Quercus: Linfield Journal of Undergraduate Research

Volume 1 Article 3

2012

Tilings with T and Skew Tetrominoes

Cynthia Lester
Linfield College

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Acknowledgements

This research was supported by the Linfield Faculty-Student Collaborative Research Grant.

1 Introduction

In this paper we answer two tiling questions involving the tiles appearing in Figure 2. In particular, we prove the following result: An $a \times b$ rectangle can be tiled by the set T (Figure 2) if and only if $a, b \geq 4$ and either one side is divisible by 4, or $a, b \equiv 2 \pmod 4$ and $a + b > 16$. We prove this result in section 2 after providing background and context for this work. In section 3 we tackle related tiling questions involving modified rectangles.

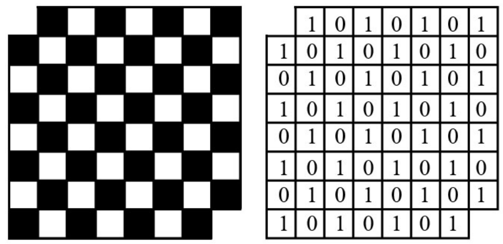
In the last several decades, tiling problems have been attracting the attention of mathematicians. Their allure is easy to understand: the questions are often simple and tangible, but the answers may require abstract mathematics. Most tiling questions take place in the integer lattice, i.e. the tiles and regions are both made of squares, like those on graph paper. One pop culture example of a tiling problem is the game Tetris, where instead one tries to completely fill the region. We say a region R is *tileable* by a tile set T and that T *tiles* R if R can be covered without gaps or overlaps by at least some of the tiles in T and all tiles used to cover R are contained in the region.

Tiling questions usually appear in the form: Can a region of some finite dimension be tiled with a given tile set? If a region can be tiled, the proof of this could be as simple as providing a tiling of the region; however, this does not imply that finding a tiling of the region is easy. If the region cannot be tiled, then the question becomes: How does one prove a region is untileable? Obviously, going through every possible tiling of the region would be tedious to both read and write. A few useful ways to prove the nonexistence of a tiling for a given region are local considerations, coloring arguments, and tile invariants.

Local considerations are physical constraints specific to a region that make it untileable by a given set. For example, an $a \times b$ rectangle missing one corner can obviously not be tiled with copies of a 2×2 square because of the missing corner (consider trying to tile the squares near the missing corner). Although local considerations can be obvious, they can also be complicated or tedious to prove.

Coloring arguments have a rich history and typically involve modular arithmetic. For example, in 1958 George Gamow and Marvin Stern posed in [2] the following well-known question: Can dominoes tile a chessboard whose upper-left and lower-right corners have been removed? If one colors a chessboard in the normal way, see Figure 1(a), then no matter where a domino is placed it covers one black square and one white square; however, this chessboard has two more black squares than white squares and therefore dominoes can never tile this region.

In the previous example, Figure 1(a), consider replacing the white squares with 0 and the black squares with 1, as in Figure 1(b). Now each domino will sum to 1 (mod 2) regardless of where it is placed on the modified chessboard; however, the region sums to 0 (mod 2). Suppose the region is tileable. Since the region has 62 squares, it must be covered by 31 dominoes. Since each domino sums to 1 (mod 2) and the region must use 31 dominoes, then the sum of the region is $31 \equiv 1 \pmod 2$; this contradicts that the region sums to $0 \pmod 2$



(a) A coloring argument. (b) A coloring argument using modular arithmetic.

Figure 1: Can dominoes tile this region?

and therefore the region cannot be tiled by dominoes.

There is only one requirement for a coloring argument: a tile in the tile set must sum to the same number modulo n, for some fixed natural number n, no matter where it is placed on the coloring. In the domino example, every domino uses exactly one black and one white square, or sums to 1 modulo 2, no matter where it is placed on the modified chessboard. Obviously, coloring arguments depend on the tile set, but they can also depend on the region.

Tile invariants also depend on both the tile set and the region. If all the tiles are made of n squares, then any region tileable by that set must have an *area* (the number of squares in the region) divisible by n. Since every region has a constant area, the number of tiles used in a tiling of the region is an invariant; specifically called the *area invariant.* The following is an example of how the area invariant can easily prove a region is untileable by a set: Let the 1×7 rectangle be the region and the 1×2 rectangle be the tile. There are seven squares in the region and two in the tile. Since two does not divide seven then this tile cannot tile the region.

All our regions and tiles live in the integer lattice and therefore are called polyominoes. A *polyomino* is a finite set of squares in the integer lattice such that each square shares at least one edge with at least one other square in that set; for example, all of the tiles in Tetris are polyominoes. Additionally, a polyomino with n squares is called a *n-omino.* In this paper we deal with *4-minos*, which are commonly referred to as *tetrominoes.*

Ribbon tiles of area n are tiles that are built sequentially from one square and naturally correspond to a binary signature of length $n - 1$. Ribbon tiles have been studied by such mathematicians as Conway and Lagarias [1], Pak [4], and Sheffield [6]. Additionally, these mathematicians have showcased many of the nice properties of ribbon tiles. Since ribbon tiles have been significantly explored, we specifically chose a tile set of mostly non-ribbon tiles.

2 Rectangles

In our tile set T, shown in Figure 2, the tiles τ_1 through τ_4 are called skew tetrominoes and the tiles τ_5 through τ_8 are called T-tetrominoes. Although subsets of T have been looked at in reference to rectangular regions, the full set T has not. Walkup proved in [7] that if an $a \times b$ rectangle can be tiled by the four T-tetrominoes then both a and b must be divisible by four. Additionally, in [3] Korn looked into tiling rectangles with various tile sets that contain the four T-tetrominoes.

Here we look at tiling rectangles with more than just the four T-tetrominoes. The set T could be thought of as just three tiles, τ_1, τ_3, and τ_5, and all possible rotations (e.g. 90°, 180°, or 270°) of these three tiles. When viewed this way it is easy to see the symmetry of this set. Moreover, there are some specific symmetries to notice: T rotated 90°, reflected over a 45° line, or flipped vertically/horizontally is still the set T. This means that T tiles an $a \times b$ rectangle if and only if T tiles a $b \times a$ rectangle.

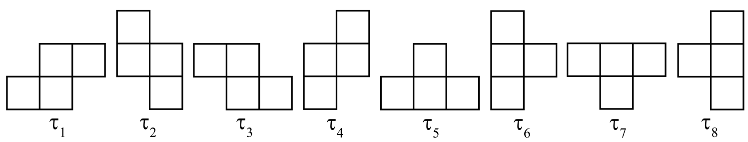


Figure 2: Tile Set T.

We define a natural partition of T: the horizontal tiles and the vertical tiles. The horizontal tiles, τ_{2n+1}, are tiles that are wider than they are tall. Similarly, the vertical tiles, τ_{2n}, are the tiles that are taller then they are wide. Notice that the horizontal tile set rotated 90° is the vertical tile set and vice versa.

Theorem 1 *An $a \times b$ rectangle can be tiled by the set T if and only if $a, b \geq 4$ and either*

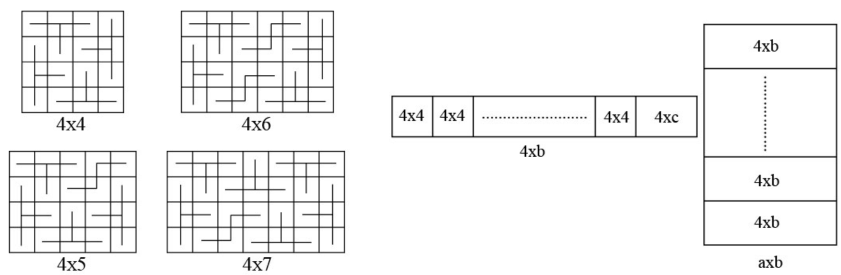
1. *one side is divisible by 4, or*
2. *$a, b \equiv 2 \pmod 4$ and $a + b > 16$.*

Since each tile in T has four squares, any rectangle with an area not divisible by four clearly cannot be tiled by T. So we restrict our attention to rectangles with areas divisible by four, which occurs when either one side is divisible by four or both sides are even. Notice that the second condition of Theorem 1 specifically excludes the 6×6, 6×10, and 10×6 rectangles.

Lemma 1 *Let $a, b \geq 4$. If either a or b is divisible by 4, then an $a \times b$ rectangle is tileable by T.*

Proof. Let a be divisible by 4. Suppose $a = 4$. If $4 \leq b \leq 7$, then $4 \times b$ is tileable by T, as shown in Figure 3(a). Let $b > 7$ and $c \in \{4, 5, 6, 7\}$, then $b = 4n + c$

for some $n \geq 1$ and some c. Now a $4 \times b$ rectangle can be decomposed into n 4×4 rectangles and one $4 \times c$ rectangle, as shown in Figure 3(b). Therefore if $b \geq 4$, then a $4 \times b$ rectangle can be tiled by T. Suppose $a > 4$, or $a = 4m$ for some $m > 1$. Then an $a \times b$ rectangle can be broken into m $4 \times b$ rectangles, see Figure 3(b), and thus $a \times b$ is tileable by T. If b is divisible by 4, then the previous arguments (after a 90° rotation) imply that T tiles $a \times b$. $\square$



(a) If $a = 4$ and $4 \leq b \leq 7$.

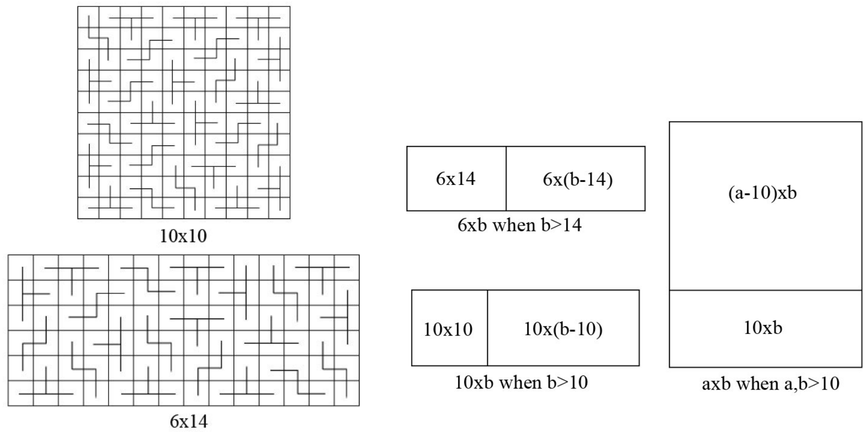
(b) The division of $a \times b$ rectangles into smaller rectangles.

Figure 3: Picture arguments for Lemma 1.

Lemma 2 *Let* $a, b \geq 4$. *If* $a, b \equiv 2$ *(mod 4) and* $a + b > 16$, *then an* $a \times b$ *rectangle is tileable by T.*

Proof. Let $a \leq b$. Then the smallest rectangles that satisfy the assumptions are the 10×10 and 6×14, which Figure 4(a) shows are tileable by T. If $b > 14$ then any $6 \times b$ rectangle can be broken down into a 6×14 and a $6 \times (b - 14)$ rectangle, as in Figure 4(b). By Lemma 1, since four divides $b - 14$ (as both b and 14 are congruent to 2 modulo 4), then the $6 \times (b - 14)$ and hence $6 \times b$ rectangle is tileable by T. Similarly, If $b > 10$, then a $10 \times b$ can be divided into two smaller rectangles, see Figure 4(b), both of which are tileable by T. Now if $a, b > 10$, then an $a \times b$ rectangle can be divided into two smaller rectangles, see Figure 4(b), which are tileable by Lemma 1 and previous parts of this proof. Therefore when $a \leq b$, an $a \times b$ rectangle is tileable by T. When $a \geq b$, simply rotating the previous arguments 90° proves the hypothesis. $\square$

Notice that Lemma 1 and Lemma 2 prove one direction of Theorem 1. Now we need to show that these are the only cases for when T can tile an $a \times b$ rectangle. We break the second direction of Theorem 1 into the next three Lemmas. We say that a tile *covers* a square, or is *placed* on a square, if that square is part of the tile. In order to generalize arguments, trominos, or 3-ominoes, may be used to represent three of the four squares that a tile must



(a) The 10×10 and 6×14 rectangles.

(b) Divisions of an $a \times b$ rectangle.

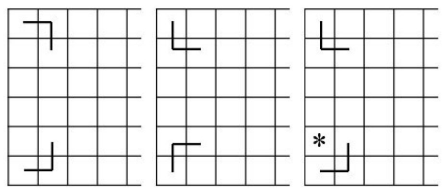
Figure 4: Picture arguments for Lemma 2.

cover. Additionally, two tiles are *adjacent* if they share an edge in the integer lattice.

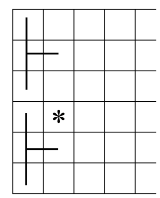
Lemma 3 *If either $a < 4$ or $b < 4$, then an $a \times b$ rectangle is not tileable by T.*

Proof. Suppose that $a < 4$. Obviously if $a = 1$, then no tile from T can be fully contained in this rectangle. Suppose $a = 2$. Consider the left most edge of the rectangle. If the rectangle is tileable then a tile must be place long this edge; however, only horizontal tiles can be placed along this edge and each leaves one of the left corner squares untileable. Now suppose $a = 3$. Consider the lower left corner square. There are only four tiles that can be placed in this square: τ_1, τ_4, τ_5, and τ_6. Clearly, if τ_4 is placed in this corner, then it forces the upper left corner square to be untileable. If either τ_1 or τ_5 is placed in this corner, then the two other squares along the left edge are untileable. If τ_6 is placed in the lower left corner square, then any tile placed adjacent to τ_6 will force an untileable region, which will be obvious by inspection. Therefore $a \times b$ is not tileable by T. Similarly if $b < 4$, then rotating all arguments 90° shows that $a \times b$ is not tileable by T. □

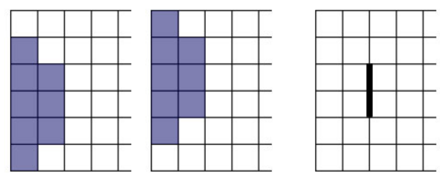
After the previous Lemma, only three rectangles remain to be proven untileable: the 6×6, the 6×10, and the 10×6 (this last case will follow directly from the untileability of the 6×10 rectangle). Consider the two left corners of a $6 \times b$, $b \geq 4$ rectangle. These two corners must be covered by two tiles, if the rectangle is tileable. There are three possibilities for these two tiles, either they are both horizontal tiles, both vertical tiles, or one is a horizontal tile and the other is a vertical tile; Figure 5(a) shows the three squares each tile must cover in the respective three cases.



(a) The three tile possibilities for the two left corners of a $6 \times b$ rectangle, up to flips and rotations. From left to right: both horizontal tiles, both vertical, one horizontal and one vertical.



(b) Using two τ_6's in the left corners. Notice that * cannot have a tile covering it without creating an untileable area.



(c) The two possible placements for the sombrero along the left edge of a $6 \times b$ rectangle. The third picture is the wall created by the two possible placements of the sombrero.

Figure 5: Arguments for the sombrero requirement along the left side of a $6 \times b$ rectangle.

If two horizontal tiles are used in the two left corners, then by inspection the four remaining squares along the left edge cannot all be tiled. If two vertical tiles are used, then they must both be τ_6, otherwise obvious tiling issues occur; however, using two τ_6's forces an adjacent square to be untileable, see Figure 5(b). So one corner must use a horizontal tile and the other a vertical tile, as seen in the right-most picture of Figure 5(a) (up to flips and rotations). Then the square marked * in Figure 5(a) must be covered by a vertical tile. This implies that exactly two adjacent vertical tiles must be used against the left edge of the rectangle (anymore would create overlapping tiles) and exactly one must be in a corner. Therefore, there are only two ways to tile this left edge: using one horizontal tile, a τ_6, and either a τ_2 or a τ_4. By inspection, the two vertical tiles used in these two tilings of the left edge create a "sombrero" (Figure 5(c)), which must be placed along the left edge. This sombrero obviously has only two possible placements and forces a wall, as seen in Figure 5(c), call it the sombrero wall.

A segment of finite length is called a *wall* if there is no valid tiling of the region in which a tile crosses that segment. In other words, if the $6 \times b$ rectangle is tileable, then no tile crosses the sombrero wall. Notice that the symmetry of this tile set implies that a sombrero must be placed along the right edge of the rectangle and hence a sombrero wall must also exist on the right side of the rectangle. Similarly, the symmetry of this set implies that sombreros must be placed along the lower and upper edges and sombrero walls must exist in the upper and lower regions of any $a \times 6$, $a \geq 4$ rectangle.

Lemma 4 *The* 6×6 *rectangle is not tileable by T.*

Proof. Suppose that the 6×6 rectangle is tileable by T. Then there are four forced sombrero walls in this rectangle; however, Figure 6 shows that these four walls separate the rectangle into two independent areas, where the inside area is a 2×2 rectangle. By Lemma 3, we cannot tile the 2×2 rectangle and therefore cannot tile the 6×6 rectangle with T. □

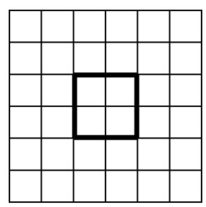
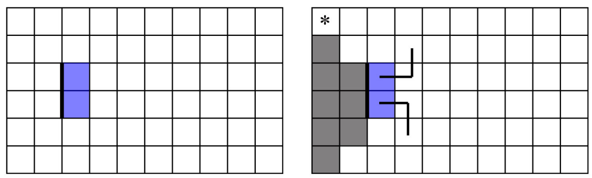


Figure 6: Picture argument for Lemma 4. The 6×6 rectangle with the four forced sombrero walls.

A tile, part of a tile, or tile placement/position is *forced* if there exists a square such that no other tile in T, or placement of said tile, can cover the square without creating an obvious untileable area. In pictures, such a square will be labeled with a number and its respective forced tile will be placed in its forced position. If a square is labeled with number n, then the forced placement of that tile is based on the previous forced placement of tiles with squares numbered strictly less than n. In other words, forced tiles (or parts of tiles) are sequential in nature and are forced based on other tiles that have been already placed (either due to lower numbers or some original tiling choice) in the region.

Lemma 5 *The* 6×10 *and* 10×6 *rectangles are not tileable by T.*

Proof. Due to the symmetry of the tile set, it suffices to show that the 6×10 rectangle is not tileable by T. Suppose the 6×10 rectangle is tileable. Consider the two squares to the right of the left sombrero wall, see Figure 7(a). If both squares are covered by one tile, the tile must be either τ_2, τ_4, or τ_6. Otherwise, the two squares are covered by two tiles. Suppose they are both covered by horizontal tiles. By inspection, the left most square in each tile must be the one adjacent to the wall and the tiles must be placed as seen in Figure 7(b), otherwise obvious untileable areas arise. Then wherever the sombrero is placed, the corner not occupied by the sombrero (marked by * in Figure 7(b)) cannot be tiled without creating an obvious untileable area. Thus if two tiles are used to cover the two shaded squares in Figure 7(a), then one of the tiles must be a vertical tile; the only type of vertical tile that can do this without creating an obvious tiling problem (after a sombrero is placed) is a vertical skew, i.e. τ_2 or τ_4. In sum, either a vertical skew is used to cover at least one of the squares adjacent to the right side of the left sombrero wall, or τ_6 is used to cover both of the squares; any other tile placed in one of these squares will create an obvious untileable region.



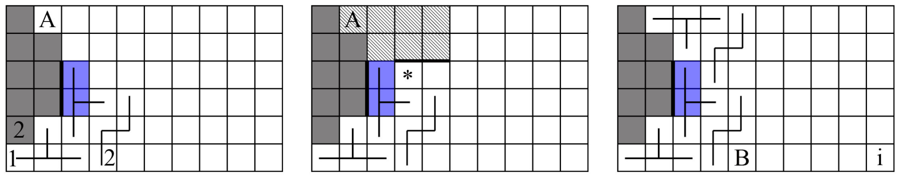
(a) The two squares right of the left sombrero wall.

(b) The placement of two horizontal tiles adjacent to the sombrero wall.

Figure 7: Determining what kind of tiles can be placed in at least one of the squares to the right of the left sombrero wall.

Suppose τ_6 is used to cover both squares, which can be seen in Figure 8(a) along with the tiles it forces, up to any flips and rotations. Consider which horizontal tiles can be placed in the square marked A of Figure 8(a) and which tiles can be placed adjacent to it. If those two tiles do not cross the horizontal wall seen in Figure 8(b), then the square marked * in Figure 8(b) cannot be covered. Hence the scenario shown in Figure 8(c) is forced. Now there are only two possibilities for what can cover the square marked B in Figure 8(c): τ_1 or τ_5. Depending on whether or not a sombrero is placed in square i and which tile is chosen to cover B, there are only two scenarios that do not lead to obvious tiling problems. These scenarios are shown in figures 8(d) and 8(e); each scenario forces tiles and leaves a square marked * untileable.

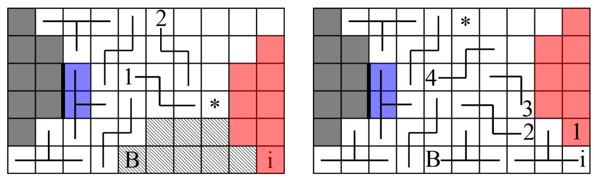
This means that τ_6 cannot cover both of the squares to the right of the left sombrero wall in Figure 7(a). Thus a vertical skew must be used to cover at least one of the two squares shown in Figure 7(a). Up to flips and rotations there are only two possible scenarios for this skew's placement, see Figure 9(a), which do not create an obvious untileable region; however, both force the creation of another wall, call this wall the skew wall, see Figure 9(b). Now by flipping all of the the previous arguments vertically (i.e. along a vertical axis), there is another forced skew wall in the 6×10 rectangle, shown in Figure 9(c). The area between these two skew walls, specifically the two squares marked by * in Figure 9(c), obviously cannot be tilled by T. Therefore the 6×10 and hence 10×6 cannot be tiled by T. $\square$



(a) The tiles that are forced if τ_6 is used to cover both blue squares, up to flips and rotations.

(b) A scenario which cannot happen.

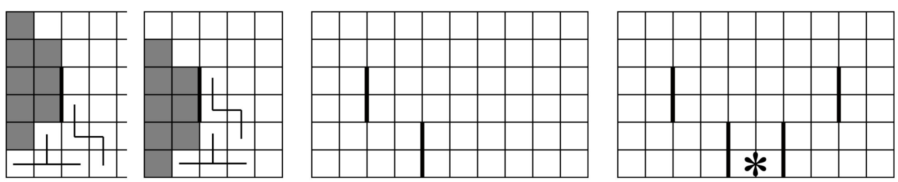
(c) All of the forced tiles based on figure 8(b).



(d) One possibility for a tile covering square B.

(e) The other possibility for a tile covering square B.

Figure 8: Arguments against τ_6 covering both squares.



(a) The two possibilities, up to flips and rotations, for the placement of a vertical skew next to the sombrero wall.

(b) Another wall, created by the possible placements of the vertical skew.

(c) The four walls in the 6×10 rectangle

Figure 9: The forced walls in the 6×10 rectangle, up to rotations and flips.

3 Modified Rectangles

A modified rectangle, also called a mutilated rectangle or M(a,b), is simply an $a \times b$ rectangle with both the upper-left and lower-right corners removed (this type of region was looked at in an unpublished paper by Hitchman and Coate). When these modified rectangles have the right area, they are tileable by T; this will end up being a corollary to Theorem 2 and Theorem 3, which are proved later in this section. This section, however, specifically focuses on two subsets of T, which partition T, call them L_1 (Figure 10) and L_2 (Figure 15). These two subsets have a special property, namely when either is rotated 90°, it is equivalent to the other tile set.

3.1 Tiling with L_1

For what values of a and b can we tile M(a,b) with the tile set L_1 (Figure 10)?

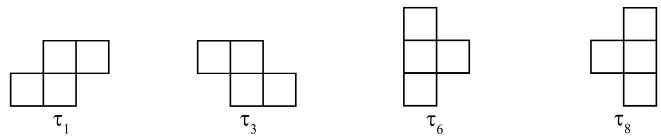


Figure 10: Tile set L_1.

Theorem 2 *Let a and b be integers strictly greater than 1. Then the set L_1 tiles M(a,b) if and only if either*
1. *$a \equiv 2 \pmod 4$ and b is odd, or*
2. *$b = 2$ and $a \equiv 1 \pmod 4$.*

Since all the tiles in L_1 use four squares, then four must divide the area of M(a,b), or $4 \mid (ab - 2)$, if M(a,b) is tileable by L_1. The only two cases of when $4 \mid (ab - 2)$ are as follows:
1. $a \equiv 2 \pmod 4$ and b is odd, or
2. a is odd and $b \equiv 2 \pmod 4$.

Obviously we restrict our attention to modified rectangles with areas divisible by four. Notice that Theorem 2 says that T cannot tile M(a,b) if $a \equiv 3$ (mod 4) or $a \equiv 1 \pmod 4$ and $b > 2$. Theorem 2 will be proved with the following four lemmas.

Lemma 6 *If $a \equiv 2 \pmod 4$ and $b > 1$ is odd, then L_1 tiles M(a,b).*

Proof. Notice that the smallest such modified rectangle, M(2,3), is actually the tile τ_1. Suppose that the statement holds for M(2,b-2) and that $b > 3$. Then M(2,b) can be divided into two sections, M(2,b-2) and M(2,3), see Figure 11(a). Therefore, M(2,b) is tileable by L_1 for all odd values of b greater than 1. Now suppose that the statement is true for M(a-4,b) and that $a > 2$. Then M(a,b) can be divided into three sections, M(a-4,b), S(b), and one τ_8, see Figure 11(b). So it suffices to show that if $b > 1$ is odd then S(b) is tileable by L_1. When $b = 3$ Figure 11(c) shows that L_1 does in fact tile S(3). Additionally, if the statement holds for S(b-2) and $b > 3$, then S(b) can be divided into three tileable sections: S(b-2), τ_1, and τ_3. □

Lemma 7 *If $a \equiv 1 \pmod 4$, $a \geq 5$, and $b = 2$ then L_1 tiles M(a,b).*

Proof. The smallest such modified rectangle, M(5,2), can be tiled by one τ_6 and one τ_8, as seen in Figure 12. Suppose the statement holds for M(a-4,2) and

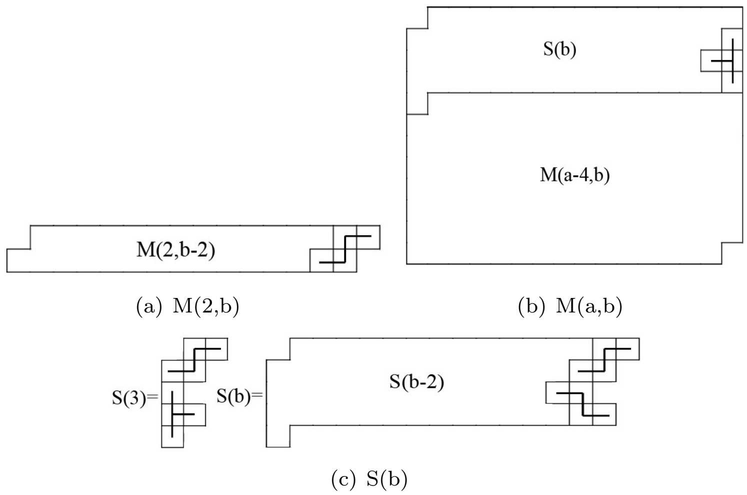


Figure 11: Picture arguments for Lemma 6.

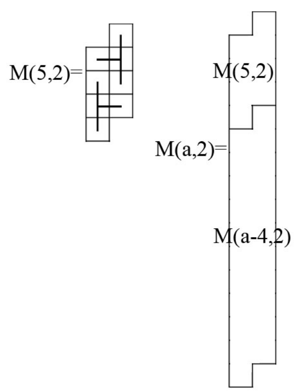


Figure 12: Picture arguments for Lemma 7.

$a > 5$. Then M(a,2) can be divided into two sections, M(a-4,2) and M(5,2), shown in Figure 12, which are both tileable. □

Notice that Lemma 6 and Lemma 7 prove one direction of Theorem 2. Now we show that these are the only two cases for when L_1 tiles M(a,b). If a or b equalled 1, then M(a,b) would be a rectangle, which by Lemma 3 cannot by tiled by T and hence cannot be tiled by L_1. Therefore, we only consider cases when both a and b are greater than one.

Lemma 8 *If $a \equiv 3 \pmod 4$ and $b \equiv 2 \pmod 4$ then L_1 cannot tile M(a,b).*

Proof. Consider the following coloring,

$$\begin{array}{ccccc} 3 & 2 & 3 & 2 & 3 \\ 1 & 4 & 1 & 4 & 1 \\ 2 & 3 & 2 & 3 & 2 \\ 4 & 1 & 4 & 1 & 4 \\ 3 & 2 & 3 & 2 & 3 \end{array}$$

Notice that with this coloring all tiles of L_1 sum to 0 (mod 5), no matter where they are placed. This means that if L_1 can tile M(a,b), then the sum of M(a,b) will equal 0 (mod 5), no matter where M(a,b) is placed on the coloring. Now consider the sum of the region M(a,b). Let the coloring defined above be expressed in arbitrary terms,

$$\begin{array}{ccccc} C_1 & C_2 & C_1 & C_2 & C_1 \\ C_4 & C_3 & C_4 & C_3 & C_4 \\ C_2 & C_1 & C_2 & C_1 & C_2 \\ C_3 & C_4 & C_3 & C_4 & C_3 \\ C_1 & C_2 & C_1 & C_2 & C_1 \end{array}$$

Notice that $C_1 + C_2 \equiv 0 \pmod 5$ and $C_3 + C_4 \equiv 0 \pmod 5$. Then the coloring of M(a,b) may be represented by Figure 13, where M(a,b) is separated into three sections A, B, and D. Notice B sums to 0 (mod 5) since all rows in B have an even number of colors, whereas A and D sum to C_1 (mod 5). This means the sum of M(a,b) is $2C_1$ (mod 5), where $C_1 \in \{1, 2, 3, 4\}$. Since for all values of C_1, $2C_1 \neq 0 \pmod 5$, then L_1 cannot tile M(a,b). □

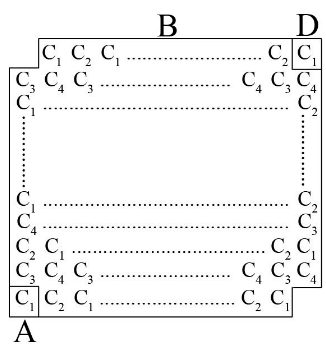


Figure 13: An arbitrary coloring of M(a,b), where M(a,b) has been divided into three sections, A, B, and D.

Lemma 9 *If $a \equiv 1$ (mod 4), $b \equiv 2$ (mod 4), and $b > 2$, then L_1 cannot tile M(a,b).*

Proof. Consider the following coloring,

$$\begin{array}{ccccc} 2 & -1 & 2 & -1 & 2 \\ -2 & 1 & -2 & 1 & -2 \\ -1 & 2 & -1 & 2 & -1 \\ 1 & -2 & 1 & -2 & 1 \\ 2 & -1 & 2 & -1 & 2 \end{array}$$

Notice that with this coloring all tiles of L_1 sum to zero, no matter where they are placed. Now consider the sum of the region M(a,b). Let the coloring defined above be expressed in arbitrary terms,

$$\begin{array}{ccccc} C_1 & C_2 & C_1 & C_2 & C_1 \\ C_4 & C_3 & C_4 & C_3 & C_4 \\ C_2 & C_1 & C_2 & C_1 & C_2 \\ C_3 & C_4 & C_3 & C_4 & C_3 \\ C_1 & C_2 & C_1 & C_2 & C_1 \end{array}$$

Notice that $C_1 + C_2 + C_3 + C_4 = 0$. The coloring of M(a,b) may be represented by Figure 14, where M(a,b) is separated into two sections A and B. Notice A has the same number of each color in each column and therefore sums to zero. On the other hand, B is an alternating row of either 2 and -1 or 1 and -2. Since $b > 2$ and b is even, then B will never sum to zero. Therefore L_1 cannot tile M(a,b). □

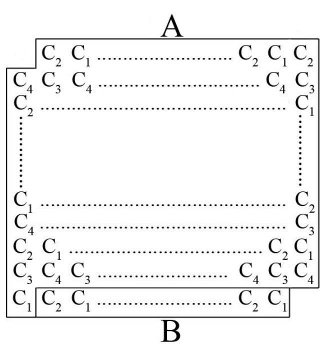


Figure 14: An arbitrary coloring of M(a,b), where M(a,b) has been divided into two sections, A and B.

3.2 Tiling with L_2

Consider the subset of T called L_2, as defined in Figure 15. For what values of a and b does this set tile M(a,b).

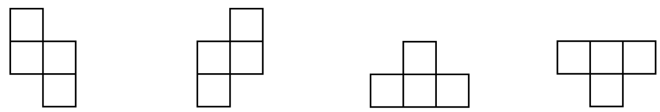


Figure 15: Tile set L_2.

Theorem 3 *Let a and b be integers strictly greater than 1. The set L_2 tiles M(a,b) if and only if either*

1. $b \equiv 2 \pmod 4$ and a is odd, or

2. $a = 2$ and $b \equiv 1 \pmod 4$.

Proof. Notice that the set L_2 reflected over a 45° line is the set L_1 and vice versa. Additionally, M(a,b) reflected over this same line is M(b,a). Then Theorem 2 completes this proof. □

Corollary 1 *T tiles M(a,b) if and only if 4 divides ab-2.*

Proof. Since all tiles in T use four squares, then T tiling M(a,b) implies that four divides the area of M(a,b), i.e. 4 divides $ab - 2$. Now notice that when M(a,b) has the right area, Theorem 2 and Theorem 3 state that M(a,b) can be tiled by either L_1 or L_2. Since L_1 and L_2 are subsets of T whose union is T, then 4 dividing $ab - 2$ implies that T tiles M(a,b). □

References

[1] J. H. Conway and J.C. Lagarias, Tiling with polyominoes and combinatorial group theory, *Journal of Combinatorial Theory*, 53: 183-208 (1990).

[2] G. Gamow and M. Stern, *Puzzle-Math*, Viking Press, New York, 1958.

[3] M. Korn, *Geometric and algebraic properties of polyomino tilings*, Ph. D Thesis, Massachusetts Institute of Technology, Cambridge MA, June 2004.

[4] C. Moore and I. Pak, Ribbon tile invariants from signed area, *Journal of Combinatorial Theory*, 98: 1-16 (2002).

[5] J. Propp, A pedestrian approach to a method of Conway, or, A tale of two cities, *Mathematics Magazine*, 70(5): 327-340, 1997.

[6] S. Sheffield, Ribbon tilings and multidimensional height functions, *Transactions of the Amer. Math. Society*, 354(12): 4789-4813 (2002).

[7] D. W. Walkup, Covering a rectangle with T-tetrominoes, *Amer. Math. Monthly*, 72: 986-988, 1965.