ to $FC(n)$. Thus, the label for the node in the j th column and the i th row in the mesh will be the concatenation of the label for the j th and the i th row. Since a simple concatenation may result in two consecutive 1's, we place one 0 in front of labels for all rows. Clearly, this is a dilation 1 embedding. Furthermore, $(n-1)$ bits are used for labeling columns (as $FC(n+1)$ has $(n+1)-2$ bits) and $(n-2)+1$ bits are used for labeling rows. Thus, total $(2n-2)$ bits are used and this gives us a dilation 1 embedding of the $f_n \times F_{n+1}$ mesh into $EFC(2n)$. Figure 6 illustrates this technique, and Figure 7 shows an example.

Now, we show that $EFC(2n)$ is the smallest extended Fibonacci cube with at least $f_n \times F_{n+1}$ nodes, [8,9] that is $f_n \times F_{n+1} > |EFC(2n-1)| = F_{2n-1}$.
 $f_n \times F_{n+1} = f_n \times (F_n + F_{n-1}) = f_n F_n + f_n F_{n-1} = f_n F_n + (f_{n-1} + f_{n-2}) F_{n-1} = f_n F_n + f_{n-1} F_{n-1} + f_{n-2} F_{n-1}$
 Since $f_n F_n + f_{n-1} F_{n-1} = F_{2n-1}$, we have
 $f_n \times F_{n+1} = F_{2n-1} + f_{n-2} F_{n-1} > F_{2n-1} = |EFC(2n-1)| \quad \square$

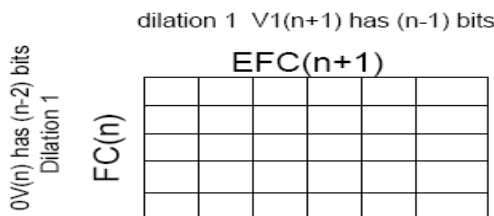


Fig. 6. Embedding extended Fibonacci meshes into EFC

For dilation 1 embeddings of square meshes into the EFCs, we have the following result :

Corollary 5 : An $f_n \times F_n$ extended Fibonacci mesh can be embedded into its optimum extended Fibonacci cube $EFC(2n-1)$ with dilation 1.

Proof : Obtained by using the same technique illustrated in the proof of Theorem 4. \square

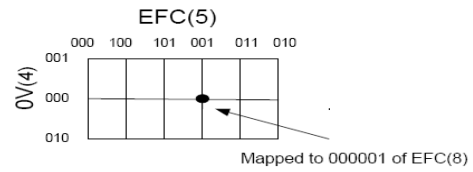


Fig.7 : Embedding of $f_4 \times F_5$ mesh into EFC(8).

Theorem 6 : Two disjoint subgraphs $f_n \times F_n$ extended Fibonacci mesh and $f_{n+1} \times F_{n+1}$ extended Fibonacci mesh can be embedded directly into $EFC(2n+1)$ with dilation 1. Two disjoint subgraphs $f_n \times F_{n+1}$ extended Fibonacci mesh and $f_n \times F_{n-1}$ extended Fibonacci mesh can be embedded into $EFC(2n)$ with dilation 1.

Theorem 7 : Two disjoint subgraphs $f_n \times F_{n+1}$ extended Fibonacci mesh and $f_n \times F_{n-1}$ extended Fibonacci mesh can be embedded into its corresponding optimum extended Lucas cube $ELC(2n+1)$ with dilation 1.

Proof : We know that two disjoint subgraphs $EFC(2n)$ and $EFC(2n-2)$ can be embedded directly into $ELC(2n+1)$. From Theorem 6 , two disjoint subgraphs $f_n \times F_{n+1}$ extended Fibonacci mesh and $f_n \times F_{n-1}$ extended Fibonacci mesh can be embedded into $EFC(2n)$. Since $|EFC(2n)| > |EFC(2n-2)|$, then the theorem are proved. \square

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