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THE RELAXED EDGE-COLORING GAME AND k-DEGENERATE GRAPHS

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ABSTRACT. The (r, d)-relaxed edge-coloring game is a two-player game using r colors played on the edge set of a graph G. We consider this game on forests and more generally, on k-degenerate graphs. If F is a forest with $\Delta(F) = \Delta$, then the first player, Alice, has a winning strategy for this game with $r = \Delta - j$ and $d \geq 2j + 2$ for $0 \leq j \leq \Delta - 1$. This both improves and generalizes the result for trees in [10]. More broadly, we generalize the main result in [10] by showing that if G is k-degenerate with $\Delta(G) = \Delta$ and $j \in [\Delta + k - 1]$, then there exists a function $h(k, j)$ such that Alice has a winning strategy for this game with $r = \Delta + k - j$ and $d \geq h(k, j)$.

1. INTRODUCTION

The *map-coloring game* was first conceived by Steven Brams and published by Martin Gardner in his column in *Scientific American* [17]. In this game two players alternate coloring the countries on a map such that no two countries with a non-trivial border receive the same color. It was not until 1991, that Bodlaender [2] reinvented it by framing the game in terms of coloring vertices of a graph.

In the original version of the game two players, Alice and Bob alternate coloring the vertices of a finite graph G using colors from a set X of r colors. Alice goes first. On each turn, the players must choose an uncolored vertex to color such that at the end of the player's turn, no adjacent vertices have the same color. Alice wins this game if all of the vertices are eventually colored. Bob wins if there comes a time when there is an uncolored vertex for which no allowable color exists. The least r such that Alice has a winning strategy for this game is called the *game chromatic number* of G, denoted $\chi_{\mathrm{g}}(G)$. This parameter has been investigated extensively in a number of papers [8, 16, 18, 20, 21, 25, 26].

Observe that in the above game, at every step every color class induces an independent set in G, and that while playing the game, the players are in the process of creating a proper coloring of the graph. We first consider a variation of this game in which the players are creating a *relaxed coloring*,

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also called *defect coloring*, of the the graph G. This variation was introduced
by Chou et al. [4] and builds on the known work [5, 6, 7, 14, 24] concerning
defect colorings of graphs. The only difference between this game and the
original version is which colors the players are allowed to use. In this version
of the game, a color $\alpha \in X$ is *legal* for an uncolored vertex u if after u
is colored α each color class $H \subseteq G$ satisfies $\Delta(H) \leq d$, where $\Delta(H)$ is
the maximum degree in H, and d is a fixed nonnegative integer set at the
beginning of the game. We call d the *defect*. As in the original game, Alice
wins if every vertex is eventually colored. Bob wins if there comes a time in
the game when there is an uncolored vertex for which no legal color exists.
We call this game the (r, d)-*relaxed coloring game*. For a fixed d, the least r
for which Alice has a winning strategy is called the d-*relaxed game chromatic
number of G*, denoted ${}^d\chi_g(G)$. If $d = 0$, we drop the initial superscript
and write simply $\chi_g(G)$. For a fixed r, the r-*game defect of G*, denoted
$\mathrm{def}_g(G, r)$, is the least d such that Alice has a winning strategy. These
parameters have been further examined in a number of papers, including
[11, 12, 13, 9, 19].

It should be noted that the game chromatic number has some interesting
non-monotone properties. It is well known that $\chi_g(K_{n,n}) = 3$ for $n \geq 2$,
but if M is any perfect matching in $K_{n,n}$, then $\chi_g(K_{n,n} - M) = n$. This
gives an example of a graph G with subgraph H such that $\chi_g(H) > \chi_g(G)$.
Extending this idea to the (r, d)-relaxed coloring game, it is well known that
${}^1\chi_g(K_{n,n}) = n$. This provides an example of a graph G such that $\chi_g(G) <$
${}^1\chi_g(G)$. In fact, it was shown in [9] that for every $m \in \mathbb{N}$, there exists a
graph G such that $m \leq \chi_g(G) < {}^1\chi_g(G)$. It remains open whether for all
nonnegative integers d there exists a graph G such that ${}^d\chi_g(G) < {}^{d+1}\chi_g(G)$.

The focus of our work in this paper is a further variation of this game.
Rather than coloring vertices, the players color edges. Of course, for a given
graph G, this could be viewed as playing the (r, d)-relaxed coloring game
game on the line graph of G. For the purposes of this paper, however, we
will consider this game in terms of edge coloring. We call this variation the
(r, d)-*relaxed edge-coloring game* and formalize it in the following way.

Let G be a finite graph and let r be a positive integer and d be a non-
negative integer. As above, d is the *defect* and X is a set of r colors. The
players alternate coloring, with Alice coloring an edge first. We say that a
color $\alpha \in X$ is *legal* for an uncolored edge e if the following conditions are
satisfied:

(1) The edge e is incident with no more than d edges already colored α.
(2) If e' is an edge incident to e and e' has already been colored α, then
e' is adjacent to no more than $d - 1$ edges colored α.

Note that if e is colored α, then at this point in the game, every edge has
at most d neighbors colored α. Alice wins if every edge is eventually legally
colored. Bob wins if there comes a time in the game when there is an
uncolored edge for which no legal color exists. For a fixed defect d, the least

r such that Alice has a winning strategy for this game is called the d-*relaxed game chromatic index* of G, denoted $^d\chi'_{\mathrm{g}}(G)$. Similarly, for a fixed r, the r-*edge-game defect* of G, denoted $\mathrm{def}'_{\mathrm{g}}(G, r)$, is the least d such that Alice has a winning strategy. This game was first introduced in [10].

To provide the appropriate context for the work in this paper, we note that for the class of forests, much is already known about the (r, d)-relaxed (vertex) coloring game. We have the following sequence of results:

Theorem 1 (Faigle et al. [16])**.** *If F is a forest, then $\chi_{\mathrm{g}}(F) \leq 4$. Moreover, there exists a forest F_0 such that $\chi_{\mathrm{g}}(F_0) = 4$.*

Theorem 2 (Chou et al. [4])**.** *If F is a forest, then $^1\chi_{\mathrm{g}}(F) \leq 3$. Moreover, there exists a forest F_1 such that $^1\chi_{\mathrm{g}}(F_1) = 3$.*

Theorem 3 (He et al. [19])**.** *If F is a forest, then $^2\chi_{\mathrm{g}}(F) \leq 2$. Moreover, there exists a forest F_2 such that $^2\chi_{\mathrm{g}}(F_2) = 2$.*

From a certain perspective, this fully describes the (r, d)-relaxed coloring game on forests, as a class. The goal of this paper is to similarly describe the (r, d)-relaxed edge-coloring game on both forests and k-degenerate graphs. We begin by considering the following results:

Theorem 4 (Cai and Zhu [3], Lam et al. [23])**.** *Let T be a tree with $\Delta(T) = \Delta$. Then $\chi'_{\mathrm{g}}(T) \leq \Delta + 2$.*

Theorem 5 (Dunn [10])**.** *Let T be a tree with $\Delta(T) = \Delta$. Then $\mathrm{def}'_{\mathrm{g}}(T, \Delta + 1) \leq 1$. Moreover, if $d \geq 1$, then $^d\chi'_{\mathrm{g}}(T) \leq \Delta + 1$.*

Theorem 6 (Dunn [10])**.** *Let T be a tree with $\Delta(T) = \Delta$. Then $\mathrm{def}'_{\mathrm{g}}(T, \Delta) \leq 3$. Moreover, if $d \geq 3$, then $^d\chi'_{\mathrm{g}}(T) \leq \Delta$.*

We note that Theorems 4, 5 and 6 can easily be extended to forests. Therefore, thinking in terms of Theorems 1–3, Theorems 4–6 lead to the following question:

Question 1. *For any forest F with $\Delta(F) = \Delta$ and $j \in [\Delta - 1]$ is there a function $h : \mathbb{N} \to \mathbb{N}$ such that whenever $d \geq h(j)$ we have that $^d\chi'_{\mathrm{g}}(F) \leq \Delta - j$?*

In Section 2 of this paper we will answer Question 1 in the affirmative with $h(j) = 2j + 2$. It should be noted that Theorem 4 was improved in [15]. It was shown that if $\Delta = 3$ or $\Delta \geq 6$, then any tree T with $\Delta(T) = \Delta$ satisfies $\chi'_{\mathrm{g}}(T) \leq \Delta + 1$. In [1], this result was extended to trees (and forests) with $\Delta = 5$. The case for $\Delta = 4$ remains open, indicating the difficulty of determining these parameters, even in the non-relaxed environment.

For the more general class of k-degenerate graphs, which will be defined in detail in Section 3, we note the following result of Cai and Zhu:

Theorem 7 (Cai and Zhu [3])**.** *Let G be a k-degenerate graph with $\Delta(G) = \Delta$. Then $\chi'_{\mathrm{g}}(G) \leq \Delta + 3k - 1$.*

As a consequence, in terms of the (r, d)-relaxed edge-coloring game, we have that $^{d}\chi'_{\mathrm{g}}(G) \leq \Delta + 3k - 1$ for all k-degenerate graphs G when $d = 0$. Also, in [10], the following result was proven:

Theorem 8 (Dunn [10]). *Let G be k-degenerate with $\Delta(G) = \Delta$. Then $\mathrm{def}'_{\mathrm{g}}(G, \Delta + k - 1) \leq 2k^2 + 4k$. Moreover, if $d \geq 2k^2 + 4k$, then $^{d}\chi'_{\mathrm{g}}(G) \leq \Delta + k - 1$.*

Theorems 7 and 8 together lead to the following question:

Question 2. *For any k-degenerate graph G with $\Delta(G) = \Delta$ and $j \in [\Delta + k - 1]$ is there a function $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that whenever $d \geq h(k, j)$ we have that $^{d}\chi'_{\mathrm{g}}(G) \leq \Delta + k - j$?*

In Section 3 we answer Question 2 in the affirmative. In particular, we will show that the following $h(k, j)$ suffices:

$$h(k, j) = \begin{cases} 2k^2 + 4k, & \text{if } j = 1; \\ 2k^2 + 4k + 2j - 4, & \text{if } j \geq 2. \end{cases}$$

We also derive corollaries for outerplanar graphs and planar graphs.

2. The Relaxed Edge-Coloring Game on Forests

Let $F = (V, E)$ be a forest with $\Delta(F) = \Delta$ for some positive integer Δ. Let $T_1, T_2, \ldots, T_n$ be all components of F containing at least one edge. For her strategy, for each $i \in [n]$ Alice chooses an arbitrary leaf $r_i \in V(T_i)$ at which she roots T_i. She then regards all edges in T_i as oriented toward r_i. Let E_0 be the set of n edges in F that are incident to a root. For each vertex $v \in V \setminus \{r_1, r_2, \ldots, r_n\}$, define $p(v)$ to be the unique outneighbor of v. Then for each edge $e \in E$, there is a unique vertex $x \in V$ such that $e = xp(x)$. We now introduce some terminology, as illustrated in Figure 1.

For every edge $e = xp(x)$ with $p(x) \neq r_i$ for some $i \in [n]$, define the *parent of e*, denoted $p(e)$, to be the edge $p(x)p^2(x)$, where $p^2(x) = p(p(x))$. We say that e is a *child of $p(e)$*. Note that, because $p(x)$ is well defined, $p(e)$ is also well defined. Whenever $p^i(e)$ is defined and $p^i(e)$ is not incident with the root, define $p^{i+1}(e) = p(p^i(e))$. Define the *descendants of e* to be

$$G(e) = \{e' \in E \mid e = p^k(e') \text{ for some positive integer } k\}.$$

For each edge $e = xp(x)$, define the *siblings of e* to be

$$B(e) = \{yp(y) \in E \mid p(y) = p(x) \text{ and } y \neq x\}$$

and $B[e] = B(e) \cup \{e\}$. Define the *children of e* to be

$$S(e) = \{yp(y) \in E \mid x = p(y)\}.$$

We call the set of all edges incident to an edge e the *neighborhood of e*, denoted $N(e)$.

Fix $j \in [\Delta - 3]$ and let X be a set of $\Delta - j$ colors. Note that $|X| \geq 3$. At any point in the game, let C and U be the set of colored and uncolored
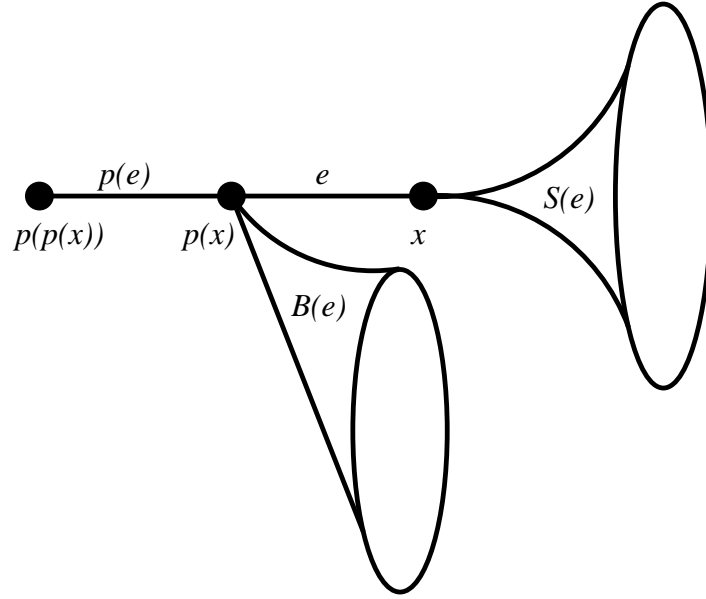
FIGURE 1. For an edge $e = xp(x)$, the vertex $p(p(x)) = p^2(x)$, the edge $p(e)$, and the sets $B(e)$ and $S(e)$.

edges, respectively. For each $\alpha \in X$, we call the set of all edges colored α the *color class of* α, denoted C_α. For a colored edge e, denote the color of e by $c(e)$.

For each colored edge e, define the *defect of* e to be the number of neighbors of e colored with $c(e)$. If e is uncolored, we set the defect of e to be zero. We denote the defect of e by $\mathrm{def}(e)$. Thus

$$\mathrm{def}(e) = \begin{cases} \left| N(e) \cap C_{c(e)} \right|, & \text{if } e \in C; \\ 0, & \text{otherwise.} \end{cases}$$

We say that color $\alpha \in X$ is *eligible* for edge e if $p(e) \notin C_\alpha$. We denote the set of eligible colors for e by $X(e)$. When coloring an edge e, Alice always chooses an eligible color. Note that since $|X| \geq 3$, this is always possible.

For any edge e, we say that $B[e]$ is *secure* if there exist edges $e_1, e_2, \ldots, e_{j+1} \in B(e)$ and a color α such that $c(e_i) = \alpha$ for $i \in [j+1]$. In other words, $B[e]$ is secure if e has $j+1$ siblings colored with the same color. Note that if $B[e]$ is secure, then the number of distinctly colored siblings of e is at most

$$|B(e) \setminus \{e_1, e_2, \ldots, e_j\}| \leq \Delta - j - 2.$$

As $|X(e)| \geq \Delta - j - 1$, there is always a legal eligible color for an uncolored edge e when $B[e]$ is secure.

We will now define the strategy that Alice will use for this game with trees. This strategy is a modification of the activation strategy developed in [10]. In response to Bob's moves, Alice designates certain edges *active*;

precisely how she chooses these edges will be explained below. When an edge e becomes active, we say that e has been *activated*. In addition, all colored edges are active. We denote the set of active edges by A, and remark that $C \subseteq A$. This set has the property that once an edge e is in A, e will remain active for the remainder of the game.

Modified Forest Strategy

Alice begins the game by coloring the unique edge in E_0 that is incident with r_1 with any color. Suppose now that Bob has just colored edge $b = xp(x)$ in T_i for some $x \in V \setminus \{r_1, r_2, \ldots, r_n\}$ and some $i \in [n]$. Alice's response has two stages: a Search Stage and a Coloring Stage. In the Search Stage, Alice finds an edge e to color. In the Coloring Stage, Alice chooses a color for e.

Search Stage
- If $b \in E_0$, then set e to be any uncolored edge in E_0, if such an edge exists, or any uncolored edge whose parent is colored.
- If $p(b) \in U$, then activate each edge along the (x, r_i)-path until reaching either an edge g with $g \in E_0 \cap U$ or $p(g) \in A$. [Note that this includes Alice activating the edge b.] If $g \in E_0 \cap U$ or $p(g) \in C$, set $e := g$. Otherwise, set $e := p(g)$.
- If $p(b) \in C$ with $c(p(b)) = c(b)$ and $p^2(b) \in U$, then set $e := p^2(b)$.
- If $p(b) \in C$ with $c(p(b)) = c(b)$, $p^2(b) \in C$, and $B(p(b)) \cap U \neq \emptyset$, then set e to be any uncolored sibling of $p(b)$.
- Otherwise, set e to be any uncolored edge in E_0, if such an edge exists, or any uncolored edge whose parent is colored.

Coloring Stage
- If $B[e]$ is secure, then color e with an eligible color for e that does not appear among the siblings of e.
- Otherwise, $B[e]$ is not secure. Let f be the last edge to be colored with a color eligible for e such that $c(f) = c(p(f))$ and $p(f) \in B(e)$. If such an edge exists, then color e with $c(f)$. If no such edge exists, then color e with any eligible color for e that minimizes $\text{def}(e)$.

We are now ready to state and prove our result for (r, d)-relaxed edge-coloring game on forests.

Theorem 9. *Let $F = (V, E)$ be a forest and $\Delta(F) = \Delta$ for some positive integer Δ. Let j be an integer with $0 \leq j \leq \Delta - 1$, and define $h(j) = 2j + 2$. Then $\text{def}'_{\text{g}}(F, \Delta - j) \leq h(j)$. Moreover, if $d \geq h(j)$ then $^d\chi'_{\text{g}}(F) \leq \Delta - j$.*

Proof. Suppose that Alice and Bob are playing the $(\Delta - j, d)$-relaxed edge-coloring game on F for some $d \geq h(j)$. Note that when either $j = \Delta - 1$

or $j = \Delta - 2$, the result is immediate. Hence, it will suffice to consider the game with color set X with $|X| \geq 3$. We will assume that Alice uses the Modified Forest Strategy.

Claim 1. *If $e \in U$, then e has at most two active children. Furthermore, when e has two active children, Alice colors e.*

Proof. Let f be the first active child of e. When Alice activates f, she also activates e. Note that while e is uncolored, Alice never colors an edge in $G(e) \backslash G(f)$ before Bob. If Bob colors an edge $b \in G(e) \backslash G(f)$, Alice activates $p(b), p^2(b), \ldots$, and so on, until she reaches e. Since e is active, Alice colors e. $\qquad\square$

Claim 2. *Suppose that Alice has chosen to color edge e with $\alpha \in X$. Then at the end of Alice's turn, $\mathrm{def}(e) \leq j + 2$.*

Proof. Since $\alpha \in X(e)$, then $p(e)$ does not contribute to the defect of e. By Claim 1, e has at most two active children; hence, e has at most two children colored α. If $B[e]$ is secure, then Alice would have chosen a color that does not appear among the siblings of e. In this case, $\mathrm{def}(e) \leq 2$. Otherwise, when $B[e]$ is not secure, there are at most j siblings of e colored α. Thus, $\mathrm{def}(e) \leq j + 2$. $\qquad\square$

Claim 3. *Suppose that e is about to be colored α and $p(e) \in U$. Then e has at most one child colored α. Furthermore, if e has a child colored α, then Bob colors e and Alice colors $p(e)$.*

Proof. If some sibling $f \in B(e)$ is the first active child of $p(e)$, then Alice colors $p(e)$ when the first edge in $G(p(e)) \setminus G(f)$ is activated. Since $p(e)$ is uncolored and e is to be colored, we conclude that e has no active children and hence no children colored. So assume that e is the first active child of $p(e)$. Note that $p(e) \in A$. If an edge in $G(p(e)) \setminus G(e)$ is then activated, Alice colors $p(e)$. Otherwise, we may assume that e has no colored siblings at the time when e is colored. Before e is colored, it is incident with at most two colored edges, which are children of e. Since $|X| \geq 3$, there is a color that does not appear on any child of e. Then, because Alice will choose a color to minimize $\mathrm{def}(e)$, Alice never chooses to color e with α if a child of e has already been colored α. So, if e has two active children before e is colored, then Alice colors e with α only when neither child is colored α. Thus, if e has a child colored α, then Bob must be coloring e with α, and since $p(e) \in A$, Alice responds by coloring $p(e)$. $\qquad\square$

Suppose $f \in S(e) \cap C_\alpha$. By Claim 3, if f has a child colored α before f is colored, then Bob must have colored f and Alice responds by coloring e. Thus $\mathrm{def}(f) = 2$ once e is colored. Otherwise, f has no children colored α before f is colored. Since e has at most two active children before e is colored, f has at most one sibling colored α before e is colored. Then $\mathrm{def}(f) \leq 2$ once e is colored.

Now consider the siblings of e. If $B[e]$ is secure, since Alice is choosing to color e with α, then α does not appear among the siblings of e. Hence, coloring e with α does not affect the defect of any edge in $B(e)$.

Finally, we consider the case that $B[e]$ is not secure.

Claim 4. *Suppose Alice has chosen to color edge e with $\alpha \in X$ and $B[e]$ is not secure. If there exists an edge $f \in B(e) \cap C_\alpha$, then $\mathrm{def}(f) \le 2j+2$ once e is colored.*

Proof. Let $E' = B(e) \cap C_\alpha$. Since $B[e]$ is not secure, we have that $|E'| \le j$. Let $f \in E'$ such that $|S(f) \cap C_\alpha|$ is maximal, and let

$$S(f) \cap C_\alpha = \{s_1, s_2, \ldots, s_m\},$$

where $i < j$ implies that s_i is colored before s_j. We show that $m \le |E'|+2$. By Claim 1, f has at most two active children before it is colored. Hence, f has at most two children colored α before f is colored. So only the following cases need be considered:

Case 1: The edge f is colored before s_1.

Since $p(s_i) = f$ for each $i \in [n]$ and $c(f) = \alpha$, Alice does not color any s_i. For each s_i that Bob colors, Alice then colors $p(e)$ if $p(e) \in U$, an edge in $E' \setminus \{f\}$ if $p(e) \in C$, or e if $E' \cap U = \emptyset$. Then at most $|E'| + 1$ children of f are colored α before Alice colors e. Hence, $m \le |E'| + 1$.

Case 2: The edge f is colored after s_1 and before s_2.

Alice does not color s_i for any $i \ge 2$. As in the previous case, when Bob colors s_i with $i \ge 2$, Alice then colors $p(e)$, an edge in $E' \setminus \{f\}$, or e. Thus, once $f \in C$, at most $|E'| + 1$ children of f are colored α. Including s_1, we have that $m \le |E'| + 2$.

Case 3: The edge f is colored after s_2.

If $p(e) \in U$, then Claim 3 implies that f has at most one child colored α before f is colored. Since f has two children colored α before f is colored, $p(e)$ must be colored before f. Furthermore, Alice colors f immediately after s_2 is colored, as s_1 and s_2 must be the first two active children of f. Once f is colored, each time Bob colors a child of f with α, Alice colors an edge in $E' \setminus \{f\}$. Therefore, once f is colored, Bob can color at most $|E'|$ children of f with α before e is colored. So $m \le |E'| + 2$.

Thus we have that

$$m = |S(f) \cap C_\alpha| \le |E'| + 2 \le j + 2$$

once e is colored. Since f was chosen to maximize $|S(f) \cap C_\alpha|$ and each $f' \in E'$ has at most j siblings colored α before e is colored, we have that

$$\mathrm{def}(f') \le |S(f) \cap C_\alpha| + j \le 2j + 2$$

for all $f' \in E'$. $\qquad\square$

Note now that if Alice is coloring edge e with α, according to the Modified Forest Strategy, Claim 2 guarantees that $\mathrm{def}(e) \leq h(j)$. We have also shown that for any edge $f \in N(e) \cap C_\alpha$, immediately after e is colored, $\mathrm{def}(f) \leq h(j)$. As Bob may adopt Alice's strategy at any point in the game, every edge is eventually colored, and Alice wins the game. Thus

$$\mathrm{def}'_{\mathrm{g}}(F, \Delta - j) \leq 2j + 2 = h(j).$$

Moreover, if the game is being playing with some defect $d > 2j + 2$, and an edge e eventually has defect at least d, then it must be through the actions of Bob that this occurs. At the time that e is uncolored, the above arguments show that it is possible to color e with an eligible color α such that coloring e does not increase the defect of any edge e' with $\mathrm{def}(e') > 2j + 2$. Thus, for any $d \geq h(j)$, we have that

$$^{d}\chi'_{\mathrm{g}}(F) \leq \Delta - j,$$

as desired. $\square$

We note that the above theorem generalizes the bound in Theorem 6 (Theorem 3 in [10]). Moreover, for $j = 0$, Theorem 9 provides that for a forest F with $\Delta(F) = \Delta$, we have that $\mathrm{def}'_{\mathrm{g}}(F, \Delta) \leq 2$. This is an improvement by 1 of Theorem 6.

3. THE RELAXED EDGE-COLORING GAME ON k-DEGENERATE GRAPHS

As discussed in Section 1, we now consider the (r, d)-relaxed edge-coloring game on the class of k-degenerate graphs. Recall that a graph $G = (V, E)$ is k-degenerate if there exists a linear ordering $L = v_1, v_2, \ldots, v_n$ of V such that for every $i \in [n]$, we have that

$$|\{\, j \mid v_i \leftrightarrow v_j \text{ and } j < i \,\}| \leq k.$$

For example, trees and forests are 1-degenerate, outerplanar graphs are 2-degenerate, planar graphs are 5-degenerate, and partial k-trees are k-degenerate.

We now develop some necessary notation and terminology for our strategy with k-degenerate graphs. Let $G = (V, E)$ be a finite graph and let L be a linear ordering of V. Once L is established, when we write $xy \in E$ we assume that $x < y$ in L. We will also assume that the edge xy is oriented $y \to x$. Let $e = xy$ be an edge in G. As in [10], we now define sets of edges relative to e using L and this orientation of $E(G)$.

$$
\begin{aligned}
P(e) &= \{\, wx \in E \mid w \in N^+(x) \,\} & \qquad P[e] &= P(e) \cup \{e\} \\
B(e) &= \{\, xv \in E \mid v \in N^-(x) \,\} & \qquad B[e] &= B(e) \cup \{e\} \\
H(e) &= \{\, uy \in E \mid u \in N^+(y) \,\} & \qquad H[e] &= H(e) \cup \{e\} \\
S(e) &= \{\, yz \in E \mid z \in N^-(y) \,\} & \qquad S[e] &= S(e) \cup \{e\} \\
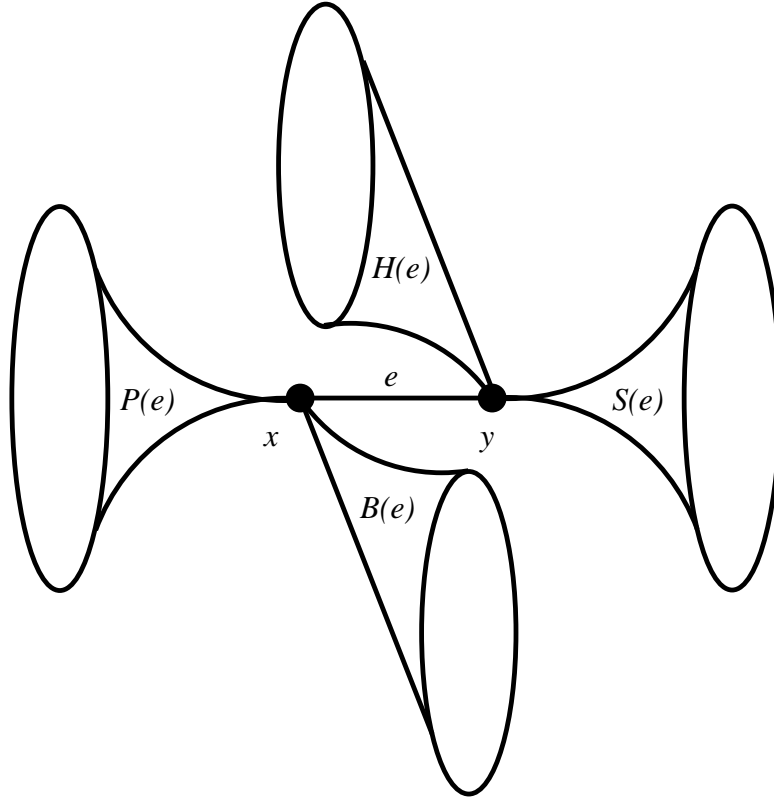R(e) &= P(e) \cup H(e) & &
\end{aligned}
$$

FIGURE 2. For an edge e and linear ordering L, sets $P(e)$,
$H(e)$, $B(e)$, and $S(e)$, relative to e and L.

We call the edges in $P(e)$ the *parents* of e and the edges in $S(e)$ the
children of e. We note that $N(e) = R(e) \cup S(e) \cup B(e)$. See Figure 2.

Consider the linear ordering $\overline{L}$ of E induced lexicographically from L. So
$xy < wz$ in $\overline{L}$ if and only if either $x < w$ in L or both $x = w$ and $y < z$ in
L. Although $\overline{L}$ is used in the strategy below, since $\overline{L}$ is determined by L, it
is L that determines the strategy.

As with the game on trees, at any time in the game we define U to
be the set of uncolored edges and C to be the set of colored edges. For
$e \in C$, let the color assigned to e be $c(e)$. Define the *defect set* of e by
$D(e) = \{ e' \in N(e) \mid c(e') = c(e) \}$. If $e \in U$, then $D(e) = \emptyset$. In either case,
the *defect* of e is defined by $\mathrm{def}(e) = |D(e)|$.

Let $F \subseteq E$. We define

$$\overline{X}(F) = \bigcup_{e' \in F} \{c(e')\},$$

where $\{c(e')\} = \emptyset$ if $e' \in U$. For an edge e, we then let $X(e) = X - \overline{X}(R(e))$. In other words, $X(e)$ is the set of colors not used on any edge in $R(e)$. As before, we call $X(e)$ the set of *eligible colors* for e.

Let $e \in E$. We define $M(e) = U \cap P[e]$. If $M(e) \neq \emptyset$, we define the *mother* of e by $m(e) = \min_{\overline{L}} M(e)$. So the mother of e, if it exists, is either the $\overline{L}$-least uncolored parent of e, or e itself. Note that if $e \in U$ then $m(e)$ must be defined since e is a candidate. For any colored edge e we define $D^*(e) \subseteq D(e)$ by

$$D^*(e) = \{\, e' \in D(e) \mid m(e') \text{ exists} \,\}.$$

Using this definition, for a colored edge e, let $F(e) = P(e) \cap (U \cup D^*(e))$. If $F(e) \neq \emptyset$, define the *father* of e by $f(e) = \min_{\overline{L}} F(e)$. So the father of e, if it exists, is either the $\overline{L}$-least uncolored parent of e, or the $\overline{L}$-least parent of e colored $c(e)$ whose mother exists.

We now assume that G is k-degenerate and that L is a linear ordering witnessing this. Let $\Delta(G) = \Delta$. Alice and Bob will be playing the (r, d)-relaxed edge-coloring game with $\Delta + k - j$ colors, for some $j \in [\Delta + k - 1]$. Extending the idea with forests above, we define $B[e]$ to be *secure* if there exist edges $e_1, e_2, \ldots, e_{j+k-1} \in B(e)$ and a color α such that $c(e_i) = \alpha$ for $i \in [j + k - 1]$. Finally, we define $l(e) = \min_{\overline{L}} B(e) \cap U$, if such an edge exists.

We are now ready to define the strategy that Alice will employ in the (r, d)-relaxed edge-coloring game on k-degenerate graphs. This is a modification and extension of the strategy developed in [10].

Modified K Strategy

Let $G = (V, E)$ be a k-degenerate graph with a linear ordering L of V witnessing this. Let X be a set of colors. Alice will again maintain an edge set A of active edges. Similar to the case with forests, we note that all colored edges are active. Alice starts by activating and coloring the least edge in $\overline{L}$. Suppose that Bob has just colored edge b. First Alice activates b. Then, as in the Modified Forest Strategy, Alice's response has two stages: a Search Stage and a Coloring Stage. In the Search Stage, Alice finds an edge e to color. The Search Stage includes an Initial Step and a Recursive Step. In the Coloring Stage, Alice chooses a color for e.

Search Stage
Initial Step

- If $f(b)$ exists and $f(b) \in U$, then set $g := f(b)$ and move to the Recursive Step.
- If $f(b)$ exists and $f(b) \in C$, then set $g := m(f(b))$ and move to the Recursive Step.
- If $f(b)$ does not exists and $l(b)$ exists, then set $e := l(b)$ and move to the Coloring Stage.

- Otherwise, set $e := \min_{\overline{L}} U$. If e is inactive, activate it. Move to the Coloring Stage.

Recursive Step

- If $g \notin A$, then activate g, set $g := m(g)$, and repeat the Recursive Step.
- Otherwise, set $e := g$ and move to the Coloring Stage.

Coloring Stage

- If $B[e]$ is secure, then color e with an eligible color for e that does not appear in $B(e)$.
- Otherwise, $B[e]$ is not secure. Let f be the last edge to be colored with a color eligible for e such that $c(f) = c(f')$ for some $f' \in B(e) \cap P(f)$. If such an edge exists, then color e with $c(f)$. If no such edge exists, then color e with any eligible color for e that minimizes $\operatorname{def}(e)$.

Note that three of the options for response in the Initial Step are to move from b to $f(b)$, $m(f(b))$, and $l(b)$. We will refer to these actions as *jumping*, *skipping*, and *sliding*, respectively. When Alice either activates or colors an edge e, we say that she is *taking action* at e. Note that Alice can take action at a edge at most twice. As before, this will play an important role in the proof of our main result, which we are now ready to state and prove.

Theorem 10. *Let G be k-degenerate with $\Delta(G) = \Delta$. Let $j \in [\Delta + k - 1]$. Let*

$$h(k, j) = \begin{cases} 2k^2 + 4k, & \text{if } j = 1; \\ 2k^2 + 4k + 2j - 4, & \text{if } j \geq 2. \end{cases}$$

Then $\operatorname{def}'_{\mathrm{g}}(G, \Delta + k - j) \leq h(k, j)$. Moreover, if $d \geq h(k, j)$ then ${}^d\chi'_{\mathrm{g}}(G) \leq \Delta + k - j$.

Proof. We begin by noting that if $\Delta - k \leq j \leq \Delta + k - 1$ and G has at least two edges, then

$$\begin{aligned} h(k, j) &= 2k^2 + 4k + 2j - 2 \\ &\geq 2k^2 + 4k + (2\Delta - 2k) - 2 \\ &= 2\Delta - 2 + 2k^2 + 2k \\ &> 2\Delta - 2. \end{aligned}$$

However, an edge e in G is incident with at most $2\Delta - 2$ other edges. Since our allowable defect is greater than this, Alice can always win. Also, if G has only one edge, the result is trivial. So for the remainder of our argument, we will assume that $j \in [\Delta - k - 1]$. Let X be a set of colors with $|X| = \Delta + k - j$. We will assume that Alice uses the Modified K Strategy. We first show throughout the game, if $e \in U$, then e has at least one legal eligible color. Note that $|P(e)| \leq k$ and $|H(e)| \leq k - 1$. First suppose that

$B[e]$ is not secure. Thus, the only colors that Alice must avoid are those used on edges in $R(e)$. So we have that

$$
\begin{aligned}
|X(e)| = \left|X - \overline{X}(R(e))\right| \\
\geq \Delta + k - j - |R(e)| \\
= \Delta + k - j - |P(e) \cup H(e)| \\
\geq \Delta + k - j - (2k - 1) \\
= \Delta - k - j + 1 \\
\geq 1.
\end{aligned}
$$

If $B[e]$ is secure, then we must show that the number of distinctly colored edges in $R(e) \cup B(e)$ is less than $\Delta + k - j$. Let m be this number. First note that since $B[e]$ is secure, then within $B(e)$, at least $k + j - 1$ edges are colored the same. So

$$
\begin{aligned}
m \leq |R(e) \cup B(e)| - (k + j - 2) \\
= |P(e) \cup B(e)| + |H(e)| - (k + j - 2) \\
\leq \Delta - 1 + (k - 1) - (k + j - 2) \\
= \Delta - j \\
< \Delta + k - j.
\end{aligned}
$$

Thus, every uncolored edge has a legal eligible color. Suppose Alice is about to color edge e with $\alpha \in X$. We consider the defect of both e and any α-colored neighbors of e. We will show that immediately after Alice colors e, none of these edges has defect greater than $h(k, j)$.

Claim 1. *If $e \in U$, then e has at most $2k$ children colored α.*

Proof. Let $e' \in S(e) \cap C_\alpha$. Note that $\alpha \in X(e)$ implies that α is not used on any edge in $H(e)$. When e' is activated or colored, Alice does not skip, as this would require that both e' and $f(e') \in H[e]$ are colored α. Also, Alice does not slide, since e is a candidate for $f(e)$. Thus, Alice takes action in $H[e]$. So we have that

$$
|S(e) \cap C_\alpha| \leq 2|H[e]| \leq 2k.
$$

$\square$

Claim 2. *If $e \in U$, then e is incident with at most $3k + j - 2$ edges colored α.*

Proof. Since $\alpha \in X(e)$, then $|R(e) \cap C_\alpha| = 0$. So it will suffice to show

$$
|S(e) \cap C_\alpha| + |B(e) \cap C_\alpha| \leq 3k + j - 2.
$$

By Claim 1, we have that

$$
|S(e) \cap C_\alpha| \leq 2k.
$$

Now, if $B[e]$ is secure, then $|B(e) \cap C_\alpha| = 0$. Otherwise, there are at most $j + k - 2$ edges in $B(e)$ colored α. Thus, in either case,

$$|B(e) \cap C_\alpha| \leq j + k - 2.$$

Hence,

$$|S(e) \cap C_\alpha| + |B(e) \cap C_\alpha| \leq 3k + j - 2,$$

as desired. □

By Claim 2, immediately after Alice colors e with α, we have that $\text{def}(e) \leq 3k + j - 2 < h(k, j)$. We will now consider the defect of a child of e already colored α.

Claim 3. *If $g \in S(e) \cap C_\alpha$, then before e is colored $\text{def}(g) \leq 2k^2 + 4k - 1$.*

Proof. Observe that since $B(g) \subset S(e)$ and $g \notin D(g)$, Claim 1 implies that $|D(g) \cap B(g)| \leq 2k - 1$. We also have that $P(g) = H[e]$, implying that $P(g) \cap D(g) = \emptyset$. Let $|D(g) \cap H(g)| = a$, noting that $0 \leq a \leq k - 1$, by our choice of L. Thus we have that

$$|D(g)| = |D(g) \cap P(g)| + |D(g) \cap B(g)| + |D(g) \cap H(g)| + |D(g) \cap S(g)|$$
$$\leq 2k - 1 + a + |D(g) \cap S(g)|.$$

Thus, it will suffice to show that $|D(g) \cap S(g)| \leq 2k^2 + 2k - a$.

Let $S = D(g) \cap S(g)$. We partition S into $\{S_1, S_2\}$. We define S_1 to be the set of edges e' where Alice responds to the activation of e' by jumping and, therefore, taking action at an edge in $H[g]$. Similarly, S_2 is the set of edges e' where Alice responds to the activation of e' by skipping. Note that Alice will not respond by sliding since e is uncolored. At first, we have that $|S_1| \leq 2k$. However, this would imply that Alice both activates and colors the edges in $H[g]$. But she can color at most one edge in $H[g]$ with α. Since

$$|C_\alpha \cap H(g)| = |D(g) \cap H(g)| = a,$$

we have that $|S_1| \leq 2k - a$.

Now let $e' \in S_2$. Let $Q = \bigcup_{h \in H[g]} P(h)$. Since Alice will skip once e' is activated, she next will take action at an edge in Q. Therefore, as Alice can take action at most twice at any given edge, we have that $|S_2| \leq 2|Q| \leq 2k^2$. Hence, we have

$$|S| = |S_1| + |S_2| \leq 2k^2 + 2k - a,$$

as desired. □

So by Claim 3 once e is colored, $\text{def}(g) \leq 2k^2 + 4k$ for any child of e already colored α. If $j = 1$, then $\text{def}(g) \leq h(k, j)$, since $h(k, 1) = 2k^2 + 4k$. Otherwise, if $j \geq 2$, we note that

$$h(k, j) = 2k^2 + 4k + 2j - 4 \geq 2k^2 + 4k.$$

Thus, in either case, $\text{def}(g) \leq h(k, j)$.

Finally, we consider the effect of coloring e with α on any sibling of e already colored α.

Claim 4. *If $g \in B(e) \cap C_\alpha$, then before e is colored $\operatorname{def}(g) \leq 2k^2 + 4k + 2j - 5$.*

Proof. Note that if $B[e]$ is secure, then $B(e) \cap C_\alpha = \emptyset$, as Alice will not choose to color e with any color already used in $B(e)$. Thus, we may assume that $B[e]$ is not secure. As in Claim 3, it will suffice to consider

$$\operatorname{def}(g) = |D(g)|$$
$$= |D(g) \cap P(g)| + |D(g) \cap B(g)| + |D(g) \cap H(g)| + |D(g) \cap S(g)|.$$

Since $P(g) = P(e)$, we have that $|D(g) \cap P(g)| = 0$. Let $|D(g) \cap H(g)| = a$, noting that $0 \leq a \leq k-1$, by our choice of L. Also, since $B[e]$ is not secure, then $|D(g) \cap B(g)| \leq j + k - 3$. It then suffices to show that

$$|D(g) \cap S(g)| \leq 2k^2 + 3k + j - 2 - a.$$

Let $S = D(g) \cap S(g)$, and partition S into $\{S_1, S_2, S_3\}$. As in Claim 3, we define S_1 to be the set of edges e' which Alice responds to the activation of e' by jumping and, therefore, taking action at an edge in $H[g]$. Similarly, S_2 is the set of edges e' which Alice responds to the activation of e' by skipping. Finally, S_3 is the set of edges e' which Alice responds to the activation of e' by sliding. As in our argument in Claim 3, $|S_1| \leq 2k - a$ and $|S_2| \leq 2k^2$.

Let $e' \in S_3$. Since Alice responds by sliding, she colors an edge in $B(g)$ with α. Including the possible move on which she colors e, we have that $|S_3| \leq k + j - 2$. Hence

$$|S| \leq 2k - a + 2k^2 + k + j - 2 = 2k^2 + 3k + j - 2 - a,$$

as desired. $\qquad \square$

Thus, by Claim 4, once e is colored α, we have that

$$\operatorname{def}(g) \leq 2k^2 + 4k + 2j - 4$$

for any sibling of e already colored α. Note that if $j = 1$, then $h(k, j) = 2k^2 + 4k$, and $\operatorname{def}(g) \leq h(k, j)$. Otherwise, if $j \geq 2$, we have $h(k, j) = 2k^2 + 4k + 2j - 4$, and again we have that $\operatorname{def}(g) \leq h(k, j)$.

We make two final observations. First, Bob can always borrow this strategy to find a legal move. Second, if $d > h(k, j)$ and an edge e eventually has defect at least d then it must be through the actions of Bob that this has occurred. At the time an edge e is uncolored, the above arguments show that it is possible to color e with an eligible color α such that coloring e did not increase the defect of any edge e' where $\operatorname{def}(e') > h(k, j)$. Thus, every edge will eventually be colored, and Alice will win the game. $\qquad \square$

We note that Theorem 10 generalizes Theorem 8 (Theorem 5 in [10]). Additionally, while Theorem 10 includes forests when $k = 1$, Theorem 9 provides a better bound. Hence, we suspect strongly that there may be room for improvement in the bound given by $h(k, j)$ in Theorem 10.

Using the fact that outerplanar graphs are 2-degenerate and planar graphs are 5-degenerate, we have the following corollaries to Theorem 10:

Corollary 11. *Let G be an outerplanar graph with $\Delta(G) = \Delta$ and let $j \in [\Delta + 1]$. Let*

$$h(j) = \begin{cases} 16, & \text{if } j = 1; \\ 2j + 12, & \text{if } j \geq 2. \end{cases}$$

Then $\operatorname{def}'_{\mathrm{g}}(G, \Delta + 2 - j) \leq h(j)$. Moreover, if $d \geq h(j)$ then ${}^{d}\chi'_{\mathrm{g}}(G) \leq \Delta + 2 - j$.

Corollary 12. *Let G be a planar graph with $\Delta(G) = \Delta$ and let $j \in [\Delta + 4]$. Let*

$$h(j) = \begin{cases} 70, & \text{if } j = 1; \\ 2j + 66, & \text{if } j \geq 2. \end{cases}$$

Then $\operatorname{def}'_{\mathrm{g}}(G, \Delta + 5 - j) \leq h(j)$. Moreover, if $d \geq h(j)$ then ${}^{d}\chi'_{\mathrm{g}}(G) \leq \Delta + 5 - j$.

Observe that taken together, Corollaries 11 and 12 generalize, and in fact include as a corollary, Corollary 1 in [10].

4. Future Work

Thinking in terms of Theorem 4 and the work in [15], it is evident that even the game chromatic index is difficult to determine for trees and forests. Thus, determining the tightness of the bounds in Theorems 5, 6, and 9 for the defect remain open problems. Additionally, the fact that the bound for the defect in Theorem 10 is not as good for trees and forests as in Theorem 9 seems to imply that the results in Theorem 10 have room for improvement. Our goal in this paper was first to show that such a bound exists. Future work will ideally lead to tighter bounds.

Finally, there are many properties of $\chi'_{\mathrm{g}}(G)$ and ${}^{d}\chi'_{\mathrm{g}}(G)$ that remain to be studied. As discussed in the Introduction, both $\chi_{\mathrm{g}}(G)$ and ${}^{d}\chi_{\mathrm{g}}(G)$ have somewhat unexpected non-monotone properties, both in terms of subgraphs and increasing the defect. It remains open to determine whether this is also the case with edge coloring. We suspect the answer is yes, but have yet to determine appropriate examples. Related to this question, it would be interesting to determine whether the difference between $\chi'_{\mathrm{g}}(G)$ and ${}^{d}\chi'_{\mathrm{g}}(G)$ can be bounded by a function of d, or if ${}^{d}\chi'_{\mathrm{g}}(G)$ can be bounded by other graph parameters. While some of these questions have been explored for $\chi_{\mathrm{g}}(G)$, very little, if any, has been studied for the relaxed version of the games, either with vertex coloring or edge coloring.

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